Capacitary function spaces and applications

Maria Pilar Silvestre Albero

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Capacitary function spaces and applications

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Maria Pilar Silvestre Albero

Directors del treball:

Dr. Joan Lluís Cerdà Martín
Catedràtic d'Anàlisi

Dr. Joaquim Martín Pedret
Titular d'universitat

Novembre 2011

Juan Luís Cerdà Martín, Full Professor at the
Department of Applied Mathematics and
Analysis of Barcelona's University, and
Joaquin Martín Pedret, Professor at
the Department of Mathematics of the
Autónoma University of Barcelona

Certify:

That this dissertation presented to the Faculty
of Mathematics of Barcelona's University by
Maria Pilar Silvestre Albero in fullfilment of
the requirements for the degree of Doctor of
Mathematics has been done by Maria Pilar.

Barcelona, 28 of November 2011

Juan Luís Cerdà Martín Joaquin Martín Pedret

To my parents

To my siblings

Never consider the study
as an obligation but as
an opportunity to enter
the beautiful and won-
derful world of learning.
Albert Einstein.

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Abstract/Resum

Abstract

The first part of the thesis is devoted to the analysis on a capacity space, with capacities as substitutes of measures in the study of function spaces. The goal is to extend to the associated function lattices some aspects of the theory of Banach function spaces, to show how the general theory can be applied to classical function spaces such as Lorentz spaces, and to complete the real interpolation theory for these spaces included in [CeClM] and [Ce].

In the second part of the thesis, we present an integral inequality connecting a function space norm of the gradient of a function to an integral of the corresponding capacity of the conductor between two level surfaces of the function, which extends the estimates obtained by V. Maz'ya and S. Costea, and sharp capacitary inequalities due to V. Maz'ya in the case of the Sobolev norm. The inequality, obtained under appropriate convexity conditions on the function space, gives a characterization of Sobolev type inequalities involving two measures, necessary and sufficient conditions for Sobolev isocapacitary type inequalities, and self-improvements for integrability of Lipschitz functions.

Resum

La primera part està dedicada a l'anàlisi d'un espai de capacitat, amb capacitats com a substituts de les mesures en l'estudi d'espais de funcions. L'objectiu és estendre als reticles de funcions associats alguns aspectes de la la teoria d'espais de funcions de Banach, mostrar com la teoria general pot ser aplicada a espais funcionals clàssics com els espais de Lorentz, i completar la teoria d'interpolació real d'aquests espais inclosos en [CeClM] i [Ce].

A la segona part de la tesi es presenta una desigualtat integral que connecta la norma del gradient d'una funció en un espai de funcions amb la integral de la corresponent capacitat del conductor entre dues superfícies de nivell de la funció, que estén les estimacions obtingudes per V. Maz'ya i S. Costea, i desigualtats capacitàries fortes de V. Maz'ya en el cas de la norma de Sobolev. La desigualtat, obtinguda sota condicions de convexitat pel espai funcional, permet una caracterització de les desigualtats de tipus Sobolev per dues mesures, condicions

necessàries i suficients per desigualtats isocapacitàries de tipus Sobolev, i la millora de l'auto-integrabilitat de les funcions de Lipschitz.

Introduction

The main concept in this thesis is the concept of capacity. A capacity, as a generic set theoretic measuring device, is intimately associated to the idea of a function space -in much the same way as Lebesgue measure is related to the usual L^p spaces.

A first model example is given by the *variational capacity*. For m be a positive integer, $1 \leq p \leq \infty$, and Ω be a domain in the Euclidean n-space $\mathbb{R}^n$, the variational capacity is defined as

$$C'_{m,p}(K) := \inf\{\|\phi\|^p_{W^{m,p}};\ \phi \in C_0^\infty,\ \phi \geq 1 \text{ on } K\},$$

where K is a compact subset of $\mathbb{R}^n$, $C_0^\infty(\mathbb{R}^n)$ denotes the class of all infinite continuously differentiable functions on $\mathbb{R}^n$ with compact support, and $W^{m,p} = W^{m,p}(\mathbb{R}^n)$ the classical *Sobolev space* with the usual norm

$$\|\phi\|^p_{W^{m,p}(\Omega)} = \sum_{|\sigma|\leq m} \int_\Omega |D^\sigma \phi|^p dx.$$

Capacity has classically entered in Analysis through removable singularity results and boundary regularity criteria. But nowadays the concept of a capacity has become much more a tool that is used in much the same way as measure is used. There is a desire to integrate with respect to a capacity as if it really were an additive set function - which is not. One way around this difficulty is to define such an integral using the distributional form of a Lebesgue integral. This was first proposed by Choquet in his seminal work on capacities [Ch], by defining for any measurable set E

$$\int_E f dC := \int_0^\infty C\{x \in E; f(x) > t\} dt,$$

where f is a non-negative function and C is a capacity. This new perspective provides a tool to extend the traditional integral of a function with respect to an additive measure - the Choquet integral.

Let (Ω, Σ) be a measurable space. Sets will always be assumed to be in Σ and functions in $\mathrm{L}_0(\Omega)$, the set of all real valued measurable functions on (Ω, Σ). A set function C defined on Σ is called a *capacity* if it satisfies at least the following properties:

(a) $C(\emptyset) = 0$,

(b) $0 \leq C(A) \leq \infty$,

(c) $C(A) \leq C(B)$ if $A \subset B$, and

(d) $C(A \cup B) \leq c(C(A) + C(B)) \qquad (c \geq 1) \qquad$ (*quasi-subadditive*).

It is called concave if

$$C(A \cup B) + C(A \cap B) \leq C(A) + C(B).$$

The *decreasing rearrangement* f_C^* of f is

$$f_C^\star(x) = \inf\{t > 0;\, C\{|f| > t\} \leq x\} \qquad (x > 0),$$

and a quasi-subadditive capacity C such that $C(A_n) \to C(A)$ whenever $A_n \uparrow A$ is called a *Fatou capacity.*

The first part of this thesis contains a wide study of the analytical and topological properties of the capacitary function spaces. The emphasis is placed upon the study of the essential functional analytic elements such that a satisfactory theory can be developed in the context of quasi-Banach spaces. The key results to study the properties of the capacitary Lebesgue and Lorentz spaces are Theorem 1.2.17 and Theorem 1.2.19, which are the extended versions of Fatou's lemma and Hölder and Minkowski's inequalities.

For a capacity C, a property is said to hold $C-$q.e. if the exceptional set has zero capacity, and we say that $\{f_n\}_{n\in\mathbb{N}} \subset \mathrm{L}_0(\Omega)$ *converges in capacity to* $f \in \mathrm{L}_0(\Omega)$ if $C\{|f_n - f| > \epsilon\} \to 0$ as $n \to \infty$, $\forall\, \epsilon > 0$. Similarly, we say that $\{f_n\}_{n\in\mathbb{N}}$ is a *Cauchy sequence in capacity* if for every $\epsilon > 0$, $C\{|f_p - f_q| > \epsilon\} \to 0$ as $p, q \to \infty$.

The capacitary Lorentz spaces $L^{p,q}(C)$ $(p, q > 0)$ are defined by the condition

$$\|f\|_{L^{p,q}(C)} := \begin{cases} \left(q \int_0^\infty t^{q-1} C\{|f| > t\}^{q/p} dt\right)^{1/q} < \infty, & q < \infty \\ \sup_{t>0} t C\{|f| > t\}^{1/p} < \infty, & q = \infty. \end{cases}$$

and it is the capacitary Lebesgue space $L^p(C) = L^{p,p}(C)$ when $q = p$.

Then, the third key result for our first objective: to set the basic properties of the Lebesgue $L^p(C)$ and Lorentz spaces $L^{p,q}(C)$ is the following one (see Theorem 1.3.11).

Theorem: A sequence $\{f_n\}_{n\in\mathbb{N}}$ is convergent in capacity to a function f if and only if it is a Cauchy sequence in capacity. In this case, the sequence has a subsequence which is $C-$q.e. convergent to f.

The key point in this result is a consequence for the capacity given by the Aoki-Rolewicz theorem, that is, if C is a capacity on (Ω, Σ) with constant $c \geq 1$ and $(2c)^{\varrho} = 2$, $f_i = \chi_{A_i}$, $i = 1, ..., n$ and $p = 1$, we obtain

$$C\Big(\bigcup_{i=1}^{\infty} A_i\Big)^{\varrho} \leq 2 \sum_{i=1}^{\infty} C(A_i)^{\varrho}.$$

We show how the general theory can be applied to function spaces. As an application of these results we show that the capacitary Lebesgue and Lorentz spaces are complete (see Theorems 1.3.12 and 1.3.15).

We study also the normability of these spaces: Let μ be a measure on (Ω, Σ) such that $\mu(\Sigma) = [0, \mu(\Omega)] \subset [0, \infty]$, and let us suppose that C is *μ-invariant*, this meaning that $C(A) = C(B)$ if $\mu(A) = \mu(B)$.

Definition: A capacity C on (Ω, Σ) is called *quasi-concave with respect to* μ if there exists a constant $\gamma \geq 1$ such that, whenever $\mu(A) \leq \mu(B)$, the following two conditions are satisfied:

(a) $C(A) \leq \gamma\, C(B)$, and

(b) $\frac{C(B)}{\mu(B)} \leq \gamma \frac{C(A)}{\mu(A)}$,

this is, for all $A, B \in \Sigma$,

$$C(B) \leq \gamma \max\Big(1, \frac{\mu(B)}{\mu(A)}\Big) C(A).$$

In the study of the normability of the capacitary Lebesgue spaces, a result is Theorem 1.4.4:

Theorem: If the capacity C is μ-invariant and quasi-concave with respect to μ, then

$$\widetilde{C}(A) := \sup\Big\{ \sum_{i=1}^{n} \lambda_i C(A_i);\ n \in \mathbb{N},\ \sum_{i=1}^{n} \lambda_i = 1,\ \lambda_i \geq 0,\ \sum_{i=1}^{n} \lambda_i \mu(A_i) \leq \mu(A) \Big\}$$

defines a concave capacity which is equivalent to C, i.e. $C \simeq \widetilde{C}$[1].

Let C be a quasi-subadditive Fatou capacity, quasi-concave with respect to μ. We will see that $L^p(C)$ is normable for $1 \leq p \leq \infty$. For that we define $\widetilde{C}$ as before and $\bar{C}$ by

$$\bar{C}(A) := \inf_{A_n \uparrow A,\, A_n \in \Sigma} \Big\{ \lim_{n \to \infty} \widetilde{C}(A_n) \Big\}.$$

[1] In all this memoir, the symbol $f \lesssim g$ will mean that there exists a universal constant $c > 0$ (independent of all parameters involved) such that $f \leq cg$, and the symbol $f \simeq g$ will mean that $f \lesssim g \lesssim f$.

Then we will show (see Proposition 1.4.7) that, $\bar{C}$ is a concave Fatou capacity equivalent to C. Therefore in Theorem 1.4.8 we see that if $\widetilde{C}$ has the Fatou property, then $\bar{C} = \widetilde{C}$, and hence, $L^p(C) = L^p(\bar{C})(1 \leq p \leq \infty)$, which is normable.

Since we are forced to work with a non-additive integral -the Choquet integral- the dual spaces are not easily identifiable. However, we will see in Theorem 1.5.5 that for every $1 \leq p \leq \infty$, q be the conjugate exponent of p, and C a quasi-subadditive Fatou capacity, the associate space of $L^p(C)$ is $L^q(C)$.

As it is known, interpolation of operators has many applications in different areas of mathematics. In interpolation theory of linear operators, couples (B_0, B_1) and (A_0, A_1) of Banach spaces, continuously contained in some Hausdorff topological vector space, and linear operators $T : A_0 + A_1 \to B_0 + B_1$ are considered. An interpolation method builds new Banach spaces A and B, $A \hookrightarrow A_0 + A_1$ and $B \hookrightarrow B_0 + B_1$, such that if $T : A_0 \to B_0$ and $T : A_1 \to B_1$ continuously, then also $T : A \to B$. We say that A and B are interpolation spaces for (A_0, A_1) and (B_0, B_1).

The classical results which provided the main impetus for the study of interpolation in se are the theorems of M. Riesz and of Marcinkiewicz. The way of proving Marcinkiewicz's theorem turned into the starting point to origin the definition of the real method of interpolation. The proof allowed to J. L. Lions and J. Peetre (see [LiP, LiP1]) to construct this method. In particular, if $\bar{A} = (A_0, A_1)$ is a couple of Banach spaces, $0 < \theta < 1$ and $0 < q \leq \infty$, the interpolation space $\bar{A}_{\theta,q}$ is the Banach space of all $f \in A_0 + A_1$ such that

$$\|f\|_{\theta,q} := \Big(\int_0^\infty (t^{-\theta} K(t, f; \bar{A}))^q \frac{dt}{t} \Big)^{1/q} < \infty,$$

where $K(t, f; \bar{A})$ is the *K-functional* defined by

$$K(t, f; \bar{A}) := \inf \Big\{ \|f_0\|_{A_0} + t\|f_1\|_{A_1};\ f = f_0 + f_1 \Big\}.$$

We refer to [BeSh], [BK] and [BeLo] for general facts concerning interpolation theory.

We will say that a set A in (Ω, Σ) is a null set for the capacity C if $C(A) = 0$, and that two capacities C_0, C_1 in (Ω, Σ) have the same null sets if for every $A \in \Omega$, $C_0(A) = 0$ if and only if $C_1(A)$.

First we obtain the description of $K(t, f, L^p(C), L^\infty(C))$ for a quasi-subadditive Fatou capacity on (Ω, Σ), $0 < p < \infty$ and $t > 0$. With this formulas real interpolation follows easily as in the classical case. In [Ce] it is proved that, for (C_0, C_1) be a couple of concave Fatou capacities on (Ω, Σ) with the same null sets and $0 < \eta < 1$, if $1 < p_0, p_1 < \infty$, $1 \leq q_0, q_1 < \infty$, $1/q = (1 - \eta)/q_0 + \eta/q_1$, and $1/p = (1 - \eta)/p_0 + \eta/p_1$, then

$$(L^{p_0,q_0}(C_0), L^{p_1,q_1}(C_1))_{\eta,q} = L^{p,q}(C_{\eta p/p_1, q/p}),$$

where $C_{\theta,q}(A) := \|\chi_A\|_{(L(C_0),L(C_1))_{\theta,q}}$. Here we extend this result to a more general class of capacities (see Theorem 2.5.12). The capacities will be still supposed to be Fatou but the Choquet integral will not be necessarily subadditive anymore, and parameters between zero and one are also allowed.

Our main problem is then interpolation with change of capacities. We want to determine for convenient parameters the interpolation space

$$(L^{p_0,q_0}(C_0), L^{p_1,q_1}(C_1))_{\eta,q}$$

and, in particular we want to study

$$(L^{p_0}(C_0), L^{p_1}(C_1))_{\eta,q}.$$

Since $L^{p_i}(C_i) = (L^{\alpha_i}(C_i), L^\infty)_{\theta_i,p_i}$ for $\alpha_i = (1-\theta_i)p_i$, we want to determine

$$((L^{\alpha_0}(C_0), L^\infty)_{\theta_0,p_0}, (L^{\alpha_1}(C_1), L^\infty)_{\theta_1,p_1})_{\eta,q}. \tag{1}$$

One, in an earlier attempt, tries to apply classical reiteration theorems but we can not do it because we have spaces with different capacities.

For $0 < p < \infty$ and w be a weight in $\mathrm{L}_0(\Omega)^+$, the Lorentz space $L^p(w)$ is defined with the quasi-norm

$$\|f\|_{L^p(w)} := \Big(\int_\Omega |f|^p w^p d\mu\Big)^{1/p}.$$

In the classical case Stein and Weiss proved that for $0 < p \leq \infty$ and w_0, w_1 weights in $\mathrm{L}_0(\Omega)^+$,

$$(L^p(w_0), L^p(w_1))_{\eta,p} = L^p(w_0^{1-\eta} w_1^\eta).$$

Moreover, we will see in Chapter 2 that $(\Lambda^p(w_0), \Lambda^p(w_1))_{\eta,p} = \Lambda^p(w)$ with

$$W = W_0^{1-\eta} W_1^\eta.$$

To deal with this problem in the case of capacities one suspects that

$$(L^p(C_0), L^p(C_1))_{\eta,p} = L^p(C_0^{1-\eta} C_1^\eta).$$

Observe that in (1) three spaces appear, namely $L^{\alpha_0}(C_0)$, $L^{\alpha_1}(C_1), L^\infty$. In [CeClM] the same happens but in the Banach case, studied previously by other authors, and then, the result follows as an application of those studies. So, it is natural to try to apply an extension of Sparr's method for triples of Banach spaces (see [AK]).

Let $n = 2$, $p_i, q_i \in (0, \infty]$ and C_i be quasi-subadditive Fatou capacities on (Ω, Σ), $i = 0, 1, 2$, with subadditivity constants $c_i \geq 1$, such that for an arbitrary set $A \subset \Omega$, then

$$C_0(A) = 0 \Longleftrightarrow C_1(A) = 0 \Longleftrightarrow C_2(A) = 0.$$

Fix $\bar{X} = (L^{p_0,q_0}(C_0), L^{p_1,q_1}(C_1), L^{p_2,q_2}(C_2))$ and denote by $\mathbb{R}^2_+$ the set of vectors $\mathbf{t} = (t_1, t_2)$ for which $t_i > 0$, $i = 1, 2$. We will try to extend the construction in [AK] to our quasi-Banach triple. As usual, for elements $x \in \Sigma(\bar{X})$, *Peetre's K-functional* of the 3-tuple $\bar{X}$ is defined for $\mathbf{t} \in \mathbb{R}^2_+$ by the formula

$$K(\mathbf{t}, x; \bar{X}) = \inf \Big\{ \|x_0\|_{L^{p_0,q_0}(C_0)} + \cdots + t_2 \|x_2\|_{L^{p_2,q_2}(C_2)}; x = \sum_{i=0}^{2} x_i \ x_i \in L^{p_i,q_i}(C_i) \Big\}.$$

Let $\varrho \in (0, 1]$ be the parameter in Aoki-Rolewicz's theorem corresponding to a common constant $c := \max(c_0, c_1, c_2)$ in the triangle inequality for the quasi-Banach spaces in $\bar{X}$, $p_i, q_i \in (0, \infty]$, $i = 0, 1, 2$. We define S_ρ, a modified *Calderón operator*, by the formula

$$(S_\varrho f)(\mathbf{t}) := \left(\int_{\mathbb{R}^2_+} \left[\min\left(1, \frac{t_1}{s_1}, \frac{t_2}{s_2}\right) f(\mathbf{s}) \right]^\varrho \frac{ds_1}{s_1} \frac{ds_2}{s_2} \right)^{1/\varrho} \quad (\mathbf{t} \in \mathbb{R}^2_+),$$

with $(2c)^\varrho = 2$ and consider the space

$$\sigma_\varrho(\bar{X}) := \Big\{ f \in \Sigma(\bar{X});\ S_\varrho(S_\varrho K(\cdot, f; \bar{X}))(\mathbf{1})^\varrho < \infty \Big\}.$$

The *interpolation space* $\bar{X}_{\Theta,q;K}$ is defined, for $\Theta = (\theta_0, \theta_1)$ with $\theta_0, \theta_1 > 0$ and $\theta_0 + \theta_1 < 1$, by the condition

$$\|f\|_{\Theta,q;K} = \|K(\cdot, f; \bar{X})\|_{\Theta,q} < \infty,$$

where

$$\|g\|_{\Theta,q} := \Big(\int_0^\infty \int_0^\infty (t_1^{-\theta_0} t_2^{-\theta_1} f(t_1, t_2))^q \frac{dt_1}{t_1} \frac{dt_2}{t_2} \Big)^{1/q} \quad (0 < q < \infty),$$

and the *J-space* $\bar{X}_{\Theta,q;J}$ is defined as

$$\|f\|_{\Theta,q;J} := \inf \Big\{ \Big(\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} (2^{-m\theta_0} 2^{-n\theta_1} J(2^m, 2^n, u_{mn}))^q \Big)^{1/q}; f = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} u_{mn} \Big\},$$

where the operator J is given by $J(\mathbf{t}, v) = J(\mathbf{t}, v; \bar{X}) = \max(\|v\|_0, t_1\|v\|_1, t_2\|v\|_2)$ and $(u_{mn}) \subset \Delta(\bar{X})$ satisfies that

$$\Big(\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} (2^{-m\theta_0} 2^{-n\theta_1} J(2^m, 2^n, u_{mn}))^q \Big)^{1/q} < \infty.$$

Definition: The *Fundamental Lemma with the operator S_ϱ is valid* for the 3-tuple $\bar{X}$ if any element $x \in \sigma_\varrho(\bar{X})$ can be represented as a series $x = \sum_{\mathbf{k}\in\mathbf{Z}^2} x_k$, absolutely convergent in $\Sigma(\bar{X})$, where $x_k \in \triangle(\bar{X})$, $J(2^{\mathbf{k}}, x_k; \bar{X}) \leq C[S_\varrho K(\cdot, x; \bar{X})](2^{\mathbf{k}})$, $2^{\mathbf{k}} = (2^{k_1}, 2^{k_2})$ and $\mathbf{k} = (k_1, k_2) \in \mathbf{Z}^2$.

Lemma: Let $\bar{X} = (L^{p_0,q_0}(C_0), L^{p_1,q_1}(C_1), L^{p_2,q_2}(C_2))$. The Fundamental Lemma with the operator S_ϱ is valid for $\bar{X}$.

In Definition 2.3.1, we say that a quasi-Banach function lattice Φ on $\mathbb{R}^2_+$ with the measure $\frac{d\mathbf{t}}{\mathbf{t}} = \frac{dt_1}{t_1}\frac{dt_2}{t_2}$ is a *parameter of the ϱ-real method* if the operator S_ϱ is bounded in Φ, $\varrho \in (0, 1]$. As usual, the interpolation spaces $K_\Theta(\bar{X})$ and $J_\Theta(\bar{X})$ are defined by the quasi-norms

$$\begin{aligned} \|f\|_{K_\Theta(\bar{X})} &= \|K(\cdot, f; \bar{X})\|_\Theta, \\ \|f\|_{J_\Theta(\bar{X})} &= \inf\Big\{\|J(\cdot, u(\cdot); \bar{X})\|_\Theta; f = \sum_k u_k \text{ convergent in } \Sigma(\bar{X}),\ u_k \in \Delta(\bar{X})\Big\}, \end{aligned}$$

and we will show in Theorem 2.3.2:

Theorem: Let $p_i, q_i \in (0, \infty]$, $i = 0, 1, 2$ and $\bar{X} = (L^{p_0,q_0}(C_0), L^{p_1,q_1}(C_1), L^{p_2,q_2}(C_2))$ be a 3-tuple for which the Fundamental Lemma with the operator S_ϱ is valid. Then, for any parameter Θ of the ϱ-real method, we have that $K_\Theta(\bar{X}) = J_\Theta(\bar{X})$.

With these ingredients, the key result is Theorem 2.4.4 that cames from the Power theorem of G. Sparr for quasi-normed abelian groups (see [Sp, Sp1]) and Theorem 2.3.2:

Theorem: Let $p_i, q_i \in (0, \infty]$, $i = 0, 1, 2$, and $0 < \mu < 1$. If $0 < \bar{q}_0, \bar{q}_1, q < \infty$ and $\frac{1}{q} = \frac{1-\mu}{\bar{q}_0} + \frac{\mu}{\bar{q}_1}$, then

$$\begin{aligned} &((L^{p_0,q_0}(C_0), L^{p_2,q_2}(C_2))_{\alpha_0,\bar{q}_0}, (L^{p_1,q_1}(C_1), L^{p_2,q_2}(C_2))_{\alpha_1,\bar{q}_1})_{\mu,q} \\ &\qquad = (L^{p_0,q_0}(C_0), L^{p_1,q_1}(C_1), L^{p_2,q_2}(C_2))_{(\theta_1,\theta_2),q}, \end{aligned}$$

where $\theta_1 = (1 - \alpha_1)\mu,\ \theta_2 = \alpha_0(1 - \mu) + \alpha_1\mu$.

As an application we extend the results on real interpolation of capacitary $L^p(C)$ spaces included in [Ce] and [CeClM] to general capacities. The main objective is Theorem 2.5.12:

Theorem: Let C_0, C_1 be a couple of quasi-subadditive Fatou capacities on (Ω, Σ) with the same null sets and $0 < \eta < 1$. If $0 < p_0, p_1 < \infty$, $0 < q_0, q_1 \leq \infty$, $\frac{1}{p} := \frac{1-\eta}{p_0} + \frac{\eta}{p_1}$ and $\frac{1}{q} := \frac{1-\eta}{q_0} + \frac{\eta}{q_1}$, then, for $C_{\theta,q}(A) := \|\chi_A\|_{(L(C_0),L(C_1))_{\theta,q}}$ $(0 < \theta < 1)$,

$$(L^{p_0,q_0}(C_0), L^{p_1,q_1}(C_1))_{\eta,q} = L^{p,q}(C_{\frac{\eta p}{p_1},q/p}).$$

For Ω be a subset of $\mathbb{R}^n$ and C a quasi-subadditive Fatou capacity on $(\Omega, \mathcal{B}(\Omega))$, a function $f : \Omega \to \mathbb{R}$ is termed *C-quasi-continuous* on Ω, denoted by $f \in QC$, if given any $\varepsilon > 0$, there exists a relatively open set $G \subset \Omega$ such that $C(G) < \varepsilon$ and f is continuous on G^c.

More estimates for the K-functional with respect to the pair $(L^{p_0}(C), L^\infty(C))$ similar to those in [Ce] are developed, but restricted to the cone of quasi-continuous functions. Proposition 2.7.4 is a result and states that, if f is a quasi-continuous function, not necessarily positive, then

$$K(t, f; L^{p_0}(C) \cap QC, L^\infty(C) \cap QC) := K_{QC}(t, f; L^{p_0}(C), L^\infty(C)) \simeq K(t, f; L^{p_0}(C), L^\infty(C)).$$

Hence, by denoting $\mathfrak{L}^{p,q}(C) = L^{p,q}(C) \cap QC$, we will show in Theorem 2.7.5:

Theorem: Suppose that $0 < \theta < 1$, $0 < p_0 < q \leq \infty$ or $0 < p_0 \leq q < \infty$ and $\frac{1}{p} := \frac{1-\theta}{p_0}$. Then

$$(\mathfrak{L}^{p_0}(C), \mathfrak{L}^\infty(C))_{\theta,q} = \mathfrak{L}^{p,q}(C).$$

A classical property of the Lebesgue measure spaces that still holds in the capacitary setting is Theorem 2.8.4: For $0 < p_0, p_1 \leq \infty$, $\alpha \in (0,1)$ and $\frac{1}{p} = \frac{1-\alpha}{p_0} + \frac{\alpha}{p_1}$,

$$L^{p_0}(C)^{1-\alpha} L^{p_1}(C)^\alpha = L^p(C)$$

with equivalent quasi-norms. Finally we extend the classical theory of Orlicz spaces to general Orlicz spaces, *capacitary Orlicz spaces* and we study their interpolation behaviour.

In the second part of this memoir we will study *Sobolev inequalities*. To show the connection of this topic with capacities let us consider the following problem. Just consider the problem of maximizing the area a of a plane domain Ω with rectifiable boundary of a fixed length l. As it is known, the disk attach the maximum. The maximizing property of the disk can be written as the isoperimetric inequality

$$4\pi a \leq l^2. \tag{2}$$

For $n \in \mathbb{N}$, the n-dimensional generalization of (2) is

$$(mes_n g)^{\frac{n-1}{n}} \leq c_n \mathcal{H}_{n-1}(g), \tag{3}$$

where g is a domain with smooth boundary δg and compact closure, and $\mathcal{H}_{n-1}$ is the $n-1$-dimensional area (see [EvGa]). How does this geometric fact concern Sobolev embedding theorems? The answer is given in [FF] and [Ma05] where it is proved that:

Theorem: Let $u \in C_0^\infty(\mathbb{R}^n)$. There holds the inequality[2]

$$\Big(\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}} dx \Big)^{\frac{n-1}{n}} \leq c_n \int |\nabla u| dx,$$

[2] ∇f denotes the usual gradient of f when it exists.

where the best constant is the same as in the isoperimetric inequality (3) and $C_0^\infty(\mathbb{R}^n)$ denotes the class of all infinite continuously differentiable functions on $\mathbb{R}^n$ with compact support.

V. Maz'ya as a fourth year undergraduate student discovered that isoperimetric and isocapacitary inequalities are equivalent to Sobolev type inequalities. It turned out that classes of domains and measures involved in embedding and compactness theorems could be completely described in terms of length, area and capacity minimizing functions. If we consider the inequality:

$$\Big(\int_\Omega |u|^q d\mu\Big)^{1/q} \leq C \int_\Omega |\nabla u| dx, \tag{4}$$

where $q \geq 1$, Ω is an open subset of $\mathbb{R}^n$, μ is an arbitrary measure on $\mathbb{R}^n$ and $u \in C_0^\infty(\Omega)$, then we have the following nice theorem.

Theorem: Inequality (4) with $q \geq 1$ holds if and only if

$$\mu(g)^{1/q} \leq C\mathcal{H}_{n-1}(g)$$

for every bounded open set g with smooth boundary $\overline{g} \subset \Omega$. This classical theorem shows the relation between isoperimetry and Sobolev estimates.

Some capacities of potential theory are very useful to obtain bounds for some classical operators. For example, Calderón's theorem shows that every $u \in W^{m,p}(\mathbb{R}^n)$ can be represented as a Bessel potential in the way $u(x) = G_m * f(x)$, for all $x \in \mathbb{R}^n$ and $f \in L^p$, where G_α is the L^1- function with Fourier transform $(1+|\xi|^2)^{-\alpha/2}$, $\xi \in \mathbb{R}^n$, $\alpha > 0$. With this, we obtain the following potential theoretic capacity:

$$C_{\alpha,p}(K) := \inf\{\|f\|_{L^p}^p;\, G_\alpha * f \geq 1 \text{ on } K,\, f \geq 0 \text{ a.e.}\}$$

for $\alpha > 0$ and $1 < p < \infty$. This capacity is often refered as Bessel capacity. Easily it follows the weak type inequality

$$C_{\alpha,p}(\{G_\alpha * f > t\}) \leq t^{-p} \int_{\mathbb{R}^n} f(x)^p dx$$

for any $f \geq 0$ a.e. on $\mathbb{R}^n$. Such an estimate have an analogy with the situation of operators between measurable functions. Here, we are thinking of the operator

$$G_\alpha : L^p \to L^{p,\infty}(C_{\alpha,p}).$$

The history of such inequalities really begins with V. Maz'ya in [Ma85], where the capacitary strong type inequality was given. We adopt here the notation

$$C_p(K) := \inf\Big\{\int |\nabla\phi|^p dx;\, \phi \in C_0^\infty(\mathbb{R}^n),\, \phi \geq 1 \text{ on } K\Big\}.$$

The strong capacitary inequality for C_p is

$$\int |u|^p dC_p \leq A \int |\nabla u|^p dx \, (1 < p < n). \tag{5}$$

The fact that C_p is a capacity and the definition of the Choquet integral make to think about the possibility to obtain on the left part of (5) a real integral on $(0, \infty)$. The solution of this problem was given first by V. Maz'ya in [Ma06].

If $\mathrm{Lip}_0(\Omega)$ is the class of all Lipschitz functions with compact support in a domain $\Omega \subset \mathbb{R}^n$, Wiener's capacity of a compact subset K of Ω,

$$\mathrm{cap}(K, \Omega) = \inf_{0 \leq f \leq 1,\, f=1 \text{ on } K} \|\nabla f\|_2^2 \qquad (f \in \mathrm{Lip}_0(\Omega)),$$

extended in the obvious way for any $p \geq 1$ as the p-capacity

$$\mathrm{cap}_p(K, \Omega) = \inf_{0 \leq f \leq 1,\, f=1 \text{ on } K} \|\nabla f\|_p^p \qquad (f \in \mathrm{Lip}_0(\Omega)),$$

was used by V. Maz'ya to obtain the Sobolev inequality

$$\int_0^\infty \mathrm{cap}_p(\overline{M}_{at}, M_t) d(t^p) \leq c(a, p) \int_\Omega |\nabla f|^p dx,$$

where M_t is the set $\{x \in \Omega; |f(x)| > t\}$ with $t > 0$. It has numerous extensions and has been applied to the theory of Sobolev type spaces on domains in $\mathbb{R}^n$, Riemannian manifolds, metric and topological spaces, to linear and non-linear partial differential equations, etc.

Some extensions to the setting of Lorentz spaces $L^{p,q}(\Omega)$ has been obtained in [CosMa], where it is proved that

$$\int_0^\infty \mathrm{cap}_{p,q}(\overline{M}_{at}, M_t) d(t^p) \lesssim \|\nabla f\|^p_{L^{p,q}(\Omega, m_n; \mathbb{R}^n)} \, (1 \leq q \leq p) \tag{6}$$

and

$$\int_0^\infty \mathrm{cap}_{p,q}(\overline{M}_{at}, M_t)^{q/p} d(t^q) \lesssim \|\nabla f\|^q_{L^{p,q}(\Omega, m_n; \mathbb{R}^n)} \, (p < q < \infty). \tag{7}$$

From (6) and (7) they derive necessary and sufficient conditions for certain two-weight inequalities involving Sobolev-Lorentz norms, extending results obtained in [Ma05] and [Ma06]. For μ and η be two Borel measures on Ω and p, q, r, s real numbers such that $1 < s < \max(p, q) \leq r < \infty$ and $q \geq 1$, V. Maz'ya and S. Costea characterize the inequality

$$\|f\|_{L^{r, \max(p,q)}(\Omega, \mu)} \leq A \Big(\|\nabla u\|_{L^{p,q}(\Omega, m_n)} + \|f\|_{L^{s, \max(p,q)}(\Omega, \eta)} \Big)$$

restricted to functions in $\mathrm{Lip}_0(\Omega)$ by requiring the condition[3]

$$\mu(g)^{1/r} \leq K\Big(\mathrm{cap}_{p,q}(\overline{g}, G)^{1/p} + \eta(G)^{1/s}\Big) \qquad (g \subset\subset G \subset\subset \Omega).$$

In the sequel we extend this result also for r.i. quasi-Banach spaces on Ω and $0 < p < 1$.

The proofs of these new Lorentz-Sobolev inequalities in [CosMa] are based on the properties

$$\|f\|^p_{L^{p,q}(\Omega,\mu)} + \|g\|^p_{L^{p,q}(\Omega,\mu)} \leq \|f+g\|^p_{L^{p,q}(\Omega,\mu)} \qquad (1 \leq q \leq p)$$
$$\|f\|^q_{L^{p,q}(\Omega,\mu)} + \|g\|^q_{L^{p,q}(\Omega,\mu)} \leq \|f+g\|^q_{L^{p,q}(\Omega,\mu)} \qquad (1 < p < q)$$

of the Lorentz (quasi-)norms, for f, g disjointly supported functions. Since the constant in the right hand side of the inequalities is one, they can be extended to an arbitrary set of disjoint functions. A thorough study in the proofs allow us to see that the limitation of these techniques is that it allows us to cover only certain particular kind of spaces because of the *lower estimates* with constant one, and it does not apply to a wider class of spaces. However, we will see that an extension is possible in the setting of (quasi-)Banach function spaces with lower estimates, independently of the value of the constant, by means of new techniques. The key point is a result due to N. J. Kalton and S. J. Montgomery-Smith on the theory of *submeasures*. Our results have the advantage that they can be applied to many examples.

In general, a Banach function space $X = X(\Omega)$ on (Ω, Σ, μ) is called a rearrangement-invariant (r.i. for short) space if $g \in X$ implies that all μ-measurable functions f with the same distribution function, that is, such that $\mu_f = \mu_g$[4], also belong to X and $\|f\|_X = \|g\|_X$.

Our aim in the second part of this thesis is to show that inequalities (6) and (7) can be extended to other function spaces $X = X(\Omega)$, with Ω endowed with the Lebesgue measure, under certain convexity conditions on the Sobolev norm or quasi-norm. In the sequel, we prove in Theorems 3.3.5 and 3.3.6 the inequality

$$\int_0^\infty t^{p-1}\mathrm{Cap}_X(\overline{M}_{at}, M_t)^p dt \leq c(a,p)\|\nabla f\|^p_X,$$

under appropiate convexity conditions, where

$$\mathrm{Cap}_X(\overline{M}_{at}, M_t) := \inf\{\|\nabla u\|_X;\ u \in W(\overline{M}_{at}, M_t)\},$$

and $W(\overline{M}_{at}, M_t) := \{u \in \mathrm{Lip}_0(M_t);\ u = 1 \text{ on a neighbourhood of } \overline{M}_{at},\ 0 \leq u \leq 1\}$.

[3]The notation $g \subset\subset G$ means that g is an open set whose closure is a compact subset of the open set G.
[4]Let f be μ-measurable, the distribution function is $\mu_f(t) := \mu\{x \in \Omega; |f(x)| > t\}, t > 0$.

In general, for a compact set $K \subset \Omega$ and an open set $G \subset \Omega$ containing K, we denote

$$W(K,G) := \{u \in \mathrm{Lip}_0(G);\ u = 1 \text{ on a neighbourhood of } K,\ 0 \le u \le 1\},$$

$$\mathrm{Cap}_X(K,G) := \inf\{\|\nabla u\|_X;\ u \in W(K,G)\}$$

and we will write $\mathrm{Cap}_X(\cdot) = \mathrm{Cap}_X(\cdot,\Omega)$ when Ω has been fixed.

Definition: A quasi-Banach function space X on (Ω,Σ) is called p-convex if there exists a constant M so that

$$\Big\|\Big(\sum_{i=1}^n |f_i|^p\Big)^{1/p}\Big\| \le M\Big(\sum_{i=1}^n \|f_i\|^p\Big)^{1/p} \quad (n \in \mathbb{N},\ \{f_i\}_{i=1}^n \subset X).$$

Definition: Let $0 < p < \infty$. A quasi-Banach function space X on (Ω,Σ) satisfies an *upper p-estimate* (a *lower $p-$estimate*) if there exists a constant M so that, for all $n \in \mathbb{N}$ and for any choice of disjointly supported elements $\{f_i\}_{i=1}^n \subset X$,

$$\Big\|\Big(\sum_{i=1}^n |f_i|^p\Big)^{1/p}\Big\| \le M\Big(\sum_{i=1}^n \|f_i\|^p\Big)^{1/p} \quad \Big(\Big(\sum_{i=1}^n \|f_i\|^p\Big)^{1/p} \le M\Big\|\Big(\sum_{i=1}^n |f_i|^p\Big)^{1/p}\Big\|\Big).$$

In particular, necessary and sufficient conditions for Sobolev type estimates in rearrangement invariant spaces involving two measures are developed, extending results of [Ma05], [Ma06] and [CosMa]. Consider μ and ν be two Borel measures on Ω, X a quasi-Banach function space on Ω, Y an r.i. space on (Ω,μ) and Z an r.i. space on (Ω,ν). Under this conditions we will prove (see Theorem 3.4.1):

Theorem: If X satisfies a lower $p-$estimate, then the following properties are equivalent:

(i) There is a constant $A > 0$ such that

$$\|f\|_{\Lambda^{1,p}(Y)} \le A(\|\nabla f\|_X + \|f\|_{\Lambda^{1,p}(Z)}) \qquad (f \in \mathrm{Lip}_0(\Omega)).$$

(ii) There exists a constant $B > 0$ such that

$$\varphi_Y(\mu(g)) \le B(\mathrm{Cap}_X(\overline{g},G) + \varphi_Z(\nu(G))) \qquad (g \subset\subset G \subset\subset \Omega).$$

Here φ_X denotes the fundamental function of the r.i. space X defined in (3.2) and $\Lambda^{1,p}(X)$ represents the Lorentz space defined by the condition

$$\|f\|_{\Lambda^{1,p}(X)} = \Big(\int_0^\infty t^{p-1}\varphi_X(\mu_f(t))^p dt\Big)^{1/p} < \infty.$$

A characterization of Sobolev type inequalities and *improvements of the integrability* of Lipschitz functions follow from our studies. Let X be a quasi-Banach function space on the domain $\Omega \subset \mathbb{R}^n$, μ a Borel measure on Ω, and Y be an r.i. space on (Ω, μ). In Theorem 3.5.1 we see that if

$$\sup \frac{\varphi_Y(\mu(g))}{\mathrm{Cap}_X(\overline{g}, G)} < \infty,$$

the supremum being taken over all sets g, G such that $g \subset\subset G \subset\subset \mathbb{R}^n$, then for every compact subset K in Ω,

$$\varphi_Y(\mu(K)) \lesssim \mathrm{Cap}_X(K).$$

As an application of Theorem 3.3.6 and Theorem 3.5.1, Theorem 3.5.2 states:

Theorem: Let $0 < p < \infty$. If X satisfies a lower $p-$estimate, then the following properties are equivalent:

(i) $\varphi_Y(\mu(K)) \lesssim \mathrm{Cap}_X(K)$ for every compact set K on Ω.

(ii) $\|f\|_{\Lambda^{1,p}(Y)} \lesssim \|\nabla f\|_X \quad (f \in \mathrm{Lip}_0(\Omega))$.

(iii) $\|f\|_{\Lambda^{1,\infty}(Y)} \lesssim \|\nabla f\|_X \quad (f \in \mathrm{Lip}_0(\Omega))$.

Moreover, for $q \geq p$, if Y is q-convex or, if Y satisfies an upper $q-$estimate and $\varphi_Y(t)/t^{1/p}$ is quasi-increasing, then, for every $f \in \mathrm{Lip}_0(\Omega)$,

$$\|f\|_{\Lambda^{1,\infty}(Y)} \lesssim \|\nabla f\|_X \Leftrightarrow \|f\|_{\Lambda^{1,p}(Y)} \lesssim \|\nabla f\|_X \Leftrightarrow \|f\|_Y \lesssim \|\nabla f\|_X .$$

In the particular case when $X = L^p$, $p \in (1, n)$, and $Y = L^s$ with $s = \frac{np}{n-p}$ we recover the well-known self-improvement of integrability of Lipschitz functions

$$\|f\|_{L^{s,p}} = \|f\|_{\Lambda^{1,p}(L^s)} \lesssim \|\nabla f\|_{L^p}.$$

To finish this chapter, we develop some extensions to the capacitary function spaces studied in the first chapter of this memoir (see Theorem 3.6.1):

Theorem: Suppose $0 < p, s, q < \infty$, and let C and $\widetilde{C}$ be two capacities on (Ω, Σ). If X satisfies a lower $q-$estimate, then the following properties are equivalent:

(i) $\|f\|_{L^{p,q}(C)} \lesssim \|\nabla f\|_X + \|f\|_{L^{s,q}(\widetilde{C})}$ for every $f \in \mathrm{Lip}_0(\Omega)$.

(ii) $C^{(p)}(g) \lesssim \mathrm{Cap}_X(\overline{g}, G) + \widetilde{C}^{(s)}(G)$ for all sets g and G such that $g \subset\subset G \subset\subset \Omega$.

Recall that $C^{(p)} := C^{1/p}$ denotes the *p-convexification* of C (see [Ce]).

As it is known, the Sobolev-Poincaré inequality is closely related with isocapacitary inequalities. So that, we finish this memoir with an study of some Sobolev inequalities of second order.

Let Ω be a domain of $\mathbb{R}^n$ with the Lebesgue measure m_n. Let f be a continuously differentiable function with compact support in Ω. The classical version of the Pólya-Szegö principle states (cf. [K])

$$\|\nabla f^\circ\|_{L^p(\Omega)} \leq \|\nabla f\|_{L^p(\Omega)},$$

where f° denotes the symmetric rearrangement of f, defined as

$$f^\circ(x) := f^*(\omega_n|x|^n) \quad (x \in \mathbb{R}^n),$$

where $\omega_n =$ measure of the unit ball in $\mathbb{R}^n$. It is well-known that the *isoperimetric and isocapacitary inequalities* are equivalent to Sobolev type inequalities (cf. [Ma85, Ma11]). A well-known principle, due to Maz'ya, and Federer and Fleming (cf. [Ma85], [FF], [Fed], and the references therein), is the equivalence between the isoperimetric inequality and the *Sobolev and Gagliardo-Nirenberg inequality*

$$\|f\|_{L^{\frac{np}{n-p}}} \leq c\||\nabla f|\|_{L^p}, \quad (f \in \mathrm{Lip}_0(\mathbb{R}^n), 1 \leq p < n).$$

For $1 < p < n$ the exact value of the constant was found by Talenti [Ta1] and Aubin [Au].

Maz'ya's work also influenced specially the most recent work of J. Martin and M. Milman. For instance, in [MMi3] the authors show some connections between symmetrization inequalities and the isocapacitary inequalities due to Maz'ya.

As it is known, symmetrization is a very useful classical tool in PDE's and the theory of Sobolev spaces, being the *symmetrization inequalities* formulated frequently as norm inequalities. A difficulty in that area is that the norm inequalities need to be proven separately for different classes of spaces. Moreover, one may lose information in the extreme cases. Normaly, the end point Sobolev embeddings usually require a different type of spaces (extrapolation spaces), and different geometries produce different types of optimal spaces.

In [MMi1] and [MMiP] new symmetrization inequalities have been developed that can be applied to provide a unified treatment of sharp Poincaré inequalities and sharp integrability of solutions of elliptic equations. Moreover, in [MMi4] higher order symmetrization inequalities are also given. On the other hand, A. Cianchi in [Ci1] has characterized second-order Sobolev-Poincaré inequalities in $\mathbb{R}^n$ with the Lebesgue measure and a Pólya-Szegö principle for second-order derivatives is established.

In [MMi5] the authors provide, using isoperimetry and symmetrization, a unified framework to study the classical and logarithmic Sobolev inequalities. In particular, they obtain new Gaussian symmetrization inequalities and connect them with logarithmic Sobolev inequalities. In those inequalities, the *isoperimetric function* appears sistematically. For second order derivatives we will see, in the appendix, that the inequalities depend on the square of the isoperimetric function.

In the appendix, we will try to obtain second order Sobolev-Poincaré inequalities and to characterize them. Let μ be a Borel measure on $\Omega = \mathbb{R}^n$, and assume that μ is given by $d\mu(x) = \varphi(x)dx$, where $\varphi \in C(\mathbb{R}^n)$, $\varphi(x) > 0$ for any $x \in \mathbb{R}^n$ and $\int_{\mathbb{R}^n} \varphi(x)dx = 1$. We define the *non-increasing rearrangement* of $f \in \mathrm{L}_0(\mathbb{R}^n)$ with respect to μ (compare with (1.2)), as

$$f^*_\mu(t) := \inf\{s \geq 0;\ \mu\{|f| > s\} \leq t\} \qquad (0 < t \leq 1).$$

Let $A \subset \mathbb{R}^n$ be a measurable set, the μ-perimeter (in the sense of De Giorgi) is defined by

$$P_\mu(A) = \sup\Big\{ \int_A div(h(x)\varphi(x))dx; h \in C^1_0(\mathbb{R}^n, \mathbb{R}^n),\ |h| \leq 1\Big\},$$

and the *isoperimetric function* I_μ is defined as the pointwise maximal function $I_\mu : [0,1] \to [0,\infty)$ such that

$$P_\mu(A) \geq I_\mu(\mu(A)),$$

holds for all Borel sets A.

We will assume that the isoperimetric function (i.e., the isoperimetric profile) I_μ is a concave continuous function, increasing on $(0, 1/2)$, symmetric about the point $1/2$ that, moreover, vanishes at zero.

M. Milman and J. Martín consider as a usual space, a Banach function space X on $(\mathbb{R}^n, \mu)$ and they show that, if Y is also an r.i. space on $(\mathbb{R}^n, \mu)$, the $X - Y$ Sobolev-Poincaré inequality depends on the boundedness of the *Hardy type operator*

$$Q_\mu g(t) = \int_t^{1/2} g(s)\frac{ds}{I_\mu(s)} := \int_t^1 g(s)\chi_{(0,1/2)}(s)\frac{ds}{I_\mu(s)},$$

as shown in [MMi1].

In [MMi3], it is shown that, if $0 \leq g \in \bar{X}(0,1)$ and supp $g \subset (0, 1/2)$, implies that [5]

$$\|Q_\mu g\|_{\bar{Y}(0,1)} \lesssim \|g\|_{\bar{X}(0,1)},$$

[5]For an r.i. Banach function space X on $(\mathbb{R}^n, \mu)$, $\bar{X}(0,1)$ is the r.i. space endowed with the Lebesgue measure given by the Luxemburg theorem (see [BeSh, Theorem 4.2]).

then for any $f \in \mathrm{Lip}(\mathbb{R}^n)$, the *Sobolev-Poincaré estimate* holds

$$\Big\| f - \int_{\mathbb{R}^n} f \, d\mu \Big\|_Y \lesssim \|\nabla f\|_X.$$

It is observed that the reverse result holds only for measure spaces of *isoperimetric Hardy type*.

Using similar techniques second order Sobolev-Poincaré inequalities are developed and related with the boundedness of some Hardy type operators involving the square of the isoperimetric profile on $(\mathbb{R}^n, \mu)$. We define a new operator $\bar{A}$ for $g \in \bar{X}(0,1)$ by

$$\frac{I_\mu(t)}{t} \int_t^{1/2} g(s) \frac{ds}{I_\mu(s)} := \bar{A}g(t),$$

and obtain a unified treatment of second order Sobolev-Poincaré inequalities in rearrangement invariant function spaces.

By $W^{2,X}(\mu)$ we denote the classical second-order Sobolev space generated by the norm in X,

$$\|\phi\|_{W^{2,X}(\mu)} = \sum_{|\sigma| \le 2} \|D^\sigma \phi\|_{X(\mathbb{R}^n,\mu)} = \sum_{|\sigma| \le 2} \|D^\sigma \phi\|_X.$$

Suppose that X and Y are r.i. spaces on $(\mathbb{R}^n, \mu)$. We will show (see Theorem A.3.9) that:

Theorem: Assume that $\overline{\alpha}_X < 1$ and $\bar{A}$ is bounded on $\bar{X}(0,1)$. The following statements are equivalent:

(i) For every $g \ge 0$ with supp $g \subset (0, 1/2)$,

$$\Big\| \int_t^1 g(s) \Big(\frac{s}{I_\mu(s)} \Big)^2 \frac{ds}{s} \Big\|_{\bar{Y}(0,1)} \lesssim \|g\|_{\bar{X}(0,1)}.$$

(ii) For every $f \in W^{2,X}(\mu)$,

$$\|f\|_Y \lesssim \Big\| f_\mu^*(t) \Big(\frac{I_\mu(t)}{t} \Big)^2 \Big\|_{\bar{X}(0,1)}.$$

(iii) For every $f \in W^{2,X}(\mu)$,

$$\|f\|_Y \lesssim \Big\| (f_\mu^{**}(t) - f_\mu^*(t)) \Big(\frac{I_\mu(t)}{t} \Big)^2 \Big\|_{\bar{X}(0,1)} + \|f\|_{L^1(\mu)}.$$

If these properties are satisfied and $f \in W^{2,X}(\mu)$[6], then

$$\Big\| f - \int_{\mathbb{R}^n} f \, d\mu \Big\|_Y \lesssim \|d^2 f\|_X + \|\nabla f\|_{L^1(\mu)}.$$

We will prove under the isoperimetric Hardy type condition in Theorem A.4.1 that: If X and Y are r.i. spaces on $(\mathbb{R}^n, \mu)$ with $\overline{\alpha}_X < 1$ and such that $\|Q_\mu g\|_{\bar{Y}(0,1)} \lesssim \|g\|_{\bar{X}(0,1)}$ for $g \geq 0$, $g \in \bar{X}(0,1)$ supported on $(0, 1/2)$, then $W^{2,X}(\mu) \hookrightarrow Y$, and for every $f \in W^{2,X}(\mu)$

$$\inf_{\Lambda \in \mathcal{P}_1} \|f - \Lambda_f\|_Y \lesssim \|d^2 f\|_X.$$

More precisely, $\|f - \Lambda_f\|_Y \lesssim \|d^2 f\|_X$ if $\Lambda_f := p_f + \int (f - p_f) \, d\mu$ with $p_f(x) := \int f \, d\mu + \sum_{i=1}^n (\int \partial_i f \, d\mu) x_i$. Finally, for the Gaussian measure γ we show in Theorem A.5.2 that, if

$$\|f - \Lambda_f\|_Y \lesssim \|d^2 f\|_X \qquad (f \in W^{2,X}(\gamma)),$$

then $Q_\gamma^2 : \bar{X}(0,1) \to \bar{Y}(0,1)$ and $Q_\gamma : \bar{X}(0,1) \to \bar{Y}(0,1)$ are bounded operators.

Topics covered in this dissertation: Statements of the problems and main results

As we said, capacities, interpolation and Sobolev inequalities are the ingredients of this dissertation. A brief description of the most important obtained results is provided. The different problems will be treated in different chapters of this dissertation and they will be contextualized in the corresponding chapters.

Chapter 1: Capacitary function spaces

Let (Ω, Σ) be a measurable space. Sets will always be assumed to be in Σ and functions in $\mathrm{L}_0(\Omega)$, the set of all real valued measurable functions on (Ω, Σ). A set function C defined on Σ is called a *capacity* if it satisfies at least the following properties:

(a) $C(\emptyset) = 0$,

(b) $0 \leq C(A) \leq \infty$,

(c) $C(A) \leq C(B)$ if $A \subset B$, and

(d) $C(A \cup B) \leq c(C(A) + C(B)) \qquad (c \geq 1) \qquad$ (*quasi-subadditive*).

It is called concave if

$$C(A \cup B) + C(A \cap B) \leq C(A) + C(B).$$

[6] $W^{2,X}(\mu)$ denotes the second-order Sobolev space with the norm generated by the norm in X.

The Choquet integral is defined as

$$\int f\,dC := \int_0^\infty C\{f > t\}\,dt$$

if $f \geq 0$ is a measurable function in the sense that $\{f > t\} \in \Sigma$ for every $t > 0$. One of the main problems is that we are forced to work with a non-additive integral, the Choquet integral, so that the dual spaces are not easily identifiable and some basic properties, such as the dominated convergence theorem, are not longer available. Therefore, we must check all the classical properties to assure their validity.

In measure theory, measure convergence is a really useful concept and its facts. The measure convergence must be understood as the convergence to zero of the measure of the set between the graphs of a sequence of measurable functions $\{f_n\}$ and f. As a capacity is an extension of a measure, our first natural question is to analyze under which conditions on the capacity, we will have the corresponding theorem of convergence of measurable functions. We will answer to this question in a general case, in Theorem 1.3.11.

For a capacity C, a property is said to hold $C-$q.e. if the exceptional set has zero capacity, and we say that $\{f_n\}_{n\in\mathbb{N}} \subset \mathrm{L}_0(\Omega)$ *converges to* $f \in \mathrm{L}_0(\Omega)$ *in capacity* if $C\{|f_n - f| > \epsilon\} \to 0$ as $n \to \infty$, $\forall\,\epsilon > 0$. Similarly, we say that $\{f_n\}_{n\in\mathbb{N}}$ is a *Cauchy sequence in capacity* if for every $\epsilon > 0$, $C\{|f_p - f_q| > \epsilon\} \to 0$ as $p, q \to \infty$.

Theorem: A sequence $\{f_n\}_{n\in\mathbb{N}}$ is convergent in capacity to a function f if and only if it is a Cauchy sequence in capacity. In this case, the sequence has a subsequence which is $C-$q.e. convergent to f.

The *decreasing rearrangement* f_C^* of f is

$$f_C^\star(x) = \inf\{t > 0;\, C\{|f| > t\} \leq x\} \qquad (x > 0),$$

and a quasi-subadditive capacity C such that $C(A_n) \to C(A)$ whenever $A_n \uparrow A$ is called a *Fatou* capacity. The *Lebesgue space* $L^p(C)$ $(p > 0)$ is defined by the condition

$$\varrho_p(f) := \begin{cases} (\int_\Omega |f|^p dC)^{1/p} < \infty, & 0 < p < \infty \\ \inf\{M > 0; C\{|f| \geq M\} = 0\} < \infty, & p = \infty. \end{cases}$$

The Fatou property allows us to prove for a general quasi-subadditive Fatou capacity C on the measurable space (Ω, Σ), the completeness of the capacitary Lebesgue and Lorentz spaces (see Theorems 1.3.12 and 1.3.15). Another interesting problem partially

analized in [CeClM] is the normability or not of the capacitary Lebesgue space $L^p(C)$ under a strong condition. This is not possible in all the desirable generalization, but we will see that it is possible under appropiate conditions.

Let μ be a measure on (Ω, Σ) such that $\mu(\Sigma) = [0, \mu(\Omega)] \subset [0, \infty]$, and let us suppose that C is *μ-invariant*, this meaning that $C(A) = C(B)$ if $\mu(A) = \mu(B)$.

Definition: A capacity C on (Ω, Σ) is called *quasi-concave with respect to μ* if there exists a constant $\gamma \geq 1$ such that, whenever $\mu(A) \leq \mu(B)$, the following two conditions are satisfied:

(a) $C(A) \leq \gamma\, C(B)$, and

(b) $\frac{C(B)}{\mu(B)} \leq \gamma \frac{C(A)}{\mu(A)}$,

this is, for all $A, B \in \Sigma$,

$$C(B) \leq \gamma \max\Big(1, \frac{\mu(B)}{\mu(A)}\Big) C(A).$$

In the study of the normability of the capacitary Lebesgue spaces with weaker conditions, a key result is Theorem 1.4.4:

Theorem: If the capacity C is μ-invariant and quasi-concave with respect to μ, then

$$\widetilde{C}(A) := \sup\Big\{ \sum_{i=1}^{n} \lambda_i C(A_i);\ n \in \mathbb{N},\ \sum_{i=1}^{n} \lambda_i = 1,\ \lambda_i \geq 0,\ \sum_{i=1}^{n} \lambda_i \mu(A_i) \leq \mu(A) \Big\}$$

defines a concave capacity which is equivalent to C, i.e. $C \simeq \widetilde{C}$[7]. As an application we show, in Theorem 1.4.8, that if $\widetilde{C}$ has the Fatou property, then $L^p(C)$ $(1 \leq p \leq \infty)$ is normable.

Chapter 2: Interpolation of capacitary Lorentz spaces

The way of proving Marcinkiewicz's theorem allowed to J. L. Lions and J. Peetre (see [LiP, LiP1]) to construct the real method of interpolation. In particular, if $\bar{A} = (A_0, A_1)$ is a couple of Banach spaces, $0 < \theta < 1$ and $0 < q \leq \infty$, the interpolation space $\bar{A}_{\theta,q}$ is the Banach space of all $f \in A_0 + A_1$ such that

$$\|f\|_{\theta,q} := \Big(\int_0^\infty (t^{-\theta} K(t, f; \bar{A}))^q \frac{dt}{t} \Big)^{1/q} < \infty,$$

[7] In all this memoir, the symbol $f \lesssim g$ will mean that there exists a universal constant $c > 0$ (independent of all parameters involved) such that $f \leq cg$, and the symbol $f \simeq g$ will mean that $f \lesssim g \lesssim f$.

where $K(t, f; \bar{A})$ is the *K-functional* defined by

$$K(t, f; \bar{A}) := \inf \Big\{ \|f_0\|_{A_0} + t\|f_1\|_{A_1};\ f = f_0 + f_1 \Big\}.$$

In the second chapter our objective is to extend the results on real interpolation of capacitary $L^p(C)$ spaces included in [Ce] and [CeClM] to general capacities, that is, to the quasi-Banach case. In those articles the concavity of the capacities is needed. We will avoid this restrictive property.

First we see that for $0 < p < \infty$, C be a quasi-subadditive Fatou capacity on (Ω, Σ), $f \in L^p(C) + L^\infty(C)$, and $t > 0$,

$$K(t, f; L^p(C), L^\infty(C)) \simeq \Big(\int_0^\infty y^{p-1} \min(C\{|f| > y\}, t^p) dy \Big)^{1/p} \simeq \Big(\int_0^{t^p} f_C^*(y)^p dy \Big)^{1/p}.$$

With these formulas, real interpolation follows easily as in the classical case (see Theorem 2.5.2): Suppose $0 < \theta < 1$, $0 < p_0 < q \leq \infty$ or $0 < p_0 \leq q < \infty$, and $\frac{1}{p} = \frac{1-\theta}{p_0}$. Then

$$(L^{p_0}(C), L^\infty(C))_{\theta,q} = L^{p,q}(C).$$

We want to determine for convenient parameters and capacities the interpolation space

$$(L^{p_0}(C_0), L^{p_1}(C_1))_{\eta,q}.$$

Since $L^{p_i}(C_i) = (L^{\alpha_i}(C_i), L^\infty)_{\theta_i,p_i}$ for $\alpha_i = (1-\theta_i)p_i$, we want to determine

$$((L^{\alpha_0}(C_0), L^\infty)_{\theta_0,p_0}, (L^{\alpha_1}(C_1), L^\infty)_{\theta_1,p_1})_{\eta,q}. \tag{8}$$

After a first look, one tries to apply classical reiteration theorems but we can not do it because we have spaces with different capacities.

For $0 < p < \infty$ and w be a weight in $\mathrm{L}_0(\Omega)^+$, the Lorentz space $L^p(w)$ is defined with the quasi-norm

$$\|f\|_{L^p(w)} := \Big(\int_\Omega |f|^p w^p d\mu \Big)^{1/p}.$$

In the classical case Stein and Weiss proved that for $0 < p \leq \infty$ and w_0, w_1 weights in $\mathrm{L}_0(\Omega)^+$,

$$(L^p(w_0), L^p(w_1))_{\eta,p} = L^p(w_0^{1-\eta} w_1^\eta).$$

Moreover, we will see in Chapter 2 that $(\Lambda^p(w_0), \Lambda^p(w_1))_{\eta,p} = \Lambda^p(w)$ with

$$W = W_0^{1-\eta} W_1^\eta.$$

Therefore in the case of capacities one suspects

$$(L^p(C_0), L^p(C_1))_{\eta,p} = L^p(C_0^{1-\eta}C_1^{\eta}).$$

Observe that in (8) three spaces appear, namely $L^{\alpha_0}(C_0)$, $L^{\alpha_1}(C_1), L^{\infty}$. It is natural to try to apply an extension of Sparr's method for triples of Banach spaces (see [AK]).

A thorough study of [AK], and the analytical and topological properties of the spaces in our problem show that it is necessary to define an appropiate Calderón operator in order to prove the key result (Theorem 2.4.4): Let $p_i, q_i \in (0, \infty]$ and C_i be quasi-subadditive Fatou capacities on (Ω, Σ), $i = 0, 1, 2$ such that for an arbitrary set $A \subset \Omega$, then

$$C_0(A) = 0 \Longleftrightarrow C_1(A) = 0 \Longleftrightarrow C_2(A) = 0,$$

that is, C_0, C_1, C_2 have the same null sets. An application of the properties of the modified Calderón operator defined in (2.5) and the Power theorem of G. Sparr (see [Sp, Sp1]) give, in particular, that:

$$((L^{\alpha_0}(C_0), L^{\infty})_{\theta_0,p_0}, (L^{\alpha_1}(C_1), L^{\infty})_{\theta_1,p_1})_{\eta,q} = (L^{\alpha_0}(C_0), L^{\alpha_1}(C_1), L^{\infty})_{\beta_1,\beta_2,q},$$

for $\beta_1 := (1-\theta_1)\eta, \beta_2 = \theta_0(1-\eta) + \eta\theta_1$, and $1/q = \frac{1-\eta}{p_0} + \frac{\eta}{p_1}$.

The capacitary Lorentz spaces $L^{p,q}(C)$ $(p, q > 0)$ are defined by the condition

$$\|f\|_{L^{p,q}(C)} := \begin{cases} \left(q\int_0^{\infty} t^{q-1}C\{|f| > t\}^{q/p}dt\right)^{1/q} < \infty, & q < \infty \\ \sup_{t>0} tC\{|f| > t\}^{1/p} < \infty, & q = \infty. \end{cases}$$

In general:

Theorem: Let $0 < \mu < 1$. If $0 < \bar{q}_0, \bar{q}_1, q < \infty$ and $\frac{1}{q} = \frac{1-\mu}{\bar{q}_0} + \frac{\mu}{\bar{q}_1}$, then

$$\begin{aligned}((L^{p_0,q_0}(C_0), L^{p_2,q_2}(C_2))_{\alpha_0,\bar{q}_0}, (L^{p_1,q_1}(C_1), L^{p_2,q_2}(C_2))_{\alpha_1,\bar{q}_1})_{\mu,q} \\ = (L^{p_0,q_0}(C_0), L^{p_1,q_1}(C_1), L^{p_2,q_2}(C_2))_{(\theta_1,\theta_2),q},\end{aligned}$$

where $\theta_1 = (1-\alpha_1)\mu$, $\theta_2 = \alpha_0(1-\mu) + \alpha_1\mu$. As an application our main objective follows in Theorem 2.5.12:

Theorem: Let C_0, C_1 be a couple of quasi-subadditive Fatou capacities with the same null sets and $0 < \eta < 1$. If $0 < p_0, p_1 < \infty$, $0 < q_0, q_1 \leq \infty$, $\frac{1}{p} := \frac{1-\eta}{p_0} + \frac{\eta}{p_1}$ and $\frac{1}{q} := \frac{1-\eta}{q_0} + \frac{\eta}{q_1}$, then, for $C_{\theta,q}(A) := \|\chi_A\|_{(L(C_0),L(C_1))_{\theta,q}}$ $(0 < \theta < 1)$,

$$(L^{p_0,q_0}(C_0), L^{p_1,q_1}(C_1))_{\eta,q} = L^{p,q}(C_{\frac{\eta p}{p_1},q/p}).$$

Chapter 3: Conductor Sobolev type estimates and isocapacitary inequalities

For $\Omega \subset \mathbb{R}^n$ be a domain, the extension of Wiener's capacity of a compact subset K of Ω for $p \geq 1$, is the p-capacity (see Example 1.2.8)

$$\mathrm{cap}_p(K,\Omega) = \inf_{0\leq f\leq 1,\, f=1 \text{ on } K} \|\nabla f\|_p^p \qquad (f \in \mathrm{Lip}_0(\Omega))^8.$$

This was used in [Ma05] to obtain the *Sobolev inequality*

$$\int_0^\infty \mathrm{cap}_p(\overline{M}_{at}, M_t) d(t^p) \leq c(a,p)\|\nabla f\|_p^p,$$

where M_t is the level set $\{x \in \Omega;\ |f(x)| > t\}$ for $t > 0$.

In [CosMa], the authors show some extensions to the setting of Lorentz spaces $L^{p,q}(\Omega)$. The proofs of the new Lorentz-Sobolev inequalities are based on the properties

$$\|f\|^p_{L^{p,q}(\Omega,\mu)} + \|g\|^p_{L^{p,q}(\Omega,\mu)} \leq \|f+g\|^p_{L^{p,q}(\Omega,\mu)} \qquad (1 \leq q \leq p)$$
$$\|f\|^q_{L^{p,q}(\Omega,\mu)} + \|g\|^q_{L^{p,q}(\Omega,\mu)} \leq \|f+g\|^q_{L^{p,q}(\Omega,\mu)} \qquad (1 < p < q)$$

of the (quasi-)norms for f, g disjointly supported functions. Since the constant in the right hand side of the inequalities is one, they can be extended to an arbitrary set of disjoint functions. Nevertheless, the limitation of these techniques is that it allows us to cover only certain particular kind of spaces because of the lower estimates with constant one. We will see that an extension is possible for (quasi-)Banach function spaces with lower estimates, independently of the value of the constant. The key point is a result on the theory of *submeasures*. Our results can be applied to many examples.

Our aim is to extend these capacitary estimates when a general function space X substitutes $L^p(\Omega)$ in the definition of cap_p. Let Ω be a domain of $\mathbb{R}^n$ endowed with the *Lebesgue measure* m_n and $X = X(\Omega)$ denotes a quasi-Banach function space on Ω. For a compact set $K \subset \Omega$ and an open set $G \subset \Omega$ containing K, the couple (K, G) is called a *conductor* and we denote

$$W(K,G) := \{u \in \mathrm{Lip}_0(G);\ u = 1 \text{ on a neighbourhood of } K,\ 0 \leq u \leq 1\}.$$

Each conductor has an X-*capacity* defined by

$$\mathrm{Cap}_X(K,G) := \inf\{\|\nabla u\|_X;\ u \in W(K,G)\}.$$

[8] $\mathrm{Lip}_0(\Omega)$ is the class of all Lipschitz functions with compact support in a domain $\Omega \subset \mathbb{R}^n$ and ∇f denotes the usual *gradient* of $f \in \mathrm{Lip}_0(\Omega)$.

We will write $\mathrm{Cap}_X(\cdot) = \mathrm{Cap}_X(\cdot, \Omega)$ when Ω has been fixed.

Definition: A quasi-Banach function space X on (Ω, Σ) is called p-convex if there exists a constant M so that

$$\Big\| \Big(\sum_{i=1}^{n} |f_i|^p \Big)^{1/p} \Big\| \leq M \Big(\sum_{i=1}^{n} \|f_i\|^p \Big)^{1/p} \quad (n \in \mathbb{N},\ \{f_i\}_{i=1}^n \subset X).$$

Definition: Let $0 < p < \infty$. A quasi-Banach function space X on (Ω, Σ) satisfies an *upper p-estimate* (a *lower $p-$estimate*) if there exists a constant M so that, for all $n \in \mathbb{N}$ and for any choice of disjointly supported elements $\{f_i\}_{i=1}^n \subset X$,

$$\Big\| \Big(\sum_{i=1}^{n} |f_i|^p \Big)^{1/p} \Big\| \leq M \Big(\sum_{i=1}^{n} \|f_i\|^p \Big)^{1/p} \quad \Big(\Big(\sum_{i=1}^{n} \|f_i\|^p \Big)^{1/p} \leq M \Big\| \Big(\sum_{i=1}^{n} |f_i|^p \Big)^{1/p} \Big\| \Big).$$

Then a new argument solves our main problem (see Theorem 3.3.6):

Theorem: Suppose $0 < p < \infty$ and let $a > 1$ be a constant. If X is a quasi-Banach function space which satisfies a lower $p-$estimate, then

$$\int_0^\infty t^p \mathrm{Cap}_X(\overline{\{|f| > at\}}, \{f > t\})^p \frac{dt}{t} \leq c_1 \|\nabla f\|_X^p \quad (f \in \mathrm{Lip}_0(\Omega)),$$

where the constant c_1 depends on $a, p, M_{(p)}(X)$ and on the quasi-subadditivity constant c of the quasi-norm in X.

Given $0 < p \leq \infty$, the Lorentz space $\Lambda^{1,p}(X)$ associated to X is defined as[9]

$$\Big\{ f \in L^0(\Omega);\ \|f\|_{\Lambda^{1,p}(X)} = \Big(\int_0^\infty t^{p-1} \big(\varphi_X(\mu_f(t))\big)^p dt \Big)^{\frac{1}{p}} < \infty \Big\}$$

with the usual changes when $p = \infty$.

As an application of Theorem 3.3.6 we obtain an unnoticed fact: It could seem that for improvements of integrability only truncations methods are needed. For instance, in [KO] it appears that inequalities of Sobolev-Poincaré-type are improved to Lorentz type scales thanks to stability under truncations, but there also p-convexity is implicitly used. It is well known that the Gagliardo-Nirenberg inequality

$$\|f\|_{L^{n/(n-1)}} \lesssim \|\nabla f\|_{L^1} \qquad (f \in \mathrm{Lip}_0(\Omega)),$$

[9]For X be an r.i. quasi-Banach function space on Ω and μ a totally σ-finite measure on Ω, φ_X is the *fundamental function* of X defined in (3.2).

allows us to see that, if $p \in (1,n)$, $s = \frac{np}{n-p}$ and $\alpha = \frac{(n-1)s}{n}$, since $\|f\|_{L^s}^s = \||f|^\alpha\|_{L^{\frac{n}{n-1}}}^{n/(n-1)}$, then $\|f\|_{L^s}^{s(n-1)/n} \lesssim \|\alpha|f|^{\alpha-1}|\nabla f|\|_{L^1} \lesssim \|f\|_{L^s}^{s/p'}\|\nabla f\|_{L^p}$, where p' is the conjugate exponent of p. Hence $\|f\|_{L^s} \lesssim \|\nabla f\|_{L^p}$. Therefore, since $L^s \hookrightarrow L^{s,\infty}$, it follows that

$$\|f\|_{L^{s,\infty}} \lesssim \|\nabla f\|_{L^p}.$$

But $\|f\|_{\Lambda^{1,\infty}(L^s)} = \|f\|_{L^{s,\infty}} \lesssim \|\nabla f\|_{L^p}$ and then, from Theorem 3.5.2, we will be able to conclude that

$$\|f\|_{L^{s,p}} = \|f\|_{\Lambda^{1,p}(L^s)} \lesssim \|\nabla f\|_{L^p} \quad (f \in \mathrm{Lip}_0(\Omega)),$$

and we obtain a self-improvement.

If $p = n$, the Trudinger inequality,

$$\left(\frac{\int_0^t f^*(s)^{\frac{n}{n-1}}}{t(1+\log\frac{1}{t})}\right)^{\frac{n-1}{n}} \lesssim \|\nabla f\|_{L^n},$$

gives the estimates

$$\varphi(\mu(K)) = \Big(1+\log\frac{1}{\mu(K)}\Big)^{\frac{1-n}{n}} \leq \mathrm{Cap}_{L^n}(K), \qquad \|f\|_{\Lambda^{1,n}(\varphi)} \lesssim \|\nabla f\|_{L^n}.$$

But,

$$\Lambda^{1,n}(\varphi) = \Big(\int_0^\infty t^{n-1}(\varphi(\mu_f(t)))^n dt\Big)^{1/n} = \Big(\int_0^1 \Big(\frac{f^*(s)}{(1+\log\frac{1}{s})}\Big)^n \frac{ds}{s}\Big)^{1/n}.$$

If $r \leq s < p$, then $L^{s,r}$ satisfies an upper p-estimate and $\varphi_{L^{s,r}}(t)/t^{1/p}$ is quasi-increasing, so that, since $\|f\|_{L^{s,\infty}} = \|f\|_{\Lambda^{1,\infty}(L^s)} \lesssim \|\nabla f\|_{L^p}$, we will see for $q \leq p$ that $\|f\|_{L^{s,\infty}} \simeq \|f\|_{\Lambda^{1,\infty}(L^{s,r})} \lesssim \|\nabla f\|_{L^p} \lesssim \|\nabla f\|_{L^{p,q}}$, and then $\|f\|_{\Lambda^{1,p}(L^{s,r})} \lesssim \|\nabla f\|_{L^p}$. Therefore, if $q \leq p$, then we obtain the self-improvement

$$\|f\|_{L^{s,p}} \simeq \|f\|_{\Lambda^{1,p}(L^{s,r})} \lesssim \|\nabla f\|_{L^{p,q}} \qquad (f \in \mathrm{Lip}_0(\Omega)).$$

In this sense, for μ be a Borel measure on Ω, and Y an r.i. space on (Ω,μ), we will show:

Theorem: Let $0 < p < \infty$. If X satisfies a lower $p-$estimate, then the following properties are equivalent:

(i) $\varphi_Y(\mu(K)) \lesssim \mathrm{Cap}_X(K)$ for every compact set K on Ω.

(ii) $\|f\|_{\Lambda^{1,p}(Y)} \lesssim \|\nabla f\|_X \quad (f \in \mathrm{Lip}_0(\Omega))$.

(iii) $\|f\|_{\Lambda^{1,\infty}(Y)} \lesssim \|\nabla f\|_X \quad (f \in \mathrm{Lip}_0(\Omega))$.

Moreover, for $q \geq p$, if Y is q-convex or, if Y satisfies an upper $q-$estimate and $\varphi_Y(t)/t^{1/p}$ is quasi-increasing, then, for every $f \in \mathrm{Lip}_0(\Omega)$,

$$\|f\|_{\Lambda^{1,\infty}(Y)} \lesssim \|\nabla f\|_X \Leftrightarrow \|f\|_{\Lambda^{1,p}(Y)} \lesssim \|\nabla f\|_X \Leftrightarrow \|f\|_Y \lesssim \|\nabla f\|_X .$$

Appendix: Second order Sobolev-Poincaré estimates

We obtain a unified treatment of second order Sobolev-Poincaré inequalities in rearrangement invariant function spaces.

Let us consider $\mathbb{R}^n$ with the Borel measure μ. We assume that μ is given by $d\mu(x) = \varphi(x)dx$, where $\varphi \in C(\mathbb{R}^n)$, $\varphi(x) > 0$ for any $x \in \mathbb{R}^n$ and $\int_{\mathbb{R}^n} \varphi(x)dx = 1$.

For a Borel set $A \subset \mathbb{R}^n$, the μ-perimeter (in the sense of De Giorgi) is defined by

$$P_\mu(A) = \sup\Big\{ \int_A div(h(x)\varphi(x))dx; h \in C_0^1(\mathbb{R}^n, \mathbb{R}^n),\ |h| \leq 1\Big\},$$

and the *isoperimetric function* I_μ is defined as the pointwise maximal function $I_\mu : [0,1] \to [0,\infty)$ such that

$$P_\mu(A) \geq I_\mu(\mu(A)),$$

holds for all Borel sets A.

M. Milman and J. Martín show that, for X and Y r.i. spaces on $(\mathbb{R}^n, \mu)$, the $X - Y$ Sobolev-Poincaré inequality depends on the boundedness of Q_μ, defined by

$$Q_\mu g(t) = \int_t^{1/2} g(s)\frac{ds}{I_\mu(s)}.$$

Suppose that X and Y are r.i. spaces on $(\mathbb{R}^n, \mu)$, and let us define a new operator $\bar{A}$ for $g \in \bar{X}(0,1)$ by

$$\frac{I_\mu(t)}{t}\int_t^{1/2} g(s)\frac{ds}{I_\mu(s)} := \bar{A}g(t).$$

We will show (see Theorem A.3.9) that:

Theorem: Suppose $\overline{\alpha}_X < 1$ and $\bar{A}$ is bounded on $\bar{X}(0,1)$. The following statements are equivalent:

(i) For every $g \geq 0$ with supp $g \subset (0, 1/2)$,

$$\Big\| \int_t^1 g(s) \Big(\frac{s}{I_\mu(s)} \Big)^2 \frac{ds}{s} \Big\|_{\bar{Y}(0,1)} \lesssim \|g\|_{\bar{X}(0,1)}.$$

(ii) For every $f \in W^{2,X}(\mu)$,

$$\|f\|_Y \lesssim \Big\| f_\mu^*(t) \Big(\frac{I_\mu(t)}{t} \Big)^2 \Big\|_{\bar{X}(0,1)}.$$

(iii) For every $f \in W^{2,X}(\mu)$,

$$\|f\|_Y \lesssim \Big\| (f_\mu^{**}(t) - f_\mu^*(t)) \Big(\frac{I_\mu(t)}{t} \Big)^2 \Big\|_{\bar{X}(0,1)} + \|f\|_{L^1(\mu)}.$$

If these properties are satisfied and $f \in W^{2,X}(\mu)$[10], then

$$\Big\| f - \int_{\mathbb{R}^n} f \, d\mu \Big\|_Y \lesssim \|d^2 f\|_X + \|\nabla f\|_{L^1(\mu)}.$$

[10] $W^{2,X}(\mu)$ denotes the second-order Sobolev space with the norm generated by the norm in X.

Part I

Capacitary function spaces and interpolation

Chapter 1

Capacitary function spaces

1.1 Introduction

The concept of a capacity has its origins in Electrostatics. Assume that K is a conductor and that we take a charge distribution on K and let the charge move until reach the equilibrium. Call μ the equilibrium distribution, so that $\mu(K)$ is the total charge. The Newtonian potential of the measure μ is defined as

$$U^\mu(x) = \int \frac{d\mu(y)}{|x-y|},$$

and it is the potential energy of a unit charge placed in x. On that situation, it takes a constant value V on K. Define, following to Wiener, the *capacity* of K by

$$C(K) = \frac{\mu(K)}{V}.$$

One may imagine that the boundary of K and any sphere of large radius surrounding K are the plates of the condenser. Letting the radius of the sphere tend to ∞, one gets an ideal condenser, the boundary of K and the point ∞. Since $U^\mu(\infty) = 0$, Wiener capacity may be understood as the capacity of this ideal condenser, and it does not depend on the charge.

Sets of zero capacity play the role of negligible sets for potential theoretic questions. The fact that a ball and its boundary have the same capacity, being positive, implies that this capacity is not an additive set function, that is, this capacity is not a measure. Frostman, a student of M. Riesz, solved the problem of the equilibrium distribution showing in his thesis (1935) that there exists a unique probability measure μ on K (the normalized equilibrium measure) such that U^μ is constant C-almost everywhere on K.

La Vallée Poussin gave an alternative description of this capacity. The capacity is the

maximal charge of a charge distribution with potential bounded by 1, that is,

$$C(K) = \sup\{\mu(K); \mu \geq 0, \text{ supp } \mu \subset K \text{ and } U^{\mu}(x) \leq 1\}.$$

In particular, a set K has positive capacity if and only if there exists a positive measure supported on K with bounded potential.

Another definition of Wiener's capacity can be given. One can show that

$$C(K) = \inf\Big\{\frac{1}{4\pi}\int |\nabla\varphi(x)|^2\,dx;\ \varphi \in \mathcal{C}_0^{\infty}(\mathbb{R}^3),\ \varphi \geq 1 \text{ on } K\Big\},$$

where $\mathcal{C}_0^{\infty}$ denotes the class of infinitely differentiable functions with compact support, and the extremal φ of this formulation coincides with U^{μ}.

It was arround the fifthies when the concept of a capacity started to be used as a generic set theoretic measuring device. Therefore, a capacity is intimately associated to the idea of a function space -in much the same way as Lebesgue measure is related to the usual L^p spaces.

The desire to integrate with respect to a capacity was solved by Choquet in [Ch] using the distributional form of a Lebesgue integral. The *Choquet integral* of f is defined by

$$\int_E f dC := \int_0^{\infty} C\{x \in E; f(x) > t\}dt,$$

where f is a non-negative function and $C(\cdot)$ is a capacity. This new view provide a convenient language to extend the traditional additive integral to a non-additive integral, the Choquet integral. So that, it turns out to be necessary to study the essential functional analytic elements such that a satisfactory theory can be developed in the context of quasi-Banach spaces. The dual spaces are not easily identifiable and some basic properties, such as the *dominated convergence theorem* or *Fubini's theorem*, are not longer available.

There are well-known characterizations of negligible sets by means of capacities, Hausdorff measures, arithmetical conditions, etc. and the significance of these concepts to existence problems for harmonic and analytic functions, boundary behaviour, convergence of expansions and to harmonic analysis. For instance, capacities, as extensions of measures, are very useful for the study of the boundedness of certain operators.

For any subset E of $\mathbb{R}^n$ and $0 < \alpha \leq n$, the *α-dimensional Hausdorff content* of E is defined by

$$H^{\alpha}(E) = \inf\Big\{\sum_{j=1}^{\infty} \ell(Q_j)^{\alpha}\Big\},$$

where the imfimum is taken over all coverings of E by countable families of cubes Q_j with sides parallel to the coordinate axes, and $\ell(Q)$ denotes the side length of the cube Q.

In the eighties, D. R. Adams proved in [A], using the BMO-H^1 duality, the strong type inequality for f be a locally integrable function on $\mathbb{R}^n$

$$\int Mf(x)dH^\alpha(x) \leq C \int |f(x)|dH^\alpha(x),\ 0 < \alpha < n,$$

where Mf is the *Hardy-Littlewood maximal function* of f. Other results in the same direction as well as some extensions can be found on [AH], [Ma85], [OV], and the references therein.

Capacities were used to analyze the non-removability of sets, where a compact set $E \subset \mathbb{C}$ is called *removable for bounded analytic functions* if, for any open set $U \supset E$, any bounded analytic function $f : U \setminus E \to \mathbb{C}$ has an analytic extension to the whole U.

In 1947, L. Ahlfors (see [Ahl]) introduced the notion of *analytic capacity* to quantify the non-removability of a set. The *analytic capacity* of a compact set E is:

$$\gamma(E) := \sup\{|f'(\infty)|; f : \mathbb{C} \setminus E \to \mathbb{C} \text{ is bounded analytic with } \|f\|_\infty \leq 1\},$$

where $f'(\infty)$ is the derivative of f at ∞, that is, $f'(\infty) = \lim_{z\to\infty} z(f(z) - f(\infty))$.

This capacity was used to adresses Painlevé's problem. L. Ahlfors proved that E is removable if and only if $\gamma(E) = 0$, and more recently, X. Tolsa in [To] gave a characterization of removable sets in terms of *Menger curvature*.

1.2 Capacities

Let (Ω, Σ) be a measurable space. From now on, sets will always be assumed to be in Σ and functions in $\mathrm{L}_0(\Omega)$, the set of all real valued measurable functions on (Ω, Σ). By $\mathrm{L}_0^+(\Omega)$ we will denote the subset of all positive functions in $\mathrm{L}_0(\Omega)$.

1.2.1 Preliminaries

Definition 1.2.1. *A set function C defined on Σ is called a* capacity *if it satisfies at least the following properties:*

(a) $C(\emptyset) = 0$,

(b) $0 \leq C(A) \leq \infty$, *and*

(c) $C(A) \leq C(B)$ *if* $A \subset B$.

In this case, (Ω, Σ, C) *is called a* capacity space.

If moreover for all measurable sets A and B on Σ

$$C(A \cup B) \leq c(C(A) + C(B)),$$

where $c \geq 1$ is a constant, we say that the capacity is *quasi-subadditive*; it is *subadditive* if $c = 1$.

Given $f \in \mathrm{L}_0(\Omega)$, the *distribution function* C_f of f is defined similarly to the case of a measure by

$$C_f(t) := C\{|f| > t\},\ t > 0 \tag{1.1}$$

and the *decreasing rearrangement* f_C^* of f is defined as

$$f_C^\star(x) = \inf\{t > 0;\ C\{|f| > t\} \leq x\},\ x > 0. \tag{1.2}$$

In this capacitary setting many of the basic properties remain true. Easily it follows that

$$\begin{aligned} f_C^\star(x) &= \sup\{t;\ C\{|f| \leq t\} > x\} \\ &= \int_0^\infty \chi_{[0, C\{|f|>t\})}(x)\, dt \\ &= \sup_{C(A)>x} \Big(\inf_{a \in A} |f(a)| \Big). \end{aligned}$$

Besides, both functions C_f and f_C^* are non-increasing. Using the most convenient of the above equivalent definitions of the non-increasing rearrangement, the following properties are easily proved:

(1) $(\chi_A)_C^\star = \chi_{[0, C(A))}$.

(2) If $s = \sum_{k=1}^N a_k \chi_{A_k}$, $A_k \cap A_j = \emptyset$ $(k \neq j)$ and $a_1 > a_2 > \cdots > a_N > 0 = a_{N+1}$, then $s_C^\star = \sum_{k=1}^N (a_k - a_{k+1}) \chi_{[0, C(A_1 \cup \cdots \cup A_k))}$.

(3) If $s = \sum_{k=1}^N b_k \chi_{F_k}$, $F_k \subset F_{k+1}$ and $b_k > 0$ for every k, then

$$s_C^\star = \sum_{k=1}^N b_k \chi_{[0, C(F_k))}.$$

(4) If $s = \sum_{k=1}^N c_k \chi_{F_k}$, $F_k \supset F_{k+1}$ and $c_k > 0$ for every k, then

$$s_C^\star = \sum_{k=1}^N b_k \chi_{(C(F_{k+1}), C(F_k))},\ (C(F_{N+1}) := 0).$$

(5) If $\psi : [0,\infty) \to [0,\infty)$ is non-decreasing and right-continuous, then

$$\psi(|f|)^\star_C = \psi(f^\star_C).$$

For instance, $(|f|^p)^\star_C = (f^\star_C)^p$ $(p > 0)$.

Let us show that the decreasing rearrangement of f is also quasi-subadditive.

Proposition 1.2.2. *Let C be a quasi-subadditive Fatou capacity on (Ω, Σ) with quasi-subadditivity constant c. Then, for $x > 0$,*

$$(f+g)^\star_C(x) \le f^\star_C\Big(\frac{x}{2c}\Big) + g^\star_C\Big(\frac{x}{2c}\Big). \tag{1.3}$$

Proof. Suppose that $\lambda := f^\star_C(x_1) + g^\star_C(x_2) < \infty$ and let $x = C_{f+g}(\lambda)$. Then

$$x = C\{|f+g| > f^\star_C(x_1) + g^\star_C(x_2)\} \le cC_f(f^\star_C(x_1)) + cC_g(g^\star_C(x_2)) \le cx_1 + cx_2,$$

so that

$$(f+g)^\star_C(cx_1 + cx_2) \le (f+g)^\star_C(x) \le \lambda = f^\star_C(x_1) + g^\star_C(x_2). \tag{1.4}$$

Taking then, $x_1 = x_2 = x/2c$ in (1.4), it follows that

$$(f+g)^\star_C(x) \le f^\star_C(x/2c) + g^\star_C(x/2c). \qquad \square$$

Proposition 1.2.3. *Under the same condition of Proposition 1.2.2, for $t_1, t_2 > 0$,*

$$(fg)^\star_C(c(t_1+t_2)) \le f^\star_C(t_1)g^\star_C(t_2).$$

Proof. Consider $a = f^\star_C(t_1)$ and $b = g^\star_C(t_2)$. We have the inclusion

$$\{t \in \Omega; |f(t)g(t)| > ab\} \subset \{t \in \Omega; |f(t)| > a\} \cup \{t \in \Omega; |g(t)| > b\}$$

which implies by the right continuity that

$$\begin{aligned} C_{fg}(ab) &= C\{t \in \Omega; |f(t)g(t)| > ab\} \le C\Big(\{t \in \Omega; |f(t)| > a\} \cup \{t \in \Omega; |g(t)| > b\}\Big) \\ &\le cC\{t \in \Omega; |f(t)| > a\} + cC\{t \in \Omega; |g(t)| > b\} \\ &\le ct_1 + ct_2. \end{aligned}$$

Hence,

$$(fg)^\star_C(c(t_1+t_2)) \le (fg)^\star_C(C_{fg}(ab)) = ab = f^\star_C(t_1)g^\star_C(t_2). \qquad \square$$

For a given capacity C, a property is said to hold *quasi-everywhere* ($C-$q.e. for short) if the exceptional set has zero capacity.

Definition 1.2.4. *Let $f_1, f_2, \ldots$ be elements in* $\mathrm{L}_0(\Omega)$. *We will say that f_n* pointwise converges *to f, and we write $f_n \to f$, when $C\{f_n \not\to f\} = 0$. Similarly, we say that $f_n \uparrow f$ when $f_n \to f$ and $C\{f_n > f_{n+1}\} = 0$.*

We will write $A_n \uparrow A$ or $A_n \downarrow A$ when $\chi_{A_n} \uparrow \chi_A$ or $\chi_{A_n} \downarrow \chi_A$ in the above sense, respectively.

Let us remember that if $f_1, f_2, \ldots$ are elements in $\mathrm{L}_0(\Omega)$, then for every $x \in \Omega$ we have that

$$\liminf_{i\to\infty} f_i(x) := \sup_j \inf_{i>j} f_i(x), \quad \limsup_{i\to\infty} f_i(x) := \inf_j \sup_{i>j} f_i(x).$$

Therefore, the corresponding facts for lim sup and lim inf follow from their definitions.

Let us introduce now the main element, the main concept in this thesis.

Definition 1.2.5. *Let $f \geq 0$. The* Choquet integral *is defined as*

$$\int f\, dC := \int_0^\infty C\{f > t\}\, dt \in [0, \infty],$$

where C is an arbitrary capacity on (Ω, Σ).

This integral is well defined and positive-homogeneous, that is,

$$\int \alpha f\, dC = \alpha \int f\, dC \qquad (\alpha > 0)$$

and such that $\int f\, dC = 0$ if and only if $f = 0$ $C-$q.e.

Since $\{t > 0;\, C\{|f| > t\} \leq x\}$ is the interval $[f_C^\star(x), \infty]$, by *Fubini's theorem*, it follows that

$$\int_0^\infty f_C^\star(x)\, dx = \int |f|\, dC.$$

Indeed,

$$\begin{aligned}\int_0^\infty f_C^\star(x)\, dx &= \int_0^\infty \int_0^\infty \chi_{[0, C\{|f|>t\})}(x)\, dt dx \\ &= \int_0^\infty \int_0^{C\{|f|>t\}} dx dt = \int_0^\infty C\{|f| > t\} dt = \int |f| dC.\end{aligned}$$

Proposition 1.2.6. *Let C be a quasi-subadditive capacity on (Ω, Σ) with quasi-subadditivity constant c. The Choquet integral, defined on non-negative functions, is quasi-subadditive with constant $2c$.*

Proof. Let $f, g \geq 0$. The relation $\{f+g>t\} \subset \{f>t/2\} \cup \{g>t/2\}$ shows that

$$\begin{aligned}\int (f+g)dC &\leq \int_0^\infty C(\{f>t/2\} \cup \{g>t/2\})dt \\ &\leq \int_0^\infty \Big(cC\{f>t/2\} + cC\{g>t/2\}\Big)dt \\ &\leq 2c\Big(\int f\,dC + \int g\,dC\Big). \qquad \square\end{aligned}$$

Fifty years ago, G. Choquet proved in [Ch] that the Choquet integral is subadditive on sets,

$$\int (\chi_A + \chi_B)\,dC \leq \int \chi_A\,dC + \int \chi_B\,dC,$$

if and only if

$$C(A \cup B) + C(A \cap B) \leq C(A) + C(B).$$

Then the Choquet integral is also subadditive on non-negative simple functions. For a direct elementary proof see [Ce] and [CeClM]. In this case we say that C is strongly subadditive or a *concave* capacity.

1.2.2 Examples

Let us present here some classical examples of capacities that appears naturally in Analysis, specially in Potential theory. The capacities of the examples extend from compact sets to other type of sets taking supremums.

Example 1.2.7. *Let us remember that the* analytic capacity $\gamma(E)$ *of* $E \subset \mathbb{C}$ *be compact is defined by*

$$\gamma(E) = \sup\{|f'(\infty)|; f : \mathbb{C} \setminus E \to \mathbb{C} \text{ is bounded analytic with } \|f\|_\infty \leq 1\}.$$

We refer to [Pa] for details concerning this capacity, which is quasi-subadditive. In [Pa] it is stated that, if $E_1, E_2, ..., E_n$ are pairwise disjoint connected domains in $\mathbb{C}$, then

$$\gamma(\cup_{i=1}^n E_i) \leq \gamma(E_1 + \cdots + E_n).$$

Example 1.2.8. *If* $\mathrm{Lip}_0(\Omega)$ *is the class of all Lipschitz functions with compact support in a domain* $\Omega \subset \mathbb{R}^n$*, Wiener's capacity of a compact subset* K *of* Ω *is*

$$\mathrm{cap}(K, \Omega) = \inf_{0 \leq f \leq 1,\, f=1 \text{ on } K} \|\nabla f\|_2^2 \qquad (f \in \mathrm{Lip}_0(\Omega)),$$

which extends in the obvious way for any $p \geq 1$ as the p-capacity

$$\operatorname{cap}_p(K,\Omega) = \inf_{0\leq f\leq 1,\, f=1 \text{ on } K} \|\nabla f\|_p^p \qquad (f \in \operatorname{Lip}_0(\Omega)).$$

Example 1.2.9. *Let Ω be a domain of $\mathbb{R}^n$ endowed with the Lebesgue measure m_n and $X = X(\Omega)$ a quasi-Banach function space on Ω, see Definition 1.3.2.*

Given a compact set $K \subset \Omega$ and an open set $G \subset \Omega$ containing K, we denote

$$W(K,G) := \{u \in \operatorname{Lip}_0(G);\ u = 1 \text{ on a neighbourhood of } K,\ 0 \leq u \leq 1\},$$

and as in [CosMa] (where $X = L^{p,q}$ and $\operatorname{Cap}_X = \operatorname{cap}_{p,q}^{1/p}$), we define

$$\operatorname{Cap}_X(K,\Omega) := \inf\{\|\nabla u\|_X;\ u \in W(K,\Omega)\}.$$

We denote by $\operatorname{Cap}_X(\cdot) = \operatorname{Cap}_X(\cdot,\Omega)$ when Ω has been fixed.

From the definition (see [Cos]), we will see in Chapter 3 that Cap_X is a capacity on Ω. Moreover, for any compact set $E \subset \Omega$ and $\varepsilon > 0$, there exists a neighbourhood G such that

$$\operatorname{Cap}_X(K,\Omega) \leq \operatorname{Cap}_X(E,\Omega) + \varepsilon \tag{1.5}$$

for every compact set K, $E \subset K \subset \Omega$. Indeed, there exists $u \in \operatorname{Lip}_0(\Omega)$, $u = 1$ in a neighbourhood of E such that

$$\|\nabla u\|_X \leq \operatorname{Cap}_X(E,\Omega) + \varepsilon.$$

Therefore, there exists G_1 an open set on Ω containing E where $u = 1$. So that, we can find a compact set K such that $E \subset K \subset G_1$ since G_1 is open and E is compact. Therefore since $u = 1$ in a neighbourhood of K, it follows that

$$\operatorname{Cap}_X(K,\Omega) \leq \|\nabla u\|_X \leq \operatorname{Cap}_X(E,\Omega) + \varepsilon.$$

Similarly, for any compact set $e \subset \Omega$ and any $\varepsilon > 0$ there exists $u \in \operatorname{Lip}_0(\Omega)$, $u \geq 1$ in a neighbourhood of e such that $\|\nabla u\|_X \leq \operatorname{Cap}_X(e,\Omega) + \varepsilon$. Since supp u is compact in the open set Ω, there exists an open set ω such that supp $u \subset \omega \subset \bar{\omega} \subset \Omega$ and then, by definition,

$$\operatorname{Cap}_X(e,\omega) \leq \|\nabla u\|_X \leq \operatorname{Cap}_X(e,\Omega) + \varepsilon.$$

Let E be an arbitrary subset of Ω. The number

$$\operatorname{Cap}_X(E,\Omega) = \sup_{K\subset E} \operatorname{Cap}_X(K,\Omega),$$

where the supremum is taken over all compact subsets contained in E is called the X*-capacity of* E *relative to* Ω, and the number

$$\bar{\mathrm{Cap}}_X(E,\Omega) = \inf_{E\subset G} \mathrm{Cap}_X(G,\Omega),$$

where the infimum runs over all open subsets of Ω containing E, is called the *outer capacity* $\bar{\mathrm{Cap}}_X(E,\Omega)$ of $E \subset \Omega$. A set $E \subset \Omega$ is *capacitable* if

$$\mathrm{Cap}_X(E,\Omega) = \bar{\mathrm{Cap}}_X(E,\Omega).$$

From the definitions, every open set in Ω is capacitable. If e is any compact set in Ω, then given $\varepsilon > 0$, by (1.5), there exists an open set G such that $\mathrm{Cap}_X(G,\Omega) \le \mathrm{Cap}_X(e,\Omega) + \varepsilon$, and consequently, all compact subsets of Ω are capacitable.

This capacity will be revisited on Chapter 3.

Example 1.2.10. *Let* (Ω,Σ,μ) *be a measure space. A set function* E *on* Σ *is called a* quasi-entropy *function if it satisfies:*

(a) $0 \le E(A) \le \infty$,

(b) $E(A)$ *if and only if* $\mu(A) = 0$,

(c) $E(A) \le E(B)$ *if* $A \subset B$,

(d) $\lim_{k\to\infty} E(A_k) = E(A)$ *if* $A_k \uparrow A$, *and*

(e) $E(A\cup B) \lesssim E(A) + E(B)$[1].

Therefore, every quasi-entropy function is a capacity.

Example 1.2.11. *Let* h *be a continuous increasing function on* $[0,\infty)$ *such that* $h(0) = 0$, *which is called a measure function in [Car], and let* μ_h *be the corresponding Hausdorff measure on* $\mathbb{R}^n$. *Let* I *or* I_k *denotes a general cube in* $\mathbb{R}^n$ *with its sides parallel to the axes.*

In many problems, the Hausdorff capacity

$$E_h(A) := \inf_{A\subset\bigcup_{k=1}^{\infty} I_k} \Big\{ \sum_{k=1}^{\infty} h(|I_k|) \Big\} \tag{1.6}$$

is more convenient than μ_h, *and it satisfies that* $E_h(A) = 0$ *if and only if* $\mu_h(A) = 0$.

[1] In all this memoir, the symbol $f \lesssim g$ will mean that there exists a universal constant $c > 0$ (independent of all parameters involved) such that $f \le cg$, and the symbol $f \simeq g$ will mean that $f \lesssim g \lesssim f$.

*If $h(t) = t^\alpha$ ($\alpha > 0$), it is customary to write H^α instead of E_h, and this capacity is called the α-*dimensional Hausdorff content *presented in the introduction of this chapter.*

If the measure function is $h(x) := x\log(1/x)$ on $[0, 1/e]$ ($h(x) := 1/e$ if $x \geq 1/e$), we obtain the Shannon entropy*, considered in [Fe]. This capacity is concave on dyadic cubes, see [CeClM].*

Example 1.2.12. *Let E be a quasi-Banach function space on the measure space (Ω, Σ, μ). The* associated capacity *C_E is defined on (Ω, Σ) as*

$$C_E(A) := \|\chi_A\|_E \qquad (A \in \Sigma). \tag{1.7}$$

This capacity is subadditive in the normed case and, as in the case of Hausdorff's capacities, there is a measure μ such that $C_E(A) = 0$ if and only if $\mu(A) = 0$. It is a quasi-entropy function.

Definition 1.2.13. *Given $x_0 \in \mathbb{R}^n$ and $\varepsilon > 0$, a real function g in $\mathbb{R}^n$ is said to be* lower semicontinuous *at x_0 if, for every $\varepsilon > 0$, there exists a neighbourhood U of x_0 such that $g(x) \geq g(x_0) - \varepsilon$ for all $x \in U$.*

$\widehat{f}$ denotes the Fourier transform *of f an integrable function.*

Example 1.2.14. *Let $1 \leq p < \infty$ and $g : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ be such that every $g(\cdot, y)$ is lower semicontinuous and every $g(x, \cdot)$ is measurable.*

Then by $C_{g,p}$ we denote the capacity defined on every $E \subset \mathbb{R}^n$ by

$$C_{g,p}(E) = \inf\Big\{ \int f(y)^p dy;\ 0 \leq f \in L^p,\ \mathcal{G}f(x) := \int g(x,y) f(y)\, dy \geq 1 \text{ on } E \Big\}.$$

For such a capacity, by a result due to Choquet, every Borel set $B \subset \mathbb{R}^n$ is capacitable.

If $g(x, y)$ is $I_\alpha(x - y)$ or $G_\alpha(x - y)$, where $\widehat{I}_\alpha(\xi) = |\xi|^{-\alpha}$, $\widehat{G}_\alpha(\xi) = (1 + |\xi|^2)^{-\alpha/2}$ and $0 < \alpha < n$, the corresponding capacities are the fundamental Riesz *and* Bessel capacities *of potential theory, $R_{\alpha,p}$ and $B_{\alpha,p}$, respectively. See [AH] and [Ma85] for an extended overview.*

Example 1.2.15. *The* variational capacity *C_p on $\mathbb{R}^n$ is defined for $1 \leq p < n$ as in [EvGa] by*

$$C_p(A) := \inf\Big\{ \int_{\mathbb{R}^n} |Df|^p dx;\ f \in K^p, A \subset \{f \geq 1\}^\circ \Big\} \quad (E^\circ \text{is the interior of } E)$$

for any $A \subset \mathbb{R}^n$ with $p^* = \frac{np}{n-p}$

$$K^p := \{f : \mathbb{R}^n \to \mathbb{R};\ f \geq 0, f \in L^{p^*}(\mathbb{R}^n), Df \in L^p(\mathbb{R}^n, \mathbb{R})\},$$

and using regularization, for a compact set K in $\mathbb{R}^n$ we have that

$$C_p(K) := \inf\Big\{\int |\nabla\varphi(x)|^p dx;\ 0 \le \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n),\ K \subset \{f \ge 1\}\Big\}.$$

It is a countably subadditive and concave set function.

1.2.3 Fatou's lemma

Let us show some extensions to capacities of well-known results in measure theory.

*If $f = g$ $C-$*q.e. and C is *subadditive*, then $\int f\,dC = \int g\,dC$ since, if $A = \{f \ne g\}$, then

$$C\{f > t\} \le C\Big((\{f > t\} \cap A^c) \cup (\{f > t\} \cap A)\Big) \le C\{g > t\}.$$

This will be also true for any quasi-subadditive capacity C such that $C(A_n) \to C(A)$ whenever $A_n \uparrow A$, that is, that C is *Fatou* or that C has the Fatou property.

Example 1.2.16. *Every entropy is a Fatou capacity, so that, the Shanon entropy in Example 1.2.11 is Fatou, and so is also C_p.*

As we assume that every quasi-Banach function space (see Definition 1.3.2) has the Fatou property, then the capacities in Examples 1.2.8 and 1.2.9 are Fatou.

As far as we know, an interesting still open problem related with the analytic capacity is to prove whether this capacity is Fatou or not.

Let us observe that, if C is a Fatou quasi-subadditive capacity, then the countable union of $C-$null sets are also $C-$null. Indeed,

$$C(A_1 \cup \cdots \cup A_n) \le c^n(C(A_1) + \cdots + C(A_n)) = 0 \text{ if } C(A_k) = 0\ (\forall\, k \in \mathbb{N}),$$

and then

$$C\Big(\bigcup_{k=1}^{\infty} A_k\Big) = \lim_{n\to\infty} C(A_1 \cup \cdots \cup A_n) = 0.$$

Observe also that if $\chi_A = \chi_B$ $C-$q.e., then since $f_n := \chi_A \to \chi_B$ $C-$q.e. it follows from the Fatou property that $C(A) = C(B)$.

From now on, we consider two functions, f and g, to be equivalent if they are equal $C-$q.e. In this case $|f|$ and $|g|$ are also equivalent and $\int |f|\,dC = \int |g|\,dC$, since $C\{|f| > t\} = C\{|g| > t\}$ for every $t \ge 0$. Thus, $\int |f|\,dC = 0$ if and only if $f = 0$ $C-$q.e.

Note that, if a Fatou capacity is subadditive, then it is σ-subadditive, since finite subadditivity in combination with the Fatou property imply countable subadditivity.

As it is known (see [Fed, Lemma 2.4.6]) the classical *Fatou lemma* states that, if $f_1, f_2, \ldots$ are non-negative μ-measurable functions where μ is a measure on (Ω, Σ), then

$$\liminf_{n\to\infty} \int f_n d\mu \geq \int \liminf_{n\to\infty} f_n d\mu.$$

Let us gather together some properties of capacities concerning sequence of functions $\{f_n\}_{n\in\mathbb{N}} \subset \mathrm{L}_0(\Omega)$ which extend the corresponding facts for measures:

Theorem 1.2.17. *The following properties are equivalent:*

(i) C *is a Fatou capacity.*

(ii) $|f| \leq \liminf_n |f_n| \implies f_C^\star \leq \liminf_n (f_n)_C^\star$.

(iii) $\int (\liminf_n |f_n|)\, dC \leq \liminf_n \int |f_n|\, dC$.

(iv) $0 \leq f_n \uparrow f \implies (f_n)_C^\star \uparrow f_C^\star$.

Proof. (iii) follows from (ii) and (i) follows from (iii) by taking $f_n = \chi_{A_n}$.

Suppose now that C satisfies (i) and that $|f| \leq \liminf_n |f_n|$. Let $A^t := \{|f| > t\}$ and $A_n^t := \{|f_n| > t\}$. Then,

$$A^t \subset \liminf_n A_n^t = \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} A_n^t$$

and by (i),

$$C(A^t) \leq \lim_m C\Big(\bigcap_{n=m}^{\infty} A_n^t\Big) \leq \liminf_n C(A_n^t),$$

so that

$$\chi_{[0,C(A^t))} \leq \liminf_n \chi_{[0,C(A_n^t))}$$

and

$$f_C^\star(x) = \int_0^\infty \chi_{[0,C(A^t))}(x)\, dt \leq \liminf_n \int_0^\infty \chi_{[0,C(A_n^t))}(x)\, dt = \liminf_n (f_n)_C^\star(x),$$

which is (ii).

Moreover, (i) follows from (ii) by taking $f_n = \chi_{A_n}$ and $f = \chi_A$.

Suppose now that C satisfies (i) and that $0 \leq f_n \uparrow f$. Then $(f_n)_C^\star \leq f_C^\star$ and hence $\lim_{n\to\infty} (f_n)_C^\star(x) \leq f_C^\star(x)$. Let $A_n^t := \{|f_n| > t\}$ and $A^t := \{|f| > t\}$. From (i) we obtain that $C(A^t) = \lim_{n\to\infty} C(A_n^t)$ and

$$f_C^\star(x) = \int_0^\infty \chi_{[0,C(A^t))}(x)\, dt = \int_0^\infty \lim_{n\to\infty} \chi_{[0,C(A_n^t))}(x)\, dt \leq \lim_{n\to\infty} (f_n)_C^\star(x). \qquad \square$$

Corollary 1.2.18. *Let C be a Fatou capacity on (Ω, Σ) and $\{f_n\}_{n\in\mathbb{N}} \subset \mathrm{L}_0(\Omega)$. Suppose that for all $x \in \Omega$*

(i) $0 \le f_1(x) \le \ldots \le f_n(x)$, *and*

(ii) $f_n(x) \overset{n\to\infty}{\longrightarrow} f(x)$.

Then, f is a measurable function and $\int_\Omega f_n dC \overset{n\to\infty}{\longrightarrow} \int_\Omega f dC$.

Proof. Observe that $\int_0^\infty f_C^\star(x)\, dx = \int f\, dC$. □

Hölder and Minkowski inequalities are fundamental in the theory of Lebesgue spaces. Let μ be a measure, $1 < p < \infty$ and q the *conjugate exponent* of p, that is, $1/p + 1/q = 1$. Then

$$\int |fg| d\mu \le \Big(\int |f|^p d\mu\Big)^{1/p} \Big(\int |g|^q d\mu\Big)^{1/q} \quad \text{(Hölder's inequality)}$$

expresses that functions in $L^q(\mu)$ give rise to bounded linear functionals on $L^p(\mu)$. This inequality is sharp in the sense that given $f \in L^p(\mu)$ there is a function $g \in L^q(\mu)$ such that the inequality becomes an equality. For this reason, improvements or extensions of Hölder's inequality must necessarily be quite delicate.

In the case of capacities let us see that we can extend these inequalities without extra assumptions.

Theorem 1.2.19. *Let C be a concave capacity on (Ω, Σ), $1 \le p \le \infty$ and q the conjugate exponent of p. Then, Hölder and Minkowski inequalities hold:*

$$\int_\Omega |fg| dC \le \Big(\int_\Omega |f|^p dC\Big)^{1/p} \Big(\int_\Omega |g|^q dC\Big)^{1/q}$$

$$\Big(\int_\Omega |f+g|^p dC\Big)^{1/p} \le \Big(\int_\Omega |f|^p dC\Big)^{1/p} + \Big(\int_\Omega |g|^p dC\Big)^{1/p}. \tag{1.8}$$

When $p = \infty$ the integral should be replaced by the essential supremum.

Proof. We write

$$|fg| = (|f|^p)^{1/p} (|g|^q)^{1-1/p}.$$

Since

$$b^\theta c^{1-\theta} = \min_{\epsilon>0} \{\theta \epsilon^{\theta-1} b + (1-\theta)\epsilon^\theta\} \ \ (b, c > 0,\ 0 \le \theta \le 1),$$

the inequality $a \le b^\theta c^{1-\theta}$ holds if and only if $a \le \theta\epsilon^{\theta-1} b + (1-\theta)\epsilon^\theta c$ for all $\epsilon > 0$. By taking $\theta = 1/p$, $a = |fg|$, $b = |f|^p$, and $c = |g|^q$ we obtain $|fg| \le \theta\epsilon^{\theta-1}|f|^p + (1-\theta)\epsilon^\theta |g|^q$.

Hence, since the capacity is concave, by [Ce, Theorem 5.1] it follows that

$$\int_\Omega |fg|\,dC \le \theta\epsilon^{\theta-1}\int_\Omega |f|^p\,dC + (1-\theta)\epsilon^\theta\int_\Omega |g|^q\,dC.$$

Denote $A = \int_\Omega |f|^p\,dC$, $B = \int_\Omega |g|^q\,dC$, and $\gamma(\epsilon) = \theta\epsilon^{\theta-1}A + (1-\theta)\epsilon^\theta B$. Then we have that $\int_\Omega |fg|\,dC \le \gamma(\epsilon)$ for all $\epsilon > 0$. But γ reaches its minimum at $\epsilon_0 = A/B$, so that

$$\int_\Omega |fg|\,dC \le \gamma(\epsilon_0) = \frac{A^\theta}{B^{\theta-1}} = \Big(\int_\Omega |f|^p\,dC\Big)^{1/p}\Big(\int_\Omega |g|^q\,dC\Big)^{1/q}.$$

Minkowski's inequality follows in the usual way. □

With the same arguments,

Corollary 1.2.20. *Let C be a quasi-subadditive capacity on (Ω,Σ) with constant $c > 1$ and $p, q \ge 1$ conjugate exponents. Then*

$$\int_\Omega |fg|dC \le 2c\Big(\int_\Omega |f|^p dC\Big)^{1/p}\Big(\int_\Omega |g|^q dC\Big)^{1/q}$$

and

$$\Big(\int_\Omega |f+g|^p dC\Big)^{1/p} \le 4c^2\Big[\Big(\int_\Omega |f|^p dC\Big)^{1/p} + \Big(\int_\Omega |g|^p dC\Big)^{1/p}\Big].$$

A natural question is whether the concavity condition is necessary to obtain Hölder's inequality with constant one. In the following example we see that this is true, concavity is necessary to get Hölder's inequality, and consequently, Minkowski's inequality.

Example 1.2.21. *Let $(\Omega,\Sigma,\mu) = (\mathbb{R},\mathcal{B}(\mathbb{R}),m)$, $\varphi(x) := x^p$ for $x \in \mathbb{R}_+$, $1 < p < \infty$, and define, for all $A \subset \mathbb{R}$, $C_\varphi(A) := \varphi(m(A)) = m(A)^p$. For $A, B \subset \mathbb{R}_+$ we have*

$$\begin{aligned} C_\varphi(A\cup B) &= (m(A\cup B))^p = (m(A)+m(B)-m(A\cap B))^p \\ &\le (m(A)+m(B))^p \le 2^p(m(A)^p + m(B)^p) \\ &= 2^p(C_\varphi(A)+C_\varphi(B)). \end{aligned}$$

Hence, C_φ is quasi-subadditive.

Let p' be the conjugate exponent of p and $C \subset \mathbb{R}_+$. We have that

$$\|\chi_C\|_{L^p(C_\varphi)} = \Big(\int_\Omega \chi_C^p dC_\varphi\Big)^{1/p} = C_\varphi(C)^{1/p} = m(C)$$

$$\|\chi_C\|_{L^{p'}(C_\varphi)} = \Big(\int_\Omega \chi_C^{p'} dC_\varphi\Big)^{1/p'} = C_\varphi(C)^{1/p'} = m(C)^{p/p'}.$$

It is easily seen that C_φ is not necessarily concave. Indeed, define $A := [0,1]$ and take $b > 1$. Then, for $\epsilon > 0$ taking B as (ϵ, b) we have that $C_\varphi(A) = 1$, $C_\varphi(B) = (b-\epsilon)^p$, $C_\varphi(A \cup B) = b^p$, and $C(A \cap B) = (1-\epsilon)^p$. Hence, if ϵ is selected so that $(b-\epsilon)^p < b^p + (1-\epsilon)^p - 1$, then we obtain

$$C_\varphi(A) + C_\varphi(B) < C_\varphi(A \cup B) + C_\varphi(A \cap B).$$

Moreover,

$$\|\chi_B\|_{L^p(C_\varphi)} \|\chi_A\|_{L^{p'}(C_\varphi)} = m(B)m(A)^{p/p'} = (b-\epsilon)$$
$$\int_\Omega \chi_A \chi_B dC_\varphi = C_\varphi(A \cup B) = (1-\epsilon)^p$$

and then, if ϵ satisfies also that $b - \epsilon < (1-\epsilon)^p$, we get that Hölder's inequality does not hold.

1.3 Capacitary Lebesgue and Lorentz spaces

From now on, C will represent a Fatou capacity on (Ω, Σ) and $c \geq 1$ its quasi-subadditivity constant.

In this section we study the completeness of the capacitary Lebesgue and Lorentz spaces.

Definition 1.3.1. *A mapping $\varrho : \mathrm{L}_0(\Omega)^+ \to [0, \infty]$ is a function* quasi-norm *if for all f, g, f_n, $n = 1, 2, \ldots$ in $\mathrm{L}_0(\Omega)^+$, $a \in \mathbb{R}_+$ and E be a measurable subset of Ω, the following conditions hold*

(a) $\varrho(f) = 0 \Longleftrightarrow f = 0$ $C-$q.e., $\varrho(af) = a\varrho(f)$, $\varrho(f+g) \lesssim \varrho(f) + \varrho(g)$.

(b) If $0 \leq g \leq f$ $C-$q.e., then $\varrho(g) \leq \varrho(f)$.

(c) If $0 \leq f_n \uparrow f$ $C-$q.e., then $\varrho(f_n) \uparrow \varrho(f)$.

(d) If $C(E) < \infty$, then $\varrho(\chi_E) < \infty$.

(e) If $C(E) < \infty$, then $\int_E f dC \leq C_E \varrho(f)$ for some constant C_E, $0 < C_E < \infty$.

Definition 1.3.2. *Let ϱ be a function quasi-norm on (Ω, Σ, C). The space $X = X(\varrho)$ defined as*

$$X := \{f \in \mathrm{L}_0(\Omega); \varrho(|f|) < \infty\}$$

is called a quasi-Banach function space. *Moreover, for $f \in X$ we define $\|f\|_X := \varrho(|f|)$.*

We are going to carefully check those property of the usual Banach function spaces that extend to our capacitary setting.

Using the same argument as in [BeSh] we first obtain the following lemma.

Lemma 1.3.3. *Let X be a quasi-Banach function space on $\mathrm{L}_0(\Omega)$ and suppose $f_n \in X$, $n = 1, 2, ...$*

(i) If $0 \leq f_n \uparrow f$ $C-$q.e., then either f is not in X and $\|f_n\| \uparrow \infty$, or $f \in X$ and $\|f_n\|_X \uparrow \|f\|_X$.

(ii) If $f_n \to f$ $C-$q.e. and $\lim_{n\to\infty} \|f_n\|_X < \infty$, then $f \in X$ and

$$\|f\|_X \leq \liminf_{n\to\infty} \|f_n\|_X.$$

Proof. See [BeSh]. □

Let us present a useful property of these capacitary Lebesgue and Lorentz spaces.

Definition 1.3.4. *A vector subspace of $\mathrm{L}_0(\Omega)$, X, endowed with a quasi-norm it is called a* quasi-normed lattice *if $|f| \leq |g|$, with $g \in X$ and $f \in \mathrm{L}_0(\Omega)$, implies $f \in X$ and $\|f\|_X \leq \|g\|_X$.*

Let $\mathrm{L}_0(C)$ be the real vector space of all measurable functions, two functions being equivalent if they coincide $C-$q.e. We endow $\mathrm{L}_0(C)$ with the topology of the convergence in capacity on every set of finite capacity and with the lattice structure given by the partial order $f \leq g$, that is, $f \leq g$ $C-$q.e.

In relation with Definition 1.3.1, we say that $E \subset \mathrm{L}_0(C)$ is a *quasi-normed capacitary function space* on (Ω, Σ, C) with constant $k > 0$ if

$$E = \{f \in \mathrm{L}_0(C);\ \varrho(f) < \infty\},$$

where $\varrho : \mathrm{L}_0^+(\Omega) \to [0, \infty]$ is a mapping which satisfies conditions (a), (b), (d) and (e) with $k = c$ in Definition 1.3.1 and such that, if $\varrho(f) < \infty$, then the support $\{f > 0\}$ is C-sigma-finite, that is, $\{f > 0\} = \bigcup_{k=1}^{\infty} \Omega_k$ with $C(\Omega_k) < \infty$ for every $k \in \mathbb{N}$. Then, we define on E the *quasi-norm* $\|f\|_E := \varrho(|f|)$, that does not depend on the representative.

Theorem 1.3.5. *For each quasi-normed capacitary function space E the following conditions are equivalent:*
(i) If $\sup_n \|f_n\|_E = M < \infty$ and $f_n \to f$ C-q.e., then $f \in E$ and $\|f\|_E \leq \liminf_n \|f_n\|_E$.
(ii) If $0 \leq f_n \uparrow f$ C-q.e., then $\lim_n \varrho(f_n) = \varrho(f)$.

Proof. To prove that (i) implies (ii), let $0 \le f_n \uparrow f$ $C-$q.e. If $\varrho(f) < \infty$, then $\varrho(f) = \|f\|_E \le \lim_n \|f_n\|_E = \varrho(f_n)$ by (i) and $\varrho(f_n) \le \varrho(f)$ $(n \in \mathbb{N})$. So that $\lim_n \varrho(f_n) = \varrho(f)$. If $\varrho(f) = \infty$, since $f_n \uparrow f$ $C-$q.e, necessarily $\lim_n \varrho(f_n) = \infty$ because $\sup_n \varrho(f_n) = M < \infty$ would imply $f \in E$ by (i).

To prove the converse, suppose that (ii) holds and that $\{f_n\}_{n\in\mathbb{N}}$ satisfies that $\sup_n \|f_n\|_E = M < \infty$ and $f_n \to f$ $C-$q.e. Define $g_n := \inf_{m\ge n} |f_m|$ $(n \in \mathbb{N})$, so that $g_n \uparrow |f|$ $C-$q.e. and $\|f\|_E = \varrho(|f|) = \lim_n \varrho(g_n)$. Since $g_n \le |f_m|$ for every $m \ge n$, it follows that $\varrho(g_n) \le \inf_{m\ge n} \varrho(|f_m|)$ and then $\|f\|_E \le \lim_n \inf_{m\ge n} \varrho(|f_m|) = \liminf_n \|f_n\|_E$. □

Conditions (i) and (ii) are called the Fatou conditions. If they hold, then we say that E has the Fatou property.

Theorem 1.3.6. *Every quasi-normed capacitary function space E on (Ω, Σ, C) is continuously imbedded in* $\mathrm{L}_0(C)$.

Proof. It is sufficient to prove that the condition $\|f_n\|_E \to 0$ for $\{f_n\}_{n\in\mathbb{N}} \subset E$ implies $f_n \to 0$ in capacity on any set Ω_0 of finite capacity.

Assume the contrary, so that, there exists a set Ω_0 with $0 < C(\Omega_0) < \infty$ and a positive number ε such that for some subsequence f_{n_k}, the inequality $|f_{n_k}(t)| > \varepsilon$ is satisfied on a set $\Omega_k \subset \Omega_0$ with capacity $C(\Omega_k) > \delta > 0$, for all $k = 1, 2, \ldots$ Then $\varepsilon\chi_{\Omega_k}(t) \le |f_{n_k}(t)|$ and so $\varepsilon\|\chi_{\Omega_k}\|_E \le \|f_{n_k}\|_E$. Since $C(\Omega_0) < \infty$ we have that

$$\frac{\varepsilon}{C_E} \int \chi_{\Omega_k} dC \le \varepsilon\|\chi_{\Omega_k}\|_E \le \|f_{n_k}\|_E,$$

and letting $k \to \infty$, it follows that $\lim_k C(\Omega_k) = 0$, which is impossible. So that, $f_n \to 0$ in capacity on any set of finite capacity. □

1.3.1 Capacitary Lebesgue spaces

The *Lebesgue space* $L^p(C)$ $(p > 0)$ is defined by the condition

$$\varrho_p(f) := \begin{cases} (\int_\Omega |f|^p dC)^{1/p} < \infty, & 0 < p < \infty \\ \inf\{M > 0; C\{|f| \ge M\} = 0\} < \infty, & p = \infty. \end{cases} \tag{1.9}$$

Proposition 1.3.7. *Let $1 \le p \le \infty$ and C be quasi-subadditive with constant $c > 1$. Then ϱ_p is a quasi-norm in $L^p(C)$. In particular, if C is concave, then ϱ_p is a norm in $L^p(C)$.*

Proof. If C is concave, by Minkowski's inequality (1.8), we obtain

$$\varrho_p(f + g) \le \varrho_p(f) + \varrho_p(g).$$

The remaining parts of (a) and (b) are obvious.

If $C(E) < \infty$, then $\varrho_p(\chi_E) = C(E)^{1/p} < \infty$. Moreover, if $f_n \uparrow f$ $C-$q.e., then $0 \le f_1 \le \ldots \le \lim_{n\to\infty} f_n = f$ $C-$q.e., $|f|^p = \lim_{n\to\infty} |f_n|^p$ $C-$q.e., and

$$\lim_{n\to\infty} \varrho_p(f_n)^p = \lim_{n\to\infty} \int_\Omega |f_n|^p dC = \int_\Omega |f|^p dC = \varrho_p(f)^p.$$

Finally assume that $C(E) < \infty$. Then, by Hölder's inequality, $\int_E f dC \le \varrho_p(f) C(E)^{1/p'}$ where p' is the conjugate exponent of p, and the proof in this case follows.

In the general case, by the corresponding Hölder's and Minkowki's inequalities the proof follows. □

Notice that, $L^p(C)$ is a quasi-normed lattice of $\mathrm{L}_0(\Omega)$ for every $p > 0$.

As for Lebesgue function spaces, there are several descriptions of these "norms":

Theorem 1.3.8.

$$\|f\|_{L^p(C)} = \|f_C^\star\|_p = \||f|^p\|_{L^1(C)}^{1/p} = \Big(p \int_0^\infty t^{p-1} C\{|f| > t\}\, dt\Big)^{1/p}.$$

Proof. Let $\psi(t) = t^p$. Then $\int_0^\infty \psi(f_C^\star(t))\, dt = \int_0^\infty \psi(|f|)_C^\star(t)\, dt$ and, if we denote $g = \psi(|f|)$, an application of Fubini's theorem gives

$$\begin{aligned} \int_0^\infty g_C^\star(t)\, dt &= \int_0^\infty \int_0^\infty \chi_{[0,C\{g>x\})}(t)\, dx\, dt \\ &= \int_0^\infty \int_0^\infty \chi_{[0,C\{g>x\})}(t)\, dt\, dx = \int_0^\infty C\{g > x\}\, dx, \end{aligned}$$

this is, $\int_0^\infty \psi(f)_C^\star(t)\, dt = \int_0^\infty C\{\psi(|f|) > t\}\, dt$.

Also, if $x = \psi(t)$, then

$$\int_0^\infty C\{|f| > t\}\, d\psi(t) = \int_0^\infty C\{|f| > \psi^{-1}(x)\}\, dx = \int_0^\infty C\{\psi(|f|) > x\}\, dx$$

and $\int_0^\infty C\{|f| > t\}\, d\psi(t) = \int_0^\infty C\{\psi(|f|) > t\} dt$. □

Many important examples of capacities are not concave and the corresponding Lebesgue spaces are quasi-normed lattices. Natural questions for these capacitary Lebesgue spaces are to find the best constant in the "triangle inequality" and to analyze when the spaces are complete.

Theorem 1.3.9. *The functional* $\|\cdot\| := \|\cdot\|_{L^p(C)}$ *is quasi-subadditive, with constant* $c_p = (2c)^{1/p}$ *if* $1 \le p < \infty$ *and* $c_p = c^{1/p} 2^{(2-p)/p}$ *if* $0 < p < 1$.

Proof. Suppose $1 \le p < \infty$. By (1.3),

$$\|f+g\|^p \le \int_0^\infty \Big(f_C^\star\Big(\frac{x}{2c}\Big) + g_C^\star\Big(\frac{x}{2c}\Big)\Big)^p\,dx = 2c\|f_C^\star + g_C^\star\|_p^p$$

and the result follows from the estimates for $L^p(\mathbb{R}^+)$.

If $p<1$, then since $a^p + b^p \le 2^{1-p}(a+b)^p$ $(a,b \ge 0)$, we conclude that

$$\|f_C^\star + g_C^\star\|_p^p \le \int_0^\infty f_C^\star(y)^p\,dy + \int_0^\infty g_C^\star(y)^p\,dy \le 2^{1-p}(\|f_C^*\|_p + \|g_C^*\|_p)^p,$$

and $\|f+g\| \le (2c)^{1/p}2^{(1-p)/p}(\|f\| + \|g\|)$. □

Now recall that, if $\|\cdot\|$ is a quasi-seminorm with constant $c \ge 1$ and $(2c)^\varrho = 2$, then, by *Aoki-Rolewicz's theorem*, there exists a $\varrho-$*seminorm* $\|\cdot\|^*$ such that

$$\|f\|^* \le \|f\|^\varrho \le 2\|f\|^*. \tag{1.10}$$

This $\varrho-$seminorm, constructed as in [BeLo, Section 3.10] by

$$\|f\|^* := \inf\Big\{\sum_{j=1}^n \|f_j\|^\varrho;\ n \ge 1,\ \sum_{j=1}^n f_j = f\Big\},$$

allows to prove (1.10) and the triangle inequality. The $\varrho-$homogeneity follows also very easily in our case since obviously, if $\sum_{j=1}^n f_j = f$, then for $\lambda \in \mathbb{R}$

$$\|\lambda f\|^* \le \sum_{j=1}^n \|\lambda f_j\|^\varrho = |\lambda|^\varrho \sum_{j=1}^n \|f_j\|^\varrho,$$

so that $\|\lambda f\|^* \le |\lambda|^\varrho \|f\|^*$. Conversely, if $\lambda \ne 0$ and $\lambda f = g_1 + \cdots + g_n$, then

$$\|f\|^* \le \sum_{j=1}^n \|\lambda^{-1} g_j\|^\varrho = |\lambda|^{-\varrho} \sum_{j=1}^n \|g_j\|^\varrho$$

and $|\lambda|^\varrho \|f\|^* \le \|\lambda f\|^*$.

It follows from (1.10) that

$$\Big\|\sum_i |f_i|\Big\| \le 2^{1/\varrho}\Big(\sum_i \|f_i\|^*\Big)^{1/\varrho} \le 2^{1/\varrho}\Big(\sum_i \|f_i\|^\varrho\Big)^{1/\varrho}. \tag{1.11}$$

In the special case $f_i = \chi_{A_i}$ and $p=1$ we obtain

$$C\Big(\bigcup_{i=1}^\infty A_i\Big)^\varrho \le 2\sum_{i=1}^\infty C(A_i)^\varrho. \tag{1.12}$$

Definition 1.3.10. *We say that* $\{f_n\}_{n\in\mathbb{N}}$ converges to f in capacity *if*

$$C\{|f_n - f| > \epsilon\} \to 0, \ as\, n \to \infty, \forall\, \epsilon > 0.$$

Similarly, we say that $\{f_n\}_{n\in\mathbb{N}}$ *is a* Cauchy sequence in capacity *if for every* $\epsilon > 0$, $C\{|f_p - f_q| > \epsilon\} \to 0$ *as* $p, q \to \infty$. *That is, when for every* $\epsilon > 0$ *and* $\eta > 0$, *there exists* $n_0 \in \mathbb{N}$ *such that*

$$C\{|f_p - f_q| > \epsilon\} < \eta \qquad (p, q \geq n_0).$$

Let us remark that, if the sequence $\{f_n\}_{n\in\mathbb{N}}$ converges to f in capacity, the relation

$$\{|f_p - f_q| > \epsilon\} \subset \{|f_p - f| > \epsilon/2\} \cup \{|f_q - f| > \epsilon/2\}$$

shows that $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in capacity. Aoki-Rolewicz's theorem allows us to prove that the converse is also true in this capacitary setting.

Theorem 1.3.11. *A sequence* $\{f_n\}_{n\in\mathbb{N}}$ *is convergent in capacity to a function* f *if and only if it is a Cauchy sequence in capacity. In this case, the sequence has a subsequence which is* $C-$q.e. convergent to f.

Proof. If $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in capacity, there exists $n_k \in \mathbb{N}$ so that

$$C\{|f_p - f_q| > 2^{-k}\} < 2^{-k} \qquad (p, q \geq n_k),$$

and we can suppose that $n_1 < n_2 < \cdots$.

We associate to $\{f_{n_k}\}_{k\in\mathbb{N}}$ the sets $A_k := \{|f_{n_k} - f_{n_{k+1}}| > 1/2^k\}$ and denote $F_m := \bigcup_{k\geq m} A_k$. If $j \geq i \geq m$, we have that $|f_{n_i} - f_{n_j}| \leq 1/2^{m-1}$ on $\Omega \setminus F_m$. In other words, the partial sequence is uniformly Cauchy on $\Omega \setminus F_m$ and then $\{f_{n_k}\}_{k\in\mathbb{N}}$ converges uniformly to f on $\Omega \setminus F_m$. The sequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ converges to a function f on $E := \bigcup_{m=1}^{\infty}(\Omega \setminus F_m)$ and, by (1.11),

$$\begin{aligned} C(\Omega \setminus E) &\leq \lim_{m\to\infty} C(F_m) = \lim_{m\to\infty} \|\chi_{F_m}\|_{L^1(C)} = \lim_{m\to\infty} \|\chi_{\bigcup_{k\geq m} A_k}\|_{L^1(C)} \\ &\leq \lim_{m\to\infty} \Big\| \sum_{k\geq m} \chi_{A_k} \Big\|_{L^1(C)} \leq \lim_{m\to\infty} 2^{1/\varrho} \Big(\sum_{k\geq m} \|\chi_{A_k}\|^{\varrho}_{L^1(C)} \Big)^{1/\varrho} \\ &= \lim_{m\to\infty} 2^{1/\varrho} \Big(\sum_{k\geq m} C(A_k)^{\varrho} \Big)^{1/\varrho} = 0. \end{aligned}$$

Since $|f_{n_k} - f_{n_j}| \to |f_{n_k} - f|$ pointwise, by the Fatou property

$$C\{|f_{n_k} - f| > \eta\} = C\{\lim_{j\to\infty} |f_{n_k} - f_{n_j}| > \eta\} \leq \lim_{j\to\infty} C\{|f_{n_k} - f_{n_j}| > \eta\} < \epsilon.$$

Finally, since $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in capacity which has a subsequence which is convergent in capacity to f, then $\{f_n\}_{n\in\mathbb{N}}$ converges to f in capacity. $\square$

The topology and the uniform structure of $L^p(C)$ are given by the *metric*

$$d(f,g) := \|f-g\|^*,$$

where $\|\cdot\|^*$ is associated to $\|\cdot\|_{L^p(C)}$ by the Aoki-Rolewicz theorem.

Theorem 1.3.12. *$L^p(C)$ $(0<p<\infty)$ is complete.*

Proof. It follows by the usual arguments of measure theory combined with (1.12): Let $\{f_n\}_{n\in\mathbb{N}} \subset L^p(C)$ be a Cauchy sequence. For each $k\in\mathbb{N}$, let $n_k > n_{k-1}$ be such that

$$\|f_m - f_n\|^p = \int |f_m - f_n|^p \, dC < \frac{1}{3^k} \qquad (m,n \geq n_k).$$

If $A_k = \{|f_{n_{k+1}} - f_{n_k}|^p > 1/2^k\}$, then $C(A_k) < 2^k/3^k$ since

$$\frac{C(A_k)}{2^k} \leq \int_{A_k} |f_{n_{k+1}} - f_{n_k}|^p \, dC < \frac{1}{3^k}.$$

Note that

$$\sum_{k=1}^{\infty} |f_{n_{k+1}}(t) - f_{n_k}(t)| < \infty \quad \forall t \notin \bigcup_{k>N} A_k$$

because $|f_{n_{k+1}}(t) - f_{n_k}(t)| \leq 1/2^{k/p}$ if $k>N$. Therefore, there exists

$$f(t) := f_{n_1}(t) + \sum_{k=1}^{\infty} \big(f_{n_{k+1}}(t) - f_{n_k}(t)\big) = \lim_k f_{n_k}(t) \quad \forall t \notin A = \bigcap_{N=1}^{\infty} \bigcup_{k>N} A_k,$$

and $C(A)=0$ since, by (1.12)

$$C(A)^{\varrho} \leq C\Big(\bigcup_{k>N} A_k\Big)^{\varrho} \leq 2\sum_{k>N} \Big(\frac{2}{3}\Big)^{\varrho k}$$

and $\sum_{k>N}(2/3)^{\varrho k} < \infty$. Put $f(t) := 0$ if $t\in A$.

As $n_k \to \infty$, $|f_{n_k}(t) - f_n(t)|^p \to |f(t) - f_n(t)|^p$ $C-$q.e. and, by the Fatou property (see Theorem 1.2.17),

$$\int |f - f_n|^p \, dC \leq \liminf_k \int |f_{n_k} - f_n|^p \, dC \leq \varepsilon$$

for n large enough. $\square$

Example 1.3.13. *Although C_p (see Example 1.2.15) is not a Caratheodory metric outer measure, it follows from a general theorem due to G. Choquet that every Borel set $B \subset \mathbb{R}^n$ is* capacitable*, this meaning that*

$$\sup\{C_p(K);\, K \subset B,\, K \text{ compact}\} = C_p(B) = \inf\{C_p(G);\, G \supset B,\, G \text{ open}\}.$$

Hence, $L^p(C_p)$ defined in (1.9) by

$$L^p(C_p) := \Big\{ f \in \mathrm{L}_0(\Omega);\, \|f\|_{L^p(C_p)} := \Big(p \int_0^\infty t^{p-1} C_p\{|f| > t\} dt \Big)^{1/p} < \infty \Big\}$$

is an example of a capacitary Lebesgue space. For more details, see [EvGa].

Remark 1.3.14. *The absence of additivity for the Choquet integral makes it difficult to give a description of the dual of $L^p(C)$. See for instance [A2, Section 4], where duality in the case of Hausdorff and Bessel capacities is studied.*

If p' is the conjugate exponent of $p \in [1, \infty]$, Hölder's inequality shows that every $g \in L^{p'}(C)^+$ defines a functional $u_g(f) := \int f g\, dC$ which is homogeneous and bounded on $L^p(C)^+$

$$u_g(f) \le 2c \Big(\int g^{p'}\, dC \Big)^{1/p'} \Big(\int f^p\, dC \Big)^{1/p},$$

but in general u_g is not additive.

1.3.2 Capacitary Lorentz spaces

The capacitary Lorentz spaces $L^{p,q}(C)$ $(p, q > 0)$ are defined by the condition

$$\|f\|_{L^{p,q}(C)} := \begin{cases} \Big(q \int_0^\infty t^{q-1} C\{|f| > t\}^{q/p} dt \Big)^{1/q} < \infty, & q < \infty \\ \sup_{t>0} t C\{|f| > t\}^{1/p} < \infty, & q = \infty. \end{cases}$$

The space $L^{p,\infty}(C)$ is called the *weak capacitary L^p space.*

Let us observe that for $p, q > 0$, $\|f\|_{L^{p,q}(C)} = 0$ if and only if $f = 0$ $C-$q.e. and equivalent functions have the same $\|\cdot\|_{L^{p,q}(C)}$-quasi-norm. Moreover, for every $\lambda \in \mathbb{R}$, $\|\lambda f\|_{L^{p,q}(C)} = |\lambda| \|f\|_{L^{p,q}(C)}$ and

$$\|f + g\|_{L^{p,q}(C)} \le 2c(\|f\|_{L^{p,q}(C)} + \|g\|_{L^{p,q}(C)}).$$

Therefore, $L^{p,q}(C)$ is a quasi-normed function space. Easily it follows that, $L^{p,q}(C)$ is a quasi-normed lattice on $\mathrm{L}_0(\Omega)$ for all $p, q > 0$.

In order to study the completeness of the Lorentz spaces $L^{p,q}(C)$, which are quasi-normed spaces by considering equivalent two functions when they coincide $C-$q.e., we fix a $\varrho-$seminorm $\|\cdot\|^*$ associated to $\|\cdot\|_{L^{p,q}(C)}$.

Theorem 1.3.15. *$L^{p,q}(C)$ $(0 < p < \infty,\ 0 < q \leq \infty)$ is complete.*

Proof. Let us start with the case $1 \leq q < \infty$, and let $\{f_n\}_{n\in\mathbb{N}} \subset L^{p,q}(C)$ be a Cauchy sequence on this space. For every $k \in \mathbb{N}$, let $n_k \in \mathbb{N}$ so that

$$\|f_n - f_m\|^q_{L^{p,q}(C)} < \frac{1}{3^k} \qquad (m, n \geq n_k).$$

Define $A_k := \{|f_{n_{k+1}} - f_{n_k}| > \frac{1}{2^k}\}$. Then $A_k \subset \{|f_{n_{k+1}} - f_{n_k}| > t\}$ if $t < \frac{1}{2^k}$ and

$$\begin{aligned}
\frac{1}{3^k} &> \int_0^\infty qt^{q-1}C\{|f_{n_{k+1}} - f_{n_k}| > t\}^{\frac{q}{p}}dt \\
&\geq \int_0^{\frac{1}{2^k}} qt^{q-1}C(A_k)^{\frac{q}{p}}dt + \int_{\frac{1}{2^k}}^\infty qt^{q-1}C\{|f_{n_{k+1}} - f_{n_k}| > t\}^{\frac{q}{p}}dt \\
&= C(A_k)^{\frac{q}{p}}t^q\Big]_0^{\frac{1}{2^k}} + \int_{\frac{1}{2^k}}^\infty qt^{q-1}C\{|f_{n_{k+1}} - f_{n_k}| > t\}^{\frac{q}{p}}dt \geq C(A_k)^{\frac{q}{p}}\frac{1}{2^{kq}}.
\end{aligned}$$

Hence, $C(A_k) < \alpha_k$, $\sum_k \alpha_k < \infty$.

Moreover, if $t \notin \bigcup_{k>N} A_k$, then $t \notin A_{N+1}$, so that $|f_{n_{N+2}}(t) - f_{n_{N+1}}(t)| \leq 1/2^{N+1}$ and then $\sum_{k=1}^\infty |f_{n_{k+1}}(t) - f_{n_k}(t)| < \infty$. Therefore, there exists

$$f(t) := f_{n_1}(t) + \sum_{k=1}^\infty (f_{n_{k+1}}(t) - f_{n_k}(t)) = \lim_{k\to\infty} f_{n_k}(t)$$

for all $t \notin A := \bigcap_{N=1}^\infty \bigcup_{k>N} A_k$, and $C(A) = 0$ since

$$C(A)^\rho \leq C\Big(\bigcup_{k>N} A_k\Big)^\rho \leq 2\sum_{k>N} C(A_k)^\rho < 2\sum_{k>N}\Big(\frac{2^{pk}}{3^{\frac{kp}{q}}}\Big)^\rho.$$

Define $f(t) := 0$ if $t \in A$.

As $n_k \to \infty$, $|f_{n_k}(t) - f_n(t)| \to |f(t) - f_n(t)|$ $C-$q.e. and then, by Theorem 1.2.17,

$$\begin{aligned}
\|f - f_n\|^q_{L^{p,q}(C)} &= \int_0^\infty qt^{q-1}C\{|f - f_n| > t\}^{\frac{q}{p}}dt \\
&= \int_0^\infty qt^{q-1}C\Big\{\lim_{k\to\infty}|f_{n_k} - f_n| > t\Big\}^{\frac{q}{p}}dt \\
&\leq \lim_{k\to\infty}\int_0^\infty qt^{q-1}C\{|f_{n_k} - f_n| > t\}^{\frac{q}{p}}dt < \epsilon^q
\end{aligned}$$

for n large enough.

A similar argument applies to the case $q = \infty$. Let $\{f_n\}_{n\in\mathbb{N}} \subset L^{p,\infty}(C)$ be a Cauchy sequence and for all $k \in \mathbb{N}$, let $n_k \in \mathbb{N}$ so that

$$\|f_n - f_m\|_{L^{p,\infty}(C)} < \frac{1}{3^k} \qquad (m, n \geq n_k),$$

and define $A_k := \{|f_{n_{k+1}} - f_{n_k}| > 1/2^k\}$. Hence, $A_k \subset \{|f_{n_{k+1}} - f_{n_k}| > t\}$ if $t < 1/2^k$.

Thus

$$\begin{aligned}
\frac{1}{3^k} &> \sup_{t>0} tC\{|f_{n_{k+1}} - f_{n_k}| > t\}^{\frac{1}{p}} \\
&= \max\Big\{ \sup_{t\geq 1/2^k} tC\{|f_{n_{k+1}} - f_{n_k}| > t\}^{1/p}, \sup_{t<1/2^k} tC\{|f_{n_{k+1}} - f_{n_k}| > t\}^{1/p}\Big\} \\
&\geq \frac{1}{2^k} C(A_k)^{\frac{1}{p}}
\end{aligned}$$

and $C(A_k) < (2/3)^{kp}$.

If $t \notin \bigcup_{k>N} A_k$, then $t \notin A_{N+1}$ and $|f_{n_{N+2}}(t) - f_{n_{N+1}}(t)| \leq 1/2^{N+1}$. Then,

$$\sum_{k=1}^{\infty} |f_{n_{k+1}}(t) - f_{n_k}(t)| \leq \sum_{k=1}^{N} |f_{n_{k+1}}(t) - f_{n_k}(t)| + \sum_{k=N+1}^{\infty} \frac{1}{2^k} < \infty.$$

Therefore, there exists

$$f(t) := f_{n_1}(t) + \sum_{k=1}^{\infty} (f_{n_{k+1}}(t) - f_{n_k}(t)) = \lim_{k\to\infty} f_{n_k}(t)$$

for every $t \notin A := \bigcap_{N=1}^{\infty} \bigcup_{k>N} A_k$, where $C(A) = 0$ since

$$C(A)^\rho \leq C\Big(\bigcup_{k>N} A_k\Big)^\rho \leq 2\sum_{k>N} C(A_k)^\rho < 2\sum_{k>N} \Big((2/3)^{kp}\Big)^\rho.$$

Put $f(t) = 0$ if $t \in A$.

As $n_k \to \infty$, $|f_{n_k}(t) - f_n(t)| \to |f(t) - f_n(t)|$ $C-$q.e. and then, by the Fatou property (see Theorem 1.2.17),

$$\begin{aligned}
\|f - f_n\|_{L^{p,\infty}(C)} &= \sup_{t>0} tC\{\lim_{k\to\infty} |f_{n_k} - f_n| > t\}^{\frac{1}{p}} \\
&\leq \lim_{k\to\infty} \sup_{t>0} tC\{|f_{n_k} - f_n| > t\}^{\frac{1}{p}} < \epsilon
\end{aligned}$$

for n large enough.

Finally, in the case $p > 0$ and $0 < q < 1$, we observe that

$$\|f\|_{L^{p,q}(C)} := \Big(\int_0^\infty qt^{q-1} C\{|f| > t\}^{\frac{q}{p}} dt\Big)^{\frac{1}{q}} = \Big(\int_0^\infty C\{|f| > u^{1/q}\}^{\frac{q}{p}} du\Big)^{\frac{1}{q}}$$

and consider a Cauchy sequence $\{f_n\}_{n\in\mathbb{N}} \subset L^{p,q}(C)$, so that for every $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ such that

$$\|f_n - f_m\|^q_{L^{p,q}(C)} = \int_0^\infty C\{|f_n - f_m| > u^{1/q}\}^{\frac{q}{p}} du < \frac{1}{3^k} \qquad (m, n \geq n_k).$$

If $A_k := \{|f_{n_{k+1}} - f_{n_k}| > \frac{1}{2^{k\varrho}}\}$, then $A_k \subset \{x \in \Omega; |f_{n_{k+1}}(x) - f_{n_k}(x)| > u^{1/q}\}$ when $u^{1/q} < \frac{1}{2^{k\varrho}}$. Hence

$$\begin{aligned}
\frac{1}{3^k} &> \int_0^{\frac{1}{2^{kq}}} C\{|f_{n_{k+1}} - f_{n_k}| > u^{\frac{1}{q}}\}^{\frac{q}{p}} du + \int_{\frac{1}{2^{kq}}}^{\infty} C\{|f_{n_{k+1}} - f_{n_k}| > u^{\frac{1}{q}}\}^{\frac{q}{p}} du \\
&\geq \int_0^{\frac{1}{2^{kq}}} C(A_k)^{\frac{q}{p}} du + \int_{\frac{1}{2^{kq}}}^{\infty} C\{|f_{n_{k+1}} - f_{n_k}| > u^{\frac{1}{q}}\}^{\frac{q}{p}} du \\
&\geq C(A_k)^{\frac{q}{p}} \frac{1}{2^{qk}},
\end{aligned}$$

and $C(A_k) < \frac{2^{kp}}{3^{\frac{kp}{q}}}$.

If $t \notin \bigcup_{k>N} A_k$, then

$$\sum_{k=1}^{\infty} |f_{n_{k+1}}(t) - f_{n_k}(t)| \leq \sum_{k=1}^{N} |f_{n_{k+1}}(t) - f_{n_k}(t)| + \sum_{k=N+1}^{\infty} \frac{1}{2^k} < \infty.$$

Therefore, there exists

$$f(t) := f_{n_1}(t) + \sum_{k=1}^{\infty} (f_{n_{k+1}}(t) - f_{n_k}(t)) = \lim_{k\to\infty} f_{n_k}(t)$$

if $t \notin A := \bigcap_{N=1}^{\infty} \bigcup_{k>N} A_k$, and $C(A) = 0$ since, by (1.11),

$$C(A)^\rho \leq 2 \sum_{k>N} C(A_k)^\rho < 2 \sum_{k>N} \Big(\frac{2^{kp} q^{\frac{p}{q}}}{3^{\frac{kp}{q}}} \Big)^\rho < \infty.$$

Put $f(t) = 0$ if $t \in A$.

As $n_k \to \infty$, $|f_{n_k}(t) - f_n(t)| \to |f(t) - f_n(t)|$ $C-$q.e. and then, by Theorem 1.2.17, it follows that

$$\|f - f_n\|_{L^{p,q}(C)} \leq \frac{1}{q^{1/q}} \Big(\int_0^{\infty} \lim_{k\to\infty} C\{|f_{n_k} - f_n| > u^{1/q}\}^{\frac{q}{p}} du \Big)^{1/q} < \epsilon$$

for n large enough. □

1.3.3 Some known results

Let us recall that in Example 1.2.14 for $1 \leq p < \infty$ and $g : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ be lower semicontinuous (recall Definition 1.2.13), by $C_{g,p}$ we denote the capacity defined on every $E \subset \mathbb{R}^n$ by

$$C_{g,p}(E) = \inf \Big\{ \int f(y)^p dy;\ 0 \leq f \in L^p,\ \int g(x,y) f(y)\, dy \geq 1 \text{ on } E \Big\}.$$

It follows from the definition that the potential $\mathcal{G}f(x) = \int g(x,y)f(y)\,dy$ satisfies the *weak-type estimate*

$$t^p C_{g,p}\{\mathcal{G}f > t\} \le \int f(x)^p\,dx,$$

and the very useful *strong type inequality* of K. Hansson

$$\int_0^\infty C_{g,p}\{\mathcal{G}f > t\}\,dt^p \le A\int f(x)^p\,dx \qquad (0 \le f \in L^p,\ 1 < p < \infty)$$

which may be represented as $\mathcal{G} : L^p(\mathbb{R}^n) \to L^{p,\infty}(C_{g,p})$ and $\mathcal{G} : L^p(\mathbb{R}^n) \to L^p(C_{g,p})$, respectively.

The dyadic version E_h^d of the Haussdorf capacity (1.6), defined using dyadic cubes D_k instead of general cubes I_k, has similar properties. For example, E_h^d is countably subadditive since it is subadditive and Fatou.

The Shannon entropy was considered by R. Feffermann to obtain also *weak-type entropic estimates*

$$tE_\varphi\{Mf > t\} \le A\Big(\int_0^\infty t^{p-1}E_\varphi\{f > t\}\,dt\Big)^{1/p}$$

for the Hardy-Littlewood maximal function. This is, $M : L^p(E_\varphi) \to L^{1,\infty}(E_\varphi)$. Observe that E_φ satisfies the Fatou property. Moreover, for $1 \le q \le \infty$ and $1 < p < \infty$, in [CeClM] we find that

$$M : L^{p,q}(E_\varphi) \to L^{p,q}(E_\varphi).$$

Similarly, in 1980, C. Calderón defined new entropies to obtain some convergence results for singular integrals.

Let E be a quasi-Banach function space on the measure space (Ω, Σ, μ) and recall the definition of C_E in (1.7). We always assume that E has the Fatou property, so that C_E is quasi-subadditive and Fatou.

For any Banach function space E, the function spaces $L^1(C_E)$ and $M(C_E) := L^{1,\infty}(C_E)$ are extremal in the sense that, if X is another Banach function space such that $\|\chi_A\|_X = \|\chi_A\|_E$ for any measurable set $A \subset \Omega$, then

$$L^1(C_E) \hookrightarrow X \hookrightarrow M(C_E) = M(E).$$

1.4 Normability

To motivate the main problem of this section let us remember that the Choquet integral is subadditive on sets,

$$\int (\chi_A + \chi_B)\,dC \le \int \chi_A\,dC + \int \chi_B\,dC,$$

if and only if

$$C(A \cup B) + C(A \cap B) \leq C(A) + C(B).$$

Then the Choquet integral is also subadditive on non-negative simple functions and C is said to be concave.

Variational capacities and those of Fuglede and Meyers are examples of *concave* capacities. Instead, for the *Hausdorff content* we have that, E_h is concave if $n = 1$, but not if $n > 1$ (see [Car]). In the case of entropies C_E associated to Banach function spaces, examples and counterexamples of concave capacities are given in [CeClM].

As we have shown, concave capacities give rise to normed L^p-spaces, since Minkowski's inequality holds with constant one. In [Ce, Theorem 5.1] we see that for $1 \leq p, q < \infty$, $L^{p,q}(C)$ is a normed space if and only if $C^{q/p}$ is concave.

As many of the classical examples of capacities are not concave, a natural objective is to try to study the possible normability of the capacitary Lorentz spaces for general capacities. We want to determine when, for a non-concave capacity C, $L^p(C)$ is *normable*, this meaning that there exists in $L^p(C)$ a norm which is equivalent to $\|\cdot\|_{L^p(C)}$.

This problem can be solved only in special cases. The difficulties of this question can be shown observing that, if $n = 1$, the Shannon entropy E_φ is concave and $L^p(E_\varphi)$ is a Banach function space on $[0, 1/e]$. But, this is not true for $L^p(E_\varphi)$ on $[0, 1/e]^n$ if $n > 1$ although it holds for $L^p(E_\varphi^d)$ and $L^p(E_\varphi) = L^p(E_\varphi^d)$.

As for usual Lorentz spaces, one could try to substitute $f_C^\star$ by

$$f^{\star\star}(t) := \frac{1}{t}\int_0^t f_C^\star(s)\, ds, \tag{1.13}$$

which is decreasing and satisfies $f^\star(t)_C \leq f^{**}(t)$, and for every $\lambda \in \mathbb{R}$ by Proposition 1.2.2

$$(\lambda f)^{**} = |\lambda| f^{\star\star}, \quad (f+g)^{\star\star} \leq 2cf^{\star\star} + 2cg^{\star\star}.$$

But this *average function* is unfortunately subadditive precisely when $L^p(C)$ $(p \geq 1)$ are normed spaces:

Theorem 1.4.1. *$f^{\star\star}$ is subadditive if and only if C is concave.*

Proof. It is clear that $C_t(A) := \min(C(A), t)$ is a Fatou capacity (remember that we assumed that C is a quasi-subadditive Fatou capacity). For a fixed $t > 0$, $f^{\star\star}(t)$ is subadditive if and only if C_t is concave, since

$$\begin{aligned}\int_0^t f_C^\star(s)\, ds &= \int_0^\infty dy \int_0^t \chi_{[0, C\{f>y\})}(s)\, ds = \int_0^\infty \min(\{C\{f>y\}, t)\, dy \\ &= \int_0^\infty C_t(\{|f| > y\}) dy = \int_\Omega |f| dC_t.\end{aligned}$$

Therefore, the theorem follows. □

For $t > 0$ we consider the capacity C_t defined previously. Let us observe briefly the most important facts concerning f^{**}.

1. As in the measure case (see [KrPS]): If $f, g \geq 0$ and $0 < \alpha < 1$, then by Hölder's inequality it follows that:

   - If C is concave, then $(f^\alpha g^{1-\alpha})^{**}(t) \leq 2(f)^{**}(t)^\alpha (g)^{**}(t)^{1-\alpha}$.
   - If C is quasi-subadditive, then $(f^\alpha g^{1-\alpha})^{**}(t) \leq 2c(f)^{**}(t)^\alpha (g)^{**}(t)^{1-\alpha}$.

   Indeed, by Proposition 1.2.3 and Holder's inequality with $p = 1/\alpha$ and $q = \frac{1}{1-\alpha}$

$$\begin{aligned}
\int_0^u (f^\alpha g^{1-\alpha})^\star_C(t)dt &= \int_0^{u/c} (f^\alpha g^{1-\alpha})^\star_C(cs)cds = c\int_0^{u/c} (f^\alpha g^{1-\alpha})^\star_C(cs/2 + cs/2)ds \\
&\leq c\int_0^{u/c} (f^\alpha)^\star_C(s/2)(g^{1-\alpha})^\star_C(s/2)ds \\
&= 2c\int_0^{u/2c} (f^\star)^\alpha_C(s)(g^\star)^{1-\alpha}_C(s)ds \\
&\leq 2c\Big(\int_0^{u/2c} f^\star_C(s)ds\Big)^\alpha \Big(\int_0^{u/2c} g^\star_C(s)ds\Big)^{1-\alpha} \\
&\leq 2c\Big(\int_0^{u} f^\star_C(s)ds\Big)^\alpha \Big(\int_0^{u} g^\star_C(s)ds\Big)^{1-\alpha}.
\end{aligned}$$

   Therefore,

$$(f^\alpha g^{1-\alpha})^{**}(u) \leq 2c\frac{1}{u}\Big(\int_0^u f^\star_C(s)ds\Big)^\alpha \Big(\int_0^u g^\star_C(s)ds\Big)^{1-\alpha} = 2cf^{**}(u)^\alpha g^{**}(u)^{1-\alpha}. \quad (1.14)$$

   In the particular case when C is concave, $c = 1$ and the conclusion follows.

2. Since $(f+g)^*_C(x) \leq f^*_C(\frac{x}{2c}) + g^*_C(\frac{x}{2c})$, it follows that

$$(f+g)^{**}(t) \leq f^{**}(\frac{t}{2c}) + g^{**}(\frac{t}{2c}).$$

   Indeed,

$$\begin{aligned}
(f+g)^{**}(t) &= \frac{1}{t}\int_0^t (f+g)^*_C(s)ds \leq \frac{1}{t}\int_0^t \Big(f^*_C(\frac{s}{2c}) + g^*_C(\frac{s}{2c})\Big)ds \\
&= \frac{2c}{t}\Big\{\int_0^{\frac{t}{2c}} f^*_C(u)du + \int_0^{\frac{t}{2c}} g^*_C(u)du\Big\} = f^{**}(\frac{t}{2c}) + g^{**}(\frac{t}{2c}).
\end{aligned}$$

We do not have a satisfactory sufficient normability condition, but let us see a restrictive one, which extends a known result for the usual Lorentz space.

In the rest of the section μ will represent a measure on (Ω, Σ) such that $\mu(\Sigma) = [0, \mu(\Omega)] \subset [0, \infty]$, and we will suppose that C is *μ-invariant*, this meaning that $C(A) = C(B)$ if $\mu(A) = \mu(B)$.

Definition 1.4.2. *A capacity C on (Ω, Σ) will be said to be* quasi-concave with respect to μ *if there exists a constant $\gamma \geq 1$ such that, whenever $\mu(A) \leq \mu(B)$, the following two conditions are satisfied:*

(a) $C(A) \leq \gamma\, C(B)$, *and*

(b) $\frac{C(B)}{\mu(B)} \leq \gamma \frac{C(A)}{\mu(A)}$,

this is, for all $A, B \in \Sigma$,

$$C(B) \leq \gamma \max\Big(1, \frac{\mu(B)}{\mu(A)}\Big) C(A).$$

Example 1.4.3. *As an example of this type of capacities consider $J : [0, \mu(\Omega)] \to \mathbb{R}$ an increasing function such that $J(t)/t$ is decreasing. It is readily seen that $C(A) := J(\mu(A))$ defines a μ-invariant and quasi-concave capacity with respect to μ. For instance, $C(A) := \varphi_X(\mu(A))$ where φ_X is the fundamental function of an r.i. space (see the definition in Chapter 3). Note that φ_X is a quasi-concave function.*

Theorem 1.4.4. *If the capacity C is μ-invariant and quasi-concave with respect to μ, then*

$$\widetilde{C}(A) := \sup\Big\{ \sum_{i=1}^{n} \lambda_i C(A_i);\ n \in \mathbb{N},\ \sum_{i=1}^{n} \lambda_i = 1,\ \lambda_i \geq 0,\ \sum_{i=1}^{n} \lambda_i \mu(A_i) \leq \mu(A) \Big\}$$

defines a concave capacity which is equivalent to C i.e. $C \simeq \widetilde{C}$.

Proof. It is clear that $\widetilde{C}(A) \geq 0$ and it is readily seen that $\widetilde{C}$ is increasing.

Let us show that

$$C(A) \leq \widetilde{C}(A) \leq 2\gamma C(A). \tag{1.15}$$

Obviously, $C(A) \leq \widetilde{C}(A)$. On the other hand, for $\varepsilon > 0$ we can find $\sum_{i=1}^{n} \lambda_i \mu(A_i) \leq \mu(A)$ with $\sum_{i=1}^{n} \lambda_i = 1$ and $\lambda_i \geq 0$ $(i = 1, ..., n)$ such that

$$\widetilde{C}(A) - \varepsilon \leq \sum_{i=1}^{n} \lambda_i C(A_i) \leq \gamma \sum_{i=1}^{n} \lambda_i \max\Big(1, \frac{\mu(A_i)}{\mu(A)}\Big) C(A) \leq 2\gamma C(A)$$

and (1.15) follows.

To prove that $\widetilde{C}$ is concave, let $0 < \theta < 1$ and $\varepsilon > 0$. Given $A, B \in \Sigma$, we can find $\sum_{i=1}^n \lambda_i \mu(A_i) \le \mu(A)$ with $\sum_{i=1}^n \lambda_i = 1$ and $\lambda_i \ge 0$ $(i = 1, ..., n)$ such that

$$(1-\theta)\widetilde{C}(A) - \frac{\varepsilon}{2} \le (1-\theta)\sum_{i=1}^n \lambda_i C(A_i)$$

and, similarly,

$$\theta\widetilde{C}(B) - \frac{\varepsilon}{2} \le \theta\sum_{j=1}^m \lambda'_j C(B_j)$$

with $\sum_{j=1}^m \lambda'_j \mu(B_j) \le \mu(B)$, $\sum_{j=1}^m \lambda'_j = 1$ and $\lambda'_j \ge 0$ $(j = 1, ..., m)$.

Then $(1-\theta)\mu(A) + \theta\mu(B) \ge \sum_{i=1}^n (1-\theta)\lambda_i \mu(A_i) + \sum_{j=1}^m \theta\lambda'_j \mu(B_j)$ and $\sum_{i=1}^n (1-\theta)\lambda_i + \sum_{j=1}^m \theta\lambda'_j = 1$. We can choose $D \in \Sigma$ such that $\mu(D) = (1-\theta)\mu(A) + \theta\mu(B)$, and then

$$(1-\theta)\widetilde{C}(A) + \theta\widetilde{C}(B) - \varepsilon \le \sum_{i=1}^n (1-\theta)\lambda_i C(A_i) + \sum_{j=1}^m \theta\lambda'_j C(B_j) \le \widetilde{C}(D),$$

so that

$$(1-\theta)\widetilde{C}(A) + \theta\widetilde{C}(B) \le \widetilde{C}(D). \tag{1.16}$$

Since C is μ-invariant, the same happens with $\widetilde{C}$, and we may define $\varphi(s) := \widetilde{C}(A)$ if $s = \mu(A)$, which is by (1.16) a concave function on $[0, \mu(\Omega)]$.

We claim that, if $x, y \ge t > 0$, then

$$\varphi(x+y-t) + \varphi(t) \le \varphi(x) + \varphi(y), \tag{1.17}$$

and the concavity of $\widetilde{C}$ follows by taking $t = \mu(A \cap B)$, $x = \mu(A)$ and $y = \mu(B)$, since then $\varphi(t) = \widetilde{C}(A \cap B)$, $\varphi(x+y-t) = \varphi(\mu(A) + \mu(B) - \mu(A \cap B)) = \varphi(\mu(A \cup B)) = \widetilde{C}(A \cup B)$, and $\varphi(x) + \varphi(y) = \widetilde{C}(A) + \widetilde{C}(B)$.

To prove the claim, we may assume that $0 < t < x \le y$ and write

$$x = (1-\tau)t + \tau(x+y-t), \quad y = (1-\tau')t + \tau'(x+y-t) \qquad (\tau, \tau' \in (0,1)).$$

Since φ is concave,

$$\Big(1 - \frac{x-t}{x+y-2t}\Big)\varphi(t) + \frac{x-t}{x+y-2t}\varphi(x+y-t) \le \varphi(x)$$

and

$$\Big(1 - \frac{y-t}{x+y-2t}\Big)\varphi(t) + \frac{y-t}{x+y-2t}\varphi(x+y-t) \le \varphi(y).$$

Finally, by addition, (1.17) follows. □

Although we do not know whether $\widetilde{C}$ has the Fatou property, we can still define on $L^p(C)$ the quasi-norm $\|f\|_{L^p(\widetilde{C})}$ which is equivalent to $\|f\|_{L^p(C)}$, since $C\{|f|>t\} \simeq \widetilde{C}\{|f|>t\}$. Let us denote by $\mathcal{S}$ the class of all simple functions and $\mathcal{S}^p(C) = \mathcal{S} \cap \bar{L}^p(C) \subset L^p(C)$.

Corollary 1.4.5. *For $1 \le p \le \infty$, on $\mathcal{S}^p(C)$ the functional $\|\cdot\|_{L^p(\widetilde{C})}$ is a norm which is equivalent to the quasi-norm $\|\cdot\|_{L^p(C)}$. Hence, if $\mathcal{S}$ is dense in $L^p(C)$, then $L^p(C)$ is normable.*

The following proposition shows that although we can not get the converse, we have an approximation to it.

Proposition 1.4.6. *If $L^1(C)$ (or $\mathcal{S}^1(C)$) is normable by a μ-invariant norm $\|\cdot\|_*$, that is $\|\chi_A\|_* = \|\chi_B\|_*$ if $\mu(A) = \mu(B)$, then C is also quasi-concave with respect to the measure μ.*

Proof. Let $\|\cdot\|_*$ be an equivalent norm on $L^1(C)$. Then $\widehat{C}(A) := \|\chi_A\|_*$ defines a new μ-invariant capacity and $\widehat{C}(A) \simeq C(A)$. Moreover $\widehat{C}$ is concave since

$$\widehat{C}(A \cup B) + \widehat{C}(A \cap B) \le \widehat{C}(A) + \widehat{C}(B).$$

We can suppose $0 \le \mu(A \cap B) < \mu(A) \le \mu(B)$ and define $\varphi(\mu(A)) := \widehat{C}(A)$.

Let $t = \mu(A \cap B)$, $x = \mu(A)$, and $y = \mu(B)$, so that $0 < t < x \le y$ and $\varphi(x+y-t)+\varphi(t) \le \varphi(x) + \varphi(y)$. In particular, if $m \in N$ and $r > 0$, then $\varphi(mr) \le m\varphi(r)$. Moreover, if $a \le b$, then there exists $m \ge 2$ such that $(m-1)a \le b \le ma$ and

$$\frac{\varphi(b)}{b} \le \frac{\varphi(ma)}{b} \le \frac{ma}{b}\frac{\varphi(a)}{a} \le \frac{m}{m-1}\frac{\varphi(a)}{a}.$$

Since $x \le y$ it follows that there is some $m \in N$ such that

$$\frac{\varphi(y)}{y} \le \frac{m}{m-1}\frac{\varphi(x)}{x},$$

which means that $\widehat{C}$ is quasi-concave respect to the measure μ, with constant $\gamma = 2$. Since $\widehat{C} \simeq C$, C is also quasi-concave with respect to μ. □

In the rest of the section C will be a quasi-subadditive Fatou capacity, quasi-concave with respect to μ. Let us see that $L^p(C)$ is normable for $1 \le p \le \infty$. Define $\widetilde{C}$ as in Theorem 1.4.4 and $\bar{C}$ by

$$\bar{C}(A) := \inf_{A_n \uparrow A,\, A_n \in \Sigma} \left\{ \lim_{n\to\infty} \widetilde{C}(A_n) \right\}.$$

Proposition 1.4.7. *The capacity $\bar{C}$ is a concave Fatou capacity equivalent to C.*

Proof. Let $\epsilon > 0$ and $A, B \in \Sigma$ arbitrary. There exists $\{A_n\}_{n\in\mathbb{N}}$ with $A_n \uparrow A$ such that $\lim_{n\to\infty} \widetilde{C}(A_n) \leq \bar{C}(A) + \epsilon$. Since $\widetilde{C} \simeq C$, there exists $c' > 0$ such that $C(A) \leq c'\widetilde{C}(A)$ for all set A, and hence

$$\frac{1}{c'} \lim_{n\to\infty} C(A_n) \leq \lim_{n\to\infty} \widetilde{C}(A_n) \leq \bar{C}(A) + \epsilon.$$

Then, by the Fatou property, $\lim_{n\to\infty} C(A_n) = C(A)$. As $\widetilde{C}$ is equivalent to C, the equivalence follows.

Moreover, there exist increasing sequences $\{A_n\}_{n\in\mathbb{N}}$, $\{B_n\}_{n\in\mathbb{N}}$ such that

$$\lim_n \widetilde{C}(A_n) \leq \bar{C}(A) + \epsilon/2,$$
$$\lim_n \widetilde{C}(B_n) \leq \bar{C}(B) + \epsilon/2.$$

Assume, without loss of generality, that $\bar{C}(A) + \bar{C}(B) < \infty$. By the concavity of $\widetilde{C}$ we have that

$$\begin{aligned} \lim_{n\to\infty} \left[\widetilde{C}(A_n \cup B_n) + \widetilde{C}(A_n \cap B_n)\right] &\leq \lim_{n\to\infty} \widetilde{C}(A_n) + \widetilde{C}(B_n) \\ = \lim_{n\to\infty} \widetilde{C}(A_n) + \lim_{n\to\infty} \widetilde{C}(B_n) &\leq \bar{C}(A) + \bar{C}(B) + \epsilon \end{aligned}$$

and then, since $A_n \cup B_n \uparrow A \cup B$ and $A_n \cap B_n \uparrow A \cap B$ by definition of $\bar{C}$ we get that

$$\lim_{n\to\infty} \left[\widetilde{C}(A_n \cup B_n) + \widetilde{C}(A_n \cap B_n)\right] \geq \bar{C}(A \cup B) + \bar{C}(A \cap B).$$

Then, the concavity follows.

Finally, let us prove the Fatou property. Let $\{A_n\}_{n\in\mathbb{N}}$ with $A_n \uparrow A$. It will be sufficient to show that $\bar{C}(A) \leq \lim_{n\to\infty} \bar{C}(A_n)$. Assume that $\lim_{n\to\infty} \bar{C}(A_n) < \infty$. For all n, there exists $(A_{n_m})_{m=1}^{\infty}$ such that $A_{n_m} \uparrow A_n$ as $m \to \infty$ and $\lim_{m\to\infty} \widetilde{C}(A_{n_m}) \leq \bar{C}(A_n) + \epsilon$. Considering the sequence of sets $B_n := A_{n_n}$, we have that $B_n \uparrow A$ as $n \to \infty$. Then

$$\bar{C}(A) \leq \lim_{n\to\infty} \widetilde{C}(B_n) \quad = \quad \lim_{n\to\infty} \widetilde{C}(A_{n_n}) \leq \lim_{n\to\infty} \bar{C}(A_n) + \epsilon. \qquad \square$$

Theorem 1.4.8. *If $\widetilde{C}$ has the Fatou property, then $\bar{C} = \widetilde{C}$. Hence, $L^p(C) = L^p(\bar{C})$ and $L^p(C)$ $(1 \leq p \leq \infty)$ is normable.*

Proof. Suppose that $\widetilde{C}$ has the Fatou property. Let $A \subset \Omega$ and $\epsilon > 0$. If $A_n \uparrow A$, then

$$\widetilde{C}(A) = \lim_{n\to\infty} \widetilde{C}(A_n) \geq \bar{C}(A)$$

and there exists $\{A_n\}_{n\in\mathbb{N}}$ with $A_n \uparrow A$ such that

$$\widetilde{C}(A) = \lim_{n\to\infty} \widetilde{C}(A_n) < \bar{C}(A) + \epsilon.$$

The proof then follows by letting $\varepsilon \to 0$. $\square$

1.5 The associate space

As we observed in Theorem 1.2.19, for a concave capacity C, Hölder's inequality asserts that

$$\int_\Omega |fg|dC \leq \|f\|_{L^p(C)}\|g\|_{L^q(C)}, \forall\, f \in L^p(C),\, g \in L^q(C),\, 1 \leq p \leq \infty,$$

where q is the conjugate exponent of p. We will see that in this case the inequality is sharp in the sense that

$$\|g\|_{L^q(C)} = \sup\Big\{\int_\Omega |fg|dC; f \in L^p(C), \|f\|_{L^p(C)} \leq 1\Big\} \tag{1.18}$$

for all $g \in L^q(C)$ and p, q conjugate exponents.

In relation with Definition 1.3.1, if ϱ is a function quasi-norm on $\mathrm{L}_0(\Omega)^+$, we define its *associate quasi-norm* ϱ' on $\mathrm{L}_0(\Omega)^+$ by

$$\varrho'(g) := \sup\Big\{\int_\Omega fgdC; f \in \mathrm{L}_0(\Omega)^+, \varrho(f) \leq 1\Big\}$$

for all $g \in \mathrm{L}_0(\Omega)^+$.

Theorem 1.5.1. *Let C be a capacity such that for all $E \subset \Omega$ with $C(E) > 0$ there exists $E_0 \subset E$, $0 < C(E_0) < \infty$. Then, if ϱ is a functional quasi-norm on $\mathrm{L}_0(\Omega)^+$, the associate quasi-norm is itself a functional quasi-norm on $\mathrm{L}_0(\Omega)^+$.*

Proof. We shall show that all the conditions in the Definition 1.3.1 are satisfied by ϱ'.

If $\varrho(f) \leq 1$, then $f < \infty$ $C-$q.e. Hence, if $g = 0$ $C-$q.e., then $\int_\Omega fgdC = 0$ and so $\varrho'(g) = 0$. Conversely, if $\varrho'(g) = 0$, then for all $f \in \mathrm{L}_0(\Omega)^+$ with $\varrho(f) \leq 1$, it follows that $\int_\Omega fgdC = 0$. If $E \subset \Omega$ is measurable with $0 < C(E) < \infty$, then $0 < \varrho(\chi_E) < \infty$.

Taking $f = \frac{\chi_E}{\varrho(\chi_E)}$ we obtain that

$$0 = \int_\Omega \frac{\chi_E}{\varrho(\chi_E)} gdC = \frac{1}{\varrho(\chi_E)}\int_E gdC,$$

and then necessarily $g = 0$ $C-$q.e. in E. Suppose that there exists E such that $C(E) > 0$ and $g \neq 0$ on E. Then, by the assumption, there exists $E_0 \subset E$ with $0 < C(E_0) < \infty$. Hence for E_0, taking $f = \frac{\chi_{E_0}}{\varrho(\chi_{E_0})}$, it follows that $\varrho(f_0) = 1$ and

$$0 = \int_\Omega \frac{\chi_{E_0}}{\varrho(\chi_{E_0})} gdC = \frac{1}{\varrho(\chi_{E_0})}\int_{E_0} gdC.$$

So that, $g = 0$ in E_0 but, since $E_0 \subset E$, necessarily $g \neq 0$ in E_0. Hence, $g = 0$ $C-$q.e. In both cases the remaining properties of (a) and (b) are easy to check.

For the Fatou property, suppose that $g_n, g \in \mathrm{L}_0(\Omega)^+$ and $0 \leq g_n \uparrow g$ $C-$q.e. Then $\varrho'(g_n) \leq \varrho'(g)$ for all $n \in \mathbb{N}$. Let us assume, without loss of generality, that $\varrho'(g_n) < \infty$ for all $n \in \mathbb{N}$. Let $\xi < \varrho'(g)$. There exists $f \in \mathrm{L}_0(\Omega)^+$ with $\varrho(f) \leq 1$ such that $\int_\Omega fgdC > \xi$. Now $0 \leq fg_n \uparrow fg$ $C-$q.e., so the *monotone convergence theorem* (see Corollary 1.2.18) shows that

$$\int_\Omega fg_n dC \to \int_\Omega fgdC.$$

Hence, there is $N \in \mathbb{N}$ such that $\int_\Omega fg_n dC > \xi$ for all $n \geq N$. It follows then that $\varrho'(g_n) > \xi$ $(n \geq N)$, which shows $\varrho'(g_n) \uparrow \varrho'(g)$ and establish property (c).

If $C(E) < \infty$, then since ϱ satisfies the property (e) in Definition 1.3.1, we obtain a constant $C_E < \infty$ for which $\int_\Omega \chi_E fdC \leq C_E \varrho(f)$, and then $\varrho'(\chi_E) \leq C_E < \infty$.

Finally, fix E such that $C(E) < \infty$ and assume that $0 < C(E)$, otherwise there is nothing to prove. In this case $C'_E = \varrho(\chi_E)$ satisfies $0 < C'_E < \infty$ and $\varrho(\frac{\chi_E}{\varrho(\chi_E)}) = 1$. Hence, for any $g \in \mathrm{L}_0(\Omega)^+$

$$\int_E gdC = C'_E \int_\Omega \frac{\chi_E}{C'_E} gdC \leq C'_E \varrho'(g)$$

which shows that property (e) holds for ϱ'. □

Given ϱ a function quasi-norm on $\mathrm{L}_0(\Omega)^+$ and ϱ' its associate quasi-norm, the Banach function space $X(\varrho')$ determined by ϱ' is called the *associate space of* X and is denoted by X'. It follows that the norm of a function g in the associate space X' is given by

$$\|g\|_{X'} = \sup\Big\{ \int_\Omega |fg|dC; f \in X, \|f\|_X \leq 1 \Big\}. \tag{1.19}$$

From now on in this section, C will denote a quasi-subadditive capacity on (Ω, Σ) such that for all $E \subset \Omega$ with $C(E) > 0$ there exists $E_0 \subset E$, $0 < C(E_0) < \infty$.

Theorem 1.5.2. *Let $1 \leq p \leq \infty$ and q be the conjugate exponent of p. If C is concave and Fatou, then for every measurable function g, the following properties are equivalent:*

(i) $g \in L^q(C)$.

(ii) $fg \in L^1(C)$ *if* $f \in L^p(C)$ *and* $A = \sup\{\int_\Omega fgdC; \|f\|_{L^p(C)} = 1\} < \infty$.

(iii) $sg \in L^1(C)$ *if s is simple and* $B = \sup\{\int_\Omega sgdC; \|s\|_{L^p(C)} = 1\} < \infty$.

It holds that $\|g\|_{L^q(C)} = A = B$.

Proof. We can follow the same arguments as in the case of measures. It is clear that $B \leq A \leq \|g\|_{L^q(C)}$ and that we can suppose that g is not zero. We begin with the case $p = 1$, $q = \infty$. Given $\varepsilon > 0$, we see that the set $E := \{|g| > B + \varepsilon\}$ has zero capacity. Indeed, if $C(E) > 0$, there exists $F \subset E$ such that $0 < C(F) < \infty$ and for

$$f = \frac{sgn\, g}{C(F)}\chi_F$$

we have that $\|f\|_{L^1(C)} = 1$ and $\int_\Omega fgdC > B + \varepsilon$, which is impossible if f is simple. Then, $\|g\|_{L^\infty(C)} \leq B$.

In the case $q < \infty$, if Ω is the union of an increasing sequence of sets $E_k \in \Sigma$ of finite capacity, then let $\{s_k\}_{k\in\mathbb{N}}$ be a sequence of simple functions, zero outside E_k, $|s_k| \leq |g|$ and such that $\lim_k s_k = g$ $C-$q.e. And denote

$$f_k = |s_k|^{q-1}\frac{sgn\, g}{\|s_k\|_q^{q-1}} \quad (k \in \mathbb{N}).$$

It is immediate that $\|f_k\|_{L^p(C)} = 1$ for all $k \in \mathbb{N}$ and

$$\|g\|_{L^q(C)} \leq \liminf_{k\to\infty} \|s_k\|_{L^q(C)} = \liminf_{k\to\infty} \int_\Omega |f_k s_k| dC \leq \liminf_{k\to\infty} \int_\Omega f_k g dC,$$

which is smaller than B if the $f_k's$ are simple functions.

In the case $q < \infty$, if Ω is not the union of an increasing sequence of sets $E_k \in \Sigma$ with finite capacity, observe that for $g \in L^q(C)$ we have that

$$\|g\|^q_{L^q(C)} = \int_\Omega |g|^q dC = \int_{\{g\neq 0\}} |g|^q dC = \int_{\bigcup_{n=1}^\infty A_n} |g|^q dC,$$

where $A_n := \{1/(n+1) \leq |g(x)| < 1/n\}$ $(n \in \mathbb{N})$ are sets with finite capacity. Let $\{s_k\}_{k\in\mathbb{N}}$ be a sequence of simple functions, zero outside A_k, $|s_k| \leq |g|$, such that $\lim_k s_k = g$ $C-$q.e. and define f_k as before. It follows that $\|f_k\|_{L^p(C)} = 1$ for all $k \in \mathbb{N}$ and

$$\|g\|_{L^q(C)} \leq \liminf_{k\to\infty} \|s_k\|_{L^q(C)} = \liminf_{k\to\infty} \int_{\bigcup_{n=1}^\infty A_n} |f_k s_k| dC \leq \liminf_{k\to\infty} \int_{\bigcup_{n=1}^\infty A_n} f_k g dC,$$

which is smaller than B if the $f_k's$ are simple functions. □

The inequality (1.18) is sharp as we announced, that is, under the same conditions of Theorem 1.5.2, $L^q(C)$ is the associate space of $L^p(C)$.

As in the classical case, we obtain:

Theorem 1.5.3. *Let X be a quasi-Banach function space on $\mathrm{L}_0(\Omega)^+$ with associate space X'. If $f \in X$ and $g \in X'$, then fg is C-integrable (that is, the Choquet integral of $|fg|$ is finite) and*

$$\int_\Omega |fg| dC \lesssim \|f\|_X \|g\|_{X'}.$$

Proof. Just follow the usual arguments, as in [BeSh, Theorem 2.4]. □

Theorem 1.5.4. *Let $1 \leq p \leq \infty$ and q be the conjugate exponent of p. Let us define for all $g \in \mathrm{L}_0(\Omega)$,*

$$A := \sup\Big\{\int_\Omega fgdC; \|f\|_{L^p(C)} = 1\Big\},\ B := \sup\Big\{\int_\Omega sgdC;\ s \text{ simple }, \|s\|_{L^p(C)} = 1\Big\}.$$

The following equivalence holds,

$$A = B \simeq 2c\|g\|_{L^q(C)}.$$

Proof. It's clear that $B \leq A \leq 2c\|g\|_{L^q(C)}$ and that we can suppose that g is not zero.

We begin with the case $p = 1$, $q = \infty$. Given $\varepsilon = (2c-1)A > 0$, we see that the set $E := \{|g| > \frac{A+\varepsilon}{2c}\}$ has zero capacity. Indeed, if $C(E) > 0$, there exists $F \subset E$ such that $0 < C(F) < \infty$ and for

$$f = \frac{sgn\, g}{C(F)}\chi_F$$

we have that $\|f\|_{L^1(C)} = 1$ and $\int_\Omega fgdC = \int_F \frac{|g|}{C(F)}dC > A$, which is impossible. Hence as $C(E) = 0$, then $|g| \leq A\, C-$q.e. Therefore, $\|g\|_{L^\infty(C)} \leq A \leq 2c\|g\|_{L^\infty(C)}$ and $A \simeq \|g\|_{L^\infty(C)}$.

In the case $q < \infty$, if Ω is the union of an increasing sequence of sets $E_k \in \Sigma$, we proceed as in Theorem 1.5.2 and we conclude that, in this case, $\|g\|_{L^q(C)} \leq B \leq A \leq 2c\|g\|_{L^q(C)}$. If Ω is not the union of an increasing sequence of sets $E_k \in \Sigma$ of finite capacity, the proof follows as in Theorem 1.5.2. □

Theorem 1.5.5. *Let $1 \leq p \leq \infty$ and q be the conjugate exponent of p. Then, $(L^p(C))' = L^q(C)$.*

Proof. If $g \in L^q(C)$, then $A := \sup\{\int_\Omega fgdC; \|f\|_{L^p(C)} = 1\} \leq 2c\|f\|_{L^q(C)} < \infty$, and then $g \in (L^p(C))'$. Conversely, if $g \in (L^p(C))'$, then $A < \infty$ and $\|g\|_{L^q(C)} \lesssim A$. Hence, $g \in L^q(C)$. □

Chapter 2

Interpolation of capacitary Lorentz spaces

2.1 Introduction

In interpolation theory of linear operators, couples (B_0, B_1) and (A_0, A_1) of (quasi-)Banach spaces, continuously contained in some corresponding Hausdorff topological vector space, and linear operators $T : A_0 + A_1 \to B_0 + B_1$ are considered. An interpolation method builds new (quasi-)Banach spaces A and B, $A \hookrightarrow A_0 + A_1$ and $B \hookrightarrow B_0 + B_1$, such that if $T : A_0 \to B_0$ and $T : A_1 \to B_1$ continuously, then also $T : A \to B$. We say that A and B are interpolation spaces for (A_0, A_1) and (B_0, B_1).

The real *Peetre's K-method*, based on the classical Marcinkiewicz theorem for every $0 < \theta < 1$ and $1 \le p \le \infty$, determines interpolation spaces $(A_0, A_1)_{\theta,p}$, $(B_0, B_1)_{\theta,p}$ by means of the K-functional

$$K(t, f; \bar{A}) := \inf \Big\{ \|f_0\|_{A_0} + t\|f_1\|_{A_1};\ f = f_0 + f_1, f_i \in A_i, i = 0, 1 \Big\}.$$

The norm in $(A_0, A_1)_{\theta,p}$ is then defined as

$$\|a\|_{\theta,p} := \Big(\int_0^\infty (t^{-\theta} K(t, a; \bar{A}))^p \frac{dt}{t} \Big)^{1/p}.$$

We refer to [BeLo], [BK] and [T1] for general facts concerning interpolation theory, and to [T2] and [T3] for general facts concerning function spaces.

The interpolation spaces for couples of Lebesgue, Lorentz and Orlicz spaces on given measure spaces have been extensively studied, including the cases of quasi-Banach spaces and $0 < p \le \infty$. It is a natural question to determine if these interpolation results extend to our capacitary setting.

In [Ce] (see also [CeClM]) it is shown that, if (C_0, C_1) is a couple of concave Fatou capacities on (Ω, Σ) with the same null sets, $0 < \eta < 1$, $1 < p_0, p_1 < \infty$, $1 \leq q_0, q_1 < \infty$, $1/p = (1-\eta)/p_0 + \eta/p_1$, and $1/q = (1-\eta)/q_0 + \eta/q_1$, then

$$(L^{p_0,q_0}(C_0), L^{p_1,q_1}(C_1))_{\eta,q} = L^{p,q}(C_{\eta p/p_1, q/p}), \tag{2.1}$$

where $C_{\theta,q}(A) := \|\chi_A\|_{(L(C_0),L(C_1))_{\theta,q}}$.

One of our goals is to extend this result. The capacities will be still supposed to be Fatou but the Choquet integral will not be necessarily subadditive anymore, and $0 < p < 1$ is also allowed (See [CeMS]). Our main problem is then interpolation with change of capacities. We want to determine, in particular, for convenient parameters the interpolation space

$$(L^{p_0}(C_0), L^{p_1}(C_1))_{\eta,q}.$$

Since $L^{p_i}(C_i) = (L^{\alpha_i}(C_i), L^\infty)_{\theta_i,p_i}$ for $\alpha_i = (1-\theta_i)p_i$, we want to determine

$$((L^{\alpha_0}(C_0), L^\infty)_{\theta_0,p_0}, (L^{\alpha_1}(C_1), L^\infty)_{\theta_1,p_1})_{\eta,q}. \tag{2.2}$$

The usual reiteration theorems do no work because we have spaces with different capacities. In the classical case Stein and Weiss proved that for $0 < p \leq \infty$ and w_0, w_1 weights in $\mathrm{L}_0(\Omega)^+$,

$$(L^p(w_0), L^p(w_1))_{\eta,p} = L^p(w_0^{1-\eta} w_1^\eta).$$

To deal with this problem in the case of capacities one suspects that

$$(L^p(C_0), L^p(C_1))_{\eta,p} = L^p(C_0^{1-\eta} C_1^\eta).$$

Observe that in (2.2) three spaces appear, namely $L^{\alpha_0}(C_0)$, $L^{\alpha_1}(C_1), L^\infty$. The problem will be solved by extending Sparr's method for triples of Banach spaces (see [AK]).

In non-linear potential theory, operators are applied to quasi-continuous functions. We will see in Section 2.7 that our results on interpolation of capacitary Lebesgue spaces still holds when we restrict them to quasi-continuous functions (see [CeMS1]).

A complex interpolation method was developed by J. L. Lions [Li3], A. P. Calderón [Ca] and S. G. Krein [Kr] (see also [KrPS]). In his seminal paper [Ca], A. P. Calderón includes an study of interpolation of Banach function spaces on a measure space, covering the concrete cases of Lebesgue, Lorentz and Orlicz spaces. This is done by defining the so-called Calderón products $X_0^{1-\theta} X_1^\theta$ as in Definition 2.8.1. In Section 8 we check how this Calderón method applies to our capacitary setting.

Last section is devoted to the analysis of capacitary Orlicz spaces. The goal is to show how the general theory can be applied, and to extend the classical interpolation theory of Orlicz spaces to capacitary Orlicz spaces.

2.2 Interpolation of quasi-Banach lattices

Let us present an extension of the interpolation method of G. Sparr for more than two quasi-Banach lattices and let us extend (2.1). For that purpose, we need first to clarify some concepts.

Definition 2.2.1. *A* vector lattice *is a pair* (X, P) *where* X *is a real vector space and* $P \subset X$ *such that* $P \cap (-P) = \{0\}$, $P + P \subset P$, $\mathbb{R}^+ P \subset P$ *and for* $x \in X$ *we have* $x \geq 0$ *if and only if* $x \in P$.

Definition 2.2.2. *A quasi-Banach couple (or a triple) of function spaces* (A_0, A_1) *is said to be a* compatible couple *(or triple) if they are continuously embedded in some Hausdorff topological vector space.*

From now on, let (Ω, Σ) be a measure space and, without loss of generality, assume that $n = 2$. A set A in (Ω, Σ, C) is called $C-$null if $C(A) = 0$.

Let $p_i, q_i \in (0, \infty]$ and C_i be quasi-subadditive Fatou capacities on (Ω, Σ), $i = 0, 1, 2$, with subadditivity constants $c_i \geq 1$, such that for an arbitrary set $A \subset \Omega$, then

$$C_0(A) = 0 \Longleftrightarrow C_1(A) = 0 \Longleftrightarrow C_2(A) = 0,$$

that is, C_0, C_1, C_2 have the same null sets. Then we know that $L^{p_i,q_i}(C_i)$ is a quasi-Banach function space, $i = 0, 1, 2$.

Moreover, $\bar{X} = (L^{p_0,q_0}(C_0), L^{p_1,q_1}(C_1), L^{p_2,q_2}(C_2))$ is a compatible 3-tuple of quasi-Banach spaces. Indeed, consider $\Sigma := L^{p_0,q_0}(C_0) + L^{p_1,q_1}(C_1) + L^{p_2,q_2}(C_2) \subset \mathrm{L}_0(\Omega)$ the space of all elements of the form $f = \sum_{i=0}^{2} f_i, f_i \in L^{p_i,q_i}(C_i)$, $i = 0, 1, 2$ with the quasi-norm

$$\|f\|_\Sigma := \inf \Big\{ \|f_0\|_{L^{p_0,q_0}(C_0)} + \ldots + \|f_2\|_{L^{p_2,q_2}(C_2)};\ f = \sum_{i=0}^{2} f_i, f_i \in L^{p_i,q_i}(C_i), i = 0, 1, 2 \Big\}. \quad (2.3)$$

It follows that Σ is a topological vector space and, if $f \in L^{p_i,q_i}(C_i)$ for $i \in \{0, 1, 2\}$, since $0 \in L^{p_i,q_i}(C_i)$ for all i, then $\|f\|_\Sigma \leq \|f\|_{L^{p_i,q_i}(C_i)}$ which means that $L^{p_i,q_i}(C_i) \hookrightarrow \Sigma$, $i = 0, 1, 2$.

It remains to show that Σ is Hausdorff, that is, if $\|f\|_\Sigma = 0$ for some $f \in \Sigma$, then $f = 0$ q.e. For that, suppose that $\|f\|_\Sigma = 0$ for some $f \in \Sigma$. Then, there exists a sequence of elements $f_k^i \in L^{p_i,q_i}(C_i)$, $i \in \{0, 1, 2\}$, $k \in \mathbb{N}$ such that $f = \sum_{i=0}^{2} f_k^i$ and $f_k^i \to 0$ in $L^{p_i,q_i}(C_i)$ as $k \to \infty$. Then, there exists a subsequence $\{f_{k_{0,n}}^0\}_{n\in\mathbb{N}}$ such that $f_{k_{0,n}}^0 \to 0$ C_0-q.e. as $k_{0,n} \to \infty$ by Theorem 1.3.11. Considering $\{f_{k_{0,n}}^i\}_{n\in\mathbb{N}}$ for $i = 0, 1, 2$, we have

$$f_{k_{0,n}}^0 \to 0 \text{ in } L^{p_0,q_0}(C_0) \text{ as } k_{0,n} \to \infty,\ f_{k_{0,n}}^i \to 0 \text{ in } L^{p_i,q_i}(C_i) \text{ as } k_{0,n} \to \infty, i \in \{1, 2\},$$
$$f_{k_{0,n}}^0 \to 0\, C_0 - \text{q.e. as } k_{0,n} \to \infty.$$

Hence, there exists a subsequence of $\{f^i_{k_{0,n}}\}_{n\in\mathbb{N}}$, $i \in \{0,1,2\}$, such that $f^1_{k_{1,n}} \to 0\, C_1 -$ q.e. as $k_{1,n} \to \infty$. Considering $\{f^i_{k_{1,n}}\}_{n\in\mathbb{N}}$, $i = 0,1,2$, we have that

$$\begin{array}{l} f^0_{k_{1,n}} \to 0 \text{ in } L^{p_0,q_0}(C_0) \text{ as } k_{1,n} \to \infty,\ f^i_{k_{1,n}} \to 0 \text{ in } L^{p_i,q_i}(C_i) \text{ as } k_{1,n} \to \infty, i \in \{1,2\} \\ f^0_{k_{1,n}} \to 0\, C_0 - \text{q.e. as } k_{1,n} \to \infty,\ f^1_{k_{1,n}} \to 0\, C_1 - \text{ q.e. as } k_{1,n} \to \infty. \end{array}$$

Finally reiterating this we have that $f^i_{k_{2,n}} \to 0$ C_i-q.e. as $k_{2,n} \to \infty, i = 0,1,2$, and moreover, $\sum_{i=0}^2 f^i_{k_{2,n}} = f$. Then $f = 0$ q.e. in Σ.

Finally we show that these spaces are quasi-Banach function lattices. It follows that $(L^{p_i,q_i}(C_i), P)$ is a vector lattice, where $P = \{f : \Omega \to \mathbb{R}^+; \|f\|_{L^{p_i,q_i}(C_i)} < \infty,\ i \in \{0,1,2\}\}$. Moreover, for $i \in \{0,1,2\}$, $L^{p_i,q_i}(C_i)$ is a quasi-Banach function lattice.

Consider now the triple $\bar{X} = (L^{p_0,q_0}(C_0), L^{p_1,q_1}(C_1), L^{p_2,q_2}(C_2))$. We will denote by X_i the $(i+1)$-component of the vector $\bar{X}$, $i = 0,1,2$, and $\Delta(\bar{X}) = X_0 \cap X_1 \cap X_2$ will denote the space of all elements common to X_0, X_1, and X_2 with the quasi-norm

$$\|f\|_{X_0\cap X_1\cap X_2} = \max\{\|f\|_{X_0}, \|f\|_{X_1}, \|f\|_{X_2}\} \quad (f \in X_0 \cap X_1 \cap X_2). \tag{2.4}$$

Since we have a triple of quasi-Banach function lattices, $\Sigma(\bar{X}) := L^{p_0,q_0}(C_0) + L^{p_1,q_1}(C_1) + L^{p_2,q_2}(C_2)$ is a quasi-Banach function space with the quasi-norm $\|\cdot\|_\Sigma$.

Let us denote by $\mathbb{R}^2_+$ the set of vectors $\mathbf{t} = (t_1, t_2)$ for which $t_i > 0$, $i = 1,2$. Then, as usual, for elements $x \in \Sigma(\bar{X})$, *Peetre's K-functional* of the 3-tuple $\bar{X}$ is defined for $\mathbf{t} \in \mathbb{R}^2_+$ by the formula

$$K(\mathbf{t}, x; \bar{X}) = \inf\Big\{\|x_0\|_{L^{p_0,q_0}(C_0)} + \cdots + t_2\|x_2\|_{L^{p_2,q_2}(C_2)}; x = \sum_{i=0}^{2} x_i\ x_i \in L^{p_i,q_i}(C_i)\Big\}.$$

As in the classical case, the K-functional is a concave function of $\mathbf{t}$.

To show that one can apply to $\bar{X}$ the methods of [AK] and [AKMNP], let $\varrho \in (0,1]$ be the parameter in Aoki-Rolewicz's theorem corresponding to a common constant $c := \max(c_0, c_1, c_2)$ in the triangle inequality for the quasi-Banach spaces in $\bar{X}$.

Define S_ρ, a modified *Calderón operator*, by the formula

$$(S_\varrho f)(\mathbf{t}) := \left(\int_{\mathbb{R}^2_+} \left[\min\left(1, \frac{t_1}{s_1}, \frac{t_2}{s_2}\right) f(\mathbf{s})\right]^\varrho \frac{ds_1}{s_1}\frac{ds_2}{s_2}\right)^{1/\varrho} \quad (\mathbf{t} \in \mathbb{R}^2_+), \tag{2.5}$$

where ϱ is such that $(2c)^\varrho = 2$ and consider the space

$$\sigma_\varrho(\bar{X}) := \Big\{f \in \Sigma(\bar{X});\ S_\varrho(S_\varrho K(\cdot, f; \bar{X}))(\mathbf{1})^\varrho < \infty\Big\}$$

which allows us to extend the construction in [AK] to our quasi-Banach triple, as we will show.

Let $\Theta = (\theta_0, \theta_1)$ with $\theta_0, \theta_1 > 0$ and $\theta_0 + \theta_1 < 1$. The *interpolation space* $\bar{X}_{\Theta,q;K}$ is defined by the condition

$$\|f\|_{\Theta,q;K} = \|K(\cdot, f; \bar{X})\|_{\Theta,q} < \infty,$$

where

$$\|g\|_{\Theta,q} := \Big(\int_0^\infty \int_0^\infty (t_1^{-\theta_0} t_2^{-\theta_1} f(t_1, t_2))^q \frac{dt_1}{t_1} \frac{dt_2}{t_2} \Big)^{1/q}$$

and $0 < q < \infty$ (with the usual change when $q = \infty$).

Also, as in the case of couples, the *J-space* $\bar{X}_{\Theta,q;J}$ is defined as

$$\|f\|_{\Theta,q;J} := \inf \Big\{ \Big(\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} (2^{-m\theta_0} 2^{-n\theta_1} J(2^m, 2^n, u_{mn}))^q \Big)^{1/q}; f = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} u_{mn} \Big\},$$

where $(u_{mn}) \subset \Delta(\bar{X})$ satisfies that

$$\Big(\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} (2^{-m\theta_0} 2^{-n\theta_1} J(2^m, 2^n, u_{mn}))^q \Big)^{1/q} < \infty,$$

and the operator J is defined as

$$J(\mathbf{t}, v) = J(\mathbf{t}, v; \bar{X}) = \max(\|v\|_0, t_1\|v\|_1, t_2\|v\|_2).$$

To use the construction in [AK] to our triple, in Section 3, we will show the following embeddings $\bar{X}_{\Theta,q,K} \hookrightarrow \bar{X}_{\Theta,\infty,K} \hookrightarrow \sigma_\varrho(\bar{X})$ for $q > 0$.

Definition 2.2.3. *We shall say that the* Fundamental Lemma with the operator S_ϱ is valid *for the* 3*-tuple* $\bar{X}$ *if any element* $x \in \sigma_\varrho(\bar{X})$ *can be represented as a series*

$$x = \sum_{\boldsymbol{k} \in \mathbf{Z}^2} x_k, \tag{2.6}$$

absolutely convergent in $\Sigma(\bar{X})$*, where* $x_k \in \triangle(\bar{X})$ *and*

$$J(2^{\boldsymbol{k}}, x_k; \bar{X}) \le C[S_\varrho K(\cdot, x; \bar{X})](2^{\boldsymbol{k}}). \tag{2.7}$$

Here and below $2^{\boldsymbol{k}} = (2^{k_1}, 2^{k_2})$*, where* $\boldsymbol{k} = (k_1, k_2) \in \mathbf{Z}^2$*, and* $C > 0$ *is a constant independent of* x *and* $\boldsymbol{k}$*.*

Lemma 2.2.4. *Let* $\bar{X}$ *be a* 3*-tuple consisting of the quasi-Banach function lattices* $L^{p_i,q_i}(C_i)$ *on* (Ω, Σ) *and* $p_i, q_i \in (0, \infty]$*,* $i = 0, 1, 2$*. Then the Fundamental Lemma with the operator* S_ϱ *is valid for* $\bar{X}$*.*

Proof. Observe that we must prove that every $f \in \sigma_\varrho(\bar{X})$ admits a representation as a sum in $\Sigma(\bar{X})$,

$$f = \sum_{\mathbf{k}\in\mathbf{Z}^2} f_{\mathbf{k}} \qquad (f_{\mathbf{k}} \in \Delta(\bar{X})),$$

where $\sum_{\mathbf{k}\in\mathbf{Z}^2} \|f_{\mathbf{k}}\|^\rho_{\Sigma(\bar{X})} < \infty$ and

$$J(2^{\mathbf{k}}, f_{\mathbf{k}}) \le C(S_\rho K(\cdot, f; \bar{X}))(2^{\mathbf{k}}).$$

To simplify some of the formulas, we will denote by $K(\cdot, f) = K(\cdot, f; \bar{X})$ for $f \in \Sigma(\bar{X})$.

First of all, we show that for any $\mathbf{k} \in \mathbf{Z}^2$ and $f \in \Sigma(\bar{X})$ there can be found non-overlapping sets $A_j(\mathbf{k})$, $j = 0, 1, 2$, such that

- $\bigcup_{j=0}^2 A_j(\mathbf{k}) = \Omega$,
- $$K(2^{\mathbf{k}}, f; \bar{X}) \approx \{\|f\chi_{A_0(\mathbf{k})}\|_{L^{p_0,q_0}(C_0)} + \ldots + 2^{k_2}\|f\chi_{A_2(\mathbf{k})}\|_{L^{p_2,q_2}(C_2)}\}. \tag{2.8}$$

For that, let us observe first that we can find $f_i \in L^{p_i,q_i}(C_i)$, $i = 0, 1, 2$, such that $f = \sum_{i=0}^2 f_i$ and

$$\|f_0\|_{L^{p_0,q_0}(C_0)} + 2^{k_1}\|f_1\|_{L^{p_1,q_1}(C_1)} + 2^{k_2}\|f_2\|_{L^{p_2,q_2}(C_2)} < 2K(2^{\mathbf{k}}, f; \bar{X}) < \infty.$$

Define the sets

$$\begin{aligned} A_0(\mathbf{k}) &= \{\omega \in \Omega; |f_0(\omega)| > |f_1(\omega)|\}, \\ A_1(\mathbf{k}) &= \{\omega \in \Omega; |f_0(\omega)| \le |f_1(\omega)|, |f_1(\omega)| > |f_2(\omega)|\}, \text{ and} \\ A_2(\mathbf{k}) &= \{\omega \in \Omega; |f_0(\omega)| \le |f_1(\omega)| \le |f_2(\omega)|\}. \end{aligned}$$

It follows then that $\bigcup_{j=0}^2 A_j(\mathbf{k}) = \Omega$ and $f(\omega) = \sum_{i=0}^2 (f\chi_{A_i(\mathbf{k})})(\omega)$ for $\omega \in \Omega$.

Moreover, since for $i \in \{0, 1, 2\}$,

$$|f\chi_{A_i(\mathbf{k})}(\omega)| \le \Big|\sum_{j=0}^2 f_j\chi_{A_i(\mathbf{k})}(\omega)\Big| \le \sum_{j=0}^2 |f_j(\omega)|\chi_{A_i(\mathbf{k})}(\omega) \le 3|f_i(\omega)|\chi_{A_i(\mathbf{k})}(\omega)$$

and these spaces $L^{p_i,q_i}(C_i)$ are vector lattices, we have that $\{f\chi_{A_i(\mathbf{k})}\}_{i=0}^2$ is an appropiate descomposition. Therefore,

$$\begin{aligned} 2K(2^{\mathbf{k}}, f; \bar{X}) &> \|f_0\|_{L^{p_0,q_0}(C_0)} + 2^{k_1}\|f_1\|_{L^{p_1,q_1}(C_1)} + 2^{k_2}\|f_2\|_{L^{p_2,q_2}(C_2)} \\ &\ge \|f_0\chi_{A_0(\mathbf{k})}\|_{L^{p_0,q_0}(C_0)} + 2^{k_1}\|f_1\chi_{A_1(\mathbf{k})}\|_{L^{p_1,q_1}(C_1)} + 2^{k_2}\|f_2\chi_{A_2(\mathbf{k})}\|_{L^{p_2,q_2}(C_2)} \\ &\ge \frac{1}{3}K(2^{\mathbf{k}}, f; \bar{X}). \end{aligned}$$

Now we shall construct the descomposition of $f \in \sigma_\varrho(\bar{X})$ satisfying (2.6) in several steps, using a special partition.

Step 1: Construction of a new familiy $\bar{A}_j(\mathbf{k})$ with the monotonicity property. For $\mathbf{k} \in \mathbf{Z}^2$ we define

$$\begin{aligned}\Omega_0(\mathbf{k}) &= \Big\{s \in \mathbf{Z}^2; 1 = \min(1, 2^{k_1-s_1}, 2^{k_2-s_2})\Big\}\\ \Omega_j(\mathbf{k}) &= \Big\{s \in \mathbf{Z}^2; 2^{k_j-s_j} = \min(1, 2^{k_1-s_1}, 2^{k_2-s_2})\Big\} \quad (j = 1, 2).\end{aligned}$$

Let

$$\bar{A}_j(\mathbf{k}) := \bigcup_{s\in\Omega_j(\mathbf{k})} A_j(\mathbf{s}) \quad (j = 1, 2), \tag{2.9}$$

$$\bar{A}_0(\mathbf{k}) := \Omega\backslash\bigcup_{i=1}^{2} \bar{A}_i(\mathbf{k}). \tag{2.10}$$

Then, $\bar{A}_j(\mathbf{k}) \supset A_j(\mathbf{k})$, $j = 1, 2$, and

$$\bar{A}_0(\mathbf{k}) \subset A_0(\mathbf{k}). \tag{2.11}$$

Let us see that there exists a constant C such that $CS_\varrho K(\cdot, f; \bar{X}) \geq K(\cdot, f; \bar{X})$. Indeed, we have that

$$\begin{aligned}(S_\varrho K(\cdot, f; \bar{X}))^\varrho &\geq \int_{t_2}^{2t_2}\int_{t_1}^{2t_1}\Big[\min\Big(1, \frac{t_1}{s_1}, \frac{t_2}{s_2}\Big)K(s_1, s_2, f)\Big]^\varrho \frac{ds_1}{s_1}\frac{ds_2}{s_2}\\ &\geq \int_{t_2}^{2t_2}\int_{t_1}^{2t_1}\Big[\min\Big(1, \frac{t_1}{2t_1}, \frac{t_2}{s_2}\Big)K(s_1, s_2, f)\Big]^\varrho \frac{ds_1}{s_1}\frac{ds_2}{s_2}\\ &\geq \int_{t_2}^{2t_2}\int_{t_1}^{2t_1}\Big[\min\Big(1, \frac{t_1}{2t_1}, \frac{t_2}{s_2}\Big)K(t_1, s_2, f)\Big]^\varrho \frac{ds_1}{s_1}\frac{ds_2}{s_2}\\ &\geq \int_{t_2}^{2t_2}\int_{t_1}^{2t_1}\Big[\min\Big(1, \frac{t_1}{2t_1}, \frac{t_2}{2t_2}\Big)K(t_1, t_2, f)\Big]^\varrho \frac{ds_1}{s_1}\frac{ds_2}{s_2}\\ &= \frac{1}{2^\varrho}K(t_1, t_2, f)^\varrho(\log 2)^2.\end{aligned}$$

Hence, $K(\cdot, f; \bar{X}) \leq \frac{2}{(\log 2)^{2/\varrho}} S_\varrho K(\cdot, f; \bar{X})$.

Moreover, it follows that, from (2.8), (2.9), (2.11) and last inequality, for a certain constant $C > 0$

$$\begin{aligned}\|f\chi_{\bar{A}_0(\mathbf{k})}\|_{L^{p_0,q_0}(C_0)} + 2^{k_1}\|f\chi_{\bar{A}_1(\mathbf{k})}\|_{L^{p_1,q_1}(C_1)} + 2^{k_2}\|f\chi_{\bar{A}_2(\mathbf{k})}\|_{L^{p_2,q_2}(C_2)}\\ \leq CS_\varrho K(\cdot, f; \bar{X})(2^{\mathbf{k}}).\end{aligned}$$

Let $\Gamma_1, \Gamma_2 : \mathbf{Z}^2 \to \mathbf{Z}^2$ be the operators

$$\Gamma_1(\mathbf{k}) := (k_1 - 1, k_2), \Gamma_2(\mathbf{k}) := (k_1, k_2 - 1).$$

Then it follows that

$$\bar{A}_j(\Gamma_i(\mathbf{k})) \subset \bar{A}_j(\mathbf{k}) \text{ for } j \neq i, \bar{A}_j(\Gamma_j(\mathbf{k})) \supset \bar{A}_j(\mathbf{k}). \tag{2.12}$$

Step 2: Construction of elements from the intersection. For any $2^{\mathbf{k}} = (2^{k_1}, 2^{k_2}) \in \mathbb{R}^2_+$ we define

$$B(\mathbf{k}) := \bar{A}_0(\mathbf{k}) \setminus \bigcup_{i=1}^{2} \bar{A}_0(\Gamma_i(\mathbf{k})). \tag{2.13}$$

Considering (2.10) and (2.12) it follows that

$$B(\mathbf{k}) = \bigcap_{i=1}^{2} \Big[\bar{A}_i(\Gamma_i(\mathbf{k})) \setminus \bigcup_{j=1}^{2} \bar{A}_j(\mathbf{k}) \Big]. \tag{2.14}$$

By definition, since $|f|\chi_{B(\mathbf{k})} \leq |f|\chi_{\bar{A}_0(\mathbf{k})}$ and $|f|\chi_{B(\mathbf{k})} \leq |f|\chi_{\bar{A}_i(\Gamma_i(\mathbf{k}))}$, $i = 1, 2$, it follows that $|f|\chi_{B(\mathbf{k})} \in \Delta(\bar{X})$ and for $i = 1, 2$, we have

$$\begin{aligned} 2^{k_i} \|f\chi_{B(\mathbf{k})}\|_{L^{p_i,q_i}(C_i)} &\leq 2^{k_i} \|f\chi_{\bar{A}_i(\Gamma_i(\mathbf{k}))}\|_{L^{p_i,q_i}(C_i)} = 2 2^{k_i - 1} \|f\chi_{\bar{A}_i(\Gamma_i(\mathbf{k}))}\|_{L^{p_i,q_i}(C_i)} \\ &\lesssim 2K(2^{\Gamma_i(\mathbf{k})}, f; \bar{X}) \lesssim 2C[S_\varrho K(\cdot, f; \bar{X})](2^{\Gamma_i(\mathbf{k})}). \end{aligned}$$

Since $S_\varrho K(\cdot, f; \bar{X})$ is a non-decreasing function and $(\Gamma_i(\mathbf{k}))_i < k_i$, we obtain for $i \in \{1, 2\}$

$$\|f\chi_{B(\mathbf{k})}\|_{L^{p_0,q_0}(C_0)} \leq \|f\chi_{\bar{A}_0(\mathbf{k})}\|_{L^{p_0,q_0}(C_0)} \lesssim C[S_\varrho K(\cdot, f; \bar{X})](2^{\mathbf{k}}), \tag{2.15}$$

$$2^{k_i} \|f\chi_{B(\mathbf{k})}\|_{L^{p_i,q_i}(C_i)} \leq 2C[S_\varrho K(\cdot, f; \bar{X})](2^{\Gamma_i(\mathbf{k})}) \lesssim 2C[S_\varrho K(\cdot, f; \bar{X})](2^{\mathbf{k}}).$$

Let

$$y_{\mathbf{k}} := |f|\chi_{B(\mathbf{k})}, \ \mathbf{k} \in \mathbf{Z}^2.$$

We have then that, $y_{\mathbf{k}} \in \Delta(\bar{X})$ and it follows, for $C' = 2C$, that

$$\begin{aligned} J(2^{\mathbf{k}}, y_{\mathbf{k}}; \bar{X}) &:= \max\Big\{ \|y_{\mathbf{k}}\|_{L^{p_0,q_0}(C_0)}, 2^{k_1} \|y_{\mathbf{k}}\|_{L^{p_1,q_1}(C_1)}, 2^{k_2} \|y_{\mathbf{k}}\|_{L^{p_2,q_2}(C_2)} \Big\} \\ &\lesssim 2C[S_\varrho K(\cdot, f; \bar{X})](2^{\mathbf{k}}) = C'[S_\varrho K(\cdot, f; \bar{X})](2^{\mathbf{k}}). \end{aligned}$$

Step 3: Construction of the required descomposition.
Let $f \in \sigma_\varrho(\bar{X})$ and $\mathbf{Z}^2 = \bigcup_{j=0}^{2} \Omega_j(\mathbf{1})$. From (2.14) and (2.15) we have

$$\begin{aligned}
\sum_{\mathbf{k}\in\mathbf{Z}^2} \|y_{\mathbf{k}}\|^\varrho_{\Sigma(\bar{X})} &\leq 2\sum_{\mathbf{k}\in\mathbf{Z}^2}\sum_{j=0}^{2} \|y_{\mathbf{k}}\|^\varrho_{X_j} \\
&= 2\Big\{ \sum_{\mathbf{k}\in\Omega_0(1,1)} \|y_{\mathbf{k}}\|^\varrho_{L^{p_0,q_0}(C_0)} + \sum_{j=1}^{2}\sum_{\mathbf{k}\in\Omega_j(1,1)} \|y_{\mathbf{k}}\|^\varrho_{L^{p_j,q_j}(C_j)} \Big\} \\
&= 2\Big\{ C^\varrho \sum_{\mathbf{k}\in\Omega_0(1,1)} [S_\varrho K(\cdot,f)](2^{\mathbf{k}})^\varrho \\
&+ (2C)^\varrho \sum_{j=1}^{2}\sum_{\mathbf{k}\in\Omega_j(1,1)} \Big(\frac{[S_\varrho K(\cdot,f)](2^{\mathbf{k}})}{2^{k_j}}\Big)^\varrho \Big\} \\
&\lesssim 2(2C)^\varrho \Big\{ \sum_{\mathbf{k}\in\Omega_0(1,1)} [S_\varrho K(\cdot,f)](2^{\mathbf{k}})^\varrho \\
&+ \sum_{j=1}^{2}\sum_{\mathbf{k}\in\Omega_j(1,1)} \Big(\frac{[S_\varrho K(\cdot,f)](2^{\mathbf{k}})}{2^{k_j}}\Big)^\varrho \Big\} < \infty
\end{aligned}$$

since

$$\begin{aligned}
&\sum_{\mathbf{k}\in\Omega_0(1,1)} [S_\varrho K(\cdot,f)](2^{\mathbf{k}})^\varrho + \sum_{j=1}^{2}\sum_{\mathbf{k}\in\Omega_j(1,1)} \Big(\frac{[S_\varrho K(\cdot,f)](2^{\mathbf{k}})}{2^{k_j}}\Big)^\varrho \\
= &\sum_{\mathbf{k}\in\Omega_0(1,1)} (\log 2)^{-2} [S_\varrho K(\cdot,f)](2^{\mathbf{k}})^\varrho \int_{2^{k_2}}^{2^{k_2+1}} \int_{2^{k_1}}^{2^{k_1+1}} \frac{ds_1}{s_1}\frac{ds_2}{s_2} \\
+ &\sum_{j=1}^{2}\sum_{\mathbf{k}\in\Omega_j(1,1)} \frac{1}{(\log 2)^2} \Big(\frac{[S_\varrho K(\cdot,f)](2^{\mathbf{k}})}{2^{k_j}}\Big)^\varrho \int_{2^{k_2-1}}^{2^{k_2}} \int_{2^{k_1-1}}^{2^{k_1}} \frac{ds_1}{s_1}\frac{ds_2}{s_2} \\
\leq &\frac{1}{(\log 2)^2} \Big\{ \sum_{\mathbf{k}\in\Omega_0(1,1)} \int_{2^{k_2}}^{2^{k_2+1}} \int_{2^{k_1}}^{2^{k_1+1}} [S_\varrho K(\cdot,f)](s)^\varrho \frac{ds_1}{s_1}\frac{ds_2}{s_2} \\
+ &\sum_{j=1}^{2}\sum_{\mathbf{k}\in\Omega_j(1,1)} \int_{2^{k_2-1}}^{2^{k_2}} \int_{2^{k_1-1}}^{2^{k_1}} \Big(\frac{[S_\varrho K(\cdot,f)](s)}{s_j}\Big)^\varrho \frac{ds_1}{s_1}\frac{ds_2}{s_2} \Big\} \\
\lesssim &\frac{1}{(\log 2)^2} \Big\{ \sum_{\mathbf{k}\in\Omega_0(1,1)} \int_{2^{k_2}}^{2^{k_2+1}} \int_{2^{k_1}}^{2^{k_1+1}} \min(1, 1/s_1, 1/s_2)^\varrho [S_\varrho K(\cdot,f)](s)^\varrho \frac{ds_1}{s_1}\frac{ds_2}{s_2} \\
+ &\sum_{j=1}^{2}\sum_{\mathbf{k}\in\Omega_j(1,1)} \int_{2^{k_2-1}}^{2^{k_2}} \int_{2^{k_1-1}}^{2^{k_1}} \Big(\min(1, 1/s_1, 1/s_2)[S_\varrho K(\cdot,f)](s)\Big)^\varrho \frac{ds_1}{s_1}\frac{ds_2}{s_2} \Big\} \\
= &\frac{1}{(\log 2)^2} S_\varrho(S_\varrho K(\cdot,f;\bar{X}))(1,1)^\varrho.
\end{aligned}$$

Thus, the finiteness of $\sum_{\mathbf{k}\in\mathbf{Z}^2}\|y_{\mathbf{k}}\|^{\varrho}_{\Sigma(\bar{X})}$ implies the existence of an element $g\in\Sigma(\bar{X})$ such that $\|S_n-g\|_{\Sigma(\bar{X})}\to 0$ as $n\to\infty$, where S_n is the n-essim partial sum. Moreover, defining $A_n=\{|S_n-S_{n+1}|>\sqrt{\epsilon}\}$ we have that

$$C(A_n)\sqrt{\epsilon}<\int_{A_n}|S_n-S_{n+1}|dC\leq 4C_\Omega\|S_n-g\|_{\Sigma(\bar{X})}.$$

Hence, $C(A_n)\to 0$ as $n\to\infty$. That is, $\{S_n\}_n$ is a Cauchy sequence in capacity and therefore, it is convergent in capacity to the function g, by Theorem 1.3.11. Moreover, since it is pointwise convergent to f in $\Sigma(\bar{X})$, we get that $f=g$ $C-$ q.e.

Below we shall show that the inequality

$$|f|\leq\sum_{\mathbf{k}\in\mathbf{Z}^2}y_{\mathbf{k}} \tag{2.16}$$

holds quasi everywhere.

If (2.16) is correct, then for $a\in\operatorname{supp} f$ we have that $f(a)>0$ and then $\sum_{\mathbf{k}\in\mathbf{Z}^2}y_{\mathbf{k}}(a)>0$. So that $a\in\operatorname{supp}[\sum_{\mathbf{k}\in\mathbf{Z}^2}y_{\mathbf{k}}]$. Moreover, the series composed of the elements

$$x_{\mathbf{k}}=f\frac{y_{\mathbf{k}}}{\sum_{\mathbf{k}\in\mathbf{Z}^2}y_{\mathbf{k}}}$$

pointwise converges to f almost everywhere, since the series $\sum_{\mathbf{k}\in\mathbf{Z}^2}y_{\mathbf{k}}$ is pointwise convergent almost everywhere. From (2.16) it follows that

$$\|x_{\mathbf{k}}\|_{\Sigma(\bar{X})}=\left\|\left|\frac{f}{\sum_{\mathbf{k}\in\mathbf{Z}^2}y_{\mathbf{k}}}\right||y_{\mathbf{k}}|\right\|_{\Sigma(\bar{X})}\leq\|y_{\mathbf{k}}\|_{\Sigma(\bar{X})}.$$

So that, the series $\sum_{\mathbf{k}\in\mathbf{Z}^2}x_{\mathbf{k}}$ is absolutely convergent in $\Sigma(\bar{X})$, and since it pointwise converges to f, we get that its sum will be equal to f.

It follows from (2.16) and the lattice property of X_i, $i=0,1,2$ that

$$\begin{aligned}J(2^{\mathbf{k}},x_{\mathbf{k}};\bar{X}) &:= \max\left(\|x_{\mathbf{k}}\|_{L^{p_0,q_0}(C_0)},2^{k_1}\|x_{\mathbf{k}}\|_{L^{p_1,q_1}(C_1)},2^{k_2}\|x_{\mathbf{k}}\|_{L^{p_2,q_2}(C_2)}\right)\\ &\leq \max\left(\|y_{\mathbf{k}}\|_{L^{p_0,q_0}(C_0)},2^{k_1}\|y_{\mathbf{k}}\|_{L^{p_1,q_1}(C_1)},2^{k_2}\|y_{\mathbf{k}}\|_{L^{p_2,q_2}(C_2)}\right)\\ &= J(2^{\mathbf{k}},y_{\mathbf{k}};\bar{X})\leq C'[S_\varrho K(\cdot,f;\bar{X})](2^{\mathbf{k}}).\end{aligned}$$

Therefore, the elements $x_{\mathbf{k}}$ where $\mathbf{k}\in\mathbf{Z}^2$ satisfy the requirements of the Lemma. It remains only to prove inequality (2.16).

It follows from the definition that, if $a\in B(\mathbf{k})$, then $|f(a)|=|f(a)\chi_{B(\mathbf{k})}(a)|=|y_{\mathbf{k}}(a)|$ and then (2.16) holds almost everywhere. Moreover, if $a\in(\operatorname{supp} f)^c$ [1], then

[1]For $A\subset\Omega$, A^c denotes the contrary of A, that is, $\Omega\setminus A$.

$f(a) = 0$ and hence (2.16) holds. So, (2.16) holds almost everywhere on $\bigcup_{\mathbf{k}\in\mathbf{Z}^2} B(\mathbf{k}) \cup (\text{supp } f)^c := \widetilde{C}$. To finish, it is enough to prove that the contrary of $\widetilde{C}$, $\bar{C} := \text{supp } f \setminus \bigcup_{\mathbf{k}\in\mathbf{Z}^2} B(\mathbf{k})$, has zero capacity. We have that

$$\begin{aligned}
\|f - f\chi_{\cup_{\mathbf{k}\in\mathbf{Z}^2} B(\mathbf{k})}\|_{\Sigma(\bar{X})} &= \|(f - f\chi_{\cup_{\mathbf{k}\in\mathbf{Z}^2} B(\mathbf{k})})\chi_{\text{supp } f}\|_{\Sigma(\bar{X})} \\
&= \Big\|(f\chi_{\text{supp } f}\chi_{\left[\cup_{\mathbf{k}\in\mathbf{Z}^2} B(\mathbf{k}) \cup (\cup_{\mathbf{k}\in\mathbf{Z}^2} B(\mathbf{k}))^c\right]} \\
&- f\chi_{\cup_{\mathbf{k}\in\mathbf{Z}^2} B(\mathbf{k})}\chi_{\text{supp } f})\Big\|_{\Sigma(\bar{X})} \\
&= \|f\chi_{\bar{C}}\|_{\Sigma(\bar{X})}.
\end{aligned}$$

Since all the capacities have the same null sets, if we prove $\|f\chi_{\bar{C}}\|_{\Sigma(\bar{X})} = 0$, then it will be proved that $\bar{C}$ has zero capacity. Take $\epsilon > 0$. From the definition of the set $\bar{A}_0(\mathbf{m})$ ($\mathbf{m}$=(m, m)), using that $f \in \sigma_\varrho(\bar{X})$, we deduce for sufficiently large m that

$$\begin{aligned}
\|f - f\chi_{\bar{A}_0(\mathbf{m})}\|_{\Sigma(\bar{X})} &\leq \sum_{i=1}^{2} 2^m \|f\chi_{\bar{A}_i(\mathbf{m})}\|_{L^{p_i,q_i}(C_i)} \frac{1}{2^m} \\
&\lesssim 2C[S_\varrho K(\cdot, f; \bar{X})](2^{\mathbf{m}})/2^m < \varepsilon/2.
\end{aligned}$$

Moreover, since $\chi_{\bigcup_{\mathbf{k}\in\mathbf{Z}^2}[B(\mathbf{k})\cap\bar{A}_0(\mathbf{m})]} \leq \chi_{\bigcup_{\mathbf{k}\in\mathbf{Z}^2} B(\mathbf{k})}$, then, by the lattice property,

$$\|f - f\chi_{\cup_{\mathbf{k}\in\mathbf{Z}^2} B(\mathbf{k})}\|_{\Sigma(\bar{X})} \leq \|f - f\chi_{\cup_{\mathbf{k}\in\mathbf{Z}^2}[B(\mathbf{k})\cap\bar{A}_0(\mathbf{m})]}\|_{\Sigma(\bar{X})}$$

and

$$\begin{aligned}
\|f - f\chi_{\cup_{\mathbf{k}\in\mathbf{Z}^2}[B(\mathbf{k})\cap\bar{A}_0(\mathbf{m})]}\|_{\Sigma(\bar{X})} &\lesssim \|f - f\chi_{\bar{A}_0(\mathbf{m})}\|_{\Sigma(\bar{X})} \\
&+ \|f\chi_{\bar{A}_0(\mathbf{m})} - f\chi_{\cup_{\mathbf{k}\in\mathbf{Z}^2}[B(\mathbf{k})\cap\bar{A}_0(\mathbf{m})]}\|_{\Sigma(\bar{X})}
\end{aligned}$$

(There is a constant associated with the quasi-norm $\|\cdot\|_{\Sigma(\bar{X})}$ in the quasi-Banach case, but finally this fact does not affect the conclusion). It is clear that in order to prove (2.16) it is enough to show that

$$\Big\|f\chi_{\bar{A}_0(\mathbf{m})} - f\chi_{\cup_{\mathbf{k}\in\mathbf{Z}^2}[B(\mathbf{k})\cap\bar{A}_0(\mathbf{m})]}\Big\|_{\Sigma(\bar{X})} < \epsilon/2. \tag{2.17}$$

To prove (2.17) we shall consider the sets

$$\Omega_{m,l} = \{\mathbf{k} = (k_1, k_2) \in \mathbf{Z}^2; l < k_i \leq m, i = 1, 2\}$$

and

$$\Omega^j_{m,l} = \{\mathbf{k} = (k_1, k_2) \in \mathbf{Z}^2; k_j = l \text{ and } l < k_i \leq m \text{ for } i \neq j\},\ j = 1, 2.$$

It follows from the definition (2.13) that

$$\bar{A}_0(\mathbf{k}) \subset B(\mathbf{k}) \cup \bigcup_{i=1}^{2} \bar{A}_0(\Gamma_i(\mathbf{k})). \tag{2.18}$$

In particular,

$$\bar{A}_0(\mathbf{m}) \subset B(\mathbf{m}) \cup \bigcup_{i=1}^{2} \bar{A}_0(\Gamma_i(\mathbf{m}))$$

and for $\Gamma_i(\mathbf{m})$, which is certain $\mathbf{k}$, we have that

$$\bar{A}_0(\Gamma_i(\mathbf{m})) \subset B(\Gamma_i(\mathbf{m})) \cup \bigcup_{j=1}^{2} \bar{A}_0(\Gamma_j\Gamma_i(\mathbf{m})).$$

Therefore

$$\bar{A}_0(\mathbf{m}) \subset B(\mathbf{m}) \cup \bigcup_{i=1}^{2} B(\Gamma_i(\mathbf{m})) \cup \bigcup_{i,j=1}^{2} \bar{A}_0(\Gamma_j\Gamma_i(\mathbf{m})).$$

Repeatedly using the embedding (2.18), we continue this process of replacing the sets $\bar{A}_0(\mathbf{s})$ for $\mathbf{s} \in \Omega_{m,l}$, and we obtain

$$\bar{A}_0(\mathbf{m}) \subset \bigcup_{\mathbf{k}\in\Omega_{m,l}} B(\mathbf{k}) \cup \bigcup_{j=1}^{2} \bigcup_{\mathbf{s}\in\Omega^j_{m,l}} \bar{A}_0(\mathbf{s}).$$

Therefore, taking into account (2.11) and the descomposition, we have

$$\begin{aligned}
\left\| f\chi_{\bar{A}_0(\mathbf{m})} - f\chi_{\cup_{\mathbf{k}\in\mathbf{Z}^2}[B(\mathbf{k})\cap\bar{A}_0(\mathbf{m})]} \right\|_{\Sigma(\bar{X})} &\lesssim \sum_{j=1}^{2} \sum_{\mathbf{s}\in\Omega^j_{m,l}} \|f\chi_{\bar{A}_0(\mathbf{s})}\|_{\Sigma(\bar{X})} \\
&\lesssim \sum_{j=1}^{2} \sum_{\mathbf{s}\in\Omega^j_{m,l}} \|f\chi_{A_0(\mathbf{s})}\|_{\Sigma(\bar{X})} \\
&\lesssim \sum_{j=1}^{2} \sum_{\mathbf{s}\in\Omega^j_{m,l}} \|f\chi_{A_0(\mathbf{s})}\|_{L^{p_0,q_0}(C_0)} \\
&\lesssim \sum_{j=1}^{2} \sum_{\mathbf{s}\in\Omega^j_{m,l}} K(2^{\mathbf{s}}, f; \bar{X}).
\end{aligned}$$

We also note that, if $\mathbf{s} \in \Omega^j_{m,l}$, then $s_j = l$ and $l < s_i \le m$ for $i \ne j$. Hence

$$\begin{aligned}\sum_{\mathbf{s}\in\Omega^j_{m,l}} K(2^{\mathbf{s}}, f; \bar{X}) &\simeq \sum_{\mathbf{s}\in\Omega^j_{m,l}} \Big(\|f\chi_{A_0(\mathbf{s})}\|_{L^{p_0,q_0}(C_0)} + \sum_{i=1}^{2} 2^{s_i}\|f\chi_{A_i(\mathbf{s})}\|_{L^{p_i,q_i}(C_i)}\Big)\\ &\leq \sum_{\mathbf{s}\in\Omega^j_{m,l}} \Big(\|f\chi_{A_0(\mathbf{s})}\|_{L^{p_0,q_0}(C_0)} + \sum_{i=1, i\neq j}^{2} 2^{m}\|f\chi_{A_i(\mathbf{s})}\|_{L^{p_i,q_i}(C_i)}\\ &+ 2^{l}\|f\chi_{A_j(\mathbf{s})}\|_{L^{p_j,q_j}(C_j)}\Big) \leq 2C'[S_\varrho K(\cdot, f; \bar{X})](2^{\Gamma_j^{m-l}(\mathbf{m})})\end{aligned}$$

since, by definition of $\Gamma_j^{m-l}(\mathbf{m})$, we know that it has the value l in the j component and m in the rest. It follows from the fact $f \in \sigma_\varrho(\bar{X})$ that, for a fixed m, as $l \to -\infty$, $2^{\Gamma_j^{m-l}(\mathbf{m})}$ (in the limit) has a zero coordinate, since $2^{-\infty} = 0$, and then, the minimum inside the integral will be zero, which means

$$[S_\varrho K(\cdot, f; \bar{X}](2^{\Gamma_j^{m-l}(\mathbf{m})}) \to 0.$$

Now (2.17) follows from this and the proof is finished. □

2.3 The Equivalence theorem

In 1964 J. L. Lions and J. Peetre [LiP1] proved one of the most important theoretical results in interpolation theory, the so-called *reiteration formula* for couples of Banach spaces $\bar{X}$:

$$(\bar{X}_{\theta_0,q_0}, \bar{X}_{\theta_1,q_1})_{\eta,q} = \bar{X}_{\theta,q}, \qquad \theta = (1-\eta)\theta_0 + \eta\theta_1, \tag{2.19}$$

where $\theta_0 \neq \theta_1$, $0 < \eta < 1$ and $\bar{X}_{\theta,q}$ is defined in an analogous way in the case of triples. This formula also holds for quasi-Banach spaces (see [BeLo]).

The classical proof of the formula (2.19) is based on the so-called *Equivalence theorem* for the $K-$ and $J-$ methods:

$$\bar{X}_{\theta,q;K} = \bar{X}_{\theta,q;J},$$

which is valid for any couple $\bar{X} = (X_0, X_1)$ of quasi-Banach spaces (cf. [BeLo, Theorem 3.11.3]).

G. Sparr defined the $K-$ and $J-$ functionals and the corresponding interpolation spaces for $(n+1)$-tuples $\bar{X} = (X_0, \ldots, X_n)$, as for couples, and tried to extend the reiteration formula (2.19) to $(n+1)$-tuples and he showed that, if an analogue of the Equivalence theorem is valid for $\bar{X}$, then an analogue of the Lions-Peetre reiteration formula is also true. But there are troubles with the Equivalence theorem for $n > 1$. Even for a good triple, such

as a triple of Hilbert spaces, the classical method of proving the reiteration theorem does not work.

In 1997 I. Asekritova and N. Krugljak showed in [AK] that the Equivalence theorem is in fact valid for any n-tuple of Banach function lattices. In particular, it holds for triples of weighted L^p spaces. The proof of the result of Asekritova-Krugljak is rather complicated and uses significantly the structure of the Banach function lattices.

Here we will check that this also holds for the quasi-Banach triple $\bar{X}$ considered in the previous section with the properties of the modified Calderón operator defined in (2.5).

Definition 2.3.1. *A quasi-Banach function lattice Φ on $\mathbb{R}^2_+$ with the measure $\frac{d\boldsymbol{t}}{\boldsymbol{t}} = \frac{dt_1}{t_1}\frac{dt_2}{t_2}$ is called a* parameter of the ϱ-real method *if the operator S_ϱ is bounded in Φ, $\varrho \in (0,1]$.*

One of the main parameters of the real method is $\Phi_{\Theta,q}$ for $1 \le q \le \infty$. In this lattice, the norm is denoted by $\|f\|_{\Theta,q}$, where $\mathbf{t}^{-\Theta} = t_1^{-\theta_0} t_2^{-\theta_1}$, $\Theta = (\theta_0, \theta_1)$, $\theta_i > 0$ $(i = 0, 1)$, $\theta_0 + \theta_1 \le 1$ and $q \in [1, \infty]$.

In order to follow, in a similar way, the construction in [AK] we have to show that for $0 < q \le \infty$ and $\theta_0 + \theta_1 = 1$, $\bar{X}_{\theta_0,\theta_1,q;K} \hookrightarrow \sigma_\varrho(\bar{X})$. For that, we show first that $\bar{X}_{\theta_0,\theta_1,\infty;K} \hookrightarrow \sigma_\varrho(\bar{X})$ and then we will see that $\bar{X}_{\theta_0,\theta_1,q;K} \hookrightarrow \bar{X}_{\theta_0,\theta_1,\infty;K}$.

$\bar{X}_{\theta_0,\theta_1,\infty;K} \hookrightarrow \sigma_\varrho(\bar{X})$: Let $f \in \bar{X}_{\theta_0,\theta_1,\infty;K}$. Then $\|f\|_{X_{\theta_0,\theta_1,\infty;K}} < \infty$, and hence

$$\begin{aligned} S_\varrho K(1,1,f;\bar{X})^\varrho &= \int_{\mathbb{R}^2_+} \Big[\min\Big(1, \frac{1}{s_1}, \frac{1}{s_2}\Big) s_1^{\theta_0} s_1^{-\theta_0} s_2^{\theta_1} s_2^{-\theta_1} K(s_1, s_2, f)\Big]^\varrho \frac{ds_1}{s_1}\frac{ds_2}{s_2} \\ &\le \int_{\mathbb{R}^2_+} \Big[\min\Big(1, \frac{1}{s_1}, \frac{1}{s_2}\Big) s_1^{\theta_0} s_2^{\theta_1}\Big]^\varrho \|f\|^\varrho_{X_{\theta_0,\theta_1,\infty,K}} \frac{ds_1}{s_1}\frac{ds_2}{s_2} < \infty, \end{aligned}$$

since for $\prod$ be the surface $\min(1, t_1, t_2) = 1$ we have that

$$\begin{aligned} \int_{\mathbb{R}^2_+} \Big[\min\Big(1, \frac{1}{s_1}, \frac{1}{s_2}\Big) s_1^{\theta_0} s_2^{\theta_1}\Big]^\varrho \frac{ds_1}{s_1}\frac{ds_2}{s_2} &\le \int_{\mathbb{R}^2_+} \Big[t_1^{-\theta_0} t_2^{-\theta_1}\Big]^\varrho \frac{dt_1}{t_1}\frac{dt_2}{t_2} \\ &= \int_{\prod} [\min(1, t_1, t_2) t_1^{-\theta_0} t_2^{-\theta_1}]^\varrho \frac{dt_1}{t_1}\frac{dt_2}{t_2} \\ &= \sum_{j=0}^{2} \int_{\min(\mathbf{t})=t_j=1} [\min(1, t_1, t_2) t_1^{-\theta_0} t_2^{-\theta_1}]^\varrho \frac{dt_1}{t_1}\frac{dt_2}{t_2} \end{aligned}$$

and all the integrals are finite.

$X_{\theta_0,\theta_1,q;K} \hookrightarrow X_{\theta_0,\theta_1,\infty;K}$: For $f \in X_{\theta_0,\theta_1,q;K}$, we have that

$$\int_{\mathbb{R}_+^2} (t_1^{-\theta_0} t_2^{-\theta_1} K(t_1, t_2, f))^q \frac{dt_1}{t_1}\frac{dt_2}{t_2} < \infty,$$

and

$$\begin{aligned}
t_1^{-\theta_0} t_2^{-\theta_1} K(t_1, t_2, f) &\lesssim \Big(\int_{t_2}^{2t_2}\int_{t_1}^{2t_1} \frac{ds_1}{s_1}\frac{ds_2}{s_2}\big(t_1^{-\theta_0} t_2^{-\theta_1} K(t_1, t_2, f)\big)^q\Big)^{1/q} \\
&= 2\Big(\int_{t_2}^{2t_2}\int_{t_1}^{2t_1} \big((2t_1)^{-\theta_0}(2t_2)^{-\theta_1} K(t_1, t_2, f)\big)^q \frac{ds_1}{s_1}\frac{ds_2}{s_2}\Big)^{1/q} \\
&\le 2\Big(\int_{t_2}^{2t_2}\int_{t_1}^{2t_1} \big(s_1^{-\theta_0} s_2^{-\theta_1} K(t_1, t_2, f)\big)^q \frac{ds_1}{s_1}\frac{ds_2}{s_2}\Big)^{1/q} \\
&\le 2\Big(\int_{t_2}^{2t_2}\int_{t_1}^{2t_1} \big(s_1^{-\theta_0} s_2^{-\theta_1} K(s_1, s_2, f)\big)^q \frac{ds_1}{s_1}\frac{ds_2}{s_2}\Big)^{1/q} \\
&\le 2\|f\|_{\bar{X}_{\theta_0,\theta_1,q;K}},
\end{aligned}$$

therefore

$$\sup_{t_1,t_2\in\mathbb{R}_+} [t_1^{-\theta_0} t_2^{-\theta_1} K(t_1, t_2, f; \bar{X})] := \|f\|_{\bar{X}_{\theta_0,\theta_1,\infty;K}} \le 2\|f\|_{\bar{X}_{\theta_0,\theta_1,q;K}}.$$

We have proved that we can apply in similar way the construction in [AK].

We define the interpolation spaces $K_\Theta(\bar{X})$ and $J_\Theta(\bar{X})$ by the quasi-norms

$$\begin{aligned}
\|f\|_{K_\Theta(\bar{X})} &= \|K(\cdot, f; \bar{X})\|_\Theta, \\
\|f\|_{J_\Theta(\bar{X})} &= \inf\Big\{\|J(\cdot, u(\cdot); \bar{X})\|_\Theta; f = \sum_k u_k \text{ convergent in } \Sigma(\bar{X}),\ u_k \in \Delta(\bar{X})\Big\}.
\end{aligned}$$

Theorem 2.3.2. *Let $p_i, q_i \in (0, \infty]$, $i = 0, 1, 2$ and $\bar{X} = (L^{p_0,q_0}(C_0), L^{p_1,q_1}(C_1), L^{p_2,q_2}(C_2))$ be a 3-tuple of quasi-Banach spaces for which the Fundamental Lemma with the operator S_ϱ is valid. Then, for any parameter Θ of the ϱ-real method, we have that*

$$K_\Theta(\bar{X}) = J_\Theta(\bar{X}).$$

Proof. The embedding $J_\Theta(\bar{X}) \hookrightarrow K_\Theta(\bar{X})$ follows from the definitions of the quasi-norms and the fact that the operator S_ϱ is bounded in Θ.

The opposite embeding follows from the fact that the Fundamental Lemma with the operator S_ϱ is valid for the 3-tuple $\bar{X}$. Let $f \in K_\Theta(\bar{X})$. By definition, this means that $K(\cdot, f; \bar{X}) \in \Theta$. Hence, since S_ϱ is bounded in Θ, we get that $S_\varrho K(\cdot, f; \bar{X}) \in \Theta$, i.e., $f \in \sigma_\varrho(\bar{X})$. Therefore a descomposition of f into a series satisfying (2.6) and the estimate (2.7) is possible.

Let

$$\mathbb{Q}_{\mathbf{k}} = \{\mathbf{s} = (s_1, s_2); 2^{k_i} \le s_i < 2^{k_i+1}, i = 1, 2\},\ \mathbf{k} \in \mathbf{Z}^2.$$

We define

$$u(\mathbf{s}) = \sum_{\mathbf{k}\in\mathbf{Z}^2} (\log 2)^{-2} x_{\mathbf{k}} \chi_{\mathbb{Q}_{\mathbf{k}}}(\mathbf{s}),$$

where $x_{\mathbf{k}}$ is the $\mathbf{k}-$summand in the decomposition (2.6) of f. Then

$$f = \sum_{\mathbf{k}\in\mathbf{Z}^2} x_{\mathbf{k}} = \int_{\mathbb{R}^2_+} u(\mathbf{s}) \frac{d\mathbf{s}}{\mathbf{s}},$$

and for any $\mathbf{s} \in \mathbb{R}^2_+$, from (2.7) and the concavity of the K-functional, we have

$$J(\mathbf{s}, u(\mathbf{s}); \bar{X}) \leq C^{'} [S_\varrho K(\cdot, f; \bar{X})](\mathbf{s}) \tag{2.20}$$

with constant $C^{'} > 0$ independent of $\mathbf{s}$ and $f \in \sigma_\varrho(\bar{X})$.

Applying $\|\cdot\|_\Theta$ to both sides of (2.20) we deduce that, from the boundedness of the operator S_ϱ in Θ, $K_\Theta(\bar{X}) \hookrightarrow J_\Theta(\bar{X})$ and this completes the proof of the theorem. □

Observe that it holds for n bigger than two and the proof follows in analogous way.

Remark 2.3.3. *It should be note (see [BK]) that in the case of couples of Banach spaces, the Equivalence theorem holds if and only if the operator S is bounded, where S is the corresponding operator in the Banach case ($\varrho = 1$).*

2.4 The Reiteration theorem

Let $H := \{(\theta_0, \theta_1); \theta_0 > 0,\ \theta_1 > 0 \text{ and } \theta_0 + \theta_1 < 1\}$, and let us remember that the spaces $\bar{X}_{\Theta,q;K} = \bar{X}_{(\theta_0,\theta_1),q;K}$ are defined for $0 < q \leq \infty$ and $\Theta = (\theta_0, \theta_1) \in H$, as the set of all $f \in \mathrm{L}_0(\Omega)$ for which

$$\|f\|_{\Theta,q;K} := \|K(\cdot, f; \bar{X})\|_{\Theta,q} < \infty,$$

where for $g = K(\cdot, f, \bar{X})$

$$\|g\|_{\Theta,q} := \Big(\int_0^\infty \int_0^\infty (t_1^{-\theta_0} t_2^{-\theta_1} g(t_1, t_2)) \Big)^q \frac{dt_1}{t_1} \frac{dt_2}{t_2} \Big)^{1/q} \qquad (q < \infty).$$

In [AKMNP], we find that for $\bar{X} = (X_0, X_1, X_2)$ be a triple of quasi-Banach function lattices, $\bar{\lambda} = (\lambda_1, \lambda_2) \in H$ and $\Theta_i = (\theta_0^i, \theta_1^i) \in H$, $i = 0, 1, 2$, if $\Theta = (1 - \lambda_1 - \lambda_2)\Theta_0 + \lambda_1\Theta_1 + \lambda_2\Theta_2$, then

$$(\bar{X}_{\Theta_0,q_0}, \bar{X}_{\Theta_1,q_1}, \bar{X}_{\Theta_2,q_2})_{\bar{\lambda},q} = \bar{X}_{\Theta,q},$$

whenever

the vectors $\Theta_0, \Theta_1, \Theta_2$ are not colinear.

Corollary 2.4.1. *Let* $\bar{X} = (L^{p_0,q_0}(C_0), L^{p_1,q_1}(C_1), L^{p_2,q_2}(C_2))$, *where* $p_i, q_i \in (0,\infty]$, $i = 0,1,2$. *Under the same conditions*

$$(\bar{X}_{\Theta_0,q_0}, \bar{X}_{\Theta_1,q_1}, \bar{X}_{\Theta_2,q_2})_{\bar{\lambda},q} = \bar{X}_{\Theta,q},\ \Theta = (1-\lambda_1-\lambda_2)\Theta_0 + \lambda_1\Theta_1 + \lambda_2\Theta_2$$

whenever

the vectors $\Theta_0, \Theta_1, \Theta_2$ *are not colinear.*

Proof. Observe that in this case we have that $L^{p_0,q_0}(C_0)$, $L^{p_1,q_1}(C_1)$ and $L^{p_2,q_2}(C_2)$ are quasi-Banach function lattices on Ω. Then the theorem follows by [AKMNP, Theorem 2.1]. □

We will use the following simple fact concerning triples $\bar{X} = (X_0, X_1, X_2)$ of arbitrary quasi-Banach spaces (cf. [AKMNP]):

Lemma 2.4.2. *If* $0 < \alpha_0, \alpha_1, \eta < 1$, $\theta_1 = \eta(1-\alpha_1)$, *and* $\theta_2 = \eta\alpha_1 + (1-\eta)\alpha_0$, *then*

$$\big((X_0,X_2)_{\alpha_0,1;K}, (X_1,X_2)_{\alpha_1,1;K}\big)_{\eta,1;K} \subset \bar{X}_{\Theta,1;K}$$

and

$$\bar{X}_{\Theta,1;J} \subset \big((X_0,X_2)_{\alpha_0,1;J}, (X_1,X_2)_{\alpha_1,1;J}\big)_{\eta,1;J}.$$

Let us remember here the *Power theorem* of G. Sparr (see [Sp]). Let $A^{[q]}$, $0 < q < \infty$, denotes the Banach space A with its norm $\|\cdot\|_A$ replaced by the functional

$$a \to \|a\|_A^q.$$

This functional does not in general define a norm on A but still it will be useful here. Within the framework of an interpolation theory for normed abelian groups such functionals can be used sistematically. If $\mathbf{A} = (A_0, A_1, ..., A_n)$ is a Banach $(n+1)-$tuple, $\mathbf{q} = (q_0, q_1, ..., q_n)$, $0 < q_i < \infty$ for $i = 0, 1, ..., n$, then we set

$$\mathbf{A}^{[q]} = (A_0^{[q_0]}, A_1^{[q_1]}, ..., A_n^{[q_n]}).$$

With $\mathbf{t} = (t_0, t_1, ..., t_n) \in \mathbb{R}^{n+1}$ and $\mathbf{s} = (s_0, s_1, ..., s_n) \in \mathbb{R}^{n+1}$ we write

$$\mathbf{ts} = (t_0s_0, t_1s_1, ..., t_ns_n)$$

and

$$\mathbf{t}/\mathbf{s} = (t_0/s_0, t_1/s_1, ..., t_n/s_n).$$

Theorem 2.4.3. *(Power theorem of G. Sparr) Let* $\eta = q\theta/\mathbf{q}$ *with* $1/q = \sum_{i=0}^n \theta_i/q_i$ *(or* $q = \sum_{i=0}^n \eta_i q_i$*). Then*

$$(\mathbf{A}_{\theta,pq;K})^{[q]} = (\mathbf{A}^{[q]})_{\eta,p;K}.$$

Proof. See [Sp1, Theorem 7.1]. □

The same holds for quasi-Banach triples.

Theorem 2.4.4. *Let $p_i, q_i \in (0,\infty]$, $i = 0,1,2$, and $0 < \mu < 1$. If $0 < \bar{q}_0, \bar{q}_1, q < \infty$ and $\frac{1}{q} = \frac{1-\mu}{\bar{q}_0} + \frac{\mu}{\bar{q}_1}$, then*

$$\begin{aligned}((L^{p_0,q_0}(C_0), L^{p_2,q_2}(C_2))_{\alpha_0,\bar{q}_0}, (L^{p_1,q_1}(C_1), L^{p_2,q_2}(C_2))_{\alpha_1,\bar{q}_1})_{\mu,q}\\ = (L^{p_0,q_0}(C_0), L^{p_1,q_1}(C_1), L^{p_2,q_2}(C_2))_{(\theta_1,\theta_2),q},\end{aligned}$$

where

$$\theta_1 = (1-\alpha_1)\mu,\ \theta_2 = \alpha_0(1-\mu) + \alpha_1\mu.$$

Proof. For $1 \le q < \infty$, the proof follows using the Power theorem for quasi-Banach couples (see [BeLo, TH.3.11.6]), by Lemma 2.4.2 and by [Sp1, Theorem 7.1](cf. [Sp1]).

For $0 < q < 1$ we have that, for $\eta := \frac{\mu q}{\bar{q}_1}$

$$[((X_0, X_2)_{\alpha_0,\bar{q}_0}, (X_1, X_2)_{\alpha_1,\bar{q}_1})_{\mu,q}]^q = ((X_0, X_2)^{\bar{q}_0}_{\alpha_0,\bar{q}_0}, (X_1, X_2)^{\bar{q}_1}_{\alpha_1,\bar{q}_1})_{\eta,1}$$

by the Power theorem for quasi-Banach spaces. We can find $0 < \beta_0, \beta_1 < 1$ such that for $s_2 := \frac{\alpha_0\bar{q}_0}{\beta_0} = \frac{\alpha_1\bar{q}_1}{\beta_1}$ and, for $s_0 := \frac{\bar{q}_0(1-\alpha_0)}{1-\beta_0}$ and $s_1 := \frac{\bar{q}_1(1-\alpha_1)}{1-\beta_1}$ it follows that

$$\begin{aligned}((X_0, X_2)^{\bar{q}_0}_{\alpha_0,\bar{q}_0}, (X_1, X_2)^{\bar{q}_1}_{\alpha_1,\bar{q}_1})_{\eta,1} &= ((X_0^{s_0}, X_2^{s_2})_{\beta_0,1}, (X_1^{s_1}, X_2^{s_2})_{\beta_1,1})_{\eta,1}\\ &= (X_0^{s_0}, X_1^{s_1}, X_2^{s_2})_{\lambda_1,\lambda_2,1},\end{aligned}$$

where $\lambda_1 := (1-\beta_1)\eta$, $\lambda_2 := \beta_0(1-\eta) + \beta_1\eta$. Last equality follows by using Lemma 2.4.2 and the Equivalence theorem. Finally, by the Power theorem of G. Sparr for triples of quasi-normed abelian groups, for $\theta_1 = \frac{\lambda_1 s_1}{q} = \mu(1-\alpha_1)$ and $\theta_2 = \frac{\lambda_2 s_2}{q} = \alpha_0(1-\mu) + \alpha_1\mu$, it follows that

$$(X_0^{s_0}, X_1^{s_2}, X_2^{s_2})_{\lambda_1,\lambda_2,1} = (X_0, X_1, X_2)^q_{\theta_1,\theta_2,q}. \qquad \square$$

Corollary 2.4.5. *Let $\bar{X} = (L^{p_0,q_0}(C_0), L^{p_1,q_1}(C_1), L^{p_2,q_2}(C_2))$, where $p_i, q_i \in (0,\infty]$, $i = 0,1,2$ and $0 < q < \infty$. Then*

$$\begin{aligned}((L^{p_0,q_0}(C_0), L^{p_2,q_2}(C_2))_{\theta_2,q}, (L^{p_1,q_1}(C_1), L^{p_2,q_2}(C_2))_{\theta_2,q})_{\theta,q}\\ = (L^{p_0,q_0}(C_0), L^{p_1,q_1}(C_1), L^{p_2,q_2}(C_2))_{(\theta_1,\theta_2),q},\end{aligned}$$

where $\theta = \frac{\theta_1}{(1-\theta_2)}$.

Proof. This follows immediately from last theorem by putting $\alpha_0 = \alpha_1 = \theta_2$, $\bar{q}_0 = \bar{q}_1 = q$ and $\mu = \theta_1/(1-\theta_2)$. □

2.5 Interpolation of capacitary Lorentz spaces

Let us remember that, if $\bar{A} = (A_0, A_1)$ is a couple of quasi-Banach spaces, $0 < \theta < 1$ and $0 < q \leq \infty$, the interpolation space $\bar{A}_{\theta,q}$ is the quasi-Banach space of all $f \in A_0 + A_1$ such that

$$\|f\|_{\theta,q} := \Big(\int_0^\infty (t^{-\theta} K(t, f; \bar{A}))^q \frac{dt}{t} \Big)^{1/q} < \infty,$$

where $K(t, f; \bar{A})$ is the K-functional,

$$K(t, f; \bar{A}) := \inf \Big\{ \|f_0\|_{A_0} + t\|f_1\|_{A_1};\ f = f_0 + f_1, f_i \in A_i, i = 0, 1 \Big\}.$$

We refer to [BeLo] and [BK] for general facts concerning interpolation theory.

Let $0 < p < \infty$. From now on, let (Ω, Σ) be a measurable space and C a quasi-subadditive Fatou capacity on (Ω, Σ). To calculate $K(t, f) = K(t, f; L^p(C), L^\infty(C))$ we will follow the usual construction (see e.g. [BeLo, Theorem 5.2.1.]), with $f^\star_C(t)$ instead of the classical non-increasing rearrangement.

Theorem 2.5.1. *Let $0 < p < \infty$ and $f \in L^p(C) + L^\infty(C)$. Then for all $t > 0$,*

$$K(t, f; L^p(C), L^\infty(C)) \simeq \Big(\int_0^\infty y^{p-1} \min(C\{|f| > y\}, t^p) dy \Big)^{1/p}$$

and

$$K(t, f; L^p(C), L^\infty(C)) \simeq \Big(\int_0^{t^p} f^*_C(y)^p dy \Big)^{1/p}.$$

Proof. Let $0 \leq f \in L^p(C) + L^\infty(C)$. For $t > 0$ given, let

$$y^* := \inf\{y > 0; C\{f > y\} \leq t^p\} = f^\star_C(t^p),$$

and consider

$$g_0(x) := \int_{y^*}^\infty \chi_{\{f>y\}}(x) dy = (f(x) - y^*)_+$$

and

$$g_1(x) := \int_0^{y^*} \chi_{\{f>y\}}(x) dy = (y^* - f(x))_-.$$

Then $f = g_0 + g_1$ and $\{g_0 > y\} = \{f > y + y^*\}$. So that

$$\begin{aligned}
\|g_0\|^p_{L^p(C)} &= \int_0^\infty py^{p-1}C\{f > y + y^*\}dy \\
&\le \int_0^{y^*} py^{p-1}C\{f > y + y^*\}dy + \int_{y^*}^\infty py^{p-1}C\{f > y\}dy \\
&\le \int_0^{y^*} py^{p-1}C\{f > y^*\}dy + \int_{y^*}^\infty py^{p-1}C\{f > y\}dy \\
&\le C\{f > y^*\}(y^*)^p + \int_{y^*}^\infty py^{p-1}C\{f > y\}dy \\
&\lesssim t^p(y^*)^p + \int_{y^*}^\infty y^{p-1}C\{f > y\}dy.
\end{aligned}$$

Hence,

$$\begin{aligned}
K(t,f) &\le \|g_0\|_{L^p(C)} + t\|g_1\|_{L^\infty(C)} \\
&\lesssim \Big(t^p(y^*)^p + \int_{y^*}^\infty y^{p-1}C\{f > y\}dy\Big)^{1/p} + ty^* \\
&\lesssim \Big(t^p(y^*)^p + \int_{y^*}^\infty y^{p-1}C\{f > y\}dy\Big)^{1/p} + \Big(t^p\int_0^{y^*} y^{p-1}dy\Big)^{1/p} \\
&\lesssim \Big(\int_{y^*}^\infty y^{p-1}C\{f > y\}dy + t^p\int_0^{y^*} y^{p-1}dy\Big)^{1/p} \\
&= \Big(\int_0^\infty y^{p-1}\min(C\{f > y\}, t^p)dy\Big)^{1/p}.
\end{aligned}$$

Moreover, as it is shown in [Ce]-(5), there exists $\Omega_f(t) \subset \Omega$ such that

$$K(t,f) \simeq \|f\chi_{\Omega_f(t)}\|_{L^p(C)} + t\|f\chi_{\Omega\setminus\Omega_f(t)}\|_{L^\infty(C)} = \|f_0\|_{L^p(C)} + t\|f_1\|_{L^\infty(C)},$$

with $f_0 := f\chi_{\Omega_f(t)}$ and $f_1 := f\chi_{\Omega\setminus\Omega_f(t)}$. Just consider $f = f_0 + f_1$ such that $\|f_0\|_{L^p(C)} + t\|f_1\|_{L^\infty(C)} \le 2K(t,f)$ and take $\Omega_f(t) = \{|f_0| \ge |f_1|\}$.

If $f = \chi_A$, then

$$K(t,\chi_A) \simeq \inf\{C(A_0) + tC(A_1);\ A = A_0 \cup A_1,\ A_0 \cap A_1 = \emptyset\} \simeq \min(C(A), t).$$

Now, since $\chi_{\{f>y\}} = \chi_{\{f_0>y\}} + \chi_{\{f_1>y\}}$ (f_0, f_1 are disjointly supported),

$$\min(C\{f > y\}, t) \simeq K(t, \chi_{\{f>y\}}) \lesssim C\{f_0 > y\} + t\|\chi_{\{f_1>y\}}\|_{L^\infty(C)}.$$

Using now that

$$\|f_1\|_{L^\infty(C)} \simeq p^{-1/p}\Big(\int_0^\infty y^{p-1}\|\chi_{\{f_1>y\}}\|_{L^\infty(C)}dy\Big)^{1/p},$$

we obtain that

$$\begin{aligned} K(t,f) &\simeq \Big(\int_0^\infty y^{p-1}C\{f_0>y\}dy\Big)^{1/p} + \Big(t^p\int_0^\infty y^{p-1}\|\chi_{\{f_1>y\}}\|_{L^\infty(C)}dy\Big)^{1/p} \\ &\simeq \Big(\int_0^\infty y^{p-1}(C\{f_0>y\}+t^p\|\chi_{\{f_1>y\}}\|_{L^\infty(C)})dy\Big)^{1/p} \\ &\gtrsim \Big(\int_0^\infty y^{p-1}\min(C\{f>y\},t^p)dy\Big)^{1/p}. \end{aligned}$$

The first description of the $K-$functional then follows.

To prove now that $K(t,f)\simeq\Big(\int_0^{t^p} f_C^*(y)^p dy\Big)^{1/p}$, let

$$f_0(x) := \begin{cases} f(x) - f_C^\star(t^p)\frac{f(x)}{|f(x)|}, & \text{if } |f(x)|>f_C^\star(t^p) \\ 0, & \text{otherwise} \end{cases}$$

and $f_1 := f - f_0$. Define $E := \{x\in\Omega; f_0(x)\neq 0\}$. Then $E=\{x\in\Omega; |f(x)|>f_C^\star(t^p)\}$, $C(E)\le t^p$ and, since $f_C^\star$ is constant on $[C(E),t^p]$,

$$\begin{aligned} K(t,f) &\le \|f_0\|_{L^p(C)} + t\|f_1\|_{L^\infty(C)} \\ &\lesssim \Big(\int_E (|f(x)|-f_C^\star(t^p))^p dC\Big)^{1/p} + tf_C^\star(t^p) \\ &\lesssim \Big(\int_0^{C(E)} (f_C^\star(s)-f_C^\star(t^p))^p ds\Big)^{1/p} + \Big(\int_0^{t^p} f_C^\star(t^p)^p ds\Big)^{1/p} \\ &\le \Big(\int_0^{t^p} (f_C^\star(s)-f_C^\star(t^p))^p ds\Big)^{1/p} + \Big(\int_0^{t^p} f_C^\star(t^p)^p ds\Big)^{1/p} \\ &\lesssim \Big\{\int_0^{t^p} (f_C^\star(s)-f_C^\star(t^p))^p ds + \int_0^{t^p} f_C^\star(t^p)^p ds\Big\}^{1/p} \\ &\lesssim \Big(\int_0^{t^p} f_C^\star(s)^p ds\Big)^{1/p}. \end{aligned}$$

Conversely, consider $f=g+h$ with $g\in L^p(C)$ and $h\in L^\infty(C)$. Then, by the properties of f_C^* and by Theorem 1.3.8, we obtain

$$\begin{aligned} \int_0^{t^p} f_C^\star(s)^p ds &= \int_0^{t^p} (|f|^p(s))_C^\star ds \lesssim \int_0^{t^p} (|g|^p+|h|^p)_C^\star(s)ds \\ &\lesssim \int_0^{t^p} [(|g|^p)_C^\star(s)+(|h|^p)_C^\star(s)]ds = \int_0^{t^p} (g_C^\star(s))^p ds + \int_0^{t^p} (h_C^\star(s))^p ds \\ &\lesssim \int_0^{t^p} (g_C^*(s))^p ds + t^p h_C^\star(0)^p \lesssim \|g_C^\star\|_{L^p(\mathbb{R})}^p + t^p\|h\|_{L^\infty(C)}^p \\ &\lesssim \|g\|_{L^p(C)}^p + t^p\|h\|_{L^\infty(C)}^p \lesssim (\|g\|_{L^p(C)} + t\|h\|_{L^\infty(C)})^p. \end{aligned}$$

Taking infimum over all descompositions it follows

$$\Big(\int_0^{t^p} f_C^\star(s)^p ds\Big)^{1/p} \lesssim K(t,f). \qquad \square$$

Once we have the description of $K(t,f;L^p(C),L^\infty(C))$ for any positive p and $f\in \mathrm{L}_0(\Omega)$, real interpolation follows easily as in [BeLo, Theorem 5.2.1]:

Theorem 2.5.2. *Suppose* $0<\theta<1$, $0<p_0<q\le\infty$ *or* $0<p_0\le q<\infty$, *and* $\frac{1}{p}=\frac{1-\theta}{p_0}$. *Then*

$$(L^{p_0}(C),L^\infty(C))_{\theta,q}=L^{p,q}(C).$$

Proof. It follows from Theorem 2.5.1 that

$$\int_0^\infty y^{p-1}\min(C\{f>y\},t^p)dy \simeq \int_0^{t^p} f_C^\star(y)^p dy.$$

So that, using Minkowski's inequality ($q/p_0\ge 1$),

$$\begin{aligned}
\|f\|_{\theta,q} &= \Big(\int_0^\infty t^{-\theta q}K(t,f)^q\frac{dt}{t}\Big)^{1/q}\\
&\simeq \Big(\int_0^\infty t^{-\theta q}\Big(\int_0^{t^{p_0}} f_C^\star(s)^{p_0}ds\Big)^{q/p_0}\frac{dt}{t}\Big)^{1/q}\\
&= \Big(\int_0^\infty \Big(t^{-\theta p_0+p_0}\int_0^1 f_C^\star(yt^{p_0})^{p_0}y\frac{dy}{y}\Big)^{q/p_0}\frac{dt}{t}\Big)^{1/q}\\
&\lesssim \Big(\int_0^1 \Big(y^{q/p_0}\int_0^\infty t^{(1-\theta)q}(f_C^\star(yt^{p_0}))^q\frac{dt}{t}\Big)^{p_0/q}\frac{dy}{y}\Big)^{1/p_0}\\
&\lesssim \Big(\int_0^\infty \Big(s^{\frac{1-\theta}{p_0}}f_C^\star(s)\Big)^q\frac{ds}{s}\Big)^{1/q} = \Big(\int_0^\infty \Big(s^{1/p}f_C^\star(s)\Big)^q\frac{ds}{s}\Big)^{1/q}.
\end{aligned}$$

Then $\|f\|_{\theta,q}\lesssim \|f\|_{L^{p,q}(C)}$ since $\|f\|_{L^{p,q}(C)}\simeq \Big(\int_0^\infty \Big(s^{1/p}f_C^*(s)\Big)^q\frac{ds}{s}\Big)^{1/q}$.

Conversely,

$$\begin{aligned}
\|f\|_{L^{p,q}(C)} &\simeq \Big(\int_0^\infty \Big(s^{1/p}f_C^\star(s)\Big)^q\frac{ds}{s}\Big)^{1/q} = \Big(\int_0^\infty \Big(s^{\frac{1-\theta}{p_0}}f_C^\star(s)\Big)^q\frac{ds}{s}\Big)^{1/q}\\
&\simeq \Big(\int_0^\infty \Big(t^{1-\theta}f_C^\star(t^{p_0})\Big)^q\frac{dt}{t}\Big)^{1/q} = \Big(\int_0^\infty \Big(t^{(1-\theta)p_0}f_C^\star(t^{p_0})^{p_0}\Big)^{q/p_0}\frac{dt}{t}\Big)^{1/q}\\
&\lesssim \Big(\int_0^\infty \Big(t^{-\theta}\Big(\int_0^{t^{p_0}} f_C^\star(s)^{p_0}ds\Big)^{1/p_0}\Big)^q\frac{dt}{t}\Big)^{1/q} = \|f\|_{\theta,q},
\end{aligned}$$

where in the last inequality we have used that $f_C^\star$ is decreasing. $\square$

In the case of a single quasi-subadditive Fatou capacity [Ce, Theorem 6.6.] is extended by reiteration:

Theorem 2.5.3. *Let $0 < p_0, p_1, q_0, q_1 < \infty$, $p_0 \neq p_1$ and $0 < \eta < 1$. Then*

$$(L^{p_0,q_0}(C), L^{p_1,q_1}(C))_{\eta,q} = L^{p,q}(C)$$

with $\frac{1}{p} = \frac{1-\eta}{p_0} + \frac{\eta}{p_1}$.

Proof. Let $0 < r < \min(p_0, p_1, q_0, q_1)$ and $(1-\theta_i) := r/p_i$ $(i = 0, 1)$. If $\theta := (1-\eta)\theta_0 + \eta\theta_1$, since $1/p = (1-\theta)/r$, then Theorem 2.5.2 gives

$$(L^{p_0,q_0}(C), L^{p_1,q_1}(C))_{\eta,q} = ((L^r(C), L^\infty(C))_{\theta_0,q_0}, (L^r(C), L^\infty(C))_{\theta_1,q_1})_{\eta,q}.$$

By reiteration (cf. [BeLo, Theorem 3.11.5]) we obtain that

$$((L^r(C), L^\infty(C))_{\theta_0,q_0}, (L^r(C), L^\infty(C))_{\theta_1,q_1})_{\eta,q} = (L^r(C), L^\infty(C))_{\theta,q}.$$

And, again from Theorem 2.5.2, $(L^{p_0,q_0}(C), L^{p_1,q_1}(C))_{\eta,q} = L^{p,q}(C)$. □

We want to consider interpolation with change of capacities so that, let (C_0, C_1) be a couple of capacities on (Ω, Σ) with the same null sets. We will denote by $\mathbf{L^1(C)} = (L^1(C_0), L^1(C_1))$ and, for every $t > 0$,

$$[C_0 + tC_1](A) := K(t, \chi_A; \mathbf{L^1(C)})$$

as in [Ce].

Remark 2.5.4. *Let us observe that, since C_0 and C_1 are capacities with the same null sets,* $\|\cdot\|_{L^\infty(C_0)} \simeq \|\cdot\|_{L^\infty(C_1)}$.

First of all, let us see that we can extend [CeClM, Lemma 6.5].

Proposition 2.5.5. *Let C_0, C_1 be two concave Fatou capacities with the same null sets and $r > 0$. Then*

$$\begin{aligned} K(t_1, t_2, f; L^{r,1}(C_0), L^{r,1}(C_1), L^\infty) &\simeq K(t_2, f; L(C_0^{(r)} + t_1 C_1^{(r)}), L^\infty) \\ &\simeq \int_0^\infty \min((C_0^{(r)} + t_1 C_1^{(r)})\{|f| > y\}, t_2)dy, \end{aligned}$$

*where $C^{(r)} := C^{1/r}$ denotes the r-*convexification *of C.*

Proof. We have that, since the power of the capacities are at least quasi-subadditive, then

$$\begin{aligned} & K(t_1, t_2, f; L^{r,1}(C_0), L^{r,1}(C_1), L^\infty) \\ \simeq\ & \inf_{f=f_0+f_1+f_2} \Big\{ \|f_0\|_{L^{r,1}(C_0)} + t_1\|f_1\|_{L^{r,1}(C_1)} + t_2\|f_2\|_{L^\infty} \Big\} \\ =\ & \inf_{f=f_0+f_1+f_2} \Big\{ \int_0^\infty C_0^{(r)}\{|f_0| > t\}dt + t_1 \int_0^\infty C_1^{(r)}\{|f_1| > t\}dt + t_2\|f_2\|_{L^\infty} \Big\} \\ \simeq\ & \inf_{f=f_0+f_1+f_2} \Big\{ \int_0^\infty \Big(C_0^{(r)}\{|f_0| > t\} + t_1 C_1^{(r)}\{|f_1| > t\} \Big) dt + t_2\|f_2\|_{L^\infty} \Big\} \\ =\ & K(t_2, f; L(C_0^{(r)} + t_1 C_1^{(r)}), L^\infty). \end{aligned}$$

Since

$$K(t_2, f; L(C_0^{(r)} + t_1 C_1^{(r)}), L^\infty) = \int_0^\infty \min((C_0^{(r)} + t_1 C_1^{(r)})\{|f| > y\}, t_2) dy,$$

the result then follows. □

Remark 2.5.6. *From [Ce], for concave capacities, we can obtain more interpolation results. For instance, since* $L^{r,1}(C) = L(C^{1/r})$,

$$(L^{r,1}(C), L^\infty)_{\theta,q} = L^{\bar{p},q}(C^{1/r}) = L^{r\bar{p},q}(C) = L^{p,q}(C)$$

if $\theta = 1 - 1/\bar{p}$ $(0 < \bar{p} < \infty)$ *or* $\theta = 1 - r/p$ $(0 < r \le p = r\bar{p})$.

Theorem 2.5.7. *Let* C_0, C_1 *be a couple of concave Fatou capacities with the same null sets and* $0 < \eta < 1$. *If* $0 < p_0, p_1 < \infty$, $0 < q_0, q_1 \le \infty$, $\frac{1}{p} := \frac{1-\eta}{p_0} + \frac{\eta}{p_1}$ *and* $\frac{1}{q} := \frac{1-\eta}{q_0} + \frac{\eta}{q_1}$, *then, for* $C_{\theta,q}(A) := \|\chi_A\|_{(L(C_0),L(C_1))_{\theta,q}}$ $(0 < \theta < 1)$,

$$(L^{p_0,q_0}(C_0), L^{p_1,q_1}(C_1))_{\eta,q} = L^{p,q}(C_{\frac{\eta p}{p_1},q/p}).$$

Proof. Let $0 < r < \min(p_0, p_1, q_0, q_1)$. Since $q_i > r$, $i = 0, 1$, then, by Remark 2.5.6, we have that

$$L^{p_i,q_i}(C_i) = (L^{r,1}(C_i), L^\infty)_{\theta_i,q_i} \quad \big(1 - \theta_i := r/p_i\big),$$

and by Theorem 2.4.4,

$$(L^{p_0,q_0}(C_0), L^{p_1,q_1}(C_1))_{\eta,q} = (L^{r,1}(C_0), L^{r,1}(C_1), L^\infty)_{\alpha_1,\alpha_2,q} = \bar{X}_{\alpha_1,\alpha_2,q},$$

where $\alpha_1 = (1 - \theta_1)\eta, \alpha_2 := \theta_0(1 - \eta) + \theta_1\eta$. Hence, by Proposition 2.5.5, it follows that

$$\|f\|^q_{\bar{X}_{\alpha_1,\alpha_2,q}} \simeq \int_0^\infty \int_0^\infty \Big(t_1^{-\alpha_1} t_2^{-\alpha_2} \int_0^\infty \min\Big(\Big(C_0^{(r)} + t_1 C_1^{(r)}\Big)\{|f| > y\}, t_2\Big) dy\Big)^q \frac{dt_2}{t_2}\frac{dt_1}{t_1},$$

where, by Theorem 2.5.1, we have that

$$\begin{aligned}
& \int_0^\infty \big(t_2^{-\alpha_2} \int_0^\infty \min\Big(\big(C_0^{(r)} + t_1 C_1^{(r)}\big)\{|f| > y\}, t_2\Big) dy\big)^q \frac{dt_2}{t_2} \\
\simeq\ & \int_0^\infty t_2^{-\alpha_2 q} K\big(t_2, f; L\big(C_0^{(r)} + t_1 C_1^{(r)}\big), L^\infty\big)^q \frac{dt_2}{t_2} \\
=\ & \|f\|^q_{(L(C_0^{(r)}+t_1C_1^{(r)}),L^\infty)_{\alpha_2,q}} = \|f\|^q_{L^{\frac{1}{1-\alpha_2},q}(C_0^{(r)}+t_1C_1^{(r)})} \\
\simeq\ & \int_0^\infty y^{q-1}\big((C_0^{(r)} + t_1 C_1^{(r)})\{|f| > y\}\big)^{(1-\alpha_2)q} dy.
\end{aligned}$$

On the other hand, since $C_i^{(r)}(A) = C_i(A)^{1/r}$,

$$\big[C_0^{(r)} + t_1 C_1^{(r)}\big](A) \simeq K\big(t, \chi_A; L(C_0)^{(r)}, L(C_1)^{(r)}\big),$$

and $K(t^r, |g|^r; X_0, X_1) \simeq K\big(t, g; X_0^{(r)}, X_1^{(r)}\big)^r$, it follows that

$$\begin{aligned} & \int_0^\infty y^{q-1}\big(\big(C_0^{(r)} + t_1 C_1^{(r)}\big)\{|f| > y\}\big)^{(1-\alpha_2)q} dy \\ \simeq & \int_0^\infty y^{q-1} K(t_1^r, \chi_{\{|f|>y\}}; L(C_0), L(C_1))^{\frac{(1-\alpha_2)q}{r}} dy \\ = & \int_0^\infty y^{q-1}(C_0 + t_1^r C_1)\{|f| > y\}^{\frac{(1-\alpha_2)q}{r}} dy. \end{aligned}$$

Hence,

$$\begin{aligned} \|f\|^q_{\bar{X}_{\alpha_1,\alpha_2,q}} & \simeq \int_0^\infty t_1^{-\alpha_1 q} \int_0^\infty y^{q-1}(C_0 + t_1^r C_1)\{|f| > y\}^{\frac{(1-\alpha_2)q}{r}} dy \frac{dt_1}{t_1} \\ & \simeq \int_0^\infty y^{q-1} \|\chi_{\{|f|>y\}}\|^{q/p}_{(L(C_0),L(C_1))_{\frac{\eta p}{p_1},q/p}} dy = \|f\|^q_{L^{p,q}(C_{\frac{\eta p}{p_1},q/p})}. \quad \square \end{aligned}$$

We present here Theorem 2.5.7 and 2.5.11 because, although they follow from some other results proved later on, they show the process followed to attach our objective. We think that the proof could be of interest for the reader.

A capacity C is called *semiadditive* if there exists a constant $c \geq 1$ such that

$$C\Big(\bigcup_{n=1}^\infty A_n\Big) \leq c \sum_{n=1}^\infty C(A_n) \quad \big(\{A_n\}_{n\in\mathbb{N}} \subset \Sigma\big),$$

and it is *σ-subadditive* if moreover $c = 1$. Observe that each concave capacity is semiadditive and every semiadditive capacity is quasi-subadditive.

Lemma 2.5.8. *Suppose that C_0 and C_1 are both (countably) semiadditive. Then:*
(i) If $\{f_k\}_{k\in\mathbb{N}}$ are non-negative disjointly supported functions, then

$$K\Big(t, \sum_{k=1}^\infty f_k; \mathbf{L^1(C)}\Big) \lesssim \sum_{k=1}^\infty K\Big(t, f_k; \mathbf{L^1(C)}\Big).$$

(ii) $K(t, f; \mathbf{L^1(C)}) \simeq \int_0^\infty [C_0 + tC_1]\{|f| > y\} dy = \|f\|_{L^1([C_0+tC_1])}$.

Proof. (i) Note that $\chi_{\{\sum_{k=1}^\infty f_k > y\}} = \chi_{\bigcup_{k=1}^\infty \{f_k > y\}} = \sum_{k=1}^\infty \chi_{\{f_k>y\}}$ and

$$\Big\|\sum_{k=1}^\infty f_k\Big\|_{L^1(C_i)} = C_i\Big(\bigcup_{k=1}^\infty \{f_k > y\}\Big) \leq c_i \sum_{k=1}^\infty \|f_k\|_{L^1(C_i)} \qquad (i = 0, 1).$$

We can consider descompositions $f_k = f_{k,0} + f_{k,1}$ (with $f_{k,i}$ non-negative, $i = 0, 1$, and $k \in \mathbb{N}$) so that, the functions $f_{k,0}$ (and also $f_{k,1}$) are disjointly supported. Hence

$$\begin{aligned} K\Big(t, \sum_{k=1}^{\infty} f_k; \mathbf{L^1}(\mathbf{C})\Big) &\leq \inf\Big\{\Big\|\sum_{k=1}^{\infty} f_{k,0}\Big\|_{L^1(C_0)} + t\Big\|\sum_{k=1}^{\infty} f_{k,1}\Big\|_{L^1(C_1)}\Big\} \\ &\lesssim \inf\Big\{\sum_{k=1}^{\infty} \|f_{k,0}\|_{L^1(C_0)} + t\sum_{k=1}^{\infty} \|f_{k,1}\|_{L^1(C_1)}\Big\} \\ &= \sum_{k=1}^{\infty} K(t, f_k; \mathbf{L^1}(\mathbf{C})). \end{aligned}$$

(ii) We may assume that $f \geq 0$. We have that $K(t, f; \mathbf{L^1}(\mathbf{C})) \simeq \|f_0\|_{L^1(C_0)} + t\|f_1\|_{L^1(C_1)}$, where f_0, f_1 are disjointly supported functions such that $f_0 + f_1 = f$. Thus, $\chi_{\{f>y\}} = \chi_{\{f_0>y\}} + \chi_{\{f_1>y\}}$ and

$$\begin{aligned} K(t, f; \mathbf{L^1}(\mathbf{C})) &\simeq \int_0^{\infty} C_0\{f_0 > y\}dy + t\int_0^{\infty} C_1\{f_1 > y\}dy \\ &= \int_0^{\infty} (C_0\{f_0 > y\} + tC_1\{f_1 > y\})dy \\ &\gtrsim \int_0^{\infty} [C_0 + tC_1]\{f > y\}dy. \end{aligned}$$

For the reverse estimate, since $f \leq \sum_{k\in\mathbf{Z}} 2^{k+1}\chi_{\{2^k<f\leq 2^{k+1}\}}$, from (i) we get

$$\begin{aligned} K(t, f; \mathbf{L^1}(\mathbf{C})) &\leq \sum_{k\in\mathbf{Z}} 2^{k+1}\,[C_0 + tC_1]\,\{2^k < f \leq 2^{k+1}\} \\ &\leq 4\sum_{k\in\mathbf{Z}} 2^{k-1}\,[C_0 + tC_1]\,\{f > 2^k\} \\ &\leq 4\int_0^{\infty} [C_0 + tC_1]\{f > y\}dy. \qquad \square \end{aligned}$$

Remark 2.5.9. *Let $E(A) = \|\chi_A\|_{L^{p,q}(C)}$ for $A \in \Sigma$, and $0 < u \leq 1$ such that $(2c)^u = 2$ if c is the quasi-subadditivity constant of C. Then*

$$L^{1,u}(E) = L^{p,u}(C) \hookrightarrow L^{p,q}(C) \hookrightarrow L^{p,r}(C) \hookrightarrow L^{p,\infty}(C) \ (0 < u < q < r < \infty).$$

Indeed,

$$\|f\|_{L^{1,u}(E)} \simeq \Big(\int_0^{\infty} y^{u-1}\|\chi_{\{|f|>y\}}\|^u_{L^{p,q}(C)}dy\Big)^{1/u},$$

where

$$\begin{aligned} \|\chi_{\{|f|>y\}}\|_{L^{p,q}(C)} &\simeq \Big(\int_0^{\infty} s^{q-1}C\{\chi_{\{|f|>y\}} > s\}^{\frac{q}{p}}ds\Big)^{1/q} \\ &= \Big(\frac{1}{q}\Big)^{1/q} C\{|f| > y\}^{\frac{1}{p}}, \end{aligned}$$

so that

$$\|f\|_{L^{1,u}(E)} \simeq \Big(q^{-u/q}\int_0^\infty y^{u-1}C\{|f|>y\}^{u/p}dy\Big)^{1/u} \simeq \|f\|_{L^{p,u}(C)},$$

and it follows, for u smaller than q, that

$$L^{p,u}(C) \hookrightarrow L^{p,q}(C).$$

Recall that, since $f_C^\star$ is decreasing, for $r>0$ it follows that

$$t^{1/p}f_C^\star(t) = \Big(\frac{p}{r}\int_0^t (s^{1/p}f_C^\star(t))^r\frac{ds}{s}\Big)^{1/r} \le \Big(\frac{p}{r}\int_0^t (s^{1/p}f_C^\star(s))^r\frac{ds}{s}\Big)^{1/r},$$

and $L^{p,r}(C) \hookrightarrow L^{p,\infty}(C)$.

Also, $L^{p,q}(C) \hookrightarrow L^{p,r}(C)$ if $q \le r < \infty$, since $\|f\|_{p,r} \le c\|f\|_{p,q}$ with $c = (p/q)^{(r-q)/rq}$ as in the usual case (cf. [BeSh, Proposition 4.2, Chapter 4]).

Remark 2.5.10. *Let C_0 and C_1 be quasi-subadditive Fatou capacities on (Ω,Σ) with the same null sets. Let $\bar X = (X_0,X_1,X_2) = (L^r(C_0),L^r(C_1),L^\infty)$ where $0<r<\infty$, $0<q,q_0,q_1<\infty$, and suppose that $0<\mu<1$. Let $\frac{1}{q} = \frac{1-\theta_0}{q_0}+\frac{\theta_1}{q_1}$. Then*

$$\bar X_{(\bar\theta_0,\bar\theta_1),q} = ((X_0,X_2)_{\theta_0,q_0},(X_1,X_2)_{\theta_1,q_1})_{\mu,q},$$

with $\bar\theta_0 = (1-\theta_1)\mu$ and $\bar\theta_1 = \theta_0(1-\mu)+\theta_1\mu$.

Indeed, by the power theorem,

$$((X_0,X_2)_{\theta_0,q_0},(X_1,X_2)_{\theta_1,q_1})^q_{\mu,q} = ((X_0,X_2)^{q_0}_{\theta_0,q_0},(X_1,X_2)^{q_1}_{\theta_1,q_1})_{\eta,1},$$

if $\eta = \mu q/q_1$. We choose $0<\beta_0,\beta_1<1$ so that $\theta_0q_0/\beta_0 = \theta_1q_1/\beta_1$, and if $s_0 = q_0(1-\theta_0)/(1-\beta_0)$, $s_1 = q_1(1-\theta_1)/(1-\beta_1)$ and $s_2 = q_0\theta_0/\beta_0 = q_1\theta_1/\beta_1$, then

$$((X_0,X_2)^{q_0}_{\theta_0,q_0},(X_1,X_2)^{q_1}_{\theta_1,q_1})_{\eta,1} = ((X_0^{s_0},X_2^{s_2})_{\beta_0,1},(X_1^{s_1},X_2^{s_2})_{\beta_1,1})_{\eta,1}.$$

From Theorem 2.3.2, Lemma 2.4.2, with $\lambda_1 = \eta(1-\beta_1)$ and $\lambda_2 = (1-\eta)\beta_0+\eta\beta_1$, it follows

$$((X_0^{s_0},X_2^{s_2})_{\beta_0,1},(X_1^{s_1},X_2^{s_2})_{\beta_1,1})_{\eta,1} = (X_0^{s_0},X_1^{s_1},X_2^{s_2})_{(\lambda_1,\lambda_2),1}.$$

An application of the Power theorem for triples of quasi-Banach spaces (cf. [Sp]) gives, for $\bar\theta_0 = \mu(1-\theta_1)$ and $\bar\theta_1 = (1-\mu)\theta_0+\mu\theta_1$,

$$(X_0^{s_0},X_1^{s_1},X_2^{s_2})_{(\lambda_1,\lambda_2),1} = \bar X^q_{(\bar\theta_0,\bar\theta_1),q}.$$

Thus as announced

$$((X_0,X_2)_{\theta_0,q},(X_1,X_2)_{\theta_1,q})_{\mu,q} = \bar X_{(\bar\theta_0,\bar\theta_1),q}.$$

Theorem 2.5.11. *Let C_0, C_1 be a couple of semiadditive capacities with the same null sets and $0 < \eta < 1$. If $0 < p_0, p_1 < \infty$, $0 < q_0, q_1 \le \infty$, $\frac{1}{p} := \frac{1-\eta}{p_0} + \frac{\eta}{p_1}$, and $\frac{1}{q} := \frac{1-\eta}{q_0} + \frac{\eta}{q_1}$, then, for $C_{\theta,q}(A) := \|\chi_A\|_{(L(C_0),L(C_1))_{\theta,q}}$ $(0 < \theta < 1)$,*

$$(L^{p_0,q_0}(C_0), L^{p_1,q_1}(C_1))_{\eta,q} = L^{p,q}(C_{\eta p/p_1, q/p}).$$

Proof. Choose $0 < r < \min(p_0, p_1, q_0, q_1)$. Then, by Theorem 2.5.2, for $(1-\theta_0)p_0 = r$ and $(1-\theta_1)p_1 = r$ we have

$$(L^{p_0,q_0}(C_0), L^{p_1,q_1}(C_1))_{\eta,q} = ((L^r(C_0), L^\infty)_{\theta_0,q_0}, (L^r(C_1), L^\infty)_{\theta_1,q_1})_{\eta,q}$$

and, if $\bar{X} = (L^r(C_0), L^r(C_1), L^\infty)$, then by the Remark 2.5.10,

$$((L^r(C_0), L^\infty)_{\theta_0,q_0}, (L^r(C_1), L^\infty)_{\theta_1,q_1})_{\eta,q} = \bar{X}_{\bar{\theta}_1,\bar{\theta}_2,q}$$

with $\bar{\theta}_1 = (1-\theta_1)\eta, \bar{\theta}_2 = \theta_0(1-\eta) + \theta_1\eta$.

Since $[C_0 + t_1^r C_1]$ is a quasi-subadditive capacity

$$K(t_1, t_2, f; \bar{X}) \simeq K(t_2, f; L^r([C_0 + t_1^r C_1]), L^\infty)$$

and, since by Theorem 2.5.1,

$$K(t, f; L^r(C), L^\infty(C)) = \Big(\int_0^\infty y^{r-1} \min(C\{|f| > y\}, t^r) dy\Big)^{1/r},$$

it follows that

$$K(t_1, t_2, f; \bar{X}) \simeq \Big(\int_0^\infty y^{r-1} \min([C_0 + t_1^r C_1]\{|f| > y\}, t_2^r) dy\Big)^{1/r}.$$

So that

$$\|f\|^q_{\bar{X}_{\bar{\theta}_1,\bar{\theta}_2,q}} = \int_{\mathbb{R}^2_+} \Big[t_1^{-\bar{\theta}_1} t_2^{-\bar{\theta}_2} \Big(\int_0^\infty y^{r-1} \min([C_0 + t_1^r C_1]\{|f| > y\}, t_2^r) dy\Big)^{1/r}\Big]^q \frac{dt_2}{t_2}\frac{dt_1}{t_1},$$

where by Theorem 2.5.2,

$$\begin{aligned}
&\int_0^\infty \Big[t_2^{-\bar{\theta}_2} \Big(\int_0^\infty y^{r-1} \min\Big([C_0 + t_1^r C_1]\{|f| > y\}, t_2^r\Big) dy\Big)^{1/r}\Big]^q \frac{dt_2}{t_2} \\
&= \int_0^\infty t_2^{-\bar{\theta}_2 q} K(t_2, f; L^r([C_0 + t_1^r C_1]), L^\infty)^q \frac{dt_2}{t_2} \simeq \|f\|^q_{(L^r([C_0+t_1^r C_1]),L^\infty)_{\bar{\theta}_2,q}} \\
&\simeq \int_0^\infty y^{q-1} [C_0 + t_1^r C_1]\{|f| > y\}^{\frac{(1-\bar{\theta}_2)q}{r}} dy.
\end{aligned}$$

Thus, since $\frac{1-\bar\theta_2}{r} = 1/p$,

$$\begin{aligned}
\|f\|^q_{\bar X_{\bar\theta_1,\bar\theta_2,q}} &\simeq \int_0^\infty t_1^{-\bar\theta_1 q}\int_0^\infty y^{q-1}[C_0+t_1^rC_1]\{|f|>y\}^{\frac{q(1-\bar\theta_2)}{r}}dy\frac{dt_1}{t_1}\\
&= \int_0^\infty\int_0^\infty \tau^{\frac{-\bar\theta_1 q}{r}}y^{q-1}[C_0+\tau C_1]\{|f|>y\}^{\frac{q(1-\bar\theta_2)}{r}}dy\frac{1}{r}\frac{d\tau}{\tau}\\
&= \frac{1}{r}\int_0^\infty y^{q-1}\int_0^\infty \tau^{\frac{-\eta q}{p_1}}[C_0+\tau C_1]\{|f|>y\}^{q/p}\frac{d\tau}{\tau}dy\\
&= \frac{1}{r}\int_0^\infty y^{q-1}\int_0^\infty \Big(\tau^{\frac{-\eta p}{p_1}}[C_0+\tau C_1]\{|f|>y\}\Big)^{q/p}\frac{d\tau}{\tau}dy,
\end{aligned}$$

and

$$\|\chi_{\{|f|>y\}}\|^{q/p}_{(L^1(C_0),L^1(C_1))_{\frac{\eta p}{p_1},q/p}} = \int_0^\infty \Big(\tau^{\frac{-\eta p}{p_1}}K(\tau,\chi_{\{|f|>y\}};L^1(C_0),L^1(C_1))\Big)^{q/p}\frac{d\tau}{\tau}.$$

Hence, if we define $C_{\theta,q}(A) := \|\chi_A\|_{(L^1(C_0),L^1(C_1))_{\theta,q}}$, then it follows that

$$\|f\|^q_{\bar X_{\bar\theta_1,\bar\theta_2,q}} \simeq \int_0^\infty y^{q-1}\|\chi_{\{|f|>y\}}\|^{q/p}_{(L^1(C_0),L^1(C_1))_{\eta p/p_1,q/p}}dy \simeq \|f\|^q_{L^{p,q}(C_{\eta p/p_1,q/p})}. \qquad \square$$

Theorem 2.5.12. *Let C_0, C_1 be a couple of quasi-subadditive Fatou capacities with the same null sets and $0<\eta<1$. If $0<p_0,p_1<\infty$, $0<q_0,q_1\le\infty$, $\frac{1}{p} := \frac{1-\eta}{p_0}+\frac{\eta}{p_1}$ and $\frac{1}{q} := \frac{1-\eta}{q_0}+\frac{\eta}{q_1}$, then, for $C_{\theta,q}(A) := \|\chi_A\|_{(L(C_0),L(C_1))_{\theta,q}}$ $(0<\theta<1)$,*

$$(L^{p_0,q_0}(C_0),L^{p_1,q_1}(C_1))_{\eta,q} = L^{p,q}(C_{\frac{\eta p}{p_1},q/p}).$$

Proof. Let $0<r<\min(p_0,p_1,q_0,q_1)$. By Theorem 2.5.2 and the Remark 2.5.10 we get, for $(1-\theta_0)p_0 = r$ and $(1-\theta_1)p_1 = r$, that

$$\begin{aligned}
&(L^{p_0,q_0}(C_0),L^{p_1,q_1}(C_1))_{\eta,q} = ((L^r(C_0),L^\infty)_{\theta_0,q_0},(L^r(C_1),L^\infty)_{\theta_1,q_1})_{\eta,q}\\
&= (L^r(C_0),L^r(C_1),L^\infty)_{\bar\theta_1,\bar\theta_2,q},\ \bar\theta_1 = (1-\theta_1)\eta, \bar\theta_2 = \theta_0(1-\eta)+\theta_1\eta.
\end{aligned}$$

Now we estimate $K(t_1,t_2,f;L^r(C_0),L^r(C_1),L^\infty)$. Easily, it follows that

$$K(t_1,t_2,f;L^r(C_0),L^r(C_1),L^\infty) \simeq K(t_2,f;L^r(C_0+t_1^rC_1),L^\infty)$$

and hence, by Theorem 2.5.1, we get that

$$K(t_1,t_2,f;L^r(C_0),L^r(C_1),L^\infty) \simeq \Big(\int_0^\infty y^{r-1}\min[(C_0+t_1^rC_1)\{|f|>y\},t_2^r]dy\Big)^{1/r}.$$

As in Theorem 2.5.11 we estimate

$$\begin{aligned}
&\|f\|^q_{\bar X_{\bar\theta_1,\bar\theta_2,q}}\\
&\simeq \int_0^\infty\int_0^\infty \Big[t_1^{-\bar\theta_1}t_2^{-\bar\theta_2}\Big(\int_0^\infty y^{r-1}\min\Big((C_0+t_1^rC_1)\{|f|>y\},t_2^r\Big)dy\Big)^{1/r}\Big]^q\frac{dt_2}{t_2}\frac{dt_1}{t_1}.
\end{aligned}$$

The proof then follows since $\frac{1-\bar\theta_2}{r} = 1/p$. $\square$

2.6 Applications to classical Lorentz spaces

Let $p, q > 0$, μ be a measure or weight on $\mathbb{R}^n$, and w be a weight on $\mathbb{R}^+$. The *classical Lorentz spaces* $\Lambda^{p,q}_\mu(w)$ are defined by the condition

$$\|f\|_{\Lambda^{p,q}_\mu(w)} = \Big(\int_0^\infty s^{q/p} f^*_\mu(s)^q w(s)\frac{ds}{s}\Big)^{1/q} < \infty,$$

where f^*_μ is the decreasing rearrangement of f with respect to μ defined as in (1.2) with μ instead of C. If $p = q$, $\Lambda^p_\mu(w) = \Lambda^{p,p}_\mu(w)$ and $\|f\|_{\Lambda^p_\mu(w)} = \Big(\int_0^\infty f^*_\mu(s)^p w(s)\,ds\Big)^{1/p}$. Moreover, if $w = 1$, $\Lambda^{p,q}_\mu(1) = L^{p,q}(\mu)$ and $\Lambda^{p,p}_\mu(1) = L^p(\mu)$.

Some basic questions concerning these spaces are the following ones:

1. Are they normed or quasi-normed function spaces?
2. Is there an imbedding $\Lambda^{p,q_0}_\mu(w) \hookrightarrow \Lambda^{p,q_1}_\mu(w)$ for $0 < q_0 < q_1 \le \infty$?
3. Find the weights for which classical operators (such as the Hardy operator $Sf(x) = \int_0^x f(x)dx$) are bounded from $\Lambda^{p_0}_{\mu_0}(w_0)$ to $\Lambda^{p_1}_{\mu_1}(w_1)$.

Two good references for these topics are [CSo] and [CRSo].

In [CRSo] it is proved that for $0 < p < \infty$, $0 < q \le \infty$, $\Lambda^{p,q}_\mu(w)$ is quasi-normed if and only if $\int_0^{2r} w(s)ds \lesssim \int_0^r w(s)ds$ for each $r > 0$. And, $\Lambda^{p,q_0}_\mu(w) \hookrightarrow \Lambda^{p,q_1}_\mu(w)$ continuously if $0 < q_0 \le q_1 \le \infty$.

Let us show in this section that classical Lorentz spaces are capacitary Lebesgue spaces. Denote $W(A) = \int_A w(t)\,dt$. By [CRSo, Proposition 2.2.5] it follows that

$$\|f\|_{\Lambda^p_\mu(w)} = \Big(\int_0^\infty f^*_\mu(s)^p w(s)ds\Big)^{1/p} = \Big(\int_0^\infty pt^{p-1}W([0,\mu\{|f| > t\}))\,dt\Big)^{1/p},$$

so that $\Lambda^p_\mu(w) = L^p(C)$ for $C(A) = W[0,\mu(A))$, which is a Fatou capacity. It follows then that $\Lambda^p_\mu(w)$ is a normed space precisely when C is concave, and this means that W is concave.

But such a remark can be also applied to new Lorentz spaces obtained from some other well known symmetrization methods of analysis as:

- Spherical symmetrization: $f^*_S(y) := f^*_\mu(\sigma_n|y|^n) = \int_0^\infty \chi_{\{|f|>s\}^*}$ if $\chi_{A^*} = \chi^*_A$. Also for the Steiner symmetrization of order k $(1 < k \le n)$.
- Multidimensional symmetrization, f^*_2, where f^*_2 is defined in [BPSo] as follows: For a set $A \subset \mathbb{R}^2$, $A^*_2 = \{(s,t);\, 0 < t < \chi^*_{E(s)}\}$, where $E(s)$ is the s-section $\{y \in \mathbb{R};\, (s,y) \in E\}$. Then s^*_2 is defined for a simple function s, and finally $f^*_2 := \lim_k (s_k)^*_2$,

$$\|f\|_{\Lambda^p_2(v)} := \|f^*_2\|_{L^p(v)}.$$

- Discrete rearrangements on trees as in [GDS].

In [BSo], S. Boza and J. Soria consider increasing transformations $A \mapsto \mathcal{R}(A)$ on measure spaces with the Fatou property, $A_n \uparrow A \Rightarrow \mathcal{R}(A_n) \uparrow \mathcal{R}(A)$, that allow to define the corresponding rearrangement of functions

$$f^*_{\mathcal{R}}(y) := \int_0^\infty \chi_{\mathcal{R}\{|f|>t\}}(y)\, dt,$$

that brings pass to unify various Lorentz spaces found in the literature, included all the mentioned above:

$$\|f\|_{\Lambda^p_{\mathcal{R}}(w)} := \|f^*_{\mathcal{R}}\|_{L^p(w)} = \Big(\int_0^\infty pt^{p-1} W(\mathcal{R}\{|f|>t\})\, dt\Big)^{1/p}.$$

Obviously $\Lambda^p_{\mathcal{R}}(w) = L^p(C_{W,\mathcal{R}})$ if we define the capacity $C_{W,\mathcal{R}}$ as

$$C_{W,\mathcal{R}}(A) = W(\mathcal{R}\{|f|>t\}),$$

and our results on capacities apply to this special case.

As a final example, let us show how interpolation of capacitary Lebesgue spaces can be used in interpolation of classical Lorentz spaces, $(\Lambda^{p_0}(w_0), \Lambda^{p_1}(w_1))_{\eta,p}$ $(0 < p_0, p_1 < \infty)$. We have shown that if C_0 and C_1 are quasi-subadditive Fatou capacities on (Ω, Σ) with the same null sets, $0 < p_0, p_1 < \infty$ and $0 < \eta < 1$, then

$$\big(L^{p_0}(C_0), L^{p_1}(C_1)\big)_{\eta,p} = L^p(C_{\eta p/p_1,1}) \qquad (1/p = (1-\eta)/p_0 + \eta/p_1),$$

where

$$C_{\theta,q}(A) := \|\chi_A\|_{(L^1(C_0),L^1(C_1))_{\theta,q}}.$$

We start from the identity $(\Lambda^1(w_0), \Lambda^1(w_1))_{\theta,1} = \Lambda^1(w)$, where $W = W_0^{1-\theta} W_1^\theta$. Consider $\Lambda^{p_j}(w_j) = L^{p_j}(C_j)$ with $C_j = W_j \circ \mathcal{R}$ $(j = 0, 1)$. Then

$$(\Lambda^{p_0}(w_0), \Lambda^{p_1}(w_1))_{\eta,p} = (L^{p_0}(C_0), L^{p_1}(C_1))_{\eta,p} = L^p(C_{\theta,1})$$

with $\theta = \eta p/p_1$.

Since $C_{\theta,1}(A) = \|\chi_A\|_{(L^1(C_0),L^1(C_1))_{\theta,1}} = \|\chi_A\|_{\Lambda^1(w)} = W \circ \mathcal{R}(A)$ and $L^p(C_{\theta,1}) = \Lambda^p(w)$, it follows that

$$(\Lambda^{p_0}(w_0), \Lambda^{p_1}(w_1))_{\eta,p} = \Lambda^p(w)$$

with

$$W = W_0^{1-\theta} W_1^\theta = W_0^{(1-\eta)p/p_0} W_1^{\eta p/p_1}.$$

2.7 Interpolation of quasi-continuous functions

Often the interest in Potential theory is in the Choquet integral of special class of functions, such as the class of quasi-continuous functions.

In this section, Ω will be a subset of $\mathbb{R}^n$ and C will be a quasi-subadditive Fatou capacity on $(\Omega, \mathcal{B}(\Omega))$. A function $f : \Omega \to \mathbb{R}$ is termed *C-quasi-continuous* on Ω if given any $\varepsilon > 0$, there exists a relatively open set $G \subset \Omega$ such that $C(G) < \varepsilon$ and f is continuous on G^c.

Note that the classical theorem of Egorov implies that any Lebesgue integrable function ϕ on Ω is m_n-quasi-continuous there. Notice also that the potential $\mathcal{G}f(x)$ for $f \in L^p$, in Example 1.2.14 is $C_{g,p}$-quasi-continuous on $\mathbb{R}^n$.

Proposition 2.7.1. *If C is an outer capacity on subsets of Ω be an open subset of $\mathbb{R}^n$, and ϕ_k is a sequence of continuous functions on Ω with compact support such that for $p > 0$,*

$$\int_\Omega |\phi_k - \phi|^p dC \to 0$$

as $k \to \infty$, then ϕ is C-quasi-continuous on Ω.

For more details about it, see [Ch] and [Ma11].

As we showed, the theory of capacities of Potential theory is very useful to obtain bounds for some classical operators. The class of quasi-continuous functions appears frequently in this framework. Therefore we will use this section to the study of a particular problem of interpolation when we restrict us to this setting.

We are interested in obtaining results on interpolation of capacitary function spaces on $\mathbb{R}^n$ of quasi-continuous functions, starting from previous results on general capacitary function spaces contained in [CeClM], [Ce] and [CeMS]. This means to obtain a result about restriction of interpolation to the subspace QC of $C-$quasi-continuous functions.

Our goal is to prove that the restriction of the K-functional of the couple $(L^p(C), L^\infty(C))$ to quasi-continuous functions $f \in QC$ is equivalent to

$$K(t, f; L^p(C) \cap QC, L^\infty(C) \cap QC).$$

Then we will apply this result to identify the interpolation space of the couple of "capacitary Lorentz spaces" $(L^{p_0,q_0}(C) \cap QC, L^{p_1,q_1}(C) \cap QC)$.

For $0 < p_0 < \infty$, consider the spaces $L^{p_0}(C)$ and $L^\infty(C)$. For every $t > 0$, we have that

$$\begin{aligned} K(t, f; L^{p_0}(C), L^\infty(C)) &\leq K(t, f; L^{p_0}(C) \cap QC, L^\infty(C) \cap QC) \\ &=: K_{QC}(t, f; L^{p_0}(C), L^\infty(C)). \end{aligned}$$

Proposition 2.7.2. *If f is non-negative, then*

$$K(t, f; L^{p_0}(C), L^\infty(C)) = \inf_{\lambda>0} \left(\|(f-\lambda)_+\|_{L^{p_0}(C)} + t\|\min(f,\lambda)\|_{L^\infty(C)} \right).$$

Proof. By definition, since $f = (f-\lambda)_+ + \min(f,\lambda)$ for all $\lambda > 0$, we have that

$$K(t, f; L^{p_0}(C), L^\infty(C)) \le \inf_{\lambda>0} \left(\|(f-\lambda)_+\|_{L^{p_0}(C)} + t\|\min(f,\lambda)\|_{L^\infty(C)} \right).$$

To prove the reversed estimate, let $\epsilon > 0$ and choose $f_0, f_1 \ge 0$ such that $f = f_0 + f_1$ and

$$\|f_0\|_{L^{p_0}(C)} + t\|f_1\|_{L^\infty(C)} \le K(t, f; L^{p_0}(C), L^\infty(C)) + \epsilon.$$

If $\bar\lambda = \|f_1\|_{L^\infty(C)}$, then $f - \bar\lambda \le f_0$ and $0 \le (f-\bar\lambda)_+ \le f_0$. Hence $\|(f-\bar\lambda)_+\|_{L^{p_0}(C)} \le \|f_0\|_{L^{p_0}(C)}$ and $\|\min(f,\bar\lambda)\|_{L^\infty(C)} \le \|f_1\|_{L^\infty(C)}$. Thus

$$\begin{aligned}
&\inf_{\lambda>0} \left(\|(f-\lambda)_+\|_{L^{p_0}(C)} + t\|\min(f,\lambda)\|_{L^\infty(C)} \right) \\
&\le \|(f-\bar\lambda)_+\|_{L^{p_0}(C)} + t\|\min(f,\bar\lambda)\|_{L^\infty(C)} \\
&\le \|f_0\|_{L^{p_0}(C)} + t\|f_1\|_{L^\infty(C)} \\
&\le K(t, f; L^{p_0}(C), L^\infty(C)) + \epsilon
\end{aligned}$$

and the estimate follows. □

Proposition 2.7.3. *If $f \in QC$ is non-negative, then*

$$K_{QC}(t, f; L^{p_0}(C), L^\infty(C)) = K(t, f; L^{p_0}(C), L^\infty(C)).$$

Proof. If $f \in QC$ is non-negative, then for all $\lambda > 0$ we have that $(f-\lambda)_+ \in QC$ and $\min(f,\lambda) \in QC$ since they are non-negative. Then, for all $\lambda > 0$,

$$K_{QC}(t, f; L^{p_0}(C), L^\infty(C)) \le \|(f-\lambda)_+\|_{L^{p_0}(C)} + t\|\min(f,\lambda)\|_{L^\infty(C)}$$

and hence

$$\begin{aligned}
K_{QC}(t, f; L^{p_0}(C), L^\infty(C)) \quad &\le \inf_{\lambda>0} \left(\|(f-\lambda)_+\|_{L^{p_0}(C)} + t\|\min(f,\lambda)\|_{L^\infty(C)} \right) \\
&= K(t, f; L^{p_0}(C), L^\infty(C))
\end{aligned}$$

by Proposition 2.7.2. □

Let us now show that $K(t, f; L^{p_0}(C), L^\infty(C)) = K(t, |f|; L^{p_0}(C), L^\infty(C))$. Obviously,

$$K(t, f; L^{p_0}(C), L^\infty(C)) \le K(t, |f|; L^{p_0}(C), L^\infty(C)).$$

To prove the reversed inequality, let $\epsilon > 0$ and choose $f_0 \in L^{p_0}(C)$, $f_1 \in L^\infty(C)$ such that $f = f_0 + f_1$ and

$$\|f_0\|_{L^{p_0}(C)} + t\|f_1\|_{L^\infty(C)} \leq K(t, f; L^{p_0}(C), L^\infty(C)) + \epsilon.$$

Define

$$s(f) := \begin{cases} 1, & \text{if } f(x) \geq 0 \\ -1, & \text{if } f(x) < 0. \end{cases}$$

Then $s(f)f = s(f)(f_0 + f_1) = s(f)f_0 + s(f)f_1$, which means that $|f| = s(f)f_0 + s(f)f_1$ with $\|f_0\|_{L^{p_0}(C)} = \|s(f)f_0\|_{L^{p_0}(C)}$ and $\|f_1\|_{L^\infty(C)} = \|s(f)f_1\|_{L^\infty(C)}$. Then

$$\begin{aligned} K(t, |f|; L^{p_0}(C), L^\infty(C)) &\leq \|s(f)f_0\|_{L^{p_0}(C)} + t\|s(f)f_1\|_{L^\infty(C)} \\ &\leq K(t, f; L^{p_0}(C), L^\infty(C)) + \epsilon \end{aligned}$$

and letting $\varepsilon \to 0$ we get that

$$K(t, |f|; L^{p_0}(C), L^\infty(C)) \leq K(t, f; L^{p_0}(C), L^\infty(C)).$$

Since $K(t, |f|; L^{p_0}(C), L^\infty(C)) = K(t, f; L^{p_0}(C), L^\infty(C))$, we conclude

$$\||f|\|_{(L^{p_0}(C), L^\infty(C))_{\theta,q}} = \|f\|_{(L^{p_0}(C), L^\infty(C))_{\theta,q}}.$$

Proposition 2.7.4. *Let f be a quasi-continuous function, not necessarily positive. Then*

$$K_{QC}(t, f; L^{p_0}(C), L^\infty(C)) \simeq K(t, f; L^{p_0}(C), L^\infty(C)).$$

Proof.

$$\begin{aligned} K_{QC}(t, f; L^{p_0}(C), L^\infty(C)) &= K_{QC}(t, f^+ - f^-; L^{p_0}(C), L^\infty(C)) \\ &\leq K_{QC}(t, f^+; L^{p_0}(C), L^\infty(C)) + K_{QC}(t, f^-; L^{p_0}(C), L^\infty(C)) \\ &\leq K_{QC}(t, |f|; L^{p_0}(C), L^\infty(C)) + K_{QC}(t, |f|; L^{p_0}(C), L^\infty(C)) \\ &= 2K_{QC}(t, |f|; L^{p_0}(C), L^\infty(C)) = 2K(t, |f|; L^{p_0}(C), L^\infty(C)) \\ &= 2K(t, f; L^{p_0}(C), L^\infty(C)) \leq 2K_{QC}(t, f; L^{p_0}(C), L^\infty(C)) \end{aligned}$$

since $|f| \in QC$. □

Thus, for $0 < \theta < 1$ and $q > 0$, we have that

$$\begin{aligned} \|f\|_{(L^{p_0}(C)\cap QC, L^\infty(C)\cap QC)_{\theta,q}} &:= \left(\int_0^\infty (t^{-\theta} K_{QC}(t, f))^q \frac{dt}{t}\right)^{1/q} \\ &\lesssim \|f\|_{(L^{p_0}(C), L^\infty(C))_{\theta,q}}. \end{aligned}$$

Hence

$$(L^{p_0}(C), L^{\infty}(C))_{\theta,q} \cap QC \hookrightarrow (L^{p_0}(C) \cap QC, L^{\infty}(C) \cap QC)_{\theta,q}$$

and therefore

$$(L^{p_0}(C), L^{\infty}(C))_{\theta,q} \cap QC = (L^{p_0}(C) \cap QC, L^{\infty}(C) \cap QC)_{\theta,q}. \tag{2.21}$$

By denoting $\mathfrak{L}^{p,q}(C) = L^{p,q}(C) \cap QC$, we obtain:

Theorem 2.7.5. *Suppose that* $0 < \theta < 1$, $0 < p_0 < q \leq \infty$ *or* $0 < p_0 \leq q < \infty$ *and* $\frac{1}{p} := \frac{1-\theta}{p_0}$. *Then*

$$(\mathfrak{L}^{p_0}(C), \mathfrak{L}^{\infty}(C))_{\theta,q} = \mathfrak{L}^{p,q}(C).$$

Proof.

$$\begin{aligned}(\mathfrak{L}^{p_0}(C), \mathfrak{L}^{\infty}(C))_{\theta,q} &= (L^{p_0}(C) \cap QC, L^{\infty}(C) \cap QC)_{\theta,q} \\ &= (L^{p_0}(C), L^{\infty}(C))_{\theta,q} \cap QC = L^{p,q}(C) \cap QC := \mathfrak{L}^{p,q}(C)\end{aligned}$$

by (2.21) and Theorem 2.5.2. □

Corollary 2.7.6. *Take* $0 < p_0, p_1, q_0, q_1 < \infty$, $p_0 \neq p_1$ *and* $0 < \eta < 1$. *Then*

$$(\mathfrak{L}^{p_0,q_0}(C), \mathfrak{L}^{p_1,q_1}(C))_{\eta,q} = \mathfrak{L}^{p,q}(C)$$

with $\frac{1}{p} := \frac{1-\eta}{p_0} + \frac{\eta}{p_1}$.

Proof. Let $0 < r < \min(p_0, p_1, q_0, q_1)$ and choose $1 - \theta_i := \frac{r}{p_i}$, $i = 0, 1$. Then, if $\theta = (1-\eta)\theta_0 + \eta\theta_1$, since $\frac{1}{p} = \frac{1-\theta}{r}$, we get

$$\begin{aligned}(\mathfrak{L}^{p_0,q_0}(C), \mathfrak{L}^{p_1,q_1}(C))_{\eta,q} &= ((\mathfrak{L}^{r}(C), \mathfrak{L}^{\infty}(C))_{\theta_0,q_0}, (\mathfrak{L}^{r}(C), \mathfrak{L}^{\infty}(C))_{\theta_1,q_1})_{\eta,q} \\ &= (\mathfrak{L}^{r}(C), \mathfrak{L}^{\infty}(C))_{\theta,q} \\ &= \mathfrak{L}^{p,q}(C) = L^{p,q}(C) \cap QC \\ &= (L^{p_0,q_0}(C), L^{p_1,q_1}(C))_{\eta,q} \cap QC\end{aligned}$$

by Theorem 2.7.5, [BeLo, Theorem 3.11.5], and Theorem 2.5.3. □

Let C_0, C_1 be quasi-suabdditive Fatou capacities on $(\Omega, \mathcal{B}(\Omega))$, $0 < p_0, p_1, q_0, q_1 < \infty$ and $0 < \eta < 1$, and define $\frac{1}{p} := \frac{1-\eta}{p_0} + \frac{\eta}{p_1}$ and $\frac{1}{q} := \frac{1-\eta}{q_0} + \frac{\eta}{q_1}$. As far as we know, an interesting still open problem in this setting is to check the existence or not of a concrete capacity $\widetilde{C}$ such that

$$(\mathfrak{L}^{p_0,q_0}(C_0), \mathfrak{L}^{p_1,q_1}(C_1))_{\eta,q} = \mathfrak{L}^{p,q}(\widetilde{C}).$$

2.8 Calderón products

A natural question in the capacitary setting is if any pair of $L^p(C)$ spaces is a Calderón product. In this section we will see that, as in the case of measures, it holds for general capacities.

Now, in analogous way to Banach function spaces on measure spaces, we can define Calderón products of quasi-normed capacitary function spaces with the same basic properties.

Definition 2.8.1. *Let X_0 and X_1 be quasi-normed capacitary function spaces on (Ω, Σ, C) and let $\alpha \in (0,1)$. The* Calderón product *of X_0 and X_1, denoted by $X = X_0^{1-\alpha} X_1^{\alpha}$, is the class of all $f \in \mathrm{L}_0(C)$ such that*

$$|f(t)| \leq \lambda |f_0(t)|^{1-\alpha} |f_1(t)|^{\alpha} \tag{2.22}$$

for every $t \in \Omega$, $\lambda > 0$, $f_0 \in X_0$, and $f_1 \in X_1$ with $\|f_0\|_{X_0} \leq 1$, $\|f_1\|_{X_1} \leq 1$.

We endow the linear space X with $\|f\|_X := \inf \lambda$, where the infimum is taken over all λ satisfying (2.22).

Note that $\{f \neq 0\}$ is C-sigma-finite and that, if we define on $\mathrm{L}_0(C)$

$$\varrho_\alpha(f) := \begin{cases} \|f\|_X & \text{, if } f \in X \\ \infty & \text{, if } f \notin X \end{cases}$$

then $X = \{f \in \mathrm{L}_0(C);\ \varrho_\alpha(f) < \infty\}$. For every $f \geq 0$ we can also write

$$\varrho_\alpha(f) = \inf \left\{ \lambda > 0;\ f \leq \lambda f_0^{1-\alpha} f_1^{\alpha},\ f_i \geq 0,\ \|f_i\|_{X_i} \leq 1\ (i = 0,1) \right\}$$

with $\inf \emptyset = \infty$.

Then ϱ_α satisfies all the required properties to define a quasi-normed capacitary function space and $\|f\|_X = \varrho_\alpha(|f|)$. Indeed, to check that $\varrho_\alpha(f) = 0$ if and only if $f = 0$ $C-$q.e., note that the condition $\|f\|_X = 0$ means that there exist $\lambda_n \to 0$ in $\mathbb{R}_+$ and functions $f_{0,n} \in X_0$ and $f_{1,n} \in X_1$ with $\|f_{0,n}\|_{X_0} \leq 1$, $\|f_{1,n}\|_{X_1} \leq 1$ such that

$$|f(t)| \leq \lambda_n |f_{0,n}(t)|^{1-\alpha} |f_{1,n}(t)|^{\alpha}.$$

Then $\{f \neq 0\}$ is sigma-finite and the sequences $y_{0,n} := \lambda_n^{1/2(1-\alpha)} |f_{0,n}|$ and $y_{1,n} := \lambda_n^{1/2\alpha} |f_{1,n}|$ $(n \in \mathbb{N})$ converge to zero in X_0 and X_1, respectively. By Theorem 1.3.6 and Theorem 1.3.11, they converge to zero in capacity on every set $A \subset \{f \neq 0\}$ of finite capacity and, by passing to subsequences, they can be supposed to be convergent to zero $C-$q.e. on A. Then

$$\lim_n \lambda_n |f_{0,n}(t)|^{1-\alpha} |f_{1,n}(t)|^{\alpha} = \lim_n \left(\lambda_n^{1/2(1-\alpha)} |f_{0,n}| \right)^{1-\alpha} \left(\lambda_n^{1/2\alpha} |f_{0,n}| \right)^{\alpha} = 0$$

$C-$q.e. on A, and $f = 0$. Hence, $\|\cdot\|_X$ is a quasi-norm.

From now on in this section, X_0 and X_1 will be two quasi-normed capacitary function spaces on (Ω, Σ, C). With any pair of quasi-normed function spaces, as we did in (2.3) and (2.4) we may canonically associate a couple of embedded spaces in the following way:

(a) $X_0 \cap X_1$ consists of the elements common to X_0 and X_1. The quasi-norm is introduced by

$$\|f\|_{X_0 \cap X_1} = \max\{\|f\|_{X_0}, \|f\|_{X_1}\} \quad (x \in X_0 \cap X_1),$$

and

(b) $X_0 + X_1$ denotes the set of elements of the form $x = u + v$, where $u \in X_0$, $v \in X_1$, and it is equipped with the quasi-norm

$$\|x\|_{X_0+X_1} = \inf\{\|u\|_{X_0} + \|v\|_{X_1}\},$$

where the infimum is taken over all elements $u \in X_0$, $v \in X_1$ whose sum is equal to x.

Proposition 2.8.2. *The space $X_0^{1-\alpha} X_1^{\alpha}$ is intermediate between X_0 and X_1, that is,*

$$X_0 \cap X_1 \subset X_0^{1-\alpha} X_1^{\alpha} \subset X_0 + X_1$$

with continuous inclusions, i.e., $X_0 \cap X_1 \hookrightarrow X_0^{1-\alpha} X_1^{\alpha} \hookrightarrow X_0 + X_1$.

Proof. If $f \in X_0 \cap X_1$, then for all t

$$|f(t)| = |f(t)|^{1-\alpha}|f(t)|^{\alpha} = \|f\|_{X_0}^{1-\alpha}\|f_1\|_{X_1}^{\alpha}\Big(\frac{|f(t)|}{\|f\|_{X_0}}\Big)^{1-\alpha}\Big(\frac{|f(t)|}{\|f\|_{X_1}}\Big)^{\alpha},$$

which implies that $f \in X_0^{1-\alpha} X_1^{\alpha}$ and

$$\|f\|_{X_0^{1-\alpha} X_1^{\alpha}} \leq \|f\|_{X_0}^{1-\alpha}\|f\|_{X_1}^{\alpha} \leq \|f\|_{X_0 \cap X_1}.$$

Moreover, if $|f(t)| \leq \lambda |f_0(t)|^{1-\alpha}|f_1(t)|^{\alpha}$, where f_0 and f_1 satisfy the required conditions in the definition, then $|f(t)| \leq \lambda\{(1-\alpha)|f_0(t)| + \alpha|f_1(t)|\}$ and so

$$\begin{aligned} \|f\|_{X_0+X_1} &\leq \lambda\|(1-\alpha)|f_0| + \alpha|f_1|\|_{X_0+X_1} \\ &\lesssim \lambda\{(1-\alpha)\|f_0\|_{X_0+X_1} + \alpha\|f_1\|_{X_0+X_1}\} \leq \lambda \end{aligned}$$

which implies that $f \in X_0 + X_1$ and

$$\|f\|_{X_0+X_1} \lesssim \|f\|_{X_0^{1-\alpha} X_1^{\alpha}}. \qquad \square$$

Theorem 2.8.3. *The space $X_0^{1-\alpha}X_1^{\alpha}$ is complete.*

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence such that $\sum_n \|f_n\|_X < \infty$. Given $\epsilon > 0$, we can find $\lambda_n > 0, f_{0,n} \in X_0, f_{1,n} \in X_1$ with $\|f_{0,n}\|_{X_0} \leq 1, \|f_{1,n}\|_{X_1} \leq 1$, and $\lambda_n \leq \|f_n\|_X + \frac{\epsilon}{2^n}$ that satisfy $|f_n(t)| \leq \lambda_n |f_{0,n}(t)|^{1-\alpha}|f_{1,n}(t)|^{\alpha}$. Then $\sum_n \lambda_n \leq \sum_n (\|f_n\|_X + \frac{\epsilon}{2^n}) < \infty$ and for all t

$$\begin{aligned}\sum_n |f_n(t)| &\leq \sum_n \lambda_n |f_{0,n}(t)|^{1-\alpha}|f_{1,n}(t)|^{\alpha} \\ &= \sum_n \lambda_n \cdot \sum_n \left(\frac{\lambda_n}{\sum_n \lambda_n}|f_{0,n}(t)|\right)^{1-\alpha}\left(\frac{\lambda_n}{\sum_n \lambda_n}|f_{1,n}(t)|\right)^{\alpha}.\end{aligned}$$

By Corollary 1.2.20 applied with exponents $\frac{1}{p} = 1-\alpha$ and $\frac{1}{q} = \alpha$ to $\bar{f}_n(t)^p := \frac{\lambda_n}{\sum_n \lambda_n}|f_{0,n}(t)|$ and $g_n(t) := \left(\frac{\lambda_n}{\sum_n \lambda_n}|f_{1,n}(t)|\right)^{\alpha}$, it follows that

$$\begin{aligned}\sum_n |f_n(t)| &\leq \sum_n \lambda_n \cdot \sum_n \left(\frac{\lambda_n}{\sum_n \lambda_n}|f_{0,n}(t)|\right)^{1-\alpha}\left(\frac{\lambda_n}{\sum_n \lambda_n}|f_{1,n}(t)|\right)^{\alpha} \\ &\leq k\sum_n \lambda_n \cdot \left(\sum_n \bar{f}_n(t)^p\right)^{1/p}\left(\sum_n g_n(t)^q\right)^{1/q} \\ &= k \cdot \sum_n \lambda_n \cdot \left(\sum_n \frac{\lambda_n}{\sum_n \lambda_n}|f_{0,n}(t)|\right)^{1-\alpha}\left(\sum_n \frac{\lambda_n}{\sum_n \lambda_n}|f_{1,n}(t)|\right)^{\alpha}.\end{aligned}$$

The functions in brackets are defined $C-$q.e. and belongs to X_0 and X_1. This implies that $\sum_n |f_n| \in X$. We write $f(t) = \sum_n f_n(t)$ for all t. Then, $|f(t)| \leq \sum_n |f_n(t)|$ and therefore $f \in X$ with

$$\|f\|_X \leq k\sum_n \|f_n\|_X.$$

Applying this inequality to $f(t) - \sum_{n=1}^N f_n(t) = \sum_{N+1}^{\infty} f_n(t)$, we obtain

$$\left\|f - \sum_{n=1}^N f_n\right\|_X \leq k\sum_{N+1}^{\infty} \|f_n\|_X,$$

which tends to zero when $N \to \infty$. Necessarily $\lim_{N\to\infty} \sum_{n=1}^N f_n = f$ $C-$q.e. □

Theorem 2.8.4. *Let $0 < p_0, p_1 \leq \infty$, $\alpha \in (0,1)$ and $\frac{1}{p} = \frac{1-\alpha}{p_0} + \frac{\alpha}{p_1}$. Then*

$$L^{p_0}(C)^{1-\alpha}L^{p_1}(C)^{\alpha} = L^p(C)$$

with equivalent quasi-norms (or equal norms in the normed case).

Proof. Let $X_0 = L^{p_0}(C)$ and $X_1 = L^{p_1}(C)$. If $f \in X_0^{1-\alpha}X_1^{\alpha}$, we consider $\lambda > 0$ such that $|f(t)| \le \lambda |f_0(t)|^{1-\alpha}|f_1(t)|^{\alpha}$ as in (2.22). Then $|f(t)|^p \le \lambda^p |f_0(t)|^{(1-\alpha)p}|f_1(t)|^{\alpha p}$ and taking $\bar{f} := |f_0|^{(1-\alpha)p}$, $g := |f_1|^{\alpha p}, p' = \frac{p_0}{(1-\alpha)p}$, $p'' := \frac{p_1}{\alpha p}$, it follows, by Hölder's inequality, that

$$\begin{aligned}
\int_\Omega |f|^p dC &\le \int_\Omega \lambda^p |f_0|^{(1-\alpha)p}|f_1|^{\alpha p} dC = \lambda^p \int_\Omega |\bar{f}||g(t)| dC \\
&\lesssim \lambda^p \Big(\int_\Omega |f_0(t)|^{p_0} dC\Big)^{\frac{(1-\alpha)p}{p_0}} \Big(\int_\Omega |f_1(t)|^{p_1} dC\Big)^{\frac{\alpha p}{p_1}} \\
&= \lambda^p \|f_0\|_{X_0}^{(1-\alpha)p} \|f_1\|_{X_1}^{\alpha p} \le \lambda^p,
\end{aligned}$$

from which we obtain the first inclusion $L^{p_0}(C)^{1-\alpha}L^{p_1}(C)^{\alpha} \hookrightarrow L^p(C)$.

Conversely, suppose $f \in L^p(C)$. Then for all t

$$\begin{aligned}
&\|f\|_{L^p(C)} \Big(\frac{|f(t)|^{p/p_0}}{\||f|^{p/p_0}\|_{L^{p_0}(C)}}\Big)^{1-\alpha} \Big(\frac{|f(t)|^{p/p_1}}{\||f|^{p/p_1}\|_{L^{p_1}(C)}}\Big)^{\alpha} \\
&= \|f\|_{L^p(C)} \frac{1}{\|f\|_{L^p(C)}^{p(\frac{1-\alpha}{p_0}+\frac{\alpha}{p_1})}} (|f(t)|^p)^{\frac{1-\alpha}{p_0}+\frac{\alpha}{p_1}} = |f(t)|
\end{aligned}$$

and therefore $f \in X = L^{p_0}(C)^{1-\alpha}L^{p_1}(C)^{\alpha}$ with $\|f\|_X \le \|f\|_{L^p(C)}$. □

Remark 2.8.5. *If $[X_0, X_1]_\alpha$ has the Fatou property, then*

$$\||x_0|^{1-\alpha}|x_1|^{\alpha}\|_{[X_0,X_1]_\alpha} \le \|x_0\|_{X_0}^{1-\alpha}\|x_1\|_{X_1}^{\alpha}$$

holds for every $x_0 \in X_0$ and $x_1 \in X_1$ (see [KrPS, Chapter 4]).

A natural question is to determine whether the space $X = X_0^{1-\alpha}X_1^{\alpha}$ has the Fatou property. Let us see that if both spaces have the Fatou property, then the general Calderón product has also the same property.

Theorem 2.8.6. *Let $0 < \alpha < 1$. If X_0 and X_1 have the Fatou property, then $X = X_0^{1-\alpha}X_1^{\alpha}$ has also the Fatou property.*

Proof. Let $0 \le f_n \uparrow f$ $C-$q.e. and $L := \lim_n \varrho_\alpha(f_n)$, so that $\varrho_\alpha(f) \ge L$ since $\varrho_\alpha(f) \ge \varrho_\alpha(f_n)$ for every n. If $L = \infty$, obviously $\varrho_\alpha(f) = \infty = L$.

Suppose then that $L < \infty$ and choose $\varepsilon > 0$. We can find

$$f_n \le \lambda_n f_{0,n}^{1-\alpha} f_{1,n}^{\alpha}, \quad \lambda_n \le L + \varepsilon, \quad \|f_{i,n}\|_{X_i} \le 1, \quad f_{i,n} \ge 0, \quad (i = 0, 1)$$

and then

$$f = \lim_n f_n \le \liminf_n \lambda_n f_{0,n}^{1-\alpha} f_{1,n}^{\alpha},$$

where $\|\liminf_n f_{i,n}\|_{X_i} \le \liminf_n \|f_{i,n}\|_{X_i} \le 1$ by Theorem 1.3.5. So that $f_i := \liminf_n f_{i,n} \in X_i$ and $\|f_i\|_{X_i} \le 1$. Moreover, $\lambda := \liminf_n \lambda_n \le L + \varepsilon$ and

$$\varrho_\alpha(f) \le \lambda.$$

Thus, $\varrho_\alpha(f) \le L + \varepsilon$ for every $\varepsilon > 0$ and hence $\varrho_\alpha(f) \le L$. □

2.8.1 Operators between Calderón products

Let (X_0, X_1) and (Y_0, Y_1) be two couples of quasi-Banach capacitary function spaces and $\theta \in (0,1)$. Denote by X_θ and Y_θ the spaces $X_0^{1-\theta} X_1^\theta$ and $Y_0^{1-\theta} Y_1^\theta$, respectively.

Definition 2.8.7. *Let X and Y be quasi-Banach capacitary function spaces. An operator $T : X \to Y$ is called* sublinear *if*

$$T(\alpha x) = \alpha T(x) \quad (\forall \alpha > 0),\ T(x+y) \le T(x) + T(y) \quad (\forall\, x, y \in X).$$

We say that T is bounded *from X to Y if there exists $C > 0$ such that*

$$\|T(x)\|_Y \le C\|x\|_X \quad (\forall\, x \in X).$$

As in the usual case, positive sublinear operators interpolate for capacitary Calderón products:

Proposition 2.8.8. *If T is a positive sublinear operator which acts boundedly from X_0 to Y_0, and from X_1 to Y_1, then it acts boundedly from X_θ to Y_θ as well.*

Proof. If $x \ge 0$ and $x \in X_\theta$, then there exist $\lambda > 0$ and $x_i \in X_i$ with $\|x_i\|_{X_i} \le 1$ $(i = 0, 1)$ such that

$$x(t) \le \lambda |x_0(t)|^{1-\theta} |x_1(t)|^\theta \text{ for all } t.$$

Equivalently, there exist $\lambda > 0$, $x_i \in X_i$, $\|x_i\|_{X_i} \le 1$ such that for all $\epsilon > 0$

$$x(t) \le \lambda\{(1-\theta)\epsilon^{-\theta}|x_0(t)| + \theta\epsilon^{1-\theta}|x_1(t)|\}.$$

Therefore, since T is positive and sublinear,

$$\begin{aligned} T(x) &\le |T(\lambda\{(1-\theta)\epsilon^{-\theta}|x_0| + \theta\epsilon^{1-\theta}|x_1|\})| \\ &\lesssim \epsilon^{-\theta}\lambda(1-\theta)T(|x_0|) + \epsilon^{1-\theta}\lambda\theta T(|x_1|). \end{aligned}$$

By defining

$$G(\epsilon) := \epsilon^{-\theta}\lambda(1-\theta)T(|x_0|) + \epsilon^{1-\theta}\lambda\theta T(|x_1|)$$

and minimizing this function of ϵ we get

$$T(x) \leq \inf_{\epsilon>0} G(\epsilon) = G\Big(\frac{T(|x_0|)}{T(|x_1|)}\Big) = \lambda T(|x_0|)^{1-\theta} T(|x_1|)^{\theta},$$

where $T(x_i) \in Y_i$ is such that there exists $C_i \in (0,\infty)$ a constant such that $\|T(x_i)\|_{Y_i} \leq C_i$. Hence, $T(x) \in Y_\theta$ and $\|T(x)\|_{Y_\theta} \lesssim C_0^{1-\theta} C_1^{\theta} \|x\|_{X_\theta}$. □

There are concrete situations where it is not necessary that $T \geq 0$ (see [Se]).

2.8.2 Rearrangement and Calderón products

Let (Ω, Σ, C) be a σ-finite capacity space and f a measurable function on Σ, integrable on sets of finite capacity. We associate to f the function f^{**} defined in (1.13).

Definition 2.8.9. *We say that two functions g and $\widetilde{g}$ are* equicapacitable *on Ω if for all $\lambda > 0$*

$$C\{x \in \Omega; |g(x)| > \lambda\} = C\{x \in \Omega; |\widetilde{g}(x)| > \lambda\}.$$

Definition 2.8.10. *Let X be a quasi-Banach lattice. We define X^* as the set*

$$X^* := \{f \in L_0(C); f^{**} \in X\}$$

with $\|f\|_{X^*} = \|f^{**}\|_X$. Then X^* is a vector space that satisfies properties (a), (b), (c) and (d) of Definition 1.3.1.

Indeed, for $f, g \in L_0(C)$ such that $f^{**}, g^{**} \in X$, by the properties of X,

$$\|f+g\|_{X^*} = \|(f+g)^{**}\|_X \leq \|f^{**} + g^{**}\|_X \lesssim \|f^{**}\|_X + \|g^{**}\|_X = \|f\|_{X^*} + \|g\|_{X^*}.$$

Moreover, for $\{f_n\}_{n\in\mathbb{N}} \subset X^*$ such that $f_n \uparrow f$ $C-$q.e., we have $(f_n)^*_C \uparrow f^*_C$. Hence, $(f_n)^{**} \uparrow f^{**}$ and $\|f_n\|_{X^*} \uparrow \|f\|_{X^*}$. Property (d) follows by the same property for X.

Let X_0 and X_1 be Banach lattices on Ω and $0 < \alpha < 1$. Then, $X_0^{1-\alpha} X_1^{\alpha}$ is a Banach lattice.

Let us study the relation between $(X_0^*)^{1-\alpha}(X_1^*)^{\alpha}$ and $X^* = (X_0^{1-\alpha} X_1^{\alpha})^*$ for $0 < \alpha < 1$. Let us see that still in this capacitary setting, this can be partially analized.

Let $f \in (X_0^*)^{1-\alpha}(X_1^*)^{\alpha}$. Then, there exist $\lambda > 0$, $g \in X_0^*$ and $h \in X_1^*$ with norm less than one such that $|f(t)| \leq \lambda |g(t)|^{1-\alpha} |h(t)|^{\alpha}$. Hence, by (1.14),

$$(|f(t)|)^{**} \leq |\lambda| (|g(t)|^{1-\alpha} |h(t)|^{\alpha})^{**} \leq 2c|\lambda| (|g(t)|^{**})^{1-\alpha} (|h(t)|^{**})^{\alpha}.$$

It follows that $\|f\|_{X^*} \lesssim |\lambda|$ and $(X_0^*)^{1-\alpha}(X_1^*)^{\alpha} \hookrightarrow (X_0^{1-\alpha} X_1^{\alpha})^*$.

The proof of $(X_0^{1-\alpha}X_1^{\alpha})^* \hookrightarrow (X_0^*)^{1-\alpha}(X_1^*)^{\alpha}$ under some additional conditions is much more complicated but still can be done it. The function f_C^* is related to $C_f(t)$ as follows:

$$C_f[f_C^*(t)] \geq t,\ f_C^*[C_f(t)] \geq t, \tag{2.23}$$

and hence

$$f_C^*\{C_f[|f(x)|]\} \geq |f(x)|. \tag{2.24}$$

Consider now the *Hardy operators* P and Q defined as

$$(Pf)(t) := \frac{1}{t}\int_0^t f(s)ds,\ (Qf)(t) := \int_t^\infty \frac{f(s)}{s}ds. \tag{2.25}$$

By definition $f^{**} = Pf_C^*$.

If $g \geq 0$, then we have for $t > 0$

$$\begin{aligned} Q(Pg)(t) &= \int_0^t g(u)\left\{\int_t^\infty \frac{ds}{s^2}\right\}du + \int_t^\infty \int_t^s \frac{g(u)}{s^2}duds \\ &= \int_0^t \frac{g(u)}{t}du + \int_t^\infty \int_u^\infty \frac{g(u)}{s^2}dsdu = (Pg)(t) + (Qg)(t). \end{aligned}$$

On the other hand, if g_1 and g_2 are non-negative functions, by Hölder's inequality it follows that

$$\begin{aligned} Q(g_1^{1-\alpha}g_2^{\alpha}) &= \int_t^\infty g_1(v)^{1-\alpha}g_2(v)^{\alpha}\frac{dv}{v} \\ &\leq 2c\left[\int_t^\infty g_1(v)\frac{dv}{v}\right]^{1-\alpha}\left[\int_t^\infty g_2(v)\frac{dv}{v}\right]^{\alpha} = 2c(Qg_1)^{1-\alpha}(Qg_2)^{\alpha}. \end{aligned}$$

Now we are ready to show that the following condition implies the desired result, where the condition is: The function f in X must have finite norm and the operators Pf and Qf should be bounded in X_0 and X_1.

Proposition 2.8.11. *If the function f in X has finite norm and the operators Pf and Qf are bounded in X_0 and X_1, then $(X_0^{1-\alpha}X_1^{\alpha})^* \hookrightarrow (X_0^*)^{1-\alpha}(X_1^*)^{\alpha}$.*

Indeed, let c be a bound for the norms of the operators P and Q in X_0 and X_1. Suppose that $f \in (X_0^{1-\alpha}X_1^{\alpha})^*$ and let us denote by $\|f\| = \|f\|_{(X_0^{1-\alpha}X_1^{\alpha})^*}$. Then, if $\lambda > \|f\|$, there exist two functions $g_1 \geq 0$ and $g_2 \geq 0$ in X_0 and X_1, respectively, such that $\|g_1\|_{X_0} \leq 1$, $\|g_2\|_{X_1} \leq 1$ and $f^{**}(t) \leq \lambda g_1(t)^{1-\alpha}g_2(t)^{\alpha}$ for all t. Let

$$h_1 = \frac{1}{c^2}Qg_1,\ h_2 = \frac{1}{c^2}Qg_2,\ h_i(0) = \infty,\ h_i(+\infty) = \lim_{t\to\infty} h_i(t)\,(i = 1, 2).$$

Then, we obtain that $Qf^{**} \leq \lambda Q(g_1^{1-\alpha} g_2^{\alpha}) \leq 2c\lambda (Qg_1)^{1-\alpha}(Qg_2)^{\alpha} = 2c^3\lambda h_1^{1-\alpha} h_2^{\alpha}$. On the other hand, since $f^{**} = Pf_C^*$, whence we find that

$$Qf^{**} = QPf_C^* = Pf_C^* + Qf_C^* \geq f_C^*$$

which combined with the preceding inequality gives

$$f_C^* \leq 2c^3\lambda h_1^{1-\alpha} h_2^{\alpha}.$$

Define now $f_1(x) := h_1\{C_f(|f(x)|)\}$ and $f_2(x) := h_2\{C_f(|f(x)|)\}$. Since $|f(x)|$ and $f_C^*(t)$ are equicapacitable, $f_i(x) = h_i\{C_f(|f(x)|)\}$ is equicapacitable $(i = 1, 2)$ with $h_i\{C_f(f_C^*(t))\}$, which is a non-increasing function since h_i is non-increasing. Consequently $(f_i)_C^* = h_i[C_f(f_C^*)]$, except perhaps at the points of discontinuity of $(f_i)_C^*$. Now the first inequality in (2.23) and the non-increasing character of h_i imply that $h_i\{C_f[f_C^*(t)]\} \leq h_i(t)$, and then $(f_i)_C^*(t) \leq h_i(t)$, except perhaps at the points of discontinuity of $(f_i)_C^*$. Hence we obtain

$$f_i^{**} = P(f_i)_C^* \leq Ph_i = \frac{1}{c^2} PQg_i.$$

Since the operators P and Q are bounded in X_0 and X_1, and their norms do not exceed c, $f_i^{**} \in X_{i-1}$ and $\|f_i^{**}\|_{X_{i-1}} \leq 1$ which implies that $f_i \in X_{i-1}^*$, $i = 1, 2$. Now from the inequality (2.24) it follows that

$$\begin{aligned} |f(x)| &\leq f_C^*\{C_f[|f(x)|]\} \\ &\leq 2c^3\lambda h_1\{C_f[|f(x)|]\}^{1-\alpha} h_2\{C_f[|f(x)|]\}^{\alpha} \\ &= 2c^3\lambda f_1(x)^{1-\alpha} f_2(x)^{\alpha}. \end{aligned}$$

Since $f_i \in X_{i-1}^*$ and $\|f_i\|_{X_{i-1}^*} \leq 1$ for $i = 1, 2$, it follows that $f \in (X_0^*)^{1-\alpha}(X_1^*)^{\alpha}$. Its norm as an element of this space does not exceed $2c^3\lambda = 2c^3(\|f\| + \epsilon)$, and since ϵ is arbitrary the desired conclusion follows,

$$(X_0^*)^{1-\alpha}(X_1^*)^{\alpha} \hookrightarrow (X_0^{1-\alpha} X_1^{\alpha})^* \hookrightarrow (X_0^*)^{1-\alpha}(X_1^*)^{\alpha}.$$

Moreover, if $X_0 = L^{p_0}(C)$, $X_1 = L^{p_1}(C)$ and $\frac{1}{p} = \frac{1-\alpha}{p_0} + \frac{\alpha}{p_1}$, then $X_0^{1-\alpha} X_1^{\alpha} = L^p(C)$. By definition $(L^p(C))^* = \{f \in L^0(C); f^{**} \in L^p(C)\}$. If $f \in (L^p(C))^*$, then, since $\|f\|_{(L^p(C))^*} < \infty$, there exist $g \in L^{p_0}(C)$ and $h \in L^{p_1}(C)$ with norms less than one such that

$$|f^{**}(t)| \leq \|f\|_{(L^p(C))^*} |g(t)|^{1-\alpha} |h(t)|^{\alpha}$$

and hence

$$L^p(C)^* \hookrightarrow ((L^{p_0}(C))^*)^{1-\alpha}((L^{p_1}(C))^*)^{\alpha} \hookrightarrow (L^{p_0}(C)^{1-\alpha} L^{p_1}(C)^{\alpha})^*,$$

where, by Theorem 2.8.4, we have that

$$L^{p_0}(C)^{1-\alpha} L^{p_1}(C)^{\alpha} = L^p(C).$$

2.9 Capacitary Orlicz spaces

From now on, let C be a quasi-subadditive Fatou capacity on (Ω, Σ) with quasi-subadditivity constant c, and $\varphi : [0, \infty) \to [0, \infty]$ an *unbounded increasing function* such that $\varphi(0) = 0$, which is neither identically zero nor identically infinite on $(0, \infty)$.

We define the *Orlicz class* $P_C(\varphi)$ to be the set of all functions $f \in \mathrm{L}_0(\Omega)$ for which

$$M^\varphi(f) := \rho_\varphi(f) = \int_\Omega \varphi(|f|)dC < \infty,$$

and

$$L^\varphi(C) := \{f \in \mathrm{L}_0(\Omega);\ \|f\|_\varphi < \infty\},$$

where

$$\|f\|_\varphi := \inf\{\lambda > 0; M^\varphi(\lambda^{-1}f) \le 1\}.$$

The space $L^\varphi(C)$ is called a *capacitary Orlicz space.*

Definition 2.9.1. *A function H on $[0, \infty)$ (or on a linear space) is called* quasi-convex *with constant $\beta \ge 1$, if*

$$H(\lambda x + (1-\lambda)y) \le \beta\{\lambda H(x) + (1-\lambda)H(y)\} \ \textit{for}\ 0 \le \lambda \le 1 \textit{ and } x, y > 0.$$

Let us observe that, as in the usual case, by the quasi-subadditivity of the Choquet integral (see Proposition 1.2.2), if φ is quasi-convex, then M^φ is also a quasi-convex function.

We say that φ satisfies the Δ_2-*condition* if there exist $s_0 > 0$ and $c > 0$ such that

$$\varphi(2s) \le c\varphi(s) < \infty \quad (s_0 \le s < \infty). \tag{2.26}$$

Let C be a finite capacity and φ a quasi-convex function with the Δ_2-condition. Then $P_C(\varphi)$ is a linear subspace of $\mathrm{L}_0(\Omega)$. Indeed, let $f \in P_C(\varphi)$, define $E := \{|f| \ge s_0\}$ and $F := \Omega \setminus E$. Then $2f \in P_C(\varphi)$ since

$$\begin{aligned}
\int_\Omega \varphi(|2f|)dC &\le \int_E \varphi(2|f|)dC + \int_F \varphi(2|f|)dC \\
&\le \int_E c\varphi(|f|)dC + \int_F \varphi(2|f|)dC \\
&\le \int_E c\varphi(|f|)dC + \int_F \varphi(2s_0)dC < \infty.
\end{aligned}$$

Suppose now that $f \in P_C(\varphi)$ and let α be any scalar. Choosing $n \in \mathbb{N}$ with $2^n \ge |\alpha|$, we see that $2^n f \in P_C(\varphi)$. Hence, since φ is increasing

$$M^\varphi(\alpha f) \le \int_\Omega \varphi(2^n|f|)dC < \infty$$

and so that $\alpha f \in P_C(\varphi)$.

On the other hand, if $f, g \in P_C(\varphi)$, then since $2f, 2g \in P_C(\varphi)$

$$M^\varphi(f+g) = M^\varphi\Big(\frac{1}{2}(2f) + \frac{1}{2}(2g)\Big) \leq 2c\beta\Big\{\frac{1}{2}M^\varphi(2f) + \frac{1}{2}M^\varphi(2g)\Big\}$$

and $f + g \in P_C(\varphi)$.

Proposition 2.9.2. *Let φ be an increasing function. Then*

$$f = 0 \quad C-\text{q.e.} \;\Leftrightarrow M^\varphi(kf) \leq 1 \;\forall k > 0.$$

Proof. If $f = 0$ $C-$q.e., then $M^\varphi(kf) = 0$ for all $k > 0$. Conversely, suppose that $M^\varphi(kf) \leq 1$ for all $k > 0$, but for some $\epsilon > 0$ we have $|f| \geq \epsilon$ on $E \subset \Omega$ with $C(E) > 0$. Then

$$M^\varphi(kf) = \int_\Omega \varphi(k|f|)dC \geq \int_E \varphi(\epsilon k)dC = C(E)\varphi(\epsilon k).$$

Since $\varphi(s) \uparrow \infty$ as $s \uparrow \infty$, we obtain a contradiction. □

Proposition 2.9.3. *Let φ be a convex function and C a concave capacity. Then, M^φ is a convex function and $\|\cdot\|_\varphi$ is a norm on $P_C(\varphi)$.*

Proof. Obviously, M^φ is a convex function.

(i) If $f = 0$ $C-$q.e., then $M^\varphi(kf) = 0$ for all $k > 0$ and hence $\|f\|_\varphi = 0$. Conversely, if $\|f\|_\varphi = 0$, then we have that for all $k > 0$, $M^\varphi(kf) \leq 1$ and then, by Proposition 2.9.2, $f = 0$ $C-$q.e.

(ii) Trivially, $\|\beta f\|_\varphi = |\beta|\|f\|_\varphi$ for any $\beta \in \mathbb{R}$.

(iii) Take now $f, g \in P_C(\varphi)$, $\gamma := \|f\|_\varphi + \|g\|_\varphi < \infty$ and define $\alpha := \frac{\|f\|_\varphi}{\gamma}$ and $\beta := \frac{\|g\|_\varphi}{\gamma}$. Then $M^\varphi(\frac{f}{\|f\|_\varphi})$, $M^\varphi(\frac{g}{\|g\|_\varphi}) \leq 1$ and by convexity

$$M^\varphi\Big(\frac{f+g}{\gamma}\Big) = M^\varphi\Big(\alpha\frac{f}{\|f\|_\varphi} + \beta\frac{g}{\|g\|_\varphi}\Big) \leq \alpha + \beta = 1,$$

so that $\|f+g\|_\varphi \leq \gamma = \|f\|_\varphi + \|g\|_\varphi$. □

Briefly let us observe that $L^p(C)$ can be seen as an Orlicz space. Certainly, if we take $\varphi(t) = t^p$, then

$$\|f\|_\varphi := \inf\Big\{\lambda > 0; \frac{1}{\lambda^p}\int_\Omega |f(x)|^p dC \leq 1\Big\}$$

and $L^\varphi(C) = L^p(C)$ with $\|f\|_{L^\varphi(C)} = \|f\|_{L^p(C)}$, for any $p \in (0, \infty)$.

We know that $L^p(C)$ is complete also when $0 < p < 1$. In this case, φ is a *p-convex function* in the following sense:

Definition 2.9.4. *Let $0 < s \leq 1$. The function $\varphi : [0, \infty) \to [0, \infty)$ is called* s-convex *(resp.* (s)-convex*) if*

$$\varphi(\alpha t_1 + \beta t_2) \leq \alpha^s \varphi(t_1) + \beta^s \varphi(t_2) \text{ for each } t_1, t_2 \in [0, \infty)$$

and all $\alpha, \beta \geq 0$ such that $\alpha^s + \beta^s = 1$ (such that $\alpha + \beta = 1$).

Observe that every convex function is 1-convex. Every (s)-convex function is s-convex, but let us see below that the converse does not hold.

Example 2.9.5. *Let $0 < p < 1$ and $\varphi(t) := t^p$. Then φ is p-convex. Indeed, if $\alpha, \beta \geq 0$ are such that $\alpha^p + \beta^p = 1$ and $t_1, t_2 \in [0, \infty)$, then*

$$\varphi(\alpha t_1 + \beta t_2) \leq (\alpha t_1)^p + (\beta t_2)^p = \alpha^p \varphi(t_1) + \beta^p \varphi(t_2),$$

but φ is not (p)-convex.

Let φ be any s-convex function, $0 < s \leq 1$ and define

$$L_\varphi(C) := \big\{ f;\ \lim_{\lambda \to 0^+} \rho_\varphi(\lambda f) = 0 \big\}.$$

Trivially, $L_\varphi(C) \subset L^\varphi(C)$.

Modular spaces were first defined by H. Nakano in 1950 (see [Nak]) on vector lattices. Independly, another version was introduced by J. Musielak and W. Orlicz around 1959 (see [Mu] and [MuO]).

Let X be a real vector space on $\mathrm{L}_0(\Omega)$. A functional $\rho : X \to [0, \infty]$ is called a *modular* if it satisfies the following conditions:

(a) $\rho(x) = 0 \Longleftrightarrow x = 0$,

(b) $\rho(-x) = \rho(x)$ for each $x \in X$, and

(c) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for $x, y \in X, \alpha, \beta \geq 0$ such that $\alpha + \beta = 1$.

A functional $\rho : X \to [0, \infty]$ is termed a *pseudo-modular* if it satisfies the weak condition (a'), that is, $\rho(0) = 0$ and, (b) and (c). The pseudo-modular ρ is said *s-convex*, $0 < s \leq 1$, if

$$\rho(\alpha x + \beta y) \leq \alpha^s \rho(x) + \beta^s \rho(y) \text{ for } x, y \in X \text{ and } \alpha, \beta \geq 0 \text{ such that } \alpha^s + \beta^s = 1.$$

From now on in this section, s will denote a positive real number, $0 < s \leq 1$.

Proposition 2.9.6. *Let φ be an s-convex function and C a concave capacity. Then, ρ_φ is an s-convex pseudo-modular on* $\mathrm{L}_0(\Omega)$.

Proof. It follows observing that φ is increasing, s-convex, and C is concave. □

Theorem 2.9.7. *If ρ is an s-convex pseudo-modular in $L_\varphi(C)$, then $L_\varphi(C) = L^\varphi(C)$ and an s-norm can be defined on $L_\varphi(C)$ as follows*

$$\|f\|_{\varphi,s} := \inf\Big\{\lambda > 0; \rho_\varphi\Big(\frac{f}{\lambda^{1/s}}\Big) \leq 1\Big\}.$$

Proof. If $f \in L^\varphi(C)$, then $\rho_\varphi(\lambda_0 f) < \infty$ for some $\lambda_0 > 0$ and, if $0 < \lambda < \lambda_0$, then

$$\rho_\varphi(\lambda f) = \rho_\varphi\Big(\frac{\lambda}{\lambda_0}\lambda_0 f\Big) = \rho_\varphi\Big(\frac{\lambda}{\lambda_0}(\lambda_0 f) + (1 - \frac{\lambda}{\lambda_0})0\Big) \leq \Big(\frac{\lambda}{\lambda_0}\Big)^s \rho_\varphi(\lambda_0 f) \to 0$$

as $\lambda \to 0$, so that $f \in L_\varphi(C)$.

Now, let us show that $\|\cdot\|_{\varphi,s}$ satisfies the properties of a norm. That, $\|f\|_{\varphi,s} = 0$ if and only if $f = 0$ $C-$q.e. follows with a direct proof. The same holds with the identity

$$\|\lambda f\|_{\varphi,s} = |\lambda|^s \|f\|_{\varphi,s} \text{ for all } \lambda \in \mathbb{R}.$$

Finally, let $f, g \in \mathrm{L}_0(\Omega)$ and $u, v > 0$ such that $\|f\|_{\varphi,s} < u$, $\|g\|_{\varphi,s} < v$. Then

$$\begin{aligned}\rho_\varphi\Big(\frac{f+g}{(u+v)^{1/s}}\Big) &= \rho_\varphi\Big(\frac{u^{1/s}}{(u+v)^{1/s}}\frac{f}{u^{1/s}} + \frac{v^{1/s}}{(u+v)^{1/s}}\frac{g}{v^{1/s}}\Big)\\ &\leq \frac{u}{u+v}\rho_\varphi\Big(\frac{f}{u^{1/s}}\Big) + \frac{v}{u+v}\rho_\varphi\Big(\frac{g}{v^{1/s}}\Big) \leq 1.\end{aligned}$$

Thus, $\|f+g\|_{\varphi,s} \leq \|f\|_{\varphi,s} + \|g\|_{\varphi,s}$. □

By Proposition 2.9.6 and Theorem 2.9.7 it follows that, if φ is s-convex and C is concave, then $L_\varphi(C) = L^\varphi(C)$ and $\|\cdot\|_{\varphi,s}$ is an s-norm. In this case, $L_\varphi(C)$ is called a *modular capacitary Orlicz space* or a *capacitary $s-$convex space.*

Remark 2.9.8.

$$\begin{aligned}\|f\|_{\varphi,s} &= \inf\Big\{(u^{1/s})^s > 0; \rho_\varphi\Big(\frac{f}{u^{1/s}}\Big) \leq 1\Big\}\\ &= \Big(\inf\Big\{u^{1/s} > 0; \rho_\varphi\Big(\frac{f}{u^{1/s}}\Big) \leq 1\Big\}\Big)^s\\ &= \Big(\inf\Big\{\lambda > 0; \rho_\varphi\Big(\frac{f}{\lambda}\Big) \leq 1\Big\}\Big)^s = \|f\|_\varphi^s.\end{aligned}$$

By Theorem 2.9.7 and Remark 2.9.8, if ρ is an s-convex pseudo-modular in $L_\varphi(C)$, then $\|\cdot\|_\varphi$ is a quasi-norm on $L^\varphi(C)$. Indeed, let $f, g \in \mathrm{L}_0(\Omega)$. Since $0 < s \leq 1$, it follows that

$$\|f+g\|_\varphi = (\|f+g\|_{\varphi,s})^{1/s} \leq 2^{1/s}\Big(\|f\|_{\varphi,s}^{1/s} + \|g\|_{\varphi,s}^{1/s}\Big) = 2^{1/s}(\|f\|_\varphi + \|g\|_\varphi).$$

Proposition 2.9.9. *Let φ be an s-convex function. Then $\|\cdot\|_\varphi$ is a quasi-norm on $L^\varphi(C)$.*

Proof. Observe that, since φ is s-convex, if $0 < a < 1$,

$$\varphi(a^{1/s}t) = \varphi(a^{1/s}t + (1-a)^{1/s}0) \leq a\varphi(t)$$

and hence, for $0 < \lambda < 1$, $\varphi(\lambda t) \leq \lambda^s\varphi(t)$. The proof of the first two properties of a quasi-norm follow trivially.

Let $f, g \in L^\varphi(C)$ and take $u^{1/s} > \|(2c)^{1/s}f\|_\varphi$ and $v^{1/s} > \|(2c)^{1/s}g\|_\varphi$. Then, by convexity and the quasi-subadditivity (see Proposition 1.2.2), defining $\theta := \frac{u}{u+v}$, it follows that

$$\begin{aligned}
M^\varphi\Big(\frac{f+g}{(u+v)^{1/s}}\Big) &\leq \int_\Omega \varphi\Big(\frac{u^{1/s}}{(u+v)^{1/s}}\frac{|f|}{u^{1/s}} + \frac{v^{1/s}}{(u+v)^{1/s}}\frac{|g|}{v^{1/s}}\Big)dC \\
&\leq \int_\Omega \Big(\theta\varphi\Big(\frac{|f|}{u^{1/s}}\Big) + (1-\theta)\varphi\Big(\frac{|g|}{v^{1/s}}\Big)\Big)dC \\
&\leq \int_\Omega \Big(\frac{\theta}{2c}\varphi\Big(\frac{(2c)^{1/s}|f|}{u^{1/s}}\Big) + \frac{1-\theta}{2c}\varphi\Big(\frac{(2c)^{1/s}|g|}{v^{1/s}}\Big)\Big)dC \\
&\leq \theta M^\varphi\Big(\frac{(2c)^{1/s}f}{u^{1/s}}\Big) + (1-\theta)M^\varphi\Big(\frac{(2c)^{1/s}g}{v^{1/s}}\Big) \leq 1.
\end{aligned}$$

So that $\|f+g\|_\varphi \leq (u+v)^{1/s} \leq 2^{1/s}(u^{1/s} + v^{1/s})$ and then,

$$\|f+g\|_\varphi \leq (4c)^{1/s}(\|f\|_\varphi + \|g\|_\varphi). \qquad \square$$

Theorem 2.9.10. *Let φ be an s-convex function. Then,*

(i) $\|f_k - f\|_{\varphi,s} \to 0$ *as* $k \to \infty$ *if and only if* $\rho_\varphi(\lambda(f_k - f)) \to 0$ *as* $k \to \infty$*, for any* $\lambda > 0$.

(ii) $\{f_k\}_k$ *is a Cauchy sequence in* $L^\varphi(C)$ *with respect to* $\|\cdot\|_{\varphi,s}$ *if and only if* $\rho_\varphi(\lambda(f_k - f_l)) \to 0$ *as* $k, l \to \infty$*, for all* $\lambda > 0$.

Proof. If $\rho_\varphi(\lambda f_k) \to 0$ as $k \to \infty$ for all $\lambda > 0$, then there exists $k_\lambda \in \mathbb{N}$ such that

$$\rho_\varphi\Big(\frac{f_k}{(\frac{1}{\lambda^s})^{1/s}}\Big) \leq 1 \text{ for each } k \geq k_\lambda \text{ and } \lambda > 0.$$

Hence, $\|f_k\|_{\varphi,s} \leq \frac{1}{\lambda^s}$ for all $k \geq k_\lambda$ and $\lambda > 0$, and so $\|f_k\|_{\varphi,s} \to 0$ as $k \to \infty$.

Conversely, if $\|f_k\|_{\varphi,s} \to 0$ as $k \to \infty$, then, given $\epsilon > 0$, there exists $k_{\lambda,\epsilon} \in \mathbb{N}$ such that $\rho_\varphi(\frac{\lambda f_k}{\epsilon^{1/s}}) \leq 1$ for all $k \geq k_{\lambda,\epsilon}$ and

$$\begin{aligned}\rho_\varphi(\lambda f_k) &= \int_\Omega \varphi\Big(\epsilon^{1/s}\Big(\frac{\lambda|f_k|}{\epsilon^{1/s}}\Big)\Big)dC \\ &\leq \int_\Omega \Big(\epsilon\varphi\Big(\frac{\lambda|f_k|}{\epsilon^{1/s}}\Big) + (1-\epsilon)\varphi(0)\Big)dC \\ &= \epsilon\rho_\varphi\Big(\frac{\lambda f_k}{\epsilon^{1/s}}\Big).\end{aligned}$$

Hence, for all $\epsilon > 0$, there exists $k_{\lambda,\epsilon} \in \mathbb{N}$ such that $\rho_\varphi(\lambda f_k) \leq \epsilon$ for all $k \geq k_{\lambda,\epsilon}$, and so $\rho_\varphi(\lambda f_k) \to 0$ as $k \to 0$ for any $\lambda > 0$.
The proof of (ii) is similar. □

Corollary 2.9.11. *If φ is an s-convex function, then for each sequence $\{f_k\}_{k\in\mathbb{N}} \subset \mathrm{L}_0(\Omega)$ and f be any measurable function, the following properties hold:*

(i) $\|f_k - f\|_\varphi \to 0$ as $k \to \infty$ if and only if $\rho_\varphi(\lambda(f_k - f)) \to 0$ as $k \to \infty$, for all $\lambda > 0$.

(ii) $\{f_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $L^\varphi(C)$ with respect to $\|\cdot\|_\varphi$ if and only if $\rho_\varphi(\lambda(f_k - f_l)) \to 0$ as $k, l \to \infty$, for all $\lambda > 0$.

Proof. It follows by Theorem 2.9.10 and Remark 2.9.8. □

Assume now that φ is an increasing convex function. Note that $(L^\varphi(C), \|\cdot\|_\varphi)$ is a quasi-normed space by Proposition 2.9.9. It is a normed space when C is concave by Proposition 2.9.3.

Theorem 2.9.12. *Let C be a concave capacity and φ an increasing convex function. Then $(L^\varphi(C), \|\cdot\|_\varphi)$ is a Banach function space.*

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $(L^\varphi(C), \|\cdot\|_\varphi)$ and $x_0 := \sup\{x \in \mathbb{R}; \varphi(x) = 0\}$. Then, $0 \leq x_0 < \infty$ since the set $\{x \in \mathbb{R}; \varphi(x) = 0\}$ is relatively compact in $\mathbb{R}$.

Moreover, by Corollary 2.9.11, there exists $k_{mn} \geq 0$ $(m, n \in \mathbb{N})$ such that

$$\int_\Omega \varphi(k_{mn}|f_n - f_m|)dC \leq 1.$$

First note that, $A_{mn} := \{\omega \in \Omega; k_{mn}|f_n(\omega) - f_m(\omega)| > x_0\} \in \Sigma$ is at most σ-finite. Indeed, if for $k \in \mathbb{N}$ we define $B_k := \{\omega \in \Omega; k_{mn}|f_n(\omega) - f_m(\omega)| > x_0 + k^{-1}\}$, then $A_{mn} = \bigcup_{k=1}^\infty B_k$ and $C(B_k) < \infty$ for all k, since

$$C(B_k)\varphi(x_0 + k^{-1}) = \int_{B_k} \varphi(x_0 + k^{-1})dC \leq \int_{B_k} \varphi(k_{mn}|f_n - f_m|)dC \leq 1.$$

Therefore, each A_{mn} is σ-finite and so is $A := \bigcup_{m,n\geq 1} A_{mn}$.

On $A^c, k_{mn}|f_n - f_m| \leq x_0$ so that, in A^c $|f_n(\omega) - f_m(\omega)| \to 0$ uniformly. Hence, there is a measurable function g_0 on A^c such that $f_n(\omega) \to g_0(\omega)$ and $|g_0| \leq x_0$, for all $\omega \in A^c$.

Let us write Ω for A temporarily. Then, for all $B \in \Sigma$ with $C(B) < \infty$ we have that

$$\begin{aligned} C(B \cap \{|f_n - f_m| \geq \epsilon|\}) &= C(B \cap \{\varphi(k_{mn}|f_n - f_m|) \geq \varphi(k_{mn}\epsilon)\}) \\ &\leq \frac{1}{\varphi(k_{mn}\epsilon)} \int_\Omega \varphi(k_{mn}|f_n - f_m|)dC \\ &\leq \frac{1}{\varphi(k_{mn}\epsilon)}. \end{aligned}$$

Since $k_{mn} \to \infty$ and $\epsilon > 0$ is fixed, $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in capacity on B. Then, by Theorem 1.3.11, this sequence has a subsequence which is pointwise convergent on B to some $\widetilde{f}$, and also on $\bigcup_k B_k$ since $C(B_k) < \infty$ for all $k \in \mathbb{N}$. Then, there exists a subsequence $\{f_{n_i}\}_{i\in\mathbb{N}}$ such that $f_{n_i} \to \widetilde{f}$ $C-$q.e.

Let $f := \widetilde{f}\chi_A + g_0\chi_{A^c}$. Hence, $f_{n_i} \to f$ $C-$q.e. But, since $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, it follows that $\|f_n\|_\varphi \to \rho$. By the Fatou property (see Theorem 1.2.17)

$$\int_\Omega \varphi\Big(\frac{|f|}{\rho}\Big)dC \leq \lim_{i\to\infty} \int_\Omega \varphi\Big(\frac{|f_{n_i}|}{\|f_{n_i}\|_\varphi}\Big)dC \leq 1.$$

Thus, $f \in L^\varphi(C)$.

By continuity, for $k \geq 0$ given,

$$\varphi(|f_{n_i} - f_{n_j}|k) \to \varphi(|f - f_{n_j}|k) \; C - \text{q.e. as } i \to \infty,$$

and if $n_0 \geq 1$ is choosen such that $n_i, n_j \geq n_0$ implies $k_{n_i n_j} \geq k$, then

$$\int_\Omega \varphi(k|f_{n_i} - f_{n_j}|)dC \leq \int_\Omega \varphi(k_{n_i n_j}|f_{n_i} - f_{n_j}|)dC \leq 1.$$

Hence, letting $n_i \to \infty$, $\|f - f_{n_j}\|_\varphi \leq k^{-1}$ and the result then follows. □

Theorem 2.9.13. *Let φ be an increasing convex function. The space $(L^\varphi(C), \|\cdot\|_\varphi)$ is a quasi-Banach function space on (Ω, Σ).*

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $(L^\varphi(C), \|\cdot\|_\varphi)$. By Corollary 2.9.11 for every $\lambda, \eta > 0$, there exists $N \in \mathbb{N}$ such that $M^\varphi(\lambda(f_n - f_m)) < \eta$ for all $n, m \geq N$.

Define for $\varepsilon > 0$, the sets $A_{nm} := \{x \in \Omega; \lambda|f_n(x) - f_m(x)| \geq \epsilon\}$ $(m, n \geq N)$. It follows that

$$C(A_{nm})\varphi(\epsilon) \leq \int_{A_{nm}} \varphi(\lambda|f_n(x) - f_m(x)|)dC < \eta \qquad (n, m \geq N).$$

Therefore, by Theorem 1.3.11, $\{\lambda f_n\}_{n\in\mathbb{N}}$ converges in capacity to some function λf and it has a subsequence $\{\lambda f_{n_k}\}_{k\in\mathbb{N}}$ which is convergent to λf $C-$q.e. Hence, from the continuity of φ,

$$\varphi(\lambda|f_n(x) - f_{n_k}(x)|) \to \varphi(\lambda|f_n(x) - f(x)|) \; C - \text{q.e. in } \Omega,$$

and by Fatou's property (see Theorem 1.2.17),

$$\begin{aligned} M^\varphi(\lambda(f_n - f)) &= \int_\Omega \lim_{k\to\infty} \varphi(\lambda|f_n(x) - f_{n_k}(x)|)dC \\ &\leq \liminf_{k\to\infty} M^\varphi(\lambda(f_n - f_{n_k})) < \eta \;\; (n \geq N). \end{aligned}$$

Thus, $\|f_n - f\|_\varphi \to 0$ as $n \to \infty$, and $f \in L^\varphi(C)$. □

Proposition 2.9.14. *Let φ be an s-convex function. Then, $f_n \to f$ in $\|\cdot\|_\varphi$ implies that there exists $\{f_{n_k}\}_{k=1}^\infty$ such that $f_{n_k} \to f$ $C-$q.e.*

Proof. By Corollary 2.9.11, $\rho_\varphi(\lambda(f_n - f)) \to 0$ as $n \to \infty$, for any $\lambda > 0$. Hence, for all $\lambda, \eta > 0$, there exists $N \in \mathbb{N}$ such that, $\rho_\varphi(\lambda(f_n - f)) < \eta$ for all $n \geq N$. Defining $A_n := \{x \in \Omega; \lambda|f_n(x) - f(x)| \geq \epsilon\}$ for $n \geq N$, it follows that

$$C(A_n)\varphi(\epsilon) \leq \int_{A_n} \varphi(\lambda|f_n(x) - f(x)|)dC < \eta \; (n \geq N).$$

So that, $\{\lambda f_n\}_{n\in\mathbb{N}}$ converges in capacity to λf and, by Theorem 1.3.11, it follows that $\{\lambda f_n\}_{n\in\mathbb{N}}$ has a subsequence $\{\lambda f_{n_k}\}_{k\in\mathbb{N}}$ convergent to λf $C-$q.e. □

Finally, let us analyze the completeness of the space $L^\varphi(C)$ with respect to the quasi-norm $\|\cdot\|_\varphi$ when φ is only s-convex. Let us see that, in general, for an s-convex function φ, we need to imposse the continuity of φ to obtain the completeness of the capacitary s-convex space.

Observe fist that not all s-convex function is continuous.

Example 2.9.15. *Let $0 < s < 1$ and $k > 1$. Define for $u \in \mathbb{R}_+$,*

$$f(u) = \left\{ u^{\frac{s}{(1-s)}} \; if \; 0 \leq u \leq 1, \; ku^{\frac{s}{(1-s)}} \; if \; u \geq 1 \right\}.$$

The function f is non-negative, discontinuous at $u = 1$, s-convex and it is not (s)-convex.

Theorem 2.9.16. *Let φ be a continuous s-convex function. Then $(L^\varphi(C), \|\cdot\|_\varphi)$ is complete.*

Proof. For all $\lambda, \eta > 0$, there exists $N \in \mathbb{N}$ such that

$$\rho_\varphi(\lambda(f_n - f_m)) < \eta \quad (m, n \geq N).$$

Thus, by defining $A_{n,m} := \{x \in \Omega;\ \lambda|f_n(x) - f_m(x)| \geq \epsilon\}$ for $\varepsilon > 0$, we have that

$$\begin{aligned} C(A_{n,m})\varphi(\epsilon) &\leq \int_{A_{n,m}} \varphi(\lambda|f_n(x) - f_m(x)|)dC \\ &\leq \rho_\varphi(\lambda(f_n - f_m)) < \eta \qquad (m, n \geq N). \end{aligned}$$

Hence, by Theorem 1.3.11, it follows that $\{\lambda f_n\}_{n\in\mathbb{N}}$ is convergent in capacity to a function λf and $\{\lambda f_n\}_{n\in\mathbb{N}}$ contains a subsequence $\{\lambda f_{n_k}\}_{k\in\mathbb{N}}$ which is convergent to λf $C-$q.e. in Ω. Hence, from the continuity,

$$\varphi(\lambda|f_n(x) - f_{n_k}(x)|) \to \varphi(\lambda|f_n(x) - f(x)|)\ C - \text{q.e. in } \Omega,$$

and by Fatou's property,

$$\rho_\varphi(\lambda(f_n - f)) \leq \liminf_{k\to\infty} \rho_\varphi(\lambda(f_n - f_{n_k})) < \eta \quad (n \geq N).$$

Thus $\|f_n - f\|_\varphi \to 0$ as $n \to \infty$, and $f \in L^\varphi(C)$. □

Example 2.9.17. *Let (Ω, Σ, μ) be a measure space, and $\psi(t) := t^{1-p}$ $(0 < p < 1)$ which is concave and continuous. Then $C_\psi(A) := \psi(\mu(A))$ defines a concave Fatou capacity (see [Ce]).*

Take now $\varphi(t) := t^p$ a continuous p-convex function. Then, the space $L^\varphi(C)$ defined by the condition $\|f\|_\varphi < \infty$ is quasi-Banach. We called it a capacitary p-convex space.

Example 2.9.18. *Let $0 < p < 1$ and C be some concave Fatou capacity. The space*

$$L^p(C) := \Big\{f \in \mathrm{L}_0(\Omega);\ \Big(\int_0^\infty pt^{p-1}C\{|f| > t\}dt\Big)^{\frac{1}{p}} < \infty\Big\}$$

is also a quasi-Banach space endowed with the quasi-norm

$$\|f\| := \inf\Big\{\lambda > 0;\ \int_\Omega \Big(\frac{|f(x)|}{\lambda}\Big)^p dC \leq 1\Big\},$$

which coincides with $\|f\|_{L^p(C)} := \Big(\int_0^\infty pt^{p-1}C\{|f| > t\}dt\Big)^{\frac{1}{p}}$.

2.9.1 Interpolation of capacitary s-convex spaces

Definition 2.9.19. *Let φ be a positive function on $\mathbb{R}_+$ such that, for every $\lambda \in \mathbb{R}_+$ with some constant $\bar{C} = C(\lambda)$, it holds*

$$\varphi(\lambda x) \leq \bar{C}\varphi(x).$$

Therefore, for certain p_0, p_1 we have that

$$\varphi(\lambda x) \leq \bar{C}\max(\lambda^{p_0}, \lambda^{p_1})\varphi(x).$$

In this case we say that φ is of lower type p_0 *and* upper type p_1.

Assume further that φ is continuous increasing with $\varphi(\mathbb{R}_+) = \mathbb{R}_+$ so that, φ^{-1} exists and is continuous increasing too. Then, in [GuP] it is proved that, if φ is of type (p_0, p_1) where $p_0 > 0$, then φ^{-1} is of type (p_1^{-1}, p_0^{-1}).

We say that a positive function ρ on $\mathbb{R}_+$ is *quasi-concave* if it is equivalent to a concave function. In [P] we find that ρ is pseudo-concave if and only if ρ is of lower type 0 and upper type one. In other words, we have with a suitable $\bar{C}$

$$\rho(\lambda x) \leq \bar{C}\max(1, \lambda)\rho(x). \tag{2.27}$$

The class of functions satisfying (2.27) will be denoted by $\mathfrak{B}(C)$.

Remark 2.9.20. *To develop the theory it is convenient to introduce the homogeneous function*

$$R(x, y) = x\rho(y/x).$$

Then ρ is in $\mathfrak{B}(1)$ if and only if R is non-decreasing in each variable separately (that is, $x \leq x'$ implies $R(x, y) \leq R(x', y)$ and $y \leq y'$ implies $R(x, y) \leq R(x, y')$). In fact, it fulfils always in the strong sense

$$x < x', y < y' \Rightarrow R(x, y) < R\big(x', y'\big).$$

Given $\rho \in \mathfrak{B}(1)$, for any two finite positive sequences $\{x_\eta\}_\eta$ and $\{y_\eta\}_\eta$, it follows that

$$\sum R(x_\eta, y_\eta) \leq 2R\big(\sum x_\eta, \sum y_\eta\big),$$

and for any two positive sequences $\{x_\eta\}_{\eta=1}^\infty$ and $\{y_\eta\}_{\eta=1}^\infty$,

$$\sum_{\eta=1}^\infty R(x_\eta, y_\eta) \leq 2R\Big(\sum_{\eta=1}^\infty x_\eta, \sum_{\eta=1}^\infty y_\eta\Big)$$

(see [P]).

Let us show that every s-convex function is of positive lower type. Indeed, if φ is s-convex, then for all $\alpha > 0$, taking $\beta = (1-\alpha^s)^{1/s}$ and $y = 0$, it follows that

$$\varphi(\alpha x) = \varphi(\alpha x + \beta 0) \leq \alpha^s \varphi(x) + \beta^s \varphi(0) = \alpha^s \varphi(x).$$

Then, the conclusion follows.

Definition 2.9.21. *A function $\rho : X \to [0,\infty]$ is called a* quasi-modular *if it satisfies the following properties:*

(a) $\rho(x) = 0 \Longleftrightarrow x = 0$,

(b) $\rho(\lambda x) \leq \rho(x)$ *if* $|\lambda| \leq 1, \rho(-x) = \rho(x)$,

(c) $\lim_{\lambda\to 0} \rho(\lambda x) = 0$ *if* $\rho(x) < \infty$,

(d) $\rho((x+y)/h) \leq k(\rho(x) + \rho(y))$ *for certain constants h and k.*

From now on, let φ be a continuous function on $\mathbb{R}_+$, $\varphi(0) = 0$ and such that $\varphi|_{(0,\infty)}$ is increasing. If φ has positive lower type, then $\rho_\varphi(f) := \int_\Omega \varphi(|f|)dC$ is a quasi-modular. Moreover the space $L^\varphi(C)$ is locally bounded.

Proposition 2.9.22. *Let φ be a continuous function with positive lower type. Then $(L^\varphi(C), \|\cdot\|_\varphi)$ is a quasi-Banach function space.*

Proof. It follows with similar techniques to the ones in Theorem 2.9.16. □

Proposition 2.9.23. *Let φ, φ_0 and φ_1 be continuous increasing functions on $\mathbb{R}_+$, where φ is of positive lower type and it can be expressed by $\varphi^{-1} = \varphi_0^{-1}\rho\left(\frac{\varphi_1^{-1}}{\varphi_0^{-1}}\right)$ with ρ quasi-concave. Assume that*

$$\int_\Omega \varphi_i(|a_i|)dC \leq C_i,\ i = 0,1, \quad |a| \leq |a_0|\rho\left(\frac{|a_1|}{|a_0|}\right).$$

Then

$$\int_\Omega \varphi(|a|)dC \leq 2c(C_0 + C_1)$$

holds, where c is the subadditivity constant associated with the capacity.

Proof. Following [GuP], put $b_i := \varphi_i(|a_i|), i = 0,1$, and $b = b_0 + b_1$. We have that $\varphi_0^{-1}, \varphi_1^{-1}$ are increasing, $b_0 \leq b$ and $b_1 \leq b$. So that $\varphi_0^{-1}(b_0) \leq \varphi_0^{-1}(b)$, $\varphi_1^{-1}(b_1) \leq \varphi_1^{-1}(b)$ and by Remark 2.9.20,

$$|a| \leq R(|a_0|, |a_1|) = R(\varphi_0^{-1}(b_0), \varphi_1^{-1}(b_1)) \leq R(\varphi_0^{-1}(b), \varphi_1^{-1}(b)) = \varphi^{-1}(b).$$

Invoking the positive lower type of φ and integrating respect to the quasi-subadditive capacity we conclude that

$$\begin{aligned}\int_\Omega \varphi(|a|)dC &\leq \int_\Omega [\varphi_0(|a_0|)+\varphi_1(|a_1|)]dC \\ &\leq 2c\left\{\int_\Omega \varphi_0(|a_0|)dC+\int_\Omega \varphi_1(|a_1|)dC\right\} \leq 2c(C_0+C_1). \qquad \square\end{aligned}$$

Remark 2.9.24. *Last proposition can be given with the following interpretation. Let X_0, X_1 be two rearrangement invariant (that is, X satisfies that, if $f \in X$ and g is measurable such that $\mu_f = \mu_g$[2], then $g \in X$ and $\|f\|_X = \|g\|_X$) quasi-Banach function spaces of measurable functions on Ω, a capacity space equipped with a positive capacity C, and ρ be a quasi-concave function. We introduce $X = X_0\rho\left(\frac{X_1}{X_0}\right)$ to be the space of those measurable functions h for which one can find a constant $\widetilde{C}$ and functions $a_0 \in X_0$ and $a_1 \in X_1$ such that*

$$|h| \leq \widetilde{C}|a_0|\rho\Big(\frac{|a_1|}{|a_0|}\Big).$$

We equip X with $\|\cdot\|_X = \inf_{\widetilde{C}} \widetilde{C}$. We can see, with the usual techniques, that $\|\cdot\|_X$ is a quasi-norm and hence, X becomes a quasi-Banach space. The proof of the completeness of X follows with similar techniques to the ones in Theorem 2.9.16.

If $\rho = \rho_\alpha$ for $0 < \alpha < 1$, then $X = X_0^{1-\alpha}X_1^\alpha$ and we get the *Calderón product.*

Let φ_i be continuous increasing functions on $\mathbb{R}_+$ and consider $X_i = L^{\varphi_i}(C), i = 0, 1$. It follows that

$$L^{\varphi_0}(C)\rho\left(\frac{L^{\varphi_1}(C)}{L^{\varphi_0}(C)}\right) \hookrightarrow L^\varphi(C),\ \varphi^{-1} = \varphi_0^{-1}\rho\left(\frac{\varphi_1^{-1}}{\varphi_0^{-1}}\right).$$

Now, we consider the same interpolation method as in [GuP]. Let $\bar{X} = (X_0, X_1)$ be any quasi-Banach couple and let ρ be a quasi-concave function.

$$\begin{aligned}< X_0, X_1, \varrho >= \Big\{a \in \Sigma(\bar{X});\ &\text{there exists } u = \{u_\nu\}_{\nu\in\mathbf{Z}},\ u_\nu \in \Delta(\bar{X}) \\ &\text{such that (12) and (13) are satisfied}\Big\},\end{aligned}$$

where

(12) $a = \sum_{\nu\in\mathbf{Z}} u_\nu$ with convergence in $\Sigma(\bar{X})$

(13) For all $F \subset \mathbf{Z}$ finite and every real sequence $\{\xi_\nu\}_{\nu\in F}, |\xi_\nu| \leq 1$ we have $\|\sum_{\nu\in F}\frac{\xi_\nu u_\nu}{\rho(2^\nu)}\|_{X_0} \leq \widehat{C}, \|\sum_{\nu\in F}\frac{2^\nu\xi_\nu u_\nu}{\rho(2^\nu)}\|_{X_1} \leq \widehat{C}$ with $\widehat{C}$ independent of F and ξ.

[2]Recall that $\mu_f(\lambda) := \mu\{x; |f(x)| > \lambda\}$, $\lambda > 0$.

We equip $< \bar{X}, \rho >=< X_0, X_1, \rho >$ with the quasi-norm

$$\|a\|_{<\bar{X},\rho>} = \inf \widehat{C}.$$

It follows that, if ρ is of lower type 0 and upper type 1, then $\|a\|_{<\bar{X},\rho>}$ is a quasi-norm and $< \bar{X}, \rho >$ is quasi-Banach function space.

Let φ_0 and φ_1 be continuous increasing functions on $\mathbb{R}_+$ such that $\varphi_i((0,\infty)) = (0,\infty), i = 0, 1$. Let $\rho \in \mathfrak{B}(1)$ and define φ by $\varphi^{-1} = \varphi_0^{-1}\rho\left(\frac{\varphi_1^{-1}}{\varphi_0^{-1}}\right)$. With similar techniques to the ones in [GuP], it follows that $L^{\varphi}(C), L^{\varphi_0}(C)$ and $L^{\varphi_1}(C)$ are quasi-Banach spaces if φ_0 and φ_1 have positive lower type.

Theorem 2.9.25. *Assume that both φ_0 and φ_1 have positive lower type and that one of them, say φ_0 has finite upper type. Assume that $\rho \in \mathfrak{B}(1)$. Then φ defined by $\varphi^{-1} = \varphi_0^{-1}\rho\left(\frac{\varphi_1^{-1}}{\varphi_0^{-1}}\right)$ satisfies $L^{\varphi}(C) \hookrightarrow < L^{\varphi}\bar{(}C), \rho >$.*

Proof. It follows similarly to the analogous in [GuP, Theorem 7.1]. □

We have to remark that we were not able to get the converse in Theorem 2.9.25 because we do not have a capacitary Fubini's theorem.

Theorem 2.9.26. *Under the same conditions of last theorem, φ defined by $\varphi^{-1} = \varphi_0^{-1}\rho\left(\frac{\varphi_1^{-1}}{\varphi_0^{-1}}\right)$ satisfies $L^{\varphi}(C) \hookrightarrow L^{\varphi_0}(C)\rho\left(\frac{L^{\varphi_1}(C)}{L^{\varphi_0}(C)}\right)$.*

Proof. Let $f \in L^{\varphi}(C)$ with norm smaller than one and define the function $\psi(t) := \varphi_0(\frac{|f|}{\rho(t)}) - \varphi_1(\frac{t|f|}{\rho(t)})$. By hypothesis, ψ is decreasing, continuous and $\lim_{t\to 0}\psi(t) > 0$, $\lim_{t\to\infty}\psi(t) < 0$. Thus, there exists a unique t such that $\psi(t) = 0$. Denote this t by $h(t)$. Then h is continuous. Defining $x = \frac{|f|}{\rho(t)}$ and $y = \frac{t|f|}{\rho(t)}$, we have that, since $\psi(t) = 0$, then $\varphi_0(x) = \varphi_1(y)$. Moreover $\varphi^{-1}(\varphi_0(x)) = |f|$ and $\varphi(|f|) = \varphi_0(x) = \varphi_1(y)$. Thus

$$\int_{\Omega} \varphi_0\left(\frac{|f|}{\rho(t)}\right) dC = \int_{\Omega} \varphi(|f|)dC \leq 1,$$

and we can write $|f|$ as an element in $L^{\varphi_0}(C)\rho\left(\frac{L^{\varphi_1}(C)}{L^{\varphi_0}(C)}\right)$. So that, the proof follows. □

Corollary 2.9.27. *Let φ_0 and φ_1 be continuous increasing functions on $\mathbb{R}_+$ with $\varphi_i((0,\infty)) = (0,\infty), i = 0, 1$, both of positive lower type and finite upper type. Define $\varphi^{-1} = \varphi_0^{-1}\rho\left(\frac{\varphi_1^{-1}}{\varphi_0^{-1}}\right)$ for ρ be a quasi-concave function in $\mathfrak{B}(1)$. Then*

$$L^{\varphi}(C) = L^{\varphi_0}(C)\rho\left(\frac{L^{\varphi_1}(C)}{L^{\varphi_0}(C)}\right) \hookrightarrow < L^{\varphi}\bar{(}C), \rho > .$$

By Theorem 2.8.4, it follows that for $0 < p_0 \leq \infty$, $0 < p_1 \leq \infty$, $\alpha \in (0,1)$ and $\frac{1}{p} = \frac{1-\alpha}{p_0} + \frac{\alpha}{p_1}$,

$$L^{p_0}(C)^{1-\alpha} L^{p_1}(C)^{\alpha} = L^p(C) \hookrightarrow < L^{p_0}(C), L^{p_1}(C), \rho_\alpha > .$$

See [P1, Theorem 2.2]. □

In the limit case it follows:

Theorem 2.9.28. *Let φ_0 be a continuous increasing function on $\mathbb{R}_+$ of positive lower type and finite upper type such that $\varphi_0((0,\infty)) = (0,\infty)$. Define $\varphi^{-1} = \varphi_0^{-1}\rho\left(\frac{1}{\varphi_0^{-1}}\right)$ for any quasi-concave function ρ in $\mathfrak{B}^+(1)$. Then*

$$L^{\varphi}(C) = L^{\varphi_0}(C)\rho\left(\frac{L^{\infty}(C)}{L^{\varphi_0}(C)}\right) \hookrightarrow < L^{\varphi_0}(C), L^{\infty}(C), \rho >$$

holds with equivalence of norms.

Proof. See [GuP, Theorem 9.1]. □

Part II

Conductor Sobolev type estimates and isocapacitary inequalities

Chapter 3

Conductor Sobolev type estimates and isocapacitary inequalities

3.1 Introduction

Recall that for $\Omega \subset \mathbb{R}^n$ be a domain, the extension of Wiener's capacity of a compact subset K of Ω for $p \geq 1$, is the p-capacity (see Example 1.2.8)[1]

$$\mathrm{cap}_p(K, \Omega) = \inf_{0 \leq f \leq 1,\, f=1 \text{ on } K} \|\nabla f\|_p^p \qquad (f \in \mathrm{Lip}_0(\Omega)).$$

It was used in [Ma05] to obtain the *Sobolev inequality*

$$\int_0^\infty \mathrm{cap}_p(\overline{M}_{at}, M_t) d(t^p) \leq c(a,p) \|\nabla f\|_p^p, \tag{3.1}$$

where M_t is the level set $\{x \in \Omega;\ |f(x)| > t\}$ for $t > 0$. Recall that for $f \in \mathrm{Lip}(\Omega)$, the usual gradient of f is defined by

$$\limsup_{d(x,y) \to 0} \frac{|f(x) - f(y)|}{d(x,y)} = |\nabla f(x)| \quad (x \in \Omega)$$

and zero at isolated points.

This "conductor inequality" is a powerful tool with applications to Sobolev type imbedding theorems, which for $p > 1$ plays the same role as the co-area formula for $p = 1$.

With its variants, (3.1) has many applications to very different areas, such as Sobolev inequalities on domains of $\mathbb{R}^n$ and on metric spaces, to linear and non-linear partial differential equations, to calculus of variations, to Markov processes, etc. (See eg. [AH], [AP], [Ci], [Da],

[1] $\mathrm{Lip}(\Omega)$ denotes the class of all Lipschitz functions on Ω and $\mathrm{Lip}_0(\Omega)$ the ones with compact support on Ω.

[DKX], [Han], [Ko84], [MMi], [MMi1], [MMi2], [Ma85], [Ma11], [Ma05], [Ma06], [MaN], [MaP], [Ra], [V99], and the references therein).

An interesting extension based on the Lorentz space $L^{p,q}(\Omega)$ $(1 < p < \infty,\ 1 \le q \le \infty)$ was obtained in [CosMa].

Our aim is to extend these capacitary estimates when a general function space X substitutes $L^p(\Omega)$ or $L^{p,q}(\Omega)$ in the definition of cap_p and $\mathrm{cap}_{p,q}$.

The proofs of these new Lorentz-Sobolev inequalities in [CosMa] are based on the properties

$$\begin{aligned} \|f\|^p_{L^{p,q}(\Omega,\mu)} + \|g\|^p_{L^{p,q}(\Omega,\mu)} &\le \|f+g\|^p_{L^{p,q}(\Omega,\mu)} \qquad (1 \le q \le p) \\ \|f\|^q_{L^{p,q}(\Omega,\mu)} + \|g\|^q_{L^{p,q}(\Omega,\mu)} &\le \|f+g\|^q_{L^{p,q}(\Omega,\mu)} \qquad (1 < p < q) \end{aligned}$$

of the Lorentz (quasi-)norms, for f, g disjointly supported functions. Using the fact that the constant in the right hand side of the inequalities is one, they can be extended to an arbitrary set of disjoint functions, and $L^{p,q}$ satisfies lower estimates with constant one (see Section 2).

A perusal in the proofs allow us to see that the limitation of these usual techniques is that it allows us to cover only certain particular kind of spaces because of the lower estimates with constant one, and it does not apply to a wider class of spaces.

However, we will see that an extension is possible in the setting of (quasi-)Banach function spaces with lower estimates, independently of the value of the constant, by means of new techniques different to the ones followed by V. Maz'ya and S. Costea. The key point is a result due to N. J. Kalton and S. J. Montgomery-Smith on the theory of *submeasures*. Our results can be applied to many examples, which include Lebesgue spaces, Lorentz spaces, classical Lorentz spaces, Orlicz spaces, mixed Norm spaces, etc.

It could seem that for improvements of integrability only truncations methods are needed. In [KO] it appears that inequalities of Sobolev-Poincaré-type are improved to Lorentz type scales thanks to stability under truncations, but there also p-convexity is implicitly used. In this sense, we will characterize Sobolev type inequalities in the setting of rearrangement invariant (r.i. for short) spaces. Under appropriate conditions on the space X (see Theorem 3.5.2) and for any $0 < p < \infty$, we show the equivalence of the following properties:

(i) For every compact set K on Ω, $\varphi_Y(\mu(K)) \lesssim \mathrm{Cap}_X(K)$.

(ii) $\|f\|_{\Lambda^{1,p}(Y)} \lesssim \|\nabla f\|_X$ $(f \in \mathrm{Lip}_0(\Omega))$.

(iii) $\|f\|_{\Lambda^{1,\infty}(Y)} \lesssim \|\nabla f\|_X$ $(f \in \mathrm{Lip}_0(\Omega))$,

where φ_Y denotes the fundamental function of Y (see (3.2)) and $\Lambda^{p,q}(Y)$ $(0 < q \leq \infty)$ is the Lorentz space defined in (3.12). Moreover, under the appropiate conditions on Y, we show that

$$\|f\|_{\Lambda^{1,\infty}(Y)} \lesssim \|\nabla f\|_X \Leftrightarrow \|f\|_{\Lambda^{1,p}(Y)} \lesssim \|\nabla f\|_X \Leftrightarrow \|f\|_Y \lesssim \|\nabla f\|_X \qquad (f \in \mathrm{Lip}_0(\Omega)).$$

In the particular case when $X = L^p$, $p \in (1, n)$, and $Y = L^s$ with $s = \frac{np}{n-p}$, we recover the well-known self-improvement of integrability of Lipschitz functions

$$\|f\|_{L^{s,p}} = \|f\|_{\Lambda^{1,p}(L^s)} \lesssim \|\nabla f\|_{L^p}.$$

As an application of the Sobolev capacitary inequalities, we derive necessary and sufficient conditions for Sobolev type inequalities in rearrangement invariant spaces involving two measures, recovering results obtained in [CosMa], [Ma05] and [Ma06] for Lorentz spaces. We show that under appropiate conditions on the r.i. spaces the following properties are equivalent:

(i) $\|f\|_{\Lambda^{1,p}(Y)} \lesssim \|\nabla f\|_X + \|f\|_{\Lambda^{1,p}(Z)} \qquad (f \in \mathrm{Lip}_0(\Omega))$.

(ii) $\varphi_Y(\mu(g)) \lesssim \mathrm{Cap}_X(\bar{g}, G) + \varphi_Z(\nu(G)) \qquad (g \subset\subset G \subset\subset \Omega)$.[2]

To finish we develop some extensions to the capacitary function spaces studied in Chapter 1. All the developments on these chapter are summarized in [CeMS2].

3.2 Preliminaries

Let (Ω, Σ, μ) be a measure space. Recall that a *quasi-Banach function space* X is a quasi-Banach linear subspace of $\mathrm{L}_0(\Omega)$ with the following properties:

(a) Lattice property: Given $g \in X$ and $f \in \mathrm{L}_0(\Omega)$ such that $|f| \leq |g|$, then $f \in X$ and $\|f\|_X \leq \|g\|_X$.

(b) Fatou property: $0 \leq f_n \uparrow f$ μ-a.e. $\Rightarrow \|f_n\|_X \uparrow \|f\|_X$.

Assume that μ is a measure on Ω. A quasi-Banach function space X on Ω is said to be *rearrangement invariant* (r.i. for short) if $f \in X$, $g \in \mathrm{L}_0(\Omega)$ and $g_\mu^* \leq f_\mu^*$ imply $g \in X$ and $\|g\|_X \leq \|f\|_X$.

[2]The notation $g \subset\subset G$ will mean that g and G are two open sets in $\mathbb{R}^n$ such that $\overline{g}$ is a compact subset of G.

Definition 3.2.1. *Let X be an r.i. quasi-Banach function space on Ω. The* fundamental function *of X (see [BeSh] and [BeR]) is defined as*

$$\varphi_X(t) := \|\chi_A\| \qquad (\mu(A) = t). \tag{3.2}$$

3.2.1 Convexity conditions

As we said in the introduction, certain convexity conditions are needed.

Definition 3.2.2. *A quasi-Banach function space X on Ω is called p-convex or p-*concave *if there exists a constant M so that*

$$\Big\|\Big(\sum_{i=1}^n |f_i|^p\Big)^{1/p}\Big\| \le M\Big(\sum_{i=1}^n \|f_i\|^p\Big)^{1/p} \quad (n \in \mathbb{N},\ \{f_i\}_{i=1}^n \subset X)$$

or

$$\Big(\sum_{i=1}^n \|f_i\|^p\Big)^{1/p} \le M\Big\|\Big(\sum_{i=1}^n |f_i|^p\Big)^{1/p}\Big\| \quad (n \in \mathbb{N},\ \{f_i\}_{i=1}^n \subset X),$$

respectively.

Definition 3.2.3. *Let $0 < p < \infty$. A quasi-Banach function space X on Ω satisfies an* upper p-estimate *or a* lower $p-$estimate *if there exists a constant M so that, for all $n \in \mathbb{N}$ and for any choice of disjointly supported elements $\{f_i\}_{i=1}^n \subset X$,*

$$\Big\|\Big(\sum_{i=1}^n |f_i|^p\Big)^{1/p}\Big\| \le M\Big(\sum_{i=1}^n \|f_i\|^p\Big)^{1/p} \tag{3.3}$$

or

$$\Big(\sum_{i=1}^n \|f_i\|^p\Big)^{1/p} \le M\Big\|\Big(\sum_{i=1}^n |f_i|^p\Big)^{1/p}\Big\|, \tag{3.4}$$

respectively. The smallest constant M in (3.3)(resp. in (3.4)) is called the upper p-estimate *(resp.* lower p-estimate*)* constant *and it will be denoted by $M^{(p)}(X)$ (resp. $M_{(p)}(X)$).*

Example 3.2.4. *Let (Ω, Σ, μ) be a measure space. For $1 \le q < p$, $L^{p,q}(\mu)$ satisfies a lower p-estimate with constant one, and for $1 < p < q < \infty$ it satisfies an upper p-estimate and a lower $q-$estimate with constants one. Easy proofs of these facts can be seen in [CosMa].*

Example 3.2.5. *Since $\sum_{i=1}^n |g_i|^p = |\sum_{i=1}^n g_i|^p$ when $\{g_i\}_{i=1}^n \subset X$ are disjointly supported, if X is $p-$concave, then X satisfies a lower p-estimate.*

Proposition 3.2.6. *If a Banach lattice X on Ω satisfies an upper, respectively, lower r-estimate for some $1 < r < \infty$, then it is p-convex, respectively q-concave, for every $1 < p < r < q < \infty$.*

In [CSo], in connection with these properties, it is observed a similar one for the classical Lorentz spaces on $\mathbb{R}^n$. The property observed by M. J. Carro and J. Soria is the following:

Definition 3.2.7. *Let $0 < p < \infty$ and X be a Banach lattice on Ω. Let $\{E_i\}_{i=1}^{\infty}$ be a collection of pairwise disjoint measurable subsets of Ω, $E_0 := \bigcup_{i=1}^{\infty} E_i$, and $f \in X$. We say that X satisfies a* weak lower $p-$estimate *if the following inequality*

$$\sum_{i=1}^{\infty} \|\chi_{E_i} f\|_X^p \leq \|\chi_{E_0} f\|_X^p$$

holds. The condition follows if and only if it holds for two disjoint sets A, B.

Proposition 3.2.8. *Let X be a Banach lattice on Ω. Then, X satisfies a lower p-estimate with $M_{(p)}(X) = 1$ if and only if X satisfies a weak lower $p-$estimate.*

Proof. Let $\{E_i\}_{i=1}^{\infty}$ be a collection of pairwise disjoint measurable subsets of Ω, $E_0 := \cup_{i=1}^{\infty} E_i$ and $f \in X$. Defining $x_i := f\chi_{E_i}$ we have that, for $n \in \mathbb{N}$

$$\Big(\sum_{i=1}^{n} \|x_i\|^p\Big)^{1/p} \leq M_{(p)}(X)\Big\|\sum_{i=1}^{n} x_i\Big\|.$$

Then, for all $n \in \mathbb{N}$, $\sum_{i=1}^{n} \|x_i\|^p \leq \|\sum_{i=1}^{n} x_i\|^p \leq \|f\chi_{E_0}\|^p$, and therefore letting $n \to \infty$ we get that

$$\sum_{i=1}^{\infty} \|x_i\|^p = \lim_{n\to\infty} \sum_{i=1}^{n} \|x_i\|^p \leq \|f\chi_{E_0}\|^p.$$

The converse follows taking, for $\{f_i\}_{i=1}^{n} \subset X$ disjointly supported, $f = \sum_{i=1}^{n} f_i$. □

We will see more facts related to these properties in the last section of this chapter.

3.2.2 Examples

For the sake of the reader convenience, let us present some examples of spaces satisfying this kind of properties.

A function $F : (0, \infty) \to (0, \infty)$ is called *quasi-increasing* (resp. *quasi-decreasing*) if $F(s) \lesssim F(t)$ (resp. $F(t) \lesssim F(s)$) for any $0 < s < t$. Moreover, F is said *quasi-superadditive* if there exists a constant $d > 0$ such that $F(x) + F(y) \leq dF(x + y)$ for all $0 < x, y < \infty$, and it is said *superadditive* when $d = 1$.

Example 3.2.9 (Classical Lorentz spaces or Weighted Lorentz spaces)**.** *In the second chapter we showed that classical Lorentz spaces are capacitary Lebesgue spaces. Now our focus is in the convexity properties of these spaces.*

The Weighed Lorentz space $\Lambda^p_\mu(w)$ *is the class of all measurable functions in* $\mathbb{R}^n$ *such that*

$$\|f\|_{\Lambda^p_\mu(w)} = \Big(\int_0^\infty f^*_\mu(x)^p w(x)dx\Big)^{1/p} < \infty,$$

where f^*_μ *denotes the decreasing rearrangement of* f *with respect to* μ *(compare with (1.2)). This spaces were introduced in 1951 by [Lo]. It is shown in [Lo] that the condition of* w *being non-increasing is necessary and sufficient for* $\|\cdot\|_{\Lambda^p_\mu(w)}$ *to be a norm.*

These spaces are r.i. and hence, by *Luxemburg's theorem*, we can reduce them to the measure space $(\mathbb{R}^+, m)$. When the measure is the Lebesgue measure, we write f^* instead of f^*_m and $\Lambda^p(w)$ instead of $\Lambda^p_m(w)$.

As in the study of capacitary Orlicz spaces, the Δ_2-condition (see (2.26)) is also useful here. Let $W(t) := \int_0^t w(s)ds$. In [CSo, Corollary 2.2] we see that $\|\cdot\|_{\Lambda^p_\mu(w)}$ is a quasi-norm if and only if W satisfies the Δ_2-condition.

These spaces were studied among others by S. Reisner [Rei], S. Ya. Novikov [No] and C. Schütt [Sch] generalizing results of J. Creekmore [Cre] relative to $L^{p,q}$. One of the first attempts to study their convexity nature was done in [Ray]. $\Lambda^p(w)$ is $p-$convex (resp. p-concave) if and only if w is, up to an admissible change, non-increasing (resp. non-decreasing), where the change from w_1 to w_2 is *admissible* if W_1 and W_2 are equivalent.

Let us observe that the distribution function of f with respect to the measure $\mu(x)dx$ is $\lambda^\mu_f(y) = \int_{\{x;|f(x)|>yx\}} \mu(x)dx$. We have that

$$\|f\|_{\Lambda^p_\mu(w)} = \Big\|y\Big(\int_0^{\lambda^\mu_f(y)} w(t)dt\Big)^{1/p}\Big\|_{L^p(dy/y)}$$

and hence, for $0 < p, q < \infty$, $\Lambda^{p,q}_\mu(w)$ is the space of all measurable functions in $\mathbb{R}^n$ such that

$$\|f\|_{\Lambda^p_\mu(w)} = \Big\|y\Big(\int_0^{\lambda^\mu_f(y)} w(t)dt\Big)^{1/p}\Big\|_{L^q(dy/y)} < \infty.$$

These spaces satisfy the following chain of embeddings

$$\cdots \Lambda^{p,1}_\mu(w) \subset \cdots \subset \Lambda^p_\mu(w) \subset \cdots \subset \Lambda^{p,q}_\mu(w) \subset \cdots \subset \Lambda^{p,\infty}_\mu(w),$$

for $p \le q$. And in the case $w = 1$, $\Lambda^{p,q}_\mu(w)$ is the *Lorentz space* $L^{p,q}(\mu)$.

Let us observe that the spaces $\Lambda^{p,q}_{\mu}(w)$ are, in fact, *weighted Lorentz spaces* if $q<\infty$, and they coincide with the spaces $\Lambda^{q}_{\mu}(w_p)$, where

$$w_p(x)=\Big(\int_0^x w(t)dt\Big)^{q/p-1}w(x).$$

In [CSo] it was proved:

Proposition 3.2.10. *Let $p_0\le p_1$. Then W^{p_1/p_0} is quasi-superadditive, that is, for every $\{t_k\}_k\subset\mathbb{R}^+$,*

$$\Big(\sum_k\Big(\int_0^{t_k}w(s)ds\Big)^{p_1/p_0}\Big)^{p_0/p_1}\le C\int_0^{\sum_k t_k}w(s)ds,$$

if and only if, $\Lambda^{p_0}_{\mu_0}(w)$ satisfies a lower p_1-estimate.

Proof. To prove the necessary condition, we use the fact that for $0<p<\infty$

$$\int_0^\infty (f^*_{\mu_0}(t))^p w(t)dt=p\int_0^\infty y^{p-1}\Big(\int_0^{\lambda^{\mu_0}_f(y)}w(t)dt\Big)dy,$$

and the Minkowski integral inequality to obtain

$$\begin{aligned}\sum_k\|f_k\|^{p_1}_{\Lambda^{p_0}_{\mu_0}(w)} &= \sum_k\Big(\int_0^\infty y^{p_0-1}\Big(\int_0^{\lambda^{\mu_0}_{f_k}(y)}w(t)dt\Big)dy\Big)^{p_1/p_0}\\ &\le \Big(\int_0^\infty y^{p_0-1}\Big(\sum_k\Big(\int_0^{\lambda^{\mu_0}_{f_k}(y)}w(t)dt\Big)^{p_1/p_0}\Big)^{p_0/p_1}dy\Big)^{p_1/p_0}\\ &\le C\Big(\int_0^\infty y^{p_0-1}\Big(\int_0^{\sum_k\lambda^{\mu_0}_{f_k}(y)}w(t)dt\Big)dy\Big)^{p_1/p_0}\\ &= C\Big(\int_0^\infty y^{p_0-1}\Big(\int_0^{\lambda^{\mu_0}_{\sum_k f_k}(y)}w(t)dt\Big)dy\Big)^{p_1/p_0}\\ &= C\|\sum_k f_k\|^{p_1}_{\Lambda^{p_0}_{\mu_0}(w_0)}.\end{aligned}$$

Conversely, $\Big(\int_0^{t_k}w(s)ds\Big)^{1/p_0}=\|f_k\|_{\Lambda^{p_0}_{\mu_0}(w)}$, where $(f_k)^*_{\mu_0}=\chi_{(0,t_k)}$. If for $x\in\mathbb{R}^n$

$$F_k(x)=\chi_{(t_{k-1},t_{k-1}+t_k)}(\mu_0(B(0,|x|))),$$

then one can easily check that $(F_k)^*_{\mu_0}(s)=(f_k)^*_{\mu_0}$ and the F_k's have pairwise disjoint supports. Therefore,

$$\begin{aligned}\sum_k\Big(\int_0^{t_k}w(s)ds\Big)^{p_1/p_0} &= \sum_k\|f_k\|^{p_1}_{\Lambda^{p_0}_{\mu_0}(w)}=\sum_k\|F_k\|^{p_1}_{\Lambda^{p_0}_{\mu_0}(w)}\\ &\le C\|\sum_k F_k\|^{p_1}_{\Lambda^{p_0}_{\mu_0}(w)}=C\Big(\int_0^{\sum_k t_k}w(s)ds\Big)^{p_1/p_0}.\square\end{aligned}$$

A considerable amount of work has been done to study the properties of these spaces. It is well-known that $\Lambda^p(w)$ is p-convex with constant one when w is decreasing, and p-concave with constant one when w is increasing. Moreover, if $0 < r < p < \infty$, then $\Lambda^p(w)$ is r-convex if and only if for some $\varepsilon > 0$, $W(t)/t^{p/r-\varepsilon}$ is quasi-decreasing (see [KaMa]), and if $r > p$, then $\Lambda^p(w)$ is not r-convex.

If w is decreasing, by [KaP, Theorem 1], for $r > p$ and $p/r + 1/s = 1$, the r-concavity constant of $\Lambda^p(w)$ is

$$\sup_{t>0} \Big[\frac{(\frac{1}{t}\int_0^t w^s)^{1/s}}{\frac{1}{t}\int_0^t w} \Big]^{1/p}.$$

Moreover, if $0 < \int_0^x w(t)dt < \infty$ and $\int_x^\infty t^{-p} w(t)dt < \infty$, then, for $p \le r < \infty$, $\Lambda^p(w)$ satisfies a lower $r-$estimate if and only if

$$t^{-p/r} \int_0^t w(s)ds$$

is quasi-increasing. In particular, for $0 < p < \infty$, $\Lambda^p(w)$ is p-concave if and only if $W(t)/t$ is quasi-increasing (see [KaMa, Theorem 7 and 8]).

The classical Lorentz spaces generalize many known spaces in the literature. If $w(t) := t^{p/q-1}$, then we obtain the *Lorentz space* $L^{q,p}(\Omega)$. If $w(t) := t^{p/q-1}(1 + \log(t))^{\lambda p}$, then we obtain the *Lorentz-Zygmund space*, that is, $\Lambda^p(w) = L^{q,p}(\mathrm{Log}L)^\lambda(\Omega)$ (see e.g. [BeR]).

More generally, a positive function b is called *slowly varying* on $(1, \infty)$ in the sense of Karamata (s.v. for short), if for each $\varepsilon > 0$, $t^\varepsilon b(t)$ is quasi-increasing and $t^{-\varepsilon} b(t)$ is quasi-decreasing. For example, the following funtions are s.v.

$$b(t) = \exp(\sqrt{\log t}) \quad \text{and} \quad b(t) = (e + \log t)^\alpha (\log(e + \log t))^\beta \quad \text{for} \quad \alpha, \beta \in \mathbb{R}.$$

If $w(t) = t^{q/p-1} b(\max(t, 1/t))^q$ with b s.v., then $\Lambda^q(w)$ is the *Lorentz-Karamata space* $L_{p,q,b}(\Omega)$ (see e.g. [Nev]).

For the classical Lorentz spaces the fundamental function is $\varphi_{\Lambda^p_\mu(w)} = W^{1/p}$. Therefore, these previous results sujest a possible relation between these properties for a quasi-Banach function space X on $\mathrm{L}_0(\Omega)$ and the behaviour of $\varphi_X(t)/t$.

Definition 3.2.11. *Given $0 < p < \infty$ and a quasi-Banach lattice X, $X^{(p)} = \{x; |x|^p \in X\}$ denotes the $p-$*convexification *of X eqquiped with the quasi-norm $\|x\|_{X^{(p)}} = \||x|^p\|^{1/p}$.*

It is clear then that $X^{(p)}$ is 1-convex (resp. 1-concave) if and only if X is $1/p$-convex (resp. $1/p$-concave). Notice also that a quasi-Banach space is normable if and only if it is $1-$convex (see [KaMa1, Preliminaries]).

Theorem 3.2.12. *Assume that X is a Banach function space on $(\mathbb{R}^+, m)$. If X has a lower $q-$estimate, then there exists $\varepsilon \geq 0$ such that $\frac{\varphi_X(t)^{q+\varepsilon}}{t}$ is quasi-increasing on $[0, \infty]$.*

Proof. Suppose first that $q = 1$ and that X has a lower $q-$estimate over $(\mathbb{R}^+, m)$. Let $0 < s < t$, $n = [\frac{t}{s}]$, $f_i := \chi_{(\frac{(i-1)t}{2n}, \frac{it}{2n}]}$, $i = 1, ..., 2n$. It follows that $f_i^* = \chi_{(0, \frac{t}{2n}]}$ and $\|f_i\|_X = \varphi_X(t/2n)$.

By the lower estimate, there exists $C > 0$ such that

$$\Big\| \sum_{i=1}^{2n} f_i \Big\|_X = \varphi_X(t) \geq C\Big(\sum_{i=1}^{2n} \|f_i\|_X \Big) = C2n\varphi_X(t/2n).$$

Since $2n = 2[\frac{t}{s}] \geq \frac{t}{s}$, then $\frac{t}{2n} \leq s$; and then,

$$\Big\| \sum_{i=1}^{2n} f_i \Big\|_X \geq C\frac{2n}{t} t\varphi_X(t/2n) = Ct\frac{\varphi_X(t/2n)}{\frac{t}{2n}} \geq Ct\frac{\varphi_X(s)}{s}.$$

Therefore, $\frac{\varphi_X(s)}{s}$ is quasi-increasing. Hence, since φ_X is increasing, it follows that there exists $\varepsilon > 0$ such that $\frac{\varphi_X^{1+\varepsilon}(t)}{t}$ is also quasi-increasing.

Let now $q \neq 1$ and suppose that X has a lower $q-$estimate. Since for a Banach lattice (see [KaMaP]) it follows:

$$\begin{aligned} q_c(X) &:= \inf\{q > 0; X \text{ is } q - \text{concave}\}, \\ &= q_d(X) = \inf\{q > 0; X \text{ satisfies a lower } q - \text{estimate}\} = q(X), \end{aligned}$$

then for all $\varepsilon > 0$, X is $q + \varepsilon$-concave. Therefore, there exists $p \in \mathbb{R}$ such that $q + \varepsilon = 1/p$ and X is $1/p$-concave. Then, $X^{(p)}$ is 1-concave and hence, it has a lower $1-$estimate. Then, by before, $\frac{\varphi_{X^{(p)}}(t)^1}{t}$ is quasi-increasing, that is, $\frac{\varphi_X(t)^{(q+\varepsilon)}}{t}$ is quasi-increasing. □

Remark 3.2.13. *If X has an upper $q-$estimate, then there exists $\varepsilon \geq 0$ such that $\frac{\varphi_X(t)^{q-\varepsilon}}{t}$ is quasi-decreasing on $[0, \infty]$.*

It is our feeling that these results can be the starting point of a project to try to develop in the earlier future.

Example 3.2.14 ($\Gamma_p(w)$)**.** *Suppose that the weight w (any non-negative function) satisfies that for $0 < p < \infty$ and $x > 0$,*

$$0 < W(x) = \int_0^x w(t)dt < \infty \ \text{ and } W_p(x) := x^p \int_x^\infty t^{-p} w(t)dt < \infty.$$

Easily it follows that

$$\Gamma_p(w) = \Big\{ f \in L_0(\Omega);\ \|f\|_{\Gamma_p(w)} := \Big(\int_0^\infty f^{**}(x)^p w(x)\, dx \Big)^{1/p} < \infty \Big\}$$

is a quasi-Banach function space with the Fatou property, where recall that

$$f^*(t) = \inf \{ \lambda > 0;\ \mu \{ x \in \Omega; |f(x)| > \lambda \} \le t \}, \quad \Omega \subset \mathbb{R}^+$$

and $f^{**}(t) := t^{-1} \int_0^t f^*(s)\, ds$.

Assume that w satisfies that $t^{-p} \int_0^t w(s) ds \lesssim \int_t^\infty s^{-p} w(s)\, ds$ and the non-degeneracy conditions $\int_0^1 s^{-p} w(s)\, ds = \int_1^\infty w(s)\, ds = \infty$. In that case, $\varphi_{\Gamma_p(w)} \simeq W_p^{1/p}$ (see [KaMa1]).

Theorem 3.2.15. *Under the previous conditions on w and $1 < p < \infty$, the following conditions are equivalent:*

(i) $\Gamma_p(w)$ *is p-convex (resp. p-concave).*

(ii) $\Gamma_p(w)$ *satisfies an upper $p-$estimate (resp. a lower $p-$estimate).*

(iii) $W_p(x)/x$ *is quasi-decreasing (resp. quasi-increasing),*

(iv) $\varphi_{\Gamma_p(w)}/x^{1/p}$ *is quasi-decreasing (resp. quasi-increasing).*

Proof. Under these conditions, by [KaMa1, Corollary 1.9] it follows that

$$\Gamma_p(w)^* \simeq (\Gamma_p(w))' = \Lambda^{p'}(v),$$

where $v(x) = V'(x)$, $1/p + 1/p' = 1$, and

$$V(x) = \Big(\int_x^\infty t^{-p} w(t) dt \Big)^{-1/(p-1)}, x > 0.$$

So that, there exists a weight v such that $\Lambda^{p'}(v)$ is a normable space and it is a predual of $\Gamma_p(w)$, that is $\Lambda^{p'}(v)^*$ is lattice isomorphic to $\Gamma_p(w)$. Thus $\Gamma_p(w)$ is p-convex or satisfies an upper p-estimate if and only if $\Lambda^{p'}(v)$ is p'-concave or satisfies a lower p'-estimate, respectively (cf. Proposition 1.d.4 in [LiZa]).

Moreover, $V(x)/x$ is quasi-increasing if and only if $W_p(x)/x$ is quasi-decreasing. Now, as we know, $\Lambda^{p'}(v)$ is p'-concave if and only if $V(x)/x$ is quasi-increasing. Therefore, (i)-(iii) are equivalent. Finally, the conditions on w yield that $\varphi^p_{\Gamma_p(w)} \simeq W_p$ and hence (iii) is equivalent to (iv). □

Moreover, if $t^{-p}\int_0^t w(s)ds \lesssim \int_t^\infty s^{-p}w(s)\,ds$ and $1 < p \le r < \infty$ then, by [KaMa1, Theorem 3.3], it follows that $\Gamma_p(w)$ satisfies a lower $r-$estimate if and only if $t^{p(1-1/r)}\int_t^\infty s^{-p}w(s)\,ds$ is quasi-increasing.

For $0 < p \le 1$ and $r \ge 1$, or for $1 < p < r < \infty$, $\Gamma_p(w)$ is $r-$concave if and only if

$$t^{p(1-1/r)-\varepsilon}\int_t^\infty s^{-p}w(s)\,ds$$

is quasi-increasing for some $\varepsilon > 0$. For more details see [KaMa1].

Example 3.2.16 (Musielak-Orlicz spaces)**.** *A two variable real function $\phi(u,t) : [0,\infty) \times \Omega \to [0,\infty)$ will be called a* Musielak-Orlicz function *if for a.a. $t \in \Omega$, $u \to \phi(u,t)$ is increasing and continuous with $\phi(0,t) = 0$, $\phi(u,t) > 0$ for $u > 0$ and $\lim_{u\to\infty}\phi(u,t) = \infty$, and for all $u \ge 0$ the function $t \to \phi(u,t)$ is Σ-measurable. If ϕ is convex with respect to u we say that it is a* Young function *(see [Ka1] and [Ka2]).*

The Musielak-Orlicz space *$L_\phi(\Omega)$ is then defined as the set of equivalence classes of measurable functions $f : \Omega \to \mathbb{R}$ such that*

$$\int_\Omega \phi(\lambda|f(t)|,t)d\mu(t) < \infty$$

for some $\lambda > 0$. Under the norm

$$\|f\|_\phi := \inf\Big\{\varepsilon > 0; \int_\Omega \phi(|f(t)|/\varepsilon,t)d\mu(t) \le 1\Big\},$$

$L_\phi(\Omega)$ is a Banach space. If ϕ does not depend on t, then $L_\phi(\Omega)$ is an Orlicz space.

Let $L_\phi(\Omega)$ be an Orlicz space over a non-atomic measure space, $1 < p \le 2 \le q < \infty$, and $\mu(\Omega) < \infty$. Then, if there exists $u \ge 0$ such that

$$\phi(\lambda u) \lesssim \lambda^q\phi(u)\,(\lambda \ge 1 \text{ and } u \ge u_0),$$

then $L_\phi(\Omega)$ is r-concave for all $r > q$ and hence, it satisfies a lower $r-$estimate. If $\mu(\Omega) = \infty$, then the above inequalities have to hold for all $u \ge 0$.

For $0 < q < \infty$, $L_\phi(\Omega)$ satisfies a lower $q-$estimate if and only if

$$\phi(\lambda u) \lesssim \lambda^q\phi(u) \text{ for all } \lambda \ge 1 \text{ and all } u.$$

Given $0 < q < \infty$ (resp. $0 < p < \infty$), it is said that ϕ satisfies *condition* Δ^q (resp. *condition* Δ^{*p}) if there exists $K > 0$ and a non-negative integrable function h such that

$$\phi(\lambda u,t) \le K\lambda^q(\phi(u,t) + h(t)) \qquad (\text{resp. } \phi(\lambda u,t) \ge K\lambda^p(\phi(u,t) - h(t)))$$

for all $\lambda \geq 1$, $u \geq 0$ and a.a. $t \in \Omega$.

Given a Musielak-Orlicz function ϕ, the *lower indice* $\alpha(\phi)$ is defined as follows

$$\alpha(\phi) := \sup\{p; \phi \in \Delta^{*p}\}.$$

If $\alpha(\phi) > 0$, it is well-known that the quasi-norm $\|\cdot\|_\phi$ is q-concave for $0 < q < \infty$. Moreover, for a quasi-normed space $(L_\phi(\Omega), \|\cdot\|_\phi)$ and $0 < q < \infty$,

$$L_\phi(\Omega) \text{ satisfies a lower } q \text{ estimate} \Leftrightarrow \phi \text{ satisfies condition } \Delta^q.$$

For more details about it, see [Ka1] and [Ka2].

Function spaces that are not rearrangement invariant may also be considered:

Example 3.2.17 (Mixed norm L^p spaces)**.** *The space $L^q(\Omega_2)[L^p(\Omega_1)]$ for $1 \leq p, q \leq \infty$, defined by the condition*

$$\|f\| := \Big(\int \Big(\int |f(x,y)|^p \, d\mu_1(x) \Big)^{q/p} d\mu_2(y) \Big)^{1/q} < \infty,$$

satisfies a lower $pq-$estimate with constant one.

Indeed, if f and g are two disjointly supported functions, it follows from [BP, Theorem 1] that $\|f+g\|^{pq} \geq \|f\|^{pq} + \|g\|^{pq}$.

In the case $L^{p_n}(\mu_n)[\ldots[L^{p_1}(\mu_1)]]$ we have a lower $p_1 \cdots p_n-$estimate with constant one.

Example 3.2.18 (Mixed norm weighted Lorentz spaces)**.** *Suppose $1 \leq p, q < \infty$ and, for a measurable function f on $\Omega = \Omega_1 \times \Omega_1$, denote $f_y^*(x,t)$ the decreasing rearrangement of f with respect to the second variable y, when the first variable x is fixed (see [BKPSo]).*

Let u and v be weights on Ω_1 and Ω_2, u such that $U(x) := \int_0^x u(t)\,dt$ is quasi-superadditive. Then the space $\Lambda^q(v)[\Lambda^p(u)]$ defined by the condition

$$\|f\|_{\Lambda^q(v)[\Lambda^p(u)]} := \Big(\int_0^\infty \Big[\Big(\int_0^\infty (f_y^*(\cdot,t))^p u(t) dt \Big)^* (s) \Big]^{q/p} v(s) ds \Big)^{1/q} < \infty$$

also satisfies a lower $pq-$estimate.

Indeed, $\Lambda^p(u)$ satisfies a lower $p-$estimate (see [CSo, Lemma 3.2]).

Let $0 \leq a \leq 1$. If we apply Hölder's inequality to the scalar product in $a^{1-\frac{1}{p}}x + (1-a)^{1-\frac{1}{p}}y$ for $1 \leq p < \infty$, we obtain

$$(|x|^p + |y|^p)^{1/p} \geq a^{1-\frac{1}{p}}x + (1-a)^{1-\frac{1}{p}}y.$$

So that, if $f, g \in \Lambda^p(u)$ are disjointly supported, then

$$\begin{aligned} M_{(p)}(\Lambda^p(u))\|f+g\|_{\Lambda^p(u)} &\geq \Big(\|f\|^p_{\Lambda^p(u)} + \|g\|^p_{\Lambda^p(u)}\Big)^{1/p} \\ &\geq a^{1-1/p}\|f\|_{\Lambda^p(u)} + (1-a)^{1-1/p}\|g\|_{\Lambda^p(u)}, \end{aligned}$$

if $0 \leq a \leq 1$.

Let now $f, g \in \Lambda^q(v)[\Lambda^p(u)]$ be disjointly supported. Then,

$$\begin{aligned} &M_{(p)}(\Lambda^p(u))\|f+g\|_{\Lambda^q(v)[\Lambda^p(u)]} \\ &= M_{(p)}(\Lambda^p(u))\Big(\int_0^\infty \Big[\Big(\int_0^\infty (f+g)^*_y(\cdot,t)^p u(t)dt\Big)^*(s)\Big]^{q/p} v(s)ds\Big)^{1/q} \\ &\geq \Big(\int_0^\infty [a^{1-1/p}\|f_y(s)\|^q_{\Lambda^p(u)} + (1-a)^{1-1/p}\|g_y(s)\|^q_{\Lambda^p(u)}]v(s)ds\Big)^{1/q} \\ &\geq a^{1-\frac{1}{pq}}\|f\|_{\Lambda^q(v)[\Lambda^p(u)]} + (1-a)^{1-\frac{1}{pq}}\|g\|_{\Lambda^q(v)[\Lambda^p(u)]}. \end{aligned}$$

Finally, choosing

$$a = \frac{\|f\|^{pq}_{\Lambda^q(v)[\Lambda^p(u)]}}{\|f\|^{pq}_{\Lambda^q(v)[\Lambda^p(u)]} + \|g\|^{pq}_{\Lambda^q(v)[\Lambda^p(u)]}}$$

it follows that

$$M_{(p)}(\Lambda^p(u))\|f+g\|_{\Lambda^q(v)[\Lambda^p(u)]} \geq \Big(\|f\|^{pq}_{\Lambda^q(v)[\Lambda^p(u)]} + \|g\|^{pq}_{\Lambda^q(v)[\Lambda^p(u)]}\Big)^{\frac{1}{pq}}.$$

Observe that, if U is superadditive, then $M_{(p)}(\Lambda^p(u)) = 1$.

Remark 3.2.19. *In Example 3.2.9, if w is decreasing (resp. increasing), then for all $q > p$, $\Lambda^p(w)$ is q-concave with constant one (resp. q-convex with constant one for all $0 < q < p$) if and only if $\Lambda^p(w)$ is isometric to L^p. See [KaP, Corollary 4].*

For a Lorentz-Karamata space, we have (see [EP])

$$\begin{aligned} \|f\|_{p,q,b} &= \Big(\int_0^\infty \Big[t^{1/p-1/q} f^*(t)b(t)\Big]^q dt\Big)^{1/q} \\ &= \Big(\int_0^\infty f^*(t)^q t^{q/p-1} b(t)^q dt\Big)^{1/q} = \|f\|_{\Lambda^q(w)}, \end{aligned}$$

where w is the weight defined as $w(s) := s^{q/p-1}b(s)^q$, $s > 0$. For $q > p$, since b is s. v., then $w^{1/q}$ and w are quasi-increasing. Then for $0 < x \leq y$

$$\begin{aligned} W(x) + W(y) &= \int_0^x w(s)ds + \int_0^y w(s)ds \\ \int_0^x w(s)ds + \int_x^{x+y} w(s-x)ds &\lesssim \int_0^x w(s)ds + \int_x^{x+y} w(s)ds = W(x+y). \end{aligned}$$

Therefore

$$\left(W(x)^{q/p}+W(y)^{q/p}\right)^{p/q}\le W(x)+W(y)\lesssim W(x+y),$$

and $W^{q/p}$ *is also quasi-superadditive. Hence, by [CSo, Lemma 3.2],* $\Lambda^p(w)$ *satisfies a lower* q*-estimate and also* $\Lambda^q(w)$*. For* $q\le p$*, by [KaMa, Theorem 6],* $\Lambda^p(w)$ *is not* q*-concave.*

A function ϕ *is said to satisfy* (RC) *if*

$$\frac{\phi(au)}{\phi(u)}+\frac{\phi((1-a)v)}{\phi(v)}\ge 1 \text{ for all } u,v>0 \text{ and } 0<a<1.$$

Assume that ϕ *is an Orlicz function, that is,* ϕ *is strictly increasing and continuous with* $\lim_{u\to\infty}\phi(u)=\infty$*,* $\phi(0)=0$ *and* $\phi(1)=1$*. When* $\mu(\Omega)<\infty$*, if* $\phi(u^{1/p})$ *satisfies (RC), by [HKT, Corollary 3.3], it follows that* $L_\phi(\Omega)$ *satisfies a lower* p*-estimate with constant one. If* $\mu(\Omega)=\infty$*, then* $\phi(u^{1/p})$ *satisfies (RC) if and only if* $L_\phi(\Omega)$ *satisfies a lower* p*-estimate with constant one.*

3.3 Sobolev capacitary inequalities

Submeasures: If $\mathcal{A}$ is an algebra of subsets on Ω, a set-function $\phi:\mathcal{A}\to\mathbb{R}$ is called *monotone* if it satisfies $\phi(\emptyset)=0$ and $\phi(A)\le\phi(B)$ whenever $A\subset B$, and that ϕ is *normalized* when $\phi(\Omega)=1$. A monotone set-function ϕ is a *submeasure* if

$$\phi(A\cup B)\le\phi(A)+\phi(B)$$

whenever $A,B\in\mathcal{A}$ are disjoint, and ϕ is a *supermeasure* if

$$\phi(A\cup B)\ge\phi(A)+\phi(B)$$

whenever $A,B\in\mathcal{A}$ are disjoint.

Definition 3.3.1. *For any* $0<p<\infty$*, we say that a monotone set-function* ϕ *satisfies an* upper p-estimate *if* ϕ^p *is a submeasure, and a* lower p-estimate *if* ϕ^p *is a supermeasure.*

In the proof of Theorem 3.3.5, we shall use [KMo, Theorem 2.2], where it is shown that, if $0<p<1$ and φ is a normalized supermeasure which satisfies an upper p-estimate, then there exists a measure μ on Ω such that $\varphi\le\mu$ and $\mu(\Omega)\le K_p$, where

$$K_p=\frac{2}{(2^p-1)^{1/p}}-1.$$

For a more complete treatment, see [KMo] and the references quoted therein.

From now on, let Ω be a domain of $\mathbb{R}^n$ endowed with the *Lebesgue measure* m_n and $X = X(\Omega)$ a quasi-Banach function space on Ω.

Given a compact set $K \subset \Omega$ and an open set $G \subset \Omega$ containing K, the couple (K, G) is called a *conductor* and we denote

$$W(K,G) := \{u \in \mathrm{Lip}_0(G);\ u = 1 \text{ on a neighbourhood of } K,\ 0 \le u \le 1\}.$$

Each conductor has an X-*capacity* defined by

$$\mathrm{Cap}_X(K,G) := \inf\{\|\nabla u\|_X;\ u \in W(K,G)\},$$

that for $X = L^{p,q}$ recovers the capacity $\mathrm{Cap}_X = \mathrm{cap}_{p,q}^{1/p}$ from [CosMa].

We will write $\mathrm{Cap}_X(\cdot) = \mathrm{Cap}_X(\cdot, \Omega)$ if Ω has been fixed.

From the definition (see [Ma85], [Ma05] and [Cos]) we have:

Theorem 3.3.2. *The set function $(K,G) \to \mathrm{Cap}_X(K,G)$, where K is a compact subset of the open set $G \subset \Omega$, enjoys the following properties:*

(i) If $K_1 \subset K_2$ are compact sets in G, $\mathrm{Cap}_X(K_1,G) \le \mathrm{Cap}_X(K_2,G)$.

(ii) If $\Omega_1 \subset \Omega_2$ are open sets and K is a compact subset of Ω_1, then

$$\mathrm{Cap}_X(K,\Omega_2) \le \mathrm{Cap}_X(K,\Omega_1).$$

(iii) If $\{K_i\}_{i=1}^\infty$ is a decreasing sequence of compact subsets of G with $K := \bigcap_{i=1}^\infty K_i$, then

$$\mathrm{Cap}_X(K,G) = \lim_{i\to\infty} \mathrm{Cap}_X(K_i,G).$$

(iv) If $\{\Omega_i\}_{i=1}^\infty$ is an increasing sequence of open subsets of Ω with $\Omega := \bigcup_{i=1}^\infty \Omega_i$ and K is a compact subset of Ω_1, then

$$\mathrm{Cap}_X(K,\Omega) = \lim_{i\to\infty} \mathrm{Cap}_X(K,\Omega_i).$$

Proof. (i) Let $u \in W(K_2,G)$, then $u \in W(K_1,G)$. Hence by definition, we get that

$$\mathrm{Cap}_X(K_1,G) \le \mathrm{Cap}_X(K_2,G).$$

(ii) Let $\Omega_1 \subset \Omega_2$ be two open subsets and K be a compact subset of Ω_1. Hence K is a compact subset of Ω_2. Let $u \in \mathrm{Lip}_0(\Omega_1)$. Extending u by zero, we have that $u \in \mathrm{Lip}_0(\Omega_2)$. Hence $W(K,\Omega_1) \subset W(K,\Omega_2)$, and therefore

$$\mathrm{Cap}_X(K,\Omega_2) \le \mathrm{Cap}_X(K,\Omega_1).$$

(iii) Let $\{K_i\}_{i=1}^{\infty}$ be a decreasing sequence of compact subsets of G with $K := \bigcap_{i=1}^{\infty} K_i$. Let $b := \lim_{i\to\infty} \mathrm{Cap}_X(K_i, G)$. Consider any $u \in W(K, G)$. There exists, by definition, an open set E in Ω such that $K \subset E \subset G$ where $u = 1$. We have that

$$K \subset E \subset \{x \in \Omega; u(x) = 1\} \subset \operatorname{supp} u \subset G$$

is compact, and hence $\{u = 1\}$ is compact. Therefore for sufficiently large j, since $K_i \downarrow K$, we have $K \subset K_j \subset E$ and then

$$\lim_{i\to\infty} \mathrm{Cap}_X(K_i, G) \quad \leq \quad \mathrm{Cap}_X(K_j, G) \leq \|\nabla u\|_X.$$

Taking infimum over all $u \in W(K, G)$ the conclusion follows.

(iv) Let $\{\Omega_i\}_{i=1}^{\infty}$ be an increasing sequence of open sets with $\Omega := \cup_{i=1}^{\infty}\Omega_i$, and K be a compact subset of Ω_1. Let $b := \lim_{i\to\infty} \mathrm{Cap}(K, \Omega_i)$. Consider any $u \in W(K, \Omega)$. Since $\operatorname{supp} u$ is a compact subset of the open set Ω then, for large i, $u \in \mathrm{Lip}_0(\Omega_i)$ and $u = 1$ in a neighbourhood of K. Hence $u \in W(K, \Omega_i)$ and $\mathrm{Cap}_X(K, \Omega_i) \leq \|\nabla u\|_X$. Therefore, $\lim_{i\to\infty} \mathrm{Cap}_X(K, \Omega_i) \leq \|\nabla u\|_X$ and taking infimum over all $W(K, \Omega)$, we see that $b \leq \mathrm{Cap}(K, \Omega)$.

Moreover, since $\Omega_i \subset \Omega$ and both are open subsets, by (ii), $\mathrm{Cap}_X(K, \Omega) \leq \mathrm{Cap}_X(K, \Omega_i)$ for all $i \in \mathbb{N}$ and then the reverse inequality follows. □

With similar techniques to the ones in [Cos] we obtain:

Proposition 3.3.3. *Let $G \subset \mathbb{R}^n$ be an open set and $\{K_i\}_{i\in\mathbb{N}}$ a sequence of compact subsets in G. If $K := \cup_{i=1}^{k} K_i \subset G$, then*

$$\mathrm{Cap}_X(K, G) \leq \sum_{i=1}^{k} \mathrm{Cap}_X(K_i, G),$$

where $k \geq 1$ is a positive integer.

Proof. Suppose that $k = 2$. Consider any $u_i \in W(K_i, G)$ and define $u := \max(u_1, u_2)$. We have that $u_1 \in \mathrm{Lip}(G)$, $u_1 = 1$ in a neighbourhood of K_1 and $u_1 = 0$ on ∂G and $u_2 \in \mathrm{Lip}(G)$, $u_2 = 1$ in a neighbourhood of K_2 and $u_2 = 0$ on ∂G. Hence $u \in \mathrm{Lip}(G)$, $u = 1$ in a neighbourhood of $K_1 \cup K_2$, $u = 0$ on ∂G. Then $u \in W(K_1 \cup K_2, G)$ where $K_1 \cup K_2$ is bounded on G. Therefore $\mathrm{Cap}_X(K_1 \cup K_2, G) \leq \|\nabla u\|_X$.

Moreover $|\nabla u| \leq \max(|\nabla u_1|, |\nabla u_2|)$. Define $f_3 := \max(|\nabla u_1|, |\nabla u_2|)$ and $f_1 := |\nabla u_1|$, $f_2 := |\nabla u_2|$. Then

$$\|\nabla u\|_X \leq \|f_3\|_X \leq \|f_1 + f_2\|_X \leq \|f_1\|_X + \|f_2\|_X,$$

$$\mathrm{Cap}_X(K_1 \cup K_2, G) \leq \|\nabla u\|_X \leq \|\nabla u_1\|_X + \|\nabla u_2\|_X$$

and, taking infimum over $W(K_i, G)$ $(i = 1, 2)$, we obtain that $\mathrm{Cap}_X(K_1 \cup K_2, G) \leq \mathrm{Cap}_X(K_1, G) + \mathrm{Cap}_X(K_2, G)$.

Suppose now that for all $k < m$ we have that $\mathrm{Cap}_X(\cup_{i=1}^k K_i, G) \leq \sum_{i=1}^\infty \mathrm{Cap}_X(K_i, G)$ and let $k = m$. Then

$$\begin{aligned}\mathrm{Cap}_X(\cup_{i=1}^m K_i, G) &= \mathrm{Cap}_X\Big(K_m \cup \bigcup_{i=1}^{k-1} K_i, G\Big) \leq \mathrm{Cap}_X(K_m, G) + \mathrm{Cap}_X\Big(\bigcup_{i=1}^{k-1} K_i, G\Big) \\ &\leq \mathrm{Cap}_X(K_m, G) + \sum_{i=1}^{k} \mathrm{Cap}_X(K_i, G) = \sum_{i=1}^{m} \mathrm{Cap}_X(K_i, G).\end{aligned}$$

□

With similar techniques to the ones in [CosMa] we get:

Proposition 3.3.4. *Let X be an r.i space satisfying a weak lower $1-$estimate. Suppose that $\Omega_1, \ldots, \Omega_k$ are k pairwise disjoint open sets and K_i are compact subsets of Ω_i for $i = 1, ..., k$. Then*

$$\mathrm{Cap}_X\Big(\bigcup_{i=1}^{k} K_i, \bigcup_{i=1}^{k} \Omega_i\Big) \geq \sum_{i=1}^{k} \mathrm{Cap}_X(K_i, \Omega_i).$$

Proof. A finite induction on k would prove it, so we can assume that $k = 2$. Let $u \in \mathrm{Lip}_0(\Omega_1 \cup \Omega_2)$ and $u_i := \chi_{\Omega_i} u$, $i = 1, 2$. Consider $i \in \{1, 2\}$ and v_i be the restriction of u to Ω_i. Then $v_i \in \mathrm{Lip}_0(\Omega_i)$. We notice that u_i can be regarded as the extension of v_i by 0 to $\Omega_1 \cup \Omega_2$.

If $u \in W(K_1 \cup K_2, \Omega_1 \cup \Omega_2)$, then $u \in \mathrm{Lip}_0(\Omega_1 \cup \Omega_2)$ and $u = 1$ in a neighbourhood of $K_1 \cup K_2$. Hence, since $v_i = u|_{\Omega_i}$, it follows that $v_i = u$ on $K_i \subset \Omega_i$ for $i \in \{1, 2\}$. Then $v_i \in \mathrm{Lip}_0(\Omega_i)$ and $v_i = 1$ on a neighbourhood of K_i, that is, $v_i \in W(K_i, \Omega_i)$.

Conversely for $i \in \{1, 2\}$, if $v_i \in W(K_i, \Omega_i)$, then $v_i \in \mathrm{Lip}_0(\Omega_i)$ and $v_i = 1$ in a neighbourhood of K_i. Therefore, since $v_i = u|_{\Omega_i}$, it follows that $u = 1$ in a neighbourhood of $K_1 \cup K_2$, $v_1 + v_2 = u$. And, since $v_i \in \mathrm{Lip}_0(\Omega_i)$, then u is Lipschitz on $\Omega_1 \cup \Omega_2$, which gives $u \in W(K_1 \cup K_2, \Omega_1 \cup \Omega_2)$.

Since Ω_1 and Ω_2 are disjoint and $u = u_1 + u_2$, with the functions u_i supported on Ω_i for $i = 1, 2$, we obtain that $u = u\chi_{\Omega_1 \cup \Omega_2} = u(\chi_{\Omega_1} + \chi_{\Omega_2}) = u_1 + u_2$ and, by the weak lower $1-$estimate

$$\|\chi_{\Omega_1} \nabla u\|_X + \|\chi_{\Omega_2} \nabla u\|_X \leq \|\chi_{\Omega_1 \cup \Omega_2} \nabla u\|_X = \|\nabla u\|_X.$$

For $i \in \{1,2\}$, in Ω_i we have that $u_i = v_i$, and then

$$\|\chi_{\Omega_1}\nabla v_1\|_X + \|\chi_{\Omega_2}\nabla v_2\|_X \quad \leq \quad \|\nabla u_1\|_X + \|\nabla u_2\|_X.$$

Therefore, since $u \in W(K_1 \cup K_2, \Omega_1 \cup \Omega_2)$, it follows that

$$\mathrm{Cap}_X(K_1 \cup K_2, \Omega_1 \cup \Omega_2) \geq \mathrm{Cap}_X(K_1, \Omega_1) + \mathrm{Cap}_X(K_2, \Omega_2)$$

and in general too. □

As we showed in Example 1.2.9, for this capacity every open and every compact set in Ω is capacitable. Recall that $X = X(\Omega)$ is a quasi-Banach function space on Ω endowed with the *Lebesgue measure* m_n.

Theorem 3.3.5. *Suppose* $0 < p < \infty$ *and let* $a > 1$ *be a constant. If* X *is a Banach function space that satisfies a lower* $p-$*estimate, then*

$$\int_0^\infty t^p \mathrm{Cap}_X\Big(\overline{\{|f| > at\}}, \{|f| > t\}\Big)^p \frac{dt}{t} \leq c\|\nabla f\|_X^p \quad (f \in \mathrm{Lip}_0(\Omega)), \tag{3.5}$$

where c *is a constant that depends on* a, p *and* $M_{(p)}(X)$.

In particular,

$$\int_0^\infty t^p \mathrm{Cap}_X(\{|f| \geq t\})^p \frac{dt}{t} \leq 2^p c\|\nabla f\|_X^p \qquad (f \in \mathrm{Lip}_0(\Omega)), \tag{3.6}$$

where c *depends on* p *and* $M_{(p)}(X)$.

Proof. Without loss of generality, assume that $\|\nabla f\|_X < \infty$, and that $f \geq 0$, since $|\nabla |f|| \leq |\nabla f|$.

Since X is a Banach function space, the set-function

$$\phi(A) := \frac{\||\nabla f|\chi_A\|_X}{\|\nabla f\|_X} \qquad (A \in \mathcal{B}(\Omega))$$

is a submeasure. Moreover, using that X satisfies a lower $p-$estimate, we conclude that, if $A_1, \cdots, A_1$ are disjoint, then

$$\phi(A_1 \cup \cdots \cup A_n) \geq \frac{1}{M_{(p)}(X)} \left(\phi^p(A_1) + \cdots + \phi^p(A_n)\right)^{1/p}. \tag{3.7}$$

Let us consider the set-function ψ, defined by

$$\psi(A) := \sup\Big\{\sum_{i=1}^n \phi^p(A_i)\Big\}, \tag{3.8}$$

the supremum being taken over all finite partitions $(A_1, \cdots, A_n)$ of A.

It follows from (3.7) and (3.8) that

$$\frac{\psi}{\big(M_{(p)}(X)\big)^p} \leq \phi^p \leq \psi, \tag{3.9}$$

and we claim that ψ is a supermeasure that satisfies an upper $\min(p, 1/p)-$ estimate. Indeed, given any $\varepsilon > 0$ and two disjoint sets A and B, choose finite partitions $A = \bigcup_{i=1}^{n_a} A_i$, $B = \bigcup_{j=1}^{n_b} B_j$ such that

$$\psi(A)(1-\varepsilon) \leq \sum_{i=1}^{n_a} \phi^p(A_i) \ \text{ and } \psi(B)(1-\varepsilon) \leq \sum_{j=1}^{n_b} \phi^p(B_j).$$

Then $\{D_k\}_{k=1}^{n_a+n_b} = \{A_k\}_{k=1}^{n_a} \cup \{B_k\}_{k=1}^{n_b}$ is a partition of $A \cup B$ which satisfies

$$\begin{aligned}\psi(A)(1-\varepsilon) + \psi(B)(1-\varepsilon) &\leq \sum_{i=1}^{n_a} \phi^p(A_i) + \sum_{j=1}^{n_b} \phi^p(B_j) \\ &\leq \sum_{k=1}^{n_a+n_b} \phi^p(D_k) \leq \psi(A \cup B)\end{aligned}$$

and ψ is a supermeasure.

Let $r = \min(p, 1/p)$. Recall that ψ satisfies an upper $r-$estimate if ψ^r is a submeasure.

Suppose first $p \geq 1$, that is $r = 1/p$, and let A, B be disjoint sets. If $(C_1, \ldots, C_n)$ is a partition of $A \cup B$, then, since ϕ is a submeasure,

$$\begin{aligned}\Big(\sum \phi^p(C_i)\Big)^{1/p} &= \Big(\sum_i \phi^p((C_i \cap A) \cup (C_i \cap B))\Big)^{1/p} \\ &\leq \Big(\sum_i \big(\phi(C_i \cap A) + \phi(C_i \cap B)\big)^p\Big)^{1/p} \\ &= \Big\|\{\phi(C_i \cap A) + \phi(C_i \cap B)\}_{i=1}^n\Big\|_{\ell^p} \\ &\leq \Big\|\{\phi(C_i \cap A)\}_{i=1}^n\Big\|_{\ell^p} + \Big\|\{\phi(C_i \cap B)\}_{i=1}^n\Big\|_{\ell^p} \\ &= \Big(\sum \phi^p(C_i \cap A)\Big)^{1/p} + \Big(\sum \phi^p(C_i \cap B)\Big)^{1/p} \\ &\leq \psi(A)^{1/p} + \psi(B)^{1/p}.\end{aligned}$$

Therefore, taking the supremum over all partitions we obtain that

$$\psi(A \cup B) = \sup \sum \phi^p(C_i) \leq \big(\psi(A)^{1/p} + \psi(B)^{1/p}\big)^p$$

and $\psi^{1/p}$ is a submeasure.

If $p < 1$ and $(C_1, \ldots, C_n)$ is a partition of $A \cup B$, then, since ϕ is a submeasure, using that

$$(x+y)^p \leq x^p + y^p \qquad (x, y \geq 0),$$

we have that

$$\begin{aligned}\Big(\sum \phi^p(C_i)\Big)^p &\leq \Big(\sum_i \big(\phi(C_i \cap A) + \phi(C_i \cap B)\big)^p\Big)^p \\ &\leq \Big(\sum \phi^p(C_i \cap A)\Big)^p + \Big(\sum \phi^p(C_i \cap B)\Big)^p \\ &\leq \psi(A)^p + \psi(B)^p.\end{aligned}$$

Therefore, taking the supremum over all partitions we obtain that

$$\psi(A \cup B) = \sup \sum \phi^p(C_i) \leq (\psi(A)^p + \psi(B)^p)^{1/p}$$

and ψ^p is a submeasure.

We normalize ψ and define

$$\varphi(A) := \frac{\psi(A)}{\psi(\Omega)},$$

a normalized supermeasure which satisfies an upper $r-$estimate. Thus, by [KMo, Theorem 2.2], there is a measure μ on Ω such that

$$\varphi \leq \mu \quad \text{and} \quad \mu(\Omega) \leq K_r. \tag{3.10}$$

Now, if $M_t := \{|f| > t\} = \{f > t\}$ for $t > 0$, the function $\gamma(t) := \mu(M_t)$ is decreasing on $(0, \infty)$ and the limits $\gamma(0)$ and $\gamma(\infty)$ exist, so that

$$\int_0^\infty (\gamma(t) - \gamma(at))\frac{dt}{t} = \lim_{\varepsilon \to 0, N \to \infty} \int_\varepsilon^N (\gamma(t) - \gamma(at))\frac{dt}{t}$$

as an improper integral.

We have

$$\mu(M_t) = \mu(M_t \setminus M_{at}) + \mu(M_{at})$$

and therefore

$$\int_0^\infty \mu(M_t \setminus M_{at})\frac{dt}{t} = \int_0^\infty (\mu(M_t) - \mu(M_{at}))\,\frac{dt}{t} = \mu(M_0)\log a.$$

By (3.9) and (3.10) we obtain

$$\begin{aligned}\int_0^\infty \mu(M_t \setminus M_{at})\frac{dt}{t} &\geq \int_0^\infty \varphi(M_t \setminus M_{at})\frac{dt}{t} = \int_0^\infty \frac{\psi(M_t \setminus M_{at})}{\psi(\Omega)}\frac{dt}{t} \\ &\geq \frac{1}{\psi(\Omega)}\int_0^\infty \phi^p(M_t \setminus M_{at})\frac{dt}{t} \\ &= \frac{1}{\psi(\Omega)}\int_0^\infty \frac{\left\| |\nabla f|\, \chi_{M_t\setminus M_{at}}\right\|_X^p}{\|\nabla f\|_X^p}\frac{dt}{t},\end{aligned}$$

so that

$$\begin{aligned}\int_0^\infty \left\| |\nabla f|\, \chi_{M_t\setminus M_{at}}\right\|_X^p \frac{dt}{t} &\leq \psi(\Omega)\mu(M_0)\log a\, \|\nabla f\|_X^p \\ &\leq K_r M_{(p)}(X)^p \log a\, \|\nabla f\|_X^p\,.\end{aligned} \tag{3.11}$$

Consider now

$$\Lambda_t(f) = \min\left\{\frac{(|f|-t)_+}{(a-1)t}, 1\right\}.$$

Since $f \in \mathrm{Lip}_0(\Omega)$, an easy computation shows that

$$|\nabla \Lambda_t(f)| = \frac{1}{(a-1)t}|\nabla f|\chi_{M_t\setminus M_{at}}$$

and obviously

$$\left\| |\nabla f|\chi_{M_t\setminus M_{at}}\right\|_X^p = (a-1)^p t^p \left\| |\nabla \Lambda_t(f)|\right\|_X^p.$$

Moreover, since $\Lambda_t(f) \in W(\overline{M}_{at}, M_t)$,

$$\left\| |\nabla f|\chi_{M_t\setminus M_{at}}\right\|_X^p \geq (a-1)^p t^p \mathrm{Cap}_X(\overline{M}_{at}, M_t)^p,$$

and the proof of (3.5) with

$$c := c(a, p, M_{(p)}(X)) = \frac{M_{(p)}(X)^p K_r \log a}{(a-1)^p}$$

ends by inserting the last estimate in the left hand side of (3.11).

If $p = 1$, then X satisfies a lower $1-$estimate and it follows from [LiZa, Proposition 1.f.7] that X is q-concave for all $q > 1$. Therefore, X can be equivalently renormed so that, with the new norm, it satisfies a lower q-estimate with constant one. Hence, the result follows with similar arguments to those in [CosMa].

The capacitary inequality (3.6) follows using (3.5) with $a = 2$ and

$$\mathrm{Cap}_X(\overline{M}_{at}) \leq \mathrm{Cap}_X(\overline{M}_{at}, M_t).$$

In this case

$$2^p c = 2^p c(2, p, M_{(p)}(X)) = M_{(p)}(X))^p K_r 2^p \log 2. \qquad \square$$

Theorem 3.3.5 can be extended to the setting of quasi-Banach spaces using Aoki-Rolewicz's Theorem (see e.g. [BeLo, Section 3.10]):

Theorem 3.3.6. *Suppose $0 < p < \infty$ and let $a > 1$ be a constant. If X is a quasi-Banach function space which satisfies a lower $p-$estimate, then*

$$\int_0^\infty t^p \mathrm{Cap}_X\Big(\overline{\{|f| > at\}}, \{f > t\}\Big)^p \frac{dt}{t} \le c_1 \|\nabla f\|_X^p \quad (f \in \mathrm{Lip}_0(\Omega)),$$

where the constant c_1 depends on $a, p, M_{(p)}(X)$ and on the quasi-subadditivity constant c of the quasi-norm in X.

In particular,

$$\int_0^\infty t^p \mathrm{Cap}_X(\{|f| \ge t\})^p \frac{dt}{t} \le 2^p c_1 \|\nabla f\|_X^p \qquad (f \in \mathrm{Lip}_0(\Omega)),$$

with c_1 depending on $p, M_{(p)}(X)$ and c.

Proof. The proof of Theorem 3.3.5 can be adapted to this case as follows.

By Aoki-Rolewicz's Theorem, if ϱ is defined as $(2c)^\varrho = 2$, there is a 1-seminorm $\|\cdot\|^*$ such that, for all $f \in X$,

$$\|f\|^* \le \|f\|_X^\varrho \le 2\|f\|^*.$$

Endowed with this 1-seminorm X satisfies a lower $p/\varrho-$estimate, since, if $f_1, \cdots, f_n$ are disjointly supported functions in X, then

$$\begin{aligned}\Big(\sum_{i=1}^n (\|f_i\|^*)^{p/\varrho}\Big)^{\varrho/p} &\le \Big(\sum_{i=1}^n \|f_i\|_X^p\Big)^{\varrho/p} \le M_{(p)}(X)^\varrho \Big\|\sum_{i=1}^n |f_i|\Big\|_X^\varrho \\ &\le 2M_{(p)}(X)^\varrho \Big\|\sum_{i=1}^n |f_i|\Big\|^*.\end{aligned}$$

Now consider

$$\psi(A) = \sup\left\{\sum_{i=1}^n \phi^{p/\varrho}(A_i)\right\}$$

with

$$\phi(A) = \frac{\||\nabla f|\,\chi_A\|^*}{\|\nabla f\|^*}.$$

With the same arguments as in Theorem 3.3.5, it can be shown that ψ is a supermeasure that satisfies an upper r-estimate and the proof ends in the same way, now with

$$c_1 := c_1(a, p, c, M_{(p)}(X)) = \frac{2^{2p/\varrho} K_r \log a M_{(p)}(X)^p}{(a-1)^p},$$

for ϱ such that $(2c)^\varrho = 2$ and $r = \min(p/\varrho, \varrho/p)$. □

Definition 3.3.7. *An extended real-valued function f is called* upper semicontinuous at a point x_0 *if for every $\varepsilon > 0$, there exists a neighbourhood U of x_0 such that $f(x) \leq f(x_0) + \varepsilon$ for all $x \in U$. The function is called* upper semicontinuous *if it is upper semicontinuous at all points in the domain.*

As a remark, we observe that the function

$$\mathrm{Cap}_X(\overline{M}_{at}, M_t)$$

in Theorem 3.3.5 is certainly a measurable function. For that is enough to show that is upper semicontinuous.

Lemma 3.3.8. *Let $f \in \mathrm{Lip}_0(\Omega)$ and $a > 1$ be a constant. Then the function $\psi : t \to \mathrm{Cap}_X(\overline{M_{at}}, M_t)$ is upper semicontinuous.*

Proof. Let $t_0 > 0$ and $\varepsilon > 0$. There exists $u_\varepsilon \in W(\overline{M_{at_0}}, M_{t_0})$ such that $\|\nabla u_\varepsilon\|_X < \mathrm{Cap}_X(\overline{M_{at_0}}, M_{t_0}) + \varepsilon$. Since $u_\varepsilon = 1$ in a neighbourhood of $\overline{M_{at_0}}$, there exists an open set g neigbourhood of $\overline{M_{at_0}}$ such that $u_\varepsilon = 1$ in g.

For $\lambda_1 > 0$ we have that $\overline{M_{at_0}} \subset \overline{M_{a(t_0-\lambda_1)}}$. Since $\overline{M_{at_0}} \subset g$, for $\lambda_1 > 0$ small enough we have that $\overline{M_{a(t_0-\lambda_1)}} \subset g$.

Since $u_\varepsilon = 0$ on ∂M_{t_0}, then supp u_ε is a compact subset of M_{t_0}. Hence, there exists an open set G such that supp $u_\varepsilon \subset G \subset\subset M_{t_0}$. For all $\lambda_2 > 0$ we have $M_{t_0+\lambda_2} \subset M_{t_0}$. So that, there exists λ_2 small enough such that $\overline{G} \subset M_{t_0+\lambda_2}$.

Then, for $\lambda \leq \min(\lambda_1, \lambda_2)$ we have that $\overline{M_{a(t_0-\lambda)}} \subset g$, and $\overline{G} \subset M_{t_0+\lambda}$. From the choice of g and G we have that $u_\varepsilon \in W(K, \Omega)$ whenever $K \subset g$ and $G \subset \Omega$. In particular, for $K = \overline{M_{a(t_0-\lambda)}}$ and $\Omega = M_{t_0+\lambda}$ we have that, $u_\varepsilon \in W(\overline{M_{a(t_0-\lambda)}}, M_{t_0+\lambda})$ being $\|\nabla u_\varepsilon\|_X < \mathrm{Cap}_X(\overline{M_{at_0}}, M_{t_0}) + \varepsilon$, and hence

$$\mathrm{Cap}_X(\overline{M_{a(t_0-\lambda)}}, M_{t_0+\lambda}) \leq \|\nabla u_\varepsilon\|_X < \mathrm{Cap}_X(\overline{M_{at_0}}, M_{t_0}) + \varepsilon.$$

Therefore, $\mathrm{Cap}_X(\overline{M_{a(t_0-\lambda)}}, M_{t_0+\lambda}) \leq \mathrm{Cap}_X(\overline{M_{at_0}}, M_{t_0})$ when $\varepsilon \to 0$ for $\lambda \leq \min(\lambda_1, \lambda_2)$.

Let t be close enough to t_0 meaning that $|t-t_0| \leq \min(\lambda_1, \lambda_2)$. If $t < t_0$, then $\overline{M_{at_0}} \subset \overline{M_{at}}$, $\overline{M_{t_0}} \subset \overline{M_t}$. There exists $\lambda > 0$ small enough such that $t = t_0 - \lambda$. Hence, by Theorem 3.3.2, (ii), since $M_{t_0+\lambda} \subset M_{t_0} \subset M_{t_0-\lambda}$

$$\begin{aligned} \mathrm{Cap}_X(\overline{M_{at}}, M_t) &= \mathrm{Cap}_X(\overline{M_{at}}, M_{t_0-\lambda}) \leq \mathrm{Cap}_X(\overline{M_{at}}, M_{t_0+\lambda}) \\ &= \mathrm{Cap}_X(\overline{M_{a(t_0-\lambda)}}, M_{t_0+\lambda}) \leq \mathrm{Cap}_X(\overline{M_{at_0}}, M_{t_0}). \end{aligned}$$

If $t > t_0$, then there exists $\lambda > 0$ such that $t = t_0 + \lambda$. There exists $n^* \in \mathbb{N}$ such that $1/(n^*+1) \leq \lambda < 1/n^*$. Using the monotonicity of Cap_X, we deduce that

$$\mathrm{Cap}_X(\overline{M_{at}}, M_{t_0}) \leq \mathrm{Cap}_X(\overline{M_{at_0}}, M_{t_0}) + \varepsilon$$

for every t close enough to t_0. Since $\bigcup_{n>n^*} M_{t_0+1/n} = M_{t_0}$, by Theorem 3.3.2, (iv),

$$\mathrm{Cap}_X(\overline{M_{at}}, M_{t_0}) = \lim_{n\to\infty} \mathrm{Cap}_X(\overline{M_{at}}, M_{t_0+1/n})$$

and the result follows. □

3.4 Sobolev-Poincaré estimates for two measure spaces

In [CosMa], characterizations for Sobolev-Lorentz type inequalities involving two measures are proved, extending results obtained in [Ma05] and [Ma06]. Here, we extend those results and derive with similar methods necessary and sufficient conditions for such Sobolev type inequalities involving two rearrangement invariant spaces subjected to appropriate convexity conditions.

Let μ be a Borel measure on Ω and let X be an r.i. quasi-Banach function space on Ω. Recall that the *distribution function* of f is defined (see (1.1) changing C by μ) as

$$\mu_f(\lambda) := \mu\{x \in \Omega; |f(x)| > \lambda\}, \ (\lambda \geq 0),$$

and the *Lorentz spaces* $\Lambda^{p,q}(X)$ associated to X are for $0 < p < \infty$

$$\Lambda^{p,q}(X) = \Big\{ f; \|f\|_{\Lambda^{p,q}(X)} = \Big(\int_0^\infty p t^{q-1} (\varphi_X(\mu_f(t)))^{q/p} dt \Big)^{1/q} < \infty \Big\} \quad (0 < q < \infty) \quad (3.12)$$

with the usual changes when $q = \infty$ (when $p = q$ we obtain the space $\Lambda^p(X)$). Notice that, if $X = L^1$, then $\Lambda^{p,q}(L^1) = L^{p,q}$.

It is well-known that for $0 < q_0 \leq q_1 \leq \infty$

$$\Lambda^{p,q_0}(X) \subset \Lambda^{p,q_1}(X).$$

Moreover, if X is a Banach space, then

$$\Lambda^1(X) \subset X \subset \Lambda^{1,\infty}(X).$$

In fact the spaces $\Lambda^{1,1}(X)$ and $\Lambda^{1,\infty}(X)$ are respectively the smallest and largest r.i. spaces with fundamental functions equal to φ_X.

From now on in this section, let μ and ν be two Borel measures on Ω and $0 < p < \infty$. Let X be a quasi-Banach function space on (Ω, m_n), Y an r.i. space on (Ω, μ), and Z be an r.i. space on (Ω, ν).

Theorem 3.4.1. *Suppose that X satisfies a lower $p-$estimate. Then, the following properties are equivalent:*

(i) There is a constant $A > 0$ such that

$$\|f\|_{\Lambda^{1,p}(Y)} \leq A(\|\nabla f\|_X + \|f\|_{\Lambda^{1,p}(Z)}) \qquad (f \in \mathrm{Lip}_0(\Omega)).$$

(ii) There exists a constant $B > 0$ such that

$$\varphi_Y(\mu(g)) \leq B(\mathrm{Cap}_X(\overline{g}, G) + \varphi_Z(\nu(G)) \qquad (g \subset\subset G \subset\subset \Omega).$$

Proof. $(i) \Rightarrow (ii)$: Choose $g \subset\subset G \subset\subset \Omega$ and consider $f \in W(\overline{g}, G)$. Since $g \subset \{f \geq 1\}$, it follows that

$$\begin{aligned} \varphi_Y(\mu(g))^p &\leq \int_0^1 \varphi_Y(\mu(\{f \geq 1\}))^p dt^p \leq \int_0^1 \varphi_Y(\mu(\{f > t\}))^p dt^p \\ &\leq p\|f\|^p_{\Lambda^{1,p}(Y)} \lesssim \|\nabla f\|^p_X + \|f\|^p_{\Lambda^{1,p}(Z)}, \end{aligned}$$

with

$$\|f\|^p_{\Lambda^{1,p}(Z)} \leq \int_0^1 t^{p-1}\varphi_Z(\nu(G))^p dt = \frac{1}{p}\varphi_Z(\nu(G))^p,$$

and (ii) follows by taking infimum over all functions $f \in W(\overline{g}, G)$.

(ii) $\Rightarrow$ (i): From $M_t = \{|f| > t\} \subset \mathrm{supp}\, f$ and $M_{at} \subset M_t$ if $a > 1$, $\varphi_Y(\mu(M_{at}))^p \lesssim \mathrm{Cap}_X(\overline{M}_{at}, M_t)^p + \varphi_Z(\nu(M_t))^p$, and Theorem 3.3.6 yields

$$\begin{aligned} \|f\|_{\Lambda^{1,p}(Y)} &= \Big(\int_0^\infty a^{p-1}s^{p-1}\varphi_Y(\mu(M_{as}))^p a ds\Big)^{1/p} \\ &\lesssim a\Big\{\Big(\int_0^\infty s^{p-1}\mathrm{Cap}_X(\overline{M}_{as}, M_s)^p ds\Big)^{1/p} \\ &+ \Big(\int_0^\infty s^{p-1}\varphi_Z(\nu(M_s))^p ds\Big)^{1/p}\Big\} \\ &\lesssim \|\nabla f\|_X + \|f\|_{\Lambda^{1,p}(Z)}. \qquad \square \end{aligned}$$

Remark 3.4.2. *Taking $X = L^{p,q}(\Omega,\mu)$, $Y = L^{s,p}(\Omega,\eta)$ and $f \in \mathrm{Lip}_0(\Omega)$, since $L^p(\Omega,\mu) = \Lambda^{1,p}(X)$ and $L^p(\Omega,\mu) \hookrightarrow L^{r,p}(\Omega,\mu)$ then, since the relation of our capacity with the ones in [CosMa] for X is $\mathrm{Cap}_X(\overline{M_{at}}, M_t) = \mathrm{cap}_{p,q}(M_{at}, M_t)^{1/p}$, then we see that Theorem 3.4.1, (i) is an extension of [CosMa, Theorem 5.1, i)].*

Analogously, for $X = L^{p,q}(\Omega,\mu)$, $Y = L^{s,q}(\Omega,\eta)$ and $f \in \mathrm{Lip}_0(\Omega)$, then we see that Theorem 3.4.1, (ii) is an extension of [CosMa, Theorem 5.1, ii)].

Let us remark that under the same assumptions of Theorem 3.4.1, one also has the equivalence of the properties

$$\|f\|_{\Lambda^{1,p}(Y)} \lesssim \|\nabla f\|_X \qquad (f \in \mathrm{Lip}_0(\Omega)),$$

and

$$\varphi_Y(\mu(g)) \lesssim \mathrm{Cap}_X(\overline{g}, G) \qquad (g \subset\subset G \subset\subset \Omega).$$

3.5 Isocapacitary and Sobolev type inequalities

Let X be an r.i. space on $\mathbb{R}^n$. Maz'ya's classical method shows that

$$\|f\|_X \lesssim \|\nabla f\|_{L^1} \qquad (f \in \mathrm{Lip}_0(\mathbb{R}^n))$$

if and only if, for every Borel set A,

$$\varphi_X(m_n(A)) \lesssim m_n^+(A),$$

where m_n^+ is *Minkowski's perimeter* (see [Ma11] or [EvGa]) defined as

$$m_n^+(A) := \liminf_{h\to 0} \frac{m_n(A_h) - m_n(A)}{h},$$

where $A_h := \{x \in \mathbb{R}^n; d_{\mathbb{R}^n}(x, A) < h\}$.

As shown in [MMi2], the following *self-improvement property* follows for $f \in \mathrm{Lip}_0(\mathbb{R}^n)$

$$\|f\|_{\Lambda^{1,\infty}(X)} \lesssim \|\nabla f\|_{L^1} \Leftrightarrow \|f\|_X \lesssim \|\nabla f\|_{L^1} \Leftrightarrow \|f\|_{\Lambda^1(X)} \lesssim \|\nabla f\|_{L^1}.$$

This *Sobolev self-improvement* obtained in the case $q = 1$ is also extended to the case $q > 1$ as

$$\|f\|_{\Lambda^{q,\infty}(X)} \lesssim \|\nabla f\|_{L^q} \Leftrightarrow \|f\|_{\Lambda^{1,q}(X)} \lesssim \|\nabla f\|_{L^q}.$$

In particular, if X is $q-$convex, then the space

$$X_{(q)} = \{f;\, |f|^{1/q} \in X\}, \quad \|f\|_{X_{(q)}} = \left\||f|^{1/q}\right\|_X^q$$

is an r.i. space and

$$\Lambda^{1,q}(X) = \Lambda^q(X_{(q)}) \subset X_{(q)} \subset \Lambda^{1,\infty}(X_{(q)}).$$

In summary, in terms of the $X_{(q)}$ *scale* of spaces, on Lipschitz functions we have the following equivalences (see [MMiP])

$$\|f\|_{\Lambda^{1,\infty}(X_{(q)}))} \lesssim \|\nabla f\|_{L^q} \Leftrightarrow \|f\|_{\Lambda^q(X_{(q)})} \lesssim \|\nabla f\|_{L^q} \Leftrightarrow \|f\|_{X_{(q)}} \lesssim \|\nabla f\|_{L^q}.$$

In this section we shall extend this result to the setting of r.i. quasi-Banach spaces. As an application of Theorem 3.3.6, we characterize *Sobolev type estimates* in terms of isocapacitary inequalities.

From now on, Ω will be a domain in $\mathbb{R}^n$, X a quasi-Banach function space on (Ω, m_n), μ a Borel measure on Ω, and Y an r.i. space on (Ω, μ). An *isocapacitary inequality* is an inequality of the form $\mathrm{Cap}_X(K) \geq J(\mu(K))$, where J is a non-negative function and K is any compact set in Ω.

Proposition 3.5.1. *If*

$$\sup \frac{\varphi_Y(\mu(g))}{\mathrm{Cap}_X(\overline{g}, G)} < \infty,$$

the supremum being taken over all sets g, G such that $g \subset\subset G \subset\subset \mathbb{R}^n$, then for every compact subset K in Ω,

$$\varphi_Y(\mu(K)) \lesssim \mathrm{Cap}_X(K).$$

Proof. Let K be a compact subset in Ω and $d := d(K, \Omega^c) > 0$. Denote $\lambda_n = 1/n$ and consider the smallest $n \in \mathbb{N}$, n^*, such that $1/n^* \leq d$. For each $n \geq n^*$, let

$$\mathcal{G}(\lambda_n) := \{x \in \Omega; d(K, x) < \lambda_n\},\ \mathcal{K}(\lambda_n) := \{x \in \Omega; d(K, x) \leq \lambda_n\}.$$

Then $\bigcap_{n \geq n^*} \overline{\mathcal{G}(\lambda_n)} = K$ and $\bigcap_{n \geq n^*} \mathcal{K}(\lambda_n) = K$. Since

$$\varphi_Y(\mu(\mathcal{G}(\lambda_k))) \lesssim \mathrm{Cap}_X(\overline{\mathcal{G}(\lambda_k)}) \qquad (k \geq n^*),$$

by the properties of Cap_X,

$$\varphi_Y(\mu(K)) \leq \lim_{k \to \infty} \varphi_Y(\mu(\mathcal{G}(\lambda_k))) \lesssim \lim_{k \to \infty} \mathrm{Cap}_X(\overline{\mathcal{G}(\lambda_k)}) = \mathrm{Cap}_X(K),$$

and the result follows. □

Theorem 3.5.2. *Let $0 < p < \infty$. If X satisfies a lower $p-$estimate, then the following properties are equivalent:*

(i) $\varphi_Y(\mu(K)) \lesssim \mathrm{Cap}_X(K)$ *for every compact set* K *on* Ω.

(ii) $\|f\|_{\Lambda^{1,p}(Y)} \lesssim \|\nabla f\|_X \quad (f \in \mathrm{Lip}_0(\Omega))$.

(iii) $\|f\|_{\Lambda^{1,\infty}(Y)} \lesssim \|\nabla f\|_X \quad (f \in \mathrm{Lip}_0(\Omega))$.

Moreover, for $q \geq p$, *if* Y *is* q*-convex or, if* Y *satisfies an upper* $q-$*estimate and* $\varphi_Y(t)/t^{1/p}$ *is quasi-increasing, then, for every* $f \in \mathrm{Lip}_0(\Omega)$,

$$\|f\|_{\Lambda^{1,\infty}(Y)} \lesssim \|\nabla f\|_X \Leftrightarrow \|f\|_{\Lambda^{1,p}(Y)} \lesssim \|\nabla f\|_X \Leftrightarrow \|f\|_Y \lesssim \|\nabla f\|_X . \tag{3.13}$$

Proof. $(i) \Rightarrow (ii)$ If $a > 1$, by Theorem 3.3.6 we have

$$\begin{aligned}
\|f\|_{\Lambda^{1,p}(X)} &\leq \Big(\int_0^\infty t^{p-1}\varphi_X(\mu(\overline{M_t}))^p dt\Big)^{1/p} \\
&\lesssim \Big(\int_0^\infty t^{p-1}\mathrm{Cap}_X(\overline{M_t})^p dt\Big)^{1/p} \\
&\leq a\Big(\int_0^\infty s^{p-1}\mathrm{Cap}_X(\overline{M_{as}}, M_s)^p ds\Big)^{1/p} \lesssim \|\nabla f\|_X.
\end{aligned}$$

$(ii) \Rightarrow (iii)$ Observe that $\Lambda^{1,p}(X) \subset \Lambda^{1,\infty}(X)$.

$(iii) \Rightarrow (i)$ Trivial using Proposition 3.5.1.

To prove (3.13), if Y is q-convex, then, by Remark 3.2.13, $\frac{\varphi_Y(t)^q}{t}$ is quasi-decreasing and

$$\|f\|_Y = \||f|^q\|_{Y_{(q)}}^{1/q} \leq \||f|^q\|_{\Lambda^q(Y_{(q)})}^{1/q} = \||f^q|\|_{\Lambda^{1,q}(Y)}^{1/q}.$$

Then (3.13) follows.

If Y satisfies an upper $q-$estimate and $\varphi_Y(t)/t^{1/p}$ is quasi-increasing, then it also satisfies an upper $p-$estimate and then, for every simple function $s = \sum_i a_i\chi_{A_i}$ with $A_i \cap A_j = \emptyset$ if $i \neq j$ we obtain

$$\begin{aligned}
\|s\|_Y &= \Big\|\sum_i a_i\chi_{A_i}\Big\|_Y = \Big\|\Big(\sum_i a_i^p\chi_{A_i}\Big)^{1/p}\Big\|_Y \\
&\leq M^{(p)}(X)\Big(\sum_i \|a_i\chi_{A_i}\|_Y^p\Big)^{1/p} = M^{(p)}(X)\Big(\sum_i |a_i|^p\varphi_Y(\mu(A_i))^p\Big)^{1/p}.
\end{aligned}$$

Since φ_Y is also the fundamental function of $\Lambda^{1,p}(Y)$ and $\varphi_Y(t)/t^{1/p}$ is quasi-increasing, we know that $\Lambda^{1,p}(Y)$ satisfies a lower $p-$estimate (see [KaMa, Theorem 8]). Hence

$$\begin{aligned}
\Big(\sum_i |a_i|^p\varphi_Y(\mu(A_i))^p\Big)^{1/p} &= \Big(\sum_i \|a_i\chi_{A_i}\|_{\Lambda^{1,p}(Y)}^p\Big)^{1/p} \\
&\leq \Big\|\Big(\sum_i a_i^p\chi_{A_i}\Big)^{1/p}\Big\|_{\Lambda^{1,p}(Y)} = \|s\|_{\Lambda^{1,p}(Y)}.
\end{aligned}$$

Then, by the Fatou property, for every positive function f we have

$$\|f\|_Y \lesssim \|f\|_{\Lambda^{1,p}(Y)}.$$

Therefore, if $\|f\|_{\Lambda^{1,p}(Y)} \lesssim \|\nabla f\|_X$, then $\|f\|_Y \lesssim \|\nabla f\|_X$. Conversely, if $\|f\|_Y \lesssim \|\nabla f\|_X$, then, since $Y \hookrightarrow \Lambda^{1,\infty}(Y)$, it follows that $\|f\|_{\Lambda^{1,\infty}(Y)} \lesssim \|\nabla f\|_X$, and we conclude that $\|f\|_{\Lambda^{1,p}(Y)} \lesssim \|\nabla f\|_X$. □

Remark 3.5.3. *Theorem 3.5.2 is to be compared with the results in [MMi2, Section 1].*

Let us consider some examples: It is well-known that the *Gagliardo-Nirenberg inequality*

$$\|f\|_{L^{n/(n-1)}} \lesssim \|\nabla f\|_{L^1} \qquad (f \in \mathrm{Lip}_0(\Omega)),$$

allows us to see that, if $p \in (1,n)$, $s = \frac{np}{n-p}$ and $\alpha = \frac{(n-1)s}{n}$, since $\|f\|_{L^s}^s = \||f|^\alpha\|_{L^{\frac{n}{n-1}}}^{n/(n-1)}$, then $\|f\|_{L^s}^{s(n-1)/n} \lesssim \|\alpha|f|^{\alpha-1}|\nabla f|\|_{L^1} \lesssim \|f\|_{L^s}^{s/p'}\|\nabla f\|_{L^p}$, where p' is the conjugate exponent of p. Hence $\|f\|_{L^s} \lesssim \|\nabla f\|_{L^p}$. Therefore, since $L^s \hookrightarrow L^{s,\infty}$, it follows that

$$\|f\|_{L^{s,\infty}} \lesssim \|\nabla f\|_{L^p}.$$

But $\|f\|_{\Lambda^{1,\infty}(L^s)} = \|f\|_{L^{s,\infty}} \lesssim \|\nabla f\|_{L^p}$ and then, since L^p satisfies a lower p-estimate, from Theorem 3.5.2, we conclude that

$$\|f\|_{L^{s,p}} = \|f\|_{\Lambda^{1,p}(L^s)} \lesssim \|\nabla f\|_{L^p} \quad (f \in \mathrm{Lip}_0(\Omega)),$$

and we have obtained a self-improvement.

If $p = n$, then we start from the *Trudinger inequality*,

$$\left(\frac{\int_0^t f^*(s)^{\frac{n}{n-1}}}{t(1+\log\frac{1}{t})}\right)^{\frac{n-1}{n}} \lesssim \|\nabla f\|_{L^n},$$

which gives the estimate

$$\varphi(\mu(K)) = \Big(1+\log\frac{1}{\mu(K)}\Big)^{\frac{1-n}{n}} \le \mathrm{Cap}_{L^n}(K),$$

and then

$$\|f\|_{\Lambda^{1,n}(\varphi)} \lesssim \|\nabla f\|_{L^n}.$$

But,

$$\Lambda^{1,n}(\varphi) = \Big(\int_0^\infty t^{n-1}(\varphi(\mu_f(t)))^n dt\Big)^{1/n} = \Big(\int_0^1 \Big(\frac{f^*(s)}{(1+\log\frac{1}{s})}\Big)^n \frac{ds}{s}\Big)^{1/n}.$$

If $r \le s < p$, then we know (see Example 3.2.4) that $L^{s,r}$ satisfies an upper p-estimate and $\varphi_{L^{s,r}}(t)/t^{1/p}$ is quasi-increasing, so that, since $\|f\|_{L^{s,\infty}} = \|f\|_{\Lambda^{1,\infty}(L^s)} \lesssim \|\nabla f\|_{L^p}$,

$$\|f\|_{L^{s,\infty}} \simeq \|f\|_{\Lambda^{1,\infty}(L^{s,r})} \lesssim \|\nabla f\|_{L^p} \lesssim \|\nabla f\|_{L^{p,q}} \quad (q \le p),$$

and then $\|f\|_{\Lambda^{1,p}(L^{s,r})} \lesssim \|\nabla f\|_{L^p}$. Therefore, if $q \le p$, then we obtain the self-improvement

$$\|f\|_{L^{s,p}} \simeq \|f\|_{\Lambda^{1,p}(L^{s,r})} \lesssim \|\nabla f\|_{L^{p,q}} \qquad (f \in \mathrm{Lip}_0(\Omega)).$$

3.6 Extension to capacitary function spaces

Let us extend the results to the capacitary function spaces considered in [Ce], [CeMS] and [CeMS1].

Let us recall that by a *capacity* C on a measurable space (Ω, Σ) we mean a set function defined on Σ satisfying at least the following properties:

(a) $C(\emptyset) = 0$,

(b) $0 \le C(A) \le \infty$,

(c) $C(A) \le C(B)$ if $A \subset B$, and

(d) Quasi-subadditivity: $C(A \cup B) \le c(C(A) + C(B))$, where $c \ge 1$ is a constant.

Then the *capacitary Lorentz spaces* $L^{p,q}(C)$ are defined by

$$\|f\|_{L^{p,q}(C)} := \Big(q \int_0^\infty t^{q-1} C\{|f| > t\}^{q/p} dt\Big)^{1/q} < \infty.$$

Hence, Theorem 3.3.6 states that, if X satisfies a lower p-estimate, then

$$\|f\|_{L^{1,p}(\mathrm{Cap}_X)} \lesssim \|\nabla f\|_X \qquad (f \in \mathrm{Lip}_0(\Omega)).$$

Recall that by $C^{(p)} := C^{1/p}$ we denote the *p-convexification* of C (see [Ce] or Proposition 2.5.5).

Theorem 3.6.1. *Suppose $0 < p, s, q < \infty$, and let C and $\widetilde{C}$ be two capacities on (Ω, Σ). If X satisfies a lower $q-$estimate, then the following properties are equivalent:*

(i) $\|f\|_{L^{p,q}(C)} \lesssim \|\nabla f\|_X + \|f\|_{L^{s,q}(\widetilde{C})}$ *for every* $f \in \mathrm{Lip}_0(\Omega)$.

(ii) $C^{(p)}(g) \lesssim \mathrm{Cap}_X(\overline{g}, G) + \widetilde{C}^{(s)}(G)$ *for all sets g and G such that $g \subset\subset G \subset\subset \Omega$.*

Proof. (i) $\Rightarrow$ (ii) Choose $g \subset\subset G \subset\subset \Omega$ and any $f \in W(\overline{g}, G)$. Then $\|f\|_{L^{p,q}(C)} \lesssim \|\nabla f\|_X + \|f\|_{L^{s,p}(\widetilde{C})}$, so that

$$C^{(p)}(g) \leq \Big(\int_0^1 C^{(p)}(\{f > t\})^q dt^q \Big)^{1/q} \lesssim \|\nabla f\|_X + \|f\|_{L^{s,p}(\widetilde{C})},$$

and $\|f\|_{L^{s,p}(\widetilde{C})} = \big(\int_0^1 \widetilde{C}\{|f| > s\}^{p/s} ds^p \big)^{1/p} \leq \widetilde{C}^{(s)}(G)$. Taking the infimum over all $f \in W(\overline{g}, G)$ we conclude that

$$C^{(p)}(g) \lesssim \mathrm{Cap}_X(\overline{g}, G) + \widetilde{C}^{(s)}(G).$$

(ii) $\Rightarrow$ (i) Consider $f \in \mathrm{Lip}_0(\Omega)$ and take for $a > 1$ and $t > 0$ the open sets, $g := M_{at}$ and $G := M_t$. By hypothesis we have $C^{(p)}(M_{at}) \lesssim \mathrm{Cap}_X(\overline{M}_{at}, M_t) + \widetilde{C}^{(s)}(M_t)$, and then, by Theorem 3.3.6,

$$\begin{aligned} \|f\|_{L^{p,q}(C)} &\lesssim \Big(\int_0^\infty s^{q-1} \mathrm{Cap}_X(\overline{M}_{as}, M_s)^q ds \Big)^{1/q} \\ &+ \Big(\int_0^\infty s^{q-1} \widetilde{C}^{(s)}(M_s)^q ds \Big)^{1/q} \\ &\leq \|\nabla f\|_X + \|f\|_{L^{s,q}(\widetilde{C})}. \qquad \square \end{aligned}$$

In a similar way,

Theorem 3.6.2. *Let $0 < p, q < \infty$. Suppose that X satisfies a lower $q-$estimate and let C be a capacity on (Ω, Σ). The following properties are equivalent:*

(i) $\|f\|_{L^{p,q}(C)} \lesssim \|\nabla f\|_X$ *for every* $f \in \mathrm{Lip}_0(\Omega)$.

(ii) $C^{(p)}(g) \lesssim \mathrm{Cap}_X(\overline{g}, G)$ *if* $g \subset\subset G \subset\subset \Omega$.

Assume that X is an r.i. quasi-Banach space. If we define $C(A) := \varphi_X(m_n(A))$, then $L^p(C) = \Lambda^p(X)$ is a Banach space.

Indeed, since φ_X is continuous except possibly at the origin, C is a Fatou capacity on (Ω, m_n) and $L^p(C)$ is complete. Moreover, C is m_n-invariant and quasi-concave with respect to m_n and, by Theorem 1.4.4 and Proposition 1.4.7, there exists a Fatou concave capacity C_1 which is equivalent to C. For such a capacity, $L^p(C_1)$ is a normed space and, by the equivalence, $L^p(C_1) \simeq \Lambda^p(X)$.

3.7 Indices of r.i. spaces

Definition 3.7.1. *Let (Ω, Σ, μ) be a measure space, and X be a Banach lattice on (Ω, Σ, μ). X is said to be* order continuous *if and only if there is an equivalent lattice norm $\|\cdot\|_1$ on X such that for all $\{x_n\}_{n=1}^{\infty} \subset X$ such that x_n converges weakly to x and $\|x_n\|_1 \longrightarrow \|x\|_1$, then $\|x_n - x\|_1 \longrightarrow 0$.*

Let us observe that every separable Banach space is order continuous.

Definition 3.7.2. *Let (Ω, Σ, μ) be a measure space. A Banach function lattice X on (Ω, Σ, μ) is called a* Köthe function space *if for each $E \in \Sigma$ with $\mu(E) < \infty$ we have that $\chi_E \in X$ and $x\chi_E \in L^1(\mu)$ for all $x \in X$.*

As in [CwNS], a Banach lattice X is said to satisfy an *equal norm upper p-estimate* (e.n.u. p-estimate for short) respectively, *equal norm lower p-estimate* (e.n.l. p-estimate for short) if there exists a constant $C < \infty$ such that every finite sequence $\{x_n\}_{n=1}^{N}$ of disjointly supported norm one elements in X satisfies

$$\Big\| \sum_{n=1}^{N} x_n \Big\|_X \leq C N^{1/p},$$

respectively,

$$N^{1/p} \leq C \Big\| \sum_{n=1}^{N} x_n \Big\|_X .$$

Any Banach lattice which satisfies an upper p-estimate satisfies an e.n.u. p-estimate. It is interesting to note that there exist Banach lattices X which do not satisfy a lower p-estimate even though they satisfy an equal norm lower p-estimate.

Let X' be the *associate or Köthe dual* of any Banach lattice X. As it is observed in [CwNS], when X is order continuous or X satisfies the Fatou property, X' is a *norming subspace* of the dual X^* (that is, $\|x\| = \sup\{|x^*(x)|; x^* \in X', \|x^*\| = 1\}$).

Definition 3.7.3. *For any Banach lattice X, the* Cwikel-Nilsson-Schetmann indices *are given by*

$$p(X) := \sup\{p > 0;\ X \text{ satisfies an upper } p\text{-estimate}\}$$

$$q(X) := \inf\{q > 0;\ X \text{ satisfies a lower } q\text{-estimate}\}.$$

Moreover, we define

$$q_e(X) := \inf\{q > 0;\ X \text{ satisfies an equal norm lower } q\text{-estimate}\}$$

$$p_e(X) := \sup\{p > 0;\ X \text{ satisfies an equal norm upper } p\text{-estimate}\}.$$

In [CwNS] we find the following result:

Lemma 3.7.4. *Let X be a Banach lattice of measurable functions and suppose that $p \in [1, \infty]$. Let p' such that $1/p + 1/p' = 1$.*

(i) If X satisfies an upper p-estimate, then X' satisfies a lower p'-estimate.

(ii) If X satisfies an e.n.u. p-estimate, then X' satisfies an e.n.l. p'-estimate.

(iii) If X satisfies a lower p-estimate, then X' satisfies an upper p'-estimate.

In each case the constant which appears in the estimate satisfied by X' is equal to the constant which appears in the estimate satisfied by X.

Remark 3.7.5. *If X' is a norming subspace of X^*, the dual of X, then obviously X is closed isometric subspace of its second Köthe dual X''. In that case it is easy to combine (i) and (iii) of the preceding lemma and to obtain that: X satisfies an upper, respectively, lower p-estimate if and only if X' satisfies a lower, respectively, upper p'-estimate.*

Then in [CwNS, Lemma 2.3] it's proved:

Lemma 3.7.6. *Let X be a Banach lattice of measurable functions. Then*

(i) $q_e(X) = q(X)$.

(ii) If the associate space X' of X is a norming subspace of the dual of X, then also $p_e(X) = p(X)$.

Proof. Clearly $q(X) \geq q_e(X)$ and $p(X) \leq p_e(X)$. For the proofs of the reverse inequalities we may assume that $q_e(X) < \infty$ and $p_e(X) > 1$. (Otherwise the results are trivial).

First, to show that $q(X) \leq q_e(X)$, it is sufficient to show that X satisfies a lower s-estimate for every number $s < \infty$ with the property that X satisfies an e.n.l q-estimate for some $q < s$: Suppose then, for such q and s, that C is the constant appearing in the lower q-estimate inequalities for X. Let $x_1, x_2, \ldots, x_N$ be any finite sequence of disjointly supported functions in X. We may suppose without loss of generality that $\|x_1\| \geq \|x_2\| \geq \ldots \geq \|x_N\|$. Then, for all $1 \leq k \leq N$,

$$\|x_k\| = \Big(\frac{1}{k}\sum_{j=1}^{k}\Big\|\frac{\|x_k\|}{\|x_j\|}x_j\Big\|^q\Big)^{1/q} \le Ck^{-1/q}\Big\|\sum_{j=1}^{k}\frac{\|x_k\|}{\|x_j\|}x_j\Big\|$$
$$\le Ck^{-1/q}\Big\|\sum_{j=1}^{k}x_j\| \le Ck^{-1/q}\Big\|\sum_{j=1}^{N}x_j\Big\|.$$

Consequently,

$$\Big(\sum_{k=1}^{N}\|x_k\|^s\Big)^{1/s} \le C\Big(\sum_{k=1}^{N}k^{-s/q}\Big)^{1/s}\Big\|\sum_{j=1}^{N}x_j\Big\|$$

and, since $\sum_{k=1}^{\infty}k^{-s/q} < \infty$, this shows that X satisfies a lower s-estimate and yields (i).

Finally, to establish $p(X) \ge p_e(X)$ and therefore (ii), we must show that, whenever X satisfies an e.n.u. p-estimate for some p and $1 < s < p$, then X satisfies an upper s-estimate. By part (ii) of [CwNS, Lemma 2.1], X' satisfies an e.n.l. p'-estimate. So, since $p' < s' < \infty$, we can apply the preceding argument to show that X' satisfies a lower s'-estimate. Then, using the remark, we see that X satisfies an upper s-estimate as required, and the proof is complete. □.

Let X be an r.i. Banach function space on (Ω, Σ, μ). It is well-known, by the *Luxemburg theorem* (see [BeSh, Theorem 4.2]), that any r.i. Banach function space X on (Ω, μ) can be represented as an r.i. space $\bar{X}(0, \mu(\Omega))$ on the interval $(0, \mu(\Omega))$ with Lebesgue measure and $\|g\|_X = \|g^*_\mu\|_{\bar{X}(0,\mu(\Omega))}$ for every $g \in X$.

For each $t > 0$, let E_t denotes the *dilation operator* defined on $\mathrm{L}_0((0,\infty))$ by

$$(E_t f)(s) = f(st)\,(0 < s < \infty). \tag{3.14}$$

Let $h_X(t)$ denotes the operator norm of $E_{1/t}$ as an operator from $\bar{X}(0,\infty)$ to $\bar{X}(0,\infty)$. The *Boyd indices* of X (see [BeSh, Chapter 3, Section 5]) are the numbers $\underline{\alpha}_X$ and $\overline{\alpha}_X$ such that

$$\underline{\alpha}_X = \sup_{0<t<1}\frac{\log(h_X(t))}{\log(t)}, \qquad \overline{\alpha}_X = \inf_{1<t<\infty}\frac{\log(h_X(t))}{\log(t)}.$$

In [LiZa, Proposition 2.b.5] we observe that for X be an Orlicz function space, the Cwikel-Nilsson-Schetmann indices are related to the *upper and lower Boyd's indices.* The same holds for Lorentz spaces. Certainly, if Y is an Orlicz function spaces, then

$$p(Y) = \frac{1}{\underline{\alpha}_Y}, \qquad q(Y) = \frac{1}{\overline{\alpha}_Y}.$$

So that, a natural question is: Is there any relation between the Boyd indices of an r.i. space and the Cwikel-Nilsson-Schetmann indices. Let us analyze briefly this question.

Every r.i. Banach function space X on $\mathrm{L}_0(\Omega)$ satisfies the Fatou property. Then, for X be an r.i. function space, it follows that $p(X) = p_e(X)$, $q(X) = q_e(X)$.

For every $0 < s < \infty$, the "new" *dilation operator* D_s is given by $D_s = E_{1/s}$, where E was defined in (3.14), that is,

$$(D_s f)(t) := f(t/s),\ 0 < s < \infty,\ 0 \le t < \infty,\ f \in \mathrm{L}_0((0,\infty)).$$

Geometrically, the operator D_s dilates the graph of $f(t)$ by the ratio $s : 1$ in the direction of the t axis. It is obvious that D_s acts as a linear operator of norm one on L^∞ and of norm s on L^1; hence, D_s is bounded on every r.i. function space X and $\|D_s\|_X \le \max(1, s)$. Clearly, $(D_s f)^* \le D_s f^*$ for every f and s and hence, $\|D_s\|$ on an r.i. function space X can be computed by considering only non-increasing functions f. Since, for every non-increasing $f \ge 0$ and every $0 < r < s < \infty$, we have $D_r f \le D_s f$, it is clear that $\|D_s\|$ is a non-decreasing function of s. Also note that, for every r and s, $D_r D_s = D_{rs}$. We have

$$\|D_{rs}\| \le \|D_r\|\|D_s\|.$$

It is necessary to observe that, if f is a measurable function, then $D_n f$ can be written as $f_1 + f_2 + \cdots + f_n$, where the f_i are mutually disjoint and each f_i has the same distribution function as f. Hence, $\frac{1}{\underline{\alpha}_X}$ is the supremum of all the numbers p which have the following property: there exists a number k so that, for every choice of an integer n and of a function f having norm one, we have

$$\|f_1 + \cdots + f_n\| \le k n^{1/p},$$

where the $\{f_i\}_{i=1}^n$ are disjointly supported and they have the same distribution function as f. Similarly, $\frac{1}{\overline{\alpha}_X}$ is the infimum of all the numbers q for which there is k so that, for every n and $\{f_i\}_{i=1}^n$ as above,

$$\|f_1 + \cdots + f_n\| \ge k^{-1} n^{1/q}.$$

After this observation it follows that, if an r.i. function space satisfies an upper p-estimate, then $p \le \frac{1}{\underline{\alpha}_X}$. Certainly, $p(X) \le \frac{1}{\underline{\alpha}_X} := p_X$, where p_X denotes the Boyd indices in [LiZa].

In general $\frac{1}{\underline{\alpha}_X} > p(X)$, $\frac{1}{\overline{\alpha}_X} < q(X)$. To see that, consider for $1 \le p < \infty$, $t > 0$ and $w(t) := \frac{1}{pt^{1/p}}$, the weighted Lorentz function space $\Lambda_p(w, (0,\infty))$ of all measurable function for which

$$\|f\|_{\Lambda_p(w,(0,\infty))} := \int_0^\infty f(t) \frac{1}{pt^{1/p}} dt < \infty.$$

For every non-increasing function f in $\Lambda_p(w,(0,\infty))$ and every $0<s<\infty$ we have that

$$\int_0^\infty \frac{f(t/s)}{pt^{1/p}}dt = (s)^{1-1/p}\int_0^\infty \frac{f(u)}{pu^{1/p}}du.$$

Hence, $\frac{1}{\underline{\alpha}_X} = \frac{p}{p-1}$ which is bigger than one for all $p>1$. Moreover, easily it follows that $\Lambda_p(w,(0,\infty))$ satisfies an upper 1-estimate but it does not satisfy an upper p-estimate for $p>1$.

Definition 3.7.7. *Let $\varphi:\mathbb{R}^+\longrightarrow\mathbb{R}^+$ such that $\varphi(1)=1$ and $\varphi\in\mathcal{C}^\infty$. Then, φ is called a* function parameter *if*

$$0<\alpha_\varphi=\inf_{t>0}\frac{t\varphi'(t)}{\varphi(t)}\leq\sup_{t>0}\frac{t\varphi'(t)}{\varphi(t)}=\beta_\varphi<1.$$

In [CSo, Proposition 4.2] it's proved:

Proposition 3.7.8. *Let φ be a function parameter and let $\{a_n\}_n$ be a sequence of positive numbers such that $\sum_n a_n<\infty$. Then, for every $1/\alpha_\varphi\leq q<\infty$, $\sum_n\varphi^q(a_n)\leq\varphi^q(\sum_n a_n)$.*

Let X be a q-convex r.i. space. It can be equivalently renormed in such a way that, X with the new norm has a fundamental function that is a function parameter. Then, let us see that $\Lambda(X_{(q)})$ satisfies a lower 1-estimate with constant one if $q\geq\frac{1}{\alpha_{\varphi_X}}$. Indeed, consider f and g be disjoint functions. By [CSo, Proposition 4.2], it follows that

$$\begin{aligned}\|f\|_{\Lambda(X_{(q)})}+\|g\|_{\Lambda(X_{(q)})} &= \int_0^\infty\varphi_X(\mu_f(t))^q dt+\int_0^\infty\varphi_X(\mu_g(t))^q dt\\ &\leq \int_0^\infty\varphi_X\Big(\mu_f(t)+\mu_g(t)\Big)^q dt\\ &= \int_0^\infty\varphi_X(\mu_{f+g}(t))^q dt=\|f+g\|_{\Lambda(X_{(q)})}.\end{aligned}$$

The conclusion follows by induction on n.

Appendix: Second order Sobolev-Poincaré estimates

Appendix A

Second order Sobolev-Poincaré estimates

A.1 Introduction

It is well-known that the symmetrization inequalities are intimally related with the isocapacitary inequalities. In [MMi3] the authors discuss the connection between Maz'ya's capacitary inequalities and the method of symmetrization by truncation. Here we will study only symmetrization inequalities, so that, we called this part an appendix instead of a chapter because here capacities do not apppear.

Symmetrization is a very useful classical tool on this area. New symmetrization inequalities have been developed in [MMi1], [MMi2], [MMi3], [MMi4] and [MMiP], and they can be applied to provide a unified treatment of sharp Poincaré inequalities, concentration inequalities and sharp integrability of solutions of elliptic equations. These inequalities combine three features: the inequalities are pointwise rearrangement inequalities, incorporate in their formulation the isoperimetric profile and are formulated in terms of oscillations. In [MMi1, MMi3] the Poincaré inequality, for *isoperimetric Hardy type* measure spaces, it is completely characterized in terms of the boundedness of a Hardy type operator from $\bar{X}(0,1)$ to $\bar{Y}(0,1)$, and in [MMi3] the authors show some connections between symmetrization inequalities and the isocapacitary inequalities due to Maz'ya.

Moreover, very recently, A. Cianchi and L. Pick [CiL] have characterized the optimal range and domain norms in the Sobolev-Poincaré inequality for the Gaussian measure and functions of bounded variation. Similarly, in [MMi5], the authors, using isoperimetry and symmetrization, obtain new Gaussian symmetrization inequalities and connect them with logarithmic Sobolev inequalities. In those inequalities, the isoperimetric function appears sistematically. For second order derivatives we will see below that the inequalities depend

on the square of the isoperimetric function.

Using similar techniques to the ones in [MMi2], [MMi5] and [MMi3], in this appendix we study second order Sobolev-Poincaré inequalities (see Theorem A.3.9) and we relate them with the boundedness of some Hardy type operators involving the square of the isoperimetric profile and of the Boyd indices of the r.i. spaces (see Section 2). Our main results are Theorem A.3.9 and Theorem A.4.1. Let us note that the previous results gave us Proposition A.4.2, where the *second order Sobolev-Poincaré inequality* follows directly from the classical Sobolev-Poincaré estimate. In the Gaussian case we obtain Theorem A.5.2, which is the converse of Theorem A.4.1. To finish, with similar techniques to the ones in [CiL], we describe the optimal range and domain for the second order Sobolev-Poincaré inequality when dealing with the Gaussian measure.

A.2 Background

From now on, as in [B08], set $\mathbb{R}^n$ and let μ be a Borel measure in $\mathbb{R}^n$ given by $d\mu(x) = \varphi(x)dx$, where $\varphi \in C(\mathbb{R}^n)$, $\varphi(x) > 0$ for any $x \in \mathbb{R}^n$ and $\int_{\mathbb{R}^n} \varphi(x)dx = 1$.

For a measurable function $u : \mathbb{R}^n \to \mathbb{R}$, the *non-increasing rearrangement* of $u \in \mathrm{L}_0(\mathbb{R}^n)$, u^*_μ, is defined as in (1.2) with μ instead of C. Since u^*_μ is decreasing, the function u^{**}_μ, defined by

$$u^{**}_\mu(t) = \frac{1}{t}\int_0^t u^*_\mu(s)\,ds,$$

is also decreasing and, moreover,

$$u^*_\mu \le u^{**}_\mu, \qquad (u+v)^{**}_\mu \le u^{**}_\mu + v^{**}_\mu.$$

By definition, we have that

$$(u^{**}_\mu)'(s) = -\frac{u^{**}_\mu(s) - u^*_\mu(s)}{s} = -\frac{u_{osc}(s)}{s}. \tag{A.1}$$

Let us denote by f^* the non-increasing rearrangement of f respect to the Lebesgue measure on $\mathbb{R}^n$, m_n.

Let $A \subset \mathbb{R}^n$ be a measurable set, the *μ-perimeter* (in the sense of De Giorgi) is defined by

$$P_\mu(A) = \sup\Big\{\int_A div(h(x))\varphi(x)dx; h \in C^1_0(\mathbb{R}^n, \mathbb{R}^n),\ |h| \le 1\Big\},$$

and the *isoperimetric function* I_μ is defined as the pointwise maximal function $I_\mu : [0,1] \to [0,\infty)$ such that

$$P_\mu(A) \ge I_\mu(\mu(A)),$$

holds for all Borel sets A. The isoperimetric profile I_μ is supposed to be a concave continuous function, increasing on $(0, 1/2)$, symmetric about the point $1/2$ that, moreover, vanishes at zero. So, the *isoperimetric profile* is such that for $0 \le t \le 1$

(1) $I_\mu(0) = I_\mu(1) = 0$,

(2) $I_\mu(t) = I_\mu(1-t)$,

(3) $I_\mu(t)$ is concave.

Since I_μ has a maximum at $t = 1/2$ and $I_\mu(0) = 0$, then $\frac{I_\mu(s)}{s}$ is decreasing on $(0, 1/2)$ and $\frac{s}{I_\mu(s)}$ is increasing.

A concave continuous function, $I : [0,1] \to [0,\infty)$, increasing on $(0, 1/2)$ and symmetric about the point $1/2$, with $I(0) = 0$, and such that $I_\mu \ge I$, will be called an *isoperimetric estimator* for $(\mathbb{R}^n, \mu)$.

Furthermore, let $W^{1,1}(\mu) = W^{1,1}(\varphi, \mathbb{R}^n)$ denotes the *weighted Sobolev space* containing all functions $u \in L^1(\mu, \mathbb{R}^n)$ with weak derivatives $u_{x_i} \in L^1(\mu, \mathbb{R}^n)$, $i = 1, ..., n$ and

$$\|f\|_{W^{1,1}(\mu)} = \|u\|_{L^1(\mu,\mathbb{R}^n)} + \||\nabla u|\|_{L^1(\mu,\mathbb{R}^n)}. \tag{A.2}$$

Theorem A.2.1. *If $I : [0,1] \to [0,\infty)$ is an isoperimetric estimator on $(\mathbb{R}^n, \mu)$, then the following statements hold and are in fact equivalent (in [MMi3, Theorem 1] the result is done for functions in $f \in \mathrm{Lip}(\mathbb{R}^n)$) :*

(i) *Isoperimetric inequality: For every Borel set $A \subset \mathbb{R}^n$*

$$I(\mu(A)) \le P_\mu(A).$$

(ii) *Ledoux's inequality:*

$$\int_0^\infty I(\mu\{|f| > s\})\, ds \le \int |\nabla f|\, d\mu \qquad \big(f \in W^{1,1}(\mu)\big).$$

(iii) *Maz'ya–Talenti's inequality:*

$$(-f_\mu^*)'(s) I(s) \le \frac{d}{ds} \int_{\{|f| > f_\mu^*(s)\}} |\nabla f|\, d\mu \qquad \big(f \in W^{1,1}(\mu)\big).$$

(iv) *Oscillation inequality:*

$$f_\mu^{**}(t) - f_\mu^*(t) \le \frac{t}{I(t)} |\nabla f|_\mu^{**}(t) \qquad \big(f \in W^{1,1}(\mu)\big). \tag{A.3}$$

(v) *Pólya–Szegö inequality:*

$$\int_0^t [(-f_\mu^*)'(s) I(s)]^*(r)\, dr \le \int_0^t |\nabla f|_\mu^*(r)\, dr \qquad \big(f \in W^{1,1}(\mu)\big).$$

Proof. As soon as we know the validity of the oscillation inequality for $f \in W^{1,1}(\mu)$, we can proceed as in [MMi3, Theorem 1] to see that the rest hold and that all are, in fact, equivalent. Therefore, we only need to develop (iv).

To distinguish, by $W^{1,1}(\mathbb{R}^n)$ we denote the first order Sobolev space with respect to $L^1(\mathbb{R}^n, m_n)$ and, by $I_{\mathbb{R}^n}$ and $P_{\mathbb{R}^n}$ the isoperimetric function and the perimeter on $(\mathbb{R}^n, m_n)$.

We proceed as follows: first we obtain (A.3) for $f \in W^{1,1}(\mathbb{R}^n)$ and then we extend to our setting.

For a measurable $f : \mathbb{R}^n \to \mathbb{R}$, let

$$f^+ = \max(f, 0) \text{ and } f^- = \min(f, 0).$$

If $f \in W^{1,1}(\mathbb{R}^n)$, then $f^+, f^- \in W^{1,1}(\mathbb{R}^n)$ and

$$\nabla f^+ = \nabla f \, \chi_{\{f>0\}} \text{ and } \nabla f^- = \nabla f \, \chi_{\{f<0\}}.$$

This implies that functions on $W^{1,1}(\mathbb{R}^n)$ remain invariant under the operation of truncation, i.e., given a measurable g and $0 < t_1 < t_2$, the truncation $g_{t_1}^{t_2}$ of g is defined by

$$g_{t_1}^{t_2} = \min\{\max\{0, g - t_1\}, t_2 - t_1\}$$

and therefore, if $g \in W^{1,1}(\mathbb{R}^n)$, then $g_{t_1}^{t_2} \in W^{1,1}(\mathbb{R}^n)$ and

$$\nabla g_{t_1}^{t_2} = \nabla g \chi_{\{t_1<g<t_2\}}.$$

On the other hand, given $f \in W^{1,1}(\mathbb{R}^n)$, the Fleming-Rishel formula states that

$$\int_{\mathbb{R}^n} |\nabla f(x)| dx = \int_{-\infty}^{\infty} P_{\mathbb{R}^n}(\{f > s\}) ds.$$

Applying this result to $|f|_{t_1}^{t_2}$, we get

$$\begin{aligned} \int_{\{t_1<|f|<t_2\}} |\nabla |f|(x)| dx &= \int_0^{\infty} P_{\mathbb{R}^n}(\{|f|_{t_1}^{t_2} > s\}) ds \\ &\geq \int_0^{\infty} I_{\mathbb{R}^n}\big(\big|\{|f|_{t_1}^{t_2} > s\}\big|\big) ds = \int_0^{t_2-t_1} I_{\mathbb{R}^n}\big(\big|\{|f|_{t_1}^{t_2} > s\}\big|\big) ds, \end{aligned} \tag{A.4}$$

where the second inequality holds thanks to the isoperimetric inequality.

Observe that, for $0 < s < t_2 - t_1$,

$$|\{|f| \geq t_2\}| \leq m_{f_{t_1}^{t_2}}(s) \leq |\{|f| > t_1\}|.$$

Consequently, by the properties of $I_{\mathbb{R}^n}$, we have

$$\int_0^{t_2-t_1} I_{\mathbb{R}^n}(m_{f_{t_1}^{t_2}}(s))ds \geq (t_2-t_1)\min(I_{\mathbb{R}^n}\big(\big|\{|f|>t_1\}\big|\big), I_{\mathbb{R}^n}\big(\big|\{|f|>t_1\}\big|\big)). \tag{A.5}$$

For $s>0$ and $h>0$, pick $t_1=f^*(s+h)$ and $t_2=f^*(s)$. Then

$$s \leq \big|\big\{|f|\geq f^*(s)\big\}\big| \leq m_{f_{t_1}^{t_2}}(s) \leq \big|\big\{|f|>f^*(s+h)\big\}\big| \leq s+h. \tag{A.6}$$

Combining (A.4), (A.5) and (A.6), since I is an isoperimetric estimator, it follows that

$$\Big(f^*(s)-f^*(s+h)\Big)\min\Big(I(s+h),I(s)\Big) \leq \int_{\{f^*(s+h)<|f|<f^*(s)\}} |\nabla|f|(x)|dx.$$

At this stage we can continue as in [MMi3, from (3.8)], and we obtain that if $f\in W^{1,1}(\mathbb{R}^n)$, then

$$f^{**}(t)-f^*(t) \leq \frac{t}{I(t)}\frac{1}{t}\int_0^t |\nabla f|^*(s)ds \qquad (0<t<1). \tag{A.7}$$

We can extend this result to our setting in the following way. We consider μ be a finite measure on $\mathbb{R}^n$ defined by $d\mu=\varphi(x)dx$, where $\varphi\in C(\mathbb{R}^n)$, $\varphi(x)>0$ for any $x\in\mathbb{R}^n$ and $\int_{\mathbb{R}^n}\varphi(x)dx=1$. Furthermore, $W^{1,1}(\mu)$ denotes the weighted Sobolev space defined in (A.2). It is plain (since we are still working on $\mathbb{R}^n$) that functions on $W^{1,1}(\mu)$ remain invariant under the operation of truncation.

Let $M\subset\mathbb{R}^n$ be a measurable set and recall that the μ-perimeter (in the sense of De Giorgi) is defined by

$$P_\mu(M)=\sup\Big\{\int_M div(h(x))\varphi(x)dx; h\in C_0^1(\mathbb{R}^n,\mathbb{R}^n),\ |h|\leq 1\Big\}.$$

The theory of sets with finite μ-perimeter is imbedded in the framework of the space of BV-functions, denoted by $BV(\varphi,\mathbb{R}^n)$, and it is defined as the set of all functions $u\in L^1(\mu,\mathbb{R}^n)$ such that

$$\|Du\|_{BV}=\sup\Big\{\int_{\mathbb{R}^n} u(x)div(h(x))\varphi(x)dx; h\in C_0^1(\mathbb{R}^n,\mathbb{R}^n),\ |h|\leq 1\Big\}<\infty.$$

Notice that if M has finite μ-perimeter, then $\chi_M\in BV(\varphi,\mathbb{R}^n)$ and $P_\mu(M)=\|D\chi_M\|_{BV}$. Moreover, if $u\in W^{1,1}(\mu)$, then $\|Du\|_{BV}=\||\nabla u|\|_{L^1(\mu,\mathbb{R}^n)}$.

On the other hand, the co-area formula for functions states that

$$\|Du\|_{BV}=\int_{-\infty}^{\infty} P_\mu(\{u>s\})ds \qquad (u\in BV(\varphi,\mathbb{R}^n))$$

and therefore,

$$\||\nabla u|\|_{L^1(\mu,\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\nabla u(x)|\varphi(x)dx = \int_{-\infty}^{\infty} P_\mu(\{u > s\})ds \qquad (u \in W^{1,1}(\varphi, \mathbb{R}^n) = W^{1,1}(\mu)).$$

Thus, using the same argument that above, we obtain that inequality (A.7) is valid for $W^{1,1}(\mu)$ functions (by considering now rearrangements with respect to the measure μ).

Hence, the oscillation inequality holds for functions in $W^{1,1}(\mu)$ and, as in [MMi3], we can see that the result follows. □

Suppose $f \in W^{1,1}(\mu)$ and let m_f be a *median* of f:

$$\mu\{f \geq m_f\} \geq 1/2 \text{ and } \mu\{f \leq m_f\} \geq 1/2.$$

It follows that $\int |f - m_f|\, d\mu \simeq \int \left| f - \int_{\mathbb{R}^n} f\, d\mu \right| d\mu$.

Let us recall the $L^1(\mu)$-*Poincaré inequality* that we will use later (see [EvGa]):

$$\int |f - m_f|\, d\mu \leq \frac{1}{2I(1/2)} \int |\nabla f(x)|\, d\mu. \tag{A.8}$$

A.3 Second order Sobolev-Poincaré inequalities

In [MMi], as an application of (A.1) and (A.3), characterizations of the Sobolev-Poincaré inequality are given. In the same direction, in this section we will obtain pointwise second order inequalities to characterize second order Sobolev-Poincaré inequalities. Our main result is Theorem A.3.9. For that, some previous results are needed and so are presented before.

Let f be a locally integrable function having weak derivatives of all orders up to 2. We denote by $d^2 f$ the vector $(D^\beta f)_{|\beta|=2}$ of all derivatives of order $\beta = 2$. It is easy to see that

$$|\nabla |d^{k-1} f|| \lesssim |d^k f|, \qquad k = 1, 2.$$

Let $W^{k,1}(\mu)$ be the corresponding Sobolev space of k-order given by $L^1(\mu, \mathbb{R}^n)$ and let us define the *Hardy type operator*

$$Q_\mu g(t) = \int_t^{1/2} g(s) \frac{ds}{I_\mu(s)} := \int_t^1 g(s)\chi_{(0,1/2)}(s) \frac{ds}{I_\mu(s)},$$

as in [MMi3].

Proposition A.3.1. *Suppose* $0 < t \leq 1/2$. *Then:*

(i) For every $f \in W^{1,1}(\mu)$,

$$|\nabla f|_\mu^{**}(t) \leq \int_t^{1/2} \left(|\nabla f|_\mu^{**}(s) - |\nabla f|_\mu^{*}(s)\right) \frac{ds}{s} + 2\|\nabla f\|_{L^1(\mu)}.$$

(ii) For every $f \in W^{2,1}(\mu)$,

$$I_\mu(t)(-f_\mu^{**})'(t) = \frac{I_\mu(t)}{t} f_{\mathrm{osc}}(t) \le Q_\mu(|d^2 f|_\mu^{**})(t) + 2\|\nabla f\|_{L^1(\mu)}.$$

(iii) For every $f \in W^{2,1}(\mu)$,

$$\Big(\frac{I_\mu(t)}{t}\Big)^2 (f_\mu^{**}(t) - f_\mu^{*}(t)) \le |d^2 f|_\mu^{**}(t) + \frac{I_\mu(t)}{t} |\nabla f|_\mu^{*}(t).$$

Proof. Let $f \in W^{1,1}(\mu)$ and denote by $g := |\nabla f|$.

(i) Just apply (A.1) to g_{osc} and note that $g_\mu^{**}(1/2) \le 2\|\nabla f\|_{L^1(\mu)}$.

(ii) From (i) and from (A.3), since $|\nabla|\nabla f|| \le |d^2 f|$, if $0 < t \le 1/2$, then

$$\begin{aligned} |\nabla f|_\mu^{**}(t) &= \int_t^{1/2} ((g_\mu^{**}(s) - g_\mu^{*}(s)) \frac{ds}{s} + g_\mu^{**}(1/2) \\ &\le \int_t^{1/2} \frac{s}{I_\mu(s)} |d^2 f|_\mu^{**}(s) \frac{ds}{s} + 2\|\nabla f\|_{L^1(\mu)}. \end{aligned}$$

Let us observe that, if $1/2 < t < 1$, then $|\nabla f|_\mu^{**}(t) \le 2\|\nabla f\|_{L^1(\mu)}$.

(iii) Write the oscillation inequality as

$$\Big(\frac{I_\mu(t)}{t}\Big)^2 (f_\mu^{**}(t) - f_\mu^{*}(t)) \le \frac{I_\mu(t)}{t} g_\mu^{**}(t)$$

and $g_\mu^{**}(t) = g_\mu^{**}(t) - g_\mu^{*}(t) + g_\mu^{*}(t)$. By using that

$$g_\mu^{**}(t) - g_\mu^{*}(t) \le \frac{t}{I_\mu(t)} |d^2 f|_\mu^{**}(t),$$

we arrive to the pointwise second order estimate (iii). □

Proposition A.3.2. *Let $0 < t \le 1/2$ and $k = 2, 3$. Then,*

$$f_{osc}(t) \lesssim \frac{t}{I_\mu(t)} \Big\{ Q_\mu^{k-1}(|d^k f|_\mu^{**})(t) + \sum_{j=1}^{k-1} \|d^j f\|_{L^1(\mu)} Q_\mu^{j-1}(\mathbf{1})(t) \Big\} \quad (f \in W^{k,1}(\mu)).$$

Proof. For $k = 2$ this is proved in Proposition A.3.1. Let $k = 3$ and $0 < t < 1/2$. It follows that

$$f_{osc}(t) \Big(\frac{I_\mu(t)}{t}\Big)^2 \lesssim \frac{I_\mu(t)}{t} \Big\{ \int_t^{1/2} |d^2 f|_\mu^{**}(s) \frac{ds}{I_\mu(s)} + \|\nabla f\|_{L^1(\mu)} \Big\}.$$

It can be proved that

$$\begin{aligned} |d^2 f|_\mu^{**}(s) &\leq \int_s^{1/2} (|d^2 f|_\mu^{**}(u) - |d^2 f|_\mu^{*}(u)) \frac{du}{u} + 2\|d^2 f\|_{L^1(\mu)} \\ &\leq \int_s^{1/2} \frac{u}{I_\mu(u)} |d^3 f|_\mu^{**}(u) \frac{du}{u} + 2\|d^2 f\|_{L^1(\mu)} \end{aligned}$$

and then, defining $Q_\mu^0 g(t) = 1$, it follows the result for $k = 3$:

$$\int_t^{1/2} |d^2 f|_\mu^{**}(s) \frac{ds}{I_\mu(s)} \lesssim Q_\mu^2(|d^3 f|_\mu^{**})(t) + \|d^2 f\|_{L^1(\mu)} Q_\mu^1(\mathbf{1})(t). \qquad \square$$

Remark A.3.3. *Proposition A.3.2 is valid for any $k \in \mathbb{N}$.*

In the case $k \geq 4$, let us suppose that for all $k \leq m-1$ we have that

$$f_{osc}(t)\Big(\frac{I_\mu(t)}{t}\Big)^{k-1} \lesssim \Big(\frac{I_\mu(t)}{t}\Big)^{k-2} \Big\{ Q_\mu^{k-1}(|d^k f|_\mu^{**})(t) + \sum_{j=1}^{k-1} \|d^j f\|_{L^1(\mu)} Q_\mu^{j-1}(\boldsymbol{1})(t) \Big\},$$

and let now $k = m$. It follows by hipothesis of induction that

$$\begin{aligned} & f_{osc}(t)\Big(\frac{I_\mu(t)}{t}\Big)^{m-1} \\ \leq\ & \frac{I_\mu(t)}{t}\Big(\frac{I_\mu(t)}{t}\Big)^{m-3} \Big\{ Q_\mu^{m-2}(|d^{m-1} f|_\mu^{**})(t) + \sum_{j=1}^{m-2} \|d^j f\|_{L^1(\mu)} Q_\mu^{j-1}(\boldsymbol{1})(t) \Big\}, \end{aligned}$$

and

$$|d^{m-1} f|_\mu^{**}(s) \lesssim Q_\mu^1(|d^m f|_\mu^{**})(t) + \|d^{m-1} f\|_{L^1(\mu)}.$$

Then

$$\begin{aligned} Q_\mu^{m-2}(|d^{m-1} f|_\mu^{**})(t) &\leq Q_\mu^{m-2}\Big(Q_\mu^1(|d^m f|_\mu^{**})(t) + \|d^{m-1} f\|_{L^1(\mu)}\Big) \\ &= Q_\mu^{m-1}(|d^m f|_\mu^{**})(t) + Q_\mu^{m-2}(\boldsymbol{1})(t) \|d^{m-1} f\|_{L^1(\mu)}, \end{aligned}$$

since Q_μ and the powers are lineal, and

$$\begin{aligned} & f_{osc}(t)\Big(\frac{I_\mu(t)}{t}\Big)^{m-1} \\ =\ & \Big(\frac{I_\mu(t)}{t}\Big)^{m-2} \Big\{ Q_\mu^{m-1}(|d^m f|_\mu^{**})(t) + \sum_{j=1}^{m-1} \|d^j f\|_{L^1(\mu)} Q_\mu^{j-1}(\boldsymbol{1})(t) \Big\}. \end{aligned}$$

Corollary A.3.4. *For $f \in W^{2,1}(\mu)$*

$$f_\mu^{**}(t) \lesssim Q_\mu^2(|d^2 f|_\mu^{**})(t) + \|\nabla f\|_{L^1(\mu)} (Q_\mu 1)(t) + \|f\|_{L^1(\mu)}.$$

Proof. By Proposition A.3.1(ii), if $0 < t \leq 1/2$,

$$\begin{aligned} f_\mu^{**}(t) &\leq \int_t^{1/2} \frac{s}{I_\mu(s)} \{Q_\mu(|d^2 f|_\mu^{**})(s) + 2\|\nabla f\|_{L^1(\mu)}\} \frac{ds}{s} + 2\|f\|_{L^1(\mu)} \\ &= Q_\mu^2(|d^2 f|_\mu^{**})(t) + 2\|\nabla f\|_{L^1(\mu)} \int_t^{1/2} \frac{ds}{I_\mu(s)} + \|f\|_{L^1(\mu)}. \end{aligned}$$

If $1/2 < t < 1$, then $f_\mu^{**}(t) \leq f_\mu^{**}(1/2) \leq 2\|f\|_{L^1(\mu)}$. □

By $X = X(\mathbb{R}^n, \mu)$ we always denote an *rearrangement invariant* (r.i. for short) Banach function space on $\mathbb{R}^n$ endowed with the probability measure μ and such that $1 \in X$. In that situation, $L^\infty(\mu) \hookrightarrow X \hookrightarrow L^1(\mu)$ (see [BeSh]).

It is well-known, by the *Luxemburg theorem* (see [BeSh, Theorem 4.2]), that any r.i. Banach function space X on $(\mathbb{R}^n, \mu)$ can be represented as an r.i. space $\bar{X}(0,1)$ on the interval $(0,1)$ with Lebesgue measure and $\|g\|_X = \|g_\mu^*\|_{\bar{X}(0,1)}$ for every $g \in X$.

Let us recall that the *Boyd indices* of X (see [BeSh, Chapter 3, Section 5]) are the numbers $\underline{\alpha}_X$ and $\overline{\alpha}_X$ such that

$$\underline{\alpha}_X = \sup_{0<t<1} \frac{\log(h_X(t))}{\log(t)}, \qquad \overline{\alpha}_X = \inf_{1<t<\infty} \frac{\log(h_X(t))}{\log(t)},$$

where $h_X(t)$ is the norm of $E_{1/t}$ defined in (3.14).

For the usual *Hardy operators* P and Q defined in (2.25)[1], it is well-known that P is bounded on $\bar{X}(0,1)$ if and only if the upper Boyd indice $\overline{\alpha}_X$ is smaller than 1, and Q is bounded if and only if the lower Boyd indice $\underline{\alpha}_X$ is bigger than 0. In fact, if $0 < a < \underline{\alpha}_X$, the operator

$$Q_a g(t) = \frac{1}{t^a} \int_0^t s^a g(s) \frac{ds}{s}$$

is also bounded on $\bar{X}(0,1)$ (cf. [BeSh, Chapter 3]).

If Y is also an r.i. space on $(\mathbb{R}^n, \mu)$, the $X - Y$ Sobolev-Poincaré inequality depends on the boundedness of the *Hardy type operator*

$$Q_\mu g(t) = \int_t^{1/2} g(s) \frac{ds}{I_\mu(s)} := \int_t^1 g(s) \chi_{(0,1/2)}(s) \frac{ds}{I_\mu(s)},$$

as shown in [MMi3]. By $W^{1,X}(\mu)$ we denote the classical Sobolev space generated by the norm in X. Let us see that the same holds for functions in $W^{1,X}(\mu)$:

[1] Recall that

$$Pg(t) = \frac{1}{t} \int_0^t g(s)\, ds, \qquad Qg(t) = \int_t^1 g(s) \frac{ds}{s}.$$

Theorem A.3.5. *If for any* $0 \le g \in \bar{X}(0,1)$ *with* supp $g \subset (0,1/2)$,

$$\|Q_\mu g\|_{\bar{Y}(0,1)} \lesssim \|g\|_{\bar{X}(0,1)}, \tag{A.9}$$

then, the following Sobolev-Poincaré *estimate holds:*

$$\Big\| f - \int_{\mathbb{R}^n} f\, d\mu \Big\|_Y \lesssim \|\nabla f\|_X \qquad (f \in W^{1,X}(\mu)). \tag{A.10}$$

Proof. Let $f \in W^{1,X}(\mu)$, and write

$$f_\mu^*(t) = \int_t^{1/2} (-f_\mu^*)'(s)ds + f_\mu^*(1/2), \quad t \in (0,1/2].$$

Thus,

$$\begin{aligned}
\|f\|_Y &= \|f_\mu^*\|_{\bar{Y}(0,1)} \lesssim \|f_\mu^* \chi_{[0,1/2]}\|_{\bar{Y}(0,1)} \\
&\le \Big\| \int_t^{1/2} (-f_\mu^*)'(s)ds \Big\|_{\bar{Y}(0,1)} + f_\mu^*(1/2)\|1\|_{\bar{Y}(0,1)} \\
&\le \Big\| \int_t^{1/2} (-f_\mu^*)'(s) I_\mu(s) \frac{ds}{I_\mu(s)} \Big\|_{\bar{Y}(0,1)} + 2\|f\|_{L^1(\mu)}\|1\|_{\bar{Y}(0,1)} \\
&\lesssim \|(-f_\mu^*)'(s) I_\mu(s)\|_{\bar{X}(0,1)} + \|f\|_{L^1(\mu)} \text{ by (A.9)} \\
&\lesssim \||\nabla f|\|_X + \|f\|_{L^1(\mu)},
\end{aligned}$$

where the last inequality follows by the Pólya–Szegö inequality in Theorem A.2.1.

Therefore, by (A.8), since $X \hookrightarrow L^1(\mu)$,

$$\begin{aligned}
\Big\| f - \int_{\mathbb{R}^n} f d\mu \Big\|_Y &\lesssim \||\nabla f|\|_X + \Big\| f - \int_{\mathbb{R}^n} f d\mu \Big\|_{L^1(\mu)} \\
&\lesssim \||\nabla f|\|_X + \||\nabla f|\|_{L^1(\mu)}. \square
\end{aligned}$$

Remark A.3.6. *In [MMi3, Section 5 and 6] J. Martín and M. Milman showed that it is possible to characterize the Sobolev-Poincaré inequality on probability spaces of isoperimetric Hardy type.*

In the case of $\mathbb{R}^n$ *with the Gaussian measure* γ*, as it is shown in [MMi3], (A.9) and (A.10) are equivalent due to the fact that* $(\mathbb{R}^n, \gamma)$ *is of isoperimetric Hardy type.*

Since the operator Q_μ is associated to the Sobolev-Poincaré inequality, we will try with Q_μ^2 when dealing with second order derivatives.

From the definition, by Fubini's theorem, if $g \ge 0$ and supp $g \subset (0,1/2)$,

$$Q_\mu^2 g(t) = \int_t^{1/2} \int_s^{1/2} g(r) \frac{dr}{I_\mu(r)} \frac{ds}{I_\mu(s)} = \int_t^{1/2} g(r) \Big(\frac{1}{I_\mu(r)} \int_t^r \frac{ds}{I_\mu(s)} \Big) dr.$$

From now on, X and Y are supposed to be r.i. spaces on $(\mathbb{R}^n, \mu)$. Observe that $Q_\mu 1 \in \bar{X}(0,1)$ means that $I_\mu(t)/t \in \bar{X}(0,1)$, $\overline{\alpha}_X < 1$ means that P is continuous on $\bar{X}(0,1)$, and then $\|f_\mu^{**}\|_{\bar{X}(0,1)} \simeq \|f\|_X$. Moreover, if $\underline{\alpha}_X > 0$, then $Q_\mu 1 \in \bar{X}(0,1)$ since $0 < s < 1/2$, $s \lesssim I_\mu(s)$.

Let us define a new operator $\bar{A}$ for $g \in \bar{X}(0,1)$ by

$$\frac{I_\mu(t)}{t} \int_t^{1/2} g(s) \frac{ds}{I_\mu(s)} := \bar{A}g(t).$$

Since $\frac{I_\mu(t)}{t}$ is decreasing on $(0,1/2)$, if $\bar{A}$ is bounded in $\bar{X}(0,1)$, then

$$\|Q_\mu 1\|_{\bar{X}(0,1)} = \Big\| \int_t^{1/2} 1 \frac{ds}{I_\mu(s)} \Big\|_{\bar{X}(0,1)} \lesssim \Big\| \frac{I_\mu(t)}{t} \int_t^{1/2} 1 \frac{ds}{I_\mu(s)} \Big\|_{\bar{X}(0,1)} \lesssim \|1\|_X,$$

and $I_\mu(t)/t \in \bar{X}(0,1)$.

Remark A.3.7. *If $\bar{A}$ is bounded on $\bar{X}(0,1)$, then it follows that Q_μ is also bounded. Therefore for $Y = X$, the X-X Poincaré inequality holds by Theorem A.3.5.*

By $W^{2,X}(\mu)$ we denote the classical *second-order Sobolev space* generated by the norm in X,

$$\|\phi\|_{W^{2,X}(\mu)} = \sum_{|\sigma| \le 2} \|D^\sigma \phi\|_{X(\mathbb{R}^n,\mu)} = \sum_{|\sigma| \le 2} \|D^\sigma \phi\|_X.$$

Proposition A.3.8. *If $\overline{\alpha}_X < 1$ and $\bar{A}$ is bounded on $\bar{X}(0,1)$, then, for every $f \in W^{2,X}(\mu)$,*

$$\Big\| \frac{I_\mu(t)}{t} |\nabla f|_\mu^{**}(t) \Big\|_{\bar{X}(0,1)} \lesssim \|d^2 f\|_X + \|\nabla f\|_{L^1(\mu)},$$

$$\Big\| (f_\mu^{**}(t) - f_\mu^*(t)) \Big(\frac{I_\mu(t)}{t} \Big)^2 \Big\|_{\bar{X}(0,1)} \lesssim \|d^2 f\|_X + \|\nabla f\|_{L^1(\mu)}.$$

Proof. Suppose that $g \ge 0$ is supported by $(0,1/2)$. Since $\frac{I_\mu(t)}{t} Qg(t) = \bar{A}\Big(\frac{I_\mu(s)}{s} g(s)\Big)(t)$, it follows that

$$\Big\| \frac{I_\mu(t)}{t} Qg(t) \Big\|_{\bar{X}(0,1)} \lesssim \Big\| \frac{I_\mu(t)}{t} g(t) \Big\|_{\bar{X}(0,1)}.$$

Since $X \hookrightarrow L^1(\mu)$, by Proposition A.3.1, (i) for all $0 < t \le 1/2$,

$$|\nabla f|_\mu^{**}(t) \le \int_t^{1/2} (|\nabla f|_\mu^{**}(s) - |\nabla f|_\mu^*(s)) \frac{ds}{s} + 2\|\nabla f\|_{L^1(\mu)}.$$

For every $f \in W^{2,X}(\mu)$, let us denote by $g = |\nabla f|$. Therefore, since $\bar{A}$ is bounded on $\bar{X}(0,1)$ and $|\nabla|\nabla f|| \leq |d^2 f|$, by the oscillation inequality it follows that

$$\begin{aligned}\Big\|\frac{I_\mu(t)}{t}|\nabla f|_\mu^{**}(t)\Big\|_{\bar{X}(0,1)} &\lesssim \Big\|\frac{I_\mu(t)}{t}(g_\mu^{**}(s) - g_\mu^*(s))\Big\|_{\bar{X}(0,1)} + \|\nabla f\|_{L^1(\mu)} \\ &\lesssim \||d^2 f|_\mu^{**}(t)\|_{\bar{X}(0,1)} + \|\nabla f\|_{L^1(\mu)}.\end{aligned}$$

By (ii) of Proposition A.3.1, if $\overline{\alpha}_X < 1$, then

$$\begin{aligned}\Big\|(f_\mu^{**}(t) - f_\mu^*(t))\Big(\frac{I_\mu(t)}{t}\Big)^2\Big\|_{\bar{X}(0,1)} &\lesssim \||d^2 f|_\mu^{**}(t)\|_{\bar{X}(0,1)} + \Big\|\frac{I_\mu(t)}{t}|\nabla f|_\mu^{**}(t)\Big\|_{\bar{X}(0,1)} \\ &\lesssim \|d^2 f\|_X + \|\nabla f\|_{L^1(\mu)}. \quad \square\end{aligned}$$

Theorem A.3.9. *Suppose that $\overline{\alpha}_X < 1$ and $\bar{A}$ is bounded on $\bar{X}(0,1)$. The following statements are equivalent:*

(i) For every $g \geq 0$ with $\operatorname{supp} g \subset (0, 1/2)$,

$$\Big\|\int_t^1 g(s)\Big(\frac{s}{I_\mu(s)}\Big)^2 \frac{ds}{s}\Big\|_{\bar{Y}(0,1)} \lesssim \|g\|_{\bar{X}(0,1)}.$$

(ii) For every $f \in W^{2,X}(\mu)$,

$$\|f\|_Y \lesssim \Big\|f_\mu^*(t)\Big(\frac{I_\mu(t)}{t}\Big)^2\Big\|_{\bar{X}(0,1)}.$$

(iii) For every $f \in W^{2,X}(\mu)$,

$$\|f\|_Y \lesssim \Big\|(f_\mu^{**}(t) - f_\mu^*(t))\Big(\frac{I_\mu(t)}{t}\Big)^2\Big\|_{\bar{X}(0,1)} + \|f\|_{L^1(\mu)}.$$

If these properties are satisfied and $f \in W^{2,X}(\mu)$, then

$$\Big\|f - \int_{\mathbb{R}^n} f\, d\mu\Big\|_Y \lesssim \|d^2 f\|_X + \|\nabla f\|_{L^1(\mu)}. \tag{A.11}$$

Proof. (i) $\Longrightarrow$(ii) If $0 < t < 1/4$, then

$$f_\mu^*(2t) = 2(2t - t)\frac{1}{2t}f_\mu^*(2t) = 2\int_t^{2t}\frac{1}{2t}f_\mu^*(2t)ds \lesssim \int_t^{1/2} f_\mu^*(s)\frac{ds}{s} = \int_t^1 f_\mu^*(s)\chi_{(0,1/2)}(s)\frac{ds}{s}$$

and for $g(s) := (\frac{I_\mu(s)}{s})^2 f_\mu^*(s)\chi_{(0,1/2)}(s)$

$$\begin{aligned}\|f_\mu^*(2t)\|_{\bar{Y}(0,1)} &\leq 2\|f_\mu^*(2t)\chi_{(0,1/2)}(t)\|_{\bar{Y}(0,1)} \lesssim \Big\| \int_t^1 f_\mu^*(s)\chi_{(0,1/2)}(s)\frac{ds}{s}\Big\|_{\bar{Y}(0,1)} \\ &= \Big\| \int_t^1 \Big(\frac{s}{I_\mu(s)}\Big)^2\Big[\Big(\frac{I_\mu(s)}{s}\Big)^2 f_\mu^*(s)\chi_{(0,1/2)}(s)\Big]\frac{ds}{s}\Big\|_{\bar{Y}(0,1)} \lesssim \|g\|_{\bar{X}(0,1)}.\end{aligned}$$

(ii) $\Longrightarrow$(i) Consider g with supp $g \subset (0,1/2)$ and let us define

$$B_\mu g(t) := \int_t^1 g(s)\Big(\frac{s}{I_\mu(s)}\Big)^2\frac{ds}{s},\ t>0.$$

Hence, since this operator is decreasing, let $f \in \mathrm{L}_0(\mathbb{R}^n)$ such that $f_\mu^*(t) = B_\mu g(t)$. Then,

$$\|f\|_Y = \|f_\mu^*\|_{\bar{Y}(0,1)} = \Big\| \int_t^1 g(s)\Big(\frac{s}{I_\mu(s)}\Big)^2\frac{ds}{s}\Big\|_{\bar{Y}(0,1)} \lesssim \Big\| \int_t^1 g(s)\Big(\frac{s}{I_\mu(s)}\Big)^2\frac{ds}{s}\Big(\frac{I_\mu(t)}{t}\Big)^2\Big\|_{\bar{X}(0,1)}.$$

Let us define now $h(u) := g(u)\frac{I_\mu(t)}{t}\frac{u}{I_\mu(u)}$ $(u \in (0,1))$. Since

$$\int_t^1 g(s)\Big(\frac{s}{I_\mu(s)}\Big)^2\frac{ds}{s}\Big(\frac{I_\mu(t)}{t}\Big)^2 = \frac{I_\mu(t)}{t}\int_t^1 h(s)\frac{ds}{I_\mu(s)} \simeq \bar{A}h(t)\ (0<t<1),$$

and $\bar{A}$ is bounded on $\bar{X}(0,1)$, then

$$\|f\|_Y = \|B_\mu g\|_{\bar{Y}(0,1)} \lesssim \|h(t)\|_{\bar{X}(0,1)} = \|g\|_{\bar{X}(0,1)}.$$

(i) $\Longrightarrow$(iii) Observe that for every $f \in W^{2,X}(\mu)$

$$\|f\|_Y = \|f_\mu^*\|_{\bar{Y}(0,1)} \leq \|f_\mu^{**}\|_{\bar{Y}(0,1)} \lesssim \|f_\mu^{**}\chi_{(0,1/2)}\|_{\bar{Y}(0,1)},$$

and we have

$$f_\mu^{**}(t)\chi_{(0,1/2)}(t) \leq \int_t^{1/2}(f_\mu^{**}(s)-f_\mu^*(s))\chi_{(0,1/2)}(s)\frac{ds}{s} + f_\mu^{**}(1/2).$$

Then, it follows that

$$\|f\|_Y \lesssim \Big\|(f_\mu^{**}(t)-f_\mu^*(t))\Big(\frac{I_\mu(t)}{t}\Big)^2\Big\|_{\bar{X}(0,1)} + \|f\|_{L^1(\mu)}.$$

(iii) $\Longrightarrow$(i) Let $g \geq 0$ with supp $g \subset (0,1/2)$ and define for all $0<t<1/2$, $h(t) = B_\mu g(t)$. Then, it follows that $h(t) \lesssim Qg(t)$.

Consider $u \in \mathrm{L}_0(\mathbb{R}^n)$ such that $u_\mu^*(t) = h(t)$. We have that $\|u\|_Y = \|h(t)\|_{\bar{Y}(0,1)}$ and then,

$$\|h(t)\|_{\bar{Y}(0,1)} \lesssim \Big\|(h_\mu^{**}(t)-h_\mu^*(t))\Big(\frac{I_\mu(t)}{t}\Big)^2\Big\|_{\bar{X}(0,1)} + \|h(t)\|_1.$$

By Fubini's theorem we can observe that

$$h_\mu^{**}(t) - h_\mu^*(t) = \frac{1}{t}\int_0^t g(s)\Big(\frac{s}{I_\mu(s)}\Big)^2 ds.$$

Therefore, since the function $s/I_\mu(s)$ is increasing on $(0,1/2)$, P is bounded on $\bar{X}(0,1)$, and

$$(h_\mu^{**}(t) - h_\mu^*(t))\Big(\frac{I_\mu(t)}{t}\Big)^2 \leq \frac{1}{t}\int_0^t g(s)ds = Pg(t),$$

then

$$\Big\| \int_t^1 g(s)\Big(\frac{s}{I_\mu(s)}\Big)^2 \frac{ds}{s}\Big\|_{\bar{Y}(0,1)} \lesssim \|g\|_{\bar{X}(0,1)} + \|h(t)\|_1.$$

Finally, let us observe that for all $0 < t < 1/2$,

$$h(t) = \int_t^1 g(s)\Big(\frac{s}{I_\mu(s)}\Big)^2 \frac{ds}{s} \lesssim Q_\mu g(t) \lesssim \bar{A}g(t)$$

and, since $X \hookrightarrow L^1(\mu)$ and $\bar{A}$ is bounded on $\bar{X}(0,1)$, then it follows that

$$\|h(t)\|_1 \leq \Big\| \int_t^1 g(s)\Big(\frac{s}{I_\mu(s)}\Big)^2 \frac{ds}{s}\Big\|_{\bar{X}(0,1)} \lesssim \|Q_\mu g(t)\|_{\bar{X}(0,1)} \lesssim \|g\|_{\bar{X}(0,1)}.$$

Therefore, (i) follows by addition of the inequalities.

To finish let us show that (iii) implies (A.11): Let $f \in W^{2,X}(\mu)$. Since P and $\bar{A}$ are bounded on $\bar{X}(0,1)$, by Proposition A.3.8

$$\|f\|_Y \lesssim \|d^2 f\|_X + \|\nabla f\|_{L^1(\mu)} + \|f\|_{L^1(\mu)}.$$

Therefore, considering now $g := f - \int f d\mu$, by (A.8), it follows that

$$\begin{aligned} \|g\|_Y &\lesssim \|d^2 f\|_X + \|\nabla f\|_{L^1(\mu)} + \|f - m_f\|_{L^1(\mu)} \\ &\lesssim \|d^2 f\|_X + \|\nabla f\|_{L^1(\mu)}. \quad \square \end{aligned}$$

Remark A.3.10. *Let us observe that, in Theorem A.3.9, without any condition on the indices and on $\bar{A}$, (i) implies (ii) and (i) implies (iii). (ii) implies (i) if $\bar{A}$ is bounded on $\bar{X}(0,1)$, and (iii) implies (i) if $\overline{\alpha}_X < 1$ and $\bar{A}$ is bounded on $\bar{X}(0,1)$.*

A.4 Second order Hardy type operators

Let us see now that the second order Sobolev-Poincaré inequality follows from the boundedness of the operator Q_μ from $\bar{X}(0,1)$ to $\bar{Y}(0,1)$. In the next section we will see that for the Gaussian measure the corresponding characterization follows.

A probability measure space $(\mathbb{R}^n, \mu)$ is of *isoperimetric Hardy type* (see e.g. [MMi3]) if for any given isoperimetric estimator I, the following properties are equivalent for every r.i. spaces $X = X(\mathbb{R}^n, \mu), Y = Y(\mathbb{R}^n, \mu)$:

(i) There exists a constant $c = c(X, Y)$ such that for any $f \in \mathrm{Lip}(\mathbb{R}^n)$,

$$\Big\| f - \int_{\mathbb{R}^n} f \, d\mu \Big\|_Y \le c \|\nabla f\|_X. \tag{A.12}$$

(ii) There exists a constant $c_1 = c_1(X, Y)$ such that for any $0 \le g \in \bar{X}(0,1)$ with supp $g \subset (0, 1/2)$,

$$\|Q_I g\|_{\bar{Y}(0,1)} \le c_1 \|g\|_{\bar{X}(0,1)}, \tag{A.13}$$

where Q_I is defined as Q_μ with I instead of I_μ.

From now on, assume that μ is such that $(\mathbb{R}^n, \mu)$ is of isoperimetric Hardy type. We will denote for $f \in \mathrm{L}_0(\mathbb{R}^n)$,

$$\Lambda_f := p + \int (f - p) \, d\mu$$

with

$$p(x) := \int f \, d\mu + \sum_{i=1}^n \Big(\int \partial_i f \, d\mu \Big) x_i.$$

Let $f \in W^{2,X}(\mu)$. Then $|\nabla f| \in W^{1,X}(\mu)$. Moreover, if we assume that Q_μ is bounded for positive functions from $\bar{X}(0,1)$ on $\bar{Y}(0,1)$, then, by Theorem A.3.5, it follows that

$$\Big\| |\nabla f| - \int |\nabla f| d\mu \Big\|_Y \lesssim \|d^2 f\|_X.$$

Let us start now from $W^{1,Y}(\mu)$ and consider $Z = Y$. We have that $\|f - \int f d\mu\|_Z \lesssim \|\nabla f\|_Y$, and then $Q_\mu : \bar{Y}(0,1) \to \bar{Z}(0,1)$ is bounded. Therefore

$$Q_\mu^2 : \bar{X}(0,1) \to \bar{Z}(0,1)$$

is bounded on positive functions supported on $(0, 1/2)$.

Hence, if $Q_\mu : \bar{X}(0,1) \to \bar{Y}(0,1)$ is bounded on positive functions supported on $(0, 1/2)$, then for the same kind of functions

$$Q_\mu^2 : \bar{X}(0,1) \to \bar{Y}(0,1).$$

Theorem A.4.1. *If $\overline{\alpha}_X < 1$ and $Q_\mu : \bar{X}(0,1) \to \bar{Y}(0,1)$ is a bounded operator (i.e. $\|Q_\mu g\|_{\bar{Y}(0,1)} \lesssim \|g\|_{\bar{X}(0,1)}$ if $g \geq 0$), then*

(a) *$W^{2,X}(\mu) \hookrightarrow Y$, and*

(b)

$$\|f - \Lambda_f\|_Y \lesssim \|d^2 f\|_X \qquad (f \in W^{2,X}(\mu)). \tag{A.14}$$

Proof. (a) Suppose $f \in W^{2,X}(\mu)$ and let $0 < t < 1$, so that, by Corollary A.3.4,

$$f_\mu^{**}(t) \lesssim Q_\mu^2(|d^2 f|_\mu^{**})(t) + \|\nabla f\|_{L^1(\mu)}(Q_\mu 1)(t) + \|f\|_{L^1(\mu)}.$$

Consequently, since $f_\mu^* \leq f_\mu^{**}$ and Q_μ is bounded from $\bar{X}(0,1)$ to $\bar{Y}(0,1)$, then $Q_\mu 1 \in \bar{Y}(0,1)$ and

$$\|f\|_Y \leq \|f_\mu^{**}\|_{\bar{Y}(0,1)} \lesssim \||d^2 f|_\mu^{**}\|_{\bar{X}(0,1)} + \|\nabla f\|_{L^1(\mu)}\|Q_\mu 1\|_{\bar{Y}(0,1)} + \|f\|_{L^1(\mu)}.$$

By condition $\bar{\alpha}_X < 1$, since $X \hookrightarrow L^1(\mu)$, it follows that

$$\|f\|_Y \lesssim \||d^2 f|_\mu^{**}\|_{\bar{X}(0,1)} + \|\nabla f\|_X + \|f\|_X \simeq \|f\|_{W^{2,X}(\mu)}.$$

(b) Suppose $f \in W^{2,X}(\mu)$ and apply Corollary A.3.4 to $g = f - \Lambda_f \in W^{2,X}(\mu)$ to obtain

$$\|g\|_Y \lesssim \|Q_\mu^2(|d^2 g|_\mu^{**})\|_{\bar{Y}(0,1)} + \|\nabla g\|_{L^1(\mu)}\|Q_\mu 1\|_{\bar{Y}(0,1)} + \|g\|_{L^1(\mu)}.$$

By the basic Poincaré inequality,

$$\|\nabla g\|_{L^1(\mu)} \leq \sum_{i=1}^n \left\|\partial_i f - \int \partial_i f \, d\mu\right\|_{L^1(\mu)} \leq \sum_{i=1}^n \|\nabla \partial_i f\|_{L^1(\mu)} \lesssim \|d^2 f\|_X$$

and similarly, using the definition of p,

$$\|g\|_{L^1(\mu)} \;=\; \left\|(f-p) - \int (f-p)\, d\mu\right\|_{L^1(\mu)} \leq \sum_{i=1}^n \|\nabla(f-p)\|_{L^1(\mu)} \lesssim \|d^2 f\|_X.$$

Therefore, it follows that $\|g\|_Y \lesssim \|d^2 f\|_X$. □

Proposition A.4.2. *If $\overline{\alpha}_X < 1$ and for every $f \in W^{1,X}(\mu)$,*

$$\left\|f - \int_{\mathbb{R}^n} f \, d\mu\right\|_Y \lesssim \|\nabla f\|_X,$$

then

$$\inf_{\Lambda \in \mathcal{P}_1} \|f - \Lambda_f\|_Y \lesssim \|d^2 f\|_X \qquad (f \in W^{2,X}(\mu)).$$

Proof. By hipothesis, by the equivalence of (A.12) and (A.13) in particular for I_μ, for every $0 \le g \in \bar{X}(0,1)$ with supp $g \subset (0,1/2)$,

$$\Big\| \int_t^1 g(s) \frac{ds}{I_\mu(s)} \Big\|_{\bar{Y}(0,1)} = \|Q_\mu g\|_{\bar{Y}(0,1)} \lesssim \|g\|_{\bar{X}(0,1)}.$$

Then, by Theorem A.4.1, for every $f \in W^{2,X}(\mu)$,

$$\inf_{\Lambda \in \mathcal{P}_1} \|f - \Lambda\|_Y \lesssim \|d^2 f\|_X. \qquad \square$$

A.5 Examples: The Gaussian measure space

As in [MMi3], we consider $\mathbb{R}^n$ endowed with the Gaussian measure $\gamma = \gamma_n$, where for $A \subset \mathbb{R}^n$

$$\gamma(A) := \int_A \phi_n(x)\, dx, \quad \phi_n(x) := \frac{1}{(2\pi)^{n/2}} e^{-|x|^2/2}.$$

In that situation the isoperimetric problem is solved by half-lines (cf. [Bor85] and [Bob]). The *isoperimetric inequality* for γ was found by V. N. Sudakov and B. S. Tsirelson [ST], and by C. Borell [Bor], and the *Gaussian isoperimetric profile (or Gaussian isoperimetric function)*, is defined as

$$I_\gamma(t) := \phi_1(\Phi^{-1}(t)) \qquad (0 \le t \le 1),$$

where $\Phi : \mathbb{R} \to (0,1)$ is the *distribution function* for γ_1, extended by $\Phi(-\infty) = 0$ and $\Phi(+\infty) = 1$

$$\Phi(r) = \int_{-\infty}^r \phi_1(s)\, ds = \int_{-\infty}^r \frac{1}{(2\pi)^{1/2}} e^{-s^2/2}\, ds.$$

The Gaussian isoperimetric profile is such that for $0 \le t \le 1$

(1) $I_\gamma(0) = I_\gamma(1) = 0$,

(2) $I_\gamma(t) = I_\gamma(1-t)$,

(3) $I''_\gamma(t) = -1/I_\gamma(t)$,

(4) $I_\gamma(t)$ is concave, and

(5) $I_\gamma(t) \simeq t(\log(1/t))^{1/2}$ on $[0,1/2]$; more precisely, $\lim_{t\to 0} \frac{I_\gamma(t)}{t(2\log\frac{1}{t})^{1/2}} = 1$.

As noticed in [MMi1], the operator $Q_{\log}$, defined on functions $g \ge 0$ supported by $(0,1/2)$, as

$$\bar{A}g(t) = Q_{\log} g(t) = \frac{I_\gamma(t)}{t} \int_t^1 g(s) \frac{ds}{I_\gamma(s)} \simeq (1+\log(1/t))^{1/2} \int_t^1 g(s)(1+\log(1/s))^{1/2} \frac{ds}{s},$$

for any $0 < a < \underline{\alpha}_X$, is dominated by Q_a since, using that $t^a(1+\log(1/t))^{1/2}$ is increasing near zero,

$$\bar{A}g(t) \simeq \frac{t^a(1+\log(1/t))^{1/2}}{t^a}\int_t^1 g(s)(1+\log(1/s))^{1/2}\frac{ds}{s} \lesssim Q_a g(t).$$

Hence, if $\underline{\alpha}_X > 0$, then $\bar{A}$ is bounded on $\bar{X}(0,1)$ (cf. [MMi5, Section 2.3]).

Remark A.5.1. *If $\underline{\alpha}_X > 0$, then also Q_γ is bounded on $\bar{X}(0,1)$, since*

$$Q_\gamma g(t) = \int_t^{1/2} g(s)\frac{ds}{I_\gamma(s)} \lesssim \frac{I_\gamma(t)}{t}\int_t^{1/2} g(s)\,\frac{ds}{I_\gamma(s)}.$$

Moreover, if $\underline{\alpha}_X > 0$, since $\bar{A}$ is bounded on $\bar{X}(0,1)$, then, by [MMi3, Theorem 5,a], it follows that the $X-Y$ Sobolev-Poincaré inequalitiy holds. And then, since $(\mathbb{R}^n,\gamma)$ is of isoperimetry Hardy type, Q_γ is bounded from $\bar{X}(0,1)$ to $\bar{Y}(0,1)$.

Note that

$$Q_\gamma^2 g(t) = \int_t^{1/2}\int_s^{1/2} g(u)\frac{du}{I_\gamma(u)}\frac{ds}{I_\gamma(s)} \simeq \bar{A}g(t) - Qg(t),$$

since

$$\int_t^r \frac{ds}{I_\gamma(s)} \simeq (-\log t)^{1/2} - (-\log r)^{1/2} \simeq \frac{I_\gamma(t)}{t} - \frac{I_\gamma(r)}{r}.$$

Condition $Q_\gamma^2 1 \in \bar{X}(0,1)$ implies $Q_\gamma 1 \in \bar{X}(0,1)$. Indeed, $Q_\gamma 1(t) = \int_t^{1/2}(I_\gamma(s))^{-1}ds$ is an unbounded continuous function which decreases to 0 on $(0,1/2)$, and then $Q_\gamma 1(a) = 1$ for some $a \in (0,1/2)$. For every $t \le a$,

$$Q_\gamma^2 1(t) = \int_t^{1/2} Q_\gamma 1(s)\frac{ds}{I_\gamma(s)} \ge Q_\gamma 1(a) Q_\gamma 1(t) = Q_\gamma 1(t)$$

and, if $a < t \le 1/2$, $Q_\gamma 1(t) \le Q_\gamma 1(a) = 1$. So $Q_\gamma 1 \le \chi_{(0,a]} Q_\gamma^2 1 + \chi_{(a,1)}$ and $\|Q_\gamma 1\|_{\bar{X}(0,1)} < \infty$.

In the Gaussian case, Theorem A.4.1 has the following converse:

Theorem A.5.2. *Suppose that*

$$\|f - \Lambda_f\|_Y \lesssim \|d^2 f\|_X \qquad (f \in W^{2,X}(\gamma)).$$

Then for every $g \ge 0$ with supp $g \subset (0,1/2)$,

$$\|Q_\gamma^2\, g\|_{\bar{Y}(0,1)} \lesssim \|g\|_{\bar{X}(0,1)},$$
$$\|Q_\gamma\, g\|_{\bar{Y}(0,1)} \lesssim \|g\|_{\bar{X}(0,1)}.$$

Proof. Take $g \geq 0$ so that supp $g \subset (0, 1/2)$ and define

$$f(x) = Q_\gamma^2 g(\Phi(x_1)) = \int_{\Phi(x_1)}^1 (Q_\gamma g)(s)\, \frac{ds}{I_\gamma(s)},$$

which is a function that only depends on x_1, and in this case

$$p(x) = \int f\, d\gamma + \Big(\int \partial_1 f\, d\gamma \Big) x_1.$$

Moreover,

$$\partial_1 f(x) = -\frac{(Q_\gamma g)(\Phi(x_1))}{I_\gamma(\Phi(x_1))} \Phi'(x_1) = -(Q_\gamma g)(\Phi(x_1))$$

and

$$\partial_1^2 f(x) = g(\Phi(x_1)), \quad |d^2 f(x)| = |\partial_1^2 f(x)| = |g(\Phi(x_1))|.$$

Consequently, $f_\gamma^*(t) = (Q_\gamma^2 g)(t)$, $|\nabla f|_\gamma^*(t) = (Q_\gamma g)(t)$, $|d^2 f|_\gamma^*(t) = g_\gamma^*(t)$. By substracting and adding Λ_f to f, we obtain

$$\|Q_\gamma^2 g\|_{\bar{Y}(0,1)} = \|f\|_Y \lesssim \|d^2 f\|_X + \|\Lambda_f\|_Y = \|g\|_X + \|\Lambda_f\|_Y,$$

where it is easily checked that

$$\|\Lambda_f\|_Y \lesssim \|f\|_{L^1(\gamma)} + \|\partial_1 f\|_{L^1(\gamma)} = \|Q_\gamma^2 g\|_1 + \|Q_\gamma g\|_1.$$

By (A.10), $\|Q_\gamma g\|_1 \lesssim \|g\|_1$, and $\|Q_\gamma^2 g\|_1 + \|Q_\gamma g\|_1 \lesssim \|g\|_1 \lesssim \|g\|_{\bar{X}(0,1)}$ and we conclude that

$$\|Q_\gamma^2 g\|_{\bar{Y}(0,1)} \lesssim \|g\|_{\bar{X}(0,1)}.$$

With a similar approach we obtain the same conclusion for Q_γ. □

A.6 Optimal second order Sobolev-Poincaré embeddings

In the appendix we showed that some of the developments done for first order derivatives can be extended, somehow with the same techniques, to higher-order derivatives. In this last section we want to obtain descriptions of the optimal range and domain for second order Sobolev-Poincaré inequalities in $\mathbb{R}^n$ with the Gaussian measure. We will follow the proofs used by A. Cianchi and L. Pick in [CiL], where they studied the optimal range and domain in the Sobolev-Poincaré inequality.

From now on, let X and Y be r.i. spaces on $(\mathbb{R}^n, \gamma)$. We say that Y is the *optimal range* for X in the Gaussian second order Sobolev-Poincaré inequality (A.14) if

(a) inequality (A.14) holds, and

(b) if Z is any r.i. space on $(\mathbb{R}^n, \gamma)$ such that (A.14) holds with Y replaced by Z, then $Y \hookrightarrow Z$.

Analogously, the space X is said to be the *optimal domain* for Y in the Gaussian second order Sobolev-Poincaré inequality (A.14) if

(a) inequality (A.14) holds, and

(b) if Z is any r.i. space on $(\mathbb{R}^n, \gamma)$ such that (A.14) holds with X replaced by Z, then $Z \hookrightarrow X$.

Finally, we say that (X, Y) is an *optimal pair* in the Gaussian second order Sobolev-Poincaré inequality (A.14) if Y is the optimal range for X and, simultaneously, X is the optimal domain for Y.

Recall that given an r.i. space X, the *associate space* of X (see (1.19)) is

$$X' := \Big\{\phi \in \mathrm{L}_0(\mathbb{R}^n); \int_{\mathbb{R}^n} |\phi(x)\psi(x)| d\gamma(x) < \infty \text{ for every } \psi \in X\Big\},$$

equipped with the norm

$$\|\phi\|_{X'} = \sup_{\|\psi\|_X \le 1} \int_{\mathbb{R}^n} |\phi(x)\psi(x)| d\gamma(x).$$

The *Lorentz-Zygmund spaces* $L^{p,r}(\mathrm{Log}L)^\alpha(0,1)$, introduced by C. Bennett and K. Rudnick (see [BeR]), are defined by the conditions

$$\|f\|_{L^{p,r}(\mathrm{Log}L)^\alpha(0,1)} := \begin{cases} \Big(\int_0^1 [t^{1/p}(\log \frac{e}{t})^\alpha f^*(t)]^r \frac{dt}{t}\Big)^{1/r} < \infty, & r < \infty \\ \sup_{0<t<1} t^{1/p}(\log \frac{e}{t})^\alpha f^*(t) < \infty, & r = \infty. \end{cases}$$

Also $\exp L^\beta(0,1) = L^{\infty,\infty;-1/\beta}(0,1)$ and $L^p(\mathrm{Log}L)^\alpha(0,1) = L^{p,p;\alpha/p}(0,1)$.

Although we got quite a characterization of inequality (A.14), let us present other descriptions of the norms of the optimal range and domain spaces in inequality (A.14). Denote by $\bar{Y}(0,1/2)$ the subspace

$$\{f \in \mathrm{L}_0(\mathbb{R}^n); \|f\chi_{(0,1/2)}\|_{\bar{Y}(0,1)} < \infty\}.$$

Theorem A.6.1. *Let X be an r.i. space such that $\overline{\alpha}_X < 1$, and let Y be the r.i. space whose associate norm is given by*

$$\|f\|_{Y'} = \|f\|_{Y'(\mathbb{R}^n,\gamma)} = \Big\|P\Big(f^{**}(s)\frac{s}{I_\gamma(s)}\frac{u}{I_\gamma(u)}\Big)(u)\Big\|_{\bar{X}'(0,1/2)} \ (f \in \mathrm{L}_0(\mathbb{R}^n)).$$

Then Y is the optimal range for X in the Gaussian second order Sobolev inequality (A.14).

Proof. It is enough to prove that

$$\Big\| P\Big(f_\gamma^{**}(s) \frac{s}{I_\gamma(s)} \frac{u}{I_\gamma(u)} \Big)(u) \Big\|_{\bar{X}'(0,1/2)}$$

defines an r.i. norm for any $f \in \mathrm{L}_0(\mathbb{R}^n)$. The positive homogeneity and non-triviality follows as usual. The triangle inequality follows by the subadditivity of the operator $f \to f_\gamma^{**}$ and the fact that $\bar{X}'(0,1/2)$ is an r.i. Banach space. The lattice and the Fatou properties follow by the properties of the decreasing average f_γ^{**} and the r.i. norm in $\bar{X}(0,1/2)$. Moreover,

$$\|1\|_{Y'} \le \Big(\frac{1/2}{I_\gamma(1/2)} \Big)^2 \Big\| \frac{1}{u} \int_0^u ds \Big\|_{\bar{X}'(0,1/2)} \lesssim \|1\|_{\bar{X}'(0,1/2)} < \infty.$$

Finally, since $\|g\|_{\bar{X}'(0,1/2)} \ge \int |g| dm_n$, we obtain that $\|g\|_{\bar{Y}'(0,1)} \ge \int |g| dm_n$.

Let $g \in \mathrm{L}_0(\mathbb{R}^n)$. By Fubini's theorem applied two times we obtain that

$$\begin{aligned} & \sup_{\|g\|_{\bar{X}(0,1/2)} \le 1} \Big\| \int_t^{1/2} \int_s^{1/2} g(u) \frac{du}{I_\gamma(u)} \frac{ds}{I_\gamma(s)} \Big\|_{\bar{Y}(0,1)} \\ = & \sup_{\|g\|_{\bar{X}(0,1/2)} \le 1} \sup_{\|f\|_{\bar{Y}'(0,1)} \le 1} \int_0^1 f_\gamma^*(t) \int_t^{1/2} \int_s^{1/2} g(u) \frac{du}{I_\gamma(u)} \frac{ds}{I_\gamma(s)} dt \\ = & \sup_{\|f\|_{\bar{Y}'(0,1)} \le 1} \Big\| P\Big(f_\gamma^{**}(s) \frac{s}{I_\gamma(s)} \frac{u}{I_\gamma(u)} \Big)(u) \Big\|_{\bar{X}'(0,1/2)} = 1. \end{aligned}$$

Therefore,

$$\Big\| \int_t^{1/2} \int_s^{1/2} g(u) \frac{du}{I_\gamma(u)} \frac{ds}{I_\gamma(s)} \Big\|_{\bar{Y}(0,1)} \le \|g\|_{\bar{X}(0,1/2)} \le \|g\|_{\bar{X}(0,1)}.$$

Hence, for $g \ge 0$ supported on $(0,1/2)$

$$\Big\| \bar{A}g - Qg \Big\|_{\bar{Y}(0,1)} \le \|g\|_{\bar{X}(0,1)}.$$

Then,

$$\|Q_\gamma g\|_{\bar{Y}(0,1)} = \Big\| \int_t^{1/2} g(s) \frac{ds}{I_\gamma(s)} \Big\|_{\bar{Y}(0,1)} \le \frac{1/2}{I_\gamma(1/2)} \Big\| \int_t^{1/2} g(s) \frac{ds}{s} \Big\|_{\bar{Y}(0,1)} \le \|g\|_{\bar{X}(0,1)}.$$

Hence, by Theorem A.4.1, it follows that

$$\|f - \Lambda_f\|_Y \lesssim \|d^2 f\|_X \qquad (f \in W^{2,X}(\gamma)).$$

To show that Y is the optimal range for X, suppose that Z is another r.i. space such that

$$\|f - \Lambda_f\|_Z \lesssim \|d^2 f\|_X \qquad (f \in W^{2,X}(\gamma)).$$

Then, by Theorem A.5.2, for all $g \geq 0$ with supp $g \subset (0, 1/2)$, it follows that

$$\|Q_\gamma^2 g\|_{\bar{Z}(0,1)} \lesssim \|g\|_{\bar{X}(0,1)} = \|g\|_{\bar{X}(0,1/2)}.$$

Therefore, since

$$\begin{aligned}
&\sup_{\|g\|_{\bar{X}(0,1/2)} \leq 1} \Big\| \int_t^{1/2} \int_s^{1/2} g(u) \frac{du}{I_\gamma(u)} \frac{ds}{I_\gamma(s)} \Big\|_{\bar{Z}(0,1)} \\
= &\sup_{\|g\|_{\bar{X}(0,1/2)} \leq 1} \sup_{\|f\|_{\bar{Z}'(0,1)} \leq 1} \int_0^{1/2} f_\gamma^*(t) \int_t^{1/2} \int_s^{1/2} g(u) \frac{du}{I_\gamma(u)} \frac{ds}{I_\gamma(s)} dt \\
= &\sup_{\|f\|_{\bar{Z}'(0,1)} \leq 1} \Big\| P\Big(f_\gamma^{**}(s) \frac{s}{I_\gamma(s)} \frac{u}{I_\gamma(u)} \Big)(u) \Big\|_{\bar{X}'(0,1/2)},
\end{aligned}$$

it follows that $\|f\|_{\bar{Y}'(0,1)} \lesssim \|f\|_{\bar{Z}'(0,1)}$. And hence, $Y \hookrightarrow Z$ and Y is the optimal range for X. □

From now on, for $u \in \mathrm{L}_0(\mathbb{R}^n)$, $h \sim u$ means that there exists a *measure preserving map* $H : (0,1) \to (0,1)$ such that $h = h_\gamma^* \circ H = u_\gamma^* \circ H$.

Lemma A.6.2. *Let Y be an r.i. space satisfying*[2]

$$\exp L^4(\mathbb{R}^n, \gamma) \hookrightarrow Y \hookrightarrow L(\mathrm{Log}L)^{1/4}(\mathbb{R}^n, \gamma) \tag{A.15}$$

and $\overline{\alpha}_Y < 1$. Define

$$\|u\|_X = \sup_{0 \leq h \sim u} \Big\| \int_s^{1/2} \int_t^{1/2} \frac{h(r)}{I_\gamma(r)} dr \frac{dt}{I_\gamma(t)} \Big\|_{\bar{Y}(0,1)} \qquad (u \in \mathrm{L}_0(\mathbb{R}^n)).$$

Then $\|\cdot\|_X$ is an r.i. norm and X is the optimal domain for Y in the Gaussian second order Sobolev embedding (A.14) .

Proof. Since $\overline{\alpha}_Y < 1$, we obtain that P is bounded on $\bar{X}(0,1)$, that is, $\overline{\alpha}_X < 1$. Since Q_γ^2 is bounded on positive functions supported on $(0, 1/2)$, then as in Theorem A.6.1 we obtain that the hipothesis of Theorem A.4.1 holds. Therefore, by Theorem A.4.1, it follows that

$$\|f - \Lambda_f\|_Y \lesssim \|d^2 f\|_X \qquad (f \in W^{2,X}(\gamma)).$$

Moreover, if Z is another r.i. space satisfying the same inequality, by Theorem A.5.2 we see that, for all $g \geq 0$ with supp $g \subset (0, 1/2)$ we obtain that $\|g\|_{\bar{X}(0,1)} := \|Q_\gamma^2 g\|_{\bar{Y}(0,1)} \leq$

[2]The spaces $\exp L^2(\mathbb{R}^n, \gamma)$ and $L(\mathrm{Log}L)^{1/2}(\mathbb{R}^n, \gamma)$ where used by A. Cianchi and L. Pick in [CiL] in their study of the Sobolev-Poincaré inequality.

$\|g\|_{\bar{Z}(0,1)}$ which implies $\bar{Z}(0,1) \hookrightarrow \bar{X}(0,1)$ and hence X is the optimal domain for Y in the Gaussian second order Sobolev embedding (A.14).

To see that $\|\cdot\|_X$ defines an r.i. norm we proceed as in [CiL]. We can see the lattice property and the triangle inequality thanks to the same properties for $\bar{Y}(0,1)$ and some classical facts in measure theory. Indeed, consider $f, g \in \mathrm{L}_0^+(\mathbb{R}^n)$ such that $f \leq g$ a.e. in $\mathbb{R}^n$. Then, for any nonnegative function $h \sim f$, there exists, by [BeSh, Chapter 2, Corollary 7.6], a measure preserving map $H : (0,1) \to (0,1)$ such that $h = h_\gamma^* \circ H = f_\gamma^* \circ H$, and, since $f_\gamma^* \leq g_\gamma^*$ in $(0,1)$, then $h \leq g_\gamma^* \circ H$. Moveover, $g_\gamma^* \circ H \sim g$ since both functions are equimeasurable (see [BeSh, Chapter 2, Corollary 7.2]). Hence,

$$\|f\|_X \leq \|g\|_X. \tag{A.16}$$

The lattice property is consequence of (A.16).

Now, let us see the triangular one. First, observe that for any simple functions f, g in $\mathbb{R}^n$ and h in $(0,1)$ such that $h \sim f + g$, there exists simple functions h_f and h_g on $(0,1)$ (see Examples 1.4, 1.6, Proposition 7.4, Corollary 7.6, and the paragraph before Example 7.7 in [BeSh, Chapter 2]) such that

$$h_f \sim f, \ h_g \sim g \text{ and } \ h = h_f + h_g. \tag{A.17}$$

Then, let now $f, g \in \mathrm{L}_0(\mathbb{R}^n)$. It is well-known, from measure theory, that there exists a sequence of nonnegative simple functions $\{f_k\}$ and $\{g_k\}$ such that

$$f_k \nearrow |f| \text{ and } g_k \nearrow |g| \text{ as } k \to \infty,$$

and, in particular,

$$\lim_{k\to\infty} (f_k + g_k)_\gamma^* = (|f| + |g|)_\gamma^* \text{ in } (0,1) \tag{A.18}$$

by the properties of the decreasing rearrangement.

Given any $h \in \mathrm{L}_0^+(0,1)$ such that $h \sim |f| + |g|$, there exists a measure preserving map H such that $h = h_\gamma^* \circ H = (|f|+|g|)_\gamma^* \circ H$. Then, defining the sequence $\{h_k\}$ by $h_k = (f_k + g_k)_\gamma^* \circ H$ for $k \in \mathbb{N}$, we obtain that

$$h_k \sim f_k + g_k \text{ for } k \in \mathbb{N},$$

and, by (A.18),

$$\lim_{k\to\infty} h_k = h \text{ in } (0,1).$$

Moreover, by the subadditivity of the average function and by definition, it follows that

$$h_k^{**}(s) = (f_k + g_k)^{**}(s) \leq f_k^{**}(s) + g_k^{**}(s) \leq f^{**}(s) + g^{**}(s) \qquad (s \in (0,1), k \in \mathbb{N}). \tag{A.19}$$

Since f_k and g_k are integrable, by (A.19) we see that h_k is also equiintegrable in $(0,1)$, that is, h_k is integrable and equimeasurable with f_k+g_k. Moreover, since the functions $\frac{1}{I_\gamma(r)}$, $\frac{1}{I_\gamma(t)}$ are bounded for $r\in(t,1/2)$ and $t\in(s,1/2)$, then the function $\frac{h(r)}{I_\gamma(r)}\frac{1}{I_\gamma(t)}$ is equiintegrable in $r\in(t,1/2), t\in(s,1/2)$. Therefore, by the monotone convergence theorem,

$$\lim_{k\to\infty}\int_s^{1/2}\int_t^{1/2}\frac{h_k(r)}{I_\gamma(r)}dr\frac{dt}{I_\gamma(t)}=\int_s^{1/2}\int_t^{1/2}\frac{h(r)}{I_\gamma(r)}dr\frac{dt}{I_\gamma(t)}\qquad (s\in(0,1/2)). \tag{A.20}$$

Since the r.i. spaces have the Fatou property (recall Section 2 of Chapter 3), then, by (A.20), it follows that

$$\Big\|\int_s^{1/2}\int_t^{1/2}\frac{h(r)}{I_\gamma(r)}dr\frac{dt}{I_\gamma(t)}\Big\|_{\bar Y(0,1)}\le\liminf_{k\to\infty}\Big\|\int_s^{1/2}\int_t^{1/2}\frac{h_k(r)}{I_\gamma(r)}dr\frac{dt}{I_\gamma(t)}\Big\|_{\bar Y(0,1)}. \tag{A.21}$$

Therefore, taking supremum in (A.21) over all $h\in \mathrm{L}_0^+(0,1)$ such that $h\sim|f|+|g|$, we obtain

$$\||f|+|g|\|_X\le\liminf_{k\to\infty}\Big\|\int_s^{1/2}\int_t^{1/2}\frac{h_k(r)}{I_\gamma(r)}dr\frac{dt}{I_\gamma(t)}\Big\|_{\bar Y(0,1)}. \tag{A.22}$$

We have sequences of functions h_k such that $h_k\sim f_k+g_k$. To finish, we need to find two sequences related by "equivalence" with the sequences $\{f_k\}$ and $\{g_k\}$. For that, let us observe that, by (A.17), there exists two sequences of functions $\{h_{f_k}\}$ and $\{h_{g_k}\}$ such that

$$h_{f_k}\sim f_k,\ h_{g_k}\sim g_k \text{ and } h_k=h_{f_k}+h_{g_k} \text{ for } k\in\mathbb{N}.$$

Furthermore, there exists two sequences of measure preserving maps $\{H_{f_k}\}$ and $\{H_{g_k}\}$ such that

$$h_{f_k}=(h_{f_k})^*\circ H_{f_k}=(f_k)^*\circ H_{f_k}\le f^*\circ H_{f_k}\sim f^* \text{ for } k\in\mathbb{N},$$

and the same for g_k replacing f_k by g_k and f by g. Therefore,

$$\begin{aligned}
&\Big\|\int_s^{1/2}\int_t^{1/2}\frac{h_k(r)}{I_\gamma(r)}dr\frac{dt}{I_\gamma(t)}\Big\|_{\bar Y(0,1)}\\
\le\ &\Big\|\int_s^{1/2}\int_t^{1/2}\frac{h_{f_k}(r)}{I_\gamma(r)}dr\frac{dt}{I_\gamma(t)}\Big\|_{\bar Y(0,1)}+\Big\|\int_s^{1/2}\int_t^{1/2}\frac{h_{g_k}(r)}{I_\gamma(r)}dr\frac{dt}{I_\gamma(t)}\Big\|_{\bar Y(0,1)}\\
\le\ &\Big\|\int_s^{1/2}\int_t^{1/2}\frac{f^*\circ H_{f_k}(r)}{I_\gamma(r)}dr\frac{dt}{I_\gamma(t)}\Big\|_{\bar Y(0,1)}+\Big\|\int_s^{1/2}\int_t^{1/2}\frac{g^*\circ H_{g_k}(r)}{I_\gamma(r)}dr\frac{dt}{I_\gamma(t)}\Big\|_{\bar Y(0,1)}\\
\le\ &\|f\|_X+\|g\|_X\qquad (k\in\mathbb{N}).
\end{aligned} \tag{A.23}$$

By (A.16), since $|f+g|\le|f|+|g|$ a.e. in $\mathbb{R}^n$, it follows that $\|f+g\|_X\le\||f|+|g|\|_X$. Hence, by (A.22) and (A.23), it follows the triangular inequality

$$\|f+g\|_X\le\|f\|_X+\|g\|_X\qquad (f,g\in \mathrm{L}_0(\mathbb{R}^n)).$$

To finish, let us see the rest of the required properties. Suppose that $\{f_k\}$ is a sequence in $\mathrm{L}_0^+(0,1)$ such that $f_k \uparrow f$ a.e. in $(0,1)$. We have that $\|f_k\|_{\bar{X}(0,1)} \le \|f_{k+1}\|_{\bar{X}(0,1)}$ for $k \in \mathbb{N}$ by the lattice property. Furthermore, if h is any function such that $h \sim f$, then $h = f_\gamma^* \circ H$ for measure-preserving transformations H. Consequently, we have that $f_k = f_{k_\gamma}^* \circ H \nearrow f_\gamma^* \circ H = h \sim f$ for $k \in \mathbb{N}$, so that $\|f_k\|_{\bar{X}(0,1)} \nearrow \|f\|_{\bar{X}(0,1)}$. By the inclusions we obtain that

$$\begin{aligned}\|1\|_{\bar{X}(0,1)} &\simeq \Big\| \int_s^{1/2} \int_t^{1/2} \frac{1}{I_\gamma(r)} dr \frac{dt}{I_\gamma(t)} \Big\|_{\bar{Y}(0,1)} = \Big\| \int_s^{1/2} \int_t^{1/2} \frac{1}{I_\gamma(r)} dr \frac{dt}{I_\gamma(t)} \Big\|_{\bar{Y}(0,1/2)} \\ &\lesssim \Big\| \chi_{(0,1/2)}(s) \Big(\frac{s}{I_\gamma(s)}\Big)^2 \Big\|_{\exp L^4(0,1)} = \Big\| \Big(\frac{s}{I_\gamma(s)}\Big)^2 \Big\|_{\exp L^4(0,1/2)} \lesssim \|1\|_{L^\infty(0,1)} < \infty\end{aligned}$$

and

$$\begin{aligned}\|f\|_{\bar{X}(0,1)} &\ge \Big\| \int_s^{1/2} \int_t^{1/2} \frac{f_\gamma^*(r)}{I_\gamma(r)} dr \frac{dt}{I_\gamma(t)} \Big\|_{\bar{Y}(0,1)} \\ &\ge \Big\| \int_s^{1/2} \int_t^{1/2} \frac{f_\gamma^*(r)}{I_\gamma(r)} dr \frac{dt}{I_\gamma(t)} \Big\|_{L(\mathrm{Log} L)^{1/4}(0,1)} \ge c\|f\|_{L^1(0,1)}. \qquad \square\end{aligned}$$

Let us recall the *Hardy-Littlewood inequality* which states for every $\phi, \psi \in \mathrm{L}_0(\mathbb{R}^n)$ that

$$\int_{\mathbb{R}^n} |\phi(x)\psi(x)| d\gamma(x) \le \int_0^1 \phi_\gamma^*(s)\psi_\gamma^*(s) ds.$$

Define the *Hardy type operator* $T_{\gamma,2}$ on $\mathrm{L}_0(0,1)$ as (cf. [CiL])

$$T_{\gamma,2} f(s) := \begin{cases} \Big(\frac{I_\gamma(s)}{s}\Big)^2 \sup_{s \le r \le 1/2} f_\gamma^*(r) \Big(\frac{r}{I_\gamma(r)}\Big)^2, & s \in (0,1/2] \\ \Big(\frac{I_\gamma(s)}{s}\Big)^2 f_\gamma^*(1/2), & s \in (1/2,1]. \end{cases}$$

Lemma A.6.3. *Let $\bar{Y}(0,1)$ and $\bar{Z}(0,1)$ be r.i. spaces. If $T_{\gamma,2}$ is bounded from $\bar{Y}'(0,1)$ to $\bar{Z}'(0,1)$, then, there exists a constant $C(Y,Z)$ such that*

$$\Big\| \int_s^{1/2} \int_t^{1/2} \frac{h(r)}{I_\gamma(r)} dr \frac{dt}{I_\gamma(t)} \Big\|_{\bar{Y}(0,1)} \le C(Y,Z) \Big\| \int_s^{1/2} \int_t^{1/2} \frac{h(r)_\gamma^*}{I_\gamma(r)} dr \frac{dt}{I_\gamma(t)} \Big\|_{\bar{Z}(0,1)}$$

for every $h \in \mathrm{L}_0^+(0,1)$.

Proof. It follows by Fubini's theorem twice applied, the boundedness of the operator, and the Hardy-Littlewood inequality. $\square$

Theorem A.6.4. *Let Y be an r.i. space such that $\exp L^4(\mathbb{R}^n,\gamma) \hookrightarrow Y$. If $T_{\gamma,2}$ is bounded on $\bar{Y}'(0,1)$ and $\overline{\alpha}_Y < 1$, then, (A.15) holds, and the optimal domain X for Y in the Gaussian second order Sobolev inequality (A.14) fulfils*

$$\|u\|_X \simeq \Big\| \int_s^{1/2} \int_t^{1/2} \frac{u_\gamma^*(r)}{I_\gamma(r)} dr \frac{dt}{I_\gamma(t)} \Big\|_{\bar{Y}(0,1)}$$

for $u \in \mathrm{L}_0(\mathbb{R}^n)$, with absolute equivalence of constants.

Proof. The function $f(s) = 1 \in \bar{Y}'(0,1)$. Therefore, $T_{\gamma,2}f \in \bar{Y}'(0,1)$, that is, $(\frac{I_\gamma(s)}{s})^2 \in \bar{Y}'(0,1)$ and hence $(1+\log(1/s)) \in \bar{Y}'(0,1)$. Therefore, $\exp L^4(0,1) \hookrightarrow \bar{Y}'(0,1)$ and $\bar{Y}(0,1) \hookrightarrow L(\mathrm{Log}L)^{1/4}(0,1)$.

Then, by definition,

$$\Big\| \int_s^{1/2} \int_t^{1/2} f_\gamma^*(r) \frac{dr}{I_\gamma(r)} \frac{dt}{I_\gamma(t)} \Big\|_{\bar{Y}(0,1)} \le \|f\|_X = \|f_\gamma^*\|_{\bar{X}(0,1)}. \qquad (f \in \mathrm{L}_0(\mathbb{R}^n)).$$

Hence, (A.14) holds for Y and X.

Conversely, by Lemmas A.6.2 and A.6.3 applied to the case $\bar{Y}(0,1) = \bar{Z}(0,1)$ we conclude that, for $f \sim h$,

$$\begin{aligned} \|f\|_X &\simeq \Big\| \int_s^{1/2} \int_t^{1/2} \frac{h(r)}{I_\gamma(r)} dr \frac{dt}{I_\gamma(t)} \Big\|_{\bar{Y}(0,1)} \lesssim \Big\| \int_s^{1/2} \int_t^{1/2} \frac{h_\gamma^*(r)}{I_\gamma(r)} dr \frac{dt}{I_\gamma(t)} \Big\|_{\bar{Y}(0,1)} \\ &\le \Big\| \int_s^{1/2} \int_t^{1/2} \frac{f_\gamma^*(r)}{I_\gamma(r)} dr \frac{dt}{I_\gamma(t)} \Big\|_{\bar{Y}(0,1)} \simeq \|f\|_Z. \end{aligned}$$

Then, $Z \hookrightarrow X$ and the proof follows. □

To calculate the optimal domain and range for certain Sobolev spaces, we will make use of the following result from [Muc].

Proposition A.6.5. *Let $1 \le p \le \infty$ and ν, ω in $\mathrm{L}_0^+(0,1)$.*

- *For every $f \in \mathrm{L}_0^+(0,1)$, $\Big\| w(s) \int_0^s f(r)dr \Big\|_{L^p(0,1)} \le C\|\nu f\|_{L^p(0,1)}$ if and only if*
$$\sup_{0<s<1} \Big\| w\chi_{(s,1)} \Big\|_{L^p(0,1)} \Big\| \frac{\chi_{(0,s)}}{\nu} \Big\|_{L^{p'}(0,1)} < \infty.$$
- *For every $f \in \mathrm{L}_0^+(0,1)$, $\Big\| w(s) \int_s^1 f(r)dr \Big\|_{L^p(0,1)} \le C\|\nu f\|_{L^p(0,1)}$ if and only if*
$$\sup_{0<s<1} \Big\| w\chi_{(0,s)} \Big\|_{L^p(0,1)} \Big\| \frac{\chi_{(s,1)}}{\nu} \Big\|_{L^{p'}(0,1)} < \infty.$$

The following proposition is a special case of a more general result in [GoBP].

Proposition A.6.6. *Let $p \in [1,\infty)$ and $\nu, \omega \in \mathrm{L}_0^+(0,1)$.*

- *For every $f \in \mathrm{L}_0(0,1)$*
$$\int_0^1 \Big(\sup_{t \le r \le 1} \Big(\frac{r}{I_\gamma(r)} \Big)^2 f_\gamma^*(r) \Big)^p w(t)dt \lesssim \int_0^1 f_\gamma^*(t)^p \nu(t)dt$$
if and only if
$$\sup_{0<s<1} \frac{\int_0^s w(t)dt}{\Big(I_\gamma(s)/s \Big)^{2p} \int_0^s \nu(t)dt} < \infty. \tag{A.24}$$

- *For every* $f \in \mathrm{L}_0(0,1)$

$$\int_0^1 \Big(\sup_{t \le r \le 1} \Big(\frac{r}{I_\gamma(r)} \Big)^2 f_\gamma^{**}(r) \Big)^p w(t)dt \lesssim \int_0^1 f_\gamma^*(t)^p \nu(t)dt$$

if and only if either $p = 1$ *and*

$$\sup_{0<s<1} \frac{s \int_s^1 \frac{w(t)}{I_\gamma(t)} dt}{\int_0^s \nu(t)dt} < \infty$$

or $1 < p < \infty$, *(A.24) holds and*

$$\sup_{0<s<1} \Big(\int_s^1 \Big(\frac{1}{I_\gamma(t)} \Big)^p w(t)dt \Big)^{1/p} \Big(\int_0^s \Big(\frac{r}{\int_0^r \nu(\rho)d\rho} \Big)^{p'} \nu(r)dr \Big)^{\frac{1}{p'}} < \infty.$$

As an application, it follows that our operator $T_{\gamma,2}$ is bounded on $L^{p,q}(\mathrm{Log}L)^\alpha(0,1)$ for $p = q = 1$ and $\alpha > 0$, and for $1 < p < \infty$, $1 \le q \le \infty$ and $\alpha \in \mathbb{R}$.

Theorem A.6.7. *(i) Let* $p \in [1,\infty)$. *Then for every* $f \in W^{2,L^p}(\gamma)$

$$\|f - \Lambda_f\|_{L^p(\mathrm{Log}L)^p} \lesssim \|d^2 f\|_{L^p}. \tag{A.25}$$

Moreover, $(L^p, L^p(\mathrm{Log}L)^p)$ *is an optimal pair in (A.25).*

(ii) For every $f \in W^{2,L^\infty}(\gamma)$

$$\|f - \Lambda_f\|_{\exp L^4} \lesssim \|d^2 f\|_{L^\infty}. \tag{A.26}$$

Moreover, $(L^\infty, \exp L^4)$ *is an optimal pair in (A.26).*

(iii) Let $\beta \in (0,\infty)$. *Then for every* $f \in W^{2,\exp L^\beta}(\gamma)$

$$\|f - \Lambda_f\|_{\exp L^{\frac{\beta}{1-\beta}}} \lesssim \|d^2 f\|_{\exp L^\beta}. \tag{A.27}$$

Moreover, $(\exp L^\beta, \exp L^{\frac{\beta}{1-\beta}})$ *is an optimal pair in (A.27).*

Proof. (i) Let us take L^p. Then the optimal range fulfils, by Theorem A.6.1 and *Hardy's inequality*

$$\|f\|_{\bar{Y}'(0,1)} = \Big\| P\Big(f_\gamma^{**}(s) \frac{s}{I_\gamma(s)} \frac{u}{I_\gamma(u)} \Big)(u) \Big\|_{L^{p'}(0,1)} \simeq \Big\| f_\gamma^*(u) \frac{1}{1+\log(1/u)} \Big\|_{L^{p'}(0,1)}$$

and, by [EKP, Theorem 2.7], it follows that since $\frac{1}{1+\log(1/s)}$ is increasing

$$\Big\| f_\gamma^*(u) \frac{1}{1+\log(1/u)} \Big\|_{L^{p'}(0,1)} \simeq \|f\|_{L^p(\mathrm{Log}L)^p(0,1)}.$$

Set now $Y = L^p(\mathrm{Log}L)^p(0,1)$. Then, $\bar{Y}'(0,1) = L^{p'}(\mathrm{Log}L)^{p'}(0,1)$ and hence, $T_{\gamma,2}$ is bounded on $\bar{Y}'(0,1)$. Therefore, by Theorem A.6.4 and Proposition A.6.5, it follows that

$$\begin{aligned}\|f\|_{\bar{X}(0,1)} &\simeq \Big\| \int_s^{1/2} \int_t^{1/2} \frac{f_\gamma^*(r)}{I_\gamma(r)} dr \frac{dt}{I_\gamma(t)} \Big\|_{L^p(\log L)^p(0,1)} \\ &\lesssim \|Q_\gamma^2 f_\gamma^*\|_{L^p(0,1/2)} \le \|f\|_{L^p(0,1)}.\end{aligned}$$

Conversely,

$$\begin{aligned}&\Big\| \int_s^{1/2} \int_t^{1/2} \frac{f_\gamma^*(r)}{I_\gamma(r)} dr \frac{dt}{I_\gamma(t)} (1+\log(1/s)) \Big\|_{L^p(0,1)} \\ \ge\ & \Big\| \chi_{(0,1/2)}(s)(1+\log(1/s)) \int_s^{2s} \int_t^{2t} \frac{f_\gamma^*(r)}{I_\gamma(r)} dr \frac{dt}{I_\gamma(t)} \Big\|_{L^p(0,1)} \ge \|f\|_{L^p(0,1)}.\end{aligned}$$

Then, L^p is the optimal domain for $L^p(\mathrm{Log}L)^p$ in inequality (A.25).

(ii) It follows similarly to [GoBP, Proposition 4.4, ii].

(iii) Consider now $\exp L^\beta = X$. By Theorem A.6.4 and Proposition A.6.6 the optimal range for X is given by

$$\begin{aligned}\|f\|_{\bar{Y}'(0,1)} &= \Big\| P\Big(f_\gamma^{**}(s) \frac{s}{I_\gamma(s)} \frac{u}{I_\gamma(u)} \Big)(u) \Big\|_{L(\log L)^{1/\beta}(0,1/2)} \simeq \Big\| f_\gamma^{**}(u) \Big(\frac{u}{I_\gamma(u)} \Big)^2 \Big\|_{L(\mathrm{Log}L)^{1/\beta}(0,1/2)} \\ &\simeq \int_0^{1/2} \Big[\frac{f_\gamma^{**}(\cdot)(\cdot)^2}{I_\gamma(\cdot)^2} \Big]^*(s)(1+\log(1/s))^{1/\beta} ds \le \int_0^{1/2} \sup_{s\le r\le 1/2} \frac{f_\gamma^{**}(r)}{(\frac{I_\gamma(r)}{r})^2} (1+\log(1/s))^{1/\beta} ds \\ &\simeq \int_0^{1/2} f_\gamma^*(s) \Big(1+\log(1/s)\Big)^{1/\beta-1} ds \le \|f\|_{L(\log L)^{1/\beta-1}(0,1)}.\end{aligned}$$

Conversely, by *Hardy-Littlewood's inequality*

$$\begin{aligned}\|f\|_{\bar{Y}'(0,1)} &\simeq \Big\| f_\gamma^{**}(s) \Big(\frac{s}{I_\gamma(s)} \Big)^2 \Big\|_{L(\mathrm{Log}L)^{1/\beta}(0,1/2)} \\ &\ge \int_0^{1/2} f_\gamma^*(s) \Big(1+\log(1/s)\Big)^{1/\beta-1} ds \simeq \|f\|_{L(\mathrm{Log}L)^{1/\beta-1}(0,1)}.\end{aligned}$$

Therefore, $\bar{Y}'(0,1) = L(\mathrm{Log}L)^{1/\beta-1}(0,1)$, so that $Y = \exp L^{\frac{\beta}{1-\beta}}$. To see that $\exp L^\beta$ is the optimal domain for $\exp L^{\frac{\beta}{1-\beta}}$ in inequality (A.27) we proceed as in [GoBP, Proposition 4.4, iii]. □

Bibliography

[A] D. R. Adams. *A note on Choquet integrals with respect to Hausdorff capacity*, Function spaces and applications (Lund, 1986), 115-124, Lecture Notes in Math. **1302**, Springer, Berlin, 1988, 26-39

[A2] D. R. Adams. *Choquet integrals in potential theory*, Publ. Mat. **42** (1998), 3-66.

[AH] D. R. Adams and L. I. Hedberg. *Function Spaces and Potential Theory*, Springer-Verlag, 1996.

[AP] D. R. Adams and M. Pierre. *Capacitary strong type estimates in semilinear problems*, Ann. Inst. Fourier (Grenoble) **41** (1991), 117-135.

[A1] R. A. Adams. *General logarithmic Sobolev inequalities and Orlicz imbeddings*, J. Funct. Anal. **34** (1979), 292-303.

[Ahl] L. Ahlfors. *Bounded analytic funcions*, Duke Math. J. **14** (1974), 1-11.

[AK] I. Asekritova and N. Krugljak. *On equivalence of K and J-methods for* $(n+1)$ *tuples of Banach spaces*, Studia Math. **122** (1997), no. 2, 99-116.

[AKMNP] I. Asekritova, N. Krugljak, L. Maligranda, L. Nikolova and L. E. Persson. *Lions-Peetre reiteration formulas for triples and their applications*, Studia Math. **145** (2001), no. 3, 219-254.

[Au] T. Aubin. *Problèmes isoperimétriques et espaces de Sobolev*, C. R. Acad. Sci. Paris. **280: 5** (1975), A279-281.

[Bar] F. Barthe, *Log-concave and spherical models in isoperimetry*, Geom. Funct. Anal. **12** (2002), no. 1, 32-55.

[BCR] F. Barthe, P. Cattiaux and C. Roberto. *Interpolated inequalities between exponential and Gaussian, Orlicz hipercontractivity and isoperimetry*, Rev. Mat. Iberoam. **22** (2006), no. 3, 993-1067.

[BKPSo] S. Barza, A. Kamínska, L. E. Persson and J. Soria. *Mixed norm and multidimensional Lorentz spaces*, Positivity **10** (2006), no. 3, 539-554.

[BPSo] S. Barza, L. E. Persson and J. Soria. *Multidimensional rearrangement and Lorentz spaces*, Acta Math. Hungar. **104** (2004), no. 3, 203-224.

[Be1] W. Beckner. *A generalized Poincairé inequality for Gaussian measures*, Proc. Amer. Math. Soc. **105** (1989), no. 2, 397-400.

[Be2] W. Beckner. *Logarithmic Sobolev inequalities and the existence of singular integrals*, Forum Math. **9** (1997), no. 3, 303-323.

[BP] A. Benedek and R. Panzone. *The spaces L^P, with mixed norm*, Duke Math. J. **28** (1961), 301-324.

[BeSh] C. Bennett and B. Sharpley. *Interpolation of Operators*, Pure and Applied Mathematics, 129. Academic Press, Inc., Boston, MA, 1988.

[BeR] C. Bennett and K. Rudnick. *On Lorentz-Zygmund spaces*, Dissertationes Math. **175** (1980), 67 pp.

[BeLo] J. Bergh and J. Löfström. *Interpolation Spaces.* An introduction. Grundlehren der Mathematischen Wissenschaften, no. 223. Springer-Verlag, Berlin-New York, 1976.

[B08] M. F. Betta, F. Brock, A. Mercaldo, and M. Posteraro. *Weighted isoperimetric inequalities on Rn and applications to rearrangements*, Math. Nachr. **281** (2008), no. 4, 466- 498.

[Bob] S. G. Bobkov. *Extremal properties of half-spaces for log-concave distributions*, Ann. Probab. **24** (1996), no. 1, 35-48.

[BZ] S. G. Bobkov and B. Zegarlinski. *Entropy bounds and isoperimetry*, Mem. Amer. Math. Soc. **176** (2005), no. 829.

[Bor] C. Borell. *The Brunn-Minkowski inequality in Gauss space*, Invent. Math. **30** (1975), 207-216.

[Bor85] C. Borell. *Intrinsic bounds on some real-valued stationary random functions*, Lectures Notes in Math. **1153** (1985), 72-95.

[BSo] S. Boza and J. Soria. *Rearrangement transformations on general measure spaces*, Indag. Math. (N.S.) **19** (2008), no. 1, 33-51.

[BK] Y. Brudnyi and N. Krugljak. *Interpolation Functors and Interpolation Spaces, V. 1*, North-Holland Co., Amsterdam, 1991.

[Ca] A. P. Calderón. *Spaces between L^1 and L^∞ and the Theorem of Marcinkiewicz*, Studia Math. **26** (1966), 273-299.

[Car] L. Carleson. *Selected Problems on Exceptional Sets*, D. Van Nostrand Company, Inc. 1967, No. 13, Inc., Princeton.

[CSo] M. J. Carro and J. Soria. *Weighted Lorentz Spaces and the Hardy Operator*, J. Funct. Anal. **112** (1993), no. 2, 480-494.

[CRSo] M. J. Carro, J. A. Raposo and J. Soria. *Recent Developments in the Theory of Lorentz Spaces and Weighted Inequalities*, Memoirs of the American Mathematical Society **187** (2007), no. 877.

[Ce] J. Cerdà. *Lorentz capacity spaces*, Contemp. Math. **445** (2007), Amer. Math. Soc., 49-55.

[CeClM] J. Cerdà, H. Coll and J. Martín. *Entropy function spaces and interpolation*, J. Math. Anal. Appl. **304** (2005), no. 1, 269-295.

[CeMS] J. Cerdà, J. Martín and P. Silvestre. *Capacitary function spaces*, Collectanea Mathematica **62** (2011), no. 1, 95-118.

[CeMS1] J. Cerdà, J. Martín and P. Silvestre. *Interpolation of quasicontinuous functions*, to appear in Józef Marcinkiewicz Centenary Volume, Banach Center Publications.

[CeMS2] J. Cerdà, J. Martín and P. Silvestre. *Conductor Sobolev type estimates and isocapacitary inequalities*, to appear in Indiana Univ. Math. J.

[Ch] G. Choquet. *Theory of capacities*, Ann. Inst. Fourier (Grenoble) **5** (1953-54), 131-295.

[Ci1] A. Cianchi. *Symmetrization and second-order Sobolev inequalities*, Ann. Mat. Pura Appl. (4) **183** (2004), no. 1, 45-77.

[Ci] A. Cianchi. *Moser-Trudinger inequalities without boundary conditions and isoperimetric problems*, Indiana Univ. Math. J **54** (2005), no. 3, 669-705.

[CiL] A. Cianchi and L. Pick. *Optimal Gaussian Sobolev embeddings*, J. Funct. Anal. **256** (2009), no. 11, 3588-3642.

[CFS] F. Cobos, L. M. Fernández-Cabrera, P. Silvestre. *New limiting real interpolation methods and their connection with the methods associated to the unit square*, to appear in Math. Nachr.

[Cos] S. Costea. *Scaling invariant Sobolev-Lorentz capacity on* $\mathbb{R}^n$, Indiana Univ. Math. J. **56** (2007), no. 6, 2641-2669.

[CosMa] S. Costea and V. Maz'ya. *Conductor inequalities and criteria for Sobolev-Lorentz two-weight inequalities*, Sobolev spaces in mathematics. II, 103-121, Int. Math. Ser. (N. Y.), **9**, Springer, New York, 2009.

[Cre] J. Creekmore. *Type and cotype in Lorentz* $L^{p,q}$ *spaces*, Nederl. Akad. Wetensch. Indag. Math.**43** (1981), no. 2, 145-152.

[CwNS] M. Cwikel, P. G. Nilsson and G. Schechtman. *Interpolation of Weighted Banach Lattices. A Characterization of Relatively Descomposable Banach Lattices*, Mem. Amer. Math. Soc. **165** (2003), no. 787.

[DKX] G. Dafni, G. Karadzhov and J. Xiao. *Classes of Carleson-type measures generated by capacities*, Math. Z. **258** (2008), no.4, 827-844.

[Da] B. Dahlberg. *Regularity properties of Riesz potentials*, Indiana Univ. Math. J **28** (1979), no.2, 257-268.

[EP] D. E. Edmunds and B. Opic. *Alternative characterisations of Lorentz-Karamata spaces*, Czechoslovak Math. J. **58** (133) (2008), no. 2, 517-540.

[EKP] D. E. Edmunds, R. Kerman and L. Pick. *Optimal imbeddings involving rearrangement-invariant quasi-norms*, J. Funct. Anal. **170** (2000), 307-355.

[EvGa] L. C. Evans and R. F. Gariepy. *Measure Theory and Fine Properties of functions*, Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992, 268 pp.

[Fed] H. Federer. *Geometric Measure Theory*, Die Grundlehren der mathematischen Wissenschaften, Band 153 Springer-Verlag New York Inc., New York 1969, 676 pp.

[FF] H. Federer, W. H. Fleming. *Normal and integral currents*, Ann. Math. (2) **72** (1960), 458-520.

[Fe] R. A. Fefferman. *A theory of entropy in Fourier analysis*, Adv. in Math. **30** (1978), no. 3, 171-201.

[Fes] G. F. Feissner. *Hypercontractive semigroups and Sobolev's inequality*, Trans. Amer. Math. Soc.**210** (1975) 51-62.

[GDS] J. L. Garcia-Domingo and J. Soria. *A decreasing rearrangement for functions on homogeneous trees*, European J. Combin. **26** (2005), no. 2, 201-225.

[GoBP] A. Gogatishvili, B. Opic and L. Pick. *Weighted inequalities for Hardy-type operators involving suprema*, Collect. Math. **57** (2006), no. 3, 227-255.

[G] L. Gross. *Logarithmic Sobolev inequalities*, Amer. J. Math. **97** (1975), no. 4, 1061-1083.

[GuP] J. Gustavsson and J. Peetre. *Interpolation of Orlicz Spaces*, Studia Math. **60** (1977), no. 1, 33-59.

[Han] K. Hansson. *Embedding theorems for Sobolev type in potential theory*, Math. Scand. **45** (1979), 77-102.

[HKT] C. Hao, A. Kamińska and N. Tomczak-Jaegermann. *Orlicz spaces with convexity or concavity constant one*, J. Math. Anal. Appl. **320** (2006), 303-321.

[KMo] N. J. Kalton and S. J. Montgomery-Smith. *Set-functions and factorization*, Arch. Math. (Basel) **61** (1993), no. 2, 183-200.

[Ka1] A. Kamińska. *Type and cotype in Musielak-Orlicz spaces*, Geometry of Banach spaces (Strobl, 1989), 165-180, London Math. Soc. Lecture Note Ser., **158**, Cambridge Univ. Press, Cambridge, 1990.

[Ka2] A. Kamińska. *Indices, convexity and concavity in Musielak-Orlicz spaces*, Funct. Approx. Comment. Math. **26** (1998), 67-84.

[KaMa] A. Kamińska and L. Maligranda. *Order convexity and concavity of Lorentz spaces* $\Lambda_{p,w}$, $0 < p < \infty$, Stud. Math. **160** (2004), no. 3, 267-286.

[KaMa1] A. Kamińska and L. Maligranda. *On Lorentz spaces* $\Gamma_{p,w}$, Israel J. Math. **140** (2004), 285-318.

[KaMaP] A. Kamińska, L. Maligranda L. E. Persson. *Indices, convexity and concavity of Calderón-Lozanovskii spaces*, Math. Scand. **92** (2003), no. 1, 141-160.

[KaP] A. Kamińska and A. Parrish. *Convexity and concavity constants in Lorentz and Marcinkiewicz spaces*, J. Math. Anal. Appl. **343** (2008), no. 1, 337-351.

[Ko84] T. Kolsrud. *Capacitary integrals in Dirichlet spaces*, Math. Scand. **55** (1984), no. 1, 95-120.

[K] V. Kolyada. *Rearrangements of functions, and embedding theorems*, Russian Math. Surveys **44** (1989), 73-117.

[KO] P. Koskela and J. Onninen. *Sharp inequalities via truncation*, J. Math. Anal. Appl. **278** (2003), no. 2, 324-334.

[Kr] S. G. Krein, *On the concept of a normal scale of spaces*, Dokl. Akad. Nauk SSSR **132** 510-513 (Russian); translated as Soviet Math. Dokl. **1** 1960, 586-589.

[KrPS] S. G. Krein, Ju. I. Petunin and E. M. Semenov. *Interpolation of linear operators*, Translations of mathematical monographs, **54** (1982), American Mathematical Society, Providence, R.I., 375 pp.

[L] M. Ledoux. *Isoperimetry and Gaussian analysis*, Lectures on probability theory and statistics (Saint-Flour, 1994), 165-294, Lecture Notes in Math. **1648**, Springer, Berlin, 1996.

[LiZa] J. Lindestrauss and L. Tzafriri. *Classical Banach Spaces II, Function Spaces*, Springer-Verlag, A Series of Modern Surveys in Mathematics.

[Li3] J. L. Lions. *The construction d'espaces d'interpolation*, C. R. Acad. Sci. Paris **251** (1960), 1853-1856.

[LiP] J. L. Lions and J. Peetre. *Propriétés d'espaces d'interpolation*, C. R. Acad. Sci. Paris **253** (1961), 1747-1749.

[LiP1] J. L. Lions and J. Peetre. *Sur une classe d'espaces d'interpolation*, Inst. Hautes Ètudes Sci. Publ. Math. **19** (1964), 5-68.

[Lo] G. G. Lorentz. *On the theory of spaces* Λ, Pacific J. Math. **1** (1951), 411-429.

[MMi4] J. Martín and M. Milman. *Higher order symmetrization inequalities and applications*, J. Math. Anal. Appl. **330** (2007), 91-113.

[MMi1] J. Martín and M. Milman. *Isoperimetry and symmetrization for Sobolev spaces on metric spaces*, C. R. Math. Acad. Sci. Paris **347** (2009), 627-630.

[MMi5] J. Martín and M. Milman. *Isoperimetry and Symmetrization for Logarithmic Sobolev inequalities*, J. Funct. Anal. **256** (2009), 149-178.

[MMi3] J. Martín and M. Milman. *Pointwise symmetrization inequalities for Sobolev functions and applications*, Adv. Math. **225** (2010), 121-199.

[MMi] J. Martín and M. Milman. *Isoperimetric Hardy type and Poincaré inequalities on metric spaces*, Around the research of Vladimir Maz'ya. I, 285-298, Int. Math. Ser. (N. Y.), **11**, Springer, New York, 2010.

[MMi2] J. Martín and M. Milman. *Sobolev inequalities, rearrangements, isoperimetry and interpolation spaces*, Contemporary Mathematics of the AMS **545** (2011), 167-193.

[MMiP] J. Martín, M. Milman and E. Pustylnik. *Sobolev inequalities: Symmetrization and self-improvement via truncation*, J. Funct. Anal. **252** (2007) 677-695.

[Ma85] V. G. Maz'ya. *Sobolev Spaces* (translated from the Russian by T. O. Shaposhnikova), Springer-Verlag, 1985.

[Ma05] V. G. Maz'ya. *Conductor and capacitary inequalities for functions on topological spaces and their applications to Sobolev type imbeddings*, J. Funct. Anal. **224** (2005), no. 2, 408-430.

[Ma06] V. G. Maz'ya. *Conductor inequalities and criteria for Sobolev type two-weight embeddings*, J. Comput. Appl. Math. **194** (2006), no. 11, 94-114.

[Ma11] V. G. Maz'ya. *Sobolev Spaces, with applications to elliptic partial differential equations*, Second, revised and augmented edition. Grundlehren der Mathematischen Wissenschaften, 342. Springer, Heidelberg, 2011, 866 pp.

[MaN] V. G. Maz'ya and Yu. Netrusov. *Some counterexamples for the theory of Sobolev spaces on bad domains*, Potential Anal. **4** (1995), no. 1, 47-65.

[MaP] V. G. Maz'ya and S. Poborchi. *Differentiable functions on Bad Domains*, World Scientific Publishing Co., Inc., River Edge, NJ, 1997, 481 pp.

[Muc] B. Muckienhoupt. *Hardy's inequality with weights*, Collection of articles honoring the completion by Antoni Zygmund of 50 years of scientific activity, I. Studia Math. **44** (1972), 31-38.

[Mu] J. Musielak and W. Orlicz. *On generalized variations*, Studia Math. **18** (1959), 11-41.

[MuO] J. Musielak and W. Orlicz. *On modular spaces*, Studia Math., **18** (1959), 49-65.

[Nak] H. Nakano. *Modulared semi-ordered spaces*, Tokyo 1950.

[Nev] J. S. Neves. *Spaces of Bessel-potential type and embeddings: the super-limiting case*, Math. Nachr. **265** (2004), 68-86.

[No] S. Ya. Novikov. *Cotype and type of Lorentz Function Spaces*, Mat. Zamet. **32**, 2 (1982), 213-221 (Russian) (English translation: Math. Notes 586-590).

[OV] J. Orobitg and J. Verdera. *Choquet integrals, Hausdorff content and the Hardy-Littlewood maximal operator*, Bull. London Math. Soc. **30** (1998), no. 2, 145-150.

[Pa] H. Pajot. *Analytic Capacity, Rectifiability, Menger Curvature and the Cauchy integral*, Lecture Notes in Math. **1799** (2002), Springer-Verlag, Berlin, 2002, 119 pp.

[P] J. Peetre. *A theory of interpolation of normed spaces*, Notas de Matemática, No. **39**, Instituto de Matemática Pura e Aplicada, Conselho Nacional de Pesquisas, Rio de Janeiro 1968, 86 pp.

[P1] J. Peetre. *Sur l'utilisation des suites inconditionellement sommables dans la théorie des espaces d'interpolation*, Rend. Sem. Mat. Padova **46** (1971), 173-190.

[Ra] M. Rao. *Capacitary inequalties for energy*, Israel. J. of Math. **61** (1988), no. 1, 179-191.

[Ray] Y. Raynaud. *On Lorentz-Sharpley spaces*, Israel Mathematical Conference Proceedings **5** (1992), 207-228.

[Rei] S. Reisner. *On the duals of Lorentz Function and Sequence Spaces*, Ind. Univ. Math. J. **31** (1982), no. 1, 65-72.

[Sch] C. Schütt. *Lorentz Spaces that are isomorphic to subspaces of* L^1, Trans. Amer. Math. Soc. **314** (1989), no. 2, 583-595.

[Se] V. A. Sestakov, *Complex interpolation in Banach spaces of measurable functions*, Vestnik Leningrad. Univ. 1974, no. 19 (Ser. Mat. Meh. Astr. vyp. 4) 64-68; English transl. in Vestnik Leningrad Univ. Math. **7** (1979).

[Sp1] G. Sparr. *Interpolation of several Banach spaces*, Ann. Mat. Pura Appl. (4) **99** (1974), 247-316.

[Sp] G. Sparr. *Interpolation of Weighted* L^p*- Spaces*, Studia Math. **62** (1978), no. 3, 229-271.

[ST] V. N. Sudakov and B. S. Tsirelson. *Extremal properties of half-spaces for spherically invariant measures*, J. Soviet. Math. **9** (1978), 918; translated from Zap. Nauch. Sem. L.O.M.I. **41** (1974), 1424.

[Ta1] G. Talenti. *Best constant in Sobolev inequality*, Ann. Mat. Pura. Appl. (4) **110** (1976), 353-372.

[To] X. Tolsa. *Painleve's problem and the semiadditivity of analytic capacity*, Acta Math. **190:1** (2003), 105-149.

[T1] H. Triebel. *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland, Amsterdam, 1978.

[T2] H. Triebel. *Theory of Function Spaces II*, Birkhäuser, Basel, 1992.

[T3] H. Triebel. *Theory of Function Spaces III*, Birkhäuser, Basel, 2006.

[V99] I. E. Verbitksky. *Nonlinear potentials and trace inequalities*, Operator Theory, Advances and Applications, Vol. 110, The Maz'ya Annniversary Collection, vol. 2, Birkhäuser (1999), 323-343.