Learning in Monetary Economics

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UNIVERSITAT POMPEU FABRA
Doctoral Thesis

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Barcelona, December 2006

Dipòsit legal: B.53875-2007
ISBN: 978-84-691-1194-9

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Acknowledgements

I am indebted to so many people for the completion of my Ph.D., that I would need another thesis chapter to mention them all.

First of all, I want to thank my supervisor Ramon Marimon, whose guidance and encouragements have been determinant in leading me towards the end of my thesis. Not only he taught me a lot about modern economics, but he also showed me, through his example and critics, how good research should be done.

I am especially grateful also to Albert Marcet, whose sharp comments helped me to improve significantly many parts of the thesis.

I am also indebted to the teachers I had at Pompeu Fabra, who shared with me their knowledge and deep understanding of all the subjects in the economics spectrum. In particular, I want to mention Kosuke Aoki, Jordi Galí and Gábor Lugosi, from whom I learnt more than I could have ever hoped for, and who have always been ready to help me.

A special thank goes to Giancarlo Gandolfo, who showed me how fascinating economics and mathematics can be. Without his support and advices at the early stages of my postgraduate career, not a single page of this thesis would have been written.

Many colleagues and friends shared with me the bad and good times of my years in Barcelona, making the formers easier, and the latters happier. Among them, a special thank is for Krisztina Molnár, for many fruitful conversations on economics, that lead to a common project that is part of this thesis, and for her friendship. Michele Caivano, Marco Cosconati, Fabrizio Ferrari, Francesco Sangiorgi, Raffaella Sadun and Giorgio Zanarone have also played a significant role at various stages of my years in Barcelona.

I am also grateful to my long time friends Alessandro, Andrea and Francesco, who have always believed in me more than myself. At last, but not the least, I want to thank my brother Andrea, for his continuous support and precious advices, and my parents Italico and Silvana, who showed me how a man should be. I hope to be able, one day, to match that standard.

Foreword

Building on the seminal paper of Marcet and Sargent (1989), in the last fifteen years adaptive learning emerged as a widely used alternative to rational expectations in modelling the formation of agents beliefs. This paradigm assumes that agents behave as econometricians: they have in mind a model of the economy, and estimate its coefficients using observed data. Most of this literature is mainly concerned about the asymptotic properties of the learning algorithms, and in particular if -and under which conditions- a rational expectations equilibrium in a given model can be achieved as a limit point of an adaptive scheme. This property has been often invoked also as an equilibrium selection device: if a rational expectations equilibrium is such that a small departure from it makes the system diverge forever, then it may not be an economically relevant equilibrium. Much less effort has been devoted to investigate the implications of the introduction of learning along the transition to the limit of the adaptive algorithm. In particular, it may have relevant consequences both from a positive point of view, helping to explain stylized facts, and from a normative point of view, possibly changing, with respect to the rational expectations framework, the advices that a benevolent planner should follow. My thesis applies the tools of the adaptive learning literature to otherwise standard monetary models with monopolistic competition, focusing on some consequences that this assumption on agents beliefs may have also along the transition to the limit of the adaptive algorithm. In the first chapter I take a positive point of view and investigate to what extent adaptive learning can help a simple flexible prices model to match certain empirical features, which are difficult to replicate in the rational expectations, staggered price Calvo model, which is by now the workhorse in most of the monetary literature. I found that when the source of stickiness in firms' pricing behavior is given by backward-looking expectations, the model generates inflation and output dynamics that are broadly consistent with several stylized facts. As a side issue, I also highlight the importance of endogenizing the free parameters that the use of adaptive algorithms introduce into the setup, linking them to some optimal behavior of the agents. In the rest of the thesis I take a normative approach, and reassess some monetary policy issues when the rational expectations hypothesis is relaxed. In the second chapter, I study the optimal monetary policy problem for a fully rational central bank, when the private sector forms its beliefs according to an adaptive learning algorithm. I show that this assumption on the expectations formation is not just an embellishment of the model, but it has potentially relevant policy implication for the monetary authority. In the third chapter I assume that a fraction of agents has rational expectations, and the rest are learners, and investigate the consequences of this framework for the design of monetary policy: in particular, I show that, when the central bank follows simple interest rate rules, the possibility of the resulting equilibrium being indeterminate or explosive is potentially affected by the presence in the economy of a fraction of agents endowed with backward-looking expectations.

Chapter 1
Adaptive Learning and Inflation Dynamics in a Flexible Price Model

In most of the recent macroeconomics literature, the sticky reaction of prices in response to changes in aggregate conditions has been modelled following the highly influential contribution of Calvo (1983). However, this approach has difficulties in accounting for some well-established stylized facts, like the sluggish and delayed response of inflation to demand shocks, and the positive correlation between real output and the rate of change of inflation. In this paper, we will investigate the possibility of a simple flexible prices and monopolistic competition model to match this features, when the expectations of the firms are formed following the adaptive learning literature. The main result is that, with reasonable parameters values, this setup can considerably improve the performance of the Calvo model, generating inflation and output dynamics that are broadly consistent with the two stylized facts above mentioned; moreover, also the inflation autocorrelation is not at odds with what is empirically observed. As a side issue, I also studied the relationship between the gain parameter in the updating scheme of the beliefs and inflation autocorrelation to show how, keeping this parameter free to assume any value, we could make the model match almost any empirical pattern, hence stressing the importance of endogenizing this coefficient, linking it to some optimal behavior of the agents.

Chapter 2
Optimal Monetary Policy when Agents are Learning

joint with Krisztina Molnár

Most studies of optimal monetary policy under learning rely on optimality conditions derived for the case when agents have rational expectations. In this paper, we derive optimal monetary policy in an economy where the Central Bank knows, and makes active use of, the learning algorithm agents follow in forming their expectations. In this setup, monetary policy can influence future expectations through its effect on learning dynamics, introducing an additional trade-off between inflation and output gap stabilization. Specifically, the optimal interest rate rule reacts more aggressively to out of equilibrium inflation expectations and noisy cost-push shocks than would be optimal under rational expectations: the Central Bank exploits its ability to "drive" future inflation expectations closer to equilibrium. This optimal policy qualitatively resembles optimal policy when the Central Bank can commit and agents have rational expectations. Monetary policy should be more aggressive in containing inflation expectations when private agents pay more attention to recent data. In particular, when beliefs are updated according to recursive least squares, the optimal policy is time-varying: after a structural break the Central Bank should be more aggressive and relax the degree of aggressiveness in subsequent periods. The policy recommendation is robust: under our policy the welfare loss if the private sector actually has rational expectations is much smaller than if the Central Bank mistakenly assumes rational expectations whereas in fact agents are learning.

Chapter 3
Monetary Policy with Heterogenous Expectations

When agents have rational expectations, monetary policy literature has emphasized the importance of the Taylor principle, namely a reaction of the Central Bank to inflation sufficiently aggressive. We assume heterogeneous expectations, in the sense that a fraction of agents has rational expectations, and the rest has backward-looking expectations, updated according to the adaptive learning literature. We address determinacy issues related to the use of different policy rules, with the interest rate responding to current, past or (expected) future values of inflation and output gap. We show that the Taylor principle retains its validity as a criterion to assess the desirability of a monetary policy rule, in the sense that it is a necessary condition for determinacy. However, the complete characterization of the determinacy region shows differences related to the timing of the endogenous variables in the Taylor rule: when the interest rate reacts to current values of inflation and output gap, the determinacy region is the same under rational and heterogeneous expectations; when it responds to past realizations (forecasts of future values) of the same variables, the determinacy region is smaller (larger) under heterogeneous expectations than under rational expectations. Finally, when the equilibrium is determinate, the learners' beliefs settle down to a stationary distribution around the rational expectation values of the endogenous variables. The policy implication is that the Central Bank should not pursue a passive monetary policy, but the computation of the maximum aggressiveness consistent with a determinate equilibrium requires a deep understanding of how the private sector forms its expectations.

Chapter 1

Adaptive Learning and Inflation Dynamics in a Flexible Price Model

1.1 Introduction

In most of the recent macroeconomics literature, the sticky reaction of prices in response to changes in aggregate conditions has been modelled following the highly influential contribution of Calvo (1983). In particular, it is assumed that, in each period, firms face a constant probability to reset prices optimally. Due to its appealing analytical tractability, this approach has become the workhorse of most of monetary policy literature (see for example Clarida et al. (1999)).

However, this model has difficulties in explaining some well-established stylized facts: in particular, two robust features of data are a sluggish and delayed response of inflation to demand shocks (e.g., see Christiano et al. (2005)), and a positive correlation between real output and the rate of change of inflation (the so-called acceleration phenomenon, see Mankiw and Reis (2002)). Both of these patterns are not replicated by the Calvo-type staggered price settings; the main theoretical reason is the fact that, despite of the stickiness of price level, inflation can respond rapidly to exogenous shocks.

In this chapter, we propose a different source of intrinsic stickiness in firms' pricing behavior: in particular, we assume that firms do not have an exact knowledge of the "true" economic model, but form their expectations according to their most recent estimates of the law of motion of the unknown aggregate variables. This approach is typical of the adaptive learning literature, which has received an increasing attention in recent years (see Evans and Honkapohja (2001) for an extensive monograph).

Up to now, most effort in this literature has been devoted to the issue of stability under learning, namely under which conditions a rational expectations equilibrium is a limiting point of the learning process; only recently there have been attempts to evaluate quantitatively the effect of introducing adaptive learning into a macro or finance model, and to test the ability of this framework to explain empirical facts[1]. This may be due to the caveat that learning can introduce too many degrees of freedom in the model, allowing the researcher to match any pattern of data just playing with the learning algorithm.

The aim of this chapter is to build a simple flexible price model of monopolistic competition, augmented by the adaptive learning formation of agents' expectations, and to investigate (via numerical simulations) whether this is able to outperform the staggered price model in replicating the above mentioned features of data. In doing so, we will try to use a learning scheme with a basis in a payoff-maximizing choice of the agents, in order to make less severe the potential criticism toward *ad hoc* learning procedures. To assess the goodness of fit of the model, we will use some techniques to evaluate calibrated dynamic general equilibrium stochastic models presented in the survey of Canova and Ortega (2000).

It is worth noting that the approach we will develop in this chapter is conceptually linked to the one of Mankiw and Reis (2002), who assume that firms are free to reset prices in each period, but that information diffuses slowly among them; in their model, in each period firms face a constant probability to update their information set.

The rest of the chapter is organized as follows. In Section 1.2 we present some stylized facts, emphasizing that they can be hardly reconciled with the Calvo model. In Section 1.3 we present the model, and discuss some issues related to the additional degrees of freedom introduced by the presence of learning. Section 1.4 shows the results of the simulations. In Section 1.5 we

[1] See Timmermann (1993) and Timmermann (1996) for applications to the stock market, Marcet and Nicolini (2003) for a model that aims to explain hyperinflations in South american countries, and Sargent (1999) for and explanation of the change in the U.S. inflation pattern.

check that our findings are robust to changes in the variable tracked by the firms, and Section 1.6 concludes.

1.2 Stylized Facts and Calvo Model

In recent work on monetary policy issues, the standard tool to model the firms' pricing behavior has been the so-called Calvo model: in each period, firms have a constant probability to reset their price optimally, and they do so taking into account that such a price will last for an unknown number of periods. Since this probability is independent of the last time a specific firm has reset its price, this approach leads to an analytically tractable framework which constitutes one of the building blocks of what Clarida et al. (1999) call the New Keynesian Science of Monetary Policy.

Besides its theoretical appeal, this approach has done well in replicating certain empirical patterns (like, for example, the high autocorrelation of inflation); nevertheless, it has shown many difficulties in explaining some well-established stylized facts that are also common wisdom of policymakers. In particular, we will concentrate on two features of the Calvo model that are at odds with empirical evidence:

- many empirical investigations (and conventional wisdom of central bankers) show that nominal shocks have a sluggish and delayed effect on inflation, and that the impulse response of inflation is hump-shaped[2]. Instead, the Calvo model is characterized by a monotonic decreasing impulse response function[3];

- another widely documented empirical fact is a positive relationship between real output and the growth rate of inflation. This pattern, that Mankiw and Reis (2002) call acceleration phenomenon, has been shown through the use of scatterplots by many economists[4], and has been confirmed by Mankiw and Reis (2002) calculating the correlation between real output and the growth rate of inflation for U.S. data. Instead, the Calvo model predicts a slightly negative value for this statistics.

[2]See Christiano et al. (1998) and (2005).

[3]For more details, see the Appendix.

[4]See, e.g. Abel and Bernanke (1998) and Blanchard (2000).

Both of these shortcomings are originated by the fact that in the baseline version of the Calvo model inflation is a purely forward looking-variable, without any intrinsic source of inertia; to overcome this problem, there have been many attempts to improve the empirical fit of the New Keynesian framework through the introduction of some source of inertia in the Phillips curve[5].

Some alternative avenues to reconcile monetary theory with data have been tried. In particular, some recent papers have abandoned the approach of assuming that firms face some kind of constraint on their possibility of resetting each period their prices; instead, they assume that prices are fully flexible, but that the information set of the firms is somehow constrained.

In this spirit, Mankiw and Reis (2002) introduced the so-called "sticky information Phillips curve": in their model, the information is supposed to spread slowly across the economy, so that each firm faces every period a constant probability to update its information set. Since it is free to reset prices every period, the firm would set prices such that its expected profit, given the latest update of the information set, is maximized. In Mankiw and Reis (2002), the authors show that this alternative assumption can outperform the Calvo model in a simple business cycle framework[6].

Another example of this new strand of literature is the model of Woodford (2001) where, following the pioneering idea of Phelps (1970) and the highly influential paper of Lucas (1972), the firms are assumed not to be able to observe correctly the level of the aggregate variables, and to take their decisions on the basis of their subjective expectations. The main difference with the Lucas' model is that the information constraint is not simply a one-period delay in the aggregate variables' observability, but, following Sims (2003), "a limited capacity of the private decision-makers to *pay attention* to all of the information in their environment". In this way, the agents' decision process is converted into a signal-extraction problem: in fact, letting q_t be the nominal income, each firm i can observe in period t only the private signal $z_t(i)$ of the form:

$$z_t(i) = q_t + v_t(i)$$

where $v_t(i)$ is an idiosyncratic noise term. Woodford uses this model to study the impulse response functions of inflation and output, and finds that

[5]For example, Rotemberg and Woodford (1997) have introduced a decision delay for some price setters, and Christiano et al. (2005) have added various other source of nominal rigidity together with price stickiness, like adjustment costs and wage stickiness.

[6]For an application of the sticky information Phillips curve to optimal monetary issues, see Ball et al. (2005).

it can replicate data patterns better than the Calvo approach, at least for a reasonable region of the parameters' space.

One of the most common ways to model the economic behavior in a world characterized by constraints on the information sets and bounded rationality is the adaptive learning[7]. This approach typically deals with agents that does not know the "correct" model of the world, but use data to estimate it like an econometrician would do. Following this line of reasoning, the most natural way to model how the individuals do their estimations is to assume that they have a mental model of the law of motion of the relevant variables in the economy (the "perceived law of motion", or PLM), and that they estimate its parameters via OLS[8]. Given these estimates, the endogenous variables will follow what is called the "actual law of motion" (ALM). Calling ϕ_t the N-dimensional vector of parameters estimates, R_t the matrix of its second moments, z_t the regressors and p_t the endogenous variables, it can be shown that, with the appropriate initial conditions, the updating algorithm:

$$\begin{aligned} \phi_t &= \phi_{t-1} + t^{-1} R_{t-1}^{-1} z_{t-1} (p_t' - z_{t-1}' \phi_{t-1}) \\ R_t &= R_{t-1} + t^{-1} (z_{t-1} z_{t-1}' - R_{t-1}) \end{aligned} \tag{1.1}$$

delivers the same sequence of estimates $\{\phi_t\}_{t=0}^{\infty}$ as the standard OLS techniques applied period by period; for this reason, it is called recursive least squares algorithm (RLS). Given these estimates, the ALM (in case of a linear model) is:

$$p_t = T^{'}(\phi_{t-1}) z_{t-1} + \varepsilon_t$$

where the function $T : \mathbb{R}^N \to \mathbb{R}^N$ is the mapping from the estimated coefficients to the actual coefficients determined by the estimates, and ε_t is a white noise. The appealing feature of this formulation is that it can be studied with the tools of stochastic approximation[9], with the result that for large classes of models the asymptotic dynamics are governed by the stability properties of a deterministic ordinary differential equation; in particular we have that a rational expectations equilibrium is (locally) stable under adaptive learning

[7]For extensive monographs, see Sargent (1993) and Evans and Honkapohja (2001).

[8]Note that the PLM may or may not be of the same functional form of the rational expectations solution of the model.

[9]For an extensive monograph on stochastic approximation, see Benveniste et al. (1990).

if it is a (locally) stable rest point of the ordinary differential equation[10][11]:

$$\frac{d\phi}{d\tau} = T(\phi) - \phi$$

Since a rational expectation equilibrium is a rest point of the T-mapping, RLS learning has often been invoked to argue that assuming rational expectations is not a too restrictive hypothesis (at least in the limit), and a lot of effort has been devoted to study the learnability of equilibriums in widely used economic models.

There are also other adaptive algorithms employed in the literature; one of the most used alternatives is to substitute in (1.1) the factor t^{-1} with a constant gain (or tracking parameter) $0 < \gamma < 1$. Since in this case the estimates always react to any new shock (also asymptotically), the system never converges to a fixed value, but under some technical conditions it can settle down as a normal distribution, whose support shrinks to zero as γ approaches zero[12].

There have been some effort in trying to reconcile inflation data with monetary models applying the adaptive learning techniques. In Orphanides and Williams (2004b), the authors assume an ad hoc model with a Phillips curve which includes a lag of inflation as an explanatory variable, and a demand relation that expresses output gap as a function of the real interest rate deviation from its equilibrium value, and study the design of the optimal policy in a setting of adaptive learning through a constant gain algorithm.

More closely related to this work is the paper of Williams (2003), who introduce adaptive learning (both in the RLS and the constant gain versions) in a standard New Keynesian framework with monopolistic competition and staggered prices à la Calvo. He analyzes to what extent the introduction of adaptive learning matters for business cycle statistics; he found that, quantitatively, this change is of second-order importance. However, in his model he still assumes that the agents optimize taking into account the Calvo constraint on the pricing resetting possibility.

Moreover, Sargent (1999) shows how a mispecification by the policymakers of the "true" structural relations of the economy, coupled with constant

[10]In the adaptive learning terminology, an equilibrium which is a stable solution of the differential equation reported below is defined an E-stable equilibrium.

[11]For the derivation of this result, see Marcet and Sargent (1989) and Evans and Honkapohja (2001), Chapter 6 and 8.

[12]For more on constant gain, see below Section 3.1.

gain learning, may lead to a system that oscillates most of the time around the high inflation equilibrium (the time-consistent one, according to Barro and Gordon (1983)), occationally moving towards the low inflation time-inconsistent (or Ramsey) equilibrium, when a suitable sequence of shocks occurs.

We aim to do a first step towards the construction of a bridge between the adaptive learning literature and the other limited information approaches, in assuming no price stickiness, and instead taking the imperfect information as the main source of inertia in the model. We will therefore introduce adaptive learning in a monopolistic competitive, flexible prices setting, with an exogenous process for nominal output, and compare its performance to that of an analogous model where firms behave according to Calvo model.

1.3 The Model

The production side is characterized by a continuum of firms that produce differentiated goods in a competitive monopolistic framework. The demand side is characterized by a representative consumer with rational expectations, who solves the following problem:

$$\max E_0 \sum_{t=0}^{\infty} \beta^t U(C_t, N_t) \tag{1.2}$$

$$s.t. \int_o^1 P_{it} C_{it} di + M_t = M_{t-1} + W_t N_t + \int_0^1 \Pi_{it} di - T_t$$

where C_{it} and P_{it} denotes the demand of good i and its price, respectively, M_t is the money stock hold at the end of period t, T_t are transfers from government, N_t is labor supply, W_t is nominal wage, Π_{it} is the profit from the sell of good i[13], and C_t represents the CES aggregate of consumption:

$$C_t = \left[\int_0^1 C_{it}^{\frac{\theta-1}{\theta}} di \right]^{\frac{\theta}{\theta-1}}$$

As shown in Dixit and Stigliz (1977), maximizing the CES index of consumption subject to a certain level of overall expenditure leads to the demand

[13] We are assuming that firms are owned by the representative consumer.

schedule for good i:

$$C_{it} = \left(\frac{P_{it}}{P_t}\right)^{-\theta} C_t$$

where:

$$P_t = \left[\int_0^1 P_{it}^{1-\theta} di\right]^{\frac{1}{1-\theta}}$$

Moreover, consumer faces a cash-in-advance constraint[14]:

$$P_t C_t \leq M_t$$

which is assumed to be binding. Doing so, we can close the model with the simplest possible specification for the demand side of the economy, i.e. the quantity theory. This is different from the specification used as a standard framework of monetary policy evaluation, which derives an IS relationship from a money-in-utility setup; otherwise, it is useful to simplify the analysis in this early stage, and such a simplifying assumption has been already used in many of the papers that propose alternatives to the Calvo model[15]. The extention of this approach to a more standard specification of the demand side of the economy is left as future work.

Assuming separability of the $U(\cdot)$ function between its arguments, it is easily shown that:

$$\frac{W_t}{P_t} = -\frac{U_N(N_t)}{U_C(C_t)} \equiv G(C_t, N_t)$$

We assume that each firm produce a differenciated good according to the strictly increasing and concave production function:

$$Y_{it} = A_t F(N_{it})$$

where A_t denotes a technology indicator, and N_{it} is firm i labor demand, and Y_{it} good i output. In equilibrium, market clearing implies:

$$Y_{it} = C_{it}, \quad Y_t = C_t$$

[14]Note that, in writing this constraint, we are implicitly assuming that at time t the money market closes before the opening of commodity markets.

[15]It is common to Mankiw and Reis (2002), Ball et al. (2005), and Woodford (2001); in a learning framework, a similar assumption has been used by Adam (2005).

so that the demand schedule can be rewritten as:

$$Y_{it} = \left(\frac{P_{it}}{P_t}\right)^{-\theta} Y_t \tag{1.3}$$

To model firms' behavior, we make the same distinction between the optimization and the inference problem underlined in Townsend (1983). In particular we assume:

- firms can freely reset prices in each period, but they can observe aggregate variables only with a one period delay; so, in each period firm i's optimization problem is given by the static expected profit maximization:

$$\max_{P_{it}} E_t^i \left[P_{it} Y_{it} - W_t N_{it}\right] \tag{1.4}$$

where $E_t^i x_t$ is firm i's (in general non rational) expectations of x_t, formed using time t information set. This maximization is done subject to the demand schedule (1.3). As is shown in Woodford (2003), maximizing (1.4) subject to (1.3) yields an optimality condition that can be expressed, once loglinearized around the full-information equilibrium, as:

$$p_{it} = E_t^i p_t + \xi E_t^i y_t$$

where $x_t \equiv \log X_t$, $y_t = \log(Y_t/Y_t^N)$ (here Y_t^N represents potential output), and ξ is a function of the elasticities of the marginal cost function with respect to its arguments. Taking logs of (1.3), and assuming that E_t^i is a linear operator, we can write:

$$E_t^i y_t = y_{it} + \theta\left(p_{it} - E_t^i p_t\right)$$

where $y_{it} = \log(Y_{it}/Y_t^N)$; using the above equation to substitute out $E_t^i y_t$, we get:

$$p_{it} = E_t^i p_t + \frac{\xi}{1 - \theta\xi} y_{it}$$

Integrating over i, and assuming homogeneous expectations, we obtain:

$$p_t = E_t^* p_t + \frac{\xi}{1 - \theta\xi} y_t \tag{1.5}$$

Equation (1.5) can be rewritten as:

$$p_t = \frac{1-\xi\theta}{1+\xi(1-\theta)}E_t^* p_t + \frac{\xi}{1+\xi(1-\theta)}q_t \tag{1.6}$$

where $q_t = y_t + p_t$. To close the model, we need a process for nominal output[16], and a rule for expectations formation; the first one is given by an AR(1) process for the growth rate of nominal output[17]:

$$\Delta q_t = (1-\rho)g + \rho\Delta q_{t-1} + u_t \tag{1.7}$$

where u_t is an i.i.d. shock. Substituting out q_t from (1.6) using (1.7)[18], we get:

$$\begin{aligned} p_t \quad = \quad & \frac{1-\xi\theta}{1+\xi(1-\theta)}E_t^* p_t + \frac{\xi(1-\rho)}{1+\xi(1-\theta)}g + \frac{\xi(1+\rho)}{1+\xi(1-\theta)}q_{t-1} \\ & -\frac{\xi\rho}{1+\xi(1-\theta)}q_{t-2} + \frac{\xi}{1+\xi(1-\theta)}u_t \end{aligned} \tag{1.8}$$

It can be easily shown that the minimum state variable (MSV) solution of the model given by (1.7)-(1.8) under rational expectations is[19]:

$$p_t = (1-\rho)g + (1+\rho)q_{t-1} - \rho q_{t-2} + \frac{\xi}{1+\xi(1-\theta)}u_t$$

- The inference problem is modelled following the literature of adaptive learning[20]. In particular, we assume that agents do not know the exact MSV solution of the model given by (1.7)-(1.8) but, instead, form their expectations according to the perceived law of motion (PLM):

$$p_t = a_{t-1} + b_{t-1}q_{t-1} + c_{t-1}q_{t-2} + \eta_t \tag{1.9}$$

[16]Because of the cash-in-advance constraint, money and nominal output are equivalent.

[17]See for analogous processes Woodford (2001), or Mankiw and Reis (2002). Moreover, Christiano et al. (1998) argue that an AR(1) process for growth rate of money is empirically plausible.

[18]Since agents do not observe contemporaneous nominal output, we write the law of motion for price level in terms of observable variables.

[19]I.e., $E_t^* = E_t$.

[20]See Evans and Honkapohja (2001).

where the vector $\phi_{t-1} \equiv (a_{t-1}, b_{t-1}, c_{t-1})'$ denotes the estimates of model parameters computed by agents using information available on aggregate variables at time t[21]; these estimates are updated according to the recursive algorithm:

$$\begin{aligned} \phi_t &= \phi_{t-1} + \gamma_t R_{t-1}^{-1} z_{t-1}(p_t - \phi'_{t-1} z_{t-1}) \\ R_t &= R_{t-1} + \gamma_t (z_{t-1} z'_{t-1} - R_{t-1}) \end{aligned} \tag{1.10}$$

where $z_t = (1, q_t, q_{t-1})'$ and $\{\gamma_t\}$ is a sequence of nonincreasing values called "gain parameters". The precise path followed by this sequence will be described in the next subsection. Equation (1.9) implies that:

$$E_t^* p_t = a_{t-1} + b_{t-1} q_{t-1} + c_{t-1} q_{t-2} \tag{1.11}$$

which can be plugged into (1.8) to obtain the actual law of motion (ALM):

$$\begin{aligned} p_t = & \left(\frac{1-\xi\theta}{1+\xi(1-\theta)} a_{t-1} + \frac{\xi(1-\rho)}{1+\xi(1-\theta)} g \right) + \left(\frac{1-\xi\theta}{1+\xi(1-\theta)} b_{t-1} + \right. \\ & \left. \frac{\xi(1+\rho)}{1+\xi(1-\theta)} \right) q_{t-1} + \left(\frac{1-\xi\theta}{1+\xi(1-\theta)} c_{t-1} - \frac{\xi\rho}{1+\xi(1-\theta)} \right) q_{t-2} \\ & + \frac{\xi}{1+\xi(1-\theta)} u_t \end{aligned} \tag{1.12}$$

Given (1.12), the T-mapping is:

$$\begin{aligned} T(a) &= \frac{1-\xi\theta}{1+\xi(1-\theta)} a + \frac{\xi(1-\rho)}{1+\xi(1-\theta)} g \\ T(b) &= \frac{1-\xi\theta}{1+\xi(1-\theta)} b + \frac{\xi(1+\rho)}{1+\xi(1-\theta)} \\ T(c) &= \frac{1-\xi\theta}{1+\xi(1-\theta)} c - \frac{\xi\rho}{1+\xi(1-\theta)} \end{aligned} \tag{1.13}$$

which can be easily shown to imply that the MSV solution under rational expectations is E-stable.

Equation (1.12), together with the process for nominal output (1.7) and the stochastic recursive algorithm (1.10), constitutes our model.

[21] I.e., the sequence $\{p_i, q_i, q_{i-1}\}_{i=1}^{t-1}$.

1.3.1 The Gain Parameter

At this stage of the analysis, it is necessary to be explicit about the form taken by the γ_t's in (1.10). In the learning literature there are two main strategies used to calibrate them, corresponding to two different hypothesis of how the agents conceive the world:

- decreasing gain: assume that it is a decreasing sequence with the properties that $\sum_{t=0}^{\infty} \gamma_t = \infty$ and $\sum_{t=0}^{\infty} \gamma_t^2 < \infty$, as we would obtain setting $\gamma_t = t^{-1}$. In this case, equation (1.10) becomes a particular case of the RLS algorithm (1.1), assigning the same weight to every observation. This last remark means that a decreasing gain is a reasonable assumption if agents think that the model's parameters are constant over time, so that each observation has the same information content;

- constant gain: assume that $\gamma_t = \gamma$, where γ is a small positive constant. In this case, our algorithm does not deliver the same estimates as the OLS anymore, since past data are downweighted. The assumption behind this behavior is that agents believe structural changes to occur, even if they are able neither to model them nor to predict in which period they will take place. As a result, they will update their estimates given their belief that more recent data embed more information on the structure of the economy. As mentioned in Section 2, this specification of the gain sequence prevents the parameters' estimates to converge to any particular value, since they will be significantly influenced by any new shock.

In what follows, we will employ the constant gain specification; before proceeding, however, there are two logical problems that we have to take into account.

First of all, in the standard adaptive learning setup, the agents take their decisions treating their expectations as if they correspond to the "true" model. If this kind of behavior can be justified in a decreasing gain case[22], it seems to contrast with the basic assumption that motivates constant gain,

[22] Actually, the fact that the ALM has time varying parameters, can arise mispecification issues in a context of decreasing gain learning, as noted in Bray and Savin (1986); for a time-varying parameters estimation approach in an adaptive learning approach, see McGough (2003).

i.e. that agents are convinced that economic structure shifts over time. In Tetlow and von zur Muehlen (2004) this issue is investigated in the context of the Sargent (1999) model; they allow policy makers to make their decisions taking into account model uncertainty in a Bayesian way[23]. Their conclusion is that this method does not yields results quantitatively different from the Sargent's more standard approach.

Another relevant question that we now should address is how to calibrate γ. In particular, this choice is potentially subject to a high degree of arbitrariness, that could allow the model to replicate any empirical pattern we want: in fact, as pointed out in Marcet and Nicolini (2003), the presence of too many degrees of freedom in designing the learning algorithms has always made this class of models hardly falsifiable, hence preventing many researchers from using them to match data. The introduction in the setup of an additional exogenous parameter, that could in principle be used to make the model behave as we want, makes this criticism particularly sound, and the "wilderness of irrationality" particularly dangerous.

A possible way out of both these pitfalls is to endogenize the value of γ, making it the outcome of some optimal choice of the agents. The way we have decided to do it is using standard game-theoretic concepts. We assume that firms are interested in minimizing the mean square error (MSE) of their forecasted inflation; in this case, we can define a *misspecified equilibrium* as a value γ^* of the gain parameter which minimizes the MSE of an individual, when all the rest of the economy update its expectations using the same γ^*. More formally, we look for a fixed point of the function:

$$\gamma^* = \arg\min_{\gamma} \frac{1}{T} \sum_{t=1}^{T} \left[p_t(\widehat{\gamma}) - E_t^* p_t(\gamma)\right]^2 \equiv f(\widehat{\gamma}) \tag{1.14}$$

where T is the time horizon taken into account (in our case, 100 periods). This is an *equilibrium* in the sense that no agent has an incentive to deviate from this strategy, and is *misspecified* in the sense that agents behave as if they perceive the economy as a time varying parameters system, when the "true" model is characterized by constant structural parameters[24]. This approach is closely related to the concept of equilibrium in learning rules of

[23]In other words, taking into account the standard errors of the estimates.

[24]Note that, as we point out below, it is not necessary that each agent believes the economy to be time varying, but only that in his opinion all the other agents have this belief.

Evans and Honkapohja (1993), and to the internal consistency requirement introduced in Marcet and Nicolini (2003).

Note that, in this way, the tracking parameter is not freely chosen anymore, but becomes a function of the other structural parameters of the economy, solving the second problem above mentioned; moreover, it implies that even if an individual agent does not think that the model is really time-varying, he will play according to the misspecified equilibrium if he thinks that the other agents will do the same[25]. Hence, this kind of approach allows us to deal also with the first problem that we discussed above[26]. If we model the choice of the gain parameter in this way, standard issues of how the agents could end up coordinating on a certain equilibrium (even with bounded rationality) arise, but they are far beyond the scope of this chapter.

As an additional remark, observe that in Orphanides and Williams (2004b), where a constant gain algorithm is implemented, γ is left as a free parameter, and the behavior of the model for different values of it is studied.

Moreover, a technical remark on the asymptotic behavior of parameters' estimates under constant gain learning is to be done: as above mentioned, under certain conditions, they will converge in distribution to a normal with an E-stable equilibrium as a mean; unfortunately, in this model one of the sufficient conditions required to obtain this result is not satisfied[27]. However, very long simulations of the model (10000 periods) show that the parameters' estimates really converge to a neighborhood of the rational expectations values after a few periods (less than the fifty that we usually discard at the beginning of each simulation), around which they keep oscillating.

It is interesting to note that the problem that is behind the construction of the best-reply function $f(\bullet)$ (i.e., the minimization of the MSE with respect to the constant gain used by an individual that cannot influence the non-stationary process that is trying to track) is similar to the framework analyzed in Chapter 4, Part I of Benveniste et al. (1990). In that context, the authors study how to derive analytically the value of the tracking parameter that minimizes the expected value of the square of the distance between the sequence of the actual values of the time-varying coefficients of the process to track, and the estimated values of these coefficients. First

[25] This is a typical coordination problem.

[26] Notice that some kind of myopic behavior from the agents' side must be assumed since, even if playing γ^* minimize the MSE in finite time, it makes convergence to the rational expectations equilibrium impossible.

[27] In particular, the law of motion of the state variables $[q_{t-1}, q_{t-2}]$ contains a unit root.

of all, they decompose the objective function into the sum of the distance between the mean dynamics of the two sequences, and the variance of their distance. Moreover, they show that the optimal choice of γ is the result of a compromise between *tracking* and *accuracy*: in other words, higher (lower) values of the gain parameter reduce (increase) the distance between the mean dynamics of the two sequences, thus reducing (increasing) the magnitude of the MSE, and increase (reduce) the variance of the distance between the two sequences, hence increasing (reducing) the MSE.

These results cannot be directly applied to our model for the same reason we outlined above, when we talked about the asymptotic behavior of parameters' estimates. However, these tools provide useful insights on how our model would behave: in fact, we observe that an increase (decrease) of $\widehat{\gamma}$, on one hand, does not influence the mean dynamics of the time-varying coefficients of p_t (which are given by the rational expectations values, as mentioned above), while on the other hand it increases (decreases) the variance of these coefficients. Loosely applying the concepts sketched above, the best-reply of the individual firm would be to reduce γ^*; hence, we can expect $f(\bullet)$ to be a decreasing function. And this is exactly what we obtained, when we computed a numerical approximation of $f(\bullet)$[28].

1.4 Numerical Results

1.4.1 Calibration Strategy

We need to calibrate five parameters (ξ, θ, ρ, g), plus the initial conditions ϕ_0 for the RLS algorithm; the chosen values are summarized in Table 1.1.

Table 1.1: Baseline calibration

ξ	θ	ρ	g	a_0	b_0	c_0
0.15	6	0.7	0	$(1-\rho)g$	0.05	-0.05

The parameter γ is a function of these other seven coefficients, as clarified in the previous section.

The value for ξ is suggested in Woodford (2003) as an empirically plausible value for U.S. economy, and is used also in Woodford (2001); it also lies in the range of values examined in Mankiw and Reis (2002). θ is chosen according to

[28] See next section.

the standard New Keynesian literature, while ρ is chosen to match empirical facts (see below). The choice to set the trend g to zero is justified by the aim to concentrate on the model's behavior at business cycle frequency.

We now need to be more explicit on the strategy adopted to set the value of the tracking parameter. First of all, note that neither a formal proof of the existence and the uniqueness of a fixed point of the expression (1.14)[29], nor an analytical expression of γ^* as a function of the parameters' vector $(\xi, \theta, \rho, g, a_0, b_0, c_0)'$ have been obtained; hence, we had to search for a numerical approximation of this equilibrium[30]. We used the following procedure: first of all, we set up a grid of 19 values (0.05, 0.1, 0.15,...,0.95), then we draw 1000 realizations of the nominal output shock $\{u_i\}_{i=1}^{150}$; then, we computed the corresponding sequences $\{p_i\}_{i=1}^{150}$ using a fixed value of $\widehat{\gamma}$. Then, we throw away the first fifty values of each sequence, to dampen the influence of initial conditions, compute the empirical MSE for each $\{p_i\}$ and for each possible γ, average across all realization, and look for the γ for which the resulting value is minimum. This procedure has been repeated for all the 19 possible values of $\widehat{\gamma}$, and we got that the only fixed point is at $\gamma^* = 0.55$.

Note that this value is much higher than those used in Orphanides and Williams (2004b)[31]; in Figure 1.1 we have plotted the evolution over time of the parameters' estimates in one of the 10000 stochastic simulations that we performed to analyze the behavior of our model (see subsections below). As we can observe in Figure 1.1, we have a very rapid convergence to a neighborhood of the rational expectations equilibrium, followed by wide oscillations around it. This last feature is due to the very high value of the tracking parameter, which makes the estimates of the parameters very sensible to any new shock.

To conclude, the expectations has been initialized at values that have the same sign as the rational expectations parameters, but that deliver the desired hump-shaped impulse response for inflation; for values closer than those to the rational expectations, would yield a peak response only two periods after the nominal shock.

[29] As is instead derived, in a simpler context, in Evans and Ramey (2006).

[30] However, since the values obtained numerically for γ^* are a monotonic decreasing function of $\widehat{\gamma}$, as explained in the previous section, this makes us conjecture that the "true" equilibrium exists and is unique.

[31] The largest value of the tracking parameter that they adopt is 0.1.

1.4.2 Persistence of Monetary Shocks

A well-known shortcoming of the Calvo staggered price framework is that, even if prices responds sluggishly to monetary shocks, inflation does not: the highest effect of the shock is experienced in the first period, and then it monotonically decays.

In a framework similar to the one developed in the previous section, this drawback of the Calvo model can be easily seen starting from the well known New Keynesian Phillips Curve[32]:

$$\pi_t = ky_t + \beta E_t \pi_{t+1} \tag{1.15}$$

where β denotes the subjective discount rate, $E_t[\cdot]$ is the rational expectations operator, and k is a function of the parameters ξ, β and α, where the latter indicates the probability that a given firm does not review its price in a given period; the exact form of this function of parameters is not important for our purposes: the only relevant point is that it assumes positive values, as shown in Woodford (2001).With a nominal demand process like (1.7), it can be shown that inflation can be written in terms of an MA(∞) process, whose coefficients are a monotonic decreasing sequence, independently of the values of the parameters[33]; this result has an immediate consequence in terms of the impulse response function: as claimed before, the peak effect of a monetary shock is reached on impact, and the inflation returns monotonically towards the pre-shock value.

This is at odds with what is empirically observed, and with what is considered conventional wisdom; in fact, there is an extensive literature[34] that stresses the fact that the maximum effect of a monetary shock is reached between 1 and 2 years after the impact of the shock.

As shown in Figure 1.2, with a value of ρ of 0.7, our model is able to generate some persistence of the nominal shock; in particular, the peak effect of the shock is reached after three periods. Even if it is not as much as in data, nevertheless it is a better performance than the Calvo model. The reason is straightforward: since agents do not know the exact model, when they observe a discrepancy between the actual and the forecasted inflation, they are not aware how much of it is due to the presence of a shock, and how much to an imprecise estimate of the parameters; hence, they react with

[32] For the derivation, see for e.g. Woodford (2001).

[33] For the proof, see the Appendix.

[34] See, e.g. Adam (2005) and Christiano et al. (2005).

more caution than what would be optimal for a Calvo-agent, and smooth their reaction over more than one period. Formally, we can observe from equation (1.6):

$$\Delta p_t \equiv \pi_t = \frac{1-\xi\theta}{1+\xi(1-\theta)}\left(E_t^* p_t - E_{t-1}^* p_{t-1}\right) + \frac{\xi}{1+\xi(1-\theta)}\Delta q_t$$

which can be rewritten as:

$$\pi_t = \frac{1-\xi\theta}{1+\xi(1-\theta)}\left(E_t^* \pi_t - E_{t-1}^* \pi_{t-1}\right) + \frac{1-\xi\theta}{1+\xi(1-\theta)}\pi_{t-1} + \frac{\xi}{1+\xi(1-\theta)}\Delta q_t \tag{1.16}$$

This equation makes explicit the fact that adaptive learning has introduced a backward-looking element in the inflation dynamics; this element would disappear only in the case in which the agent comes from a period of no forecasting error (i.e., in the case of $\pi_{t-1} = E_{t-1}^* \pi_{t-1}$).

Even if the model can generate more realistic dynamics, it still has to be improved to match data better. Moreover the impulse response of Figure 1.2 is clearly dependent on the initial conditions; so we could consider different initial conditions, like extracting them from an assumed prior distribution, or introducing a "training period"[35], i.e. a period during which the economy is run in the rational expectations equilibrium, and after which the agents use OLS to estimate the coefficients; this estimates would constitute the initial conditions for the impulse response exercise. Unfortunately the model has problems in replicating an enough hump-shaped impulse response function for inflation with initial conditions too close to the rational expectations equilibrium, since the backward-looking element above mentioned is too weak to ensure enough delay in the inflation response to the shock. However, at least two periods of delay in the peak effect of the shock can be obtained for a wide range of initial conditions.

We are also interested in checking whether the inflation inertia generated by adaptive learning (and captured by equation (1.16)) is enough to match empirical data on first order autocorrelation of inflation; in particular, we consider the value of 0.76 reported in Mankiw and Reis (2002), who compute first order autocorrelation of the CPI using Hodrick-Prescott filtered U.S. data. Since the model cannot be solved analytically, the population value of any statistic is not available. Hence, the only way is to perform stochastic simulations, and then to compare the value of the statistic for this simulated

[35]See Williams (2003).

sample with the empirical one. In doing so, a standard problem arises: since we want to assess how "close" is our model to reality, we need a metric. Our choice, following the approach outlined in Canova and Ortega (2000), is to perform 10000 stochastic simulations of the model, reporting not only the mean of the simulated statistic of interest (i.e., the first order autocorrelation of inflation), as is usually done, but to use information on all the simulated distribution. In other words, we check whether the actual value lies between the 5^{TH} and the 95^{TH} percentiles[36]. It turns out (see Figure 1.3) that 0.76 almost coincides with the 95^{TH} percentile of the simulated distribution, denoting the capacity of adaptive learning to generate a realistic degree of inflation inertia, even without any other source of rigidity.

This result is more remarkable, if we take into account the extremely high value of γ we are using: in fact, we would expect that a learning scheme so sensitive to every forecast error would induce a more erratic behavior of inflation expectations, hence dampening the possibility of the model of replicating the empirical value of inflation autocorrelation. More properly, we can think about increasing the value of γ as a trade-off: in fact, a value too small of the tracking parameter would nullify the role of learning[37], while a value too large would make the expectations "overreact", introducing noise in the inflation process. To confirm this conjecture, we performed stochastic simulations of the model for different values of the constant gain, and for each of them we computed the median and the 95^{TH} percentile, plotting the results in Figure 1.4. A quick inspection of the figure shows that the highest values of both are reached when $\gamma = 0.05$; for every level of the tracking larger than this, the first order autocorrelation generated by the model starts declining monotonically. In Figure 1.5 we have plotted the evolution over time of the parameters' estimates of one of the simulations conducted with $\gamma = 0.05$; it is evident just by comparing Figure 1.5 and Figure 1.1 how the smaller value of the constant gain let the estimates vary in smoother waves than those observed for $\gamma = 0.55$, when we had a very "nervous" behavior. Moreover, the estimates of the coefficients on q_{t-1} and q_{t-2} tend to change in a negligible way and to keep close to the rational expectations equilibrium value, when agents have had enough time to learn.

The pattern displayed in Figure 1.4 could help explaining why Orphanides

[36] In a sense, we are taking our model as a null hypothesis; for a survey of these techniques to evaluate calibrated dynamic stochastic general equilibrium models, see Canova and Ortega (2000).

[37] In the limit, for $\gamma = 0$, we would have $\phi_t = \phi_{t-1}$.

and Williams (2004b) observe that inflation persistence is *increasing* in the value of γ: actually, they assumed magnitudes of the gain parameter so small[38] that their simulations moved along the increasing side of Figure 1.4. Actually, we see that, setting exogenously the constant gain[39] would have allowed us to obtain a better performance in replicating the empirical patterns.

The fact that our approach was able to generate a first order autocorrelation of inflation broadly consistent with data, even with an endogenously determined value for the constant gain, strengthen the result.

1.4.3 Acceleration Phenomenon

The hump-shaped impulse response function of inflation is not the only empirical pattern which is difficult to reconcile with the Calvo model; another example is the widely documented positive and significant correlation between the level of real output and the growth rate of inflation[40]. This same pattern is reported also in Mankiw and Reis (2002), who computed this correlation for Hodrick-Prescott filtered U.S. data for CPI inflation, obtaining a value of 0.38. On the other hand, the Calvo model does not exhibit this pattern; the main reason is the interaction between a monotonically decreasing impulse response of inflation, and a positive response of real output (at least in the short run) to the nominal shock. These two features generate (after a positive shock) the contemporaneous presence for many periods of decreasing inflation and high output, thus explaining the negative correlation. To check formally this intuition, Mankiw and Reis (2002) calculated the population cross-correlation $corr(y_t, \pi_{t+2} - \pi_{t-2})$ for a Calvo-type staggered price model for a wide range of values of the key parameters, always obtaining negative numbers.

To check whether our model can instead match this feature of the data, we followed a procedure analogous to that outlined in the previous subsection, performing 10000 stochastic simulations, and then computing the simulated distribution of $corr(y_t, \pi_{t+2} - \pi_{t-2})$ (which is plotted in Figure 1.6); the mean of this distribution is 0.4, very "close" to the actual value of 0.38. To evaluate how "close" it is, we need a metric; we choose to check whether 0.38 lies

[38] The largest value they assume is 0.1.

[39] In particular, giving it a value of 0.05.

[40] See, e.g. Abel and Bernanke (1998) and Blanchard (2000), who use a scatterplot to document this phenomenon.

between the 5^{TH} and the 95^{TH} percentiles of the simulated distribution. These percentiles are equal to 0.24 and 0.54, which implies that our model is broadly consistent with this empirical pattern. Moreover, we obtained that no one of the draws has delivered a negative value of $corr(y_t, \pi_{t+2} - \pi_{t-2})$[41], denoting a stark difference with respect to the Calvo model.

1.5 An Alternative Specification

The previous results have been obtained assuming that firms form their expectations on current prices regressing p on nominal output lagged of one and two periods (plus a constant). It seems reasonable, since individual profits depend on the difference between individual and aggregate price level, so that we can expect agents to estimate the law of motion of the relevant aggregate variable (i.e., the price level), and then forecasting its current value.

As a robustness check, we will try also a different approach. In particular, we want to check whether the previous results[42] have been driven by the fact that the price level is a nonstationary variable; hence, we will assume now that firms estimate the law of motion of the inflation rate (which is stationary), and that they forecast its current level using the most recent estimates of this law of motion; then, given the value of p_{t-1} (which is known at time t) and the identity:

$$E_t^* \pi_t \equiv E_t^* p_t - p_{t-1}$$

they obtain $E_t^* p_t$, which is used in their decision process. To derive the PLM, note that from equation (1.8) we get:

$$\begin{aligned} \pi_t &= \frac{1-\xi\theta}{1+\xi(1-\theta)} E_t^* p_t + \frac{\xi(1-\rho)}{1+\xi(1-\theta)} g + \frac{\xi(1+\rho)}{1+\xi(1-\theta)} q_{t-1} \\ &\quad - \frac{\xi\rho}{1+\xi(1-\theta)} q_{t-2} + \frac{\xi}{1+\xi(1-\theta)} u_t - p_{t-1} \\ &= \frac{1-\xi\theta}{1+\xi(1-\theta)} E_t^* \pi_t + \frac{\xi(1-\rho)}{1+\xi(1-\theta)} g + \frac{\xi(1+\rho)}{1+\xi(1-\theta)} q_{t-1} \\ &\quad - \frac{\xi\rho}{1+\xi(1-\theta)} q_{t-2} - \frac{\xi}{1+\xi(1-\theta)} p_{t-1} + \frac{\xi}{1+\xi(1-\theta)} u_t \end{aligned}$$

[41] Note that, on the contrary of what is observed for inflation persistence, the acceleration phenomenon is a robust feature of the model, since it holds for a wide range of values of ρ.

[42] In particular, the very high value of the tracking parameter.

which implies a PLM of the form:

$$\pi_t = \tilde{a}_{t-1} + \tilde{b}_{t-1} q_{t-1} + \tilde{c}_{t-1} q_{t-2} + \tilde{d}_{t-1} p_{t-1} + \eta_t$$

and the ALM:

$$\begin{aligned} \pi_t = & \left(\frac{1-\xi\theta}{1+\xi(1-\theta)} \tilde{a}_{t-1} + \frac{\xi(1-\rho)}{1+\xi(1-\theta)} g \right) + \left(\frac{1-\xi\theta}{1+\xi(1-\theta)} \tilde{b}_{t-1} + \right. \\ & \left. \frac{\xi(1+\rho)}{1+\xi(1-\theta)} \right) q_{t-1} + \left(\frac{1-\xi\theta}{1+\xi(1-\theta)} \tilde{c}_{t-1} - \frac{\xi\rho}{1+\xi(1-\theta)} \right) q_{t-2} \\ & \left(\frac{1-\xi\theta}{1+\xi(1-\theta)} \tilde{d}_{t-1} - \frac{\xi}{1+\xi(1-\theta)} \right) p_{t-1} + \frac{\xi}{1+\xi(1-\theta)} u_t \end{aligned} \quad (1.17)$$

It is easy to see that this model has a unique rational expectations equilibrium given by:

$$\pi_t = (1-\rho)g + (1+\rho)q_{t-1} - \rho q_{t-2} - p_{t-1} + \frac{\xi}{1+\xi(1-\theta)} u_t$$

which is equivalent to say:

$$p_t = (1-\rho)g + (1+\rho)q_{t-1} - \rho q_{t-2} + \frac{\xi}{1+\xi(1-\theta)} u_t$$

In other words, the two forecasting strategy are equivalent under rational expectations.

To see what changes under constant gain learning, we repeated the same exercises described in the previous section. First of all, we looked for the equilibrium value of the tracking parameter in this new context, obtaining $\gamma^* = 0.15$. A couple of remarks are now necessary:

- this value is considerably lower than the 0.55 obtained in the previous section. The reasons why it is so are still an open question. A possible explanation would be that now agents are trying to forecast a stationary variable;

- the best reply function is monotonically decreasing also under this alternative specification, making us conjecture that this is a common feature of this kind of models.

Even with these different assumptions on how firms forecast the aggregate level of prices, the adaptive learning approach does well in accounting for inflation dynamics[43]. In fact, setting the initial condition of $\tilde{d}$ in a small enough neighborhood of the rational expectations value[44], the shape of the impulse response function is analogous to the one obtained previously. Moreover, the 5^{TH} and the 95^{TH} percentiles of the simulated distribution of the first order autocorrelation of inflation are 0.57 and 0.79, respectively, so that they include the actual value of 0.76.

Also the correlation between the level of real output and the growth rate of inflation implied by the model is consistent with the one empirically observed: in fact, the 5^{TH} and the 95^{TH} percentiles of the simulated distribution are 0.18 and 0.46, respectively, so that 0.38 is included between them, but zero is not.

1.6 Conclusions and Future Research

The starting point of this chapter is the difficulty of the Calvo model to replicate some well-established empirical facts. In particular, this model *per se* is not able to generate an hump-shaped impulse response function for inflation, nor a positive correlation between real output and inflation growth. To reconcile this approach with empirics, additional sources of inertia have been introduced by the New Keynesian literature.

However, a new line of research has been recently developed, whose key points are flexible prices coupled with some form of boundedly rational behavior; this chapter shares this modelling strategy, and aims to investigate the properties of a simple flexible prices, monopolistic competitive setup augmented by non-rational expectations, modelled following the adaptive learning approach.

The main result is that, with reasonable parameters values, this setup can considerably improve the performance of the Calvo model, generating inflation and output dynamics that are broadly consistent with the two stylized facts above mentioned; moreover, also the inflation autocorrelation is not at odds with what is empirically observed.

[43]In all the simulations, we used the same set of realizations of the shock that we used in the baseline case, in order to make the comparison consistent.

[44]In particular, for $\tilde{d} \in (-0.6, -1.5)$.

As a side issue, we studied the relationship between the constant gain and inflation autocorrelation to show how, keeping this parameter free to assume any value, we could make the model match almost any empirical pattern, hence stressing the importance of endogenizing this coefficient, linking it to some optimal behavior of the agents.

As future research, we could possibly extend the model modifying the demand side under two respects: first of all, we could make the consumer's problem fully dynamic, dispensing with the cash-in-advance constraint and introducing a riskless bond in the budget constraint, in order to obtain an Euler equation as an optimization condition, from which we could derive an IS schedule; on the other hand, we could introduce adaptive learning also on the consumer's side, in a way consistent with what we assume for the firms.

Another interesting issue would be to remove the exogeneity assumption for money, and to suppose instead that the monetary authority pursues an optimal policy; Orphanides and Williams (2004b) went in this direction, but considered only the optimal rule in the restricted class of linear rules, while, as they pointed out, the "true" optimal rule is a nonlinear function of the states of the system (including the time-t estimates). Moreover, they used an exogenously determined constant gain, when it would be preferable to implement an endogenous one.

1.7 Appendix

In this section we will prove the statements made in the text about the impulse response function of the Calvo price setting model.

First of all, we will solve the model formed by the New Keynesian Phillips Curve (equation (1.15)) and the nominal demand process that we have used throughout this chapter (equation (1.7)), finding the MA(∞) representation of the inflation process. Then, we will prove that the coefficients of this MA(∞) representation (and, therefore, the impulse response function) are a monotonic decreasing sequence.

To begin with, recall the New Keynesian Phillips Curve:

$$\pi_t = ky_t + E_t\pi_{t+1}$$

where we have set $\beta = 1$ for notational simplicity; since β is usually calibrated at 0.99, it does not seem a restrictive assumption; k is a function of structural parameters which assumes positive values. For any arbitrary sequence$\{q_t\}_{t=0}^{\infty}$ for the nominal output,the only stationary solution is[45]:

$$p_t = \lambda p_{t-1} + (1-\lambda)^2 \sum_{j=0}^{\infty} \lambda^j q_{t+j} \tag{1.18}$$

We now assume that q_t follows the process given by (1.7), which can be represented in an MA(∞) form as[46]:

$$\Delta q_t = \sum_{j=0}^{\infty} \rho^j u_{t-j}$$

so that:

$$q_t = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \rho^j u_{t-j-k}$$

To derive the process of inflation, we follow Mankiw and Reis (2002) and guess that it is stationary, so that can be represented in the MA(∞) form:

$$\pi_t = \sum_{j=0}^{\infty} \varphi_j u_{t-j}$$

and the price level is the non-stationary process:

$$p_t = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \varphi_j u_{t-j-k}$$

where the $\{\varphi_j\}$ are unknown. Plugging this guess into equation (1.18), and taking into account the particular process considered for nominal output, we get:

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \varphi_j u_{t-j-k} = \lambda \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \varphi_j u_{t-1-j-k} + (1-\lambda)^2 \sum_{j=0}^{\infty} \lambda^j \sum_{i=0}^{\infty} \sum_{k=\max\{j-i,0\}}^{\infty} \rho^i u_{t+j-i-k}$$

[45]See the appendix of Mankiw and Reis (2002).

[46]We omit the constant term in the process for nominal output, since we calibrate it to zero throughout the chapter.

Matching coefficients, we get:

$$\varphi_0 = (1-\lambda)^2 \sum_{j=0}^{\infty} \lambda^j \sum_{i=0}^{j} \rho^i = \frac{1-\lambda}{1-\rho\lambda} \tag{1.19}$$

for the coefficient on u_t, and for an arbitrary $m > 1$:

$$\varphi_m = (\lambda - 1) \sum_{j=0}^{m-1} \varphi_j + \left[(1-\lambda)^2/(1-\rho)\right] \left[1/(1-\lambda) - \rho^{m+1}/(1-\rho\lambda)\right] \tag{1.20}$$

Now that the sequence of MA coefficients $\{\varphi_j\}$ as been characterized, it is possible to show that it is monotonic decreasing. First of all, we will show that $\varphi_1 - \varphi_0 < 0$; in fact, using (1.20):

$$\begin{aligned} \varphi_1 - \varphi_0 &= (\lambda - 2)\varphi_0 + \frac{(1-\lambda)^2}{(1-\rho)(1-\lambda)} - \frac{(1-\lambda)^2\rho^2}{(1-\rho)(1-\rho\lambda)} \\ &= (\lambda - 2)\frac{1-\lambda}{1-\rho\lambda} + \frac{(1-\lambda)^2}{(1-\rho)(1-\lambda)} - \frac{(1-\lambda)^2\rho^2}{(1-\rho)(1-\rho\lambda)} \end{aligned}$$

Simple algebra shows that:

$$\begin{aligned} \varphi_1 - \varphi_0 &= \frac{(1-\lambda)\left[-(1-\rho)^2 + \lambda - 2\lambda\rho + \lambda\rho^2\right]}{(1-\rho)(1-\rho\lambda)} \\ &= \frac{(1-\lambda)\left[-(1-\rho)^2 + \lambda(1-\rho)^2\right]}{(1-\rho)(1-\rho\lambda)} \end{aligned}$$

Since $\lambda < 1$, $\lambda(1-\rho)^2 - (1-\rho)^2$ is less than zero, and so is $\varphi_1 - \varphi_0$.
Now, observe that for any $m \geq 2$, from (1.20) we obtain:

$$\varphi_m - \varphi_{m-1} = (\lambda - 1)\left(\sum_{j=0}^{m-1} \varphi_j - \sum_{j=0}^{m-2} \varphi_j\right) + \frac{(1-\lambda)^2\left(\rho^m - \rho^{m+1}\right)}{(1-\rho)(1-\rho\lambda)}$$

or, equivalently:

$$\varphi_m = \lambda\varphi_{m-1} + \frac{(1-\lambda)^2\rho^m}{(1-\rho\lambda)} \tag{1.21}$$

Now we will show directly that also $\varphi_2 - \varphi_1$ is less than zero; in fact, from (1.21) we get:

$$\varphi_2 - \varphi_1 = (\lambda - 1)\,\varphi_{1+}\frac{(1-\lambda)^2\rho^2}{(1-\rho\lambda)}$$

Using equation (1.20) for $m = 2$ to substitute out φ_1, and equation (1.19) to substitute out for φ_0, we can write:

$$\varphi_2 - \varphi_1 = \left(\rho^2 + 1 - \lambda\right) \frac{1-\lambda}{1-\rho\lambda} + \frac{(1-\lambda)^2}{(1-\rho\lambda)(1-\rho)}\rho^2 - \frac{1-\lambda}{1-\rho}$$

Simple algebra shows that this expression is negative if and only if $\rho^2 (2 - \lambda - \rho) + 2\rho\lambda - \lambda - \rho \equiv H(\lambda, \rho)$ is negative; but this function is always negative valued, provided that $\rho < 1$, which is a stationarity condition always assumed. In fact, we have that:

$$\frac{\partial}{\partial\lambda} H(\lambda, \rho) = -\rho^2 - 1 + 2\rho$$

which is negative whenever $\rho \neq 1$[47]; so, it is sufficient to check that $H(\cdot)$ is negative valued when λ has the minimum admittable value (i.e., zero). Note that:

$$H(0, \rho) = \rho^2 (2-\rho) - \rho \gtreqless 0 \Leftrightarrow \rho (2-\rho) - 1 \gtreqless 0$$

But the last expression is $-\rho^2 - 1 + 2\rho$, which we have already see that is negative; thus, we conclude that $\varphi_2 - \varphi_1$ is negative whenever $\rho < 1$.

To prove that also the rest of the sequence is decreasing, we proceed by induction: we assume that $\varphi_{m-1} - \varphi_{m-2} < 0$ for an arbitrary $m \geq 3$; we want to show that $\varphi_m - \varphi_{m-1} < 0$ as well. Using equation (1.20) we get:

$$\varphi_m - \varphi_{m-1} = \lambda(\varphi_{m-1} - \varphi_{m-2}) + \frac{(1-\lambda)^2}{(1-\rho\lambda)} \left(\rho^m - \rho^{m-1}\right)$$

Since $\varphi_{m-1} - \varphi_{m-2} < 0$ for the induction hypothesis, and $\rho^m - \rho^{m-1} < 0$ because $0 \leq \rho < 1$, we conclude that $\varphi_m - \varphi_{m-1} < 0$.

[47] In fact, the discriminant of this quadratic expression is zero, so that it has only one root at $\rho = 1$.

1.8 Figures

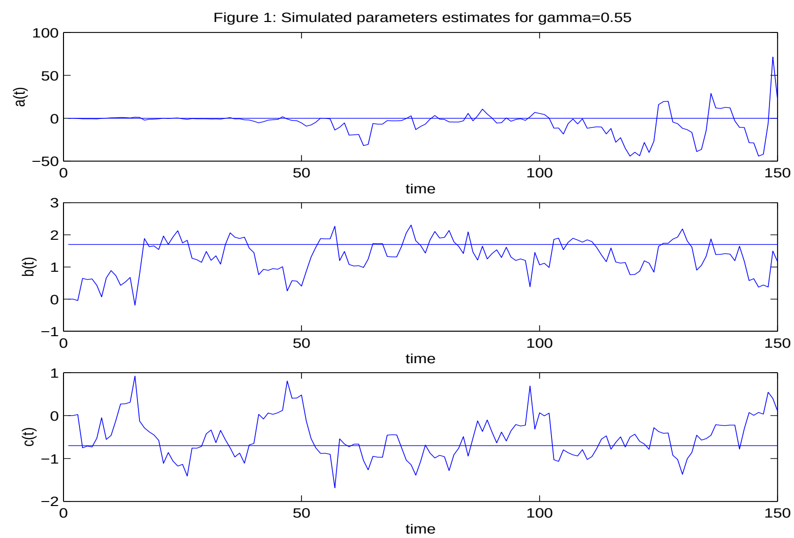

Figure 1.1: Simulated parameters' estimates for γ=0.55.

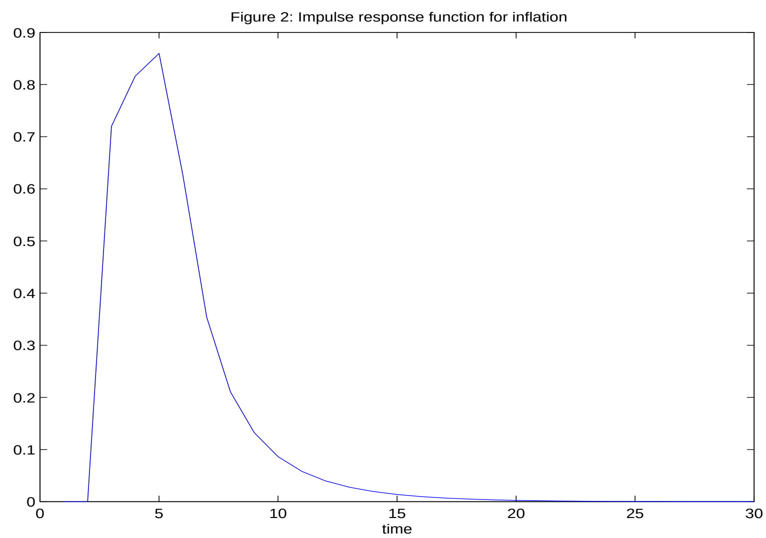

Figure 1.2: Impulse response function for inflation

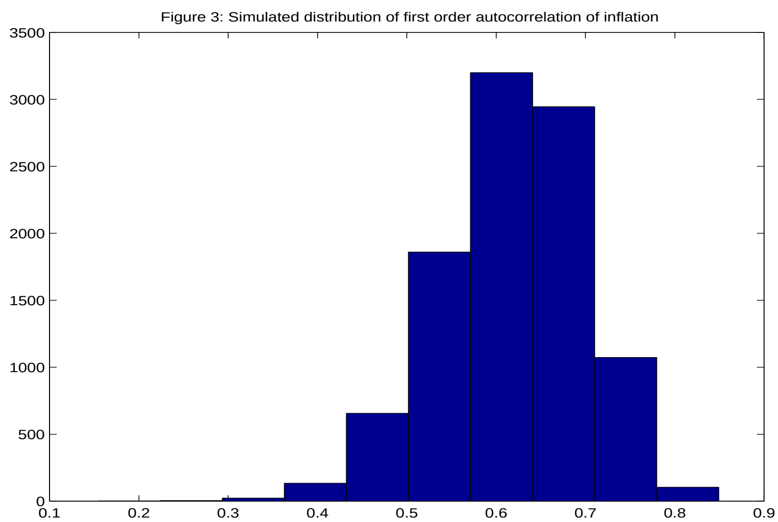

Figure 1.3: Simulated distribution of first order autocorrelation of inflation

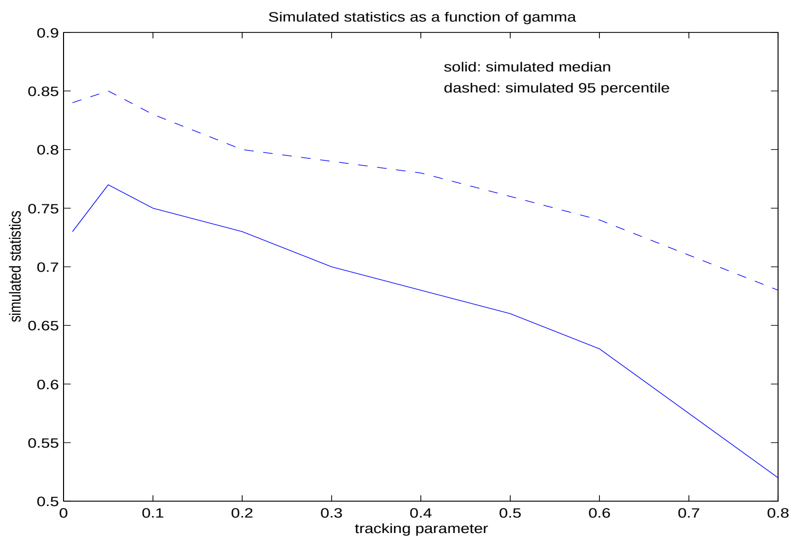

Figure 1.4: Simulated statistics as a function of γ

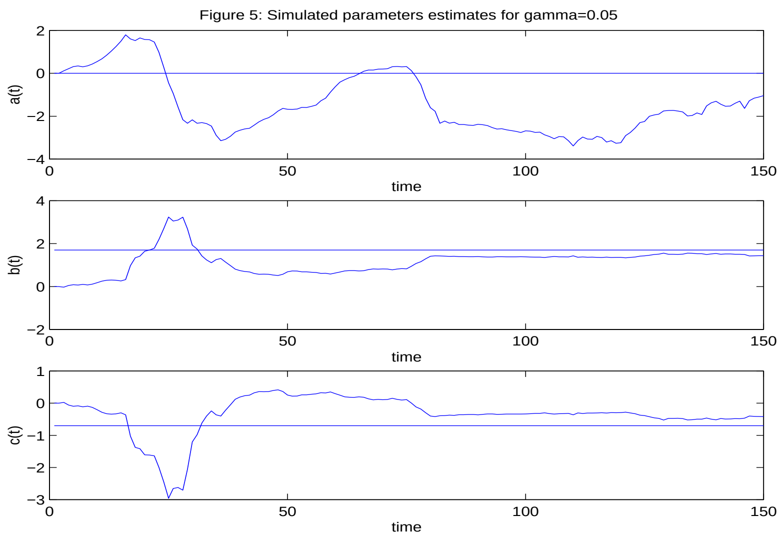

Figure 1.5: Simulated parameters' estimates for γ=0.05.

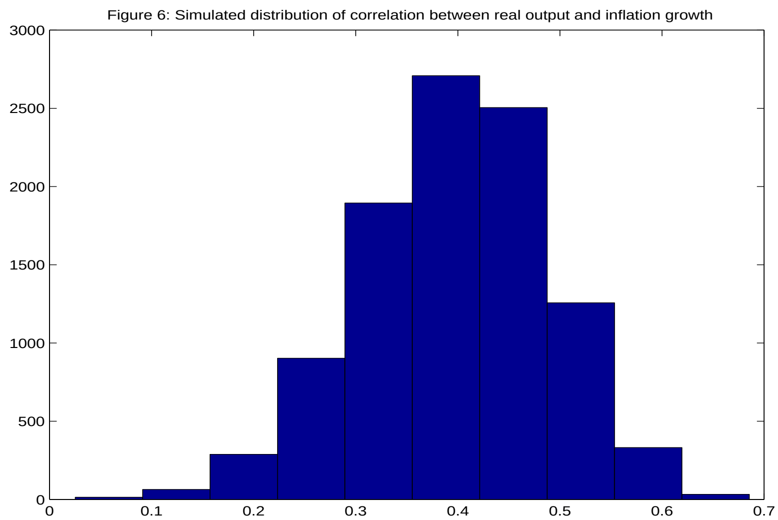

Figure 1.6: Simulated distribution of correlation between real output and inflation growth

Chapter 2

Optimal Monetary Policy when Agents are Learning

2.1 Introduction

> *Monetary policy makers can affect private-sector expectations through their actions and statements, but the need to think about such things significantly complicates the policymakers' task.* (Bernanke (2004))

How should optimal monetary policy be designed? A particularly influential framework used in studying this question is the dynamic stochastic general equilibrium economy where money has real effects due to nominal rigidities, sometimes referred to as the "New Keynesian" model. Many papers have explored optimal monetary policy in this framework, under the assumption that both agents and policymakers have rational expectations.[1] More recently, the literature has started to explore the robustness of these optimal policies when some of the assumptions of the standard New Keynesian setup are relaxed.[2] An important aspect of this robustness analysis is to model more carefully the process through which the private sector forms expectations. This issue is particularly relevant given that there is a large

[1]See Clarida et al. (1999) for a survey on this literature, and Woodford (2003) for an extensive treatise on how to conduct monetary policy *via* interest rate rules.

[2]Wieland (2000a) and Wieland (2000b) look at the effects of parameter uncertainty; Aoki (2006) explores monetary policy with data uncertainty, Levin et al. (2003) and Hansen and Sargent (2001) study model uncertainty.

body of evidence which suggests that agents' forecasts are not consistent with the paradigm of rational expectations.[3] In response, a growing theoretical literature explores the robustness of the optimal policies, which were derived under rational expectations, when instead agents update their expectations according to a learning algorithm.[4] A typical result in this literature is that interest rate rules that are optimal under rational expectations may lead to instability under learning.

Earlier research uses either ad hoc policy rules, as for example Orphanides and Williams (2005a), or optimality conditions derived under rational expectations, like Evans and Honkapohja (2003a), Evans and Honkapohja (2003b) and Evans and Honkapohja (2006). In this chapter, we take a normative approach, and address the issue of how in a New Keynesian setup, a rational Central Bank should optimally conduct monetary policy, if the private sector forms expectations following an adaptive learning model.

We are able to analytically derive optimal monetary policy in our theoretical model. One important feature of the optimal policy is that the Central Bank should act more aggressively towards inflation that what a rational expectations model suggests. Earlier work in the literature that uses ad hoc rules has shown similar results computationally (see Ferrero (2003), Orphanides and Williams (2004b), and Orphanides and Williams (2005a)); here we establish that these results extend to the case when the central bank uses the optimal policy, and provide a formal proof. The intuition for the result is that aggressively driving inflation close to equilibrium helps private agents to learn the true equilibrium value of inflation at a faster pace. As is well-known, even with rational expectations the central bank cares about price stability due to nominal rigidities. When, in addition expectations of nominal variables are sluggish because of learning, our results show that monetary policy should be even more aggressive towards inflation. Being aggressive towards inflation generates a welfare cost in terms of an increased volatility of the output gap. We show analytically that the optimal policy in-

[3]See Roberts (1997), Forsells and Kenny (2002) and Adam and Padula (2003).

[4]For an early contribution to adaptive learning applied to macroeconomics, see Cagan (1956), Phelps (1967), for early applications to the Muth market model see Fourgeaud et al. (1986) and Bray and Savin (1986). The modern literature on this topic was initiated by Marcet and Sargent (1989), who were the first to apply stochastic approximation techniques to study the convergence of learning algorithm. Important earlier contributions to the literature on convergence to the rational equilibrium are Bray (1982) and Evans (1985).

volves a more volatile output gap then the rational expectations benchmark; this holds true even if the Central Bank puts a high weight on output gap stabilization.

A second important feature of the optimal policy is that it is time consistent, and qualitatively resembles the commitment solution under rational expectations in the sense that the optimal policy is unwilling to accommodate noisy shocks. As a consequence the impulse response of a cost push shock is also similar to the commitment case. The contemporaneous impact of a cost push shock on inflation is small (compared to the case of discretionary policy rational expectations), and inflation reverts to the equilibrium in a sluggish manner. In both instances this pattern comes from the Central Bank's ability to directly manipulate private expectations, even if the channels used are quite different. Under commitment the policy maker uses a *credible promise about the future* to obtain an immediate decline in inflation expectations and thus in inflation; the inertia in the optimal solution is due to the commitments carried over from previous periods. In contrast, under learning the pattern results from the sluggishness of expectations: the Central Bank influences private sector's belief through its *past actions*, and the inertia comes from the past realizations of the endogenous variables. We observe a smaller initial response of inflation relative to the rational expectations discretionary case because optimal policy reacts less to the cost push-shock to ease private agents learning. In this sense, we can say that the ability to manipulate future private sector expectations through the learning algorithm plays a role similar to a commitment device under rational expectations, hence easing the short-run trade-off between inflation and output gap.

An analogous investigation, when the model is characterized by a Phillips Curve à la Lucas and private agents follow a constant gain algorithm is performed in Sargent (1999), Chapter 5. A parallel paper of Gaspar et al. (2005) provides a numerical solution to optimal monetary policy under constant gain learning in the New Keynesian framework with indexation to lagged inflation among firms. They show that an optimally behaving Central Bank aims to decrease the limiting variance of the private sector's inflationary expectations and show that optimal policy qualitatively resembles the commitment solution under rational expectations. In their framework private agents estimate the persistence of inflation. Another important result they find is that, when the degree of estimated persistence is high the central bank should be more aggressive.

The ability to derive analytical solutions allows us to contribute to this literature in several respects. We derive that optimal policy should be more aggressive when private agents heavily discount past data and place more weight on current data. Under constant gain learning this implies that the incentive to decrease volatility of inflationary expectations is more pronounced when the gain parameter is higher. The intuition behind this is: under constant gain learning expectations remain volatile even in the limit, and this limiting variance is higher with a high gain parameter; this volatility in expectations causes welfare losses even in the limit, so it is optimal to conduct monetary policy against it. We also show that optimal policy at the same time allows for higher volatility in output gap expectations. The reason for this is that optimal policy allows for higher variability of the output gap, which translates to higher volatility of output gap expectations. Of course, allowing a higher variance in output gap also causes welfare losses. We analytically determine the extent to which output gap losses should be tolerated.

Our next contribution is to derive optimal policy under decreasing gain learning. We show that our main results are robust to the changing the gain parameter: (1) optimal policy is aggressive on inflation even at the cost of higher output gap volatility, (2) optimal policy under learning qualitatively resembles optimal policy under rational expectations when the Central Bank is able to commit. A new result is that when beliefs are updated according to a decreasing gain algorithm, the optimal policy is time-varying, reflecting the fact that the incentives for the Central Bank to manipulate agents' beliefs evolve over time. After a structural break, for example the appointment of a new central bank governor, the Central Bank should be more aggressive in containing inflationary expectations and decrease the extent of this aggressiveness in subsequent periods. The intuition for this result is that in the first periods after the appointment of a new governor, agents pay more attention to monetary policy actions (place more weight on current data), therefore an optimally behaving central bank should make active use of this by aggressively driving private sector expectations close to the equilibrium inflation.

Finally, we show that when the Central Bank is uncertain about the nature of expectation formation (within a set relevant for the US economy) the optimal learning rules derived in this chapter are more robust than the time consistent optimal rule derived under rational expectations. Optimal learning rules provide smaller expected welfare losses even if the Central Bank assigns only a very small probability to learning and a very high probability to rational expectations in how it believes the private sector forms its expectations.

The rest of the chapter is organized as follows. in Section 2.2 we briefly recall the discretionary optimal policy when expectations are rational, and analyze optimal policy under constant gain learning ; Section 2.3 relaxes the assumption that expectations follow constant gain learning, and show that our main results remain valid under decreasing gain learning. Section 2.4 relaxes the assumption that the policy maker can perfectly observe the fundamental shocks and the beliefs of the agents, and argues that the optimal policy rule derived in the previous Sections is robust to uncertainty about the agents' expectations formation mechanism. Section 2.5 concludes.

2.2 The Baseline Model

We will consider the baseline version of the New Keynesian model, which is by now the workhorse in monetary economics; in this framework, the economy is characterized by two structural equations[5]. The first one is an IS equation:

$$x_t = E_t^* x_{t+1} - \sigma^{-1}(r_t - E_t^* \pi_{t+1} - \overline{rr}_t) + g_t \tag{2.1}$$

where x_t, r_t and π_t denote time t output gap[6], short-term nominal interest rate and inflation, respectively; σ is a parameter of the household's utility function, representing risk aversion, g_t is an exogenous demand shock and $\overline{rr}_t$ is the natural real rate of interest, i.e. the real interest rate that would hold in the absence of any nominal rigidity. Note that the operator E_t^* represents the (conditional) agents' expectations, which are not necessarily rational. The above equation is derived loglinearizing the household's Euler equation, and imposing the equilibrium condition that consumption equals output minus government spending .

The second equation is the so-called New Keynesian Phillips Curve (NKPC):

$$\pi_t = \beta E_t^* \pi_{t+1} + \kappa x_t + u_t \tag{2.2}$$

where β denotes the subjective discount rate, κ is a function of structural parameters, and $u_t \sim N(0, \sigma_u^2)$ is a white noise cost-push shock[7]; this relation

[5]For the details of the derivation of the structural equations of the New Keynesian model see, among others, Yun (1996), Clarida et al. (1999) and Woodford (2003).

[6]Namely, the difference between actual and natural output.

[7]Note that the cost-push shock is usually assumed to be an AR(1); we instead assume it to be iid to make the problem more easily tractable, see below.

is obtained under the assumption that the supply side of the economy is characterized by a continuum of firms that produce differentiated goods in a monopolistically competitive market, and that prices are staggered à la Calvo (Calvo (1983))[8]. The coefficient κ is decreasing in the level of stickiness: the longer are prices fixed in expectation the smaller is the effect of the output gap on inflation.

The standard New Keynesian literature imposes the existence of rational expectations (RE), namely that $E_t^* = E_t$). Under this assumption, the full commitment solution of the optimal monetary policy turns out to be time inconsistent, even if the Central Bank (CB) does not have a target for output gap larger than zero. In other words, even if we rule out the possibility of the inflation bias discussed in Barro and Gordon (1983) and all the subsequent literature, there are potential welfare gains associated with the presence of a credible commitment device for the CB. Hence, the time-consistent discretionary solution is suboptimal, giving rise to what is sometimes called as stabilization bias. There is, however, a crucial difference with the traditional inflation bias problem: the discretion and the commitment solution are not only different in the coefficients of the equilibrium laws of motion of aggregate variables, but even the functional form of these laws of motion differs between the two cases; in particular, under discretion inflation and output gap are linear functions of the cost-push shock only, under commitment an additional dependence on lagged values of output gap is introduced[9].

The loss function of the Central Bank (CB) is given by:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left(\pi_t^2 + \alpha x_t^2\right) \tag{2.3}$$

where α is the relative weight put by the CB on the objective of output gap stabilization[10].

[8] In other words, the probability that firm i in period t can reset the price is constant over time and across firms.

[9] See Woodford (2003), Clarida et al. (1999) and McCallum and Nelson (1999).

[10] As is shown in Rotemberg and Woodford (1997), equation (2.3) can be seen as a quadratic approximation to the expected household's utility function; in this case, α is a function of structural parameters.

2.2.1 Benchmark: discretionary solution under rational expectations and under learning

Let's assume that the CB takes the private sector beliefs as given. In Kreps (1998) terminology, this is equivalent to suppose that the monetary authority is an anticipated utility maximizer.

The policy problem is to choose a time path for the nominal interest rate r_t [11] to engineer a law of motion of the target variables π_t and x_t such that the social welfare loss (2.3) is minimized, subject to the structural equations (2.1) and (2.2), and given the private sectors expectations.

$$\begin{aligned} &\min_{\{\pi_t, x_t, r_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \left(\pi_t^2 + \alpha x_t^2\right) \\ &\text{s.t. } (2.1), (2.2) \\ &\qquad E_t^* \pi_{t+1},\ E_t^* x_{t+1} \text{ given for } \forall\ t \end{aligned} \tag{2.4}$$

Because there are no endogenous state variables, problem (2.4) reduces to a sequence of static optimization problems. As shown in Clarida et al. (1999), the optimality condition to this problem (at time t) is

$$\frac{\kappa}{\alpha}\pi_t + x_t = 0. \tag{2.5}$$

Combining (2.5) with the structural equations, one can derive the following law of motion for inflation and output gap:

$$\pi_t^{EH} = \frac{\alpha\beta}{\alpha+\kappa^2} E_t^* \pi_{t+1} + \frac{\alpha}{\alpha+\kappa^2} u_t \tag{2.6}$$

$$x_t^{EH} = -\frac{\kappa\beta}{\alpha+\kappa^2} E_t^* \pi_{t+1} - \frac{\kappa}{\alpha+\kappa^2} u_t\ . \tag{2.7}$$

and the interest rate rule that implements this allocations:

$$r_t = \overline{rr}_t + \delta_\pi^{EH} E_t^* \pi_{t+1} + \delta_x^{EH} E_t^* x_{t+1} + \delta_g^{EH} g_t + \delta_u^{EH} u_t \tag{2.8}$$

[11] We have chosen the nominal interest rate to be the instrument variable for easier interpretation (as in real life it is usually a primary instrument of central banks). We could have equally chosen π_t or x_t.

where:

$$\begin{aligned}\delta_{\pi}^{EH} &= 1+\sigma\tfrac{\kappa\beta}{\alpha+\kappa^2}\\ \delta_{x}^{EH} &= \sigma\\ \delta_{g}^{EH} &= \sigma\\ \delta_{u}^{EH} &= \sigma\tfrac{\kappa}{\alpha+\kappa^2}\ .\end{aligned}$$

Throughout the chapter we denote the coefficients by EH referring to the paper Evans and Honkapohja (2003a) (EH hereafter), where the authors derive a rule analogous to (2.8). In the terminology introduced in Evans and Honkapohja (2003a), Evans and Honkapohja (2003b), this is an *expectations-based reaction function*; they show that this rule guarantees not only determinacy under RE, but also convergence to the RE equilibrium when expectations E_t^* evolve according to least squares learning.

If the agents have RE (i.e., if $E_t^* = E_t$), Clarida et al. (1999) show that the solution of (2.4) yields:

$$\begin{aligned}\pi_t^{RE} &= \frac{\alpha}{\kappa^2+\alpha}u_t\\ x_t^{RE} &= -\frac{\kappa}{\kappa^2+\alpha}u_t\ .\end{aligned}$$

Under RE, the assumption that the monetary authority takes private sector beliefs as given has a precise motivation in terms of lack of credibility[12]: if the CB is free to reoptimize every period, agents take it into account ignoring any promise it makes on the future. As a result, the discretionary RE equilibrium has the property that the CB has no incentive to change its policy (it is time consistent).

If private agents follow learning, a fully rational CB could do better than (2.8). In the next section we show how optimal monetary policy is modified when the CB optimizes taking into account its effect on private expectations.

2.2.2 Constant Gain Learning

We now assume that private sector's expectations are formed according to the adaptive learning literature[13]; we assume that agents do not know the exact process followed by the endogenous variables, but recursively estimate a Perceived Law of Motion (PLM) consistent with the law of motion that the

[12]In the literature this case is known as optimal policy under discretion.

[13]For an extensive monograph on this paradigm, see Evans and Honkapohja (2001).

CB would implement under RE. As explained above, the optimal allocations of the discretion and the commitment solution under RE have different functional forms, and are therefore associated with different PLMs. For analytical simplicity, in this chapter we will restrict our attention to the discretionary case. In particular, we assume that agents believe that inflation and output gap are continuous invariant functions of the cost-push shock only, $\pi_t = \pi(u_t)$ and $x_t = x(u_t)$[14]; this hypothesis, together with the iid nature of the shock, implies that the conditional and unconditional expectations of inflation and output gap coincide, and are perceived by the agents as constants. Hence, it is natural to assume that agents estimate them using their sample means. Throughout this section we will assume that expectations evolve following the algorithm[15]:

$$E_t^* \pi_{t+1} \equiv a_t = a_{t-1} + \gamma \left(\pi_{t-1} - a_{t-1} \right) \tag{2.9}$$

$$E_t^* x_{t+1} \equiv b_t = b_{t-1} + \gamma \left(x_{t-1} - b_{t-1} \right) \tag{2.10}$$

where $\gamma \in (0, 1)$ is the gain parameter, constant through time.

The use of constant gain algorithms to track structural changes is well known from the statistics and engineering literature[16]. Analogously, private agents would be likely to use constant gain algorithms if they confidently believe structural changes to occur. This algorithm implies that past data are geometrically downweighted, in other words agents 'trust more' recent data. This approach is closely related to using a fixed sample length, or rolling window regressions.

In Section 2.3 we will relax this assumption, and examine how optimal policy changes when agents follow decreasing gain learning.

To analyze the optimal control problem faced by the CB, we use the standard Ramsey approach, namely we suppose that the policymakers take the structure of the economy (equations (2.1) and (2.2)) as given; moreover, we assume that the CB knows how private agents' expectations are formed, and takes into account its ability to influence the evolution of the beliefs. Hence, the CB problem can be stated as follows:

---

[14] In the terminology of Evans and Honkapohja (2001) Chapter 11, the PLM is a noisy steady state.

[15] To be precise, in the algorithms (2.9), (2.10) the observations are weighted geometrically, while in the normal sample average they all receive equal weight.

[16] See for example Benveniste et al. (1990), Part I. Chapters 1. and 4.

$$\min_{\{\pi_t, x_t, r_t, a_{t+1}, b_{t+1}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \left(\pi_t^2 + \alpha x_t^2\right) \tag{2.11}$$

s.t. (2.1), (2.2), (2.9), (2.10)
a_0, b_0 given

This optimization problem is linear quadratic, the Bellman equation holds, thus the resulting policy is time consistent [17].

The first order conditions at every $t \geq 0$ are:

$$\lambda_{1t} = 0 \tag{2.12}$$

$$2\pi_t - \lambda_{2t} + \gamma\lambda_{3t} = 0 \tag{2.13}$$

$$2\alpha x_t + \kappa\lambda_{2t} - \lambda_{1t} + \gamma\lambda_{4t} = 0 \tag{2.14}$$

$$E_t\left[\frac{\beta}{\sigma}\lambda_{1t+1} + \beta^2\lambda_{2t+1} + \beta(1-\gamma)\lambda_{3t+1}\right] = \lambda_{3t} \tag{2.15}$$

$$E_t\left[\beta\lambda_{1t+1} + \beta(1-\gamma)\lambda_{4t+1}\right] = \lambda_{4t} \tag{2.16}$$

where λ_{it}, $i = 1, ..., 4$ denote the Lagrange multipliers associated to (2.1), (2.2), (2.9) and (2.10), respectively. The necessary conditions for an optimum are the first order conditions, the structural equations (2.1)-(2.2) and the laws of motion of private agents' beliefs, (2.9)-(2.10). Combining equation (2.12) and (2.16), we get:

$$\lambda_{4t} = \beta(1-\gamma) E_t\left[\lambda_{4t+1}\right]$$

which can be solved forward, implying that the only bounded solution is:

$$\lambda_{4t} = 0 \tag{2.17}$$

If we put together equations (2.12)-(2.15) and (2.17), we derive the following optimality condition:

$$\frac{\kappa}{\alpha}\pi_t + x_t = \beta E_t\left[\beta\gamma x_{t+1} + (1-\gamma)\left(\frac{\kappa}{\alpha}\pi_{t+1} + x_{t+1}\right)\right] \tag{2.18}$$

[17] A problem solved at t is said to be time consistent for $t+1$ if the continuation from $t+1$ on of the optimal allocation chosen at t solves in $t+1$; moreover, in period zero it is time consistent if the problem in period t is time consistent for $t+1$ for all $t \geq 0$.

Inflation-Output Gap Tradeoff

We can solve forward (2.18), obtaining the unique bounded solution:

$$\frac{\kappa}{\alpha}\pi_t + x_t = \beta^2\gamma E_t\left[\sum_{s=1}^{\infty}\left[\beta\left(1-\gamma\right)\right]^{s-1}x_{t+s}\right] \qquad (2.19)$$

From this relation we can isolate two trade-offs faced by the CB in designing the optimal policy. When $\gamma = 0$, namely when expectations are constant and, consequentely, cannot be manipulated by the monetary authority, (2.19) collapses to:

$$\frac{\kappa}{\alpha}\pi_t + x_t = 0\ , \qquad (2.20)$$

which is identical to the optimality condition derived in the RE optimal monetary policy literature when the CB sets the optimal plan taking private sector's expectations as given (i.e., in the discretionary case). When a cost-push shock is present, (2.20) represents a well known *intratemporal trade-off* between stabilization of inflation at t and output gap at t: because of the nonzero term u_t in the Phillips Curve (2.2), π_t and x_t cannot be set contemporaneously equal to zero in every period. Clarida et al. (1999) describe (2.20) as implying a 'lean against the wind' policy: in other words, if output gap (inflation) is above target, it is optimal to deflate the economy (contract demand below capacity).

Under learning (i.e., when $\gamma > 0$), it turns out that the CB faces an additional *intertemporal trade-off* between optimal behavior at t and stabilization of output gap at $t+1$, generated by its ability to manipulate future values of a. In fact (2.19) implies that, for a given positive value of x_t, the optimal disinflation is less harsh with respect to the one implied by (2.5), provided that future output gaps are also expected to be positive. A smaller deflation in turn guarantees that future inflationary expectations will be closer to the rational expectations equilibrium of inflation, zero. As a result, the CB renounces to optimally stabilize the economy in period t, in exchange for a reduction in future inflation expectations that allows an ease in the future inflation output gap trade-off embedded in the Phillips Curve.

Let us summarize our first result for later reference:

Result 1. *Learning introduces an intertemporal trade-off not present under rational expectations.*

Optimal allocations

We can combine the conditions for an optimum to characterize analytically the optimal allocations implemented by the CB; the results are summarized in the following Proposition.

Proposition 1. *There exists a unique solution of the control problem* (2.11)*, and the policy function for inflation associated to it has the form:*

$$\pi_t = c_\pi^{cg} a_t + d_\pi^{cg} u_t \tag{2.21}$$

The coefficient c_π^{cg} can be characterized as follows:

$$\begin{aligned} \text{-}if\ \gamma &\in (0,1),\ we\ have\ that\ 0 < c_\pi^{cg} < \frac{\alpha\beta}{\alpha+\kappa^2}, \\ \text{-}if\ \gamma &= 0,\ i.e.\ if\ expectations\ are\ constant,\ we\ have\ that\ c_\pi^{cg} = \frac{\alpha\beta}{\alpha+\kappa^2}, \end{aligned}$$

and:

$$d_\pi^{cg} = \frac{\alpha}{\kappa^2 + \alpha + \alpha\beta^2\gamma^2(\beta - c_\pi^{cg}) + \beta\gamma\,(1-\gamma)\,(\alpha\beta - (\kappa^2+\alpha)\,c_\pi^{cg})}$$

Proof. See the Appendix. □

Following the adaptive learning terminology, we call (2.21) the Actual Law of Motion (ALM) of inflation.
Under the optimal policy (OP) a positive a_t increases current inflation, but less than proportionally, since $\frac{\alpha\beta}{\alpha+\kappa^2} < 1$. As is shown in the Appendix, c_π^{cg} depends on all the structural parameters; in particular, its dependence on the constant gain γ is not necessarily monotonic. In fact, a higher value of γ has two effects on c_π^{cg}: on one hand, it increases the effect of current inflation on future expectations, increasing the incentive for the CB to use this influence (i.e., it would determine a lower c_π^{cg}); on the other hand, it reduces the impact of current expectations on future expectations, thus reducing the benefits from a reduction of the expectations, so that there is an incentive to set a higher c_π^{cg}. In Figure 2.1 we show a numerical example with the calibration found in Woodford (1999), i.e. with $\beta = 0.99$, $\sigma = 0.157$, $\kappa = 0.024$ and $\alpha = 0.04$; in this case, the first effect dominates, so that c_π^{cg} is a monotonically decreasing function of γ.

Using the structural equation (2.2) we can derive the optimal allocation of the output gap:

$$x_t = c_x^{cg} a_t + d_x^{cg} u_t \tag{2.22}$$

where:

$$\begin{aligned} c_x^{cg} &= \frac{c_\pi^{cg} - \beta}{\kappa} \\ d_x^{cg} &= \frac{d_\pi^{cg} - 1}{\kappa} \end{aligned}$$

$c_\pi^{cg} < \frac{\alpha\beta}{\alpha+\kappa^2}$ (see Proposition 1) implies $c_x^{cg} < -\frac{\kappa\beta}{\alpha+\kappa^2}$; if the private sector expects inflation to be positive, the optimal CB response will imply a negative output gap, i.e. the policymaker will contract economic activity (using the interest rate instrument) in order to attain an actual inflation sufficiently smaller than the expected one. Using (2.21) and (2.22) in (2.1) we can derive the nominal interest rate:

$$r_t = \overline{rr}_t + \delta_\pi^{cg} a_t + \delta_x^{cg} b_t + \delta_g^{cg} g_t + \delta_u^{cg} u_t \tag{2.23}$$

where:

$$\begin{aligned} &\delta_\pi^{cg} = 1 - \sigma \tfrac{c_\pi^{cg} - \beta}{\kappa} \\ &\delta_x^{cg} = \sigma \\ &\delta_g^{cg} = \sigma \\ &\delta_u^{cg} = -\sigma \tfrac{d_\pi^{cg} - 1}{\kappa} \end{aligned}$$

The interest rate rule (2.23) is an *expectations-based reaction function*, which is characterized by a coefficient on inflation expectations that is decreasing in c_π^{cg}: an optimal ALM for inflation that requires a more aggressive undercutting of inflation expectations (a lower c_π^{cg}) calls for a more aggressive behavior of the CB when it sets the interest rate (a higher coefficient on inflation expectations in the rule (2.23)). Moreover, the coefficients on b_t and g_t are such that their effects on the output gap in the IS curve are fully neutralized.

Since $c_{\pi,t}^{cg} < \beta$ (see Proposition 1) $\delta_{\pi,t}^{cg}$ is always bigger than 1. In response to a rise in expected inflation optimal policy should raise the nominal interest rate sufficiently to increase the real interest rate. An increase in the real rate has a negative effect on current output; this reflects the intertemporal substitution of consumption. Then a contraction in output will decrease current inflation through the Phillips Curve (2.2), and consequently through equation (2.9) inflationary expectations in the next period will decrease. This criterion -also known as the "Taylor principle"- is emphasized in Clarida et al. (1999) under the discretionary rational expectations solution; since this holds both under RE and learning it provides a very simple criterion for evaluating monetary policy[18].

Plugging (2.21) into (2.9), we get:

$$\begin{aligned} a_{t+1} &= a_t + \gamma(c_\pi^{cg} - 1)a_t + \gamma d_\pi^{cg} u_t \\ &= \left(1 - \gamma(1 - c_\pi^{cg})\right) a_t + \gamma d_\pi^{cg} u_t \end{aligned}$$

which is a stationary[19] AR(1); thus, as is well-known in the literature on adaptive learning, the contemporaneous presence of random shocks in the ALM and of constant gain specification of the updating algorithm, prevents the expectations from converging asymptotically to a precise value: instead, we have that $a_t \sim N\left(0, \frac{\gamma^2 \left(d_\pi^{cg}\right)^2}{1-\left(1-\gamma(1-c_\pi^{cg})\right)^2}\sigma_u^2\right)$.

2.2.3 Comparison with the EH rule

In this section we state results regarding how optimal monetary policy under constant gain learning differs from rules used earlier in the literature, where the CB is treated as an anticipated utility maximizer (i.e., it considers expectations as given in the optimization problem); in particular we refer to rule (2.8), derived in EH.

It is clear that the coefficients on the output gap expectations and on the demand shock are the same in rule (2.8) as in rule (2.23), while the other two coefficients are typically different. Proposition 1 implies $\delta_\pi^{cg} > \delta_\pi^{EH}$: the interest rate response of OP to out-of-equilibrium inflation expectations is more aggressive than the interest rate response of EH. This is due to the fact that when the CB takes into account its ability to influence agents' beliefs, it optimally chooses to undercut future inflation expectations more than what it would do otherwise.

[18] Clarida et al. (2000) estimate that the pre-Volcker era violated this simple criterion.

[19] In fact, since $0 < c_\pi^{cg} < 1$, it immediately follows that $0 < (1 - \gamma(1 - c_\pi^{cg})) < 1$.

From Proposition 1 it also follows that $\delta_u^{cg} > \delta_u^{EH}$: optimal policy reacts more aggressively also to cost push shocks. After a positive cost push shock the optimally behaving CB raises the interest rate more aggressively than in the case of an anticipated utility maximizer CB; this in turn decreases output, which has a negative effect on inflation. Thus conducting an aggressive interest rate rule in response to the cost push shock, decreases the influence of the cost push shock on inflation, and this in turn will ease agents learning about the true equilibrium level of inflation.

An analogous difference emerges when we compare the allocations implemented by the two different interest rate rules; under constant gain learning optimal allocations are characterized by (2.21) and (2.22), while EH allocations are given by (2.6) with $E_t^*\pi_{t+1} = a_t$.

From Proposition 1 we know that the feedback coefficient under optimal policy c_π^{cg} is smaller than under the EH rule, in order to undercut inflation expectations more. Also the response to the cost push shock is of lesser magnitude when (2.23) is used instead of (2.8) (in fact, $c_\pi^{cg} < \frac{\alpha\beta}{\kappa^2+\alpha}$ implies that $d_\pi^{cg} < \frac{\alpha}{\kappa^2+\alpha}$), because the CB is less willing to accommodate noisy shocks, in order to make easier for the private sector to learn what is the long-term value of the conditional expectations of inflation.

On the other hand, under OP both coefficients in the ALM of x_t are higher in absolute value than under EH, hence allowing a higher feedback from out of equilibrium expectations and noisy cost push shocks to the output gap.

The difference between (2.8) and (2.23) can be summarized as follows:

Result 2. *When the CB takes into account its influence on private agents learning it is optimal to decrease the effect of out of equilibrium expectations on inflation (engineering an aggressive interest rate reaction to inflationary expectations) and increase the effect of out of equilibrium expectations on the output gap compared to the EH policy; moreover, it accommodates less the effect of noisy shocks to inflation compared to the EH policy, even if it translates into a more volatile output gap.*

Similarity to the commitment solution

From Result 2 it follows that the impact of a given nonzero cost push shock drives inflation (output gap) closer to (further from) target when agents are learning, relative to the discretionary RE case. Interestingly, this behavior *qualitatively resembles the optimal RE equilibrium under commitment* within a simple class of policy rules derived in Clarida et al. (1999): if the CB can commit to a policy rule that is a linear function of u_t, the solution can be characterized, when compared to the discretionary equilibrium, by inequalities analogous to the ones summarized in the results stated above. However, the (constrained) commitment solution differs from the discretionary one only when the cost-push shock is an AR(1); if u -and consequently, the equilibrium processes for inflation and output gap- is iid, the two solutions coincide, since future (rational) expectations of the agents cannot be manipulated by the CB. Instead, if expectations are backward-looking, the future beliefs can be manipulated also when the shock is iid: the current actions of the CB influence future beliefs through (2.9) and (2.10) even if the shock is iid.

In both instances this behavior results from the CB's ability to directly manipulate private expectations, even if the channels used are quite different. In fact, under commitment the policy maker uses a *credible promise on the future* to obtain an immediate decline in inflation expectations and thus in inflation. Under learning we observe a smaller initial response of inflation relative to the RE discretionary case because optimal policy reacts less to the cost push-shock to ease private agents learning. In this sense, we can say that the ability to manipulate future private sector expectations through the learning algorithm plays a role similar to a commitment device under RE, hence easing the short-run trade-off between inflation and output gap.

Another similarity to the commitment solution is the sluggish behavior of inflation after an initial cost push shock. The source of inertia under RE commitment and learning is quite different. Under commitment the policy maker carries commitments made in the past (in other words commits to behave in a past dependent way). Under learning the pattern results from the sluggishness of expectations.

As a result of these two similarities, the impulse response function of inflation to a cost push shock will be also similar under OP and RE commitment. Figure 2.2 displays the impulse response function of inflation to a unit shock under OP and discretionary RE policy. In the optimal RE discretionary policy, inflation rises on impact and immediately reverts to the steady state once the shock dies out. Instead, under learning the policy maker engineers a smaller initial response of inflation; in subsequent periods inflation gradually converges back to the steady state value. Clarida et al. (1999) and Gali (2003) show a *similar disinflation path for the Ramsey policy*: a smaller initial inflation compared to the discretionary case, in exchange for a more persistent deviation from the steady state later[20] This behavior of Ramsey policy leads to welfare gains over discretion due to the convexity of the loss function; this preference for slower but milder adjustment to shocks is at the heart of the stabilization bias.

The similarity to the RE commitment solution resembles the analysis carried out in Sargent (1999), Chapter 5, which shows that in the Phelps problem under adaptive expectations[21], the optimal monetary policy drives the economy close to the Ramsey optimum. Moreover, when the discount factor β equals 1, optimal policy under learning replicates the Ramsey equilibrium. In our case, optimal policy under learning cannot replicate the commitment solution even for β going to 1. This result follows from the particular nature of the gains from commitment; commitment calls for an ALM with a different functional form with respect to the discretionary case[22]. In the Phelps problem, on the other hand, the Phillips Curve is such that the discretion and commitment outcome of inflation has the same functional form, but different coefficients. However, also in our case an increase in the discount factor makes the optimal disinflationary path under learning getting closer to the commitment solution. This can be seen in Table 2.1, where we summarize the behavior of inflation in response to a unit cost push shock when the model's parameters are calibrated as in Woodford (1999), apart from β which takes several values. As β goes to 1 the initial response of inflation is milder and the path back to the steady state longer.

Welfare Loss Analysis

To have a quantitative feeling of the welfare gains that the use of the optimal rule (2.23) instead of the EH rule (2.8) implies, we present a numerical welfare loss analysis.

Since welfare losses in utility terms are hard to interpret we report consumption equivalents: for a given monetary policy rule we calculate the cumulative utility losses resulting from deviations from the steady state allocation and then express what is the equivalent percentage decrease of the

[20] A difference is that commitment policy under RE engineers a sequence of negative inflation after the first period, while a positive sequence under learning.

[21] Phelps (1967) formulated a control problem for a natural rate model with rational Central Bank and private agents endowed with a mechanical forecasting rule, known to the Central Bank.

[22] See Clarida et al. (1999).

Table 2.1: Paths of inflation for different βs after an initial cost push shock

beta	0.5	0.6	0.7	0.8	0.9	1.0
1	0.99	0.99	0.98	0.98	0.96	0.91
2	0.44	0.52	0.61	0.69	0.75	0.73
3	0.24	0.33	0.44	0.55	0.66	0.66
10	0.00	0.01	0.04	0.12	0.27	0.33
50	0.00	0.00	0.00	0.00	0.00	0.01

Woodford (1999) calibration. Cost push shock $u_0 = 1$ in the first period, starting from $a_0 = 0$, $\pi_0 = 0$, $x_0 = 0$, with $\gamma = 0.2$

steady state consumption that results in the same cumulative utility loss (For details of the calculation see the Appendix.). We use the calibration of Woodford (1999): $\beta = 0.99$, $\kappa = 0.024$, $\alpha = 0.048$ and $\sigma = 0.157$. We perform Monte Carlo with a simulation length 10,000 and a cross sectional sample size of 1000, with the initial condition $a_0 = b_0 = 0$. Cost push shocks are drawn from a normal distribution with 0 mean and variance 0.1[23].

Table 2.2 reports consumption equivalents for a range of tracking parameters. For small tracking parameters the results are in the range of Lucas' original estimates[24]: consumption losses resulting from cyclical fluctuations are small. For higher tracking parameters the consumption equivalents are also higher, which results from the fact in the presence of a cost push shock, constant gain learning does not settle down to RE, but converges to a limiting distribution and the limiting variance of inflation expectation increases in γ (keeping other coefficients constant). This is illustrated in Figure 2.3. A higher variance of inflationary expectations in turn results in higher variance of inflation and output gap both under OP (see equation (2.21) and (2.22)) and under EH (see equation (2.6)), thus a higher welfare loss equivalent permanent consumption reduction.

Since both inflation and output gap variance can be expressed as a linear function of the variance of the cost push shock, clearly the absolute value of consumption equivalents are also increasing with the variance of the cost push shock, but the ratio of consumption equivalents under OP and EH are not sensitive to the choice of σ_u^2[25].

[23] Note that the demand shock does not appear in the actual law of motion of the endogenous variables.

[24] See Lucas (1987).

[25] We performed the welfare loss analysis also in the case of $\sigma_u^2 = 0.6$, an estimate of

Optimal policy decreases consumption equivalents relative to the rule (2.8) (see the third column in Table 2.2). Even for tracking parameters below 0.05[26] the gain from using an optimal interest rate rule (2.23) compared to the EH rule (2.8) is around $1-3\%$. The gain in consumption equivalents is higher the higher is the gain parameter. For a very high tracking parameter $\gamma = 0.9$ the welfare loss in consumption terms of not using the optimal rule is twice as large as under OP. This follows from the fact that, optimal policy takes into account that expectations have a limiting variance while the EH policy considers expectations to be fixed[27]. As a result, optimal policy aims to decrease the limiting variance of inflationary expectations while EH does not, and the higher is the tracking parameter the bigger is the decrease in the limiting variance OP engineers compared to EH (see Figure 2.3).

Table 2.2: Consumption equivalents using OP and EH under constant gain learning

Tracking parameter	p^{OP}	p^{EH}	p^{OP}/p^{EH}
0.0187	0.0129	0.0129	0.9990
0.05	0.0148	0.0151	0.9759
0.08	0.0171	0.0185	0.9243
0.1	0.0188	0.0213	0.8830
0.3	0.0371	0.0619	0.5996
0.5	0.0554	0.1122	0.4935
0.9	0.0910	0.2217	0.4106

Woodford (1999) calibration

It is interesting to examine the composition of welfare losses coming from inflation variation and output gap variation. For this we calculate the equivalent permanent consumption decrease for welfare losses caused by only inflation variation or output gap variation respectively, and report the ratios of OP and EH in Table 2.3. The table demonstrates Results 2: optimal policy

Milani (2005), obtaining the same ratio of consumption equivalents under OP and EH.

[26]Estimates for the US are typically in this range. 0.0187 is the estimation of Milani (2005) with Bayesian estimation, for a calibration of the tracking parameter see Orphanides and Williams (2004a).

[27]It is worth noting that the EH rule is designed to ensure learnability of the optimal RE in a decreasing gain environment, and its performance under constant gain is never considered in the EH paper; however, it can be useful to employ a constant gain version of their rule to illustrate potential advantages of fully optimal monetary policy.

focuses on decreasing inflation variation even at the cost of higher output gap variation. The higher is the tracking parameter, the higher is the incentive of the Central Bank to focus on lowering inflation variance and allowing for an increase in output gap deviation from the flexible price equilibrium. For $\gamma = 0.9$ compared to EH an optimally behaving Central Bank engineers a 78% lower welfare loss in inflation when it properly conditions on expectation formation, permitting at the same time 15 times more variation in output gap.

Table 2.3: Ratio of consumption equivalents of losses due to inflation and output gap variations using OP and EH under constant gain learning

Tracking parameter	Inflation	Output gap
0.0187	0.9962	1.2296
0.05	0.9441	3.6263
0.08	0.8511	7.0185
0.1	0.7853	9.0290
0.3	0.4187	15.6711
0.5	0.3073	16.0060
0.9	0.2286	15.5719

Woodford (1999) calibration

Moreover, it is worth noting that the use of the rule (2.8) under constant gain learning allows for the autocorrelation of inflation to rise, thus increasing the persistence of a shock's effect on inflation expectations. This problem arises from the relatively weak response to inflation expectations which feeds back to current inflation and, in turn, into subsequent expectations and inflations. The optimal rule's strong feedback to inflation expectations dampens this interaction between inflation and expectations[28].

This section has shown that optimal policy under learning is characterized by a more aggressive interest rate reaction to out-of-equilibrium expectations and to the cost push shock than would be optimal when the CB does not make active use of its influence on expectations. This aggressive behavior

[28]It can be easily derived that the autocorrelation of inflation under constant gain with EH is $E\pi_t^{EH}\pi_{t-1}^{EH} = \left(\frac{\alpha\beta}{\alpha+\kappa^2}\right)^2 \left(1-\gamma+\gamma\frac{\alpha\beta}{\alpha+\kappa^2}\right)\sigma^2_{a_{EH}} + \frac{\alpha\beta}{\alpha+\kappa^2}\left(\frac{\alpha}{\alpha+\kappa^2}\right)^2\gamma\sigma_u^2$ while under the optimal rule $E\pi_t^{OP}\pi_{t-1}^{OP} = \left(c_\pi^{cg}\right)^2\left(1-\gamma+\gamma c_\pi^{cg}\right)\sigma^2_{a_{OP}} + c_\pi^{cg}\left(d_\pi^{cg}\right)^2\gamma\sigma_u^2$. We have already seen that $\sigma^2_{a_{OP}} < \sigma^2_{a_{EH}}$, $c_\pi^{cg} < \frac{\alpha\beta}{\alpha+\kappa^2}$ and $d_\pi^{cg} < \frac{\alpha}{\alpha+\kappa^2}$, thus $E\pi_t^{OP}\pi_{t-1}^{OP} < E\pi_t^{EH}\pi_{t-1}^{EH}$.

guarantees that inflation will deviate less from its equilibrium value, thus private agents can learn the true equilibrium level of inflation faster than under EH policy. Helping inflationary expectations is beneficial, even at the cost of allowing higher deviations in output gap expectations and a higher output gap volatility. Welfare gains from using the optimal policy are particularly pronounced when private agents use a high tracking parameter (i.e. discount more past data) for forecasting. This result indicates that properly conditioning on private agents expectation formation is especially important in a nonconvergent environment, i.e. when agents follow constant gain learning.

2.3 Decreasing Gain Learning

In this section we relax the assumption of constant gain learning and show that our main results remain valid also with decreasing gain learning (henceforth DG) and show that the time varying nature of expectations imply that during the transition the optimal policy should be time varying even in a stationary environment.

Using a constant gain parameter γ is appropriate when agents believe structural changes to occur. If instead the private sector confidently believes that the environment is stationary it is more reasonable to model their learning behavior with a decreasing gain rule, namely an algorithm of the form:

$$E_t^* \pi_{t+1} \equiv a_t = a_{t-1} + t^{-1}(\pi_{t-1} - a_{t-1}) \tag{2.24}$$

$$E_t^* x_{t+1} \equiv b_t = b_{t-1} + t^{-1}(x_{t-1} - b_{t-1}) \tag{2.25}$$

where the only difference with (2.9)-(2.10) is the substitution of γ with t^{-1}. An updating scheme of this form is equivalent[29] to estimating inflation and output gap every period with OLS[30].

The problem of the CB is now:

[29]Under certain conditions on the values used to initialize the algorithm, see Evans and Honkapohja (2001).

[30]Note that, since inflation and output gap are assumed by the learners to be constant, the OLS is just the sample averages of the two.

$$\min_{\{\pi_t, x_t, r_t, a_{t+1}, b_{t+1}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t (\pi_t^2 + \alpha x_t^2) \tag{2.26}$$

$$\text{s.t. } (2.1), (2.2), (2.24), (2.25) \tag{2.27}$$
$$a_0, b_0 \text{ given}$$

The optimization can be solved in a way analogous to the constant gain case; hence, the dynamics of the system can be summarized by the optimality condition:

$$\frac{\kappa}{\alpha}\pi_t + x_t = \beta E_t \left[\beta \frac{1}{t+1} x_{t+1} + \frac{\kappa}{\alpha}\pi_{t+1} + x_{t+1} \right] \tag{2.28}$$

Iterating it forward we get:

$$\frac{\kappa}{\alpha}\pi_t + x_t = E_t \left[\sum_{s=1}^{\infty} \beta^{s+1} \frac{1}{t+s} x_{t+s} \right] .$$

Similarly to Section 2.2 our result is that learning introduces an intratemporal tradeoff between inflation and output that is not present under RE in an economy without a cost push shock and an additional intertemporal tradeoff that is not present in general under rational expectations (Result 1). From the latter it follows that during the transition for a given positive value of x_t, the optimal disinflation is less harsh with respect to the one implied by (2.5) (optimizing taking expectations as given) provided that the series on the right hand side is expected to be positive. The intuition behind is that when the CB makes active use of the expectation formation, it renounces its ability to optimally stabilize the economy in period t, in exchange for a reduction in future inflation expectations (in absolute value) and this allows an ease in the future inflation-output gap trade-off embedded in the Phillips Curve.

To derive the optimal allocations, we can use (2.2) to substitute out x_t in (2.28), then using the evolution of inflationary expectations (2.24)we get:

$$E_t [\pi_{t+1}] = A_{11,t}\pi_t + A_{12,t}a_t + P_{1,t}u_t \tag{2.29}$$

where:

$$
\begin{aligned}
A_{11,t} &\equiv \frac{\kappa^2+\alpha+\alpha\beta^2\frac{1}{t+1}\left(1+\beta\frac{1}{t+1}\right)}{\alpha\beta(1+\beta\frac{1}{t+1})+\kappa^2\beta} \\
A_{12,t} &\equiv -\frac{\alpha\beta\left[1-\beta\left(1-\frac{1}{t+1}\right)\left(1+\beta\frac{1}{t+1}\right)\right]}{\alpha\beta(1+\beta\frac{1}{t+1})+\kappa^2\beta} \ . \\
P_{1,t} &\equiv -\frac{\alpha}{\alpha\beta(1+\beta\frac{1}{t+1})+\kappa^2\beta}
\end{aligned}
$$

Hence, at an optimum, the dynamics of the economy can be summarized by stacking equations (2.24), (2.25) and (2.29), and obtaining the trivariate system:

$$E_t y_{t+1} = A_t y_t + P_t u_t \tag{2.30}$$

where $y_t \equiv [\pi_t, a_t, b_t]'$, and:

$$A_t \equiv \begin{pmatrix} A_{11,t} & A_{12,t} & 0 \\ \frac{1}{t+1} & 1-\frac{1}{t+1} & 0 \\ \frac{\frac{1}{t+1}}{\kappa} & -\frac{\beta\frac{1}{t+1}}{\kappa} & 1-\frac{1}{t+1} \end{pmatrix} \ , \quad P_t = \begin{pmatrix} P_{1,t} \\ 0 \\ -\frac{\frac{1}{t+1}}{\kappa} \end{pmatrix} .$$

We can find the solution with the method of undetermined coefficients, with the guess[31]:

$$\pi_t = c^{dg}_{\pi,t} a_t . + d^{dg}_{\pi,t} u_t \tag{2.31}$$

The sequence $\left\{c^{dg}_{\pi,t}\right\}$ must satisfy the non-linear, non-autonomous first order difference equation:

$$c^{dg}_{\pi,t} = \frac{c^{dg}_{\pi,t+1}\left(1-\frac{1}{t+1}\right)-A_{12,t}}{A_{11,t}-c^{dg}_{\pi,t+1}\frac{1}{t+1}} \tag{2.32}$$

and the sequence $\left\{d^{dg}_{\pi,t}\right\}$ is defined as:

$$d^{dg}_{\pi,t} = \frac{P_{1,t}}{c^{dg}_{\pi,t+1}\frac{1}{t+1}-A_{11,t}} .$$

[31] This guess corresponds to the unique solution under constant gain learning. A proof of uniqueness of a bounded solution for decreasing gain learning is not worked out completely yet.

Of course, there exist infinite sequences that satisfy equation (2.32), one for each initial value $c^{dg}_{\pi,0}$. However, since the boundary conditions require π_t to stay bounded, we will concentrate on the solutions that do not explode. The properties of the coefficients in (2.31) are characterized in the following Proposition.

Proposition 2. *Let* $\left\{c^{dg}_{\pi,t}\right\}$ *be defined by* (2.32)*, and assume it is bounded; then,* $\lim_{t\to\infty} c^{dg}_{\pi,t}$ *exists, and is given by:*

$$\lim_{t\to\infty} c^{dg}_{\pi,t} = \frac{\alpha\beta}{\alpha+\kappa^2}$$

Moreover, for any $t < \infty$, we have:

$$c^{dg}_{\pi,t} < \frac{\alpha\beta}{\alpha+\kappa^2}$$

Proof. See the Appendix. □

Thus Result 2 holds during the transition: when the CB takes into account its influence on expectations it is optimal to decrease the effect of out-of-equilibrium expectations on inflation compared to equation (2.6), in order to undercut future inflation expectations by a larger amount. This relaxes the future inflation-output gap trade-off embedded in the Phillips Curve. The ALM for output gap is:

$$x_t = c^{dg}_{x,t} a_t + d^{dg}_{x,t} u_t \tag{2.33}$$

where:

$$\begin{aligned} c^{dg}_{x,t} &= \frac{c^{dg}_{\pi,t} - \beta}{\kappa} \\ d^{dg}_{x,t} &= \frac{d^{dg}_{\pi,t} - 1}{\kappa} \end{aligned}$$

If the private sector expects inflation to be positive, the optimal CB will contract economic activity more than EH[32] (using the interest rate instrument);

[32]From $c^{dg}_{\pi,t} < \frac{\alpha\beta}{\alpha+\kappa^2}$ it follows that $c^{dg}_{x,t} < -\frac{\kappa\beta}{\alpha+\kappa^2}$. Compare with the ALM under EH (2.6).

the CB is ready to pay a short-term cost represented by a wider current output gap in order to contain future inflationary expectations.

Note that the policy function does not depend on the period when the cb optimizes, even if it is not time invariant. Thus, the optimal policy characterized above is time consistent, in the sense of Lucas and Stokey (1983) and Alvarez et al. (2004).

The nominal interest rate rule is as follows:

$$r_t = \overline{rr}_t + \delta^{dg}_{\pi,t} a_t + \delta^{dg}_x b_t + \delta^{dg}_g g_t + \delta^{dg}_{ut} u_t \tag{2.34}$$

where:

$$\begin{aligned} \delta^{dg}_{\pi,t} &= 1 - \sigma \tfrac{c^{dg}_{\pi,t} - \beta}{\kappa} \\ \delta^{dg}_x &= \sigma \\ \delta^{dg}_g &= \sigma \\ \delta^{dg}_{ut} &= -\sigma \tfrac{d^{dg}_{\pi,t} - 1}{\kappa} \end{aligned}$$

Since $c^{dg}_{\pi,t} < \beta$ (see Proposition 2) $\delta^{dg}_{\pi,t}$ is always bigger than 1. In response to a rise in expected inflation optimal policy should raise the nominal interest rate sufficiently to increase the real interest rate. The following proposition pertains to the characteristics of the optimal rule compared to the EH rule (2.8):

Proposition 3. *Assume that $t < \infty$; then:*
-$\delta^{dg}_{\pi,t} > \delta^{EH}_{\pi}$, $\delta^{dg}_{ut} > \delta^{EH}_{u}$.
Moreover, we have:
-$\lim_{t\to\infty} \delta^{dg}_{\pi,t} = \delta^{EH}_{\pi}$, $\lim_{t\to\infty} \delta^{dg}_{ut} = \delta^{EH}_{u}$.

Proof. See the Appendix. □

Result 2 under CG is parallelled by our results under DG: the optimal interest rate rule should react more aggressively to out of equilibrium expectations than the EH rule. A CB that knows how its behavior affects private sector expectations should contain more inflationary expectations than a CB that takes expectations as given.

An interesting result is that the coefficient on inflation expectations in the interest rate rule (2.34) is *time-varying*, reflecting the fact that the Central Bank's incentives to manipulate agents' beliefs evolve over time. This implies that during the transition optimal policy should be time varying even in a stationary environment.

In Figure 2.5, we show how this coefficient depends on time when the parameters are calibrated according to Woodford (1999). $\delta^{dg}_{\pi,t}$ is always above its limiting level (see analytical proof in Proposition 3), moreover, it decreases over time. Numerical analysis on the grid $\beta = 0.99$ and $\alpha \in [0.01, 2]$, $\kappa \in [0.01, 0.5]$ shows that this decreasing behavior of $\delta^{dg}_{\pi,t}$ is a robust feature of the model [33]. We find that after the 4th period (from the 4th to the 5th period and so on) $\delta^{dg}_{\pi,t}$ is always decreasing, while in the first 4 periods $\delta^{dg}_{\pi,t}$ might be increasing (hump-shaped) for a combination of low values of α and high values of κ (see Figure 2.7) [34]. We summarize our new results as:

Result 3. *Optimal policy is time varying even in a stationary environment. Initially it reacts more aggressively to both out of equilibrium expectations and cost push shocks and dampens its aggressiveness later.*

To get an intuition, suppose that a structural break occurs. For example there is a policy change because a new central bank governor is appointed, agents know that monetary policy has changed and try to learn how this affects the equilibrium. In this situation is convenient for the CB to react more aggressively to out-of-equilibrium inflation beliefs in the first periods, when agents pay more attention to new information and the CB's possibilities of influencing private expectations are greater. This behavior is beneficial even at the cost of larger short-term losses in terms of output gap variability. As time passes, the expectations will be influenced to a lesser extent by the last realization of inflation, hence determining a CB reaction that closely resembles the optimizing behavior when policymakers cannot manipulate expectations.

The inequality $\delta^{dg}_{ut} > \delta^{EH}_{u}$ is parallel to the second part of Result 2: during the transition the optimal policy engineers more aggressive interest rate movements in response to cost push shock variations than EH, and this way it accommodates less the effect of noisy shocks on inflation compared to EH. δ^{dg}_{ut} is positive and decreasing over time (see Figure 2.6)[35]. In response to a positive cost push shock, the Central Bank raises interest rate to contract the output and thus reduce inflation, and future inflationary expectations.

[33] We have chosen the grid to include typical calibrated values for the US and the EURO area.

[34] In fact, $\delta^{dg}_{\pi,t}$ is always decreasing also for other calibrations widely adopted in the New Keynesian Literature, like those taken from Clarida et al. (2000) and McCallum and Nelson (1999).

[35] Since $\delta^{dg}_{u,t} < 1$ from (2.34) it follows that the change of $\delta^{dg}_{u,t}$ through time is identical to that of $\delta^{dg}_{\pi,t}$.

The asymptotic properties of the ALM (2.31),(2.33) depend on the limiting behavior of a_t, which is given by the stochastic recursive algorithm:

$$a_{t+1} = a_t + (t+1)^{-1} \left((c_{\pi t}^{dg} - 1)a_t + d_{\pi,t}^{dg} u_t \right) \tag{2.35}$$

We study its properties in the Appendix, where we use the stochastic approximation techniques[36] to prove the following Proposition:

Proposition 4. *Let a_t evolve according to (2.35); then, $a_t \to 0$ a.s.*

This result, together with the boundedness of $c_{\pi,t}^{dg}$, implies that $c_{\pi,t}^{dg} a_t$ goes to zero almost surely; moreover, it is easy to see that $d_{\pi,t}^{dg} \to \frac{\alpha}{\kappa^2+\alpha}$, so that we can conclude that $\pi_t \to \frac{\alpha}{\kappa^2+\alpha} v$ almost surely, where v is a random variable with the same probability distribution as u_t. The equilibrium corresponds to the discretionary rational expectations equilibrium. Optimal policy 'helps' private agents to learn the rational discretionary REE[37].

From Proposition 3 it follows that the optimal policy converges to the EH policy; since expectations converge to a constant it is intuitive that in the limit OP behaves as if expectations were fixed. Bellow we provide a numerical analysis on how the difference during the transition translates into welfare losses. Similarly to Section 2.2 we report consumption equivalents[38].

Tables 2.4 and 2.5 show that similarly to the constant gain case in the long run OP engineers a lower consumption equivalent then the EH policy, and OP engineers lower variation of inflation at the cost of allowing higher variation in output. The last row of the first column in Table 2.4 shows that if we start the economy from the RE equilibrium, $a_0 = 0$, in the long run the consumption equivalent of OP is about 10% lower then that of EH. Table 2.5 reports the composition of these losses: the optimal policy engineers an inflation variation 20 percent lower then EH and allows a 3-9 times higher welfare loss due to output gap variations.

These long run gains of OP result from the different transition path towards the RE equilibrium this policy engineers compared to EH.

[36]For an extensive monograph on stochastic approximation, see Benveniste et al. (1990); the first paper to apply these techniques to learning models is Marcet and Sargent (1989).

[37]Note that the PLM of private agents does not nest the commitment REE, only the discretionary REE, so agents have a 'chance' to learn only the latter.

[38]We report the permanent percentage decrease in the steady state consumption that is equivalent to the cumulative welfare losses up to time T under OP as a ratio of the same measure under EH (See Appendix.). Results are obtained by Monte Carlo simulations.

Let us first examine the path of expectations. Both OP and EH are E-stable under learning, so guarantee that expectations converge to the discretionary REE, the difference is the speed of convergence. Figure 2.8 shows a typical realization of the evolution of expectations under OP and EH. We can observe that inflation expectations converge faster and output gap expectations converge more slowly with our rule than with the EH one. This is a consequence of the intertemporal tradeoff (Result 1): when the CB does take into account its influence on the learning algorithm, it has an incentive to undercut future inflation beliefs. The way the central bank can achieve this, is to keep inflation close to its RE value; since inflationary expectations are formed as averages of past inflation data, this policy undercuts future inflation expectations. Because of the intratemporal tradeoff between inflation and output, the cost of keeping inflation closer to its RE value is a wider output gap and consequently a slower convergence of b to its RE value.

We report how the ratios of OP and EH consumption equivalents evolve during the transition in Table 2.4. In the first periods the optimal interest rate rule (2.34) yields *ex-post* higher cumulative welfare losses expressed in consumption terms than the EH rule; later, however, our rule starts generating smaller welfare losses. These findings are consistent with our finding that a CB that follows the optimal rule (2.34) reacts to out-of-equilibrium inflation expectations more aggressively than in the EH case, in order to undercut more future expectations, even if this means allowing a wider output gap in the short run. This implies that in the first periods, when this more aggressive behavior has not generated yet a pay-off in terms of a smaller $|a|$ sufficient to offset the costly output gap variability, our rule performs worse than the EH one; as soon as inflation expectations become small enough, this initial disadvantage is more than compensated. This pattern is magnified by the time-varying behavior of $\delta_{\pi,t}^{dg}$ that we characterized above: the coefficient on inflation expectations in (2.34) is particularly large in the first periods, hence determining large output gap variations and large welfare losses in the short run, and large gains from the contraction of $|a|$ in the medium and long run.

Since the main advantage of OP is that it helps private agents' inflationary expectations to converge faster, the advantage of OP over EH increases the further away initial expectations are from the RE values. The different columns of Table 2.4 report ratios of consumption equivalents for different initial inflationary expectations. The higher is a_0 the bigger is the consumption cost of OP compared to EH in the first periods: OP allows for higher

Table 2.4: Path of cumulative consumption equivalent ratios under decreasing gain, using OP and EH

	p^{OP}/p^{EH}			
T	$a_0 = 0$	$a_0 = 1$	$a_0 = 2$	$a_0 = 3$
1	2.086	4.145	4.327	4.362
5	1.511	2.241	2.325	2.344
10	1.279	1.574	1.609	1.617
20	1.104	1.116	1.117	1.118
26	1.057	0.993	0.986	0.984
27	1.050	0.978	0.969	0.967
40	0.997	0.841	0.821	0.817
43	0.989	0.820	0.799	0.795
49	0.975	0.786	0.763	0.758
10,000	0.899	0.583	0.542	0.533

Woodford (1999) calibration

welfare losses in order to keep inflation closer to the SS in order to help inflationary expectations converge faster. As time goes on, inflation expectations converge closer to 0 under OP than under EH; the further away a_0 is from the equilibrium, also the further away future inflation expectations remain from the equilibrium under EH. Consequently the inflation output gap tradeoff remains worse under EH and consumption equivalents remain also higher then under OP. The bigger is a_0 the bigger is the gain in decreasing inflation variation of OP over EH, and the higher is the output gap variation OP allows compared to EH (See Table 2.5)[39].

In this section we have proved that our main results do not depend on what type of learning algorithm private agents follow. Our new results are that under decreasing gain learning optimal policy should be time varying: more aggressive on inflation initially and less in subsequent periods. In the limit, expectations converge to the discretionary RE equilibrium, and optimal policy is equivalent to the one derived under the assumption of constant expectations. Numerical simulations confirmed the relevance of the welfare gains induced by the implementation of the optimal policy.

[39] Similarly to Section 2.2 ratios of consumption equivalents do not depend on the choice of σ_u^2.

Table 2.5: Ratio of consumption equivalents of losses due to inflation and output gap variations using OP and EH under decreasing gain learning

	$a_0 = 0$	$a_0 = 1$	$a_0 = 2$	$a_0 = 3$
	Inflation			
p^{OP}	0.016	0.054	0.166	0.353
p^{EH}	0.019	0.125	0.439	0.961
p^{OP}/p^{EH}	0.838	0.432	0.379	0.368
	Output gap			
p^{OP}	0.029	0.411	1.547	3.434
p^{EH}	0.005	0.031	0.110	0.241
p^{OP}/p^{EH}	6.044	13.188	14.100	14.279

Woodford (1999) calibration

2.4 Robustness Analysis

Up to now, we have supposed that the CB perfectly observes all the relevant state variables of the system, namely the exogenous shocks and the agents' beliefs. In this section we show that our main results extend to a more general framework, where either the shocks or the expectations are not observable. In particular, to make the problem non-trivial, throughout this section we modify the structural equations (2.1) and (2.2) with the introduction of unobservable shocks, so that the model is now given by:

$$x_t = E_t^* x_{t+1} - \sigma^{-1}(r_t - E_t^* \pi_{t+1} - \overline{rr}_t) + g_t + e_{x,t} \tag{2.36}$$

and:

$$\pi_t = \beta E_t^* \pi_{t+1} + \kappa x_t + u_t + e_{\pi,t} \tag{2.37}$$

where we assume that the CB can observe π_t and x_t only with a lag, and that $e_{x,t}$ and $e_{\pi,t}$ are independent white noise that are not observable, not even with a lag. The rest of the setup is identical to Section 2.3.

2.4.1 Measurement Error in the Shocks

We start with the case in which the monetary authority can observe g_t and u_t only with an error; in particular, we assume that it receives the noisy signals g_t^* and u_t^*, where:

$$\begin{aligned} g_t^* &= g_t + \epsilon_t, \qquad \epsilon_t \sim N(0, \sigma_\epsilon^2) \\ u_t^* &= u_t + \eta_t, \qquad \eta_t \sim N(0, \sigma_\eta^2) \end{aligned}$$

Note that the shocks do not depend on the policy followed by the CB; hence, the *separation principle* applies, namely, the optimization of the welfare criterion and the estimation of the realizations of the shocks can be solved as separate problems. As is well known, the above signal-extraction problem implies that the expected values of the shocks given the signals are[40]:

$$
\begin{aligned}
E\left[g_t \mid g_t^*\right] \equiv E_t^{CB} g_t = \tfrac{\sigma_g^2}{\sigma_\epsilon^2+\sigma_g^2} g_t^* \equiv \zeta_g g_t^* \\
E\left[u_t \mid u_t^*\right] \equiv E_t^{CB} u_t = \tfrac{\sigma_u^2}{\sigma_\eta^2+\sigma_u^2} u_t^* \equiv \zeta_u u_t^*
\end{aligned}
$$

Note that in the above equation we are not using all the available information; to keep the problem simpler, we assume that the posterior beliefs of the CB, $E_t^{CB} g_t$ and $E_t^{CB} u_t$, are not fully rational, but are instead conditional only on the signals. In other words:

$$
\begin{aligned}
E_t^{CB} g_t \equiv E\left[g_t \mid g_t^*\right] \\
E_t^{CB} u_t \equiv E\left[u_t \mid u_t^*\right]
\end{aligned}
$$

Moreover, the separation principle implies that certainty equivalence holds in designing the optimal interest rate rule, which turns out to be identical to (2.34), with g_t and u_t replaced by $E_t^{CB} g_t$ and $E_t^{CB} u_t$, respectively:

$$
\begin{aligned}
r_t &= \overline{rr}_t + \delta_{\pi,t}^{dg} a_t + \delta_x^{dg} b_t + \delta_g^{dg} \zeta_g g_t^* + \delta_{ut}^{dg} \zeta_u u_t^* \\
&= \overline{rr}_t + \delta_{\pi,t}^{dg} a_t + \delta_x^{dg} b_t + \delta_g^{dg} \zeta_g g_t + \delta_g^{dg} \zeta_g \epsilon_t + \delta_{ut}^{dg} \zeta_u u_t + \delta_{ut}^{dg} \zeta_u \eta_t
\end{aligned}
$$

We can combine the above equation with (2.36) and (2.37) to obtain the ALM for inflation and output gap:

$$
\begin{aligned}
\pi_t = \mu_{at}^1 a_t + \mu_g^1 g_t + \mu_\epsilon^1 \epsilon_t + \mu_{ut}^1 u_t + \mu_{\eta t}^1 \eta_t + \kappa e_{x,t} + e_{\pi,t} \\
x_t = \mu_{at}^2 a_t + \mu_g^2 g_t + \mu_\epsilon^2 \epsilon_t + \mu_{ut}^2 u_t + \mu_{\eta t}^2 \eta_t + e_{x,t}
\end{aligned}
$$

where:

$$
\begin{aligned}
&\mu_{at}^1 = c_{\pi,t}^{dg}, \quad \mu_{at}^2 = c_{x,t}^{dg} \\
&\mu_g^1 = \kappa\left(1-\zeta_g\right), \quad \mu_g^2 = 1-\zeta_g \\
&\mu_\epsilon^1 = -\kappa\zeta_g, \quad \mu_\epsilon^2 = -\zeta_g \\
&\mu_{ut}^1 = \left(d_{\pi,t}^{dg}-1\right)\zeta_u + 1, \quad \mu_{ut}^2 = \left(\tfrac{d_{\pi,t}^{dg}-1}{\kappa}\right)\zeta_u \\
&\mu_{\eta t}^1 = \left(d_{\pi,t}^{dg}-1\right)\zeta_u, \quad \mu_{\eta t}^2 = \left(\tfrac{d_{\pi,t}^{dg}-1}{\kappa}\right)\zeta_u
\end{aligned}
$$

40 E.g., see Hamilton (1994).

As a consequence of the measurement error, inflation and output gap now depend on a wider set of state variables; however, it is easy to see that the main findings of the preceding section go through in this modified environment. First of all, the separation principle trivially implies that when the CB takes into account the effect of its decisions on future beliefs, the optimal policy is more aggressive against out-of-equilibrium inflation expectations, compared to the case in which the private sector's expectations are considered as exogenously given[41]; moreover, the analysis of convergence of learning algorithms to the optimal discretionary RE equilibrium[42] does not change in this modified environment.

2.4.2 Heterogenous Forecasts

As argued in Honkapohja and Mitra (2005) (HM hereafter), the hypothesis that the CB can perfectly observe private sector's expectations is subject to several criticisms. For example, private sector expectations and their forecasts produced by different institutions do not necessarily coincide. In this case, the CB could use a proxy for the agents' beliefs. In what follows, we assume that the optimal interest rate rule takes the same form as (2.34), but the agents' forecasts for inflation and output gap, a_t and b_t, are replaced by the CB internal forecasts, a_t^{CB} and b_t^{CB}[43]; in particular, we suppose that the CB and the private sector forecasts have the same form, and are updated according to the same algorithm, which is given by (2.24)-(2.25). The only difference is given by the initial beliefs. Note that this setup corresponds to a situation where the CB, in solving its optimization problem, knows the adaptive algorithm used by the agents to form their expectations, but cannot observe the actual values of these expectations; instead, the CB has a tight prior on a_0 and b_0[44], and forms its internal forecasts accordingly. Plugging

[41]For a description of the optimal policy when the CB does not consider its effect on future beliefs, and there is measurement error in the shocks, see Evans and Honkapohja (2003a) section 4.2.

[42]Note that the optimal RE equilibrium is now different from the baseline case, since inflation and output gap depend also on g_t, ϵ_t, η_t, and the unobservable shocks $e_{x,t}$ and $e_{\pi,t}$.

[43] This approach is developed in HM, where it is applied to the EH rule and to a simple Taylor rule. Evans and Honkapohja (2003b) use this method in a setup where the CB follows the expectations based interest rule derived in Evans and Honkapohja (2006).

[44]In other words, it believes that $a_0 = a_0^{CB}$ and $b_0 = b_0^{CB}$ with probability one, where a_0^{CB} and b_0^{CB} are given.

the interest rate rule into the structural equations (2.36) and (2.37), we get the ALM:

$$\begin{aligned} \pi_t &= \nu_a^1 a_t + \nu_{a^{CB}t}^1 a_t^{CB} + \nu_b^1 b_t + \nu_{b^{CB}}^1 b_t^{CB} + \nu_{ut}^1 u_t + \kappa e_{x,t} + e_{\pi,t} \\ x_t &= \nu_a^2 a_t + \nu_{a^{CB}t}^2 a_t^{CB} + \nu_b^2 b_t + \nu_{b^{CB}}^2 b_t^{CB} + \nu_{ut}^2 u_t + e_{x,t} \end{aligned} \tag{2.38}$$

where:

$$\begin{aligned} &\nu_a^1 = \beta + \kappa\sigma^{-1}, \quad \nu_a^2 = \sigma^{-1} \\ &\nu_{a^{CB}t}^1 = -\kappa\sigma^{-1}\left(1 - \sigma\frac{c_{\pi,t}^{dg} - \beta}{\kappa}\right), \quad \nu_{a^{CB}t}^2 = -\sigma^{-1}\left(1 - \sigma\frac{c_{\pi,t}^{dg} - \beta}{\kappa}\right) \\ &\nu_b^1 = \kappa, \quad \nu_b^2 = 1 \\ &\nu_{b^{CB}}^1 = -\kappa, \quad \nu_{b^{CB}}^2 = -1 \\ &\nu_{ut}^1 = d_{\pi,t}^{dg}, \quad \nu_{ut}^2 = d_{x,t}^{dg} \end{aligned}$$

Again, our main results are unaffected by this change in the CB information set, both for $t < \infty$ and for $t \to \infty$. In fact, since the parameters in the optimal rule are the same as in rule (2.34), the results summarized in Proposition 3 are still valid. On the other hand, we can study E-stability of the system extending Proposition 2 in HM to a time-varying environment. In particular, it is easy to show[45]:

Corollary 1. *Consider the model (2.38); it is E-stable if and only if the corresponding model with homogenous expectations is E-stable.*

Since E-stability of the homogenous expectations model is ensured by Proposition 4, we conclude that also system (2.38) is E-stable, and it converges to the optimal discretionary RE equilibrium[46].

2.4.3 Policy Advice

We have seen that assuming adaptive learning instead of RE fundamentally changes the Ramsey solution for optimal policy. However assuming that the CB knows the exact learning algorithm followed by private agents, is clearly a strong assumption to make. Therefore in this section we examine what is the policy recommendation when the monetary authority is uncertain about the nature of private sector expectations.

[45] The proof is available from the authors upon request.

[46] In fact, the system we are analyzing falls into the class for which E-stability and convergence of real time learning are equivalent, see Evans and Honkapohja (2001).

In particular we aim to define consistent policy advice on a relevant set of private agents' expectation formation. First we investigate the issue of how monetary policy in the US should behave when the FED is uncertain about expectation formation, but its uncertainty is restricted to a set which would be reasonable given the empirical evidence. Then we examine the policy recommendation on a set of more volatile learning algorithms.

Empirical evidence on learning is relatively scarce, and mainly focuses on the US. For the US constant gain algorithms with a small tracking parameter is a good approximation for the data. With Bayesian estimation of the New Keynesian model with adaptive learning Milani (2005) finds γ to be 0.0187, he also finds γ to be stable through time. Orphanides and Williams (2005b) take a different approach, they calibrate γ on the Survey of Professional Forecasters and find that tracking parameters between 0.01 and 0.04 fit well survey expectations.

Let us conduct an experiment, and suppose that the FED is uncertain about how private sector forms its expectations, but relying on the empirical literature listed above it can define a relevant set of expectations to be: constant gain with a small gain, and RE[47]. Moreover, it has no probability distribution over these two possible realizations of the agents' expectations formation mechanism; instead, we use robust control theory and look for the policy that minimizes the maximum loss[48]. We perform numerical Monte Carlo analysis to examine welfare losses when private expectations are taken from this set and the CB interest rate rule is either an optimal rule for a given small gain parameter or the EH rule (2.8), derived in Evans and Honkapohja (2003a). In order to present the results in a compact way, for a given private expectation we compare consumption equivalents of using the optimal rule to the consumption equivalents of using a wrong rule and report increases in consumption equivalents. Note that when the structural equations are given by (2.36) and (2.37), the ALM for inflation and output gap depend also on the non observable shocks $e_{\pi,t}$ and $e_{x,t}$; hence, the values of the welfare losses are functions of the variances of these shocks too. However, it is easy to see

[47]Rational expectations means, substituting the interest rate rule in the IS curve (2.36), and then using it and the Phillips Curve (2.37) to solve for the fixed point in expectations. Under all interest rate rules listed above, this results $E_t\pi_{t+1} = E_t x_{t+1} = 0$.

[48]For an extensive treatise on the use of robust control techniques in economics, see Hansen and Sargent (2006).

that the ranking between different rules is independent of these variances[49]. Thus, since any parametrization of the non observable terms would be equally arbitrary and would deliver the same results in terms of relative performance of alternative rules, in tables 2.6-2.7 we consider the extreme case with the variances of $e_{\pi,t}$ and $e_{x,t}$ that go to zero.

Table 2.6 reports results when initial expectations coincides with RE: $a_0 = 0$. In this case, constant gain expectations with a small gain will be really close to the rational forecasts. We can think of this economy as populated by agents who are making only very small mistakes compared to the rational forecasts.

Table 2.6: Consumption equivalents under the optimal or a wrong rule, initial inflation expectations at RE

Expectations	0.0187	0.02	0.03	0.04	RE
Interest rate rule					
$\gamma = 0.0187$	0.01302	0.01307	0.01353	0.01412	0.01265
$\gamma = 0.02$	0.01302	0.01307	0.01353	0.01412	0.01265
$\gamma = 0.03$	0.01302	0.01307	0.01352	0.01410	0.01265
$\gamma = 0.04$	0.01303	0.01308	0.01353	0.01409	0.01265
EH	0.01303	0.01308	0.01359	0.01426	0.01265
Maximum percentage increase	EH	EH	EH	EH	$\gamma = 0.04$
compared to optimal rule	0.09	0.12	0.47	1.18	0.02

Woodford (1999) calibration. Starting from RE: $a_0 = 0$.
Consumption equivalents for a given underlying private sector expectation formation and a given interest rate rule.

The main result is that the worst case scenario is using the EH rule when private agents are learning. A min-max rule (following Hansen and Sargent (2006)), which minimizes the maximum loss is always a learning rule.

Under RE all of these rules lead to a determinate equilibrium. The EH rule provides smaller losses than optimal learning rules (see last line of table 2.6), and the reason for this is that learning rules allow for too high volatility in the output gap [50].

[49]In other words, the variances of $e_{\pi,t}$ and $e_{x,t}$ do not alter the ordinal properties of the loss function; to see this, an inspection of equation (2.47) in the Appendix suffices.

[50]We would like to note, that since learning rules decrease volatility of inflation and allow for higher volatility in the output gap, for small values of alpha (a small weight on

However, losses under RE caused by mistakenly using an optimal learning rule are smaller than losses due to using the EH rule when agents are learning.

When private agents are learning and the FED uses a bad learning rule, consumption equivalents increase but the loss is always smaller then losses of using the EH rule. The bigger is the misperception of the monetary policy about γ the bigger is the increase in consumption equivalents. When for example agents follow constant gain with $\gamma = 0.04$ and the central bank uses an optimal rule with $\gamma = 0.03$ consumption equivalent is 0.03% higher than it is when the optimal interest rate rule is used. While if the FED uses $\gamma = 0.02$ which is further from the true tracking parameter, the loss increases to 0.17%. The percentage increase in loss achieved using the EH rule is 1.18%, which is bigger then with any of the learning rules.

When we initialize the economy at the RE equilibrium, beliefs stay close to the RE. This way our analysis does not take into account an advantage of the optimal learning rules, which is that it helps private agents to learn faster the rational expectations forecasts. Therefore in table 2.7 we report numerical results for $a_0 = 1$. Our results show that the gain of using a learning rule over the EH rule is much bigger in this case, since the EH rule increases consumption equivalents compared to the optimal policy by $3 - 11\%$. Learning rules on the other hand result in smaller losses under learning, even if they are misspecified.

We now assume that the monetary authority is able to formulate a probability distribution over the mechanism used by the private sector to form its forecasts. In particular, let's assume that the prior of the FED is that with probability p private agents follow constant gain learning with a given tracking parameter, and with probability $1 - p$ agents have RE. Then we can calculate the expected welfare loss of using EH p times the consumption equivalent under constant gain learning with EH rule, and $1 - p$ times consumption equivalent of using EH under RE. Then we can find a cut-off value of p for which the expected loss in consumption terms of using the OP rule is less than the welfare loss of the EH rule.

A surprising result is that the cutoff value of p is between $1 - 1.5\%$[51]. This means that it is optimal to use the learning rule even if the CB attributes only a very small $1 - 2\%$ probability (or higher) to agents following learning

output gap in the welfare loss function) learning rules even outperform the discretionary rule under rational expectations (EH).

[51] Cutoff values for p are slightly lower when we initialize the economy out of RE.

Table 2.7: Consumption equivalents under the optimal or a wrong rule, initial inflation expectations out of RE

Expectations	0.0187	0.02	0.03	0.04	RE
Interest rate rule					
$\gamma = 0.0187$	0.1249	0.1240	0.1175	0.1120	0.0127
$\gamma = 0.02$	0.1250	0.1240	0.1173	0.1116	0.0127
$\gamma = 0.03$	0.1257	0.1245	0.1167	0.1102	0.0127
$\gamma = 0.04$	0.1271	0.1258	0.1170	0.1098	0.0127
EH	0.1287	0.1282	0.1246	0.1215	0.0127
Maximum percentage increase	EH	EH	EH	EH	$\gamma = 0.04$
compared to optimal rule	2.99	3.37	6.75	10.64	0.02

Woodford (1999) calibration. Starting from RE: $a_0 = 1$.
Consumption equivalents for a given underlying private sector expectation formation and a given interest rate rule.

and a very high probability to RE.

In sum, our "policy advice" for the FED is to choose an optimal learning rule even if it attributes only very small probability to learning.

Evans and Ramey (2006) shows that in an economy with a high probability of structural changes it is optimal to use constant gain algorithms with high tracking parameters. Therefore we could perform the same analysis on a set of constant gain algorithms with much higher tracking parameters. In this case the results[52] are even more against the use of the EH rule. For example, when agents follow CG with $\gamma = 0.2$ and $a_0 = 0$ using the EH rule results in 40% higher losses compared to the optimal learning rule. This is intuitive, because the higher is the gain parameter, the bigger is the influence that monetary policy can have on expectations if the CB makes active use of the learning algorithm; thus, the EH rule makes a bigger mistake when it does not take this into account[53].

Interestingly, different assumptions on the initial beliefs yields different solutions to the min-max problem: the robust rule is always a learning rule, as we stressed above, but when $a_0 = 0$ the min-max corresponds to the op-

[52] Available from the authors upon request.

[53] The cutoff value of p is between 1-2 percent. Thus the CB should use a learning rule even if it attributes only a very small probability to learning.

timal rule when γ is the largest in the set, and when $a_0 = 1$ to the optimal rule when γ is the smallest. This reflects the tension between *tracking* and *accuracy* present in any adaptive algorithm, and analyzed in Benveniste et al. (1990): a higher γ implies a better tracking of the mean dynamics of the underlying process, and a larger variance around this mean dynamics. If the process starts close to the mean dynamics (in our model, the RE equilibrium), the second effect would prevail over the first one, inducing a positive relation between γ and the welfare losses; consequently, for any rule used by the CB, the maximum losses are attained when the gain parameter is the largest possible (0.04), and the robust rule coincides with the optimal one for that value of γ. If the process starts far from the mean dynamics, the opposite line of reasoning applies.

In this subsection we presented numerical evidence suggesting that, when the CB is insecure as to whether agents have RE or are learning, it should use an optimal learning rule, unless it attaches a tiny probability to agents following learning; we are aware that a full-fledged robustness analysis would require a larger set of possible expectations' formation mechanisms, but this is beyond the scope of this paper.

2.5 Conclusions

In this paper we analyzed the optimal monetary policy problem faced by a CB that tries to exploit its ability to influence future beliefs of the agents, when they follow adaptive learning to form their expectations.

We have shown that in this framework the implications for policymaking go beyond the asymptotic learnability criterion. In the short run the optimal policy under learning resembles more the commitment solution under rational expectations then the discretionary solution under rational expectations. Both the commitment solution under rational expectations and the Ramsey solution under learning aims to anchor inflation expectations, thus it accommodates less the effect of noisy supply shocks on inflation. The intuition behind is simple and stems from the presence of a new intertemporal inflation output trade-off, that is not present under rational expectations. Under learning the central bank has to take into account how its policy affects future inflation expectations, since out of equilibrium expectations worsen the future inflation output trade-offs. As a result optimal policy is aggressive towards inflation, in order to induce private agents to learn faster the equilibrium expected value of inflation.

In the long run the equilibrium depends on how private agents learn. Even though during the transition optimal policy resembles the commitment solution under rational expectations, in our setup it drives expectations to the discretionary rational expectations solution. The reason for this is that agents expectation formation does not nest the commitment solution under rational expectations. Under rational expectations commitment calls for an ALM with a different functional form than the discretionary case (see **?**).

For future research it would be very interesting to explore the possibility whether optimal policy under adaptive learning car drive the economy to the commitment solution under rational expectations. This question is particularly interesting as from the backward looking nature of these learning algorithms it follows that such policies are time consistent, so the commitment solution could be reached in a time consistent fashion.

A large body of research in learning focused on how to design rules that are stable under learning; a typical result is that a strong reaction to out of equilibrium inflation expectations is necessary. We would like to note that under optimal policy examining E-stability is not necessary, since optimal policy naturally chooses an E-stable solution; moreover optimal policy is similar to the consensus reached in earlier papers on the desirability of the monetary policy being aggressive towards inflation, in order to anchor private agents' inflation expectations.

An additional message of our paper for policy conduct is that optimal policy should closely monitor private sectors expectations. Actually this is what is happening in real life: central banks do pay close attention to private expectations. Under rational expectations this is not justified, since expectations are pinned down by the model and the monetary policy rule, however once we depart from rationality expectations become a natural state variable.

Since optimal policy depends on the way private agents learn, we think a particularly important area of future research would be to estimate how agents learn. Even if expectations are not rational, expectations should be endogenous, one should allow agents to abandon their ad hoc learning rule if they can do better. Empirical research on examining differences in learning behavior in different environments is still missing. It would be also interesting to examine how monetary policy should be conducted with endogenous expectation formation, in other words when private agents would change their expectation formation depending on their perception about the underlying economy. Endogenous expectation formation could be formulated for example along the lines of Marcet and Nicolini (2003) where agents dynamically switch between different predictors depending on the last forecast error. An alternative way would be to model expectation formation as in Molnar (2006) where agents do not change the predictor used, but always use a weighted average of the forecasts generated by different predictors, and adjust the weight on predictors dynamically depending on the relative forecasting performance.

2.6 Appendix

2.6.1 Constant Gain Learning

In this section we give the outline of the derivation of the inflation law of motion (2.21), and prove Proposition 1.

We start from the optimality condition (2.18), that we recall below:

$$\frac{\kappa}{\alpha}\pi_t + x_t = \beta E_t\left[\beta\gamma x_{t+1} + (1-\gamma)\left(\frac{\kappa}{\alpha}\pi_{t+1} + x_{t+1}\right)\right]$$

Using the Phillips curve (2.2) and the evolution of inflationary expectations (2.9), we get:

$$E_t\left[\pi_{t+1}\right] = A_{11}\pi_t + A_{12}a_t + P_1 u_t \tag{2.39}$$

where:

$$\begin{aligned} A_{11} &\equiv \frac{\kappa^2 + \alpha + \alpha\beta^2\gamma\left(1-\gamma\left(1-\beta\right)\right)}{\alpha\beta\left(1-\gamma\left(1-\beta\right)\right) + \kappa^2\beta\left(1-\gamma\right)} \\ A_{12} &\equiv -\frac{\alpha\beta\left(1-\beta\left(1-\gamma\right)\left(1-\gamma\left(1-\beta\right)\right)\right)}{\alpha\beta\left(1-\gamma\left(1-\beta\right)\right) + \kappa^2\beta\left(1-\gamma\right)} \\ P_1 &\equiv -\frac{\alpha}{\alpha\beta\left(1-\gamma\left(1-\beta\right)\right) + \kappa^2\beta\left(1-\gamma\right)} \end{aligned} .$$

Hence, at an optimum, the dynamics of the economy can be summarized by stacking equations (2.9), (2.10) and (2.39), and obtaining the trivariate system:

$$E_t y_{t+1} = A y_t + P u_t \tag{2.40}$$

where $y_t \equiv [\pi_t, a_t, b_t]'$, and:

$$A \equiv \begin{pmatrix} A_{11} & A_{12} & 0 \\ \gamma & 1-\gamma & 0 \\ \frac{\gamma}{\kappa} & -\frac{\beta\gamma}{\kappa} & 1-\gamma \end{pmatrix}, \quad P = \begin{pmatrix} P_1 \\ 0 \\ -\frac{\gamma}{\kappa} \end{pmatrix}.$$

The three boundary conditions of the above system are:

$$\begin{gathered} a_0,\ b_0 \text{ given} \\ \lim_{s\to\infty}\left|E_t\pi_{t+s}\right| < \infty \end{gathered} . \tag{2.41}$$

The last one is due to the fact that, if there exists a solution to the problem (2.11) when the possible stochastic processes $\{\pi_t, x_t, r_t, a_{t+1}, b_{t+1}\}$ are restricted to be bounded, then this would also be the minimizer in the unrestricted case[54].

Since A is block triangular, its eigenvalues are given by $1-\gamma$ and by the eigenvalues of:

$$A_1 \equiv \begin{pmatrix} A_{11} & A_{12} \\ \gamma & 1-\gamma \end{pmatrix} \tag{2.42}$$

In the following Lemma 1 we show that A_1 has one eigenvalue inside and one outside the unit circle, which implies (together with $(1-\gamma) \in (0,1)$) that we can invoke Proposition 1 of Blanchard and Kahn (1980) to conclude that the system (2.40)-(2.41) has one and only one solution. In other words, there exists one and only one stochastic process for each of the three variables of y such that (2.41) is satisfied. Moreover, note that $y_{1t} \equiv [\pi_t, a_t]'$ does not depend on b_t; therefore, the processes for inflation and a that solve (together with the process for b) the system (2.40)-(2.41) are also a solution of the subsystem:

$$E_t y_{1t+1} = A_1 y_{1t} + (P_1, 0)' \, u_t$$

together with the boundary conditions:

$$a_0 \text{ given,} \quad \lim_{s\to\infty} |E_t \pi_{t+s}| < \infty$$

By Lemma 1, we can invoke Proposition 1 of Blanchard and Kahn (1980) to conclude that the law of motion for inflation can be written in the form:

$$\pi_t = c_\pi^{cg} a_t + d_\pi^{cg} u_t$$

as stated in Proposition 1.

Lemma 1. *Let A_1 be given by equation (2.42) in the text; then it has an eigenvalue inside and one outside the unit circle.*

Proof. First of all, we recall a result of linear algebra that we will use in the proof, i.e. that a necessary and sufficient condition for a 2 by 2 matrix to have an eigenvalue inside and one outside the unit circle, is that[55]:

$$|\mu_1 + \mu_2| > |1 + \mu_1 \mu_2|$$

[54] The proof is available from the authors upon request.

[55] LaSalle (1986).

where μ_1, μ_2 are the eigenvalues of the matrix; in the case of A_{11}, the above condition can be written equivalently:

$$\frac{\kappa^2+\alpha+\alpha\beta^2\gamma\left(1-\gamma\left(1-\beta\right)\right)}{\kappa^2\beta\left(1-\gamma\right)+\alpha\beta\left(1-\gamma\left(1-\beta\right)\right)}+1-\gamma>$$

$$1+\frac{\kappa^2+\alpha+\alpha\beta^2\gamma\left(1-\gamma\left(1-\beta\right)\right)}{\kappa^2\beta\left(1-\gamma\right)+\alpha\beta\left(1-\gamma\left(1-\beta\right)\right)}\left(1-\gamma\right)+\frac{\alpha\beta\left(1-\beta\left(1-\gamma\right)\left(1-\gamma\left(1-\beta\right)\right)\right)}{\kappa^2\beta\left(1-\gamma\right)+\alpha\beta\left(1-\gamma\left(1-\beta\right)\right)}\gamma$$

where we have used the fact that the trace is equal to the sum of the eigenvalues, and that the determinant is equal to the product. After simplifying the above inequality, we get:

$$-\gamma>-\gamma\left(\frac{\kappa^2+\alpha+\alpha\beta^2\gamma\left(1-\gamma\left(1-\beta\right)\right)-\alpha\beta\left(1-\beta\left(1-\gamma\right)\left(1-\gamma\left(1-\beta\right)\right)\right)}{\kappa^2\beta\left(1-\gamma\right)+\alpha\beta\left(1-\gamma\left(1-\beta\right)\right)}\right)$$

so that all we have to prove is that:

$$\frac{\kappa^2+\alpha+\alpha\beta^2\gamma\left(1-\gamma\left(1-\beta\right)\right)-\alpha\beta\left(1-\beta\left(1-\gamma\right)\left(1-\gamma\left(1-\beta\right)\right)\right)}{\kappa^2\beta\left(1-\gamma\right)+\alpha\beta\left(1-\gamma\left(1-\beta\right)\right)}>1$$

Some tedious algebra shows that this is equivalent to the following expression:

$$\kappa^2\left(1-\beta\left(1-\gamma\right)\right)+\alpha\left(1-\beta\right)\left(1-\beta\left(1-\gamma\left(1-\beta\right)\right)\right)>0$$

which is always true, since β and γ are supposed smaller than one. □

We now prove the rest of Proposition 1. First of all, we can guess that inflation follows the ALM (2.21) and use the optimality condition (2.39) and the method of undetermined coefficients to verify that c_π^{cg} must satisfy the following quadratic expression:

$$p_2\left(c_\pi^{cg}\right)^2+p_1c_\pi^{cg}+p_0=0$$

where:

$$\begin{aligned}
p_2 &\equiv \gamma\left[\kappa^2\beta\left(1-\gamma\right)+\alpha\beta\left(1-\gamma\left(1-\beta\right)\right)\right]\\
p_1 &\equiv \left(1-\gamma\right)\left[\kappa^2\beta\left(1-\gamma\right)+\alpha\beta\left(1-\gamma\left(1-\beta\right)\right)\right]-\left[\kappa^2+\alpha+\alpha\beta^2\gamma\left(1-\gamma\left(1-\beta\right)\right)\right]\\
p_0 &\equiv \alpha\beta\left(1-\beta\left(1-\gamma\right)\left(1-\gamma\left(1-\beta\right)\right)\right)
\end{aligned}$$

and that:

$$d_\pi^{cg}=\frac{\alpha}{\kappa^2+\alpha+\alpha\beta^2\gamma^2(\beta-c_\pi^{cg})+\beta\gamma\left(1-\gamma\right)\left(\alpha\beta-\left(\kappa^2+\alpha\right)c_\pi^{cg}\right)}$$

The polynomial in c_π^{cg} can be equivalently rewritten as follows:

$$c_\pi^{cg} = -\frac{p_0 + p_2\left(c_\pi^{cg}\right)^2}{p_1} \equiv f(c_\pi^{cg})$$

We will prove that the function $f(\cdot)$, defined on the interval $[0,1]$, is a contraction, so that it admits one and only one fixed point; moreover, since the two roots of the quadratic expression have the same sign (it is due to the fact that both p_2 and p_0 are positive), it follows that the other candidate value for c_π^{cg} is greater than one, which is not compatible with the boundary conditions[56].

First of all, we show that $f(\cdot)$, when defined on the interval $[0,1]$, takes values on the same interval.

Lemma 2. *$f(c_\pi^{cg})$ is strictly monotone increasing on the interval $[0,1]$.*

Proof. Note that:

$$f'(c_\pi^{cg}) = \frac{2\gamma[\alpha\beta(1-\gamma(1-\beta)) + \kappa^2\beta(1-\gamma)]}{\kappa^2 + \alpha + \alpha\beta^2\gamma\left(1-\gamma\left(1-\beta\right)\right) - (1-\gamma)[\kappa^2\beta\left(1-\gamma\right) + \alpha\beta\left(1-\gamma\left(1-\beta\right)\right)]} c_\pi^{cg}$$

which is positive if and only if the denominator is positive:

$$\kappa^2 + \alpha + \alpha\beta^2\gamma\left(1-\gamma\left(1-\beta\right)\right) - (1-\gamma)[\kappa^2\beta\left(1-\gamma\right) + \alpha\beta\left(1-\gamma\left(1-\beta\right)\right)] \lesseqgtr 0$$

After rearranging:

$$\kappa^2\left(1-\beta(1-\gamma)^2\right) + \alpha[1-\beta(1-\gamma)(1-\gamma(1-\beta))] + \alpha\beta^2\gamma\left(1-\gamma\left(1-\beta\right)\right) \lesseqgtr 0$$

which is always positive. Thus we have proved that $f(c_\pi^{cg})$ is strictly monotone increasing on the interval [0,1]. □

Lemma 3. *$f(c_\pi^{cg}) : [0,1] \to [0,1]$*

Proof. Since $f(c_\pi^{cg})$ is strictly monotone increasing it suffices to show that $f(0) > 0$ and $f(1) < 1$.

$$f(0) = \frac{\alpha\beta\left(1-\beta\left(1-\gamma\right)\left(1-\gamma\left(1-\beta\right)\right)\right)}{\kappa^2 + \alpha + \alpha\beta^2\gamma\left(1-\gamma\left(1-\beta\right)\right) - (1-\gamma)[\kappa^2\beta\left(1-\gamma\right) + \alpha\beta\left(1-\gamma\left(1-\beta\right)\right)]}$$

[56]Since it would imply an exploding inflation.

where the denominator is positive (see the preceding proof), and also the numerator is trivially positive. Thus $f(0) > 0$.

$$f(1) = \frac{\gamma\left[\kappa^2\beta\left(1-\gamma\right)+\alpha\beta\left(1-\gamma\left(1-\beta\right)\right)\right]+\alpha\beta\left(1-\beta\left(1-\gamma\right)\left(1-\gamma\left(1-\beta\right)\right)\right)}{\kappa^2+\alpha+\alpha\beta^2\gamma\left(1-\gamma\left(1-\beta\right)\right)-\left(1-\gamma\right)\left[\kappa^2\beta\left(1-\gamma\right)+\alpha\beta\left(1-\gamma\left(1-\beta\right)\right)\right]}$$

After rearranging, we get:

$$f(1) \lesseqgtr 1 \Longleftrightarrow 0 \lesseqgtr \kappa^2\left(1-\beta\left(1-\gamma\right)\right)+\alpha\left(1-\beta\right)\left(1-\beta\left(1-\gamma\left(1-\beta\right)\right)\right)$$

but, as we argued above, the RHS of the last inequality is always positive; hence, $f(1) < 1$. □

To show that $f(\cdot)$ is a contraction, it suffices to show that its derivative is bounded above by a number smaller than one: in fact, by the Mean Value Theorem, we now that for any a, b, there exists a $c \in (a, b)$ such that:

$$|f(a) - f(b)| \leq |f'(c)|\,|a-b|$$

and if $|f'(c)| \leq M < 1$ for any $c \in [0,1]$, we have the definition of a contraction.

Lemma 4. *For any $x \in [0,1]$, $0 < f'(x) \leq f'(1) < 1$.*

Proof. First of all, note that:

$$f'(x) = \frac{2\gamma[\alpha\beta(1-\gamma(1-\beta))+\kappa^2\beta(1-\gamma)]}{\kappa^2+\alpha+\alpha\beta^2\gamma\left(1-\gamma\left(1-\beta\right)\right)-\left(1-\gamma\right)\left[\kappa^2\beta\left(1-\gamma\right)+\alpha\beta\left(1-\gamma\left(1-\beta\right)\right)\right]}x$$

is positive and increasing in x, so that $\max_{x\in[0,1]} f'(x) = f'(1)$; after some algebraic manipulation, we get:

$$f'(1) \lesseqgtr 1 \Longleftrightarrow \left(1-\beta\gamma\right)\beta\left(1-\gamma\left(1-\beta\right)\right)+\beta\gamma\left(1-\gamma\left(1-\beta\right)\right)-1 \lesseqgtr \frac{\kappa^2}{\alpha}\left(1-\beta\left(1-\gamma^2\right)\right)$$

Since $\beta, \gamma \in (0,1)$, we have:

$$\left(1-\beta\gamma\right)\beta\left(1-\gamma\left(1-\beta\right)\right)+\beta\gamma\left(1-\gamma\left(1-\beta\right)\right)-1 < 1-\beta\gamma+\beta\gamma\left(1-\gamma\left(1-\beta\right)\right)-1 < 0$$

so that $f'(1)$ will be smaller than one ($\frac{\kappa^2}{\alpha}\left(1-\beta\left(1-\gamma^2\right)\right)$ is always positive). □

Moreover, we prove the following result.

Lemma 5. *Let $f(\cdot)$ be defined as above; then,* $f\left(\frac{\alpha\beta}{\kappa^2+\alpha}\right) \leq \frac{\alpha\beta}{\kappa^2+\alpha}$.

Proof. Note that:

$$f\left(\frac{\alpha\beta}{\kappa^2+\alpha}\right) = \frac{\alpha\beta\left(1-\beta\left(1-\gamma\right)\left(1-\gamma\left(1-\beta\right)\right)\right)}{\kappa^2+\alpha+\alpha\beta^2\gamma\left(1-\gamma\left(1-\beta\right)\right)-(1-\gamma)[\kappa^2\beta\left(1-\gamma\right)+\alpha\beta\left(1-\gamma\left(1-\beta\right)\right)]} + $$
$$+\frac{\gamma\left[\kappa^2\beta\left(1-\gamma\right)+\alpha\beta\left(1-\gamma\left(1-\beta\right)\right)\right]}{\kappa^2+\alpha+\alpha\beta^2\gamma\left(1-\gamma\left(1-\beta\right)\right)-(1-\gamma)[\kappa^2\beta\left(1-\gamma\right)+\alpha\beta\left(1-\gamma\left(1-\beta\right)\right)]}\left(\frac{\alpha\beta}{\kappa^2+\alpha}\right)^2$$
$$\gtreqless \frac{\alpha\beta}{\kappa^2+\alpha}$$

if and only if:

$$\frac{\left(\kappa^2+\alpha\right)\alpha\beta\left(1-\beta\left(1-\gamma\right)\left(1-\gamma\left(1-\beta\right)\right)\right)+\gamma\left[\kappa^2\beta\left(1-\gamma\right)+\alpha\beta\left(1-\gamma\left(1-\beta\right)\right)\right]\frac{\alpha\beta}{\kappa^2+\alpha}}{\kappa^2+\alpha+\alpha\beta^2\gamma\left(1-\gamma\left(1-\beta\right)\right)-(1-\gamma)[\kappa^2\beta\left(1-\gamma\right)+\alpha\beta\left(1-\gamma\left(1-\beta\right)\right)]} \gtreqless 1$$

For $\gamma = 0$ it is easy to verify that $f\left(\frac{\alpha\beta}{\kappa^2+\alpha}\right) = \frac{\alpha\beta}{\kappa^2+\alpha}$. If $\gamma > 0$, since the $\frac{\alpha\beta}{\alpha+\kappa^2} < \beta$, the LHS of the above inequality is smaller than:

$$\frac{\left(\kappa^2+\alpha\right)\alpha\beta\left(1-\beta\left(1-\gamma\right)\left(1-\gamma\left(1-\beta\right)\right)\right)+\beta\gamma\left[\kappa^2\beta\left(1-\gamma\right)+\alpha\beta\left(1-\gamma\left(1-\beta\right)\right)\right]}{\kappa^2+\alpha+\alpha\beta^2\gamma\left(1-\gamma\left(1-\beta\right)\right)-(1-\gamma)[\kappa^2\beta\left(1-\gamma\right)+\alpha\beta\left(1-\gamma\left(1-\beta\right)\right)]}$$

which is equal to one; in fact:

$$\frac{\left(\kappa^2+\alpha\right)\left(1-\beta\left(1-\gamma\right)\left(1-\gamma\left(1-\beta\right)\right)\right)+\beta\gamma[\kappa^2\beta\left(1-\gamma\right)+\alpha\beta\left(1-\gamma\left(1-\beta\right)\right)]}{\kappa^2+\alpha+\alpha\beta^2\gamma\left(1-\gamma\left(1-\beta\right)\right)-(1-\gamma)[\kappa^2\beta\left(1-\gamma\right)+\alpha\beta\left(1-\gamma\left(1-\beta\right)\right)]} \gtreqless 1$$

is equivalent to:

$$-\left(\kappa^2+\alpha\right)\beta\left(1-\gamma\right)\left(1-\gamma\left(1-\beta\right)\right)+\left(1-\gamma\left(1-\beta\right)\right)\left[\alpha\beta\left(1-\gamma\left(1-\beta\right)\right)+\kappa^2\beta\left(1-\gamma\right)\right] \gtreqless \alpha\beta^2\gamma\left(1-\gamma\left(1-\beta\right.\right.$$

But the LHS can simplified as:

$$\kappa^2\left(\beta\left(1-\gamma\right)\left(1-\gamma\left(1-\beta\right)\right)-\beta\left(1-\gamma\right)\left(1-\gamma\left(1-\beta\right)\right)\right)+\alpha\beta\left(1-\gamma\left(1-\beta\right)\right)\left(1-\gamma\left(1-\beta\right)-(1-\gamma\right.$$

which is equal to:

$$\alpha\beta^2\gamma\left(1-\gamma\left(1-\beta\right)\right)$$

Summing up, we showed that (if $\gamma > 0$) the following holds:

$$\frac{\left(\kappa^2+\alpha\right)\left(1-\beta\left(1-\gamma\right)\left(1-\gamma\left(1-\beta\right)\right)\right)+\beta\gamma[\kappa^2\beta\left(1-\gamma\right)+\alpha\beta\left(1-\gamma\left(1-\beta\right)\right)]}{\kappa^2+\alpha+\alpha\beta^2\gamma\left(1-\gamma\left(1-\beta\right)\right)-(1-\gamma)[\kappa^2\beta\left(1-\gamma\right)+\alpha\beta\left(1-\gamma\left(1-\beta\right)\right)]} = 1$$

which implies that:

$$f\left(\frac{\alpha\beta}{\kappa^2+\alpha}\right) < \frac{\alpha\beta}{\kappa^2+\alpha}$$

□

We are now ready to prove the Proposition.

Proof of Proposition 1. Combining the Lemmas 3 and 4 we obtain that $f(\cdot)$ is a contraction when defined on the interval $[0,1]$; moreover, by Lemma 5 we get that f, when defined on $[0, \frac{\alpha\beta}{\kappa^2+\alpha}]$, takes values on the same interval. This result, together with Lemma 4 and with the inequality $\frac{\alpha\beta}{\kappa^2+\alpha} < 1$, implies that $f(\cdot)$ is a contraction also when defined on the interval $[0, \frac{\alpha\beta}{\kappa^2+\alpha}]$ and, therefore, that the optimal c_{π}^{cg} must be between zero and $\frac{\alpha\beta}{\kappa^2+\alpha}$.

Finally, note that when $\gamma = 0$, $f(c_{\pi}^{cg})$ collapses to $\frac{\alpha\beta}{\kappa^2+\alpha}$, which completes the proof. □

2.6.2 Decreasing Gain Learning

In this section we prove Propositions 2 and 4.

Proof of Proposition 2. To prove the first part of the statement, note that if we solve forward the following difference equation:

$$c_{\pi t}^{dg} = \beta c_{\pi t+1}^{dg} + \frac{\alpha\beta}{\kappa^2+\alpha}(1-\beta)$$

we obtain one and only one bounded solution, i.e.:

$$c_{\pi t}^{dg} = \frac{\alpha\beta}{\kappa^2+\alpha} \quad \forall t$$

Moreover, we can rewrite the difference equation defining $c_{\pi t}^{dg}$ as:

$$G_t \equiv A_{11,t}c_{\pi,t}^{dg} - c_{\pi,t+1}^{dg} = -\frac{1}{t+1}c_{\pi,t+1}^{dg} - A_{12,t} + \frac{1}{t+1}c_{\pi,t}^{dg}c_{\pi,t+1}^{dg} \equiv F_t$$

If c_{π}^{dg} is bounded, it is easy to show that F has a limit:

$$\lim_{t\to\infty} F_t = -\lim_{t\to\infty} A_{12,t} = \frac{\alpha}{\kappa^2+\alpha}(1-\beta)$$

We can also show that the difference equation defined by G converges to:

$$\beta^{-1}c^{dg}_{\pi,\tau} - c^{dg}_{\pi,\tau+1}$$

Summing up, in the limit we have that c^{dg}_{π} evolves according to:

$$c^{dg}_{\pi\tau} = \beta c^{dg}_{\pi\tau+1} + \frac{\alpha\beta}{\kappa^2+\alpha}(1-\beta)$$

which, as we argued before, has one and only one bounded solution:

$$c^{dg}_{\pi\tau} = \frac{\alpha\beta}{\kappa^2+\alpha}$$

We prove the second part of the statement by contradiction. Assume that there exists a $T < \infty$ such that $c^{dg}_{\pi T} \geq \frac{\alpha\beta}{\alpha+\kappa^2}$; we show that this implies $c^{dg}_{\pi t} > \frac{\alpha\beta}{\alpha+\kappa^2}$ for any $t > T$. First of all, we can write:

$$\frac{c^{dg}_{\pi,T+1}\left(1-\frac{1}{T+1}\right) - A_{12,T}}{A_{11,T} - c^{dg}_{\pi,T+1}\frac{1}{T+1}} = c^{dg}_{\pi T} \geq \frac{\alpha\beta}{\alpha+\kappa^2}$$

Rearranging and simplifying, this turns out to be equivalent to:

$$\left(1-\frac{1}{T+1}\left(1-\frac{\alpha\beta}{\alpha+\kappa^2}\right)\right)c^{dg}_{\pi T+1} \geq \frac{\alpha\beta}{\alpha+\kappa^2}A_{11,T} + A_{12,T} \qquad (2.43)$$

Note that the RHS is equal to:

$$\begin{aligned}
\frac{\alpha\beta}{\alpha+\kappa^2}A_{11,T} + A_{12,T} &= \frac{\alpha\beta}{\alpha\beta(1+\beta\frac{1}{t+1})+\kappa^2\beta}\left[\beta\left(1+\beta\frac{1}{t+1}\right)\left(1-\frac{1}{T+1}\left(1-\frac{\alpha\beta}{\alpha+\kappa^2}\right)\right)\right] \\
&= \frac{\alpha\beta}{\alpha+\kappa^2\left(1+\beta\frac{1}{t+1}\right)^{-1}}\left(1-\frac{1}{T+1}\left(1-\frac{\alpha\beta}{\alpha+\kappa^2}\right)\right) \\
&> \frac{\alpha\beta}{\alpha+\kappa^2}\left(1-\frac{1}{T+1}\left(1-\frac{\alpha\beta}{\alpha+\kappa^2}\right)\right)
\end{aligned}$$

where the last inequality is due to the fact that $\left(1+\beta\frac{1}{t+1}\right)^{-1} < 1$; putting together the last inequality and (2.43), we get:

$$c^{dg}_{\pi T+1} > \frac{\alpha\beta}{\alpha+\kappa^2}$$

Then, we can apply the above argument to $c^{dg}_{\pi T+2}$ as well and, proceeding by induction, conclude that $c^{dg}_{\pi t} > \frac{\alpha\beta}{\alpha+\kappa^2}$ for any $t > T$. An immediate consequence is that $\lim_{t\to\infty} c^{dg}_{\pi t} > \frac{\alpha\beta}{\alpha+\kappa^2}$, which is a contradiction with the result stated in first part of the Proposition, namely $\lim_{t\to\infty} c^{dg}_{\pi t} = \frac{\alpha\beta}{\alpha+\kappa^2}$. Hence, we have showed that there is no $t < \infty$ such that $c^{dg}_{\pi t} \geq \frac{\alpha\beta}{\alpha+\kappa^2}$. □

Finally, we prove Proposition 4. First of all, we briefly describe some results of stochastic approximation[57] that we will exploit in the proof.

Let's consider a stochastic recursive algorithm of the form:

$$\theta_t = \theta_{t-1} + \gamma_t Q\left(t, \theta_{t-1}, X_t\right) \tag{2.44}$$

where X_t is a state vector with an invariant limiting distribution, and γ_t is a sequence of gains; the stochastic approximation literature shows how, provided certain technical conditions are met, the asymptotic behavior of the stochastic difference equation (2.44) can be analyzed using the associated deterministic ODE:

$$\frac{d\theta}{d\tau} = h\left(\theta(\tau)\right) \tag{2.45}$$

where:

$$h\left(\theta\right) \equiv \lim_{t\to\infty} EQ\left(t, \theta, X_t\right)$$

E represents the expectations taken over the invariant limiting distribution of X_t, for any fixed θ. In particular, it can be shown that the set of limiting points of (2.44) is given by the stable resting points of the ODE (2.45).

Proof of Proposition 4. Note that our equation (2.35) is a special case of (2.44), where the technical conditions are easily shown to be satisfied; moreover, it is also easy to see that:

$$h\left(a\right) = \lim_{t\to\infty} (c^{dg}_{\pi,t} - 1)a = \left(\frac{\alpha\beta}{\alpha+\kappa^2} - 1\right) a$$

which has a unique possible resting point at $a^* = 0$. Since $\frac{\alpha\beta}{\alpha+\kappa^2} < 1$, we have that a^* is globally stable, which proves the statement. □

[57] Ljung (1977), Benveniste et al. (1990) provide a recent survey.

2.6.3 Comparison with EH Rule

Proof of Proposition 3. First of all, note that:

$$\delta_{\pi,t}^{dg} \gtrless \delta_{\pi}^{EH} \Longleftrightarrow \sigma\frac{\beta - c_{\pi,t}^{dg}}{\kappa} \gtrless \sigma\frac{\kappa\beta}{\alpha+\kappa^2}$$

where the second inequality can be rewritten as:

$$\frac{\beta}{\kappa} - \frac{\kappa\beta}{\alpha+\kappa^2} \gtrless \frac{c_{\pi,t}^{dg}}{\kappa}$$

Rearranging the terms, we get:

$$\delta_{\pi,t}^{dg} \gtrless \delta_{\pi}^{EH} \Longleftrightarrow \frac{\alpha\beta}{\alpha+\kappa^2} \gtrless c_{\pi,t}^{dg}$$

Since we have shown in Proposition 2 that $t < \infty$ implies $c_{\pi,t}^{dg} < \frac{\alpha\beta}{\alpha+\kappa^2}$, we conclude that $\delta_{\pi t}^{dg} > \delta_{\pi}^{EH}$. Using a similar argument, it is easy to show that:

$$\delta_{ut}^{dg} \gtrless \delta_{u}^{EH} \Longleftrightarrow \frac{\alpha}{\alpha+\kappa^2} \gtrless d_{\pi,t}^{dg}$$

which implies, since

$$d_{\pi}^{cg} = \frac{\alpha}{\kappa^2 + \alpha + \alpha\beta^2\gamma^2(\beta - c_{\pi}^{cg}) + \beta\gamma\left(1-\gamma\right)\left(\alpha\beta - \left(\kappa^2+\alpha\right)c_{\pi}^{cg}\right)} < \frac{\alpha}{\alpha+\kappa^2},$$

that $\delta_{ut}^{dg} > \delta_{u}^{EH}$ whenever $t < \infty$. Finally, note that Proposition 2 also showed that $\lim_{t\to\infty} c_{\pi,t}^{dg} = \frac{\alpha\beta}{\alpha+\kappa^2}$, which trivially yields $\lim_{t\to\infty} \delta_{\pi,t}^{dg} = \delta_{\pi}^{EH}$ and $\lim_{t\to\infty} \delta_{ut}^{dg} = \delta_{u}^{EH}$. □

2.6.4 Derivations of Consumption Equivalents

In this section we follow derivations of Adam and Billi (2006).

Woodford (2003) chapter 6 shows that the second order approximation of the representative agents discounted utility flow is given by

$$U = -\bar{Y}U_c L^P \ , \tag{2.46}$$

where $\bar{Y}$ denotes the steady state level of output associated with zero inflation in the absence of disturbances, U_c is the marginal utility of consumption at $\bar{Y}$ and

$$L^P = \frac{1}{2}\frac{\sigma+\omega}{\alpha}\sum_{i=1}^{\infty}\beta^i(\pi_{t+i}^2 + x_{t+i}^2) \ ,$$

where (σ) is the households Arrow-Pratt Measure of relative risk aversion and ω is the elasticity of a firm's real marginal cost with respect to its own output, L^P denotes L generated under a policy rule P.

Assuming a permanent reduction in consumption from $\bar{Y}$ by $p \geq 0$ percent, a second order approximation of the utility loss is

$$\frac{1}{1-\beta}\left(-U_c\bar{Y}\frac{p}{100} + \frac{U_{cc}}{2}\left(\bar{Y}\frac{p}{100}\right)^2\right) = \frac{-U_c\bar{Y}}{1-\beta}\left(\frac{p}{100} - \frac{U_{cc}\bar{Y}}{2U_c}\left(\bar{Y}\frac{p}{100}\right)^2\right) =$$
$$\frac{-U_c\bar{Y}}{1-\beta}\left(\frac{p}{100} + \frac{\sigma}{2}\left(\bar{Y}\frac{p}{100}\right)^2\right) \ ,$$

where U_{cc} is the second derivative of utility of utility with respect to consumption evaluated at $\bar{Y}$. Equating this utility loss to (2.46), the welfare loss generated under policy rule P gives

$$\frac{p}{100} + \frac{\sigma}{2}\left(\bar{Y}\frac{p}{100}\right)^2 - (1-\beta)(L^P) = 0 \ .$$

The percentage loss in steady state consumption equivalent to the decrease in utility generated by following rule P is

$$p = 100\sigma\left(-1 + \sqrt{1 + \frac{2(1-\beta)(L^P)}{\sigma}}\right) \ .$$

Since x and π are expressed in percentage points we have to rescale the losses and use

$$p = 100\sigma\left(-1 + \sqrt{1 + \frac{2(1-\beta)(L^P)\sigma}{100^2}}\right) \ . \tag{2.47}$$

2.7 Figures

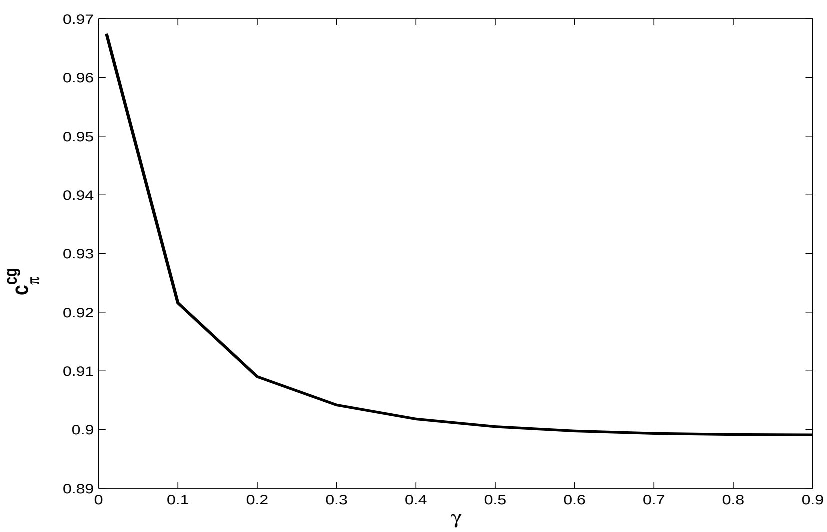

Figure 2.1: Feedback parameter in the ALM for inflation as a function of γ.

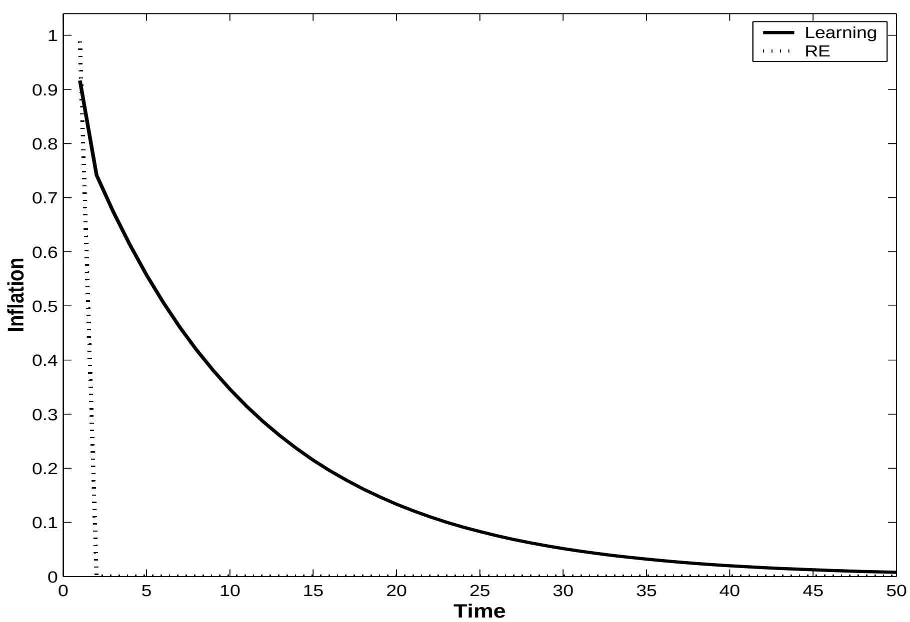

Figure 2.2: Impulse response of inflation for an initial cost-push shock $u = 1$. Solid line: optimal policy under learning and private agents following learning with $\gamma = 0.9$. Dashed line: optimal discretionary policy under RE with private agents have rational expectations. Initial conditions: $a_0 = 0$, $\pi_0 = 0$, $x_0 = 0$.

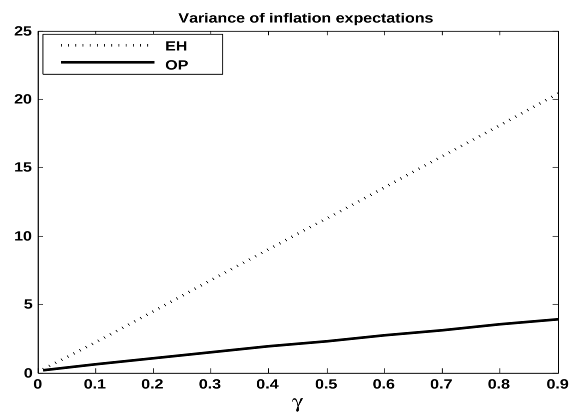

Figure 2.3: Variance of inflationary expectations

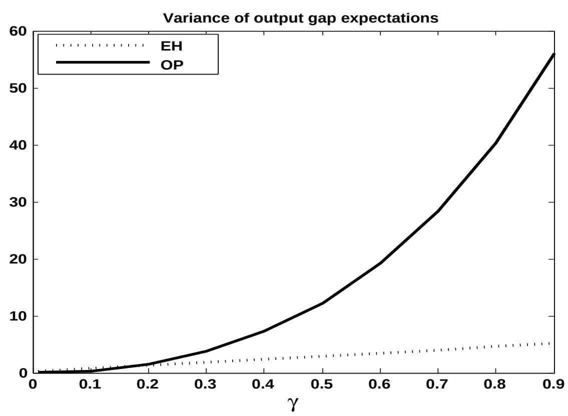

Figure 2.4: Variance of output gap expectations

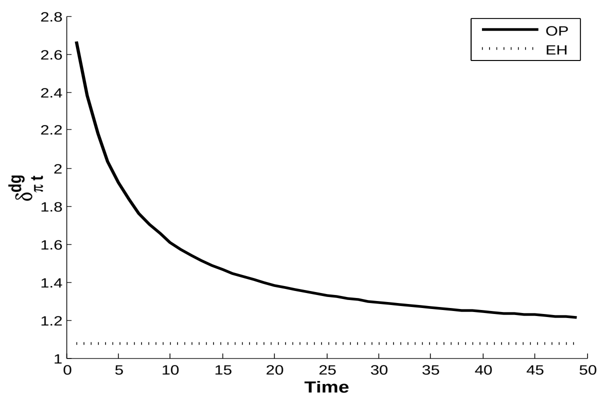

Figure 2.5: Interest rate rule coefficient on inflation expectations under decreasing gain learning.

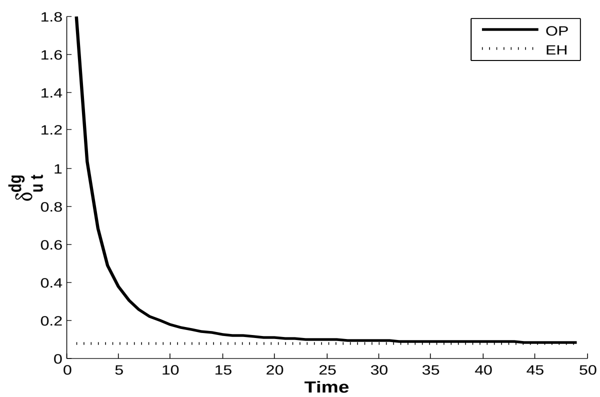

Figure 2.6: Interest rate rule coefficient on the cost push shock under decreasing gain

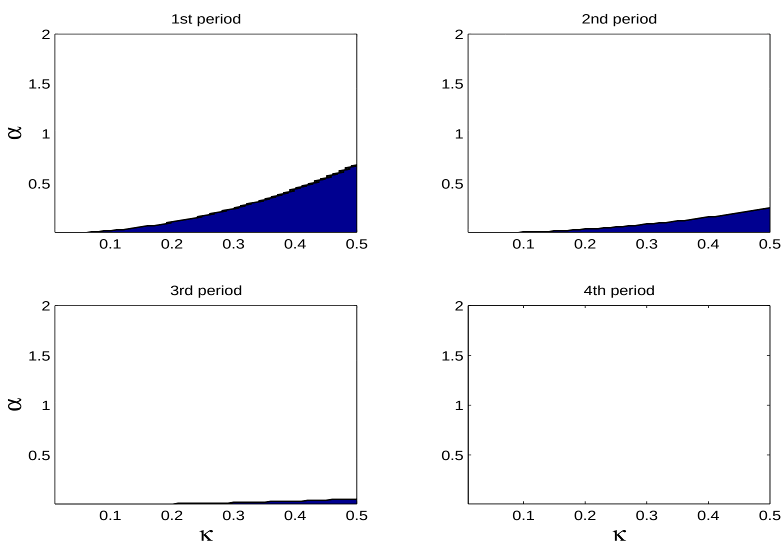

Figure 2.7: Values of α and κ for which δ_{π}^{dg} is increasing in the first 4 periods. From the 4th period on δ_{π}^{dg} is always decreasing. ($\beta = 0.99$)

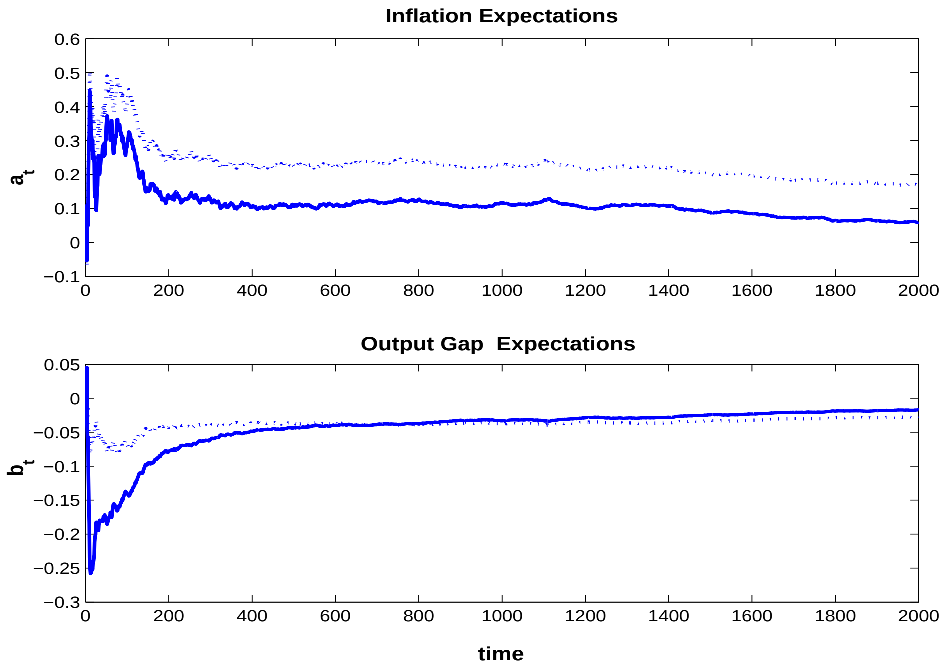

Figure 2.8: Evolution inflation and output gap expectations under the optimal (solid line) and the EH rule (dashed line), when private agents follow decreasing gain learning

Chapter 3

Monetary Policy with Heterogenous Expectations

3.1 Introduction

Since the seminal paper of Taylor (1993), it has become standard practice to assume that the monetary authority uses as instrument the short term nominal interest rate, and moves it in reaction to realized (or expected) changes on inflation and output gap. The performance of this type of simple monetary policy rules (also known as "Taylor rules"[1]) in a dynamic stochastic general equilibrium microfounded framework, where money has real effects due to nominal rigidities. This kind of models typically involves private sector expectations of future realizations of endogenous variables; modelling the formation process of these expectations is therefore a crucial issue.

When private agents have rational expectations (RE), many researchers[2] have focused on the possibility that, under certain conditions, a policy rule gives rise to a multiplicity of equilibria, characterized by self-fulfilling beliefs. This situation can give rise to large fluctuations, with negative consequences on social welfare, and is therefore deemed as undesirable. A typical result of this strand of literature is that a property that characterizes a "good" monetary policy is a response of the interest rate to inflation sufficiently

[1]Throughout the rest of this chapter, we treat the terms "Taylor rule", "interest rate rule" and "policy rule" as interchangeable.

[2]See, among others, Bernanke and Woodford (1997), Woodford (1999) and (2003), Clarida et al. (1999).

strong; in particular, if the Taylor rule shows no dependence on the output gap, determinacy requires a response to inflation more than proportional. This aggressive behavior is termed as "Taylor principle".

Over the last decade adaptive learning emerged as an alternative approach to modelling private sector expectations[3]. In particular, the main focus has been on the conditions that ensure stability under learning of the relevant RE equilibrium, namely, on the possibility to achieve RE as the limit of an adaptive learning scheme, when the initial beliefs of the agents are out of equilibrium. Examples in this line are Evans and Honkapohja (2003a), Evans and Honkapohja (2003b), Evans and Honkapohja (2006) and Honkapohja and Mitra (2005).

Bullard and Mitra (2002) try to combine the two approaches: they assess under which conditions different monetary policy rules are desirable, in the sense that they satisfy two key requirements: determinacy of the equilibrium, when agents have RE, and learnability of the RE equilibrium if agents' beliefs evolve according to an adaptive learning algorithm. Their results show that, under many (but not all) specifications of the interest rate rule, the Taylor principle retains its validity as a criterion to assess the quality of the monetary policy, since it is necessary to ensure both determinacy and learnability. Their analysis is conducted in the two extreme cases, namely when the agents are either all rational or all learners

However, as argued also in Nunes (2004), there is ample empirical evidence documenting that the dynamics of private sector beliefs, when proxied by surveys, are better explained by a combination of rational and backward-looking expectations. Early results in this spirit can be found in Roberts (1997) and (1998), where the expectations of US agents collected in the Michigan and Livingston surveys are shown to be inconsistent with the hypothesis of purely rational expectations; instead, they provide evidence in favor of an intermediate degree of rationality, with a fraction of the agents endowed with a simple form of backward-looking expectations. More recently, Carroll (2003) uses the same surveys to test a model where households derive their expectations from news media, which, in turn, report expectations of professional forecasters; the results he gets point in the direction of intermediate rationality[4]. More recent examples for US data are Adam and Padula (2003) and Erceg and Levine (2003). Also in the Euro area households expectations

[3]For an extensive monograph on adaptive learning, see Evans and Honkapohja (2001).

[4]In a related paper, Branch (2001) shows that the Michigan survey is consistent with the hypothesis of heterogeneous agents who form their forecasts according to different non-

show this form of intermediate rationality, as documented in Forsells and Kenny (2002).

The aim of this chapter is to analyze a framework where the expectations formation scheme is fairly general: in particular, we assume that a fraction of private agents has RE, and the rest has backward-looking expectations, updated according to the adaptive learning literature. We consider the baseline version of the New Keynesian model, which is by now the workhorse in monetary economics, and modify the structural equations to allow for heterogeneous expectations. To close the model we assume that the Central Bank follows a Taylor-type rule rule, with the interest rate responding to present, past or (expected) future values of inflation and output gap; we evaluate the determinacy of the associated equilibrium and the stability of the beliefs of the backward-looking agents, comparing our results with those derived for the polar cases in the existing literature.

Our first contribution is the conclusion that in this extended framework the Taylor principle retains its validity as a criterion to assess the desirability of a monetary policy rule: in fact, irrespectively of the particular Taylor rule used and of the fraction of backward-looking agents, the Taylor principle is a necessary condition for determinacy. However, when we fully characterize the determinacy properties of the model under the heterogeneous expectations hypothesis, the particular specification assumed for the Taylor rule matters. If the interest rate reacts to current values of inflation and output gap, the equilibrium is determinate under the same conditions valid when all agents are rational; instead, the introduction of backward-looking agents increases the set of values of the policy coefficients associated with determinacy if the interest rate responds to the forecasts of future inflation and output gap, and reduces it if the interest rate responds to past inflation and output gap.

Moreover, the dynamics when determinacy fails to hold can be quite different from the homogeneous rational expectations model: in fact, if the fraction of backward-looking agents is sufficiently large, the model shows instability (i.e., explosive equilibrium), instead of indeterminacy (i.e., multiple bounded equilibria).

Finally, we find a tight link between the asymptotic properties of the learning agents' beliefs, and the determinacy properties of the model: in particular, when the equilibrium is determinate, the learners' beliefs settle

rational predictors; the fraction of agents that chooses a predictor depends on its relative mean squared error.

down to a stationary distribution around the rational expectation values of the endogenous variables.

The results of this chapter have a twofold implication for policymaking. On one hand, the standard prescription that a Central Bank should follow the Taylor principle holds under fairly general assumptions on the expectations formation mechanism of the private sector, and on the specification of the interest rate rule. On the other hand, given that the policymaker should be aggressive, how much aggressiveness is desirable[5] depends on the degree of rationality embedded in the agents' beliefs, and on the timing of the endogenous variables in the Taylor rule; hence, a deep understanding of how the private sector forms its expectations should be a priority for any monetary policy authority.

The rest of the chapter is organized as follows. Section 3.2 outlines the model under general assumptions on the beliefs of the private sector, while Section 3.3 reviews the results obtained in the literature when expectations are rational. Section 3.4 states our main results about the dynamic properties of the system when expectations are heterogeneous. Section 3.5 investigates how the beliefs of the backward-looking agents behave asymptotically, and Section 3.6 concludes.

3.2 The Model

We consider the baseline version of the New Keynesian model; the economy is characterized by two structural equations[6]. The first one is an IS equation:

$$x_t = E_t^* x_{t+1} - \sigma^{-1}(r_t - E_t^* \pi_{t+1}) + \overline{rr}_t \tag{3.1}$$

where x_t, r_t and π_t denote time t output gap[7], short-term nominal interest rate and inflation, respectively; σ is a parameter of the household's utility function, representing the intertemporal elasticity of substitution, and $\overline{rr}_t$ is the natural real rate of interest, i.e. the real interest rate that would hold in absence of any nominal rigidity. Note that the operator E_t^* represents the (conditional) agents' expectations, which are not necessarily rational. The above equation is derived loglinearizing the household's Euler equation.

[5]In the context of this chapter "desirable" has no link with optimality issues; it simply implies that determinacy conditions hold.

[6]See, among others, Yun (1996), Clarida et al. (1999) and Woodford (2003).

[7]Namely, the difference between actual and natural output.

The second equation is the so-called New Keynesian Phillips Curve (NKPC):

$$\pi_t = \beta E_t^* \pi_{t+1} + kx_t + u_t \tag{3.2}$$

where β denotes the subjective discount rate, k is a function of structural parameters, and u_t is a cost-push shock[8]; this relation is obtained assuming that the supply side of the economy is characterized by a continuum of firms that produce differentiated goods in a monopolistically competitive market, and that prices are staggered à la Calvo: in other words, in each period firm i can reset the price with a constant probability $1-\theta$, and with probability θ it keeps the same price as in the previous period. If firms take this structure into account when deciding the optimal price, it can be shown[9] that the aggregate inflation is given by (3.2). The exogenous shocks are assumed to follow the process:

$$\begin{pmatrix} \overline{rr}_t \\ u_t \end{pmatrix} = \begin{pmatrix} \rho_r & 0 \\ 0 & \rho_u \end{pmatrix} \begin{pmatrix} \overline{rr}_{t-1} \\ u_{t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_t^r \\ \varepsilon_t^u \end{pmatrix} \tag{3.3}$$

where $0 < \rho_i < 1$ and $\varepsilon_t^i \sim N(0, \sigma_i^2)$, for $i = r, u$.

As is common practice in recent monetary policy literature, we assume that the instrument of the central bank (CB) is the short term nominal interest rate, which is set according to a rule; several rules have been proposed in the literature, with the interest rate reacting to current, past and future expected values of the endogenous variables[10]:

$$r_t = r\left(E_t^* y_{t+1}, y_t, y_{t-1}\right)$$

where $y_t = [x_t, \pi_t]'$.

To close the model, we need to specify: (i) the formation process of private sector expectations, and (ii) the functional form of $r(\cdot)$. For (i) we postulate that a fraction $1-\mu$ of the agents have nonrational expectations, updated through adaptive learning, and a fraction μ has RE[11]. In other words, the

[8]For interpretations of this shock, see among others Clarida et al. (1999), Erceg et al. (2000), Woodford (2003).

[9]See Yun (1996).

[10]In this chapter we consider only rules that do not include an explicite dependence on the lagged interest rate, following the original paper of Taylor (1993); however, extending the analysis to take into account the empirically documented interest rate smoothing by monetary authority is a relevant issue for future research.

[11]This coexistence of rational and learners is already present in Molnar (2006) and Nunes (2004).

expectations are defined as:

$$\begin{aligned} E_t^*\pi_{t+1} &= \mu E_t\pi_{t+1} + (1-\mu)\,\widehat{E}_t\pi_{t+1} \\ E_t^*x_{t+1} &= \mu E_t x_{t+1} + (1-\mu)\,\widehat{E}_t x_{t+1} \end{aligned}$$

where the learners beliefs are updated according to the constant gain algorithm:

$$\widehat{E}_t\pi_{t+1} \equiv a_t = a_{t-1} + \gamma(\pi_{t-1} - a_{t-1}) \tag{3.4}$$

$$\widehat{E}_t x_{t+1} \equiv b_t = b_{t-1} + \gamma(x_{t-1} - b_{t-1}) \tag{3.5}$$

with $\gamma \in (0,1)$. Note that learners have a Perceived Law of Motion (PLM) that is consistent with the law of motion that CB would implement in the discretionary optimal RE solution[12]: in other words, the conditional expectations of both inflation and output gap are assumed to be constant. Structural equations can be rewritten accordingly as:

$$x_t = \mu E_t x_{t+1} + (1-\mu)\,b_t - \sigma^{-1}(r_t - \mu E_t\pi_{t+1} - (1-\mu)\,a_t) + \overline{rr}_t \tag{3.6}$$

and:

$$\pi_t = \mu\beta E_t\pi_{t+1} + (1-\mu)\,\beta a_t + kx_t + u_t \tag{3.7}$$

Turning to (ii), we follow most of recent literature in assuming a linear specification[13]:

$$r_t = \alpha_f \Psi E_t^* y_{t+1} + \alpha_c \Psi y_t + \alpha_l \Psi y_{t-1}$$

where $\alpha \equiv [\alpha_f, \alpha_c, \alpha_l]'$ is a point in the closed three-dimensional simplex, and $\Psi \equiv [\psi_x, \psi_\pi]$ is a vector of policy coefficients. For analytical simplicity we concentrate on the extreme cases of α being one of the vertices of the simplex. Thus, the function $r(\cdot)$ can be indexed by $i = f, c, l$:

$$r_t = r_i\left(E_t^* y_{t+1}, y_t, y_{t-1}\right) = \begin{cases} \Psi E_t^* y_{t+1}, & \text{if } i = f \\ \Psi y_t, & \text{if } i = c \\ \Psi y_{t-1}, & \text{if } i = l \end{cases} \tag{3.8}$$

[12]See Clarida et al. (1999).

[13]When expectations are not rational, we have to make explicit assumptions on the forecasts of y targeted by the monetary authority, since they could be either the CB expectations, or the private sector's beliefs; in this chapter we follow Bernanke and Woodford (1997) in assuming that the interest rate reacts to the predictions formulated by the private sector.

3.3 Traditional Results

In this section we briefly recall the main findings on equilibrium determinacy in RE New Keynesian models under interest rate rules. Hence, we set $\mu = 1$, which implies that $E_t^* = E_t$, and that the economy is described by the system:

$$\begin{array}{l} x_t = E_t x_{t+1} - \sigma^{-1}(r_t - E_t \pi_{t+1}) + \overline{rr}_t \\ \pi_t = \beta E_t \pi_{t+1} + k x_t + u_t \\ r_t = r_i \left(E_t y_{t+1}, y_t, y_{t-1} \right) \end{array} \tag{3.9}$$

plus the law of motion of the shocks (3.3).

Let's define with w_t the vector $[x_t, \pi_t, r_t]'$, and with $w_i^R \equiv \left\{ w_{it}^R \right\}_{t=0}^{\infty}$ a rational expectations equilibrium associated with the economy in the case $\mu = 1$, namely a stochastic process that satisfies the system (3.9)-(3.3) at all t, with the interest rate set according to (3.8).

Definition 1. *The determinacy region $D_i^R \subset \mathbb{R}_+^2$ associated to the economy (3.9)-(3.3) is defined as:*

$$D_i^R = \left\{ \Psi \in \mathbb{R}_+^2 : \text{ there exists one and only one bounded } w_i^R \right\}$$

Note that the concept of determinacy that we use in this chapter is local in nature: the model is log-linearized around a steady state, which implies that the results we obtain are reasonable only in a neighborhood of the steady state.

Extending some well-known results, one can prove the following Proposition.

Proposition 5. *Let $k < \frac{\sigma}{\beta} (1 - \beta^2)$; then, the determinacy region is given by:*

$$D_i^R = \underline{D}_i^R \cap \overline{D}_i^R$$

where $\underline{D}_i^R$, $\overline{D}_i^R \subseteq \mathbb{R}_+^2$ are half-spaces defined as:

(i) $\underline{D}_i^R = \underline{D}^R = \left\{ \Psi \in \mathbb{R}_+^2 : \ k (\psi_\pi - 1) + (1 - \beta) \psi_x > 0 \right\}$ for $i = f, c, l$;

(ii) $\overline{D}_i^R = \left\{ \Psi \in \mathbb{R}_+^2 : \ k (\psi_\pi - 1) + (1 + \beta) \psi_x < 2\sigma (1 + \beta) \right\}$ for $i = f, l$, and $\overline{D}_c^R = \mathbb{R}_+^2$ for $i = c$.

Proof. This Proposition simply reorganize the results presented in Bullard and Mitra (2002), with the difference that the condition $k < \frac{\sigma}{\beta} (1 - \beta^2)$ rules out the possibility that, for $i = l$, determinacy regions includes also the portion of parameters' space given by $\left(\mathbb{R}^2_+ - \underline{D}^R\right) \cap \left(\mathbb{R}^2_+ - \overline{D}^R_i\right)$; we could dispose of this condition at the expense of reduced analytical tractability. □

In other words, we have that a necessary condition for determinacy is a reaction to inflation sufficiently strong to rule out self-fulfilling expectations (i.e., the Taylor principle); moreover, if the interest rate responds to future (past) endogenous variables, a monetary policy too hawkish leads to indeterminacy (explosiveness).

3.4 Determinacy with Heterogeneous Expectations

In this section we go back to the more general case $0 < \mu \leq 1$. We can stack together equations (3.6)-(3.8) and (3.4)-(3.5), obtaining the system:

$$A_i w_t = B_i E_t w_{t+1} + C_i \theta_t + \eta_t$$

$$\theta_{t+1} = (1-\gamma)\,\theta_t + \gamma y_t$$

where $\theta_t = [b_t, a_t]'$, $\eta_t = [\overline{rr}_t, u_t]'$ and the matrices A_i, B_i and C_i depend on the interest rule adopted. The above system can be rewritten compactly as:

$$M_i z_t = N_i E_t z_{t+1} + P\eta_t \tag{3.10}$$

where $z_t = [w_t', \theta_t']'$ and:

$$M_i = \begin{pmatrix} A_i & -C_i \\ \gamma I & (1-\gamma)\,I \end{pmatrix}, \quad N_i = \begin{pmatrix} B_i & 0 \\ 0 & I \end{pmatrix}, \quad P = \begin{pmatrix} I \\ 0 \end{pmatrix}$$

where I is the identity matrix. Note that, if $\mu = 1$, the system collapses to the standard New Keynesian model with RE; its determinacy properties have been analyzed in the previous section.

If instead $\mu \in (0,1)$, the economy defined by (3.10)-(3.3) has richer dynamics. Let's define with $z_i^H \equiv \left\{z_{it}^H\right\}_{t=0}^{\infty}$ a stochastic process that solves the system (3.10)-(3.3) at all t, with the interest rate set according to (3.8). We seek to characterize the set D_i^H, which is defined as follows.

Definition 2. *The determinacy region associated to the economy (3.10)-(3.3), denoted by D_i^H, is defined as:*

$$D_i^H = \left\{\Psi \in \mathbb{R}^2_+ : \text{ there exists one and only one bounded } z_i^H\right\}$$

A first important result is that, if the Phillips curve is not too steep, the Taylor principle remains a sensible criterion to assess the desirability of a policy rule; in fact, a CB that wants to stabilize the economy has to be tough on inflation not only when all the private sector is characterized by RE, but also when there is an arbitrary large fraction of backward-looking agents as is stated in the following Proposition.

Proposition 6. *Let $k < \underline{k}$, where:*

$$\underline{k} = \left[\sigma\left(1-\beta\right)\left(\frac{1+\beta\mu}{\beta}+\beta\gamma\frac{1-\mu}{2-\gamma}\right)\right]\frac{\beta\left(1+\mu\left(1-\gamma\right)\right)}{\beta\left(1+\mu\left(1-\gamma\right)\right)+\left(1-\beta\right)\left(1-\gamma\right)\left(1-\mu\right)}$$

Then $D_i^H \subseteq \underline{D}^R$, where $\underline{D}^R$ is the set defined in Proposition 5.

Proof. See the Appendix. □

Note that the above result holds for every i: independently of the particular Taylor rule used, a unique bounded equilibrium cannot be implemented by a "passive" monetary policy. On the other hand, as implied by the following Proposition, the upper contour of the determinacy region depends crucially on the type of rule that the CB is committed to.

Proposition 7. *Let $k < \underline{k}$; then, the following holds:*
(i) $D_c^H = D_c^R$;
(ii) $D_f^H \supset D_f^R$;
(iii) $D_l^H \subset D_l^R$.

Proof. See the Appendix. □

In other words, if the CB adopts the forward-looking (backward-looking) Taylor rule, the determinacy region is larger (smaller) than in the homogeneous expectations case. To get an intuition, it can be useful to start from the reason why, in the model analyzed in the previous section, a policy too hawkish drives the economy out of the determinacy region when $i = f$. As emphasized in Bernanke and Woodford (1997) and in Woodford (2003) among others, when the interest rate reacts too aggressively to changes in inflation forecasts, it determines a "policy overkill" effect that ends up inducing an oscillating equilibrium driven purely by self-fulfilling expectations. The introduction in the setup of a backward-looking term, given by the learners beliefs, helps to pin down the equilibrium, dampening sunspot fluctuations, and even eliminating them when $\Psi \in D_f^H - D_f^R$. When the Taylor rule is already backward-looking (i.e., if $i = l$) this mechanism works in the opposite direction: if the rule is aggressive, a positive (negative) value of past inflation results in an high (low) level of the nominal and real interest rate, depressing (expanding) economic activity and yielding a negative (positive) contemporaneous inflation, which feeds back in the subsequent period's policy rule. Hence, the system enters a cycle, which is explosive if the rule is too aggressive; when an additional backward-looking component is introduced in the economy, the waves of the cycle are amplified, widening the region of the policy coefficients space associated with explosive dynamics.

3.4.1 Dynamics out of the Determinacy Region

In this section we characterize the equilibrium dynamics out of the determinacy region; in particular, when the interest rate responds to contemporaneous values of inflation and output gap, we can prove the following result.

Proposition 8. *Let $i = c$, $k < \underline{k}$ and $\Psi \in \mathbb{R}_+^2 - \underline{D}^R$, where $\underline{D}^R$ is the set defined in Proposition 5; then there exists a $\overline{\mu}_i \in (0,1)$ such that for any $\mu \in (\overline{\mu}_i, 1]$ the system (3.10)-(3.3) is indeterminate, and for any $\mu \in (0, \overline{\mu}_i)$ it is explosive.*

Proof. See the Appendix. □

This result stems again from the backward-looking nature of learners beliefs, which works against possible sunspot equilibria; hence, if the proportion of adaptive agents is sufficiently high, a loose monetary policy would not trigger self-fulfilling expectations, but would put the economy on an explosive path, with very different welfare implications. We have not been able to prove a similar result for $i = f, l$ so far, but numerical simulations support the idea that it holds also when the Taylor rule is backward or forward looking.

On the other hand, the consequences of an overly aggressive Taylor rule when $i = f, l$ are analogous to the homogeneous RE case: if the rule is forward-looking, the equilibrium is indeterminate; if it is backward-looking, the economy enters an explosive cycle.

3.5 Dynamics of Learners Beliefs

In most of the existing literature on learning and monetary policy, the hypothesis of homogeneous expectations makes determinacy and learnability two separate problems: when agents are rational, the concern is determinacy, when they are learning, the concern is convergence to RE. In our framework, we can naturally analyze the two issues together.

In particular, having characterized the conditions for determinacy, we now answer the question of what are the corresponding asymptotic properties of the learners beliefs; we show them in the following Proposition.

Proposition 9. *Let's assume that the policy rule is such that a bounded solution to (3.10)-(3.3) exists and is unique, i.e.* $\Psi \in D_i^H$*; then* θ_t *converges in distribution to a process* $\theta^* \sim N(0, \Sigma_i)$*, where the variance-covariance matrix* Σ_i *is a function of the structural parameters.*

Proof. See the Appendix. □

The intuition is straightforward: if the system stays bounded, then it cannot admit a diverging path of the backward looking beliefs. When Ψ is out of the determinacy region, the results are less clear-cut. On one hand, if the system is explosive, it is possible to show that also θ explodes, but this poses the question of the plausibility of the assumption that learners keep forming the expectations using a PLM consistent with constant values of inflation and output gap; on the other hand, if the system is indeterminate, the issue of learnability of stationary sunspots emerges. The study of this possibility, for example using the results shown in Honkapohja and Mitra (2004), is beyond the scope of this chapter.

3.6 Conclusions

A well-known result in monetary economics is that, when expectations are rational, a simple interest rate rule can lead to an indeterminate or explosive solution; in particular, the reaction of the interest rate should obey the Taylor principle, namely it should be sufficiently aggressive to rule out sunspot equilibria and, when the interest rate responds to realized past (expected future) values of inflation and output gap, it should not be too aggressive, in order to avoid a "policy overkill" effect that would trigger explosive (indeterminate) cycles.

The main contribution of this chapter is to show that, in a generalized framework where a fraction of the agents has rational expectations, and the rest has backward-looking expectations updated according to the adaptive learning literature, the design of the monetary policy should be concerned about the same issues: on one hand, it should follow the Taylor principle, on the other, when the interest rate responds to realized past (expected future) values of inflation and output gap, it should not be too aggressive. However, the magnitude of this upper bound differs from the case of homogeneous rational expectations: it is more binding when the Taylor rule is backward-looking, and less binding when it is forward-looking.

When the Taylor rule fails to hold, the model can show either instability or indeterminacy, depending on how large is the fraction of backward-looking agents.We also demonstrate that when the equilibrium is determinate, the learners' beliefs settle down to a stationary distribution around the rational expectation values of the endogenous variables.

The results of this chapter have a twofold implication for policymaking. On one hand, the standard prescription that a Central Bank should follow the Taylor principle holds under fairly general assumptions on the expectations formation mechanism of the private sector, and on the specification of the interest rate rule. On the other hand, given that the policymaker should be aggressive, how much aggressiveness is desirable depends on the degree of rationality embedded in the agents' beliefs, and on the timing of the endogenous variables in the Taylor rule; hence, a deep understanding of how the private sector forms its expectations should be a priority for any monetary policy authority.

3.7 Appendix

3.7.1 Determinacy

First of all, we state the following Lemma, which summarizes some well known results.

Lemma 6. *Let λ_1, λ_2 lie in the complex plane, then*[14]*:*

- *the λ_i's ($i = 1, 2$) are both inside the unit circle if and only if the following conditions are satisfied:*

$$\begin{aligned} |\lambda_1 + \lambda_2| &< |1 + \lambda_1\lambda_2| \\ |\lambda_1\lambda_2| &< 1 \end{aligned}$$

- *the λ_i's ($i = 1, 2$) are both outside the unit circle if and only if the following conditions are satisfied:*

$$\begin{aligned} |\lambda_1 + \lambda_2| &< |1 + \lambda_1\lambda_2| \\ |\lambda_1\lambda_2| &> 1 \end{aligned}$$

- *the λ_i's ($i = 1, 2$) are one inside and one outside the unit circle if and only the following condition is satisfied:*

$$|\lambda_1 + \lambda_2| > |1 + \lambda_1\lambda_2|$$

Proof. See LaSalle (1986). □

To prove Propositions 6 and 7, we begin noting that existence and uniqueness of bounded solutions to the system (3.10)-(3.3) hinge on the eigenvalues of the matrix $Q_i \equiv N_i^{-1}M_i$; in particular, since there are three predetermined and two forward-looking variables, we can invoke the results of Blanchard and Kahn (1980) to say:

- if Q_i has three eigenvalues inside and two outside the unit circle, then there exists one and only one bounded solution to (3.10)-(3.3);
- if Q_i has more than three eigenvalues inside the unit circle, then there exist a multiplicity of bounded solutions to (3.10)-(3.3);

[14]In what follows, we rule out the non-generic case of λ_i's on the unit circle.

- if Q_i has less than three eigenvalues inside the unit circle, then (3.10)-(3.3) admits no bounded solution.

When $i = c, f$, the matrix M_i has a row of zeros, and we can concentrate on the eigenvalues of the matrix $\widehat{Q}_i \equiv \widehat{N}_i^{-1}\widehat{M}_i$, where $\widehat{M}_i$ and $\widehat{N}_i$ are given by:

$$\widehat{M}_i = \begin{pmatrix} \widehat{A}_i & -\widehat{C}_i \\ \gamma I & (1-\gamma) I \end{pmatrix}, \quad \widehat{N}_i = \begin{pmatrix} \widehat{B}_i & 0 \\ 0 & I \end{pmatrix}$$

and $\widehat{A}_i$, $\widehat{B}_i$ and $\widehat{C}_i$ are obtained from the system (3.6)-(3.7), after using the Taylor rule to substitute out r.

Proof of Proposition 6. We want to show that, if $\Psi \in D_i^H$, then $\Psi \in \underline{D}^R$. This is equivalent to say that, if $\Psi \notin \underline{D}^R$, then $\Psi \notin D_i^H$. We prove this statement separately for the three possible values of i.

- $i = c$. Note that the eigenvalues of $\widehat{Q}_c$ are the roots of the characteristic polynomial:

$$p_c(\lambda) \equiv a_{c4}\lambda^4 + a_{c3}\lambda^3 + a_{c2}\lambda^2 + a_{c1}\lambda + a_{c0}$$

where[15]:

$$\begin{aligned}
a_{c4} &= 1 \\
a_{c3} &= -\frac{k + \sigma(1 + \beta + 2\beta\mu(1-\gamma)) + \beta\psi_x}{\beta\mu\sigma} \\
a_{c2} &= \frac{1}{\beta\mu^2\sigma}\left[\sigma + \sigma\mu(2(1-\gamma) + \beta(2 + \mu(1 - \gamma(4-\gamma)))) + \psi_x + \right. \\
&\quad \left. +2\beta\mu\psi_x(1-\gamma) + k(2\mu(1-\gamma) + \psi_\pi)\right] \\
a_{c1} &= -\frac{1}{\beta\mu^2\sigma}\left[\sigma\left(2 + \mu + \beta\mu + \gamma^2\mu(1 - \beta + 2\beta\mu) - \gamma\left(1 + 3\mu + \beta\left(1 - \mu + 2\mu^2\right)\right)\right)\right. \\
&\quad \left. +\psi_x(2 - \gamma(2-\beta) + \beta\mu(1 - \gamma(3-\gamma))) + k(\mu + \gamma(1 - \mu(3-\gamma) - 2\psi_\pi) + 2\psi_\pi)\right] \\
a_{c0} &= \frac{1}{\beta\mu^2\sigma}\left[(-1 + \gamma(1 - \beta(1-\mu)))(\sigma(-1 + \gamma\mu) - \psi_x(1-\gamma)) - k(1-\gamma)\right. \\
&\quad \left.(-\psi_\pi + \gamma(-1 + \mu + \psi_\pi))\right]
\end{aligned}$$

[15] The Mathematica routines used to obtain the expressions for the coefficients are available from the author upon request.

We can use the solution formula of the quartic equation to group the four roots of $p_c(\cdot)$ in two couples, characterized by the following conditions:

$$\begin{aligned}
\lambda_{c1}\lambda_{c2} &= \frac{1}{2\beta\mu\sigma}\left[k(1-\gamma)+\sigma(1-\gamma+\beta(1+\gamma-2\gamma\mu))+(1-\gamma)\left(\beta\psi_x\right.\right.\\
&\quad\left.\left.-\sqrt{k^2+(\beta\psi_x-\sigma(1-\beta))^2+2k(\sigma+\beta\sigma+\beta\psi_x-2\beta\sigma\psi_\pi)}\right)\right]\\
\lambda_{c1}+\lambda_{c2} &= \frac{1}{2\beta\mu\sigma}\left[k+\sigma(1+\beta+2\beta\mu(1-\gamma))+\beta\psi_x\right.\\
&\quad\left.-\sqrt{k^2+(\beta\psi_x-\sigma(1-\beta))^2+2k(\sigma+\beta\sigma+\beta\psi_x-2\beta\sigma\psi_\pi)}\right]
\end{aligned}$$

and:

$$\begin{aligned}
\lambda_{c3}\lambda_{c4} &= \frac{1}{2\beta\mu\sigma}\left[k(1-\gamma)+\sigma(1-\gamma+\beta(1+\gamma-2\gamma\mu))+(1-\gamma)\left(\beta\psi_x\right.\right.\\
&\quad\left.\left.+\sqrt{k^2+(\beta\psi_x-\sigma(1-\beta))^2+2k(\sigma+\beta\sigma+\beta\psi_x-2\beta\sigma\psi_\pi)}\right)\right]\\
\lambda_{c3}+\lambda_{c4} &= \frac{1}{2\beta\mu\sigma}\left[k+\sigma(1+\beta+2\beta\mu(1-\gamma))+\beta\psi_x\right.\\
&\quad\left.+\sqrt{k^2+(\beta\psi_x-\sigma(1-\beta))^2+2k(\sigma+\beta\sigma+\beta\psi_x-2\beta\sigma\psi_\pi)}\right]
\end{aligned}$$

To determine for each λ_{cj} whether it lies inside or outside the unit circle, we combine the above conditions with Lemma 6[16]. Let's start from the group $(\lambda_{c3},\lambda_{c4})$; the condition $\lambda_{c3}+\lambda_{c4}\gtrless 1+\lambda_{c3}\lambda_{c4}$ is easily shown to be equivalent to:

$$0\gtrless-\gamma\left(k+\sigma(1-\beta)+\beta\psi_x+\sqrt{k^2+(\beta\psi_x-\sigma(1-\beta))^2+2k(\sigma+\beta\sigma+\beta\psi_x-2\beta\sigma\psi_\pi)}\right)$$

The RHS of the above inequality is always negative, so we conclude[17]:

$$\lambda_{c3}+\lambda_{c4}>1+\lambda_{c3}\lambda_{c4}$$

[16] An inspection of the signs of the characteristic polynomial's coefficients, combined with the Descartes' rule, allows us to conclude that all the roots of $p(\cdot)$ lie in the right half of the complex plane.

[17] We have implicitly assumed that the radicand in the RHS of the inequality is positive; the case of a negative radicand, and hence of an imaginary term, can be easily accomodated.

Hence, for any value of μ, and independently of the Taylor principle being satisfied or not, one eigenvalue of the group $(\lambda_{c3}, \lambda_{c4})$ lies inside the unit circle, and the other outside.

Let's consider the other group; the condition $\lambda_{c1} + \lambda_{c2} \gtrless 1 + \lambda_{c1}\lambda_{c2}$ is equivalent to:

$$0 \gtrless -\gamma \left(k + \sigma(1-\beta) + \beta\psi_x - \sqrt{k^2 + (\beta\psi_x - \sigma(1-\beta))^2 + 2k(\sigma + \beta\sigma + \beta\psi_x - 2\beta\sigma\psi_\pi)} \right)$$

To determine the sign of the RHS, note that, if $\psi_\pi = \frac{-(1-\beta)\psi_x}{k} + 1$ (i.e., if $k(\psi_\pi - 1) + (1-\beta)\psi_x = 0$):

$$k^2 + (\beta\psi_x - \sigma(1-\beta))^2 + 2k(\sigma + \beta\sigma + \beta\psi_x - 2\beta\sigma\psi_\pi) = (k + \sigma(1-\beta) + \beta\psi_x)^2$$

which means that:

$$k + \sigma(1-\beta) + \beta\psi_x - \sqrt{k^2 + (\beta\psi_x - \sigma(1-\beta))^2 + 2k(\sigma + \beta\sigma + \beta\psi_x - 2\beta\sigma\psi_\pi)} = 0$$

Consequently, if $\psi_\pi < \frac{-(1-\beta)\psi_x}{k} + 1$ (i.e., if the Taylor principle does not hold):

$$k^2 + (\beta\psi_x - \sigma(1-\beta))^2 + 2k(\sigma + \beta\sigma + \beta\psi_x - 2\beta\sigma\psi_\pi) > (k + \sigma(1-\beta) + \beta\psi_x)^2$$

and:

$$0 < -\gamma \left(k + \sigma(1-\beta) + \beta\psi_x - \sqrt{k^2 + (\beta\psi_x - \sigma(1-\beta))^2 + 2k(\sigma + \beta\sigma + \beta\psi_x - 2\beta\sigma\psi_\pi)} \right)$$

which is equivalent to say that $\lambda_{c1} + \lambda_{c2} < 1 + \lambda_{c1}\lambda_{c2}$, namely that λ_{c1} and λ_{c2} lie both either inside or outside the unit circle. Note that this conclusion holds for any value of μ. Putting together the result for $(\lambda_{c1}, \lambda_{c2})$ and the one for $(\lambda_{c3}, \lambda_{c4})$, we prove that when $k(\psi_\pi - 1) + (1-\beta)\psi_x < 0$, i.e. when $\Psi \notin \underline{D}^R$, the system (3.10)-(3.3) admits either infinite or zero bounded solutions.

- $i = f$. The proof goes through analogously to the case $i = c$, with the characteristic polynomial given by:

$$p_f(\lambda) \equiv a_{f4}\lambda^4 + a_{f3}\lambda^3 + a_{f2}\lambda^2 + a_{f1}\lambda + a_{f0}$$

where[18]:

$$
\begin{aligned}
a_{f4} &= 1 \\
a_{f3} &= \frac{k\left(\psi_{\pi}-1\right)+\sigma\left(-1-\beta\left(1+2\mu\left(1-\gamma\right)\right)+2\beta\mu\left(1-\gamma\right)\right)+\psi_{x}+2\beta\mu\left(1-\gamma\right)\psi_{x}}{\beta\mu\left(\sigma-\psi_{x}\right)} \\
a_{f2} &= \frac{1}{\beta\mu^{2}\left(\sigma-\psi_{x}\right)}\left[\sigma+\sigma\mu\left(2\left(1-\gamma\right)+\beta\left(2+\mu\left(1-\gamma\left(4-\gamma\right)\right)\right)\right)+\right. \\
&\quad \left.+\left(2-2\gamma\left(1-\beta\right)+\beta\mu\left(1-\gamma\left(4-\gamma\right)\right)\right)\psi_{x}-2k\left(1-\gamma\right)\left(\psi_{\pi}-1\right)\right] \\
a_{f1} &= -\frac{1}{\beta\mu^{2}\left(\sigma-\psi_{x}\right)}\left[\sigma\left(2+\mu+\beta\mu+\gamma^{2}\mu\left(1-\beta+2\beta\mu\right)-\gamma\left(1+3\mu+\beta\left(1-\mu+2\mu^{2}\right)\right)\right)\right. \\
&\quad \left.+\left(\mu+\gamma\left(1+\mu\left(-3+\gamma+2\beta\left(1-\gamma\right)\left(1-\mu\right)\right)\right)\right)\psi_{x}+k\left(\gamma+\mu-\gamma\mu\left(3-\gamma\right)\right)\left(\psi_{\pi}-1\right)\right] \\
a_{f0} &= \frac{1}{\beta\mu^{2}\left(\sigma-\psi_{x}\right)}\left[\left(-1+\gamma\left(1-\beta\left(1-\mu\right)\right)\right)\left(\sigma\left(-1+\gamma\mu\right)+\gamma\left(1-\mu\right)\psi_{x}\right)-k\gamma\left(1-\gamma\right)\right. \\
&\quad \left.\left(1-\mu\right)\left(\psi_{\pi}-1\right)\right]
\end{aligned}
$$

- $i = l$. The system is governed by the eigenvalues of Q_l; since it is a 5×5 matrix, it's eigenvalues cannot in general be derived analytically, so that the type of analysis carried out when $i = c, f$ is not applicable. However, we can draw conclusions studying the coefficients of the characteristic polynomial:

$$p_l(\lambda) \equiv a_{l5}\lambda^5 + a_{l4}\lambda^4 + a_{l3}\lambda^3 + a_{l2}\lambda^2 + a_{l1}\lambda + a_{l0}$$

[18] The Mathematica routines used to obtain the expressions for the coefficients are available from the author upon request.

where:

$$
\begin{aligned}
a_{l5} &= -1 \\
a_{l4} &= \frac{k+\sigma\left(1+\beta+2\beta\mu\left(1-\gamma\right)\right)}{\beta\mu\sigma} \\
a_{l3} &= \frac{1}{\beta\mu^2\sigma}\left[\beta\mu\psi_x - 2k\mu\left(1-\gamma\right)-\sigma\left(1+\mu\left(2-2\gamma+\beta\left(2+\mu\left(1-\gamma\left(4-\gamma\right)\right)\right)\right)\right)\right] \\
a_{l2} &= \frac{1}{\beta\mu^2\sigma}\left[\sigma\left(2+\mu+\beta\mu+\gamma^2\mu\left(1-\beta+2\beta\mu\right)-\gamma\left(1+3\mu+\beta\left(\mu-1+2\mu^2\right)\right)\right)\right. \\
&\quad \left.-\psi_x\left(1+2\beta\mu\left(1-\gamma\right)\right)+k\left(\mu+\gamma-\gamma\mu\left(3-\gamma\right)-\psi_\pi\right)\right] \\
a_{l1} &= \frac{1}{\beta\mu^2\sigma}\left[\sigma\left(1-\gamma\mu\right)\left(-1+\gamma-\beta\gamma+\beta\gamma\mu\right)+\psi_x\left(2-\gamma\left(2-\beta\right)+\beta\mu\left(1-\gamma\left(3-\gamma\right)\right)\right)\right. \\
&\quad \left.+k\left(1-\gamma\right)\left(2\psi_\pi-\gamma\left(1-\mu\right)\right)\right] \\
a_{l0} &= \frac{1}{\beta\mu^2\sigma}\left[\left(1-\gamma\right)\left(\psi_x\left(-1+\gamma\left(1-\beta\left(1-\mu\right)\right)\right)-k\psi_\pi\left(1-\gamma\right)\right)\right]
\end{aligned}
$$

Using the Descartes' Rule of Signs we obtain that there exists exactly one negative real root, while the other four are either real positive, or complex conjugates; moreover, note that:

$$
\begin{aligned}
p_l\left(1\right) &= -\frac{1}{\beta\mu^2\sigma}\gamma^2\left[k\left(\psi_\pi-1\right)+\left(1-\beta\right)\psi_x\right] \\
p_l\left(0\right) &= -\frac{1}{\beta\mu^2\sigma}\left(1-\gamma\right)\left[k\psi_\pi\left(1-\gamma\right)+\left(1-\gamma+\beta\gamma\left(1-\mu\right)\right)\psi_x\right] \\
p_l\left(-1\right) &= \frac{1}{\beta\mu^2\sigma}\left[2\sigma\left(1+\mu\left(1-\gamma\right)\right)\left(\left(1+\beta\mu\right)+\beta\gamma\frac{1-\mu}{2-\gamma}\right)+\left(k-\beta\psi_x\right)\gamma\left(1-\mu\right)\right. \\
&\quad \left.-\left(2-\gamma\right)\left(k\left(\psi_\pi-\mu\right)+\left(1+\beta\mu\right)\psi_x\right)\right]
\end{aligned}
$$

Examining the above relations, one can show that:

$$
\begin{aligned}
p_l\left(1\right) &\gtrless 0 \Longleftrightarrow k\left(\psi_\pi-1\right)+\left(1-\beta\right)\psi_x \lesseqgtr 0 \\
p_l\left(0\right) &< 0 \\
p_l\left(1\right) &> 0 \Longrightarrow p_l\left(-1\right) > 0
\end{aligned}
$$

where the last implication makes use of the hypothesis $k < \underline{k}$. Hence, if the Taylor principle does not hold ($k\left(\psi_\pi-1\right)+\left(1-\beta\right)\psi_x < 0$), we have that the negative real root lies inside the unit circle (since $p_l\left(-1\right)$ and $p_l\left(0\right)$ have the opposite sign) and there is either one or three real positive roots in the unit circle (since $p_l(0)$ and $p_l(1)$ have the opposite sign); in turn, this implies that the roots configuration in the complex plane is one of the following:

- four roots inside the unit circle (one real negative, three real positive) and one outside;

-two roots inside the unit circle (one real negative, one real positive) and three outside;

- four roots inside the unit circle (one real negative, one real positive, and two complex conjugates) and one outside.

In any of these cases, the condition of existence and uniqueness of bounded solution is not met. □

Proof of Proposition 7. We prove the three statements separately.

(i) By Proposition5, we know that $D_c^R = \underline{D}^R$; moreover, we have already shown in the proof of Proposition 6 that two of the eigenvalues of $\widehat{Q}_c$, namely λ_{c3} and λ_{c4}, are always one inside and one outside the unit circle. Hence, all we need to prove is that, when the Taylor principle holds, also λ_{c1} and λ_{c2} are one inside and one outside the unit circle. Recall that $\lambda_{c1} + \lambda_{c2} \gtrless 1 + \lambda_{c1}\lambda_{c2}$ is equivalent to:

$$0 \gtrless -\gamma \left(k + \sigma(1-\beta) + \beta\psi_x - \sqrt{k^2 + (\beta\psi_x - \sigma(1-\beta))^2 + 2k(\sigma + \beta\sigma + \beta\psi_x - 2\beta\sigma\psi_\pi)} \right)$$

where the radicand is such that:

$$k^2 + (\beta\psi_x - \sigma(1-\beta))^2 + 2k(\sigma + \beta\sigma + \beta\psi_x - 2\beta\sigma\psi_\pi) \gtrless (k + \sigma(1-\beta) + \beta\psi_x)^2$$

if and only if:

$$\psi_\pi \lesseqgtr \frac{-(1-\beta)\psi_x}{k} + 1$$

In other words, when the Taylor principle holds (i.e., $\psi_\pi > \frac{-(1-\beta)\psi_x}{k} + 1$), the term:

$$-\gamma \left(k + \sigma(1-\beta) + \beta\psi_x - \sqrt{k^2 + (\beta\psi_x - \sigma(1-\beta))^2 + 2k(\sigma + \beta\sigma + \beta\psi_x - 2\beta\sigma\psi_\pi)} \right)$$

is always negative, which in turns implies that $\lambda_{c1} + \lambda_{c2} > 1 + \lambda_{c1}\lambda_{c2}$; by Lemma 6, we conclude that when the Taylor principle holds, the system admits one and only one bounded solution, i.e. $D_c^H = D_c^R = \underline{D}^R$.

(ii) We begin noting that the characteristic polynomial of $\widehat{Q}_f$ is such that the following relations hold[19]:

$$\begin{aligned}
p_f(1) &= \frac{1}{\beta\mu^2(\sigma-\psi_x)}\gamma^2\left[k(\psi_\pi-1)+(1-\beta)\psi_x\right]\\
p_f(0) &= \frac{1}{\beta\mu^2(\sigma-\psi_x)}\left[(1-\gamma+\beta\gamma(1-\mu))((1-\gamma\mu)\sigma-\gamma(1-\mu)\psi_x)-\right.\\
&\quad \left. k(\psi_\pi-1)\gamma(1-\mu)(1-\gamma)\right]\\
p_f(-1) &= \frac{1}{\beta\mu^2(\sigma-\psi_x)}\left[(2-\gamma+\beta\gamma+2\beta\mu(1-\gamma))((1+\mu(1-\gamma))2\sigma-\right.\\
&\quad \left.(\gamma+2\mu(1-\gamma))\psi_x)-k(\psi_\pi-1)(2-\gamma)(\gamma+2\mu(1-\gamma))\right]
\end{aligned}$$

In the case of homogenous (rational) expectations, the above equations collapse to:

$$\begin{aligned}
p_f(1)\big|_{\mu=1} &= \frac{1}{\beta(\sigma-\psi_x)}\gamma^2\left[k(\psi_\pi-1)+(1-\beta)\psi_x\right]\\
p_f(0)\big|_{\mu=1} &= \frac{1}{\beta(\sigma-\psi_x)}(1-\gamma)^2\sigma\\
p_f(-1)\big|_{\mu=1} &= \frac{1}{\beta(\sigma-\psi_x)}\left[(2-\gamma)(1+\beta)((2-\gamma)2\sigma-\right.\\
&\quad \left.(2-\gamma)\psi_x)-k(\psi_\pi-1)(2-\gamma)^2\right]
\end{aligned}$$

Note that $p_f(-1)\big|_{\mu=1}>0$ if and only if $\Psi\in\overline{D}_f^R$; next, we show that the set:

$$\overline{D}_f^H=\left\{\Psi\in\mathbb{R}_+^2:\ p_f(-1)>0\right\}$$

is such that $\overline{D}_f^H\supseteq\overline{D}_f^R$, where $\overline{D}_f^H=\overline{D}_f^R$ if and only if $\mu=1$. To do so, we consider the hyperplanes defined by $p_f(-1)=0$ and $p_f(-1)\big|_{\mu=1}=0$, solve them with respect to ψ_π, obtaining the linear equations:

$$\begin{aligned}
\psi_{\pi,f}^R &= \delta_{0,f}^R+\delta_{1,f}^R\psi_x\\
\psi_{\pi,f}^H &= \delta_{0,f}^H+\delta_{1,f}^H\psi_x
\end{aligned}$$

It is easy to show that $\psi_{\pi,f}^H-\psi_{\pi,f}^R>0$ when $0<\mu<1$, and (obviously) $\psi_{\pi,f}^H-\psi_{\pi,f}^R=0$ when $\mu=1$; this means that for any value of ψ_x, the

[19]In the rest of the proof we assume that $\sigma>\psi_x$; the opposite case can be easily accomodated.

corresponding value of ψ_π on the hyperplane corresponding to $p_f(-1) = 0$ is larger than the one on the on the hyperplane corresponding to $p_f(-1)|_{\mu=1} = 0$. This conclusion, together with the fact that $p_f(-1)$ is decreasing in ψ_π (for any value of μ) leads to the conclusion that $\overline{D}_f^H \supseteq \overline{D}_f^R$. We now prove that $D_f^H = \underline{D}^R \cap \overline{D}_f^H$. First of all, note that when $\Psi \notin \overline{D}_f^H$, the signs of $p_f(1)$, $p_f(0)$ and $p_f(-1)$ can be either -,+,+, or -,-,+[20]. In the first case, there is a negative real root in the unit circle ($p_f(0)$ and $p_f(-1)$ have opposite signs), a negative real root outside the unit circle (since $p_f(0)$, which is equal to the product of the roots of $p_f(\cdot)$, is positive), and two other roots which can be: (a) real positive, in which case they have to be either both inside or both outside the unit circle ($p_f(0)$ and $p_f(1)$ have the same sign); (b) real negative, in which case they have to be either both inside or both outside the unit circle ($p_f(0)$ and $p_f(-1)$ have opposite signs, and one of the first two roots is already between 0 and -1); (c) complex conjugates, in which case they have to be either both inside or both outside the unit circle. In all these configurations, the determinacy condition (two eigenvalues inside and two outside the unit circle) fails to hold. When the signs are -,-,+, instead, there is a positive real root in the unit circle ($p_f(0)$ and $p_f(1)$ have opposite signs), and three other roots which can be[21]: (a) three real negative, in which case they have to be either all outside the unit circle, or two inside and one outside ($p_f(0)$ and $p_f(-1)$ have the same sign); (b) two real negative, and one real positive, in which case the positive root has to be outside the unit circle, and the negative ones have to be either both inside or both outside the unit circle; (c) two complex conjugates and one real, in which case the real root must be negative ($p_f(0)$ is negative) and outside the unit circle ($p_f(0)$ and $p_f(-1)$ have the same sign), and the complex conjugates ones have to be either both inside or both outside the unit circle. In all these configurations, the determinacy condition (two eigenvalues inside and two outside the unit circle) fails to hold. Having examined all the possible cases, we conclude that $\Psi \notin \overline{D}_f^H \Longrightarrow \Psi \notin D_f^H$. Combining this result with Proposition 6, we have that $D_f^H \subseteq \underline{D}^R \cap \overline{D}_f^H$.

Next, we show that the four roots of $p_f(\lambda)$ are such that, if $\Psi \in D_f^R$, the

[20]On one hand, as a consequence of the hypothesis $k < \underline{k}$, $\Psi \notin \overline{D}_f^H$ implies that the Taylor principle holds (namely, $p_f(1) > 0$); on the other hand, it is possible to show that, as ψ_π increases, $p_f(1)$ becomes negative before $p_f(0)$.

[21]In what follows, we repeatedly use the fact that $p_f(0)$, which is equal to the product of the roots of $p_f(\cdot)$, is negative.

following relations:

$$\begin{aligned}(|\lambda_{f1}+\lambda_{f2}|)^2 &> (|1+\lambda_{f1}\lambda_{f2}|)^2 \\ (|\lambda_{f3}+\lambda_{f4}|)^2 &> (|1+\lambda_{f3}\lambda_{f4}|)^2\end{aligned} \tag{3.11}$$

hold for any $0<\mu\leq 1$. For $\mu=1$, they are trivially true[22]; we want to show that they are true also when $0<\mu\leq 1$. Note that, for a complex number $X=A+iB$, the expression $(|X|)^2$ is equal to A^2+B^2, and for a real number $(|X|)^2=(X)^2$. We start computing:

$$\begin{aligned}\lambda_{f1}\lambda_{f2} &= \frac{1}{2\beta\mu(\sigma-\psi_x)}\left[\sigma(1-\gamma+\beta(1+\gamma-2\gamma\mu))-k(1-\gamma)(\psi_\pi-1)-2\beta\gamma(1-\mu)\psi_x\right. \\ &\quad \left.(1-\gamma)\left(-\psi_x-\sqrt{k^2(\psi_\pi-1)^2-2k(\psi_\pi-1)(\sigma+\beta\sigma-\psi_x)+(\psi_x-\sigma(1-\beta))^2}\right)\right] \\ \lambda_{f1}+\lambda_{f2} &= \frac{1}{2\beta\mu(\sigma-\psi_x)}\left[\sigma(1+\beta+2\beta\mu(1-\gamma))-k(\psi_\pi-1)-\psi_x-2\beta\mu(1-\gamma)\psi_x\right. \\ &\quad \left.-\sqrt{k^2(\psi_\pi-1)^2-2k(\psi_\pi-1)(\sigma+\beta\sigma-\psi_x)+(\psi_x-\sigma(1-\beta))^2}\right]\end{aligned}$$

and we distinguish two cases: $\lambda_{f1}\lambda_{f2}$ and $\lambda_{f1}+\lambda_{f2}$ are real, or $\lambda_{f1}\lambda_{f2}$ and $\lambda_{f1}+\lambda_{f2}$ are complex[23]. In the first case, we derive:

$$\begin{aligned}\frac{\partial}{\partial\mu}(\lambda_{f1}+\lambda_{f2})^2 &= 2(\lambda_{f1}+\lambda_{f2})\left[\frac{(2\beta(\sigma-\psi_x))^2(1-\gamma)\mu}{(2\beta\mu(\sigma-\psi_x))^2}-\right. \\ &\quad \left.-\frac{(2\beta(\sigma-\psi_x))^2\mu}{(2\beta\mu(\sigma-\psi_x))^2}(\lambda_{f1}+\lambda_{f2})\right]\end{aligned}$$

and:

$$\begin{aligned}\frac{\partial}{\partial\mu}(1+\lambda_{f1}\lambda_{f2})^2 &= 2(1+\lambda_{f1}\lambda_{f2})\left[\frac{(2\beta(\sigma-\psi_x))^2(1-\gamma)\mu}{(2\beta\mu(\sigma-\psi_x))^2}-\right. \\ &\quad \left.-\frac{(2\beta(\sigma-\psi_x))^2\mu}{(2\beta\mu(\sigma-\psi_x))^2}(1+\lambda_{f1}\lambda_{f2})\right]\end{aligned}$$

[22]In fact, they ensure determinacy of the equilibrium which, by Proposition 5, holds for any $\Psi\in D_f^R$ when $\mu=1$.

[23]It is easy to show the case of one expression real, and one complex, is not possible.

Letting:

$$\Lambda^r_{1,2} \equiv (\lambda_{f1} + \lambda_{f2})^2 - (1 + \lambda_{f1}\lambda_{f2})^2$$

we have:

$$\frac{\partial}{\partial \mu}\left[\Lambda^r_{1,2}\right] = \frac{2}{\mu}\left[(1-\gamma)\left((\lambda_{f1}+\lambda_{f2}) - (1+\lambda_{f1}\lambda_{f2})\right) - \Lambda^r_{1,2}\right] \tag{3.12}$$

which is a first order, linear nonhomogeneous ordinary differential equation, whose general solution is:

$$\Lambda^r_{1,2} = \frac{1}{\mu}\left(K^r_{1,2}\ln\mu + C^r_{1,2}\right) \tag{3.13}$$

where $C^r_{1,2}$ is a constant of integration, and $K^r_{1,2} = (1-\gamma)\,\mu\left((\lambda_{f1}+\lambda_{f2}) - (1+\lambda_{f1}\lambda_{f2})\right)$ is a constant independent of μ. Plugging (3.13) into (3.12), we get:

$$\frac{\partial}{\partial \mu}\left[\Lambda^r_{1,2}\right] = -\frac{1}{\mu^2}\left[C^r_{1,2} + K^r_{1,2}\,(1-\ln\mu)\right]$$

It is easy to show that:

$$C^r_{1,2} = \left.\Lambda^r_{1,2}\right|_{\mu=1} > 0$$

where the last inequality holds by assumption, and that $K^r_{1,2} > 0$ whenever $\Psi \in \underline{D}^R$; the signs of the constants $C^r_{1,2}$ and $K^r_{1,2}$ imply that, for $0 < \mu \le 1$, the derivative $\frac{\partial}{\partial\mu}\left[\Lambda^r_{1,2}\right]$ is negative. Hence, when $\Psi \in \underline{D}^R$ the expression $\Lambda^r_{1,2}$ is always positive for $0 < \mu \le 1$.

Let's turn to the case of $\lambda_{f1}\lambda_{f2}$ and $\lambda_{f1}+\lambda_{f2}$ complex; we can write:

$$\begin{aligned}\frac{\partial}{\partial\mu}\left(|\lambda_{f1}+\lambda_{f2}|\right)^2 \;=\;& 2\mathrm{Re}\left(|\lambda_{f1}+\lambda_{f2}|\right)\left[\frac{\left(2\beta\left(\sigma-\psi_x\right)\right)^2(1-\gamma)\,\mu}{\left(2\beta\mu\left(\sigma-\psi_x\right)\right)^2} - \right.\\ &\left. -\frac{\left(2\beta\left(\sigma-\psi_x\right)\right)^2\mu}{\left(2\beta\mu\left(\sigma-\psi_x\right)\right)^2}\mathrm{Re}\left(|\lambda_{f1}+\lambda_{f2}|\right)\right] - 2\left(\mathrm{Im}\left(|\lambda_{f1}+\lambda_{f2}|\right)\right)^2\frac{1}{\mu}\end{aligned}$$

and:

$$\begin{aligned}\frac{\partial}{\partial\mu}\left(|1+\lambda_{f1}\lambda_{f2}|\right)^2 \;=\;& 2\mathrm{Re}\left(|1+\lambda_{f1}\lambda_{f2}|\right)\left[\frac{\left(2\beta\left(\sigma-\psi_x\right)\right)^2(1-\gamma)\,\mu}{\left(2\beta\mu\left(\sigma-\psi_x\right)\right)^2} - \right.\\ &\left. -\frac{\left(2\beta\left(\sigma-\psi_x\right)\right)^2\mu}{\left(2\beta\mu\left(\sigma-\psi_x\right)\right)^2}\mathrm{Re}\left(|1+\lambda_{f1}\lambda_{f2}|\right)\right] - 2\left(\mathrm{Im}\left(|1+\lambda_{f1}\lambda_{f2}|\right)\right)^2\frac{1}{\mu}\end{aligned}$$

Letting $\Lambda^c_{1,2} \equiv (|\lambda_{f1} + \lambda_{f2}|)^2 - (|1 + \lambda_{f1}\lambda_{f2}|)^2$, the above formula implies:

$$\frac{\partial}{\partial \mu}\left[\Lambda^c_{1,2}\right] = \frac{2}{\mu}\left[(1-\gamma)\,\mathrm{Re}\left((\lambda_{f1} + \lambda_{f2}) - (1 + \lambda_{f1}\lambda_{f2})\right) - \Lambda^c_{1,2}\right]$$

Proceeding as in the real case, we get:

$$\frac{\partial}{\partial \mu}\left[\Lambda^c_{1,2}\right] = -\frac{1}{\mu^2}\left[C^c_{1,2} + K^c_{1,2}\,(1 - \ln \mu)\right]$$

where $C^c_{1,2} = \Lambda^c_{1,2}\big|_{\mu=1} > 0$ and $K^c_{1,2} = (1-\gamma)\,\mu\mathrm{Re}((\lambda_{f1} + \lambda_{f2}) - (1 + \lambda_{f1}\lambda_{f2})) > 0$ whenever $\Psi \in \underline{D}^R$. Hence, when $\Psi \in \underline{D}^R$ the expression $\Lambda^c_{1,2}$ is always positive for $0 < \mu \leq 1$.

Along the same line of reasoning, we get that the expression:

$$(|\lambda_{f3} + \lambda_{f4}|)^2 - (|1 + \lambda_{f3}\lambda_{f4}|)^2$$

is positive whenever $\Psi \in \overline{D}^R_f$. We conclude that conditions (3.11) hold for $0 < \mu \leq 1$, when $\Psi \in D^R_f$; combining this result with the one proved above, we get:

$$\underline{D}^R \cap \overline{D}^H_f \supseteq D^H_f \supseteq D^R_f$$

By continuity of the eigenvalues, we can also say[24]:

$$\underline{D}^R \cap \overline{D}^H_f \supseteq D^H_f \supset D^R_f$$

which proves the statement (ii) of Proposition 7.

(iii) We begin proving that the set:

$$\overline{D}^H_l = \left\{\Psi \in \mathbb{R}^2_+ : p_l\,(-1) > 0\ \right\}$$

where $p_l\,(-1)$ has been defined in the proof of Proposition 6, is such that $\overline{D}^H_l \sqsubseteq \overline{D}^R_l$, where $\overline{D}^H_l = \overline{D}^R_l$ if and only if $\mu = 1$. To do so, we proceed as in the case $i = f$, and compute the hyperplanes defined by the linear equations:

$$\begin{aligned} \psi^R_{\pi,f} &= \delta^R_{0,f} + \delta^R_{1,f}\psi_x \\ \psi^H_{\pi,l} &= \delta^H_{0,l} + \delta^H_{1,l}\psi_x \end{aligned}$$

[24] We conjecture that $D^H_f \subseteq \underline{D}^R \cap \overline{D}^H_f$, which is also supported by numerical simulations, but we could not prove it yet.

It is easy to show that $\psi^H_{\pi,l} - \psi^R_{\pi,f} < 0$ when $0 < \mu < 1$, and (obviously) $\psi^H_{\pi,l} - \psi^R_{\pi,f} = 0$ when $\mu = 1$; this means that for any value of ψ_x, the corresponding value of ψ_π on the hyperplane corresponding to $p_l(-1) = 0$ is larger than the one on the on the hyperplane corresponding to $p_l(-1)|_{\mu=1} = 0$. This conclusion, together with the fact that $p_l(-1)$ is decreasing in ψ_π (for any value of μ) leads to the conclusion that $\overline{D}_l^H \sqsubseteq \overline{D}_l^R$. Next, we show that if $\Psi \notin \overline{D}_l^H$, then $\Psi \notin D_l^H$. To begin with, note that $\Psi \notin \overline{D}_l^H \iff p_l(-1) < 0$; this fact, together with $p_l(0) < 0$, implies that the negative real root lies outside the unit circle. To analyze the other four roots, we can study the auxiliary polynomial $h(\nu) = p_l(\lambda)$, where $1 + \nu = \lambda$. Clearly, any negative root of $h(\cdot)$ corresponds to a root of $p_l(\lambda)$ smaller than one; the coefficients of $h(\cdot)$ are:

$$
\begin{aligned}
h_5 &= a_{l5} \\
h_4 &= 5a_{l5} + a_{l4} \\
h_3 &= 10a_{l5} + 4a_{l4} + a_{l3} \\
h_2 &= 10a_{l5} + 6a_{l4} + 3a_{l3} + a_{l2} \\
h_1 &= 5a_{l5} + 4a_{l4} + 3a_{l3} + 2a_{l2} + a_{l1} \\
h_0 &= a_{l5} + a_{l4} + a_{l3} + a_{l2} + a_{l1} + a_{l0}
\end{aligned}
$$

Using the Descartes' Rule of Signs we obtain that there are at most three roots of $h(\cdot)$ in the left half of the complex plane, and at most two in the right half, or, equivalently, at most three roots of $p_l(\cdot)$ smaller than one, and at most two larger than one[25]; if all the roots are real, we have that exactly two roots of $p_l(\cdot)$ are larger than one, and three are smaller, which implies[26] that there are at least three roots outside the unit circle, and the determinacy condition fails to hold; if four roots are complex, we can have either zero, two or four of them inside the unit circle, but in any of these cases the determinacy condition fails to hold. If two roots are complex conjugates, they can be either both inside or both outside the unit circle, and also two of the real roots can be either both inside or both outside the unit circle[27]; together with the existence of a negative real root smaller than minus one, this implies again that we can have either zero, two or four roots inside the

[25] This result, for the moment, is only numerical.

[26] Recall that we already know that one root is smaller than -1.

[27] This can be shown observing that, when $\Psi \notin \overline{D}_l^H$, $p_l(1)$, $p_l(0)$ and $p_l(0)$ have all the same sign (they are all negative).

unit circle, but in any of these cases the determinacy condition fails to hold. Having examined all the possible cases, we conclude that $\Psi \notin \overline{D}_l^H \Longrightarrow \Psi \notin D_l^H$. Combining this result with the one proved in Proposition 6, we get $\Psi \notin \overline{D}_l^H \cap \underline{D}^R \Longrightarrow \Psi \notin D_l^H$. Since we also showed that $\overline{D}_l^H \subset \overline{D}_l^R$, which is equivalent to say that $\overline{D}_l^H \cap \underline{D}^R \subset D_l^R$, we conclude that $D_l^H \subset D_l^R$. □

We now prove Proposition 8.

Proof of Proposition 8. We have shown in the proof of Proposition 6 that if $\psi_\pi < \frac{-(1-\beta)\psi_x}{k} + 1$ (i.e., if the Taylor principle does not hold), we have $\lambda_{c1} + \lambda_{c2} < 1 + \lambda_{c1}\lambda_{c2}$; by Lemma 6, this means that $(\lambda_{c1}, \lambda_{c2})$ lie both inside the unit circle, if $\lambda_{c1}\lambda_{c2} < 1$, or both outside, if $\lambda_{c1}\lambda_{c2} > 1$. Observe that:

$$\lambda_{c1}\lambda_{c2} \gtrless 1$$

is equivalent to:

$$\begin{aligned} 0 \quad \gtrless \quad & 2\beta\mu\sigma(1+\gamma) - (1-\gamma)\left(k + \sigma + \beta\psi_x - \sqrt{k^2 + (\beta\psi_x - \sigma(1-\beta))^2 + \\ +2k(\sigma + \beta\sigma + \beta\psi_x - 2\beta\sigma\psi_\pi)}\right) - \beta\sigma(1+\gamma) \end{aligned}$$

Let the RHS of the above inequality be defined as $g(\mu)$; note that $\frac{d}{d\mu}g = 2\beta\sigma(1+\gamma) > 0$, and that:

$$\begin{aligned} g(1) \quad = \quad & 2\beta\gamma\sigma - (1-\gamma)\left(k + \sigma + \beta\psi_x - \sqrt{k^2 + (\beta\psi_x - \sigma(1-\beta))^2 + \\ +2k(\sigma + \beta\sigma + \beta\psi_x - 2\beta\sigma\psi_\pi)}\right) \\ > \quad & 0 \end{aligned}$$

where the last inequality follows from the fact that $2\beta\gamma\sigma > 0$, and that, as argued in the proof of Proposition 6:

$$k + \sigma(1-\beta) + \beta\psi_x - \sqrt{k^2 + (\beta\psi_x - \sigma(1-\beta))^2 + 2k(\sigma + \beta\sigma + \beta\psi_x - 2\beta\sigma\psi_\pi)} < 0$$

when the Taylor principle is violated. Moreover, we have:

$$
\begin{aligned}
g\left(0\right) &= -\left(1-\gamma\right)\left(k+\sigma+\beta\psi_x-\sqrt{k^2+\left(\beta\psi_x-\sigma\left(1-\beta\right)\right)^2+2k\left(\sigma+\beta\sigma+\beta\psi_x-2\beta\sigma\psi_\pi\right)}\right) \\
&\quad -\beta\sigma\left(1+\gamma\right) \\
&< -\left(1-\gamma\right)\left(k+\sigma+\beta\psi_x-\sqrt{k^2+\left(\beta\psi_x-\sigma\left(1-\beta\right)\right)^2+ \right.\\
&\quad \left.\overline{+2k\left(\sigma+\beta\sigma+\beta\psi_x-2\beta\sigma\psi_\pi\right)}\right) \\
&< -\left(1-\gamma\right)\left(k+\sigma\left(1+\beta\right)+\beta\psi_x-\sqrt{k^2+\left(\beta\psi_x-\sigma\left(1-\beta\right)\right)^2+2k\left(\sigma+\beta\sigma+\beta\psi_x\right)}\right) \\
&= -\left(1-\gamma\right)\left(k+\sigma\left(1+\beta\right)+\beta\psi_x-\sqrt{\left(k+\sigma\left(1+\beta\right)+\beta\psi_x\right)^2+\left(\beta\psi_x-\sigma\left(1-\beta\right)\right)^2}\right.\\
&\quad \left.\overline{-\left(\beta\psi_x+\sigma\left(1+\beta\right)\right)^2}\right) \\
&= -\left(1-\gamma\right)\left(k+\sigma\left(1+\beta\right)+\beta\psi_x-\sqrt{\left(k+\sigma\left(1+\beta\right)+\beta\psi_x\right)^2-4\beta\sigma\left(\beta\psi_x+\sigma\right)}\right) \\
&< 0
\end{aligned}
$$

Summing up, we showed that:
(i) $\frac{d}{d\mu}g > 0$,
(ii) $g\left(0\right) < 0$,
(iii) $g\left(1\right) > 0$.

Thus, we can invoke the Intermediate Value Theorem to conclude that there exists one and only one $\overline{\mu} \in (0,1)$ such that, for any $\mu \in (\overline{\mu}, 1]$:

$$g\left(\mu\right) > 0 \Longleftrightarrow \lambda_1\lambda_2 < 1$$

and for any $\mu \in (0, \overline{\mu})$:

$$g\left(\mu\right) < 0 \Longleftrightarrow \lambda_1\lambda_2 > 1$$

Putting together this result with the fact that $(\lambda_{c3}, \lambda_{c4})$ lie always one inside and one outside the unit circle[28], we conclude that, for $\mu \in (\overline{\mu}, 1]$ ($\mu \in (0, \overline{\mu})$) violation of the Taylor principle implies that three eigenvalues of $\widehat{Q}_c$ lie inside (outside) the unit circle; hence, the Proposition is proved. □

[28]See the proof of Proposition 6.

3.7.2 Learnability

Proof of Proposition 9. Let's start from the case $i = c, f$. Note that, if the system is determinate, we can apply the results of Blanchard and Kahn (1980) to write the resulting equilibrium in the form:

$$z_t = R_i \theta_t + S_i \eta_t$$

Focusing on the upper block of the above system, we have:

$$y_t = R_i^1 \theta_t + S_i^1 \eta_t \tag{3.14}$$

which we can use in the law of motion for θ obtaining:

$$\theta_t = \left(I_2 - \gamma \left(I_2 - R_i^1\right)\right) \theta_{t-1} + \gamma S_i^1 \eta_{t-1}$$

Furthermore, we can write the conditional (at t) expectations of y_{t+s}, $s \geq 0$, as:

$$E_t y_{t+s} = R_i^1 \left(I_2 - \gamma \left(I_2 - R_i^1\right)\right)^s \theta_t + S_i^1 E_t \eta_{t+s}$$

which stays bounded, as $s \to \infty$, iff $(I_2 - \gamma (I_2 - R_i^1))$ has all the eigenvalues inside the unit circle; note that this is also the condition that ensures that θ_t converges to a stationary and ergodic distribution $\theta^* \sim N(0, \Sigma_i)$, where the variance-covariance matrix is given by:

$$vec\left(\Sigma_i\right) = \left(I_4 - \left(I_2 - \gamma\left(I_2 - R_i^1\right)\right) \otimes \left(I_2 - \gamma\left(I_2 - R_i^1\right)\right)\right)^{-1} \gamma^2 \left(S_i^1 \otimes S_i^1\right) vec\left(\Omega\right)$$

where Ω is the variance-covariance matrix of the exogenous shocks η.

In the case $i = l$, the same line of reasoning applies, with the only difference that now the vector of predetermined variables is given by $\widehat{\theta}_t = [\theta_t', r_t]'$, so that the limiting multivariate normal distribution is the three-dimensional process $\widehat{\theta}^* \sim N(0, \Sigma_l)$; hence, θ^* is simply the marginal of $\widehat{\theta}^*$. □

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