DEPARTAMENTO DE ALGEBRA

SOBRE GRUPOS RADICALES LOCALMENTE FINITOS CON MIN-p PARA TODO PRIMO-p

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Sobre grupos radicales localmente finitos con min-p para todo primo p

Memoria presentada por
TATIANA PEDRAZA AGUILERA

para optar al Grado de Doctor en
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ADOLFO BALLESTER BOLINCHES

Valencia, Noviembre 2002

D. ADOLFO BALLESTER BOLINCHES, Profesor Titular del Departament d'Algebra de la Universitat de València

CERTIFICO: que la presente memoria *"Sobre grupos radicales localmente finitos con min-p, para todo primo p"*, ha sido realizada bajo mi dirección por DÑA. TATIANA PEDRAZA AGUILERA, en el Departament d'Algebra de la Universitat de València, y constituye su tesis para optar al grado de Doctor en Ciencias Matemáticas.

Y para que así conste, autorizo la presentación de la referida tesis doctoral, firmando el presente certificado.

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"Si he llegado a ver más lejos, es porque me subí a hombros de gigantes. No sé lo que pareceré a los ojos del mundo, pero a los míos es como si hubiese sido un muchacho que juega en la orilla del mar y se divierte de tanto en tanto encontrando un guijarro más pulido o una concha más hermosa, mientras el inmenso océano de la verdad se extendía, inexplorado frente a mi."

Isaac Newton.

"Nadie nos echará nunca del paraíso que Cantor ha creado para nosotros."

David Hilbert.

Resumen

Un subgrupo propio M de un grupo G se dice que es maximal en G si no existe ningún subgrupo propio de G distinto de él que lo contenga. En 1885, Frattini en su estudio sobre los subgrupos maximales de un grupo finito define el subgrupo $\phi(G)$ de un grupo G como la intersección de G y sus subgrupos maximales. Este subgrupo es conocido como el subgrupo de Frattini de G y es un subgrupo relevante en el estudio de los grupos nilpotentes. Si G es un grupo finito entonces G posee subgrupos maximales y, por tanto, el subgrupo de Frattini de un grupo finito siempre es propio en el grupo. Más aún, si G es un grupo finito entonces $\phi(G)$ es nilpotente. Es de sobra conocido que en el universo de los grupos finitos las condiciones $G' \leq \phi(G)$ y $G/\phi(G)$ nilpotente son ambas equivalentes a la nilpotencia de G. Otras propiedades de los grupos finitos que son equivalentes a la nilpotencia son, por ejemplo, la normalidad de todos los subgrupos de Sylow, la subnormalidad de cada subgrupo, la centralidad de todo factor principal o la normalidad de todo subgrupo maximal.

En grupos infinitos la situación es muy diferente puesto que no está asegurada la existencia de subgrupos maximales. Es decir, existen grupos infinitos que no poseen subgrupos maximales y por tanto su subgrupo de Frattini coincide con todo el grupo. En estos casos el subgrupo de Frattini del grupo proporciona escasa información sobre el grupo. Como consecuencia en grupos infinitos no podemos obtener caracterizaciones de la nilpotencia en términos del subgrupo de Frattini.

En 1975, Tomkinson introduce una variación de los subgrupos maximales, los subgrupos mayores, y a partir de ella un subgrupo característico $\mu(G)$ con propiedades similares a las del subgrupo de Frattini de un grupo finito. Usando este subgrupo de tipo Frattini, en esta tesis doctoral presentamos una caracterización de una clase de grupos nilpotentes generalizados en el universo $c\bar{\mathfrak{L}}$

de todos los grupos radicales localmente finitos con min-p para todo primo p, análoga a la caracterización de los grupos finitos nilpotentes. La clase de grupos nilpotentes generalizados considerada es la clase $\mathfrak{B}$ formada por todos los $c\bar{\mathfrak{L}}$-grupos tales que cada subgrupo propio tiene clausura normal propia. Esta clase de grupos está situada entre la clase de los $c\bar{\mathfrak{L}}$-grupos nilpotentes y los $c\bar{\mathfrak{L}}$-grupos localmente nilpotentes y resulta ser la extensión natural de la clase de los grupos nilpotentes finitos en el universo $c\bar{\mathfrak{L}}$.

En esta misma línea, continuamos con el estudio de la clase $\mathfrak{B}$, obteniendo un gran número de resultados que extienden los resultados conocidos de grupos finitos nilpotentes. Más aún, a través de la caracterización de los $\mathfrak{B}$-grupos, podemos probar que en todo $c\bar{\mathfrak{L}}$-grupo G, el $\mathfrak{B}$-radical coincide con el subgrupo de Fitting de G, $F(G)$, es decir, coincide con el producto de todos los normales nilpotentes de G. En general, el subgrupo de Fitting de un grupo infinito proporciona escasa información sobre la estructura del grupo. En cambio, en el universo $c\bar{\mathfrak{L}}$, este subgrupo juega un papel muy importante al adquirir todas las propiedades del $\mathfrak{B}$-radical. Este hecho nos permitirá además obtener resultados sobre el subgrupo de Fitting de un $c\bar{\mathfrak{L}}$-grupo análogos a los resultados de grupos finitos. Por último, estudiamos los inyectores asociados a la clase $\mathfrak{B}$ en el universo de grupos $c\bar{\mathfrak{L}}$, obteniendo una descripción similar a la caracterización de los inyectores nilpotentes de un grupo finito resoluble.

Otro punto importante en el análisis de la clase $\mathfrak{B}$ es el estudio de su versión local, la clase $\mathfrak{B}_p$. Así, en una segunda parte, obtenemos que esta clase de grupos es la generalización natural de la clase de los grupos finitos p-nilpotentes. En este sentido, extendemos algunos resultados de grupos finitos al universo de grupos $c\bar{\mathfrak{L}}$. En particular, surgen propiedades que relacionan el $\mathfrak{B}_p$-radical de un $c\bar{\mathfrak{L}}$-grupo G y un nuevo subgrupo característico que se define como intersección de un cierto tipo de subgrupos mayores de G. Este subgrupo resulta ser una versión local del subgrupo de tipo Frattini $\mu(G)$ introducido por Tomkinson. Además, caracterizamos los inyectores asociados a la clase de grupos p-nilpotentes generalizados $\mathfrak{B}_p$.

La última parte del trabajo está dedicada al estudio de $c\bar{\mathfrak{L}}$-grupos $G = AB$ factorizados por dos subgrupos A y B pertenecientes a la clase de grupos nilpotentes generalizados $\mathfrak{B}$, así como a la clase de los $c\bar{\mathfrak{L}}$-grupos localmente nilpotentes. Obtenemos, en este sentido, resultados que extienden algunos de los teoremas ya conocidos de productos de grupos nilpotentes en el universo finito.

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Preface

A group is said to be *locally finite* if every finite subset of G generates a finite subgroup. The class of locally finite groups is placed near the cross-roads of finite group theory and the general theory of infinite groups. Many theorems about finite groups can be phrased in such a way that their statements still make sense for locally finite groups. However, in general, Sylow's Theorems do not hold in the class of locally finite groups and there are a number of generic examples which show that locally finite groups can be very varied and complex. If we restrict our attention to locally finite-soluble groups with min-p for all primes p then the Sylow π-subgroups are very well behaved if π or its complementary in the set of all primes is finite. The conjugacy of Sylow p-subgroups in these groups is a very strong condition which have guaranteed the successful development of formation theory and interesting results on Fitting classes in the universe $c\bar{\mathfrak{L}}$ of all radical locally finite groups with min-p for all primes p. Moreover, using an extension of the Frattini subgroup introduced by Tomkinson, it has been proved a Gaschütz-Lubeseder type theorem characterizing saturated formations in this universe.

It is therefore appropriate to study the class $c\bar{\mathfrak{L}}$ of all radical locally finite groups with min-p for all primes p in more detail. In this thesis we have obtained results which help to understand better the groups in this class.

Consequently, the unspoken rule is that all groups considered in the three chapters of this thesis belong to the class $c\bar{\mathfrak{L}}$. The work is organized as follows.

In Chapter 1, we explore the class $\mathfrak{B}$ of generalized nilpotent groups in the universe $c\bar{\mathfrak{L}}$. We obtain that this class behaves in the universe $c\bar{\mathfrak{L}}$ as the nilpotent groups in the finite universe and we determine the structure of $\mathfrak{B}$-groups explicitly. Moreover, we show that the largest normal $\mathfrak{B}$-subgroup of a $c\bar{\mathfrak{L}}$-group is the Fitting subgroup. This fact allows us to prove some results concerning the Fitting subgroup of a $c\bar{\mathfrak{L}}$-group which are extensions of the finite ones. The aim of the last section is to study the injectors associated to the class $\mathfrak{B}$. In fact, we obtain a description of the $\mathfrak{B}$-injectors similar to the characterization of nilpotent injectors of a finite soluble group.

Chapter 2 is devoted to study the local version of the class $\mathfrak{B}$. This is a natural generalization of the class of finite p-nilpotent groups. We extend some results of finite groups to the above universe using a local version of a Frattini-like subgroup. In particular, some properties appear relating the Frattini and Fitting subgroups. The injectors associated to this class of generalized p-nilpotent groups are also characterized.

Finally, Chapter 3 is concerned with the structure of a radical locally finite group with min-p for all p, $G = AB$, factorized by two subgroups A and B in the class $\mathfrak{B}$. We extend the well-known results of finite products of nilpotent groups to the above universe.

We have introduced a Chapter 0 establishing the notation and terminology. It also presents many of the well-known results that will be used throughout this thesis. Notation that is not specifically cited here is consistent with that used in [13, 14, 22, 23].

Chapter 0

Preliminaries

The main purpose of this introductory chapter is to establish the notation and terminology and also list many results which will be used throughout this thesis. In this chapter we shall be concerned with infinite groups in general, not necessarily locally finite ones.

0.1 Fundamental concepts

A *group theoretical class* or *class of groups* $\mathfrak{X}$ is a class in the usual sense, consisting of groups, with two additional properties:

(a) $H \cong G \in \mathfrak{X}$ implies that $H \in \mathfrak{X}$,

(b) $\mathfrak{X}$ contains the trivial group.

Groups which belong to a class $\mathfrak{X}$ are referred to as $\mathfrak{X}$*-groups*. Standard examples of group theoretical classes are the classes of finite groups, abelian groups, nilpotent groups and soluble groups. For a deeper discussion of group classes and closure operations we refer the reader to [23]. We only mention explicitly the closure operation **L**: If $\mathfrak{X}$ is a class of groups then $G \in \mathbf{L}\mathfrak{X}$ if and only if every finite subset of G is contained in an $\mathfrak{X}$-group. We say that G is a *locally $\mathfrak{X}$-group* in this case. For example, the group G is *locally finite* if every finitely generated subgroup of G is finite. Analogously, the group G is *locally soluble* if every finitely generated subgroup of G is soluble.

There are a number of standard methods which can be used to construct examples of locally finite groups. These essentially depend on building the group up from its finite subgroups. An example of an infinite locally finite group which plays a very important role is the *quasicyclic group*:

Example 0.1. Let p be a prime. Then the complex pth roots of unity form a group $G = \langle x_i : x_1^p = 1, x_{i+1}^p = x_i; i = 1, 2, \dots \rangle$. G is an infinite abelian p-group which is the union of an ascending chain of cyclic p-groups of orders $p, p^2, \dots$. We call this group a *quasicyclic p-group* or *Prüfer p-group*, and denote it by C_p^∞. This group also can be thought of as the set of elements of p-power order in the additive abelian group $\mathbb{Q}/\mathbb{Z}$, or as a direct limit of cyclic groups of orders p^i. Every proper subgroup of C_p^∞ is a finite cyclic group. Thus G is clearly a locally finite group.

A *periodic* or *torsion group* is a group each of whose elements has finite order. Clearly the class of locally finite groups is a subclass of the class of periodic groups, although these classes are different. Probably one of the most important classes of periodic groups is the class of periodic locally soluble groups. Every group in this class is locally finite ([13, (1.1.5)]).

If π is a nonempty set of primes, a *π-number* is a positive integer whose prime divisors belong to π. An element of a group is called a *π-element* if its order is a π-number, and should every element be a π-element, the group is called a π-group (in particular it is a periodic group). We will denote π' the set $\mathbb{P}\backslash\pi$. The most important case is $\pi = \{p\}$, when we speak of p-groups and p'-groups.

We recall that if G is a finite group and π is a set of primes then a Hall π-subgroup of G is a π-subgroup whose order is relatively prime to its index and the Sylow p-subgroups of a finite group are precisely the Hall p-subgroups. However a finite group need not contain Hall π-subgroups as the alternating group A_5 shows. We present now what is perhaps the usual definition of Sylow subgroup in infinite group theory. Unlike with finite group theory we shall make no distinction between Sylow and Hall subgroup. If G is a group and π is a nonempty set of primes, G always contains maximal π-subgroups by Zorn's Lemma. A maximal π-subgroup of G will be called a *Sylow π-subgroup of G*, and the set of all Sylow π-subgroups of a group G is denoted by $Syl_\pi(G)$, with the understanding that $Syl_\pi(G)$ consists of the trivial group if G is a π'-group. Again if $\pi = \{p\}$ we omit the bracers and refer them as *Sylow p-subgroups* and if $\pi = \mathbb{P}\backslash\{p\}$, we simply say *Sylow p'-subgroups.*

Suppose that H is a subgroup of a group G and suppose that α is an ordinal number. An *ascending series* from H to G is a set of subgroups $\{V_\sigma : \sigma \leq \alpha\}$ of G such that

(A1) $V_0 = H$,

(A2) $V_\alpha = G$,

(A3) $V_\sigma \lhd V_{\sigma+1}$ for all $\sigma < \alpha$,

(A4) $V_\lambda = \bigcup_{\sigma<\lambda} V_\sigma$ for all limit ordinals $\lambda \leq \alpha$.

In this case we call H an *ascendant subgroup* of G, unless the series involved is of finite length in which case, of course, H is said to be *subnormal* in G. A *descending series* from G to H is a set of subgroups $\{\Lambda_\sigma : \sigma \leq \alpha\}$ of G such that

(D1) $\Lambda_0 = G$,

(D2) $\Lambda_\alpha = H$,

(D3) $\Lambda_{\sigma+1} \lhd \Lambda_\sigma$ for all $\sigma < \alpha$,

(D4) $\Lambda_\tau = \bigcap_{\sigma<\tau} \Lambda_\sigma$ for all limit ordinals $\tau \leq \alpha$.

In this case we call H a *descendant subgroup* of G, unless the series involved is of finite length in which case H is again subnormal in G. It is easily seen that subnormality and ascendance are preserved under taking homomorphic images, but this is not true for descendance. The subgroups V_σ (respectively Λ_σ) are called the terms of the series and the factor groups $V_{\sigma+1}/V_\sigma$ (respectively $\Lambda_\sigma/\Lambda_{\sigma+1}$) are called the factors of the series. Moreover, an ascending (respectively descending) series of is called *normal* if each term of the series is a normal subgroup of G. More generally, if Σ is a linearly ordered set, the above concepts are particular cases of a *series* from H to G with order type Σ, when the linearly ordered set Σ or its reverse is well-ordered. But, as we will see later, seriality and ascendance are equivalent in the class of groups that we will consider throughout this work. Finally, a series between 1 and G is simply called a *series in* G. For a fuller treatment of serial subgroups we refer the reader to [23].

Let $\mathcal{X}$ be a subgroup theoretical property. An ascending normal series $\{V_\sigma : \sigma \leq \alpha\}$ in a group G is called an *ascending $\mathcal{X}$-series* if $V_{\sigma+1}/V_\sigma$ has the property $\mathcal{X}$ as a subgroup of G/V_σ for each $\sigma \leq \alpha$. Groups with have an ascending $\mathcal{X}$-series are called *hyper-$\mathcal{X}$ groups*. For example, when $\mathcal{X}$ is centrality we obtain the class of *hypercentral groups*. This class is one of the many generalizations of the class of nilpotent groups. In particular, when $\mathcal{X}$ is the property of being a normal $\mathfrak{X}$-subgroup (where $\mathfrak{X}$ is a class of groups) we obtain the class of *hyper-$\mathfrak{X}$ groups*. For instance, a group G is *hyperfinite* if it has an ascending normal series whose factors are finite. Likewise, a group G is *hyperabelian* if it has an ascending normal series whose factors are abelian. In similar fashion, a descending normal series in a group G is called a *descending $\mathcal{X}$-series* if every factor of the series has the property $\mathcal{X}$ as a subgroup of the appropriate factor group. Groups with have a descending $\mathcal{X}$-series are called *hypo-$\mathcal{X}$ groups*. In particular, when $\mathcal{X}$ is centrality we obtain the class of *hypocentral groups*.

The class of *locally nilpotent groups* is a rather broad class of groups which is a generalization of the class of nilpotent groups. It is easy to prove that all hypercentral locally finite groups are locally nilpotent, but the converse is not true. Moreover, every locally finite-nilpotent group is the direct product of its Sylow subgroups. This basic structure reduces the study of locally finite-nilpotent groups, in some sense, to the study of locally finite p-groups. Recall that the product of two normal nilpotent subgroups is nilpotent (this is Fitting's Theorem). The corresponding statement holds for locally nilpotent groups. Furthermore, an arbitrary product of normal locally nilpotent subgroups of a group is also locally nilpotent (and normal). Hence every group G has a unique largest normal locally nilpotent subgroup called the *Hirsch-Plotkin radical* of G, denoted by $\rho(G)$ (see [22, (12.1.3)]). In fact, the Hirsch-Plotkin radical contains all ascendant locally nilpotent subgroups of a group ([22, (12.1.4)]).

A number of other radicals play important roles in the theory of finite (and hence locally finite) groups. For instance, it is easily seen that the product of an arbitrary number of normal π-subgroups of a group is again a π-group, for each set of primes π. Hence every group G has a unique largest normal π-subgroup which is denoted by $O_\pi(G)$. If G is an arbitrary group, the *upper $\pi'\pi$-series* is generated by repeatedly applying $O_{\pi'}$ and O_π. The first few terms are denoted by $O_{\pi'}(G)$, $O_{\pi'\pi}(G)$, $O_{\pi'\pi\pi'}(G)$ and so on. For instance, $O_{\pi'\pi}(G)/O_{\pi'}(G) = O_\pi(G/O_{\pi'}(G))$.

Define the *upper Hirsch-Plotkin series* of a group G to be the ascending series $1 = R_0 \leq R_1 \leq \dots$ in which $R_{\alpha+1}/R_\alpha$ is the Hirsch-Plotkin radical of G/R_α. If G coincides with a term of their upper Hirsch-Plotkin series then we call G a *radical group*. Thus radical groups are precisely those groups which have an ascending series with locally nilpotent factors.

There are numerous properties of infinite groups which are designed because they are properties enjoyed by finite groups. Such finiteness conditions have played an extremely important role in the development of group theory over the past fifty years. The property of being locally finite is, of course, such a finiteness condition. Now we define another important example and related finiteness condition. A group G is said to have the *minimum condition on subgroups*, or simply, *min*, if each non-empty set of subgroups of G has a minimal element; that is, if $\mathcal{S} = \{H_i : i \in I\}$ is a set of subgroups of G then there exists $K \in \mathcal{S}$ such that if $L \in \mathcal{S}$ and $L \leq K$ then $L = K$. It is easily seen that this property is identical with the *descending chain condition on subgroups*, that is, every descending chain of subgroups terminates in finitely many steps.

Certainly all finite groups have the minimum condition. Also, for each prime p, the quasicyclic p-group has min, since the proper subgroups of such group are all finite. Furthermore, if G is an abelian group then G has the minimum condition if and only if G is a finite direct product of quasicyclic p-groups and finite cyclic groups ([13, (1.5.5)]).

Now we define another important example of groups with min : *Chernikov groups*. First of all we recall one of the most important classes of abelian groups: the *divisible groups*. An abelian group G is called *divisible* if for every $x \in G$ and every positive integer n there exists $y \in G$ such that $x = ny$. In particular, the unique divisible finite group is the trivial group. The divisible groups have a number of prominent properties which are characteristic for them and their structure is completely known.

A group G is said to be a *Chernikov group* if it is a finite extension of an abelian group with the minimum condition. Such groups are named in honor of S. N. Chernikov, who made an extensive study of groups with the minimum condition. It follows from the above structural result that a group G is Chernikov if and only if it has a normal divisible abelian subgroup N of finite index, and N is a direct product of only finitely many quasicyclic groups. For example, the group C_p^∞ has an automorphism of order 2, namely the inversion automorphism, so we can form the Chernikov group $[C_p^\infty]C_2$. It is called the *locally dihedral p-group* and it is interesting the case when $p = 2$.

Now we introduce the notion of the divisible part of a group. It will be very useful throughout this work. Suppose that G is a group. If G has a unique largest divisible abelian subgroup N containing all other divisible abelian subgroups, then we call N the *radicable part* (or sometimes the *divisible part*) of G, and denote this subgroup by G^0. Of course, a group need not have a radicable part. However if G^0 exists in the group G then it is clearly a characteristic subgroup of G. Every Chernikov group G has a radicable part and it is precisely the finite residual of G. In particular, the radicable part of a finite group is trivial.

There is a more general condition than the minimum condition which is of interest in this work and we now proceed to describe this. Let p be a prime. The group G is said to satisfy the *minimum condition on p-subgroups* (*min-p* for short) if every non-empty subset of p-subgroups of G has a minimal element. It is easy to check that G has min-p if and only if every descending chain of p-subgroups terminates in finitely many steps.

In the following section we outline some of the main results on locally finite groups satisfying min-p for all primes p. The best reference here is the book of Dixon [13].

0.2 Basic results

We begin by recalling a very simple result concerning divisible subgroups. It will be useful in this work.

Lemma 0.1. [23, (3.29.1)] *Let R be a divisible abelian subgroup of a group G and let F be a finite subgroup of G. If $R^F = R$, then $R = [R, F]C_R(F)$.*

We present now two strong structural results for locally finite-soluble groups satisfying min-p for all p that we shall require throughout this text. The second one was obtained by Kargapolov in 1961.

Lemma 0.2. [13, (2.5.13)] *Suppose G is a locally finite-soluble group with min-p for all primes p. Then $G/O_{\pi'}(G)$ is a soluble Chernikov group for every finite set of primes π.*

Recall that a group G is called *residually finite* if it contains a collection of normal subgroups $\{N_i\}_{i\in I}$ such that G/N_i is finite for all $i \in I$ and $\bigcap_{i\in I} N_i = 1$.

Theorem 0.1. [13, (2.5.14)] *Let G be a locally finite-soluble group satisfying min-p for all primes p. Then G has a radicable part G^0 such that G/G^0 is residually finite and the Sylow p-subgroups of G/G^0 are finite for all primes p.*

As it is well known Sylow proved in 1872 his now famous results concerning the Sylow p-subgroups of a finite group. Hall generalized Sylow's work in 1928. He showed that a finite group G has Hall π-subgroups for every set of primes π if and only if G is soluble and went on to prove that in this case the Hall π-subgroups form a unique conjugacy class. In general, Sylow's theorems do not hold in the class of locally finite groups. However, using the above results, it is possible to obtain conjugacy results in the class of locally finite-soluble groups with min-p for all p.

Theorem 0.2. [13, (3.1.1)] *Let G be a locally finite-soluble group satisfying min-p for all primes p. If π is a finite set of primes then the Sylow π'-subgroups of G are all conjugate in G.*

Theorem 0.3. [13, (3.1.3)] *Suppose G is locally finite-soluble and satisfies min-p for all primes p. If π is a finite set of primes then the Sylow π-subgroups of G are all conjugate in G.*

We present also some elementary but useful consequences on the behaviour of the Sylow subgroups of a locally soluble group with min-p, for all p, with respect to factor groups, normal subgroups and serial subgroups.

Proposition 0.1. [13, (3.1.6)] *Suppose G is locally finite-soluble and satisfies min-p for all primes p. Suppose π is a finite set of primes and that N is a normal subgroup of G. If $P \in Syl_\pi(G)$ and $Q \in Syl_{\pi'}(G)$ then*

(i) $P \cap N \in Syl_\pi(N)$ *and all the Sylow π-subgroups of N have this form.*

(ii) $PN/N \in Syl_\pi(G/N)$ *and all the Sylow π-subgroups of G/N have this form.*

(iii) $Q \cap N \in Syl_{\pi'}(N)$ *and all the Sylow π'-subgroups of N have this form.*

(iv) $QN/N \in Syl_{\pi'}(G/N)$ *and all the Sylow π'-subgroups of G/N have this form.*

(v) $G=PQ$.

Lemma 0.3. [13, (5.1.8)] *Let G be a group, N a normal subgroup of G and P a π-subgroup of G for some set of primes π. Suppose that $P \cap N \in Syl_\pi(N)$ and $PN/N \in Syl_\pi(G/N)$. Then $P \in Syl_\pi(G)$.*

Lemma 0.4. [12, (2.7)] *Let G be a countable periodic locally soluble group with min-p for all primes p. Suppose π is a finite set of primes. If $P \in Syl_\pi(G)$ and H is a serial subgroup of G then $P \cap H \in Syl_\pi(H)$.*

Moreover, Proposition 0.1 and Lemma 0.3 allows us to obtain the following useful consequence:

Corollary 0.1. [13, (5.1.9)] *Let G be a locally finite-soluble satisfying min-p for all primes p. Let π be a finite set of primes. Suppose that H and K are subgroups of G satisfying $K \leq N_G(H)$ and P is a π-subgroup of G such that $P \cap H \in Syl_\pi(H)$ and $P \cap K \in Syl_\pi(K)$. Then*

$$P \cap HK = (P \cap H)(P \cap K) \in Syl_\pi(HK).$$

Recall that if H and K are two normal subgroups of a group G such that K is contained in H we say that H/K is a *chief factor of* G if H/K is a minimal normal subgroup of G/K. Concerning their structure we have the following results due to Mal'cev.

Lemma 0.5. [13, (1.2.4)] *Suppose that G is a periodic locally soluble group. Then the chief factors of G are elementary abelian p-groups.*

Lemma 0.6. [13, (1.2.6)] *The chief factors of a locally nilpotent group are central of prime order.*

As a consequence, if G is a periodic locally soluble group with min-p for all primes p, we may conclude the finiteness of all its chief factors.

0.3 The Frattini-like subgroup and the class $c\bar{\mathfrak{L}}$

The Frattini subgroup $\phi(G)$ of a group G is defined to be the intersection of G and its maximal subgroups. In a finite group G, the Frattini subgroup is always a proper subgroup of G and it enjoy some important properties. The situation is quite different in infinite groups, mainly due to the fact of G having insufficient maximal subgroups or even none at all. In 1975, Tomkinson introduced a characteristic subgroup $\mu(G)$ with properties similar to those of the Frattini subgroup of a finite group. We recall now the definitions which appear in Tomkinson's papers [26] and [29].

Let U be a proper subgroup of a group G. Consider a properly ascending chain

$$U = U_0 < U_1 < \ldots < U_\alpha = G$$

of subgroups from U to G, then we shall say that the ordinal α is the *type* of the chain. We define $m(U)$ to be the least upper bound of the types α of all such chains. Then $m(U)$ can be considered as a measure of how far U is from G. Clearly, $m(U) = 1$ if and only if U is a maximal subgroup of G. If G is either infinite cyclic or a quasicyclic p-group, then $m(1) = \omega$.

A proper subgroup M of G is said to be a *major subgroup* of G if $m(U) = m(M)$ whenever $M \leq U < G$. Roughly speaking, this means that every proper subgroup containing M is as far from G as M is. If M is a major subgroup of G we denote $M \in \mathcal{M}_j(G)$. The major subgroups of a group satisfying the maximal condition (in particular, an infinite cyclic group) are just the maximal subgroups, whereas every proper subgroup of a quasicyclic p-group is a non-maximal major subgroup.

The intersection of all major subgroups of G is denoted by $\mu(G)$. When G is finitely generated, $\mu(G)$ coincides with the Frattini subgroup $\phi(G)$ of G. The following result of Tomkinson, although elementary, will be of basic importance.

Lemma 0.7. **[26, Lemma 2.3]** *Every proper subgroup U of a group G is contained in a major subgroup of G.*

As a consequence, $\mu(G)$ is always a proper subgroup of G. Moreover, Tomkinson shows that if G is a Chernikov group then $\mu(G)$ is a very small subgroup.

Proposition 0.2. **[27, (1.2)]** *If G is a Chernikov group then $\mu(G)$ is finite.*

Recall that if p is a prime, a divisible abelian p-group A which is $\mathbb{Z}G$-module for some group G is said to be *divisibly irreducible* if every submodule of A is finite. In this way, a group G is said to be *semiprimitive* if it is the split extension, $G = [D]M$, of a faithful divisibly irreducible $\mathbb{Z}M$-module D by a finite soluble group M. In such a group, M is a major subgroup of G and $\mathrm{Core}_G(M) = 1$. Moreover, $\mathrm{C}_G(D) = D$ and if G is a Chernikov group then D is the Fitting subgroup of G ([29, (2.3)]). Note that this concept seems to be the natural extension, in infinite groups, of the concept of a primitive group in finite soluble groups. It will be widely used in this work.

From this point on all groups occurring will be in *the class* $c\bar{\mathfrak{L}}$ composed of all radical locally finite groups satisfying min-p for all primes p. Since every group in the class $c\bar{\mathfrak{L}}$ is locally soluble because it is radical, all results of Section 0.2 can be applied to the groups in the class $c\bar{\mathfrak{L}}$. In particular, by Lemma 0.2, the Sylow p-subgroups of a $c\bar{\mathfrak{L}}$-group G are Chernikov groups for every prime p.

Furthermore, a locally soluble group G with min-p for all primes p is radical if and only if G is hyperfinite. Therefore the following result concerning serial subgroups can be applied to the class $c\bar{\mathfrak{L}}$.

Lemma 0.8. **[13, (7.2.11)]** *Suppose that G is a hyperfinite group. Then every serial subgroup of G is ascendant.*

On the other hand, since every $c\bar{\mathfrak{L}}$-group G is radical and satisfies min-p for all primes p it follows that G contains minimal normal subgroups. Moreover, by Lemma 0.5, every minimal normal subgroup of G is abelian. In particular, each non-trivial homomorphic image of G contains a non-trivial normal abelian subgroup. Therefore, applying [23, (1.22)], G is hyperabelian. Consequently, every $c\bar{\mathfrak{L}}$-group is hyperabelian and hence the following result can also be applied to the class $c\bar{\mathfrak{L}}$.

Lemma 0.9. **[23, (2.17)]** *Let G be a hyperabelian group and let $F(G)$ be its Fitting subgroup. Then $C_G(F(G)) \leq F(G)$.*

Finally, we present a result, due to Ballester-Bolinches and Camp-Mora, concerning major subgroups of $c\bar{\mathfrak{L}}$-groups, that appears to be crucial throughout this thesis.

Theorem 0.4. **[3, Theorem 1]** *Let G be a $c\bar{\mathfrak{L}}$-group and let M be a major subgroup of G. Then:*

(a) *If M is a maximal subgroup of G, then $G/Core_G(M)$ is a finite soluble primitive group.*

(b) *If* M *is not a maximal subgroup of* G, *then* $G/Core_G(M)$ *is a semiprimitive group.*

Chapter 1

On a class of generalized nilpotent groups

1.1 Introduction

A group G is said to be *nilpotent* if it has a central series of finite length. It is well-known that there are numerous properties of finite groups which are equivalent to nilpotence. For instance, subnormality of each subgroup, normality of all Sylow subgroups, centrality of every chief factor and normality of all maximal subgroups. If the attention is restricted to locally finite-soluble groups, the first three properties are sufficient to ensure local nilpotence and the latter three ones are enjoyed by each locally nilpotent group.

It is also well-known that, for finite groups G, the conditions $G' \leq \phi(G)$ and $G/\phi(G)$ nilpotent are both equivalent to nilpotence. Taking into account that the Frattini subgroup $\phi(G)$ of a group G is defined as the intersection of G with all its maximal subgroups, it is rather clear that the condition $G' \leq \phi(G)$ is a weak property for infinite groups, even for locally finite groups, because an infinite group can have insufficient maximal subgroups or even none at all. For instance, let G be the standard wreath product of a Prüfer p-group and a Prüfer q-group. It is easy to show that $G = \phi(G)$. However G is not locally nilpotent if $p \neq q$. Thus $G' \leq \phi(G)$ does not imply local nilpotence and hence nilpotence.

Our first main objective in this chapter is to use the Frattini-like subgroup $\mu(G)$ introduced by Tomkinson to study the class $\mathfrak{B}$ of generalized nilpotent groups in the universe $c\bar{\mathfrak{L}}$. We show that in the class $c\bar{\mathfrak{L}}$, $\mathfrak{B}$-groups are to infinite groups as nilpotent groups to finite groups, obtaining a complete characterization of the class $\mathfrak{B}$, through the Frattini-like subgroup, analogously to the finite one for nilpotent groups and the Frattini subgroup.

We present structure results, some of which extend well known results and concepts from the finite universe. Furthermore, in the first section of this chapter, we ensure the existence of the $\mathfrak{B}$-radical in every group G belonging to the class $c\bar{\mathfrak{L}}$, showing that, in fact, it is the Fitting subgroup of G. Consequently, the Fitting subgroup in a $c\bar{\mathfrak{L}}$-group G share all the good properties of the $\mathfrak{B}$-radical of G. Finally, we describe the relationships which exist, in the universe $c\bar{\mathfrak{L}}$, between the class $\mathfrak{B}$ and some of the main classes of generalized nilpotent groups.

In the second section we prove some results concerning the Fitting subgroup of a $c\bar{\mathfrak{L}}$-group which are extensions of the finite ones. Since the class $\mathfrak{B}$ is a $c\bar{\mathfrak{L}}$-Fitting class, in the last section we proceed to describe the injectors associated to this class. We obtain a similar characterization of the $\mathfrak{B}$-injectors in the class $c\bar{\mathfrak{L}}$ to the nilpotent injectors of a finite soluble group. We may prove that, in fact, the maximal $\mathfrak{B}$-subgroups of a $c\bar{\mathfrak{L}}$-group containing the Fitting subgroup are precisely the $\mathfrak{B}$-injectors.

1.2 The class $\mathfrak{B}$

Let $\mathfrak{B}$ be the class of all $c\bar{\mathfrak{L}}$-groups in which every proper subgroup has a proper normal closure. This is a class of generalized nilpotent groups in the universe $c\bar{\mathfrak{L}}$ because every nilpotent $c\bar{\mathfrak{L}}$-group is in $\mathfrak{B}$ and every finite $\mathfrak{B}$-group is nilpotent. Moreover, this class contains the class of all $c\bar{\mathfrak{L}}$-groups for which every subgroup is descendant.

As an attempt to extend the concept of chief factor of a group, we define the concept of a δ-chief factor of a $c\bar{\mathfrak{L}}$-group.

Definition 1.1. Let G be a $c\bar{\mathfrak{L}}$-group and consider two normal subgroups H, K of G such that K is contained in H. We say that H/K is a *δ-chief factor of* G if H/K is either a minimal normal subgroup of G/K or a divisibly irreducible $\mathbb{Z}G$-module, that is, H/K has not proper infinite G-invariant subgroups.

Now we can establish our main result. It shows that $\mathfrak{B}$-groups play the same role in the class $c\bar{\mathfrak{L}}$ as finite nilpotent groups do in the class of all finite groups. Moreover, we obtain a complete characterization of the $\mathfrak{B}$-groups G, through the Frattini-like subgroup $\mu(G)$, analogously to the finite one for nilpotent groups and the Frattini subgroup. Let us first prove two preliminary lemmas.

Lemma 1.1. *Let G be either a finite primitive soluble group or a semiprimitive group. If G is a $\mathfrak{B}$-group, then G is abelian.*

Proof. Suppose first that G is a finite primitive soluble group and let M be a maximal subgroup of G with trivial core. Since G is a $\mathfrak{B}$-group and M is a proper subgroup of G, it follows that $\langle M^G\rangle$ is also a proper subgroup of G. Hence, since M is maximal in G, we have that $M = \langle M^G\rangle$. This implies that M is a normal subgroup of G. Therefore $M = 1$ is a maximal subgroup of G and thus G is a cyclic group of order p for some prime p.

Now, suppose that $G = [D]M$ is a semiprimitive group, where D is a faithful divisibly irreducible $\mathbb{Z}M$-module and M is a finite soluble group with trivial core. Denote $T = \langle M^G \rangle$, the normal closure of M in G. Since G is a $\mathfrak{B}$-group, it follows that T is a proper subgroup of G. Hence $T = (D \cap T)M$ and $D \cap T$ is a proper subgroup of D. Moreover, as $D \cap T \lhd G$ and D is divisibly irreducible it follows that $D \cap T$ is finite. Notice that T is a finite subgroup of G because M is also finite. Consequently M has only finitely many conjugates in G and $|G : N_G(M)|$ is finite. Consider now $N_G(M) = N_D(M)M$ and assume that $N_D(M) = D$. Then M is normal in G. Since $\mathrm{Core}_G(M) = 1$, we have that $M = 1$ and G is abelian.

Therefore we may assume that $N_D(M)$ is a proper subgroup of D. The fact that D is a normal subgroup of G implies that $N_D(M)$ is M-invariant and hence it is finite. Since M is also finite, we have that $N_G(M)$ is finite and hence G is a finite group. This contradicts the fact that G is a semiprimitive group and the lemma is proved. □

Lemma 1.2. *Let G be a $c\bar{\mathfrak{L}}$-group. If G is a Chernikov $\mathfrak{B}$-group, then G is nilpotent.*

Proof. Let M be a major subgroup of G. Denote $M_G = \mathrm{Core}_G(M)$. By Theorem 0.4, we have that either G/M_G is a finite primitive soluble group or G/M_G is a semiprimitive group. Consequently, G/M_G is abelian by Lemma 1.1. Since $G/\mu(G)$ is isomorphic to a subgroup of the cartesian product $\mathrm{Cr}_{M \in \mathcal{M}_j(G)} G/M_G$, we have that $G/\mu(G)$ is also an abelian group.

Let $G = G^0 A$, where A is a finite subgroup of G and G^0 is the radicable part of G. Then $A\mu(G)$ is a normal subgroup of G because $G/\mu(G)$ is abelian. Moreover, as G is a Chernikov group, it follows that $\mu(G)$ is finite by Proposition 0.2. Therefore $A\mu(G)$ is also a finite subgroup of G. Now, applying Lemma 0.1 we have that $G^0 = [G^0, A\mu(G)]C_{G^0}(A\mu(G))$. Since $A\mu(G)$ is a normal subgroup of G, it follows that $[G^0, A\mu(G)]$ is contained in $A\mu(G)$ and hence $[G^0, A\mu(G)]$

is finite. As a consequence $G^0 = C_{G^0}(A\mu(G))$. Since $G = G^0(A\mu(G))$ and G^0 is an abelian group, it follows that G^0 is contained in the center of G. Therefore $G/Z(G)$ is a finite group. Moreover $G/Z(G)$ is a $\mathfrak{B}$-group. Then $G/Z(G)$ is nilpotent and so G is nilpotent. □

Theorem 1.1. *Let G be a group in the class $c\bar{\mathfrak{L}}$. The following statements are pairwise equivalent:*

(i) *G is a $\mathfrak{B}$-group.*

(ii) *$G/\mu(G)$ is a $\mathfrak{B}$-group.*

(iii) *$G' \leq \mu(G)$.*

(iv) *Every major subgroup of G is a normal subgroup of G.*

(v) *G is a direct product of nilpotent Sylow subgroups.*

(vi) *G is locally nilpotent and the radicable part of G is central.*

(vii) *Every δ-chief factor of G is central.*

Proof. (i) implies (ii). This is clear from the fact that the class $\mathfrak{B}$ is closed under taking epimorphic images.

(ii) implies (iii). Note that if M is a major subgroup of G, then $\mu(G) \leq M_G$ and hence G/M_G is isomorphic to a quotient of $G/\mu(G)$. Since $G/\mu(G)$ is a $\mathfrak{B}$-group, we have that G/M_G is also a $\mathfrak{B}$-group. Therefore, by Lemma 1.1, G/M_G is an abelian group. Consequently $G/\mu(G)$ is also abelian and then $G' \leq \mu(G)$.

(iii) implies (iv). Since $G/\mu(G)$ is an abelian group, it follows that $M/\mu(G)$ is a normal subgroup of $G/\mu(G)$ for every major subgroup M of G. Consequently every major subgroup of G is a normal subgroup of G.

(iv) implies (i). Let H be a proper subgroup of G. Then, by Lemma 0.7, H is contained in a major subgroup M of G. Since M is a normal subgroup of G, we have that $\langle H^G \rangle \leq M$. In particular, $\langle H^G \rangle$ is a proper subgroup of G. Thus G is a $\mathfrak{B}$-group.

(iii) implies (v). Since $G' \leq \mu(G)$, it follows that $G/\mu(G)$ is an abelian group. In particular, $G/\mu(G)$ is a locally nilpotent group and then, by [26, (5.2)], G is locally nilpotent. Thus, if we denote by G_p the unique Sylow p-subgroup of G for each prime p, then $G = \mathrm{Dr}_p G_p$. We prove that G_p is a nilpotent group for each prime p. Since G is a $c\bar{\mathfrak{L}}$-group, we have that G_p is a Chernikov group for each prime p by Lemma 0.2. Moreover, G_p is isomorphic to $G/G_{p'}$, where $G_{p'} = \mathrm{Dr}_{q \neq p} G_q$. Since (i) is equivalent to (iii), we have that G is a $\mathfrak{B}$-group and consequently G_p is also a $\mathfrak{B}$-group for each prime p. By Lemma 1.2, G_p is nilpotent for each prime p.

(v) implies (iii). Let M be a major subgroup of G and denote $M_G = \mathrm{Core}_G(M)$. Applying Theorem 0.4, if M is a maximal subgroup of G then G/M_G is a finite primitive soluble group. By hypothesis, G is locally nilpotent. In particular, G/M_G is locally nilpotent and therefore G/M_G is nilpotent. Consequently, G/M_G is a cyclic group of order p for some prime p and thus G/M_G is an abelian group.

On the other hand, assume that M is a non-maximal major subgroup of G. Then G/M_G is a semiprimitive group. By [29, (2.3)], $F(G/M_G) = (G/M_G)^0 = \mathrm{C}_{G/M_G}(F(G/M_G))$ and $F(G/M_G)$ is a p-group for some prime p. Moreover, G/M_G is a direct product of nilpotent Sylow subgroups. Consequently, $F(G/M_G) = G/M_G$ and G/M_G is a quasicyclic p-group because G/M_G is divisibly irreducible. In both cases we have proved that if M is a major subgroup of G, then G/M_G is an abelian group. As a consequence, $G/\mu(G)$ is also abelian and thus G' is contained in the subgroup $\mu(G)$.

(v) implies (vi). It is clear that G is locally nilpotent. For each prime p, G has a normal Sylow p-subgroup G_p and $G = \mathrm{Dr}_p G_p$. Moreover, G_p is Chernikov for all primes p by Lemma 0.2. Let G^0 be the radicable part of G. Then $G^0 \cap G_p = G_p^0$ for all primes p. Therefore $G^0 = \mathrm{Dr}_p G_p^0$. Since G_p is nilpotent, we have that $G_p^0 \leq \mathrm{Z}(G_p)$ by [13, (1.5.12)]. Consequently $G_p^0 \leq \mathrm{Z}(G)$ for all primes p and $G^0 \leq \mathrm{Z}(G)$.

(vi) implies (v). Arguing as above, we have that $G = \mathrm{Dr}_p G_p$ and $G^0 = \mathrm{Dr}_p G_p^0$. Since $G_p^0 \leq \mathrm{Z}(G_p)$ and G_p is Chernikov, it follows that $G_p/\mathrm{Z}(G_p)$ is nilpotent and so G_p is nilpotent for all primes p. Hence (v) holds.

(vi) implies (vii). Let G be a locally nilpotent group such that $G^0 \leq \mathrm{Z}(G)$ and consider a δ-chief factor H/K of G. If H/K is a minimal normal of G/K then, by [22, (12.1.6)], H/K is central.

Suppose now that H/K is a divisibly irreducible $\mathbb{Z}G$-module. In particular, H/K is a divisible subgroup of G/K and H/K is contained in $(G/K)^0$, the radicable part of G/K. On the other hand, if we denote $T/K = (G/K)^0$, it follows from Theorem 0.1 that the Sylow p-subgroups of $G/(G^0K)$, and hence the Sylow p-subgroups of $T/(G^0K)$, are finite for each prime p. Furthermore, $T/(G^0K)$ is a locally nilpotent group and hence it is the direct product of its Sylow subgroups. Hence every Sylow subgroup of $T/(G^0K)$ is a finite divisible group and so it is trivial. Therefore $T/K = (G/K)^0 = (G^0K)/K$. Consequently, $H/K \leq (G^0K)/K \leq (\mathrm{Z}(G)K)/K \leq \mathrm{Z}(G/K)$ as required.

(vii) implies (vi). By hypothesis every δ-chief factor of G is central. In particular every chief factor of G is central. It follows from [13, (6.2.4)] that G is locally nilpotent. Moreover, the Sylow subgroups of G are Chernikov groups. Hence, to prove that $G^0 \leq \mathrm{Z}(G)$, we may assume that G is a Chernikov p-group for some prime p. Then $G = G^0A$ where A is a finite subgroup of G and G^0 is the radicable part of G. Since G^0 is a divisible abelian p-group of finite rank, it follows from [27, (1.3)] that there is a finite normal subgroup C of G contained in G^0 such that G^0/C is a direct product of divisibly irreducible $\mathbb{Z}G$-modules, say

$$G^0/C = (G_1/C) \times (G_2/C) \times \cdots \times (G_n/C).$$

Since G_i/C is a δ-chief factor of G for all $i \in \{1, \cdots, n\}$ then $G_i/C \leq \mathrm{Z}(G/C)$

for all $i \in \{1, \cdots, n\}$ and, $G^0/C \leq \mathrm{Z}(G/C)$. Hence the commutator $[G^0, G]$ is contained in C and $[G^0, G]$ is a finite group. Furthermore, applying Lemma 0.1 we have that $G^0 = [G^0, A]\mathrm{C}_{G^0}(A)$. But, since the commutator $[G^0, A]$ is a finite group, it is clear that $G^0 = \mathrm{C}_{G^0}(A)$. Therefore, as G^0 is abelian and $G = G^0 A$, we conclude that $G^0 \leq \mathrm{Z}(G)$. □

Assume that a group G has the minimum condition on subgroups. If G is a $\mathfrak{B}$-group, then G is a direct product of nilpotent Sylow subgroups by Theorem 1.1. Since G has min, only finitely many of these Sylow subgroups are non trivial. Hence G is actually a nilpotent group.

Corollary 1.1. *Let G be a group in the class $c\bar{\mathfrak{L}}$. Assume G satisfies the minimal condition on subgroups. Then G is a $\mathfrak{B}$-group if and only if G is a nilpotent group.*

In particular, if the group G is the split extension of a quasicyclic 2-group by its involution (that is, the *locally dihedral 2-group*), then G is a locally nilpotent group, in fact it is hypercentral, but it is not a $\mathfrak{B}$-group. Notice that for instance the subgroup generated by the involution is a major subgroup of the group G which is not normal in G.

Consider the set $\{p_i\}_{i\geqslant 1}$ of all prime numbers in their natural order. Let G_i be the split extension of the cyclic group $\langle x_i \rangle$ of order p_i^i by its automorphism y_i of order p_i^{i-1} which maps x_i to $x_i^{p_i+1}$. Then G_i is nilpotent of class i. Let $G = Dr_{i=1}^{\infty} G_i$, then G is a $\mathfrak{B}$-group which is not nilpotent (see [26]).

These examples show that the class $\mathfrak{B}$ is intermediate between the classes of nilpotent $c\bar{\mathfrak{L}}$-groups and locally nilpotent $c\bar{\mathfrak{L}}$-groups.

Our aim now is to continue the study of the class $\mathfrak{B}$ in the universe $c\bar{\mathfrak{L}}$, as it enjoys very interesting properties of nilpotent type. It is well-known that for finite groups G, the Frattini subgroup $\phi(G)$ is nilpotent. This result can be extended to $c\bar{\mathfrak{L}}$-groups using Tomkinson's subgroup.

Theorem 1.2. *Let G be a group in the class $c\bar{\mathfrak{L}}$. Then $\mu(G)$ is a $\mathfrak{B}$-group with finite Sylow subgroups.*

Proof. By [26, (5.3)] we have that $\mu(G)$ is locally nilpotent. Next we see that every Sylow p-subgroup of $\mu(G)$ is nilpotent for each prime p. Let p be a prime and let P be a Sylow p-subgroup of $\mu(G)$. By Lemma 0.2, we know that $G/O_{p'}(G)$ is a Chernikov group and, by Proposition 0.2, $\mu(G/O_{p'}(G))$ is finite. Since $\mu(G)O_{p'}(G)/O_{p'}(G)$ is contained in $\mu(G/O_{p'}(G))$, it follows that $\mu(G)O_{p'}(G)/O_{p'}(G)$ is finite and so is $PO_{p'}(G)/O_{p'}(G)$. Therefore P is finite and nilpotent. We conclude by Theorem 1.1 that $\mu(G)$ is a $\mathfrak{B}$-group. □

Our next result analyzes the behaviour of $\mathfrak{B}$ as a class of $c\bar{\mathfrak{L}}$-groups. Recall that a class $\mathfrak{F}$ of $c\bar{\mathfrak{L}}$-groups is said to be a *$c\bar{\mathfrak{L}}$-formation* if it satisfies the following properties:

(i) If $G \in \mathfrak{F}$ and N is a normal subgroup of G, then $G/N \in \mathfrak{F}$.

(ii) If $\{N_i\}_{i\in I}$ is a collection of normal subgroups of $G \in c\bar{\mathfrak{L}}$ such that $G/N_i \in \mathfrak{F}$ for every $i \in I$ and $\bigcap_{i\in I} N_i = 1$, then $G \in \mathfrak{F}$.

Theorem 1.3. *$\mathfrak{B}$ is a subgroup-closed $c\bar{\mathfrak{L}}$-formation.*

Proof. First we prove that every subgroup of a $\mathfrak{B}$-group is also a $\mathfrak{B}$-group. Let G be a $\mathfrak{B}$-group and let H be a subgroup of G. Since G is locally nilpotent, we have that H is also locally nilpotent. Let H_p be the Sylow p-subgroup of H for each prime p. Then H_p is contained in the unique Sylow p-subgroup G_p of G. By Theorem 1.1, we have that G_p is nilpotent. Then H_p is also nilpotent for each prime p. According to Theorem 1.1, H is a $\mathfrak{B}$-group.

On the other hand, it is clear that $\mathfrak{B}$ is closed under taking epimorphic images. Let $\{N_i\}_{i\in I}$ be a collection of normal subgroups of $G \in c\bar{\mathfrak{L}}$ such that $G/N_i \in \mathfrak{B}$ for every $i \in I$ and $\bigcap_{i\in I} N_i = 1$. Since G/N_i is locally nilpotent for all $i \in I$, we know that G is locally nilpotent ([13, (6.2.11)]). Let G^0 be the radicable part of G. Then G^0N_i/N_i is contained in $(G/N_i)^0$, which is central in G/N_i for all $i \in I$. Therefore $[G, G^0] \leq N_i$ for all $i \in I$. This implies that $[G, G^0] = 1$ and G^0 is contained in $\mathrm{Z}(G)$. Applying Theorem 1.1, we have that G is a $\mathfrak{B}$-group. □

Theorem 1.3 allows us to show the existence of the $\mathfrak{B}$-radical in every group G belonging to the class $c\bar{\mathfrak{L}}$. It is known that the product of two normal nilpotent subgroups is nilpotent ([22, (5.2.8)]). The corresponding statement holds for locally nilpotent groups and is of great importance. Moreover in any group G there is a unique maximal normal locally nilpotent subgroup (called the Hirsch-Plotkin radical) containing all normal locally nilpotent subgroups of G (see [22, (12.1.3)]). We obtain analogous results for the class $\mathfrak{B}$ by defining the corresponding radical subgroup associated to this class.

Theorem 1.4. *Let G be a $c\bar{\mathfrak{L}}$-group. Assume that H and K are two normal $\mathfrak{B}$-subgroups of G. Then HK is a $\mathfrak{B}$-group.*

Proof. We know from Theorem 1.1 that H and K are locally nilpotent with nilpotent Sylow subgroups. By [13, (12.1.2)] HK is locally nilpotent. Let p be a prime and let H_p and K_p be the Sylow p-subgroups of H and K respectively. Then it is clear that H_p and K_p are normal in G. Let S_p be the Sylow p-subgroup of HK. We have that $S_p \cap H = H_p$, $S_p \cap K = K_p$ and $S_p = H_pK_p$. Therefore S_p is a nilpotent group by Fitting's Theorem. □

Lemma 1.3. *Let G be a Chernikov $c\bar{\mathfrak{L}}$-group. Assume that $G = \bigcup_{i\in I} G_i$, where G_i is a normal $\mathfrak{B}$-subgroup of G for each $i \in I$ and $\{G_i : i \in I\}$ is a totally ordered set by inclusion. Then G is a $\mathfrak{B}$-group.*

Proof. Let G^0 be the radicable part of G and let A be a finite subgroup of G such that $G = G^0A$. If $a \in A$, then there exists $\alpha_a \in I$ such that $a \in G_{\alpha_a}$. Since A is finite, it follows that $\{G_{\alpha_a} : a \in A\}$ is finite and so we can choose a maximal element, K say. From the fact that $\{G_i : i \in I\}$ is totally ordered, we can conclude that $G_{\alpha_a} \leq K$ for each $a \in A$. Therefore A is contained in K and the result follows from Theorem 1.4. $\square$

Theorem 1.5. *Every group $G \in c\bar{\mathfrak{L}}$ has a unique largest normal $\mathfrak{B}$-subgroup, denoted by $\delta(G)$.*

Proof. Consider the non-empty set $\mathcal{S} = \{B \triangleleft G : B \in \mathfrak{B}\}$. Let $\mathcal{C} = \{B_i : i \in I\}$ be a chain in $\mathcal{S}$. We show that $X = \bigcup_{i \in I} B_i$ is an upper bound for $\mathcal{C}$ in $\mathcal{S}$. First, it is clear that X is a normal subgroup of G. By Lemma 0.2, $X/O_{p'}(X)$ is a Chernikov group for all primes p, and $X/O_{p'}(X)$ is the union of the elements of the chain $\{B_i O_{p'}(X)/O_{p'}(X) : i \in I\}$. By Lemma 1.3, $X/O_{p'}(X)$ is a $\mathfrak{B}$-group. Since $\bigcap_p O_{p'}(X) = 1$ and $\mathfrak{B}$ is a $c\bar{\mathfrak{L}}$-formation by Theorem 1.3, we conclude that X belongs to $\mathfrak{B}$. It is now clear that X is a subgroup which is an upper bound for $\mathcal{C}$. We may now apply Zorn's Lemma to produce a maximal element of $\mathcal{S}$, $\delta(G)$ say. Let N be a normal $\mathfrak{B}$-subgroup of G. By Theorem 1.4, $\delta(G)N$ is a normal $\mathfrak{B}$-subgroup of G. The maximality of $\delta(G)$ in $\mathcal{S}$ implies that $\delta(G) = \delta(G)N$ and $N \leq \delta(G)$. Consequently, $\delta(G)$ is the unique largest normal $\mathfrak{B}$-subgroup of G. $\square$

From Theorems 1.3 and 1.5, we have that the product of arbitrarily many normal $\mathfrak{B}$-subgroups of a group $G \in c\bar{\mathfrak{L}}$ belongs to the class $\mathfrak{B}$. In particular, the subgroup $\delta(G)$ is the product of all normal $\mathfrak{B}$-subgroups of G.

Note that for every group $G \in c\bar{\mathfrak{L}}$, the Fitting subgroup $F(G)$ is contained in $\delta(G)$ by Theorem 1.5. Now if $N \in \mathfrak{B}$ is a normal subgroup of G, then every Sylow subgroup of N is a normal nilpotent subgroup of G. Consequently N is contained in the Fitting subgroup of G, $F(G)$. This implies that $\delta(G) = F(G)$ for every group $G \in c\bar{\mathfrak{L}}$. It is known that in general the Fitting subgroup in an infinite group gives little information about the structure of the group. However in this case it plays an important role as it inherits the properties of the $\mathfrak{B}$-

radical.

We say that a subclass $\mathfrak{F}$ of $c\bar{\mathfrak{L}}$ is a $c\bar{\mathfrak{L}}$-*Fitting class* if it satisfies the following properties:

(i) If $G \in \mathfrak{F}$ and H is a normal subgroup of G, then $H \in \mathfrak{F}$.

(ii) If $G = \langle H_i : i \in I \rangle \in c\bar{\mathfrak{L}}$ and, for each $i \in I$ the subgroup H_i is a normal $\mathfrak{F}$-subgroup of G, then $G \in \mathfrak{F}$.

Therefore, it is clear that $\mathfrak{B}$ is an example of a $c\bar{\mathfrak{L}}$-Fitting class. Perhaps, it is worth noting at this point a well-known result of Bryce and Cossey concerning saturated formations and Fitting classes. It asserts that, in the finite soluble universe, a subgroup-closed Fitting formation is saturated ([14, (XI.1.2)]). Thus we can formulate the following question: Does Bryce and Cossey's Theorem hold in the class $c\bar{\mathfrak{L}}$?. We will see that the answer to this question is negative by proving that the class $\mathfrak{B}$ is not a saturated $c\bar{\mathfrak{L}}$-formation.

In finite groups a formation $\mathfrak{F}$ is called *saturated* if a group $G \in \mathfrak{F}$ whenever $G/\phi(G) \in \mathfrak{F}$. It is also well-known that, in the finite soluble universe, the saturated formations are precisely the same classes as the so-called *locally defined* ones. The main difficulty in extending the definition of saturated formation to infinite groups lies in the aforementioned deficiencies of the Frattini subgroup. Therefore, the definition of saturated formation is done locally in infinite groups to avoid situations where a given group has no maximal subgroups.

However, Ballester-Bolinches and Camp-Mora ([3]) show that it is possible to characterize $c\bar{\mathfrak{L}}$-saturated formations, which are the locally defined ones, by means of the Frattini-like subgroup $\mu(G)$.

Definition 1.2. [3, **Definition** 1] A $c\bar{\mathfrak{L}}$-formation $\mathfrak{F}$ is said to be E_μ-closed if $\mathfrak{F}$ enjoys the following properties:

(i) A $c\bar{\mathfrak{L}}$-group G is in $\mathfrak{F}$ if and only if $G/\mu(G)$ is in $\mathfrak{F}$.

(ii) A semiprimitive group G is an $\mathfrak{F}$-group if and only if it is the union of an ascending chain $\{G_i : i \in \mathbb{N}\}$ of finite $\mathfrak{F}$-subgroups.

Theorem 1.6. **[3, Theorem A]** *Let $\mathfrak{F}$ be a $c\bar{\mathfrak{L}}$-formation. Then $\mathfrak{F}$ is E_μ-closed if and only if $\mathfrak{F}$ is a saturated $c\bar{\mathfrak{L}}$-formation.*

Applying this result, to see that the class $\mathfrak{B}$ is not saturated, it will be enough to prove that in the class $c\bar{\mathfrak{L}}$, $\mathfrak{B}$ is not E_μ-closed.

Example 1.1. Consider $G = D_{2^\infty} = [C_{2^\infty}]\langle\alpha\rangle$ the locally dihedral 2-group and the subgroups of C_{2^∞}:

$$\Omega_i(C_{2^\infty}) = \{\ g \in C_{2^\infty} : o(g) \mid 2^i\} \simeq C_{2^i}, \text{ for each } i \geq 1.$$

Then the group G can be expressed as $G = \bigcup_{i\geq 1} G_i$, where $G_i = \Omega_i(C_{2^\infty})\langle\alpha\rangle$, for each natural number i and $\{G_i : i \geq 1\}$ is an ascending chain. Notice that G_i is a finite 2-group for every $i \geq 1$. Hence G_i is a nilpotent group and then a $\mathfrak{B}$-group for each $i \geq 1$. However G is not a $\mathfrak{B}$-group. Therefore we have that G is a semiprimitive group which is the union of an ascending chain of finite $\mathfrak{B}$-subgroups but it is not in the class $\mathfrak{B}$.

This proves that, in the universe $c\bar{\mathfrak{L}}$, the class $\mathfrak{B}$ is not E_μ-closed and so it is not a saturated formation. Consequently, Bryce and Cossey's Theorem is not longer true in infinite groups as the class $\mathfrak{B}$ shows.

We recall now some classes of generalized nilpotent groups, that is, group theoretical properties that are possessed by all nilpotent groups and which for finite groups imply nilpotence. A group G is called a *Fitting group* if $G = F(G)$, that is, if G is a product of normal nilpotent subgroups. Obviously, in the universe $c\bar{\mathfrak{L}}$, every group G in the class $\mathfrak{B}$ is a Fitting group since $F(G) = \delta(G)$.

We say that a group G is a *Baer group* if it is generated by its abelian subnormal subgroups, or equivalently, if every finitely generated subgroup of G is subnormal. It is known that a Fitting group is a Baer group but the converse is not true, even in the locally finite universe. On the other hand, we present an example of a locally finite group which is a Fitting group but it is not in the class $\mathfrak{B}$.

Example 1.2. Let $G = M(\mathbb{Q}, GF(p))$ be the McLain group determined by the set $\mathbb{Q}$ of all rational numbers and the field $GF(p)$ (see [22, (12.1.9)]). It is clear that G is not in $c\bar{\mathfrak{L}}$. Moreover, it is known that G is the product of its normal abelian subgroups and hence is a Fitting group. Let H be the subgroup generated by the set of all $1 + e_{\alpha\beta}$, such that $\alpha \in \mathbb{Q}$, $\beta \in X$ and $\alpha < \beta$, where $X = \{x^2 : x \in \mathbb{Q}\} \cup \{-x^2 : x \in \mathbb{Q}\}$. We have that H is a proper subgroup of G and $G = \langle H^G \rangle$. Therefore, G is a locally finite Fitting group which is not a $\mathfrak{B}$-group.

However, in the universe $c\bar{\mathfrak{L}}$, the relationship between these classes of generalized nilpotent groups is quite different as the following result shows.

Theorem 1.7. *Let G be a $c\bar{\mathfrak{L}}$-group. Then the following statements are pairwise equivalent:*

(i) *G is a $\mathfrak{B}$-group.*

(ii) *G is a Fitting group.*

(iii) *G is a Baer group.*

(iv) *G is hypocentral with hypocentral length $\leq \omega$.*

Proof. It is clear that (i) implies (ii) and that (ii) implies (iii).

(iii) implies (i). Let G be a Baer group and consider a major subgroup M of G. If M is maximal in G, it follows by [22, (12.1.5)] that M is normal in G. Then G/M_G is a cyclic group of order p, for some prime p, and hence it is abelian. Therefore, $G' \leq M_G$. Suppose that M is not maximal in G. Then G/M_G is a semiprimitive Baer group. We shall prove that G/M_G is abelian. To see this, we may assume that $M_G = 1$. Then $G = [D]M$, where D is a divisibly irreducible abelian p-group for some prime p and M is a finite soluble group with trivial core. Denote $M_i = \Omega_i(D)M$, where $\Omega_i(D)$ is the subgroup generated by the elements of D of order dividing p^i. Then $G = \langle M_i : i \in \mathbb{N}\rangle$ and M_i is finite and hence subnormal in G for all $i \in \mathbb{N}$ because G is a Baer group. Therefore there exists a proper normal subgroup T_i of G such that $M_i \leq T_i$. Since D is divisibly irreducible and $T_i = M(T_i \cap D)$, it follows that T_i is finite. Therefore, every subgroup of T_i is subnormal in T_i. Consequently, T_i is nilpotent and then M_i is nilpotent for all $i \in \mathbb{N}$. We conclude that $M_i \leq \delta(G)$ for all $i \in \mathbb{N}$ and hence G is a $\mathfrak{B}$-group. Consequently G is abelian by Lemma 1.1 and then G/M_G is abelian for all major subgroups M of G. By Theorem 1.1, G is a $\mathfrak{B}$-group.

(i) implies (iv). Let G be a $\mathfrak{B}$-group and let p be a prime. Since $G/O_{p'}(G)$ is Chernikov by Lemma 0.2 then, applying Lemma 1.2, $G/O_{p'}(G)$ is nilpotent. Hence, there exists $n_p \in \mathbb{N}$ such that $\gamma_{n_p}(G) = [G, \overset{(n_p)}{\ldots}, G] \leq O_{p'}(G)$ for each prime p, where $\gamma_i(G)$ denotes the i-th term of the lower central series of G. Therefore, $\gamma_\omega(G) = \bigcap_{\beta<\omega} \gamma_\beta(G) \leq \gamma_{n_p}(G) \leq O_{p'}(G)$ for every prime p and then $\gamma_\omega(G) = 1$.

(iv) implies (i). $G/O_{p'}(G)$ is a Chernikov hypocentral group and then it is nilpotent for every prime p. In particular, since $\mathfrak{B}$ is a formation, we have that G is a $\mathfrak{B}$-group.

□

On the other hand, it is well-known that every hypercentral group is locally nilpotent but, in general, the converse is not true (see [22, (12.2)]). However, it is not difficult to see that, in the class $c\bar{\mathfrak{L}}$, hypercentrality is equivalent to local nilpotence.

Proposition 1.1. *Suppose that G is in the class $c\bar{\mathfrak{L}}$. Then G is hypercentral if and only if G is locally nilpotent.*

Proof. Suppose that G is locally nilpotent. To prove that G is hypercentral it is enough to prove, by [23, (1.22)], that for every normal subgroup N of G such that G/N is non-trivial we have that $Z(G/N)$ contains a non-trivial normal subgroup. Since G/N satisfies min-p, there exist a minimal normal subgroup A/N of G/N. Moreover, applying Lemma 0.6, $A/N \leq Z(G/N)$. We conclude that $Z(G/N) \neq 1$ and hence G is hypercentral. □

As a consequence of the above result, a $c\bar{\mathfrak{L}}$-group G *is locally nilpotent if and only if every subgroup of G is ascendant in G* (see [22, (12.2)]).

It is known that every descendant subgroup of a $c\bar{\mathfrak{L}}$-group is ascendant (Lemma 0.8), but the converse is not true as the locally dihedral 2-group shows. Since the class $\mathfrak{B}$ is a generalization of nilpotence that is stronger than local nilpotence in the universe $c\bar{\mathfrak{L}}$, it would be desirable to obtain a corresponding result to the locally nilpotent one by using descendant subgroups. In [4] Ballester-Bolinches and Camp-Mora have been able to do this.

Theorem 1.8. [4, Theorem 5] *Let G be a $c\bar{\mathfrak{L}}$-group and let H be a subgroup of G. Then H is a descendant $\mathfrak{B}$-subgroup of G if and only if H is contained in $F(G)$.*

Corollary 1.2. *A $c\bar{\mathfrak{L}}$-group G is a $\mathfrak{B}$-group if and only if every subgroup of G is descendant in G.*

It is also well-known that a finite group is nilpotent if and only if every subgroup is subnormal. For infinite groups the situation is different and the property that every subgroup is subnormal is weaker than nilpotence. We conclude this section by showing that these two properties are identical in the universe $c\bar{\mathfrak{L}}$.

Theorem 1.9. *A $c\bar{\mathfrak{L}}$-group G is nilpotent if and only if every subgroup of G is subnormal in G.*

Proof. It is easy to see that, in general, every subgroup of a nilpotent group is subnormal. Conversely, let G be a $c\bar{\mathfrak{L}}$-group such that every subgroup is subnormal. In particular, applying Corollary 1.2, G is a $\mathfrak{B}$-group and therefore $G^0 \leq \mathrm{Z}(G)$ by Theorem 1.1. On the other hand, G/G^0 is a periodic group with every subgroup subnormal which is residually finite by Theorem 0.1. It follows from [24] that G/G^0 is nilpotent. Since $G^0 \leq \mathrm{Z}(G)$, we conclude that G is nilpotent.

□

1.3 The Fitting subgroup

As we have already prove, the subgroup generated by all normal $\mathfrak{B}$-subgroups of a $c\bar{\mathfrak{L}}$-group G is the Fitting subgroup of G, which is also the largest normal $\mathfrak{B}$-subgroup of G. In fact, it follows from Theorem 1.8 that the Fitting subgroup of a group G is actually generated by all descendant $\mathfrak{B}$-subgroups of G. It is known that the Fitting subgroup in an infinite group gives little information about the structure of the group. However in this case it plays an important role as it inherits the properties of the $\mathfrak{B}$-radical. The aim of this section is to look more closely at the properties of the Fitting subgroup of a $c\bar{\mathfrak{L}}$-group and its behaviour as radical. We present some results which extend well known results and concepts from the finite universe.

It is well-known that the Fitting subgroup of a finite group G is the intersection of the centralizers of all chief factors of G (see [14, (A.13.8)]). This result is also true for the Hirsch-Plotkin radical of a periodic locally soluble group (see [13, (1.3.5)], [13, (6.2.4)]). As one might expect, if G is a $c\bar{\mathfrak{L}}$-group, there is an important connection between $F(G)$ and the centralizers of the δ-chief factors as the following theorem shows.

Theorem 1.10. *Suppose that G is a $c\bar{\mathfrak{L}}$-group. Then $F(G)$ is the intersection of the centralizers of all δ-chief factors of G.*

Proof. Let H/K be a δ-chief factor of G. We show that $F(G)$ is contained in $C_G(H/K)$. Since $F(G)K/K \leq F(G/K)$, we may assume that $K = 1$. If H is a minimal normal subgroup of G, then it follows from [13, (1.3.5)] and [13, (6.2.4)] that the Hirsch-Plotkin radical of G, $\rho(G)$, is contained in $C_G(H)$. Consequently, since $F(G) \leq \rho(G)$, we have that $F(G)$ is also contained in $C_G(H)$ as required. Suppose now that H is a divisibly irreducible $\mathbb{Z}G$-module. In particular, H is a divisible subgroup of G and so $H \leq G^0$. On the other hand, $F(G)$ is a $\mathfrak{B}$-group. Then, applying Theorem 1.1, we have that $G^0 = (F(G))^0 \leq Z(F(G))$. Therefore $F(G) \leq C_G(H)$ and we may conclude that $F(G) \leq \bigcap\{C_G(H/K) : H/K$ is a δ-chief factor of G$\}$. Next, we denote $T = \bigcap\{C_G(H/K) : H/K$ is a δ-chief factor of G$\}$. We prove that $T \leq F(G)$. Since T is a normal subgroup of G, we need only to show that T is in the class $\mathfrak{B}$. As it has been said above, $\rho(G)$ is the intersection of the centralizers of all chief factors of G and, consequently, T is contained in $\rho(G)$. It follows that T is a locally nilpotent group. It remains to prove that T has nilpotent Sylow subgroups. Let T_p be the Sylow p-subgroup of T for the prime p. According to Lemma 0.2, T_p is a Chernikov group. Then ${T_p}^0$ has finite rank. We argue by induction on the rank of ${T_p}^0$. If ${T_p}^0 = 1$ then T_p is a finite p-group and, consequently, it is nilpotent. Then we may assume that ${T_p}^0$ is non-trivial. Consider the non-empty set $\mathcal{S} = \{A \leq {T_p}^0 :\ A$ is a non-trivial divisible G-invariant subgroup of $G\}$. Since G satisfies min-p and $\mathcal{S}$ is a non-empty set of p-subgroups of G, there exists a minimal element A in $\mathcal{S}$. Then A is a divisibly irreducible $\mathbb{Z}G$-module and hence it is a δ-chief factor of G. Consequently A is contained in the center of T. On the other hand, we have that $T/A = \bigcap\{C_{G/A}((H/A)/(K/A)) : (H/A)/(K/A)$ is a δ-chief factor of G/A$\}$. Moreover, since A is a non-trivial divisible subgroup, the rank of $T_p^0/A = (T_p/A)^0$ is less than the rank of T_p^0. By induction, T_p/A is nilpotent (note that T_p/A is the unique Sylow p-subgroup of T/A). Since $A \leq Z(T_p)$, we have that T_p is nilpotent. Therefore T is a $\mathfrak{B}$-group, as we wanted to see. $\square$

Let G be a $c\bar{\mathfrak{L}}$-group and consider M a major subgroup of G. Denote $M_G = \mathrm{Core}_G(M)$. Then, applying Theorem 0.4, the factor group G/M_G is either a finite soluble primitive group, if M is a maximal subgroup of G, or a semiprimitive group, and therefore a Chernikov group, if M is not maximal in G. This result motivates the following definition:

Definition 1.3. Suppose that G is a $c\bar{\mathfrak{L}}$-group and let M be a major subgroup of G. We define

$$D_M/M_G = \begin{cases} \mathrm{Soc}(G/M_G), & \text{if } M \text{ is a maximal subgroup of } G \\ (G/M_G)^0, & \text{if } M \text{ is not a maximal subgroup of } G \end{cases}$$

It is easily seen that in both cases $D_M/M_G = F(G/M_G)$, $(D_M/M_G) \cap (M/M_G) = 1$ and $C_{G/M_G}(D_M/M_G) = D_M/M_G$ for every major subgroup M of G.

Theorem 1.11. *Let G be a $c\bar{\mathfrak{L}}$-group. Then*

$$F(G) = \bigcap\{D_M : M \text{ is a major subgroup of } G\}.$$

Proof. Denote $R = \bigcap\{D_M : M$ is a major subgroup of $G\}$. Suppose that N is a nilpotent normal subgroup of G and consider a major subgroup M of G. Then $NM_G/M_G \leq F(G/M_G) = D_M/M_G$ and hence $N \leq D_M$ for every major subgroup M of G. We conclude that N is contained in R and thus $F(G) \leq R$.

We prove now that $R \leq F(G)$. Let M be a major subgroup of G. Since $R/(R \cap M_G) \cong RM_G/M_G \leq D_M/M_G$ is abelian we have $R' \leq M$. Therefore $R' \leq \mu(G) \cap R$ and hence [26, (5.1)] implies that R is locally nilpotent. Let p be a prime. Then $(R^0)_p$ is normal in G. If $(R^0)_p \leq M$ the $[R, (R^0)_p] \leq (R^0)_p \leq M$.

Assume now that $(R^0)_p$ is not contained in M. We have that $(R^0)_p M_G/M_G$ is a non-trivial divisible normal subgroup of G/M_G which is contained in D_M/M_G. If $(R^0)_p M_G/M_G$ is a proper subgroup of D_M/M_G then $(R^0)_p M_G/M_G$ is finite, a contradiction. It follows that $(R^0)_p M_G/M_G = D_M/M_G$. Since $\mathrm{C}_G(D_M/M_G) = D_M$ it follows that $R \leq D_M = \mathrm{C}_G((R^0)_p/((R^0)_p \cap M_G))$ and hence $[R,(R^0)_p] \leq (R^0)_p \cap M_G$. Therefore $[R,(R^0)_p] \leq (R^0)_p \cap M$ for each major subgroup M of G. It follows that $[R,(R^0)_p] \leq \mu(G) \cap (R^0)_p$. However $\mu(G)O_{p'}(G)/O_{p'}(G) \leq \mu(G/O_{p'}(G))$ which is finite by Proposition 0.2. Therefore $\mu(G) \cap (R^0)_p \cong (\mu(G) \cap (R^0)_p)O_{p'}(G)/O_{p'}(G)$ is also finite. Thus $[R,(R^0)_p]$ is finite and hence trivial, since it is a divisible group. This holds for all primes p and hence $R^0 \leq \mathrm{Z}(R)$. Theorem 1.1 implies that R is a normal $\mathfrak{B}$-subgroup of G so that $R \leq F(G)$, the $\mathfrak{B}$-radical of G, as required. □

Some consequences of the above theorem, concerning the influence of the structure of $G/\mu(G)$ on that of G, are now given. A classical result in finite groups asserts that $F(G)/\phi(G) = F(G/\phi(G))$, where $\phi(G)$ is the Frattini subgroup of G. The corresponding result in our context is:

Corollary 1.3. *Let G be a c$\bar{\mathfrak{L}}$-group. Then $F(G/\mu(G)) = F(G)/\mu(G)$.*

Proof. According to Theorem 1.2, $\mu(G)$ is a $\mathfrak{B}$-subgroup of G and then it is contained in $F(G)$. Obviously, $F(G)/\mu(G)$ is a normal $\mathfrak{B}$-subgroup of $G/\mu(G)$ and thus it is contained in $F(G/\mu(G))$. Conversely, let $T/\mu(G) = F(G/\mu(G))$. By Theorem 1.10, $F(G/\mu(G))$ is the intersection of the centralizers of all δ-chief factors of $G/\mu(G)$ and, therefore, T centralizes every δ-chief factor H/K of G such that $\mu(G) \leq K \leq H$. Note that D_M/M_G is a δ-chief factor of G such that $\mu(G) \leq M_G \leq D_M$ for each major subgroup M of G. Consequently, $T \leq \mathrm{C}_G(D_M/M_G) = D_M$ for every major subgroup M of G. Then, it follows from Theorem 1.11 that $T \leq F(G)$. □

On the other hand, it is known that if G is a finite group and N is a normal subgroup of G such that $N/(N \cap \phi(G))$ is nilpotent, then N is nilpotent. This result has been extended to a class of infinite groups by Tomkinson ([26]) replacing nilpotent by locally nilpotent and $\phi(G)$ by $\mu(G)$. The following result is in the same line.

Corollary 1.4. *Let N be a descendant subgroup of a $c\bar{\mathfrak{L}}$-group G containing $\mu(G)$. Then $N/\mu(G)$ is a $\mathfrak{B}$-group if and only if N is a $\mathfrak{B}$-group.*

Proof. Suppose that $N/\mu(G) \in \mathfrak{B}$. Applying Theorem 1.8 we have that $N/\mu(G) \leq F(G/\mu(G))$. Moreover, $F(G/\mu(G)) = F(G)/\mu(G)$ by Corollary 1.3. Therefore, $N \leq F(G)$ and consequently $N \in \mathfrak{B}$ because $\mathfrak{B}$ is subgroup-closed (Theorem 1.3). □

As a consequence of Theorem 1.1, we obtain the following description of the Fitting subgroup of a $c\bar{\mathfrak{L}}$-group G, which will be useful to find the injectors associated to the class $\mathfrak{B}$.

Theorem 1.12. *Suppose that G is a $c\bar{\mathfrak{L}}$-group. Then $F(G) = \rho(G) \cap C_G(G^0)$, where $\rho(G)$ denotes the Hirsch-Plotkin radical of G.*

Proof. Denote $X = \rho(G) \cap C_G(G^0)$. It is clear that X is locally nilpotent. Moreover, $G^0 = X^0$ because $G^0 \leq X$ and then $X^0 \leq Z(X)$. According to Theorem 1.1, X is a $\mathfrak{B}$-subgroup of G and thus X is contained in $F(G)$. On the other hand, $F(G)$ is a locally nilpotent normal subgroup of G and therefore it is contained in $\rho(G)$, the Hirsch-Plotkin radical of G. Moreover, G^0 is a normal abelian subgroup and thus $G^0 \leq F(G)$. Consequently, $G^0 = (F(G))^0$. Since $F(G)$ is a $\mathfrak{B}$-group, it follows from Theorem 1.1 that $G^0 \leq Z(F(G))$. We conclude that $F(G) \leq \rho(G) \cap C_G(G^0)$. □

As the locally dihedral 2-group shows, it is not true that the ascendant $\mathfrak{B}$-subgroups of a $c\bar{\mathfrak{L}}$-group G lie in the Fitting subgroup of G. However we can give a sufficient condition on an ascendant $\mathfrak{B}$-subgroup of G to be contained in $F(G)$.

Corollary 1.5. *Let G be a $c\bar{\mathfrak{L}}$-group. Suppose A is an ascendant $\mathfrak{B}$-subgroup of G such that $G^0 \leq A$. Then $A \leq F(G)$.*

Proof. By Theorem 1.1, A is an ascendant locally nilpotent subgroup of G and thus, applying [22, (12.1.4)], it is contained in $\rho(G)$, the Hirsch-Plotkin radical of G. On the other hand, $G^0 \leq A$ and therefore $G^0 = A^0$. Since $A \in \mathfrak{B}$, it follows from Theorem 1.1 that $A \leq C_G(G^0)$. Then we conclude, by Theorem 1.12, that A is contained in $F(G)$. □

It is known that for finite groups $\phi(N) \leq \phi(G)$, the Frattini subgroup of G, whenever N is normal in G. This result is clearly false for the subgroup $\mu(G)$ as is shown by taking G a quasicyclic p-group and N the subgroup of order p^2. Even for normal subgroups of finite index very little information was given by Tomkinson ([28]) in this direction. We obtain that, in our universe, the Fitting subgroup always satisfies this property.

Lemma 1.4. *Let G be a $c\bar{\mathfrak{L}}$-group and suppose that M is a non-maximal major subgroup of G. Then G^0 is not contained in M and hence $MG^0 = G$.*

Proof. Let M be a non-maximal major subgroup of G. Then G/M_G is semiprimitive and hence Chernikov. If $G^0 \leq M$ then $G^0 \leq M_G$ so G/M_G has finite Sylow subgroups by Kargapolov's Theorem (Theorem 0.1). Hence G/M_G is finite, a contradiction. Thus $G^0 \not\leq M$. Moreover, if we have that MG^0 is a proper subgroup of G, then MG^0 is a major subgroup of G which contains G^0. By the previous argument, it follows that MG^0 is a maximal subgroup of G and thus $m(M) = m(MG^0) = 1$. Consequently, M is maximal in G, a contradiction. We conclude that $MG^0 = G$, and the lemma follows. □

Theorem 1.13. *Let G be a $c\bar{\mathfrak{L}}$-group. Then $\mu(F(G)) \leq \mu(G)$.*

Proof. Let $X = F(G)$ and consider M a major subgroup of G such that $\mu(X)$ is not contained in M. If M is non-maximal in G, it follows from Lemma 1.4

that $MG^0 = G$ and, therefore, $D_M = G^0 M_G$. Moreover, since $G^0 \leq X \leq D_M$ by Theorem 1.11, we have that $D_M = XM_G$. Now, assume that M is maximal in G. This implies that D_M/M_G is a minimal normal subgroup of G/M_G. Since X is not contained in M and $X \leq D_M$ by Theorem 1.11, it follows that $D_M/M_G = XM_G/M_G$. In both cases, $\mu(D_M/M_G) = 1$ and, consequently, $\mu(XM_G/M_G) = 1$. Therefore, $\mu(X) \leq M_G$ because $\mu(X)(X \cap M_G)/(X \cap M_G) \leq \mu(XM_G/M_G) = 1$. This contradicts our assumption and thus $\mu(X) \leq M$ for each major subgroup M of G. Consequently, $\mu(F(G)) \leq \mu(G)$, and the theorem is proved. □

1.4 $\mathfrak{B}$-injectors

In infinite groups there are of course several generalizations of the definition of Fitting class based on normal, ascendant or serial subgroups. Using the first definition, we have already shown that $\mathfrak{B}$ is a $c\bar{\mathfrak{L}}$-Fitting class. On the other hand, applying Theorem 1.8, the subgroup generated by descendant $\mathfrak{B}$-subgroups of a $c\bar{\mathfrak{L}}$-group is a $\mathfrak{B}$-group. This latter result motivates another definition of Fitting class. We say that a subclass $\mathfrak{F}$ of $c\bar{\mathfrak{L}}$ is a *$c\bar{\mathfrak{L}}$-Fitting class* if it satisfies the following properties:

1. If $G \in \mathfrak{F}$ and H is a descendant subgroup of G, then $H \in \mathfrak{F}$.

2. If $G = \langle H_i : i \in I \rangle \in c\bar{\mathfrak{L}}$ and, for each $i \in I$ the subgroup H_i is a descendant $\mathfrak{F}$-subgroup of G, then $G \in \mathfrak{F}$.

Then, it is clear that $\mathfrak{B}$ is a $c\bar{\mathfrak{L}}$-Fitting class with this broader definition using descendant subgroups. Note that this is not true if we replace, in the definition of Fitting class, descendant subgroups by serial subgroups (which are in fact ascendant in a $c\bar{\mathfrak{L}}$-group). For instance, the locally dihedral 2-group is an example of a join of serial $\mathfrak{B}$-subgroups which is not a $\mathfrak{B}$-group.

Suppose that $\mathfrak{F}$ is a subclass of $c\bar{\mathfrak{L}}$ and $G \in c\bar{\mathfrak{L}}$. An *$\mathfrak{F}$-injector* of G is a subgroup V of G such that for all descendant subgroups H of G we have that $V \cap H \in \mathrm{Max}_{\mathfrak{F}}(H)$. We denote the set of all $\mathfrak{F}$-injectors of the group G by $\mathrm{Inj}_{\mathfrak{F}}(G)$. The nilpotent injectors of a finite soluble group have been characterized (see the book of Doerk and Hawkes [14]). A similar characterization of the locally nilpotent injectors of a radical $\mathfrak{L}$-group has been obtained by Dixon using serial subgroups (see [13, section (7.4)]). The aim of this section is to obtain the corresponding description of the $\mathfrak{B}$-injectors in the class $c\bar{\mathfrak{L}}$. First we obtain an analogous result to [13, (7.4.3)], characterizing the maximal $\mathfrak{B}$-subgroups containing the Fitting subgroup. The key point is that $\mathrm{C}_G(F(G)) \leq F(G)$ for every $c\bar{\mathfrak{L}}$-group G. This is a consequence of Lemma 0.9 because every $c\bar{\mathfrak{L}}$-group is hyperabelian.

Theorem 1.14. *Let G be a $c\bar{\mathfrak{L}}$-group. Suppose that V is a $\mathfrak{B}$-subgroup of G containing the Fitting subgroup $F(G)$ of G. Then V is a maximal $\mathfrak{B}$-subgroup of G containing $F(G)$ if and only if $V_p \in Syl_p(C_G((F(G))_{p'}) \cap C_G(G^0))$.*

Proof. By Theorem 1.1, $V = \mathrm{Dr}_p V_p$, where V_p is the unique Sylow p-subgroup of V for each prime p. Since $(F(G))_{p'} \leq V_{p'}$ and $V_p \leq \mathrm{C}_G(V_{p'})$ for each prime p, it follows that $V_p \leq \mathrm{C}_G((F(G))_{p'})$. On the other hand, $G^0 \leq F(G) \leq V$ and, therefore, $G^0 = V^0$. Applying Theorem 1.1, $V \leq \mathrm{C}_G(G^0)$ and, in particular, $V_p \leq \mathrm{C}_G(G^0)$ for each prime p. Then, we have that $V_p \leq \mathrm{C}_G((F(G))_{p'}) \cap \mathrm{C}_G(G^0)$ for each prime p. Suppose $V_p \leq W_p \in \mathrm{Syl}_p(\mathrm{C}_G((F(G))_{p'}) \cap \mathrm{C}_G(G^0))$. We shall show that $[W_p, W_q] = 1$ if $p \neq q$. Since $(F(G))_p \leq V_p \leq W_p \leq \mathrm{C}_G((F(G))_{p'})$, we have that $F(G)W_p = (F(G))_{p'}W_p$ and hence W_p is normal in $F(G)W_p$. Using the conjugacy of the Sylow p-subgroups of a $c\bar{\mathfrak{L}}$-group, we deduce that W_p is a characteristic subgroup of $F(G)W_p$. Furthermore, $[W_p, W_q] \leq [\mathrm{C}_G((F(G))_{p'}), \mathrm{C}_G((F(G))_{q'})] \leq \mathrm{C}_G(F(G)) \leq F(G)$ and then W_q normalizes $F(G)W_p$. Now, using symmetry and the fact that W_q is a characteristic subgroup of $F(G)W_q$, we have that $[W_p, W_q] \leq W_p \cap W_q = 1$. Thus $V = \mathrm{Dr}_p V_p \leq \mathrm{Dr}_p W_p$.

On the other hand, W_p is a locally nilpotent group such that $W_p \leq C_G(G^0)$ and then $(W_p)^0 \leq Z(W_p)$. Applying Theorem 1.1, we have that W_p is a $\mathfrak{B}$-group for each prime p and hence $\mathrm{Dr}_p W_p \in \mathfrak{B}$ because the class $\mathfrak{B}$ is N-closed (Theorem 1.5). By maximality of V as a $\mathfrak{B}$-subgroup of G, it follows that $V = \mathrm{Dr}_p V_p = \mathrm{Dr}_p W_p$ and consequently $V_p = W_p \in \mathrm{Syl}_p(C_G((F(G))_{p'}) \cap C_G(G^0))$, as we wanted to see. Conversely, we shall show that maximal $\mathfrak{B}$-subgroups containing the Fitting subgroup always exist using Zorn's Lemma. Let $\{G_i\}_{i \in I}$ be a chain of $\mathfrak{B}$-subgroups of G such that $F(G) \leq G_i$ for each $i \in I$ and consider $C = \bigcup_{i \in I} G_i$. We shall see that C is a $\mathfrak{B}$-group. Since $G^0 \leq F(G) \leq G_i$ and $(G_i)^0 \leq Z(G_i)$ by Theorem 1.1, it follows that $G^0 \leq Z(G_i)$ for each $i \in I$. As a consequence, $G^0 \leq Z(C)$ and, in particular, $C^0 \leq Z(C)$. Moreover, C is the union of locally nilpotent groups and thus it is locally nilpotent. Applying Theorem 1.1, C is a $\mathfrak{B}$-group. This implies, by Zorn's Lemma, that there exists maximal $\mathfrak{B}$-subgroups of G containing $F(G)$. If $V \leq W$, a maximal $\mathfrak{B}$-subgroup of G, then $V_p \leq W_p$. However, by the above, $W_p \in Syl_p(C_G((F(G))_{p'}) \cap C_G(G^0))$ so $W_p = V_p$ for all primes p and hence $W = V$. □

We can show that, in fact, the maximal $\mathfrak{B}$-subgroups of a $c\bar{\mathfrak{L}}$-group containing the Fitting subgroup are precisely the $\mathfrak{B}$-injectors. First we obtain a preliminary result. It deals with the situation in which we take a product of a $\mathfrak{B}$-subgroup of a descendant subgroup with the Fitting subgroup.

Lemma 1.5. *Let G be a $c\bar{\mathfrak{L}}$-group and W a $\mathfrak{B}$-subgroup of G. Suppose H is a descendant subgroup of G and that $F(H) \leq W \leq H$. Then $WF(G)$ is a $\mathfrak{B}$-subgroup of G.*

Proof. Let $\{H_\sigma : \sigma \leq \alpha\}$ be a descending series from G to H. Since $W \leq H$ we have that $[W, F(G) \cap H_\sigma] \leq [H_{\sigma+1}, H_\sigma] \leq H_{\sigma+1}$ and so $[W, F(G) \cap H_\sigma] \leq F(G) \cap H_{\sigma+1}$. Consequently, $WF(G) \cap H_{\sigma+1} = W(F(G) \cap H_{\sigma+1})$ is normal in $W(F(G) \cap H_\sigma) = WF(G) \cap H_\sigma$. Hence $\{WF(G) \cap H_\sigma : \sigma \leq \alpha\}$ is a descending series from $WF(G)$ to $WF(G) \cap H$. On the other hand, applying Theorem

1.8 and the fact that $\mathfrak{B}$ is subgroup-closed, it is easily seen that $F(G) \cap H = F(H)$. Thus $WF(G) \cap H = W(F(G) \cap H) = WF(H) = W$. Therefore W is a descendant subgroup of $WF(G)$. Since the join of descendant $\mathfrak{B}$-subgroups is a $\mathfrak{B}$-group by Theorem 1.8, we conclude that $WF(G)$ is a $\mathfrak{B}$-group. □

Theorem 1.15. *The maximal $\mathfrak{B}$-subgroups of a $c\bar{\mathfrak{L}}$-group G which contain the Fitting subgroup are precisely the $\mathfrak{B}$-injectors of G.*

Proof. Consider V a maximal $\mathfrak{B}$-subgroup of G containing $F(G)$. Since $G^0 \le F(G) \le V$ and $V^0 \le Z(V)$ by Theorem 1.1, it follows that V is contained in $C_G(G^0)$. Denote $C = C_G(G^0)$. We shall show that V is a maximal locally nilpotent subgroup of C. Suppose that W is a locally nilpotent subgroup of $C = C_G(G^0)$ such that $V \le W$. Then, $W^0 \le Z(W)$ and, by Theorem 1.1, W is a $\mathfrak{B}$-group. By maximality of V, we have that $V = W$. Moreover, $\rho(C) = \rho(G) \cap C = F(G)$ by Theorem 1.12. Therefore, V is a maximal locally nilpotent subgroup of C containing $\rho(C)$, the Hirsch-Plotkin radical of C. Applying [13, (7.4.5)], this implies that V is a locally nilpotent injector of C with the broadest definition using serial subgroups. In particular, $V \cap D \in \mathrm{Max}_{\mathrm{L}\mathfrak{N}}(D)$ for every descendant subgroup D of C. Let H be a descendant subgroup of G. We proceed to show that $V \cap H \in \mathrm{Max}_{\mathfrak{B}}(H)$. Suppose that W is a $\mathfrak{B}$-subgroup of G such that $V \cap H \le W \le H$. It is easy to show, by Theorem 1.8, that $F(H) = F(G) \cap H$. Therefore, by Lemma 1.5, $WF(G)$ is a $\mathfrak{B}$-subgroup of G. Since $G^0 \le WF(G)$, it follows from Theorem 1.1 that $WF(G) \le C_G(G^0)$. In particular, W is contained in C. On the other hand, $H \cap C$ is a descendant subgroup of C and then $V \cap (H \cap C) = V \cap H \in \mathrm{Max}_{\mathrm{L}\mathfrak{N}}(H \cap C)$. However, W is locally nilpotent, by Theorem 1.1, and $V \cap H \le W \le H \cap C$ because W is contained in C. Consequently, $W = V \cap H$ and thus $V \cap H \in \mathrm{Max}_{\mathfrak{B}}(H)$. We conclude that $V \in \mathrm{Inj}_{\mathfrak{B}}(G)$, which is the desired conclusion. In particular, applying these results, we can see that the $\mathfrak{B}$-injectors of a $c\bar{\mathfrak{L}}$-group are finitely conjugate (see [13, (5.3.6)]) and hence isomorphic by [13, (5.3.8)]. □

Chapter 2

A local approach to a class of locally finite groups

2.1 Introduction

A property of groups is said to be "local" if it is generalized in a form referring to a prime. An interesting problem in this context is to find out whether the original property can be described as the conjunction of all the local properties for all primes p. For instance, if we consider the property of finite groups of being nilpotent groups, a local version is that of being p-nilpotent groups. Recall that a finite group G is said to be p-nilpotent, for some prime p, if it has a normal p'-subgroup N such that G/N is p-subgroup. Obviously every finite nilpotent group is p-nilpotent and a finite group which is p-nilpotent for all prime p is nilpotent. In the same line, a similar approach to the class of locally nilpotent groups has been obtained by using the same definition of p-nilpotent groups (see [13, (1.3.5)]).

Our purpose in this chapter is to introduce and study a local version of another class of generalized nilpotent groups, the class $\mathfrak{B}$. In the first section we establish the local version of $\mathfrak{B}$-groups, called *$\mathfrak{B}_p$-groups*. We show that these groups play the same role in the universe $c\bar{\mathfrak{L}}$ as finite p-nilpotent groups do in the finite universe. In particular, some properties appear relating the p-Fitting subgroup and a new characteristic subgroup which is defined by intersections of certain types of major subgroups. This subgroup can be considered as the local version of the Frattini-like subgroup $\mu(G)$ introduced by Tomkinson. Moreover, in the second section we study the injectors associated to this class of generalized p-nilpotent groups in the universe $c\bar{\mathfrak{L}}$.

2.2 The subgroup $\mu_p(G)$ and the class $\mathfrak{B}_p$

A $c\bar{\mathfrak{L}}$-group G is said to be *p-nilpotent*, for some prime p, if it contains a normal Sylow p'-subgroup Q. In this case $Q = O_{p'}(G)$ and G/Q is a p-group. Recall that we have already defined $O_{p'p}(G)/O_{p'}(G) = O_p(G/O_{p'}(G))$. It is easy to see that if G is p-nilpotent then $G = O_{p'p}(G)$. Moreover, in every $c\bar{\mathfrak{L}}$-group G, $O_{p'p}(G)$ is the largest normal p-nilpotent subgroup of G.

This section is devoted to the study of a class of generalized p-nilpotent groups, in the universe $c\bar{\mathfrak{L}}$, which is a local version of the class $\mathfrak{B}$. Groups in this class are called *$\mathfrak{B}_p$-groups*. They play the same role in the universe $c\bar{\mathfrak{L}}$ as finite p-nilpotent groups do in the finite one. We shall prove that this class can be associated in a natural way to a local version of Tomkinson's Frattini-like subgroup. We begin with the following definition.

Definition 2.1. Let G be a $c\bar{\mathfrak{L}}$-group. We say that G is a *$\mathfrak{B}_p$-group*, for some prime p, if G is p-nilpotent and the Sylow p-subgroups of G are nilpotent.

It is clear that a finite group is a $\mathfrak{B}_p$-group if and only if it is p-nilpotent. Moreover, applying Theorem 1.1 and [13, (1.3.5)], a $c\bar{\mathfrak{L}}$-group G is in the class $\mathfrak{B}$ if and only if G is a $\mathfrak{B}_p$-group for every prime p.

Proposition 2.1. *Let G be a $c\bar{\mathfrak{L}}$-group and let p be a prime. Then G is a $\mathfrak{B}_p$-group if and only if G is p-nilpotent and P^0 is central in P for every Sylow p-subgroup P of G.*

Proof. Suppose that G is a $\mathfrak{B}_p$-group and let P be a Sylow p-subgroup of G. Since P is nilpotent it follows from [13, (1.5.12)] that P^0 is central in P. Conversely, suppose that G is p-nilpotent and for every Sylow p-subgroup P of G, P^0 is central in P. Obviously P is locally nilpotent and it is a Chernikov group by Lemma 0.2. Since $P^0 \leq Z(P)$ we conclude that P is nilpotent. □

Recall that a class $\mathfrak{F}$ of $c\bar{\mathfrak{L}}$-groups is said to be a *$c\bar{\mathfrak{L}}$-formation* if $\mathfrak{F}$ is Q-closed and it satisfies that every $c\bar{\mathfrak{L}}$-group G containing normal subgroups $\{N_i\}_{i\in I}$ such that $\bigcap_{i\in I} N_i = 1$ and $G/N_i \in \mathfrak{F}$ for every $i \in I$ also belongs to $\mathfrak{F}$. We can easily see that $\mathfrak{B}_p$ is a subgroup-closed $c\bar{\mathfrak{L}}$-formation and this fact will allow to show the existence of its corresponding radical

Theorem 2.1. *$\mathfrak{B}_p$ is a subgroup-closed $c\bar{\mathfrak{L}}$-formation for every prime p.*

Proof. The proof is straightforward using the same arguments to those used in Theorem 1.3 and the fact that the class of all p-nilpotent groups is a subgroup-closed $c\bar{\mathfrak{L}}$-formation ([13, (6.2.6)]). □

Theorem 2.2. *Let G be a $c\bar{\mathfrak{L}}$-group. Assume that H and K are two normal $\mathfrak{B}_p$-subgroups. Then HK is a $\mathfrak{B}_p$-group.*

Proof. Since H and K are p-nilpotent normal subgroups, it follows that HK is p-nilpotent. Let P be a Sylow p-subgroup of HK. By Corollary 0.1 we have that $P = (P \cap H)(P \cap K)$ because $P \cap H$ and $P \cap K$ are Sylow p-subgroups of

H and K, respectively, by Proposition 0.1. Since H and K are $\mathfrak{B}_p$-groups, we have that $P \cap H$ and $P \cap K$ are normal nilpotent subgroups of P. Consequently, we conclude that P is nilpotent by Fitting's Theorem ([22, (5.2.8)]). □

Theorem 2.3. *Every group $G \in c\bar{\mathfrak{L}}$ has a unique largest normal $\mathfrak{B}_p$-subgroup denoted by $\delta_{p'p}(G)$ and called the $\mathfrak{B}_p$-radical of G. Moreover, $F(G) = \bigcap_p \delta_{p'p}(G)$.*

Proof. We argue as in Lemma 1.3. It is easy to prove that if G is a Chernikov $c\bar{\mathfrak{L}}$-group which is the union of a totally ordered set of normal $\mathfrak{B}_p$-subgroups then G is a $\mathfrak{B}_p$-group. Using this fact and Theorem 2.1, we may apply Zorn's Lemma to construct a maximal normal $\mathfrak{B}_p$-subgroup of G, denoted by $\delta_{p'p}(G)$. It follows from Theorem 2.2 that $\delta_{p'p}(G)$ is the unique largest normal $\mathfrak{B}_p$-subgroup of G. Finally, since a $c\bar{\mathfrak{L}}$-group G is in the class $\mathfrak{B}$ if and only if G is a $\mathfrak{B}_p$-group for every prime p, we deduce that the Fitting subgroup of G can be described as the intersection of the $\mathfrak{B}_p$-radicals because it is the largest normal $\mathfrak{B}$-subgroup of G. □

It is known that in a finite group G, the p-nilpotent radical $O_{p'p}(G)$ is the intersection of the centralizers of all p-chief factors of G. Moreover, this result has been generalized in [13, (6.2.4)] to periodic locally soluble groups. We obtain an analogous result in $c\bar{\mathfrak{L}}$-groups connecting the subgroup $\delta_{p'p}(G)$ and the δ-chief factors which are p-groups.

Remark 2.1. Let T be a normal subgroup of a $c\bar{\mathfrak{L}}$-group and let P be a Sylow p-subgroup of T. Then $P^0 = (T^0)_p$, the unique Sylow p-subgroup of T^0.

Theorem 2.4. *Suppose that G is a $c\bar{\mathfrak{L}}$-group. Then $\delta_{p'p}(G)$ is the intersection of the centralizers of all δ-chief factors of G which are p-groups.*

Proof. The proof runs parallel to the proof of Theorem 1.10, the only difference being in the use of the above remark. Let H/K be a δ-chief factor of G such that H/K is a p-group. We prove that $\delta_{p'p}(G)$ is contained in

$C_G(H/K)$. Since $\delta_{p'p}(G)K/K \leq \delta_{p'p}(G/K)$, we may assume that $K = 1$. If H is a minimal normal subgroup of G, then $O_{p'p}(G)$ is contained in $C_G(H)$ by [13, (6.2.4)]. Therefore $\delta_{p'p}(G) \leq C_G(H)$ because $\delta_{p'p}(G) \leq O_{p'p}(G)$. Suppose now that H is a divisibly irreducible $\mathbb{Z}G$-module. In particular $H \leq G^0$ and hence $H \leq (G^0)_p$, the unique Sylow p-subgroup of G^0. On the other hand, $(G^0)_p \leq P$ where P is a Sylow p-subgroup of $\delta_{p'p}(G)$. Then, it follows from Proposition 2.1 that $(G^0)_p \leq P^0 \leq Z(P)$. In particular, H centralizes P. Furthermore, H centralizes $O_{p'}(\delta_{p'p}(G))$. Since $\delta_{p'p}(G) = PO_{p'}(\delta_{p'p}(G))$, we conclude $\delta_{p'p}(G) \leq C_G(H)$ as required. Conversely, let $T = \bigcap\{C_G(H/K) :$ H/K is a δ-chief factor of G which is a p-group$\}$. We prove that $T \leq \delta_{p'p}(G)$. As it has been said above, $O_{p'p}(G)$ is the intersection of the centralizers of all p-chief factors of G and, therefore, T is contained in $O_{p'p}(G)$. Consequently T is p-nilpotent. It remains to prove that every Sylow p-subgroup of T is nilpotent. Let T_p be a Sylow p-subgroup of T. According to Lemma 0.2, T_p is a Chernikov group. In particular $(T_p)^0$ has finite rank. We argue by induction on the rank of $(T_p)^0$. If $(T_p)^0 = 1$ then T_p is a finite p-group and, consequently, it is nilpotent. Then we may assume that $(T_p)^0$ is non-trivial. Consider the set $\mathcal{S} = \{A \leq (T_p)^0 :$ A is a non-trivial divisible normal subgroup of $G\}$. Applying Remark 2.1, we have that $(T_p)^0 = (T^0)_p$, the unique Sylow p-subgroup of T^0, and therefore $(T_p)^0$ is normal in G. Consequently, $(T_p)^0 \in \mathcal{S}$ and hence $\mathcal{S}$ is non-empty set of p-subgroups of G. Since G satisfies min-p, there exists a minimal element A in $\mathcal{S}$. Then A is a divisibly irreducible $\mathbb{Z}G$-module and hence it is a δ-chief factor of G. Consequently, $T \leq C_G(A)$, that is, A is contained in the center of T. On the other hand, we have that $T/A = \bigcap\{C_{G/A}((H/A)/(K/A)) :$ $(H/A)/(K/A)$ is a δ-chief factor of G/A$\}$. Moreover, since A is a non-trivial divisible subgroup, the rank of $(T_p)^0/A = (T_p/A)^0$ is less than the rank of $(T_p)^0$. By induction, T_p/A is nilpotent (note that T_p/A is a Sylow p-subgroup of T/A). Since $A \leq Z(T_p)$, we have that T_p is nilpotent. Therefore T is a $\mathfrak{B}_p$-group and so $T \leq \delta_{p'p}(G)$, as we wanted to see. □

In [20] Lafuente introduces a new characteristic subgroup $\phi_p(G)$ (where p is a prime) of a finite group G, satisfying that $\phi(G) = \bigcap_p \phi_p(G)$. Using this subgroup he obtains some results concerning maximal subgroups, Frattini and Fitting subgroups. Following this line of thought, we extend this concept to the universe $c\bar{\mathfrak{L}}$. A local version of Tomkinson's subgroup appears.

Definition 2.2. Let p be a prime and let G be a $c\bar{\mathfrak{L}}$-group. Denote $\mu_p(G) = \bigcap\{M$ major subgroup of $G : D_M/M_G$ is a p-group$\}$ if the intersection set is non-empty and $\mu_p(G) = G$ in other case.

Obviously, we deduce from the definition that $\mu_p(G)$ is a characteristic subgroup of G for every prime p and $\mu(G) = \bigcap_p \mu_p(G)$.

The subgroup $\phi_p(G)$ of a finite group G is contained in $O_{p'p}(G)$ ([20, (1.3)]). Moreover, in [20] it is proved that if G is a finite group then $O_{p'p}(G/\phi_p(G)) = O_{p'p}(G)/\phi_p(G)$ for some prime p. The following result and its corollary are the "$c\bar{\mathfrak{L}}$"-version.

Proposition 2.2. *Let T be a normal subgroup of a $c\bar{\mathfrak{L}}$-group G containing $\mu_p(G)$. If $T/\mu_p(G)$ is p-nilpotent, then T is p-nilpotent.*

Proof. Let $T_{p'}$ be a Sylow p'-subgroup of T. Then $T_{p'}$ is a Sylow p'-subgroup of $T_{p'}\mu_p(G)$. Since $T/\mu_p(G)$ is p-nilpotent, it follows that $T_{p'}\mu_p(G)$ is a normal subgroup of G. Moreover Sylow p'-subgroups of $T_{p'}\mu_p(G)$ are conjugate by Theorem 0.2. This implies that $G = \mathrm{N}_G(T_{p'})\mu_p(G)$. Assume that $\mathrm{N}_G(T_{p'})$ is a proper subgroup of G. Then there exists a major subgroup M of G such that $\mathrm{N}_G(T_{p'}) \leq M$. In particular $G = M\mu_p(G)$. Since M is a proper subgroup of G, it follows that $\mu_p(G)$ is not contained in M and consequently D_M/M_G is a p'-group. Assume that $(D_M/M_G) \cap (TM_G/M_G) = 1$. Then $TM_G/M_G \leq \mathrm{C}_{G/M_G}(D_M/M_G) = D_M/M_G$ and hence TM_G/M_G is a p'-group. This implies that $TM_G/M_G = T_{p'}M_G/M_G \leq M/M_G$ and therefore T is contained in M, a contradiction. Consequently, $(D_M/M_G) \cap (TM_G/M_G)$ is a non-trivial normal p'-subgroup of G/M_G

contained in TM_G/M_G. Hence $(D_M/M_G)\cap(TM_G/M_G) \leq T_{p'}M_G/M_G \leq M/M_G$ and so $(D_M/M_G)\cap(TM_G/M_G)$ is contained in $(M/M_G)\cap(D_M/M_G) = 1$, a contradiction. Consequently $\mathrm{N}_G(T_{p'}) = G$. In particular $T_{p'}$ is a normal subgroup of T and hence T is p-nilpotent. □

Corollary 2.1. *Let N be a normal subgroup of a $c\bar{\mathfrak{L}}$-group G. Then N is p-nilpotent if and only if $N/N \cap \mu_p(G)$ is p-nilpotent. In particular, $\mu_p(G)$ is contained in $O_{p'p}(G)$ and $O_{p'p}(G/\mu_p(G)) = O_{p'p}(G)/\mu_p(G)$.*

Our next objective is to get a similar result changing $O_{p'p}(G)$ by $\delta_{p'p}(G)$. The following extension of [20, (1.4)] turns out to be crucial.

Lemma 2.1. *Let G be a $c\bar{\mathfrak{L}}$-group and let p be a prime. Then $\mu_p(G)/O_{p'}(G) = \mu(G/O_{p'}(G))$. Therefore $\mu_p(G)/O_{p'}(G)$ is a finite p-group and the Sylow p-subgroups of $\mu_p(G)$ are nilpotent.*

Proof. Suppose first that $\mu_p(G) = G$, that is, for every major subgroup M of G, D_M/M_G is a p'-group. We will show that G is a p'-group. Suppose that $O_{p'}(G)$ is a proper subgroup of G. Let M be a major subgroup M of G containing $O_{p'}(G)$. On the other hand, applying Corollary 2.1, we have that $G = O_{p'p}(G)$ and so $G/O_{p'}(G)$ is a p-group. Consequently, G/M_G is a p-group, which contradicts that D_M/M_G is a non-trivial p'-group. We conclude that $O_{p'}(G) = G$ and the result is true in this case.

Assume now that $\mu_p(G)$ is a proper subgroup of G. We begin by proving that $O_{p'}(G) \leq \mu_p(G)$. Let M be a major a subgroup of G such that D_M/M_G is a p-group. Suppose that $O_{p'}(G)$ is not contained in M. Then $O_{p'}(G)M_G/M_G$ is a non-trivial normal p'-subgroup of G/M_G contained in $\mathrm{C}_{G/M_G}(D_M/M_G) = D_M/M_G$ which is a p-group, a contradiction. This implies that $O_{p'}(G) \leq M$ and therefore $O_{p'}(G) \leq \mu_p(G)$. We show that $\mu_p(G)/O_{p'}(G) \leq \mu(G/O_{p'}(G))$. Let $M/O_{p'}(G)$ be a major subgroup of $G/O_{p'}(G)$. Suppose that $\mu_p(G)$ is not contained in M. Then D_M/M_G is a p'-group. Since $\mu_p(G) \leq O_{p'p}(G)$ by Corollary

2.1, we have that $\mu_p(G)/O_{p'}(G)$ is a p-group and so $\mu_p(G)M_G/M_G$ is a non-trivial p-group. It follows that $\mu_p(G)M_G/M_G \leq C_{G/M_G}(D_M/M_G) = D_M/M_G$, a contradiction. Therefore $\mu_p(G)$ is contained in M and we conclude that $\mu_p(G)/O_{p'}(G) \leq \mu(G/O_{p'}(G))$. Let $T/O_{p'}(G) = \mu(G/O_{p'}(G))$. We shall prove that $T \leq \mu_p(G)$. Let M be a major subgroup of G such that D_M/M_G is a p-group. Then $O_{p'}(G) \leq M$ because $O_{p'}(G) \leq \mu_p(G)$. Therefore $M/O_{p'}(G)$ is a major subgroup of $G/O_{p'}(G)$ and so $T/O_{p'}(G) \leq M/O_{p'}(G)$. We conclude that T is contained in $\mu_p(G)$ and hence $\mu(G/O_{p'}(G)) \leq \mu_p(G)/O_{p'}(G)$ as claimed.

Applying Lemma 0.2 we have that $G/O_{p'}(G)$ is a Chernikov group. Therefore $\mu(G/O_{p'}(G))$ is finite by Proposition 0.2. On the other hand, by Corollary 2.1, $\mu_p(G)$ is p-nilpotent. Consequently $O_{p'}(G)$ is the Sylow p'-subgroup of $\mu_p(G)$ and every Sylow p-subgroup of $\mu_p(G)$ is finite and so nilpotent. □

As a consequence of the above lemma, we obtain that the subgroup $\mu_p(G)$ of a group G is contained in $\delta_{p'p}(G)$. We can say much more than this.

Theorem 2.5. *Let G be a $c\bar{\mathfrak{L}}$-group and let p be a prime. Then:*

(i) $\mu_p(G)$ *is contained in* $\delta_{p'p}(G)$.

(ii) $\delta_{p'p}(G/\mu_p(G)) = \delta_{p'p}(G)/\mu_p(G) = F(G/\mu_p(G))$.

(iii) $C_G(\delta_{p'p}(G)/\mu_p(G)) = \delta_{p'p}(G)$.

Proof. *(i)* The above lemma shows that $\mu_p(G)$ is a $\mathfrak{B}_p$-group and so $\mu_p(G)$ is contained in $\delta_{p'p}(G)$.

(ii) Let $T/\mu_p(G) = \delta_{p'p}(G/\mu_p(G))$. Applying Theorem 2.4 to $G/\mu_p(G)$ we deduce that T centralizes every δ-chief factor H/K of G such that H/K is a p-group and $\mu_p(G) \leq K \leq H$. Note that D_M/M_G is a δ-chief factor of G for every major subgroup M of G. Moreover, if D_M/M_G is a p-group we have that $\mu_p(G) \leq M_G$. Consequently, $T \leq C_G(D_M/M_G) = D_M$ for every major subgroup M of G such that D_M/M_G is a p-group. By Theorem 2.6 we

conclude that $T \leq \delta_{p'p}(G)$. Since the other inclusion is obvious, it follows that $\delta_{p'p}(G/\mu_p(G)) = \delta_{p'p}(G)/\mu_p(G)$. On the other hand, it is clear that $F(G/\mu_p(G))$ is contained in $\delta_{p'p}(G/\mu_p(G))$. We prove that $\delta_{p'p}(G)/\mu_p(G)$ is abelian. We may assume that $\mu_p(G) \neq G$, since otherwise $\delta_{p'p}(G)/\mu_p(G) = 1$ is abelian. Let M be a major subgroup of G such that D_M/M_G is a p-group. Applying Theorem 2.6, it follows that $\delta_{p'p}(G)/M_G \leq D_M/M_G$, which is an abelian group by definition. Since $\mu_p(G) = \bigcap\{M_G : D_M/M_G \text{ is a } p\text{-group}\}$ it follows that $\delta_{p'p}(G)/\mu_p(G)$ is abelian and hence it is contained in $F(G/\mu_p(G))$. We conclude that $\delta_{p'p}(G)/\mu_p(G) = F(G/\mu_p(G))$.

(iii) Applying Lemma 0.9, we have that $C_{G/\mu_p(G)}(F(G/\mu_p(G)))$ is contained in $F(G/\mu_p(G))$. Since $F(G/\mu_p(G)) = \delta_{p'p}(G)/\mu_p(G)$ is abelian, it follows that $C_{G/\mu_p(G)}(F(G/\mu_p(G))) = F(G/\mu_p(G))$. This means that $C_G(\delta_{p'p}(G)/\mu_p(G)) = \delta_{p'p}(G)$. □

Corollary 2.2. *Let N be a normal subgroup of a $c\bar{\mathfrak{L}}$-group G. Then N is a $\mathfrak{B}_p$-group if and only if $N/N \cap \mu_p(G)$ is a $\mathfrak{B}_p$-group.*

Corollary 2.3. *Let G be a $c\bar{\mathfrak{L}}$-group and let p be a prime. Then $C_G(\delta_{p'p}(G)) \leq \delta_{p'p}(G)$.*

Corollary 2.4. *Let G be a $c\bar{\mathfrak{L}}$-group. Then $\delta_{p'p}(G/O_{p'}(G)) = \delta_{p'p}(G)/O_{p'}(G)$ for every prime p.*

Proof. Our proof starts with the observation that $O_{p'}(G)$ is a $\mathfrak{B}_p$-group and so it is contained in $\delta_{p'p}(G)$. Let $T/O_{p'}(G) = \delta_{p'p}(G/O_{p'}(G))$. We shall prove that $T \leq \delta_{p'p}(G)$. From Lemma 2.1 and Theorem 2.5(i) we see that $\mu_p(G)/O_{p'}(G) = \mu(G/O_{p'}(G)) \leq \mu_p(G/O_{p'}(G)) \leq \delta_{p'p}(G/O_{p'}(G))$ and hence $\mu_p(G) \leq T$. Thus the group $T/\mu_p(G)$ is isomorphic to a quotient of $\delta_{p'p}(G/O_{p'}(G))$ and so it is a $\mathfrak{B}_p$-group. From Theorem 2.5(ii) we conclude that $T/\mu_p(G) \leq \delta_{p'p}(G/\mu_p(G)) = \delta_{p'p}(G)/\mu_p(G)$ and hence $T \leq \delta_{p'p}(G)$. □

Taking into account that the Fitting subgroup of a $c\bar{\mathfrak{L}}$-group G is the intersection of the members of the set $\{D_M : M$ is a major subgroup of $G\}$ (Theorem 1.11), it seems to be natural the following characterization of the $\mathfrak{B}_p$-radical of G.

Theorem 2.6. *Let G be a $c\bar{\mathfrak{L}}$-group and let p be a prime. Then*

$$\delta_{p'p}(G) = \bigcap\{D_M : D_M/M_G \text{ is a } p\text{-group}\}.$$

(In the case that the intersection above is the empty set, we have that $\delta_{p'p}(G) = G$)

Proof. Denote $T = \bigcap\{D_M : D_M/M_G$ is a p-group$\}$. Let N be a normal $\mathfrak{B}_p$-subgroup of G. We prove that $N \leq T$. Consider a major subgroup M of G such that D_M/M_G is a p-group. Then NM_G/M_G is a $\mathfrak{B}_p$-group and hence it has a unique Sylow p'-subgroup, $(NM_G/M_G)_{p'}$, which is normal in G/M_G. In particular $(NM_G/M_G)_{p'} \leq C_{G/M_G}(D_M/M_G) = D_M/M_G$. This implies that $(NM_G/M_G)_{p'} = 1$ and so NM_G/M_G is a p-group. Consequently, NM_G/M_G is nilpotent because it is a $\mathfrak{B}_p$-group. Therefore $NM_G/M_G \leq F(G/M_G) = D_M/M_G$ and hence $N \leq D_M$ for every major subgroup M of G. We conclude that N is contained in T. In particular, $\delta_{p'p}(G) \leq T$.

We prove now that $T \leq \delta_{p'p}(G)$. Assume that $\mu_p(G)$ is a proper subgroup of G. Let M be a major subgroup of G such that D_M/M_G is a p-group. Then $T/(T \cap M_G) \cong TM_G/M_G \leq D_M/M_G$ is abelian. Therefore $T/(T \cap \mu_p(G))$ is abelian and so it is p-nilpotent. Applying Corollary 2.1, it follows that T is p-nilpotent. It remains to prove, by Proposition 2.1, that if P is a Sylow p-subgroup of T then $P^0 \leq Z(P)$. According to Remark 2.1, $P^0 = (T^0)_p$, the unique Sylow p-subgroup of T^0 and hence P^0 is normal in G. Let M be a major subgroup of G with D_M/M_G p-group. If $P^0 \leq M$ then $[T, P^0] \leq P^0 \leq M$. Assume now that P^0 is not contained in M. Then P^0M_G/M_G is a non-trivial divisible normal subgroup of G/M_G. Moreover $P^0M_G/M_G \leq TM_G/M_G \leq D_M/M_G$.

If $P^0M_G/M_G = D_M/M_G$ then $T \leq D_M = C_G(P^0M_G/M_G) = C_G(P^0/(P^0 \cap M_G))$. Consequently $[T, P^0] \leq P^0 \cap M$. Asumme now that P^0M_G/M_G is a proper subgroup of D_M/M_G. Since D_M/M_G is divisibly irreducible, we have that P^0M_G/M_G is finite, a contradiction. Therefore, $[T, P^0] \leq P^0 \cap M$ for every major subgroup M of G such that D_M/M_G is a p-group and hence $[T, P^0] \leq P^0 \cap \mu_p(G)$. Moreover, $(P^0 \cap \mu_p(G))O_{p'}(G)/O_{p'}(G) \leq \mu_p(G)/O_{p'}(G) = \mu(G/O_{p'}(G))$ by Lemma 2.1. Since $\mu(G/O_{p'}(G))$ is finite by Proposition 0.2, it follows that $P^0 \cap \mu_p(G)$ is finite. In particular, $[T, P^0]$ is finite and hence trivial, since it is a divisible group. We conclude that $P^0 \leq \mathrm{Z}(P)$ and hence $T \leq \delta_{p'p}(G)$. Assume now that $\mu_p(G) = G$. Since $\mu_p(G) \leq O_{p'p}(G)$ by Corollary 2.1, we have that $G = O_{p'p}(G)$. Moreover, $G/O_{p'}(G) = \mu(G/O_{p'}(G))$ by Lemma 2.1 and hence $G/O_{p'}(G) = 1$. Therefore G is a p'-group. We conclude in this case that $G = \delta_{p'p}(G)$. □

In Chapter 1 we obtain a description of the Fitting subgroup of a $c\bar{\mathfrak{L}}$-group in terms of the locally nilpotent radical and the radicable part. Now we obtain the corresponding description of the $\mathfrak{B}_p$-radical of a $c\bar{\mathfrak{L}}$-group. This result will be very useful in the characterization of the $\mathfrak{B}_p$-injectors in the next section.

Lemma 2.2. *Let G be a group in the class $c\bar{\mathfrak{L}}$ and let p be a prime. Then $\delta_{p'p}(G) = O_{p'p}(G) \cap C_G((G^0)_p)$, where $(G^0)_p$ is the unique Sylow p-subgroup of G^0.*

Proof. Write $X = O_{p'p}(G) \cap C_G((G^0)_p)$. Obviously X is a normal p-nilpotent subgroup of G. Let P be a Sylow p-subgroup of X. Since G^0 is abelian we have that it is contained in X and so $(X^0)_p = (G^0)_p$. Moreover, according to the above remark, $P^0 = (X^0)_p$. Therefore $P^0 = (G^0)_p$. On the other hand, $P \leq X \leq C_G((G^0)_p)$. We conclude that $P^0 \leq \mathrm{Z}(P)$ and hence X is a $\mathfrak{B}_p$-group by Proposition 2.1. Conversely, every $\mathfrak{B}_p$-group is p-nilpotent and so $\delta_{p'p}(G) \leq O_{p'p}(G)$. Let us prove that $\delta_{p'p}(G) \leq C_G((G^0)_p)$. Let P be a Sylow p-subgroup of $\delta_{p'p}(G)$. Applying Proposition 2.1, we have that $P^0 \leq \mathrm{Z}(P)$. Since $G^0 \leq \delta_{p'p}(G)$

it follows that $G^0 = (\delta_{p'p}(G))^0$. As before, Remark 2.1 gives $P^0 = (G^0)_p$ and consequently $(G^0)_p \leq Z(P)$. On the other hand, $O_{p'}(\delta_{p'p}(G))$ is the Sylow p'-subgroup of $\delta_{p'p}(G)$ and so $\delta_{p'p}(G) = PO_{p'}(\delta_{p'p}(G))$ by Proposition 0.1. Since $(G^0)_p$ centralizes $O_{p'}(\delta_{p'p}(G))$ we conclude that $(G^0)_p$ centralizes $\delta_{p'p}(G)$ and so $\delta_{p'p}(G) \leq C_G((G^0)_p)$, which completes the proof. □

2.3 $\mathfrak{B}_p$-injectors

In infinite groups the definition of Fitting class is done in terms of the different generalizations of subnormality. In this work we will be concerned with Fitting classes defined using descendant subgroups. Working with this concept, it was shown in Section 1.4 that $\mathfrak{B}$ is a $c\bar{\mathfrak{L}}$-Fitting class. Moreover, it was obtained a similar description of the $\mathfrak{B}$-injectors in the class $c\bar{\mathfrak{L}}$ to the nilpotent injectors in the class of all finite soluble groups. We are now interested in studying the $\mathfrak{B}_p$-injectors in the class $c\bar{\mathfrak{L}}$. In this sense, we need to prove an analogous result to Theorem 1.8 in the class $\mathfrak{B}_p$ to ensure that this class of groups is a $c\bar{\mathfrak{L}}$-Fitting class with the definition using descendant subgroups.

Theorem 2.7. *Let H be a descendant $\mathfrak{B}_p$-subgroup of a $c\bar{\mathfrak{L}}$-group G. Then H is contained in $\delta_{p'p}(G)$.*

Proof. Since every serial subgroup of a $c\bar{\mathfrak{L}}$-group is ascendant by Lemma 0.8, it follows that H is an ascendant subgroup of G. Then, by [23, (1.31)], $H \leq O_{p'p}(G)$ because H is p-nilpotent. In particular H is descendant in $O_{p'p}(G)$. If $H \leq \delta_{p'p}(O_{p'p}(G))$ then $H \leq \delta_{p'p}(G)$. Therefore, there is no loss of generality in assuming that G is p-nilpotent. Applying Proposition 0.1, we have that $G = PO_{p'}(G)$ for every Sylow p-subgroup P of G. The task is now to show that if P is a Sylow p-subgroup of G then $F(P)$ is a Sylow p-subgroup of $\delta_{p'p}(G)$. Since G is a $c\bar{\mathfrak{L}}$-group , we have that P is a Chernikov group by Lemma 0.2 and hence, applying Corollary 1.1, $F(P)$ is nilpotent. It follows that $F(P)O_{p'}(G)$ is in the class $\mathfrak{B}_p$ and so $F(P)O_{p'}(G)$ is contained in $\delta_{p'p}(G)$ because it is normal in $PO_{p'}(G) = G$. On the other hand, $P \cap \delta_{p'p}(G)$ is nilpotent and so it is contained in $F(P)$. Consequently, $F(P) = P \cap \delta_{p'p}(G)$ and thus $\delta_{p'p}(G) = F(P)O_{p'}(G)$. We deduce that $F(P)$ is a Sylow p-subgroup of $\delta_{p'p}(G)$. On the other hand, since H is p-nilpotent it follows $H = P_H O_{p'}(H)$ where P_H is a Sylow p-subgroup of H. Moreover, there exists a Sylow p-subgroup P of G such that $P_H \leq P$ and hence $P_H = P \cap H$. Consequently, P_H is a nilpotent descendant subgroup of P and hence, applying Theorem 1.8, $P_H \leq F(P)$. Moreover $O_{p'}(H) \leq O_{p'}(G)$ by [23, (1.31)]. We conclude that $H = P_H O_{p'}(H) \leq F(P)O_{p'}(G) = \delta_{p'p}(G)$. □

Corollary 2.5. *Let H be a descendant subgroup of a $c\bar{\mathfrak{L}}$-group G. Then $\delta_{p'p}(G) \cap H = \delta_{p'p}(H)$.*

Proof. Obviously $\delta_{p'p}(G) \cap H$ is a normal $\mathfrak{B}_p$-subgroup of H and so it is contained in $\delta_{p'p}(H)$. Conversely, since H is descendant in G it follows that $\delta_{p'p}(H)$ is a descendant $\mathfrak{B}_p$-subgroup of G and hence, by Theorem 2.7, it is contained in $\delta_{p'p}(G)$. □

As a consequence of Theorem 2.7, the subgroup generated by descendant $\mathfrak{B}_p$-subgroups of a $c\bar{\mathfrak{L}}$-group is a $\mathfrak{B}_p$-group. We conclude that $\mathfrak{B}_p$ is a $c\bar{\mathfrak{L}}$-Fitting class. Note that this is not true if we replace, in the definition of a Fitting class, descendant subgroups by serial subgroups (which are in fact ascendant in a $c\bar{\mathfrak{L}}$-group). For instance, the locally dihedral 2-group is an example of a join of serial $\mathfrak{B}_2$-subgroups which is not a $\mathfrak{B}_2$-group.

Moreover, this group has ascendant $\mathfrak{B}_2$-subgroups which are not contained in the $\mathfrak{B}_2$-radical. Therefore, it is not true that ascendant $\mathfrak{B}_p$-subgroups of a $c\bar{\mathfrak{L}}$-group G lie in the $\mathfrak{B}_p$-radical of G. However, we can give a sufficient condition on an ascendant $\mathfrak{B}_p$-subgroup of G to be contained in the $\mathfrak{B}_p$-radical.

Proposition 2.3. *Suppose that A is an ascendant $\mathfrak{B}_p$-subgroup of a $c\bar{\mathfrak{L}}$-group G such that $(G^0)_p \leq A$, where $(G^0)_p$ is the unique Sylow p-subgroup of G^0. Then $A \leq \delta_{p'p}(G)$.*

Proof. According to [23, (1.31)], $A \leq O_{p'p}(G)$ because A is p-nilpotent. If we prove that $A \leq C_G((G^0)_p)$, the assertion follows by Lemma 2.2. By hypothesis $(G^0)_p \leq A$, so $(G^0)_p = (A^0)_p$. Moreover $(A^0)_p \leq P^0 \leq Z(P)$ where P is a Sylow p-subgroup of A. This implies that $P \leq C_G((G^0)_p)$. On the other hand, applying [23, (1.31)], $O_{p'}(A) \leq O_{p'}(G) \leq C_G((G^0)_p)$. Since $A = PO_{p'}(A)$ we conclude that $A \leq C_G((G^0)_p)$ and the proof is complete. □

Let $\mathfrak{F}$ be a subclass of $c\bar{\mathfrak{L}}$ and $G \in c\bar{\mathfrak{L}}$. Recall that an $\mathfrak{F}$-*injector* of G is a subgroup V of G such that for all descendant subgroups H of G we have that $V \cap H$ is $\mathfrak{F}$-maximal in H. The following result describes the injectors for the Fitting class of all p-nilpotent groups.

Theorem 2.8. *Let G be a $c\bar{\mathfrak{L}}$-group. Then*

$$Inj_{\mathfrak{S}_{p'}\mathfrak{S}_p}(G) = \{PO_{p'}(G) : P \text{ is a Sylow } p\text{-subgroup of } G\}.$$

Therefore the p-nilpotent injectors of G are exactly the maximal p-nilpotent subgroups of G containing $O_{p'p}(G)$ and they form a conjugacy class of subgroups of G.

Proof. Let P be a Sylow p-subgroup of G. We prove that $PO_{p'}(G)$ is a p-nilpotent injector of G. Let us first show that $PO_{p'}(G) \in \mathrm{Max}_{\mathfrak{S}_{p'}\mathfrak{S}_p}(G)$. Obviously $PO_{p'}(G)$ is p-nilpotent. Let W be a p-nilpotent subgroup of G such that $PO_{p'}(G) \leq W$. It follows that P is a Sylow p-subgroup of W and so $W = PO_{p'}(W)$. Therefore, it remains to prove that $O_{p'}(W) = O_{p'}(G)$. Since $O_{p'}(W)/O_{p'}(G) = O_{p'}(W/O_{p'}(G))$, there is no loss of generality in assuming that $O_{p'}(G) = 1$ and proving that $O_{p'}(W) = 1$. In particular, $O_p(G) = O_{p'p}(G)$. Moreover, by Lemma 0.2, G is a Chernikov group, that is $G = G^0A$ where A is a finite subgroup of G. We can certainly assume that $G^0 \neq 1$, since otherwise G is finite and soluble and then the result is true (see [18]). Since $O_{p'}(G) = 1$, we have that G^0 is a p-group. Therefore $O_{p'}(W) \cong O_{p'}(W)G^0/G^0$ is finite. Suppose, contrary to our claim, that $O_{p'}(W) \neq 1$. Write $A_p = O_p(G) \cap A$. Then $G^0 A_p = O_p(G)$. Since $O_p(G) \leq P \leq W$ it follows that $A_p \leq W$ and hence A_p normalizes $O_{p'}(W)$. From a result of coprime action of finite groups ([14, (A.12.5)]) we have that $O_{p'}(W) = [O_{p'}(W), A_p]\mathrm{C}_{O_{p'}(W)}(A_p)$. Moreover, $[O_{p'}(W), A_p] \leq O_p(G) \cap O_{p'}(W) = 1$. We deduce that $O_{p'}(W) = \mathrm{C}_{O_{p'}(W)}(A_p)$, that is $O_{p'}(W) \leq \mathrm{C}_G(A_p)$. On the other hand, $G^0 = [G^0, O_{p'}(W)]\mathrm{C}_{G^0}(O_{p'}(W))$ by Lemma 0.1. Since $G^0 \leq P \leq W$ we have that G^0 normalizes $O_{p'}(W)$ and so $[G^0, O_{p'}(W)] \leq G^0 \cap O_{p'}(W) = 1$. Consequently $G^0 = \mathrm{C}_{G^0}(O_{p'}(W))$, that is $O_{p'}(W) \leq \mathrm{C}_G(G^0)$. Since $G^0 A_p = O_p(G) = O_{p'p}(G)$ we conclude that $O_{p'}(W) \leq \mathrm{C}_G(O_p(G)) \leq O_p(G) \neq 1$, a contradiction. Therefore $O_{p'}(W) = 1$ and thus we have proved that $PO_{p'}(G)$ is a maximal p-nilpotent subgroup of G. We are now in position to show that if H is a descendant subgroup of G then $PO_{p'}(G) \cap H$ is $\mathfrak{S}_{p'}\mathfrak{S}_p$-maximal in H by the above argument. Since $P \cap H$ is a Sylow p-subgroup of H by Lemma 0.4, we have that $(P \cap H)O_{p'}(H)$ is a maximal p-nilpotent subgroup of H. Moreover, $O_{p'}(H) \leq O_{p'}(G)$. Therefore $(P \cap H)O_{p'}(H) = (O_{p'}(H)P) \cap H \leq (O_{p'}(G)P) \cap H$ and hence $(P \cap H)O_{p'}(H) = (O_{p'}(G)P) \cap H$ because $(O_{p'}(G)P) \cap H$ is p-nilpotent. We conclude that $(O_{p'}(G)P) \cap H$ is $\mathfrak{S}_{p'}\mathfrak{S}_p$-maximal in H and so we have that $PO_{p'}(G)$ is a p-nilpotent injector of G. We will be done if we show that every p-nilpotent injector of G appears in that way. Let I be a p-nilpotent injector of G. In particular we can write $I = O_{p'}(I)I_p$ where I_p is a Sylow p-subgroup of I. We show that $O_{p'}(I) = O_{p'}(G)$. Clearly, since I is a p-nilpotent injector of G, it follows that $O_{p'p}(G) \leq I$ and thus $O_{p'}(G) \leq O_{p'}(I)$. Because $O_{p'}(I)/O_{p'}(G) = O_{p'}(I/O_{p'}(G))$, there is no loss of generality in assuming that $O_{p'}(G) = 1$. Arguing as above we deduce that $O_{p'}(I) = 1$. Therefore $I = O_{p'}(G)I_p$ is contained in the p-nilpotent subgroup $O_{p'}(G)P$ where P is a Sylow p-subgroup of G. We conclude, by maximality of

I, that $I = O_{p'}(G)P$, which is our claim. In particular, we have obtained that the p-nilpotent injectors of G are conjugate in G. Furthermore, by the same method as above, it can be proved that every maximal p-nilpotent subgroup V of G containing $O_{p'p}(G)$ can be described as $V = PO_{p'}(G)$ where P is a Sylow p-subgroup of G, and consequently V is a p-nilpotent injector of G. Therefore, the maximal p-nilpotent subgroups of a group containing the p-nilpotent radical are precisely the p-nilpotent injectors. □

Note that, using the same arguments to those used in the proof of Theorem 2.8, we can replace descendant subgroups by ascendant subgroups to obtain the same description for p-nilpotent injectors with the definition of injector using ascendant subgroups.

We now proceed to show that the $\mathfrak{B}_p$-injectors of a $c\bar{\mathfrak{L}}$-group can be described in terms of the p-nilpotent injectors of one of its subgroups. We require a preliminary result. It deals with the situation in which we take a product of a $\mathfrak{B}_p$-subgroup of a descendant subgroup with the $\mathfrak{B}_p$-radical.

Lemma 2.3. *Let G be a $c\bar{\mathfrak{L}}$-group and let W be a $\mathfrak{B}_p$-subgroup of G. Suppose that H is a descendant subgroup of G and $\delta_{p'p}(H) \leq W \leq H$. Then $W\delta_{p'p}(G)$ is a $\mathfrak{B}_p$-subgroup of G.*

Proof. By Corollary 2.5 we have that $\delta_{p'p}(G) \cap H = \delta_{p'p}(H)$. Moreover, applying Theorem 2.7, the join of descendant $\mathfrak{B}_p$-subgroups is a $\mathfrak{B}_p$-group. Using these facts, the result follows by the same method as in Lemma 1.5. □

Theorem 2.9. *Let G be a $c\bar{\mathfrak{L}}$-group. Then the $\mathfrak{B}_p$-injectors of G are exactly the p-nilpotent injectors of $C = C_G((G^0)_p)$. In particular the $\mathfrak{B}_p$-injectors of G are conjugate in G.*

Proof. Denote $C = C_G((G^0)_p)$ and let V be a p-nilpotent injector of C. We show that V is a $\mathfrak{B}_p$-injector of G. According to Lemma 2.2, $\delta_{p'p}(G) = O_{p'p}(G) \cap$

$C = O_{p'p}(C)$. Of course, $O_{p'p}(C) \leq V$ because V is a p-nilpotent injector of C. It follows that $\delta_{p'p}(G) \leq V$. Let H be a descendant subgroup of G and suppose that W is a $\mathfrak{B}_p$-subgroup of H such that $V \cap H \leq W$. From Corollary 2.5, we have that $\delta_{p'p}(H) = \delta_{p'p}(G) \cap H \leq V \cap H \leq W \leq H$. This clearly forces that $W\delta_{p'p}(G)$ is a $\mathfrak{B}_p$-group by Lemma 2.3. Since $G^0 \leq \delta_{p'p}(G)$ we have that $(G^0)_p$ is contained in a Sylow p-subgroup P of $W\delta_{p'p}(G)$. Consequently $(G^0)_p \leq P^0 \leq \mathrm{Z}(P)$ because $W\delta_{p'p}(G)$ is a $\mathfrak{B}_p$-group. Thus $P \leq C$. Furthermore, $O_{p'}(W\delta_{p'p}(G)) \leq C$. Since $W\delta_{p'p}(G) = O_{p'}(W\delta_{p'p}(G))P$ we conclude that $W\delta_{p'p}(G) \leq C$. In particular, W is contained in C. On the other hand, $V \cap H = V \cap (H \cap C)$ is a maximal p-nilpotent subgroup of $H \cap C$ because V is a p-nilpotent injector of C. Since $V \cap H \leq W \leq H \cap C$ it follows that $V \cap H = W$. Therefore $V \cap H$ is a maximal $\mathfrak{B}_p$-subgroup of H, and we have proved that $V \in \mathrm{Inj}_{\mathfrak{B}_p}(G)$. Conversely, suppose that V is a $\mathfrak{B}_p$-injector of G. Let us first observe that V is contained in C. Clearly $G^0 \leq \delta_{p'p}(G) \leq V$. It follows that $(G^0)_p$ is normal in V, and consequently $O_{p'}(V) \leq \mathrm{C}_G((G^0)_p)$. Moreover $(G^0)_p \leq V_p$ where V_p is a Sylow p-subgroup of V and ${V_p}^0 \leq \mathrm{Z}(V_p)$. Therefore $V_p \leq C$. Since $V = O_{p'}(V)V_p$ we conclude that $V \leq C$. We may now prove that V is a p-nilpotent injector of C. Let D be a descendant subgroup of C and suppose that W is a p-nilpotent subgroup of D such that $V \cap D \leq W$. Using Remark 2.1, it is easy to check that W is a $\mathfrak{B}_p$-group. Since D is also descendant in G we deduce that $V \cap D = W$ because V is a $\mathfrak{B}_p$-injector of G. We conclude that V is a p-nilpotent injector of C as required. □

Finally, we can show that, in fact, the maximal $\mathfrak{B}_p$-subgroups of a $c\bar{\mathfrak{L}}$-group containing the $\mathfrak{B}_p$-radical are precisely the $\mathfrak{B}_p$-injectors.

Theorem 2.10. *The maximal $\mathfrak{B}_p$-subgroups of a $c\bar{\mathfrak{L}}$-group G which contain the $\mathfrak{B}_p$-radical are precisely the $\mathfrak{B}_p$-injectors of G.*

Proof. Obviously, every $\mathfrak{B}_p$-injector of G is a maximal $\mathfrak{B}_p$-subgroup of G and contains the $\mathfrak{B}_p$-radical of G. Conversely, let V be a maximal $\mathfrak{B}_p$-subgroup of

G such that $\delta_{p'p}(G) \leq V$. We wish to show that $V \in \mathrm{Inj}_{\mathfrak{B}_p}(G)$. By Theorem 2.9, we are reduced to proving that V is a p-nilpotent injector of $\mathrm{C}_G((G^0)_p)$. Write $C = \mathrm{C}_G((G^0)_p)$. Arguing as in the proof of Theorem 2.9, it is easily seen that $V \leq C$. We proceed to show that V is a maximal p-nilpotent subgroup of C. Clearly V is p-nilpotent because it is a $\mathfrak{B}_p$-group. Suppose that W is a p-nilpotent subgroup of C such that $V \leq W$. Using Remark 2.1, it is clear that W is a $\mathfrak{B}_p$-group. Therefore, since V is a maximal $\mathfrak{B}_p$-subgroup of G, it follows that $W = V$ and hence V is a maximal p-nilpotent subgroup of C. Furthermore, applying Lemma 2.2, $\delta_{p'p}(G) = O_{p'p}(G) \cap C = O_{p'p}(C)$. Therefore V is a maximal p-nilpotent subgroup of C containing $O_{p'p}(C)$. As a consequence of Theorem 2.8, we deduce that V is a p-nilpotent injector of C, which is the desired conclusion. □

Chapter 3

On products of generalized nilpotent groups

3.1 Introduction

If A and B are subgroups of a written group G, the product AB of A and B is defined to be the subset of all elements of G with the form ab where $a \in A$ and $b \in B$. It is well known from elementary group theory that AB is a subgroup if and only if $AB = BA$, i.e., the subgroups A and B are *permutable*. Should it happen that AB coincides with the group G, with the result that $G = AB = BA$, then G is said to be *factorized* by its subgroups A and B.

Factorized groups have played a significant part in the theory of groups over the past fifty years. The first prominent result is a theorem of Itô (1955) which states that every product of two abelian groups is metabelian, i.e., soluble with derived ≤ 2 (see [1, Ch.2]). After the appearance of Itô's Theorem attention shifted to finite groups that are products of a pair of nilpotent groups, the conjecture being that such subgroups are soluble. The motivation here was provided by the well-known Burnside's Theorem which states that a product of two subgroups with prime power orders is soluble. The outcome of this line of investigation was the celebrated result of Kegel and Wielandt ([1, (2.4.3)]) which shows that if A and B are nilpotent subgroups of a finite group $G = AB$, then G is soluble. Such groups have been widely studied by several authors (see [1, Ch.2]). To date Kegel and Wielandt's result has not been generalized to infinite groups. In fact, very little is known about the structure of a product of two nilpotent groups, even in the locally finite universe. Some progress in this direction was obtained in [15, 17], replacing nilpotence by local nilpotence.

The aim of this chapter is to investigate the structure of a $c\bar{\mathfrak{L}}$-group, $G = AB$, factorized by two subgroups A and B belonging to the class $\mathfrak{B}$ of generalized nilpotent groups and also in the class of locally nilpotent groups. We extend some results of products of finite nilpotent groups to the universe $c\bar{\mathfrak{L}}$.

3.2 Results

A group G is said to be *metanilpotent*, or $G \in \mathfrak{N}^2$, if there exists a nilpotent normal subgroup N of G such that the factor group G/N is nilpotent. It is well known that a product of two nilpotent groups is not metanilpotent in general.

Example 3.1. Let $G = \Sigma_4 = [C_2 \times C_2]\Sigma_3$ be the symmetric group of degree four. Consider $A = [C_2 \times C_2]C_2$ and $B = C_3$. Obviously A and B are nilpotent subgroups of $G = AB$. However, $F(G) = C_2 \times C_2$ and so $G/F(G)$ is isomorphic to the group Σ_3 which is not nilpotent. We conclude that $G = \Sigma_4$ is not metanilpotent.

However, in 1972 Maier shows that if the factors are finite and modular, then the group is metanilpotent ([21, Theorem 1]). Recall that a group G is said to be *modular* if the subgroup lattice of G is modular (see [25]). The following two results are extensions of Maier's one in the universe $c\bar{\mathfrak{L}}$.

We say that a group G is *locally nilpotent-by-locally nilpotent*, or $G \in (\mathbf{L}\mathfrak{N})^2$, if there exists a locally nilpotent normal subgroup N of G such that G/N is locally nilpotent. Analogously, we say that a group $G \in \mathfrak{B}^2$ if there exists a normal $\mathfrak{B}$-subgroup N of G such that G/N is a $\mathfrak{B}$-group. The following fact turns out to be crucial in the proof of our results.

Remark 3.1. Applying Theorem 1.3, it is easy to see that $\mathfrak{B}^2$ is a $c\bar{\mathfrak{L}}$-formation and, by Corollary 1.4, if G is a $c\bar{\mathfrak{L}}$-group such that $G/\mu(G) \in \mathfrak{B}^2$, then $G \in \mathfrak{B}^2$. The same is true for the class $(\mathbf{L}\mathfrak{N})^2$ (see [13, (6.2.11)]) and ([3, Theorem A]): $(\mathbf{L}\mathfrak{N})^2$ is a $c\bar{\mathfrak{L}}$-formation which is closed under extensions by Tomkinson's subgroup. Consequently a $c\bar{\mathfrak{L}}$-group G belongs to $\mathfrak{F}$ if and only if $G/\mathrm{Core}_G(M) \in \mathfrak{F}$ for every major subgroup M of G, where $\mathfrak{F}$ is either $(\mathbf{L}\mathfrak{N})^2$ or $\mathfrak{B}^2$.

Theorem 3.1. *Let $G = AB$ be a $c\bar{\mathfrak{L}}$-group, where A and B are modular locally nilpotent groups. Then $G \in (\mathbf{L}\mathfrak{N})^2$.*

Proof. Let M be a major subgroup of G and denote $M_G = \mathrm{Core}_G(M)$. It is clear that G/M_G is the product of the modular locally nilpotent subgroups AM_G/M_G and BM_G/M_G. Hence there is no loss of generality in assuming that $M_G = 1$ and then G is either a finite primitive soluble group or G is a semiprimitive group by Theorem 0.4. In the first case, we know that G is metanilpotent by Maier's Theorem. Consequently we may assume that G is a semiprimitive group. In particular, G is a Chernikov group. Then $G = [G^0]M$, where G^0 is a divisibly irreducible abelian p-group for some prime p such that $C_G(G^0) = G^0$ and M is a finite soluble group. Applying [17, (1.3)], we can assume that A is a p-group. Moreover, by [1, (3.2.10)], $G^0 = A^0B^0$. Let us first assume that A is non-abelian. By Iwasawa's theorem ([25, (2.4.14)]), A has finite exponent and then $A^0 = 1$. Since A is a Chernikov group, we have that A is finite. Consequently G^0 is contained in B and so in B_p, the unique Sylow p-subgroup of

B. If B_p is non-abelian, applying [25, (2.4.14)], B_p is finite and so $G^0 = 1$, a contradiction. We conclude that B_p is abelian. Therefore $B_p \leq C_G(G^0) = G^0$ and so $B_p = G^0$. Since $B_{p'} \leq C_G(B_p) = B_p$, we have that $B_{p'} = 1$. Consequently $B = B_p$ is a normal subgroup of G and the result follows. Assume now that A is an abelian group. Since B is a locally nilpotent group satisfying min, we know that B is a direct product of its Sylow subgroups and finitely many of them are non-trivial. Moreover, by [25, (2.4.14)], each non-abelian Sylow subgroup of B is finite and so nilpotent. Hence B is actually a nilpotent group. Since G is an $\mathfrak{S}_1$-group, it follows from [1, (7.2.2)] that AG^0 is normal in G and hence $AG^0 \leq O_p(G)$. Consequently, $G = O_p(G)B$ and we conclude that $G \in (\mathbf{L}\mathfrak{N})^2$. Therefore we have proved that $G/M_G \in (\mathbf{L}\mathfrak{N})^2$ for each major subgroup M of G and hence $G \in (\mathbf{L}\mathfrak{N})^2$ by Remark 3.1. □

One may ask whether theorem above is still true if we replace locally nilpotence by the concept of $\mathfrak{B}$-group. The answer to this question is affirmative as the following result show.

Theorem 3.2. *Let $G = AB$ be a $c\bar{\mathfrak{L}}$-group, where A and B are modular $\mathfrak{B}$-groups. Then $G \in \mathfrak{B}^2$.*

Proof. Using the same arguments as in the proof of Theorem 3.1, we may assume that G is either a finite primitive soluble group or G is a semiprimitive group. Again if G is finite the result follows by Maier's Theorem. Therefore we can assume that G is a semiprimitive group. In particular, G is a Chernikov group. Again $G = [G^0]M$, where G^0 is a divisibly irreducible abelian p-group for some prime p such that $C_G(G^0) = G^0$, M is a finite soluble group and we may assume that A is a p-group and $G^0 = A^0B^0$. If A is non-abelian we proceed as in Theorem 3.1. Assume now that A is an abelian group. Now, B, being a Chernikov $\mathfrak{B}$-group, is actually nilpotent by Corollary 1.1. Let B_p be the unique Sylow p-subgroup of B. If B_p is non-abelian, applying [25, (2.4.14)], B_p is finite and so $B^0 = 1$. Then $G^0 \leq A$ and hence $A \leq C_G(G^0) = G^0$. Therefore $A = G^0$

is a normal subgroup of G and the result follows. Assume now that B_p is abelian. By [15, (2.6)], AB_p is a Sylow p-subgroup of G, $B_{p'}$ is a Sylow p'-subgroup of G and $G = (AB_p)B_{p'}$. If $B_{p'} = 1$, applying Itô's Theorem ([1, (2.1.1)]), we conclude that G is metabelian and in particular $G \in \mathfrak{B}^2$. Hence we may assume that $B_{p'} \neq 1$. Let $\rho(G)$ be the Hirsch-Plotkin radical of G. By Theorem 3.1, $G/\rho(G)$ is locally nilpotent and, moreover, it is finite because G/G^0 is finite. Since $(\rho(G))_{p'} \leq O_{p'}(G) = 1$, we have that $\rho(G)$ is a p-group and so it is contained in the Sylow p-subgroup AB_p of G. On the other hand, since $G/\rho(G)$ is locally nilpotent, we deduce that $(AB_p)/\rho(G)$ is normal in $G/\rho(G)$. Consequently, AB_p is a locally nilpotent normal subgroup of G and hence $AB_p \leq \rho(G)$. This implies that $AB_p = \rho(G)$ and so $G = \rho(G)B_{p'}$. As a consequence $B_{p'}$ is a nilpotent finite subgroup of G. Let $M_{p'}$ be a Sylow p'-subgroup of M. Of course, $M_{p'}$ is a Sylow p'-subgroup of G. Therefore, there is no loss of generality in assuming that $M_{p'} = B_{p'}$. It follows that $M_{p'} = M_{q_1} \times M_{q_2} \cdots \times M_{q_s}$ is the product of its Sylow subgroups, where $q_j \neq p$ for all $j \in \{1, \ldots, s\}$. Suppose that $s > 1$ and let $B_j = B_p M_{q_j}$ for all $j \in \{1, \ldots, s\}$. We now proceed to show that $AB \in \mathfrak{B}^2$ by induction on $|B : B^0|$. Let $j \in \{1, \ldots, s\}$. Since $B^0 \leq B_p \leq B_j$, we have that $(B_j)^0 = B^0$. Moreover, $B_j \lneqq B$ because $s > 1$. We deduce that $|B_j : (B_j)^0| < |B : B^0|$ and hence, by induction, $AB_j \in \mathfrak{B}^2$, that is, $(AB_j)/F(AB_j) \in \mathfrak{B}$. Moreover, $F(AB_j) = (AB_j)^0 = G^0$ because $G^0 \leq \rho(G) \leq AB_j$. Consequently, $AB_j/G^0 = (\rho(G)M_{q_j})/G^0$ is a nilpotent finite group. In particular, $\rho(G)/G^0$ centralizes $M_{q_j}G^0/G^0$. This implies that $[\rho(G), M_{q_j}] \leq G^0$ and then $[\rho(G) \cap M, M_{q_j}] \leq G^0 \cap M = 1$. This clearly forces that M_{q_j} centralizes $\rho(G) \cap M = M_p$ for all $j \in \{1 \ldots s\}$. Therefore $M_{p'}$ centralizes M_p and hence M is nilpotent. We conclude that $G/F(G)$ is nilpotent and so $G \in \mathfrak{B}^2$. Assume now that $s = 1$, that is $M_{p'} = M_q$ where $q \neq p$. Since M_q is a finite nilpotent group, every maximal subgroup of M_q is normal. Let N_1, N_2 be two maximal subgroups of M_q, where $N_1 \neq N_2$. Then $M_q = N_1 N_2$. We can now proceed analogously to the proof of the case $s > 1$ to show that N_i centralizes $M \cap \rho(G) = M_p$, for

$i = 1, 2$. Therefore $M_q \leq C_G(M_p)$ and hence M is nilpotent. We obtain that $G/F(G)$ is nilpotent and consequently $G \in \mathfrak{B}^2$. If M_q has a unique maximal subgroup, then we deduce that $M_q = B_q$ is abelian and hence $B = B_p \times B_q$ is also abelian. By Itô's Theorem ([1, (2.1.1)]), we conclude that $G = AB$ is metabelian and, in particular, $G \in \mathfrak{B}^2$, which is our claim. □

Let $G = AB$. If A and B are normal nilpotent subgroups of G, then G is nilpotent by a result of Fitting. However, if A and B are normal supersoluble subgroups of $G = AB$, then G need not be supersoluble even in the finite case (see [2]). This fact motivates the interest in the study of factorized groups whose factors are connected by certain permutability properties (see [2, 9, 11]). Following the terminology of [9], we consider a functor f which to every group G associates a family of subgroups $f(G)$ such that $f(\alpha(G)) = \alpha(f(G))$ for every homomorphism α of G. The subgroups A and B in a group G are called a *mutually f-permutable* pair if A permutes with all members of $f(B) \cup \{B\}$ and B permutes with all member of $f(A) \cup \{A\}$. Further, A and B are called a *totally f-permutable* pair if every member of $f(A) \cup \{A\}$ permutes with every member of $f(B) \cup \{B\}$. Examples of functors include: $f(G) = s_n(G)$, the family of subnormal subgroups of G, $f(G) = \mathrm{der}(G)$, the terms of the derived series of G and $f(G) = s(G)$, the family of all subgroups of G. Thus the results of Asaad and Shaalan [2] involve totally s-permutable pairs of subgroups A and B. Beidleman, Galoppo, Heineken and Manfredino (see [9]) investigate the structure of a group $G = AB$ which is the product of either a certain mutually f-permutable pair or a certain totally f-permutable pair of subgroups A and B. They extend Asaad and Shaalan's results to infinite groups considering totally s-permutable and totally s_n-permutable pairs. Moreover, they prove that if $G = AB$ is a group factorized by the totally s_n-permutable pair of soluble subgroups A and B of derived length d_A and d_B respectively, and $d_A \leq d_B$, then G is soluble of derived length $d_G \leq 2d_A + d_B$.

The above result leads us to consider mutually der-permutable pairs in the universe $c\bar{\mathfrak{L}}$.

Theorem 3.3. *Let $G = AB$ be a $c\bar{\mathfrak{L}}$-group factorized by two $\mathfrak{B}$-subgroups A and B. If A permutes with B', and B permutes with A' (in particular if A and B are a mutually der-permutable pair), then $G \in \mathfrak{B}^2$.*

Proof. Assume first that G is a finite primitive soluble group. Then G has a unique minimal normal subgroup N and a maximal subgroup M such that $G = NM$, $\mathrm{Core}_G(M) = 1$, $N \cap M = 1$, N is a p-group for some prime p and $\mathrm{C}_G(N) = N$. Suppose that $G = AB$ is a product of two nilpotent groups A and B such that A permutes with B', and B permutes with A'. We prove that G is metanilpotent. Applying Gross' Lemma ([1, (2.5.2)]) we may assume that B is a p-group and A is a p'-group. In particular, $N \leq B$ and $A \cap B = 1$. Denote $T = AB'$. Then $(B')^G = (B')^A \leq AB' = T$. Assume that $(B')^G \neq 1$. This means that $N \leq (B')^G \leq T$ and so N is really contained in B' because B' is a Sylow p-subgroup of T. Hence $B = NM \cap B = N(M \cap B) = B'(M \cap B) = M \cap B$ because $B' \leq \phi(B)$. This is a contradiction. Consequently $(B')^G = 1$. In particular $B' = 1$ and hence B is abelian. Since $N \leq B$, it follows that $B \leq \mathrm{C}_G(N) = N$ and so $B = N$. In particular, we deduce that G is metanilpotent.

Suppose now that $G = AB$ is a semiprimitive group which is a product of two $\mathfrak{B}$-subgroups A and B such that A permutes with B', and B permutes with A'. Then $G = [G^0]M$, where $D = G^0$ is a divisibly irreducible abelian p-group for some prime p such that $\mathrm{C}_G(D) = D$ and M is a finite soluble group. In particular, A and B are Chernikov groups and thus, applying Proposition 0.2, $\mu(A)$ and $\mu(B)$ are finite. Moreover, since A and B are $\mathfrak{B}$-groups, it follows from Theorem 1.1 that $A' \leq \mu(A)$ and $B' \leq \mu(B)$. Therefore A' and B' are finite. Denote $N = (A')^G \cap D$. Since D is divisibly irreducible and N is normal in G it follows that either $N = D$ or N is finite. Suppose that N is a finite subgroup of D. Then it is easy to prove that $G/N = [D/N](MN/N)$ is a semiprimitive group satisfying the hypothesis of the theorem. If we prove that $G/N \in \mathfrak{B}^2$, then $G \in \mathfrak{B}^2$ because $F(G/N) = D/N = F(G)/N$. As a consequence, there is no loss of generality in assuming that if N is a proper subgroup of D, then $N = 1$. In the same manner, if we denote $L = (B')^G \cap D$, we can assume without loss of generality that if $L \neq D$, then $L = 1$. Suppose now that $N = D$. Then $D \leq (A')^G = (A')^B \leq A'B$ and hence $D \leq (A'B)^0$. By [1, (3.2.10)] and the fact that A' is finite we have that $(A'B)^0 = B^0$ and therefore $D = B^0$. Since $B^0 \leq \mathrm{Z}(B)$ by Theorem 1.1, we deduce that $B \leq \mathrm{C}_G(D) = D$ and thus $B = D$. We conclude that B is a normal subgroup of G and consequently $G \in \mathfrak{B}^2$. Similar arguments apply to the case $L = D$, yielding $A = D$ and $G \in \mathfrak{B}^2$. Then, we may assume that $N = 1 = L$. This implies that $(A')^G \leq \mathrm{C}_G(D) = D$ and $(B')^G \leq \mathrm{C}_G(D) = D$ and hence $(A')^G = 1 = (B')^G$. In particular, $A' = 1 = B'$ and therefore A and B are, in fact, abelian subgroups of G. By Itô's Theorem, we conclude that G is metabelian and, in particular, $G \in \mathfrak{B}^2$. Let M be a major subgroup of G. Denote $M_G = \mathrm{Core}_G(M)$. By Theorem 0.4, we have that either G/M_G is a finite primitive soluble group or G/M_G is a semiprimitive group. Moreover G/M_G satisfies the hypothesis of the theorem. By the above argument, we have that $G/M_G \in \mathfrak{B}^2$ for every major subgroup M of G. Therefore, applying Remark 3.1, we conclude that $G \in \mathfrak{B}^2$, which is our claim. □

We also may prove that the converse of Theorem 3.3 is true in the finite case.

Remark 3.2. The finite metanilpotent groups are exactly those groups $G = AB$ which are a product of two nilpotent subgroups A and B such that A permutes with B', and B permutes with A'.

Proof. Assume that G is a finite group which is metanilpotent. Then G is soluble. If A denotes the nilpotent residual of G, that is, the smallest normal subgroup A of G with G/A nilpotent, it follows that A is also nilpotent. Now, if B is a Carter subgroup of G (see [14, (III, 4.6)]), we have that $G = AB$. Moreover, it is clear that A permutes with B' and B permutes with A'. Consequently, a finite group G is metanilpotent if and only if G is a product of two nilpotent subgroups A and B such that A permutes with B', and B permutes with A'. □

A classical (and considerably hard) topic in the study of a product of two nilpotent finite groups has been trying to obtain bounds for the derived length of this kind of groups. In 1973, Pennington shows that there are strong restrictions on the $(c+d)$th term of the derived series of a finite group $G = AB$ where c and d are the classes of the nilpotent subgroups A and B, respectively (see [1, (2.5.3)]). This result has been generalized by Franciosi, De Giovanni and Sysak to periodic radical groups ([15, Theorem D]). Now we obtain a similar result of this kind, using the derived lenghts of the subgroups A and B instead of their classes, under some restrictions on the permutability of their derived series. Let us denote by d_G the derived length of a soluble group G.

Theorem 3.4. *Let $G = AB$ be a $c\bar{\mathfrak{L}}$-group factorized by the soluble $\mathfrak{B}$-groups A and B. Let d_A and d_B be the derived lengths of A and B respectively. If A permutes with B', and B permutes with A' (in particular, if A and B are a mutually der-permutable pair), then the (d_A+d_B)th-term $G^{(d_A+d_B)}$ of the derived series of G is a π-group in $\mathfrak{B}$, where $\pi = \pi(A) \cap \pi(B)$.*

Proof. Let the $c\bar{\mathfrak{L}}$-group $G = AB$ be the product of two soluble $\mathfrak{B}$-subgroups A and B such that A permutes with B', and B permutes with A'. Let d_A and d_B be the derived lengths of A and B respectively. By Lemma 0.2, $G/O_{p'}(G)$ is a soluble Chernikov group for every prime p. Moreover, $G/O_{p'}(G)$ satisfies the hypothesis. If we prove that $(G/O_{p'}(G))^{(d_A+d_B)}$ is a π-group in the class $\mathfrak{B}$ for every prime p, where $\pi = \pi(A) \cap \pi(B)$ then $G^{(d_A+d_B)}$ will be a π-group in the class $\mathfrak{B}$ because $\mathfrak{B}$ is a formation. Then, there is no loss of generality in assuming that G is a Chernikov group such that $O_{p'}(G) = 1$ for some prime p. Note that in a $c\bar{\mathfrak{L}}$-group G, the Fitting subgroup $F(G)$ is non-trivial. Since $O_{p'}(G) = 1$, we have that $F(G)$ is a p-group.

Suppose first that $p \notin \pi$. We may assume that $p \in \pi(A)$. Since A and B are locally nilpotent, it follows from [15, (2.6)] that A_pB_p is a Sylow p-subgroup of G, where A_p and B_p are the unique Sylow p-subgroups of A and B respectively. Moreover, since $p \notin \pi(B)$, we have that $B_p = 1$ and thus A_p is a Sylow p-subgroup of G. In particular, $F(G) \leq A_p$ and hence $A_{p'} \leq C_G(F(G))$. Since every $c\bar{\mathfrak{L}}$-group is hyperabelian, it follows from Lemma 0.9 that $C_G(F(G)) \leq F(G)$ and thus $A_{p'} = 1$. We deduce that A is a Sylow p-subgroup of G. On the other hand, it follows from Theorem 3.3 that $G/F(G) \in \mathfrak{B}$, whence the Sylow p-subgroup $A/F(G)$ is normal in $G/F(G)$. Consequently A is normal in G. In this case, we can write $d_G \leq d_{G/A} + d_B$ and being $G/A \cong B$ we deduce that $G^{(d_A+d_B)} = 1$ and the result follows.

Suppose now that $p \in \pi$. Let M be a major subgroup of G and denote $M_G = \mathrm{Core}_G(M)$. By Theorem 0.4, either G/M_G is a finite primitive soluble group or G/M_G is a semiprimitive group. Moreover, it satisfies the hypothesis. If we prove that $(G/M_G)^{(d_A+d_B)}$ is a π-group in the class $\mathfrak{B}$ for each major subgroup M of G then $(G/\mu(G))^{(d_A+d_B)}$ will be a π-group in the class $\mathfrak{B}$ because $\mathfrak{B}$ is a formation. Since $\mu(G)$ is a p-group, from Corollary 1.4 we will deduce that $G^{(d_A+d_B)}$ is a π-group in the class $\mathfrak{B}$. Then, without restriction of generality we can assume that G is either a finite primitive soluble group or a semiprimitive group. If G is a finite primitive soluble group, it follows from Gross' lemma that A and B have coprime orders. This contradicts the fact that $p \in \pi(A) \cap \pi(B)$. Consequently, we can assume that G is a semiprimitive group, that is, $G = [G^0]M$, where $D = G^0$ is a divisibly irreducible abelian p-group for some prime p such that $C_G(D) = D$ and M is a finite soluble group. As in the proof of Theorem 3.3, we consider $N = (A')^G \cap D$ and $L = (B')^G \cap D$. Since D is divisibly irreducible, then $N = D$ or N is finite and the same conclusion can be drawn for L. If $N = D$ then it follows, by the same method as in the proof of Theorem 3.3, that $B = D$ is a normal subgroup of G. Therefore we deduce that $G^{(d_A+d_B)} = 1$. In the same manner, if $L = D$ we can see that $A = D$ is normal in G and thus $G^{(d_A+d_B)} = 1$.

Suppose now that N is a finite subgroup of G. Then G/N is a semiprimitive group satisfying the hypothesis. Suppose that the result holds for G/N, that is, $(G/N)^{(d_A+d_B)}$ is a π-group in the class $\mathfrak{B}$. Obviously, since $N \leq D$ is a p-group and $p \in \pi$, we have that $G^{(d_A+d_B)}$ is a π-group. Now we prove that $G^{(d_A+d_B)} \in \mathfrak{B}$. Let $X = G^{(d_A+d_B)} \cap D$. Then X is a normal subgroup of G contained in D. Since D is divisibly irreducible, then either $X = D$ or X is finite. Suppose that $X = D$. Then $N \leq D \leq G^{(d_A+d_B)}$ and hence $G^{(d_A+d_B)} = D(M \cap G^{(d_A+d_B)})$. On the other hand, since $G^{(d_A+d_B)}N/N \in \mathfrak{B}$, its radicable part D/N is central in $G^{(d_A+d_B)}N/N$ by Theorem 1.1. Therefore $[D, G^{(d_A+d_B)}] \leq N$. In particular, since N is finite, $[D, M \cap G^{(d_A+d_B)}]$ is finite and therefore, applying Lemma 0.1, $M \cap G^{(d_A+d_B)} \leq \mathrm{C}_G(D) = D$. We conclude that $D = G^{(d_A+d_B)}$. Then $G^{(d_A+d_B)}$ is abelian and, in particular, it is a $\mathfrak{B}$-group. Assume now that X is finite and consider the semiprimitive group G/X. Since $(G^{(d_A+d_B)}/X) \cap (D/X) = 1$ we have that $G^{(d_A+d_B)}/X \leq \mathrm{C}_{G/X}(D/X) = D/X$ and hence $G^{(d_A+d_B)} \leq D$. In particular we conclude that $G^{(d_A+d_B)}$ is abelian and then it is a $\mathfrak{B}$-group. As a consequence of the above argument, there is no loss of generality in assuming that if N is a proper subgroup of D then $N = 1$. Using the same reasoning, we can assume without loss of generality that if $L \neq D$ then $L = 1$. This leaves the case $N = 1 = L$. Then $(A')^G \leq \mathrm{C}_G(D) = D$ and $(B')^G \leq \mathrm{C}_G(D) = D$ and, consequently, $A' = 1 = B'$. Then A and B are in fact abelian subgroups of G. By Itô's Theorem, we conclude that G is metabelian. That is $G^{(d_A+d_B)} = 1$ where $d_A = d_B = 1$, and the proof is complete. □

As a consequence of Theorem 3.4, we obtain that a $c\bar{\mathfrak{L}}$-group G which is a product of two soluble $\mathfrak{B}$-subgroups A and B such that A permutes with B', and B permutes with A' is soluble of derived length less or equal than $d_A + d_B$ provided that $\pi(A) \cap \pi(B) = \emptyset$.

It was proved by Kegel ([19]) that if the finite group $G = AB = AC = BC$ is the product of three nilpotent subgroups A, B and C, then G is nilpotent. This result has been extended by Franciosi, De Giovanni and Sysak to periodic radical groups under the condition that al least one of the subgroups is hyperabelian (see [15, Theorem B]). Our next theorem extends these results to $c\bar{\mathfrak{L}}$-groups and the class $\mathfrak{B}$ of generalized nilpotent groups.

Theorem 3.5. *Let the $c\bar{\mathfrak{L}}$-group $G = AB = AC = BC$ be the product of three $\mathfrak{B}$-subgroups A, B and C. Then G is a $\mathfrak{B}$-group.*

Proof. Let M be a major subgroup of G and denote $M_G = \mathrm{Core}_G(M)$. By Theorem 0.4, either G/M_G is a finite primitive soluble group or G/M_G is a semiprimitive group. Moreover, G/M_G satisfies the hypothesis. If we prove that $G/M_G \in \mathfrak{B}$ for each major subgroup M of G then $G/\mu(G)$ is a $\mathfrak{B}$-group. Consequently, G is a $\mathfrak{B}$-group by Corollary 1.3. Hence, there is no loss of generality in assuming that $M_G = 1$. Suppose first that G is a finite group such that $G = AB = AC = BC$, where A, B and C are nilpotent subgroups of G. By Kegel's Theorem, G is a nilpotent group. Then, we may assume that $G = AB = AC = BC$ is a semiprimitive group which is the product of three $\mathfrak{B}$-subgroups A, B and C. Since G has min, it follows from Corollary 1.1 that A, B and C are actually nilpotent groups. Moreover, the soluble-by-finite group G is an $\mathfrak{S}_1$-group. Therefore, it follows from [1, (6.6.7)] that G is nilpotent and so it is a $\mathfrak{B}$-group. □

A subgroup of a factorized group $G = AB$ will usually not be the product of a subgroup of A and a subgroup of B. Thus, a subgroup S of a factorized group $G = AB$ is said to be *factorized* if $S = (A \cap S)(B \cap S)$ and $A \cap B \leq S$. It is known that the Fitting subgroup of a finite product of two nilpotent groups is factorized (see [1, (2.5.7)]). We obtain the corresponding result in our universe.

Theorem 3.6. *Let the $c\bar{\mathfrak{L}}$-group $G = AB$ be the product of two $\mathfrak{B}$-subgroups A and B. Then the Fitting subgroup of G, $F(G)$, is factorized.*

Proof. Suppose that G is a finite group. We have that A and B are nilpotent groups and thus, applying [1, (2.5.7)], $F(G)$ is factorized. Assume now that G is a semiprimitive group. Since G satisfies min, we have that A and B are in fact nilpotent subgroups of G by Corollary 1.1. Moreover, G is a soluble $\mathfrak{S}_1$-group. Therefore, it follows from [1, (6.3.10)] that $F(G)$ is factorized. Let M be a major subgroup of G and denote $M_G = \mathrm{Core}_G(M)$. We have that either G/M_G is a finite primitive soluble group or G/M_G is a semiprimitive group. Applying the above arguments to G/M_G, we have that $D_M/M_G = F(G/M_G)$ is factorized for every major subgroup M of G. Thus, according to [1, (1.1.2)(iii)], D_M is factorized for every major subgroup M of G. But $F(G) = \bigcap\{D_M :$ M is a major subgroup of G$\}$ by Theorem 1.11. We conclude from [1, (1.1.2)(i)] that $F(G)$ is factorized. □

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