Spontaneous generation of geometry and its possible consequences

Daniel Puigdomènech Bourgon

Spontaneous generation of geometry and its possible consequences

Daniel Puigdomènech Bourgon

Departament d'Estructura i Constituents de la Matèria
i
Institut de Ciències del Cosmos

Universitat de Barcelona

Advisor: Prof. Domènec Espriu

A thesis submitted for the degree of

PhilosophiæDoctor (PhD) in Physics
(Programa de Doctorat en Física)

September 2012

Abstract

This thesis is composed of two distinct but related research topics. In the first one, a seemingly viable theoretical framework by which gravity emerges dynamically using the mechanism of spontaneous symmetry breaking of some more fundamental theory is developed. We start from a theory without any predefined metric, only an affine connection is included from the start. We demand invariance of the theory under a global $SO(D) \times GL(D)$ symmetry. The relevant degrees of freedom are those of two non-standard species of fermions coupled to the (spin) affine connection. We show that whenever these fermions condensate in a bilinear acquiring a vacuum expectation value different from zero, the original symmetry of the theory is spontaneously broken down to the diagonal subgroup $SO(D)$ and Goldstone bosons appear as a result. The exact value of the vacuum is computed for vanishing connection and the equations of motion investigated. Then we allow small perturbations above this vacuum and calculate the effective action induced by these perturbations. We show, both for two and four dimensions, that the new degrees of freedom emerging in the effective theory correspond precisely to those of the graviton. The relation between the metric excitations and the spin connection appears as an equation of motion of the model, which is implemented perturbatively considering small deviations with respect to a zero connection.

In the two-dimensional case a heat kernel calculation is attempted to derive the effective action. The final result is proven to be covariant in spite of the apparent loss of covariance in the intermediate steps. Nonetheless, a concrete result cannot be obtained due to the fact that the expansion parameter used in the heat kernel calculation does not univocally correspond to an expansion in derivatives of the fields involved. One would have to

calculate the infinite series of terms to reconstruct the numerical factors in front of each term in the effective action. In light of this, we turn to a one-loop diagrammatic calculation assuming conformally flat perturbations to obtain the final result which corresponds to the Einstein-Hilbert (EH) theory with a cosmological constant plus higher dimensional operators. The number of divergences appears to be very limited, presumably due to the scarcity of counterterms one can write without making use of a metric. Despite not constituting a proof of renormalizability our results point towards this possibility.

In the four-dimensional case, the diagrammatic calculation is extended using a more general perturbation with four degrees of freedom. Making the assumption that divergences are independent of the precise realization of the symmetry (i.e. whether the global symmetry is broken or unbroken in the vacuum), the number of possible counterterms, despite growing with the dimensionality, stays very limited due to the lack of a metric in the unbroken phase. The number of divergences also grows but all of them can still be identified and reabsorbed in the available counterterms. Although the calculations are substantially more complicated in this case the nice properties found in the two dimensional case seem to persist. The resulting effective theory is again that of Einstein-Hilbert with a cosmological constant Λ, the value of which is not determined *a priori* by the model, plus higher dimensional operators. It would be natural to assume the value of Λ to be related to the symmetry breaking scale, however the model possesses enough freedom to fine-tune it to any observed value (the smallness problem derived from the small value of Λ that is preferred by observations is therefore not solved). The divergent pieces combine themselves to reproduce the volume form (as in $D = 2$) and also the curvature term, which in $D = 4$ is expressible entirely in terms of a differential form and hence requires no metric. Thus the curvature term $\sqrt{g}R$ can univocally be reconstructed. Any other divergence found to the order calculated (that is $\mathcal{O}(R^2)$) can be identified with particular pieces of the Gauss-Bonnet term.

The second topic of study is somehow motivated by the natural appearance of the cosmological constant in the effective theory. A non-zero cosmological constant is a desirable thing from the observational point of view but its origin is still unclear. Its dynamical generation in models of emergent gravity, in spite of some claims to the contrary, constitutes a hint that it could be an intrinsic property of space-time itself rather than an effective description at very large scales. Therefore we investigate the effect of Λ in the propagation of gravitational waves (GW) and the possibility of detection of such effects assuming its presence at any scale. We expose a complete study of the wave equation and its solutions in the linearized theory of gravity for different gauge choices when Λ is included. These coordinate choices are studied in full detail. The importance of the number of terms retained in the linearization process is also addressed. The final solutions are expressed in terms of Friedmann-Robertson-Walker (FRW) cosmological coordinates, those in which the universe is isotropic and homogeneous. These are the relevant coordinates for observation. We show how the effects of the cosmological constant, although being very small, are not totally negligible.

Pursuing this line of thought, we finally investigate the observational effects of the cosmological constant in the detection of gravitational waves in pulsar timing arrays (PTA). Using the wave solutions derived for de Sitter space-time we compute the statistical significance of the timing residuals induced by GW, originated in far away violent phenomena, in the measured periods of an array of real pulsars. The results show a dependency of the significance in the angle subtended by the line from the observer (Earth) to the pulsars and the line from the observer to the source. A large enhancement is found for a particular value of the angle in contraposition to the results in flat space-time where no enhancement is observed. The position of this peak depends strongly on the value of the cosmological constant and therefore, although our results are very preliminary, could represent an alternative way of determining the value of Λ while being a direct confirmation of the 'local' existence of the cosmological constant.

A l'Yvette Kosta, per haver estat una peça fonamental del meu esdevenir durant aquesta tesis.

I a la memòria del meu pare, Albert Puigdomènech, els motius són innombrables.

To Yvette Kosta, for being a fundamental part of who I have become during this thesis.

And to the memory of my father, Albert Puigdomènech, the reasons are countless.

Agraïments

Primer de tot m'agradaria agrair a Domènec Espriu el seu guiatge durant aquesta tesi. En un viatge com aquest sempre hi ha bons moments, fins i tot de tant en tant descobreixes alguna cosa, però també hi ha entrebancs i moments difícils. Em plau que la nostra feina plegats hagi fluït d'una manera molt natural durant tot el procés. Ha sigut per mi un treball d'equip i el més important, ha sigut divertit. També m'agradaria agrair a Jorge Alfaro la seva simpatia i hospitalitat. Treballar amb ell a Santiago de Xile va ser una experiència molt positiva i enriquidora. També vull agrair a Pepe Bernabeu la seva col·laboració amb nosaltres. Aquesta col·laboració em va permetre ampliar la visió que tenia del camp i treballar en temes interessants i diferents dels que havíem estat fent fins aquell moment.

Arribats a aquest punt és just recordar tota la gent que d'una manera o altra m'ha ajudat a arribar a aquest dia. Jaume Carbonell del LPSC de Grenoble per deixar-me jugar per allà molt abans que jo tingués la més remota idea de què era la recerca. Mees de Roo per un magnífic any a Groningen on vaig aprendre de què anava la física teòrica. Eric Bergshoeff, Elisabetta Pallante i la gent del departament de teòrica de Groningen. També la gent del departament d'ECM, els meus companys de doctorat, Francesco Aprile, Octavi Boada, Jordi Salvadó i tants altres. Realment ha sigut divertit. En Diego Villuendas i en Roger Mor també han contribuït, a la seva manera, a aquesta tesi. Les discussions de física davant d'una canya amb Bruno Juliá són realment apreciades.

Finalment vull agrair a la meva mare el seu suport incondicional en tot el que faig. Parlar amb ella quan estic capficat sempre em retorna l'equilibri.

Acknowledgements

First of all I would like to specially thank Domènec Espriu for his guidance during this thesis. In such a long journey there are lots of good moments, eventually one even discovers something, but there are also hiccups and difficulties to overcome. I am glad that our work together has flowed very naturally all the way. It felt like a team effort and most importantly it has been fun. I would also like to thank Jorge Alfaro for his kindness and hospitality. Working with him in Santiago de Chile was a very positive and enriching experience. I also want to thank Pepe Bernabeu for his collaboration with us. It allowed me to widen the view of the field and to work in something very interesting and different from what we were doing at that time.

At this point it is only fair to remember all the people that at in a way or another have helped me reach this day. Jaume Carbonell from the LPSC in Grenoble for letting me play around in the LPSC long before I knew anything about research. Mees de Roo for such a good year in Groningen where I really learnt what theoretical physics were about. Eric Bergshoeff, Elisabetta Pallante and the guys at the theoretical physics department in Groningen. Also the people from ECM, my fellow PhD students, Franscesco Aprile, Octavi Boada, Jordi Salvadó, and many others. It really has been fun. Diego Villuendas and Roger Mor also contributed, in their way, to this thesis. The discussions about physics in front of a *caña* with Bruno Juliá are much appreciated.

Finally I would like to thank my mother for being so supportive in everything I do. Talking to her when I am concerned always gives me back my balance.

Contents

Chapter 1

Introduction

Gravitation has ruled the development of the Universe since the very dawn of time. Long before humankind started wondering about the apparent motion of the stars and planets, its machinery was already keeping everything running as we observe it today. The desire to understand the observed phenomena has been present in every generation of mankind; there have always been great minds pushing the boundaries of what was known. Probably among the first documented dissertations about gravity we find the work of Aristotle in the 4th century BC [1]. For him the motion of bodies depended on their composition in terms of the 'elements' and their position tended to the 'natural place', reaching it without need of pull or push. His insight was literally centuries away from our present knowledge but nonetheless it shows a remarkable will to make sense of Nature. In the 7th Century the Indian mathematician Brahmagupta stated 'Bodies fall towards the earth as it is in the nature of the earth to attract bodies, just as it is in the nature of water to flow' [2]. This persistent will to understand Nature led to the first modern attempts to find systematic explanations to the observed behavior of bodies. Galileo Galilei was the first to assert that all bodies are accelerated in the same manner towards the earth, contrary to Aristotelian thoughts [3]. That was the early 16th century. His work was pointing towards the first true understanding of gravitation. However, we had to wait almost a century for Isaac Newton to realize the key was in the relation between the distances of the masses. With the inverse-square law published in the Principia [4], Newton was the first man capable of translating into a single solid mathematical description two apparently very different phenomena such as the fact that apples fall to the ground as well as the fact that the Earth revolves around the Sun. This revolution in the understanding of gravity was bound to endure more than two centuries.

When Albert Einstein wrote in 1915 the theory of General Relativity (GR) he not only made a giant leap forward in the understanding of gravitation, he notoriously changed the rules of the game. Two masses did not exert force on each other, the concept of force being lost. They simply curved the space-time around them making other masses 'roll' along these curves. The notion of space-time, with the inclusion of time as just another dimension of the reality we perceive, was one of the most crucial findings of the 20th century. This change of perspective was so deep and brilliantly confirmed by early observations [5] that still nowadays GR, as originally formulated by Einstein, is the best and most complete description we have at hand of the gravitational interaction. Some of its most striking predictions have been confirmed; no one doubts about the existence of black holes for instance. Some other predictions such as gravitational waves still await for experimental evidence to be confirmed (although indirect evidence from energy balance in some neutron binaries exists [6]).

Of course Einstein did not solve all questions related to gravitation. At a time where the quantum world was quickly unraveling, GR did not provide an easy way to accommodate these new ideas into the gravitational scales. In fact every attempt to find the true quantum nature of gravity has run into trouble. There is no explanation as for why gravitation is so weak compared to the other fundamental interactions. There is ambiguity on whether a cosmological constant, i.e. a vacuum energy, should be included in the formulation and how to ultimately justify its value. And most importantly an ultraviolet (UV) completion for most proposals has been lacking so far. Among these proposals, probably the one that has drawn more attention, and in a way has been the most successful, is string theory. A consistent quantum theory of gravity can be constructed in its framework although at present it seems to be yielding more questions than the ones it aims to answer.

It was the late sixties when physicists started considering the possibility that the difficulties in quantizing gravity were in reality due to the lack of fundamental degrees of freedom to quantize. That the gravitational interaction is not fundamental as such. At the root of some proposals back at that time there was the idea that gravity emerges as a low energy effective theory. Probably at the kick off of these theories we find the work by Zel'dovich and a little later by Sakharov [7]. The first author worked out the effect of the vacuum quantum fluctuations on the cosmological constant resulting in a value different from zero for the latter, and the second one complemented this work by assessing how quantum fluctuations above this non-trivial vacuum in a field theory could in general yield the dynamics of Einstein-like effective theories. The technical difficulties they encountered relented the progress of the field for some years. Meanwhile, already in the early seventies, Salam and coworkers studied conformal group symmetries in the framework of non-linear realizations [8]. Although they did not aim at solving the Quantum Gravity problem, some insight was gained on regarding general covariance as a spontaneously violated symmetry, and in the same sense gravitons as Goldstone bosons. Ogievetsky and coworkers pursued the group theory approach a little after [9]. Using already the analogy with quantum field theories (QFT) such as the meson chiral theory, they were able to prove how theories invariant under certain desirable symmetries (affine and conformal symmetries for instance) could result, after undergoing a spontaneous symmetry breaking, in effective theories whose equations of motion were those of Einstein theory.

The literature is extensive and quite some proposals have seen the light since the seventies. Breaking of different symmetry groups (Lorentz, diffeomorphisms,...) has been studied and the particle yield of the corresponding effectives theories worked out. As common features for any well-behaved QFT following this program we find the generation of a curvature-like term; this is, Einstein-Hilbert (EH); and a cosmological term as well, although this is disputed by some authors (such as Tomboulis [10]). Major stumbling blocks have to do with the interpretation of the theories before the symmetry breaking and with the ultimate UV behavior of the effective theories. Since the eighties, when the Weinberg-Witten theorem was published, even the generation of massless spin-two bosons as Goldstone bosons was questioned as a matter of principle.

As mentioned, some of these proposals have gone so far as suggesting that there are no fundamental degrees of freedom at all associated to the gravitational interaction, gravity being a sort of 'collective' or 'entropic' effect. Some, if not all, of these proposals fail to reproduce the known properties of gravity or to be specific enough to be falsified.

An issue one has to bear in mind is how to justify the inclusion of some predefined metric structure before the symmetry breaking. Most proposals rely on a predefined notion of geometry. Its inclusion eases the calculations and as a result one can obtain effective metrics according to those in GR but the interpretation of this *a priori* input is obscure given that the corresponding degrees of freedom should be available only after the symmetry breaking. Apparently, only Russo and Amati [11] in the early nineties, and also Wetterich [12], proposed a model in which the geometrical degrees of freedom were generated dynamically from a theory without any predefined metric structure. Their results were compelling and already pointed it should be possible to reproduce in a renormalizable fashion the EH action from a fundamental theory with no metric. Their proposals, however, lacked an extensive study of the symmetry breaking mechanism and ultimately of the particular properties of the effective theory obtained, mostly due to their technical complexity. Shortly after the publication of our model, a proposal along these lines by Tumanov and Vladimirov [13] appeared. In a way it is the most similar proposal to ours in the literature, although they include explicitly a vierbein from the start in the fundamental theory.

Following all this knowledge, the point of view adopted in this thesis is to work out a mechanism by which gravity, understood as EH gravity, is consistently obtained from a theory without any predefined metric structure. The real significance of the fundamental theory that yields this result is unknown. As most proposals, we aim at providing a plausible explanation to the 'why' admitting we do not understand or try to make sense of the 'where from'.

The line of thought we use is to exploit the present knowledge of effective quantum field theories developed during decades in the field of particle physics. In particular, we will construct an analog of Quantum Chromodynamics (QCD) and its effective low energy modelization chiral theory. Looking closely at the properties of these theories and studying the mechanism by which the effective action is obtained; that is, spontaneous symmetry breaking, we realize most of the key features present in particle physics allow for an analog in gravitation. Exploiting this analogy to the end we will be able to construct a fundamental theory that will be the equivalent of QCD for gravity, and which yields as an effective theory nothing but EH gravitation (which would be the analog of the effective chiral theory). The guiding principles to construct such a theory will be covariance, locality and renormalization group (RG) relevance. Without the first two no information can be extracted form the calculations and the third one is the guide behind the perturbative calculation. Since we truly want gravity to emerge from scratch, the fundamental theory will not contain a metric whatsoever. All the geometrical degrees of freedom will be obtained dynamically. However, one needs to assume some priors. An affine connection defining parallel transport of vectors on a manifold will be included from the start and later determined via equations of motion in terms of the spontaneously generated metric. This defines the differentiable pseudo-topological manifold constituting our starting point.

A key feature of the program developed in this thesis relies in the impossibility of constructing an unlimited number of counterterms in terms of the fundamental degrees of freedom, which contain no metric before the breaking. This, unavoidably, limits the number of divergences in the broken phase too. And gives a hint of the plausible renormalizability of the theory.

With all these ingredients we carry out a one-loop perturbative calculation obtaining as effective action, both in $D = 2$ and in $D = 4$, precisely EH plus a cosmological constant. So not only GR with no divergences is obtained but also an adjustable cosmological constant emerges in a natural way.

The question of whether the cosmological constant is a part of the Einstein equations traces back to Einstein himself. He famously referred to its inclusion as the 'biggest blunder of my life'. Later on with the confirmation that the universe is exponentially expanding, the presence of a vacuum energy (unavoidable if gravity is emergent) is a convenient way to accommodate the observations.

This is the linking point between the first part of this thesis and the second. Since we naturally generate a cosmological constant it is natural to interpret its presence as something necessary and not merely optional. Under our point of view, the role of the cosmological constant will be fundamental, meaning we will take it to be an intrinsic property of space-time rather than an effective description only relevant at very large scales.

In our work, we will be interested in the effect that Λ has in the propagation of gravitational waves, an ingredient of Einstein theory that has eluded confirmation so far, and in the possibility of assessing its influence in 'local' systems.

The study of the relevance of the cosmological constant in local measurements (meaning measurements that involve sub-cosmological scales) has received growing attention in the last decade. Sereno and Jetzer [14] in 2006 set a loose bound on the value of Λ from the precession of a gyroscope, the change in the mean motion and the periastron shift of a massive body and, finally, gravitational redshift within the solar system. Although not being competitive with other cosmological estimations these bounds suggest the universality of the cosmological constant. Between 2007 and 2009 different groups investigated the influence of Λ in the bending of light from distant objects. Very different results were obtained. First Khriplovich and Pomeransky [15] found no indication of any influence, Sereno [16] concluded one year after that the effect was very small while Rindler and Ishak [17] finally amended the previous work by stating that, although small, the effect was appreciable. Bernabeu and coworkers [18] in 2010 published a study of the linearized Einstein equations in the presence of Λ finding some very interesting solutions that have motivated part of this thesis.

The inclusion of Λ in Einstein equations has an obvious and immediate consequence. Even in the absence of a source it produces a non-trivial curvature of space-time (de Sitter). Therefore, it is expectable that the propagation of waves differs form that of flat space-time (Minkowski). The logic behind the usual treatment of gravitational waves is to perform the study in the linearized version of Einstein equations. That is possible since one considers waves to be a small perturbation above the background. Then one is to make a coordinate choice and solve the equations, which in the linearized version decouple and are easily solvable. When the cosmological constant is added, new terms appear in the linearized equations. How many of this terms must be retained is one of the questions answered in this work. We also perform a careful study of the importance of the coordinate choice, crucial for the present discussion. A gauge choice is needed to solve the equations but the meaning of the different coordinate frames is not always clear. In fact, the only coordinate system we can make sense of is the cosmological one, i.e. that one in which the Universe appears uniform and isotropic. However, this particular choice makes it impossible to linearize the equations in the present set up. The precise meaning of this linearization will be clear later. To solve the equations we must resort to the usual choice, the Lorenz gauge. Or alternatively, to a gauge choice we will name Λ gauge. In these choices the equations are closely related to those of flat space-time and can be solved. It can be shown these coordinates correspond to nothing but different parametrizations of a Schwarzschild-deSitter (SdS) space-time. Once the wave functions are found one would like to transform them into FRW coordinates so we can make observable predictions. The change of coordinates taking the solutions from SdS to FRW is an intricate one but can be worked out nonetheless. The transformed wave functions get modifications in their amplitude and in their arguments proportional to fractional powers of Λ due to the change of coordinates and proportional to integer powers of the cosmological constant due to the higher-order terms retained in the equations. The dispersion relation of the waves is modified acquiring both an effective frequency and an effective wave number*. Their amplitude grows with the distance and they are redshifted as they move away from the source, but in a different way from the usual gravitational redshift of electromagnetic radiation.

Finally we focus on pulsar timing arrays (PTA), one of the most promising methods to obtain the first direct observations of gravitational waves. There are other possible types of experiments capable of obtaining GW signals in the coming years. Ground based GW detectors such as LIGO can reach sensitivities down to $\sim 10^{-23}$ with optimal sensitivity in the region between 10 Hz and 10^3 Hz [19]. The space mission LISA will reach a similar sensitivity in the range 10^{-2} Hz to 10^{-3} Hz but will actually be able to set relevant bounds on a more extended range of frequencies [20] (if this mission eventually flies). The theoretical framework developed in this thesis, however, is aimed to be useful for experiments such as the International Pulsar Timing Array project or the Square Kilometer Array project [21, 22]. These are sensitive to lower frequencies $\nu < 10^{-4}$ Hz, and although for a time their sensitivities only reached $\sim 10^{-10}$ going up to $\sim 10^{-15}$ for $\nu \sim 10^{-10}$ Hz, they are expected to have collected enough data in the upcoming years to improve notoriously this sensitivity.

PTA are suitable detectors for very low frequency gravitational waves form different sources, as super massive black hole binary (SMBHB) mergers or the relic gravitational wave background. To obtain the signal a number of pulsars is simultaneously observed recording the variations in the time of arrival (TOA) of the electromagnetic signal of the pulsar, these are the timing residuals. These correlated signals are isolated and can be compared to theoretical models to infer if they are caused by gravitational waves passing through the whole system. In the present study we do not consider the timing residuals in the periods caused by the Earth proper motion or any cause other than that of gravitational radiation. The usual theoretical approach to calculate timing residuals for waves is to use plane waves propagating in flat space-time in the models and to include the effect of the expansion of the universe only through a redshift in the frequency, that is, frequencies are redshifted *ad hoc* to account for the expansion of the Universe. Our goal is to determine the changes that occur in the results when

*This is not saying the graviton has a mass, but it reflects the properties of its propagation as seen by an observer equipped with a Lorentz metric.

the more realistic wave front beforementioned, including the effect of Λ, is used. We find notorious differences. In particular, the angular distribution of the pulsars with respect to the source will be fundamental in the relevance of the observations. A great enhancement is found for a particular value of the angle subtended by the line observer-pulsar and the line observer-source when Λ is included. This peak depends noticeably on the value of the cosmological constant. And, if observed, could represent an independent measurement of the value of the cosmological constant at sub-cosmological scales. We carry out a detailed study of the statistical significance of the timing residual caused by GW in a real array of pulsars and determine its relevant dependencies concluding that the cosmological constant has a clear effect on these observations. Its value could eventually be 'locally' measured. In fact, this enhancement could enormously facilitate the first direct measurement of GW.

Chapter 2

Gravity as an emergent phenomenon

Einstein formulated General Relativity almost a century ago and we still have little or no clue as to the true quantum nature of gravity.

String theory has been for many years one of the most prolific and promising proposals to construct a consistent perturbative quantum theory of gravitation. The price to pay, however, is a radical modification of quantum field theory, including the acceptance that we live in a world with more than four dimensions. String theory in its present formulation is also incapable of selecting a unique vacuum, in particular it does not shed light at present on the fact that we live in a world where $\langle g_{\mu\nu} \rangle \neq 0$. Other modifications of gravity that include extra dimensions, although extremely interesting from a conceptual and phenomenological point of view, typically lack a ultraviolet completion and therefore should probably find their ultimate justification in specific compactifications of string theory.

Less popular alternatives, but of considerable interest nonetheless, are the search for non-trivial ultraviolet fixed points in gravity (asymptotic safety [23]) and the notion of induced gravity [7]. The former approach is the one pursued by exact renormalization group practitioners [24] and by lattice and numerical techniques such as Lorentzian triangulation analysis [25]. Some problems at the root of these proposals are the lack of an accurate derivation of the fixed points and the interpretation of the space-time at sub-Planckian scales. Ultimately, some authors argue that any attempt to probe the energy scales involved would lead to the formation of black holes (a sort of natural cut-off) emptying these proposals of any falsifiability [26]. Lattice analysis only require some pre-metric input, in particular a notion of causality (hence transport of a time-like vector) in Lorentzian gravity formulation. These theories are discretized and their continuum limit is not always straightforward. Lorentzian triangulation yielded very interesting results in two dimensions [25]. Their extension to four dimensions is possible, and although smooth manifolds can be obtained, there is still a long way to go before safe conclusions can be drawn. Induced gravity advocates that a possible explanation of the relative weakness of gravity as compared to other interactions is that it is a residual or induced force, a subproduct of all the rest of matter and interaction fields. A usual problem of this approach is the obtention of the wrong sing for Newton's constant [7] and it is unclear that it may yield massless gravitons at all. All these proposals also rely on the introduction of a metric from the very beginning. How to justify its inclusion, even if it is just a trivial metric, is an unresolved issue.

Another interesting proposal is to consider gravity as an entropic force caused by changes in the information associated with the positions of material bodies [27]. This is a bold proposal that deserves consideration, but seems at the moment far too speculative.

It has been pointed out several times in the literature (see e.g. [10, 28]) that gravitons should perhaps be considered as Goldstone bosons of some broken symmetry. This is exactly the point of view adopted in Chapters 3 and 4. This idea goes probably back to early papers by Salam and coworkers [8], and Ogievetsky and coworkers [9], if not earlier [29], but a concrete proposal has been lacking so far (see however [11, 12, 13]). By concrete proposal we mean some field theory that does not contain the graviton field as an elementary degree of freedom. Ideally it should not even contain the tensor $\eta_{\mu\nu}$ as this already implies the use of some background metric. Indeed we would like to see the metric degrees of freedom emerging dynamically, like the pions appear dynamically after chiral symmetry breaking in QCD. Furthermore, if possible, we would like the underlying theory to be in some sense 'simpler' than gravity, in particular it should be renormalizable. One could then pose questions that are left unanswered in gravity, such as the fate of black hole singularities and the counting of degrees of freedom.

The dynamical generation of geometry, combined with the usual renormalization group arguments have rather interesting consequences. Geometry and distance are induced rather than fundamental concepts. At sufficiently short scales, when the effective action does not make sense anymore, the fundamental degrees of freedom emerge. Below that scale there is not even the notion of distance: in a sense that is the shortest scale that can exist. This precludes the existence of an ultraviolet fixed point advocated by some [23] but also indicates that at short distances gravity is non-Wilsonian as suggested by others [30] in an holographic context.

2.1 The low-energy effective action of QCD

The four-dimensional chiral Lagrangian is a non-renormalizable theory describing accurately pion physics at low energies. It has a long history, with the first formal studies concerning renormalizability being due mostly to Weinberg [31] and later considerably extended by Gasser and Leutwyler [32]. The chiral Lagrangian contains a (infinite) number of operators

$$\mathcal{L} = f_\pi^2 \mathrm{Tr}\, \partial_\mu U \partial^\mu U^\dagger + \alpha_1 \mathrm{Tr}\, \partial_\mu U \partial^\mu U^\dagger \partial_\nu U \partial^\nu U^\dagger + \alpha_2 \mathrm{Tr}\, \partial_\mu U \partial_\nu U^\dagger \partial^\mu U \partial^\nu U^\dagger + \ldots, \quad (2.1)$$

$$U \equiv \exp i\tilde{\pi}/f_\pi, \qquad \tilde{\pi} \equiv \pi^a \tau^a/2,$$

organized according to the number of derivatives

$$\mathcal{L} = \mathcal{O}(p^2) + \mathcal{O}(p^4) + \mathcal{O}(p^6) + \ldots . \quad (2.2)$$

Pions are the Goldstone bosons associated to the (global) symmetry breaking pattern of QCD

$$SU(2)_L \times SU(2)_R \to SU(2)_V . \quad (2.3)$$

The above Lagrangian is the most general one compatible with the symmetries of QCD and their breaking. Locality, symmetry and relevance (in the renormalization group sense) are the guiding principles to construct $\mathcal{L}$. Renormalizability is not; in fact if we cut-off the derivative expansion at a given order the theory requires counterterms beyond that order no matter how large the order is. Note that, although the symmetry has been spontaneously broken, the effective Lagrangian still has the full symmetry $U \to LUR^\dagger$ with L and R being $SU(2)$ matrices belonging to the left and right groups respectively.

The lowest order, tree level contribution to pion-pion scattering derived from the previous Lagrangian is $\sim p^2/f_\pi^2$. Simple counting arguments show that the one-loop chiral corrections are $\sim p^4/(16\pi^2 f_\pi^4)$. Thus the counting parameter in the loop (chiral) expansion in $D = 4$ is

$$\frac{p^2}{16\pi^2 f_\pi^2}. \tag{2.4}$$

Each chiral loop gives an additional power of p^2.

At each order in perturbation theory the divergences that arise can be eliminated by redefining the coefficients in the higher order operators

$$\alpha_i \to \alpha_i + \frac{c_i}{\epsilon}. \tag{2.5}$$

In addition to the pure pole in ϵ, logarithmic non-local terms necessarily appear. For instance in a two-point function they appear in the combination

$$\frac{1}{\epsilon} + \log \frac{-p^2}{\mu^2}, \tag{2.6}$$

p being the external momentum. Note that the cut provided by the log is actually absolutely required by unitarity. All coefficients in the chiral Lagrangian are nominally of $\mathcal{O}(N)$ (being N the number of fermions). Loops are automatically suppressed by powers of N, because $f_\pi^2 \sim N$ appears in the denominator, but they are enhanced by logs at low momenta.

We have also acquired experience from chiral Lagrangians in the use of the equations of motion in an effective theory: at any order in the chiral expansion we can use the equations of motion derived from previous orders. For instance, using that at the lowest order $U \Box U^\dagger - (\Box U) U^\dagger = 0$ (from the $\mathcal{O}(p^2)$ Lagrangian), one can reduce the number of operators at $\mathcal{O}(p^4)$.

2.2 Is gravity a Goldstone phenomenon?

The $D = 4$ Einstein-Hilbert action shares several remarkable aspects with the pion chiral Lagrangian. It is a non-renormalizable theory as well as it is also described, considering the most relevant operator (we ignore here for a moment the cosmological constant), by a dimension-two operator containing in both cases two derivatives of the dynamical variable. Both Lagrangians contain necessarily a dimensionful constant in four dimensions: M_P, the Planck mass, is the counterpart of the constant f_π in the pion Lagrangian (of course the value of both constants is radically different). Both theories are non-linear and, finally, both describe the interactions of massless quanta. The Einstein-Hilbert action is

$$\mathcal{L} = M_P^2 \sqrt{-g} R + \mathcal{L}_{matter}, \tag{2.7}$$

where as just mentioned R contains two derivatives of the dynamical variable which is the metric $g_{\mu\nu}$

$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\alpha} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{\mu\nu} - \Gamma^\alpha_{\beta\nu}\Gamma^\beta_{\mu\alpha}, \tag{2.8}$$

$$\Gamma^\gamma_{\alpha\beta} = \frac{1}{2} g^{\gamma\rho} \left(\partial_\beta g_{\rho\alpha} + \partial_\alpha g_{\rho\beta} - \partial_\rho g_{\alpha\beta} \right). \tag{2.9}$$

In the chiral language, the Einstein-Hilbert action would be $\mathcal{O}(p^2)$, i.e. most relevant, if we omit the presence of the cosmological constant which accompanies the identity operator. Arguably, locality, symmetry and relevance in the renormalization group sense (and not renormalizability) are the ones that single out Einstein-Hilbert action in front of e.g. R^2.

Unlike the chiral Lagrangian, the Einstein-Hilbert Lagrangian, or extensions that include higher derivative terms, has a local gauge symmetry. Indeed, gravity can be (somewhat loosely) described as the result of promoting a global symmetry (Lorentz) to a local one (for a detailed discussion on the gauge structure of gravity see e.g. [33]). This means that the gauge symmetry that is present in gravity will in practice reduce the number of degrees of freedom that are physically relevant.

Exactly like the chiral Lagrangian, the Einstein-Hilbert action requires an infinite number of counterterms

$$\mathcal{L} = M_P^2 \sqrt{-g} R + \alpha_1 \sqrt{-g} R^2 + \alpha_2 \sqrt{-g} (R_{\mu\nu})^2 + \alpha_3 \sqrt{-g} (R_{\mu\nu\alpha\beta})^2 + \dots \; . \tag{2.10}$$

The divergences can be absorbed order by order by redefining the coefficients α_i just as done in the previous subsection for the pion effective Lagrangian. Power counting in gravity appears, at least superficially, quite similar to the one that can be implemented in pion physics. Of course, the natural expansion parameter is a tiny number in normal circumstances, namely

$$p^2/16\pi^2 M_P^2 \quad \text{or} \quad \nabla^2/16\pi^2 M_P^2, \qquad R/16\pi^2 M_P^2, \tag{2.11}$$

making quantum effects usually quite negligible. There are some subtleties when matter fields are included (see [34] for a discussion).

Like in the pion chiral Lagrangian non-local logarithmic pieces accompany the divergences. In position space they look like

$$\frac{1}{\epsilon} + \log \frac{\nabla^2}{\mu^2}, \tag{2.12}$$

where ∇ is the covariant derivative on symmetry grounds, ∇^2 reducing to $-p^2$ in flat space-time. These non-localities are due to the propagation of strictly massless non-conformal modes, such as the graviton itself. Therefore they are unavoidable in quantum gravity. Notice that the coefficients of these non-local terms are entirely predictable from the universal properties of gravity.

Let us use 'chiral counting' arguments to derive the relevant quantum corrections to Newton's law (up to a constant). The propagator at tree level gets modified by one-loop 'chiral-like' corrections

$$\frac{1}{p^2} \quad \rightarrow \quad \frac{1}{p^2}\left(1 + A\frac{p^2}{M_P^2} + B\frac{p^2}{M_P^2}\log\frac{p^2}{M_P^2}\right). \tag{2.13}$$

Consider now the interaction of a point-like particle with an static source ($p^0 = 0$) and let us Fourier transform the previous expression for the loop-corrected propagator in order to get the potential in the non-relativistic limit. We recall that

$$\int d^3x\ e^{i\vec{p}\cdot\vec{x}}\,\frac{1}{p^2} \sim \frac{1}{r}, \quad \int d^3x\ e^{i\vec{p}\cdot\vec{x}}\,1 \sim \delta(\vec{x}), \quad \int d^3x\ e^{i\vec{p}\cdot\vec{x}}\,\log p^2 \sim \frac{1}{r^3}, \tag{2.14}$$

with $r = |\vec{x}|$. Thus quantum corrections to Newton's law are of the form

$$\frac{GMm}{r}\left(1 + K\delta(\vec{x}) + C\frac{G\hbar}{c^3}\frac{1}{r^2} + \dots\right). \tag{2.15}$$

We have restored for a moment $\hbar$ and c to make evident that C is a pure number. The contribution proportional to $\delta(\vec{x})$ is of course non-observable, even as a matter of principle. It comes from the contact divergent term (that may eventually collect contributions from arbitrarily high frequency modes). C, however, is calculable. It depends only on the infrared properties of the theory.

A long controversy regarding the value of C exists in the literature [35, 36, 37]. The result now accepted as the correct one, $C = 41/10\pi$ [38] is obtained by considering the inclusion of quantum matter fields and considering the on shell scattering matrix. Note that quantum corrections make gravity more attractive (by a really tiny amount) at long distances than predicted by Newton's law. In addition to quantum corrections there are post-newtonian classical corrections that are not discussed here (see [34]).

There are in the literature definitions of an 'effective' or 'running' Newton constant [39]. A class of diagrams is identified that dresses up G and turns it into a distance (or energy)-dependent constant $G(r)$. Unfortunately it is not clear that these definitions are gauge invariant; only physical observables (such as a scattering matrix) are guaranteed to be. Nevertheless, the renormalization group analysis derived from this 'running' coupling constant are of course very interesting and may bear relevance to the issue of asymptotic safety mentioned in the introduction of this chapter.

2.3 The model

We have given in the previous sections arguments why the Einstein-Hilbert action could be viewed as the most relevant term, in the sense of the renormalization group, of an effective theory describing the long distance behavior of some underlying dynamics.

Here we want to pursue this line of thought further. As a logical possibility, without making any particularly strong claim of physical relevance, we shall investigate a formulation inspired as much as possible in the chiral symmetry breaking of QCD. It should have the following characteristics:

1. No a priori notion of metric should exist, only an affine connection defining parallel transport of tangent vectors v^a on a manifold.
2. The Lagrangian should be manifestly independent of the field $g_{\mu\nu}(x)$.
3. The graviton field should appear as the Goldstone boson of a suitably broken global symmetry.
4. The breaking should be triggered by a fermion condensate.

A model along these lines was considered some time ago by Russo and others [11, 12], and more recently by Tumanov [13]. Our proposal appears to be perturbatively renormalizable and leads to finite calculable predictions, unlike the ones in [11, 12, 13].

As announced, we seek inspiration in the effective Lagrangians of QCD at long distances [40, 41], discussed in Section 2.1. Consider the matter part Lagrangian of

QCD with massless quarks (2 flavors)

$$\mathcal{L} = i\bar{\psi}\ \partial\!\!\!/\psi = i\bar{\psi}_L\ \partial\!\!\!/\psi_L + i\bar{\psi}_R\ \partial\!\!\!/\psi_R. \tag{2.16}$$

This theory has a global $SU(2)\times SU(2)$ symmetry that forbids a mass term M. However after chiral symmetry breaking pions appear and they must be included in the effective theory. Then it is possible to add the following term

$$-M\bar{\psi}_L U\psi_R - M\bar{\psi}_R U^\dagger\psi_L, \tag{2.17}$$

that is invariant under the full global symmetry $\psi_L \to L\psi_L,\ \psi_R \to R\psi_R,\ U \to LUR^\dagger$.

Chiral symmetry breaking is triggered by a non-zero fermion condensate $\langle\bar{\psi}\psi\rangle \neq 0$. In order to determine the value of this condensate, and in particular whether it is zero or not, one is to solve a 'gap'-like equation in some modelization of QCD, or on the lattice. The final step is to integrate out the fermions using the self-generated effective mass as an infrared regulator. This reproduces the chiral effective Lagrangian discussed in Section 2.1, although the low-energy constants α_i obtained in this way are not necessarily the real ones, as the chiral quark model is only a modelization of QCD.

The idea is now to find out a field theory with the characteristics outlined above that can yield gravity as an effective theory. We shall use Euclidean conventions. There is only one possible 'kinetic' term bilinear in fermions that is invariant under Lorentz $\times$ *Diff* (actually $SO(D)$ rather than Lorentz) and it is local and hermitian*. It is

$$\mathcal{L}_0 = i\bar{\psi}_a\gamma^a\nabla_\mu\chi^\mu + i\bar{\chi}^\mu\gamma^a\nabla_\mu\psi_a. \tag{2.18}$$

To define ∇_μ only an affine connection is needed

$$\nabla_\mu\chi^\mu = \partial_\mu\chi^\mu + \omega_\mu^{ab}\sigma_{ab}\chi^\mu + \Gamma^\nu_{\mu\nu}\chi^\mu. \tag{2.19}$$

Here $a, b...$ are tangent space indices, while $\mu, \nu, ...$ are world indices. The coordinates on the manifold shall be denoted by x^μ and of course there is no way of raising or lowering indices because there is no metric. Only δ_{ab} as invariant tensor of the tangent space is admissible. ψ_a and χ^μ are independent spinor fields. The field χ^μ is expected to have a spin 1/2 as well as 3/2 component. No attempt has been made to project out the 1/2 component.

*Actually what we really should require is that the continuation to Minkowski space is hermitian.

Note that no metric is needed at all to define the action if we assume that χ^μ behaves as a contravariant spinorial vector density under *Diff*. Then, $\Gamma^\mu_{\nu\rho}$ does not enter in the covariant derivative, only the spin connection ω^{ab}_μ. If one keeps this spin connection fixed, i.e. we do not consider it to be a dynamical field for the time being, there is no invariance under general coordinate transformations, but only under the global group $G = SO(D) \times GL(D)$. Notice once more that the spin connection is the only geometrical quantity introduced.

We would like to find a non-zero value for the fermion condensate

$$\langle \bar{\psi}_a \chi^\mu + \bar{\chi}^\mu \psi_a \rangle \sim A^\mu_a \neq 0. \tag{2.20}$$

Because the broken theory has still the full global symmetry $SO(D) \times GL(D)$, it is of course irrelevant in which direction the condensate points; all the vacua will be equivalent. we can choose $A^\mu_a = \delta^\mu_a$ without loss of generality.

A large number of Goldstone bosons are produced in the breaking. The original symmetry group $G = SO(D) \times GL(D)$ has $\frac{D(D-1)}{2} + D^2$ generators. After the breaking $G \to H$, with $H = SO(D)$ there are D^2 broken generators. It remains to be discussed in the following chapters how many of those actually couple to physical states.

In order to trigger the appearance of a vacuum expectation value (v.e.v.) we have to include some dynamics to induce the symmetry breaking. The model we propose is to add the interaction piece

$$S_I = \int d^D x \mathcal{L}_I = \int d^D x (iB^a_\mu (\bar{\psi}_a \chi^\mu + \bar{\chi}^\mu \psi_a) + c \det(B^a_\mu)). \tag{2.21}$$

Note that the interaction term also behaves as a density thanks to the covariant Levi-Civita symbol hidden in the determinant of B^a_μ so no metric is needed. Note that (2.21) is non-hermitian, but the continuation to Minkowski is: B^a_μ upon continuation changes like an Euclidean mass does $B^a_\mu \to iB^a_\mu$. Since the field B^a_μ is auxiliary, it is clear that we are dealing with a four-fermion interaction; fermions are the only dynamical fields.

Upon use of the equations of motion for the auxiliary field B^a_μ

$$\bar{\psi}_a \chi^\mu + \bar{\chi}^\mu \psi_a = -ic \, \frac{1}{(D-1)!} \epsilon_{aa_2 \ldots a_D} \epsilon^{\mu\mu_2 \ldots \mu_D} B^{a_2}_{\mu_2} \ldots B^{a_D}_{\mu_D} \tag{2.22}$$

and thus

$$\langle \bar{\psi}_a \chi^\mu + \bar{\chi}^\mu \psi_a \rangle \neq 0 \Rightarrow \langle B^a_\mu \rangle \neq 0. \tag{2.23}$$

If a non-zero value for the fermion condensate appears then the field B_μ^a necessarily acquires a non-zero expectation value B_μ^a (the reciprocal is not necessarily true, but it will be true too in our case). As we will see such condensation will happen in $D = 2$ (in the large N limit) and it will be also present in $D = 4$ even for finite values of N.

2.4 Energy-momentum tensor and symmetries

Although the above theory is 'topological' inasmuch as it is described by an action that does not contain a metric (albeit it depends on a connection), the energy-momentum tensor understood as the Noether currents of translation invariance is non-vanishing

$$T_\nu^\mu = i\bar{\chi}^\mu \gamma^a \partial_\nu \psi_a + i\bar{\psi}_a \gamma^a \partial_\nu \chi^\mu - \delta_\nu^\mu L. \tag{2.24}$$

Note that no metric is needed to define T_ν^μ. In the absence of the external connection T_ν^μ is traceless as expected given that the theory is formally conformal, but we will see later that it will not remain so at the quantum level as anomalous dimensions develop.

Traditionally a major stumbling block in the program that will be developed in Chapters 3 and 4 has been the so-called Weinberg-Witten theorem [42] (see also [43]). The apparent pathology of theories intending to generate dynamically gauge bosons (including gravitons in this category) lies in the fact that the energy-momentum tensor has to be identically zero if massless particles with spin ≥ 1 appear and one insists in the energy momentum tensor being Lorentz covariant. However, our results, while not constituting a mathematical proof, indicate that one can indeed get, both in $D = 2$ and in $D = 4$, an effective low-energy theory with massless composite gravitons, so it is legitimate to ask why the Weinberg-Witten theorem would not apply. Note something peculiar to this proposal; namely the energy-momentum tensor (2.24) does not have tangent (Lorentz) indices. In fact Lorentz indices are of an internal nature in the present approach as we will see below. The connection between Lorentz and world indices appears only after a vierbein is dynamically generated. But then one is exactly in the same situation as General Relativity where the applicability of [42] is excluded.

The free action (2.18), without considering the interaction term, is also invariant under the symmetry

$$\psi_a \to \psi'_a = (\delta_a^b - \frac{1}{D}\gamma_a \gamma^b)\psi_b. \tag{2.25}$$

Another invariance of the free action is provided by redefining, in Fourier space,

$$\chi^\mu(k) \to \psi'^\mu = P^\mu_\nu \chi^\nu(k), \tag{2.26}$$

where $k_\mu P^\mu_\nu = 0$. These two invariances difficult considerably the heat kernel derivation of an effective action for the field B^a_μ that will be discussed for $D = 2$ in the next chapter.

2.5 Free propagator and renormalizability

Note the peculiar 'free' kinetic term $\gamma^a \otimes k_\mu$. It is of course reminiscent of the Dirac equation, but it is not quite identical (Dirac needs a metric or a $n$bein to be defined). Let us assume that after the introduction of the interaction term $\sim \det B$, the field B^a_μ indeed develops a v.e.v. that we conventionally take to be

$$\langle B^a_\mu \rangle = M \delta^a_\mu. \tag{2.27}$$

Any other direction would be equivalent. The only substantial fact is whether M is zero or not. Via (2.23) this v.e.v for B^a_μ translates into a v.e.v for $\bar\psi_a \chi^\mu + \bar\chi^\mu \psi_a$. From (2.21) we see that the scale M plays the role of a dynamically generated mass for the fermions (not unlikely the 'constituent mass' in chiral dynamics, except that here it will be possible, as we will see, to determine exactly its relation to the fundamental parameters of the model).

Below we write explicitly the propagator of the fermion field. It can be written (in any number of dimensions) as

$$\Delta^{-1}(k)_{ij} = \frac{-i}{M}\left(\delta_{ij} - \frac{\gamma_i(\not k - iM)k_j}{k^2 + M^2}\right), \tag{2.28}$$

with $k^2 = \sum_i k_i^2$. In the previous expression we use a matrix notation since (2.28) is derived using the solution (2.27). For the free theory world and tangent indices can be interchanged, as it befits a flat metric. The covariance of the results, not evident at all from these expressions, will be discussed in the next chapter.

This is an appropriate point to discuss the renormalizability of the model. Naively, because the coupling constant c is dimensionless in $D = 2$, we would expect the model to be renormalizable. However this expectation is jeopardized by the behavior of the propagator. Indeed the diagonalization of (2.28) gives eigenvaules such as M, $k + iM$

and $k - iM$. Therefore the propagator does not behave, in general, as $1/k$ and therefore the usual counting rules simply do not apply.

There is however a further twist to the issue of renormalizability. The model proposed does not contain a metric and therefore the number of counterterms that one can write is extremely limited. For instance, a mass term for the B field is impossible. Higher dimensional operators would require powers of $\sqrt{g}$ to preserve the $Diff$ invariance that the model has (when w is a dynamical variable), but there is no metric. In fact the metric will be generated *after* the breaking, but the counterterms of a field theory do not depend on whether there is spontaneous breaking of a global symmetry or not.

In the previous discussion we are not considering the divergences coming from graviton loops. These will be suppressed by powers of N, if $N \to \infty$. On the contrary, if $N \to 0$ those divergences will be relevant, but then there are no fermions and one is left with gravity with all its usual UV problems. A deeper discussion on this issue can be found in the following chapters.

2.6 Counterterms

To close this chapter let us review in detail what are the possible counterterms both in two and in four dimensions that may appear upon quantization of the Lagrangian formed by (2.18) and (2.21). The only terms one can include in the action before the symmetry breaking are those constructed without a metric. To illustrate them it will be useful to resort to the language of differential forms.

In $D = 2$ two two-forms can be constructed with the ingredients we have at hand. Since formally we do not have a $n$bein before the symmetry breaking we can only use the auxiliary field B^a, understood as a 1-form, and the spin connection to construct these terms. They read

$$\mathcal{L}_V = B^a \wedge B^b \epsilon_{ab}; \qquad \mathcal{L}_R = \mathbf{d}\mathbf{w}^{ab}\epsilon_{ab}. \tag{2.29}$$

These two can be integrated without having to appeal to any metric. Although not apparent at first sight, we will see in the following chapter they correspond to the volume form and the curvature form. The first one will renormalize the cosmological constant and the second one Newtons constant*. Any other term would need a metric to be constructed.

In $D = 4$ we can write analogous terms to (2.29)

$$\mathcal{L}_V = B^a \wedge B^b \wedge B^c \wedge B^d \epsilon_{abcd}; \qquad \mathcal{L}_R = \mathbf{D}\mathbf{w}^{ab} \wedge B^c \wedge B^d \epsilon_{abcd}, \tag{2.30}$$

where $\mathbf{D}$ is the exterior derivative. Again, as we will see in Chapter 4, the first one will renormalize the cosmological constant and the second one the curvature term, which in this dimensionality will explicitly absorb divergences that are produced upon integration of the fermionic degrees of freedom. One could also write the topological term in four dimensions, the Gauss-Bonnet term, which is also a legitimate counterterm. However, since it is not expected to play a significant role in the final action we will not write it explicitly here.

In summary, the lack of counterterms makes us believe that the theory is renormalizable after all, at least in the large N limit. Indeed this expectation is supported by the explicit one-loop calculation (see Chapters 3 and 4) where the only divergences that appear can be absorbed by a very limited number of counterterms. We find this quite remarkable.

*Although in $D = 2$ the curvature form is a total derivative and we do not expect to see it in the calculations.

Chapter 3

A two-dimensional toy model

We presented in Chapter 2 all the characteristics the model should have form the start. Let us now particularize to the case $D = 2$ and $2N$ species of fermions. We shall compute the exact value of the vacuum in this dimensionality and explore the possible counterterms available. A heat kernel calculation of the effective action is attempted. The case of two dimensions is a particularly simple one, the maximum number of degrees of freedom for a perturbation above the vacuum is just one. In other words, in two dimensions we can only have a conformally flat metric in the effective theory. With such a perturbation we perform an explicit one-loop calculation of the effective action obtaining Liouville theory plus a cosmological constant.

3.1 Gap equation

If $w_\mu = 0$ then one can use homogeneity and isotropy arguments to look for constant solutions of the gap equation associated to the following effective potential obtained after integration of the fermions

$$V_{eff} = c\,\det(B^a_\mu) - 2N \int \frac{d^D k}{(2\pi)^D} \mathrm{tr}(\log(\gamma^a k_\mu + i B^a_\mu)). \tag{3.1}$$

Note that the $2N$ preceding the integral comes from the $2N$ species of fermions present. As it is explained later on in this section and in Chapter 4, we consider $2N$ fermions to be able to explore the properties of the effective theory in the different limits of the value on N. As for the flat measure used for the integration, this corresponds to the functional trace of the differential operator*. As such, the trace is independent of the particular basis that is used to compute it. Any other basis, if used correctly, would yield the same result.

Deriving (3.1) with respect to B_μ^a we obtain

$$c\,\frac{D}{D!}\epsilon_{aa_2\ldots a_D}\epsilon^{\mu\mu_2\ldots\mu_D}B_{\mu_2}^{a_2}\ldots B_{\mu_D}^{a_D} - 2Ni\ \mathrm{tr}\int\frac{d^Dk}{(2\pi)^D}(\gamma^a k_\mu + iB_\mu^a)^{-1}|_a^\mu = 0. \tag{3.2}$$

In $D = 2$ this equation is particularly simple

$$c\epsilon_{ab}\epsilon^{\mu\nu}B_\nu^b - 2Ni\ \mathrm{tr}\int\frac{d^Dk}{(2\pi)^D}(\gamma\otimes k + iB)^{-1}|_a^\mu = 0. \tag{3.3}$$

The 'gap equation' to solve for constant values of B_{ij} is

$$cB_{ij} + \frac{N}{2\pi}B_{ij}\log\frac{\det B}{\mu^2} = 0. \tag{3.4}$$

A logarithmic divergence has been absorbed in c. Notice that the equations are invariant under the permutation

$$B_{ij} \to B_{\sigma(i)\sigma(j)}, k_i \to k_{\sigma(i)}, \quad \sigma\epsilon S_2. \tag{3.5}$$

This equation has a non-trivial solution that we can always choose, as indicated before, to be $B_{ij} \sim \delta_{ij}$. We thus see that the dynamical mass for the fermions is indeed generated hence justifying *a posteriori* the propagator introduced in the previous chapter. The solution for the dynamical mass is

$$M = \mu e^{-\pi c(\mu)/N}. \tag{3.6}$$

Plugging this back in the effective potential we obtain

$$V_{eff} = -\frac{\mu^2 e^{-2\pi c(\mu)}}{2\pi}. \tag{3.7}$$

Upon continuation to Minkowski space-time this term is to be identified with the cosmological constant. At this level M is an observable and as such it should be a renormalization group invariant. This is guaranteed if c runs according to the rather trivial beta function

$$\mu\frac{dc}{d\mu} = \frac{N}{\pi}. \tag{3.8}$$

*Note that 'plane waves' are eigenmodes of the differential operator $\gamma^a\nabla_\mu$ if the connection w_μ^{ab} is set to zero. The connection itself is treated perturbatively in the subsequent.

Note that the coefficient of this term is related to the coefficient of the logarithmic divergence and hence it is universal. In the previous, we introduced the usual mass scale μ to preserve the correct dimensionality of the D-dimensional integral as dimensional regularization is used. For the solution to actually exist we have to require $c > 0$ if $M > \mu$. If $\mu > M$ the solution exists only if $c < 0$. Therefore $c > 0$ will be the case we are interested in on physical grounds.

The above effective potential and ensuing gap equation are exact in the limit where the number of fermions, N, is infinite. In fact we expect that it is exact *only* in this limit, as in $D = 2$ the phenomenon of spontaneous breaking of a continuous symmetry can take place only in the $N \to \infty$ limit.

For non-zero connection ($w_\mu \neq 0$) the gap equation is not applicable and one needs to derive the full effective action. Then one would minimize the fields B_μ^a as a function of w_μ. This is discussed in Sections 3.3 and 3.4.

3.2 Possible counterterms in $D = 2$

Before tackling the perturbative derivation of the effective action it is important to list the possible invariants one can write in this theory without making use of a metric, as discussed in Section 2.6. In $D = 2$ we have two invariants that could be constructed without having to appeal to a metric, namely $\mathcal{L}_0$ (2.18) and $\mathcal{L}_I$ (2.21). The latter, upon use of the parametrization $B_\mu^a = M\delta_\mu^a e^{-\frac{\sigma}{2}}$, reduces to

$$\frac{1}{2!}\int B_\mu^a B_\nu^b \epsilon^{\mu\nu}\epsilon_{ab}\, d^2x = \; M^2 \int \sqrt{g}\, d^2x, \tag{3.9}$$

i.e. is the cosmological term. In addition, there is the curvature term which in $D = 2$, and in terms of the connection, is simply $\int \mathbf{dw}$, thus purely topological; therefore we do not expect it to appear in the perturbative calculation. Then, apart from the free kinetic term for the fermions, there is only one invariant term that can be written down without a metric. Or what it is tantamount, only one possible counterterm remains to absorb any divergence appearing in the perturbative calculation after integrating out the fermions. This fact enforces the renormalizability of the $D = 2$ model in the large N limit in spite of the bad ultraviolet behavior of the integrals. This argument will be supported by the explicit calculations presented in the subsequent sections.

3.3 Heat kernel derivation of the effective action

Let us now attempt to derive the effective action for the fields B_μ^a and the external affine connection w_μ that eventually we will allow to become a dynamical variable too. Hereafter we want to perform a double minimization with respect to these fields. This will be an exact procedure for $N = \infty$ and provide a guidance in the general case. Of course the really interesting question is what happens for $D > 2$.

We would expect that this double minimization will provide us with two equations whose meaning would be schematically the following: One of them would provide a relation between the field B_μ^a (associated to the zweibein) and the affine connection w_μ. If the present model is to describe in its broken phase $D = 2$ gravity, this relation would be analogous to the relation of compatibility between the metric and the connection that appears when the Palatini formalism [44] is used in General Relativity and the equations of motion for the connection w_μ are derived. The remaining equation, after imposition of the previous compatibility condition, should then be equivalent to Einstein's equations.

However, in $D = 2$ gravity is rather peculiar and indeed the condition

$$w_\mu^{ab} = e_\nu^a \partial_\mu E^{\nu b} + e_\nu^a E^{\sigma b} \Gamma^\nu_{\sigma\mu}, \tag{3.10}$$

where E_a^μ is the inverse zweibein $E_a^\mu e_\mu^b = \delta_a^b$, holding in any number of dimensions, does not follow from any variational principle (see e.g. [45]). There are several ways to understand this fact, but perhaps the simplest one is to realize that Einstein-Hilbert in $D = 2$ depends on w_μ only through the two-form $\mathbf{dw}$ which is linear in the affine connection w_μ. In fact the scalar curvature term $\sqrt{g}R$ does not contain in $D = 2$ any coupling between $g_{\mu\nu}$ and w_μ. Adding higher derivatives does not really help as the Riemann tensor contains only an independent component that can be ultimately related to the scalar curvature. We shall see below that this peculiarity of two-dimensional gravity is faithfully reproduced in our proposal.

The starting point of the derivation of the effective action in two dimensions is the differential operator

$$D_\mu^a = \gamma^a(\partial_\mu + w_\mu \sigma_3) + B_\mu^a. \tag{3.11}$$

We consider the expansion around a fixed background preserving $SO(2)$ but not the full symmetry group G. We will take $B_\mu^a = M\delta_\mu^a$ where M will be determined via the gap equation discussed in the previous section, which corresponds to a solution of the equation of motion at the lowest order in a weak field and derivative expansion, in the spirit of effective Lagrangians. To go beyond this approximation we have to consider x-dependent fluctuations around this vacuum and include the external field w_μ. We shall decompose

$$B^a_\mu = \xi_L{}^a{}_b \bar{B}^b_\nu \xi_R^{-1\nu}{}_\mu, \tag{3.12}$$

where $\xi_L \in SO(2)$, $\xi_R \in GL(2)$ and $\bar{B}^a_\mu$ is a solution of the gap equation, $M\delta^a_\mu$ in our case. It is technically advantageous to absorb the matrices ξ_L and ξ_R in the fermion fields (in QCD this is the so-called 'constituent' quark basis [40, 41]). Then the differential operator to deal with will be

$$\mathcal{D}^b_\mu = \xi^{\dagger}_{La}{}^b \gamma^a(\partial_\rho + w_\rho \sigma_3)\xi_R{}^\rho{}_\mu + \bar{B}^b_\mu. \tag{3.13}$$

To evaluate the effective action generated by the integration of the fermion fields one possibility is to write the log of the fermion determinant as

$$W = -\frac{1}{2}\int_0^\infty \frac{dt}{t} \mathrm{tr}\left\langle x|e^{-tX}|x\right\rangle, \tag{3.14}$$

where

$$X_{\mu\nu} \equiv \mathcal{M}^\dagger \mathcal{M}, \tag{3.15}$$

with

$$\mathcal{M} = \mathcal{D}^b_\nu, \qquad \mathcal{M}^\dagger = -\mathcal{D}_{\mu b} \tag{3.16}$$

and

$$\mathcal{D}^b_\mu = \xi^{\dagger}_{La}{}^b \gamma^a(\partial_\rho + w_\rho \sigma_3)\xi_R{}^\rho{}_\mu + \bar{B}^b_\mu,\ \mathcal{D}_{\nu b} = \xi^{\dagger}_{R\nu}{}^\sigma(\partial_\sigma - w_\sigma\sigma_3)\gamma_a \xi_L{}^a{}_b - \bar{B}_{\nu b}. \tag{3.17}$$

$X_{\mu\nu}$ has both world and Dirac indices (the latter not explicitly written). Note that as previously discussed $\mathcal{M}$ is not hermitian, but of course $X_{\mu\nu} = \mathcal{M}^\dagger\mathcal{M}$ is. We could have also considered the determinant of $\mathcal{M}\mathcal{M}^\dagger$ which is of course identical, but it is important to maintain a covariant appearance as long as possible (note that there is no metric so far and no way of lowering or raising indices). The final result has to be of course covariant, since our starting point is, but using, as we shall do, a plane basis to evaluate the traces in the heat kernel expansion breaks in principle this covariance in intermediate steps.

Once $W(w, B)$ is known we can differentiate with respect w_μ and obtain the relation between the zweibein and the spin connection using the logic behind the Palatini formalism.

The starting point of the heat kernel derivation is the evaluation of

$$\mathrm{tr}\langle x|e^{-tX_{\mu\nu}}|x\rangle = \frac{1}{t^{\frac{D}{2}}}\int \frac{d^D k}{(2\pi)^D}\mathrm{tr}\ e^{[-D\xi^{\top}_{R\mu}{}^{\sigma}\xi^{\rho}_{R\nu}k_\sigma k_\rho + i\sqrt{t}\mathcal{D}_{\mu b}\xi_L^{-1b}{}_a\gamma^a k_\rho \xi^{\rho}_{R\nu} + i\sqrt{t}\xi^{\top}_{R\mu}{}^{\sigma}k_\sigma\gamma_a\xi_L^{-1a}{}_b\mathcal{D}^b{}_\nu - tX_{\mu\nu})]} \tag{3.18}$$

in $D = 2 - \epsilon$, where for convenience we have rescaled k_μ and a plane wave basis resolution of the identity has been used. For simplicity let us call the exponent on the right-hand side of the previous equation $X(\sqrt{t})$. Then the way to proceed is to expand the exponential $e^{X(\sqrt{t})}$ in powers of $\sqrt{t}$. Only even powers of $\sqrt{t}$ (and thus of k) will contribute at the end to the series, so the first non-trivial term will be of order t. We define

$$F_n\left(X(0), \dot{X}(0), \ddot{X}(0)\right) \equiv \frac{d^{(n)}}{(d\sqrt{t})^n}e^{X(\sqrt{t})}\,|_{\sqrt{t}=0} \tag{3.19}$$

then

$$\mathrm{tr}\left\langle x|e^{-tX_{\mu\nu}}|x\right\rangle \propto \mathrm{tr}\ \sum_n F_n\frac{(\sqrt{t})^n}{n!} = F_0 + \frac{t}{2}F_2 + \frac{t^2}{24}F_4 + \mathcal{O}(t^4). \tag{3.20}$$

This expansion is quite tedious and to perform it we used repeatedly the well-known formula

$$\frac{d}{dt}e^{A(t)} = \int_0^1 da\ e^{(1-a)A(t)}\frac{dA(t)}{dt}e^{aA(t)}. \tag{3.21}$$

Note that the invariances discussed in the previous section introduce zero modes in the exponent and hence integrals that are not damped for large values of the momentum k. Of course they are no true zero modes of the full theory, just of the kinetic term, but the technical complications that they bring about are notable.

However, it is pleasant to see that a formally covariant result emerges. If we neglect w_μ and we take the matrices ξ to be constant it is not difficult to see that the lowest non-trivial order of the heat kernel calculation gives

$$W = \frac{3M^2}{16\pi}\int d^2x\sqrt{\mathrm{Det}[(\xi^{\sigma}_{R\,\mu}\xi^{\dagger\rho}_{R\,\mu})^{-1}]}, \tag{3.22}$$

where a summation over μ is to be understood and where M^2 is the dynamically generated mass. This is just a cosmological term with $g^{\sigma\rho} = \sum_\mu \xi^{\sigma}_{R\,\mu}\xi^{\dagger\rho}_{R\,\mu}$. One can likewise verify that other pieces in the effective action are covariant. The coefficient of the cosmological constant term obtained at the lowest order in the heat kernel expansion does not agree with the one obtained through the gap equation. We shall see later why this is so.

Since the most general metric in two dimensions is conformally flat we can reconstruct the full covariant action from this particular choice. This simplifies notably the derivation of the effective action. We take $B^a{}_\mu(x) = \xi^a_{Lb}\bar{B}^b{}_\rho \xi_R^{-1\rho}{}_\mu(x) = M\phi^{-1}\delta^a_\mu$. The expressions that follow are specific to this gauge.

At second order in the heat kernel expansion (order $(\sqrt{t})^2$) the corresponding piece of the effective action reads

$$\begin{aligned} W^{(2)} = \int d^2x\ \phi^{-2} \Bigg[& \frac{3M^2}{16\pi}\left(\frac{2}{\epsilon} - \gamma - \log\left(\frac{M^2}{\mu^2}\right) + \log(8\pi) + 4\right) \\ & + \frac{(\partial_\mu\phi)^2}{4\pi}\left(\frac{2}{\epsilon} - \gamma - \log\left(\frac{M^2}{\mu^2}\right) + \log(8\pi) - \frac{5}{3}\right) \\ & + \frac{w^2\phi^2}{4\pi}\left(\frac{2}{\epsilon} - \gamma - \log\left(\frac{M^2}{\mu^2}\right) + \log(8\pi)\right)\Bigg]. \end{aligned} \tag{3.23}$$

We can take one step further and calculate the contribution to order t^2

$$\begin{aligned} W^{(4)} = \int d^2x\ \phi^{-2} \Bigg[& -\frac{3M^2}{32\pi}\left(\frac{2}{\epsilon} + \log(8\pi) - \log\left(\frac{M^2}{\mu^2}\right) - \gamma + \frac{5}{18}\right) \\ & -\frac{\phi^4(\partial_\mu w_\mu)^2}{4\pi M^2} + \frac{\phi^3\left(2w_\nu\partial_\mu\phi\partial_\mu w_\nu - 3w_\mu\partial_\mu\phi\partial_\nu w_\nu\right)}{3\pi M^2} \\ & \frac{\phi^2\left(\partial_\mu\phi\right)^2 w^2}{6\pi M^2} - \frac{w_\mu w_\nu\partial_\mu\phi\partial_\nu\phi}{\pi M^2} \\ & + \frac{\phi^2 w^2}{8\pi} - \frac{\phi^4 w^4}{4\pi M^2} - \frac{5(\partial_\mu\phi\partial_\mu\phi)}{48\pi} \\ & + \frac{\phi}{M^2}\frac{\partial_\mu\phi\partial_\nu\phi\partial_\mu\partial_\nu\phi}{3\pi} + \frac{\phi}{M^2}\frac{\partial_\mu\phi\partial_\mu\phi\partial_\nu\partial_\nu\phi}{15\pi} \\ & - \frac{7\phi^3}{M^2}\frac{\partial_\mu\partial_\mu\partial_\nu\partial_\nu\phi}{60\pi} - \frac{\partial_\mu\phi\partial_\mu\phi\partial_\nu\phi\partial_\nu\phi}{5\pi M^2}\Bigg]. \end{aligned} \tag{3.24}$$

The calculation of the fourth-order coefficients in the heat kernel expansion just shown is already a formidable task and we will not attempt to go beyond.

If we look at the results of the expansion at second order it is interesting to see that the terms that are generated are the ones expected from the point of view of general relativity. There is a cosmological term (proportional to ϕ^{-2}, which in covariant form corresponds to $\sqrt{g}$), and a Liouville term (proportional to $\phi^{-2}(\partial_\mu\phi)^2$, which in covariant form is non-local: $\sqrt{g}R\ \nabla^{-2}\sqrt{g}R$). In addition there is a term proportional to w^2 (which once written in a covariant form would be $\sqrt{g} g^{\mu\nu} w_\mu w_\nu$). Note that the Einstein term itself is topological in $D = 2$ and it is not expected to show up. However, in spite of these satisfactory results, we notice that the cosmological term does not quite coincide with the one previously derived, via the gap equation, and the Liouville term is apparently divergent casting doubts on the renormalizability of the model. We note that like in the chiral Lagrangian, the effective theory still possesses the full symmetry group G.

Yet it is easy to see that the above results are by necessity incomplete. For instance, the same operator ϕ^{-2} gets a contribution from the terms of order t and from t^2, ditto for Liouville. This comes from the fact that because the operator $X(\sqrt{t})$ contains terms linear in M and the heat kernel expansion is effectively an expansion in inverse powers of M, a given order in t does not correspond to a given order in derivatives or external fields. Therefore although the heat kernel calculation gives an interesting guidance to the form of the effective action and it shows the reappearance of covariance, the precise values of the coefficients of the different operators cannot be extracted from it. To solve this difficulty we turn to a diagrammatic calculation.

3.4 Diagrammatic calculation

Let us recapitulate. The heat-kernel calculation is plagued by two problems. The first one is related to the zero modes of the kinetic term, which increase considerably the difficulty of the calculations. The other one lies in the fact that the expansion is ill-defined in the sense of relevance of the subsequent orders. In a way, the heat-kernel fails to provide exact coefficients for the different operators but gives an accurate catalogue of the possible terms one could expect.

In this section we derive the Feynman rules of our toy model and proceed to calculate the exact contributions of the zero-, one- and two-point functions. As will be shown, we obtain finite contributions except for the cosmological term which nevertheless can be renormalized. The theory appears to be perfectly renormalizable in spite of the apparent bad power counting (due to the zero modes of the propagator).

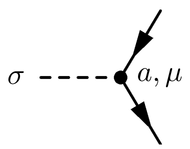
3.4.1 Feynman rules

We start by writing the generating functional of the theory in the conformal gauge and in Euclidean space. From it, we can read off the Feynman rules for the one- and two-point functions we are interested in. We know that the diagrammatic expansion is not covariant, but once we have convinced ourselves that covariance is recovered, we can use this method to identify specific coefficients. In this section it will be convenient to express the conformal gauge in the form

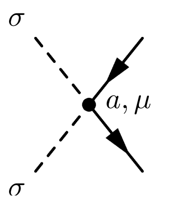
$$B^a_\mu(x) = M e^{-\sigma(x)/2} \delta^a_\mu. \tag{3.25}$$

The first term in the expansion of the exponential provides the dynamically generated mass for the fermions. Incidentally, this formalism is clearly quite reminiscent of chiral dynamics.

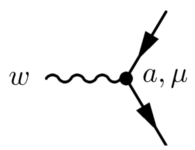
The interaction vertices are



$$i\frac{1}{2}M\delta^a_\mu$$



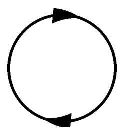
$$-\,i\frac{1}{8}M\delta^a_\mu \tag{3.26}$$



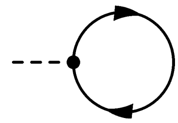
$$-\,i\gamma^a\sigma_3$$

3.4.2 Zero-, one- and two-point functions

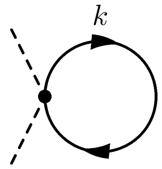
With the rules described in the previous subsection and the propagator derived in Section 2.5 we can calculate the exact contributions of the zero-, one- and two-point functions of the theory. Since the theory is non-standard, and it has a non-familiar set of Feynman rules, we will provide below the diagrams, after transcribing the Feynman rules, and the final result. Note that because there are two species of fermions the result from the Feynman diagrams has to be multiplied by a factor 2. Let us first consider one-point irreducible diagrams containing the σ field as external one



$$
\begin{aligned}
&= -\operatorname{Tr}\left[\int \frac{d^D k}{(2\pi)^D}\Delta^{-1}(k)^{\mu}{}_{a}\delta^{a}{}_{\mu}\right] \\
&= \frac{iM}{2\pi}\left(\frac{2}{\epsilon}-\gamma-\log\left(\frac{M^2}{\mu^2}\right)+\log(4\pi)\right),
\end{aligned}
$$

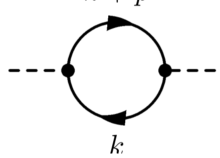


$$
\begin{aligned}
&= -\operatorname{Tr}\left[\int \frac{d^D k}{(2\pi)^D}\Delta^{-1}(k)^{\mu}{}_{a}\frac{i}{2}M\delta^{a}{}_{\mu}\right] \\
&= -\frac{M^2}{2\pi}\frac{1}{2}\left(\frac{2}{\epsilon}-\gamma-\log\left(\frac{M^2}{\mu^2}\right)+\log(4\pi)\right),
\end{aligned} \tag{3.27}
$$



$$
\begin{aligned}
&= -\operatorname{Tr}\left[\int \frac{d^D k}{(2\pi)^D}\Delta^{-1}(k)^{\mu}{}_{a}\frac{-i}{8}M\delta^{a}{}_{\mu}\right] \\
&= \frac{M^2}{2\pi}\frac{1}{8}\left(\frac{2}{\epsilon}-\gamma-\log\left(\frac{M^2}{\mu^2}\right)+\log(4\pi)\right).
\end{aligned}
$$

There is another diagram with two external scalar legs



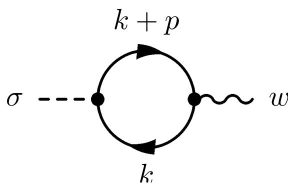
$$
\begin{aligned}
&= -\operatorname{Tr}\left[\int \frac{d^D k}{(2\pi)^D}\frac{i}{2}M\delta^{a}{}_{\mu}\Delta^{-1}(k)^{\mu}{}_{b}\frac{i}{2}M\delta^{b}{}_{\nu}\Delta^{-1}(k+p)^{\nu}{}_{a}\right] \\
&= \frac{M^2}{4\pi}\frac{1}{2}\left[\frac{2}{\epsilon}-\gamma-\log\left(\frac{M^2}{\mu^2}\right)+\log(4\pi)-2\right]-\frac{1}{48\pi}p^2
\end{aligned} \tag{3.28}
$$

from the M^2-terms in (3.27) and (3.28) we can already infer the total contribution to the cosmological term

$$
\frac{M^2 e^{-\sigma}}{2\pi}\left(\frac{2}{\epsilon}-\gamma-\log\left(\frac{M^2 e^{-\sigma}}{\mu^2}\right)+\log(4\pi)+1\right). \tag{3.29}
$$

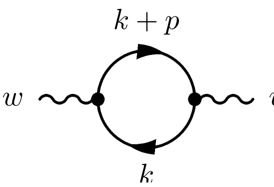
The divergence can be absorbed in the redefinition of the coupling constant, c. This result fully agrees with the one derived via the gap equation previously. In addition we observe that the p^2 piece in the last diagram will correspond in position space to the Liouville term. As it can be seen it is finite.

Next we look at the two-point function that mixes a σ-field with w-field. This could yield a R-type term but since in two dimensions gravity is topological we do not expect to see such term. Indeed, the diagram gives zero



$$= - \operatorname{Tr}\left[\int \frac{d^D k}{(2\pi)^D} \frac{i}{2} M \delta^a{}_\mu \Delta^{-1}(k)^\mu{}_b (-i\gamma^b \sigma_3) \Delta^{-1}(k+p)^\nu{}_a\right]$$
$$= 0. \tag{3.30}$$

Finally we calculate the last of the two-point functions possible. Again we obtain a finite result



$$= - \operatorname{Tr}\left[\int \frac{d^D k}{(2\pi)^D} (-i\gamma^a \sigma_3) \Delta^{-1}(k)^\mu{}_b (-i\gamma^b \sigma_3) \Delta^{-1}(k+p)^\nu{}_a\right]$$
$$= \frac{\delta^{\mu\nu}}{2\pi} - \frac{p^\mu p^\nu}{6\pi M^2}. \tag{3.31}$$

We see with relief that even if the ultraviolet behavior of each and one of the integrals is very bad, the final result hints to the renormalizability of the theory. After renormalizing the only coupling constant c of the theory the final result is perfectly finite.

3.4.3 Effective action

Let us now put all the pieces together and use the lowest order equations of motion for the field B^a_μ; or what is tantamount, for the dynamically generated mass M, to write the effective action. The result is

$$S_{\text{eff}} = \int d^2x \, \left[-\frac{M^2}{2\pi} e^{-\sigma} + \frac{1}{48\pi} \partial_\mu \sigma \partial_\mu \sigma + \frac{(\partial_\mu w_\mu)^2}{6\pi M^2} - \frac{w^2}{2\pi} + ...\right], \tag{3.32}$$

with M given by (3.6). This is the final result of this chapter. Now comparing this action with two-dimensional EH action in Euclidian conventions

$$S = -\int d^2x\sqrt{g}(-2\Lambda), \tag{3.33}$$

we see that the cosmological term has the 'wrong' sign.

Several comments are in order. First we recall that the effective action is written in the conformal gauge for the metric, but it is trivial to recover a full covariant form. Secondly, we note that there is no coupling between metric and connection, as befits the Palatini formalism in two dimensions where, exceptionally, metric and connection are unrelated. One can apply a variational principle to the affine connection w_μ in the above effective action, obtaining some equations of motion at $\mathcal{O}(p^2)$, but in $D = 2$ they do not provide any information on the conformal factor σ.

One is then left with a cosmological and a Liouville term, as corresponds to two-dimensional gravity [46]. The dots in (3.32) correspond to higher curvatures that we have not attempted to compute. In general they will be non-zero. Notice that the expansion is valid as long as the characteristic momenta fulfill $k < M$. Since M is the mass scale related to the two-dimensional cosmological constant, this would correspond to scales larger than the horizon.

Chapter 4

Extension of the model to four dimensions

We turn now to the far more interesting case of $D = 4$. We proceed to compute the vacuum of the theory and investigate the possible counterterms again. This time we skip the heat kernel calculation and focus in the diagrammatic approach. The number of possible degrees of freedom in this dimensionality grows up to six. A symmetric perturbation will, in general, have up to ten but we have at our hand the gauge condition allowing us to reduce the number to six. We shall consider only four of them to keep the calculations manageable; i.e. we use a general diagonal perturbation to perform most computations. However, we will resort to the far less general case of a conformal parametrization of the perturbation to compute some higher n-point function diagrams. In this dimensionality it is possible to work out the explicit relation between the connection and the vierbein after the breaking. Not surprisingly, we obtain precisely the usual relation of General Relativity. As we will see, in four dimensions one can write only one more counterterm than in $D = 2$ without making use of a metric, therefore the number of divergences is still under control and all the divergent terms can be identified, after the use of the equations of motion, with known objects in GR. We relegate to the Appendix the explicit calculation of the divergent term corresponding to the Gauss-Bonnet topological term, or at least to the part of it that can be reconstructed at the order we compute. Since covariance must be preserved no other divergence is expected to appear, even in higher dimensional terms we have not attempted to compute. The long distance effective theory possesses two free parameters which in principle could be adjusted to the values of M_p and Newton's constant via fine-tuning.

4.1 Gap equation

It was shown in the previous chapter that a consistent and apparently renormalizable model reproducing gravity at long distances could be built in $D = 2$. The same model can be considered in $D = 4$. Recall the free Lagrangian density is

$$\mathcal{L}_0 = i\bar{\psi}_a\gamma^a \left(\partial_\mu + iw_\mu^{bc}\sigma_{bc}\right)\chi^\mu + i\bar{\chi}^\mu\gamma^a \left(\partial_\mu + iw_\mu^{bc}\sigma_{bc}\right)\psi_a, \tag{4.1}$$

where ψ_a and χ^μ are two species of fermions transforming, respectively, under Lorentz (a, b... are the tangent indices, which can be considered internal ones for our purposes) and diffeomorphisms (μ, ν... are world indices labeling the manifold coordinates x^μ, globally defined on the manifold, with tangent vectors taken to be orthonormal with respect to the tangent space $SO(D)$ metric). A spin connection is added to the derivative to preserve the Lorentz$\times$*Diff* symmetry* under local coordinate transformations. It is important to notice that again there is no need to have a metric defined on the manifold as long as χ^μ transforms as a spinorial density because then $\Gamma^\mu_{\nu\rho}$ does not enter the covariant derivative, only w_μ^{ab}. If we keep this spin connection fixed there is no invariance under general coordinate transformations, but only under the global group $G = SO(D) \times GL(D)$. Notice once more that the spin connection is the only geometrical quantity introduced.

The interaction term in the model, in Euclidean conventions, is provided by

$$\mathcal{L}_I = iB_\mu^a(\bar{\psi}_a\chi^\mu + \bar{\chi}^\mu\psi_a) + c\ \det\left(B_\mu^a\right), \tag{4.2}$$

which obviously does not require any metric to be formulated either. We will assume that we have $2N$ species of the previous fermions but we will not add an additional index to avoid complicating the notation. We emphasize that Lorentz symmetry acts as an internal symmetry at this point.

*We actually use Euclidean conventions but still refer to $SO(D)$ as Lorentz symmetry. Note that (4.1) is not the usual Dirac coupling of fermions to a connection (that requires a metric). The field χ^μ has a spin 1/2 and 3/2 components in general, although this statement makes little sense unless a metric in the manifold is defined.

The object of the interaction (4.2) is to trigger the spontaneous breaking of the global symmetry via fermion condensation. Repeating the procedure in Chapter 3 we use the equations of motion for the auxiliary field B_μ^a

$$\bar{\psi}_a \chi^\mu + \bar{\chi}^\mu \psi_a = -ic \, \frac{1}{(D-1)!} \epsilon_{aa_2 \ldots a_D} \epsilon^{\mu\mu_2 \ldots \mu_D} B_{\mu_2}^{a_2} \ldots B_{\mu_D}^{a_D} \tag{4.3}$$

and thus

$$\langle \bar{\psi}_a \chi^\mu + \bar{\chi}^\mu \psi_a \rangle \neq 0 \Rightarrow \langle B_\mu^a \rangle \neq 0. \tag{4.4}$$

A non-trivial vacuum is assured. Small perturbations above this vacuum will yield the effective theory of the quantum excitations of the theory. We will use a perturbative approach corresponding to a weak field expansion around the solution for $w_\mu^{ab} = 0$; the value of the connection appears implicitly on the left-hand side of (4.3). We shall first consider the case $w_\mu^{ab} = 0$.

If $w_\mu^{ab} = 0$ the vacuum of the theory is expected to be translational invariant, i.e. we should obtain a constant value for B_μ^a, possibly zero. This constant value is obtained from the gap equation derived from the effective potential

$$V_{eff} = c \, \det(B_\mu^a) - 2N \int \frac{d^D k}{(2\pi)^D} \mathrm{tr}(\log(\gamma^a k_\mu + i B_\mu^a)). \tag{4.5}$$

Deriving (4.5) with respect to B_μ^a we obtain

$$c \, \frac{D}{D!} \epsilon_{aa_2 \ldots a_D} \epsilon^{\mu\mu_2 \ldots \mu_D} B_{\mu_2}^{a_2} \ldots B_{\mu_D}^{a_D} - 2Ni \, \mathrm{tr} \int \frac{d^D k}{(2\pi)^D} (\gamma^a k_\mu + i B_\mu^a)^{-1} |_a^\mu = 0. \tag{4.6}$$

This equation has a general non-trivial solution corresponding to $B_\mu^a = M \delta_\mu^a$ (or any $SO(D) \times GL(D)$ global transformations of this). This is analogous to the more familiar phenomenon of chiral symmetry breaking in strong interactions and any value of B_μ^a in the $SO(D) \times GL(D)$ orbit is equivalent. For simplicity we will take $B_\mu^a = M \delta_\mu^a$ and then the gap equation reduces to an equation for M. In $D = 4$

$$\begin{aligned} cM^3 - 2N \int \frac{d^D k}{(2\pi)^D} \frac{M}{k^2 + M^2} &= 0 \\ cM^3 + N \frac{M^3}{8\pi^2} \left(\frac{2}{\epsilon} - \log \frac{M^2}{4\pi\mu^2} - \gamma + 1 \right) &= 0, \end{aligned} \tag{4.7}$$

whose formal solution is

$$M^2 = \mu^2 e^{8\pi^2 c(\mu)/N}, \tag{4.8}$$

where $\mu\frac{dc}{d\mu} = -\frac{N}{4\pi^2}$, making M a renormalization group invariant. In the previous, we introduced the usual mass scale μ to preserve the correct dimensionality of the D-dimensional integral as dimensional regularization is used. Again for the solution to actually exist we have to require $c > 0$ if $M > \mu$. If $\mu > M$ the solution exists only if $c < 0$. Therefore $c > 0$ will be the case we are interested in on physical grounds.

Note that B^a_μ has the right structure to be identified as the vierbein, and as it was shown in Chapter 3, it consistently reemerges in the $D = 2$ effective theory to form the determinant of the spontaneously generated metric.

The free fermion propagator of the theory in the broken phase can then be easily found after replacing B^a_μ by its vacuum expectation value. With a $D = 4$ matrix notation

$$\Delta^{-1}(k)^i{}_j = \frac{-i}{M}\left(\delta^i{}_j - \frac{\gamma^i(\not{k} - iM)k_j}{k^2 + M^2}\right). \tag{4.9}$$

A particularity of $D = 2$ was that the most general form for B^a_μ (in Euclidean conventions) is a conformal factor times a scale M times a δ^a_μ. This means that perturbations around the minimum of the potential can only have one physical degree of freedom, the conformal parameter. The other degrees of freedom in B^a_μ can be removed by suitable coordinate transformations and are thus unphysical (recall that the microscopic theory is fully generally covariant –even without a metric).

The main difference of the $D = 4$ case with respect to the $D = 2$ case is that the maximum number of possible physical degrees of freedom for a perturbation around the value $B^a_\mu = M\delta^a_\mu$ grows up to six instead of one, making the calculation much more complex. Clearly, considering a uni-parametric family of perturbations is far too simple in $D = 4$ and does not yield enough information to find the long distance effective action unambiguously. To by-pass this difficulty, but still keeping the calculation manageable, we have chosen to restrict our considerations to diagonal perturbations, where

$$B^i_j(x) = M\delta^i_j e^{-\frac{\sigma_i(x)}{2}} \quad (\text{ no sum over } i). \tag{4.10}$$

This form contains four degrees of freedom (rather than six) but is rich enough for our purposes*. The validity of our conclusions rely on the assumption that the effective action should be covariant (exactly as the microscopic theory is). This was actually

*Note that these perturbations do not correspond to pure gauge degrees of freedom as they lead to non-zero values for the curvature, which is gauge invariant; i.e. they necessarily involve physical degrees of freedom checked in the $D = 2$ case using heat kernel techniques. Here we have performed partial checks but we have to assume that covariance holds to draw our conclusions.

Note that once a dynamical value for e^a_μ is generated we can write terms such as $M\bar{\psi}_a e^a_\mu \chi^\mu$, where e^a_μ would correspond the vierbein. A large number of Goldstone bosons are produced. The original symmetry group $G = SO(4) \times GL(4)$ has $\frac{4(4-1)}{2} + 4^2$ generators. After the breaking $G \to H$, with $H = SO(4)$, there are sixteen broken generators, as expected. Since the metric must be symmetric, at most ten Goldstone bosons can enter the perturbation. Four of those can be removed by a gauge choice, leaving the before mentioned six. Finally within each gauge choice a residual gauge freedom will in general allow for the removal of four more. In this respect our counting is analogous to the one in General Relativity. The final number of physical degrees of freedom will be two.

Another difference with respect to the $D = 2$ case is that the integrals involved in the perturbative calculation have potentially a much worse ultraviolet behavior in $D = 4$. We postpone to Sections 4.4 and 4.5 the explicit calculations that indicate that the nice characteristics found in the $D = 2$ model, in particular renormalizability, seem to persist in the $D = 4$ case. However the ultimate reason for the apparent renormalizability lies in the very limited number of counterterms that can be written without a metric (and the usual assumption that the ultraviolet behavior is unaltered by the phenomenon of spontaneous symmetry breaking).

4.2 Possible counterterms in $D = 4$

Now in $D = 4$ we can write, in addition to $\mathcal{L}_0$ and $\mathcal{L}_I$, one more counterterm

$$\mathcal{S}_R = \frac{1}{2} \int R_{[\mu\nu]ab} B^a_\rho B^b_\sigma \epsilon^{\mu\nu\rho\sigma} \, d^4x \tag{4.11}$$

where $R_{[\mu\nu]ab} = [\nabla_{\mu ac}, \nabla_{\nu cb}]$. After integrating the fermion fields only

$$\mathcal{S}_D = \frac{1}{4!} \int B^a_\mu B^b_\nu B^c_\rho B^d_\sigma \epsilon_{abcd} \epsilon^{\mu\nu\rho\sigma} \, d^4x, \tag{4.12}$$

which was already in present in $\mathcal{L}_I$, and $\mathcal{S}_R$ can appear as genuine divergences if general covariance is preserved. We will denote by $\mathcal{L}_D$ and $\mathcal{L}_R$ the respective Lagrangian densities.

There is another counterterm one could write without making use of a metric, namely the Gauss-Bonnet topological invariant in $D = 4$, which is of $\mathcal{O}(p^4)$ in the usual momentum counting.

We did not include the term $\mathcal{S}_R$ in our action to start with because it does not contain the fermionic fields. It does not modify the equation of motion (4.3) for the auxiliary field B^a_μ or the gap equation (4.7) either, if the connection w_μ is set to zero as we did in the previous section (recall that we use a weak field expansion and $w_\mu = 0$ is used to determine the vacuum). However, we see that $\mathcal{S}_R$ is an allowed counterterm in $D = 4$ and therefore it needs to be included in the initial action. In fact, any divergence in the theory must be reabsorbable in the two terms $\mathcal{S}_D$ and $\mathcal{S}_R$, as they are the only local counterterms one can write before the symmetry breaking, i.e. before the generation of the metric.

When the auxiliary field B^a_μ is identified with the vierbein, the parametrization (4.10) and the equations of motion are used the two counterterms reduce to

$$M^4 \int \sqrt{g}\; d^4x, \qquad M^2 \int \sqrt{g} R\; d^4x, \tag{4.13}$$

respectively; i.e. the familiar cosmological and Einstein terms. This will be explained in more detail in the next section.

4.3 Equations of motion in four dimensions

Let us write explicitly what the $D = 4$ counterterms look like once we replace B^a_μ by its vacuum expectation value plus perturbations around it. To keep the notation simple, let us consider the case in (4.10) when $\sigma_i(x) = \sigma(x)$ (conformally flat metric). After substituting the solution $B^a_\mu = M e^a_\mu = M e^{-\sigma/2} \delta^a_\mu$ we have

$$\mathcal{L}_D = \frac{1}{4!} B^a_\mu B^b_\nu B^c_\rho B^d_\sigma \epsilon_{abcd} \epsilon^{\mu\nu\rho\sigma} = M^4 e^{-2\sigma} \tag{4.14}$$

and

$$\frac{1}{2} R_{[\mu\nu]ab} B^a_\rho B^b_\sigma \epsilon^{\mu\nu\rho\sigma} = \frac{1}{2}(\partial_\mu w_\nu{}^{\mu\nu} - \partial_\nu w_\mu{}^{\mu\nu} + w_\mu{}^{\mu c} w_\nu{}^{c\nu} - w_\nu{}^{\mu c} w_\mu{}^{c\nu}) e^{-\sigma} M^2. \tag{4.15}$$

Note that because in the vacuum solution $e^a_\mu = \delta^a_\mu$ we can use indistinctively greek and latin indices; they are lowered and raised with a metric proportional to the identity.

Let us now work out the equations of motion for the full Lagrangian $\mathcal{L}_0+\mathcal{L}_I+\mathcal{L}_R$. In Section 4.1 we already discussed the equations of motion for the field B_μ^a when $w_\mu^{ab}=0$. In addition we have

$$\begin{aligned}\frac{\delta\mathcal{L}}{\delta w_\mu{}^{ab}} =&\partial_\rho\left(\frac{\delta(\mathcal{L}_0+\mathcal{L}_I+\mathcal{L}_R)}{\delta\partial_\rho w_\mu{}^{ab}}\right)-\frac{\delta(\mathcal{L}_0+\mathcal{L}_I+\mathcal{L}_R)}{\delta w_\mu{}^{ab}}=0\\ =&\frac{1}{2}\left(-\partial_a\sigma\delta_b^\mu+\partial_b\sigma\delta_a^\mu-\delta_a^\mu w_{\nu b}{}^\nu-\delta_b^\mu w_\nu{}^\nu{}_a+w_{ab}{}^\mu+w_b{}^\mu{}_a\right)e^{-\sigma}\\ &-\frac{1}{M^2}(\bar{\psi}_c\gamma^c\sigma_{ab}\chi^\mu+\bar{\chi}^\mu\gamma^c\sigma_{ab}\psi_c)=0.\end{aligned} \tag{4.16}$$

To solve (4.16) we will only consider the lowest order term in the $1/M^2$ expansion, following the usual counting rules in effective Lagrangians based on a momentum expansion. The solution for the connection in a conformally flat metric is then

$$w_\mu{}^{ab}=\frac{1}{2}(\partial^a\sigma\delta_\mu^b-\partial^b\sigma\delta_\mu^a). \tag{4.17}$$

This is the relation one obtains from the usual condition between the spin connection and the vierbein in General Relativity (4.18), characteristic of the Palatini formalism [44]

$$w_\mu^{ab}=e_\nu^a\partial_\mu E^{\nu b}+e_\nu^a E^{\rho b}\Gamma_{\mu\rho}^\nu \tag{4.18}$$

particularized to a conformally flat metric given by $e_\mu^a=\delta_\mu^a e^{-\frac{\sigma}{2}}$ ($E^{\rho b}$ is the inverse vierbein). Making use of (4.17) in (4.15), that is on shell, we are now allowed to identify the curvature in terms of the scalar field σ

$$\mathcal{L}_R|_{(\text{on shell})}=M^2\sqrt{g}R=\frac{3}{2}\left(\Box\sigma-\frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma\right)e^{-\sigma}M^2. \tag{4.19}$$

Note that in the particular case of a vierbein corresponding to a conformally flat metric one can integrate by parts either of the terms in (4.19) to obtain the other one

$$\sqrt{g}R=\frac{3}{2}M^2(\Box\sigma-\frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma)e^{-\sigma}\quad=\frac{3}{4}M^2(\Box\sigma)(1-\sigma+\frac{\sigma^2}{2}-\frac{\sigma^3}{6}+...). \tag{4.20}$$

This term plus a constant times (4.14) are the only divergences that should appear in the final effective theory upon integration of the fermionic fields for this particular type of perturbations above the vacuum (i.e. those interpretable as a conformally flat metric).

As previously mentioned we shall consider a more general type of perturbations; namely, we will use the diagonal parametrization of the perturbations around the vacuum solution given by (4.10). This is not the most general one in $D=4$, but it is enough for our purposes. After the identification of B_μ^a with the vierbein, this corresponds to a metric

$$g_{\mu\nu} = \begin{pmatrix} e^{-\sigma_1(x)} & 0 & 0 & 0 \\ 0 & e^{-\sigma_2(x)} & 0 & 0 \\ 0 & 0 & e^{-\sigma_3(x)} & 0 \\ 0 & 0 & 0 & e^{-\sigma_4(x)} \end{pmatrix}. \tag{4.21}$$

This parametrization provides enough generality to the calculation. We can now derive the equivalent expression to (4.17) for the general diagonal perturbation using (4.18) to obtain *

$$\begin{aligned} w_\mu{}^{ab} =& \frac{1}{2}\left[\begin{pmatrix} e^{\frac{\sigma_1}{2}} & 0 & 0 & 0 \\ 0 & e^{\frac{\sigma_2}{2}} & 0 & 0 \\ 0 & 0 & e^{\frac{\sigma_3}{2}} & 0 \\ 0 & 0 & 0 & e^{\frac{\sigma_4}{2}} \end{pmatrix}^{a\rho} \begin{pmatrix} \partial_\rho\sigma_1 e^{-\frac{\sigma_1}{2}} & 0 & 0 & 0 \\ 0 & \partial_\rho\sigma_2 e^{-\frac{\sigma_2}{2}} & 0 & 0 \\ 0 & 0 & \partial_\rho\sigma_3 e^{-\frac{\sigma_3}{2}} & 0 \\ 0 & 0 & 0 & \partial_\rho\sigma_4 e^{-\frac{\sigma_4}{2}} \end{pmatrix}^{b}_{\ \mu} \right. \\ & \left. - \begin{pmatrix} e^{\frac{\sigma_1}{2}} & 0 & 0 & 0 \\ 0 & e^{\frac{\sigma_2}{2}} & 0 & 0 \\ 0 & 0 & e^{\frac{\sigma_3}{2}} & 0 \\ 0 & 0 & 0 & e^{\frac{\sigma_4}{2}} \end{pmatrix}^{b\rho} \begin{pmatrix} \partial_\rho\sigma_1 e^{-\frac{\sigma_1}{2}} & 0 & 0 & 0 \\ 0 & \partial_\rho\sigma_2 e^{-\frac{\sigma_2}{2}} & 0 & 0 \\ 0 & 0 & \partial_\rho\sigma_3 e^{-\frac{\sigma_3}{2}} & 0 \\ 0 & 0 & 0 & \partial_\rho\sigma_4 e^{-\frac{\sigma_4}{2}} \end{pmatrix}^{a}_{\ \mu} \right]. \end{aligned} \tag{4.22}$$

Making use of the equations of motion one can compute the corresponding $\mathcal{L}_R$ for the general case and expand it in the σ fields. The result up to two sigma fields reads

$$\begin{aligned} \mathcal{L}_R|_{\text{(on shell)}} =& M^2\sqrt{g}R = M^2\left[\partial_3^2\sigma_4 + \partial_2^2\sigma_4 + \partial_1^2\sigma_4 + \partial_4^2\sigma_3 + \partial_2^2\sigma_3 + \partial_1^2\sigma_3\right. \\ & +\partial_4^2\sigma_2 + \partial_3^2\sigma_2 + \partial_1^2\sigma_2 + \partial_4^2\sigma_1 + \partial_3^2\sigma_1 + \partial_2^2\sigma_1 \\ & -\frac{1}{2}\left(\partial_3\sigma_1\partial_3\sigma_2 + \partial_4\sigma_1\partial_4\sigma_2 + \partial_2\sigma_1\partial_2\sigma_3 + \partial_4\sigma_1\partial_4\sigma_3 + \partial_2\sigma_1\partial_2\sigma_4 + \partial_3\sigma_1\partial_3\sigma_4\right. \\ & \left.+\partial_1\sigma_2\partial_1\sigma_3 + \partial_4\sigma_2\partial_4\sigma_3 + \partial_1\sigma_2\partial_1\sigma_4 + \partial_3\sigma_2\partial_3\sigma_4 + \partial_1\sigma_3\partial_1\sigma_4 + \partial_2\sigma_3\partial_2\sigma_4\right) \\ & \left.+\mathcal{O}(\sigma^3)\right]. \end{aligned} \tag{4.23}$$

More details on the calculation of (4.23) can be found in the Appendix. Ignoring for a moment the Gauss-Bonnet invariant, the divergent terms from the perturbative calculation for the general perturbation should match on shell either with (4.23) or with

$$\mathcal{L}_D|_{\text{(on shell)}} = M^4\sqrt{g} = M^4 e^{-\frac{\sum_i\sigma_i}{2}}. \tag{4.24}$$

*Again we emphasize that although it may seem strange to see latin indices in the derivatives this should not confuse the reader. After the symmetry breaking a vierbein is generated relating world indices with tangent space ones through δ_μ^a. In expression (4.22) we have compiled the entries for w_μ^{ab} in a bi-matrix form, but they *should not* be multiplied; only the index ρ is summed up.

An extension to the most general perturbation with the full six degrees of freedom should be possible but would require much more effort, which we consider unnecessary at this point as the above parametrization provides enough redundancy. Since the coefficients for the terms in the effective action are universal there should be no loss of generality in the present approach. This of course assumes that general covariance is kept all along the derivation of the effective action and by the regulator, as it should be the case in dimensional regularization.

So far we have explained how the $D = 2$ model can be consistently extended to $D = 4$ preserving the key features. We study small perturbations around a constant vacuum expectation value for the field B^a_μ (which does not need to be small itself) corresponding to the solution of the gap equation for $w^{ab}_\mu = 0$. In such a theory one can write a limited number of counterterms without making use of a metric. These counterterms are consistent with the usual terms of GR once used the equations of motion. With all these ingredients we are ready to move to the actual perturbative derivation of the effective action.

4.4 One-loop structure for a general diagonal perturbation

The effective action that describes perturbations above the trivial vacuum

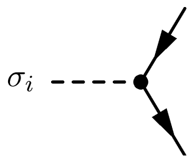
$$w^{ab}_\mu = 0, \qquad B^a_\mu = M\delta^a_\mu, \tag{4.25}$$

will be given by a polynomial expansion in powers of $w_\mu(x)$, $\sigma_i(x)$ and their derivatives obtained after integration of the fundamental degrees of freedom. In this section we will derive this effective action diagrammatically.

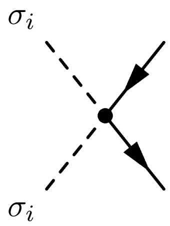
We shall use the diagonal perturbation (4.10) with four degrees of freedom for the vierbein perturbations. For simplicity, we will calculate only the one-point and two-point functions for this rather general case and then particularize to the conformal case ($\sigma_i(x) = \sigma(x)$) to compute some three-point functions.

Since perturbation theory in this model has some peculiar features (note in particular the behavior of the fermion propagator) in what follows we shall provide enough details so that the diagrammatic calculation can be reproduced.

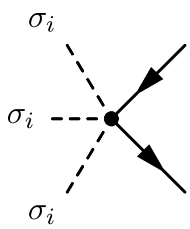
Starting from the Lagrangian density $\mathcal{L}_0 + \mathcal{L}_I$ described in Section 4.1 (note that $\mathcal{L}_R$ plays no role whatsoever in the integration of the $2N$ species of fermions), and using a parametrization of B^a_μ given by (4.10), the interaction vertices are



$$i\frac{1}{2}M\delta^i{}_\mu$$



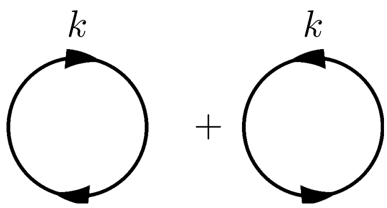
$$-\,i\frac{1}{8}M\delta^i{}_\mu \tag{4.26}$$



$$i\frac{1}{48}M\delta^i{}_\mu$$

4.4.1 One-point and two-point functions for the fields σ_i

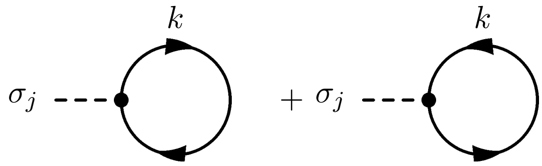
With the rules described above and using the propagator (4.9) we can calculate the first one-loop diagrams for $D = 4 - \epsilon$. We will not include the factor N in the diagrammatic results presented below. The vacuum bubble diagram is



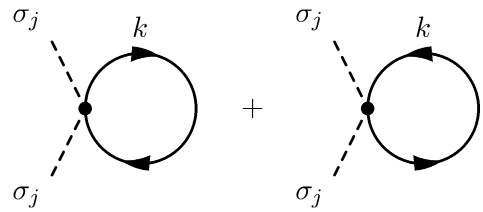
(4.27)

$$= -2\mathrm{Tr}\left[\int \frac{d^D k}{(2\pi)^D}\Delta^{-1}(k)^\mu{}_a(-i)\delta^a{}_\mu\right] = -\frac{M^3}{2\pi^2}\left(\frac{2}{\epsilon} - \log\left(\frac{M^2}{4\pi\mu^2}\right) - \gamma + 1\right).$$

We also compute the one-point function for the different vertices

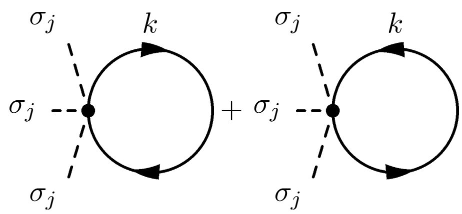


$$= -\sum_{j=1}^{4} \sigma_j \frac{2}{2} \mathrm{Tr}\left[\int \frac{d^D k}{(2\pi)^D} \Delta^{-1}(k)^{\mu}{}_{j} (i) M \delta^{j}{}_{\mu}\right] = \sum_{j=1}^{4} \frac{\sigma_j M^4}{16\pi^2}\left(\frac{2}{\epsilon} - \log\left(\frac{M^2}{4\pi\mu^2}\right) - \gamma + 1\right), \tag{4.28}$$



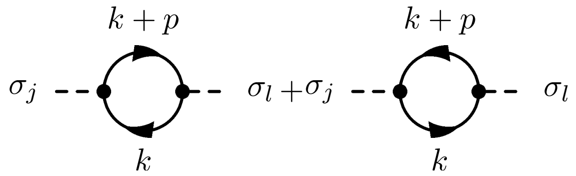
$$\begin{aligned} &= -\sum_{j=1}^{4} \sigma_j^2 \frac{2!\ 2}{2!\ 2\cdot 2} \mathrm{Tr}\left[\int \frac{d^D k}{(2\pi)^D} \Delta^{-1}(k)^{\mu}{}_{j} (-i) M \delta^{j}{}_{\mu}\right] \\ &= -\sum_{j=1}^{4} \sigma_j^2 \frac{M^4}{32\pi^2}\left(\frac{2}{\epsilon} - \log\left(\frac{M^2}{4\pi\mu^2}\right) - \gamma + 1\right). \end{aligned} \tag{4.29}$$

Let us, for this particular diagram, clarify what the origin of the numerical factor is. In the numerator, the 2! comes from the combinatorial possible connections of the external fields σ_i. The other 2 is due to the two species of fermions and it is present in all diagrams. In the denominator, $2!2\cdot 2$ comes from the vertex. Since it is a one-point function there are no additional factors, however for n-point functions the corresponding $n!$ will be present in the denominator.



$$\begin{aligned} &= -\sum_{j=1}^{4} \frac{3!\ 2}{3!\ 2\cdot 2\cdot 2} \mathrm{Tr}\left[\int \frac{d^D k}{(2\pi)^D} \Delta^{-1}(k)^{\mu}{}_{j} (i) M \delta^{j}{}_{\mu}\right] \\ &= \sum_{j=1}^{4} \frac{\sigma_j^3 M^4}{64\pi^2}\left(\frac{2}{\epsilon} - \log\left(\frac{M^2}{4\pi\mu^2}\right) - \gamma + 1\right). \end{aligned} \tag{4.30}$$

Next diagram is the two-point function

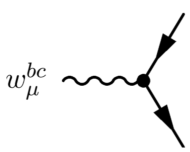


$$
\begin{aligned}
&= -\sum_{j=1}^{4}\sum_{l=1}^{4}\sigma_j\sigma_l\frac{2!\ 2}{2!\cdot 2\cdot 2}\mathrm{Tr}\left[\int\frac{d^Dk}{(2\pi)^D}iM\delta^l{}_\mu\Delta^{-1}(k)^\mu{}_j iM\delta^j{}_\nu\Delta^{-1}(k+p)^\nu{}_l\right]\\
&= \sum_{j=1}^{4}\sum_{\substack{l=1\\ l\neq j}}^{4}\left[-\frac{\sigma_j\sigma_l M^4}{16\pi^2}\left(\frac{2}{\epsilon}-\log\left(\frac{M^2}{4\pi\mu^2}\right)-\gamma+\frac{2}{3}\right)-\frac{\sigma_j\sigma_l p^2M^2}{48\pi^2}\left(\frac{2}{\epsilon}-\log\left(\frac{M^2}{4\pi\mu^2}\right)-\gamma-\frac{1}{3}\right)\right.\\
&\quad\left.+\frac{\sigma_j\sigma_l(p_j^2+p_l^2)M^2}{48\pi^2}\left(\frac{2}{\epsilon}-\log\left(\frac{M^2}{4\pi\mu^2}\right)-\gamma-\frac{1}{2}\right)\right]\\
&\quad+\sum_{j=1}^{4}\left[+\frac{\sigma_j^2M^4}{32\pi^2}+\frac{\sigma_j^2M^2p^2}{32\pi^2}-\frac{\sigma_j^2M^2p_j^2}{32\pi^2}\right]+\mathcal{O}(p^4).
\end{aligned}
\tag{4.31}
$$

The numerical factor in this case is composed by 2! 2 in the numerator from the possible contractions of external fields times the two species of fermions. And 2! in the denominator from the diagram being a two-point function and finally the 1/2 from each vertex. We will not elaborate on the combinatorial factors anymore but we write all factors explicitly, even if the notation may be a bit cumbersome, in order to facilitate the check of our results. By $\mathcal{O}(p^4)$ we mean finite higher order in p^2 contributions.

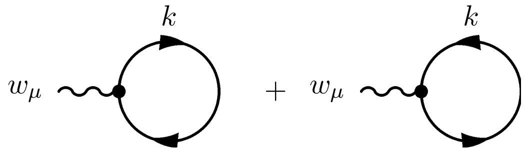
4.4.2 Diagrams with w_μ^{ab}

Now we turn to the diagrams that contain a field w_μ^{ab}. The corresponding vertex is



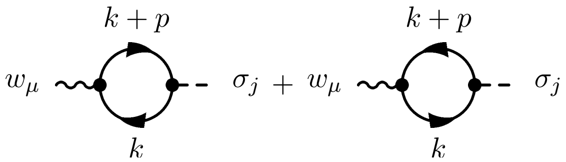
$$i\gamma^a w_\mu^{bc}\sigma_{bc}=\frac{\gamma^a}{4}[\gamma_b,\gamma_c]. \tag{4.32}$$

The one- and two-point functions yield



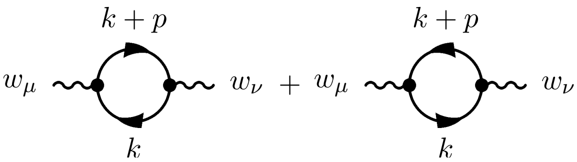
$$= -2\mathrm{Tr}\left[\int \frac{d^D k}{(2\pi)^D} i\gamma^a \sigma_{bc} \Delta^{-1}(k)^\mu{}_a\right] = 0, \tag{4.33}$$

then



$$= -\sum_{j=1}^{4} \sigma_j \frac{2}{2!\cdot 2}\mathrm{Tr}\left[\int \frac{d^D k}{(2\pi)^D} i\gamma^a \sigma_{bc} \Delta^{-1}(k)^\mu{}_d iM\delta^d{}_\nu \Delta^{-1}(k+p)^\nu{}_a\right] = 0. \tag{4.34}$$

Suggesting that diagrams containing only one field w^{ab}_μ are zero. For two w^{ab}_μ fields we have



$$\begin{aligned}
&= -\frac{2!\cdot 2}{2!}\mathrm{Tr}\left[\int \frac{d^D k}{(2\pi)^D} i\gamma^a \sigma_{bc} \Delta^{-1}(k)^\mu{}_d i\gamma^d \sigma_{ef} \Delta^{-1}(k+p)^\nu{}_a\right] \\
&= \left(\frac{2}{\epsilon} - \log\left(\frac{M^2}{4\pi\mu^2}\right) - \gamma + \frac{1}{4}\right)\left[\frac{M^2}{4\pi^2}(\delta_{be}\delta^\nu_c\delta^\mu_f - \delta_{be}\delta_{cf}\delta^{\mu\nu} + \delta_{bf}\delta_{ce}\delta^{\mu\nu} - \delta_{bf}\delta^\nu_c\delta^\mu_e\right. \\
&\quad \left. +\delta^\nu_b\delta_{cf}\delta^\mu_e - \delta^\nu_b\delta_{ce}\delta^\mu_f)\right] + \frac{M^2}{16\pi^2}\left(\delta_{bf}\delta^\nu_e\delta^\mu_c - \delta_{be}\delta^\mu_c\delta^\nu_f + \delta^\mu_b\delta_{ce}\delta^\nu_f - \delta^\mu_b\delta^\nu_e\delta_{cf}\right) \\
&\quad + \frac{1}{\epsilon}F^{\mu\nu}{}_{bcef}(p^2) + \mathcal{O}(p^2) \\
&= \left(\frac{2}{\epsilon} - \log\left(\frac{M^2}{4\pi\mu^2}\right) - \gamma + \frac{1}{4}\right)\left[\frac{M^2}{4\pi^2}D^{\mu\nu}{}_{bcef}\right] + \frac{M^2}{16\pi^2}E^{\mu\nu}{}_{bcef} + \frac{1}{\epsilon}F^{\mu\nu}{}_{bcef}(p^2) + \mathcal{O}(p^2).
\end{aligned} \tag{4.35}$$

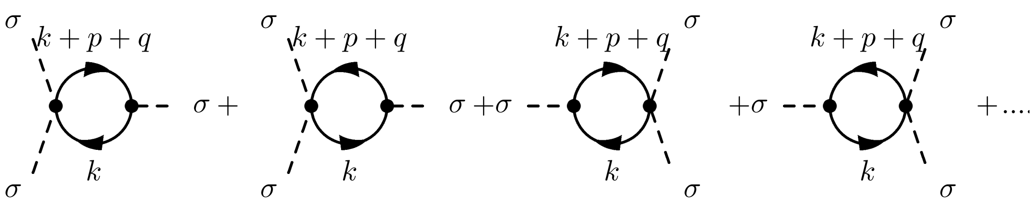
Where $F^{\mu\nu}{}_{bcef}$ is a complicated structure composed of external momenta and Kronecker deltas of order $\mathcal{O}(p^2)$. This divergence is of higher order, in the $1/M^2$ expansion, than the one of $D^{\mu\nu}{}_{bcef}$. Now, taking into account that $w^{bc}_\mu = -w^{cb}_\mu$, we can show that

$$D^{\mu\nu}{}_{bcef} w^{bc}_\mu w^{ef}_\nu = 4w^{\nu b}_\mu w^{\mu b}_\nu - 2w^{be}_\mu w^{be}_\mu = 0; \quad E^{\mu\nu}{}_{bcef} w^{bc}_\mu w^{ef}_\nu = 4w^{\mu b}_\mu w^{b\nu}_\nu. \tag{4.36}$$

More details can be found in the Appendix where we will show how the combination of the divergences proportional to ${F^{\mu\nu}}_{bcef}$ appearing in this diagram combine to reproduce exactly the $\mathcal{O}(p^4)$ terms of the Gauss-Bonnet invariant.

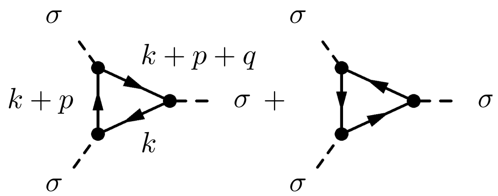
4.4.3 Other two-point and three-point functions

In order to keep the calculations simple, we particularize to the case $B^a_\mu = Me^{-\frac{\sigma(x)}{2}}\delta^a_\mu$. The previous results, (4.27–4.36) are all valid taking $\sigma_i = \sigma$, $i = 1, 2, 3, 4$. With this simplification we can easily further compute more diagrams. For the field σ we have



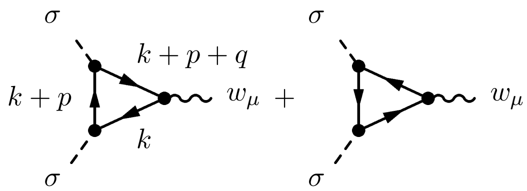
$$
\begin{aligned}
=&-\frac{2\ 2\ 2}{2!\ 2!\cdot 2\cdot 2\cdot 2}\mathrm{Tr}\left[\int\frac{d^Dk}{(2\pi)^D}(-i)M{\delta^a}_\mu{\Delta^{-1}(k)^\mu}_b iM{\delta^b}_\nu{\Delta^{-1}(k+p+q)^\nu}_a\right]\\
&-\frac{2\ 2\ 2}{2!\ 2!\cdot 2\cdot 2\cdot 2}\mathrm{Tr}\left[\int\frac{d^Dk}{(2\pi)^D}(-i)M{\delta^a}_\mu{\Delta^{-1}(k+p)^\mu}_b iM{\delta^b}_\nu{\Delta^{-1}(k+p+q)^\nu}_a\right]\\
&-\frac{2\ 2\ 2}{2!\ 2!\cdot 2\cdot 2\cdot 2}\mathrm{Tr}\left[\int\frac{d^Dk}{(2\pi)^D}(-i)M{\delta^a}_\mu{\Delta^{-1}(k+q)^\mu}_b iM{\delta^b}_\nu{\Delta^{-1}(k+p+q)^\nu}_a\right]\\
=&\frac{9M^4}{16\pi^2}\left(\frac{2}{\epsilon}-\log\left(\frac{M^2}{4\pi\mu^2}\right)-\gamma+\frac{1}{3}\right)\\
&+\frac{M^2(p^2+(p+q)^2+q^2)}{32\pi^2}\left(\frac{2}{\epsilon}-\log\left(\frac{M^2}{4\pi\mu^2}\right)-\gamma-\frac{2}{3}\right)+\mathcal{O}(p^4).
\end{aligned}
\tag{4.37}
$$

And also



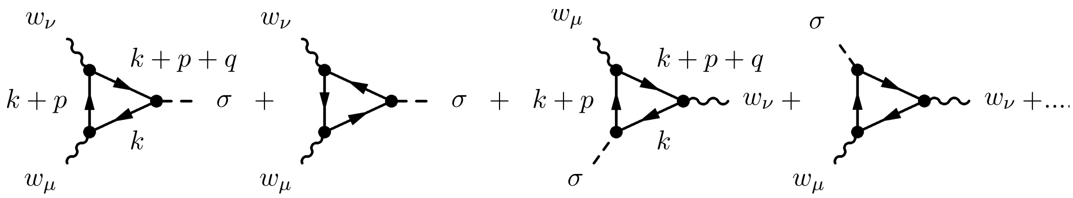
$$
\begin{aligned}
=&-\frac{3!\ 2}{3!\ 2\cdot 2\cdot 2}\mathrm{Tr}\left[\int\frac{d^Dk}{(2\pi)^D}iM{\Delta^{-1}(k)^\mu}_b iM{\delta^b}_\nu{\Delta^{-1}(k+p+q)^\nu}_d iM{\delta^d}_\rho{\Delta^{-1}(k+p)^\rho}_a\right]\\
&-\frac{3!\ 2}{3!\ 2\cdot 2\cdot 2}\mathrm{Tr}\left[\int\frac{d^Dk}{(2\pi)^D}iM{\Delta^{-1}(k)^\mu}_b iM{\delta^b}_\nu{\Delta^{-1}(k+p+q)^\nu}_d iM{\delta^d}_\rho{\Delta^{-1}(k+q)^\rho}_a\right]\\
=&\frac{3M^4}{8\pi^2}\left(\frac{2}{\epsilon}-\log\left(\frac{M^2}{4\pi\mu^2}\right)-\gamma-\frac{2}{3}\right)-M^2\left(\frac{p^2}{32\pi^2}+\frac{(p+q)^2}{32\pi^2}+\frac{q^2}{32\pi^2}\right)+\mathcal{O}(p^4).
\end{aligned}
\tag{4.38}
$$

On the other hand, for the field w_μ^{ab} we can compute



$$
\begin{aligned}
&= -\frac{2!\;2}{3!\;2\cdot 2}\mathrm{Tr}\left[\int \frac{d^Dk}{(2\pi)^D} i\gamma^a\sigma_{bc}\Delta^{-1}(k)^\mu{}_d iM\delta^d{}_\nu\Delta^{-1}(k+p+q)^\nu{}_e iM\delta^e{}_\rho\Delta^{-1}(k+p)^\rho{}_a\right]\\
&\quad -\frac{2!\;2}{3!\;2\cdot 2}\mathrm{Tr}\left[\int \frac{d^Dk}{(2\pi)^D} i\gamma^a\sigma_{bc}\Delta^{-1}(k)^\mu{}_d iM\delta^d{}_\nu\Delta^{-1}(k+p+q)^\nu{}_e iM\delta^e{}_\rho\Delta^{-1}(k+q)^\rho{}_a\right] = 0.
\end{aligned}
\tag{4.39}
$$

And finally



$$
\begin{aligned}
&= -\frac{3\;2!\;2}{3!\cdot 2}\mathrm{Tr}\left[\int \frac{d^Dk}{(2\pi)^D} i\gamma^a\sigma_{bc}\Delta^{-1}(k)^\mu{}_d\gamma^d\sigma_{ef}\Delta^{-1}(k+p+q)^\nu{}_g iM\delta^g{}_\rho\Delta^{-1}(k+p)^\rho{}_a\right]\\
&\quad -\frac{3\;2!\;2}{3!\cdot 2}\mathrm{Tr}\left[\int \frac{d^Dk}{(2\pi)^D} i\gamma^a\sigma_{bc}\Delta^{-1}(k)^\mu{}_d i\gamma^d\sigma_{ef}\Delta^{-1}(k+p+q)^\nu{}_g iM\delta^g{}_\rho\Delta^{-1}(k+q)^\rho{}_a\right]\\
&= \left(\frac{2}{\epsilon} - \log\left(\frac{M^2}{4\pi\mu^2}\right) - \gamma - \frac{3}{4}\right)\left[-\frac{M^2}{4\pi^2}D^{\mu\nu}{}_{bcef}\right] - \frac{M^2}{16\pi^2}E^{\mu\nu}{}_{bcef} + \mathcal{O}(p^2).
\end{aligned}
\tag{4.40}
$$

With $D^{\mu\nu}{}_{bcef}$ and $E^{\mu\nu}{}_{bcef}$ being the same as in (4.35).

4.5 Summary of divergences

In the previous section we obtained the results of the one-, two- and three-point functions for a general diagonal perturbation, sometimes particularizing to a conformally flat metric to ease the notation. Let us now summarize the results.

The divergent part of diagrams (4.27–4.30) together with the M^4 piece of diagram (4.31) add up in the effective action to*

$$
\frac{M^4 e^{-\frac{\sum_i \sigma_i}{2}}}{8\pi^2}\left(\frac{2}{\epsilon} - \log\left(\frac{M^2}{\mu^2}\right)\right). \tag{4.41}
$$

*Note that factors $1/n!$, where n is the number of identical external legs, and a sign flip, are needed to reconstruct the term in the effective action from the diagrammatic calculation.

Note that the dimensionality of this term matches (4.14). Furthermore, it can be proved that the divergent terms in (4.31) proportional to M^2p^2 are precisely those corresponding to (4.23) in momentum space, thus allowing us to recover the first orders of $\mathcal{L}_R$ for the general diagonal perturbation, which on shell correspond to $\sqrt{g}R$.

Diagram (4.35) has one divergent term proportional to $F^{\mu\nu}{}_{bcef}$ which is of higher order, i.e. $\mathcal{O}(\sqrt{g}R^2)$. Before addressing this apparent new divergence let us particularize to the case of a conformally flat perturbation above the vacuum. Taking $\sigma_i = \sigma$, (4.27–4.30) plus (4.31) add up in the effective action to

$$\frac{M^4 e^{-2\sigma}}{8\pi^2}\left(\frac{2}{\epsilon} - \log\left(\frac{M^2 e^{-\sigma}}{4\pi\mu^2}\right) - \gamma + \frac{3}{2}\right) = \frac{M^4 e^{-2\sigma+\frac{\sigma}{2}\epsilon}}{8\pi^2}\left(\frac{2}{\epsilon} - \log\left(\frac{M^2}{4\pi\mu^2}\right) - \gamma + \frac{3}{2}\right). \tag{4.42}$$

An important remark is in order at this point. Note the peculiar form of $e^{-2\sigma+\frac{\sigma}{2}\epsilon}$: this factor corresponds to the determinant of a conformally flat metric in $D = 4 - \epsilon$ dimensions and it is a remanent of the fact that we used dimensional regularization to calculate the momentum integrals. Of course $\lim_{\epsilon\to 0}\sqrt{g_D} = \sqrt{g}$ (where g_D is the determinant of the D-dimensional metric), but this is telling us that in order to regularize our integrals it is not enough to add a mass scale to match the dimensionality; an ϵ power of the determinant of the metric is also needed to ensure diffeomorphism invariance. That is $\mu^2 \to \mu^2 e^{-\sigma}$. Then (4.42) would read

$$\frac{M^4 e^{-2\sigma}}{8\pi^2}\left(\frac{2}{\epsilon} - \log\left(\frac{M^2}{4\pi\mu^2}\right) - \gamma + \frac{3}{2}\right). \tag{4.43}$$

Continuing with the divergences, diagrams (4.37) and (4.38) contain terms of order M^4 that are the subsequent orders of the expansion of (4.42) in terms of σ. As for the terms of order p^2, one has to express them in position space. The result of diagram (4.31) for instance is

$$-\frac{\sigma p^2 \sigma M^2}{16\pi^2}\left(\frac{2}{\epsilon} - \log\left(\frac{M^2}{4\pi}\right) - \gamma - \frac{2}{3}\right) \tag{4.44}$$

that in position space reads

$$\frac{\sigma\Box\sigma M^2}{16\pi^2}\left(\frac{2}{\epsilon} - \log\left(\frac{M^2}{4\pi}\right) - \gamma - \frac{2}{3}\right). \tag{4.45}$$

The next diagrams we consider are (4.37) plus (4.38)

$$\frac{M^2\sigma^2(p^2 + (p+q)^2 + q^2)\sigma}{32\pi^2}\left(\frac{2}{\epsilon} - \log\left(\frac{M^2}{4\pi}\right) - \gamma - \frac{5}{3}\right), \tag{4.46}$$

or in position space

$$-\frac{3M^2\sigma^2\Box\sigma}{32\pi^2}\left(\frac{2}{\epsilon}-\log\left(\frac{M^2}{4\pi}\right)-\gamma-\frac{5}{3}\right). \tag{4.47}$$

Now it is clear that the full calculation at order M^2p^2 resums to the following term in the effective action

$$\frac{M^2\Box\sigma e^{-\sigma}}{32\pi^2}\left(\frac{2}{\epsilon}-\log\left(\frac{M^2}{4\pi\mu^2}\right)-\gamma+\frac{1}{3}\right). \tag{4.48}$$

Note that this term has the same structure that $\sqrt{g}R$ for a conformally flat metric. This divergence can be absorbed by redefining $\mathcal{L}_R$ and using the equations of motion. This is already telling us that the theory is renormalizable only on shell*. Namely when the spin connection w_μ^{ab} corresponds to the Levi-Civita one. In our approach this identification is forced by the use of the equations of motion.

For a general diagonal perturbation one has to consider the momentum dependent $O(p^2)$ divergent pieces in (4.31) and similar diagrams with more external scalar legs. As a check we can see that the momentum dependent terms with two σ_i fields faithfully reproduce the $O(\sigma^2)$ piece in the curvature term (4.23) thus confirming the general covariance of the effective action. Details are relegated to the Appendix.

Let us now retake the issue of the apparent new divergences emerging from (4.35). To see if they really contribute to the final effective action we have to express them in terms of the σ fields using the available equations of motion. Then in principle, they must either vanish or correspond to a valid counterterm. We argued in Section 4.1 that there is a third possible counterterm in $D = 4$, the Gauss-Bonnet term, which is a total derivative and should not contribute to the dynamics. In the Appendix it is shown how the lower order divergence vanishes and how the higher order term indeed corresponds to a piece of the Gauss-Bonnet term.

4.6 Effective action and physical constants

We are now ready to write the effective action we obtain on shell; that is once the spin connection is set to the value obtained after use of the equations of motion and the

*Please note that this is quite unrelated to the well-known fact that pure gravity at one-loop is finite on shell. The latter result corresponds to performing a one-loop calculation with gravitons. Here instead we integrate the microscopic degrees of freedom that supposedly generate the gravitons after spontaneous symmetry breaking and generation of the metric degrees of freedom.

gap equation is used. We shall present details only for a vierbein corresponding to a conformally flat metric but as previously discussed we have a good check of its validity for the divergent parts of a general diagonal perturbation above the vacuum.

We recall our conventions. We have used Euclidean conventions so that the (emerging) metric has signature (+,+,+,+). The effective action at long distances is defined by the functional integral

$$\int [dg] \exp\left(-S[g]\right), \tag{4.49}$$

where

$$g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu = \frac{1}{M^2} \eta_{ab} B^a_\mu B^b_\nu, \tag{4.50}$$

according to our discussion in Section 4.1.

The effective action obtained after the diagrammatic calculation of the previous sections is

$$\begin{aligned} S_{eff} = \int \; d^4x \; & \left(c' M^4 e^{-2\sigma} - N \frac{M^4}{8\pi^2} e^{-2\sigma} \left(\log\left(\frac{M^2}{\mu^2}\right) - \frac{3}{2} \right) \right. \\ & \left. + A' M^2 \Box\sigma e^{-\sigma} - N \frac{M^2}{32\pi^2} \Box\sigma e^{-\sigma} \left(\log\left(\frac{M^2}{\mu^2}\right) - \frac{28}{3} \right) \right) + ..., \end{aligned} \tag{4.51}$$

where $c' = c + \frac{N}{8\pi^2}\left(\frac{2}{\epsilon} + \log 4\pi - \gamma\right)$ and $A' = A + \frac{N}{8\pi^2}\left(\frac{2}{\epsilon} + \log 4\pi - \gamma\right)$ are renormalized coupling constants that have absorbed the divergences. The $\overline{MS}$ subtraction scheme is assumed. Note that the finite part of the term proportional to M^2 has received a contribution from the diagrams containing only w^{ab}_μ fields, see the Appendix. Making use of the solution (4.8) of the gap equation we can write the previous expression as

$$S_{eff} = \int d^4x \; \left(N \tfrac{M^4}{16\pi^2} e^{-2\sigma} + A' \Box\sigma e^{-\sigma} M^2 - N \tfrac{M^2}{32\pi^2} \Box\sigma e^{-\sigma} \left(\log\left(\tfrac{M^2}{\mu^2}\right) - \tfrac{28}{3} \right) \right) + ..., \tag{4.52}$$

The resulting effective theory thus describes a geometry with a cosmological term. Sometimes it is stated in the literature, see the first reference of [10], that if gravity is an emergent phenomenon and gravitons are Goldstone bosons all interactions should be of a derivative nature and the cosmological constant problem would be in a sense solved. This is not so, as we see a cosmological terms is generated necessarily (both in $D = 2$ and $D = 4$), at least in the present approach.

The previous result is not exact of course. The effective action is in fact an infinite series containing higher order derivatives, starting with terms of $O(\sqrt{g}R^2)$ and so on, which are represented by the dots in the previous expression. In fact, as we have seen, a counterterm proportional to Gauss-Bonnet (of order $\mathcal{O}(\sqrt{g}R^2)$) is required; finite terms will appear too. The effective action should also contain a non-local finite piece corresponding to the conformal anomaly (of dimension four in $D = 4$ [47]). The conformal anomaly was indeed reproduced in the previous chapter in $D = 2$ [48]. Note that any dimension four term that is generated will be accompanied by a factor of N. The dimension six terms will be of $O(N/M^2)$ and so forth. It would be natural to redefine the constant A' to include this factor of N in order to keep the counting of powers of N homogeneous.

Appealing to covariance arguments we can now express (4.51) in terms of invariants

$$S_{eff} = \int d^4x \left[\frac{N}{16\pi^2} M^4 \sqrt{g} + \left(A' - \frac{N}{48\pi^2} \left(\log \left(\frac{M^2}{\mu^2} \right) - \frac{28}{3} \right) \right) M^2 \sqrt{g} R + ... \right]. \tag{4.53}$$

Next we recall that the classical Einstein action corresponding to the Euclidean conventions is [49]

$$S = -\frac{M_P^2}{32\pi} \int d^4x \sqrt{g}(R - 2\Lambda). \tag{4.54}$$

Now identifying

$$\begin{aligned} \frac{N}{16\pi^2} M^4 &= 2\Lambda \frac{M_P^2}{32\pi}, \\ M^2 \left(A' - \frac{N}{48\pi^2} \left(\log \left(\frac{M^2}{\mu^2} \right) - \frac{28}{3} \right) \right) &= -\frac{M_P^2}{32\pi}, \end{aligned} \tag{4.55}$$

we indeed obtain

$$S_{eff} = -\frac{M_p^2}{32\pi} \int d^4x \sqrt{g}(R - 2\Lambda) + \mathcal{O}(p^4). \tag{4.56}$$

As we see from the previous discussion, the integration of the fermions (assumed to be the fundamental degrees of freedom in the theory) yields a positive cosmological constant. As for the value of M_P^2, the Planck mass squared, the sign is not really automatically defined. More on this later.

4.6.1 Fine-tuning and running of the constants

To ensure that the action is renormalization group invariant, thus observable, the following beta function for each free constant in the theory must be obeyed

$$\begin{aligned} \mu \frac{dc'}{d\mu} &= -\frac{N}{4\pi^2}, \\ \mu \frac{dA'}{d\mu} &= -\frac{N}{24\pi^2}. \end{aligned} \tag{4.57}$$

This running has nothing to do with the one generated by graviton exchange and it is thus unrelated to the presence or absence of asymptotic safety that some authors advocate for gravity. At scales $\mu \gg M$ the relevant degrees of freedom are not gravitons, but the $2N$ fermions appearing in the microscopic Lagrangian. On the other hand, at the moment that fermions become the relevant degrees of freedom, geometry loses its meaning. There is then no 'shorter' distance than M^{-1}, or at the very least this regime cannot be probed. Note that to realize our physical assumption of having the fermions as fundamental degrees of freedom we should have $c > 0$ as discussed in Section 4.1.

These equations do not reflect the complete running of the dimensionless couplings associated to $\mathcal{L}_D$ and $\mathcal{L}_R$, i.e. the constants associated to the cosmological and Einstein-Hilbert terms, but only the one obtained at leading order in N. That is, the 'graviton' loops are not included here; they are suppressed by one power of N if N is large. To see this last statement we recall that the usual power counting rules show that the exchange of the vierbein degrees of freedom would be accompanied by a factor of M_P^{-2}, suppressed by $1/N$. Leaving these corrections aside, we note that the two free couplings of the theory have a running that is opposed in sign to the one found in $D = 2$.

It is probably useful to appeal to the QCD analogy. At long distances strong interactions are well described by the pion chiral Lagrangian, parametrized by f_π or the $O(p^4)$ coefficients, generically named low energy constants (LEC). The LEC are a complicated function of α_s, the coupling constant of QCD. The microscopic theory proposed in this thesis is the analogous of QCD, while the resulting effective theory (4.56) is the counterpart of the chiral effective Lagrangian. Then M_P and Λ are the LEC of the present theory. The running of α_s does not have an immediate translation on the LEC while in the present model, because of its simplicity, the consequences of the running in the microscopic particle reflects directly in Λ and M_P. But in addition these constants have an additional running (analogous to doing pion loops in the chiral Lagrangian). The counting of powers of N disentangles both types of running.

At some scale, $k \sim M$ the effective theory stops making sense. At that moment the relevant degrees of freedom change and, as a result, the metric disappears. Exactly in the same way as for large momentum transfers we do not see pions but quarks. Of course, if there is no metric there is no geometry and, in particular, the notion of distance disappears altogether at length scales below M^{-1}. From this point of view, gravity is non-Wilsonian.

Let us now try to make contact with the value that the LEC take in gravity. Clearly, there is enough freedom in the theory (by adjusting A' and M) to reproduce any values of Λ and M_P. But we also want higher order terms to be small for the effective theory to make sense in a reasonable range of momenta. We may even get rid of all of the high order ($\mathcal{O}(p^4)$ and beyond) if we take $M \to \infty$ and at the same time we take $N \to 0$ in a prescribed form. Then, in the actual limit, which corresponds to a 'quenched' approximation, we exactly reproduce Einstein-Hilbert Lagrangian, with a cosmological constant, and nothing else. Of course in this limit, the presumed fundamental degrees of freedom disappear completely and we have all the way up to $\mu = \infty$ Einstein's theory —with all its ultraviolet problems; there are no fundamental degrees of freedom providing form factors to cut off the offending divergent integrals*.

Of course the $N \to 0$ limit is just the opposite one to the one we have used. All our diagrammatic results are exact in the $N \to \infty$ limit and presumably get large corrections as N approaches zero, but the general features of the model should survive.

Note that M is a fixed quantity in the model and if M_P^2 increases, Λ decreases. Taking the actual observed or estimated values of these two parameters we get the value $NM^4 \sim 10^{18}$ m^{-4}, which is a very low scale. One may think that this may already represent unacceptably large corrections from higher order operators. However, this is not necessarily so because the bounds on R^2 terms are very weak. For instance, the bound $k < 10^{74}$ has been quoted for a generic coefficient [50] k of the $\mathcal{O}(p^4)$ terms. Thus, a relatively low scale for M cannot be really excluded observationally by studying gravitational effects alone and one should be aware of this. However, our own intuition tells us that M should be much larger than the value quoted above as the notion of metric certainly makes sense at much shorter distances. We can increase the value of M as much as we want by decreasing the value of N, as previously indicated. We shall not elaborate further on this as it seems too premature a speculation.

Finally we note that the sign of Newton's constant is not determined a priori in this theory due to the subtraction required from the counterterm in $\mathcal{L}_{\mathcal{R}}$. This ties nicely with some of the early discussions on induced gravity [7].

*Note that resolving the vertices singularities is not enough to mitigate the divergences of gravity as a loop of e.g. Dirac fermions generates itself new divergences of $\mathcal{O}(p^4)$. It is the combination of this with the absence of a metric tensor in the unbroken phase that might help, as in the mechanism proposed here.

Chapter 5

Gravitational waves in the presence of a cosmological constant

This chapter and the following one constitute the second part of this thesis. After setting a theoretical framework by means of which one obtains gravity equipped with a microscopic cosmological constant, it is only sensible to try to investigate its implications. In fact, as it is well-known, cosmological observations [51] indicate that we live in a Universe that is de Sitter, at least at very large scales, with $\Lambda \neq 0$. What is not so clear is whether $\Lambda \neq 0$ is an effective property valid only at very large scales or, on the contrary, a fundamental property of space-time. In this respect, recall we argued in the previous chapters that Λ could very well be an intrinsic property of space-time, and as such, its effects should be observable, as a matter of principle, at any scale.

In practice, the smallness of the cosmological constant obtained from fits to the current ΛCDM cosmological models [51] ($\Lambda \simeq 10^{-52}$ m^{-2}) may lead us to believe that it is totally unobservable except at the largest distances. However, the issue of the relevance of the cosmological constant in local measurements (meaning measurements that involve sub-cosmological scales, such as for instance galaxy clusters) has received growing attention [14, 18]. One interesting possibility is assessing the influence of Λ on the bending of light from distant objects. At present there are rather diverging results on the subject giving rather different results concerning the relevance of Λ ranging from zero [15] or very small [16] to appreciable ones [17]. The effect of Λ on the photon propagation, including frequency shift, Shapiro time delay and deflection of light, is currently under consideration [52].

The importance of these studies cannot be overemphasized. The presence of a non-zero cosmological constant contributing around 70% to the energy and matter budget of the universe, seemingly making the Universe globally a de Sitter space-time, is one of the intriguing puzzles of Physics in our time. Observations capable of confirming or refuting the relevance of Λ at redshift $z < 1$ are clearly of utmost importance.

The studies of what has been termed 'local gravity with a cosmological constant' rely on an approximate solution, valid at first order in Λ, obtained after linearizing Einstein equations. These solutions have recently been studied in detail in [18] using different gauge choices. It has been found that in the Lorenz gauge one can in addition require time independence of the metric solutions. After an additional coordinate transformation these solutions correspond to the linearized version of the Schwarzschild-de Sitter exact solution of Einstein equations. The modification to the Newtonian limit in such coordinates was also discussed in detail [18]. There are some subtleties related to the physical interpretation of the different coordinate systems that we shall review below.

Here we propose to study a different problem. Namely, how Λ influences the properties of gravitational waves (GW). As of today, gravitational waves are an unambiguous prediction of General Relativity that has not been tested directly. They are 'observed' indirectly as they are the missing ingredient needed to restore the energy balance of some astrophysical binary systems [6]. There are three types of experiments potentially capable of yielding a non-zero signal in the coming years. Let us summarize their physical and astrophysical reach here:

Ground based GW detectors such as LIGO [19] can reach sensitivities down to $\sim 10^{-23}$ with optimal sensitivity in the region between 10 Hz and 10^3 Hz. The space mission LISA [20] will reach a similar sensitivity in the range 10^{-2} Hz to 10^{-3} Hz but will actually be able to set relevant bounds on a more extended range of frequencies. Finally the International Pulsar Timing Array project [21] or the Square Kilometer Array project [22] are sensitive to lower frequencies $\nu < 10^{-4}$ Hz but reach only a sensitivity of $\sim 10^{-10}$ going up to $\sim 10^{-15}$ for $\nu \sim 10^{-10}$ Hz. These sensitivity ranges are targeted to specific astrophysical phenomena and are expected to provide detectable signals to confirm the existence of GW in the coming decades.

Given the present difficulties in asserting the very existence of GW it may seem academic to try to find modifications due to the presence of a cosmological constant that is small today. However, it should be borne in mind that in the inflationary epoch the value of Λ was much larger than at present so these effects might be of relevance for primordial GW. As we will discuss in this chapter the effect of Λ could also be of some relevance for GW traveling very long distances and for pulsar timing array projects. On the other hand, some of the results presented here we believe are of interest to understand the issue of the gauge choice in the presence of Λ for the linear theory. Finally, it seems interesting in its own right to attempt to understand wave propagation in de Sitter space-time if Λ is indeed a fundamental parameter of nature.

Understanding the choice of coordinates throughout this program will prove to be essential in order to make sense of the solutions found for the GW. Some coordinates are suitable for the resolution of the wave equations while some others do not even lead to a wave equation linearized in Λ. Moreover, the only coordinate frame in which we can make predictions that can be compared to observations is the one where the Universe appears isotropic and homogeneous. Any solution considered has to be ultimately transformed into these coordinates to make observable predictions. We proceed first by solving Einstein equation in a frame that has spherical symmetry and hence is adequate to describe local astrophysical phenomena such as central forces, gravitational collapse, etc. We will make the approximation that waves coming from far away violent phenomena such as super massive black hole binary mergers or supernovae are nearly spherical in these coordinates. In this frame, the wave equation (containing modifications due to Λ) is well defined and the modified linearized wave solutions easily found. In fact, we show that these coordinates are related by a time independent coordinate transformation to those corresponding to a linearized version of the Schwarzschild-de Sitter metric, which in virtue of Birkhoff's theorem is unique. On the other hand, the relevant coordinates for observation are the FRW coordinates. We work out the transformation of the solutions from one coordinate frame to the other one in order to extract the possible observational consequences caused the modified GW. These subtleties obviously appear only if $\Lambda \neq 0$

This chapter is organized as follows. In Section 5.1 we discuss the linearization of Einstein equations, including a discussion on different gauges and how they affect the wave equation for the gravitational field $h_{\mu\nu}$. In Section 5.2 we discuss different coordinate realizations of de Sitter space-time and their relation. In Section 5.3 we construct background solutions retaining terms of order $\Lambda h_{\mu\nu}$. This discussion is extended in Section 5.4 to include GW solutions that 'feel' the presence of Λ. In Section 5.5 we analyze the detectability of the effects previously calculated.

Some of the subjects discussed here appear to have received little attention in the past although there is an extensive literature on gravitational waves [53]. The effect of Λ on GW has been considered in [54, 55]. Physical consequences appear to have been extracted in the context of primordial gravitational waves [56] and only indirectly in what concerns the evolution of the modes and the power spectrum.

5.1 Linearization in the presence of Λ

Einstein equations, derived from the Einstein-Hilbert action, read

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = -\kappa T_{\mu\nu} \tag{5.1}$$

where $R_{\mu\nu}$ is the Ricci tensor for $g_{\mu\nu}$, $\Lambda > 0$ is the cosmological constant and $\kappa T_{\mu\nu}$ is the source term. $T_{\mu\nu}$ is the usual stress-energy tensor of matter in the gravitational field generated by $g_{\mu\nu}$ and κ is the dimensionful constant coupling matter and gravity. However, throughout this chapter we will consider $T_{\mu\nu} = 0$. The inclusion of the cosmological constant term leads to curvature even in the absence of any source

$$R = 4\Lambda. \tag{5.2}$$

We consider the linearized theory where the metric is written as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{5.3}$$

$\eta_{\mu\nu}$ being the Minkowski metric and $h_{\mu\nu} << 1$. The Ricci tensor to first order in the small perturbation $h_{\mu\nu}$ reads

$$R_{\mu\nu} = \frac{1}{2} \left(\Box h_{\mu\nu} + h_{,\mu\nu} - h^{\lambda}_{\mu,\nu\lambda} - h^{\lambda}_{\nu,\mu\lambda} \right), \tag{5.4}$$

indices being lowered and raised with $\eta_{\mu\nu}$ and $h = \eta^{\mu\nu} h_{\mu\nu}$. The theory is invariant under coordinate transformations $x^\mu \to x^\mu + \xi^\mu(x)$. For infinitesimal transformations the perturbation metric $h_{\mu\nu}$ transforms as $h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$. A gauge choice is possible, amounting to selecting a particular class of coordinates, and in fact such a choice is necessary if the perturbation $h_{\mu\nu}$ is to be quantized. In order to discuss GW two different gauge choices are particularly appropriate.

5.1.1 Lorenz gauge

In order to describe perturbations around flat space-time it is customary to employ the Lorenz gauge.

$$\partial_\mu h^\mu_\nu = \frac{1}{2}\partial_\nu h, \tag{5.5}$$

or

$$\partial_\mu \tilde{h}^\mu_\nu = 0, \tag{5.6}$$

where

$$\tilde{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h \tag{5.7}$$

is the trace reversed version of $h_{\mu\nu}$.

In this gauge, expression (5.4) is simplified

$$R_{\mu\nu} = \frac{1}{2}\Box h_{\mu\nu}, \tag{5.8}$$

and we obtain the equation of motion

$$\Box\left(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h\right) + 2\Lambda h_{\mu\nu} = -2\Lambda\eta_{\mu\nu} \tag{5.9}$$

which has always to be considered together with the Lorenz gauge condition (5.5).

Whether the term of order $\mathcal{O}(h\Lambda)$ has to be considered or not depends on the relative magnitude of h and Λ. There will be situations where the inclusion of this term is justified and may lead to observable consequences. We shall postpone the rest of the discussion on this issue to Sections 5.3 and 5.4. Note, nonetheless, that if the $\Lambda h_{\mu\nu}$ term on the left-hand side is omitted there is a residual gauge freedom within the Lorenz gauge. If we perform a linear coordinate transformation

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu, \tag{5.10}$$

Equation (5.5) is fulfilled as long as ξ^μ is an harmonic function, i.e. $\Box\xi^\mu = 0$. These residual coordinate transformations are sometimes termed ‘coordinate waves’ for rather obvious reasons. Note also that whether this is a symmetry of the equations of motion or not, depends on the terms retained in the linearization; the term $\Lambda h_{\mu\nu}$ breaks this residual coordinate invariance.

5.1.2 Λ gauge

It will be useful to consider an alternative gauge choice [57], which we will term Λ gauge. This is given by the gauge condition

$$\partial_\mu \tilde{h}^\mu_\nu = -\Lambda \eta_{\nu\mu} x^\mu. \tag{5.11}$$

In this gauge the linearized equations of motion look slightly different

$$\Box \left(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h \right) - 2\Lambda h_{\mu\nu} = 0. \tag{5.12}$$

In particular we note that the term independent of $h_{\mu\nu}$ on the right-hand side of (5.9) is absent. There is a set of coordinate transformations that can be performed without leaving the gauge orbit (5.11); these are transformations $x'^\mu = x^\mu + \xi^\mu$ with

$$\Box \xi^\mu = -\Lambda \xi^\mu. \tag{5.13}$$

Note that again in the Λ gauge this set of residual coordinate transformations allows for the removal of the unphysical degrees of freedom.

The connection between the two gauge choices in the linear theory is easily made when the terms $\Lambda h_{\mu\nu}$ are omitted. It is implemented via the following change of coordinates

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu = \left(1 - \frac{\Lambda}{12} x^2 \right) x^\mu. \tag{5.14}$$

This change of coordinates transforms a solution of $\Box \tilde{h}_{\mu\nu} = 0$ in the Λ gauge (coordinates x) to a solution of $\Box \tilde{h}_{\mu\nu} = -2\Lambda \eta_{\mu\nu}$ in Lorenz gauge (coordinates x'). Note the simplicity of the equation for linear perturbations in the Λ gauge if the term of order $\Lambda h_{\mu\nu}$ is omitted. All reference to the cosmological constant is eliminated.

Summing up, whenever the linear term in Λ is dropped from the equations of motion one has enough freedom to eliminate up to eight degrees of freedom from the solutions being ultimately left with the usual two physical ones, regardless of the gauge choice.

However, if the term of order $\Lambda h_{\mu\nu}$ is retained, i.e. in the Lorenz gauge the term $2\Lambda h_{\mu\nu}$ on the left-hand side of (5.9) or the analogous $-2\Lambda h_{\mu\nu}$ in the Λ gauge are kept, there is no residual symmetry whatsoever. Let us take for example (5.9) in the Lorenz gauge; as we will see in detail in Section 5.4 this generates a mass term and therefore apparently more physical degrees of freedom appear associated to $h_{\mu\nu}$. This is not a gauge artifact, as we will see it is an artifact of the linearization process.

5.2 De Sitter space-time

De Sitter space-time can be described by many coordinate systems. A convenient choice of coordinates is Schwarzschild-de Sitter (SdS). These provide a time-independent metric in a gauge that is none of the two previously discussed

$$ds^2 = \left[1 - \frac{\Lambda}{3}\hat{r}^2\right] d\hat{t}^2 - \left[1 - \frac{\Lambda}{3}\hat{r}^2\right]^{-1} \hat{r}^2 + \hat{r}^2 d\Omega^2. \tag{5.15}$$

and clearly shows the presence of the de Sitter horizon. We note that this metric admits an expansion in integer powers of Λ. Note also that in this metric the spatial part does not quite correspond to spherical coordinates.

At the opposite extreme, one can select a metric that depends only on time and is position independent. It is the Friedmann-Robertson-Walker (FRW) metric

$$ds^2 = dT^2 - \exp(2\sqrt{\frac{\Lambda}{3}}T)d\vec{X}^2. \tag{5.16}$$

This metric incorporates the physical principles of cosmological homogeneity and isotropy as it does not depend on the position. The coordinates X^i have a clear physical meaning, they are comoving coordinates anchored in space that expand with the universe. These are the natural coordinates where our world appears homogeneous and isotropic. It is easy to see that the FRW metric does not fulfill any linearized Einstein equation, even for very early times $t << 1/\sqrt{\Lambda}$ when is very close to the Minkowski metric. In fact, no metric that depends only on time can be a solution of the linearized Einstein equations; incompatibilities appear immediately for any gauge choice.

One should therefore accept that the linearized Einstein equations in the presence of Λ cannot be imposed in the physically relevant comoving coordinate system*. This of course has implications on GW as the very concept of 'wave' does require a wave equation, which is just impossible in linearized FRW coordinates. On the other hand, the wave equation $\Box\tilde{h}_{\mu\nu} = 0$ found in the Λ gauge is expressed in a set of coordinates whose meaning is yet to be interpreted. Therefore the simplicity of this equation is deceiving.

*Note we mean that $\eta_{\mu\nu} + h^{\Lambda}_{\mu\nu}$ being a linearized version of the FRW metric will never be a solution of any linearized Einstein equations. Nonetheless, some authors [58] study small perturbations above the exact FRW background, which is an entirely different program.

We will argue in the next section that the coordinates implied by the choice of the Λ gauge or of Lorenz gauge are closely related to SdS coordinates. Then the way to proceed is to find a solution for GW in the Lorenz gauge, a coordinate system where linearization of the Einstein equations is consistent, and then transform the solution to FRW coordinates in order to extract observable consequences.

Both the SdS metric and the FRW metric are valid (but rather different) descriptions of de Sitter geometry. One can work out the exact transformation between the two coordinate systems

$$\begin{aligned}\hat{r} =& e^{T\sqrt{\Lambda/3}}R \\ \hat{t} =& \sqrt{\frac{3}{\Lambda}}\log\left(\frac{\sqrt{3}}{\sqrt{3-\Lambda e^{2T\sqrt{\Lambda/3}}R^2}}\right)+T\end{aligned} \tag{5.17}$$

where T and R are respectively the cosmological time and comoving coordinates whose physical realization is clear. This transformation is valid inside the cosmological horizon, i.e. $R<\frac{1}{\sqrt{\Lambda}}$. Applying (5.17) to (5.15) we obtain

$$ds^2 = dT^2 - \exp(2\sqrt{\frac{\Lambda}{3}}T)d\vec{X}^2. \tag{5.18}$$

Now it is immediate to see that the FRW metric does not fulfill any linearized Einstein equation, even if $t << 1/\sqrt{\Lambda}$ as it is not expandable in integer powers of Λ. The same transformations for the linearized version of the metrics gives

$$\begin{aligned} ds^2 &= \left[1-\frac{\Lambda}{3}\hat{r}^2\right]d\hat{t}^2 - \left[1+\frac{\Lambda}{3}\hat{r}^2\right]\hat{r}^2 + \hat{r}^2 d\Omega^2. \\ &\downarrow \\ ds^2 &= dT^2 - \left[1+2\sqrt{\frac{\Lambda}{3}}\;T+2\frac{\Lambda}{3}T^2\right](dR^2+R^2d\Omega^2). \end{aligned} \tag{5.19}$$

which will only reasonably approximate the expansion of FRW for values of $R\sim T<<\frac{1}{\sqrt{\Lambda}}$. Note that, although the last metric in (5.19) is linearized, it does not fulfill any linearized Einstein equations.

The previous transformation provides the relationship between a framework where the Einstein equations can be consistently linearized and the actual coordinate system in which we observe. The solutions easily found in the linearized theory have to be transformed to the physically meaningful coordinate system in order to make predictions. It is at this point that non-trivial effects related to Λ will appear. They are discussed in Section 5.4. Of course, given the current value of Λ, these effects will be small. We believe nonetheless, that these corrections are conceptually important. Note also that (5.17) involves $\sqrt{\Lambda}$ and not Λ, yielding corrections that are potentially much more relevant for observation than those of order $\mathcal{O}(\Lambda)$.

Equation (5.16) is just one of the many possible cosmological FRW metrics. Other possibilities such as a power law cosmological scale factor do not correspond to a de Sitter space-time and therefore there is no obvious change of coordinates that allows to reexpress a GW, i.e. a solution to a wave equation, in that physically meaningful coordinate system.

5.3 Background solutions

We shall work consistently in the linearized approximation both for the background modification $h^{\Lambda}_{\mu\nu}$ and for gravitational wave perturbations $h^{W}_{\mu\nu}$. Namely, the metric can be written as $g_{\mu\nu} = \eta_{\mu\nu} + h^{\Lambda}_{\mu\nu} + h^{W}_{\mu\nu}$, where $h^{\Lambda,W}_{\mu\nu} \ll 1$. To keep the notation simple we shall only use the superscript Λ when confusion with wave perturbations $h^{W}_{\mu\nu}$ is possible. In this section we will be concerned with background linearized solutions when the cosmological constant Λ is present.

The value of the cosmological constant has presumably not been the same throughout the history of the universe. In early epochs, perhaps following an inflationary period, its value is believed to have been much larger [59]. This fact suggests that it may be necessary in some circumstances to retain the term $\Lambda h^{\Lambda}_{\mu\nu}$. Likewise it will be necessary for consistency to keep terms of order $\Lambda h^{W}_{\mu\nu}$ as the magnitudes of $h^{W}_{\mu\nu}$ and Λ are unrelated.

In what follows we proceed without making any assumptions on the value of Λ; we will just assume that the perturbation that induces on the background metric $h_{\mu\nu}$ is small enough for the linearized approximation to be meaningful.

5.3.1 Lowest order solutions

First we turn to the lowest order solutions already discussed in [18], which correspond to neglecting terms of $\mathcal{O}(\Lambda h_{\mu\nu})$. In the Lorenz gauge this amounts to solving the following equation

$$\begin{aligned} \Box \tilde{h}_{\mu\nu} =& -2\Lambda\eta_{\mu\nu} \\ \partial_\mu \tilde{h}^\mu_\nu =& 0. \end{aligned} \tag{5.20}$$

Linearization limits the validity of the solution to values of the coordinates such that $x^2 << 1/\Lambda$.

Before discussing the solutions to (5.20) we take a look at the equations in the Λ gauge

$$\begin{aligned} \Box \tilde{h}_{\mu\nu} =& 0 \\ \partial_\mu \tilde{h}^\mu_\nu =& -\Lambda\eta_{\nu\mu}x^\mu. \end{aligned} \tag{5.21}$$

Note once more that the linearized equations are not invariant under gauge transformations. In the Lorenz gauge the cosmological constant is regarded as a gravitational source, it appears in the equations of motion, whereas in the Λ gauge all dependency in the cosmological constant at this order appears through the gauge condition only and in a way it can be interpreted as a consequence of the coordinate choice*. The connection between the two gauge choices in the linear theory has already been discussed.

We can easily solve Equations (5.21) to find the traceless solution

$$\tilde{h}_{\mu\nu} = -\frac{\Lambda}{18}\left(4x_\mu x_\nu - \eta_{\mu\nu}x^2\right). \tag{5.22}$$

If we require that the solution is proportional to Λ and involves only the coordinates x^μ this is the unique solution. In addition, (5.22) is the only one that is Lorentz-covariant (note that $\eta_{\mu\nu}$ is the underlying metric and there is no other four-vector at our disposal).

It is worth noticing that although there is a residual freedom in this gauge, no transformation can turn this solution into a static metric: *The Λ gauge is explicitly incompatible with the solutions being static.*

We now transform the solution back to the Lorenz gauge using (5.14). We find

$$h_{\mu\nu} = \frac{\Lambda}{9}\left(x_\mu x_\nu + 2\eta_{\mu\nu}x^2\right). \tag{5.23}$$

Without the $\Lambda h_{\mu\nu}$ term the equation of motion is actually invariant under residual transformations. The number of physical degrees of freedom therefore is reduced to two. This is the only covariant-looking solution in the Lorenz gauge but only one of

* This of course does not mean that the consequences of Λ can be removed by a wise coordinate transformation but it does mean that it disappears from the equations of motion themselves.

the infinite number of solutions reachable by non-covariant residual transformations. The most general form of such transformations is

$$\xi'^{\mu} = \begin{pmatrix} A(t^2+r^2)t \\ \left(B_1t^2+B_2x^2+B_3(y^2+z^2)\right)x \\ \left(B_1t^2+B_2y^2+B_3(x^2+z^2)\right)y \\ \left(B_1t^2+B_2z^2+B_3(x^2+y^2)\right)z \end{pmatrix}, \tag{5.24}$$

where $2B_1 - 6B_2 - 4B_3 = 0$. In particular we find the values of these constants that allow us to reproduce the static solution of [18].

$$A = -\frac{\Lambda}{18}, \quad B_1 = -\frac{\Lambda}{9}, \quad B_2 = -\frac{\Lambda}{18}, \quad B_3 = \frac{\Lambda}{36}; \tag{5.25}$$

One should ask at this point what are these coordinates. We already know that they cannot correspond to cosmological coordinates. In fact the resulting metric is neither homogeneous nor isotropic although it preserves the symmetry among the three axes. The answer becomes obvious once one discovers that one of the possible residual gauge transformations eliminates the time dependence of the metric. A generalization of Birkhoff's theorem [60] states that there is a unique static solution with spherical symmetry which is the Schwarzschild-de Sitter metric previously discussed, or more precisely the first order of it in the Λ expansion. Since Schwarzschild-de Sitter does not fulfill the Lorenz gauge condition, a time-independent coordinate transformation must also be involved. Let us explicitly show this point using a succession of coordinate transformations linear in Λ.

The first step is to transform (5.23) to a static solution. We start from

$$\begin{aligned} ds^2 = &\left[1+\frac{\Lambda}{9}(3t^2-2r^2)\right]dt^2 - \left[1-\frac{\Lambda}{9}(-2t^2+2r^2+x^{i2})\right]dx^{i2} \\ &-\frac{2\Lambda}{9}t\ x^i\ dt\ dx^i + \frac{2\Lambda}{9}x^i\ x^j\ dx^i\ dx^j \end{aligned} \tag{5.26}$$

where $i = 1,2,3$ and $i \neq j$. After the following change of coordinates

$$\begin{aligned} x =& x' + \frac{\Lambda}{9}\left(-t'^2 - \frac{x'^2}{2} + \frac{(y'^2+z'^2)}{4}\right)x' \\ y =& y' + \frac{\Lambda}{9}\left(-t'^2 - \frac{y'^2}{2} + \frac{(x'^2+z'^2)}{4}\right)y' \\ z =& z' + \frac{\Lambda}{9}\left(-t'^2 - \frac{z'^2}{2} + \frac{(x'^2+y'^2)}{4}\right)z' \\ t =& t' - \frac{\Lambda}{18}(t'^2+r'^2)t' \end{aligned} \tag{5.27}$$

the metric transforms into the static solution to order Λ found in [18],

$$ds^2 = \left[1 - \frac{\Lambda}{3} r'^2\right] dt'^2 - \left[1 - \frac{\Lambda}{6}(r'^2 + 3x_i'^2)\right] dx_i'^2. \tag{5.28}$$

Note that this solution is still in the Lorenz gauge; we only performed a residual gauge transformation that is allowed in this gauge. Since our starting solution is only valid to order Λ, in any change of coordinates, either exact or linear, we only keep terms linear in the cosmological constant. We can further transform (5.28) to obtain a fully spherically symmetric solution. Under the following change

$$\begin{aligned} x' &= x'' + \frac{\Lambda}{12} x''^3 \\ y' &= y'' + \frac{\Lambda}{12} y''^3 \\ z' &= z'' + \frac{\Lambda}{12} z''^3 \\ t' &= t'', \end{aligned} \tag{5.29}$$

we obtain

$$ds^2 = \left[1 - \frac{\Lambda}{3} r''^2\right] dt''^2 - \left[1 - \frac{\Lambda}{6} r''^2\right] (dr''^2 + r''^2 d\Omega^2), \tag{5.30}$$

which does not obey (5.20) anymore. We can now perform another coordinate transformation to obtain the SdS metric to order Λ

$$\begin{aligned} r'' &= \hat{r} + \frac{\Lambda}{12} \hat{r}^3 \\ t'' &= \hat{t} \end{aligned} \tag{5.31}$$

$$ds^2 = \left[1 - \frac{\Lambda}{3} \hat{r}^2\right] d\hat{t}^2 - \left[1 + \frac{\Lambda}{3} \hat{r}^2\right] d\hat{r}^2 + \hat{r}^2 d\Omega^2. \tag{5.32}$$

This is the linearized Schwarzschild-de Sitter metric. Essentially the background solution (5.23) is the SdS metric in a set of coordinates related to SdS by time independent transformations.

5.3.2 Next-order solutions

Let us now relax the approximation of the previous section and retain terms proportional to $\Lambda h_{\mu\nu}$. In particular we will be interested later in terms of order $\Lambda h^W_{\mu\nu}$ that will influence the propagation of gravitational waves.

In the Lorenz gauge this requires the simultaneous fulfillment of the two sets of Equations (5.5) and (5.9). We note that because of the dimensionality of Λ any solution of the previous equations containing Λ and constructed with the only available (Lorentz-)covariant vector x^μ must necessarily be even under a change of sign of all coordinates $x^\mu \to -x^\mu$. Solutions odd in x^μ exist but they require the involvement of parameters other than the coordinates and Λ (a wave vector, for instance, see Section 5.3).

The most general solution of this equation can be written as a superposition of both complex and real exponentials

$$h_{\mu\nu} = \int \frac{d^4k}{(2\pi)^4}\delta(k^2 - 2\Lambda)\left(E_{\mu\nu}\cos kx + D_{\mu\nu}\sin kx + \frac{\eta_{\mu\nu}}{4}\left(A\cosh kx + B\sinh kx\right)\right) - \eta_{\mu\nu}, \tag{5.33}$$

with $E_{\mu\nu}$ and $D_{\mu\nu}$ traceless, i.e. $E^\mu_\mu = D^\mu_\mu = 0$. In the previous expression $E_{\mu\nu}$, $D_{\mu\nu}$, A and B are in principle all independent functions of k provided that the two following gauge conditions are met

$$\int \frac{d^4k}{(2\pi)^4}\delta(k^2 - 2\Lambda)\left(k_\mu E^\mu_\nu \sin kx + \frac{k_\nu}{4} A\sinh kx\right) = 0 \tag{5.34}$$

$$\int \frac{d^4k}{(2\pi)^4}\delta(k^2 - 2\Lambda)\left(k_\mu D^\mu_\nu \cos kx - \frac{k_\nu}{4} B\cosh kx\right) = 0. \tag{5.35}$$

Clearly the integrands involved have to fall off sufficiently fast for large values of k for the integrals to exist.

This solution has ten degrees of freedom to start with. Nine come from $E_{\mu\nu}$ and $D_{\mu\nu}$ after removal of the trace. Another one comes from the coefficients A, B. Note that both A and B are needed to provide a full degree of freedom and likewise for $E_{\mu\nu}$ and $D_{\mu\nu}$. Using the gauge condition we can eliminate four of them, leaving six independent degrees of freedom. Unlike (5.23), the above solution does not admit any residual gauge transformation to further eliminate degrees of freedom. Any attempt to perform a residual gauge transformation would take the solution 'off shell', i.e. the equations of motion would not be obeyed.

On the other hand we have to ensure that $h_{\mu\nu} << 1$; However, in general this does not eliminate any degree of freedom, it is just a requirement of the linearized theory. This translates in requiring the first term in the expansion of the hyperbolic cosine to cancel the $-\eta_{\mu\nu}$ piece in (5.33), or in other words

$$\int \frac{d^4k}{(2\pi)^4}\delta(k^2 - 2\Lambda)A(k) = 4. \tag{5.36}$$

Since (5.33) is the most general solution to the equations we must be able to recover the solutions in the previous section by performing an expansion in Λ. To do so we only have to choose the right form for $E_{\mu\nu}(k)$, $D_{\mu\nu}(k)$, $A(k)$ and $B(k)$. As mentioned previously, to reach a Lorentz-covariant formulation such as (5.23) in the Lorenz gauge we can safely assume that $D_{\mu\nu}$ and B are zero as the resulting metric must satisfy $h_{\mu\nu}(x) = h_{\mu\nu}(-x)$, as discussed. In addition $A(k)$ can only be a constant on Lorentz covariance grounds. We will take it to be $A(k) \equiv \frac{A'}{k^2} = \frac{A'}{2\Lambda}$. Also $E_{\mu\nu}$ needs to be a (traceless) Lorentz-covariant tensor, namely $E_{\mu\nu}(k) \equiv \frac{E}{2\Lambda}\left(k_\mu k_\nu - \frac{\eta_{\mu\nu}}{2}\Lambda\right)$. The proportionality coefficient between E and A' comes from the gauge condition (5.34). Finally, as also indicated previously, the integrals require a finite support to be well defined and this should be implemented in a Lorentz-invariant way too; a sharp cut-off will be used below, although this is not crucial at all. Expanding (5.33),

$$
\begin{aligned}
h_{\mu\nu} =& \int \frac{d^4k}{(2\pi)^4}\delta(k^2-2\Lambda)\Big(E_{\mu\nu}(k)\cos kx + \frac{\eta_{\mu\nu}}{4}A(k)\cosh kx\Big) - \eta_{\mu\nu} \\
=& \int \frac{d^4k}{(2\pi)^4}\delta(k^2-2\Lambda)\left(E_{\mu\nu}(k)\left(1-\frac{(k\cdot x)^2}{2}+\dots\right)\right. \\
&\left. +\frac{\eta_{\mu\nu}}{4}A(k)\left(1+\frac{(k\cdot x)^2}{2}+\dots\right)\right) - \eta_{\mu\nu},
\end{aligned}
\tag{5.37}
$$

and using the definitions given above,

$$
\begin{aligned}
h_{\mu\nu} \simeq& \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\sqrt{2\Lambda+\vec{k}^2}}\left(\frac{E}{2\Lambda}\left(k_\mu k_\nu - \frac{\eta_{\mu\nu}}{2}\Lambda\right)\left(1-\frac{(k\cdot x)^2}{2}\right)\right. \\
&\left. +\frac{\eta_{\mu\nu}}{4}\frac{A'}{2\Lambda}\left(1+\frac{(k\cdot x)^2}{2}\right)\right) - \eta_{\mu\nu}.
\end{aligned}
\tag{5.38}
$$

Now we introduce the cut-off, $\sqrt{2\Lambda}$. Already condition (5.36) dictates the value for $A' = \frac{32\pi^2}{C}$, where $C = \frac{1}{\Lambda}\int_0^{\sqrt{2\Lambda}} d|\vec{k}|\frac{\vec{k}^2}{\sqrt{2\Lambda+\vec{k}^2}}$. Then the solution reads

$$
\begin{aligned}
h_{\mu\nu} \simeq& \int_0^{\sqrt{2\Lambda}} \frac{d|\vec{k}|}{2\pi^2}\frac{\vec{k}^2}{2\sqrt{2\Lambda+\vec{k}^2}}\left(-\frac{E}{2\Lambda}\left(k_\mu k_\nu - \frac{\eta_{\mu\nu}}{2}\Lambda\right)\frac{(k\cdot x)^2}{2} + \frac{\eta_{\mu\nu}}{4}\frac{16\pi^2}{\Lambda C}\frac{(k\cdot x)^2}{2}\right) \\
=& \int_0^{\sqrt{2\Lambda}} \frac{d|\vec{k}|}{2\pi^2}\frac{\vec{k}^2}{2\sqrt{2\Lambda+\vec{k}^2}}\left(-E\left(\frac{\Lambda}{24}\left(\eta_{\mu\nu}x^2+2x_\mu x_\nu\right) - \frac{\Lambda}{16}\eta_{\mu\nu}x^2\right) + \eta_{\mu\nu}x^2\frac{\pi^2}{C}\right) \\
=& \frac{\Lambda C}{4\pi^2}\left(-E\left(\frac{\Lambda}{24}\left(\eta_{\mu\nu}x^2+2x_\mu x_\nu\right) - \frac{\Lambda}{16}\eta_{\mu\nu}x^2\right) + \eta_{\mu\nu}x^2\frac{\pi^2}{C}\right).
\end{aligned}
\tag{5.39}
$$

The value of E is fixed via the gauge condition (5.34) to $E = -\frac{16\pi^2}{3C\Lambda}$, leaving the perturbation in the form

$$h_{\mu\nu} \simeq \frac{\Lambda}{9}\left(x_\mu x_\nu + 2\eta_{\mu\nu}x^2\right), \tag{5.40}$$

which is precisely (5.23).

5.4 Wave-like solutions

In this section we will finally investigate the effects of the cosmological constant in the propagation of GW in the appropriate coordinate system.

5.4.1 Lowest order solutions

We write $h_{\mu\nu} = h^\Lambda_{\mu\nu} + h^W_{\mu\nu}$. The term $h^\Lambda_{\mu\nu}$ is the solution we just found, $h^W_{\mu\nu}$ will be a perturbation on the metric induced by some source of GW. The same decomposition holds for the trace reversed metric $\tilde{h}_{\mu\nu}$. Waves are usually considered in the transverse traceless gauge [61]

$$\tilde{h}^{W\mu}_{\mu} = h^{W\mu}_{\mu} = 0, \qquad \partial_\mu h^{W\mu}_{\nu} = \partial_\mu \tilde{h}^{W\mu}_{\nu} = 0. \tag{5.41}$$

This is compatible with the Λ gauge condition as the right-hand side of (5.11) is unchanged when considering $\tilde{h}^\Lambda_{\mu\nu} + \tilde{h}^W_{\mu\nu}$ provided that (5.11) is fulfilled by $h^\Lambda_{\mu\nu}$. This also makes clear that, at this order, the gauge condition involves the perturbation associated to the background and not the metric perturbation associated to a gravitational wave.

Since the proper equations of motion in the Lorenz gauge at this order, neglecting $\mathcal{O}(\Lambda h_{\mu\nu})$, are just $\Box h_{\mu\nu} = \Box h^\Lambda_{\mu\nu} + \Box h^W_{\mu\nu} = 0$, being the latter an independent perturbation, it is obvious that

$$\Box h^W_{\mu\nu} = 0, \tag{5.42}$$

and the gravitational wave solutions are in these coordinate systems functionally identical to those existing in flat space.

Note that because the $\Lambda h^\Lambda_{\mu\nu}$ has been neglected, the remaining residual gauge invariance allows for a removal of four of the six degrees of freedom in $h^W_{\mu\nu}$ and the analogy with wave propagation in Minkowski space is complete.

In the case of the lowest order equations the full solution of (5.20) is

$$h_{\mu\nu} = h^\Lambda_{\mu\nu} + h^W_{\mu\nu} = \frac{\Lambda}{9}\left(x_\mu x_\nu + 2\eta_{\mu\nu}x^2\right) + E^W_{\mu\nu}\cos kx + D^W_{\mu\nu}\sin kx \tag{5.43}$$

where $E^W = D^W = 0$, $k_\mu E^{\mu W}_\nu = k_\mu D^{\mu W}_\nu = 0$ and $k^2 = 0$.

We want to see now how plane waves such as the ones in (5.43) look like in the new coordinate system. Transformation (5.17) acts both on the polarization tensors and on the arguments of the sine and cosine. For the polarization tensors we can always cut the expansion in Λ and keep terms only up to a certain order. However, the transformation on the arguments yields terms of the type $Z^3 w\Lambda$ which in general can be relevant. The sine and cosine can not be expanded, we have to transform the argument exactly; we shall later evaluate the error caused by retaining only the lowest order terms in the arguments.

For the polarization tensors, since we transform them independently of the arguments, it is easy to see qualitatively what the corrections to the polarization tensors will be. On dimensional grounds alone, all corrections will be of order $\mathcal{O}(\sqrt{\Lambda}Z)$ or at most $\mathcal{O}(\Lambda Z^2)$, being these quantities in the region of validity of the approximation very small.

Nonetheless, the transformed wave-like solution to order $\sqrt{\Lambda}$ is

$$
\begin{aligned}
h^{W_{FRW}}_{\mu\nu} = &\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & E_{11}\left(1+2\sqrt{\frac{\Lambda}{3}}T\right) & E_{12}\left(1+2\sqrt{\frac{\Lambda}{3}}T\right) & 0 \\ 0 & E_{12}\left(1+2\sqrt{\frac{\Lambda}{3}}T\right) & -E_{11}\left(1+2\sqrt{\frac{\Lambda}{3}}T\right) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \times \\
&\cos\left(w(T-Z) + w\sqrt{\frac{\Lambda}{3}}\left(\frac{Z^2}{2} - TZ\right) + \mathcal{O}(\Lambda)\right) + \mathcal{O}(\Lambda) \\
&+ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & D_{11}\left(1+2\sqrt{\frac{\Lambda}{3}}T\right) & D_{12}\left(1+2\sqrt{\frac{\Lambda}{3}}T\right) & 0 \\ 0 & D_{12}\left(1+2\sqrt{\frac{\Lambda}{3}}T\right) & -D_{11}\left(1+2\sqrt{\frac{\Lambda}{3}}T\right) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \times \\
&\sin\left(w(T-Z) + w\sqrt{\frac{\Lambda}{3}}\left(\frac{Z^2}{2} - TZ\right) + \mathcal{O}(\Lambda)\right) + \mathcal{O}(\Lambda)
\end{aligned} \tag{5.44}
$$

The term $w(T-Z)$ dominates the argument of the trigonometric functions and it can be checked numerically that the error made by omitting terms of order Λ or higher is $\leqslant 10^{-3}$ for the purposes of next section.

5.4.2 Next-order solutions

As we have argued before, it is not justified to neglect the term of order $\Lambda h^W_{\mu\nu}$ in this case, as unlike for the case of the background, the magnitude of the two quantities is unrelated. We can add a wave-like piece to the solution (5.33)

$$\begin{aligned} h_{\mu\nu} =& h^\Lambda_{\mu\nu} + h^W_{\mu\nu} \\ =& \int \frac{d^4k}{(2\pi)^4}\delta(k^2-2\Lambda)\left(E_{\mu\nu}\cos kx + D_{\mu\nu}\sin kx + \frac{\eta_{\mu\nu}}{4}\left(A\cosh kx + B\sinh kx\right)\right) - \eta_{\mu\nu} \\ &+ E^W_{\mu\nu}\cos kx + D^W_{\mu\nu}\sin kx. \end{aligned} \tag{5.45}$$

This will always be a solution of (5.5) and (5.9) as long as $E^W = D^W = 0$, $k_\mu E^{W\,\mu}_{\nu} = k_\mu D^{W\,\mu}_{\nu} = 0$ and $k^2 = 2\Lambda$. However, now we are not allowed to perform any gauge transformation, at least at the next-order level. We can still use the gauge condition and the traceless condition to eliminate five degrees of freedom from the wave. We are left with a massive wave with five degrees of freedom. The polarization vectors of which, for a wave propagating in the z direction ($k_1 = k_2 = 0$), can be written as

$$E^W_{\mu\nu} = \begin{pmatrix} E_{00} & \frac{\sqrt{w^2-2\Lambda}}{w}E_{13} & \frac{\sqrt{w^2-2\Lambda}}{w}E_{23} & \frac{w}{\sqrt{w^2-2\Lambda}}E_{00} \\ \frac{\sqrt{w^2-2\Lambda}}{w}E_{13} & E_{11} & E_{12} & E_{13} \\ \frac{\sqrt{w^2-2\Lambda}}{w}E_{23} & E_{12} & -E_{11} - E_{00}\frac{2\Lambda}{w^2-2\Lambda} & E_{23} \\ \frac{w}{\sqrt{w^2-2\Lambda}}E_{00} & E_{13} & E_{23} & \frac{w^2}{w^2-2\Lambda}E_{00} \end{pmatrix}. \tag{5.46}$$

And a similar expression for $D^W_{\mu\nu}$. At the exact level this is as far as one can go but in order to understand the meaning of these massive waves we turn again to an expansion in powers of Λ. We will proceed in two steps. First we expand the solution in powers of Λ and collect terms order by order. Then, using the same reasoning in the equations of motion, we can use an approximate residual invariance to rewrite the polarization tensors as the usual GW in Minkowski space-time plus an order Λ contribution with the extra degrees of freedom.

The polarization vectors (5.46) can then be written as

$$\begin{aligned} E^W_{\mu\nu} =& \begin{pmatrix} E_{00} & E_{13} & E_{23} & E_{00} \\ E_{13} & E_{11} & E_{12} & E_{13} \\ E_{23} & E_{12} & -E_{11} & E_{23} \\ E_{00} & E_{13} & E_{23} & E_{00} \end{pmatrix} + \begin{pmatrix} 0 & -\frac{\Lambda}{w^2}E_{13} & -\frac{\Lambda}{w^2}E_{23} & \frac{\Lambda}{w^2}E_{00} \\ -\frac{\Lambda}{w^2}E_{13} & 0 & 0 & 0 \\ -\frac{\Lambda}{w^2}E_{23} & 0 & -E_{00}\frac{2\Lambda}{w^2} & 0 \\ \frac{\Lambda}{w^2}E_{00} & 0 & 0 & \frac{2\Lambda}{w^2}E_{00} \end{pmatrix} + \mathcal{O}(\Lambda^2) \\ \equiv& E^{(0)}_{\mu\nu} + E^{(1)}_{\mu\nu} + \mathcal{O}(\Lambda^2). \end{aligned} \tag{5.47}$$

The same decomposition applies to $D^W_{\mu\nu}$. This expansion makes explicit the contributions of Λ at a given order. We want to expand

$$h^W_{\mu\nu} = h^{(0)}_{\mu\nu} + h^{(1)}_{\mu\nu} + \mathcal{O}(\Lambda^2), \tag{5.48}$$

where the superscript refers to the order in Λ. The functions sine and cosine can also be expanded around a massless wave with coordinate-dependent amplitudes [55]

$$\begin{aligned} h^W_{\mu\nu} &= E^W_{\mu\nu}\cos kx + D^W_{\mu\nu}\sin kx \\ &\simeq \left[\left(E^W_{\mu\nu} - \frac{\Lambda z}{w}D^W_{\mu\nu}\right)\cos w(t-z) + \left(D^W_{\mu\nu} + \frac{\Lambda z}{w}E^W_{\mu\nu}\right)\sin w(t-z)\right] \end{aligned} \tag{5.49}$$

or what is tantamount

$$\begin{aligned} h^W_{\mu\nu} = &\left[\left(E^{(0)}_{\mu\nu} + E^{(1)}_{\mu\nu} - \frac{\Lambda z}{w}D^{(0)}_{\mu\nu}\right)\cos w(t-z)\right. \\ &+ \left.\left(D^{(0)}_{\mu\nu} + D^{(1)}_{\mu\nu} + \frac{\Lambda z}{w}E^{(0)}_{\mu\nu}\right)\sin w(t-z)\right] \\ &+ \mathcal{O}(\Lambda^2). \end{aligned} \tag{5.50}$$

We see that the massive wave we started with can be written at linear order in the cosmological constant in terms of a massless wave where all dependency in Λ appears only through the polarization tensors

$$h^W_{\mu\nu} = E^W_{\mu\nu}\cos w(t-z) + D^W_{\mu\nu}\sin w(t-z) + \mathcal{O}(\Lambda^2), \tag{5.51}$$

where $E^W_{\mu\nu}$ and $D^W_{\mu\nu}$ can be read from (5.50). The above is a valid solution of $\Box h^W_{\mu\nu} + 2\Lambda h^W_{\mu\nu} = 0$ only to order Λ (included), which means we can expand the equations of motion to the same order without loss of validity

$$\Box h^{(0)}_{\mu\nu} + \Box h^{(1)}_{\mu\nu} + 2\Lambda h^{(0)}_{\mu\nu} + \mathcal{O}(\Lambda^2) = 0 \tag{5.52}$$

Now we can split the problem and solve order by order

$$\begin{aligned} \Box h^{(0)}_{\mu\nu} &= 0 \\ \Box h^{(1)}_{\mu\nu} + 2\Lambda h^{(0)}_{\mu\nu} &= 0 \end{aligned} \tag{5.53}$$

Due to the fact that (5.52) is not exact, the solution to it can admit a residual gauge transformation that will take the solution 'off shell' some order beyond the order we consider. For the transformed solution

$$\begin{aligned} \Box h'^{(0)}_{\mu\nu} &= 0 \\ \Box h'^{(1)}_{\mu\nu} + 2\Lambda h'^{(0)}_{\mu\nu} &= 0. \end{aligned} \tag{5.54}$$

The first equation in (5.54) is analogous to (5.42), i.e. residual transformations on $h^{(0)}_{\mu\nu}$ are not restricted. To order zero we obtain GW analogous to the ones in flat space (in the present set of coordinates, that is). But in this case the transformation propagates to the following order through the second equation in (5.54) making necessary to find the transformed $h'^{(1)}_{\mu\nu}$.

It is not difficult to see that the following polarization tensor fulfills the necessary requirements of tracelessness as well as the gauge condition ($k_\mu E^{W\mu}_{\nu} = k_\mu D^{W\mu}_{\nu} = 0$)

$$E^W_{\mu\nu} = \begin{pmatrix} \frac{\Lambda}{w^2}E_{00} & -\frac{\Lambda}{w^2}E_{13} & -\frac{\Lambda}{w^2}E_{23} & \frac{\Lambda}{w^2}E_{00} \\ -\frac{\Lambda}{w^2}E_{13} & E_{11} - \frac{\Lambda z}{w}D_{11} & E_{12} - \frac{\Lambda z}{w}D_{12} & -\frac{\Lambda}{w^2}E_{13} \\ -\frac{\Lambda}{w^2}E_{23} & E_{12} - \frac{\Lambda z}{w}D_{12} & -E_{11} + \frac{\Lambda z}{w}D_{11} & -\frac{\Lambda}{w^2}E_{23} \\ \frac{\Lambda}{w^2}E_{00} & -\frac{\Lambda}{w^2}E_{13} & -\frac{\Lambda}{w^2}E_{23} & \frac{\Lambda}{w^2}E_{00} \end{pmatrix}. \tag{5.55}$$

$D_{\mu\nu}$ is similarly obtained from (5.50). Notice the presence of the usual components (of $\mathcal{O}(1)$) in the polarization tensor in the x, y entries of the metric.

To this order in Λ we obtain massless waves with coordinate-dependent modified amplitudes which depend on Λ. We can see that the extra degrees of freedom due to the form of the linearized equations of motion for non-zero Λ will only couple to matter fields proportionally to Λ thanks to the coupling $h^W_{\mu\nu}T^{\mu\nu}$ and thus will be irrelevant in practice.

5.4.3 Transformed next-order solutions

Now we are ready to apply the series of coordinate transformations (5.27, 5.29, 5.31, 5.17) to the wave-like solution (5.51) that we found in the previous subsection in order to obtain a physical expression in FRW coordinates. Recall the waves in the general Lorenz gauge read

$$h^W_{\mu\nu} = E^W_{\mu\nu}(\Lambda, z)\cos w(t-z) + D^W_{\mu\nu}(\Lambda, z)\sin w(t-z), \tag{5.56}$$

where $E^W_{\mu\nu}$ can be read off from (5.55). From (5.56) it is clear the only modification with respect to the plane waves of the lower order is in the polarization tensors, being already of order Λ. This suggests that all the new modifications to order Λ of the next-order waves are due to the change of coordinates. Explicitly the transformed waves to order Λ read

$$h_{\mu\nu}^{W_{FRW}} =$$

$$\left[\begin{pmatrix} \frac{\Lambda}{w^2}E_{00} & -\frac{\Lambda}{w^2}E_{13} & -\frac{\Lambda}{w^2}E_{23} & \frac{\Lambda}{w^2}E_{00} \\ -\frac{\Lambda}{w^2}E_{13} & E_{11}-\frac{\Lambda Z}{w}D_{11} & E_{12}-\frac{\Lambda Z}{w}D_{12} & -\frac{\Lambda}{w^2}E_{13} \\ -\frac{\Lambda}{w^2}E_{23} & E_{12}-\frac{\Lambda Z}{w}D_{12} & -E_{11}+\frac{\Lambda Z}{w}D_{11} & -\frac{\Lambda}{w^2}E_{23} \\ \frac{\Lambda}{w^2}E_{00} & -\frac{\Lambda}{w^2}E_{13} & -\frac{\Lambda}{w^2}E_{23} & \frac{\Lambda}{w^2}E_{00} \end{pmatrix} + \right.$$

$$\left.\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & E_{11}\left(2\sqrt{\frac{\Lambda}{3}}T+\frac{2\Lambda}{9}T^2+\frac{5\Lambda}{18}Z^2\right) & E_{12}\left(2\sqrt{\frac{\Lambda}{3}}T+\frac{2\Lambda}{9}T^2+\frac{5\Lambda}{18}Z^2\right) & 0 \\ 0 & E_{12}\left(2\sqrt{\frac{\Lambda}{3}}T+\frac{2\Lambda}{9}T^2+\frac{5\Lambda}{18}Z^2\right) & -E_{11}\left(2\sqrt{\frac{\Lambda}{3}}T+\frac{2\Lambda}{9}T^2+\frac{5\Lambda}{18}Z^2\right) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \mathcal{O}(\Lambda^{3/2})\right] \times$$

$$\cos\left(w(T-Z)+w\sqrt{\frac{\Lambda}{3}}\left(\frac{Z^2}{2}-TZ\right)-\frac{1}{18}w\Lambda\left(T^3+T^2Z-5TZ^2+2Z^3\right)+\mathcal{O}(\Lambda^{3/2})\right)$$

$$+\left[\begin{pmatrix} \frac{\Lambda}{w^2}D_{00} & -\frac{\Lambda}{w^2}D_{13} & -\frac{\Lambda}{w^2}D_{23} & \frac{\Lambda}{w^2}D_{00} \\ -\frac{\Lambda}{w^2}D_{13} & D_{11}+\frac{\Lambda Z}{w}E_{11} & D_{12}+\frac{\Lambda Z}{w}E_{12} & -\frac{\Lambda}{w^2}D_{13} \\ -\frac{\Lambda}{w^2}D_{23} & D_{12}+\frac{\Lambda Z}{w}E_{12} & -D_{11}-\frac{\Lambda Z}{w}E_{11} & -\frac{\Lambda}{w^2}D_{23} \\ \frac{\Lambda}{w^2}D_{00} & -\frac{\Lambda}{w^2}D_{13} & -\frac{\Lambda}{w^2}D_{23} & \frac{\Lambda}{w^2}D_{00} \end{pmatrix} + \right.$$

$$\left.\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & D_{11}\left(2\sqrt{\frac{\Lambda}{3}}T+\frac{2\Lambda}{9}T^2+\frac{5\Lambda}{18}Z^2\right) & D_{12}\left(2\sqrt{\frac{\Lambda}{3}}T+\frac{2\Lambda}{9}T^2+\frac{5\Lambda}{18}Z^2\right) & 0 \\ 0 & D_{12}\left(2\sqrt{\frac{\Lambda}{3}}T+\frac{2\Lambda}{9}T^2+\frac{5\Lambda}{18}Z^2\right) & -D_{11}\left(2\sqrt{\frac{\Lambda}{3}}T+\frac{2\Lambda}{9}T^2+\frac{5\Lambda}{18}Z^2\right) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \mathcal{O}(\Lambda^{3/2})\right] \times$$

$$\sin\left(w(T-Z)+w\sqrt{\frac{\Lambda}{3}}\left(\frac{Z^2}{2}-TZ\right)-\frac{1}{18}w\Lambda\left(T^3+T^2Z-5TZ^2+2Z^3\right)+\mathcal{O}(\Lambda^{3/2})\right). \tag{5.57}$$

5.5 Detectability

Let us now do some order of magnitude estimates to evaluate the effect of the corrections induced by $\Lambda \neq 0$ on the propagation of gravitational waves.

For the polarization tensors we have not attempted to derive the Λ-order corrections in full detail, although this is possible, because already the most relevant correction, i.e. $\sqrt{\Lambda}ZE_{\mu\nu}^{(0)}$, has to be some orders of magnitude smaller than $E_{\mu\nu}^{(0)}$ for the approximation to be valid. For example for a coordinate value of the order of a typical distance to a supernova, 10^{23} m, the quantity $\sqrt{\Lambda}Z \sim 10^{-3}$ ($\Lambda \sim 10^{-52}$ m^{-2} $\sim 10^{-35}$ s^{-2}). This already means a small correction to an amplitude that has so far escaped detection and which presumably will not be measured with sufficient precision to discern the effect of the Λ-order effects in the foreseeable future. However, conceptually it is an interesting result.

It is more interesting to work out the corrections to the dispersion relation for (5.44). As previously, let us consider waves that propagate in the Z direction and are monochromatic. The maxima of the wave will be reached when

$$w(T-Z)+w\sqrt{\frac{\Lambda}{3}}\left(\frac{Z^2}{2}-TZ\right)=n\pi, \tag{5.58}$$

or

$$Z_{\max}(n,T)=T-\frac{n\pi}{w}-\frac{T^2}{2}\sqrt{\frac{\Lambda}{3}}+\frac{n^2\pi^2}{2w^2}\sqrt{\frac{\Lambda}{3}}. \tag{5.59}$$

From (5.59) we can also calculate the phase velocity of the wave which is defined as

$$v_p(T)\equiv\frac{dZ_{\max}}{dT}=1-T\sqrt{\frac{\Lambda}{3}}+\mathcal{O}(\Lambda). \tag{5.60}$$

We see that in comoving coordinates the phase velocity is smaller than 1. This does not mean that the waves slow down. We can calculate the velocity in 'ruler' distance. For a fixed time we have

$$\begin{aligned} -dl^2 &= -\left(1+T\sqrt{\frac{\Lambda}{3}}\right)dZ^2 \\ \frac{dl}{dT} &= \frac{d}{dT}\left[\left(1+T\sqrt{\frac{\Lambda}{3}}\right)dZ_{\max}\right]=1. \end{aligned} \tag{5.61}$$

It is also interesting to rewrite the trigonometric functions of the wave defining $w_{\text{eff}}(Z)\equiv w\left(1-Z\sqrt{\frac{\Lambda}{3}}\right)$ and $k_{\text{eff}}(Z)\equiv w\left(1-\frac{Z}{2}\sqrt{\frac{\Lambda}{3}}\right)$. The cosine then reads

$$\cos\left[Tw\left(1-Z\sqrt{\frac{\Lambda}{3}}\right)-Zw\left(1-\frac{Z}{2}\sqrt{\frac{\Lambda}{3}}\right)\right]=\cos\left(w_{\text{eff}}T-k_{\text{eff}}Z\right). \tag{5.62}$$

Note that the transformed wave corresponds to a usual wave with an effective frequency dependent on the coordinate Z. The wave becomes redshifted as it propagates away from the source.

To see explicitly the effect of Λ in the propagation of a wave described in comoving coordinates we plot (Figure 1) one of the h_{++} components of the wave for a given instant ($T=0$ for simplicity). A wave with a physical frequency ranging $10^3\text{Hz} < w < 10^{-10}\text{Hz}$ cannot be practically plotted in the relevant Z-range. To see the effect in a few cycles we take $w = 4 \cdot 10^{-16}$Hz, which does not affect the overall magnitude of the correction. We plot the wave for $\Lambda = 10^{-52}m^{-2}$ and for $\Lambda = 10^{-51}m^{-2}$ to assess the influence of Λ on the wave propagation. Then we plot $h_{++} \sim \left(1 + \frac{5}{18}\Lambda Z^2\right) \cos\left[-Zw\left(1 - \frac{Z}{2}\sqrt{\frac{\Lambda}{3}}\right)\right]$.

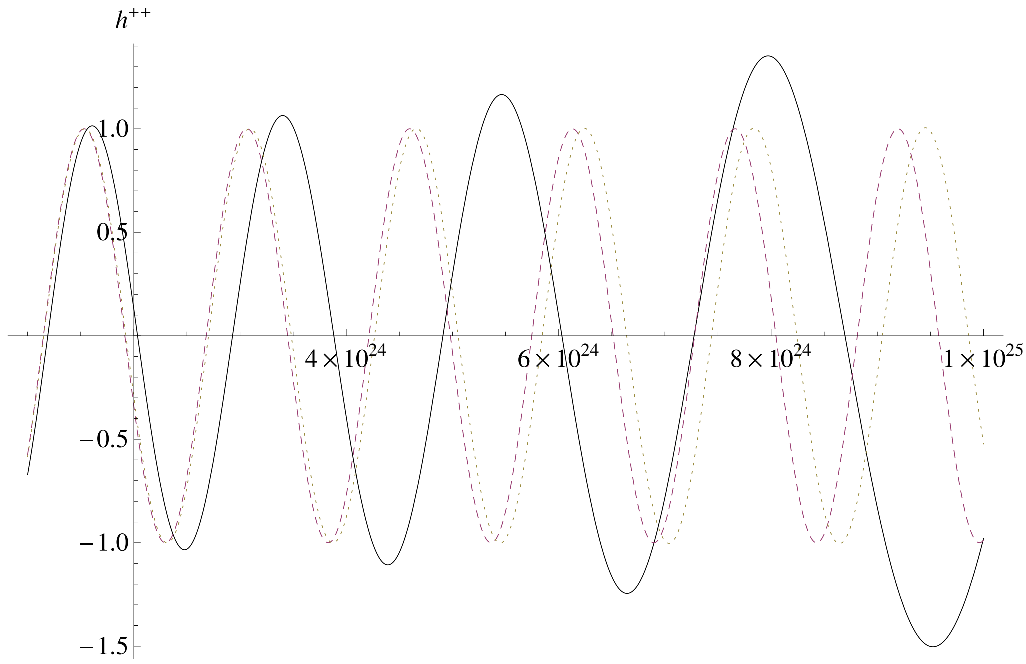


Figure 5.1: Dependency of the amplitude and wavelength on the coordinate distance Z (expressed in meters) for a constant value of T and for different values of Λ: The dashed line corresponds to $\Lambda = 0$, the dotted line to $\Lambda = 10^{-52}m^{-2}$ and the solid line to $\Lambda = 10^{-51}m^{-2}$.

From these results we can already draw some conclusions. The genuine corrections due to the mass-like term in (5.9) remain unchanged in the transformed waves if we cut the expansion to order $\mathcal{O}(\Lambda)$. Moreover they are of order $\frac{\Lambda Z}{w}$, which is in practice irrelevant unless the value of Λ is much greater than the current value. However, transformation (5.17) induces modifications to the wave, both in the amplitude and the phase, of order $\sqrt{\Lambda}$ and Λ. This modifications result in a simultaneous increase of the wavelength and of the amplitude with the coordinate Z. As shown in Figure 1, the most interesting region for detection would be that of events (supernovae and black hole mergers for example) happening at a distance $Z \sim 10^{23} - 10^{25}$m away, for which the correction $\sqrt{\frac{\Lambda}{3}}Z \sim 10^{-1} - 10^{-3}$ is not negligible and is well within the validity range of the approximation. In fact to have this type of correction into account seems probably essential to properly account for the measurements of this type of phenomena in pulsar arrays.

Chapter 6

Local measurement of Λ using pulsar timing arrays

Pulsar timing arrays (PTA) are one of the most promising candidates to offer the first direct detection of gravitational waves. They have been collecting data already for almost a decade and they are expected to obtain signals in the next years. The idea behind PTA is to detect the correlated disruption of the periods measured for a significant number of pulsars due to the passing of a gravitational wave through the system [62, 63, 64, 65]. The frequency range sensitive to this method is $10^{-9}s^{-1} \leq w \leq 10^{-7}s^{-1}$ [62], and the timing residual is expected to follow a power law [63, 66]. A key problem in making predictions for these signals is modeling in a realistic way the wave functions produced in the different sources, in particular the value of the amplitude of the metric perturbation h is a free parameter in principle. Some bounds in the range of $10^{-17} \leq h \leq 10^{-15}$ have been set already [66].

If $\Lambda \neq 0$ gravitational waves (GW) propagate in a de Sitter space-time not in flat Minkowskian space-time. The general practice is simply to account for the expansion of the universe by using a redshifted frequency according to the distance of the source [65]. In this work we go beyond this exceedingly simple approximation and use an approximate solution of the GW equation in de Sitter previously derived [67] and see that the conclusions change.

We assume that Λ is somehow an intrinsic property of space-time rather than an effective description valid at extremely large scales. If so, it is expected to be present at virtually all scales, with the exception of gravitationally bound objects such as galaxies or local groups of galaxies. If Λ is a fundamental constant of nature surely there should be a way of determining locally its value. By 'locally' here we mean at redshifts $z << 1$. This question has been addressed in [14, 18]. We will see that GW may open a nice window to realize this program. In fact, our results suggests that the currently observed value of Λ may actually facilitate the first direct detection of GW under some circumstances.

This chapter is organized as follows. In Section 6.1 the wave functions used are presented, the way in which the timing residuals are calculated is defined and a brief explanation of the coordinate systems involved is included. Section 6.2 is devoted to present our numerical analysis. In Section 6.3 we discuss the possibility of using this method to get some results on the value of the cosmological constant.

6.1 Gravitational waves and timing residuals with $\Lambda \neq 0$

In Minkowski space-time, gravitational waves obey the simple wave equation $\Box h = 0$. It is possible to show [67] that in de Sitter space-time with $\Lambda \neq 0$ and within the linearized approximation one can find solutions of the linearized Einstein equations in the traceless Lorenz gauge (TT gauge [61]) which obey the same equation of motion

$$\Box h^{SdS}_{\mu\nu} = 0. \tag{6.1}$$

Spherical massless waves are solution of this equation away from the source

$$h^{SdS}_{\mu\nu} = \frac{1}{r}\left(E_{\mu\nu}\cos[w(t-r)] + D_{\mu\nu}\sin[w(t-r)]\right). \tag{6.2}$$

However, as shown in [67], this simple linearized solution only holds in a specific set of coordinates, the Schwarzschild-de Sitter (SdS) coordinates. This is easily seen by considering a linearized background solution (rather than wave-like solutions) and realizing that their unique static solution is the (linearized) Schwarzschild-de Sitter metric [67].

Although constituting a perfectly valid solution for gravitational waves, SdS coordinates are not adequate to make observational predictions. The proper isotropic and homogeneous coordinates are the Friedmann-Robertson-Walker (FRW) ones* and the

*Note that the FRW metric cannot be approximated to obey any linearized Einstein equation, see [67] for a detailed discussion.

solution (6.2) in such coordinates, neglecting $\mathcal{O}(\Lambda)$ and higher, reads

$$\begin{aligned} h_{\mu\nu}^{FRW} =& \frac{E_{\mu\nu}}{R}\left(1+\sqrt{\frac{\Lambda}{3}}T\right)\cos\left[w(T-R)+w\sqrt{\frac{\Lambda}{3}}\left(\frac{R^2}{2}-TR\right)\right] \\ &+\frac{D_{\mu\nu}}{R}\left(1+\sqrt{\frac{\Lambda}{3}}T\right)\sin\left[w(T-R)+w\sqrt{\frac{\Lambda}{3}}\left(\frac{R^2}{2}-TR\right)\right], \end{aligned} \tag{6.3}$$

where R is the usual radial FRW comoving coordinate and T is cosmological time. Note that the linearization process that has been used makes sense as long as $\Lambda T^2,\ \Lambda R^2 << 1$ and also that in the TT gauge the only spatial components of the metric that are different from zero are the X, Y entries of the polarization tensors $E_{\mu\nu}$, $D_{\mu\nu}$. Although some temporal components of $E_{\mu\nu}$ and $D_{\mu\nu}$ are also non-zero in these coordinates, they are several orders of magnitude smaller than the spatial ones and therefore will not be relevant for the present study.

We note that the phase velocity of propagation of the GW in such coordinates is not $v_p = 1$ but $v_p \sim 1 - \sqrt{\frac{\Lambda}{3}}T + \mathcal{O}(\Lambda)$ [67]. On the other hand, with respect to the ruler distance travelled (computed with g_{ij}) the velocity is still 1 (up to terms in Λ of higher order to those considered).

Consider the set up depicted in Figure 6.1 describing the relative situation of a GW source (possibly a very massive black hole binary), the Earth and a nearby pulsar

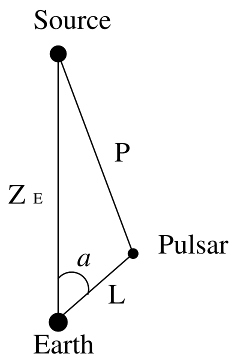


Figure 6.1: Relative coordinates of the GW source (R=0), the Earth (located at $Z = Z_E$) with respect to the GW source and the pulsar located at a coordinates $\vec{P} = (P_X, P_Y, P_Z)$ referred to the source. The Z direction is chosen to be defined by the source-Earth axis. Angles α and β are the polar and azimuthal angles of the pulsar with respect to this axis.

The timing residual [68] induced by (6.3) will be given by

$$H(T_E, L, \alpha, \beta, Z_E, w, \varepsilon, \Lambda) = -\frac{L}{2c}\hat{n}^i\hat{n}^j \int_{-1}^{0} dx\; h_{ij}^{FRW}(T_E + \frac{L}{c}x, \vec{P} + L(1+x)\hat{n}) \quad (6.4)$$

along the null geodesic from the pulsar to the earth, where we assume* $\varepsilon \sim |E_{ij}| \sim |D_{ij}|$, $i, j = X, Y$ and the unit vector $\hat{n}$ is given by $(-\sin\alpha\cos\beta, -\sin\alpha\sin\beta, \cos\alpha)$. In deriving the previous timing residual we have neglected the (non-zero) time components of $E_{\mu\nu}, D_{\mu\nu}$ that, as previously indicated, are several orders of magnitude smaller. The speed of light has been restored. We have assumed that from the pulsar to the Earth the electromagnetic signal follows the trajectory given by the line of sight $\vec{R}(x) = \vec{P} + L(1+x)\hat{n}$. Since we assume that within the Galaxy $\Lambda = 0$, L is also the ruler distance. Explicitly

$$\vec{R}(x) = \vec{P} + L(1+x)\hat{n} = (-xL\sin\alpha\cos\beta, -xL\sin\alpha\sin\beta, Z_E + xL\cos\alpha) \quad (6.5)$$

or in modulus

$$R(x) = \sqrt{Z_E^2 + 2xLZ_E\cos\alpha + x^2L^2} \simeq Z_E + xL\cos\alpha, \quad (6.6)$$

since we are considering $L << Z_E$. This approximation does not affect in any significant way the results below. We do not consider here the known contribution to the timing residual H from the Earth's peculiar motion either. The integral is of course independent of the angle β for any single pulsar but it will depend on the relative angles when several pulsars are averaged.

Let us consider the arguments of the trigonometric functions in (3) and define

$$\begin{aligned}\Theta(x, T_E, L, \alpha, \beta, Z_E, w, \Lambda) \equiv\ & w(T_E + \frac{L}{c}x - \frac{Z_E}{c} - x\frac{L}{c}\cos\alpha) \\ & + w\sqrt{\frac{\Lambda}{3}}\left(\frac{(\frac{Z_E}{c} + x\frac{L}{c}\cos\alpha)^2}{2} - \left(T_E + \frac{L}{c}x\right)\left(\frac{Z_E}{c} + x\frac{L}{c}\cos\alpha\right)\right).\end{aligned} \quad (6.7)$$

Then

$$\begin{aligned}H(T_E, L, \alpha, \beta, Z_E, w, \varepsilon, \Lambda) =\ & -\frac{1}{2}\frac{L\varepsilon}{c}\left(\sin^2\alpha\cos^2\beta + 2\sin\alpha\sin\beta\cos^2\beta - \sin^2\alpha\sin^2\beta\right) \\ & \int_{-1}^{0} dx \frac{1}{(Z_E + xL\cos\alpha)}\left(1 + \sqrt{\frac{\Lambda}{3}}(T_E + \frac{L}{c}x)\right)(\cos\Theta + \sin\Theta).\end{aligned} \quad (6.8)$$

*This approximation is unessential and can be easily removed.

At this point one should ask whether the observationally preferred exceedingly small value of the cosmological constant [51] affects the timing residuals from a pulsar at all. To answer this question we take reasonable values of the parameters both for the GW and one pulsar location and plot the resulting timing residuals as a function of the angle α. The comparison is shown in Figure 6.2. The figure speaks by itself and it strongly

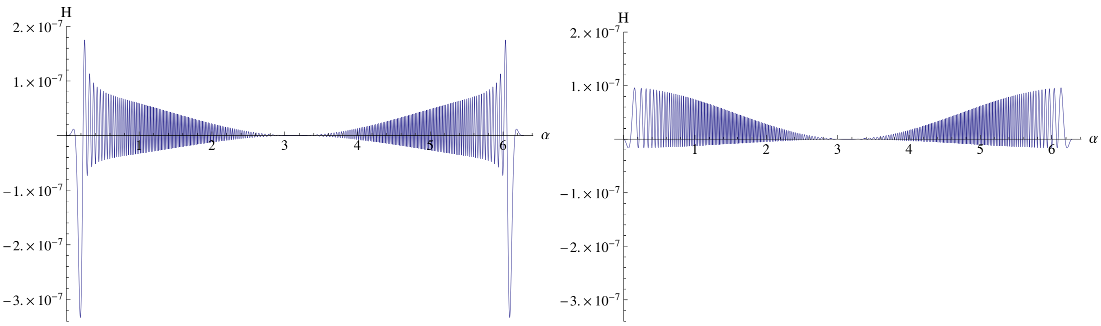


Figure 6.2: On the left the raw timing residual for $\Lambda = 10^{-35}s^{-2}$ as a function of the angle α subtended by the source and the measured pulsar as seen from the observer. On the right the same timing residual for $\Lambda = 0$. In both cases we take $\varepsilon = 1.2 \times 10^9 m$ and $T_E = \frac{Z_E}{c} s$ for $Z_E = 3 \times 10^{24} m$; with these values $|h| \sim \frac{\varepsilon}{R} \sim 10^{-15}$ which is within the expected accuracy of PTA [66].

suggests that the angular dependency of the timing residual is somehow influenced by the value of the cosmological constant, in spite of its small value. Another feature that catches the eye immediately is an enhancement of the signal for a specific small angle α (corresponding generally to a source of low galactic latitude, or a pulsar nearly aligned (but not quite as otherwise $E_{ij}\hat{n}^i\hat{n}^j$ is zero for TT waves) with the source.

To understand this enhancement let us analyze the behavior of the integral

$$I = \int_{-1}^{0} dx(\cos\Theta + \sin\Theta), \tag{6.9}$$

with Θ defined in (6.7) as the prefactors in (6.8) are not relevant for the discussion. The result can be expressed as a combination of Fresnel functions, and sines and cosines. In the limit where $\Lambda \to 0$ the Fresnel functions go to a constant and the behavior is the usual for trigonometric functions. In this respect, the Fresnel functions are responsible for the position and magnitude of the enhancement. This is clearly seen when I^2 is plotted* as a function of the angular separation α between the source and the pulsar.

*We plot I^2 rather than I to deal with a positive quantity.

I^2 always shows a maximum, the position of which is quite stable under changes of most of the parameters involved. It turns out to only depend strongly on the value of Λ and on the distance to the source. It actually depends on the time scales involved rather than on the distance to the source but since the time of arrival of the wave to the local system is directly related to the distance, the dependency is correlated. This is evidenced in Figure 6.3 which shows plots of I^2 for different values of the frequency, distance to the pulsar, distance to the source and cosmological constant. In Figure 6.3.a the following reasonable values, $Z_E = 3 \times 10^{24} m$, $w = 10^{-8} s^{-1}$, $T_E = \frac{Z_E}{c} s$ and $L = 10^{19} m$ are used. In b) there is a change in the distance to the pulsar. In c) we change the frequency. In d) we keep the distance to the source fixed and use the time at the end of an hypothetical 3 year observation. In e) we change the distance to the source one order of magnitude (therefore time also changes). Finally in f) the cosmological constant is changed. It is clear that the most dramatic changes occur when either the distance to the source or the value of the cosmological constant are modified.

6.2 Significance of the timing residuals

Now we would like to make a more detailed study of this possible signal. For that we use the ATNF pulsar catalogue [69]. As it is well-known pulsars are remarkably stable clocks whose periods are known to a very high accuracy, up to $10^{-14} s$ in some cases. However to achieve this extreme precision requires some hypothesis that are not appropriate for the physical situation we are considering and we will assume the more modest precision of $10^{-7} s$ that we take as observational uncertainty.

For each pulsar we have the galactic latitude (ϕ), the galactic longitude (θ) and the distance (L). We transform these coordinates to (α,β), where α as already explained is the angular separation between the line Earth-GW source and the line Earth-pulsar. β corresponds to the azimuthal angle of the pulsar referred to the plane perpendicular to the line Earth-source.

The statistical significance of the timing residual will be

$$\sigma = \sqrt{\frac{1}{N_p N_t} \sum_{i=1}^{N_p} \sum_{j=1}^{N_t} \left(\frac{H(T_E^{i,j}, L_i, \alpha_i, \beta_i, Z_E, w, \epsilon, \Lambda)}{\sigma_t} \right)^2} \tag{6.10}$$

where σ_t is the accuracy with which we are able to measure the pulsar signal period. We take $\sigma_t = 10^{-7}s$ as mentioned. The index i running from 1 to N_p labels the pulsars included in the average.

In the statistical average we assume an observation time span of approximately three years, starting at the time the signal is $10^{16}s$ old (time of arrival at our Galaxy). We assume that we perform observations every eleven days. That is $N_t = 101$; $10^{16}s \leq T_E \leq 1.00000001 \times 10^{16}s$. Since the coalescence times of super massive black hole binaries (SMBHB) can be of the order of $10^7 s$ [70] (that is a much shorter time scale than the time of arrival of the perturbation to the local system) it is justified to use $T_E = \frac{Z_E}{c}$. Form Figure 6.3d one can also see that the position of the enhancement is not significantly altered in the time span on observation.

We turn to the angular dependence of the significance. In the following $\sigma(\alpha)$ is plotted keeping α as a free parameter (note that it is not summed up), that is, using a set of 5 fixed pulsars supposed to be exactly at the same angular separation from a source the position of which we vary $0 \leq \alpha \leq 2\pi$ (in this respect this is still a theoretical exercise). This could be done for any set of five pulsars, since, as shown in the previous section, the position of the peak does not depend on the values L_i and β_i. However, we used the following set of real pulsars which are all close to each other at a distance $L \sim 10^{20}m$. Since there are over 600 pulsars it is not difficult to find clusters with a

Pulsars from the ATNF Catalogue
J0024-7204E
J0024-7204D
J0024-7204M
J0024-7204G
J0024-7204I

Table 6.1: List of pulsars whose L_i and β_i we used to calculate $\sigma(\alpha)$ for an hypothetical source at angular separation α.

similar α, albeit possibly with very different values of L and β.

$$\sigma(\alpha) = \sqrt{\frac{1}{5 \cdot 101} \sum_{i=1}^{5} \sum_{j=1}^{101} \left(\frac{H(T_E^{i,j}, L_i, \alpha, \beta_i, 10^{24}, 10^{-8}, 1.2 \times 10^{9}, 10^{-35})}{10^{-7}} \right)^2}. \quad (6.11)$$

Length units are given in meters, frequencies in s^{-1}. We observe a huge peak at

$\alpha \sim 0.19$rad (see Figure 6.4). If a source is located at such angular separation from the average angular position of the five pulsars chosen for observation, the significance could be boosted some 50 times. Let us compare it to the same calculation taking $\Lambda = 0$ and redshifted frequency $w_{eff} = \frac{w}{(1+z)}$; $z \sim 0.008$, which is the corresponding redshift for an object $10^{24}m$ away calculated using both matter and energy densities. No peak is observed.

Now we take a list of observed pulsars well distributed in the galaxy. The angles (α, β) are calculated for all of them considering two hypothetical sources of GW. One located at galactic coordinates $\theta_{S1} = 300°$, $\phi_{S1} = -35°$ and another located at $\theta_{S2} = 4°$, $\phi_{S2} = 10°$. We order them from the lowest α to the largest. We group them in sets of five pulsars. We consider 27 sets of 5 pulsars; that is a list of 135 pulsars. For each set we calculate the significance

$$\sigma_k = \sqrt{\frac{1}{5 \cdot 101} \sum_{i=1}^{5_k} \sum_{j=1}^{101} \left(\frac{H(T_E^{i,j}, L_i, \alpha_i, \beta_i, 10^{24}, 10^{-8}, 1.2 \times 10^9, 10^{-35})}{10^{-7}} \right)^2} \tag{6.12}$$

and plot it as a function of the average angle of the set, $\bar{\alpha}_k = \sum_{i=1}^{5_k} \frac{\alpha_i}{5}$ with $1 \leq k \leq 27$. Note this is different from (6.11); here we choose two hypothetical fixed sources and a long list of pulsars grouped by their angular separation α to these sources. This could be a realistic calculation once real sources are considered.

The results obtained are plotted in Figure 6.5. In both cases a very noticeable peak is observed at the expected angle.

The reason why the peak for Source 2 is lower than the peak for Source 1 is that Source 2 is located close but not at the precise angular separation of a real cluster of pulsars. This is meant to illustrate that even in that case a significant enhancement of the signal can be achieved.

Finally, the dependency of σ on the frequency

$$\sigma(w) = \sqrt{\frac{1}{N_p \cdot 101} \sum_{i=1}^{N_p} \sum_{j=1}^{101} \left(\frac{H(T_E^{i,j}, L_i, \alpha_i, \beta_i, 10^{24}, w, 1.2 \times 10^9, 10^{-35})}{10^{-7}} \right)^2}, \tag{6.13}$$

has also been investigated. Some of our preliminary checks indicated that no differences at all were observed in the power spectrum when the value of Λ was changed and that, as expected [63, 64, 66], the signal follows a power law $\sigma \sim \frac{1}{w}$. However, let us take a closer look at the dependency on the frequency for a short list of pulsars located at the right angular separation to observe the peak. We have already seen the significance grows notoriously in this angular region. Figure 6.6 (middle) shows the frequency dependence of the signal for fifteen pulsars at the right spot with respect to Source 1. As it can be clearly seen, the signal significance grows enormously again for $\Lambda = 10^{-35}s^{-2}$ and apparently does not follow a power law. For the same short list of pulsars and for $\Lambda = 0$ the signal falls back to smaller values and its envelope shapes towards a power law. In Figure 6.6 (top) we also present the same plot for fifteen pulsars located at an angular separation of around $\alpha \sim 1.1$rad, that is away from the peak. In this case we see no differences between the different values of the cosmological constant as well as a clear power law behavior. The magnitude of the signal is compatible with that of the fifteen pulsars at the peak separation when $\Lambda = 0$.

6.3 Measuring the cosmological constant

We have seen in the previous that there is an enhancement in the timing residual for a particular value of the angle α when GW propagating in de Sitter space-time are measured. Among all the dependencies, and when the distance to the source is well-known, the most relevant appears to be the one related to the value of the cosmological constant Λ. The position of the peak depends strongly on the value of Λ. It moves towards the central values of the angle for larger values of the cosmological constant.

The values of Λ as a function of the position at which the peak would be found are plotted in Figure 6.7 (dots) using the positions found in the plots for $\sigma(\alpha)$ (6.11) for different values of the cosmological constant. This calculation was carried out using two independent numerical methods in order to make sure that one is free of numerical instabilities (this is a necessary precaution as large numbers are involved).

We argued in Section 2 that the position of the peak is determined by the Fresnel functions one obtains when calculating the timing residuals. Indeed the integral I in (6.9), which captures the crucial effect, gives a prefactor times a combination of Fresnel functions times a combination of trigonometric functions. The latter are featureless; however the prefactor becomes quite large for a specific value of the parameters involved. This particular value renders the Fresnel function close to zero and the product is a number close to 2. Away from this point the net result is small.

Using the series expansion of the Fresnel functions at first order we are able to obtain an approximate analytical expression for the relation $\Lambda(\alpha)$; that is for the value of the cosmological constant that (all other parameters being fixed) gives a strong enhancement of the significance σ at a given angle α

$$\Lambda(\alpha) = \frac{12c^2 \sin^4\left(\frac{\alpha}{2}\right)}{\left((cT_E - Z_E)\cos\alpha + Z_E\right)^2} \simeq \frac{12c^2 \sin^4\left(\frac{\alpha}{2}\right)}{Z_E^2}, \tag{6.14}$$

which is also shown in Figure 6.7 (line). We have used the fact that, taking into account the duration of a black-hole merger, $cT_E \simeq Z_E$. Equation (6.14) is a clear prediction that could be eventually tested. In fact, this effect could also facilitate enormously the detection of GW coming from massive binary black holes by carefully selecting and binning groups of pulsars, although the possibility of measuring Λ locally certainly looks to us more exciting.

Throughout this work we have considered only the effect of Λ on GW and the way they affect pulsar timing residuals and we have neglected the effect of matter or matter density. In fact, the main effect of the latter would be through the familiar redshift in the frequency of GW. Frequency does not play a crucial role in the previous discussion provided that is low enough to be detectable in PTA. It is probably useful to remind the reader once more that Λ is assumed to be an intrinsic property of space-time, present to all scales, except close to the Galaxy. It would be easy to implement more realistic models in our study, if reasonably well-defined ones were available. In fact, these uncertainties constitute strong reasons to try to measure Λ locally.

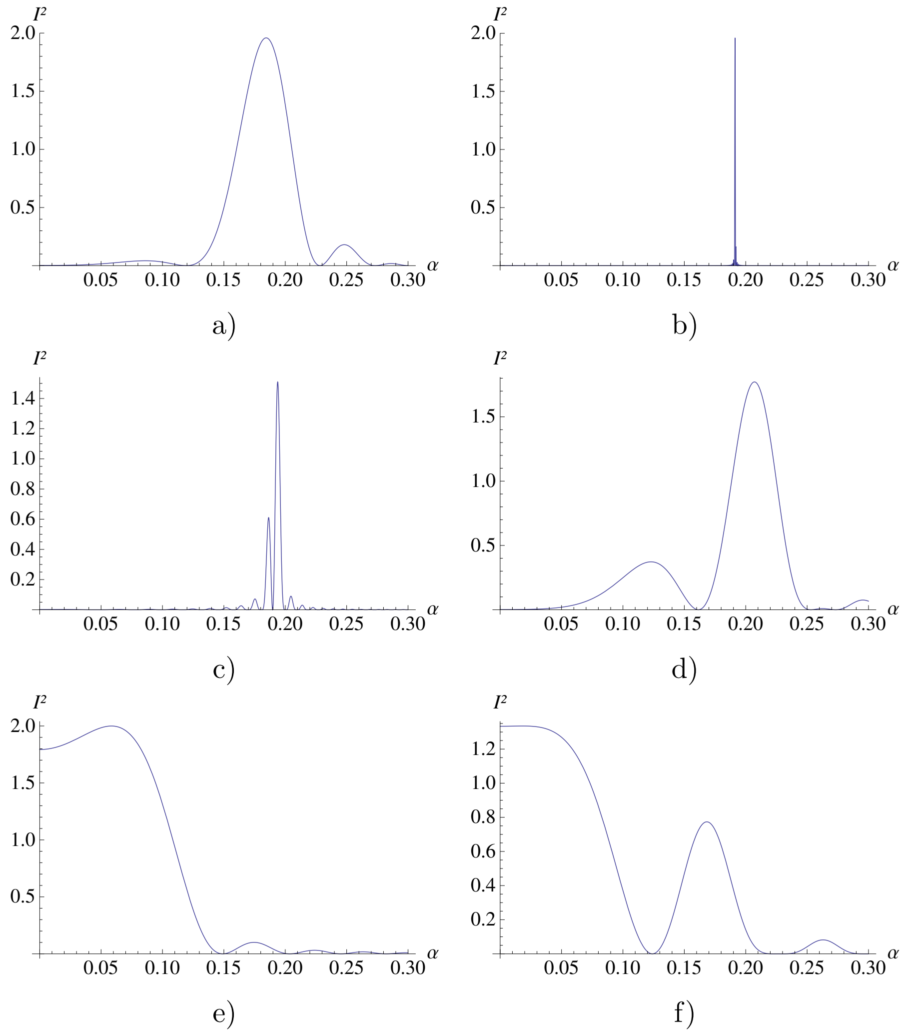


Figure 6.3: Integral I^2 plotted for different values of the parameters involved. a) Corresponds to the reasonable values $w = 10^{-8}s^{-1}$, $L = 10^{19}m$, $Z_E = 3 \times 10^{24}m$, $\Lambda = 10^{-35}s^{-2}$ and $T_E = 10^{16}s$. b) Change in pulsar distance to $L = 10^{21}m$. c) Change in frequency to $w = 10^{-7}s^{-1}$. d) Change in time to $T_E = (10^{16} + 10^8)s$. e) Change in time and distance to the source to $Z_E = 3 \times 10^{23}m$ and $T_E = 10^{15}s$. f) Change in the cosmological constant to $\Lambda = 10^{-36}s^{-2}$.

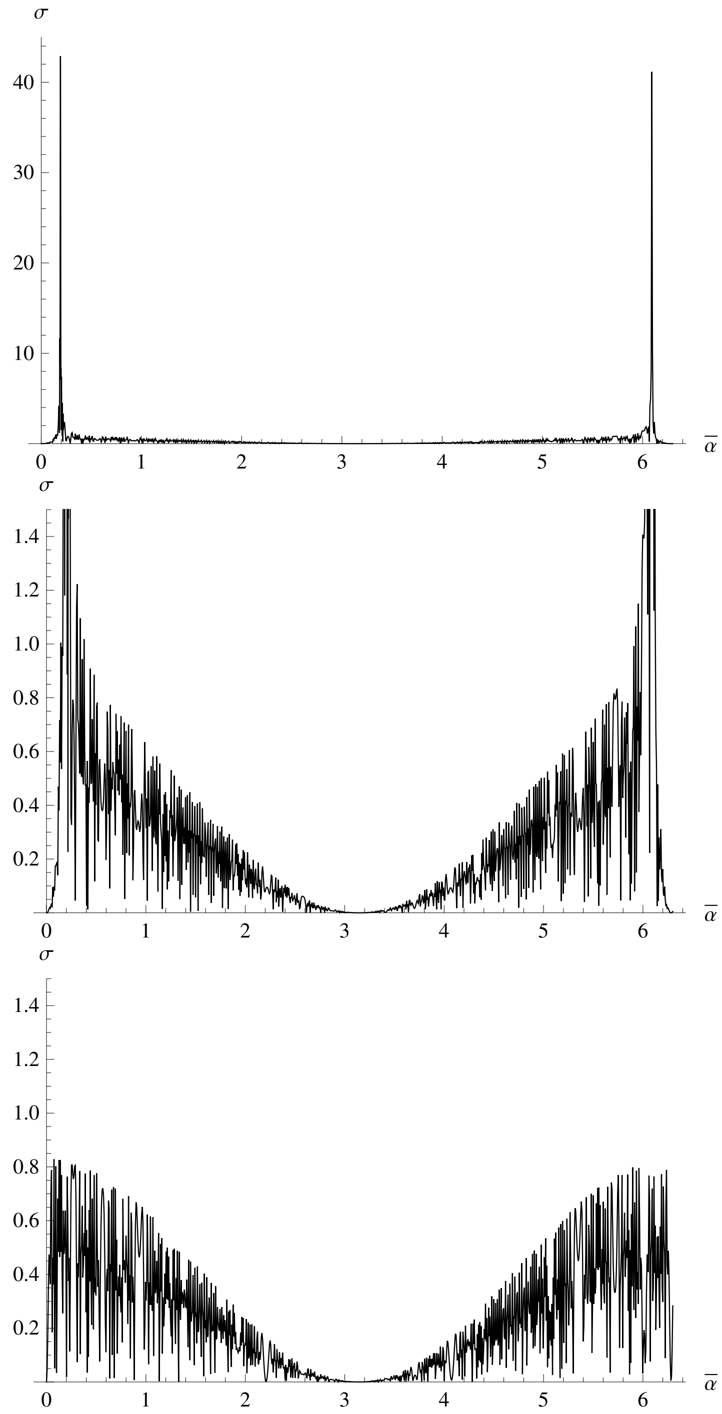


Figure 6.4: $\sigma(\alpha)$ for $\Lambda = 10^{-35}s^{-2}$ (Top). Zoom on the lower values for $\Lambda = 10^{-35}s^{-2}$ (middle), and comparison to $\Lambda = 0$ (bottom).

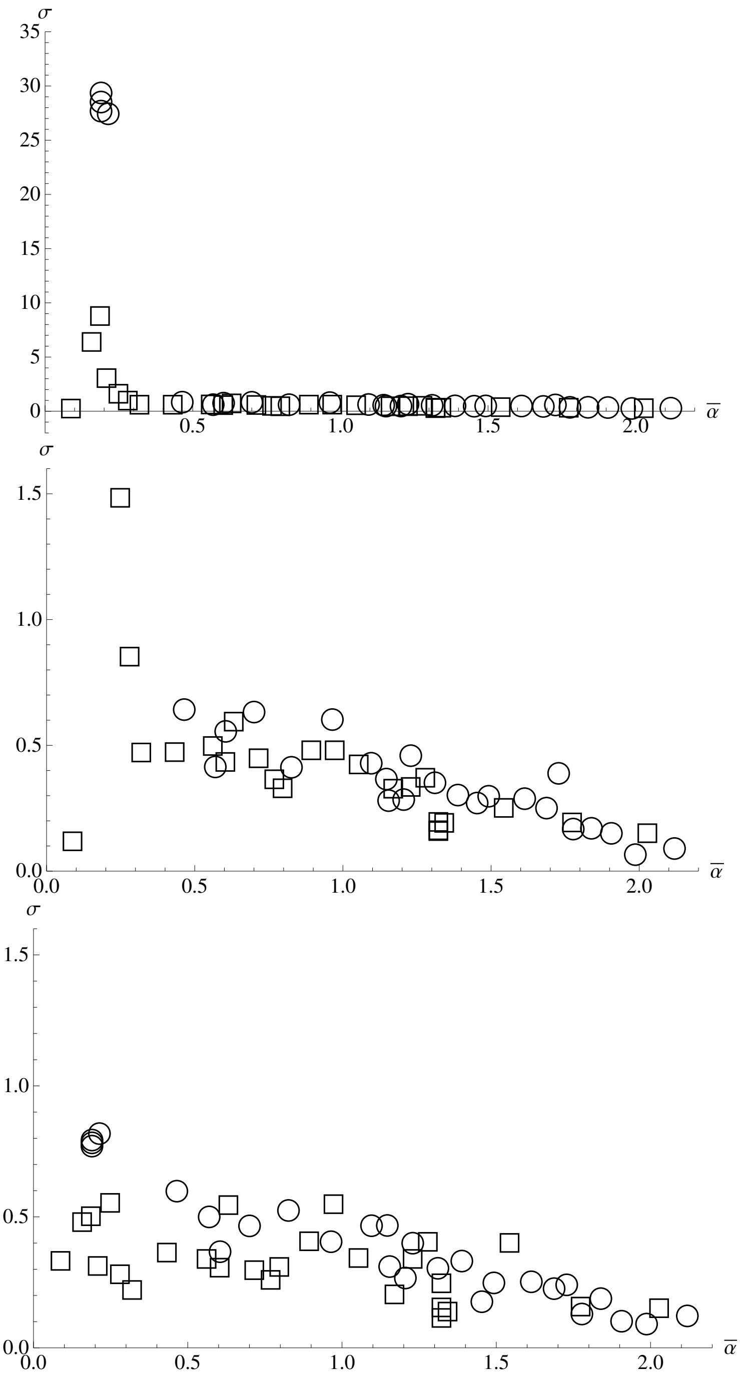


Figure 6.5: Plot of $\sigma_k(\bar{\alpha}_k)$, $k = 1, 27$. $\Lambda = 10^{-35}s^{-2}$. Circles correspond to Source 1 and squares to Source 2. Full range is showed on top, zoom on the lower values for $\Lambda = 10^{-35}s^{-2}$ and comparison to $\Lambda = 0$ show on middle and bottom respectively.

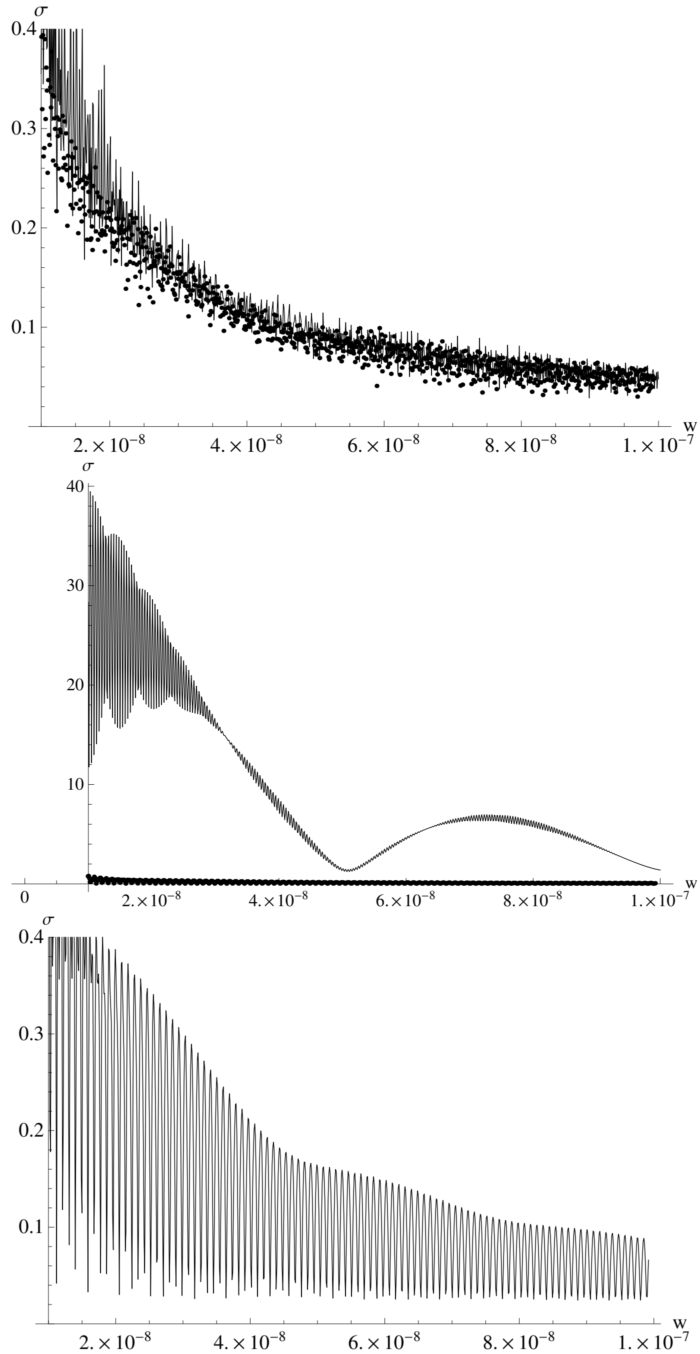


Figure 6.6: $\sigma(w)$ for 15 pulsars away from the peak angular region for Source 1 (top), the solid line corresponds to $\Lambda = 10^{-35}s^{-2}$ and dots correspond to $\Lambda = 0$. $\sigma(w)$ for 15 pulsars at the peak angular region for the same source (middle). Solid line corresponds to $\Lambda = 10^{-35}s^{-2}$ and the data close to the horizontal axis correspond to $\Lambda = 0$. Zoom on the $\Lambda = 0$ case (bottom).

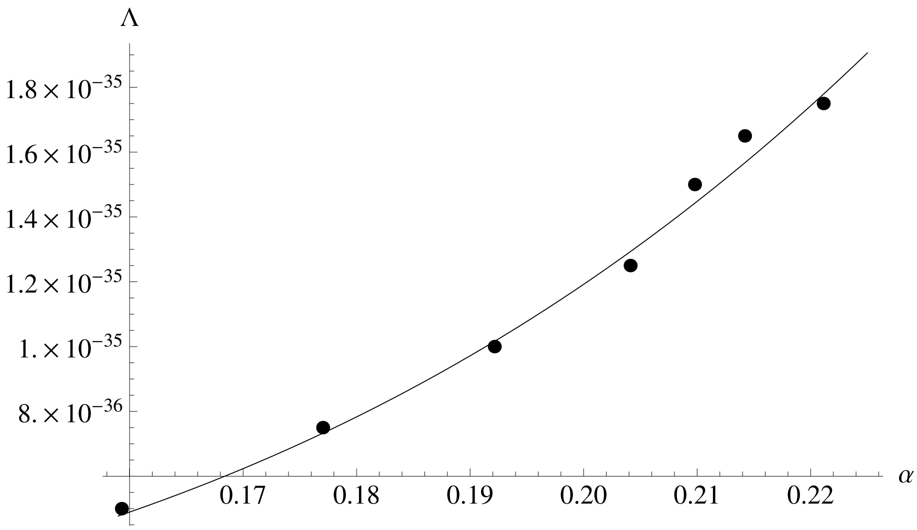


Figure 6.7: $\Lambda(\alpha)$ obtained numerically from the positions of the peaks in the $\sigma(\alpha)$ plots for different values of the cosmological constant (dots) and obtained analytically from an approximation of the Fresnel functions involved in the timing residual (line).

Chapter 7

Summary and Outlook

This chapter is devoted to sum up the main results and conclusions of this thesis. In Chapters 2, 3 and 4 we propose a model where gravity emerges from a theory without any predefined metric. The minimal input is provided by assuming a differential manifold structure endowed with an affine connection. Nothing more. The Lagrangian can be defined without having to appeal to a particular metric or vierbein.

Gravity and distance are induced rather than fundamental concepts in our proposal. At sufficiently short scales, when the effective action does not make sense anymore, the physical degrees of freedom are fermionic. At such short scales there is not even the notion of distance and hence the scale at which the symmetry is restored is the shortest distance there can be.

The relative technical simplicity of this proposal constitutes its main virtue when compared with previous proposals [11, 12, 13], where even semiquantitative discussions appear impossible. We have been able to derive in full detail the effective action and in the case of $D = 4$ make predictions such as the existence of a cosmological constant, $\Lambda \neq 0$, the value of which is not fixed by the calculations but could, in principle, be adjusted to any observed value. Also the obtention of Einstein-Hilbert theory, free of divergences at the classical level, as the low energy effective theory of the model. And more importantly, the fact that it is possible to start from a theory with no metric whatsoever and obtain at the end unambiguous consistent results.

The usual obstruction to emergent gravity due to the Weinberg-Witten theorem can be circumvented in the present proposal thanks to the particular structure of the model. Lorentz indices are of an internal nature in our approach, therefore the energy momentum tensor associated to the Lagrangian of the fundamental theory is not a Lorentz covariant rank-two tensor and falls out of the assumptions of the theorem. After the breaking one is in the same situation of GR where the applicability of the theorem is excluded.

A very important aspect of the model is the apparent improvement of the ultraviolet behavior. After integration of the fundamental degrees of freedom all the divergences that appear to the order we have computed in the external fields can be absorbed in the redefinition of the cosmological constant (in $D = 2$ and $D = 4$) and the Planck mass (in $D = 4$; as seen from the effective theory point of view, even though the respective counterterms do not have this meaning in the underlying theory). With the running dictated by the corresponding beta functions, both quantities are renormalization group invariant. In addition, in the four dimensional case, the Gauss-Bonnet invariant is renormalized too. This mitigation of the short-distance divergences happens in spite of the bad ultraviolet behavior of the propagator and the ultimate reason, we think, is that these are the *only* counterterms that can be written without having to assume an underlying metric that does not exist before spontaneous symmetry breaking takes place.

At long distances the fluctuations around the broken vacuum are the relevant degrees of freedom and are described by an effective theory whose lowest dimensional operators are just those of ordinary, either $D = 2$ or $D = 4$, gravity. They of course exhibit the usual divergences of quantum gravity but this now poses in principle no problem as we know that at very high energies this is not the right theory. For $k \sim M$ one starts seeing the fundamental degrees of freedom. Gravitons are the Goldstone bosons of a broken global symmetry. We already argued how the barrier of the Weinberg-Witten no-go theorem could be overcome.

In a sense the fundamental fermions resolve the point-like 3-graviton, 4-graviton, etc. interactions into extended form factors and this is the reason for the mitigation of the terrible ultraviolet behavior of quantum gravity. However this is only part of the story, because the resolution of the vertices could be equally achieved by using Dirac fermions coupled to gravity (or any other field for that matter). This would in fact be just a reproduction of the old program of induced gravity [7] and therefore not that interesting. The really novel point in this proposal is that the microscopic fermion action does not contain any metric tensor at all. Then not only is the metric and its fluctuations –the gravitons– spontaneously generated, but the possible counterterms are severely limited in number.

We stop short of making any strong claims about the renormalizability of the model. We can just say that, from our calculations and our experience with the model, renormalizability is a plausible hypothesis (our results actually amount to an heuristic proof in the large N limit). Likewise we do not insist in that the one presented is the *sole* possibility to carry out the present program, although it looks fairly unique. Clearly a number of issues need further study before the present proposal can be taken seriously but we think that the results presented here are interesting enough.

A number of extensions and possible applications come to our mind. Perhaps the most intriguing one from a physical point of view would be to investigate in this framework singular solutions in GR such as black holes. A more in-depth study of the renormalizability issue is certainly required too as there are issues related to the renormalization group to be addressed in the present setting. The issue of the minimal distance at which gravity looses its meaning is one of the most interesting. Observationally, the validity of GR is tested to some scale of the order of the millimeter [71]. From there to the Plank scale there is an enormous room where modifications of gravity might eventually be envisaged. The inclusion of matter fields in the model is also an appealing topic, and certainly, trying to make contact with other interactions could yield interesting results. The relation between our results and Lorentzian triangulation analysis is an attractive possible line of study too. Even more exotic ideas such as tunneling between geometries or higher dimensional realizations could be worth investigating.

In Chapter 5 we investigate the effect of the cosmological constant in the propagation of gravitational waves in a linearized theory of gravity. The presence of Λ leads unavoidably to the curvature of the background space-time in which the waves propagate. Within the linearized approximation (which is the only framework where one can properly speak of 'waves') this leads to a decomposition $g_{\mu\nu} \simeq \eta_{\mu\nu} + h^{\Lambda}_{\mu\nu} + h^{W}_{\mu\nu}$, including a modification of the background (corresponding to the curvature) and a wave-like perturbation.

To see the way the propagation of the waves is affected, one has first to understand the implications that the different coordinate choices (gauge choices) have in the resolution of the equations of motion as well as the importance of the terms of different order retained in the linearization. One is free to choose any particular gauge to solve the equations, however since the linearized Einstein equations are not invariant under general coordinate transformations their form will depend on the gauge choice. We argue that the above procedure of linearization is consistent in some coordinate systems but not in others. In particular, it is inconsistent to linearize the equations in the familiar Friedmann-Robertson-Walker cosmological coordinates (the metric only depends on time). Note here that by inconsistency of the linearization in FRW we mean that $\bar{g}_{\mu\nu} = \eta_{\mu\nu} + h^{\Lambda}_{\mu\nu}$ being a linearized version of the FRW metric will never fulfill any linearized Einstein equations. However, some authors [58] have studied small perturbations above the exact FRW background, which is a linearization process with respect to the small perturbation $h^{W}_{\mu\nu}$, but constitutes an approach completely different from ours. That is, appropriate to discuss cosmological metric perturbations but useless to describe local perturbations such as black hole collapse for instance.

Einstein equations can however be consistently linearized in certain coordinates (those of the Lorenz gauge for instance) where the calculations are notoriously easier; after studying the symmetries of these coordinates and the relation between the different gauge choices used in this thesis, one reaches the conclusions that, in virtue of Birkhoff's theorem, they are all different parametrizations of a linearized version of the SdS metric, expanded to first order in Λ. For these coordinates the analysis of gravitational waves follows a pattern very similar to the one in Minkowski space-time. In the case where the $\Lambda h_{\mu\nu}$ term is dropped the residual gauge freedom allows for the removal of four additional degrees of freedom in the general solution, leaving the wave-like component with the usual two physical degrees of freedom of waves propagating in flat space-time.

On the contrary, if the term $\Lambda h_{\mu\nu}$ is retained in the equations of motion the situation changes. Even in the Lorenz gauge the invariance under residual gauge transformations is lost. Again it is not hard to find the most general solution to the linearized equations composed of a background and a wave-like components. We prove the background solution to be consistent with the result previously found if Λ is small. Since there is no residual invariance, the wave-like solution has to be interpreted as a 'massive' wave with five degrees of freedom (the gauge condition and the trace condition amount to five constraints as seen by a Minkowski observer). However, we can make use of the approximate residual invariance at the leading order in Λ to rewrite the solution as massless gravitational waves with position-dependent modified amplitudes that change very slowly given the current values of Λ. There are only two $\mathcal{O}(1)$ polarizations; the remaining degrees of freedom (up to the five independent ones required for a massive spin two wave) are of $\mathcal{O}(\Lambda)$ and couple extremely weakly to matter sources. Of course we are not saying that gravitons are physically massive, they do have five degrees of freedom when their propagation is studied in a Minkowski frame, but one has to bear in mind this is just an artifact of the linearization process in which the residual gauge freedom is lost.

Finally, one has to transform these solutions to the physically significant FRW coordinates in order to extract observable consequences. At this point modifications of $\mathcal{O}(\sqrt{\Lambda})$ appear. Numerically these can be quite relevant for certain gravitational waves traveling from far away sources and the effect of Λ can absolutely have a detectable impact on pulsar timing arrays. Waves are modified both in the phase and the amplitude; in cosmological coordinates they are redshifted in a prescribed way and their amplitude grows as they move away from the source.

To close this thesis, in Chapter 6 we investigate the local effects of the cosmological constant for the detection of gravitational waves in PTA. The gravitational wave function is usually modeled as a massless, either plane or spherical, wave traveling in flat space-time. The expansion of the universe is accounted for by including a redshift in the frequency. Major problems are related to modeling the source and assessing the strain of the amplitudes of the waves. Here we obviate these by just assuming a spherical wave and focus in the fact that the waves propagate in a de Sitter space-time rather than in flat space-time.

We use a wave solution derived in FRW coordinates, which we expect to be considerably more realistic than the redshifted usual waves. With this, we calculate the timing residuals induced in the signal of known pulsars in our Galaxy, predicting a particular value of the angle subtended between the source and the pulsar where an enhanced significance of the timing residual is observed. We argue that the position of this peak depends strongly on the value of the cosmological constant. This peak is absent when the calculations are carried out with usual, Minkowski solution, redshifted waves. We propose two hypothetical sources at two distinct positions for which we calculate the timing residuals significance using a real set of pulsars. The peak is observed at the predicted angular position. Finally we obtain the angular dependency of the value of the cosmological constant using the position of the peak for different values of Λ and analytically from the Fresnel functions involved in the calculation. This method could represent an independent way to determine the value of the cosmological constant.

Although being very compelling, these results are preliminary. Further study is needed to assess the feasibility of detection of GW using this theoretical framework. The usage of even more realistic wave fronts, if available, could help confirming the results obtained here. Identifying real sources that could meet the requirements of our model would also lead to more realistic results.

Overall, this thesis deals with the fact that gravity, understood as Einstein Hilbert gravity, may very well be just an effective description valid for phenomena relevant at very different scales. From the very large structure of the Universe to the Solar system; and perhaps to the scale of the more elusive quantum nature of the interactions. On the other hand, gravitational waves are an unambiguous prediction of GR. If one is capable of carrying out a program by which GR is obtained as the low energy effective theory of some more fundamental theory and this effective theory comes naturally equipped with a cosmological constant, it is just natural to try to make sense of the effects that the cosmological constant has in the propagation of waves.

Let us emphasize the dual relevance of the results related to PTA. They would not only constitute, if observed, an indication that the cosmological constant is indeed an intrinsic property of space-time but, since the statistical significance of the signal is apparently boosted, they could facilitate greatly the first detection of gravitational waves. This is a relevant result on its own.

Although this thesis does not provide definite answers as of the true quantum nature of gravity or to the issue of gravitational radiation within strong gravitational fields; hopefully it takes us closer to a deeper understanding of the effective description of the most elusive of the fundamental interactions. And hopefully it also provides us with a more realistic modeling of one of its untested predictions, that might even lead to the first detection of gravitational waves.

7.1 List of publications

The original work contained in this thesis is based on the following publications by the author:

- Local measurement of Λ using pulsar timing arrays, (with D. Espriu), e-Print: arXiv:1209.3724 [gr-qc] (2012).
- Spontaneous generation of geometry in four dimensions, (with J. Alfaro and D. Espriu), Phys. Rev. D 86, 025015 (2012).
- Gravitational waves in the presence of a cosmological constant, (with J. Bernabeu and D. Espriu), Phys. Rev. D 84, 063523 (2011).
- The emergence of geometry: a two-dimensional toy model, (with J. Alfaro and D. Espriu), Phys. Rev. D 82, 045018 (2010).
- Gravity as an Effective theory, (with D. Espriu), Acta Physica Polonica B, 40, 12 (2009).

Chapter 8

Resum en català

8.1 Introducció

La gravitació ha governat el desenvolupament de l'Univers des del principi dels temps. Molt abans que l'ésser humà comencés a preguntar-se pel moviment aparent de les estrelles, la seva maquinària ja dictava, impassible, el moviment dels cosos tal i com l'observem avui dia. El desig d'entendre els fenòmens naturals ha acompanyat totes i cadascuna de les generacions de l'home; sempre hi ha hagut ments brillants empenyent els límits del coneixement. Probablement entre les primeres dissertacions documentades sobre la gravitació trobem el treball d'Aristòtil al segle quart aC [1]. Per ell, el moviment dels cossos depenia de la seva composició en termes dels 'elements' i les seves posicions tendien al 'lloc natural' sense necessitat de forces. La seva visió estava literalment segles enllà del coneixement actual, però tanmateix il·lustra una forta determinació per donar explicació als fenòmens de la Natura. Al segle setè dC, el matemàtic indi Brahmagupta deixà escrit que 'els cossos cauen cap a la terra perquè és en la natura de la terra atraure els cossos, de la mateixa manera que fluir és en la natura de l'aigua' [2]. Aquest desig persistent d'entendre la Natura va portar fins als primers intents de trobar explicacions sistemàtiques a la fenomenologia dels cosos. Galileu Galilei va ser el primer d'adonar-se que tots els cosos són accelerats de la mateixa manera cap a la terra, contràriament als pensaments Aristotèlics [3]. Això passava al segle setze. Els seus treballs començaven a representar un enteniment més profund de la matèria. Amb tot, no va ser fins un segle més tard, amb la publicació del Principa [4], que Newton va materialitzar la primera formulació consistent de la gravitació. Newton s'adonà que el secret estava en la relació entre les masses i la seva distància. Amb la llei de l'invers del quadrat va ser el primer d'unificar en una sola descripció matemàtica, fenòmens aparentment tan diferents com el moviment dels astres o com el fet que una poma caigui a terra del pomer. Aquesta revolució en la comprensió de la gravetat va ser prou significativa com per durar més de dos segles.

Quan Albert Einstein va escriure la teoria de la Relativitat General el 1915, no només va donar un pas de gegant en la comprensió de la gravetat, va canviar significativament les regles del joc. Dues masses deixaven d'exercir força l'una sobre l'altre, el concepte de força estava perdut. Els cossos massius simplement corben l'espai-temps al seu voltant fent que altres cosos 'rodolin' per aquestes corbes. La noció d'espai-temps, amb la inclusió del temps com una dimensió més de la realitat que percebem va suposar un dels canvis conceptuals més profunds del segle vint. Aquest canvi de perspectiva va ser tan notable i brillantment provat amb experiments primerencs [5], que encara avui dia la teoria de la Relativitat General, formulada com Einstein ho va fer originalment, és la més completa descripció de la interacció gravitatòria de la que disposem. Algunes de les seves prediccions han sigut provades; ningú dubta avui dia de l'existència dels forats negres. En canvi altres, com les ones gravitatòries, esperen pacients la confirmació experimental, tot i que en el cas de les ones hi ha evidències indirectes de la seva existència en el balanç energètic de sistemes binaris d'estrelles de neutrons [6].

És clar que Einstein no va donar resposta a totes les preguntes relacionades amb la gravitació. En un temps en què el món quàntic s'estava desenvolupant a gran velocitat, la Relativitat General no proporcionava una manera intuïtiva i natural d'incorporar les idees quàntiques a la interacció gravitatòria. De fet, qualsevol intent de trobar la verdadera naturalesa quàntica de la gravetat ha acabat sempre en carrers sense sortida. No hi ha una explicació a per què la gravetat és tan més feble que qualsevol de les altres interaccions fonamentals. Hi ha ambigüitat respecte a la necessitat d'incloure una constant cosmològica o energia del buit en la formulació i de com justificar-ho en cas d'incloure-la. I més fonamentalment, no hi ha hagut cap proposta que donés una solució al problema de l'estructura ultraviolada de la teoria. En aquest sentit, probablement la proposta que ha cridat més l'atenció de la comunitat, i d'alguna manera ha tingut més èxit, és la teoria de cordes. En la seva formulació és possible construir una teoria quàntica de la gravitació consistent, tanmateix, a dia d'avui sembla generar més preguntes que respostes a les preguntes que originalment pretenia respondre.

Cap al final dels anys seixanta els físics varen començar a considerar la possibilitat que les dificultats per quantitzar la gravetat tal vegada eren degudes en realitat a la manca de graus de llibertat fonamentals per quantitzar associats a la gravetat. Que la interacció gravitatòria no és fonamental com a tal. A l'arrel d'algunes propostes d'aquella època hi ha la idea que la gravetat no és més que una descripció efectiva vàlida a baixes energies. Probablement els primers en treballar en aquesta línia foren Zel'dovich i una mica més tard Sakharov [7]. El primer va estudiar l'efecte de les fluctuacions quàntiques en la constant cosmològica i com aquestes fluctuacions produïen que la constant adquirís un valor diferent de zero. El segon va complementar aquesta feina estudiant com fluctuacions quàntiques sobre aquest valor diferent de zero produïen en general teories efectives de l'estil de la d'Einstein. Les dificultats tècniques que van trobar van alentir el desenvolupament del camp durant alguns anys. Tan sols una mica més tard, a l'inici dels setanta, Salam i col·laboradors van estudiar el grup de simetria conforme en el marc de realitzacions no lineals [8]. Tot i que la seva idea no era atacar el problema de la gravetat quàntica, van acabar contribuint en la comprensió del fenomen de ruptura espontània de simetria de la covariància general, i en aquest sentit com els gravitons es podien interpretar com bosons de Goldstone. Ogievetsky i col·laboradors van prosseguir la investigació des del punt de vista de la teoria de grups una mica més tard [9]. Fent l'analogia amb la teoria quiral van provar que teories invariants sota certs grups de simetria desitjables (simetria afí o grup conforme per exemple) produïen, després de patir trencaments espontanis de simetria, teories efectives les equacions del moviment de les quals eren precisament les mateixes que les de la teoria d'Einstein.

La literatura és extensa i força propostes han vist la llum des dels setanta. El trencament de diferents grups de simetria (Lorentz, difeomorfismes...) s'ha estudiat, i els productes d'aquests diferents trencaments en les teories efectives s'han investigat. Com a característiques comunes a totes aquestes propostes trobem que per qualsevol teoria quàntica de camps ben comportada es genera un terme tipus curvatura i un de tipus cosmològic després del trencament espontani de simetria. Alguns dels problemes, però, tenen a veure amb la interpretació de les teories fonamentals abans del trencament, amb el comportament a l'ultraviolat de les teories efectives, i des dels vuitanta, quan es va publicar el teorema de Weinberg-Witten, fins i tot la generació dinàmica de bosons de Goldstone d'espí igual a dos està sota sospita.

Com ja s'ha dit, algunes propostes han arribat al punt de dubtar de l'existència de graus de llibertat fonamentals associats amb la gravetat, interpretant aquesta com un fenomen 'col·lectiu' o 'entròpic'. La majoria d'aquestes propostes són incapaces de reproduir les propietats conegudes de la gravetat i són en general suficientment ambigües per no poder ser falsades.

Una altra qüestió que s'ha de tenir present és la dificultat de justificar la inclusió d'una mètrica en la teoria fonamental abans del trencament de simetria. El funcionament de la majoria de propostes rau en la inclusió de certa noció de geometria des del principi. Aquesta simplifica els càlculs per obtenir mètriques efectives més complexes però es fa difícil d'entendre si els corresponents graus de llibertat només han d'estar disponibles després del trencament de simetria. Aparentment només Russo i Amati [11], i també Wetterich [12], al principi dels noranta van proposar un model on tots els graus de llibertat geomètrics apareixen dinàmicament. Sense assumir cap mena de mètrica obtenen interessants resultats però la complicació tècnica dels seus models els fan inviables a l'hora de fer prediccions concretes. Un model de Tumanov i Vladimirov [13] amb aquestes característiques va ser publicat poc després que el nostre model. D'alguna manera és la proposta més similar a la nostra que hem trobat a la literatura tot i que ells inclouen explícitament a la teoria fonamental el concepte de *vierbein*.

Amb tot el coneixement acumulat durant aquests anys, el punt de vista adoptat en aquesta tesi és el de treballar el mecanisme de trencament espontani de simetria per obtenir de manera consistent la teoria d'Einstein-Hilbert d'una teoria més fonamental que no inclou de partida cap noció de mètrica o geometria. El significat últim d'aquesta teoria fonamental ens és desconegut. Admetem que com la majoria de propostes, apuntem a donar una explicació al 'per què' sense pretendre entendre el 'd'on'.

Seguint aquesta línia de pensament, utilitzem el coneixement existent sobre teories efectives en el camp de la teoria quàntica de camps de partícules. En particular busquem trobar una analogia entre la teoria de Cromodinàmica Quàntica, amb la seva teoria efectiva de baixes energies la teoria Quiral, i la gravetat que ens permeti obtenir la teoria d'Einstein com a teoria vàlida a baixes energies. Repassant les propietats de la Cromodinàmica Quàntica un s'adona que les característiques fonamentals tenen un clar anàleg en termes de gravetat. Explotant aquestes analogies serem capaços de construir l'equivalent gravitatori de la Cromodinàmica Quàntica, i de trobar, a través del mecanisme de ruptura espontània de simetria, l'equivalent gravitatori del model quiral, que no serà més que la gravetat d'Einstein. Els principis que ens guiaran per construir la teoria seran la covariància, la localitat i la rellevància, en el sentit del grup de renormalització. Sense els dos primeres no es pot extreure informació rellevant dels càlculs i el tercer serà la guia al darrere del càlcul pertorbatiu. Com que realment volem veure com emergeixen tots els graus de llibertat de manera dinàmica, la teoria de partida no contindrà cap tipus de mètrica. Tanmateix, alguna informació de partida és necessària. Inclourem al model una connexió afí que definirà el transport paral·lel dels vectors en la varietat diferencial que considerarem de partida. D'aquesta manera definim una varietat pseudotopològica diferenciable com a punt de partida.

Una de les característiques claus del programa que desenvoluparem en aquesta tesi recau en la impossibilitat de construir un nombre il·limitat de contratermes en termes dels graus de llibertat fonamentals, degut al fet que no disposem abans del trencament d'una mètrica per fer-ho. Aquest fet, inevitablement, limita enormement el nombre de divergències que es poden generar en els càlculs. De fet aquest és el punt clau que indica que la teoria podria ser renormalitzable.

Amb tots aquests ingredients durem a terme un càlcul pertorbatiu a un *loop* obtenint com a acció efectiva, tant en dues com en quatre dimensions, precisament la teoria d'Einstein-Hilbert equipada de manera natural amb una constant cosmològica.

La qüestió de si s'ha d'incloure una constant cosmològica en les equacions d'Einstein és una incògnita que arriba al mateix Einstein, que la considerà 'el major error de la meva vida'. Més tard, amb la confirmació observacional que l'Univers s'expandeix exponencialment, la presència d'una energia del buit és una manera molt convenient d'acomodar les observacions.

Aquest és precisament el punt d'unió entre la primera part de la tesi i la segona. El fet de generar de manera natural una constant cosmològica ens fa pensar que la seva inclusió és un fet necessari i no opcional. Des del nostre punt de vista, el paper de la constant cosmològica és fonamental en el sentit que no és tan sols una descripció efectiva vàlida a molt grans escales, aquelles cosmològiques, sinó un part intrínseca de l'estructura de l'espai-temps. Com a tal, els seus efectes han de ser rellevants a qualsevol escala.

En aquesta segona part ens centrarem en l'estudi de l'efecte que té la constant cosmològica en la propagació d'ones gravitatòries, un ingredient de la teoria general de la relativitat que ha escapat de moment la confirmació empírica. I en la possibilitat de determinar el seu efecte en sistemes 'locals'.

L'estudi de la rellevància de la constant cosmològica en sistemes locals (per locals volem dir que involucrin escales sub-cosmològiques) ha rebut una creixent atenció en els últims anys. Sereno i Jetzer [14] el 2006 van determinar, sense massa precisió, el valor de la constant cosmològica a partir de l'estudi de la precessió de giroscopis i del corriment al roig gravitatori dins el sistema solar. Entre el 2007 i el 2009 diferents grups van investigar la influència de la constant cosmològica en la curvatura de la llum procedent d'objectes llunyans. Resultats molt dispars foren obtinguts, des de zero fins a clarament apreciable. Primer Khriplovich i Pomeransky [15] van determinar que no hi havia cap efecte. Sereno [16] més tard va arribar a la conclusió que l'efecte existia però era ínfim. Finalment Rindler i Ishak [17] van concloure que tot i que l'efecte era petit podia ser apreciable, en principi, en les observacions. Bernabeu i col·laboradors [18], el 2010, publicaren un estudi de les equacions d'Einstein linealitzades en la presència de la constant cosmològica obtenint resultats molt interessants, que en part han motivat un capítol d'aquesta tesi.

La inclusió de Λ en les equacions d'Einstein té una conseqüència immediata i òbvia. Fins i tot en absència de cap font produeix una curvatura de l'espai-temps (de Sitter). Per tant és d'esperar que la propagació d'ones gravitatòries en aquest cas difereixi del cas d'un espai-temps pla (Minkowski). La lògica al darrere del tractament usual de les ones gravitatòries és el de tractar-les com a petites pertorbacions sobre l'espai-temps de fons pla. Per trobar les seves funcions d'ona s'utilitza la versió linealitzada de les equacions d'Einstein. Per poder resoldre aquestes, hom ha de fer una tria de coordenades (o tria de *gauge*). Quan la constant cosmològica es té en compte, nous termes apareixen a les equacions linealitzades. Quants i quins d'aquests termes s'han de retenir a l'hora de resoldre les equacions és una de les qüestions que responem en aquesta tesi. També farem un estudi extensiu de la importància de les diferents tries de coordenades i la seva relació. La tria de coordenades, o el que és el mateix, la tria d'un *gauge* és obligatòria per poder trobar les solucions però la interpretació física d'aquestes coordenades no és sempre clara. De fet, l'únic sistema de coordenades que sabem interpretar és el cosmològic, aquell en què l'Univers és isòtrop i homogeni. Tanmateix, en aquest particular sistema de coordenades és impossible linealitzar les equacions d'Einstein sobre la mètrica plana. Per resoldre les equacions usarem la tria de *gauge* habitual, el *gauge* de Lorentz, o alternativament un altre *gauge* que anomenem el *gauge* Λ. Es pot demostrar que les coordenades corresponents a questes tries de *gauge* no són més que reparametritzacions d'un espai-temps d'Scharzschild-deSitter (SdS). Una vegada obtingudes les funcions d'ona haurem de transformar-les a coordenades FRW per poder-ne extreure prediccions observacionals. El canvi de coordenades de SdS a FRW és complicat, però es pot derivar analíticament. Les ones transformades adquireixen modificacions tant en la seva amplitud com en la seva relació de dispersió*. La seva amplitud creix amb la distància i pateixen un corriment al roig (diferent al corriment al roig gravitatori usual de l'electromagnetisme) a mesura que s'allunyen de la font.

Finalment ens centrem en la mesura dels períodes de cadenes de púlsars (PTA en anglès), un dels mètodes de detecció d'ones gravitatòries més prometedor per obtenir la primera mesura directa de les ones. Hi ha altres mètodes de detecció capaços de donar la primera detecció. Alguns d'ells són: detectors situats a terra com LIGO, que poden arribar a sensibilitats de 10^{-23} amb una franja òptima de freqüències de $10\text{Hz} < \nu < 10^3\text{Hz}$ [19]. La missió espacial LISA n'és un altre. S'espera que aconsegueixi sensibilitats semblants a les anteriors però en el rang de freqüències $10^{-2}\text{Hz} < \nu < 10^{-3}\text{Hz}$ [20] (si mai arriba a volar). Tanmateix, l'enfoc d'aquesta darrera part de la tesi és el de proporcionar un marc teòric que pugui ser d'utilitat per experiments com el International Pulsar Timing Array [21] o el Square Kilometer Array project [22]. Aquests són sensibles en un rang de freqüències més baix, $\nu < 10^{-4}\text{Hz}$, i tot i que de moment les seves sensibilitats estan a l'ordre de 10^{-10} (10^{-15} per ν 10^{-10}Hz) s'espera que amb l'acumulació d'estadística puguin millorar substancialment aquests valors en els propers anys.

Els PTA són detectors adequats per ones gravitatòries de freqüències molt baixes. Aquestes poden provenir de fenòmens llunyans tan diferents com la fusió de forats negres super massius o el fons de radiació gravitatòria primordial. Per obtenir el senyal és monitoritza el període d'un nombre adequat de púlsars durant un cert temps i se n'estudien les petites variacions. Aquests senyals correlats són aïllats i comparats als models teòrics per veure si són producte del pas d'ones gravitatòries per tot el sistema. En aquesta tesi només considerarem els efectes deguts a les ones gravitatòries,

*Amb això no volem dir que el gravitó sigui massiu, però reflecteix les propietats de la seva propagació vistes per un observador Lorentzià.

sense tenir en compte altres distorsions com les degudes al moviment peculiar de la terra. L'enfoc teòric habitual a la literatura és el de considerar ones propagant-se en un espai-temps pla i afegir a mà l'efecte de l'expansió de l'Univers a traves d'una freqüència efectiva correguda al roig. La nostra proposta és la d'usar les funcions d'ona obtingudes en el Capítol 5, que incorporen de manera més realista l'efecte de Λ, per determinar els efectes de la constant cosmològica en la detecció d'ones en els PTA. La conclusió és que hi ha notables diferències quan es té en compte correctament la constant cosmològica. En particular, la distribució angular dels púlsars respecte de la font serà clau en la rellevància de les observacions. S'observa un important repunt de la significació estadística per un determinat valor de l'angle entre els púlsars i la font. Aquest augment en la significació podira representar una manera alternativa de mesurar no tan sols el valor de la constant cosmològica, sinó tembé les ones gravitatòries en si mateixes, que recordem no han sigut detectades encara.

8.2 Conclusions i perspectives futures

En aquesta secció es recullen els resultats i conclusions d'aquesta tesi. Als Capítols 2, 3 i 4 proposem un model on la gravetat emergeix dinàmicament d'una teoria sense cap mena de mètrica predefinida. Els ingredients de partida són una varietat diferenciable equipada amb una connexió afí i $2N$ fermions. Res més. El Lagrangià pot ésser definit sense necessitat de cap mètrica.

La gravetat i la distància són induïts i no fonamentals en el model. A escales suficientment petites, quan la teoria efectiva perd el sentit, els graus de llibertat són aquells dels fermions. La distància on la simetria es restaura és fonamentalment la distància més petita on es pot parlar de geometria.

La relativa simplicitat del nostre model constitueix la seva gran virtut en comparació amb propostes com [11, 12, 13], on qualsevol mena de resultat quantitatiu es feia impossible. En canvi nosaltres hem pogut derivar en tot detall l'acció efectiva. En el cas de $D = 4$ podem predir sense ambigüitat l'existència d'una constant cosmològica, el valor de la qual no ve prefixat pel model però és ajustable al qualsevol valor observacional. I l'existència d'un terme de curvatura d'Einstein-Hilbert obtingut lliure de divergències. La realització del nostre programa teòric demostra que és possible obtenir una teoria efectiva consistent i amb poder predictiu sense haver d'introduir d'entrada cap mena de noció de mètrica.

Una de les obstruccions habituals per als programes de gravetat emergent és el teorema de Weinberg-Witten. Tot i no constituir una prova formal, creiem que el fet que els índexs Lorentz siguin de natura interna en la nostra teoria fa que quedem fora de les assumpcions del teorema. Aquest prohibeix l'aparició de bosons sense massa d'espí dos en teories emergents si el tensor energia moment és covariantment conservat. En el nostre cas no ho és, perquè ni tan sols és un tensor Lorentz de rang dos.

Un aspecte important del model és l'aparent millora del comportament de la teoria a l'ultraviolat. Un cop integrats els graus de llibertat fonamentals, totes les divergències que apareixen fins a l'ordre que hem calculat poden ser absorbides tan sols redefinint la constant cosmològica i la constant de Planck (la curvatura). Tal com dicta el grup de renormailtzació, els resultats són invariants sota aquest, sempre i quan es respectin les corresponents funcions beta que han sigut calculades. En el cas de quatre dimensions, a més de les divergències corresponents al terme cosmològic i a la constant de Newton, hem trobat divergències que poden ser associades a peces determinades del terme de Gauss-Bonnet, que també és un contraterme vàlid. Aquesta notable millora del comportament a altres energies es deu al fet que el nombre de contratermes que és possible escriure sense fer ús de cap mètrica és molt limitat.

Tot i que és temptador, no declarem obertament la teoria com a normalitzable. Tan sols podem dir que la nostra experiència amb el model i els càlculs realitzats semblen indicar que efectivament ho és. Tampoc podem dir que aquest és l'únic Lagrangià de partida que permet portar a terme aquest programa, tot i que veiem difícil construir-ne cap altre.

Se'ns ocorren un bon nombre d'extensions i possibles aplicacions. Tal vegada la més interessant seria la investigació de solucions singulars, tipus forat negre, en el marc de la teoria. També caldria un estudi més detallat de la renormalizabilitat de model. El tema de la distància més curta permesa per la restauració de la simetria és sens dubte una línia de recerca molt interessant. La Relativitat General ha sigut provada empíricament fins a escales de l'ordre del mil·límetre [71], d'allà fins a l'escala de Planck hi ha una immensitat d'espai on modificacions de la gravetat podrien ser, sinó testades, intuïdes. També pensem en la inclusió de camps de matèria ordinaris en el model i com fer contacte amb les altres interaccions fonamentals. La relació entre el nostre model i propostes com les triangulacions Lorentzianes serien, sens dubte, de gran interès. Fins i tot idees més exòtiques com l'extensió del model a dimensions més altes o l'efecte túnel entre geometries podrien ser investigades.

Respecte a la segona part de la tesi, en el Capítol 5 investiguem l'efecte de la constant cosmològica en la propagació d'ones gravitatòries en el marc de la teoria linealitzada d'Einstein. La presència de Λ porta inevitablement a la curvatura de l'espai-temps en què les ones es propaguen. En la teoria linealitzada d'Einstein això porta a una descomposició de la mètrica en termes de $g_{\mu\nu} = \eta_{\mu\nu} + h^{\Lambda}_{\mu\nu} + h^{W}_{\mu\nu}$, que inclou una modificació de la mètrica de fons i una pertorbació deguda a les ones gravitatòries.

Per veure de quina manera es veuen afectades les ones, hom primer ha d'entendre les implicacions que tenen els diferents sistemes de coordenades (tries de *gauge*) en la resolució de les equacions linealitzades. La tria de *gauge* és lliure en principi, tanmateix, algunes tries simplifiquen enormement la resolució de les equacions ja que aquestes en depenen explícitament. Argumentem que el procediment de linealització només és consistent en alguns sistemes de coordenades però no en altres. En particular, $\bar{g}_{\mu\nu} = \eta_{\mu\nu} + h^{\Lambda}_{\mu\nu}$ sent una versió linealitzada de la mètrica de FRW mai serà solució de cap linealització de les equacions d'Einstein. Notis que això no és el mateix que fer petites pertorbacions (quedant-se a ordre lineal) sobre una mètrica FRW exacte, cosa que és possible però un programa totalment diferent al que portem a terme en aquesta tesi.

Les equacions d'Einstein les linealitzem en el *gauge* de Lorentz, on l'anàlisi de les ones gravitatòries és molt similar al de l'espai Minkowskià. Després d'un detallat estudi tant d'aquestes coordenades com de les corresponents al *gauge* de Λ, concloem que ambdues són diferents parametritzacions de les coordenades d'SdS. Tan sols cal una transformació de coordenades que no depèn del temps per relacionar-les. En virtut del teorema de Birkhoff's està garantit que aquestes coordenades són les úniques que posseeixen una simetria esfèrica i no depenen del temps. En el cas d'ometre termes del tipus $\Lambda h^{W}_{\mu\nu}$, la llibertat residual dintre del *gauge* de Lorentz en permet eliminar tots els graus de llibertat no físics fins a deixar-ne tan sols dos, els usuals d'una ona gravitatòria. En el cas de retenir el terme $\Lambda h^{W}_{\mu\nu}$, aquesta invariància residual es perd i l'ona s'ha d'interpretar com a 'massiva' en el sentit d'un observador Lorentzià. En realitat aquest fet és un artefacte de l'aproximació lineal i els gravitons acabaran tenint tan sols els dos graus de llibertat físics. Les ones modificades adquireixen modificacions tant els tensors de polarització com el la seva relació de dispersió. Aquestes modificacions són d'ordre $\mathcal{O}(\Lambda)$.

També demostrem com les solucions per la mètrica de fons en el cas de retenir el terme $\Lambda h^{W}_{\mu\nu}$ corresponen precisament, després d'una expansió en la constant cosmològica, a les solucions trobades quan el terme era omès.

Tot i poder obtenir fàcilment les equacions d'ona en SdS el problema rau en la interpretació de les coordenades. Les úniques que entenem, com ja hem dit, són les de FRW. Per tant treballem el canvi de coordenades entre SdS i FRW i transformem les solucions trobades. En la transformació noves modificacions s'afegeixen a les anteriors, altra vegada tant en els tensors de polarització com en la relació de dispersió, en aquest cas les modificacions són d'ordre $\mathcal{O}(\sqrt{\Lambda})$. Aquestes modificacions poden ser numèricament prou importants per ones viatjant grans distàncies a l'Univers.

L'efecte de Λ és definitivament rellevant per observacions en PTA. Les solucions finals, expressades en FRW, es veuen modificades tant en l'amplitud, que creix amb la distància a la font, com en la freqüència, que corre al roig de manera prescrita pels resultats, a mida que s'allunya de la font.

Finalment, per tancar la tesi, en el Capítol 6 investiguem els efectes locals de la constant cosmològica en la detecció d'ones gravitatòries en els PTA. Utilitzem les solucions que acabem de mencionar per modelitzar les desviacions en la recepció del senyal dels púlsars. Fins ara els especialistes en el camp ha utilitzat ones planes per modelitzar aquest efecte, incloent a mà l'efecte de la constant cosmològica a traves d'un corriment al roig de les freqüències. Veiem que en utilitzar les funcions d'ona més realistes els resultats canvien dramàticament. En particular trobem un pic molt notable en la significació estadística del senyal representada en funció de l'angle entre la línia que defineixen la terra i el púlsar i la línia entre la terra i la font. Finalment trobem la relació entre la posició angular del pic i el valor de la constant cosmològica. Aquest mètode podria representar una manera alternativa de determinar el valor de Λ.

Tot i ser prometedors, aquests resultats són força preliminars. Es requereix un estudi més profund per determinar les possibilitats reals de detecció d'ones gravitatòries usant aquest model teòric. L'ús de fronts d'ona encara més realistes, si n'hi ha de disponibles, podria ajudar a confirmar els resultats aquí obtinguts. La identificació de fonts d'ones reals que compleixin els requeriments del nostre model també aportarien resultats més acurats.

En resum aquesta tesi tracta del fet que la gravetat, entesa com la gravetat d'Einstein, podria ben bé ser una descripció efectiva, vàlida només a baixes energies, però tanmateix capaç de descriure fenòmens rellevants a escales tan diverses com la gran estructura de l'univers, com el sistema solar, i tal vegada també en el límit en què les interaccions quàntiques prenen la batuta. D'altra banda, les ones gravitatòries són una predicció inequívoca de la teoria d'Einstein. Si som capaços d'obtenir, de la manera que ho fem, una teoria emergent que correspon a la d'Einstein i que a més ve equipada de manera natural amb la constant cosmològica, és natural que ens preguntem quina és l'efecte de la seva presència en les equacions i, últimament, en la propagació de les ones gravitatòries en l'Univers que ens envolta.

Volem emfatitzar la doble rellevància dels resultats relacionats amb les observacions als PTA. No només representen una oportunitat de mesurar de manera alternativa el valor de la constant cosmològica. També poden representar una ajuda definitiva per detectar les ones gravitatòries en si. Aquest és un resultat que mereix menció per si sol.

Tot i que aquesta tesi no dóna respostes definitives al problema de la gravetat quàntica, ni a què passa amb la radiació gravitatòria en presència de camps gravitatoris forts, pensem que ens apropa una mica més a un coneixement més profund de la més esquiva de les interaccions fonamentals. I tan de bo ens proveeixi amb una modelització més realista d'una de les seves prediccions, les ones, que ens porti a la seva primera detecció.

8.3 Llista de publicacions

La feina original continguda en aquesta tesi està basada en les següents publicacions de l'autor:

- Local measurement of Λ using pulsar timing arrays, (amb D. Espriu), e-Print: arXiv:1209.3724 [gr-qc] (2012).
- Spontaneous generation of geometry in four dimensions, (amb J. Alfaro i D. Espriu), Phys. Rev. D 86, 025015 (2012).
- Gravitational waves in the presence of a cosmological constant, (amb J. Bernabeu i D. Espriu), Phys. Rev. D 84, 063523 (2011).
- The emergence of geometry: a two-dimensional toy model, (amb J. Alfaro i D. Espriu), Phys. Rev. D 82, 045018 (2010).
- Gravity as an Effective theory, (amb D. Espriu), Acta Physica Polonica B, 40, 12 (2009).

References

[1] H. Land, The Order of Nature in Aristotle's Physics: Place and the Elements (1998). 1, 105

[2] R.V. Waghmare, P.S. Avhale and S.B. Kolhe, The great mathematician Brahmagupta, Golden Research Thoughts, Vol - II , Issue - I (2012). 1, 105

[3] Galileo Galilei, Dialogues Concerning Two New Sciences by Galileo Galilei. Translated from the Italian and Latin into English by Henry Crew and Alfonso de Salvio. With an Introduction by Antonio Favaro (New York: Macmillan, 1914). 1, 105

[4] Newton's Principia : the mathematical principles of natural philosophy / by Sir Isaac Newton, http://archive.org/details/100878576. 1, 105

[5] Dyson, F. W.; Eddington, A. S., Davidson C. (1920). "A determination of the deflection of light by the Sun's gravitational field, from observations made at the total eclipse of 29 May 1919". Philos. Trans. Royal Soc. London 220A: 291–333. 2, 106

[6] See, for instance: W. Rindler, Relativity: Special, General, and Cosmological, Oxford University Press (Second Edition, (2006) ISBN-13: 978-0-19-856731-8); J. HJ. Taylor, Rev. Mod. Phys. 66:711 (1994); J. M. Weisberg and J. H. Taylor, ASP Con. Ser. 328:25 (2005). 2, 58, 106

[7] Ya.B. Zeldovich, JETP Lett., 6, 316 (1967); A. Sakharov, SOv. Phys. Dokl., 12, 1040 (1968); O. Klein, Phys. Scr. 9, 69 (1974); S. Adler, Rev. Mod. Phys., 54, 729 (1982). 2, 9, 10, 55, 98, 107

[8] C.J. Isham, A. Salam and J.A. Strathdee, Annals Phys. 62:98-119 (1971). 3, 10, 107

[9] A.B. Borisov and V.I. Ogievetsky, Theor. Math. Phys. 21:1179 (1975); E.A. Ivanov and V.I. Ogievetsky, Lett. Math. Phys. 1:309-313 (1976). 3, 10, 107

[10] P. Kraus and E.T. Tomboulis, Phys. Rev. D 66, 045015 (2002); V. A. Kostelecky and R. Potting, Phys. Rev. D 79:065018,2009; G. E. Volovik, Physica B 162 (1990) 222-230. 3, 10, 52

[11] D. Amati and J. Russo, Phys. Lett. B 248, 44 (1990); J. Russo, Phys. Lett. B 254, 61 (1991). 4, 10, 15, 97, 108, 112

[12] A. Hebecker, C. Wetterich, Phys. Lett. B 574:269-275 (2003); C. Wetterich, Phys. Rev. D 70: 105004 (2004). 4, 10, 15, 97, 108, 112

[13] D.D., A. Tumanov, A. Vladimirov, Phys. Rev. D 84, 124042 (2011). 4, 10, 15, 97, 108, 112

[14] M. Sereno and P. Jetzer, Phys. Rev. D 73, 063004 (2006) [arXiv:astro-ph/0602438]; A. Balaguera-Antolinez, C. G. Boehmer and M. Nowakowski, Class. Quant. Grav. 23, 485 (2006) [arXiv:gr-qc/0511057]; L. Iorio, Adv. in Astronomy, Article ID 268647 (2008), doi:10.1155/2008/268647; New Astron. 14, 196 (2008) [arXiv:0808.0256 [gr-qc]]; Y. Suto, Progr. Theoretical Physics 90, 1173 (1993); also in arXiv:astro-ph/9609014; P.J. Peebles, Astrophysical Sp. Sci., 45, 3 (1976); The Large Scale Structure of The Universe (Princeton Univ. Pr. 1980); G. S. Adkins, J. McDonnell and R. N. Fell, Phys. Rev. D 75, 064011 (2007) [arXiv:gr-qc/0612146]; See, for instance: W. Rindler, Relativity: Special, General, and Cosmological (Oxford University Press (Second Edition, (2006) ISBN-13: 978-0-19-856731-8; W. Rindler, Phys. Lett. A 245, 363 (1998); the spherically-symmetric Schwarzschild-de Sitter solution was first discussed in F. Kottler, Ann. Phys. 361, 401 (1918). 5, 57, 82, 110

[15] I.B. Khriplovich and A.A. Pomeransky, In. J. Mod. Phys. D 17, 2255 (2008). 6, 57, 110

[16] M. Sereno, Phys. Rev. Lett. 102, 021301 (2009); Phys. Rev. D 77, 043004 (2008). 6, 57, 110

[17] W. Rindler and M. Ishak, Phys. Rev D 76, 043006 (2007); M. Ishak, W. Rindler, J. Dossett, J. Moldenhauer and C. Allison, Mon. Not. R. Astron. Soc. 388, 1279 (2008); M. Ishak, W. Rindler, Gen. Rel. Grav. 42:2247 (2010). 6, 57, 110

[18] J. Bernabeu, C. Espinoza and N. E. Mavromatos, Phys. Rev. D 81:084002 (2010). 6, 57, 58, 65, 67, 68, 82, 110

[19] J. Abadie et al, LIGO Collaboration, Nucl. Instrum. Meth. A624(2010)2234. 7, 58, 111

[20] ESA-SCI(3000)11, version 1.04 (2000). 7, 58, 111

[21] G. Hobs et al, Cl. Quant. Grav. 27:8 (2010). 7, 58, 111

[22] Science with the SKA, eds. C. Cavilli, S. Rawlingd, New Astronomy Reviews, Vol. 48, Elsevier (2004). 7, 58, 111

[23] S. Weinberg, in General Relativity: An Einstein centenary survey, edited by S. W.Hawking and W.Israel (Cambridge University Press, 1979), chapter 16, p. 790. 9, 11

[24] D.F. Litim, Phys. Rev. Lett. 92:201301 (2004); AIP Conf. Proc. 841, 322 (2006); e-Print: arXiv:0810.367; A. Codello, R. Percacci and C. Rahmede, Annals Phys. 324:414-469 (2009); M. Reuter and F. Saueressig, Lectures given at First Quantum Geometry and Quantum Gravity School, Zakopane, Poland (2007), arXiv:0708.1317. 9

[25] J. Ambjorn, J. Jurkiewicz and R. Loll, Phys. Rev. Lett. 85:924-927 (2000). 9, 10

[26] G. Dvali and C. Gomez, Self-Completeness of Einstein Gravity, arXiv:1005.3497 [hep-th]; G. Dvali and C. Gomez, Self-Completeness of Einstein Gravity, Phys. Rev. D 84:024039 (2011). 10

[27] E. Verlinde, JHEP 1104 (2011) 029. 10

[28] S.C. Zhang and J. Hu, Science 294, 823 (2001); C. Barcelo and M. Visser, Int. J. Mod. Phys. D10 (2001) 799-806; H. Grosse, H. Steinacker and M. Wohlgenannt, JHEP 0804 (2008) 023 B. McElrath, arXiv:0812.2696; F. Girelli, S. Liberati and L. Sindoni, Phys. Rev. D 79:044019 (2009); T. Padmanabhan, Int. J. Mod. Phys. D 17

(2008) 591-596; P. Horava, JHEP 0903, 020 (2009);M.M. Anber, U. Aydemir and J.F. Donoghue, Phys. Rev. D 81 (2010) 084059; C. Xu and P. Horava, Phys. Rev. D 81, 104033 (2010); E.W. Mielke, Phys. Rev. D 83, 044004 (2011); L. Sindoni, arXiv:1110.0686. 10

[29] H.C. Ohanian Phys. Rev. D 184, 1305 (1969). 10

[30] See e.g.: I. Heemskerk and J. Polchinski, JHEP 1106:031 (2011); G. Dvali and C. Gomez, arXiv:1005.3497. 11

[31] S. Weinberg, Phys. Rev. Lett. 18:188-191 (1967); Phys. Rev. 166:1568-1577 (1968); Physica A96:327 (1979). 11

[32] J. Gasser and H. Leutwyler, J. Gasser and H. Leutwyler, Annals Phys.158:142 (1984); Nucl.Phys.B250:465 (1985). 11

[33] R. Percacci, Lectures given at 5th International School on Field Theory and Gravitation, Cuiaba, MT, Brazil (2009). e-Print: arXiv:0910.5167. 13

[34] J.F. Donoghue, in Proceedings of the Advanced School on Effective Theories, Almunecar, Spain (1995), e-Print: gr-qc/9512024; Helv. Phys. Acta 69:269-275 (1996); J.F. Donoghue and T. Torma, Phys. Rev. D 54:4963-4972 (1996). 13, 15

[35] I.B. Khriplovich and G.G. Kirilin, in Proceedings of 5th International Conference on Symmetry in Nonlinear Mathematical Physics (SYMMETRY 03), Kiev, Ukraine (2003). Published in eConf C0306234:1361-1366,2003, J. Exp. Theor. Phys. 98:1063-1072 (2004). e-Print: gr-qc/0402018; J. Exp. Theor. Phys. 95:981-986 (2002); Zh. Eksp. Teor. Fiz. 95:1139-1145 (2002). 14

[36] N.E.J Bjerrum-Bohr, J. F. Donoghue and B. R. Holstein, Phys. Rev. D 67: 084033 (2003), Erratum-ibid. D 71: 069903 (2005). 14

[37] H. W. Hamber and S. Liu, Phys. Lett. B 357:51-56 (1995); I. J. Muzinich and S. Vokos, Phys. Rev. D 52:3472-3483 (1995); Arif A. Akhundov, S. Bellucci and A. Shiekh, Phys. Lett. B 395:16-23 (1997). 14

[38] N.E.J. Bjerrum-Bohr, Quantum gravity, effective fields and string theory (Ph.D. Thesis), e-Print: hep-th/0410097. 14

[39] G. Narain and R. Percacci, e-Print: arXiv:0910.5390. 15

[40] A. Manohar and H. Georgi, Nucl. Phys. B 234, 189 (1984). 15, 27

[41] D. Espriu, E. de Rafael and J. Taron, Nucl. Phys. B 345, 22 (1990); Erratum-ibid.B355, 278 (1991). 15, 27

[42] S. Weinberg and E. Witten, Phys. Lett. B 96:59 (1980). 18

[43] A. Jenkins, arXiv:0904.04532v2. 18

[44] A. Palatini, Rend. Circ. Mat. Palermo 43, (1919) 203. 26, 41

[45] M. Han, Y. Ma, Y. Ding and L. Qin, Mod.Phys.Lett.A20:725-732 (2005). 26

[46] A.M. Polyakov, Phys. Lett. 103B, 207 (1981). 34

[47] See e.g.: M. Asorey, E. V. Gorbar and I.L. Shapiro, Class. Quantum Grav. 21, 163 (2004). 53

[48] J. Alfaro, D. Espriu, D. Puigdomènech, Phys. Rev. D 82, 045018 (2010). 53

[49] S.M. Christensen and M.J. Duff, Nucl. Phys B170, 480-506 (1980); Phys. Lett 76B, 571 (1978). 53

[50] K.S. Stelle, Gen. Rel. Grav. 9, 353 (1978). 55

[51] A. G. Riess et al. [Supernova Search Team Collaboration], Astron. J. 116, 1009 (1998) [arXiv:astro-ph/9805201]. S. Perl- mutter et al. [Supernova Cosmology Project Collaboration], Astrophys. J. 517, 565 (1999) [arXiv:astro-ph/9812133]. A. G. Riess, Publ. Astron. Soc. Pac. 112, 1284 (2000) [arXiv:astro-ph/0005229]. E. Linder and S. Perlmutter, Phys. World 20N12 (2007) 24; M. Kowalski et al. [Supernova Cosmology Project Collaboration], Astrophys. J. 686, 749 (2008) [arXiv:0804.4142 [astro-ph]]; W. M. Wood-Vasey et al. [ESSENCE Collaboration], Astrophys. J. 666, 694 (2007) [arXiv:astro-ph/0701041]. A. G. Riess and M. Livio, Astrophys. J. 648, 884 (2006) [arXiv:astro-ph/0601319]. A. G. Riess et al. [Supernova Search Team Collaboration], Astrophys. J. 607, 665 (2004) [arXiv:astro-ph/0402512]. D. N. Spergel et al. [WMAP Collaboration], Astrophys. J. Suppl.

170, 377 (2007) [arXiv:astro-ph/0603449]; D. J. Eisenstein et al. [SDSS Collaboration], Astrophys. J. 633, 560 (2005) [arXiv:astro-ph/0501171]; H. J. Seo and D. J. Eisenstein, Astrophys. J. 598, 720 (2003) [arXiv:astro-ph/0307460]; and references therein; L. Fu et al., Astronomy & Astrophysics 479, 9 (2008); L. Guzzo et al., Nature 451, 541 (2008). 57, 85

[52] J. Bernabeu, C. Espinoza, N. E. Mavromatos and V. Mitsou, in preparation. 58

[53] See, for instance, H. Colins, Gravity's Shadow: the search for gravitational waves, University of Chicago Press (2004). 60

[54] J. Bicak and J. Podolsky, J. Math. Phys. 40, 4506 (1999), arXiv: gr-qc/9907049. 60

[55] J. Näf, P. Jetzer and M. Sereno, Phys. Rev. D 79:024014 (2009), arXiv: 0810.5426 [astro-ph]. 60, 74

[56] D. Baumann, P. J. Steinhardt, K. Takahashi and K. Ichiki, Phys.Rev. D76:084019 (2007). 60

[57] Lu Hui-qing, Chin. Astron. Astrophys. 12, 186-190, (1988). 62

[58] J. Bardeen, Phys. Rev. D 22, 1882–1905 (1980). 63, 100

[59] A. H. Guth, Phys. Rev. D 23: 347 (1981); A. D. Linde, Phys. Lett. B108: 389 (1982); A. Albrecht and P. J. Steinhardt, Phys. Rev. Lett. 48: 1220 (1982). 65

[60] W. Rindler, Phys. Lett. A 245, 363 (1998). 67

[61] See e.g. R. H. Price and Y. Yang, Am. J. Phys., Vol. 76, No. 10 (2008). 71, 82

[62] G. Hobbs, Pulsars as gravitational detectors, arXiv:1006.3969. 81

[63] G. Hobbs, Accepted for publication in the JPCS issue for the Amaldi 7 proceedings Journal-ref: Class. Quant. Grav. 25:114032 (2008); J.Phys. Conf. Ser.122:012003 (2008). 81, 88

[64] G. Hobbs et al, Class. Quant. Grav. 27:084013 (2010). 81, 88

[65] K. J. Lee et al, Gravitational wave astronomy of single sources with a pulsar timing array, DOI: 10.1111/j.1365-2966.2011.18622.x. 81

[66] F. A. Jenet et al, ApJ 653:1571-1576 (2006). 81, 85, 88

[67] J. Bernabeu, D. Espriu and D. Puigdomènech, Phys. Rev. D 84, 063523 (2011). 81, 82, 83

[68] X. Deng, L. S. Finn, DOI: 10.1111/j.1365-2966.2010.17913.x. 84

[69] ATNF pulsar catalogue, www.atnf.csiro.au/research/pulsar/psrcat; Manchester, R. N., Hobbs, G. B., Teoh, A. & Hobbs, M., AJ, 129, 1993-2006 (2005). 86

[70] D. A. Brown et al, Class. Quantum Grav. 24 S595-S605 (2007). 87

[71] J. Beringer et al. (Particle Data Group), Phys. Rev. D86, 010001 (2012) 99, 113

Appendix A

Explicit calculations of Chapter 4

In this appendix we include the explicit calculation of the different terms appearing in (4.35) showing how they correspond on shell to different terms in the action. We also include, for completeness, how the result of (4.31) in the diagonal parametrization of the metric used in the text yields precisely (4.23).

A.1 $D^{\mu\nu}{}_{bcef}$, $E^{\mu\nu}{}_{bcef}$ and $F^{\mu\nu}{}_{bcef}$

We saw in Section 4.2 that diagram (4.35) contains three different terms, two of them being divergent, let us show how they either cancel or can be accommodated in the available counterterms. Let us write them down together with the w_μ^{ab} fields.

$$\begin{aligned} D^{\mu\nu}{}_{bcef} w_\mu^{bc} w_\nu^{ef} =& 4 w_\mu^{\nu b} w_\nu^{\mu b} - 2 w_\mu^{be} w_\mu^{be} = 2(w_1^{12})^2 - 2(w_1^{21})^2 + 2(w_1^{13})^2 - 2(w_1^{31})^2 \\ &+ 2(w_1^{14})^2 - 2(w_1^{41})^2 + ... + 2(w_4^{34})^2 - 2(w_4^{43})^2 = 0. \end{aligned} \tag{A.1.1}$$

So this divergence cancels regardless of the parametrization we choose.

Let us write the second one now considering a conformally flat parametrization of the metric

$$g_{\mu\nu} = e^{-\sigma(x)} \delta_{\mu\nu}. \tag{A.1.2}$$

Then we have

$$E^{\mu\nu}{}_{bcef} w_\mu^{bc} w_\nu^{ef} = 4 w_\mu^{\mu b} w_\nu^{b\nu}. \tag{A.1.3}$$

Making use of Equation (4.17), that is

$$w_\mu{}^{ab} = \frac{1}{2}(\partial^a \sigma \delta_\mu^b - \partial^b \sigma \delta_\mu^a), \tag{A.1.4}$$

we obtain

$$E^{\mu\nu}{}_{bcef}w^{bc}_{\mu}w^{ef}_{\nu} = -9[(\partial_1\sigma)^2 + (\partial_2\sigma)^2 + (\partial_3\sigma)^2 + (\partial_4\sigma)^2] = -9\partial_\mu\sigma\partial_\mu\sigma. \tag{A.1.5}$$

Recall this term appeared both in (4.35) and in (4.40). Summing their contributions in the way we did to reconstruct the effective action for the σ field diagrams we obtain

$$\frac{9M^2}{32\pi^2}\partial_\mu\sigma\partial_\mu\sigma(1-\sigma+...) \to \frac{9M^2}{32\pi^2}\partial_\mu\sigma\partial_\mu\sigma e^{-\sigma} \to \frac{9M^2}{32\pi^2}e^{-\sigma}\Box\sigma. \tag{A.1.6}$$

This is just a finite contribution to $\sqrt{g}R$ and it is reflected in (4.51)

The Gauss-Bonnet term corresponding to such a metric perturbation reads

$$\begin{aligned}
\mathcal{L}_{GB} = &\sqrt{g}(R^2 + 4R_{\mu\nu}R^{\mu\nu} - R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}) \\
= &-4\partial_3\partial_4\sigma\partial_3\partial_4\sigma + 4\partial_4^2\sigma\partial_3^2\sigma - 4\partial_2\partial_4\sigma\partial_2\partial_4\sigma + 4\partial_4^2\sigma\partial_2^2\sigma - 4\partial_2\partial_3\sigma\partial_2\partial_3\sigma + 4\partial_3^2\sigma\partial_2^2\sigma \\
&- 4\partial_1\partial_4\sigma\partial_1\partial_4\sigma + 4\partial_4^2\sigma\partial_1^2\sigma - 4\partial_1\partial_2\sigma\partial_1\partial_2\sigma + 4\partial_2^2\sigma\partial_1^2\sigma - 4\partial_1\partial_3\sigma\partial_1\partial_3\sigma + 4\partial_3^2\sigma\partial_1^2\sigma \\
&- 3\partial_4\sigma\partial_4\sigma\partial_4^2\sigma - \partial_4^2\sigma\partial_3\sigma\partial_3\sigma - 4\partial_4\sigma\partial_3\sigma\partial_3\partial_4\sigma - \partial_4\sigma\partial_4\sigma\partial_3^2\sigma - 3\partial_3\sigma\partial_3\sigma\partial_3^2\sigma - \partial_4^2\sigma\partial_2\sigma\partial_2\sigma \\
&- \partial_3^2\sigma\partial_2\sigma\partial_2\sigma - 4\partial_4\sigma\partial_2\sigma\partial_2\partial_4\sigma - 4\partial_3\sigma\partial_2\sigma\partial_2\partial_3\sigma - \partial_4\sigma\partial_4\sigma\partial_2^2\sigma - \partial_3\sigma\partial_3\sigma\partial_2^2\sigma - 3\partial_2\sigma\partial_2\sigma\partial_2^2\sigma \\
&- \partial_4^2\sigma\partial_1\sigma\partial_1\sigma - \partial_3^2\sigma\partial_1\sigma\partial_1\sigma - \partial_2^2\sigma\partial_1\sigma\partial_1\sigma - 4\partial_4\sigma\partial_1\sigma\partial_1\partial_4\sigma - 4\partial_3\sigma\partial_1\sigma\partial_1\partial_3\sigma - 4\partial_2\sigma\partial_1\sigma\partial_1\partial_2\sigma \\
&- \partial_4\sigma\partial_4\sigma\partial_1^2\sigma - \partial_3\sigma\partial_3\sigma\partial_1^2\sigma - \partial_2\sigma\partial_2\sigma\partial_1^2\sigma - 3\partial_1\sigma\partial_1\sigma\partial_1^2\sigma.
\end{aligned} \tag{A.1.7}$$

The last term we have to explore is the piece $\frac{1}{\epsilon}F^{\mu\nu}{}_{bcef}$. Let us explicitly write this term together with the w^{bc}_μ fields

$$\begin{aligned}
\frac{1}{\epsilon}F^{\mu\nu}{}_{bcef}w^{bc}_\mu w^{ef}_\nu = &\frac{1}{\epsilon}\Bigg[-\frac{\delta^{\mu\nu}\delta_{be}p_cp_f}{12\pi^2} + \frac{\delta^{\mu\nu}\delta_{bf}p_ep_c}{12\pi^2} + \frac{\delta^{\mu\nu}\delta_{ce}p_bp_f}{12\pi^2} - \frac{\delta^{\mu\nu}\delta_{cf}p_bp_e}{12\pi^2} - \frac{\delta^\mu_b\delta^\nu_e\delta_{cf}p^2}{12\pi^2} \\
&+\frac{\delta^\mu_b\delta^\nu_e p_cp_f}{12\pi^2} + \frac{\delta^\mu_b\delta^\nu_f\delta_{ce}p^2}{12\pi^2} - \frac{\delta^\mu_b\delta^\nu_f p_ep_c}{12\pi^2} - \frac{\delta^\mu_e\delta_{bf}p^\nu p_c}{12\pi^2} + \frac{\delta^\mu_e\delta^\nu_b p_cp_f}{12\pi^2} \\
&-\frac{\delta^\mu_e\delta^\nu_c p_bp_f}{12\pi^2} + \frac{\delta^\mu_e\delta_{cf}p^\nu p_b}{12\pi^2} + \frac{\delta^\mu_c\delta^\nu_e\delta_{bf}p^2}{12\pi^2} - \frac{\delta^\mu_c\delta^\nu_e p_bp_f}{12\pi^2} - \frac{\delta^\mu_c\delta^\nu_f\delta_{be}p^2}{12\pi^2} \\
&+\frac{\delta^\mu_c\delta^\nu_f p_bp_e}{12\pi^2} + \frac{\delta^\mu_f\delta_{be}p^\nu p_c}{12\pi^2} - \frac{\delta^\mu_f\delta^\nu_b p_ep_c}{12\pi^2} + \frac{\delta^\mu_f\delta^\nu_c p_bp_e}{12\pi^2} - \frac{\delta^\mu_f\delta_{ce}p^\nu p_b}{12\pi^2} \\
&+\frac{\delta_{be}\delta^\nu_c p^\mu p_f}{12\pi^2} - \frac{\delta_{be}\delta_{cf}p^\mu p^\nu}{12\pi^2} - \frac{\delta_{bf}\delta^\nu_c p^\mu p_e}{12\pi^2} + \frac{\delta_{bf}\delta_{ce}p^\mu p^\nu}{12\pi^2} - \frac{\delta^\nu_b\delta_{ce}p^\mu p_f}{12\pi^2} \\
&+\frac{\delta^\nu_b\delta_{cf}p^\mu p_e}{12\pi^2}\Bigg] w^{bc}_\mu w^{ef}_\nu \\
= &\frac{1}{3\pi^2}\frac{1}{\epsilon}\Big[\partial_c w^{bc}_\mu\partial_e w^{be}_\mu + w^{\mu c}_\mu\Box w^{\nu c}_\nu - \partial_c w^{\mu c}_\mu\partial_e w^{\nu e}_\nu - \partial_c w^{bc}_\mu\partial_e w^{\mu e}_b \\
&+\partial_c w^{bc}_\mu\partial^\nu w^{\mu b}_\nu + \partial^\mu w^{b\nu}_\mu\partial_e w^{eb}_\nu + \frac{1}{2}\partial^\mu w^{bc}_\mu\partial^\nu w^{bc}_\nu\Big].
\end{aligned} \tag{A.1.8}$$

$$\frac{1}{\epsilon}F^{\mu\nu}{}_{bcef}w^{bc}_{\mu}w^{ef}_{\nu} = \frac{6}{\pi^2}\frac{1}{\epsilon}\left[-\partial_4^2\sigma\partial_1^2\sigma + \partial_1\partial_4\sigma\partial_1\partial_4\sigma - \partial_3^2\sigma\partial_4^2\sigma + \partial_3\partial_4\sigma\partial_3\partial_4\sigma \right.$$
$$-\partial_2^2\sigma\partial_4^2\sigma + \partial_2\partial_4\sigma\partial_2\partial_4\sigma - \partial_2^2\sigma\partial_3^2\sigma + \partial_2\partial_3\sigma\partial_2\partial_3\sigma \qquad \text{(A.1.9)}$$
$$\left.-\partial_3^2\sigma\partial_1^2\sigma + \partial_1\partial_3\sigma\partial_1\partial_3\sigma - \partial_2^2\sigma\partial_1^2\sigma + \partial_1\partial_2\sigma\partial_1\partial_2\sigma\right].$$

Now it is easy to see that (A.1.9) corresponds to the second and third lines in (A.1.7). This divergent contribution is part of the Gauss-Bonnet term, which, although being a total derivative, is a valid counterterm. The rest of (A.1.7) contains three sigma fields and is not present in (4.35) as it should be the case. These remaining terms would be generated in the triangular diagram with three external w^{ab}_{μ} fields and would come with a divergent coefficient (we have not computed such diagram). Note also that both (A.1.9) and the first two lines of (A.1.7) can be integrated by parts to make them vanish. This happens because in the conformally flat metric perturbations terms from the diagrams with two and three external w fields can not be related to each other integrating by parts. Therefore, they must vanish independently as the Gauss-Bonnet term is a total derivative after all.

This is a particularity of the conformally flat parametrization of the metric perturbation and would not hold for a generally diagonal parametrization. In that case terms generated in the two point function can be transformed into the terms appearing in the three point function by integration by parts and one would require a full calculation to show there is a match with an independent calculation of the Gauss-Bonnet term.

A.2 $\sqrt{g}R$ for the general diagonal parametrization of the metric

Let us consider a generally diagonal metric with four degrees of freedom

$$g_{\mu\nu} = \begin{pmatrix} e^{-\sigma_1(x)} & 0 & 0 & 0 \\ 0 & e^{-\sigma_2(x)} & 0 & 0 \\ 0 & 0 & e^{-\sigma_3(x)} & 0 \\ 0 & 0 & 0 & e^{-\sigma_4(x)} \end{pmatrix}. \qquad \text{(A.2.1)}$$

We saw that the corresponding expression for the curvature is Equation (4.23)

$$\begin{aligned}\mathcal{L}_R|_{\text{(on shell)}} =& M^2\sqrt{g}R = M^2\left[\partial_3^2\sigma_4+\partial_2^2\sigma_4+\partial_1^2\sigma_4+\partial_4^2\sigma_3+\partial_2^2\sigma_3+\partial_1^2\sigma_3\right.\\ &+\partial_4^2\sigma_2+\partial_3^2\sigma_2+\partial_1^2\sigma_2+\partial_4^2\sigma_1+\partial_3^2\sigma_1+\partial_2^2\sigma_1\\ &-\frac{1}{2}\left(\partial_3\sigma_1\partial_3\sigma_2+\partial_4\sigma_1\partial_4\sigma_2+\partial_2\sigma_1\partial_2\sigma_3+\partial_4\sigma_1\partial_4\sigma_3+\partial_2\sigma_1\partial_2\sigma_4+\partial_3\sigma_1\partial_3\sigma_4\right.\\ &\left.+\partial_1\sigma_2\partial_1\sigma_3+\partial_4\sigma_2\partial_4\sigma_3+\partial_1\sigma_2\partial_1\sigma_4+\partial_3\sigma_2\partial_3\sigma_4+\partial_1\sigma_3\partial_1\sigma_4+\partial_2\sigma_3\partial_2\sigma_4\right)\\ &\left.+\mathcal{O}(\sigma^3)\right].\end{aligned} \tag{A.2.2}$$

We consider now the divergent part proportional to M^2 of the result of (4.31)

$$\begin{aligned}&\frac{2}{\epsilon}\sum_{j=1}^{4}\sum_{\substack{l=1\\ l\neq j}}^{4}\left[-\frac{\sigma_j\sigma_l p^2M^2}{48\pi^2}+\frac{\sigma_j\sigma_l(p_j^2+p_l^2)M^2}{48\pi^2}\right]\\ =&\frac{2}{\epsilon}\frac{M^2}{48\pi^2}\left[-p^2\left(\sigma_1\sigma_2+\sigma_1\sigma_3+\sigma_1\sigma_4+\sigma_2\sigma_3+\sigma_2\sigma_4+\sigma_3\sigma_4\right)\right.\\ &+\sigma_1\sigma_2(p_1^2+p_2^2)+\sigma_1\sigma_3(p_1^2+p_3^2)+\sigma_1\sigma_4(p_1^2+p_4^2)+\sigma_2\sigma_3(p_2^2+p_3^2)\\ &\left.+\sigma_2\sigma_4(p_2^2+p_4^2)+\sigma_3\sigma_4(p_3^2+p_4^2)\right]\\ =&\frac{2}{\epsilon}\frac{M^2}{48\pi^2}\left[-\sigma_1\sigma_2(p_3^2+p_4^2)-\sigma_1\sigma_3(p_2^2+p_4^2)-\sigma_1\sigma_4(p_2^2+p_3^2)-\sigma_2\sigma_3(p_1^2+p_4^2)\right.\\ &\left.-\sigma_2\sigma_4(p_1^2+p_3^2)-\sigma_3\sigma_4(p_1^2+p_2^2)\right].\end{aligned} \tag{A.2.3}$$

This last expression, when expressed in position space, corresponds exactly to (A.2.2) except for a numerical factor and minus the second derivatives of the fields which are total derivatives and do not appear in the perturbative calculation.