Some Developments on Mechanism Design and Auctions

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Chapter 1

On efficient trading mechanisms between one seller and n buyers

1.1 Introduction

Myerson and Satterthwaite (1983) (MS henceforth) analyze an environment in which an agent (agent 0, a potential seller) owns an indivisible object and faces another agent (agent 1, a potential buyer) who is interested in the object. Each agent has a valuation of the object which is private information and views the other agent's valuation as a random variable. The buyer regards the valuation of the seller, v_0, as distributed over $[a_0, b_0]$ according to the cumulative function F_0; the seller views the valuation of the buyer, v_1, as distributed over $[a, b]$ according to the cumulative function F. The two probability distributions are independent and admit strictly positive and continuous densities. The agents are risk-neutral and have quasilinear utilities: the expected payoff of agent i ($i = 0, 1$) is $v_i y_i + t_i$, where y_i is his probability to obtain the good and t_i is the expected payment he receives. This implies that the final allocation is ex post Pareto-efficient if and only if the good is allocated to the agent with the highest valuation and no money is wasted.

MS assume that gains from trade are possible (that is, $a_0 < b$) but are not common knowledge (that is, $a < b_0$) and consider the problem of designing a mechanism having a Bayes-Nash equilibrium (BNE henceforth) such that (i) each type of the two agents obtains nonnegative net expected gains from participation given his private information (interim individual rationality constraints) and (ii) the outcome of this BNE is the ex post efficient allocation. Corollary 1 in MS is a non-existence result: it proves that, *whatever* F and F_0 are, if $(a_0, b_0) \cap (a, b) \neq \emptyset$ (which is true if and only if $a < b_0$ and $a_0 < b$) then no mechanism has an individually rational BNE resulting in the ex post efficient allocation.

In several more recent papers some of the above assumptions are modified and some existence results arise; among others, see Cramton *et al.* (1987), Matsuo (1989), Gresik (1991), McAfee (1991) and McAfee and Reny (1992). In Makowski and Mezzetti (1993) (MM henceforth) the seller faces $n \geq 2$ potential buyers; each of them has a valuation for the good which is distributed over $[a, b]$ according to the cumulative function F and is independent of the other random variables. MM detect the necessary and sufficient condition for the existence of an individually rational, incentive compatible and ex post efficient mechanism - from now on such a kind of mechanism will be referred to as "efficient mechanism". When such a condition is satisfied they prove that a modified second price auction is an efficient mechanism.

Under the assumption that the seller faces $n \geq 2$ buyers, section 1.2 proves that if b_0 is not much larger than the expected revenue from a first price auction with no reserve price then there exists an efficient mechanism which is independent of F_0. More precisely, the mechanism designer does not need to know F_0 in order to define such a mechanism - unlike the ones proposed by MM. This is important because, as Cramton *et al.* (1987) emphasize, the planner's information about the environment may not be very precise; yet, it is still possible to implement the ex post efficient allocation if only F is known.

Generally in Bayesian mechanisms the information structure has to be common knowledge among all agents and the planner. Mookherjee and Reichelstein (1992) argue against Bayesian mechanisms because of these informational assumptions which they consider very demanding. Different beliefs between agents and the mechanism designer may generate sharp differences between the expected and the actual outcome; this would occur, for example, in the modified second price auction suggested by MM to implement the efficient allocation. The mechanism we examine in section 1.2, on the other hand, requires common knowledge of the function F but not of F_0 (when b_0 is not too large). In other words, in order to compute their equilibrium strategies and to be willing to play the mechanism, the agents are not required to have common beliefs about v_0 nor the planner needs to share their beliefs. This substantially weakens the informational assumptions, therefore making the implementability of the ex post efficient allocation less demanding.

An efficient mechanism which does not require the knowledge of F_0 is clearly not as satisfactory as an efficient mechanism which is independent of both F and F_0. In a more general context d'Aspremont and Gérard-Varet (1979) attack this kind of problem by assuming that each agent privately observes the parameters characterizing his own preferences while the other agents only know the possible values of these parameters (the supports). They consider direct mechanisms and prove that requiring that truthful revelation is an equilibrium for any profile of parameters (which means, for any game that may arise) is equivalent to require that truthful revelation is a weakly dominant strategy for every profile of parameters. Furthermore, their theorem 2(b) [which follows from theorem 3 in Green and Laffont (1977)] can be invoked to establish that, in our setting, any mechanism allocating the good to the highest valuation agent and such that truthful revelation is a weakly dominant strategy in any game which may arise is a Groves mechanism.[1] However, no Groves mechanism satisfies the ex post budget balance condition in our environment, while the mechanism which is described in section 1.2 does so and also satisfies the interim individual rationality constraints [d'Aspremont

[1]See section 23.C in Mas-Colell *et al.* (1995) for an introduction to Groves mechanisms.

and Gérard-Varet (1979) do not impose such constraints].[2]

In section 1.3 we show that a simple multi-unit extension displays similar properties to the one-unit model. In section 1.4, instead, we inquire whether efficient mechanisms exist in a partially public good environment. For a public good an inefficiency result similar to the Myerson-Satterthwaite theorem holds: if gains from trade are uncertain then no efficient mechanism exists.[3] We show that in our partially public good setting such a result is isolated: if the good is not perfectly public then the existence of efficient mechanisms cannot be ruled out just because gains from trade are not common knowledge; rather, it depends on the environment parameters: probability distributions, number of agents and the degree of "privateness" of the good.

1.2 An efficient mechanism which is independent of F_0

1.2.1 Preferences, information and existence of efficient mechanisms

A seller owns an indivisible good to which he attaches valuation v_0. He faces $n \geq 2$ potential buyers and each buyer i, $i = 1, \ldots, n$, has a valuation v_i for the good. Each agent's valuation is private information and it is regarded by the other agents as an independent random variable. The seller's valuation v_0 is drawn from the probability distribution F_0 with support $[a_0, b_0]$; each buyer i's valuation v_i is drawn from the distribution F with support $[a, b]$. Both F_0 and F have continuous and strictly positive (over the supports) densities f_0 and f, respectively. To avoid trivial cases we assume that

[2] Kosmopoulou (1999) proves that if an efficient mechanism exists then there also exists an ex post individually rational Groves mechanism which implements the efficient allocation of the good in weakly dominant strategies but it is only ex ante, and not ex post, budget balanced. Moreover, the planner needs to know both F and F_0 in order to design such a mechanism.

[3] See Mailath and Postlewaite (1990).

gains from trade are uncertain: $(a_0, b_0) \cap (a, b) \neq \emptyset$ (i.e., $0 \leq a_0 < b$ and $0 \leq a < b_0$). To simplify the exposition we also assume $a_0 \leq a$; however, all the following results can easily be extended to the case in which $a_0 > a$.

Theorem 1 in MM establishes that an efficient mechanism exists if and only if the following inequality is satisfied (which requires $b_0 < b$):

$$\int_{b_0}^{b} [1 - F^n(z)]dz - n \int_{a}^{b_0} F_0(z)F^{n-1}(z)[1 - F(z)]dz - n \int_{b_0}^{b} F^{n-1}(z)[1 - F(z)]dz \geq 0 \quad (1.1)$$

MM prove that if (1.1) holds then a second price auction (in which also the seller bids) augmented by suitably defined transfer functions is an efficient mechanism; the definition of such transfer functions requires knowledge of both F and F_0. The main purpose of this section is to prove that if b_0 is smaller than a given $\bar{b}$, determined by F, then it is possible to define an efficient mechanism which does not depend on F_0. In such a case, therefore, the regulator does not need to know F_0 in designing the efficient mechanism and the buyers do not need to have the same common prior about v_0 as the planner.

1.2.2 The sequential mechanism

Before describing our proposed mechanism we need to introduce some notation. Suppose the seller auctions off the good through a first price sealed bid auction with a reserve price r. Maskin and Riley (1996a,b) prove that this game has a unique BNE; such equilibrium is symmetric among buyers and is described as follows. If $r \geq a$ then for any v_i in $[a, b]$ each buyer i with valuation v_i bids $\beta(v_i; r) = v_i - \frac{\int_r^{v_i} F^{n-1}(z)dz}{F^{n-1}(v_i)}$ if $v_i \geq r$ and $\beta(v_i; r) = 0$ if $v_i < r$. On the other hand, if $r < a$ then for any v_i in $[a, b]$ each buyer i with valuation v_i bids $\beta(v_i; r) = v_i - \frac{\int_a^{v_i} F^{n-1}(z)dz}{F^{n-1}(v_i)}$: any reserve price smaller than a does not affect the equilibrium bids. Let $R(r)$ denote the seller's expected revenue as a function of r: $R(r) \equiv \int_r^b \beta(z; r)dF^n(z)$.

Now assume that if the owner of the object wants to sell it then he is restricted to use a first price auction with a reserve price r that he can choose. As we mentioned above, in the unique BNE of this bidding game each buyer bids according to the function β; hence the object is not sold if each buyer has a valuation below r and the expected payoff for a seller with valuation v_0 from setting a reserve price r is $R(r)+v_0F^n(r)$. Since the bidding function β is strictly increasing in the valuation, if the object is sold then it is the highest valuation buyer who obtains it; hence ex post efficiency is guaranteed if each seller with valuation v_0 sets $r = v_0$. It is not difficult to verify that if $v_0 \in [a, b)$ then the reserve price which maximizes the seller's payoff is strictly larger than v_0 [see proposition 3 in Riley and Samuelson (1981)]. Thus, leaving to the seller complete freedom in choosing r does not lead to the efficient allocation; this is a typical (and well known) allocative distortion in unregulated monopolistic markets. To induce the seller to set the reserve price equal to his valuation, assume that upon choosing r he has to pay $\frac{T(r)}{n}$ to each potential buyer (i.e., including the ones who do not actually bid because each of them has a valuation below r); T is the function defined on $[a, b]$ as follows,[4] for some constant A to be determined:

$$T(r) = R(r) - A + rF^n(r) - \int_a^r F^n(z)dz \text{ for any } r \in [a, b] \tag{1.2}$$

Definition 1.1 *In any FPAWF (first price auction with fine) mechanism the seller decides whether to auction off the good or not. If he decides to auction it off, then he has to use a first price auction in which his strategic variable is $r \in [a, b]$; upon choosing r he pays $\frac{T(r)}{n}$ to each buyer.*

A remarkable feature of these mechanisms is that they do not depend on F_0; hence the mechanism designer does not need to know the probability distribution for v_0 in order to define the mechanism. For what concerns the buyers, we are going to see that only for some values of A they need to know F_0 (in order to verify the validity

[4]There is no loss of generality in restricting r to lie in $[a, b]$, since any $r < a$ is equivalent to $r = a$ ($r < a$ has the same effect on the bidding function as $r = a$) and, similarly, any $r > b$ is equivalent to $r = b$.

of their participation constraints). Indeed, there exist infinitely many FPAWF games differing because of the value of A. Let FPAWF$_1$ denote the FPAWF mechanism in which $A = R(a)$ in (1.2) and T_1 is the resulting fine function; then we can state and prove the following[5]

Proposition 1.1 *Assume* $R(a) + \int_a^{b_0} F^n(z)dz \geq b_0$. *Then FPAWF$_1$, a mechanism which is independent of F_0, implements the ex post efficient allocation in unique Perfect Bayesian Equilibrium (PBE).*

Proof. We first prove that in any PBE of FPAWF$_1$ the agents play according to the strategies (S^e, B^e) which we define below. Notice that the outcome of these strategies is the ex post efficient allocation.

S^e: Any type of seller auctions off the good; any seller with valuation $v_0 \geq a$ chooses $r = v_0$ and any seller of type $v_0 < a$ sets $r = a$.

B^e: If the seller auctions off the good with a reserve price r then any buyer i with valuation v_i bids $\beta(v_i; r)$.

Recall that for any r chosen by the seller the resulting first price auction has a unique BNE and in such equilibrium each buyer i with valuation v_i bids $\beta(v_i; r)$ (strategy B^e); this is true independently of the buyers' beliefs about v_0, since neither the rules of the first price auction nor the buyers' preferences depend on v_0. Therefore each type v_0 of seller knows that - in any PBE - choosing the reserve price r yields him an expected payoff of $V(r; v_0) \equiv R(r) + v_0 F^n(r) - T_1(r) = R(a) + (v_0 - r)F^n(r) + \int_a^r F^n(z)dz$. Integrating $\int_a^r F^n(z)dz$ by parts and rearranging yields $V(r; v_0) = R(a) + \int_a^r (v_0 - z)dF^n(z)$. Since $\int_a^r (v_0 - z)dF^n(z) = \int_a^b (v_0 - z)dF^n(z) - \int_r^b (v_0 - z)dF^n(z)$, we find that $V(r; v_0)$ is equal to the constant term $R(a) + \int_a^b (v_0 - z)dF^n(z)$ plus $\int_r^b (z - v_0)dF^n(z)$. The latter quantity

[5]In proposition 1.1 we refer to definition 8.2 in Fudenberg and Tirole (1991) of Perfect Bayesian Equilibrium in multi-stage games with observed actions and incomplete information. Any FPAWF game is a multi-stage game with observed actions and incomplete information according to the definition given in subsection 8.2.3 in Fudenberg and Tirole (1991), but for the fact that we have a continuum of types instead of finitely many. This however does not invalidate the applicability of the above mentioned definition [see example 8.3 in Fudenberg and Tirole (1991)].

is the expected social surplus generated by a first price auction given r and v_0; therefore in this modified first price auction the seller's interest coincides with society's welfare. Since maximizing $V(r; v_0)$ with respect to r is equivalent to maximizing $\int_r^b (z - v_0) dF^n(z)$ we find

$$\frac{\partial V(r; v_0)}{\partial r} = (v_0 - r)[F^n(r)]'$$

Thus, each seller of type $v_0 \geq a$ chooses $r = v_0$; if instead $v_0 < a$ then $\frac{\partial V(r;v_0)}{\partial r} < 0$ for any $r > a$ and r is set equal to a. This is strategy S^e but for the participation choice, to which we turn in few lines.

Strategy S^e determines the buyers' beliefs about v_0 after any reserve price which is an equilibrium move, that is after any $r \in [a, b_0]$. About off-equilibrium reserve prices in $(b_0, b]$,[6] we let the buyers view v_0 as uniformly distributed over $[a_0, b_0]$ if the seller chooses $r > b_0$.[7] If the seller does not auction off the good then the game ends and no beliefs need to be specified. As we remarked at the beginning of this proof, we do not need to check any (continuation) equilibrium condition for B^e given the above beliefs since, given r, any first price auction is independent of v_0.

The participation constraints are the only additional conditions FPAWF_1 needs to satisfy in order to implement the efficient allocation in unique PBE. If a seller of type v_0 did not auction off the good then his payoff would be v_0; hence, for any $v_0 \in [a_0, b_0]$ the following inequality needs to be satisfied: $V[c(v_0); v_0] = R(a) + \int_a^{c(v_0)} F^n(z)dz \geq v_0$, where $c(v_0) \equiv \max\{v_0, a\}$. Since this condition is tighter the larger v_0 is, it is sufficient to check the inequality

$$R(a) + \int_a^{b_0} F^n(z)dz \geq b_0 \tag{1.3}$$

which is true by assumption. Moving to the buyers' participation constraints, the expected equilibrium payoff of any buyer i of type v_i is equal to the expected transfer

[6] Observe that if $R(a) + \int_a^{b_0} F^n(z)dz \geq b_0$ then $b_0 < b$, hence $(b_0, b]$ is not empty.

[7] Therefore, strictly speaking, FPAWF_1 has infinitely many PBE differing only because of buyers' beliefs following out-of-equilibrium reserve prices. However, we proved that the equilibrium strategies are given by (S^e, B^e) in any PBE; hence the equilibrium outcome is the ex post efficient allocation in any PBE.

$E_{v_0}\frac{T_1[c(v_0)]}{n}$ she receives from the seller plus the expected payoff from playing the first price auction. Any type of buyer obtains a non-negative payoff from the auction, but type a earns exactly 0 in the auction because with probability 1 some other buyer has a higher valuation. Thus each type of buyer's participation constraint holds if and only if the individual rationality constraint of any buyer with valuation a is satisfied, which occurs if and only if the expected transfer she obtains from the seller is non-negative. From $T_1(a) = 0$ and $T_1'(r) \geq 0$ in $[a, b]$ follows $T_1(r) \geq 0$ for any $r \in [a, b]$, thus the expected value of T_1 is non-negative. ■

Proposition 1.1 proves that if (1.3) holds then it is possible to implement (in unique PBE) the ex post efficient allocation through a mechanism which is independent of F_0: the mechanism designer and the agents do not need to know F_0 (let alone agree about it). This contrasts with the modified second price auction proposed by MM to implement the efficient allocation, a game which cannot be defined without knowing F_0. In that mechanism slightly different priors about v_0 between the planner and the buyers may heavily alter the equilibrium strategies and the outcome of the game. Indeed, it is relatively straightforward to prove that if the buyers have F_0 as a common prior distribution about v_0 and the mechanism designer's prior is $\tilde{F}_0 \neq F_0$, then no BNE of the mechanism proposed by MM induces the ex post efficient allocation.[8] On the other hand, FPAWF_1 does not suffer from a similar drawback as long as (1.3) is satisfied.

1.2.3 The simultaneous-move mechanism

In any FPAWF first the seller sets a reserve price, then the buyers play a first price auction with that reserve price. Because of this sequential structure FPAWF mechanisms are not Bayesian mechanisms and therefore, strictly speaking, proposition 1.1 is not an improvement on the mechanism proposed by MM. However, it is possible to design a (direct) simultaneous-move mechanism which is independent of F_0 and is an efficient

[8] A formal proof of this claim is available upon request. MM also provide a direct efficient mechanism; the same criticisms mentioned above apply to such a mechanism.

mechanism if and only if (1.3) holds. In such a mechanism, which we call SWF_1, each agent reports to the planner a valuation (possibly dishonestly) from the set of his possible types.

Definition 1.2 *In mechanism SMF_1 (simultaneous-move first price auction with fine) the seller reports a number $w_0 \in [a_0, b_0]$ and each buyer i announces a number $w_i \in [a, b]$; reports are simultaneous. The good is allocated to the agent announcing the highest valuation. The seller pays $\frac{T_1[c(w_0)]}{n}$ to each buyer, where $c(w_0) = \max\{w_0, a\}$; if the seller reports the highest valuation then no other transfer occurs, but if buyer i reports the highest valuation then she pays $\beta[w_i; c(w_0)]$ to the seller.*

Observe that the outcome of SMF_1 under truthful reporting (which means $w_i = v_i$ for $i = 0, 1, \ldots, n$) is exactly the same as the equilibrium outcome of FPAWF_1 both in terms of allocation of the good and of transfers among the agents; therefore truthful reporting in SMF_1 results in the efficient allocation. Proposition 1.2 below, whose proof is a slight modification of the standard proof of the Revelation Principle [see Myerson (1979)], establishes that in SMF_1 truthful revelation is a BNE which satisfies the participation constraints if (1.3) holds. Mechanism SMF_1 may have several BNE but, by theorem 1 in Palfrey and Srivastava (1991), (essentially) unique implementation of the ex post efficient allocation is achievable by suitably augmenting SMF_1, still in a way which does not require knowledge of F_0.

Proposition 1.2 *If (1.3) holds then SMF_1 is an efficient mechanism which does not depend on F_0.*

Proof. As we observed above, under truthful revelation SMF_1 leads to the same outcome as the unique PBE in FPAWF_1. Therefore, if the buyers report truthfully then the expected payoff to type v_0 of seller from announcing w_0 is equal to $V[c(w_0); v_0]$; from the proof of proposition 1.1 follows that it is optimal for the seller to report $w_0 = v_0$ and that his participation constraint is satisfied if and only if (1.3) holds (here we mimic the standard proof of the Revelation Principle). Turning to incentive compatibility for buyers, observe that in FPAWF$_1$ each buyer bids after the seller announced a reserve price; on the contrary, in SMF$_1$ moves are simultaneous: each buyer reports without observing the seller's report w_0. Because of this difference in timing, the standard proof of the Revelation Principle does not apply to establish incentive compatibility for buyers - this is why we explicitly prove this proposition. From the definition of SMF$_1$ follows that, under truthful revelation of the other agents, the payoff to buyer i with valuation v_i from announcing w_i is

$$\begin{aligned} & \int_{a_0}^{w_i} \{v_i - \beta[w_i; c(v_0)]\} F^{n-1}(w_i) f_0(v_0) dv_0 + E_{v_0} \frac{T_1[c(v_0)]}{n} \\ = \; & [v_i - \beta(w_i; a)] F^{n-1}(w_i) F_0(a) + \int_a^{w_i} [v_i - \beta(w_i; v_0)] F^{n-1}(w_i) f_0(v_0) dv_0 + E_{v_0} \frac{T_1[c(v_0)]}{n} \end{aligned}$$

By definition of β, both the term $[v_i - \beta(w_i; a)]F^{n-1}(w_i)F_0(a)$ and the argument of the integral are maximized at $w_i = v_i$; moreover, the derivative of the integral with respect to the upper extreme of integration is equal to $(v_i - w_i)F^{n-1}(w_i)f_0(w_i)$. Hence, the whole expression is maximized at $w_i = v_i$. Individual rationality is guaranteed because $v_i - \beta[v_i; c(v_0)] \geq 0$ and $T_1[c(v_0)] \geq 0$ for any v_i and v_0. To conclude, $w_i = v_i$ for $i = 0, 1, \ldots, n$ is a BNE of SMF$_1$ in which the participation constraints are satisfied - if (1.3) holds - and its outcome is the ex post efficient allocation. ■

Remark 1 Mechanisms FPAWF$_1$ and SMF$_1$ can dispense, to some extent, with the exact knowledge of b_0. To see this, after defining $h(x) = R(a) + \int_a^x F^n(z)dz - x$ for $x \in [a, b]$, consider the equation $h(x) = 0$. Since $h(a) > 0$, $h(b) < 0$ and $h'(x) = F^n(x) - 1 < 0$ in (a, b), we conclude that there exists a unique solution $\bar{b}$ to the equation $h(x) = 0$ and that $a < R(a) < \bar{b} < b$. Then, to apply proposition 1.1 or proposition 1.2 it is sufficient to know that b_0 is not larger than $\bar{b}$, even without knowing exactly b_0. Also notice that a_0 does not matter since we assumed it is smaller than a.

Remark 2 An interesting implication of proposition 1.2 is that (1.3) implies (1.1) (that can also be proved directly). Hence, for any given distribution F, if b_0 is not much larger than the expected revenue of an auction with no reserve price (i.e., if $b_0 \leq \bar{b}$) then any distribution F_0 on $[a_0, b_0]$ satisfies (1.1). Unlike in the one buyer-one seller case, therefore, it is not true that $(a_0, b_0) \cap (a, b) \neq \emptyset$ prevents the existence of an efficient mechanism *whatever* the probability distributions are. On the contrary, an efficient mechanism surely exists if b_0 is not much larger than $R(a) > a$.

Remark 3 As it is well known, $R(a)$ converges to b if n tends to infinity.[9] Hence, for *any* given distribution F and $b_0 < b$, there is a natural number $\tilde{n}$ such that if $n > \tilde{n}$ then there exists an efficient (sequential or simultaneous-move) mechanism which is independent of F_0 [$R(a)$ converges to b, hence (1.3) holds if $b > b_0$ and n is large; then propositions 1.1 and 1.2 can be applied].

Example Let F be uniform on $[0, 1]$; then $R(0) = \frac{n-1}{n+1}$, hence $\bar{b} \geq \frac{n-1}{n+1}$. From $h(x) = \frac{n-1}{n+1} + \frac{x^{n+1}}{n+1} - x$ follows $\bar{b} \simeq 0.3473$ if $n = 2$ and $\bar{b} \simeq 0.6837$ if $n = 5$.

1.2.4 An efficient FPAWF mechanism when (1.3) fails

If (1.3) is violated then FPAWF_1 does not implement the efficient allocation anymore: in its unique PBE the participation constraint of type b_0 of seller fails. Clearly, by setting A sufficiently above $R(a)$ in the definition (1.2) of the fine function we can induce any type of seller to participate. This however lowers the buyers' expected payoffs and may violate the individual rationality constraints of low valuation buyers. Proposition 1.3 proves that the participation constraints of any type of buyer and seller can be satisfied within the class of FPAWF mechanisms if and only if an efficient mechanism exists.

Proposition 1.3 *There exists an FPAWF mechanism implementing the efficient allocation in unique PBE if and only if efficient mechanisms exist, that is if and only if (1.1) holds.*

Proof. Let T_2 be the fine function in which $A = b_0 - \int_a^{b_0} F^n(z)dz$ and let FPAWF_2 be the resulting FPAWF mechanism. Since FPAWF_2 differs from FPAWF_1 only because of

[9] See Holt (1980), for example.

the constant term in the fine function, the proof of proposition 1.1 applies to establish that FPAWF$_2$ has a unique PBE and that the outcome of such a PBE is the ex post efficient allocation. Thus, FPAWF$_2$ implements the efficient allocation in unique PBE if and only if the participation constraints are satisfied. Equality $A = b_0 - \int_a^{b_0} F^n(z)dz$ takes care of the participation constraint of each type of seller, and we know from the proof of proposition 1.1 that any type of buyer's individual rationality constraint holds if and only if $E_{v_0}\frac{T_2[c(v_0)]}{n} \geq 0$. Computing $E_{v_0}T_2[c(v_0)] = \int_{a_0}^{b_0} T_2[c(v_0)]f_0(v_0)dv_0$ we see that it is non-negative if and only if (1.1) holds:

$$\begin{aligned}
E_{v_0}T_2[c(v_0)] &= R(a)F_0(a) + \int_a^{b_0} [R(v_0) + v_0F^n(v_0) - \int_a^{v_0} F^n(z)dz]f_0(v_0)dv_0 \\
&\quad -[b_0 - \int_a^{b_0} F^n(z)dz] \\
&= \int_a^{b_0} \left\{ b - \int_a^b F^n(z)dz - n\int_{v_0}^b F^{n-1}(z)[1-F(z)]dz \right\} f_0(v_0)dv_0 \\
&\quad +R(a)F_0(a) - b_0 + \int_a^{b_0} F^n(z)dz \\
&= R(a)F_0(a) + [b - \int_a^b F^n(z)dz][1-F_0(a)] - b_0 + \int_a^{b_0} F^n(z)dz \\
&\quad -n\int_a^{b_0}\int_{v_0}^b F^{n-1}(z)[1-F(z)]dzf_0(v_0)dv_0 \\
&= \left\{ b - \int_a^b F^n(z)dz - n\int_a^b F^{n-1}(z)[1-F(z)]dz \right\} F_0(a) + \int_{b_0}^b [1-F^n(z)]dz \\
&\quad +n\left\{ F_0(a)\int_a^b F^{n-1}(z)[1-F(z)]dz - \int_a^{b_0} F_0(z)F^{n-1}(z)[1-F(z)]dz \right\} \\
&\quad -n\int_{b_0}^b F^{n-1}(z)[1-F(z)]dz - F_0(a)[b - \int_a^b F^n(z)dz] \\
&= \int_{b_0}^b [1-F^n(z)]dz - n\int_a^{b_0} F_0(z)F^{n-1}(z)[1-F(z)]dz \\
&\quad -n\int_{b_0}^b F^{n-1}(z)[1-F(z)]dz \qquad \blacksquare
\end{aligned}$$

1.3 A multi-unit case

In this section the model analyzed in section 1.2 is extended by assuming that the seller owns $q > 1$ units of the good. We suppose that each buyer is interested in at most one unit, while the seller may consume all of the units he owns. Although this assumption appears odd, it is formally equivalent to suppose that the seller initially owns nothing but he is able to produce up to q units of the good at a constant marginal cost v_0 and derives no utility from consuming the good.[10] Then, saying that the seller ends up with $j \leq q$ units (which yields him a gross payoff equal to jv_0) means that he actually produced only $q - j$ units and bear a total cost of $(q - j)v_0$.[11] We suppose there are $n \geq q$ buyers; if it were $n < q$ then at least $q - n$ units would go to the seller and we would have a setting with n units and n buyers. The agents' informations are the same as in section 1.2.

Lemma 1 in MM provides the necessary and sufficient condition in order for a feasible allocation to be implementable (in individually rational BNE) in a one-good setting. However, such a lemma can be adapted to deliver the necessary and sufficient condition for the existence of an efficient mechanism in the present multi-unit environment. That condition is written in terms of the functions P_0 and P defined as follows: P_0 is the function of v_0 describing how many units of the good a seller with valuation v_0 expects to receive in the efficient allocation; P is the function of v_i describing the probability for any buyer i with valuation v_i to obtain one unit of the good in the efficient allocation - P does not need any subscript since the buyers are ex ante symmetric. Recalling that ties have zero probability we find

$$P_0(v_0) \equiv q\binom{n}{0}F^n(v_0) + (q-1)\binom{n}{1}F^{n-1}(v_0)[1 - F(v_0)] + \dots$$

[10]Similarly, in the one-unit case we may think that the seller can produce just one unit at a cost v_0.

[11]If each buyer's gross payoff from consuming more than one unit were given by her valuation times the number of units she receives, then there would be no difference with respect to the model of section 1.2. Indeed, ex post efficiency would require that the highest valuation agent consumes all the q units and we would have a one-good model in which the good is the bundle of the q units.

$$
\begin{aligned}
& +\binom{n}{q-1} F^{n-q+1}(v_0)[1-F(v_0)]^{q-1} \\
= & \sum_{j=0}^{q-1}(q-j)\binom{n}{j} F^{n-j}(v_0)[1-F(v_0)]^{j}
\end{aligned}
$$

$$
\begin{aligned}
P(v_i) \equiv & F_0(v_i)\left\{\binom{n-1}{0} F^{n-1}(v_i)+\binom{n-1}{1} F^{n-2}(v_i)[1-F(v_i)]+\ldots\right. \\
& \left.+\binom{n-1}{q-1} F^{n-q}(v_i)[1-F(v_i)]^{q-1}\right\} \\
= & F_0(v_i) \sum_{j=0}^{q-1}\binom{n-1}{j} F^{n-1-j}(v_i)[1-F(v_i)]^{j}
\end{aligned}
$$

Proposition 1.4 *(i) In this q-unit model an efficient mechanism exists if and only if $b_0 < b$ and*

$$\int_{b_0}^{b}[q-P_0(z)]dz - n\int_{a}^{b} P(z)[1-F(z)]dz \geq 0 \tag{1.4}$$

(ii) If $n = q$ and gains from trade are uncertain, that is $(a_0, b_0) \cap (a, b) \neq \emptyset$, then no efficient mechanism exists whatever F and F_0 are.

Proof. (i) We adapt lemma 1 in MM to this multi-unit model by taking into account that the seller now owns q units of good. It can be verified that all the steps in the proof of lemma 1 in MM go through if we replace $1 - P_0(v_0)$ with $q - P_0(v_0)$; as a result, $-b_0[q - P_0(b_0)]$ substitutes $-b_0[1 - P_0(b_0)]$ in the statement of the lemma. In this way we find (since both P_0 and P are increasing functions) that an efficient mechanism exists if and only if

$$naP(a) - b_0[q - P_0(b_0)] + n\int_{a}^{b} z[1-F(z)]dP(z) - \int_{a_0}^{b_0} F_0(z)zdP_0(z) \geq 0$$

Inequality (1.4) is obtained by integrating by parts $\int_a^b z[1-F(z)]dP(z)$ and observing that $F_0(z)P_0'(z) = nP(z)f(z)$.[12]

[12] Also theorem 3.1 in Makowski and Mezzetti (1994) could be used to obtain (1.4) (but it requires more involved computations).

(ii) If $n = q$ then inequality (1.4) reduces to $-q\int_a^{b_0} F_0(z)[1 - F(z)]dz \geq 0$, which is false if $(a_0, b_0) \cap (a, b) \neq \emptyset$. ■

Proposition 1.4(i) generalizes theorem 1 in MM [observe that (1.4) reduces to (1.1) if $q = 1$] and proposition 1.4(ii) generalizes the Myerson-Satterthwaite theorem: no efficient mechanism exists when $n = q > 1$ and $(a_0, b_0) \cap (a, b) \neq \emptyset$. This is not surprising: since the buyers' valuations are independent and the seller's marginal cost is constant, when $n = q$ this model looks like q one seller-one buyer models.

As in the one-unit (and $n > 1$ buyers) case, when $n > q > 1$ we can prove that if b_0 is not too larger than a then there exists an efficient mechanism which is independent of F_0. Let FPAWF$_{1q}$ be the mechanism in which the seller can sell the goods only through a q-unit first price auction with reserve price $r \in [a, b]$ and the function T_{1q} defined below determines the amount he must pay to the buyers upon choosing r;[13] more precisely, the seller pays $\frac{T_{1q}(r)}{n}$ to each potential buyer. Then we can state proposition 1.5, a more general result than proposition 1.1; its proof is omitted as it is virtually the same as the proof of proposition 1.1.

$$T_{1q}(r) = R_q(r) - R_q(a) + rP_0(r) - \int_a^r P_0(z)dz \text{ for any } r \in [a, b]$$

Proposition 1.5 *If $R_q(a) + \int_a^{b_0} P_0(z)dz \geq qb_0$ then FPAWF$_{1q}$, a mechanism which is independent of F_0, implements the efficient allocation in unique PBE in this q-unit model.*

To verify that proposition 1.1 is a special case of proposition 1.5 just observe that $P_0(v_0)$, the number of units a seller with type v_0 expects to receive in the efficient

[13] In a q-unit first price auction with reserve price r each buyer submits a sealed bid; each of the q highest bidders (provided her bid exceeds r) receives one unit of good and pays her own bid to the seller. This game has a unique symmetric BNE [see Weber (1983); Maskin and Riley (1989) claim that such a BNE is actually the unique BNE]. In that equilibrium each buyer i with valuation $v_i \geq r$ bids $\beta_q(v_i; r) = v_i - \int_r^{v_i} \frac{P(z)/F_0(z)}{P(v_i)/F_0(v_i)}dz$ and $\beta_q(v_i; r) = 0$ if $v_i < r$. The seller's expected revenue in such a BNE is $R_q(r) \equiv \int_r^b \beta_q(z; r)dP_0(z)$.

allocation, equals $F^n(v_0)$ when $q = 1$ and compare the inequality in the statement of proposition 1.5 with (1.3). Moreover, propositions 1.2, 1.3 and the three remarks about propositions 1.1 and 1.2 can be generalized to this environment.

1.4 Non-private good

In this section we relax a major assumption in the model introduced in section 1.2: we do not suppose that the good which can be traded is a private good; rather, we analyze a *partially* public good model. Formally, assume that the seller can produce a certain good at a cost v_0 as suggested in footnote 10. If buyer i obtains the good then her gross payoff from consuming it equals v_i and, moreover (the new assumption), each buyer $j \neq i$ earns a gross payoff of αv_j with $\alpha \in (0, 1)$. The agents' informations are exactly as in section 1.2. In this section we study the conditions under which an efficient mechanism exists in this setting.

Clearly, if α were equal to 0 then we would be back to the private good environment of section 1.2, for which we know that inequality (1.1) is necessary and sufficient in order for an efficient mechanism to exist. On the other hand, if it were $\alpha = 1$ then the good would be public. Mailath and Postlewaite (1990) prove that in a public good setting no efficient mechanism exists if v_0 is common knowledge and gains from trade are uncertain; moreover, such a result extends to the case in which the seller privately observes v_0. We consider intermediate cases with $\alpha \in (0, 1)$; hence buyer j enjoys a benefit if buyer $i \neq j$ wins the good but, for a given own payment, she prefers to buy herself the good.[14]

In the ex post efficient allocation (neglecting ties) the good is produced if and only if $(1 - \alpha) \max \{v_1, \ldots, v_n\} + \alpha \sum_{i=1}^{n} v_i > v_0$ and in such a case it is obtained by buyer i if

[14] A similar model is suitable, for example, when a same office is shared by several people - say students. If student 1 has a computer on her desk then she gets some benefit from that because she can use it whenever she wants. Moreover, the other students in the office can use that computer when student 1 is not there; thus each other student gets a fraction of the utility increase she would enjoy if she had a computer on her own desk. Oliva (1997) uses a similar functional specification to represent the cost reductions generated by technology transfers within a group of two firms. Jehiel, Moldovanu and Stacchetti (1999) analyze a very general model of auctions with externalities.

$v_i = \max\{v_1, \ldots, v_n\}$. In order to establish whether an efficient mechanism exists, once again we adapt lemma 1 in MM to the present environment. Now let $P_0(v_0; \alpha)$ denote the probability that the good is *not* produced in the efficient allocation given that the seller's cost is v_0; $P(v_i; \alpha)$ is the "expected quantity" of good any buyer i with valuation v_i receives in the efficient allocation. If α were equal to 0 then $P(v_i; 0) = F^{n-1}(v_i)F_0(v_i)$ would be the probability for a buyer i with valuation v_i to win the good. When $\alpha \in (0, 1)$, instead, $P(v_i; \alpha)$ is an expectation of the numbers 0, α, and 1 in which the weight α receives is the probability, given v_i, that the good is produced but is obtained by a buyer $j \neq i$. Once this is taken into account, lemma 1 in MM implies that an efficient mechanism exists if and only if [since both $P_0(.; \alpha)$ and $P(.; \alpha)$ are increasing functions]

$$\begin{aligned} 0 &\leq naP(a; \alpha) - b_0[1 - P_0(b_0; \alpha)] + n \int_a^b z[1 - F(z)]dP(z; \alpha) - \int_{a_0}^{b_0} zF_0(z)dP_0(z; \alpha) \\ &= -b_0 - n \int_a^b P(z; \alpha)d[z - zF(z)] + \int_{a_0}^{b_0} P_0(z; \alpha)d[zF_0(z)] \end{aligned} \tag{1.5}$$

where the equality is obtained after integration by parts of $\int_a^b z[1 - F(z)]dP(z; \alpha)$ and $\int_{a_0}^{b_0} zF_0(z)dP_0(z; \alpha)$.

Observe that if we let $x' = [1 + \alpha(n-1)]a$ and $x" = [1 + \alpha(n-1)]b$ then the efficient production decision is straightforward if and only if $(a_0, b_0) \cap (x', x") = \varnothing$; to simplify the exposition we suppose $a_0 \leq x'$ (similarly, in sections 1.2 and 1.3 it is assumed $a_0 \leq a$). If $b_0 \leq x'$ then the good should always be produced and in that case the following mechanism implements the ex post efficient allocation in unique (symmetric) BNE. First each buyer pays αa to the seller; then a first price auction with no reserve price (and no fines) is run and the highest bidder wins the good by paying her own bid to the seller. In the present setting such a game has a unique symmetric BNE; it prescribes that any buyer i with valuation v_i bids $(1 - \alpha)\beta(v_i; a)$, which is strictly increasing in v_i.[15] The following proposition deals with the case in which the overlap between the intervals (a_0, b_0) and $(x', x")$ is not very large and it is analogous to remark 2 for the

[15] The function β was defined in subsection 1.2.2. Notice that all the participation constraints are satisfied.

present setting.

Proposition 1.6 *Given any $\alpha < 1$, if b_0 is not too larger than x' then, for any probability distribution F_0 on $[a_0, b_0]$, there exists an efficient mechanism.*

Proof. Assume $b_0 = x' + \varepsilon$ with $\varepsilon \in (0, b-a)$; then it is surely efficient to produce the good if at least one buyer's valuation is above $a+\varepsilon$. Hence $P(z;\alpha) = Q(z;\alpha) \equiv F^{n-1}(z) + \alpha[1 - F^{n-1}(z)]$ if $z > a+\varepsilon$: a buyer with valuation $z > a+\varepsilon$ wins the good with probability $F^{n-1}(z)$; if she does not win then some other buyer obtains the good as production occurs anyway. As in section 1.2, $R(a) = b - \int_a^b [nF^{n-1}(z) - (n-1)F^n(z)]dz$ is the expected revenue from a first price auction (with no reserve price) of a private good and it can be verified that

$$-x' - n\int_a^b Q(z;\alpha)d[z - zF(z)] = (1-\alpha)[R(a) - a]$$

Using such equality and $\int_{a_0}^{b_0} P_0(z;\alpha)d[zF_0(z)] \geq 0$ we obtain

$$\begin{array}{c} -b_0 - n\int_a^b P(z;\alpha)d[z - zF(z)] + \int_{a_0}^{b_0} P_0(z;\alpha)d[zF_0(z)] \\ \geq -x' - \varepsilon - n\int_a^b Q(z;\alpha)d[z - zF(z)] + n\int_a^{a+\varepsilon}[Q(z;\alpha) - P(z;\alpha)]d[z - zF(z)] \\ = -\varepsilon + (1-\alpha)[R(a) - a] + n\int_a^{a+\varepsilon}[Q(z;\alpha) - P(z;\alpha)]d[z - zF(z)] \end{array} \quad (1.6)$$

Since $1 \geq Q(z;\alpha) - P(z;\alpha) \geq 0$ for any z, if $M \geq f(z)$ for any $z \in [a,b]$ (recall that f is continuous on $[a,b]$) then

$$\begin{array}{l} n\int_a^{a+\varepsilon}[Q(z;\alpha) - P(z;\alpha)][1 - F(z) - zf(z)]dz \geq \\ n\int_a^{a+\varepsilon}[Q(z;\alpha) - P(z;\alpha)][-zf(z)]dz > -nbM\varepsilon \end{array} \quad (1.7)$$

Therefore, in view of (1.6) and (1.7), inequality (1.5) necessarily holds for any F_0 on $[a_0, b_0]$ if $\varepsilon = b_0 - x' \leq \frac{(1-\alpha)[R(a)-a]}{1+nbM}$.[16] ■

[16]Notice that both inequalities in (1.7) are in general quite coarse, hence $x' + \frac{(1-\alpha)[R(a)-a]}{1+nbM}$ is likely to be "too pessimistic" as an upper bound for b_0.

When $\alpha = 1$ no efficient mechanism exists if $(a_0, b_0) \cap (x', x") \neq \varnothing$; proposition 1.6 says that such an extreme result is isolated in our setting. Indeed, for *any* $\alpha < 1$ the existence of an efficient mechanism cannot be ruled out just because gains from trade are not common knowledge; rather, it is guaranteed - independently of the shape of F_0 on $[a_0, b_0]$ - if the overlap between (a_0, b_0) and $(x', x")$ is not too large. More generally, if $b_0 > x' + \frac{(1-\alpha)[R(a)-a]}{1+nbM}$ then (1.5) may still hold, depending on F_0, F, n and α.

Example Let F be uniform on $[0, 1]$; then $R(0) = \frac{n-1}{n+1}$ and, since $Q(z;\alpha) \geq P(z;\alpha)$ for any z, it is clear that $n \int_0^{\varepsilon} [Q(z;\alpha) - P(z;\alpha)]d(z - z^2) \geq 0$ if $\varepsilon \leq \frac{1}{2}$. Hence, from (1.6) follows that (1.5) holds for any possible F_0 on $[0, b_0]$ if $b_0 \leq \min\left\{\frac{(1-\alpha)(n-1)}{n+1}, \frac{1}{2}\right\}$.

The proof to proposition 1.6 and the above example suggest that the smaller is α (the more private the good is) the more likely is an efficient mechanism to exist. This claim is established for the following example.

Example Let F be uniform on $[0, 1]$, F_0 uniform on $[0, b_0]$ and $n = 2$. Then we can compute exactly $b_0^{\bullet}(\alpha)$, the largest value of b_0 consistent with the existence of an efficient mechanism, given α $[b_0^{\bullet}(\alpha) \geq \frac{1-\alpha}{3}$ by the above example]. Indeed, after some manipulations[17] inequality (1.5) can be written as $2(1 - \alpha) - 6b_0 + 2b_0^2 + \frac{b_0^3}{1+\alpha} \geq 0$ for $b_0 \in (0, 1)$; $b_0^{\bullet}(\alpha)$ is the unique solution in $(0, 1)$ to the equation $2(1-\alpha)-6x+2x^2+\frac{x^3}{1+\alpha} = 0$. Since $b_0^{\bullet}$ is a strictly decreasing function,[18] in this example it is confirmed that the conditions for an efficient mechanism to exist are less restrictive the smaller is α in $(0, 1)$. For instance, $b_0^{\bullet}(0) = 0.396$, $b_0^{\bullet}(0.5) = 0.178$ and $b_0^{\bullet}(0.95) = 0.017$.

1.5 Conclusions

When agents bargain under private information some of them may be tempted to misreport the own private information in order to obtain more favorable terms of trade. It is

[17] In carrying out the computations the following equalities are useful: $P(v_i;\alpha) = \int_0^{v_i} F_0(v_i + \alpha z)dz + \alpha \int_{v_i}^1 F_0(\alpha v_i + z)dz$ and $P_0(v_0;\alpha) = \frac{v_0^2}{1+\alpha}$.

[18] Use the implicit function theorem after observing that $b_0^{\bullet}(\alpha) < \frac{1}{2}$ for any α, since $2(1 - \alpha) - 6\frac{1}{2} + 2(\frac{1}{2})^2 + \frac{(\frac{1}{2})^3}{1+\alpha} < 0$.

well known that this may prevent the implementability of the ex post efficient allocation if participation has to be voluntary. In this chapter we examined some trading models in which the production side is represented by a unique agent facing several agents willing to consume the good(s) he can produce. We extended a one-unit private good model already analyzed in the literature to a multi-unit case and derived the necessary and sufficient condition for the existence of an efficient mechanism in that environment. Then we moved to a setting of non-private good and showed, in a simple but general model, that the inefficiency result which holds in a public good environment is not robust to any degree of "privateness" of the good. If the good is not perfectly public then, even though gains from trade are uncertain, the existence of an efficient mechanism depends on the parameters of the model: probability distributions, etc. These results are summarized in the following table.

	Economic environment	**Existence of an efficient mechanism**
1	one-unit seller, one buyer	impossible (Myerson and Satterthwaite)
2	one-unit seller, $n > 1$ buyers	depends on F_0, F and n (Makowski and Mezzetti)
3	q-unit seller, $n \leq q$ buyers	impossible
4	q-unit seller, $n > q$ buyers	depends on F_0, F, n and q
5	public good	impossible (Mailath and Postlewaite)
6	partially public good	depends on F_0, F, n and α

Inefficiency appears to be a necessary feature of the environments in which the production and consumption technologies are such that it is feasible to satiate all the buyers, as when there are as many units of good available as buyers or when the good is public: cases 1, 3 and 5. On the other hand, in cases 2, 4 and 6 it is not feasible to satiate all the buyers. They will compete among themselves to obtain the available good(s) and eventually some buyer will still have a positive marginal utility from the consumption of the good. Then an efficient mechanism may exist, depending on the environment parameters.[19] An interesting topic for future research is to investigate the validity of these results in more general models.

In some environments, therefore, efficient mechanisms exist and it is important to know how they have to be designed. Often it is necessary to assume that the planner and the agents have common priors about the parameters which are private information; this assumption is sometimes viewed as a restrictive one and its failure may generate different outcomes with respect to the ones which are expected when it holds. The main contribution of this chapter is a new mechanism proposed to implement (when it is possible) in unique PBE the ex post efficient allocation in the private good(s) case. More important, however, is that under some conditions this mechanism does not need to assume any specification for the probability distribution of the seller's valuation; hence the agents and the planner are not required to have any (let alone common) prior about v_0. This partially solves the above problem by substantially reducing the amount of information both the regulator and the agents need to have about the economic environment, thus making the implementation of the ex post efficient allocation less demanding.

[19] Consistently with this remark, Williams (1999) shows that in a multilateral bargaining model for private goods an efficient mechanism is more likely to exist when a large fraction of one side of the market is bound to be not satiated in its trading willingness [see theorem 4 in Williams (1999) and the remarks following its proof].

Chapter 2

Optimal Two-Object Auctions with Synergies

2.1 Introduction

This chapter deals with the design of the revenue-maximizing auction when an agent has two indivisible goods to sell and each buyer has superadditive values for the objects. Myerson (1981) provided the solution to the revenue maximization problem for the single object case, but few not recent papers investigated multi-object selling mechanisms when the same buyer may consume several objects and observes a specific signal for each good [see Maskin and Riley (1989) and Branco (1996) about optimal multiunit auctions when each buyer observes a unique signal]. Examples are Palfrey (1983), McAfee and McMillan (1988) and McAfee, McMillan and Whinston (1989). Following the US spectrum auction, however, in the last years several papers concentrated on optimal multi-object selling mechanisms when each buyer's private information is multidimensional; among these, Armstrong (1996), Rochet and Choné (1998), Armstrong (1999), Armstrong and Rochet (1999), Armstrong (2000) and Avery and Hendershott (2000).[1] All of these papers

[1] Even though it is not strictly related to our topic (since we assume private values), we should mention a literature about efficient multi-object auctions with interdependent valuations; see for example

assume that each buyer's gross payoff from consuming more than one good is equal to the sum of her single valuations for those goods.[2] The analysts of the FCC auction, however, emphasized that synergies associated with winning more than one licence played an important role in determining the bidders' gross payoffs.[3] Therefore, in some settings, the existence of synergies should be taken into account by a revenue-maximizing (or welfare-maximizing) seller.

In our model there are n buyers, two goods on sale and each buyer privately observes two signals determining the value to her of each item; each signal may be high or low. Synergies appear in a simple form: if a same buyer receives both goods then her gross surplus is the sum of her valuations for each single good increased by $\alpha > 0$ representing a synergic effect.[4] The goal of this chapter is to give a first cut in detecting the revenue-maximizing auction in a setting in which the synergic surplus is the same for each buyer and is common knowledge.

After formally presenting the model (in section 2.2) we briefly review the results for the case without synergies ($\alpha = 0$), which has been analyzed by Armstrong (2000) (henceforth Ar). In the optimal auction when $\alpha = 0$ each good m ($m = 1, 2$) is sold to a buyer with a high valuation for it, provided there is at least one such buyer; hence, a buyer with low value for good m never obtains it if a high valuation buyer for that good

Perry and Reny (1999), Dasgupta and Maskin (2000) and Jehiel and Moldovanu (2000). Bikhchandani (1999) examines simultaneous first and second price auctions for several objects when the buyers' valuations are common knowledge.

[2] Actually, Armstrong (1996) assumes additive separability in the buyers' utility functions only in the examples he works out.

[3] See for example McAfee and McMillan (1996). Ausubel *et al.* (1997) empirically tested for the existence of synergies in the FCC auction; their results suggest that synergies were a significant factor in determining prices, even though their importance was not overwhelming.

[4] Krishna and Rosenthal (1996) and Branco (1997) use the same analytic formulation to capture synergies. These papers inquire the features of specific selling mechanisms as simultaneous or sequential second price auctions or English auctions - in more complex environments with respect to our - without finding the optimal mechanism and avoiding multidimensional issues.

is around.[5] In the terms of Ar, the optimal auction is weakly efficient.[6]

Another result in the setting with $\alpha = 0$ is that if each buyer's valuations for the two goods are strongly and positively correlated then it is optimal to use separate auctions: Each good m is allocated only as a function of the buyers' valuations for it, thus ignoring the values for good $3-m$. The buyers' valuations for object $3-m$ matter in the allocation of good m if correlation is weak or negative and all the buyers value poorly good m, which is then sold to a buyer with a high valuation for object $3-m$. In both cases "independence at the top" is optimal: If $n_m \geq 1$ buyers value highly good m, then this is randomly and fairly allocated among them neglecting their valuations for good $3-m$.

When $\alpha > 0$ we find that independence at the top is suboptimal (hence separate auctions should never be used), since the seller tends to allocate both goods to a same buyer in order to generate and extract the synergic surplus. For instance, suppose the realized buyers' types are only HH (a type of buyer with a high valuation for each of the two goods on sale) and HL (a buyer with a high valuation for object 1 and a low value for good 2); then both the mechanisms described above sell object 2 to a type HH and allocate randomly good 1 among *all* the buyers. We show that for any $\alpha > 0$ this is inferior with respect to selling both goods to a same buyer with type HH as in the latter way the seller generates and extracts the synergic surplus with probability 1 rather than with probability $\frac{1}{n}$ (and without tightening any binding constraint). More generally, when there is at least a type HH in the auction then no other type obtains any good. Having $\alpha > 0$ implies that in any optimal mechanism, given the probability for each

[5] This is also a well known property of the one-good model when the buyers' valuations are i.i.d. over a binary support. Adapting the analysis of Myerson (1981) to a one-good model with i.i.d. discretely distributed valuations reveals that when the cardinality of the support is larger than two then a buyer with a given valuation always beats a buyer with a lower value if and only if the so called "virtual valuation function" is monotone increasing (but with discrete values this condition may fail even though the probability distribution for each buyer's valuation yields a monotone hazard rate). If that is not so, then the seller treats in the same way ("bunches") buyers with different types. See subsection 2.3.4 for more on this.

[6] The qualifier "weakly" refers to the fact that for some parameter values the seller withholds one or both objects when all the buyers' valuations for this (these) object(s) are low (yet, strictly positive) even though he attaches zero value to each good.

type of buyer to obtain good m ($m = 1, 2$), it is impossible to increase the probability that the goods are sold to a same buyer. In other words, in any optimal mechanism the probability that the synergy is produced is maximized given the probability distributions according to which good 1 and good 2 are allocated among the buyers.

When α is sufficiently large the seller allocates both objects to a same buyer not only when at least a buyer's type is HH but also for any possible profile of realized valuations. This fact may lead to the failure of the weak efficiency property holding when $\alpha = 0$, according to which a low valuation buyer for good m never receives it if another buyer values highly good m. To be clear, in general a mechanism is said to be weakly efficient if whenever the objects are sold they are allocated in a way that maximizes social surplus - which coincides with the sum of the buyers' gross surpluses. It turns out that when $\alpha > 0$ revenue-maximizing mechanisms may violate weak efficiency in several ways. In some cases the synergic surplus is not generated because the goods are not allocated to a same buyer (even though α is relatively high). In other cases good m is sold to a buyer with a low valuation for it even though some other buyer values highly good m. It is clear that when the buyers' types are HL and LH there is a tension between selling both goods to a same buyer in order to extract the synergic surplus and selling object 1 to a type HL and good 2 to a type LH. Not surprisingly, given incomplete information, this dilemma is not always efficiently solved. On the other hand, no similar tension is apparent when the buyers' types are HL and LL (or LH and LL); yet, in some cases good 1 (good 2) is sold to a type LL.

In short, when $\alpha > 0$ weak efficiency may fail because synergies are generated too rarely and also because no buyer with a high value for good m receives it when all the other buyers' types are LL. Subsection 2.3.4 provides an explanation of the latter result, but the following is a possible interpretation. If α is large enough to imply that the goods are always sold as a single unit, then the seller faces a one-good (the bundle) selling problem in which each buyer's valuation for the bundle is equal to α plus the sum of her valuations for the two single objects. Here the probability distribution for each buyer's valuation is derived from the original (bivariate) probability distribution and it induces non-monotone virtual valuations (mentioned in footnote 5) if correlation is strong and positive - which means that a same buyer is unlikely to observe different signals for the two goods. In such a case the seller should treat in the same way buyers with different valuations for the bundle, like types HL and LL.

A limitation of the present analysis is that it only allows a binary distribution for each buyer's valuation for good m ($m = 1, 2$) and it only allows for two objects. Allowing for more general (discrete) distributions and/or for more than two objects is conceivable, although this would significantly increase the number of different cases to consider. It would be very interesting to solve the problem for continuously distributed valuations; unfortunately this appears hard even when there are no synergies.[7] Ar finds the optimal auction for a specific case with $n \geq 2$ buyers in which the valuations are continuously distributed over two rays in the positive orthant of $\Re^2$ (and $\alpha = 0$). In that environment the optimal auction is not weakly efficient and Ar expects that feature to extend to more general settings, about which he conjectures that "numerical simulations will provide the most tractable method of generating insights into this problem".

The plan of this chapter is as follows. Next section formally introduces the model; section 2.3 solves the revenue maximization problem and provides some comments. Section 2.4 concludes and suggests possible extensions; proofs are left to the appendix.

2.2 The model

2.2.1 Preferences and information

An agent (the seller) owns two indivisible objects which are worthless to him and faces $n \geq 2$ agents (the buyers) who are interested in these objects; the seller wishes to

[7] Rochet and Choné (1998) characterize the optimal mechanism when the seller faces an only buyer with continuously distributed multi-dimensional private information (but no synergies). They employ control techniques which appear hard to generalize to the case of several buyers. Their results, moreover, are not constructive.

maximize his expected revenue from the sale of the two goods. Letting v^i and w^i denote buyer i's valuation ($i = 1, \ldots, n$) for good 1 and good 2 respectively, we assume there exists a positive number α such that buyer i's expected payoff from participating in any selling mechanism is equal to

$$v^i \{\text{prob to win good 1}\} + w^i \{\text{prob to win good 2}\} + \alpha \{\text{prob to win both goods}\} - t^i$$

where t^i is her expected payment to the seller. In words, buyer i's gross surplus from consuming both goods is not simply $v^i + w^i$ but rather $v^i + w^i + \alpha$ with $\alpha > 0$ due to a synergic effect; α is common knowledge, it is the same for each buyer and is independent of a buyer's valuations for the objects.

The valuations v^i and w^i are privately observed by buyer $i = 1, \ldots, n$ and can take on values in $\{v_L, v_H\}$ and $\{w_L, w_H\}$ respectively, with $v_H > v_L > 0$ and $w_H > w_L > 0$; moreover, ex ante (v^i, w^i) and $(v^{i'}, w^{i'})$ are i.i.d. bivariate random variables for any two different buyers i and i'. Maskin and Riley (1984) show that when the buyers are ex ante symmetric the seller does not lose revenue in letting a buyer's probability to win good m ($m = 1, 2$) and her payment be a function of her type only and not of her identity. Thus, henceforth we drop the reference to a buyer's identity and refer to a generic buyer with valuations $(v, w) \in \{v_L, v_H\} \times \{w_L, w_H\}$. A buyer's type is jk if her valuation for good 1 is v_j and her value for object 2 is w_k, $j, k = L, H$. Let n_{jk} denote the number of buyers with type jk participating in the auction; clearly $n_{HH} + n_{HL} + n_{LH} + n_{LL} = n$.

In order to reduce the number of different cases which can arise, we assume that the goods are symmetric in the sense that $v_L = w_L = s > 0$, $v_H = w_H = s + \Delta > s$,[8] and $\Pr\{(v, w) = (s + \Delta, s)\} = \Pr\{(v, w) = (s, s + \Delta)\}$; in such a case there is no loss of generality in letting $\Delta = 1$, thus $v_H = w_H = s + 1$. The following is the probability distribution for (v, w) ($h > 0$, $q > 0$, $l > 0$ and $h + 2q + l = 1$):

[8] These assumptions simplify exposition. Actually, only $v_H - v_L = w_H - w_L$ is really needed for our results to hold.

	$w = s$	$w = s + 1$
$v = s$	l	q
$v = s + 1$	q	h

We also let $s \geq \frac{h+q}{l}$ and we will prove that under this assumption both goods are sold for any realized profile of buyers' valuations. In other words, the well known possibility of ex post inefficiency due to the seller withholding a good (or both) when all the buyers have low valuation(s) is ruled out if $s \geq \frac{h+q}{l}$ and we can focus on other kinds of inefficiencies.[9] The environment analyzed in Ar differs from the present one because he does not restrict to symmetric goods and because he assumes $\alpha = 0$. While allowing for a positive α makes the model more cumbersome, restricting to symmetric goods is helpful to narrow down the class of mechanisms which can be optimal; among other things, it implies that the subconstrained problem (as defined in subsection 2.3.2 below) always provides the solution to the complete problem.

2.2.2 Mechanisms

By the virtue of the Revelation Principle we maximize the seller's expected revenue within the class of direct mechanisms. Therefore the seller commits to a rule which, for any possible $n-$tuple of buyers' reports of types, specifies which good(s) he sells, to whom, and the payment he requires from each type of buyer. Such a rule needs to satisfy the appropriate incentive compatibility and participation constraints.

Let x_{jk} denote the probability that a buyer reporting type jk obtains *only* good 1, $j, k = L, H$, under truthtelling of the other buyers. The quantity x_{jk} is a "reduced form" probability in the sense that it depends on the buyer's report jk but not on her

[9]Solving the model without assuming $s \geq \frac{h+q}{l}$ would increase the number of cases which may arise and there would also exist optimal mechanisms which are not efficient because of the reason just described.

opponents' reports; it is obtained from "non-reduced form" probabilities by averaging out the (sincere) reports of the other buyers.[10] Similarly, we use y_{jk} (z_{jk}) to denote the probability that a buyer announcing jk receives *only* good 2 (*both* goods) when the others report truthfully; t_{jk} is the expected payment the seller requires from such buyer, $j,k = L, H$. Type jk's expected payoff under truthful reporting is therefore

$$v_j x_{jk} + w_k y_{jk} + (v_j + w_k + \alpha) z_{jk} - t_{jk}$$

The incentive compatibility constraints are summarized by (2.1) below; for the sake of clarity we also write down both the incentive constraints which will be relevant in the following and the participation constraint for type LL:

$$\begin{aligned} v_j(x_{jk} - x_{j'k'}) + w_k(y_{jk} - y_{j'k'}) + (v_j + w_k + \alpha)(z_{jk} - z_{j'k'}) &\geq t_{jk} - t_{j'k'} \qquad (2.1) \\ jk, j'k' &= HH, HL, LH, LL \end{aligned}$$

$$v_H x_{HH} + w_H y_{HH} + (v_H + w_H + \alpha) z_{HH} - t_{HH} \geq v_H x_{HL} + w_H y_{HL} + (v_H + w_H + \alpha) z_{HL} - t_{HL} \qquad (2.2)$$

$$v_H x_{HH} + w_H y_{HH} + (v_H + w_H + \alpha) z_{HH} - t_{HH} \geq v_H x_{LH} + w_H y_{LH} + (v_H + w_H + \alpha) z_{LH} - t_{LH} \qquad (2.3)$$

$$v_H x_{HH} + w_H y_{HH} + (v_H + w_H + \alpha) z_{HH} - t_{HH} \geq v_H x_{LL} + w_H y_{LL} + (v_H + w_H + \alpha) z_{LL} - t_{LL} \qquad (2.4)$$

$$v_H x_{HL} + w_L y_{HL} + (v_H + w_L + \alpha) z_{HL} - t_{HL} \geq v_H x_{LL} + w_L y_{LL} + (v_H + w_L + \alpha) z_{LL} - t_{LL} \qquad (2.5)$$

$$v_L x_{LH} + w_H y_{LH} + (v_L + w_H + \alpha) z_{LH} - t_{LH} \geq v_L x_{LL} + w_H y_{LL} + (v_L + w_H + \alpha) z_{LL} - t_{LL} \qquad (2.6)$$

$$v_L x_{LL} + w_L y_{LL} + (v_L + w_L + \alpha) z_{LL} - t_{LL} \geq 0 \qquad (2.7)$$

The seller's revenue is given by the sum of the transfers he obtains from the buyers. As the buyers are ex ante symmetric, the expected revenue R is equal to n times the

[10] For example, if $n = 2$ then we could let $x_{jkj'k'}$ denote the probability for a buyer reporting jk to receive only good 1 when the other buyer announces $j'k'$; then $x_{jk} = hx_{jkHH} + qx_{jkHL} + qx_{jkLH} + lx_{jkLL}$.

expected revenue from any given buyer:

$$\frac{R}{n} = ht_{HH} + qt_{HL} + qt_{LH} + lt_{LL}$$

When maximizing $\frac{R}{n}$ with respect to $\{x_{jk}, y_{jk}, z_{jk}\}_{j,k=L,H}$ under the incentive and participation constraints it should be taken into account that the above variables need to satisfy some feasibility conditions arising from the fact that there is just one unit of each good to sell; such conditions are analogous to the resource constraints which appear in subsection 3.1 in Ar. In our setting the fact that good 1 (2) is sold to a type jk is represented through the variable x_{jk} (y_{jk}) or z_{jk}, depending on whether it is sold alone or together with good 2 (1). This makes harder to write the resource constraints with respect to the constraints which are imposed in Ar. Nevertheless, we can avoid considering them explicitly by proceeding as follows. First, we describe any mechanism by specifying how it allocates the goods for any possible $n-$tuple of reports, other than computing the implied values of $\{x_{jk}, y_{jk}, z_{jk}\}_{j,k=L,H}$. Moreover, when proving the optimality of a mechanism with respect to a possible variation of the sale policy we specify the profiles of buyers' reports for which the mechanism is modified in order to produce such a variation. This cannot undermine feasibility and allows to avoid considering resource constraints written in terms of $\{x_{jk}, y_{jk}, z_{jk}\}_{j,k=L,H}$. In other words, we describe each auction "explicitly" in terms of non-reduced form probabilities and then examine how varying the latter probabilities affects reduced form probabilities and in turn the seller's revenue.

To see an example of how this method works, suppose that for a given profile of reports with $n_{HH} \geq 1$ and $n_{LH} \geq 1$ good 1 is randomly and fairly allocated among types HH (each of them receives it with probability $\frac{1}{n_{HH}}$) and that each type LH wins good 2 with probability $\frac{\beta}{n_{LH}}$ ($0 < \beta \leq 1$); this gives a contribution to y_{LH} equal to

$$\frac{(n-1)!h^{n_{HH}}q^{n_{HL}}q^{n_{LH}-1}l^{n_{LL}}}{n_{HH}!n_{HL}!(n_{LH}-1)!n_{LL}!}\frac{\beta}{n_{LH}}$$

This is the probability for a buyer of type LH that the given profile of reports occurs under truthtelling (by using the multinomial distribution) times the probability to win object 2 in such a case. For the given profile we are examining, consider reducing β by $\Delta\beta > 0$ while increasing by $\Delta\beta$ the probability that the same buyer of type HH winning good 1 obtains also good 2. Then y_{LH} decreases; more precisely, $\Delta y_{LH} = -\frac{(n-1)!h^{n_{HH}}q^{n_{HL}}q^{n_{LH}-1}l^{n_{LL}}}{n_{HH}!n_{HL}!(n_{LH}-1)!n_{LL}!}\frac{\Delta\beta}{n_{LH}}$. Likewise, x_{HH} (the probability that a type HH gets *only* good 1) decreases and the probability z_{HH} that a type HH wins *both* goods increases:

$$\Delta z_{HH} = \frac{(n-1)!h^{n_{HH}-1}q^{n_{HL}}q^{n_{LH}}l^{n_{LL}}}{(n_{HH}-1)!n_{HL}!n_{LH}!n_{LL}!}\frac{\Delta\beta}{n_{HH}} = -\Delta x_{HH}$$

The middle term is the probability for a buyer of type HH that the given profile of reports occurs (under truthtelling) times the increase in the probability to win both goods under such profile. Thus, $\Delta z_{HH} = -\Delta x_{HH} = -\frac{q}{h}\Delta y_{LH} > 0$. This makes easy to evaluate the profitability of reducing β since the seller's revenue function and the constraints he faces are linear in $\{x_{jk}, y_{jk}, z_{jk}\}_{j,k=L,H}$ (after substituting for t_{HL}, t_{LH} and t_{LL} by using some binding incentive and participation constraints).

In the proofs (which are found in the appendix) a similar argument is - not explicitly - used several times, although we report only the ratios among the variations in the reduced form probabilities which are considered.

2.3 Solving the revenue maximization problem

2.3.1 Results for the model with no synergies

In this subsection we briefly review the known results when there are no synergies in order to evaluate, later, the effects of $\alpha > 0$. Ar proves that, under the assumptions we made on the parameters, depending on the correlation degree between v and w the seller should use one of the two following mechanisms. In the first one the goods are sold separately as it occurs in two independent one-good auctions. For good m ($m = 1, 2$)

this implies that (i) if $n_m \geq 1$ buyers have (report) a high valuation for good m then each of them obtains it with probability $\frac{1}{n_m}$; (ii) if all the buyers value poorly object m then each buyer receives it with probability $\frac{1}{n}$. This is called mechanism I to recall it sells the goods through independent auctions.

The second mechanism displays some bundling. For any given good m, nothing changes with respect to separate auctions when at least one buyer has a high value for object m. If instead any buyer values poorly good m then two cases may occur: when all the buyers have type LL then each of them wins object m with probability $\frac{1}{n}$; when $n_{3-m} \geq 1$ buyers value highly good $3-m$ then object m is allocated among those buyers: each of them receives it with probability $\frac{1}{n_{3-m}}$. Therefore the probability to win good m for a buyer with a low valuation for that good is increasing in her value of good $3-m$. This is called mechanism B to recall it entails a degree of bundling. Corollary 1 in Ar shows that the choice between I and B is only determined by the correlation between v and w and does not depend on s:

Proposition 2.1 (Armstrong (2000)) *Let* $s \geq \frac{h+q}{l}$ *and* $\alpha = 0$. *Mechanism I is optimal if* $\frac{h}{2} \geq q\frac{h+q}{l+q}$ *(that is, if correlation between each* v^i *and* w^i *is positive and strong); if instead* $\frac{h}{2} < q\frac{h+q}{l+q}$ *then mechanism B is optimal.*

As we mentioned in the introduction, a mechanism is weakly efficient if *whenever the objects are sold* they are allocated in a way that maximizes social surplus. When $\alpha = 0$ the efficiency of a mechanism is judged object-by-object, as each buyer's gross payoff is the sum (over m) of her valuation for good m times the probability to obtain it. Hence, by proposition 2.1 the optimal mechanism is weakly efficient when $\alpha = 0$: in both mechanism I and B good m is always sold to a buyer with a high valuation for it if there is at least one such buyer in the auction.

When $\alpha > 0$, on the other hand, weak efficiency cannot be judged object-by-object because the synergy is generated if and only if the same buyer obtains both goods. When all the buyers have a same type, maximizing social surplus is equivalent to selling both goods to a same buyer in order to generate the synergy; if instead buyers' types are different then weak efficiency requires that:

(i) if $n_{HH} \geq 1$ (this means that at least one type HH participates in the auction) then both objects are sold to a same type HH;

(ii) if $n_{HH} = 0$, $n_{HL} \geq 1$ and $n_{LH} \geq 1$ (there is no type HH and at least one buyer's type is jk, $jk = HL, LH$) then both goods are allocated to a same buyer with type HL or LH if $\alpha > 1$;[11] if instead $\alpha \leq 1$ then good 1 is sold to a type HL and good 2 is sold to a type LH (recall that $\Delta = 1$);

(iii) if $n_{HL} \geq 1$, $n_{LL} \geq 1$ and $n_{HH} = n_{LH} = 0$ (there are only buyers with type HL or LL) then both goods are sold to a same buyer with type HL;

(iv) if $n_{LH} \geq 1$, $n_{LL} \geq 1$ and $n_{HH} = n_{HL} = 0$ (there are only buyers with type LH or LL) then both goods are allocated to a same type LH.

2.3.2 The subconstrained problem

Weierstrass' theorem can be used to prove that for all parameter values there exists a global maximum point in the seller's maximization problem.[12] To find it, we start by observing that the participation constraint of type LL, inequality (2.7), binds in the optimum (this can be proved as in a scalar model); then the incentive constraint which prevents type jk from reporting LL guarantees that type jk's participation constraint is met, $jk = HH, HL, LH$.

To deal with the incentive constraints, following Ar we consider a subconstrained maximization problem in which non-downward truthtelling constraints are absent. More clearly, we neglect all the incentive constraints but (2.2) to (2.6) - (2.2) to (2.6) prevent buyers with high valuation(s) from reporting low valuation(s) - and maximize the seller's

[11] Here weak efficiency does not discriminate between types HL and types LH since $v_H - v_L = w_H - w_L$.

[12] Indeed, setting $t_{jk} < 0$ for some jk is suboptimal because then each participation constraint would be slack and the seller could obtain a higher revenue by slightly (and uniformly) increasing each t_{jk}. Hence, we can safely assume that t_{jk} is bounded below (by 0) and above (by $v_j + w_k + \alpha$), as x_{jk}, y_{jk} and z_{jk} are, $j, k = L, H$.

expected revenue under these sole (incentive compatibility) constraints. The resulting subconstrained problem is called problem HH because it includes three constraints for type HH and one constraint each for type HL and type LH. It turns out that when the goods are symmetric, as we are assuming, the neglected constraints are satisfied at the solution of problem HH (we check this ex post); hence solving problem HH provides the solution to the original maximization problem as in the one-good two-type model neglecting the truthtelling constraint of the low type yields the solution to the complete problem.

Inequalities (2.5) and (2.6) bind in the optimum to problem HH (again, by Weierstrass' theorem there exists a solution to problem HH) since otherwise the seller could profitably increase t_{HL} and/or t_{LH}. From (2.5)-(2.7) written as equalities we find $t_{LL} = sx_{LL} + sy_{LL} + (2s+\alpha)z_{LL}$, $t_{HL} = (s+1)x_{HL} + sy_{HL} + (2s+1+\alpha)z_{HL} - x_{LL} - z_{LL}$ and $t_{LH} = sx_{LH} + (s+1)y_{LH} + (2s+1+\alpha)z_{LH} - y_{LL} - z_{LL}$ which we substitute into the (per buyer) expected revenue $\frac{R}{n}$ and into (2.2)-(2.4) to get, letting $p = (t_{HH}, x_{HH}, y_{HH}, z_{HH}, \ldots, x_{LL}, y_{LL}, z_{LL})$ [D is the set of feasible values for $(t_{HH}, x_{HH}, y_{HH}, z_{HH}, \ldots, x_{LL}, y_{LL}, z_{LL})$]

$$\max_{p\in D} ht_{HH} + q[(s+1)x_{HL} + sy_{HL} + (2s+1+\alpha)z_{HL} - x_{LL} - z_{LL}] +$$
$$q[sx_{LH} + (s+1)y_{LH} + (2s+1+\alpha)z_{LH} - y_{LL} - z_{LL}] + l[s(x_{LL} + y_{LL}) + (2s+\alpha)z_{LL}]$$

subject to

$$(s+1)x_{HH} + (s+1)y_{HH} + (2s+2+\alpha)z_{HH} - y_{HL} - z_{HL} - x_{LL} - z_{LL} \geq t_{HH} \quad (2.8)$$

$$(s+1)x_{HH} + (s+1)y_{HH} + (2s+2+\alpha)z_{HH} - x_{LH} - z_{LH} - y_{LL} - z_{LL} \geq t_{HH} \quad (2.9)$$

$$(s+1)x_{HH} + (s+1)y_{HH} + (2s+2+\alpha)z_{HH} - x_{LL} - y_{LL} - 2z_{LL} \geq t_{HH} \quad (2.10)$$

From the expressions of t_{HL}, t_{LH} and t_{LL} follows that the seller always extracts the synergic surplus from the buyers when it arises for type HL or LH or LL; the same moreover is true for type HH, since necessarily at least one among (2.8)-(2.10) binds in the solution to problem HH. Therefore, no type of buyer can ever appropriate the synergic surplus; that is not surprising as the value α of the synergy is common knowledge and it is common knowledge whether it is generated or not.[13] This provides the seller with some incentive to allocate both goods to a same buyer in order to gain the synergic surplus and such incentive is stronger the larger is α. This chapter basically investigates how that incentive modifies the optimal auction with respect to the case of $\alpha = 0$.

Letting λ_1 (λ_2 and λ_3, respectively) denote the multiplier for constraint (2.8) [(2.9) and (2.10), respectively] and $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, the lagrangian function for problem HH is

$$\begin{aligned} L(p,\lambda) &= ht_{HH} + (\lambda_1+\lambda_2+\lambda_3)[(s+1)(x_{HH}+y_{HH}) + (2s+2+\alpha)z_{HH} - t_{HH}] + \\ &\quad q(s+1)x_{HL} + (qs-\lambda_1)y_{HL} + [q(2s+1+\alpha)-\lambda_1]z_{HL} + (qs-\lambda_2)x_{LH} + \\ &\quad q(s+1)y_{LH} + [q(2s+1+\alpha)-\lambda_2]z_{LH} + (ls-q-\lambda_1-\lambda_3)x_{LL} \\ &\quad +(ls-q-\lambda_2-\lambda_3)y_{LL} + [l(2s+\alpha)-2q-\lambda_1-\lambda_2-2\lambda_3]z_{LL} \end{aligned}$$

Since this maximization problem is a linear programming problem, the well known saddle-point theorem [see theorem 1.D.5 in Takayama (1985)] applies to establish the following lemma, upon which we rely to find the solution to problem HH.

Lemma 2.1 (the saddle-point theorem) *For any $\alpha \in \Re$, $\bar{p} \in D$ solves problem HH if and only if there exists $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3) \geq 0$ such that $(\bar{p}, \bar{\lambda})$ is a saddle point for L, that is $L(p,\bar{\lambda}) \leq L(\bar{p},\bar{\lambda}) \leq L(\bar{p},\lambda)$ for any $p \in D$ and any $\lambda \in \Re^3_+$.*

[13] The same would occur if the synergic surplus depended on the buyer's type (that is, if α_{jk} were the surplus for type jk) as long as $(\alpha_{HH}, \alpha_{HL}, \alpha_{LH}, \alpha_{LL})$ were common knowledge. Assuming this, however, would make the analysis which follows more involved.

2.3.3 Solution of the model with positive synergies

In this section we exploit lemma 2.1 to find the solution to problem HH and then we prove it also solves the complete maximization problem.

Lemma 2.1 implies $\frac{\partial L}{\partial t_{HH}} = 0$ since t_{HH} lives in $\Re$; thus $\lambda_1 + \lambda_2 + \lambda_3 = h$. Using this result we prove that when at least one buyer has type HH then no other type should obtain any good. Specifically, both goods are randomly and fairly allocated to a same type HH if $n_{HH} \geq 1$.

Lemma 2.2 *For any parameter values with $\alpha > 0$, when at least a type HH participates in the auction ($n_{HH} \geq 1$) then each type HH receives both goods with probability $\frac{1}{n_{HH}}$.*

Lemma 2.2 implies that, when $\alpha > 0$, nor mechanism I or mechanism B ever solves problem HH. The reason is that both of them display "independence at the top", in the sense that if $n_1 \geq 2$ buyers value highly good 1 then each of them receives it with probability $\frac{1}{n_1}$, neglecting their valuations for good 2. However, since $\alpha > 0$, if these buyers' values for object 2 differ then it is better to sell both goods to a same buyer with type HH; in this way no binding constraint is tightened and the synergic surplus is extracted with probability 1. On the contrary, in mechanisms I and B good 1 is allocated among types HH and HL. Ar proves that when $\alpha = 0$ the seller never gains - in the subconstrained problem - from letting the probability to win good 1 (2) for type HH differ with respect to type HL (LH); when $\alpha > 0$, instead, lemma 2.2 says that there is a strict incentive to distort these probabilities in favor of type HH.[14]

Before completely describing a mechanism it is useful to observe that, as the goods are always sold (because $s \geq \frac{h+q}{l}$), if all the buyers report a same type jk then both objects are allocated to a same buyer since the coefficient of z_{jk} in the lagrangian function is larger than the sum between the coefficients of x_{jk} and y_{jk}, $jk = HL, LH, LL$. Moreover, lemma 2.2 describes the optimal sale policy when at least one buyer's type is HH; hence

[14]Actually, when $\alpha = 0$ non-distorted probabilities help in making the solution to problem HH a solution to the complete problem for the largest range of parameter values; when the goods are symmetric there is no such an effect.

the remaining degrees of freedom in defining a mechanism concern the profiles of reports of types such that $n_{HH} = 0$ and at least two buyers have different types. Thus, we describe a mechanism only by specifying how the goods are allocated when the different types showing up in the auction are HL and LH; HL and LL; LH and LL; HL, LH and LL. In doing that, we keep in mind that for any mechanism introduced below it is understood that (i) when $n_{HH} \geq 1$ a randomly selected buyer of type HH obtains both goods and (ii) if $n_{jk} = n$ for some jk then a randomly selected buyer receives both goods.

The following two mechanisms are in a sense linked to I and B (introduced in subsection 2.3.1), respectively; because of this fact we denote them I1 and B1.

Mechanism I1 If $n_{HL} \geq 1$, $n_{LH} \geq 1$ and $n_{HH} = 0$ (both types HL and LH show up in the auction, possibly together with type LL), then good 1 is (randomly) allocated among types HL and item 2 is allocated among types LH: each type HL obtains good 1 with probability $\frac{1}{n_{HL}}$ and any type LH wins object 2 with probability $\frac{1}{n_{LH}}$.

If $n_{HL} \geq 1$, $n_{LL} \geq 1$ and $n_{LH} = n_{HH} = 0$ (only types HL and LL are present) then good 2 is (randomly) allocated among *all* the buyers; if it is received by a type HL then the same buyer also wins good 1; if instead a type LL obtains good 2 then good 1 is randomly allocated among types HL. Thus each type LL wins good 2 with probability $\frac{1}{n}$; each type HL wins both objects with probability $\frac{1}{n}$ and she wins only good 1 with probability $\frac{n_{LL}}{n}\frac{1}{n_{HL}}$.

Similarly, if $n_{LH} \geq 1$, $n_{LL} \geq 1$ and $n_{HL} = n_{HH} = 0$ then good 1 is allocated among all the buyers; if it is obtained by a type LH then the same buyer also wins object 2, otherwise good 2 is allocated among types LH.

Mechanism B1 If $n_{HL} \geq 1$, $n_{LH} \geq 1$ and $n_{HH} = 0$ then good 1 is allocated among types HL and good 2 is allocated among types LH, exactly as in I1.

If $n_{HL} \geq 1$, $n_{LL} \geq 1$ and $n_{LH} = n_{HH} = 0$ then both goods are sold to a same type HL. Similarly, when $n_{LH} \geq 1$, $n_{LL} \geq 1$ and $n_{HL} = n_{HH} = 0$ both objects are allocated to a same type LH.

Notice that I1 is not weakly efficient if $\alpha > 0$: when only types HL and LL (LH and LL) are in the auction, with positive probability the goods are *not* sold to a same type HL (LH). Mechanism B1, on the other hand, is weakly efficient if and only if $\alpha \leq 1$.

Lemma 1 in Ar helps in computing the values of $\{x_{jk}, y_{jk}, z_{jk}\}_{jk=HL,LH,LL}$ for I1 and B1. The same lemma and our lemma 2.2 imply $x_{HH} = 0$, $y_{HH} = 0$ and $z_{HH} = \frac{1-(1-h)^n}{nh}$ in I1, B1 and in any other mechanism which is described below.

Mechanism I1

$$x_{HL} = \frac{(1-h)^n - (1-h)(l+q)^{n-1}}{nq} \qquad y_{HL} = 0 \qquad z_{HL} = \frac{(l+q)^{n-1}}{n}$$

$$x_{LH} = 0 \qquad y_{LH} = \frac{(1-h)^n - (1-h)(l+q)^{n-1}}{nq} \qquad z_{LH} = \frac{(l+q)^{n-1}}{n}$$

$$x_{LL} = \frac{(l+q)^{n-1} - l^{n-1}}{n} \qquad y_{LL} = \frac{(l+q)^{n-1} - l^{n-1}}{n} \qquad z_{LL} = \frac{l^{n-1}}{n}$$

Mechanism B1

$$x_{HL} = \frac{(1-h)^n - 2(l+q)^n + l^n}{nq} \qquad y_{HL} = 0 \qquad z_{HL} = \frac{(l+q)^n - l^n}{nq}$$

$$x_{LH} = 0 \qquad y_{LH} = \frac{(1-h)^n - 2(l+q)^n + l^n}{nq} \qquad z_{LH} = \frac{(l+q)^n - l^n}{nq}$$

$$x_{LL} = 0 \qquad y_{LL} = 0 \qquad z_{LL} = \frac{l^{n-1}}{n}$$

Mechanisms I1 and B1 are somewhat linked to I and B, respectively, because - when $n_{HH} = 0$ - for any given type $jk = HL, LH, LL$ of buyer participating in the auction the probability to win good m (either alone or with object $3-m$) given her opponents' types is the same in I as in I1 and in B as in B1. The difference is that *given* these probabilities, in I1 and B1 it is maximized the probability that a same buyer wins both goods; clearly, the synergic effect is the root of this result as it delivers lemma 2.2. The same principle applies to the mechanisms which are introduced below: given the probability that type jk ($j, k = L, H$) has to obtain object m ($m = 1, 2$), it is maximized the probability that both goods are allocated to a same buyer.

It is worthwhile to observe, however, that if $\alpha = 0$ then I1 (B1) is optimal when I (B) is optimal. To prove this claim it is sufficient to verify that (i) in I1 and B1 (as in I and B) good m is allocated to a buyer with a high value for it if such a buyer participates in the auction; (ii) the probability to win good 1 for a buyer with type LH or LL is the same in I1 (B1) as in I (B); (iii) a similar result holds for good 2 and types HL and LL.[15]

As lemma 2.3 below establishes, for large values of α it is quite often convenient to sell both goods to a same buyer; hence nor I1 or B1 ever solves problem HH when α is large and the three following mechanisms are needed.

Mechanism WI1 If $n_{HL} \geq 1$, $n_{LH} \geq 1$ and $n_{HH} = 0$ then the goods are allocated exactly as in I1.

If $n_{HL} \geq 1$, $n_{LL} \geq 1$ and $n_{LH} = n_{HH} = 0$ then both goods are sold to a same buyer. With probability $\theta \in (0, 1)$ the group of buyers with type HL is selected and with probability $1 - \theta$ the group of types LL is selected;[16] within the selected group the buyer winning both goods is randomly chosen. Thus each type HL obtains both goods with probability $\frac{\theta}{n_{HL}}$; for each type LL such probability is $\frac{1-\theta}{n_{LL}}$.

When $n_{LH} \geq 1$, $n_{LL} \geq 1$ and $n_{HL} = n_{HH} = 0$ a similar allocation rule is adopted: each type LH (LL) wins both goods with probability $\frac{\theta}{n_{LH}}$ ($\frac{1-\theta}{n_{LL}}$).

In the next two mechanisms the goods are *always* sold as a single unit.

Mechanism B2 If $n_{HL} \geq 1$, $n_{LH} \geq 1$ and $n_{HH} = 0$, then any buyer with type HL or LH wins both goods with probability $\frac{1}{n_{HL}+n_{LH}}$. If instead only types HL and LL (LH and LL) are in the auction, then any type HL (LH) receives both items with probability $\frac{1}{n_{HL}}$ ($\frac{1}{n_{LH}}$).

Mechanism WI2 If $n_{HH} = 0$ then the goods are allocated to a same buyer who is randomly selected among *all* the buyers. In other words, when $n_{HH} = 0$ each buyer obtains both objects with probability $\frac{1}{n}$ *independently* of her own type.

[15] More briefly, by using the notation in Ar it is sufficient to verify that the following inequalities are satisfied both for I, I1 and B, B1: $h(x_{HH} + z_{HH}) + q(x_{HL} + z_{HL}) = h(y_{HH} + z_{HH}) + q(y_{LH} + z_{LH}) = \frac{1-(l+q)^n}{n}$ [condition (i)]; $x_{LH} + z_{LH} = \rho^A_{LH}$, $x_{LL} + z_{LL} = \rho^A_{LL}$ [condition (ii)]; $y_{HL} + z_{HL} = \rho^B_{HL}$, $y_{LL} + z_{LL} = \rho^B_{LL}$ [condition (iii)].

[16] The value of θ is such that each of the three constraints (2.8)-(2.10) binds. Details are provided in the proof to lemma 2.3(iii) in the appendix.

In B2 the objects are always allocated to a same buyer with the highest realized sum of valuations (hence no type LL ever wins any good unless $n_{LL} = n$). Because of this reason, Ar calls this mechanism "the pure bundling auction"[17] and proves that it is never optimal in his setting. The reason is that the optimal auction is weakly efficient if $\alpha = 0$, while B2 is not so if $\alpha = 0$: when $n_{HL} \geq 1$, $n_{LH} \geq 1$ and $n_{HH} = 0$ it generates a surplus equal to $\max\{v_L + w_H, v_H + w_L\}$ which is smaller than $v_H + w_H$, the surplus generated if good 1 is sold to a type HL and good 2 is sold to a type LH. Clearly, when $\alpha > 0$ is large B2 has chances to be optimal because it always generates and extracts the synergic surplus.

Mechanism B2 is weakly efficient when $\alpha > 1$, while that is true for B1 if $\alpha \leq 1$. Mechanisms WI1 and WI2, on the other hand, are never weakly efficient as they allocate with positive probability both goods to a type LL even though her opponents' types are HL or LH. The reduced form probabilities for B2, WI1 and WI2 are as follows

Mechanism B2

$$\begin{array}{lll} x_{HL} = 0 & y_{HL} = 0 & z_{HL} = \frac{(1-h)^n - l^n}{2qn} \\ x_{LH} = 0 & y_{LH} = 0 & z_{LH} = \frac{(1-h)^n - l^n}{2qn} \\ x_{LL} = 0 & y_{LL} = 0 & z_{LL} = \frac{l^{n-1}}{n} \end{array},$$

Mechanism WI1

$$\begin{array}{lll} x_{HL} = \frac{(1-h)^n - 2(l+q)^n + l^n}{nq} & y_{HL} = 0 & z_{HL} = \frac{2(q+l)^n - l^n}{n(1-h)} \\ x_{LH} = 0 & y_{LH} = \frac{(1-h)^n - 2(l+q)^n + l^n}{nq} & z_{LH} = \frac{2(q+l)^n - l^n}{n(1-h)} \\ x_{LL} = 0 & y_{LL} = 0 & z_{LL} = \frac{2(q+l)^n - l^n}{n(1-h)} \end{array}$$

[17] Palfrey (1983) assumes continuously distributed valuations for $G \geq 2$ goods and no synergies. He compares the English auction for the bundle of G goods (the pure bundling auction) to G separate English auctions. It turns out that for small values of n the seller prefers to bundle the goods rather than selling them separately. Chakraborty (1999) obtains further results for the case of $G = 2$.

Mechanism WI2

$$\begin{array}{lll} x_{HL}=0 & y_{HL}=0 & z_{HL}=\frac{(1-h)^{n-1}}{n} \\ x_{LH}=0 & y_{LH}=0 & z_{LH}=\frac{(1-h)^{n-1}}{n} \\ x_{LL}=0 & y_{LL}=0 & z_{LL}=\frac{(1-h)^{n-1}}{n} \end{array}$$

We can now state lemma 2.3 which describes the solution to problem HH.

Lemma 2.3 *Let* $s \geq \frac{h+q}{l}$ *and* $\alpha \geq 0$. *(i) Mechanism I1 solves problem HH if* $\alpha \leq \min\left\{\frac{(h+q)l-q}{2ql}, \frac{1}{1-h}\right\}$ *[notice that* $\frac{(h+q)l-q}{2ql} \leq \frac{1}{1-h}$ *if and only if* $hl \leq 2q$*].*

(ii) Assume $hl \leq 2q$*; then mechanism B1 is optimal in problem HH if* $\frac{(h+q)l-q}{2ql} < \alpha \leq 1+\frac{h}{2q}$ *and B2 solves problem HH if* $\alpha > 1+\frac{h}{2q}$*.*

(iii) Let $hl > 2q$*; then mechanism WI1 solves problem HH if* $\frac{1}{1-h} < \alpha \leq \frac{2}{1-h}$ *and WI2 is optimal in problem* HH *if* $\alpha > \frac{2}{1-h}$*.*

Given the solution to problem HH, to prove that it also solves the complete (not subconstrained) maximization problem we need to check that all of mechanisms mentioned in lemma 2.3 satisfy the incentive constraints which have been neglected in problem HH. Actually, this is always the case: in each of the above mechanisms those constraints hold even though they have not been imposed when solving problem HH.[18] As a consequence we have the following

Proposition 2.2 *For any parameter values such that* $s \geq \frac{h+q}{l}$ *and* $\alpha \geq 0$ *the solution to problem* HH *also solves the complete revenue maximization problem. Hence, lemma 2.3 describes the optimal auction as a function of the parameter values.*

2.3.4 Comments

A first remark about the above results is that B1 or B2 is optimal if $(h+q)l \leq q$ as that implies $hl < 2q$ and $\min\left\{\frac{(h+q)l-q}{2ql}, \frac{1}{1-h}\right\} \leq 0$. Since $hl > q(1-l)$ requires that the

[18] As we anticipated above, this result is delivered by the assumption of symmetric goods and would hold even if s were not larger than $\frac{h+q}{l}$.

correlation between each v^i and w^i is positive and sufficiently strong, we conclude that the pure bundling auction B2 or a weaker form of it (B1) is optimal when correlation is weak or negative; hence, also under independently distributed valuations.

As we remarked in subsection 2.3.1, when $\alpha = 0$ the choice between mechanism I and mechanism B only depends on the correlation in the probability distribution and not on s.[19] On the other hand, under positive synergies the buyers' preferences, as represented by the parameter α, affect the format of the optimal auction. It is also worthwhile to notice that the number of buyers does not matter in determining the optimal auction. Yet, it seems reasonable to conjecture that - as in Ar - n would matter if the goods were very asymmetric.

Lemma 2.3(ii)-(iii) establishes that when α is sufficiently large and the buyers' types are LH and HL then selling good 1 to a type HL and good 2 to a type LH is not a good idea as the synergic surplus is not generated and the seller cannot extract it; allocating the two objects to a same buyer is more profitable. Since increasing z_{HL} and z_{LH} tightens constraints (2.8) and (2.9), when $n_{HL} \geq 1$, $n_{LH} \geq 1$ and $n_{HH} = 0$ the seller does not bundle the goods if just $\alpha > 1$ as weak efficiency requires, but only if $\alpha > \min\left\{\frac{2}{1-h}, 1 + \frac{h}{2q}\right\}$.

The results in lemma 2.3(iii) are maybe more surprising: If $\alpha > \frac{1}{1-h}$ and correlation is sufficiently strong and positive ($hl > 2q$) then a type LL receives both goods with positive probability when the other buyers' types are HL or LH (mechanisms WI1 and WI2). This may look strange, since the surplus produced from selling the objects to a type LL is "obviously" smaller with respect to selling them to a type HL or LH. Ar considers an example in which the buyers' valuations are continuously distributed over two rays in the positive orthant of $\Re^2$ and there are no synergies; in that case good m is inefficiently allocated if the buyers' valuations for good $3-m$ are sufficiently different. In mechanisms WI1 and WI2, instead, good 1 (2) is inefficiently allocated when $n_{HL} \geq 1$,

[19] Armstrong and Rochet (1999) obtain a similar result in a bi-dimensional screening model in which the planner faces a unique agent.

$n_{LL} \geq 1$ and $n_{HH} = n_{LH} = 0$ ($n_{LH} \geq 1$, $n_{LL} \geq 1$ and $n_{HH} = n_{HL} = 0$), which means that all the buyers' valuations for good 2 (1) are low (recall that here each valuation has a binary support).

To get a simple intuition of why this selling policy may maximize revenue assume $hl > 2q$ and $\alpha > 1 + \frac{h}{2q}$ (hence $\alpha > \frac{2}{1-h}$). Lemma 2.3(ii) states that B2 is suboptimal as $hl > 2q$; we now show why WI2 is superior to B2 without using saddle-point arguments. In B2, constraints (2.8) and (2.9) bind while (2.10) is slack: type HH strictly prefers to reveal her own type rather than reporting LL but she is indifferent between a truthful report and announcing HL or LH. When $n_{HL} \geq 1$, $n_{LL} \geq 1$ and $n_{HH} = n_{LH} = 0$ ($n_{LH} \geq 1$, $n_{LL} \geq 1$ and $n_{HH} = n_{HL} = 0$) B2 allocates both goods to a same type HL (LH). Now consider moving away from B2 by selling the objects with positive probability to a same type LL rather than to a type HL (LH). This entails a reduction in z_{HL} (z_{LH}) and an increase in z_{LL}; more precisely, $\Delta z_{HL} = \Delta z_{LH} = -\varepsilon$ and $\Delta z_{LL} = \frac{2q}{l}\varepsilon$ for some $\varepsilon > 0$ (we are exploiting the argument introduced at the end of subsection 2.2.2). As a consequence, the left hand side of both (2.8) and (2.9) is increased and t_{HH} increases as (2.10) was initially slack: $\Delta t_{HH} = (1 - 2\frac{q}{l})\varepsilon > 0$ (as $hl > 2q$). Moreover, from (2.5)-(2.7) written as equalities we see that both t_{HL} and t_{LH} decrease while t_{LL} increases: since types HL and LH receive less goods in expected value, the payment which can be extracted from them is smaller; the opposite argument applies to type LL. Indeed, $\Delta t_{HL} = \Delta t_{LH} = -(2s+\alpha+1)\varepsilon - \frac{2q}{l}\varepsilon < 0$ and $\Delta t_{LL} = (2s+\alpha)\frac{2q}{l}\varepsilon > 0$. The change in the expected revenue per buyer is $\Delta(\frac{R}{n}) = h\Delta t_{HH} + q(\Delta t_{HL} + \Delta t_{LH}) + l\Delta t_{LL} = \frac{\varepsilon}{l}(hl - 2q)$; thus $\frac{R}{n}$ increases if we move away from B2 toward WI2 by slightly increasing ε above 0 as we assumed $hl > 2q$ (similar arguments can be put forward when comparing WI1 to B1). Clearly, $hl > 2q$ if and only if q is small enough with respect to h and l, which says that in expected value the reductions in t_{HL} and t_{LH} are more than counterbalanced by the increases in t_{HH} and t_{LL}.

Basically, therefore, the problem of minimizing the cost of the incentive constraints (2.8)-(2.10) induces an inefficient allocation of the goods when $hl > 2q$. Notice that

$hl > 2q$ does not imply that it is optimal to sell both goods with probability 1 to a type LL when $n_{HL} \geq 1$, $n_{LL} \geq 1$ and $n_{HH} = n_{LH} = 0$ and when $n_{LH} \geq 1$, $n_{LL} \geq 1$ and $n_{HH} = n_{HL} = 0$. The reason is that in such a case z_{LL} would be larger than z_{HL} and z_{LH}, hence (2.10) would bind while (2.8) and (2.9) would not. Then it would be possible to increase t_{HH}, t_{HL} and t_{LH} by slightly reducing z_{LL} and increasing z_{HL} and z_{LH}; $\frac{R}{n}$ would be higher because the associated decrease in t_{LL} would not counterbalance these increases.

Since $hl > 2q$ is a condition which does not depend on α, it should be explained why the weakly inefficient mechanisms WI1 and WI2 are never optimal when $\alpha = 0$. The reason is that the seller has no incentive to allocate *both* goods with positive probability to a type LL when $\alpha = 0$: no synergic surplus is lost by reducing z_{HL} and z_{LH} and simultaneously increasing x_{HL}, y_{LL}, y_{LH} and x_{LL}; when $hl > 2q$ the seller would better follow this strategy, converging to mechanism I ($hl > 2q$ implies $\frac{h}{2} > q\frac{h+q}{l+q}$, see proposition 2.1).

An alternative way of explaining the optimality (for some parameter values) of the weakly inefficient mechanism WI2 exploits a simple remark: When the goods are always sold as a single item (because α is large), an only object - the bundle - is on sale and each buyer's private information is summarized by a one-dimensional variable: her valuation for the bundle. If b^i is buyer i's valuation ($i = 1, \ldots, n$) for the bundle then $b^i \in \{b_L, b_M, b_H\} = \{2s + \alpha, 2s + 1 + \alpha, 2s + 2 + \alpha\}$ with $p(b_L) = l$, $p(b_M) = 2q$ and $p(b_H) = h$.[20] Let z_u denote the (reduced form) probability for a buyer reporting b_u to win the bundle, $u = L, M, H$. As it is well known, any incentive compatible mechanism satisfies the monotonicity constraints $z_H \geq z_M \geq z_L$. The techniques developed in Myerson (1981) can be adapted to this setting to prove that if the virtual valuation function $J(b) = b - \frac{1-P(b)}{p(b)}$ is increasing in b, then the buyer with the highest realized valuation should obtain the bundle whenever it is sold and this implies that the monotonicity

[20] We use p to denote the probability mass function for each b^i; P is the cumulative distribution function: $P(b_L) = l$, $P(b_M) = l + 2q$ and $P(b_H) = 1$.

constraints ($z_H \geq z_M \geq z_L$) are met. If instead J is not monotone increasing then the seller should bunch different types (each type in the bunching region has the same probability to receive the bundle); therefore with positive probability the bundle is inefficiently allocated.

It turns out that $J(b_H) = 2s+2+\alpha$, $J(b_M) = 2s+1+\alpha-\frac{h}{2q}$ and $J(b_L) = 2s+\alpha-\frac{1-l}{l}$; thus the virtual valuation function is monotone if and only if $J(b_M) \geq J(b_L)$ which is equivalent to $2q \geq hl$. Indeed, lemma 2.3(ii) establishes that when $2q \geq hl$ (and α is large) the pure bundling auction B2 is used; hence the bundle is obtained by a buyer with the highest realized valuation for it, which means that it is efficiently allocated. If instead $hl > 2q$ then $J(b_M) < J(b_L)$; indeed, by lemma 2.3(iii) (when α is large) WI2 is optimal, in which types b_M (types HL and LH in the two-good model) are bunched with types b_L (types LL in the two-good model): here the objects are not efficiently allocated. Observe, however, that the goods are inefficiency allocated when $n_{HL} \geq 1$, $n_{LL} \geq 1$ and $n_{HH} = n_{LH} = 0$ or $n_{LH} \geq 1$, $n_{LL} \geq 1$ and $n_{HH} = n_{HL} = 0$ *also* if α is not so large that they are always sold as a single object (provided $hl > 2q$), as it occurs when mechanism WI1 is optimal.

Up to now we assumed that objects are symmetric. It turns out that relaxing this assumption makes "more likely" that the optimal mechanism is weakly inefficient if the goods are always sold as a single item. To fix the ideas, let $\Delta v > \Delta w$, $q_1 = \Pr\{(v,w) = (v_H, w_L)\}$ and $q_2 = \Pr\{(v,w) = (v_L, w_H)\}$. In a scalar model, if valuations are discretely distributed with support $\{b_1, \ldots, b_U\}$ ($b_u < b_{u+1}$) then the virtual valuation $J(b_u)$ is equal to $b_u - (b_{u+1} - b_u)\frac{1-P(b_u)}{p(b_u)}$. Since $b_{u+1} - b_u$ may vary with u, a monotone hazard rate does not always imply that J is monotone increasing. Given that $\Delta v > \Delta w$, the valuations for the bundle are $b_1 = v_L + w_L + \alpha$, $b_2 = v_L + w_H + \alpha$, $b_3 = v_H + w_L + \alpha$ and $b_4 = v_H + w_H + \alpha$. The virtual valuation function is monotone increasing if and only if $J(b_3) \geq J(b_2) \geq J(b_1)$, which occurs if and only if $q_2\Delta w \geq l(h + q_1)(\Delta v - \Delta w)$ and $q_1(1-l)(\Delta v - \Delta w) \geq hq_2\Delta w$. It is easy to see that here strong and positive correlation in the distribution of (v,w) is not required in order for the optimal mechanism to be weakly inefficient. For instance, let $h = q_1 = q_2 = l = \frac{1}{4}$. Under this distribution v is independent of w and it is optimal to allocate the bundle efficiently if and only if $\frac{\Delta v}{3} \leq \Delta w \leq \frac{3\Delta v}{4}$. In this example the hazard rate $\frac{1-P(b)}{p(b)}$ is monotone decreasing, yet the virtual valuation function is not monotone increasing if the ratio $\frac{\Delta w}{\Delta v}$ is smaller than $\frac{1}{3}$ or larger than $\frac{3}{4}$.

2.4 Conclusions

This chapter analyzed optimal two-object auctions when each buyer's utility is super-additive. A first result is that many degrees of freedom existing in the model with no synergies disappear as positive synergies provide an incentive for the seller to allocate both objects to a same buyer. Formally, in any optimal mechanism, if good 1 (2) is allocated within a given set S_1 (S_2) of buyers according to a given probability distribution p_1 (p_2), then it is maximized the probability that both goods are sold to a same buyer *given* S_1, S_2, p_1 and p_2. Furthermore, for any $\alpha > 0$ the goods are always sold as a single item to a type HH when such a type of buyer is in the auction. For these reasons no mechanism put forward in Ar when $\alpha = 0$ is optimal if $\alpha > 0$: in those mechanisms the probability of generating and extracting the synergic surplus is suboptimally low. However, the optimal mechanisms when α is positive but close to 0 are optimal also if $\alpha = 0$: by the maximum theorem, the solution to the revenue maximization problem is upper-hemi-continuous with respect to α.

The optimal mechanism is often not weakly efficient. Specifically, I1 is optimal when $\alpha > 0$ is small (under strong and positive correlation) even though it generates too rarely the synergic surplus. When α is large, WI2 (or WI1) is optimal (still under strong and positive correlation) even though a type LL may win both goods when facing types HL or LH. Thus, while Ar shows that weak efficiency is consistent with revenue maximization in a two-object auction if the valuations have binary supports, we find that such a result is not robust to the presence of synergies. The weak inefficiency of WI2 and WI1 can be viewed as due to the interplay among the incentive constraints for type HH. However, as we stressed in subsection 2.3.4, synergies make the model closer to a single-object setting, for which we know that inefficiency arises when the virtual valuation function is not monotone; that is "more likely" to occur if the goods are asymmetric.

In sum, how should two goods be sold when the buyers' utilities for them are super-additive? When α is quite large and each buyer's valuations are negatively or weakly correlated then both goods are allocated to a same buyer with the highest realized sum of valuations (mechanism B2). If instead correlation is strong and positive then it is optimal to bunch types HL, LH and LL (WI2); this reduces the consumption of types HL and LH with respect to B2 and any type LL obtains the goods with positive probability not only when $n_{LL} = n$. When α is not large the above sale policies are amended (depending on the correlation degree) by not always bundling the goods. This sometimes implies "underproduction" of the synergic surplus.

The present model may be extended along many directions, for example by allowing for more than two possible valuations for each good. Solving the problem with continuously distributed values would be very interesting but it appears difficult even though some results exist when there are no synergies. An interesting extension may allow for more than two objects and different synergies depending on how many and which goods are obtained by the same buyer. This may capture the differences between local synergies and global synergies, a perceived dichotomy in the FCC auction [see Ausubel *et al.* (1997)].

2.5 Appendix

Proof of lemma 2.2 We prove that if a mechanism is such that for some profile of buyers' reports with $n_{HH} \geq 1$ no type HH obtains both goods then the value of the lagrangian function L can be increased; therefore, by lemma 2.1 the mechanism does not solve problem HH.

Suppose first that for some profile of buyers' reports with $n_{HH} \geq 1$ no type HH receives any good. That may occur because (i) no good is sold at all; (ii) only one good, say good 1, is sold, say to a type HL; (iii) the goods are sold to different types of buyer, say good 1 to a type HL and good 2 to a type LH; (iv) both goods are sold to a same buyer, say to a type HL.

In any of the above cases the value of L is increased by selling both goods to a same buyer of type HH. In case (i) this is obvious, since z_{HH} is increased and $\frac{\partial L}{\partial z_{HH}} > 0$; in case (ii), allocating the goods to a same type HH rather than good 1 to a type HL increases z_{HH} by some $\varepsilon > 0$ and decreases x_{HL} by $\frac{h}{q}\varepsilon$: $\Delta L = \varepsilon h(2s + 2 + \alpha) - \frac{h}{q}\varepsilon q(s + 1) = \varepsilon h(s + 1 + \alpha) > 0$. In case (iii), $\Delta z_{HH} = \varepsilon > 0$ and $\Delta x_{HL} = \Delta y_{LH} = -\frac{h}{q}\varepsilon$; hence $\Delta L = \varepsilon h(2s+2+\alpha) - 2\frac{h}{q}\varepsilon q(s+1) = \varepsilon h\alpha > 0$. Last, in case (iv) we have $\Delta z_{HH} = \varepsilon > 0$ and $\Delta z_{HL} = -\frac{h}{q}\varepsilon$, thus $\Delta L = \varepsilon h(1 + \frac{\lambda_1}{q}) > 0$.

Now assume that for some profile of buyers' reports with $n_{HH} \geq 1$ some type HH receives an only good, say good 1; again, we show that the value of L can be increased by allocating both goods to a same type HH. There are three possible cases: (i) good 2 is not sold at all; (ii) good 2 is sold to a type HH who is not the same buyer receiving good 1; (iii) good 2 is allocated to a buyer with a different type, say a type LH. Now we argue (about) as above: in case (i) z_{HH} is increased by some $\varepsilon > 0$ and x_{HH} is decreased by ε: $\Delta L = h\varepsilon(s+1+\alpha) > 0$. In case (ii) we set $\Delta z_{HH} = \varepsilon > 0$, $\Delta x_{HH} = -\varepsilon$ and $\Delta y_{HH} = -\varepsilon$; hence $\Delta L = h\varepsilon\alpha > 0$. Finally, in case (iii) it is $\Delta z_{HH} = \varepsilon > 0$, $\Delta x_{HH} = -\varepsilon$ and $\Delta y_{LH} = -\frac{h}{q}\varepsilon$ (this is the example which was examined at the end of subsection 2.2.2); thus $\Delta L = h\varepsilon\alpha > 0$. ■

Next lemma helps in proving lemma 2.3 by providing the conditions under which different allocations are optimal in problem HH when two or three different types of buyer show up in the auction.

Lemma 2.4 *(i) If* $n_{HL} \geq 1$, $n_{LH} \geq 1$ *and* $n_{HH} = n_{LL} = 0$ *then*

- *(a) sell good* 1 *among buyers of type HL and good* 2 *among types LH if* $\min\{\lambda_1, \lambda_2\} \geq q(\alpha - 1)$

- *(b) sell both goods to a same buyer of type HL (LH) if* $\lambda_1 \leq \min\{\lambda_2, q(\alpha-1)\}$ ($\lambda_2 \leq \min\{\lambda_1, q(\alpha-1)\}$)

- *(c) sell both goods to a same buyer of type HL or LH if* $\lambda_1 = \lambda_2 \leq q(\alpha-1)$.

(ii) If $n_{HL} \geq 1$, $n_{LL} \geq 1$ *and* $n_{HH} = n_{LH} = 0$ *then*

- *(a) sell both goods to a same type HL if* $(l+q)\lambda_1 + q\lambda_2 \leq q(1+h)$ *and* $(l+q)\lambda_1 \leq q(\alpha l + h + q)$

- *(b) allocate both goods to a same buyer which is selected with probability* $\theta \in (0,1)$ *among types HL and with probability* $1-\theta$ *among types LL if* $(l+q)\lambda_1 + q\lambda_2 = q(1+h)$ *and* $(l+q)\lambda_1 \leq q(\alpha l + h + q)$

- *(c) if* $(l+q)\lambda_1 = q(\alpha l + h + q)$ *and* $(l+q)\lambda_1 + q\lambda_2 \leq q(1+h)$ *then allocate good 2 randomly among all the buyers; if it is received by a type HL then the same buyer also wins good 1; if instead a type LL obtains good 2 then good 1 is allocated among types HL.*

(iii) If $n_{LH} \geq 1$, $n_{LL} \geq 1$ *and* $n_{HH} = n_{HL} = 0$ *then*

- *(a) sell both goods to a same type LH if* $(l+q)\lambda_2 + q\lambda_1 \leq q(1+h)$ *and* $(l+q)\lambda_2 \leq q(\alpha l + h + q)$

- *(b) allocate both goods to a same buyer which is selected with probability* $\theta \in (0,1)$ *among types LH and with probability* $1-\theta$ *among types LL if* $(l+q)\lambda_2 + q\lambda_1 = q(1+h)$ *and* $(l+q)\lambda_2 \leq q(\alpha l + h + q)$

- *(c) if* $(l+q)\lambda_2 = q(\alpha l + h + q)$ *and* $(l+q)\lambda_2 + q\lambda_1 \leq q(1+h)$ *then allocate good 1 randomly among all the buyers; if it is received by a type LH then the same buyer also wins good 2; if instead a type LL obtains good 1 then good 2 is allocated among types LH.*

(iv) If $n_{HL} \geq 1$, $n_{LH} \geq 1$, $n_{LL} \geq 1$ *and* $n_{HH} = 0$ *then*

- *(a) allocate good* 1 *among types* HL *and good* 2 *among types* LH *if* $\min\{\lambda_1, \lambda_2\} \geq q(\alpha-1)$ *and* $\lambda_1+\lambda_2 \leq 1+l+h-\alpha l$

- *(b) sell both goods to a same type* HL *or* LH *if* $\lambda_1=\lambda_2 \leq q(\alpha-1)$ *and* $(l+q)\lambda_1+q\lambda_2 \leq q(1+h)$

- *(c) allocate both goods to a same type* HL *or* LH *or* LL *if* $\lambda_1=\lambda_2 \leq q(\alpha-1)$ *and* $(l+q)\lambda_1+q\lambda_2 = q(1+h)$.

Proof. (ia) Suppose good 1 is allocated among types HL and good 2 is sold among types LH; this is the best way of allocating the objects to buyers with different types given that $n_{HL} \geq 1$, $n_{LH} \geq 1$ and $n_{HH}=n_{LL}=0$. Increasing the probability to sell the goods as a single object to a type HL requires to decrease both x_{HL} and y_{LH} by $\varepsilon > 0$ and to increase z_{HL} by ε. Since $\Delta L = [q(\alpha-1)-\lambda_1]\varepsilon$, this is not profitable if $\lambda_1 \geq q(\alpha-1)$. Similarly, if $\lambda_2 \geq q(\alpha-1)$ then the seller should not reduce the probability that types HL obtain object 1 to increase the probability that a same type LH receives both goods.

(ib,ic) Reasoning as in part (ia), if $z_{HL} > 0$ ($z_{LH} > 0$) then reducing z_{HL} (z_{LH}) and increasing x_{HL} and y_{LH} is not profitable if $\lambda_1 \leq q(\alpha-1)$ [$\lambda_2 \leq q(\alpha-1)$]; reducing z_{HL} (z_{LH}) to increase z_{LH} (z_{HL}) decreases L if $\lambda_1 \leq \lambda_2$ ($\lambda_2 \leq \lambda_1$).

(iia) We consider three alternatives to the policy of selling both objects to a same type HL: (i) selling both goods to a same type LL with positive probability; (ii) selling only item 2 among types LL (therefore allocating only good 1 among types HL) with positive probability; (iii) selling only good 1 to a type LL and only object 2 to a type HL with positive probability. The first alternative is implemented by reducing z_{HL} by $\varepsilon > 0$ and increasing z_{LL} by $\frac{q}{l}\varepsilon$; this implies $\Delta L = -\varepsilon[q(2s+1+\alpha)-\lambda_1]+\frac{q}{l}\varepsilon[l(2s+\alpha)-1+l-\lambda_3]$ which has the same sign as $(l+q)\lambda_1+q\lambda_2-q(1+h)$. Hence, the seller is indifferent between allocating the goods to a same buyer among types HL or among types LL if and only if $(l+q)\lambda_1+q\lambda_2=q(1+h)$. The second alternative implies reducing z_{HL} by ε and increasing x_{HL} by ε and y_{LL} by $\frac{q}{l}\varepsilon$; then $\Delta L = -\varepsilon[q(2s+1+\alpha)-\lambda_1-q(s+1)]+$

$\frac{q}{l}\varepsilon(ls - q - \lambda_2 - \lambda_3)$ which has the same sign as $(l+q)\lambda_1 - q(\alpha l + h + q)$. Therefore, the seller is indifferent between selling both objects to a type HL and allocating good 1 to a type HL and good 2 among types LL if and only if $(l+q)\lambda_1 = q(\alpha l + h + q)$. Finally, in the third alternative $\Delta z_{HL} = -\varepsilon < 0$, $\Delta y_{HL} = \varepsilon$ and $\Delta x_{LL} = \frac{q}{l}\varepsilon$; hence $\Delta L = \varepsilon[qs - \lambda_1 - q(2s + 1 + \alpha) + \lambda_1] + \frac{q}{l}\varepsilon(ls - q - \lambda_1 - \lambda_3) < 0$.

(iib) From the proof of part (iia) we know that the seller is indifferent between selling both goods to a type LL or to a type HL if $(l+q)\lambda_1 + q\lambda_2 = q(1+h)$. Furthermore, allocating with positive probability only object 2 among types LL and only item 1 among types HL by not always selling the two goods to a same type HL or LL is unprofitable if $(l+q)\lambda_1 \leq q(\alpha l + h + q)$.

(iic) We know that if good 1 is allocated among types HL, then varying the probability that good 2 is allocated among types LL rather than to the same type HL who wins good 1 has no effect on L if $(l+q)\lambda_1 = q(\alpha l + h + q)$. Moreover, reducing the probability that object 1 is sold to a type HL in favor of types LL is equivalent to reduce x_{HL} by ε while increasing z_{LL} by $\frac{q}{l}\varepsilon$ and reducing y_{LL} by $\frac{q}{l}\varepsilon$; then $\Delta L = \frac{\varepsilon q}{l}(\alpha l + \lambda_2 + q - 1)$. Exploiting the equality $\alpha l = \frac{(l+q)\lambda_1}{q} - h - q$ we find that $\Delta L \leq 0$ if and only if $(l+q)\lambda_1 + q\lambda_2 \leq q(1+h)$.

(iii) We omit the proof to this part as it is just a relabeling of the proof to part (ii).

(iva) From part (ia) we know that no modification (of the proposed selling policy) involving only types HL and LH is profitable as long as $\min\{\lambda_1, \lambda_2\} \geq q(\alpha - 1)$. Selling only good 1 (say) to a type LL decreases L ($\Delta x_{HL} = -\varepsilon$, $\Delta x_{LL} = \frac{q}{l}\varepsilon$). Selling with positive probability both objects to a same type LL entails reducing both x_{HL} and y_{LH} by ε while increasing z_{LL} by $\frac{q}{l}\varepsilon$; then $\Delta L = -\varepsilon 2q(s+1) + \frac{q}{l}\varepsilon[l(2s+\alpha) - 1 + l - \lambda_3]$, which has the same sign as $\lambda_1 + \lambda_2 - 1 - h - l + \alpha l$.

(ivb) In view of part (ic), no modification involving only types HL and LH increases L if $\lambda_1 = \lambda_2 \leq q(\alpha - 1)$. It can be verified that this condition also implies that selling with positive probability only one good to a type LL is not profitable. If both items are allocated with positive probability to a same type LL then $\Delta z_{HL} = -\varepsilon < 0$ (or

$\Delta z_{LH} = -\varepsilon$) and $\Delta z_{LL} = \frac{q}{l}\varepsilon > 0$, hence $\Delta L = \frac{\varepsilon}{l}[(q+l)\lambda_1 + q\lambda_2 - q(1+h)]$.

(ivc) The proof to part (ivb) shows that the seller is indifferent between allocating both goods to a type HL or to a type LH or to a type LL if and only if $(q+l)\lambda_1 + q\lambda_2 = q(1+h)$ and $\lambda_1 = \lambda_2$. The best way of selling the objects separately is to allocate item 1 among types HL and good 2 among types LH. Then $\Delta z_{HL} = -\varepsilon$ (or $\Delta z_{LH} = -\varepsilon$, or $\Delta z_{LL} = -\frac{q}{l}\varepsilon$), $\Delta x_{HL} = \Delta y_{LH} = \varepsilon$ and $\Delta L \leq 0$ if and only if $\lambda_1 \leq q(\alpha - 1)$ (as $\lambda_1 = \lambda_2$). ■

Proof of lemma 2.3 Lemma 2.4 takes for granted that both goods are always sold. That is actually optimal if, when $x_{jk} > 0$ ($y_{jk} > 0$ or $z_{jk} > 0$) then $\frac{\partial L}{\partial x_{jk}} \geq 0$ ($\frac{\partial L}{\partial y_{jk}} \geq 0$ or $\frac{\partial L}{\partial z_{jk}} \geq 0$), $j, k = L, H$. This is the case for any mechanism which is mentioned in the present lemma, given the values of the multipliers which are provided below and given that $s \geq \frac{h+q}{l}$.

(i) We prove that mechanism I1 solves problem HH if $\alpha \leq \min\left\{\frac{(h+q)l-q}{2ql}, \frac{1}{1-h}\right\}$. To this purpose set $\lambda_1 = \lambda_2 = q\frac{\alpha l+q+h}{l+q}$ and $\lambda_3 = h - 2\lambda_1$; $\lambda_3 \geq 0$ as $\alpha \leq \frac{(h+q)l-q}{2ql}$. Having $\lambda_1 > 0$, $\lambda_2 > 0$ and $\lambda_3 \geq 0$ is consistent with lemma 2.1 as (2.8)-(2.10) bind in I1. Indeed, from the heuristic description of I1 follows that any type HL (LH) has the same probability to win good 2 (1) as any type LL: $y_{HL} + z_{HL} = y_{LL} + z_{LL}$ and $x_{LH} + z_{LH} = x_{LL} + z_{LL}$; hence (2.8)-(2.10) bind.

Given the above values for λ_1, λ_2 and λ_3, by using lemma 2.4 we verify that if $n_{HH} = 0$ and at least two different types of buyer show up in the auction then the allocation prescribed by I1 maximizes L. By lemma 2.4(ia), 2.4(iic), 2.4(iiic) and 2.4(iva) we need to check the inequalities $q\frac{\alpha l+q+h}{l+q} \geq q(\alpha - 1)$, $(l+2q)q\frac{\alpha l+q+h}{l+q} \leq q(1+h)$ and $2q\frac{\alpha l+q+h}{l+q} \leq 1 + l + h - \alpha$; these are satisfied since $\alpha \leq \frac{1}{1-h}$.

(ii) Assume $hl \leq 2q$ and $\frac{(h+q)l-q}{2ql} < \alpha \leq 1 + \frac{h}{2q}$; set $\lambda_1 = \lambda_2 = \frac{h}{2}$ and $\lambda_3 = 0$. In B1 constraints (2.8) and (2.9) bind while (2.10) is slack as $y_{HL} + z_{HL} + x_{LL} + z_{LL} = x_{LH} + z_{LH} + y_{LL} + z_{LL} > x_{LL} + y_{LL} + 2z_{LL}$: the probability to win good 2 (good 1) for a type HL (LH) is higher than for a type LL; indeed, a type LL never receives any object unless $n_{LL} = n$. From lemma 2.4(ia), 2.4(iia), 2.4(iiia) and 2.4(iva) follows that B1 is optimal if $\frac{h}{2} \geq q(\alpha - 1)$, $(l + 2q)\frac{h}{2} \leq q(1 + h)$, $(l + q)\frac{h}{2} \leq q(\alpha l + h + q)$ and $\alpha l \leq 1 + l$. These inequalities hold because $hl \leq 2q$ and $\frac{(h+q)l-q}{2ql} < \alpha \leq 1 + \frac{h}{2q}$.

If instead $\alpha > 1 + \frac{h}{2q}$ but still $hl \leq 2q$, then we prove that B2 solves problem HH by setting again $\lambda_1 = \lambda_2 = \frac{h}{2}$ and $\lambda_3 = 0$. As in B1, (2.8) and (2.9) bind while (2.10) does not in B2 (actually, both $y_{HL} + z_{HL}$ and $x_{LH} + z_{LH}$ increase in B2 with respect to B1); hence the values of the multipliers are consistent with lemma 2.1. In view of lemma 2.4(ic), 2.4(iia), 2.4(iiia) and 2.4(ivb) we have to check that $\frac{h}{2} \leq q(\alpha - 1)$, $(l+2q)\frac{h}{2} \leq q(1+h)$, $(l+q)\frac{h}{2} \leq q(\alpha l+h+q)$ and $(l+2q)\frac{h}{2} \leq q(1+h)$; these inequalities follow from $1 + \frac{h}{2q} < \alpha$ and $hl \leq 2q$.

(iii) Assume $hl > 2q$ and $\frac{1}{1-h} < \alpha \leq \frac{2}{1-h}$; then WI1 is optimal in problem HH. In order to prove this claim set $\lambda_1 = \lambda_2 = q\frac{1+h}{1-h}$ and $\lambda_3 = h - 2\lambda_1$; $\lambda_3 > 0$ since $hl > 2q$. The value of θ in WI1 is determined in order to let (2.8)-(2.10) bind. This occurs if and only if $z_{LH} = z_{HL} = z_{LL}$ (as $x_{LH} = x_{LL} = y_{HL} = y_{LL} = 0$) and, since $z_{LH} = z_{HL}$, we just take care of the equality $z_{HL} = z_{LL}$. Using lemma 1 in Ar we find $z_{HL} = \frac{q^{n-1}}{n} + \theta[\binom{n-1}{1}\frac{q^{n-2}l}{n-1} + \binom{n-1}{2}\frac{q^{n-3}l^2}{n-2} + \cdots + \binom{n-1}{n-1}l^{n-1}] = \theta\frac{(l+q)^n - l^n}{nq} + (1-\theta)\frac{q^{n-1}}{n}$ and $z_{LL} = \frac{l^{n-1}}{n} + 2(1-\theta)\left[\binom{n-1}{1}\frac{l^{n-2}q}{n-1} + \binom{n-1}{2}\frac{l^{n-3}q^2}{n-2} + \cdots + \binom{n-1}{n-1}q^{n-1}\right] = (2\theta - 1)\frac{l^{n-1}}{n} + 2(1-\theta)\frac{(l+q)^n - q^n}{nl}$. There exists a unique value of θ such that $z_{HL} = z_{LL}$; that value lies in $(0, 1)$. To be exact, $\theta = \frac{2q(l+q)^n - (1-h)q^n - ql^n}{(1-h)[(l+q)^n - l^n - q^n]}$ and $z_{HL} = z_{LL} = \frac{2(l+q)^n - l^n}{n(1-h)}$.

By lemma 2.4(ia), 2.4(iib), 2.4(iiib) and 2.4(iva) the conditions $q\frac{1+h}{1-h} \geq q(\alpha - 1)$, $(l + 2q)q\frac{1+h}{1-h} = q(1 + h)$, $(l + q)q\frac{1+h}{1-h} \leq q(\alpha l + h + q)$ and $2q\frac{1+h}{1-h} \leq 1 + l + h - \alpha l$ are necessary and sufficient in order for WI1 to be optimal. It turns out that the first one and the fourth one are equivalent to $\alpha \leq \frac{2}{1-h}$; the third one is implied by $\alpha > \frac{1}{1-h}$.

Now assume still $hl > 2q$ but $\frac{2}{1-h} < \alpha$ and set again $\lambda_1 = \lambda_2 = q\frac{1+h}{1-h}$ and $\lambda_3 = h - 2\lambda_1$. In WI2, (2.8)-(2.10) bind as each type HL, LH and LL is treated in the same way; hence any type HL (LH) has the same probability to win good 2 (good 1) as any type LL.

The main difference between WI1 and WI2 concerns the allocation of the goods when $n_{HL} \geq 1$, $n_{LH} \geq 1$ and $n_{HH} = 0$. In that case WI2 allocates both goods to a same buyer who is randomly chosen among *all* the buyers in the auction; lemma 2.4(ic) and 2.4(ivc) require $q\frac{1+h}{1-h} \leq q(\alpha - 1)$ and $(l + 2q)q\frac{1+h}{1-h} = q(1 + h)$ which hold since $\alpha \geq \frac{2}{1-h}$. The two mechanisms do not allocate the goods in the same way when $n_{HL} \geq 1$, $n_{LL} \geq 1$ and $n_{LH} = n_{HH} = 0$ and when $n_{LH} \geq 1$, $n_{LL} \geq 1$ and $n_{HL} = n_{HH} = 0$: in WI1 the value of θ is determined as we have seen above, while in WI2 it is equal to $\frac{n_{HL}}{n}$ or $\frac{n_{LH}}{n}$, respectively. The conditions for optimality, however, are the same [described by lemma 2.4(iib) and 2.4(iiib)] and they are satisfied for WI2 as they hold for WI1 with the same multipliers and a smaller α. ■

Proof of proposition 2.2 In problem HH several incentive constraints are neglected. We need to verify that they are satisfied in any mechanism which is mentioned in lemma 2.3. In the proof to lemma 2.3 we have seen that (2.8) and (2.9) bind in any such mechanism; the same is true for (2.5) and (2.6). For any other incentive constraint we write down the condition under which it holds

jk	$j'k'$	inequality (2.1) given jk and $j'k'$
HH	LL	$y_{HL} + z_{HL} \geq y_{LL} + z_{LL}$
HL	HH	$y_{HH} + z_{HH} \geq y_{HL} + z_{HL}$
HL	LH	$y_{LH} + x_{LL} \geq x_{LH} + y_{LL}$
LH	HH	$x_{HH} + z_{HH} \geq x_{LH} + z_{LH}$
LH	HL	$x_{HL} + y_{LL} \geq y_{HL} + x_{LL}$
LL	HH	$x_{HH} + y_{HH} + 2z_{HH} \geq y_{HL} + z_{HL} + x_{LL} + z_{LL}$
LL	HL	$x_{HL} + z_{HL} \geq x_{LL} + z_{LL}$
LL	LH	$y_{LH} + z_{LH} \geq y_{LL} + z_{LL}$

For any of mechanisms I1, B1, WI1, B2 and WI2 the values of $\{x_{jk}, y_{jk}, z_{jk}\}_{j,k=L,H}$ which characterize that mechanism satisfy all of the above inequalities. ■

Chapter 3

Optimal Auctions Under Collusion of Buyers with Correlated Valuations

3.1 Introduction

The mechanism design literature has devoted a lot of attention to the problem of finding the expected revenue maximizing auction for an indivisible good when each buyer's willingness to pay is unknown to the seller. The seminal paper by Myerson (1981) covers the case of risk-neutral buyers with independently (and possibly not identically) distributed private values. Many following papers have extended his early analysis to more general settings.[1]

Most of the papers about this topic assume that buyers behave according to some Bayes-Nash equilibrium (BNE henceforth) of the mechanism designed by the seller. Yet, if communication among buyers is feasible then they may be expected to form a

[1] In Matthews (1983) and Maskin and Riley (1984) the buyers are risk averse; Crémer and McLean (1985, 1988) assume correlated valuations; Bulow and Klemperer (1996) and Branco (1997) deal with interdependent values [actually, also Myerson (1981) allows for a specific case of interdependent valuations]. Also see the survey by Klemperer (1999) and the references therein.

coalition to collectively manipulate the sale mechanism. Indeed, anecdotal evidence [see for example Graham and Marshall (1987) and Hendricks and Porter (1989)] suggests that collusion in auctions is quite widespread. This is particularly important if collusion considerably reduces the revenue and modifies the allocation of the good with respect to noncooperative behavior. In this chapter we analyze the effects of collusion in a specific auction setting.

Myerson (1981) shows that, in the standard environment described in section 1.2, under (Bayes-)Nash behavior a first price auction with a suitable reserve price r maximizes expected revenue (as many other sale mechanisms do) if the probability distribution F satisfies the monotone virtual valuation condition mentioned in chapter 2. Such an optimal auction leaves positive rents to buyers with high valuations: the expected payoff of any buyer with valuation $r + \varepsilon > r$ is $\int_r^{r+\varepsilon} F^{n-1}(z)dz > 0$. Nevertheless, collusion increases the buyers' expected payoffs with respect to noncooperative behavior [see for instance McAfee and McMillan (1992)]. When the valuations are (even slightly) correlated, on the other hand, Crémer and McLean (1985, 1988) show that the seller can design a mechanism which extracts (in individually rational BNE) all of the buyers' expected surplus *as if* he could observe the valuations.[2] This implies that no type of buyer gains anything from participating in the auction; therefore under correlated values the buyers' incentives to collude are likely to be stronger than when the valuations are independently distributed.

This chapter examines - assuming positively correlated values - the effects of collusion on the seller's revenue, his optimal reaction, the efficiency of the resulting allocation and whether collusion awards rents to the buyers. For the sake of tractability we focus on a setting with two risk-neutral and ex ante symmetric buyers; each of them privately

[2] In Crémer and McLean (1985, 1988) the buyers' valuations are discretely distributed. McAfee and Reny (1992) assume continuously distributed valuations and provide conditions on the joint density under which full surplus extraction is achievable. McAfee, McMillan and Reny (1989) examine a common value environment in which the buyers' signals are correlated (even though they are not so, conditional on the true, unknown, value of the object); they prove that the seller can extract almost all of the rents.

observes her own valuation for the good, which may be high (type H) or low (type L).

Section 3.2 introduces an above mentioned result: Under correlated valuations, if the buyers play noncooperatively then the seller can obtain the same expected revenue as a perfectly informed monopolist. In section 3.3 we model collusion as in Laffont and Martimort (2000) (henceforth LM). Specifically, we assume that an uninformed third party proposes to the buyers a side-contract to collectively manipulate their reports into the sale mechanism. This new agent aims at maximizing the sum of the buyers' expected payoffs but ignores their valuations (collusion occurs under asymmetric information) and therefore needs to provide each type of buyer with incentives in order for her not to lie in the side mechanism. Furthermore, each buyer accepts the collusive agreement only if by doing that she expects to obtain a higher payoff than from playing the sale mechanism noncooperatively.[3]

Even though it takes place under asymmetric information, collusion restricts the set of allocations which the seller can implement and it turns out that he cannot extract anymore the full surplus. Specifically, full surplus extraction requires ex post efficiency in the allocation of the good and, under collusion, that implies strictly positive rents for each buyer of type H. As in the setting with uncorrelated valuations and no collusion, the seller can appropriate these rents by refusing to sell to any type L and for some parameter values this is profitable for him. Therefore it is not anymore possible to screen the buyers' types at no cost: collusion generates the well known trade-off between ex post efficiency and revenue maximization which disappeared under correlation and Nash behavior. As LM remark, introducing collusion restores continuity between the correlated and uncorrelated environments. It is worth noticing that the revenue loss due to collusion is negligible when correlation is very strong: in that case the seller nearly extracts the entire surplus from the buyers, as if they played noncooperatively (but see the caveat at the end of this section and in subsection 3.4.3). Also with uncorrelated

[3] Since the side mechanism is designed by an uninformed third party, no information is leaked by its selection as it would occur if one buyer offered it. In this way a problem of information signalling is avoided. LM's approach concentrates on collusion issues and provides an upper bound on its effects.

valuations the revenue is not reduced by collusion, but it is well known that in such a setting the expected revenue is strictly smaller than under complete information even if buyers play noncooperatively. Hence collusion is very effective when correlation is not zero but is not very strong.

We highlight the differences between the above model and the one in which the buyers freely share their private information at the collusion stage, which means that the coalition is formed under symmetric information;[4] this makes collusion more powerful. In this setting the seller is more likely to withhold the good when facing two buyers with type L with respect to the previous environment and his revenue loss is bounded away from zero also when correlation is very strong.

This chapter is basically an application of the collusion model in LM to an auction setting; nevertheless, their results are not really analogous to ours. LM consider the problem of implementing the efficient production level of a public good when two agents have positively correlated (privately observed) valuations for the public good and may collude. They solve this problem when correlation is very weak or very strong. Under weak correlation it does not matter whether the coalition forms under symmetric or asymmetric information, as the planner cannot exploit the fact that the colluding agents ignore each other's valuation. On the other hand, when correlation is strong the planner can almost completely undermine the effects of collusion under asymmetric information. While the latter result holds in our setting as well, we find that the seller (weakly) prefers asymmetric to symmetric information for *any* degree of positive correlation. A further difference is that in LM production is shut down when correlation is strong and the agents have different types, as this event is ex ante quite unlikely (given strong and positive correlation) and a no-collusion constraint is considerably relaxed. In our model nothing like that can be optimal: a type H always wins the good when facing a type L. The common root of these two differences is the different shape of a no-collusion

[4]This was sometimes assumed in early studies of collusion; see for example Green and Laffont (1979), Robinson (1985) and von Ungern-Sternberg (1988).

constraint.

In an auction environment one more issue arises with respect to a public good setting. If the buyers have different types (one type H and one type L) and collude by claiming they have a same type, then there is a chance that the good will be allocated to type L even though a type H is around. Thus, the effectiveness of collusion between buyers with different types depends on the possibility to transfer the good ex post in case it is "misallocated within the coalition". If that is feasible, then it turns out that the seller never gains from the fact that the coalition forms under asymmetric information. More than the intrinsic results (which are similar to those obtained under symmetric information within the coalition and no ex post transfer of the good) we observe that the modelling "details" are pretty important: Collusion is quite effective if the buyers are able to commit to ex post efficiency within the coalition and its effects are not mitigated by asymmetric information at the collusion stage.

In section 3.4 we perform some robustness checks on the optimal sale mechanism. Buyer i's acceptance of a side mechanism depends on the equilibrium payoff she would obtain if the sale mechanism were played noncooperatively, which in turn may depend on buyer $3-i$'s beliefs following buyer i's veto of the collusive agreement. Thus the proposed side contract may depend on what a buyer is expected to learn if her opponent rejects the side mechanism. Actually, in subsection 3.4.1 we show that the proposed side mechanism does not depend on such beliefs, hence the same can be said of the optimal sale mechanism. A further (and related) robustness check involves the game beginning after the third party's proposal of a side mechanism. There each buyer may signal her own private information in a preplay cheap-talk stage and thereby affect the subsequent play. Yet, we find that the optimal sale mechanism is always strongly ratifiable according to the definition of Cramton and Palfrey (1995). Unfortunately, for some parameter values a multiplicity problem arises: the optimal sale mechanism has a non-truthful BNE which each type H prefers to the truthful BNE; by contrast, each type L earns a negative payoff in it. This problem can be eliminated by imposing a suitable constraint on the sale mechanism, clearly at the cost of a smaller revenue. In this setting the known approach of resorting to a sequential mechanism to knock out "undesired" equilibria, without reducing the planner's welfare, may help when the sale mechanism is played noncooperatively. However, this is not sufficient to eliminate the non-truthful equilibrium when the buyers play the side mechanism.[5]

3.2 The model and the full surplus extraction outcome

3.2.1 Preferences and information

A risk neutral seller owns an indivisible object which he values 0 and faces two potential buyers. Buyer i privately observes her own valuation for the good $v^i \in V^i \equiv \{v_L, v_H\}$ $(i = 1, 2)$ with $v_L = s > 0$ and $v_H = s + 1$;[6] $v = (v^1, v^2)$ and $V \equiv V^1 \times V^2$. Each buyer's payoff is equal to her valuation times her probability to win the good minus her expected payment. The probability distribution for (v^1, v^2) from the seller's viewpoint is ($h > 0$, $q > 0$, $l > 0$ and $h + 2q + l = 1$)

	$v^2 = v_L = s$	$v^2 = v_H = s + 1$
$v^1 = v_L = s$	l	q
$v^1 = v_H = s + 1$	q	h

Buyer i's $(i = 1, 2)$ beliefs about v^{3-i} are consistent with this probability distribution: she attaches probability $p_{HH} = \frac{h}{h+q}$ to the event $v^{3-i} = v_H$ if $v^i = v_H$ and probability $p_{HL} = \frac{q}{q+l}$ to the same event if $v^i = v_L$ (p_{LH} and p_{LL} are defined likewise). Correlation between v^1 and v^2 is measured by $\rho \equiv hl - q^2$; if $\rho = 0$ then v^1 and v^2 are independently distributed, otherwise they are - positively or negatively - correlated.

[5] LM's optimal mechanism under strong correlation is plagued by the same problem, even though LM do not remark it.

[6] As we observed in chapter 2, there is no loss of generality in assuming $v_H - v_L = 1$.

We will focus on positive correlation, which is sometimes proposed as a more realistic assumption with respect to the assumption of independent valuations. An example in Crémer and McLean (1985) is about an auction for oil drilling rights when two competitors have carried out geological tests which provide information about the profitability of the tract on sale. Since the results of the tests are likely to be positively correlated, by observing her own value each agent has a more detailed information about the opponent's valuation with respect to an outside observer as the seller. We retain however the assumption of private values.

3.2.2 Extracting the whole surplus

Consider the seller's problem of designing the revenue maximizing mechanism in the above setting assuming that the buyers behave noncooperatively. In view of the Revelation Principle he restricts to direct mechanisms. Let $x_{w^i w^{3-i}}$ and $t_{w^i w^{3-i}}$ denote the probability for a buyer to win the good and her payment to the seller, respectively, when she reports $w^i \in V^i$ and her opponent reports $w^{3-i} \in V^{3-i}$. A mechanism is defined by the four-tuples $t = (t_{HH}, t_{HL}, t_{LH}, t_{LL}) \in \Re^4$ and $x = (x_{HH}, x_{HL}, x_{LH}, x_{LL}) \geq 0$ such that $x_{HH} \leq \frac{1}{2}$, $x_{HL} + x_{LH} \leq 1$ and $x_{LL} \leq \frac{1}{2}$; the restrictions on x are obvious feasibility conditions.[7]

Crémer and McLean (1985) prove that under correlated valuations any allocation rule x is implementable in BNE and the seller can extract from both types of buyer the whole surplus generated by such allocation. To see how this is attained, consider any feasible x and write the participation and incentive constraints as follows, with $k_L \geq 0$ and $k_H \geq 0$ [the payoff to type L (H) is not divided by $l + q$ ($q + h$) but this has no consequences]

$$l(sx_{LL} - t_{LL}) + q(sx_{LH} - t_{LH}) \quad = \quad 0 \tag{3.1}$$

[7] Unlike in chapter 2, here we use non-reduced form probabilities: as there are only two buyers this does not entail a significant increase in complexity compared to clarity.

$$l(sx_{HL} - t_{HL}) + q(sx_{HH} - t_{HH}) = -k_L \quad (3.2)$$

$$q[(s+1)x_{HL} - t_{HL}] + h[(s+1)x_{HH} - t_{HH}] = 0 \quad (3.3)$$

$$q[(s+1)x_{LL} - t_{LL}] + h[(s+1)x_{LH} - t_{LH}] = -k_H \quad (3.4)$$

Given any x, (3.1)-(3.4) is a linear system of equations in $t = (t_{HH}, t_{HL}, t_{LH}, t_{LL})$ and the determinant of the matrix of the unknowns equals ρ^2. Therefore, in a correlated environment, for any sale policy as described by x there exists a (unique) $t \in \Re^4$ solving (3.1)-(3.4). The expected payoff to each type of buyer equals 0, hence the seller earns the entire surplus generated by trade. In other words, his expected revenue R is equal to the expected social surplus, as it is clear by substituting (3.1) and (3.3) into R:

$$R = 2lt_{LL} + 2q(t_{HL} + t_{LH}) + 2ht_{HH} = 2lsx_{LL} + 2q[sx_{LH} + (s+1)x_{HL}] + 2h(s+1)x_{HH}$$

Hence, R is maximized in $x^\bullet_{HH} = x^\bullet_{LL} = \frac{1}{2}$, $x^\bullet_{HL} = 1$ and $x^\bullet_{LH} = 0$, the allocation which maximizes social surplus. Given $x^\bullet$, the transfers which solve (3.1)-(3.4) are $t^\bullet_{HH} = \frac{s}{2} + \frac{(1-l)l}{2\rho} - \frac{q}{\rho}k_L$, $t^\bullet_{LL} = \frac{s}{2} - \frac{q^2}{2\rho} - \frac{q}{\rho}k_H$, $t^\bullet_{HL} = s - \frac{(1-l)q}{2\rho} + \frac{h}{\rho}k_L$ and $t^\bullet_{LH} = \frac{ql}{2\rho} + \frac{l}{\rho}k_H$.

Therefore in the truthful BNE of $(x^\bullet, t^\bullet)$ the seller obtains the same expected revenue as under complete information.[8] However, observe that if $k_L = 0$ and $k_H = 0$ then the sale mechanism has other BNE in addition to truthtelling; in one of them both types of buyer 1 (2) report L and both types of buyer 2 (1) report H. To avoid multiplicity, the seller could arbitrarily fix $k_L \geq 0$ and $k_H \geq 0$ and use the sequential mechanism proposed in Brusco (1998) or exploit the following result, which states that it is possible to implement the full surplus extraction outcome in unique BNE.

Proposition 3.1 *If $k_H = k^\bullet_H \equiv b\delta$ and $k_L = k^\bullet_L \equiv \frac{q+l}{2} - \delta$ with a small $\delta > 0$ and $b \in (\frac{q}{l}, \frac{h}{q})$, then truthtelling is the unique BNE of $(x^\bullet, t^\bullet)$.*

The expressions for $t^\bullet$ reveal how mechanism $(x^\bullet, t^\bullet)$ screens the buyers' types at no

[8] Crémer and McLean (1988) characterize the information structures in which this goal can be achieved when the buyers' valuations are discretely distributed.

cost: if k_L and k_H increase, then each type has a stronger incentive to report truthfully and we see that $t^\bullet_{HH}$ and $t^\bullet_{LL}$ decrease while $t^\bullet_{HL}$ and $t^\bullet_{LH}$ increase. In words, given the report w^2 of buyer 2, the payment of buyer 1 is relatively low if $w^1 = w^2$ with respect to the case of $w^1 \neq w^2$: a buyer is punished in the relatively unlikely case (given $\rho > 0$) that her value differs from the opponent's valuation and is rewarded in the opposite case.[9]

The starting point for our analysis is (about) the following: Assume that both buyers accept to play the sale mechanism when each of them only knows her own valuation but, before reporting to the seller, v^1 and v^2 become common knowledge between the buyers. Then, if $v^1 = v^2 = v_H$, by announcing LL the buyers can Pareto improve their payoffs with respect to truthful reporting. Indeed, the sale policy would not change ($x^\bullet_{HH} = x^\bullet_{LL}$) but their payments would be smaller ($t^\bullet_{LL} < t^\bullet_{HH}$ when $k_L = k^\bullet_L$ and $k_H = k^\bullet_H$). Mechanism $(x^\bullet, t^\bullet)$ is therefore not robust to collusion if buyers freely share their private information and they can coordinate their reports.

3.3 Collusion under asymmetric information

We model collusion as in LM by considering a game with the following timing. First each buyer learns her own valuation for the good on sale; then the seller proposes a sale mechanism (x, t). If both buyers accept to play (x, t) then[10] a benevolent and uninformed third party proposes them a (direct) side mechanism to collusively manipulate their reports into (x, t). If both buyers accept the collusion mechanism then each of them reports a valuation (possibly dishonestly) to the third party who enforces, as a function of the reports, zero-sum side transfers between the buyers and (possibly manipulated) announcements into (x, t). If instead at least one buyer vetoes the side mechanism then

[9] The first-best revenue cannot be achieved if $\rho = 0$ as setting $x_{LL} > 0$ implies that the expected payoff of type H is strictly positive; see subsection 7.1.2 in Fudenberg and Tirole (1991).

[10] If both buyers veto (x, t) then the game ends. If only one buyer accepts (x, t) then the seller makes her a take-or-leave-it offer: the proposed price is s if $s \geq (h + q)(s + 1)$, otherwise it is $s + 1$. As usual, a buyer obtains a payoff equal to 0 if she does not participate in the auction.

(x, t) is played noncooperatively. The buyers' decisions about accepting or rejecting the side mechanism are simultaneous.

We denote by $z = (z^1, z^2) \in V$ the buyers' reports to the side mechanism, which is described by (ϕ, τ). Let ϕ_z designate the manipulated reports into (x, t) as a function of z (we assume these manipulated reports can be chosen stochastically, as this convexifies the third party's feasible set); τ_z^i is the payment of buyer i in the side mechanism, $i = 1, 2$; $\tau_z^1 + \tau_z^2 = 0$ for any $z \in V$. If $(w^1, w^2) = \phi_{z^1 z^2}$ is a non-stochastic third party's report into (x, t) then we let $x^1(\phi_{z^1 z^2}) \equiv x_{w^1 w^2}$ and $x^2(\phi_{z^1 z^2}) \equiv x_{w^2 w^1}$; similarly, $t^1(\phi_{z^1 z^2}) \equiv t_{w^1 w^2}$ and $t^2(\phi_{z^1 z^2}) \equiv t_{w^2 w^1}$. The extension to stochastic reports by the third party is straightforward.[11]

The third party designs (ϕ, τ) to maximize the sum of the buyers' expected payoffs subject to incentive compatibility (because it ignores v^1 and v^2) and participation constraints with respect to (x, t) played noncooperatively. If collusion took place under symmetric information (that is, if the third party observed v^1 and v^2) then a Pareto optimum for the coalition given (x, t) would always be reached; under asymmetric information that is not necessarily true.

After the third party proposed (ϕ, τ) a two stage game is played: in its first stage each buyer accepts or rejects (ϕ, τ); in the second stage the buyers report types either into (x, t) or into (ϕ, τ) depending on their decisions at the first stage. In the following we refer to the "game of coalition formation" as the one which starts with the third party's proposal of a side mechanism. We are interested in (collusive continuation) equilibria of the coalition formation game in which both buyers accept (ϕ, τ), thus no learning occurs along the equilibrium path.[12] We look for the optimal (x, t) knowing that in the game of coalition formation the third party will optimally design (ϕ, τ) and that the above

[11] To be rigorous, the Revelation Principle applies to the third party's design of the side mechanism but does not apply to the seller's design of the sale mechanism. Thus we should allow the seller to use non-direct sale mechanisms. Nevertheless, as in LM (see their proposition 3) it can be proved that any perfect Bayesian equilibrium outcome arising from a non-direct sale mechanism can be obtained as a perfect Bayesian equilibrium outcome induced by a direct sale mechanism.

[12] Observe that there also exists an equilibrium in which each buyer rejects any side mechanism: if buyer i is vetoing any side mechanism, then rejecting is a best reply for buyer $3 - i$.

mentioned two stage game between the buyers follows the proposal of (ϕ, τ).

In designing (ϕ, τ) the third party needs to take into account each buyer's incentives to accept the side mechanism. If buyer i vetoes (ϕ, τ) (an out-of-equilibrium move) then buyer $3-i$ may change her own beliefs about v^i with respect to prior beliefs and, as long as the noncooperative play of (x,t) is affected by buyer $3-i$'s beliefs about v^i, this may alter buyer i's incentive to accept the collusive agreement. Hence, the side mechanism which is offered by the third party may depend on what each buyer is expected to infer following the rejection of (ϕ, τ) by her opponent. Equivalently, there may exist different equilibria in the game of coalition formation depending on a buyer's updated beliefs after the other buyer vetoed the side mechanism. In this section we assume that, after a buyer i rejected the side mechanism, buyer $3-i$ does not change her own beliefs about v^i; in subsection 3.4.1 we allow for more general beliefs.

3.3.1 Weakly collusion-proof mechanisms

In this subsection we introduce the third party's design problem and examine the conditions under which the seller can prevent collusion.

We use σ to denote a generic strategy profile in (x,t); $\sigma = (LH, LL)$, for example, is the (pure-strategy) profile in which both types of buyer 1 report truthfully and buyer 2 always claims L; $\sigma^\bullet = (LH, LH)$ is the truthful reporting profile. Let $\bar{p}^i$ represent a belief system of buyer $3-i$ about v^i which may differ from prior beliefs. More clearly, $\bar{p}^i_{kj}$ is the probability that buyer $3-i$ attaches, according to the belief system $\bar{p}^i$, to the event $v^i = v_k$ given that her own type is j ($j, k = L, H$); $\bar{p}^i_{kj}$ may not be equal to p_{kj} as defined in subsection 3.2.1. Let $BNE[(x,t)_{\bar{p}^i,p}]$ denote the set of BNE of (x,t) when buyer i has prior beliefs (p) about v^{3-i} and buyer $3-i$'s beliefs about v^i are given by $\bar{p}^i$. Clearly, (x,t) is an incentive compatible mechanism if and only if $\sigma^\bullet \in BNE[(x,t)_{p,p}]$. Finally, $U^i_j(\sigma)$ is the payoff to type j of buyer i when σ is played in (x,t), computed with prior beliefs; $U^\bullet_j \equiv U^1_j(\sigma^\bullet) = U^2_j(\sigma^\bullet)$, hence $U^\bullet_j = p_{Lj}(v_j x_{jL} - t_{jL}) + p_{Hj}(v_j x_{jH} - t_{jH})$, $j = L, H$.

Assume that if buyer i refuses the side mechanism then buyer $3-i$'s beliefs about v^i are given by $\bar{p}^i$ and $\sigma^i \in BNE[(x,t)_{\bar{p}^i,p}]$ is played in (x,t). Then each type j of buyer i accepts (ϕ,τ) if her payoff u_j^i in the truthful BNE of (ϕ,τ) (as we remarked in footnote 11, there is no loss of generality in letting the third party propose a direct side mechanism) is at least as high as $U_j^i(\sigma^i)$, her payoff if σ^i is played in (x,t).[13] Truthtelling is a BNE in (ϕ,τ) if and only if (3.6) below holds; we assume that (3.5) and (3.6) are necessary and sufficient in order for each type of each buyer to accept (ϕ,τ).

$$\begin{cases} u_j^1 \equiv p_{Lj}[v_j x^1(\phi_{jL}) - t^1(\phi_{jL}) - \tau_{jL}^1] + p_{Hj}[v_j x^1(\phi_{jH}) - t^1(\phi_{jH}) - \tau_{jH}^1] \geq U_j^1(\sigma^1) & j = L, H \\ u_j^2 \equiv p_{Lj}[v_j x^2(\phi_{Lj}) - t^2(\phi_{Lj}) - \tau_{Lj}^2] + p_{Hj}[v_j x^2(\phi_{Hj}) - t^2(\phi_{Hj}) - \tau_{Hj}^2] \geq U_j^2(\sigma^2) & j = L, H \end{cases} \tag{3.5}$$

$$\begin{cases} u_j^1 \geq p_{Lj}[v_j x^1(\phi_{kL}) - t^1(\phi_{kL}) - \tau_{kL}^1] + p_{Hj}[v_j x^1(\phi_{kH}) - t^1(\phi_{kH}) - \tau_{kH}^1] & j,k = L, H \\ u_j^2 \geq p_{Lj}[v_j x^2(\phi_{Lk}) - t^2(\phi_{Lk}) - \tau_{Lk}^2] + p_{Hj}[v_j x^2(\phi_{Hk}) - t^2(\phi_{Hk}) - \tau_{Hk}^2] & j,k = L, H \end{cases} \tag{3.6}$$

Definition 3.1 *Given (x,t), a (collusive continuation) equilibrium of the game of coalition formation is made up of (i) beliefs systems $\bar{p}^1$ and $\bar{p}^2$, (ii) associated equilibria $\sigma^1 \in BNE[(x,t)_{\bar{p}^1,p}]$ and $\sigma^2 \in BNE[(x,t)_{\bar{p}^2,p}]$ respectively and (iii) a side mechanism that maximizes the third party's objective function*

$$\begin{aligned} &h[(s+1)x^1(\phi_{HH}) - t^1(\phi_{HH}) + (s+1)x^2(\phi_{HH}) - t^2(\phi_{HH})] + q[sx^1(\phi_{LH}) \\ &-t^1(\phi_{LH}) + (s+1)x^2(\phi_{LH}) - t^2(\phi_{LH}) + (s+1)x^1(\phi_{HL}) - t^1(\phi_{HL}) \\ &+sx^2(\phi_{HL}) - t^2(\phi_{HL})] + l[sx^1(\phi_{LL}) - t^1(\phi_{LL}) + sx^2(\phi_{LL}) - t^2(\phi_{LL})] \end{aligned}$$

subject to budget balance ($\tau_z^1 + \tau_z^2 = 0$ for any $z \in V$) and (3.5)-(3.6) (i.e., the side mechanism is incentive compatible and is unanimously accepted).

Clearly, different equilibria of the game of coalition formation may arise depending on the beliefs systems $\bar{p}^1$ and $\bar{p}^2$ and on the BNE σ^1 and σ^2. In the following of this

[13] Both $U_j^i(\sigma^i)$ and u_j^i are computed by using prior beliefs because buyer i expects buyer $3-i$ to accept the side mechanism, hence she has no reason to hold non-prior beliefs about v^{3-i}.

section we assume that if buyer i vetoes (ϕ, τ) then buyer $3-i$ does not update her own beliefs about v^i, that is $\bar{p}^1 = \bar{p}^2 = p$, and that $\sigma^1 = \sigma^2 = \sigma^\bullet$.

Given any (x, t), the seller can find the solution to the third party's maximization problem and compute his resulting expected revenue; thus he needs to design (x, t) to maximize R taking into account the foreseeable [given (x, t)] reports manipulations which are implemented by the third party. However, a principle similar to the Revelation Principle applies here: Any (perfect Bayesian) equilibrium outcome that the seller can achieve through a mechanism (x, t) which is prone to collusive behavior for some couple of types $z \in V$ (that means $\phi_z \neq z$) can be obtained as an equilibrium outcome by designing a sale mechanism which is not changed through the process of coalition formation. In other words, without loss of generality the seller can design a sale mechanism such that the coalition does not lie to him (that is, $\phi_z = z$ for any $z \in V$) and no monetary side transfers occur between buyers. Next definition identifies such mechanisms when buyers hold prior beliefs and $\sigma^\bullet$ is played in (x, t) if a buyer vetoes the side mechanism. Let $(\phi^\bullet, \tau^\bullet)$ denote the null side mechanism: $\phi_z^\bullet = z$ and $\tau_z^{\bullet 1} = \tau_z^{\bullet 2} = 0$ for any $z \in V$.

Definition 3.2 (Weakly collusion-proof mechanisms) *Mechanism (x, t) is weakly collusion-proof if and only if it is incentive compatible and there exists an equilibrium in the coalition formation game in which $\bar{p}^1 = \bar{p}^2 = p$, $\sigma^1 = \sigma^2 = \sigma^\bullet$, the third party offers $(\phi^\bullet, \tau^\bullet)$ and both buyers accept to play (truthfully) $(\phi^\bullet, \tau^\bullet)$.*

Hence (x, t) is weakly collusion-proof if, when $\bar{p}^1 = \bar{p}^2 = p$ and $\sigma^1 = \sigma^2 = \sigma^\bullet$, the third party's problem is solved by recommending to not misreport to the seller and implementing no transfers between buyers. As we mentioned above, the reason of this definition is that any equilibrium outcome arising from a mechanism (x, t) which allows collusive behavior [i.e., a side mechanism $(\phi, \tau) \neq (\phi^\bullet, \tau^\bullet)$ is selected by the third party] can be achieved with a weakly collusion-proof sale mechanism which is the composition of (x, t) with (ϕ, τ).[14] The notion of weakly collusion-proof mechanism is relatively weak,

[14]LM provide a rigorous proof of this claim (see their proposition 3) which basically exploits the argument proving the Revelation Principle.

as the game of coalition formation may have different equilibria when $\bar{p}^1$ (σ^1) and/or $\bar{p}^2$ (σ^2) differ from p ($\sigma^\bullet$). In subsection 3.4.1, however, we prove that the optimal weakly collusion-proof sale mechanism is such that the third party proposes $(\phi^\bullet, \tau^\bullet)$ in all the equilibria of the coalition formation game.

The following lemma characterizes the conditions under which (x, t) is weakly collusion-proof.[15] Such conditions can be interpreted as the constraints which collusion imposes in the design of (x, t).

Lemma 3.1 *A mechanism (x, t) is weakly collusion-proof if and only if it is incentive compatible and there exists $\varepsilon \in [0, 1)$ such that, for any $w \in V$,*

$$\begin{aligned}
2(s+1)x_{HH} - 2t_{HH} &\geq (s+1)x_{w^1w^2} - t_{w^1w^2} + (s+1)x_{w^2w^1} - t_{w^2w^1} \\
(s+1)x_{HL} - t_{HL} + sx_{LH} - t_{LH} &\geq (s+1)x_{w^1w^2} - t_{w^1w^2} + sx_{w^2w^1} - t_{w^2w^1} + \varepsilon\frac{h}{q}(x_{LH} - x_{w^2w^1}) \\
2sx_{LL} - 2t_{LL} &\geq sx_{w^1w^2} - t_{w^1w^2} + sx_{w^2w^1} - t_{w^2w^1} + \varepsilon q\frac{2x_{LL} - x_{w^1w^2} - x_{w^2w^1}}{l + \frac{\rho}{q}\varepsilon}
\end{aligned} \tag{3.7}$$

If $\varepsilon > 0$ then the incentive constraint of type H in the side mechanism is binding.

Broadly speaking, ε is the multiplier of the incentive constraint of type H in the third party's maximization problem. The seller has some flexibility in choosing ε because $(\phi^\bullet, \tau^\bullet)$ satisfies the necessary and sufficient conditions for optimality in the third party's problem for any $\varepsilon \in [0, 1)$. In subsection 3.3.3 we deal with collusion under symmetric information, which would occur if the buyers could credibly commit to truthfully reveal their valuations within the coalition. In such a setting the side mechanism does not need to satisfy any incentive constraint; then ε is equal to 0 and, according to (3.7), types are misreported into (x, t) whenever by doing that it is possible to increase the sum of the buyers' payoffs. In a sense, therefore, $\varepsilon > 0$ represents the effect of asymmetric information between buyers at the collusion stage, something which the seller may be expected to exploit.

[15] We skip the proof of lemma 3.1 as it closely mimics the proof of proposition 4 in LM.

3.3.2 The optimal weakly collusion-proof mechanism

We now maximize the expected revenue subject to the no-collusion condition (3.7) in addition to the standard participation and incentive constraints [inequalities (3.8)-(3.11)]:[16]

$$\max_{(x,t)} \frac{R}{2} = ht_{HH} + qt_{HL} + qt_{LH} + lt_{LL}$$

subject to

$$l(sx_{LL} - t_{LL}) + q(sx_{LH} - t_{LH}) \geq 0 \tag{3.8}$$

$$l(sx_{LL} - t_{LL}) + q(sx_{LH} - t_{LH}) \geq l(sx_{HL} - t_{HL}) + q(sx_{HH} - t_{HH}) \tag{3.9}$$

$$q[(s+1)x_{HL} - t_{HL}] + h[(s+1)x_{HH} - t_{HH}] \geq 0 \tag{3.10}$$

$$q[(s+1)x_{HL} - t_{HL}] + h[(s+1)x_{HH} - t_{HH}] \geq q[(s+1)x_{LL} - t_{LL}] + h[(s+1)x_{LH} - t_{LH}] \tag{3.11}$$

$$2(s+1)x_{HH} - 2t_{HH} \geq (s+1)(x_{HL} + x_{LH}) - t_{HL} - t_{LH} \tag{3.12}$$

$$2(s+1)x_{HH} - 2t_{HH} \geq 2(s+1)x_{LL} - 2t_{LL} \tag{3.13}$$

$$(s+1)x_{HL} + sx_{LH} - t_{LH} - t_{HL} \geq (2s+1)x_{HH} - 2t_{HH} + \frac{h}{q}\varepsilon(x_{LH} - x_{HH}) \tag{3.14}$$

$$(s+1)x_{HL} + sx_{LH} - t_{LH} - t_{HL} \geq (2s+1)x_{LL} - 2t_{LL} + \frac{h}{q}\varepsilon(x_{LH} - x_{LL}) \tag{3.15}$$

$$2sx_{LL} - 2t_{LL} \geq 2sx_{HH} - 2t_{HH} + 2\varepsilon q \frac{x_{LL} - x_{HH}}{l + \frac{\ell}{q}\varepsilon} \tag{3.16}$$

$$2sx_{LL} - 2t_{LL} \geq s(x_{HL} + x_{LH}) - t_{HL} - t_{LH} + \varepsilon q \frac{2x_{LL} - x_{HL} - x_{LH}}{l + \frac{\ell}{q}\varepsilon} \tag{3.17}$$

The solution to this maximization problem, $(\hat{x}, \hat{t})$, is described by the following proposition 3.2. The results in propositions 3.2 and 3.3 below rely on the assumption that the buyers report truthfully in $(\phi^{\bullet}, \tau^{\bullet})$. Even though truthtelling is a BNE in $(\phi^{\bullet}, \tau^{\bullet})$, in subsection 3.4.3 we point out a multiplicity problem in mechanism $(\hat{x}, \hat{t})$ when $\hat{x}_{LL} = \frac{1}{2}$.

[16] We do not write the constraint preventing a couple of buyers with types HL from reporting LH as it turns out to be (trivially) satisfied in the optimal weakly collusion-proof mechanism.

Proposition 3.2 *(i) For any fixed $\varepsilon \in [0,1)$ an "optimal mechanism given ε" $(\hat{x},\hat{t})$ exists; $\hat{x}_{HH} = \frac{1}{2}$, $\hat{x}_{HL} = 1$ and $\hat{x}_{LH} = 0$; $\hat{x}_{LL} = \frac{1}{2}$ if*

$$(sl - q)(\rho + q) > h\rho(1 - \varepsilon) \tag{3.18}$$

and $\hat{x}_{LL} = 0$ otherwise. The transfers $\hat{t}$ are determined by (3.8), (3.11), (3.13) and (3.15) written with equality.

(ii) No "globally optimal" mechanism exists if $sl > q$, as in that case the revenue is strictly increasing in ε and $\varepsilon \in [0,1)$.

(iii) If $\hat{x}_{LL} = \frac{1}{2}$ then the payoff to each type H is $\frac{q}{2(h+q)} + \frac{h\rho(1-\varepsilon)}{2(\rho+q)(h+q)} > 0$, thus the seller never extracts the full surplus; yet, by setting ε close to 1 he can nearly achieve the first best revenue when q is close to 0.

When the buyers may collude the seller cannot extract the full surplus as under Nash behavior, since the no-collusion constraints (3.12)-(3.17) narrow down the set of mechanisms among which he can choose. More precisely, (3.13) and (3.15) bind; thus t_{HH} cannot be very large otherwise (3.13) fails and the buyers would report LL if they had types HH; similarly, $t_{HL} + t_{LH}$ cannot be too large because of (3.15). On the other hand, when $\rho > 0$ is close to 0 (and $k_H = k_H^\bullet$, $k_L = k_L^\bullet$) then both $t_{HH}^\bullet$ and $t_{LH}^\bullet$ are very large. Setting $x_{LL} = 0$ relaxes both (3.13) and (3.15) [as well as (3.11)] and it is profitable when (3.18) fails; therefore collusion may reduce the social surplus generated by the revenue-maximizing mechanism. In other words, the well known trade-off between revenue maximization and ex post efficiency appears, as in the setting with independent values and Nash behavior; in this sense, as LM emphasize, collusion restores continuity between the correlated and uncorrelated environments.[17] Observe that (3.18) fails when s is close to 0 and/or l is close to 0: if it is unlikely that $(v^1, v^2) = (s, s)$ then setting

[17] Robert (1991) shows that such a result arises under Nash behavior if the buyers are risk-averse and/or have limited budget. Kosmopoulou and Williams (1998) make a similar point about the existence of efficient mechanisms in a quite general setting: If no efficient mechanism exists when the agents observe uncorrelated signals, then non-existence is robust to the introduction of a small amount of correlation in the agents' signals if the transfers the planner can use are bounded.

$x_{LL} = 0$ implies a small loss for the seller compared to the rents that would be left to type H. On the other hand, if (3.18) holds (for example because s is large) then the good is efficiently allocated; yet, since type H earns a strictly positive expected payoff the seller does not gain the whole surplus as if he could prevent collusion for free.

Proposition 3.2(iii) states that the seller almost achieves the first best revenue if v^1 and v^2 are strongly correlated: in such a case collusion has almost no effect. The reason of this result is the following. Since x_{LH} is set equal to 0, by using (3.8) written with equality it turns out that the payoff to type H from reporting L is equal to $\frac{qx_{LL}-\frac{\rho}{l}t_{LH}}{h+q}$. This implies that, when $q \simeq 0$, setting $x_{LL} = \frac{1}{2}$ does not award significative rents to type H at least for what the term $\frac{qx_{LL}}{h+q}$ is concerned (intuitively, by reporting L type H may obtain the good only if the other buyer claims L; this is unlikely if q is close to 0, hence reporting L is not very appealing for type H for what concerns winning the good). Actually, if $\varepsilon \simeq 1$ then t_{LH} is about 0, hence the term $\frac{-\rho t_{LH}}{l(h+q)}$ is close to 0. Indeed, if $q \simeq 0$ and $\varepsilon \simeq 1$ then setting $x_{LL} = \frac{1}{2}$ considerably relaxes (3.15) by making its right hand side very negative, which allows to set a large t_{HL} and t_{LH} close to 0 in a way which is consistent with (3.11) written with equality.[18] If ε is not close to 1 then t_{HL} cannot be sufficiently large to extract almost all of the rents from type H; moreover, t_{HH} is not large because $t_{LL} = t_{HH}$ [by (3.13)] cannot be much larger than $\frac{s}{2}$ otherwise (3.8) requires a largely negative t_{LH}.

Collusion has no bite at all when the valuations are independently distributed. The reason is that under uncorrelated valuations and no collusion the optimal transfers are not uniquely determined: $t \in \Re^4$ only needs to satisfy two linear equations (the participation constraint of type L and the incentive constraint of type H). Hence, there exist infinitely many optimal t in that setting and it turns out that one of them satisfies (3.12)-(3.17) (also with $\varepsilon = 0$).[19] Given this result it is not surprising that, if we let

[18] Consistently with this remark, the proof of proposition 3.2(i) in appendix shows that the multiplier of (3.11) is close to 0 if q is about 0.

[19] Consistently with this fact, if $\rho = 0$ then the multipliers of the binding no-collusion constraints, (3.13) and (3.15), are zero [again, see the proof of proposition 3.2(i)].

$\pi \equiv \Pr\{v^1 = s+1\} = \Pr\{v^2 = s+1\}$ when $\rho = 0$, then (3.18) reduces to $s(1-\pi) > \pi$; this is the inequality implying $\hat{x}_{LL} = \frac{1}{2}$ rather than $\hat{x}_{LL} = 0$ under uncorrelated values and Nash behavior [see subsection 7.1.2 in Fudenberg and Tirole (1991)]. As in Laffont and Martimort (1997), if $\rho = 0$ then the seller can deter collusion at zero cost, a quite different result with respect to what occurs in the correlated case. Recall however that, absent collusion, when $\rho = 0$ the seller cannot extract the whole surplus from the buyers as when the valuations are correlated.

As we remarked in the introduction, this chapter applies the collusion model in LM to an auction setting. There are several analogies between our results and those in LM (they have just been pointed out), but also some remarkable differences. The most striking one is related to the effects of buyers' asymmetric information within the coalition. In our environment $\varepsilon = 1$ (broadly speaking) weakly dominates any smaller ε, while in LM it is strictly better to set $\varepsilon = 0$ when correlation is weak. Our finding arises because (3.15) binds and, as $\hat{x}_{LH} = 0$, the larger is $\varepsilon > 0$ the more (3.15) is relaxed if $\hat{x}_{LL} = \frac{1}{2}$.[20] This says that the seller is better off if the coalition forms under asymmetric information rather than with symmetric information. The reason why in LM this "intuitive" result does not always hold is that a constraint which is analogous to (3.15) (call it constraint A) binds and it is relaxed the higher is $\varepsilon(y_{LL} - y_{HL})$ (y_{jk} is the level of public good when the agents report types j and k). Under weak correlation the no-collusion constraints imply $y_{HL} \geq y_{LL}$; thus constraint A is relaxed by setting $\varepsilon = 0$, as if the agents had symmetric information when colluding.

A second difference is linked to the first one: in LM the public good is not produced at all when correlation is strong and the types are HL or LH, while we find that it is always optimal to set $x_{HL} = 1$ and $x_{LH} = 0$. The reason is that, in LM, under strong correlation it is feasible to set $y_{LL} > y_{HL}$; hence constraint A is relaxed by not producing when types are different ($y_{HL} = 0$), producing about the efficient level when both agents

[20] This explains why proposition 3.2(ii) states that no optimal mechanism exists if $sl > q$: ε cannot be maximized in [0,1), though the sup of the possible revenues can be arbitrarily closely approximated.

have type L and setting ε close to 1. This is not very expensive for the planner from an ex ante point of view, since $2q$ (the probability that the buyers have different types) is small when correlation is strong. Our constraint (3.15) instead is relaxed the higher is $\varepsilon(x_{LL} - x_{LH})$; x_{LH} is optimally set equal to 0 under Nash behavior, hence a fortiori $\hat{x}_{LH} = 0$ under collusion as this relaxes (3.15). The variable x_{HL}, on the other hand, appears in the binding no-collusion constraints only to relax (3.15); hence its optimal value is 1 as without collusion.

3.3.3 The case of symmetric information

As we mentioned above, the fact that collusion occurs under asymmetric information is captured by $\varepsilon > 0$ in (3.7) and proposition 3.2(ii) establishes that the seller weakly prefers ε close to 1 to any smaller ε. To better evaluate the effects of asymmetric information it is useful to examine the case in which collusion takes place under symmetric information. In this subsection we assume that the buyers can freely share their private information when forming the coalition or, equivalently, that the third party owns a technology allowing for credible disclosure of information.[21] Then collusion occurs under symmetric information and lemma 3.1 is amended by writing (3.7) with $\varepsilon = 0$.

Now the optimal mechanism $(\bar{x}, \bar{t})$ is more often inefficient than $(\hat{x}, \hat{t})$, as the condition for $\bar{x}_{LL} = \frac{1}{2}$ is more restrictive than (3.18); actually, it is (3.18) written with $\varepsilon = 0$:

$$(sl - q)(\rho + q) > h\rho \tag{3.19}$$

Therefore the seller is more likely to sell the good when both buyers have type L if the coalition forms under asymmetric information; this is intuitively sensible since the latter environment is a closer setting to noncooperative behavior (which implies $x^{\bullet}_{LL} = \frac{1}{2}$ for any parameter values) than collusion under symmetric information.

[21] Each buyer learns the opponent's valuation only when the coalition forms, hence the participation and incentive constraints (3.8)-(3.11) still need to be satisfied.

More interestingly, in this environment the revenue is bounded away (below) from the first best revenue for *any* degree of correlation: from proposition 3.2(iii) follows that the payoff to type H if $\varepsilon = 0$ and $\bar{x}_{LL} = \frac{1}{2}$ is equal to $\frac{\rho}{2(\rho+q)} + \frac{q^2}{2(\rho+q)(h+q)}$. The reason is that now setting $x_{LL} = \frac{1}{2}$ does not relax anymore (3.15) as it occurs when $\varepsilon > 0$; this was very powerful for small q. As a consequence, if $x_{LL} = \frac{1}{2}$ then t_{HL} cannot be sufficiently large to extract (almost) the full surplus from type H.

Observe that if q is close to 0 then (3.19) reduces to (about) $sl > 1 - l$. If the seller faced a single buyer whose valuation equals s with probability l and is $s + 1$ with probability $1 - l$, then he should sell the good to type L if and only if $sl > 1 - l$. Thus, the two buyers are about treated as if they were a single buyer when collusion takes place under symmetric information and $q \simeq 0$; this means that collusion completely eliminates the gains which the seller could obtain from facing more than one buyer. Therefore, especially under strong correlation, it is quite important to establish whether the coalition forms under symmetric or asymmetric information. In the latter case the first best outcome is nearly attained [by proposition 3.2(iii)], while that is impossible in the former setting as the seller virtually faces a single buyer.

3.3.4 Ex post efficiency within the coalition

An auction setting displays a peculiar difference with respect to the public good environment examined in LM: The buyers can modify the allocation of the good determined by the seller. Our previous analysis implicitly assumed that the good cannot be transferred between buyers after the seller has allocated it to a given buyer. To see what this implies, suppose $v^1 = s+1$, $v^2 = s$ and $x_{LL} = \frac{1}{2}$; if the buyers collude by announcing LL then there is a chance that buyer 2 (a type L) wins the good. Collusion is more effective if the third party is able to let type H always receive the good whenever it is sold and the buyers have different types. In this way the good is never misallocated within the coalition, hence collusion is harder to prevent when buyers have different types. In this subsection we suppose that the seller cannot prevent the transfer of the good within the coalition and examine how this new assumption affects the results of the model.

Since the good can be transferred ex post by the third party, the incentive and participation constraints in the side mechanism need to be amended accordingly. Specifically, given any (x,t) and (ϕ,τ) the payoff to type H of buyer 1 in the side mechanism under truthtelling is

$$u^1_{eH} \equiv p_{LH}[v_H x^1(\phi_{HL}) + v_H x^2(\phi_{HL}) - t^1(\phi_{HL}) - \tau^1_{HL}] + p_{HH}[v_H x^1(\phi_{HH}) - t^1(\phi_{HH}) - \tau^1_{HH}]$$

and her incentive constraint is

$$u^1_{eH} \geq p_{LH}[v_H x^1(\phi_{LL}) - t^1(\phi_{LL}) - \tau^1_{LL}] + p_{HH}[-t^1(\phi_{LH}) - \tau^1_{LH}]$$

Likewise, in writing the payoff and the incentive constraints for type L of buyer $i = 1,2$ and for type H of buyer 2 in the side mechanism, we need to take into account that if the buyers report different types to the third party then the one who reported H receives the good if it is sold. The third party's objective function is modified as follows (clearly, the budget balance constraint is not altered)

$$\begin{aligned}
&h[(s+1)x^1(\phi_{HH}) - t^1(\phi_{HH}) + (s+1)x^2(\phi_{HH}) - t^2(\phi_{HH})] + q[(s+1)x^1(\phi_{LH}) \\
&-t^1(\phi_{LH}) + (s+1)x^2(\phi_{LH}) - t^2(\phi_{LH}) + (s+1)x^1(\phi_{HL}) - t^1(\phi_{HL}) \\
&+(s+1)x^2(\phi_{HL}) - t^2(\phi_{HL})] + l[sx^1(\phi_{LL}) - t^1(\phi_{LL}) + sx^2(\phi_{LL}) - t^2(\phi_{LL})]
\end{aligned}$$

It turns out that in this setting a result which is analogous to lemma 3.1 states that (x,t) is weakly collusion-proof if and only if it is incentive compatible and there exists $\varepsilon \in [0,1)$ such that, for any $w \in V$,

$$\begin{aligned}
2(s+1)x_{HH} - 2t_{HH} &\geq (s+1)x_{w^1w^2} - t_{w^1w^2} + (s+1)x_{w^2w^1} - t_{w^2w^1} \\
(s+1)(x_{HL} + x_{LH}) - t_{HL} - t_{LH} &\geq (s+1)(x_{w^1w^2} + x_{w^2w^1}) - t_{w^1w^2} - t_{w^2w^1} \\
2sx_{LL} - 2t_{LL} &\geq sx_{w^1w^2} - t_{w^1w^2} + sx_{w^2w^1} - t_{w^2w^1} + \varepsilon q \frac{2x_{LL} - x_{w^1w^2} - x_{w^2w^1}}{l + \frac{\rho}{q}\varepsilon}
\end{aligned} \tag{3.20}$$

The difference with respect to (3.7) is that now the no-collusion constraints are written with $\varepsilon = 0$ when buyers have different types (as if they had symmetric information at the collusion stage); moreover, if buyers report HL or LH in the side mechanism now the good (if it is sold) is allocated to the buyer who reported H. Therefore, in the revenue maximization problem constraints (3.8) to (3.13), (3.16) and (3.17) are imposed but (3.14) and (3.15) are substituted by

$$(s+1)(x_{HL}+x_{LH}) - t_{LH} - t_{HL} \geq 2(s+1)x_{HH} - 2t_{HH} \tag{3.21}$$

$$(s+1)(x_{HL}+x_{LH}) - t_{LH} - t_{HL} \geq 2(s+1)x_{LL} - 2t_{LL} \tag{3.22}$$

The solution $(\tilde{x}, \tilde{t})$ to this problem is described in the following[22]

Proposition 3.3 *(i) If*

$$(s+1)\rho + sq > h \tag{3.23}$$

then $\tilde{x}_{LL} = \frac{1}{2}$*; if instead (3.23) fails then* $\tilde{x}_{LL} = 0$*. In either case* $\tilde{x}_{HH} = \frac{1}{2}$, $\tilde{x}_{HL} = 1$, $\tilde{x}_{LH} = 0$ *and* $\tilde{t}$ *is determined by (3.8), (3.11), (3.13) and (3.22) written with equality.*

(ii) If $\tilde{x}_{LL} = \frac{1}{2}$ *then each type* H *earns a rent equal to* $\frac{l}{2(q+l)} > 0$*, therefore the seller never obtains (or is close to obtain) the first best revenue.*

The results of proposition 3.3 are qualitatively close to those found for the model with collusion under symmetric information and no commitment to ex post efficiency within the coalition (see subsection 3.3.3). However, in the present setting the payoff of each type H if $\tilde{x}_{LL} = \frac{1}{2}$ is higher with respect to the above model.[23] Also observe that (3.23) implies (3.19) which in turn is stronger than (3.18): to countervail the higher

[22] Actually, also (3.8)-(3.11) should be amended by taking into account that when the buyers report different types in the side mechanism then type H will receive the good if it is sold. That however would result in no substantial modification of the mechanism which is described in proposition 3.3. Rather than finding $\tilde{x}_{HL} = 1$ and $\tilde{x}_{LH} = 0$, we would obtain the condition $\tilde{x}_{HL} + \tilde{x}_{LH} = 1$, $\tilde{x}_{HL} \geq 0$ and $\tilde{x}_{LH} \geq 0$, knowing that however type L would never win the good when facing a type H.

[23] That is not true if $\rho = 0$: even though now the good can be transferred within the coalition, the revenue is still not reduced by collusion if $\rho = 0$ (the same result was found when the allocation determined by the seller cannot be modified).

collusion effectiveness the seller is more likely to withhold the good if both buyers have type L.

Maybe it is more interesting to observe that in this environment the seller is not better off if collusion occurs under asymmetric information. Indeed, setting $\varepsilon > 0$ relaxes no binding constraint [because (3.22) is written as if the buyers had symmetric information when colluding], which instead occurs when the good cannot be transferred within the coalition. Hence, if the buyers are able to allocate the good efficiently within the coalition then it does not matter whether collusion takes place under symmetric or asymmetric information. Proposition 3.2(ii) establishes an opposite result when the good cannot be transferred between the buyers.

3.4 Robustness

In the game of coalition formation first the third party proposes a side mechanism (ϕ, τ); then each buyer announces whether she accepts or refuses (ϕ, τ); after these announcements the buyers report either in (x, t), if some buyer vetoed (ϕ, τ), or in the side mechanism otherwise. In section 3.3 we predicted that if the sale mechanism is $(\hat{x}, \hat{t})$ [or $(\tilde{x}, \tilde{t})$ when the good can be transferred within the coalition] then the third party proposes $(\phi^\bullet, \tau^\bullet)$, both buyers accept it and they report truthfully in $(\phi^\bullet, \tau^\bullet)$. We now try to test the robustness of these predictions at the various stages of the game of coalition formation.

3.4.1 Strong collusion-proofness

A weakly collusion-proof mechanism (x, t) is such that there exists an equilibrium in the coalition formation game in which the third party proposes $(\phi^\bullet, \tau^\bullet)$ and both buyers accept it, given that if buyer i rejects $(\phi^\bullet, \tau^\bullet)$ then buyer $3 - i$ keeps having prior beliefs about v^i and $\sigma^\bullet$ is played in (x, t). However, following buyer i's rejection of $(\phi^\bullet, \tau^\bullet)$ a different equilibrium of the sale mechanism may be played, possibly under non-prior beliefs for buyer $3-i$; that could alter the participation constraints of buyer i in the side mechanism design problem and induce the third party to select a non-null side mechanism. This problem is avoided if the sale mechanism satisfies the following definition [from Laffont and Martimort (1997)].

Definition 3.3 (Strongly collusion-proof mechanisms) *A mechanism (x,t) is strongly collusion-proof if and only if (i) it is weakly collusion-proof and (ii) there is no equilibrium of the coalition formation game in which the third party offers $(\phi,\tau) \neq (\phi^\bullet,\tau^\bullet)$ and each type of both buyers accepts (ϕ,τ).*

If (x,t) is strongly collusion-proof then it is "robust" to any equilibrium of the game of coalition formation. Clearly, a weakly collusion-proof mechanism is not necessarily strongly collusion-proof. Nevertheless, proposition 3.4 below establishes that both $(\hat{x},\hat{t})$ and $(\tilde{x},\tilde{t})$ are strongly collusion-proof; hence $(\hat{x},\hat{t})$ and $(\tilde{x},\tilde{t})$ satisfy a relatively strong implementation concept. The proof shows that if we do not impose $\bar{p}^1 = \bar{p}^2 = p$ and $\sigma^1 = \sigma^2 = \sigma^\bullet$ then either (i) the game of coalition formation has no equilibrium or (ii) the buyers' participation constraints in the third party's design problem are not altered. In the latter case $(\phi^\bullet,\tau^\bullet)$ is feasible and maximizes the third party's payoff given that $(\hat{x},\hat{t})$ and $(\tilde{x},\tilde{t})$ satisfy (3.7) and (3.20), respectively.[24]

Proposition 3.4 *Both $(\hat{x},\hat{t})$ and $(\tilde{x},\tilde{t})$ are strongly collusion-proof.*

3.4.2 Ratifiability

Strong collusion-proofness tests the robustness of $(\phi^\bullet,\tau^\bullet)$, in the design of the side mechanism, to the various BNE of the sale mechanism the buyers may coordinate on after one of them vetoed $(\phi^\bullet,\tau^\bullet)$ (and allowing for arbitrary beliefs for the non-deviant buyer). We now consider a different, although related, issue. As we mentioned at the

[24] Actually, in some cases there exists a side mechanism $(\phi,\tau) \neq (\phi^\bullet,\tau^\bullet)$ which is accepted by both buyers but yields them, and to the third party, the same payoffs as $(\phi^\bullet,\tau^\bullet)$.

beginning of section 3.3, a two stage game is entered *after* $(\phi^{\bullet}, \tau^{\bullet})$ is proposed by the third party. In the first stage each buyer i makes a preplay announcement (veto or accept) which may signal some information about v^i; in the second stage buyers report types either in (x, t) or in the side mechanism $(\phi^{\bullet}, \tau^{\bullet})$. In any case, however, in the second stage (x, t) is actually played since $(\phi^{\bullet}, \tau^{\bullet})$ is the null side mechanism; the first stage is therefore a sort of cheap-talk stage in which buyers may signal their types. We focused above on unanimous ratification of $(\phi^{\bullet}, \tau^{\bullet})$ and on the truthful BNE of (x, t) (supported by prior beliefs) in case $(\phi^{\bullet}, \tau^{\bullet})$ is vetoed; yet, a buyer's acceptance decision of $(\phi^{\bullet}, \tau^{\bullet})$ may be affected if she expects her opponent to infer non-prior beliefs from her veto of $(\phi^{\bullet}, \tau^{\bullet})$. The notion of strong ratifiability of a mechanism against itself provided by Cramton and Palfrey (1995) can be used to test whether $(\hat{x}, \hat{t})$ and $(\tilde{x}, \tilde{t})$ (or better, their respective truthful BNE) are robust to the preplay announcements at the first stage. In presenting ratifiability we follow LM and use some notation which has been introduced at the beginning of subsection 3.3.1. We denote with i the buyer who contemplates rejecting $(\phi^{\bullet}, \tau^{\bullet})$.

Definition 3.4 *Given an incentive compatible (x, t), a belief system $\bar{p}^i$ is a credible veto system of $\sigma^{\bullet}$ if there exists $\sigma \in BNE[(x, t)_{\bar{p}^i, p}]$ and refusal probabilities r^i_j $(j = L, H)$ such that $r^i_L + r^i_H > 0$ and*

(i) $\bar{p}^i_{jk} = \frac{p_{jk} r^i_j}{p_{Lk} r^i_L + p_{Hk} r^i_H}$, $j, k = L, H$

(ii) $r^i_j = 1$ *for any* j *such that* $U^i_j(\sigma) > U^{\bullet}_j$ *and* $r^i_j = 0$ *for any* j *such that* $U^i_j(\sigma) < U^{\bullet}_j$.

Thus, buyer $3 - i$'s beliefs following rejection of buyer i are required to satisfy a consistency condition similar to the one underlying the definition of perfect sequential equilibrium in Grossman and Perry (1986). In words, the non-deviant buyer $3 - i$ rationalizes a veto of buyer i by finding beliefs about v^i which are consistent with i's incentive to veto. In our context, if buyer i vetoes $(\phi^{\bullet}, \tau^{\bullet})$ then she is actually vetoing $\sigma^{\bullet}$. Hence, if buyer $3 - i$ observes buyer i rejecting $(\phi^{\bullet}, \tau^{\bullet})$, then she should infer that her opponent's type is j such that i will improve her own payoff over $U^{\bullet}_j$ by not playing

$\sigma^\bullet$ in (x,t); if it is common knowledge that σ is played in (x,t) after a veto of buyer i, that means $U^i_j(\sigma) > U^\bullet_j$. On the other hand, types j of buyer i who are going to lose if σ is played $[U^i_j(\sigma) < U^\bullet_j]$ should not have vetoed $\sigma^\bullet$; thus they receive zero probability in the revised beliefs of buyer $3-i$. Buyer $3-i$'s belief system $\bar{p}^i$ is consistent with bayesian updating given the prior beliefs p and the above argument, as embodied in refusal probabilities.[25] Observe that σ is required to be a BNE of (x,t) when buyer $3-i$ updated her beliefs about v^i as indicated above. The types j such that $r^i_j > 0$ make up the so called *credible veto set.*

Definition 3.5 *An incentive compatible mechanism (x,t) is strongly ratifiable if no credible veto system exists or if, for any given credible veto system $\bar{p}^i$ and associated credible veto set, there exists $\dot{\sigma} \in BNE[(x,t)_{\bar{p}^i,p}]$ such that $U^i_j(\dot{\sigma}) = U^\bullet_j$ for any j belonging to the credible veto set.*

This definition says that (x,t) is strongly ratifiable if no subset of V^i can credibly veto $(\phi^\bullet, \tau^\bullet)$ and be sure to obtain a strictly higher payoff than if $\sigma^\bullet$ is played.[26] It is quickly proved that $(\tilde{x},\tilde{t})$ and $(\hat{x},\hat{t})$ (when $\hat{x}_{LL} = 0$) are strongly ratifiable. Indeed, the proof of proposition 3.4 in appendix establishes that for these mechanisms there exist no $\bar{p}^i$ and $\sigma \in BNE[(x,t)_{\bar{p}^i,p}]$ such that $U^i_j(\sigma) > U^\bullet_j$ for some j. Hence, no type in any credible veto set may strictly gain from vetoing $\sigma^\bullet$.

Matters are less straightforward for $(\hat{x},\hat{t})$ when $\hat{x}_{LL} = \frac{1}{2}$; in the remaining of this section this mechanism is simply referred to as "$(\hat{x},\hat{t})$". First observe that, for any $\bar{p}^i$, $\bar{\sigma} = (LL, LL)$ is the unique BNE of $(\hat{x},\hat{t})_{\bar{p}^i,p}$ in which some type j of buyer i obtains a

[25] Cramton and Palfrey (1995) emphasize that strong ratifiability satisfies the following "rational expectations" condition: When agent $3-i$ observes a veto of agent i, then she believes that the types who may have vetoed belong to a subset S of types such that each type in S (and only the types in S) benefits from veto given that agent $3-i$ believes that the types in S veto. By contrast, such a condition is not always satisfied by somewhat related notions as the intuitive criterion [Cho and Kreps (1987)], divinity and universal divinity [Banks and Sobel (1987)]. Following an out-of-equilibrium move of an agent, for each of these refinements a set S' of types who may have deviated is determined, but S' is not necessarily the set of types who benefit from deviation when the other agent believes that her opponent's type is in S'.

[26] Actually, Cramton and Palfrey (1995) define strong ratifiability for a more general setting. In their words, if (x,t) satisfies definition 3.5 then it is "strongly ratifiable against itself".

higher payoff than $U_j^\bullet$ [see the payoff matrices of $(\hat{x}, \hat{t})$ in the proof to proposition 3.4]; more precisely, $U_H^i(\bar{\sigma}) = \frac{1}{2} - \frac{hq(1-\varepsilon)}{2(\rho+q)} > U_H^\bullet = \frac{q}{2(h+q)} + \frac{h\rho(1-\varepsilon)}{2(\rho+q)(h+q)}$ while $U_L^i(\bar{\sigma}) = \frac{-hq(1-\varepsilon)}{2(\rho+q)} < U_L^\bullet = 0$. Hence, letting $r_H^i = 1$ and $r_L^i = 0$ we see that $\bar{p}_{HL}^i = \bar{p}_{HH}^i = 1$ is a credible veto system of $\sigma^\bullet$ according to definition 3.4. Yet, it turns out that $\sigma^\bullet$ is an equilibrium of the sale mechanism even with the modified beliefs: $\sigma^\bullet \in BNE[(\hat{x}, \hat{t})_{\bar{p}^i, p}]$. Since, by definition, $U_H^i(\sigma^\bullet) = U_H^\bullet$, mechanism $(\hat{x}, \hat{t})$ is strongly ratifiable in view of definition 3.5 because $\sigma^\bullet$ plays the role of $\dot{\sigma}$ for the credible veto system $\bar{p}_{HL}^i = \bar{p}_{HH}^i = 1$ and there exists no other credible veto system.

3.4.3 Multiplicity in $(\hat{x}, \hat{t})$

Even though the two previous subsections provide positive results, the fact that $\bar{\sigma}$ is BNE of $(\hat{x}, \hat{t})_{p,p}$ reduces the confidence in mechanism $(\hat{x}, \hat{t})$. Indeed, even though each buyer accepts $(\phi^\bullet, \tau^\bullet)$, the profile (LL, LL) is an equilibrium when the side mechanism $(\phi^\bullet, \tau^\bullet)$ is played. Moreover, such an equilibrium is "focal" in the following sense. By checking the payoff matrices of $(\hat{x}, \hat{t})$ in the proof to proposition 3.4 it can be verified that truthtelling is strictly dominant for type L; therefore $(\hat{x}, \hat{t})$ is a strategically non-trivial game only from the point of view of 1_H (type H of buyer 1) and 2_H (type H of buyer 2). If these agents have prior beliefs, as they have after unanimous acceptance of $(\phi^\bullet, \tau^\bullet)$, then 1_H is indifferent between reporting L and playing H *if* 2_H plays H (given that 2_L claims L). If instead 2_H plays L with positive (even small) probability, then 1_H strictly prefers to report L; in turn, this implies that also 2_H wishes to play L. In other words, after deleting announcement H for 1_L and 2_L because it is strictly dominated, both for 1_H and 2_H reporting H is weakly dominated by misreporting. Thus, $\bar{\sigma}$ seems strategically more stable than $\sigma^\bullet$; in addition, it Pareto dominates $\sigma^\bullet$ from the point of view of 1_H and 2_H. Therefore, when the seller proposes $(\hat{x}, \hat{t})$ it appears more sensible to expect that $\bar{\sigma}$ is played in $(\phi^\bullet, \tau^\bullet)$ rather than $\sigma^\bullet$. Recall, however, that both 1_L and 2_L earn a negative payoff in $\bar{\sigma}$.

This result undermines the confidence in $(\hat{x}, \hat{t})$ not only because the revenue is smaller in $\bar{\sigma}$ than in $\sigma^{\bullet}$. Indeed, if also 1_L and 2_L believe that $\bar{\sigma}$ will be played in $(\phi^{\bullet}, \tau^{\bullet})$, then they expect to earn a non-stochastic payoff equal to $\frac{s}{2} - \hat{t}_{LL} < 0$ from playing the auction. Therefore, by proposing $(\hat{x}, \hat{t})$ the seller should expect that no type L is going to participate in the auction; that would induce a quite different outcome (and a different revenue) with respect to unanimous participation and truthtelling. Also notice that this result does not depend on setting ε close to 1: it arises also if the transfers are obtained with $\varepsilon = 0$. LM's optimal mechanism under strong correlation is plagued by the same multiplicity problem; LM do not remark it because they consider the issues of strong ratifiability and multiplicity only under weak correlation.

One way to get rid of this problem is to impose additional constraints in the seller's design problem of (x, t) in order to eliminate equilibria which 1_H and 2_H prefer to $\sigma^{\bullet}$; clearly, this reduces the revenue. In a model with risk-averse agents and without collusion, Demski and Sappington (1984) adopted a similar approach to knock out a non-truthful BNE which Pareto dominates (from the agents' point of view) the truthful equilibrium. We follow this approach in next subsection. Before that, we need to recall that in related settings (still abstracting from collusion issues) Ma (1988), Ma, Moore and Turnbull (1988) and Brusco (1998) used mechanisms with larger strategy spaces with respect to direct mechanisms in order to eliminate "undesired" equilibria. More precisely, they consider multi-stage games and, by appealing to suitable extensive form refinements as sequential equilibrium or perfect BNE, they are able to uniquely implement the outcome which is obtained in the truthtelling equilibrium of the optimal direct mechanism. In these cases, therefore, the planner's welfare is not reduced by requiring unique implementation. Likewise, in our context we could consider the mechanism defined by Brusco (1998) to obtain a sale mechanism M with a unique perfect BNE when it is played noncooperatively; the equilibrium outcome would be the same as the outcome of $\sigma^{\bullet}$ played in $(\hat{x}, \hat{t})$. Moreover, M would be strongly ratifiable. However, we believe that also M would not be very reliable. The reason is easy to explain: when the buyers report into $(\phi^{\bullet}, \tau^{\bullet})$ (a direct mechanism) truthful reporting is not the only equilibrium since also (LL, LL) is an equilibrium in $(\phi^\bullet, \tau^\bullet)$. In other words, even though M is a sale mechanism with a unique (perfect or sequential) equilibrium outcome when it is played noncooperatively, the undesired equilibrium is not eliminated in $(\phi^\bullet, \tau^\bullet)$; indeed, (LL, LL) is a BNE of $(\phi^\bullet, \tau^\bullet)$ and it induces the same outcome as (LL, LL) played in $(\hat{x}, \hat{t})$. At the end of next subsection we provide some comments about attempting to get rid of untruthful equilibria by resorting to a multi-stage collusion mechanism.

3.4.4 A "robust" mechanism

The problem with mechanism $(\hat{x}, \hat{t})$ is that (for any buyers' beliefs) $\bar{\sigma} = (LL, LL)$ is a BNE of $(\hat{x}, \hat{t})$ if it is played noncooperatively [hence $\bar{\sigma}$ is a BNE also in the side mechanism $(\phi^\bullet, \tau^\bullet)$] and $U_H^i(\bar{\sigma}) > U_H^\bullet$. This problem can be avoided by imposing, when designing the sale mechanism, the following constraint on top of (3.8)-(3.17)

$$q[(s+1)x_{HL} - t_{HL}] + h[(s+1)x_{HH} - t_{HH}] \geq (h+q)[(s+1)x_{LL} - t_{LL}] \qquad (3.24)$$

Inequality (3.24) does not rule out $\bar{\sigma}$ as a BNE of the sale mechanism; rather, it guarantees that the payoff to type H in $\bar{\sigma}$ is not larger than $U_H^\bullet$, her payoff when $\sigma^\bullet$ is played. Next proposition describes the mechanism $(\check{x}, \check{t})$ which maximizes expected revenue subject to (3.8)-(3.17) and (3.24). It turns out that $\sigma^\bullet$ is not the unique BNE of $(\check{x}, \check{t})_{p,p}$, but $U_H^i(\sigma) \leq U_H^\bullet$ and $U_L^i(\sigma) \leq U_L^\bullet$ for any $\bar{p}^i$ and $\sigma \in BNE[(\check{x}, \check{t})_{\bar{p}^i,p}]$. This implies that $(\check{x}, \check{t})$ is strongly ratifiable and it can be shown that when playing the side mechanism $(\phi^\bullet, \tau^\bullet)$ the buyers cannot coordinate on an equilibrium which some type strictly prefers to $\sigma^\bullet$. In this sense $(\check{x}, \check{t})$ is a "robust" mechanism.[27]

[27] Instead of imposing (3.24) we could have required that type H prefers to play H rather than L if her opponent reports L: $(s+1)x_{HL} - t_{HL} > (s+1)x_{LL} - t_{LL}$ (actually, weak inequality is necessary to get existence) but then we would have obtained the same mechanism described by proposition 3.5 below. Likewise, we could have imposed that type L prefers to play H when her opponent reports L: $sx_{HL} - t_{HL} \geq sx_{LL} - t_{LL}$. Yet, since it is optimal to set $x_{HL} = 1$, $sx_{HL} - t_{HL} \geq sx_{LL} - t_{LL}$ implies $(s+1)x_{HL} - t_{HL} > (s+1)x_{LL} - t_{LL}$. Hence, imposing (3.24) is the least costly way to obtain a "robust" mechanism.

Proposition 3.5 *(i) If*

$$s(q+l) > h+q \tag{3.25}$$

then $\check{x}_{LL} = \frac{1}{2}$*; if instead (3.25) fails then* $\check{x}_{LL} = 0$*. In either case* $\check{x}_{HH} = \frac{1}{2}$*,* $\check{x}_{HL} = 1$*,* $\check{x}_{LH} = 0$ *and* $\check{t}$ *is determined by (3.8), (3.11), (3.13) and (3.24) written with equality.*

(ii) There exists no BNE of $(\check{x}, \check{t})_{p,p}$ *which some type of buyer strictly prefers to* $\sigma^{\bullet}$*.*

The proof of proposition 3.5(ii) is actually straightforward, since the transfers in $(\check{x}, \check{t})$ are the same as in $(\tilde{x}, \tilde{t})$ (the optimal mechanism if the buyers commit to ex post efficiency within the coalition). To be exact, $(\check{x}, \check{t})$ differs with respect to $(\tilde{x}, \tilde{t})$ in the condition under which the good is sold when both buyers have type L [(3.25) instead of (3.23)]. However, in the proof to proposition 3.4 we established that $(\tilde{x}, \tilde{t})_{p,p}$ has no BNE which some type of buyer prefers to $\sigma^{\bullet}$; such a proof applies to $(\check{x}, \check{t})_{p,p}$ as well. Also notice that in $(\check{x}, \check{t})$ it does not matter whether collusion takes place under symmetric or asymmetric information.

We infer that imposing (3.24) weakens the seller's position in a way which is about equivalent to not being able to prevent the transfer of the good within the coalition. Observe that (3.18) (with ε close to 1) does not imply (3.25): if $\frac{q}{l} < s \leq \frac{h+q}{q+l}$ then $\hat{x}_{LL} = \frac{1}{2}$ but imposing (3.24) induces the seller to withhold the good when both buyers have type L. In such a case his revenue loss with respect to the truthful equilibrium of $(\hat{x}, \hat{t})$ is equal to $sl - q$; on the other hand, if $s > \frac{h+q}{q+l}$ then the revenue loss due to (3.24) is $\frac{\rho}{l+q}$.[28]

To conclude we offer some remarks about attempting to rule out untruthful equilibria by considering a collusion mechanism which is different from the side mechanism we have described in section 3.3. Following the literature cited in the previous subsection, we may try to design a multi-stage collusion mechanism having a unique sequential equilibrium or perfect BNE when $(\hat{x}, \hat{t})$ is the sale mechanism. This does not appear to be an easy

[28]We should also remark that we "solved" the multiplicity problem within the class of symmetric mechanisms. However, in principle, asymmetric mechanisms may help, as it occurs in Demski and Sappington (1984).

task and, moreover, before doing that we should inquire whether the third party is interested in solving the multiplicity problem. Recall that if the buyers report truthfully in $(\phi^\bullet, \tau^\bullet)$ when $(\hat{x}, \hat{t})$ is the sale mechanism, then each type H gets an expected payoff equal to about $\frac{q}{2(h+q)}$ and each type L earns 0; yet, as we know, the fact that $\bar{\sigma}$ is an equilibrium of $(\phi^\bullet, \tau^\bullet)$ is a threat to the seller's revenue. If the seller avoids that threat by imposing (3.24) on the sale mechanism, then each type L still obtains a payoff equal to 0; each type H earns instead $\frac{l}{2(l+q)} > \frac{q}{2(h+q)}$ if (3.25) holds and 0 otherwise. Thus the third party is better off with $(\check{x}, \check{t})$ rather than with (the truthful equilibrium of) $(\hat{x}, \hat{t})$ if $\check{x}_{LL} = \frac{1}{2}$. The only case in which it prefers $(\hat{x}, \hat{t})$ to $(\check{x}, \check{t})$ is when $\hat{x}_{LL} = \frac{1}{2}$ but $\check{x}_{LL} = 0$, which occurs if $\frac{q}{l} < s \leq \frac{h+q}{l+q}$; for these parameter values each type of buyer earns 0 in $(\check{x}, \check{t})$. Thus, when $\frac{q}{l} < s \leq \frac{h+q}{l+q}$ both the seller and the buyers prefer $(\hat{x}, \hat{t})$ to $(\check{x}, \check{t})$. Apart from this case, the third party should not help the seller to solve the multiplicity problem in a less expensive (for the seller) way with respect to satisfying (3.24). Designing a collusion mechanism with a unique sequential equilibrium or perfect BNE when $\frac{q}{l} < s \leq \frac{h+q}{l+q}$ appears to be a complicated task (among other things, recall that the collusion mechanism needs to be budget balanced) and it is out of the scope of this chapter.

3.5 Appendix

Proof of proposition 3.1 The following are the payoff matrices for type L and H, respectively, in mechanism $(x^\bullet, t^\bullet)$. For example, the entry in the first matrix corresponding to row H and column HL is the expected payoff to type L from announcing H when the opponent always misreports her own valuation (type L of the opponent reports H and type H claims L)

type L	LL	LH	HL	HH
L	$\frac{q^2}{2\rho} + \frac{q}{\rho}k_H$	0	$\frac{q(q-l)}{2\rho} + \frac{q-l}{\rho}k_H$	$-\frac{ql}{2\rho} - \frac{l}{\rho}k_H$
H	$\frac{(1-l)q}{2\rho} - \frac{h}{\rho}k_L$	$-\frac{k_L}{l+q}$	$\frac{(1-l)(q-l)}{2\rho} + \frac{q(l-h)}{\rho(l+q)}k_L$	$-\frac{(1-l)l}{2\rho} + \frac{q}{\rho}k_L$

type H	LL	LH	HL	HH
L	$\frac{hl}{2\rho}+\frac{q}{\rho}k_H$	$-\frac{k_H}{q+h}$	$\frac{l(h-q)}{2\rho}+\frac{q(h-l)}{\rho(q+h)}k_H$	$-\frac{ql}{2\rho}-\frac{l}{\rho}k_H$
H	$\frac{h(2l+q)}{2\rho}-\frac{h}{\rho}k_L$	0	$\frac{(2l+q)(h-q)}{2\rho}+\frac{q-h}{\rho}k_L$	$-\frac{q(2l+q)}{2\rho}+\frac{q}{\rho}k_L$

As we remarked in subsection 3.2.2, if $k_H = k_L = 0$ then (HH, LL) is a BNE (in it, both types of buyer 1 report H and both types of buyer 2 report L) as well as (LL, HH). Yet, if $k_H = k_H^\bullet$ and $k_L = k_L^\bullet$ as defined in the statement of proposition 3.1, then for type H it is strictly dominant to report H and type L strictly prefers truthtelling when her opponent plays LH or HH. Therefore, by iterative deletion of strictly dominated strategies we find that (LH, LH) is the unique BNE of $(x^\bullet, t^\bullet)$ if $(k_H, k_L) = (k_H^\bullet, k_L^\bullet)$. ■

Proof of proposition 3.2(i) Fix any given $\varepsilon \in [0, 1)$. By theorem 1.D.5 in Takayama (1985), in our setting the first order conditions are necessary and sufficient for optimality. In the lagrangian function, let λ_1 denote the multiplier of constraint (3.8), λ_2 is the multiplier of constraint (3.9), ... and λ_{10} is the multiplier of (3.17). Then we find, since $t \in \Re^4$

$$\begin{cases} \frac{\partial L}{\partial t_{HH}} = h + q\lambda_2 - h\lambda_3 - h\lambda_4 - 2\lambda_5 - 2\lambda_6 + 2\lambda_7 + 2\lambda_9 = 0 \\ \frac{\partial L}{\partial t_{HL}} = q + l\lambda_2 - q\lambda_3 - q\lambda_4 + \lambda_5 - \lambda_7 - \lambda_8 + \lambda_{10} = 0 \\ \frac{\partial L}{\partial t_{LH}} = q - q\lambda_1 - q\lambda_2 + h\lambda_4 + \lambda_5 - \lambda_7 - \lambda_8 + \lambda_{10} = 0 \\ \frac{\partial L}{\partial t_{LL}} = l - l\lambda_1 - l\lambda_2 + q\lambda_4 + 2\lambda_6 + 2\lambda_8 - 2\lambda_9 - 2\lambda_{10} = 0 \end{cases} \tag{3.26}$$

Assuming that only multipliers λ_1, λ_4, λ_6 and λ_8 are different from 0, the unique solution to (3.26) is $\hat{\lambda}_1 = \frac{1}{q+l}$, $\hat{\lambda}_4 = \frac{q}{\rho+q}$, $2\hat{\lambda}_6 = \frac{h\rho}{\rho+q}$ and $\hat{\lambda}_8 = \frac{\rho q}{\rho+q}$. These values imply $\frac{\partial L}{\partial x_{HH}} > 0$ and $\frac{\partial L}{\partial x_{HL}} > \max\left\{\frac{\partial L}{\partial x_{LH}}, 0\right\}$; hence $\hat{x}_{HH} = \frac{1}{2}$, $\hat{x}_{HL} = 1$ and $\hat{x}_{LH} = 0$. To determine $\hat{x}_{LL}$ we compute $\frac{\partial L}{\partial x_{LL}} = sl\lambda_1 - q(s+1)\lambda_4 - 2(s+1)\lambda_6 - (2s+1-\frac{h}{q}\varepsilon)\lambda_8 = \frac{(sl-q)(\rho+q)-h\rho(1-\varepsilon)}{\rho+q}$.

If (3.18) fails then $\hat{x}_{LL} = 0$ and the constraints corresponding to λ_1, λ_4, λ_6 and λ_8 [i.e., (3.8), (3.11), (3.13) and (3.15)] written as equalities yield, given $\hat{x}$, $\hat{t}_{HH} = \frac{s+1}{2}$, $\hat{t}_{HL} = s + 1$ and $\hat{t}_{LH} = \hat{t}_{LL} = 0$. Also the neglected constraints (3.9), (3.10), (3.12), (3.14), (3.16) and (3.17) hold in this mechanism, therefore it is optimal when (3.18) fails.

If instead (3.18) holds then $\hat{x}_{LL} = \frac{1}{2}$ and (3.8), (3.11), (3.13) and (3.15) written with equality imply $\hat{t}_{HH} = \hat{t}_{LL} = \frac{s}{2} + \frac{hq(1-\varepsilon)}{2(\rho+q)}$, $\hat{t}_{LH} = \frac{-hl(1-\varepsilon)}{2(\rho+q)}$ and $\hat{t}_{HL} = s + \frac{h+q}{2q} - \frac{h(\rho+hq)(1-\varepsilon)}{2q(\rho+q)}$. Since also the neglected constraints are met, this mechanism is optimal if (3.18) holds.

(ii) From $\hat{x}_{LH} = 0$ follows $\hat{x}_{LH} - \hat{x}_{LL} \leq 0$. If $sl \leq q$ then (3.18) fails for any $\varepsilon \in [0, 1)$, hence $\hat{x}_{LL} = \hat{x}_{LH} - \hat{x}_{LL} = 0$ and the exact value of ε does not matter. If instead $sl > q$ then (3.18) holds if ε is close to 1; thus $\hat{x}_{LL} = \frac{1}{2}$, $\hat{x}_{LH} - \hat{x}_{LL} = -\frac{1}{2}$ and ε should be as close as possible to 1 in order to relax (3.15). Since $\hat{\lambda}_8 > 0$, the revenue is strictly increasing in ε and no "globally optimal" mechanism exists.

(iii) By considering $\varepsilon = 1$ we obtain the sup of the revenue. If $sl \leq q$ then $\hat{x}_{LL} = 0$ and the good is not efficiently allocated; hence the first best outcome is not achieved. If instead $sl > q$ then $\hat{x}_{LL} = \frac{1}{2}$ but the expected payoff of type H equals $\frac{q[(s+1)\hat{x}_{HL} - \hat{t}_{HL}] + h[(s+1)\hat{x}_{HH} - \hat{t}_{HH}]}{h+q} = \frac{q(\rho+q) + h\rho(1-\varepsilon)}{2(\rho+q)(h+q)} \simeq \frac{q}{2(h+q)} > 0$; this reduces the revenue with respect to full surplus extraction. Yet, such a reduction is negligible if q is close to 0 (i.e., if v^1 and v^2 are strongly correlated). Hence, in that case the seller almost obtains the first best revenue: the good is efficiently allocated and each type's rent is nearly 0. ■

Proof of proposition 3.3(i) As in the proof to proposition 3.2(i), λ_1 is the multiplier of (3.8), ... and λ_{10} is the multiplier of (3.17) [λ_7 and λ_8 are the multipliers of (3.21) and (3.22), respectively]; again, since $t \in \Re^4$, we obtain (3.26) from $\frac{\partial L}{\partial t} = 0$. As above, we suppose that only multipliers λ_1, λ_4, λ_6 and λ_8 are different from 0, which yields $\tilde{\lambda}_1 = \frac{1}{q+l}$, $\tilde{\lambda}_4 = \frac{q}{\rho+q}$, $2\tilde{\lambda}_6 = \frac{h\rho}{\rho+q}$ and $\tilde{\lambda}_8 = \frac{\rho q}{\rho+q}$; hence $\frac{\partial L}{\partial x_{HH}} > 0$, $\frac{\partial L}{\partial x_{HL}} > \max\left\{\frac{\partial L}{\partial x_{LH}}, 0\right\}$ and $\frac{\partial L}{\partial x_{LL}} = sl\lambda_1 - q(s+1)\lambda_4 - 2(s+1)(\lambda_6 + \lambda_8) = \frac{l}{q+\rho}[(s+1)\rho + sq - h]$. Thus $\tilde{x}_{HH} = \frac{1}{2}$, $\tilde{x}_{HL} = 1$, $\tilde{x}_{LH} = 0$ and $\tilde{x}_{LL} = 0$ if (3.23) fails. In this case (3.8), (3.11), (3.13) and (3.22) written with equality imply $\tilde{t}_{HH} = \frac{s+1}{2}$, $\tilde{t}_{HL} = s + 1$ and $\tilde{t}_{LH} = \tilde{t}_{LL} = 0$. Since also the neglected constraints are met by this mechanism, $(\tilde{x}, \tilde{t})$ is optimal when (3.23) fails.

If $(s + 1)\rho + sq > h$ then $\tilde{x}_{LL} = \frac{1}{2}$ and (3.8), (3.11), (3.13) and (3.22) written with equality give $\tilde{t}_{HH} = \tilde{t}_{LL} = \frac{s}{2} + \frac{q}{2(l+q)}$, $\tilde{t}_{LH} = \frac{-l}{2(l+q)}$ and $\tilde{t}_{HL} = s + \frac{1-h}{2(l+q)}$ (given that $\tilde{x}_{HH} = \frac{1}{2}$, $\tilde{x}_{HL} = 1$ and $\tilde{x}_{LH} = 0$). The neglected no-collusion constraints hold as $2\tilde{t}_{HH} = 2\tilde{t}_{LL} = \tilde{t}_{HL} + \tilde{t}_{LH}$: the total sum the buyers are required to pay to receive the good does not depend on their reports, a fact which deters collusion. Moreover, the expected payoff of each type H is $\frac{l}{2(q+l)} > 0$ and also (3.9) is satisfied.

(ii) If l is close to 0 then also ρ and q are about 0 (as $\rho > 0$); thus (3.23) fails and $\tilde{x}_{LL} = 0$. Therefore the seller never gets close to extract the full surplus. ■

Proof of proposition 3.4 We examine what buyer i can obtain by rejecting the side mechanism if we do allow arbitrary beliefs $\bar{p}^i$ of buyer $3-i$ about v^i after a veto of buyer i and do not require $\sigma^i = \sigma^\bullet$. Often, that does not alter the buyers' participation constraints in the side mechanism. In such a case $(\phi^\bullet, \tau^\bullet)$ still solves the third party's problem since it is feasible and $(\hat{x}, \hat{t})$ and $(\tilde{x}, \tilde{t})$ satisfy (3.7) and (3.20), respectively. In one case the buyers' participation constraints in the side mechanism are tightened; then no feasible side mechanism exists, hence the game of coalition formation has no equilibrium.

First of all, notice that in both $(\hat{x}, \hat{t})$ and $(\tilde{x}, \tilde{t})$ truthtelling is always strictly dominant for type L. Let us start by examining $(\hat{x}, \hat{t})$ when $\hat{x}_{LL} = 0$ [the same following argument applies to $(\tilde{x}, \tilde{t})$ when $\tilde{x}_{LL} = 0$, since $(\hat{x}, \hat{t}) = (\tilde{x}, \tilde{t})$ if $\hat{x}_{LL} = \tilde{x}_{LL} = 0$]. We report below the payoff matrices for type L and type H in $(\hat{x}, \hat{t})$ when $\hat{x}_{LL} = 0$. For example, the entry in the left table corresponding to row H and column L (that is, -1) is the payoff to type L if she claims H and her opponent reports L.

type L	L	H
L	0	0
H	-1	$-\frac{1}{2}$

type H	L	H
L	0	0
H	0	0

Observe that the payoff to type H is 0 regardless of both buyers' reports; the payoff to type L is 0 if she plays L (which is strictly dominant for her) regardless of the opponent's report. Therefore $U^i_j(\sigma) = 0 = U^\bullet_j$, $j = L, H$, for any belief system $\bar{p}^i$ and $\sigma \in BNE[(\hat{x}, \hat{t})_{\bar{p}^i, p}]$.

The payoff matrices for $(\tilde{x}, \tilde{t})$ with $\tilde{x}_{LL} = \frac{1}{2}$ are

type L	L	H
L	$-\frac{q}{2(q+l)}$	$\frac{l}{2(q+l)}$
H	$-\frac{1-h}{2(q+l)}$	$-\frac{q}{2(q+l)}$

type H	L	H
L	$\frac{l}{2(q+l)}$	$\frac{l}{2(q+l)}$
H	$\frac{l}{2(q+l)}$	$\frac{l}{2(q+l)}$

Again, the payoff to type H is constant, equal to $\frac{l}{2(l+q)} = U_H^\bullet$, with respect to both her report and the opponent's report. Type L of buyer i obtains a higher expected payoff than $U_L^\bullet = 0$ if the probability (given her information $v^i = v_L$) that buyer $3-i$ reports H is higher than $\frac{q}{q+l}$. Yet, since for type L of buyer $3-i$ truthful reporting is strictly dominant, there exists no $\bar{p}^i$ and $\sigma \in BNE[(\tilde{x}, \tilde{t})_{\bar{p}^i,p}]$ such that $U_L^i(\sigma) > 0$ (recall that the deviating buyer i has prior beliefs about v^{3-i}). There actually exist $\sigma \in BNE[(\tilde{x}, \tilde{t})_{\bar{p}^i,p}]$ such that $U_L^i(\sigma) < 0$, but for sure type L would reject a side mechanism yielding her a negative payoff even under the threat that the noncooperative play of the sale mechanism gives her a more negative payoff: she would rather quit the game. Hence it looks reasonable to fix to $0 = U_L^\bullet$ the payoff to each type L from rejecting the side mechanism.

We now examine $(\hat{x}, \hat{t})$ when $\hat{x}_{LL} = \frac{1}{2}$:

type L	L	H
L	$-\frac{hq(1-\varepsilon)}{2(q+\rho)}$	$\frac{hl(1-\varepsilon)}{2(q+\rho)}$
H	$-\frac{h+q}{2q} + \frac{h(\rho+hq)(1-\varepsilon)}{2q(q+\rho)}$	$-\frac{hq(1-\varepsilon)}{2(q+\rho)}$

type H	L	H
L	$\frac{1}{2} - \frac{hq(1-\varepsilon)}{2(\rho+q)}$	$\frac{hl(1-\varepsilon)}{2(q+\rho)}$
H	$\frac{q-h}{2q} + \frac{h(\rho+hq)(1-\varepsilon)}{2q(q+\rho)}$	$\frac{1}{2} - \frac{hq(1-\varepsilon)}{2(\rho+q)}$

Since type L of buyer i reports L (and also type L of buyer $3-i$ reports L), her payoff is equal to $0 = U_L^\bullet$ if type H of buyer $3-i$ plays H and it is negative otherwise; hence, arguing as above, the participation constraint of type L of buyer i in the side mechanism should be $u_L^i \geq 0$. The only BNE of $(\hat{x}, \hat{t})_{\bar{p}^i,p}$ in which type H of buyer i earns a payoff which is not equal to $U_H^\bullet \simeq \frac{q}{2(h+q)}$ is $\bar{\sigma} = (LL, LL)$: $U_H^i(\bar{\sigma}) \simeq \frac{1}{2} > \frac{q}{2(h+q)}$. However, letting $\bar{\sigma}$ play the role of σ^1 (or σ^2) in definition 3.1 is troublesome because

$U_L^1(\bar{\sigma}) = -\frac{hq(1-\varepsilon)}{2(q+\rho)} < 0$ and, as we made clear above, we do think that any type of buyer would quit the auction rather than accepting a side mechanism which yields her a negative payoff.

For the sake of completeness (or curiosity) we may consider the third party's design problem with the following participation constraints: $u_L^1 \geq 0 = U_L^\bullet$, $u_H^1 \geq U_H^1(\bar{\sigma})$, $u_L^2 \geq 0 = U_L^\bullet$ and $u_H^2 \geq U_H^\bullet$; in this way we are violating definition 3.1 because we implicitly assume that the BNE of $(\hat{x}, \hat{t})$ which is played following a veto of buyer 1 depends on the type of buyer 1 ($\bar{\sigma}$ if $v^1 = v_H$ and $\sigma^\bullet$ if $v^1 = v_L$). Nevertheless, in this case no equilibrium of the game of coalition formation exists; hence $(\hat{x}, \hat{t})$ is strongly collusion-proof. Indeed, if an incentive compatible side mechanism $(\ddot{\phi}, \ddot{\tau})$ satisfied the participation constraints written above [$(\phi^\bullet, \tau^\bullet)$ does not since it violates inequality $u_H^1 \geq U_H^1(\bar{\sigma})$], then it would be feasible also when $\sigma^1 = \sigma^2 = \sigma^\bullet$ and $\bar{p}^1 = \bar{p}^2 = p$ as in that case the participation constraint of type H of buyer 1 is relaxed. Since the third party's payoff would be higher with $(\ddot{\phi}, \ddot{\tau})$ than with $(\phi^\bullet, \tau^\bullet)$, it would be contradicted the fact that $(\phi^\bullet, \tau^\bullet)$ is optimal for the third party when $\sigma^1 = \sigma^2 = \sigma^\bullet$ and $\bar{p}^1 = \bar{p}^2 = p$. ■

Proof of proposition 3.5(i) Once again, λ_1 is the multiplier of constraint (3.8), ..., λ_{10} is the multiplier of (3.17) and λ_{11} is the multiplier of (3.24). From $\frac{\partial L}{\partial t} = 0$ we find

$$\begin{cases} \frac{\partial L}{\partial t_{HH}} = h + q\lambda_2 - h\lambda_3 - h\lambda_4 - 2\lambda_5 - 2\lambda_6 + 2\lambda_7 + 2\lambda_9 - h\lambda_{11} = 0 \\ \frac{\partial L}{\partial t_{HL}} = q + l\lambda_2 - q\lambda_3 - q\lambda_4 + \lambda_5 - \lambda_7 - \lambda_8 + \lambda_{10} - q\lambda_{11} = 0 \\ \frac{\partial L}{\partial t_{LH}} = q - q\lambda_1 - q\lambda_2 + h\lambda_4 + \lambda_5 - \lambda_7 - \lambda_8 + \lambda_{10} = 0 \\ \frac{\partial L}{\partial t_{LL}} = l - l\lambda_1 - l\lambda_2 + q\lambda_4 + 2\lambda_6 + 2\lambda_8 - 2\lambda_9 - 2\lambda_{10} + (h+q)\lambda_{11} = 0 \end{cases} \tag{3.27}$$

Consider the solution to (3.27) in which $\check{\lambda}_1 = \frac{1}{q+l}$, $\check{\lambda}_4 = \frac{q(h+q)}{h(q+l)}$, $\check{\lambda}_{11} = \frac{\rho}{h(q+l)}$ and each other multiplier is equal to 0; then $\check{x}_{HH} = \frac{1}{2}$, $\check{x}_{HL} = 1$ and $\check{x}_{LH} = 0$. Moreover, $\frac{\partial L}{\partial x_{LL}} = ls\lambda_1 - q(s+1)\lambda_4 - (s+1)(h+q)\lambda_{11} = \frac{l}{q+l}[s(q+l) - h - q]$; thus $\check{x}_{LL} = 0$ if (3.25) fails and $\check{x}_{LL} = \frac{1}{2}$ otherwise. In the first case (3.8), (3.11) and (3.24) written with equality [in addition to constraints (3.13) and (3.15)] imply $\check{t}_{HH} = \frac{s+1}{2}$, $\check{t}_{HL} = s + 1$ and $\check{t}_{LH} = \check{t}_{LL} = 0$; instead, $\check{t}_{HH} = \check{t}_{LL} = \frac{s}{2} + \frac{q}{2(l+q)}$, $\check{t}_{LH} = \frac{-l}{2(l+q)}$ and $\check{t}_{HL} = s + \frac{1-h}{2(l+q)}$ in the second case [using (3.8), (3.11) and (3.24) written with equality in addition to (3.13) and (3.16)]; in both cases (3.13) binds even though $\check{\lambda}_6 = 0$. Observe that we obtained the same transfers as in mechanism $(\tilde{x}, \tilde{t})$, derived in proposition 3.3. ■

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