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**OPTICAL SOLITONS IN QUADRATIC  
NONLINEAR MEDIA AND  
APPLICATIONS TO ALL-OPTICAL  
SWITCHING AND ROUTING DEVICES**

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Barcelona, january 1998

## **Part I**

# **Part I: Solitons in Quadratic Nonlinear Media**

This first part is intended to settle down the basic concepts and tools around which this Thesis has been constructed so that it may serve as reference throughout the Thesis. At the time when this Thesis work started, active research in the Quadratic Solitons topic had just begun and therefore many of the ideas which at present fit well in this basics part had to be progressively discovered during the course of the Thesis work, thus actually constituting as well contributions of this Thesis.

In the first of the chapters forming this part (Chapter 2) the basic concepts about light propagation are revisited with special emphasis on nonlinear effects together with a brief outlook to the soliton concept and its properties. Given the importance of numerical methods in the study of quadratic nonlinear solitary wave propagation, in its the last section, Chapter 2 provides a short guide through the properties of some beam propagation algorithms and the premises needed to take into account when simulating beam propagation systems.

Chapter 3 develops the soliton concept in second-order nonlinear media and describes the procedures to be followed to find the families of bright solitary waves existing in the absence of Poynting vector walk-off with the aid of numerical methods. It includes a detailed description of the basic features displayed by the solutions, along with an intuitive model which has proven consistent with the observations as well as useful for to some extent being capable of predicting the approximate soliton behavior. The analysis is completed with some references to other types of solitary waves for comparison and with discussion of stability. Finally the excitation of the solutions is considered in terms of the intuitive model and checked out by extensive numerical simulations.

# Chapter 2

## Basics

### 2.1 Basic theory of light propagation

A revision is carried out in the present section of some basic concepts and tools related to light propagation problems with special emphasis on those which are relevant to quadratic nonlinear media.

#### 2.1.1 The nature of light

‘Rays of light’ wrote Isaac Newton in his *Treatise on Optics*, ‘are very small bodies emitted from shining substances’. To support his intuition Newton mainly called on the straight-line path light propagation in uniform media, and the elastic collisions suffered at boundaries. His definition however left unexplained known phenomena as diffraction, interference or the spreading of light around obstacles which a contemporary scientific by the name of Christian Huygens meant to justify by viewing light as a ‘wave motion’ spreading out from the source in all directions.

While scientists would waste efforts trying to settle on whose side right was, only time and more discoveries finally unveiled that it actually was on both, since through equatorially opposed as it seemed, both theories complemented each other to give a full description of the true nature of light.

Owing mainly to the great intuition of Michael Faraday (1791-1867) and the genius of James Clerk Maxwell (1831-1879) we know today that light is merely one form of electromagnetic

energy of very high frequency and therefore it propagates as an *electromagnetic wave*. Furthermore from the quantum theory of light pioneered by Planck, Einstein and Bohr during the first two decades of the twentieth century, we learned that electromagnetic energy is *quantized*, that is, it can only be imparted to or taken from the electromagnetic field in discrete amounts called *photons*.

Thus the modern concept of light contains elements of both Newton's and Huygen's descriptions. Light is said to have a *dual nature* by which concept is meant that while certain phenomena, such as interference exhibit the wave character of light, other phenomena such as the photoelectric effect display the particle aspect of light.

But what is really light? although it appears a rather philosophical question, a quite convenient description considers most effects regarding *propagation* to be governed by Maxwell equations while the phenomena related to *interaction* of light with matter or the emission and absorption of light to belong to the realm of *quantum theory*. Propagation of faint light signals, containing a reduced number of photons is also governed by quantum electrodynamics but the topics here discussed are not concerned with that case.

Therefore regarding the light propagation problems addressed in this Thesis, while formal expressions shall mainly rely on Maxwell equations, intuitive pictures built up out of some concepts borrowed from quantum theory will be used to get a feeling on the nature and mechanisms of nonlinear effects. In this mood, next section presents Maxwell theory of electromagnetic wave propagation along with some considerations to adapt it to the specific problem of quadratic nonlinear media.

### 2.1.2 Basic theory of electromagnetic wave propagation

Electromagnetic propagation in macroscopic media is governed by the general equations derived by Maxwell in 1864, and that in modern notation write

$$\nabla \cdot \vec{\mathcal{D}}(\vec{r}, t) = \rho_f(\vec{r}, t) , \quad (2.1)$$

$$\nabla \cdot \vec{\mathcal{B}}(\vec{r}, t) = 0 , \quad (2.2)$$

$$\nabla \times \vec{\mathcal{E}}(\vec{r}, t) = -\frac{\partial}{\partial t} \vec{\mathcal{B}}(\vec{r}, t) , \quad (2.3)$$

$$\nabla \times \vec{\mathcal{H}}(\vec{r}, t) = \vec{J}_f(\vec{r}, t) + \frac{\partial}{\partial t} \vec{\mathcal{D}}(\vec{r}, t), \quad (2.4)$$

where  $\rho_f$  and  $\vec{J}_f$  denote respectively the free charges and currents contained in the volume under consideration and the vectorial magnitudes  $\vec{\mathcal{E}}$  and  $\vec{\mathcal{B}}$  represent respectively the electric and magnetic fields while  $\vec{\mathcal{D}}$ ,  $\vec{\mathcal{H}}$  are the auxiliary fields commonly referred to as electric displacement vector and the magnetic intensity inside the volume. The latter pair of vectorial magnitudes relate to the former through the properties of the materials filling the volume under study as follows,

$$\vec{\mathcal{D}}(\vec{r}, t) = \epsilon_0 \vec{\mathcal{E}}(\vec{r}, t) + \vec{\mathcal{P}}(\vec{r}, t), \quad (2.5)$$

$$\vec{\mathcal{H}}(\vec{r}, t) = \frac{\vec{\mathcal{B}}(\vec{r}, t)}{\mu_0} - \vec{\mathcal{M}}(\vec{r}, t), \quad (2.6)$$

with  $\epsilon_0$  and  $\mu_0$  being respectively electric and magnetic permittivities of the vacuum. For the sake of clarity in notation, in what follows the spatial and temporal dependence of all functions is assumed implicit.

Through equality (2.5) the effect of the additional electric field produced by the bounded charges distribution generated inside the material as a consequence of the application of the electric field is introduced into the equations giving the vector  $\vec{\mathcal{P}}$  such volumetric distribution. Likewise, equality (2.6) takes account of magnetic dominia formed in the material by the incident magnetic field being  $\vec{\mathcal{M}}$  the volumetric distribution of such dominia.

In all propagation problems addressed in this Thesis the volumes under study won't contain neither free charges nor free currents and all materials will be non-magnetic whereupon  $\vec{\mathcal{M}} = 0$ , and equations (2.1-2.4) are written in a more convenient form as

$$\nabla \cdot \vec{\mathcal{E}} = -\frac{\nabla \cdot \vec{\mathcal{P}}}{\epsilon_0}, \quad (2.7)$$

$$\nabla \cdot \vec{\mathcal{H}} = 0, \quad (2.8)$$

$$\nabla \times \vec{\mathcal{E}} = -\mu_0 \frac{\partial}{\partial t} \vec{\mathcal{H}}, \quad (2.9)$$

$$\nabla \times \vec{\mathcal{H}} = \epsilon_0 \frac{\partial}{\partial t} \vec{\mathcal{E}} + \frac{\partial}{\partial t} \vec{\mathcal{P}}. \quad (2.10)$$

Combining the curl of (2.9) with (2.10) and using that the velocity of light on vacuum,  $c$ ,

verifies  $c = \sqrt{\mu_0 \varepsilon_0}$ , one obtains the wave equation

$$\nabla^2 \vec{\mathcal{E}} - \nabla (\nabla \cdot \vec{\mathcal{E}}) - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \vec{\mathcal{E}} = \mu_0 \frac{\partial^2}{\partial t^2} \vec{\mathcal{P}}, \quad (2.11)$$

where it is seen that the material's polarization vector generated as a result of the material's bounded charges reorientation produced by the incident field, in turn acts as a source for the same electric field.

### 2.1.3 The constitutive relation

The equilibrium eventually reached between applied electric field and the polarization vector induced into the material may be shown to be such that the *constitutive relation* between the polarization vector and the electric field vector may be written far from the material's resonances in a generic case as

$$\begin{aligned} \vec{\mathcal{P}}(\vec{r}, t) = & \vec{\mathcal{P}}^{(0)}(\vec{r}, t) + \sum_{n=1}^{\infty} \varepsilon_0 \int_{-\infty}^{\infty} d\vec{r}_1 \int_{-\infty}^{\infty} dt_1 \dots \int_{-\infty}^{\infty} d\vec{r}_n \int_{-\infty}^{\infty} dt_n \times \\ & \chi^{(n)}(\vec{r}, t; \vec{r}_1, t_1; \dots; \vec{r}_n, t_n) \times \vec{\mathcal{E}}(\vec{r}_1, t_1) \dots \vec{\mathcal{E}}(\vec{r}_n, t_n), \end{aligned} \quad (2.12)$$

where  $\vec{\mathcal{P}}^{(0)}(\vec{r}, t)$  is the dipole orientation found in absence of any applied electric field and at optical frequencies averages to zero. The *electrical susceptibility*  $\chi^{(n)}$  is a function characterizing the material's behavior with respect to electromagnetic radiation and in general is a tensor of rank  $n$  in order to account for anisotropic effects. The integration over the spatial variables is eliminated by considering that the usual interaction lengths are very much smaller than the optical wavelengths so that one has

$$\vec{\mathcal{P}}(\vec{r}, t) = \sum_{n=1}^{\infty} \varepsilon_0 \int_{-\infty}^{\infty} \chi^{(n)}(\vec{r}, t; \vec{r}_1, t_1; \dots; \vec{r}_n, t_n) \times \vec{\mathcal{E}}(\vec{r}_1, t_1) \dots \vec{\mathcal{E}}(\vec{r}_n, t_n) dt_1 \dots dt_n. \quad (2.13)$$

The temporal integration accounts for delayed response effects related to dispersion and has important implications as for wave propagation.

Very often the fields behavior may to a great extent of accuracy be described by retaining

only linear terms in the constitutive relation such that

$$\vec{\mathcal{P}}(\vec{r}, t) = \vec{\mathcal{P}}_L(\vec{r}, t) = \varepsilon_0 \int_{-\infty}^{\infty} \chi^{(1)}(\vec{r}, t_1) \vec{\mathcal{E}}(\vec{r}, t - t_1) dt_1 . \quad (2.14)$$

In the frequency domain the above yields

$$\vec{\mathcal{P}}_L(\vec{r}, \omega) = \varepsilon_0 \chi^{(1)}(\vec{r}, \omega) \vec{\mathcal{E}}(\vec{r}, \omega) , \quad (2.15)$$

and usually the polarization vector is written

$$\vec{\mathcal{P}} = \vec{\mathcal{P}}_L + \vec{\mathcal{P}}_{NL} = \varepsilon_0 \chi^{(1)}(\omega) \vec{\mathcal{E}}(\omega) + \vec{\mathcal{P}}_{NL}(\omega) , \quad (2.16)$$

so that the nonlinear polarization  $\vec{\mathcal{P}}_{NL}$  gathers all nonlinear effects in the material's response.

Writing down explicitly each of the  $\vec{\mathcal{P}}(\vec{r}, t)$  components it is obtained

$$\mathcal{P}_1 = \sum_{i_1=1}^3 \dots \sum_{i_n=1}^3 \varepsilon_0 \int_{-\infty}^{\infty} \chi_{1i_1 \dots i_n}^{(n)} \mathcal{E}_{i_1} \dots \mathcal{E}_{i_n} dt_1 \dots dt_n , \quad (2.17)$$

$$\mathcal{P}_2 = \sum_{i_1=1}^3 \dots \sum_{i_n=1}^3 \varepsilon_0 \int_{-\infty}^{\infty} \chi_{2i_1 \dots i_n}^{(n)} \mathcal{E}_{i_1} \dots \mathcal{E}_{i_n} dt_1 \dots dt_n , \quad (2.18)$$

$$\mathcal{P}_3 = \sum_{i_1=1}^3 \dots \sum_{i_n=1}^3 \varepsilon_0 \int_{-\infty}^{\infty} \chi_{3i_1 \dots i_n}^{(n)} \mathcal{E}_{i_1} \dots \mathcal{E}_{i_n} dt_1 \dots dt_n . \quad (2.19)$$

If the medium has a center of symmetry, i.e. it is *centrosymmetric*, a change in sign of every coordinate axis must entail respective sign changes in the respective  $\mathcal{E}_i$  and  $\mathcal{P}_i$  components which only is possible for an odd value of  $n$ . It is concluded therefore that for the polarization vector induced in the material to depend upon even powers of the electric field, the medium should lack a center of symmetry.

Hence, in centrosymmetric materials the most relevant term of the nonlinear polarization is cubic in the electric field dependence whereupon

$$\begin{aligned} \vec{\mathcal{P}}_{NL}(\vec{r}, t) &= \varepsilon_0 \int_{-\infty}^{\infty} \chi^{(3)}(\vec{r}, t_1, t_2, t_3) \vec{\mathcal{E}}(\vec{r}, t - t_1, t - t_2, t - t_3) \\ &\quad \vec{\mathcal{E}}(\vec{r}, t - t_1, t - t_2, t - t_3) \vec{\mathcal{E}}(\vec{r}, t - t_1, t - t_2, t - t_3) dt_1 dt_2 dt_3 . \end{aligned} \quad (2.20)$$



A stronger nonlinear response is usually obtained through the second order susceptibility but then noncentrosymmetric media are required. One has

$$\vec{\mathcal{P}}_{NL}(\vec{r}, t) = \epsilon_0 \int_{-\infty}^{\infty} \chi^{(2)}(\vec{r}, t_1, t_2) \vec{\mathcal{E}}(\vec{r}, t - t_1, t - t_2) \vec{\mathcal{E}}(\vec{r}, t - t_1, t - t_2) dt_1 dt_2 . \quad (2.21)$$

The constitutive relation in the frequency domain can be readily shown to be

$$\vec{\mathcal{P}}_{NL}(\varpi = \varpi_1 + \varpi_2) = \sum \epsilon_0 \chi^{(2)}(-\varpi_1, -\varpi_2) \vec{\mathcal{E}}(\varpi_1) \vec{\mathcal{E}}(\varpi_2) , \quad (2.22)$$

where summation has to be performed over all possible permutations of the pair  $(\varpi_1, \varpi_2)$  such that  $\varpi = \varpi_1 + \varpi_2$ . Analogous expressions can be written for other nonlinearity orders.

That suggests the use of a contracted notation for the susceptibility tensors as follows

$$\vec{\mathcal{P}}_{NL}(\varpi_\sigma) = \epsilon_0 \sum \chi^{(n)}(-\varpi_\sigma; \varpi_1, \dots, \varpi_n) \vec{\mathcal{E}}(\varpi_1) \dots \vec{\mathcal{E}}(\varpi_n) , \quad (2.23)$$

where in all cases  $\varpi_\sigma = \varpi_1 + \dots + \varpi_n$  and summation is extended over all possible permutations of the  $(\varpi_1, \dots, \varpi_n)$  set such that  $\varpi_\sigma = \varpi_1 + \dots + \varpi_n$ . Frequencies to the right hand side of the ‘;’ punctuation sign are meant to be the input frequencies while to the left hand side the generated frequencies arising from the light-matter interaction are written.

#### 2.1.4 Susceptibility

Steaming from the principles of time invariance and causality, the nonlinear susceptibilities,  $\chi_{\mu i_1 \dots i_n}^{(n)}(-\varpi_\sigma; \varpi_1, \dots, \varpi_n)$ , exhibit *intrinsic permutation symmetry* which implies invariance upon the  $n!$  permutations of the  $n$  pairs  $(i_1, \varpi_1), \dots, (i_n, \varpi_n)$ . That allows to write (2.23) as

$$\vec{\mathcal{P}}_{NL}(\varpi_\sigma) = \epsilon_0 \mathbf{S}(\varpi_1, \dots, \varpi_n) \chi^{(n)}(-\varpi_\sigma; \varpi_1, \dots, \varpi_n) \vec{\mathcal{E}}(\varpi_1) \dots \vec{\mathcal{E}}(\varpi_n) , \quad (2.24)$$

where in all cases  $\varpi_\sigma = \varpi_1 + \dots + \varpi_n$ , and  $\mathbf{S}(\varpi_1, \dots, \varpi_n)$  accounts for all possible permutations of the  $(\varpi_1, \dots, \varpi_n)$  set such that  $\varpi_\sigma = \varpi_1 + \dots + \varpi_n$ .

This property becomes an obvious one if from a physical view point the susceptibility is understood as the function regulating absorption or generation of photons inside the material in such a way that frequencies with negative sign in the  $\chi^{(n)}(-\varpi_\sigma; \varpi_1, \dots, \varpi_n)$  expression are

those at which photons are generated while those with positive sign give the absorbed photons.

It is a common practice to write the field and the polarization in the convenient fasorial form

$$\vec{\mathcal{E}}(t) = \frac{1}{2} \sum_{\varpi \geq 0} \left[ \vec{E}(\varpi) e^{-j\varpi t} + \vec{E}(-\varpi) e^{j\varpi t} \right], \quad (2.25)$$

$$\vec{\mathcal{P}}(t) = \frac{1}{2} \sum_{\varpi \geq 0} \left[ \vec{P}(\varpi) e^{-j\varpi t} + \vec{P}(-\varpi) e^{j\varpi t} \right], \quad (2.26)$$

whereupon (2.24) becomes

$$\vec{P}_{NL}(\varpi_\sigma) = \varepsilon_0 \mathbf{K}(\varpi_1, \dots, \varpi_n) \chi^{(n)}(-\varpi_\sigma; \varpi_1, \dots, \varpi_n) \vec{E}(\varpi_1) \dots \vec{E}(\varpi_n), \quad (2.27)$$

with  $\mathbf{K}(\varpi_1, \dots, \varpi_n) = 2^{1-n} \mathbf{S}(\varpi_1, \dots, \varpi_n)$  so that it accounts for the 2 factors as well.

As seen, in a centrosymmetric material the nonlinear susceptibility is basically order 3. Specifically following the above one has

$$\vec{P}_{NL}(\varpi) = \frac{3}{4} \varepsilon_0 \chi^{(3)}(\varpi; \varpi, \varpi, -\varpi) \left| \vec{E}(\varpi) \right|^2 \vec{E}(\varpi). \quad (2.28)$$

When the invariance includes the pair  $(\mu, \varpi_\sigma)$  it is said that the susceptibility tensor enjoys *overall permutation symmetry* which is actually an approximation valid only when all optical frequencies occurring in expression (2.24) are far removed from the medium resonances, so that it is transparent to all the relevant frequencies. That is seen by considering that when this property holds, for the general  $n$  wave interaction in which  $\varpi_\sigma = \varpi_1 + \dots + \varpi_n$  one may write

$$\begin{aligned} \chi^{(n)}(-\varpi_1; \varpi_\sigma, -\varpi_2, -\varpi_3, \dots, -\varpi_n) &= \chi^{(n)}(-\varpi_2; \varpi_\sigma, -\varpi_1, -\varpi_3, \dots, -\varpi_n) = \\ &\dots = \chi^{(n)}(-\varpi_n; \varpi_\sigma, -\varpi_1, -\varpi_2, \dots, -\varpi_{n-1}), \end{aligned} \quad (2.29)$$

indicating that for each photon generated at the frequency  $\varpi_1$  corresponding photons at the  $(\varpi_2, \dots, \varpi_n)$  frequencies need to be generated so that the material is not holding any energy that causes an energy band transition to be missed out.

Perhaps the most important consequence of overall permutation symmetry is the *Manley-Rowe relations* which as describing the exchange of power between electromagnetic waves in

a purely reactive nonlinear medium can be considered a kind of conservation of energy law between the frequencies taking part in the interaction. Formally it is expressed through

$$\frac{W_\sigma}{\varpi_\sigma} = \frac{W_1}{\varpi_1} + \dots + \frac{W_n}{\varpi_n} , \quad (2.30)$$

where the  $W_i$  represent the power input to unit volume of the medium at the frequency  $\varpi_i$ . If one divides the whole above expression by Planck's constant,  $\hbar$ , obtaining in the denominator the energy carried by each photon at the frequency  $\varpi_i$  and taking into account that since the medium is transparent, for any photon generated at the input frequency  $\varpi_\sigma$  corresponding photons at the  $n$  input frequencies ( $\varpi_1, \dots, \varpi_n$ ) are absorbed; an alternative lecture of the Manley-Rowe relation is that it states a conservation of the averaged number of photons of either frequency inside the medium. Being that so, it follows that the processes of generation of photons at each of the frequencies ( $\varpi_1, \dots, \varpi_n$ ) must be  $n$  times less intense than the corresponding process of generation of the  $\varpi_\sigma$  frequency, that is

$$\chi^{(n)}(-\varpi_\sigma; \varpi_1, \dots, \varpi_n) = n\chi^{(n)}(-\varpi_1; \varpi_\sigma, -\varpi_2, \dots, -\varpi_n) . \quad (2.31)$$

With an eye on further analysis to be carried out, it is taken as an example the quadratic nonlinearity,  $n = 2$  case, with  $\varpi$  and  $2\varpi$  inputs. One has

$$\chi^{(2)}(-2\varpi; \varpi, \varpi) = 2\chi^{(2)}(-\varpi; -\varpi, 2\varpi) . \quad (2.32)$$

Now taking into account that  $\mathbf{S}(\varpi, \varpi) = 1$  while  $\mathbf{S}(-\varpi, 2\varpi) = 2$  yielding  $\mathbf{K}(\varpi, \varpi) = 1/2$  and  $\mathbf{K}(-\varpi, 2\varpi) = 1$  one finally arrives to

$$\vec{P}_{NL}(\varpi) = \varepsilon_0 \chi^{(2)} \vec{E}^*(\varpi) \vec{E}(2\varpi) , \quad (2.33)$$

$$\vec{P}_{NL}(2\varpi) = \varepsilon_0 \chi^{(2)} \vec{E}(\varpi) \vec{E}(\varpi) , \quad (2.34)$$

where to simplify the notation  $\chi^{(2)} = \chi^{(2)}(-\varpi; -\varpi, 2\varpi)$ . Mind that to account for the

anisotropy present in the system  $\chi^{(2)}$  is in the more general case a 3 rank tensor so that

$$(P_{NL}(\varpi))_i = \varepsilon_0 \chi_{ijk}^{(2)} E_j^*(\varpi) E_k(2\varpi) , \quad (2.35)$$

$$(P_{NL}(2\varpi))_k = \varepsilon_0 \chi_{kij}^{(2)} E_i(\varpi) E_j(\varpi) , \quad (2.36)$$

where by virtue of the overall permutation symmetry, one has  $\chi_{ijk}^{(2)} = \chi_{kij}^{(2)} = \chi^{(2)}$ .

The kind of anisotropy most widely encountered in light propagation problems is that in which a symmetry axis, known as *optical axis*, may be identified so that the anisotropy displayed by the material gets the more specific denomination of *birefringence*. Birefringent materials feature two distinct modes of propagation depending on the incident field polarization,  $\vec{\mathcal{E}}$ , with respect to the optical axis,  $\vec{a}$ , and the fields direction of propagation,  $\vec{k}$ , so that if  $\vec{\mathcal{E}}$  is normal to the plane defined by  $(\vec{a}, \vec{k})$  light is said to propagate as an *ordinary wave* whereupon it is affected by a refractive index  $n_o$ , whereas if  $\vec{\mathcal{E}}$  lays in the plane determined by  $(\vec{a}, \vec{k})$  the propagating wave is known as *extraordinary wave* and feels a refractive index  $n_e(\theta)$  with  $\theta$  the angle between  $\vec{\mathcal{E}}$  and  $\vec{a}$ . Extraordinary waves are characterized by a *Poynting vector walk-off* effect due to the distinct directions of constant phase planes propagation which usually follow that imposed by the input, and the direction in which the energy flux is transferred which is that of the optical axis. That causes ordinary and extraordinary input beams to progressively separate in a linear medium so that they walk each other off, hence the name.

In birefringent materials, through judicious supply of the interacting waves so that the proper elements of the susceptibility tensor are used, two are basically the types of quadratic vectorial wave interaction that may be encountered. Namely

**Type I:** which is the one to use in Lithium Niobate,  $LiNbO_3$ , a very common material in optical technology. The wave interaction takes place between ordinary fundamental and extraordinary second harmonic as sketched below

$$2\varpi(e) + \varpi(o) \rightarrow \varpi(o) . \quad (2.37)$$

$$\varpi(o) + \varpi(o) \rightarrow 2\varpi(e) , \quad (2.38)$$

Thus the polarization vector as a function of the electric field writes

$$\vec{P}_{NL}(\varpi^{(o)}) = \varepsilon_0 \chi_{eff}^{(2)} \vec{E}^* (\varpi^{(o)}) \vec{E} (2\varpi^{(e)}) , \quad (2.39)$$

$$\vec{P}_{NL}(2\varpi^{(e)}) = \varepsilon_0 \chi_{eff}^{(2)} \vec{E} (\varpi^{(o)}) \vec{E} (\varpi^{(o)}) , \quad (2.40)$$

where the superindices label each frequency depending if it is ordinary ( $o$ ) or extraordinary ( $e$ ) and the values of  $\chi_{eff}^{(2)}$  are found tabulated for various crystal types in for example [134].

**Type II:** which through use of three interacting waves gives rise to more versatile settings suitable for materials like Potassium Titanyl Phosphate, *KTP*, for example. Extraordinary waves at fundamental and at second harmonic frequencies and an ordinary fundamental are the waves taking part in the interaction as follows,

$$2\varpi(e) + \varpi(o) \rightarrow \varpi(e) , \quad (2.41)$$

$$2\varpi(e) + \varpi(e) \rightarrow \varpi(o) , \quad (2.42)$$

$$\varpi(o) + \varpi(e) \rightarrow 2\varpi(e) , \quad (2.43)$$

yielding for the polarization vectors

$$\vec{P}_{NL}(\varpi^{(e)}) = \varepsilon_0 \chi_{eff}^{(2)} \vec{E}^* (\varpi^{(o)}) \vec{E} (2\varpi^{(e)}) , \quad (2.44)$$

$$\vec{P}_{NL}(\varpi^{(o)}) = \varepsilon_0 \chi_{eff}^{(2)} \vec{E}^* (\varpi^{(e)}) \vec{E} (2\varpi^{(e)}) , \quad (2.45)$$

$$\vec{P}_{NL}(2\varpi^{(e)}) = \varepsilon_0 \chi_{eff}^{(2)} \vec{E} (\varpi^{(o)}) \vec{E} (\varpi^{(e)}) , \quad (2.46)$$

where the values of  $\chi_{eff}^{(2)}$  for various crystal types are again found in the tables [134].

### 2.1.5 Slowly-varying envelope approximation (SVEA)

Using expression (2.16) and under the usual assumption that  $\vec{P}_{NL} \ll \vec{P}_L$  so that the polarization vector is almost parallel to  $\vec{E}$  and from (2.7) one has  $\nabla \vec{E} \simeq 0$ , the wave equation valid

for the majority of light propagation problems writes

$$\nabla^2 \vec{\mathcal{E}} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \left( 1 + \chi^{(1)*} \right) \vec{\mathcal{E}} = \mu_0 \frac{\partial^2}{\partial t^2} \vec{\mathcal{P}}_{NL}, \quad (2.47)$$

where ‘\*’ denotes convolution thus taking into account delay effects in the material’s response. The proper expression for  $\vec{\mathcal{P}}_{NL}$  must then be found for each specific propagation problem.

Due to the formal analogies encountered, generally speaking, light propagation problems are usually classified attending to the number of transverse variables involved as next indicated. Note that the 1 after the ‘+’ sign refers to the propagation variable  $z$  while the number before the ‘+’ sign indicates the number of transverse variables that are allowed to vary in the system.

**Plane wave or (0 + 1) case :** when the problem considers the propagation of a monochromatic beam in a bulk material whereupon the *envelope function*,  $q(z)$ , depends only on the propagation variable,  $z$ , so that the electric field is expressed

$$\vec{\mathcal{E}}(\vec{r}, t) = \frac{\hat{e}}{2} \left[ q(z) e^{jn k_0 z} e^{-j\omega_0 t} + c.c. \right], \quad (2.48)$$

$$\vec{\mathcal{E}}(\vec{r}, t) = \frac{\hat{e}}{2} \left[ q(z) e^{jn k_0 z} e^{-j\omega_0 t} + c.c. \right], \quad (2.49)$$

with  $n$  the material’s refractive index,  $k_0$  the propagation constant of light in vacuum,  $\omega_0$  the carrier frequency and  $\hat{e}$  a unity vector.

**(1 + 1) case :** Physically this case corresponds either to pulse propagation in fibers or channel waveguides whereupon the transverse variable is time  $t$ ; or else to monochromatic beam propagation in planar waveguides being  $x$  the transverse variable considered assuming waveguide confinement along the  $y$  direction.

For pulse propagation problems the field is usually written

$$\vec{\mathcal{E}}(\vec{r}, t) = \frac{\hat{e}}{2} \left[ F(x, y) \int_{-\Delta\omega/2}^{\Delta\omega/2} Q(z, \omega) e^{j\omega t} d\omega e^{jkz} e^{-j\omega_0 t} + c.c. \right], \quad (2.50)$$

where  $F(x, y)$  represents the spatial distribution of the plane wave mode of the structure with propagation constant  $k$ . The main  $t$  dependence of the field is assumed to be realized

through the term  $e^{-j\varpi_0 t}$  so that the envelope function,  $q(z, t)$ , whose Fourier Transform is given by  $Q(z, \varpi)$ , only accounts for small deviations. The frequency band,  $\Delta\varpi$ , and the carrier frequency,  $\varpi_0$ , are in most realistic situations such that  $\frac{\Delta\varpi}{\varpi_0} = \epsilon \ll 1$ , or equivalently  $\frac{\partial q/\partial t}{\varpi_0} = \epsilon \ll 1$ . The envelope function is designed to include as well deviations from the dispersion relation affecting slightly the  $z$  dependence so that the so-called *slowly-varying envelope approximation (SVEA)*  $\frac{\partial q/\partial z}{k_0} = \epsilon \ll 1$  holds.

For beam propagation problems the ansatz is used

$$\vec{\mathcal{E}}(\vec{r}, t) = \frac{\hat{e}}{2} \left[ F(y) \int_{-\Delta k/2}^{\Delta k/2} Q(z, \bar{k}) e^{j\bar{k}x} d\bar{k} e^{jkz} e^{-j\varpi_0 t} + c.c. \right], \quad (2.51)$$

where  $F(y)$  gives the modal distribution along the vertical  $y$  axis and  $\frac{\Delta k}{k} = \epsilon \ll 1$  or equivalently  $\frac{\partial q/\partial x}{k} = \epsilon \ll 1$  with  $q(z, x)$  inverse Fourier transform of  $Q(z, k)$  known as the *paraxial approximation*. The SVEA as applied to this case reads  $\frac{\partial q/\partial z}{k} = \epsilon \ll 1$ .

**(2 + 1) case** : involving either pulse propagation in planar waveguides with  $x$  and  $t$  transverse variables expressing the field as

$$\vec{\mathcal{E}}(\vec{r}, t) = \frac{\hat{e}}{2} \left[ F(y) \int_{-\Delta k/2}^{\Delta k/2} \int_{-\Delta\varpi/2}^{\Delta\varpi/2} Q(z, \varpi, \bar{k}) e^{j\varpi t} e^{j\bar{k}x} d\varpi d\bar{k} e^{jkz} e^{-j\varpi_0 t} + c.c. \right]; \quad (2.52)$$

or else monochromatic beam propagation in bulk media with  $x$  and  $y$  with the ansatz

$$\vec{\mathcal{E}}(\vec{r}, t) = \frac{\hat{e}}{2} \left[ \int_{-\Delta k_x/2}^{\Delta k_x/2} \int_{-\Delta k_y/2}^{\Delta k_y/2} Q(z, \bar{k}_x, \bar{k}_y) e^{j\bar{k}_x x} e^{j\bar{k}_y y} d\bar{k}_x d\bar{k}_y e^{j(k_x+k_y)z} e^{-j\varpi_0 t} + c.c. \right]. \quad (2.53)$$

**(3 + 1) case** : which comprises pulse propagation in bulk media with ansatz

$$\vec{\mathcal{E}}(\vec{r}, t) = \frac{\hat{e}}{2} \left[ \int_{-\Delta\varpi/2}^{\Delta\varpi/2} \int_{-\Delta k_x/2}^{\Delta k_x/2} \int_{-\Delta k_y/2}^{\Delta k_y/2} Q(z, \varpi, \bar{k}_x, \bar{k}_y) e^{j\bar{k}_x x} e^{j\bar{k}_y y} e^{j\varpi t} d\bar{k}_x d\bar{k}_y d\varpi e^{j(k_x+k_y)z} e^{-j\varpi_0 t} + c.c. \right]. \quad (2.54)$$

The results for the formally simpler 1+1 case can be extended to consider 2+1 and 3+1 geometries. The analytical derivations and numerical experiments carried out in this Thesis are mainly centered in 1+1 configurations.

### 2.1.6 Monochromatic beam propagation in linear media. Diffraction.

A monochromatic beam may be expressed as a sum of plane waves as follows

$$\vec{\mathcal{E}}(x, z, t) = \frac{\hat{e}}{2} \left[ \int_{-\Delta k_x/2}^{\Delta k_x/2} Q(z, \bar{k}) e^{j\bar{k}x} d\bar{k} e^{jkz} e^{-j\varpi_0 t} + c.c. \right], \quad (2.55)$$

where it is assumed  $\frac{\Delta k_x}{k} = \epsilon \ll 1$ . For the sake of simplicity only one transverse variable is considered but the result can be easily extended to consider more transverse variables.

Substituting the ansatz (2.55) into the wave equation (2.47) for light propagation in linear media, i.e.  $\vec{\mathcal{P}}_{NL} = 0$ , one gets

$$\frac{\partial^2 q}{\partial x^2} + \frac{\partial^2 q}{\partial z^2} + 2jk_0 \frac{\partial q}{\partial z} - k_0^2 q + \frac{\varpi_0^2}{v} q = 0, \quad (2.56)$$

where  $q(z, x)$  stands for the inverse Fourier transform of  $Q(z, k)$  and  $v$  is the velocity of the monochromatic wave inside the medium at frequency  $\varpi_0$  given by  $v = \sqrt{\mu_0 \epsilon_0 \epsilon_r}$  with  $\epsilon_r$  the material's permittivity such that  $\epsilon_r = (1 + \chi^{(1)}(\varpi_0))$ .

To account for the orders of magnitude present in the equation one can divide by  $k_0^2$  to obtain [135]

$$\frac{\partial^2 q / \partial x^2}{k_0^2} + \frac{\partial^2 q / \partial z^2}{k_0^2} + 2j \frac{\partial q / \partial z}{k_0} - q + \frac{\varpi_0^2}{vk_0^2} q = 0. \quad (2.57)$$

To leading order one has

$$-q + \frac{\varpi_0^2}{vk_0^2} q = 0 \longrightarrow k_0^2 = \frac{\varpi_0^2}{v}, \quad (2.58)$$

and then it must be  $\frac{\partial q / \partial z}{k_0} \sim \epsilon^2 \longrightarrow \frac{\partial^2 q / \partial z^2}{k_0^2} \sim \epsilon^4$  so to have to order  $\epsilon^4$

$$\frac{\partial^2 q / \partial x^2}{k_0^2} + 2j \frac{\partial q / \partial z}{k_0} = 0, \quad (2.59)$$



which in the general form

$$j \frac{\partial q}{\partial z} = -\frac{1}{2k_0} \nabla_t q, \quad (2.60)$$

with  $\nabla_t$  standing for transverse Laplacian, is known as the *paraxial equation* and describes the diffraction of a beam inside an homogeneous linear medium.

### 2.1.7 Pulse propagation in linear media. Dispersion.

When finite delay effects are present in the structure they may be included into the wave equation as follows from (2.47) setting  $\vec{\mathcal{P}}_{NL} = 0$

$$\nabla^2 \vec{\mathcal{E}} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \vec{\mathcal{E}} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \int_{-\infty}^{\infty} \chi^{(1)}(t-t_1) \vec{\mathcal{E}}(t_1) dt_1 = 0. \quad (2.61)$$

To see how this impacts on pulse propagation the electric field is conveniently expressed through

$$\vec{\mathcal{E}}(z, t) = \frac{\hat{e}}{2} \left[ q(z, t) e^{jk_0 z} e^{-j\omega_0 t} + c.c. \right], \quad (2.62)$$

where  $q(z, t)$  is assumed to be a slowly varying function in both variables.

With the usual definitions  $v_g = (k'|\omega_0)^{-1}$  and  $k'' = k''|\omega_0$  giving  $k(\omega)$  the dispersion relation and following the procedure outlined in Appendix A, pulse propagation in the presence of delay effects in the material can be written in standard form [135]

$$j \frac{\partial q}{\partial z} + j \frac{1}{v_g} \frac{\partial q}{\partial t} - \frac{1}{2} k'' \frac{\partial^2 q}{\partial t^2} = 0, \quad (2.63)$$

in a frame of reference moving with velocity  $v_g$ ,  $\bar{z} = t + z/v_g$ , one obtains

$$j \frac{\partial q}{\partial \bar{z}} = \frac{1}{2} k'' \frac{\partial^2 q}{\partial t^2}, \quad (2.64)$$

whereupon the parameter  $v_g$  is known as the *group velocity* of the wave while the parameter  $k''$  as governing the spreading out of the initial pulse on propagation gets the name of *group velocity dispersion (GVD)*. Mind that since it was assumed  $e^{jk_0 z}$  for the propagation constant it is obtained a wave traveling to the negative  $z$  side.

Hence the word *dispersion* designates the effect by which different frequencies propagate

with different velocities depending such difference on the dispersion relation  $k(\omega)$ .

Dispersion may be due to the requirements posed by boundary conditions in the structure when only one propagation mode is excited, whereupon after a medium where such a dispersion is encountered it gets the name *waveguide dispersion*, and also due to the fact that for different frequencies energy must propagate according to different propagation modes yielding *intermodal dispersion*. Delayed response on the material is also a source for dispersion known as *material dispersion* and characterized by a frequency dependence on the material's susceptibility.

The only difference observed between equations describing diffraction and dispersion effects (2.60) and (2.64) respectively, is that while the spreading ratio of a beam depends uniquely on the beam shape, the pulse spreading process depends also on the value of the GVD present in the system so that in particular it may have different signs. When  $k''$  is positive, causing red shifted frequency components to travel faster than blue shifted ones, the propagation is said to take place in the *normal GVD* regime while when  $k''$  is negative giving rise to the opposite effect, the dispersion is called *anomalous GVD* and then diffraction and dispersion lead to similar behaviors on propagation.

Hence the standard formulation

$$j \frac{\partial q}{\partial z} = \frac{\rho}{2} \frac{\partial^2 q}{\partial \theta^2}, \quad (2.65)$$

with  $z$  the propagation variable which in the temporal case is assumed in the frame of reference moving with the group velocity, and  $\rho = -1/k_0$ ,  $\theta = x$  in the spatial beam propagation case and for temporal pulse propagation,  $\rho = k''$ ,  $\theta = t$  allows simultaneous formal treatment of both dispersion and diffraction effects.

## 2.2 Light propagation in nonlinear media

In the present section, through a general revision of nonlinear wave propagation problems, quadratic nonlinear effects are introduced into the propagation equation to finally arrive to the normalized form of the governing equations for the propagation in second-order media, constituting the basis of the work developed in subsequent sections and chapters.

Nonlinear effects are readily included into the above standard formulation for the propaga-

tion equation accounting for diffraction and dispersion by setting

$$j \frac{\partial q_\sigma}{\partial z} - \frac{\rho}{2} \frac{\partial^2 q_\sigma}{\partial \theta^2} = \frac{\mu_0}{2Nk_0} \frac{\partial^2}{\partial t^2} \vec{\mathcal{P}}_{NL}(\varpi_\sigma; \varpi_1, \dots, \varpi_n) . \quad (2.66)$$

Assuming that the nonlinear polarization does not display relevant dispersion effects, one has  $\partial^2 \vec{\mathcal{P}}_{NL} / \partial t^2 = -\varpi_0^2$ , and the above yields

$$j \frac{\partial q_\sigma}{\partial z} - \frac{\rho}{2} \frac{\partial^2 q_\sigma}{\partial \theta^2} = -\frac{\mu_0 \varpi_0^2}{2Nk_0} \vec{\mathcal{P}}_{NL}(\varpi_\sigma; \varpi_1, \dots, \varpi_n) . \quad (2.67)$$

Using the standard notation introduced in section [2.1.3]

$$j \frac{\partial q_\sigma}{\partial z} - \frac{\rho}{2} \frac{\partial^2 q_\sigma}{\partial \theta^2} = -\frac{\mu_0 \varepsilon_0 \varpi_0^2}{2Nk_0} \chi^{(n)}(\varpi_\sigma; \varpi_1, \dots, \varpi_n) q_1 \dots q_n e^{j\Delta k z} e^{-j\Delta \omega t} , \quad (2.68)$$

where strictly speaking the  $q_i$  at the right hand side are as well unknown functions although usually since the nonlinearity is small, the problem is simplified by considering that the solution is to first order the linear solution  $q_i^{(L)}$  and that the nonlinear corrections  $q_i^{(NL)}$  are obtained through

$$j \frac{\partial q_\sigma^{(NL)}}{\partial z} - \frac{\rho}{2} \frac{\partial^2 q_\sigma^{(NL)}}{\partial \theta^2} = -\frac{k_0}{2N} \chi^{(n)}(\varpi_\sigma; \varpi_1, \dots, \varpi_n) q_1^{(L)} \dots q_n^{(L)} e^{j\Delta k z} e^{-j\Delta \omega t} , \quad (2.69)$$

where  $\Delta \omega = -\varpi_\sigma + \varpi_1 + \dots + \varpi_n = 0$  while  $\Delta k = -k_\sigma + k_1 + \dots + k_n$ , with  $k_i$  the linear propagation constants. In general  $\Delta k \neq 0$  due both to dispersion and to boundary conditions requirements whereupon it regulates the efficiency of the nonlinear interaction in such a way that only those frequencies that verify  $\Delta k \sim 0$  are efficiently generated in the material out of the input fields. The *phase matching* requirement as it is usually known, is understood from the physical viewpoint as the enhancement of nonlinear interaction that takes place between fields that oscillate harmonically in the propagation variable as giving rise to coherent addition.

## 2.2.1 Third-order nonlinear media

### The plane wave case: nonlinear phase shift

In this section the analysis of nonlinear light propagation problems is started up by considering the simplest case in which the envelope function only depends on the propagation variable, and then the field is written

$$\vec{\mathcal{E}}(\vec{r}, t) = \frac{\hat{e}}{2} \left[ q(z) e^{jn_k z} e^{-j\omega_0 t} + c.c. \right]. \quad (2.70)$$

The nonlinear wave equation (2.47) then simply yields

$$j \frac{\partial q}{\partial z} = -\frac{k_0}{2n} \chi^{(n)}(\omega_\sigma; \omega_1, \dots, \omega_n) |q|^2 q e^{j\Delta k z} e^{-j\Delta \omega t}. \quad (2.71)$$

In a centrosymmetric material in which the nonlinear polarization to leading order is cubic in the electric field one has

$$jq_z = \mp \frac{3k_0}{8n} \chi^{(3)} |q|^2 q, \quad (2.72)$$

where the minus and plus signs correspond respectively to self-focusing and self-defocusing nonlinearities leading to

$$q(z) = A \exp \left( \pm j \frac{3k_0}{8n} \chi^{(3)} A^2 z \right). \quad (2.73)$$

Therefore, as a consequence of the nonlinearity present in the system, a monochromatic wave gets a phase shift which is known as the *nonlinear phase shift*. In this particular case since the amount of phase shift experienced by a plane wave is proportional to its own intensity given by  $I = A^2$  it is also known as *self-phase modulation (SPM)* effect.

### The 1+1 case

When dispersive or diffractive effects become relevant, the propagation velocities at each part of the pulse are impacted by the redistribution of frequencies produced by the self-phase modulation in a way well described by the following diagram, which considers a self-focusing nonlinearity:

	BACK	FRONT	$\xrightarrow{\text{propagation}}$
SPM (self-focusing)	$f \uparrow$	$f \downarrow$	
normal GVD	$v \downarrow$	$v \uparrow$	ENHANCED SPREADING
anomalous GVD or diffraction	$v \uparrow$	$v \downarrow$	COMPENSATION

The diagram may be summarized by saying that while in the normal GVD regime SPM enhances the velocity difference between front and back parts of the pulse, when the GVD is anomalous the front part tends to 'wait' for the back part while the back part hurries up to catch the front part so that compensation and even pulse shortening may take place in the presence of the nonlinearity. The spatial case in which the linear effect considered is the spreading of the beam, i.e. diffraction, is formally analogous to the case of anomalous GVD but in the physical analogy  $f$  must be replaced by  $k$  and  $v$  stands for transverse velocity at which a given ray detaches from the  $z$  axis. A dual diagram may be constructed for the self-defocusing case.

The compensation effect having place in the anomalous GVD regime is reflected by the fact that the propagation equation taking into account both dispersion and nonlinearity in this regime takes the form of the *Nonlinear Schrödinger Equation (NLSE)*, namely

$$j\bar{q}_z + \frac{\rho}{2}\bar{q}_{\theta\theta} + |\bar{q}|^2\bar{q} = 0, \quad (2.74)$$

where  $\bar{q} = \sqrt{8N/(3k_0\chi^{(3)})}$  and  $\rho = k''$  in the temporal case and  $\rho = 1/(nk_0)$  in the spatial case. The NLSE is an integrable system supporting solitary wave solutions which because of its strong stability and its particle-like behavior are usually referred to as *solitons*.

### 2.2.2 Second-order media

#### The plane wave case

A convenient starting point for the study of light propagation in quadratic nonlinear media is provided by consideration of the second harmonic generation problem in the plane wave approximation.

If the nonlinearity is small, to zeroth order the field is composed only by the injected fundamental frequency linear plane wave

$$\overrightarrow{\mathcal{E}}^{(0)} = \frac{\widehat{e}}{2} \left[ A_1 e^{jk_1 z} e^{-j\omega t} + c.c. \right]. \quad (2.75)$$

The nonlinear corrections to this field are assumed of order  $\epsilon \ll 1$ . Subindices 1 and 2 are used for quantities related respectively to the fundamental and the second harmonic frequencies while the superindex indicates the order of the solution.

Through the  $\chi^{(2)}$  nonlinearity this zeroth order field gives rise to a nonlinear polarization that recalling (2.34) can be expressed as

$$\overrightarrow{P}^{(0)}_{NL}(2\omega) = \epsilon_0 \chi^{(2)} A_1^2 e^{2jk_1 z}, \quad (2.76)$$

so that to first order the equation for the generated second harmonic is

$$\frac{\partial^2}{\partial z^2} E_2(z) + k_2^2 E_2(z) = -\mu_0 \frac{\partial^2}{\partial t^2} \overrightarrow{P}^{(0)}_{NL} = 4k_0^2 \chi^{(2)} A_1^2 e^{2jk_1 z}, \quad (2.77)$$

with solution

$$E_2^{(0)}(z) = \frac{4k_0^2}{4k_1^2 - k_2^2} \chi^{(2)} A_1^2 \left( e^{jk_2 z} - e^{2jk_1 z} \right). \quad (2.78)$$

Letting the wavevector mismatch between the harmonics be  $\Delta k = 2k_1 - k_2$ , the second harmonic generated field is more conveniently written

$$E_2^{(0)}(z) = j \frac{2k_0}{n_1 + n_2} \chi^{(2)} A_1^2 \frac{\sin\left(\frac{\Delta k}{2} z\right)}{\Delta k/2} e^{j\frac{\Delta k}{2} z} \simeq j \frac{k_0}{n_1} \chi^{(2)} A_1^2 \frac{\sin\left(\frac{\Delta k}{2} z\right)}{\Delta k/2} e^{j\frac{\Delta k}{2} z}, \quad (2.79)$$

where use has been made of the fact that in most practical situations  $n_1 \simeq n_2$ . Expression (2.79) describes the periodical exchange of energy between fundamental and second harmonic as they propagate and its dependence upon the wavevector mismatch.

At this point the initial assumption of small nonlinear contribution to the total field must be revised for in the case of very small wavevector mismatch, the second harmonic as obtained

from (2.79) taking the limit when  $\Delta k \rightarrow 0$  is

$$q_2(z) \xrightarrow{\Delta k \rightarrow 0} j \frac{k_0}{n_1} \chi^{(2)} A_1^2 z . \quad (2.80)$$

According to the above the second harmonic should be as big as the fundamental after a propagated distance  $L = n_1 / (\chi^{(2)} A_1^2 k_0)$  violating the initial assumption of weak nonlinear effects. Therefore for small wavevector mismatches the mutual interaction between fundamental and second harmonic need to be taken into account leading to a system of coupled equations.

Returning to the large phase mismatch limit, lower order nonlinear corrections at the fundamental frequency field can be shown to arise from the interaction with the generated second harmonic given by (2.79). From this interaction a nonlinear polarization at frequency  $\varpi$  is produced which verifies

$$\overrightarrow{P^{(1)}}_{NL}(\varpi) = \varepsilon_0 \chi^{(2)} \left( E_1^{(0)}(z) \right)^* E_2^{(0)}(z) . \quad (2.81)$$

Combining (2.75) with (2.79) one arrives to

$$\overrightarrow{P^{(1)}}_{NL}(\varpi) = \varepsilon_0 \left( \chi^{(2)} \right)^2 |A_1|^2 A_1 \frac{k_0}{n_1 \Delta k} \left( e^{j(k_2 - k_1)z} - e^{jk_1 z} \right) . \quad (2.82)$$

If the assumed asymptotic expansion for the field holds, this nonlinear polarization when included into the equation for the fundamental field should give rise to an  $\epsilon$ -order correction to the linear monochromatic wave at fundamental frequency, assumed the solution to zeroth-order. What it is actually obtained through this procedure rather than a  $\epsilon$ -order correction, is a term at fundamental frequency that amplifies during propagation stemming from the term  $e^{jk_1 z}$  in resonance with the assumed zeroth-order solution. That is indicative of the fact that the monochromatic plane wave solution actually does not exhaust the behavior of the linear system to zeroth-order. The contribution of the generated nonlinear polarization (2.82) must then be included within the zeroth-order solution. That is done by allowing  $\epsilon$ -order variations in the initially assumed constant amplitude,  $A_1$ , so that the field is expressed

$$\overrightarrow{\mathcal{E}^{(0)}} = \frac{\hat{e}}{2} \left[ q_1(z) e^{jk_1 z} e^{-j\varpi t} + c.c. \right] , \quad (2.83)$$

and by appropriate choice of the value of  $\partial q_1/\partial z$  being such that  $\frac{\partial q_1/\partial z}{k_1} \sim O(\epsilon)$  and hence appearing in the equation for the fundamental field to order  $\epsilon$ , one can get rid of the resonance. Substitute the new ansatz for the field at the fundamental frequency into the wave equation and get

$$\frac{\partial^2 q_1/\partial z^2}{k_1^2} + 2j \frac{\partial q_1/\partial z}{k_1} = \left(\frac{k_0}{k_1}\right)^2 \left(\chi^{(2)}\right)^2 |A_1|^2 q_1 \frac{k_0}{n_1 \Delta k} \left(1 - e^{-j\Delta k z}\right). \quad (2.84)$$

Since the wavevector mismatch is assumed moderately large, the second term at the right hand side has little effect leading to order  $\epsilon$  to the following equation

$$\frac{\partial q_1}{\partial z} = -\frac{j}{2} \frac{k_0^2}{n_1^2 \Delta k} \left(\chi^{(2)}\right)^2 |A_1|^2 q_1, \quad (2.85)$$

whose solution is

$$q_1(z) = q_1(z=0) \exp\left(-\frac{j}{2} \frac{k_0^2}{n_1^2 \Delta k} \left(\chi^{(2)}\right)^2 |A_1|^2 z\right), \quad (2.86)$$

stating that in quadratic nonlinear media for moderately large wavevector mismatches a fundamental frequency field as a result of the interaction with the second harmonic itself generates, is affected by a *nonlinear phase shift* that when compared with that occurring in third order nonlinear media is stronger not only because the  $\chi^{(2)}$  coefficients in the susceptibility expansion are larger but because the effect is proportional to its square power. That effect which has been taken to be called *Cascading* since it acts like an effective third order nonlinearity,  $\chi_{eff}^{(3)} \sim \chi^{(2)} : \chi^{(2)}$ , resulting from the successive actuation of the  $\chi^{(2)}$  nonlinearity, entails a significant reduction in the power requirements for nonlinear phase shift based devices such as for example interferometric switches. On the other side by analogy with the phenomena encountered when to the nonlinear effect dispersive or diffractive effects are added, one may expect that nonlinear compensation may take place allowing for solitary wave propagation.

### The 1+1 case

For characterization of the quadratic response when dispersive or diffractive effects are included recall the results in (2.35-2.36) and introduce them into the general equation for diffrac-



tion/dispersion (2.65) to get for the type I of wave interaction

$$j \frac{\partial (q_1)_o}{\partial z} - \frac{\rho_1}{2} \frac{\partial^2 (q_1)_o}{\partial \theta^2} = -\frac{k_0}{2N_1} \chi_{eff} (q_1^*)_o (q_2)_e e^{-j\Delta kz}, \quad (2.87)$$

$$j \frac{\partial (q_2)_e}{\partial z} - \frac{\rho_2}{2} \frac{\partial^2 (q_2)_e}{\partial \theta^2} = -\frac{k_0}{N_2} \chi_{eff} (q_1^2)_o e^{j\Delta kz}.$$

Analogously, for the type II of wave interaction it is obtained

$$j \frac{\partial (q_1)_o}{\partial z} - \frac{\rho_1}{2} \frac{\partial^2 (q_1)_o}{\partial \theta^2} = -\frac{k_0}{2N_1} \chi_{eff} (q_1^*)_e (q_2)_e e^{-j\Delta kz}, \quad (2.88)$$

$$j \frac{\partial (q_1)_e}{\partial z} - \frac{\rho_2}{2} \frac{\partial^2 (q_1)_e}{\partial \theta^2} = -\frac{k_0}{2N_2} \chi_{eff} (q_1^*)_o (q_2)_e e^{-j\Delta kz},$$

$$j \frac{\partial (q_2)_e}{\partial z} - \frac{\rho_3}{2} \frac{\partial^2 (q_2)_e}{\partial \theta^2} = -\frac{k_0}{N_3} \chi_{eff} (q_1)_o (q_1)_e e^{j\Delta kz},$$

where the subindex 1 denotes quantities related to the fundamental field whereas 2 refers to the second harmonic for the type I of wave interaction. For the type II subindices 1 and 2 indicate fundamental frequency ordinary and extraordinary waves respectively, and 3 is for the second harmonic extraordinary wave. The propagation constants are expressed  $k_\nu = N_\nu k_0$  with  $N_\nu$ , the respective effective refractive indices and  $k_0$  the propagation constant of light in vacuum,  $k_0 = \omega_0/c$ . Effective values of the nonlinear coefficients,  $\chi_{eff}$ , need to be found tabulated in every given configuration and wave interaction. Additional subindices have been used to explicitly show the vectorial nature of the wave interaction with (e) for extraordinary wave and (o) for ordinary wave. Also for type I  $\rho_1 = k_1^{-1}$ ,  $\rho_2 = k_2^{-1}$  for the spatial case and  $\rho_1 = k''(\varpi)$ ,  $\rho_2 = k''(2\varpi)$  for the temporal case. For type II  $\rho_1 = 1/k_1$ ,  $\rho_2 = 1/k_2$ ,  $\rho_3 = 1/k_3$  for the spatial case and  $\rho_1 = (k''(\varpi))_o$ ,  $\rho_2 = (k''(\varpi))_e$ ,  $\rho_3 = (k''(2\varpi))_e$  for the temporal case. Finally  $\Delta k$  stands for the wavevector mismatch between the waves so that for type I it is  $\Delta k = 2k_1 - k_2$  while  $\Delta k = k_1 + k_2 - k_3$  for type II. Steaming from its function as regulator of the efficiency of the wave interaction is the key role played by the wavevector mismatch in the features of the propagation. As a characteristic of the wave propagating structure, there exist several ways in which it may be controlled, the most relevant being illustrated in next section.

### 2.2.3 Wavevector mismatch. Concept and techniques.

For any control of the wavevector mismatch to be performed, one needs to be capable of causing controlled variations of the propagation constants in the system. One way to do it is choosing from the dispersion curve of the material two harmonic working frequencies that meet the phase matching requirements of the proposed application. That of course will only be possible for certain materials and certain requirements and one needs to dispose of tunable sources and to be ready to change the working frequencies for any phase matching requirement.

A more versatile while still simple idea, although presenting some difficulties in practice, is based on the distinct dependence of the refractive indices on temperature variations featured by some materials at different frequencies which allows obtention of practically any value of phase mismatch by temperature control without needing to change the working frequencies [102].

Other phase matching techniques take advantage of the angle dependence of the refractive index which affects the extraordinary wave in both type I and type II wave interactions to control the wavevector mismatch through the input fields propagation angle with respect to the crystal's optical axis. Concurrence of two extraordinary waves in the Type II of wave interaction makes it the preferred one because it provides two degrees of freedom in wavevector mismatch adjustment [51].

The two first methods outlined as allowing the launch of the fields in the direction of the optical axis so that the phase planes and the energy flux for the extraordinary ray follow the same direction, are very convenient to avoid the anisotropic effects in the fields propagation that appear when using the third method which in exchange offers versatility and room-temperature operation at adequate working wavelength ranges.

Still a fourth method known as *quasi-phase matching (QPM)* based upon the harmonics relative phase correction realized at regular intervals using a structural periodicity built into the nonlinear medium, can be used which combines the advantages of the previous three so that the coupling between the fundamental and second harmonic waves can be chosen to achieve non-critical phase matching at a suitable temperature range without Poynting vector walk-off between the interacting beams. Furthermore, QPM offers the possibility to select from the nonlinear quadratic response tensor the largest second-order coefficients with optimized overlap between the guided modes involved. The periodical  $\chi_{eff}^{(2)}$  variation may be produced either

by simply alternating stacks of thin plates, through periodic electric fields or by generation of proper ferroelectric domains. For a detailed description of the QPM method and its tolerances see [136] and references therein.

## 2.3 The $\chi^{(2)}$ problem: evolution equations

### 2.3.1 Normalized parameters

Looking for a nicer expression of the equations governing simultaneous fundamental and second harmonic beam propagation in nonlinear quadratic media (2.87), use is made of the following normalizations

$$\xi = \frac{z}{k_1 \eta^2}, \quad s = \frac{\theta}{\eta}, \quad (2.89)$$

$$a_1 = q_1 k_0^2 \eta^2 \sqrt{\frac{N_1}{2N_2}} \chi_{eff}, \quad a_2 = q_2 \frac{k_0^2 \eta^2}{2} \chi_{eff}, \quad (2.90)$$

with  $\eta$  a scaling factor which is usefully set to equal the input beam width. Using these normalized parameters one obtains for the type I of wave interaction

$$\begin{aligned} j \frac{\partial}{\partial \xi} a_1 - \frac{r}{2} \frac{\partial^2}{\partial s^2} a_1 + a_1^* a_2 \exp(-j\beta\xi) &= 0, \\ j \frac{\partial}{\partial \xi} a_2 - \frac{\alpha}{2} \frac{\partial^2}{\partial s^2} a_2 + a_1^2 \exp(j\beta\xi) &= 0, \end{aligned} \quad (2.91)$$

where  $r = -1$ ,  $\alpha = -k_1/k_2$  and the effective wavevector mismatch hereafter referred to as simply wavevector mismatch, is given by  $\beta = k_1 \eta^2 \Delta k$ . In practice, under most relevant experimental conditions  $|N_1 - N_2| \sim 10^{-4}$  so that  $\alpha \simeq -0.5$ .

One arrives to the same formal expression for pulse propagation through the normalizations

$$\xi = \frac{|k_1''| z}{\eta^2}, \quad s = \frac{\theta}{\eta}, \quad (2.92)$$

$$a_1 = q_1 \frac{k_0 \eta^2}{\sqrt{2N_1 N_2} |k_1''|} \chi_{eff}, \quad a_2 = q_2 \frac{k_0 \eta^2}{2N_1 |k_1''|} \chi_{eff}, \quad (2.93)$$

then  $r = \pm 1$  depending on whether the dispersion is normal or anomalous,  $\alpha = k_2''/k_1''$  and  $\beta = \eta^2 \Delta k / |k_1''|$ .

The need for a redefinition of the  $\Delta k$  parameter stems mainly from the fact that the  $\Delta k$  value gives only partial information about whether or not the conditions in the structure correspond to a efficient nonlinear interaction. This is so because each transverse wavevector component of the broad spectrum of focused beams experiences a different wavevector mismatch. The effective wavevector mismatch given by the second-harmonic generation efficiency depends as well on the diffraction/dispersion properties of the beam and for the case studied here it is conveniently given by the normalized expressions  $\beta = k_1 \eta^2 \Delta k / \beta = \eta^2 \Delta k / |k_1''|$ .

In this model the anisotropic nature of the wave interaction just enters the problem through the value of the effective nonlinear coefficients,  $\chi_{eff}$ , which are to be determined through the tables for each given setup. However, in practice the effect of anisotropy is also manifested through appearance of a walk-off effect between the waves at  $\varpi$  and at  $2\varpi$ , that is, they propagate in slightly different directions so that in a linear case they progressively separate or walk each other off. This effect is taken into account in the governing equations by including a first order derivative in the transverse variable. The equations for the type I of wave interaction finally are

$$\begin{aligned} j \frac{\partial}{\partial \xi} a_1 - \frac{r}{2} \frac{\partial^2}{\partial s^2} a_1 + a_1^* a_2 \exp(-j\beta\xi) &= 0, \\ j \frac{\partial}{\partial \xi} a_2 - \frac{\alpha}{2} \frac{\partial^2}{\partial s^2} a_2 - j\delta \frac{\partial}{\partial s} a_2 + a_1^2 \exp(j\beta\xi) &= 0. \end{aligned} \quad (2.94)$$

with  $\delta$  the normalized walk-off parameter such that  $\delta = k_1 \eta \rho$  in the spatial case,  $\delta = -\eta \rho / |k_1''|$  in the temporal case, being  $\rho$  walk-off angle or group velocity mismatch respectively.

For a rigorous derivation of equations (2.94) which takes into account the different fields polarizations and propagation modes see [85].

Equations (2.94) are for the 1+1 case but are readily extended to consider 2+1 geometries to obtain

$$\begin{aligned} j \frac{\partial a_1}{\partial \xi} - \frac{r}{2} \nabla_{\perp}^2 a_1 + a_1^* a_2 \exp(-j\beta\xi) &= 0, \\ j \frac{\partial a_2}{\partial \xi} - \frac{\alpha}{2} \nabla_{\perp}^2 a_2 - i\delta \hat{\delta} \cdot \nabla_{\perp} a_2 + a_1^2 \exp(-j\beta\xi) &= 0, \end{aligned} \quad (2.95)$$

Considering the nature of the type II of wave interaction, the normalized governing equations

yield

$$\begin{aligned}
j \frac{\partial a_1}{\partial \xi} - \frac{r_1}{2} \nabla_{\perp}^2 a_1 + a_2^* a_3 \exp(-j\beta\xi) &= 0, \\
j \frac{\partial a_2}{\partial \xi} - \frac{r_2}{2} \nabla_{\perp}^2 a_2 - i\delta_2 \widehat{\delta}_2 \cdot \nabla_{\perp} a_2 + a_1^* a_3 \exp(-j\beta\xi) &= 0, \\
j \frac{\partial a_3}{\partial \xi} - \frac{r_3}{2} \nabla_{\perp}^2 a_2 - i\delta_3 \widehat{\delta}_3 \cdot \nabla_{\perp} a_3 + a_2 a_3 \exp(j\beta\xi) &= 0,
\end{aligned} \tag{2.96}$$

where  $a_1$  and  $a_2$  stand respectively for ordinary and extraordinary fundamental waves and  $a_3$  represents the second harmonic extraordinary wave;  $\delta_2$  and  $\delta_3$  are the respective walk-off parameters describing the detachment from the  $z$  axis felt by fundamental and second harmonic extraordinary waves respectively. For the spatial case  $r_1 = -1$ ,  $r_2 = -k_1/k_2$ ,  $r_3 = -k_1/k_3$  while for the temporal case  $r_1 = \pm 1$ , accounting respectively for normal and anomalous GVD,  $r_2 = -k_2''/k_1''$ ,  $r_3 = -k_3''/k_1''$ ; and the effective wavevector mismatch is  $\beta = \eta^2 (k_1 + k_2 - k_3) / k_1''$ .

Recall that the above equations have been obtained in the assumption that mainly the propagation has  $z$  direction so that the fields  $q_i$  feature a slowly varying nature and both the SVEA and the paraxial approximation hold. Therefore care should be taken since large detachments from the  $z$  axis may render the model physically irrelevant.

Even though equations (2.95, 2.96) have been shown to hold equally both for spatial beam propagation and for the temporal pulse propagation case by mere change of the physical significance of the parameters in the equation; attending to its more pronounced relevance for the design of practical applications, very often the analysis, derivations and interpretations in physical terms will be focused on the spatial beam propagation case.

The type II of wave interaction although often more interesting from the viewpoint of practical applications leads to more complicated expressions and qualitative speaking the type I captures most of the important effects. Hence the study will be mainly centered in the type I wave interaction with only some references to the type II case. For this same reason mainly the one dimensional case is treated leaving some considerations about 2+1 geometries to the last chapter.

To sum up the light propagation problem which unless stated otherwise will be considered in this Thesis is that of monochromatic light beams traveling in a planar waveguide whose guiding medium has a large quadratic nonlinearity under conditions for type I wave interaction.

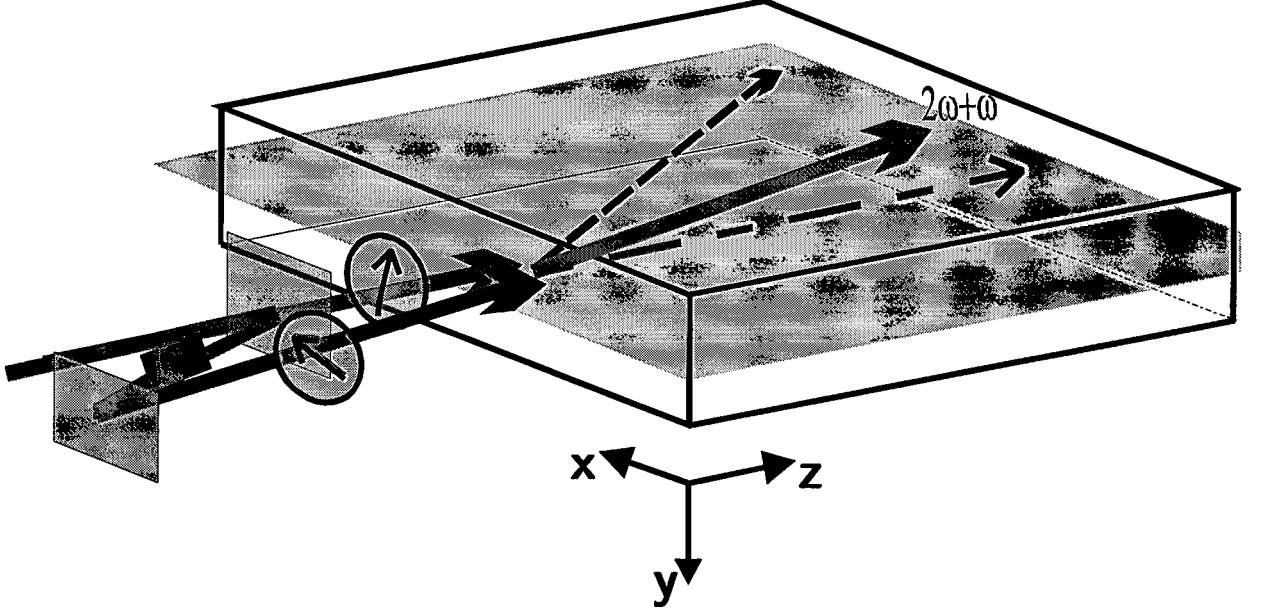


Figure 2-1: Sketch of the basic setup in which most often the investigations in the present Thesis rely. Through some considerations this simple model may be extended for treatment of more complicated setups.

The corresponding normalized governing equations (2.94) are such that the transverse modal structure of the waves along the finite transverse dimension and the specific  $\chi^{(2)}$  coefficients involved in the wave interaction are both included in the normalization of the fields through effective nonlinear coefficients. Figure 2-1 sketches such a setup.

Exponential terms in (2.94) are conveniently eliminated by using the change of variables  $\hat{a}_2 = a_2 \exp(-j\beta\xi)$  so that the normalized evolution equations are expressed

$$\begin{aligned}
 j \frac{\partial a_1}{\partial \xi} - \frac{r}{2} \nabla_{\perp}^2 a_1 + a_1^* \hat{a}_2 &= 0, \\
 j \frac{\partial \hat{a}_2}{\partial \xi} - \beta \hat{a}_2 - \frac{\alpha}{2} \nabla_{\perp}^2 \hat{a}_2 - i \delta \hat{\delta} \cdot \nabla_{\perp} \hat{a}_2 + a_1^2 &= 0.
 \end{aligned} \tag{2.97}$$

In the type II of wave interaction the change is  $\hat{a}_3 = a_3 \exp(-j\beta\xi)$  and the equations

$$\begin{aligned}
 j \frac{\partial a_1}{\partial \xi} - \frac{r_1}{2} \nabla_{\perp}^2 a_1 + a_2^* \hat{a}_3 &= 0, \\
 j \frac{\partial a_2}{\partial \xi} - \frac{r_2}{2} \nabla_{\perp}^2 a_2 - i \delta_2 \hat{\delta}_2 \cdot \nabla_{\perp} a_2 + a_1^* \hat{a}_3 &= 0,
 \end{aligned} \tag{2.98}$$

$$j \frac{\partial \hat{a}_3}{\partial \xi} - \beta \hat{a}_3 - \frac{r_3}{2} \nabla^2 \hat{a}_3 - i \delta_3 \hat{\delta}_3 \cdot \nabla_{\perp} \hat{a}_3 + a_1 a_2 = 0.$$

### 2.3.2 Conserved quantities

The equations (2.97) constitute an infinite-dimensional Hamiltonian dynamical system with hamiltonian

$$\mathcal{H} = -\frac{1}{2} \int \left\{ r |\nabla_{\perp} a_1|^2 + \frac{\alpha}{2} |\nabla_{\perp} \hat{a}_2|^2 - \beta |\hat{a}_2|^2 + j \frac{\delta}{2} \hat{\delta} \cdot (\hat{a}_2 \nabla_{\perp} \hat{a}_2^* - \hat{a}_2^* \nabla_{\perp} \hat{a}_2) + (a_1^{*2} \hat{a}_2 + a_1^2 \hat{a}_2^*) \right\} dr_{\perp}, \quad (2.99)$$

other conserved quantities in the system are the energy flow

$$I = I_1 + I_2 = \int \{ |a_1|^2 + |\hat{a}_2|^2 \} dr_{\perp}, \quad (2.100)$$

and the beam momentum

$$\vec{\mathcal{J}} = \vec{\mathcal{J}}_1 + \vec{\mathcal{J}}_2 = \frac{1}{4j} \int \{ 2(a_1^* \nabla_{\perp} a_1 - a_1 \nabla_{\perp} a_1^*) + (\hat{a}_2^* \nabla_{\perp} \hat{a}_2 - \hat{a}_2 \nabla_{\perp} \hat{a}_2^*) \} dr_{\perp}. \quad (2.101)$$

If the type II of wave interaction is considered the conserved quantities read respectively

$$\mathcal{H} = - \int \left\{ \begin{aligned} & \frac{r_1}{2} |\nabla_{\perp} a_1|^2 + \frac{r_2}{2} |\nabla_{\perp} a_2|^2 + \frac{r_3}{2} |\nabla_{\perp} \hat{a}_3|^2 - \beta |\hat{a}_3|^2 + j \frac{\delta_2}{2} \hat{\delta}_2 \cdot (\hat{a}_2 \nabla_{\perp} \hat{a}_2^* - \hat{a}_2^* \nabla_{\perp} \hat{a}_2) \\ & + j \frac{\delta_3}{2} \hat{\delta}_3 \cdot (\hat{a}_3 \nabla_{\perp} \hat{a}_3^* - \hat{a}_3^* \nabla_{\perp} \hat{a}_3) + (a_1^* a_2 \hat{a}_3 + a_1 a_2 \hat{a}_3^*) \end{aligned} \right\} dr_{\perp}, \quad (2.102)$$

$$I = I_1 + I_2 + I_3 = \int \left\{ \frac{1}{2} (|a_1|^2 + |a_2|^2) + |a_3|^2 \right\} dr_{\perp}, \quad (2.103)$$

$$\vec{\mathcal{J}} = \vec{\mathcal{J}}_1 + \vec{\mathcal{J}}_2 + \vec{\mathcal{J}}_3 = \frac{1}{4j} \int \left\{ \sum_{i=1}^2 (a_i^* \nabla_{\perp} a_i - a_i \nabla_{\perp} a_i^*) + (\hat{a}_3^* \nabla_{\perp} \hat{a}_3 - \hat{a}_3 \nabla_{\perp} \hat{a}_3^*) \right\} dr_{\perp}. \quad (2.104)$$

### 2.3.3 Limit of large wavevector mismatch

The self-focusing nature of the wave mixing process is exposed by the fact that in the large wavevector mismatch limit ( $\beta \gg 1$ ) assuming small conversion to the second harmonic ( $a_2 \ll$

$a_1$ ) one gets the interesting result

$$a_2 \simeq \frac{a_1^2}{\beta}, \quad (2.105)$$

consistent with the assumption, and

$$j \frac{\partial a_1}{\partial \xi} - \frac{r}{2} \nabla_1^2 a_1 + \frac{1}{\beta} |a_1|^2 a_1 \simeq 0, \quad (2.106)$$

the Nonlinear Schrödinger Equation which in 1+1 geometries is known to support stable soliton propagation. This impacts the properties of solitary waves found in  $\chi^{(2)}$  media while given that the NLSE in 2+1 configurations is susceptible to lead to beam collapse it poses a question about  $\chi^{(2)}$  2+1 trapping. Intuitively speaking, the difference may be understood by considering that the rate at which a self phase modulation with  $\chi^{(3)}$  origin causes a beam to focus is proportional to the intensity  $I = |E|^2$  while a  $\chi^{(2)}$  nonlinearity will cause a focusing rate proportional to the field amplitude  $|E|$ . In both cases the rate at which the beam diffracts is proportional to  $\eta^{-2}$  being  $\eta$  the beam width and since energy is conserved, the product  $I \eta^{-2}$  may be assumed to be roughly constant so that finally the beam diffraction rate can be taken to be proportional to  $I$ . Being that so, while in a  $\chi^{(3)}$  medium the rate of diffraction matches that for self-focusing, in a  $\chi^{(2)}$  medium the former will always be larger than the latter thus avoiding beam collapse.

It must be emphasized however that to date, most experimentally relevant solitons in quadratic media form for small and negative wavevector mismatches, at phase matching and in general under conditions where (2.106) does not hold. Such solitons exhibit new properties, dynamics and features and are to be treated accordingly.

### 2.3.4 Characteristic lengths and typical experimental values

Simple interpretation of the phenomena taking place on the propagation of light beams in quadratically nonlinear media is given by the characteristic lengths of the evolution in linear propagation, namely

- the *diffraction or dispersion length*

$$l_d = \frac{k_1 \eta^2}{2}, \quad (2.107)$$



giving the distance at which the wave diffracts or dispersates. Note that since  $l_{d_2} = 2l_{d_1}$ , stemming from  $\lambda_1 = 2\lambda_2$ , the second harmonic diffracts/dispersates two times more slowly than the fundamental.

- the *coherence length*

$$l_c = \frac{\pi}{|\Delta k|}, \quad (2.108)$$

characterizing the length in which the relative phase between the harmonics changes sign.

- the *walk-off length*

$$l_w = \frac{\eta}{\rho}, \quad (2.109)$$

with  $\rho$  the angle between the fields propagation direction and the optical axis, it characterizes the distance after which typically ordinary and extraordinary waves become detached from each other by a beam or pulse width. Recall that due to the assumptions made in its derivation only small walk-off angles will be correctly described by the model given by (2.94).

The system parameters can be expressed in terms on these characteristic lengths as follows

$$\delta = \pm \frac{2l_d}{l_w}, \quad \beta = \text{sgn}(\Delta k) \frac{2\pi l_d}{l_c}, \quad (2.110)$$

The mutual trapping of beams in the  $\chi^{(2)}$  crystal is governed by the interplay between these linear lengths, and the nonlinear length determined by the light intensity and by the input conditions. As a process based upon balanced effects, self-trapping is favored by the occurrence of comparable lengths into the system so that  $\beta = \pm 3$ , and  $\delta = \pm 1$ . Typical experimental conditions to yield these values, while being representative of the actual parameters involved in the experimental observation of solitons in KTP cut for Type II wave interaction [103], are beam widths of the order of  $\eta \sim 15\mu\text{m}$ ,  $\lambda \sim 1\mu\text{m}$  for the working wavelength and walk-off angles of some  $\rho \sim 0.1^\circ - 1^\circ$ . For such parameters the value of  $\delta$  falls in the range  $\delta \sim 0.15 - 1.5$ , and one needs a coherence length of some  $l_c \sim 2.5\text{cm}$ , to have  $\beta \sim \pm 3$ . For typical materials and wavelengths this coherence length corresponds to a refractive index difference of  $\Delta n \sim 10^{-4}$  between the fundamental and second harmonic waves. The above parameters yield a diffraction

length of  $l_d \sim 1mm$  so that normalized propagation distances of  $\xi \sim 20$  mean a few centimeters long waveguide, which just approaches the limit susceptible to be reached by existing technology.

To grasp the order of magnitude of the actual optical intensities associated to the normalized quantities used in the Thesis and the power levels required for the formation of solitons in quadratic nonlinear media, notice that for the typical parameters and for the nonlinear coefficients of KTP, a normalized energy flow defined as in (2.100) of some  $I \sim 50$  leads to an actual peak intensity on the order of  $\sim 10GW/cm^2$ . Existing inorganic materials (e.g. potassium niobate,  $KNbO_3$ ), semiconductors (e.g. gallium arsenide,  $AsGa$ ) and organic materials (e.g. N-4-nitrophenil-(L)-prolinol,  $NPP$ , or dimethyl amino stilbazolium tosylate,  $DAST$ ) [52], [58] with larger quadratic nonlinear coefficients would lower the power requirements. Needless to say, this is so provided the appropriate nonlinear coefficients are accessible at suitable wavelengths and phase-matching geometries with reasonable walk-off and low absorption, and provided that long enough, mechanically stable, high quality samples can be fabricated. Finally it is worth remarking that QPM techniques also allow significant power requirements reductions by addressing the largest nonlinear coefficients in the  $\chi^{(2)}$  tensor under suitable operation conditions and hence this technique holds promise for extended use in the future [136].

### 2.3.5 Physical interpretation

With the purpose of gaining some qualitative insight into the physical effects leading to the formation and propagation properties of solitons, the fields are written in the form  $a(s, \xi) = U(s, \xi) \exp(-j\psi(s, \xi))$  with  $U$  amplitude and  $\psi$  phase profiles respectively, being real functions. Substitution into the governing equations (2.94) yields by equating imaginary parts

$$U_{1\xi} = r\psi_{1s}U_{1s} + \frac{r}{2}\psi_{1ss}U_1 + U_1U_2 \sin(2\psi_1 - \psi_2 + \beta\xi), \quad (2.111)$$

$$U_{2\xi} = (\alpha\psi_{2s} + \delta)U_{2s} + \frac{\alpha}{2}\psi_{2ss}U_2 - U_1^2 \sin(2\psi_1 - \psi_2 + \beta\xi). \quad (2.112)$$

Multiply both sides of these equalities by  $2U_1$  and  $2U_2$  respectively to find for the evolution of the 'local' energy in each harmonic and in each point of the transverse beam profile

$$\frac{\partial (U_1^2)}{\partial \xi} = r(\psi_{1s}U_1^2)_s + 2U_1^2U_2 \sin(\Delta\phi), \quad (2.113)$$

$$\frac{\partial (U_2^2)}{\partial \xi} = ((\alpha\psi_{2_s} + \delta) U_2^2)_s - 2U_1^2 U_2 \sin(\Delta\phi), \quad (2.114)$$

where  $\Delta\psi = 2\psi_1 - \psi_2 + \beta\xi$  is the *nonlinear phase mismatch* between harmonics.

Physical interpretation of the above equations in the spatial case goes as follows. Clearly, the second term incorporates the effect of inter-harmonics energy exchange whereas the first term accounts for energy that travels in the transverse direction without changing its frequency, something that might be referred to as *internal transverse energy flux*. This internal transverse energy flux is such that considering bright shaped amplitude profiles, it drags energy to the positive side of the transverse coordinate (right hand side in Fig 2-1) for positive values of the corresponding phase front profile transverse derivative, whereas for negative values it drags energy to the negative side (left hand side in Fig 2-1) of the transverse coordinate. In other words, what these equations are evidencing is that variations of ‘local’ energy in one individual point of the beam transverse profile are due either to exchange with the other harmonic within the same transverse point or to exchange with neighboring transverse points within the same harmonic.

For travelling wave solutions, such that  $U(s, \xi) = U(s - v\xi)$ , since  $U_\xi = -vU_s$ , the above expressions yield

$$2U_1^2 U_2 \sin(\Delta\psi) = -((r\psi_{1_s} + v) U_1^2)_s, \quad (2.115)$$

$$2U_1^2 U_2 \sin(\Delta\psi) = ((\alpha\psi_{2_s} + (\delta + v)) U_2^2)_s, \quad (2.116)$$

indicating that for proper maintenance of the transverse travelling profile, a balance must be reached between energy dragging and interharmonics energy exchange. Picking up real terms in the governing equations for fields of the form  $a(s, \xi) = U(s, \xi) \exp(-j\psi(s, \xi))$ , one gets

$$-\frac{r}{2}U_{1_{ss}} - \left(\psi_{1_\xi} - \frac{r}{2}(\psi_{1_s})^2\right) U_1 + U_1 U_2 \cos(2\psi_1 - \psi_2 + \beta\xi) = 0, \quad (2.117)$$

$$-\frac{\alpha}{2}U_{2_{ss}} - \left(\psi_{2_\xi} - \frac{\alpha}{2}(\psi_{1_s})^2 - \delta\psi_{2_s}\right) U_2 + U_1^2 \cos(2\psi_1 - \psi_2 + \beta\xi) = 0, \quad (2.118)$$

which allow for obtention of a generic expression for the evolution of the phase profiles

$$\psi_{1_\xi} = -\frac{r}{2} \frac{U_{1_{ss}}}{U_1} + \frac{r}{2} (\psi_{1_s})^2 + U_2 \cos(\Delta\psi), \quad (2.119)$$

$$\psi_{2\xi} = -\frac{\alpha}{2} \frac{U_{2ss}}{U_2} + \frac{\alpha}{2} (\psi_{2s})^2 + \frac{U_1^2}{U_2} \cos(\Delta\psi) + \delta\psi_{2s}. \quad (2.120)$$

The first terms in these equations represent the effect of each harmonic linear diffraction whereas the last terms come from the nonlinear parametric interaction between the harmonics and for solitary wave propagation must be such that they counteract the effect of diffraction yielding a constant phase front for both harmonics. Phase variations in the first steps of propagation are mainly driven by the imbalance between the harmonics' amplitudes which results in the last terms in equations (2.119), (2.120) being the more relevant.

## 2.4 Numerical methods

Only in very few selected cases the solution to the propagation equations (2.94) can be built through analytic techniques which exposes the key importance of numerical beam propagation methods in trying to elucidate the characteristics of the beams evolution. Hence in this section a brief revision of the main beam propagation techniques susceptible of being applied to the quadratic nonlinear propagation problem is presented along with some modifications designed for a better adaptation of the numerical scheme to take into account the special features of the problem.

Numerical beam propagation methods can be cast into two main categories, namely *finite-difference* methods and *pseudospectral* methods. Generally speaking, pseudospectral methods are faster by up to an order of magnitude to achieve the same accuracy. From these the one method that has been used extensively to solve the propagation problem in nonlinear media is the *Split-step Fourier* method whose principles are very briefly outlined right away.

### 2.4.1 Split-step Fourier algorithm

To understand the philosophy behind the split-step Fourier method the propagation equation is expressed through

$$\frac{\partial a}{\partial z} = \left[ \widehat{D} + \widehat{N} \right] a, \quad (2.121)$$

where the linear operator  $\widehat{D}$  is meant to gather all linear operations while nonlinear operations are held inside the nonlinear operator  $\widehat{N}$ . The first operator is susceptible to be very easily solved

using Fourier decomposition while the second one is not. Therefore what it is proposed from the Split-step Fourier method is to obtain an approximate solution of the whole system given generically by (2.121) through separate evaluation of linear and nonlinear effects in separate steps so that mathematically next step of the propagation is obtained through

$$a(z+h) = \exp(\widehat{D}h) \exp(\widehat{N}h) a(z), \quad (2.122)$$

while the exact numerical expression should rather be

$$a(z+h) = \exp\left(\left(\widehat{D} + \widehat{N}\right)h\right) a(z), \quad (2.123)$$

but it won't allow the use of Fourier Transform.

Being  $\widehat{D}$  and  $\widehat{N}$  noncommutative operators the error committed when using (2.122) instead of (2.123) is evaluated through

$$\exp(\widehat{D}h) \exp(\widehat{N}h) = \exp\left[h\left(\widehat{D} + \widehat{N}\right) + \frac{h^2}{2}\left(\widehat{D}\widehat{N} - \widehat{N}\widehat{D}\right) + \frac{h^3}{12}\dots\right],$$

so that to order  $h^2$  the following approximation between numerically exact and split step calculations holds

$$\exp\left(\left(\widehat{D} + \widehat{N}\right)h\right) \sim \exp(\widehat{D}h) \exp(\widehat{N}h).$$

Due to cancellation of the second term in the expansion, accuracy to order  $h^3$  is obtained if the exponential operators are divided so that

$$\exp\left(\left(\widehat{D} + \widehat{N}\right)h\right) \sim \exp\left(\frac{\widehat{D}}{2}h\right) \exp(\widehat{N}h) \exp\left(\frac{\widehat{D}}{2}h\right),$$

and hence in the numerics a split-step Fourier method in symmetrized form is used.

For implementing the linear step the usual considerations regarding the characteristics of the Discrete Fourier Transform (DFT) have been taken into account. The amount of energy that results reflexed at the extrema of the transverse variable calculating window has been minimized by numerical adjust of the parameters of a damping function that has been multiplied to the field vector of samples after every numerical step. Large enough limits in the transverse variable,

$-slen < s < slen$ , need to be used both for the damping not to affect the relevant part of the propagating fields and to guarantee good spectral accuracy since the spectral step verifies  $pk = 1/(2slen)$ . Values around  $slen \sim 60 - 100$  have been found to match the requirements. Once the calculating window limits have been settled, both a large enough spectral calculating window  $-klen < k < klen = Npk$  and enough transverse variable accuracy  $ps = 2slen/N$  need to be ensured through a sufficient number of samples considered in the DFT calculation,  $N$ . Usual DFT algorithms achieve better efficiency when  $N = 2^n$  with  $n = 1, 2, 3, ..$  so that  $N = 512, 1024, 2048$  have been usual in the simulations.

As for the nonlinear step, Runge-Kutta methods give the better accuracy but being it four order enough memory room need to be ensured to hold in four samples in every step with ensuing memory reading and writing sequences. Because every step is taken out of only two previous samples a combination of Forward Cauchy, Centered Differences and Backward Cauchy methods has been used so that a trade-off is met between accuracy, speed and computer memory requirements.

## 2.5 Summary

Towards characterization of the light propagation behaviour in quadratic nonlinear structures, the basic aspects of light propagation theory have been revisited in the chapter hereby concluded. Together with long established issues about linear light propagation such as the concepts of beam diffraction and pulse dispersion, third-order nonlinear media propagation, and the properties of second-order media; some new concepts regarding solitonic propagation in quadratic nonlinear structures and new techniques developed for exploitation of its properties have been presented.

Anisotropic effects inherent to the second-order nonlinear response have been discussed along with the most common types of wave interaction encountered in practice, namely type I involving two waves, ordinary and extraordinary; and type II in which two extraordinary waves and one ordinary wave take part in the interaction.

It has been highlighted the key role played by the linear wavevector mismatch between harmonics and the most important techniques for its control, i.e. temperature and angle tuning

and Quasi-Phase-Mismatch (QPM), have been outlined.

The normalized equations governing the propagation of fundamental and second harmonic waves in a quadratic nonlinear crystal have been presented along some usual considerations that allow to obtain them such as the slowly-varying envelope approximation (SVEA) and the phenomenological treatment of anisotropic effects through use of normalized nonlinear coefficients and the walk-off parameter. Attention has been drawn to consider that on account of the hypothesis made to derive the normalized equations only moderately slow transverse travelling structures will have any physical relevance.

The conserved quantities of the evolution of the dynamical system defined by the normalized equations, i.e. the hamiltonian, energy flow and beam momentum, have been identified and its expressions provided.

The so-called NLSE limit of the system of normalized equations for large values of the effective wavevector mismatch and second harmonic amplitude very much smaller than the fundamental's has also been discussed along with its implications for soliton propagation in second-order media.

Proven very useful to understand the features of the propagation, the characteristic lengths of the evolution, namely the diffractive, coherence and walk-off lengths, have been introduced while through supply of some practical values, a feeling is intended to be given of the normalized parameters approximate range of reasonable values.

Finally, interpretation of the normalized governing equations in terms of local energy exchanges between harmonics within the same transverse point and between transverse points within the same harmonic has been presented and discussed.