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Theta correspondences and arithmetic intersections

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A mis padres y mi hermano, por su apoyo durante estos años.

Contents

Abstract

The thesis is mainly divided into two parts. In essence, the first one is an extension of the paper [Ter22]. Using the regularized Siege-Weil formula of [GQT14] we obtain an explicit expression for the truncated integral of the Siegel theta function. The main application of this result is an explicit formula for the integral of the logarithm of the Borcherds forms. The result involves different zeta values and coefficients of Eisenstein series. It completes the work of [Kud03]. Besides the aforementioned formula for the integral of the theta function, a detailed analysis of the Siegel theta function near the infinity is required.

Chapter two is an extension of the work with Antonio Cauchi in [CT]. The purpose of this part is twofold. On the one hand, under some conditions, we show that the multiplicity of the Shalika model of unramified representations for the group $\mathrm{GU}(2,2)$ is one. Using this result and following the ideas of [Sak06], we are able to find an expression of the Shalika functional in terms of the Satake parameter of a representation in GSp_4. On the other hand, we use this result and to establish a relationship between a zeta integral for a group $\mathrm{GU}(2,2)$ and a twisted standard $L-$function of GSp_4, where the relation between the involved automorphic representations is given by the theta correspondence.

KEYWORDS: Automorphic forms, special values of automorphic L-series, periods of automorphic forms, automorphic representations over local and global fields, theta series.

MSC2020: 11F03, 11F67, 11F70, 32N15, 11F27.

Resumen

Esta tesis está principalmente dividida en dos partes. En esencia, la primera parte es una extensión de la publicación [Ter22]. Gracias a la formula de Siegel-Weil regularizada de [GQT14], obtenemos una expresión explicita para la integral truncada de la función theta de Siegel. La principal aplicación de este resultado es una formula explicita para la integral del logaritmo del producto de Borcherds. El resultado obtenido consiste en valores de funciones zeta y coeficientes de series de Eisentein, completando el trabajo de [Kud03]. Además de usar la formula para la integral de la función theta anteriormente mencionada, el resultado requiere de un analisis de la función theta de Siegel cerca del infinito.

El capítulo dos es una extensión del trabajo con Antonio Cauchi en [CT]. Esta parte tiene dos objetivos. Por una parte, bajo ciertas condiciones, obtenemos que la multiplicidad del modelo de Shalika para representaciones no ramificadas del grupo $\mathrm{GU}(2,2)$ es uno. Usando este resultado y siguiendo las ideas de [Sak06], encontramos una expresion para el funcional de Shalika usando el parámetro de Satake de una representación de GSp_4. Por otra parte, usamos este resultado para establecer una relacción entre una integral zeta para el grupo $\mathrm{GU}(2,2)$ y la función L twisted standard de GSp_4, donde la relacción entre las representaciones automorfas implicadas es dada por la correspondencia theta.

KEYWORDS: Formas automorfas, valores especiales de funciones L, periodos de formas automorfas, representaciones automorfas sobre cuerpos locales y globales, funciones theta.

MSC2020: 11F03, 11F67, 11F70, 32N15, 11F27.

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Chapter 1

Borcherds forms and theta correspondences

1.1 Introduction

The theory of the theta correspondence dates back to the XX century. The classical theta functions were the first examples of classical modular forms of half-integer weight. Since these functions are defined in relation to certain representations of subgroups of the Heisenberg group, these functions have many implications in number theory, both analytic and algebraic. This introduction will illustrate a correspondence between modular forms and then we will consider a generalization to the setting of weakly holomorphic modular forms. To conclude, we will explain the generalization of these correspondences due to [How79], [How89b], [How89a], [Wei64], [Wei65].

Let (V, q) be a rational quadratic space of signature $(2, n)$ and L an integral lattice of V. The *Siegel theta function* is the vector valued function

$$\begin{aligned} \Theta_L:\ D_V \times \mathcal{H} &\to \mathbb{C}[L'/L], \\ (z, \tau) &\mapsto \sum_{\mu \in L'/L} \theta^{Sieg}(z, \tau)_\mu \mathfrak{e}_\mu, \end{aligned}$$

where D_V is the Hermitian symmetric domain associated to V, $\theta^{Sieg}(z,\tau)_\mu = \sum_{\lambda\in\mu+L} e^{2\pi i (q(\lambda_z)\tau + q(\lambda_{z^\perp})\overline{\tau})}$ and $\{e_\mu\}_{\mu\in L'/L}$ is a basis of the coset L'/L. Considering this function as a kernel integral, it defines a correspondence between vector valued modular forms and $\mathcal{C}^\infty(X_V(L))$; the $\mathcal{C}^\infty-$functions defined on the Shimura variety $X_V(L)$ associated to V with level determined by L. More concretely

$$\begin{aligned} M_{1-\frac{n}{2}, L'/L} &\to \mathcal{C}^\infty(X_V(L)), \\ f &\mapsto \int_{\mathcal{H}} \langle f(\tau), \Theta(z, \tau)\rangle \, v d\mu(\tau), \end{aligned}$$

where $d\mu(\tau) = \frac{dudv}{v^2}$ is the hyperbolic measure and $\langle\cdot,\cdot\rangle$ is the standard inner product in $\mathbb{C}[L'/L]$ defined by

$$\left\langle \sum_{\mu\in L'/L} a_\mu \mathfrak{e}_\mu, \sum_{\nu\in L'/L} b_\nu \mathfrak{e}_\nu \right\rangle = \sum_{\gamma\in L'/L} a_\gamma \bar{b}_\gamma.$$

However, the space $M_{1-\frac{n}{2}L'/L}$ is often trivial. For example when $n > 2$, we have $M_{1-\frac{n}{2}L'/L} = \{0\}$. The above correspondence was extended by Harvey and Moore to *weakly holomorphic modular forms*; vector valued modular forms with singularities on the cusps. Considering the same theta function, the integral is modified so that the divergent terms are removed and the transformation properties of the resulting function are preserved. Let us informally discuss how the integral of a weakly holomorphic modular form may diverge and hence, how is regularized. Let $f \in M^!_{1-\frac{n}{2}, L}$ be a weakly holomorphic form with Fourier expansion

$$f(\tau) = \sum_{\mu \in L'/L} \sum_{n >> -\infty} a(n, \mu) q^n \mathfrak{e}_\mu.$$

First of all, since the function $\langle f(\tau), \Theta(z, \tau) \rangle\, v$ is continuous, the convergence problems in τ arise when we are near the cusp ∞. Let us consider the following truncated fundamental domain:

$$\mathcal{F}_2^T := \{\tau = u + iv \in \mathcal{H},\ s.t.\ 1 < v < T,\ -1/2 < u < 1/2\},$$

We have

$$\int_{\mathcal{F}_2^T} \langle f(\tau), \Theta(z, \tau) \rangle\, v d\mu(\tau) = \int_1^T \sum_{\mu \in L'/L} \sum_{\lambda \in \mu + L} e^{-4\pi q(\lambda_z) v} c(\mu, q(\lambda)) \frac{dv}{v},$$

where the equality follows because we have first integrated the variable u between $-1/2$ and $1/2$. Due to the exponential behaviour of the previous function, we will divide the analysis of the divergence depending on $q(\lambda)$. When $q(\lambda) \neq 0$, the convergence of the following limit is proved directly:

$$\lim_{T \to \infty} c(\mu, q(\lambda)) \int_1^T e^{-4\pi q(\lambda_z) v} \frac{dv}{v},$$

Then, the non-constant terms of both the theta function and the weakly holomorphic modular form do not contribute to the divergence of the integral. On the contrary, when $q(\lambda) = 0$, the integral

$$c(0,0) \lim_{T \to \infty} \int_1^T \frac{dv}{v}$$

diverges if $c(0,0) \neq 0$. In order to keep track of the divergence replace the previous integral by

$$\int_1^T \frac{dv}{v^{1+s}} = \frac{c(0,0)}{s}(1 - T^s)$$

as a function of complex variable s. The previous change defines the *regularized theta lift*:

$$\Phi(f)(z) := \mathrm{CT}_{s=0} \lim_{T \to \infty} \int_{\mathcal{H}^T} \langle f(\tau), \Theta(z, \tau) \rangle\, v^{1-s} d\mu(\tau).$$

By the previous discussion it is well defined and moreover, using the integral expression for the constant term, preserves the transformation properties inherited by the theta function. The regularization process not only extends the input of the theta function, but also extends the domain. The function $\Phi(f)(z)$ is no longer $\mathcal{C}^\infty$, in fact it diverges on the subset

$$\mathcal{Z}(\gamma, m) = \sum_{\substack{\mu \in \gamma + L \\ q(\mu) = m}} \mathcal{H}_\mu \subset \mathcal{H},$$

where $\mathcal{H}_\mu := \{x \in \mathcal{H},\ s.t.\ x \perp \mu\}$. We may seen the previous subsets as divisors of the Shimura variety associated to V and L. We point out that these divisors, called *Heegner divisors*, play a relevant role in the arithmetic geometry of the Shimura varieties. For example, when $n = 1$, they are the $CM-$points of the modular curve, which are known for playing an essential role in number theory.

The $\mathcal{C}^\infty-$functions do not have a clear algebraic geometric meaning. To overcome this problem, in [Bor98] Borcherds defined a function $\Psi(f)(z) :\ D_V \to \mathbb{C}$ satisfying

$$\log|\Psi(f)(z)| = -\frac{\Phi(f)(z)}{4} - \frac{c_0(0)}{2}\left(\log|y| + \Gamma'(1)/2 + \log\sqrt{2\pi}\right).$$

It is a meromorphic modular form of weight $\frac{c(0,0)}{2}$ and level determined by the lattice L. This fact allows us to use this explicit construction to study questions related to the algebraic geometry of the Shimura variety. In fact, it can be normalized so that it is a rational section, see [Pin89].

Every classical theta correspondence between modular forms is generalized by means of the *global theta correspondence*. In fact, for certain pair of groups (G, H) (for example $(\mathrm{Sp}_n, O(V))$), there is a well defined map

$$\{cuspidal\ automorphic\ forms\ of\ H\} \to \{automorphic\ forms\ of\ G\}$$
$$f \mapsto \int_{[H]} f(h)\theta(g, h, \varphi)dh,$$

where $\theta(g, h, \varphi)$ is the so-called *theta function*, a function taking values on $G(\mathbb{A}) \times H(\mathbb{A}) \times \mathcal{S}$, where $\mathcal{S}$ is certain space of Schwartz functions. In fact, depending on the Schwartz function φ, we may recover several classical theta functions. For example, when φ is the so-called *Gaussian*, the resulting theta function (and hence the theta lift) will be intimately linked to the Siegel theta function described above. The correspondence defined by the previous integrals defines the *global theta correspondence*, a map at the level of automorphic representations:

$$\{cuspidal\ automorphic\ representations\ of\ H\} \to \{automorphic\ representations\ of\ G\}.$$

This generalization allows us to use algebraic methods to understand the theta correspondence. At the level of admissible representations of topological groups, there exists an analogous map; the so-called *local theta correspondence*. It sends admissible representations of one group to admissible representation of another. Furthermore, both constructions are defined so that there is a local-global compatibility with the global theta correspondence.

The main goal of this chapter will be to determine the integral of the logarithm of the Borcherds forms over the modular curve without level, i.e.

$$\int_{X^{mod,\hat{T}}} \log ||\Psi(f)(z)||_{Pet} d\mu(z),$$

where $f \in M^!_{-1/2,\mathbb{Z}^3}$, $||\cdot||_{Pet}$ is the Peterson norm and $d\mu$ is the hyperbolic measure. As we explained above, the function $\Psi(f)$ is closely related to the regularized theta lift, hence the computation can be reduced to

$$\int_{X^{mod,\hat{T}}} \int^{\bullet}_{X^{mod}} \theta^{Sieg}(\tau, z) f(\tau) d\mu(\tau) d\mu(z),$$

where $\int^{\bullet}_{X^{mod}}$ is the regularization proposed before. As in [Kud03], our computation requires the computation of the integral of the theta function. This computation, the so-called *Siegel-Weil* formula, has been widely studied. In fact, it relates integrals of theta functions (or regularized versions of those functions) with values of Eisenstein series (or residues and constant terms). This formula has been used for several applications. For example, it provides a relation between the theta correspondence and automorphic $L-$functions, furthermore it plays a key role to determine invariants defined by the arithmetic geometry. Recently it has been generalized to the setting of generating series, providing us new techniques to relate the $L-$functions with the geometry of Shimura varieties, see [GS19], [LZ22] and [DY19].

In addition to the Siegel-Weil formula (more concretely the regularized version of it, given by [GQT14]), our computation requires to consider the behaviour of the theta function near the cusp. As we did above with the regularized theta function, we consider the constant term separately. We unfold the resulting integral with the sum of the constant coefficient, allowing us to use the regularized Rankin-Selberg method studied in [Zag81].

This chapter is organized as follows: The first section is an introduction to the geometric setting, where we will describe the GSpin Shimura varieties and how to recover the classical modular curves from those definitions. In the second section we will study the singular theta correspondence and Borcherds products. Section 3 is devoted to giving a brief introduction to the theory of automorphic representations. This topic will be covered in detail in the following chapter. In the fourth section we will study the theory of the local and global theta correspondence, showing why are both notions compatible and further, showing how to relate the singular theta lift with the global theta lift. In section 5 we study the regularized version of the Siegel-Weil formula, showing the proof of [KR94]. Although it is not the version used in this dissertation, the proof of this theorem provides a general point of view of how these kind of problems may be approached. The final section is devoted to showing the original results of this chapter, where, using the preceding theory, we compute the integral of the Borcherds forms over the modular curve without level.

1.2 Geometric setting

1.2.1 GSpin Shimura varieties

Throughout this section we fix (V, q) a rational quadratic space of dimension m and of signature $(p^+, p^-) = (2, p)$ with $p \geq 1$. We let $(\cdot, \cdot)$ be the symmetric bilinear form associated to q. The general Spin group $\widetilde{\mathrm{H}} := \mathrm{GSpin}_V$ is defined as the central extension of the special orthogonal group $SO(V)$ that fits in the following exact sequence:

$$1 \to \mathbb{G}_m \to \widetilde{\mathrm{H}} \to SO(V) \to 1. \tag{1.0}$$

Using the quadratic form q we factor the vector space $V(\mathbb{R})$ into the direct sum of two vector spaces

$$V(\mathbb{R}) = V^+(\mathbb{R}) \oplus V^-(\mathbb{R}), \tag{1.0}$$

where $V^+(\mathbb{R})$ is a totally positive definite quadratic space and $V^-(\mathbb{R})$ is a totally negative definite quadratic space. The group $\mathrm{SO}(2) \oplus \mathrm{SO}(p)$ is a maximal compact subgroup of $\mathrm{SO}(V)(\mathbb{R})$. Following [Mil05, prop. 1.6, p. 12] we construct a Hermitian symmetric domain as follows:

$$D_V = \mathrm{SO}(V)(\mathbb{R}) / \left(\mathrm{SO}(2) \oplus \mathrm{SO}(p)\right).$$

Let us fix $\mathrm{K} < \widetilde{\mathrm{H}}(\mathbb{A}_f)$ a compact open subgroup. We define the Shimura variety associated to $\widetilde{\mathrm{H}}$ with level K by the following double quotient:

$$\mathrm{X_K} = \widetilde{\mathrm{H}}(\mathbb{Q}) \setminus D_V \times \widetilde{\mathrm{H}}(\mathbb{A}_f)/\mathrm{K}. \tag{1.0}$$

It is an scheme defined over a number field called *reflex field* [Mil05, def. 11.1, p. 107]. According to [Mil05, prop. 5.13, p. 57] this double quotient is isomorphic to

$$\bigsqcup_{g \in \mathcal{C}} \Gamma_g \setminus D_V, \tag{1.0}$$

where $\mathcal{C}$ is a set of representatives of $\widetilde{\mathrm{H}}(\mathbb{Q})_+ \setminus \widetilde{\mathrm{H}}(\mathbb{A}_f)/\mathrm{K}$ and $\Gamma_g = g\mathrm{K}g^{-1} \cap \widetilde{\mathrm{H}}(\mathbb{Q})_+$ with $\widetilde{\mathrm{H}}(\mathbb{Q})_+ = \widetilde{\mathrm{H}}(\mathbb{Q}) \cap \widetilde{\mathrm{H}}(\mathbb{R})^+$ the intersection of the rational points of $\widetilde{\mathrm{H}}$ and the identity component of the real points of $\widetilde{\mathrm{H}}$.

1.2.2 Models of the Hermitian symmetric space.

There are three different ways to realize the Hermitian symmetric domain D and we will refer to each of them as *model of the Hermitian symmetric domain.* Depending on the situation, each provides a different point of view. For further details on this topic we refer the reader to [Bru02, sec. 2.3, p. 76].

Definition 1.2.1. Let (V, q) be a quadratic space of signature $(2, p)$, we define the *Grassmanian* as

$$\mathrm{Gr}(V) := \{\textit{positive definite planes } U \subset V(\mathbb{R})\}.$$

The space $\mathrm{Gr}(V)$ is a complex variety with two connected components, see [Bru02, p. 76]. We fix one of them and, for the sake of clarity, we denote it also by $\mathrm{Gr}(V)$.

Proposition 1.2.2. $\mathrm{Gr}(V) \simeq \mathrm{SO}(V)(\mathbb{R})^+ / (\mathrm{SO}(2) \oplus \mathrm{SO}(n))$ *as* $\mathcal{C}^\infty-$*manifolds.*

Proof. We can find the proof in [Hof17, sec. 2.1.1, p.12]. □

For the forthcoming exposition we denote the projective space associated to the vector space $V(\mathbb{C})$ by

$$\mathbb{P}V(\mathbb{C}) := (V(\mathbb{C}) \setminus \{0\}) / \mathbb{C}^\times.$$

The map $\pi : V(\mathbb{C}) \setminus \{0\} \to \mathbb{P}V(\mathbb{C})$ will denote the canonical projection.

Definition 1.2.3. The *projective cone model* is defined as the following subset of $\mathbb{P}V(\mathbb{C})$:

$$\mathcal{K} := \{Z \in \mathbb{P}V(\mathbb{C}) \ s.t. \ (Z, Z) = 0, \ (Z, \overline{Z}) > 0\}.$$

By definition, $\mathcal{K}$ is an analytic open sub-variety on a projective variety of dimension $n+1$ with two connected components. We proceed as before and denote one of them by $\mathcal{K}$.

Proposition 1.2.4. *There exists an isomorphism of* $\mathcal{C}^\infty-$*manifolds of the form*

$$\mathrm{Gr}(V) \simeq \mathcal{K}.$$

Proof. Let $U \in \mathrm{Gr}(V)$, there is an orthogonal basis so that $U = \langle e_1, e_2 \rangle$. This choice determines a unique element $[Z] = [e_1 + ie_1] \in \mathcal{K}$.

Conversely, for $[Z] = [X + iY] \in \mathcal{K}$, the vectors X and Y span an element of $\mathrm{Gr}(V)$. □

Remark 1.2.5. *The projective cone model inherits the complex structure from* $\mathbb{P}V(\mathbb{C})$*. Therefore, the spaces* $\mathrm{Gr}(V)$ *and* $\mathrm{SO}(V)(\mathbb{R}) / (\mathrm{SO}(2) \oplus \mathrm{SO}(n))$ *may also be considered as complex varieties.*

The group $\mathrm{SO}(V)(\mathbb{R})$ acts on $\mathrm{Gr}(V)$ via the action on V. The action of $\mathrm{SO}(V)(\mathbb{R})$ on the projective cone is defined extending the above action by $\mathbb{C}-$linearity form V to $V(\mathbb{C})$, i.e.

$$g[Z] := [gZ],$$

for any $g \in \mathrm{SO}(V)(\mathbb{R})$.

Let us suppose there are two isotropic vectors $l, l' \in V$ so that $(l, l') = 1$. We consider the vector subspace $K(\mathbb{R}) := V(\mathbb{R}) \cap l^\perp \cap l'^\perp$. The restriction of the quadratic form to $K(\mathbb{R})$ endows this vector subspace with structure of Lorentzian space, i.e. a quadratic space of signature $(1, n-1)$.

Definition 1.2.6. The *tube domain model* is defined as the subset

$$\mathbb{H} := \{z = x + iy \in K(\mathbb{C}) \ s.t \ q(y) > 0\}.$$

This space has also two connected components, we select one of them and denote it by $\mathbb{H}$.

Proposition 1.2.7. *There is an isomorphism of complex manifolds of the form*

$$\mathbb{H} \simeq \mathcal{K}.$$

Proof. Let $z = x + iy \in K(\mathbb{C})$. In one direction, we define the following map

$$\begin{aligned} &\mathbb{H} \to \mathcal{K}, \\ &z \mapsto Z(z) := [z + l' - q(z)l]. \end{aligned}$$

Conversely, given $[Z] = [X + iY] \in \mathcal{K}$ we may find a unique representative of $[Z]$ the form $(z, 1, b)$, with $z \in K(\mathbb{C})$ and $b \in \mathbb{C}$. The assignment $[Z] \to z$ defines the inverse of the former map, obtaining the isomorphism between complex manifolds. □

The action of $\mathrm{SO}(V)(\mathbb{R})$ on $\mathcal{K}$ defines an action of $\mathrm{SO}(V)(\mathbb{R})$ on $\mathbb{H}$. More concretely, given $\sigma \in \mathrm{SO}(V)$ and $z \in \mathbb{H}$, the element σz is defined by making the following diagram commutative

$$\begin{tikzcd} \mathcal{K}^+ \arrow[r, "{[Z] \to [\sigma Z]}"] & \mathcal{K}^+ \\ \mathbb{H} \arrow[u, "{z \to [Z(z)]}"] \arrow[r, "z \to \sigma z"] & \mathbb{H} \arrow[u] \end{tikzcd}$$

1.2.3 Modular subgroups

From now on, given any integral lattice $L \subset V$, we denote its set of isometries by $\mathrm{SO}(L)$. The dual lattice of L is defined by

$$L' := \{l \in V(\mathbb{R}),\ s.t.\ (\lambda, l) \in \mathbb{Z},\ for\ all\ \lambda \in L\}.$$

Throughout this chapter we will consider the following group:

$$\Gamma_L := \{g \in \mathrm{SO}(L),\ s.t.\ g\ acts\ trivially\ on\ L'/L\}.$$

Definition 1.2.8. $\Gamma \subseteq \Gamma_L$ is a *modular subgroup* if $[\Gamma : \Gamma_L]$ is finite.

Any modular group acts on D via the isomorphisms 1.2.2 and 1.2.4. Given any $\Gamma \subseteq \Gamma_L$, the quotient

$$\mathrm{X}_{\Gamma_L} := \Gamma_L \setminus D,$$

is an algebraic variety which admits a compactification $\mathrm{X}^*_{\Gamma_L}$ via the Baily-Borel theory, see [BB66].

1.2.4 Special divisors

One of the advantages of working with GSpin Shimura varieties is that, due to the description of the Hermitian domain in terms of a quadratic space, their subvarieties and respectively their embeddings are described using linear algebra.

Fix $W \subset V$ a negative definite subspace of V dimension one. Using the projetive cone model, see proposition 1.2.3, the space

$$D_W := \{x \in D,\ s.t.\ x \perp W\}.$$

is a codimension 1 submanifold of D. Note that this definition implicitly requires propositions 1.2.2 and 1.2.4.

Definition 1.2.9. The *Heegner divisors* of D attached to $\gamma \in L'/L$ and $m \in \mathbb{Z}_{<0}$ are defined by the following sum:

$$\mathcal{Z}(\gamma, m) = \sum_{\substack{\mu \in \gamma + L \\ q(\mu) = m}} D_\mu \subset D.$$

Remark 1.2.10. *Since Γ_L preserves the sub-variety $\mathcal{Z}(\gamma, m)$, Heegner divisors are in fact algebraic divisors of the algebraic variety X_{Γ_L}. This construction may be used to describe sub-Shimura varieties of co-dimension up to n.*

1.2.5 The modular curve.

This subsection is devoted to describing the modular curve in the setting introduced in the previous sections. From now on, we fix V as the following vector space:

$$V := \{z \in M_{2\times 2}(\mathbb{Q}),\ s.t.\ \mathrm{tr}(z) = 0\}.$$

We endow V with the quadratic form given by $q(\cdot) = -2\det(\cdot)$, obtaining an isotropic quadratic space of signature $(2,1)$, which we will denote by (V,q). The tuple given by

$$e_1 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ e_2 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ e_3 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

forms an orthogonal basis of (V,q) so that $q(e_1) = q(e_2) = 2$ and $q(e_3) = -2$. Let us denote by $(\cdot,\cdot)$ the bilinear form defined by q. The projective cone model associated to (V,q) is defined by:

$$\mathcal{K} = \left\{ Z \in \mathbb{P}M_{2\times 2}(\mathbb{C}),\ s.t.\ \mathrm{tr}\,(Z) = 0,\ \mathrm{tr}\left(Z\overline{Z}\right) > 0 \right\}.$$

Let us choose

$$l := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ l' = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The Lorentzian vector space associated to V, l and l' is

$$K(\mathbb{R}) = V(\mathbb{R}) \cap l^{\perp} \cap l'^{\perp} = \langle e_1 \rangle.$$

Therefore, one connected component of the tube domain model can be realized as

$$\mathbb{H} = \{z = x + iy\ s.t.\ x, y \in \mathbb{R},\ y > 0\},$$

which is the *Poincaré half plane*, usually denoted by $\mathcal{H}$. Moreover since $\mathrm{SO}(V)(\mathbb{R}) \simeq \mathrm{SL}_2(\mathbb{R})$ there is a well defined action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$.

In this chapter we will mainly work with the modular curve without level. It is obtained taking $L = \mathbb{Z}^3$. In fact, using the description of the quadratic space given in [Fun02, (3.1), p. 295] the lattice L satisfies

$$L'/L \simeq \left(\mathbb{Z} \oplus \frac{1}{2}\mathbb{Z} \oplus \mathbb{Z}\right) / \left(\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}\right) \simeq \frac{1}{2}\mathbb{Z}/\mathbb{Z}.$$

Then, $\Gamma_L = \mathrm{SL}_2(\mathbb{Z})$, obtaining

$$SL_2(\mathbb{Z}) \setminus \mathcal{H} \simeq SL_2(\mathbb{Q}) \setminus \mathcal{H} \times SL_2(\mathbb{A}_f)/K, \tag{1.0}$$

where $K = \prod_p SL_2(\mathbb{Z}_p)$.

1.3 Complex automorphic forms

1.3.1 Singular theta lift and Borcherds products

The singular theta correspondence is an assignment between *weakly holomorphic modular forms* and functions defined over a GSpin Shimura variety with singularities over Shimura subvarieties. This construction can be viewed as an intermediate step in constructing Borcherds products.

Definition 1.3.1. The *real metaplectic group* $\mathrm{Mp}_2(\mathbb{R})$ is the following double cover of $\mathrm{SL}_2(\mathbb{R})$: the elements are given as pairs $(M, \phi(\tau))$ where $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ and $\phi(\tau)$ is a holomorphic square root of $c\tau + d \in \mathbb{C}$. Given $(M_1, \phi_1(\tau))$, $(M_2, \phi_2(\tau)) \in \mathrm{Mp}_2(\mathbb{R})$ the group law in $\mathrm{Mp}_2(\mathbb{R})$ is defined by $(M_1, \phi_1(\tau)) \cdot (M_2, \phi_2(\tau)) = (M_1M_2, \phi_1(M_2\tau)\phi_2(\tau))$.

Remark 1.3.2. *The real metaplectic group is generated by the elements*

$$S = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau}\right), \ T = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1\right).$$

For any integral lattice $L \subset V$ there exists a representation

$$\rho_L : \ \mathrm{Mp}_2(\mathbb{R}) \to \mathrm{End}(\mathbb{C}[L'/L]), \tag{1.0}$$

defined throughout the action of the generators of the metaplectic group. More concretely, fixing representatives $\{\mathfrak{e}_\mu\}$ of L'/L the representation can be realized as follows:

$$\rho_L(T)\mathfrak{e}_\mu = e^{q(\mu)}\mathfrak{e}_\mu,$$

$$\rho_L(S)\mathfrak{e}_\mu = \frac{\sqrt{i}^{n-2}}{\sqrt{|L'/L|}} \sum_{\nu\in L'/L} e^{-(\mu,\nu)}\mathfrak{e}_\nu.$$

This representation is usually called *Weil representation*. In subsequent sections we will show that it is a piece of a canonical representation also called Weil representation.

Definition 1.3.3. Let $k \in \frac{1}{2}\mathbb{Z}$. A *weakly holomorphic modular form* of weight k is a smooth function $f : \mathcal{H} \to \mathbb{C}[L'/L]$, holomorphic on $\mathcal{H}$, meromorphic on the cusp ∞ and satisfying the following relation:

$$f(\tau) = \phi(\tau)^{-2k}\rho_L^*(M, \phi(\tau))^{-1} f(M\tau),$$

where $\rho_L^*(M, \phi(\tau))$ is the dual of the Weil representation. The space of weakly holomorphic modular forms is denoted by $M^!_{k,L}$.

Remark 1.3.4. *Using the transformation law and holomorphicity of weakly holomorphic modular forms, we may express every $f \in M^!_{k,L}$ using its Fourier expansion*

$$f(\tau) = \sum_{\mu\in L'/L} \sum_{n>>-\infty} a(n,\mu) q^n \mathfrak{e}_\mu,$$

where $q := e^{2\pi i\tau}$.

By propositions 1.2.4 and 1.2.7 there is an isomorphism of the form

$$\omega :\ \mathbb{H} \to \mathrm{Gr}(V).$$

Recall that the elements of $\mathrm{Gr}(V)$ are positive definite subspaces of V. Then, given $v \in \mathrm{Gr}(V)$ we may factor the vector space V as

$$V = v \oplus v^\perp.$$

The map ω and the above factorization allows us to define the *Gaussian*:

$$\begin{aligned} \phi :\ & V(\mathbb{R}) \times \mathbb{H} \times \mathcal{H} \to \mathbb{C}, \\ & (x, z, \tau) \mapsto e^{2\pi i(q(x_z)\tau + q(x_{z^\perp})\overline{\tau})}, \end{aligned}$$

where $x_z := x_{\omega(z)}$. It is a rapidly decreasing function on $V(\mathbb{R})$.

Definition 1.3.5. The *Siegel theta function* is defined by

$$\begin{aligned} \Theta_L :\ & \mathbb{H} \times \mathcal{H} \to \mathbb{C}[L'/L], \\ & (z,\tau) \mapsto \sum_{\mu\in L'/L} \theta(z,\tau)_\mu \mathfrak{e}_\mu, \end{aligned}$$

where $\theta(z,\tau)_\mu = \sum_{\lambda\in\mu+L} e^{2\pi i(q(\lambda_z)\tau + q(\lambda_{z^\perp})\overline{\tau})}$.

Remark 1.3.6. *Since the Gaussian is rapidly decreasing, the Siegel theta function converges absolutely.*

Proposition 1.3.7. *Given $A = (\gamma, \phi(\tau)) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Mp}_2(\mathbb{Z})$, the Siegel theta function satisfies the following relation:*

$$\Theta_L(z, \gamma\tau) = \phi(\tau)\overline{\phi(\tau)}^n \rho_L(A)\Theta_L(z,\tau).$$

Proof. We can find the proof in [Bru02, thm. 2.1, p. 40]. □

Remark 1.3.8. *Set $\mathrm{O}(L) := \{g \in \mathrm{O}(V)(\mathbb{R}),\ s.t.\ gL = L\}$. The Siegel theta function $\Theta_L(z,\tau)$ is $\mathrm{O}(L)-$invariant.*

We denote by $\langle\cdot,\cdot\rangle$ the standard inner product in $\mathbb{C}[L'/L]$ defined by

$$\left\langle \sum_{\mu\in L'/L} a_\mu \mathfrak{e}_\mu, \sum_{\nu\in L'/L} b_\nu \mathfrak{e}_\nu \right\rangle = \sum_{\gamma\in L'/L} a_\gamma \bar{b}_\gamma.$$

Definition 1.3.9. Given $k\in\frac{1}{2}\mathbb{Z}$ and $f\in M^!_{k,L}$, the *singular theta lift* of f is defined by the following expression:

$$\Phi(f)(z) := \mathrm{CT}_{s=0}\lim_{T\to\infty}\int_{\mathcal{H}^T}\langle f(\tau),\Theta(z,\tau)\rangle\, v^{1-s}d\mu(\tau),$$

where $\mathcal{H}^T := \{\tau = u+iv\in\mathcal{H},\ s.t.\ v<T\}$.

For the sake of clarity, throughout the dissertation we will use the following notation:

$$\int^{\bullet}\langle f(\tau),\Theta(z,\tau)\rangle\, v d\mu(\tau) = \mathrm{CT}_{s=0}\lim_{T\to\infty}\int_{\mathcal{H}^T}\langle f(\tau),\Theta(z,\tau)\rangle\, v^{1-s}d\mu(\tau).$$

Theorem 1.3.10. *For any $k\in\frac{1}{2}\mathbb{Z}$ and $f\in M^!_{k,L}$ so that $f(\tau)=\sum_{\mu\in L'/L}\sum_{m>>-\infty}a(n,\mu)q^n\mathfrak{e}_\mu$ with $a(n,\mu)\in\mathbb{Z}$ when $n\le 0$, there exists a function*

$$\Psi(f):\ \mathbb{H}\to\mathbb{C},$$

such that

1. *$\Psi(f)$ is a meromorphic modular form of weight $c(0,0)/2$ with respect to the group Γ_L.*
2. *The divisor of $\Psi(f)$ is given by*

$$\mathrm{div}(\Psi(f)) = \sum_{\mu\in L'/L}\sum_{n<0} c(n,\mu)\mathcal{Z}(n,\mu).$$

3. *The function $\Psi(f)$ satisfies the following relation:*

$$\log|\Psi(f)(z)| = -\frac{\Phi(f)(z)}{4} - \frac{c_0(0)}{2}\left(\log|y| + \Gamma'(1)/2 + \log\sqrt{2\pi}\right),$$

where $z = x+iy$.

Proof. We can find the proof of this theorem in [Bor98, thm. 13.3, p. 48]. □

1.4 Automorphic representations

In this subsection we will briefly recall the basic theory of automorphic forms and representations, introducing crucial concepts for the forthcoming discussion. Let us denote by G a reductive group defined over $\mathbb{Q}$ and $K = K^\infty K_f$ a maximal compact subgroup of $G(\mathbb{A})$ so that K^∞ is a maximal compact subgroup of $G(\mathbb{R})$ and K_f is an open compact subgroup of $G(\mathbb{A}_f)$. Furthermore, we fix a set of simple roots Δ of G determining a Borel subgroup B. In this subsection we will consider the following group:

$$G(\mathbb{A})^1 := \cap_{\xi\in X^*(G)}\ker\left(|\cdot|\circ\xi:\ G(\mathbb{A})\to\mathbb{R}_{>0}\right).$$

1.4.1 Background

Let $i':\ G\to\mathrm{GL}_n$ be a closed embedding, we define

$$\begin{aligned} i:\ & G\to\mathrm{SL}_{2n},\\ & g\mapsto i(g) := \begin{pmatrix} i'(g) & \\ & i(g^{-1})^t\end{pmatrix}.\end{aligned}$$

Definition 1.4.1. Given $g \in G(\mathbb{A})$ and a place p of $\mathbb{Q}$, we set

$$||g||_p = \sup_{1 \le i,j \le 2n} |i(g)_{ij}|_p.$$

The *adelic norm* of g is given by

$$||g|| = \prod_{p\ place} ||g||_p.$$

Since $i(g_p) \in \mathrm{SL}_{2n}(\mathbb{Z}_p)$ for almost all p, the adelic norm is a well defined function.

Let $A_G < G(\mathbb{R})$ be the identity component of the $\mathbb{R}-$points of the maximal $\mathbb{Q}-$split torus of Z_G. For each $t \in \mathbb{R}_{>0}$ a *Siegel set* is a subset of $G(\mathbb{A})$ of the form

$$\mathcal{G}(t) := \Omega A_{T_0}(t) K, \tag{1.0}$$

where $\Omega = B(\mathbb{A}) \cap G(\mathbb{A})^1$ is a compact subset and

$$A_{T_0}(t) = \{x \in A_{T_0},\ s.t.\ |\alpha(x)| \ge t,\ for\ all\ \alpha \in \Delta\},$$

with T_0 the maximal split torus of G^{der} contained in T. For $x \in \mathcal{G}(t)$ we denote by $x = x_\Omega x_{a(t)} x_K$ the factorization given by (1.4.1). The construction of Siegel sets allows us to understand the quotient $A_G G(\mathbb{Q}) \setminus G(\mathbb{A})$. In fact, combining [GH22, (2.19) p. 59] and [GH22, thm. 2.7.2, p. 61], one has that there is a $t \in \mathbb{R}_{>0}$ satisfying

$$G(\mathbb{Q}) \setminus \mathcal{G}(t) \simeq A_G G(\mathbb{Q}) \setminus G(\mathbb{A}),$$

where the above map is an homeomorphism of topological spaces. From now on we will denote by $[G] := A_G G(\mathbb{Q}) \setminus G(\mathbb{A})$. The Siegel sets help us to control "how big" is the $\mathbb{R}-$component of each element, which mirrors the purely imaginary direction of the modular forms i.e. the only possible divergence is in this direction. In order to see the similarities between the classical and the adelic cases, we can compare the asymptotic behaviour of the classical cusp forms with the following definitions.

Definition 1.4.2. A function $\varphi : G(\mathbb{A}) \to \mathbb{C}$ is of *moderate growth* if there are constants $c, r \in \mathbb{R}_{>0}$ satisfying $|\varphi(g)| \le c||g||^r$. A function $f : [G] \to \mathbb{C}$ is *rapidly decreasing* if it is smooth and for all Siegel sets and all $r \in \mathbb{R}_{>0}$ there is a constant $c \in \mathbb{R}_{>0}$ so that $|f(x)| \le c\alpha(x_{a(t)})^{-r}$ for all $x \in \mathcal{G}(t)$ and for all $\alpha \in \Delta$.

Remark 1.4.3. *The rapidly decreasing functions decrease to zero faster than the inverse of any polynomial. Therefore, these functions are integrable and the product of a moderate growth function with a rapidly decresing function is also integrable. This property allows us to make a wider range of constructions for rapidly decreasing functions than for moderate growth functions; such as theta lifts or zeta integrals.*

Definition 1.4.4. A moderate growth smooth function $\varphi :\ G(\mathbb{A}) \to \mathbb{C}$ is an *automorphic form* on G if it is

- left $G(\mathbb{Q})-$invariant,
- $K-$finite,
- $Z(\mathfrak{g})-$finite.

We denote the space of automorphic forms by $\mathcal{A}([G])$, which is a $(\mathfrak{g}, K^\infty) \times G(\mathbb{A}_f)-$module.

Definition 1.4.5. An automorphic form f is *cuspidal* if for every parabolic P of G satisfies

$$\int_{[N]} f(ng)dn = 0,$$

where N is the unipotent radical of P. We denote the space of cusp forms by $\mathcal{A}_0([G])$, which is a $(\mathfrak{g}, K^\infty) \times G(\mathbb{A}_f)-$module.

Remark 1.4.6. *Cusp forms are rapidly decreasing functions, therefore $\mathcal{A}_0([G]) \subset L^2([G])$, see [MW08, sec. 1.2.18, p. 39] for the details.*

Definition 1.4.7. An *automorphic representation* is an irreducible $(\mathfrak{g}, K^\infty) \times G(\mathbb{A}_f)-$sub-quotient of $\mathcal{A}([G])$. A *cuspidal automorphic representation* is an irreducible $(\mathfrak{g}, K^\infty) \times G(\mathbb{A}_f)-$sub-quotient of $\mathcal{A}_0([G])$.

Remark 1.4.8. *In the general literature two different notions of automorphic representations are treated. One of them is the one stated before, see 1.4.7, and the other describes a L^2-automorphic representation as a $G(\mathbb{A})-$sub-quotient of the unitary representation $L^2([G])$. The L^2-automorphic representations are $G(\mathbb{A})-$representations. The space $\mathcal{A}_0([G])$ provides the relation between the above two concepts: both spaces $\mathcal{A}_0([G])$ and $L^2([G])$ have the same unique $[G]-$invariant bilinear form. While the space $L^2([G])$ is complete with respect to the aforementioned bilinear form, the space $\mathcal{A}_0([G])$ is not a Hilbert space. Let $L^2_{cusp}([G])$ be the subspace of $L^2([G])$ whose elements satisfy the definition 1.4.5. On the one hand, by [GH22, thm. 6.5.1, p. 150], the $K-$finite elements of $L^2_{cusp}([G])$ coincide with the cusp forms $\mathcal{A}_0([G])$, i.e.*

$$L^2([G])^{fin}_{cusp} = \mathcal{A}_0([G]). \tag{1.0}$$

On the other hand, [GH22, p. 149] shows the following factorization

$$L^2_{cusp}([G]) = \widehat{\bigoplus}_{\pi} L^2(\pi), \tag{1.0}$$

where $\widehat{\bigoplus}$ is the direct sum of Hilbert spaces, $L^2_{cusp}(\pi)$ is the $\pi-$isotypic subspace of $L^2([G])$ and the sum is over all the isomorphism classes of cuspidal automorphic representations of G in the L^2-sense. Therefore, taking the $K-$finite vectors in both sides of (1.4.8) and using (1.4.8), we obtain the first implication of the equality (1.4.8): the cuspidal representations are sub $(\mathfrak{g}, K^\infty) \times G(\mathbb{A}_f)-$modules of $\mathcal{A}([G])$. The opposite role is played by the so-called Eisenstein series, which are automorphic forms describing the sub-quotients of $\mathcal{A}([G])$ explicitly.

Theorem 1.4.9. *Given an automorphic representation π, we have*

$$\pi = \otimes_{p\ place} \pi_p,$$

where for almost all the non-archimedian places p, π_p is an unramified representation. Furthermore if π is a cuspidal automorphic representation, the representations π_p are unitary in the sense of 2.3.6

Proof. For the first statement we refer the reader to [GH22, thm. 5.7.1, p. 135]. Using the equalities (1.4.8) and (1.4.8) we obtain the second statement, see [Fla79, thm. 4, p. 182] for more details. □

In the classical setting, the Fourier expansion is a tool that provides a way to work with modular forms in a more explicit way. Furthermore, it encodes relevant arithmetic information; many results and conjectures concerning the properties of the Fourier coefficients have provided the ground for modern number theory, for example see [Pet30] or [KRY06]. In the adelic case there is an analogous and compatible definition.

Definition 1.4.10. Let P be a parabolic subgroup of G, U its maximal unipotent subgroup and $\psi : [U] \to \mathbb{C}^1$ a unitary character. For any $\varphi \in \mathcal{A}([G])$, its *Fourier coefficient* is defined by

$$W_\psi(\varphi, g) = \int_{[N]} \varphi(ug)\overline{\psi(u)}du,$$

where du denotes the Haar measure of $U(\mathbb{A})$.

Proposition 1.4.11. *For any parabolic P with maximal unipotent subgroup U we have*

$$\varphi(g) = W_{id}(\varphi, g) + \sum_{\psi\colon [U]\to\mathbb{C}^1} W_\psi(\varphi, g) + \sum_{\psi^{(1)}\colon [U^{(1)}]\to\mathbb{C}^1} W_{\psi^{(1)}}(\varphi, g) + ... + \sum_{\psi^{(n)}\colon [U^{(n)}]\to\mathbb{C}^1} W_{\psi^{(n)}}(\varphi, g),$$

where $U^{(1)} = U$, $U^{(i+1)} = [U^{(i)}, U^{(i)}]$ and n is the smallest integer satisfying $[U^{(n-1)}, U^{(n-1)}] = id$.

Proof. We refer the reader to [FGKP18, thm. 7.18, p. 132]. □

Remark 1.4.12. *The non vanishing of $W_\psi(\varphi, g)$ defines an injective intertwining map of the form*

$$\pi \to \mathrm{Ind}_{U(\mathbb{A})}^{G(\mathbb{A})}\psi.$$

Realizing automorphic representations inside induced representations as described above allows us to obtain information about these representations and their $L-$functions. An example of this phenomenon is studied in detail in subsection 2.6.20.

1.4.2 Eisenstein series

One of the most relevant examples of automorphic forms are the so-called Eisenstein series. Those functions are the harmonic counterpart of the cusp forms: *the cusp forms are part of the discrete spectrum and the Eisenstein series are part of the continuous spectrum* see [GH22, (9.5), p. 215]. Summarizing the previous statement in a few words, in the case of classical modular forms, both cusp forms and the Eisenstein series are eigenfunctions of the Laplacian. The difference between those families of functions is that on the one hand there is a continuous sequence of Eisenstein series $\{E_r\}$ so that their eigenvalues $\{s_r\}$ define a continuous sequence of real numbers. In contrast, the eigenvalues $\{s_i\}$ of any sequence of cusp forms $\{\varphi_i\}$ form a discrete subset of $\mathbb{R}$. The Eisenstein series are also relevant for many other reasons; for example their explicit and continuous construction allows us to define the zeta integrals. These functions are used to provide the analytic continuation of the $L-$functions in many cases. Another remarkable property that will be exploited throughout this dissertation is the relation with the theta functions via the so-called *Siegel-Weil formula*.

Let P be a parabolic subgroup of G and π a cuspidal automorphic representation or a character of $P(\mathbb{A})$.

Definition 1.4.13. For any $\Phi(g) \in \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}\pi$ we define an *Eisenstein series* by

$$E(g, \Phi) = \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} \Phi(\gamma g).$$

At a later stage we will define the Eisenstein series depending on a complex number s to stress out the continuous construction of them (in fact, this might be seen as taking a continuous sequence of representations).

Remark 1.4.14. *If the Eisenstein series is absolutely convergent, it determines an intertwining map of the form*

$$E(g, \cdot) : \ \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}\pi \to \mathcal{A}([G]).$$

In general, the representation $\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}\pi$ is not irreducible and the image of $E(g, \cdot)$ is not an automorphic representation. If $\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}\pi$ has an irreducible quotient denoted by I and there is a map A which factors throughout I, the representation $\mathrm{im}\ A|_I$ is an automorphic representation. The previous picture may be summarized in the following diagram:

$$\begin{tikzcd}
\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}\pi \arrow[r, "E(g,\cdot)"] \arrow[d] & \mathcal{A}([G]) \\
I \arrow[r, "A"] & \mathrm{im}\ A|_I \arrow[u, "\subset"']
\end{tikzcd}$$

The map A is realized via residues or constant terms of Eisenstein series. See section 1.6.4, where explicit examples of this phenomenon are considered.

Example 1.4.15. *The classical Eisentein series can be interpolated using the previous discussion. In fact, set*

$$B(\mathbb{A}) = \left\{ \begin{pmatrix} v & \\ & v^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix}, \ v \in \mathbb{A}^\times, \ u \in \mathbb{A} \right\},$$

the $\mathbb{A}-$*points of the upper triangular Borel subgroup of* SL_2*. Using the Iwasawa decomposition we set a character*

$$\begin{aligned} \chi_s : \ \mathrm{SL}_2(\mathbb{Q}) \setminus \mathrm{SL}_2(\mathbb{A}) &\to \mathbb{C}^\times, \\ g = bk &\mapsto |v|^{2s}, \end{aligned}$$

for each $s \in \mathbb{C}$*. Given* $w \in \mathbb{Z}$*, we fix a factorizable section of the form*

$$f_{s,w} = f_{s,\infty,w} \otimes_{p \nmid \infty} f_{s,p} \in \mathrm{Ind}_{B(\mathbb{A})}^{G(\mathbb{A})} \chi_s,$$

so that $f_{s,\infty,w}(b_\infty) = \chi_{s,\infty}(b_\infty)$*,* $f_{s,\infty,w}(k_\infty g_\infty) = e^{iw\theta} f_{s,\infty}(g_\infty)$ *for* $k_\infty = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ *and* $f_{s,p}(b_p k_p) = \chi_{s,p}(b_p)$ *for every non-archimedean place* p *and* $k_p \in \mathrm{SL}_2(\mathbb{Z}_p)$*. Let* $\tau = u + iv \in \mathcal{H}$*, the element* $g_\tau = \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \begin{pmatrix} v & \\ & v^{-1} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{A})$ *satisfies*

$$E(g_\tau, f_{s,w}) = e^{iw\theta} \sum_{(c,d)=1} \frac{v^s}{(c\tau + d)^w |c\tau + d|^{2s-w}}.$$

Therefore, the adelic definition of the Eisenstein series for SL_2 *may be seen as an interpolation between the holomorphic and non holomorphic classical Eisenstein series.*

1.5 Theta correspondence

The theta correspondence allows us to define automorphic representations of a group G from automorphic representations of another group H. This construction fits into the local-global principle of 1.4.9. While the global theta correspondence has a purely analytic construction, the local theta correspondence is defined in terms of representations. Therefore, the ideas used to approach problems in both correspondences are of a different nature. The connection between these two methods comes from the Howe duality conjecture.

In this section we fix a rational symplectic vector space $(W, \langle \cdot, \cdot \rangle)$ of dimension $2n$ and a quadratic space (V, q) of dimension m. The basis $e_1, ..., e_n, e_1^*, ..., e_n^*$ of W so that $\langle e_i, e_j^* \rangle = \delta_{ij}$ and $\langle e_i, e_j \rangle = \langle e_i^*, e_j^* \rangle = 0$ for $1 \leq i, j, \leq n$ defines the following isomorphism of algebraic groups:

$$\mathrm{Sp}(W) \simeq \mathrm{Sp}_{2n}.$$

For any place p we fix $K_p = \mathrm{Sp}(W)(\mathbb{Z}_p)$, the maximal compact open subgroup of $\mathrm{Sp}(W)(\mathbb{Q}_p)$.

1.5.1 Metaplectic group and Weil representation

This subsection is devoted to treating a very short overview of the construction of the metaplectic group and of the Weil representation. In spite of the fact that this dissertation is not strictly necessary for the main objectives of the thesis, it is essential to understand why the metaplectic group must be introduced to define the theta correspondence. For a detailed discussion of the topic we refer the reader to [Swe90, chap. 1, p. 10], [Kud96, chap. 1, p. 3] and [IP21, sec. 3.2, p. 35].

Let $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ be a $\mathbb{C}-$valued smooth function of variable $(x_1, .., x_n) \in \mathbb{R}^n$. Given $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$, we denote by

$$(D^\alpha f)(x_1, ..., x_n) := \left(\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} ... \frac{\partial^{\alpha_n}}{\partial x_1^{\alpha_n}} f \right) (x_1, ..., x_n).$$

Furthermore

$$x^\alpha := x_1^{\alpha_1} ... x_n^{\alpha_n},$$

is the product of $\alpha-$powers of the coordinate functions.

Definition 1.5.1. A *Schwartz function* on $\mathbb{R}^n$ is a smooth function $\varphi : \mathbb{R}^n \to \mathbb{C}$ so that for every $\alpha, \beta \in \mathbb{N}^n$

$$\sup_{x\in\mathbb{R}^n} \left| x^\alpha (D^\beta f)(x) \right| < \infty.$$

The $\mathbb{C}-$vector space of Schwartz functions on $\mathbb{R}^n$ is denoted by $\mathcal{S}\left(\mathbb{R}^n\right)$.

Definition 1.5.2. For any $p \in \mathbb{N}$, a function $\varphi : \mathbb{Q}_p^n \to \mathbb{C}$ is a *Schwartz function* on $\mathbb{Q}_p$ if it is locally constant and compactly supported. We denote the $\mathbb{C}-$vector space of $p-$adic Schwartz functions by $\mathcal{S}(\mathbb{Q}_p^n)$.

Definition 1.5.3. Given W a symplectic vector space, we say that $W = X \oplus Y$ is a *polarization* of W if X and Y are maximally isotropic subspaces of W.

For the time being, we fix a place p of $\mathbb{Q}$ and a polarization $W_p = X_p \oplus Y_p$ of the $p-$adic symplectic space $(W_p := W \otimes_{\mathbb{Q}} \mathbb{Q}_p, \langle \cdot, \cdot \rangle)$. The *Heisenberg group* associated to W_p is defined by

$$\mathrm{H}(W_p) := W_p \oplus \mathbb{Q}_p,$$

with multiplication law of the form $(w_1, t_1) \cdot (w_2, t_2) := \left(w_1 + w_2, t_1 + t_2 + \frac{1}{2} \langle w_1, w_2 \rangle\right)$. The Stone, von Neumann theorem [Kud96, thm 1.1, p. 3] shows that the irreducible smooth representations of $\mathrm{H}(W_p)$ are classified up to isomorphism by their non trivial central characters. Fix a unitary character $\psi : \mathbb{Q}_p \to \mathbb{C}^1$ and extend it to $\mathrm{H}(X_p)$ by $\psi_{X_p}(w, t) := \psi(t)$. It is straightforward that the unitary representation

$$\begin{aligned} \rho :\ \mathrm{H}(W_p) &\to \mathrm{Aut}\left(\mathrm{Ind}_{\mathrm{H}(X_p)}^{\mathrm{H}(W_p)} \psi_{X_p}\right) \simeq \mathrm{Aut}\left(\mathcal{S}(Y_p)\right), \\ (w, t) &\mapsto \Big[f(v, s) \mapsto f\left((w, t) \cdot (v, s)\right) \Big], \end{aligned}$$

has central character ψ. The group $\mathrm{Sp}(W_p)(\mathbb{Q}_p)$ acts on the representation ρ by means of the action of $\mathrm{Sp}(W_p)(\mathbb{Q}_p)$ on $\mathrm{H}(W_p)$, i.e. given $g \in \mathrm{Sp}(W_p)(\mathbb{Q}_p)$ and $(w, t) \in \mathrm{H}(W_p)$ the action is defined by $g \cdot \rho(w, t) := \rho(gw, t)$. According to [Kud96, p. 4]

$$\begin{aligned} {}^g\rho :\ \mathrm{H}(W_p) &\to \mathrm{Aut}(\mathcal{S}(Y_p)), \\ (w, t) &\mapsto g \cdot \rho(w, t), \end{aligned}$$

is a representation of $\mathrm{H}(W_p)$ with central character ψ. Then by the previously mentioned Stone, von Neumann theorem, there exists a unitary operator $A(\cdot)$ so that

$$ {}^g\rho = A(g) \rho A(g)^{-1}. \tag{1.-1}$$

The map A is unique up to a non-zero constant and then $A : \mathrm{Sp}(W_p)(\mathbb{Q}_p) \to \mathrm{Aut}(\mathcal{S}(Y_p))$ is a projective representation. By [Swe90, def. 1.6.7, p. 35], the map A may be refined to a projective representation of the form

$$\mathrm{Sp}(W_p)(\mathbb{Q}_p) \to \mathrm{Aut}(\mathcal{S}(Y_p))/\pm 1. \tag{1.-1}$$

The main goal of the theory of the Weil representation is to overcome the $\pm 1-$indeterminacy. To that end, we will construct an assignment of the form

$$\begin{aligned} \mathrm{Sp}(W_p)(\mathbb{Q}_p) &\to \mathrm{Aut}(\mathcal{S}(Y_p)), \\ g &\mapsto \omega_p(g) \end{aligned}$$

satisfying a cocycle condition, i.e. for any $g_1, g_2 \in \mathrm{Sp}(W_p)(\mathbb{Q}_p)$ (1.5.1) must satisfy

$$\omega_p(g_1 g_2) = \tilde{c}(g_1, g_2) \omega_p(g_1) \omega_p(g_2),$$

with $\tilde{c}(g_1, g_2) \in \pm 1$, see [Swe90, lem. 1.6.7, p. 35] and [Kud96, prop. 2.3, p. 8] for the details of the construction of ω_p. This cocycle condition is required because when we consider the group $\mathrm{Sp}(W_p)(\mathbb{Q}_p) \times \pm 1$

with multiplication law $(g_1,\xi_1)\cdot(g_2,\xi_2) = (g_1g_2, \xi_1\xi_2\tilde{c}(g_1,g_2))$, the map

$$\begin{aligned}\omega_p :\ \mathrm{Sp}(W_p)(\mathbb{Q}_p)\times\{\pm 1\} &\to \mathrm{Aut}(\mathcal{S}(Y_p)),\\ (g,\xi) &\mapsto \Big[\varphi \mapsto \xi\omega_p(g)\varphi\Big],\end{aligned}$$

is a representation. In fact, the multiplication law of $\mathrm{Sp}(W_p)(\mathbb{Q}_p)\times \pm 1$ and the cocycle condition are compatible i.e.

$$\begin{aligned}\omega_p\left((g_1,\xi_1)\cdot(g_2,\xi_2)\right)\varphi &= \omega_p\left(g_1g_2,\xi_1\xi_2\tilde{c}(g_1,g_2)\right)\varphi\\ &= \xi_1\xi_2\tilde{c}(g_1,g_2)\omega_p(g_1g_2)\varphi = \omega_p(g_1,\xi_1)\omega_p(g_2,\xi_2)\varphi.\end{aligned}$$

In addition, the definition of the map ω_p involves the construction of a topology compatible with the one of $\mathrm{Sp}(W_p)(\mathbb{Q}_p)$, making $\mathrm{Sp}(W_p)(\mathbb{Q}_p)\times\pm 1$ a twofold cover of $\mathrm{Sp}(W_p)(\mathbb{Q}_p)$. Summarizing the above discussion, the theory of the Weil representation constructs a compatible topological group refinement of the map A and an extension of the topological group $\mathrm{Sp}(W_p)(\mathbb{Q}_p)$. This allows us to study the intertwining maps between representations of the form (1.5.1) from a representation theoretic perspective.

Definition 1.5.4. Let

$$\widetilde{\mathrm{Sp}}(W_p)(\mathbb{Q}_p) := \{(g,s),\ s.t.\ g\in \mathrm{Sp}(W_p)(\mathbb{Q}_p)\times U(\mathcal{S}(Y_p)),\ s=\pm\omega_v \text{ given by (1.5.1)}\},$$

where $U(\mathcal{S}(Y_p))$ is the set of unitary operators of $\mathcal{S}(Y_p)$. We endow $\widetilde{\mathrm{Sp}}(W_p)(\mathbb{Q}_p)$ with multiplication law given by $(g_1,s_1)\cdot(g_2,s_2) = (g_1g_2, s_1s_2)$. Furthermore, we regard this group as a topological group with the product topology, $\mathrm{Sp}(W_p)(\mathbb{Q}_p)$ is endowed with the topology inherited from the local field $\mathbb{Q}_p$ and the strong operator topology on $U(\mathcal{S}(Y_p))$. The resulting topological group is called the *metaplectic group*.

Remark 1.5.5. *The metaplectic group $\widetilde{\mathrm{Sp}}(W_p)(\mathbb{Q}_p)$ is a twofold topological cover of $\mathrm{Sp}(W_p)(\mathbb{Q}_p)$, where the projection map is given by*

$$\begin{aligned}\widetilde{\mathrm{Sp}}(W_p)(\mathbb{Q}_p) &\to \mathrm{Sp}(W_p)(\mathbb{Q}_p),\\ (g,s) &\mapsto g.\end{aligned}$$

Definition 1.5.6. We define the group $\mathrm{Mp}(W_p)(\mathbb{Q}_p)$ as $\mathrm{Sp}(W_p)(\mathbb{Q}_p)\times\mu_2$ with multiplication law

$$(g_1,\xi_1)\cdot(g_2,\xi_2) = (g_1g_2,\xi_1\xi_2\tilde{c}(g_1,g_2)).$$

Remark 1.5.7. *The cocycle condition (1.5.1) yields the group isomorphism*

$$\begin{aligned}t:\ \mathrm{Mp}(W_p)(\mathbb{Q}_p) &\to \widetilde{\mathrm{Sp}}(W_p)(\mathbb{Q}_p),\\ (g,\xi) &\mapsto (g,\xi\omega_p(g)).\end{aligned}$$

If we endowed $\mathrm{Mp}(W_p)(\mathbb{Q}_p)$ with the product topology, the map t would not be continuous. Hence, we consider $\mathrm{Mp}(W_p)(\mathbb{Q}_p)$ with the pullback topology, allowing us to regard $\mathrm{Mp}(W_p)(\mathbb{Q}_p)$ as a topological group. It is clearly isomorphic to $\widetilde{\mathrm{Sp}}(W_p)(\mathbb{Q}_p)$ as a topological group. The algebraic description of $\mathrm{Mp}(W_p)(\mathbb{Q}_p)$ is simpler than the one of $\widetilde{\mathrm{Sp}}(W_p)(\mathbb{Q}_p)$ and more useful for certain manipulations. Therefore, from now on we will use the topological group isomorphism (1.5.7) without any comment, referring to $\mathrm{Mp}(W_p)(\mathbb{Q}_p)$ as the local metaplectic group.

Remark 1.5.8. *The local metaplectic group $\mathrm{Mp}(W_p)(\mathbb{Q}_p)$ fits in a short exact sequence of the form:*

$$1\to\{\pm 1\}\to \mathrm{Mp}(W_p)(\mathbb{Q}_p)\to \mathrm{Sp}(W_p)(\mathbb{Q}_p)\to 1.$$

Lemma 1.5.9. *The cocylce $\tilde{c}(\cdot,\cdot)$ evaluated on elements of K_p is trivial, i.e. the morphism*

$$\begin{aligned}K_p &\to \mathrm{Mp}(W_p)(\mathbb{Q}_p),\\ k_p &\to (k_p,1),\end{aligned}$$

is an isomorphism onto its image.

Proof. See [Swe90, p. 39]. □

Remark 1.5.10. *The previous lemma allows us to consider the subgroups K_p as subgroups of the metaplectic group.*

Definition 1.5.11. The *local Weil representation* is defined as follows:

$$\begin{aligned}\omega_p : \ \mathrm{Mp}(W_p)(\mathbb{Q}_p) &\to \mathrm{Aut}(\mathcal{S}(Y_p)),\\ (g,\xi) &\mapsto \xi\omega_p(g).\end{aligned}$$

We point out that the Weil representation ω_p depends on the unitary character ψ we chose at the beginning of the subsection. Since this dependence will not be relevant in our dissertation, we avoid it from the notation and we will always consider the following characters:

Definition 1.5.12. Throughout this exposition we fix the character $\psi_\infty(x) = e^{2\pi i x}$ for the archimedian place and the characters $\psi_p : \mathbb{Q}_p \to \mathbb{C}^1$ such that whose restriction to $\mathbb{Z}_p$ is trivial and $\psi(p^{-1}) = e^{\frac{-2\pi i}{p}}$ for the $p-$adic places.

1.5.2 P-adic points of the Siegel parabolic subgroup

We denote by P the Siegel parabolic subgroup of $\mathrm{Sp}(W_p)$, i.e. the standard maximal parabolic subgroup fixing the subspace spanned by $e'_1, ..., e'_n$. See the introduction of section 1.5 to recall the notation. This parabolic subgroup has a Levi decomposition $P = MN$ so that

$$M(\mathbb{Q}_p) = \left\{ m(a) := \left(\begin{smallmatrix} a & \\ & {}^t a^{-1}\end{smallmatrix}\right),\ s.t.\ a \in \mathrm{GL}_n(\mathbb{Q}_p)\right\},$$

$$N(\mathbb{Q}_p) = \left\{ n(b) := \left(\begin{smallmatrix} 1 & b \\ & 1\end{smallmatrix}\right),\ s.t.\ b =^t b \in \mathrm{M}_n(\mathbb{Q}) = \mathrm{Sym}_n(\mathbb{Q}_p)\right\}.$$

For a non-archimedian place p, the maximal compact open subgroup of $\mathrm{Sp}(W_p)(\mathbb{Q}_p)$ is given by $K_p := \mathrm{Sp}(W_p)(\mathbb{Z}_p)$. The Iwasawa decomposition of $\mathrm{Sp}(W_p)$ allows us to factor the group as

$$\mathrm{Sp}(W_p)(\mathbb{Q}_p) = P(\mathbb{Q}_p)K_p.$$

According to the above factorizations we write the elements $g_p \in \mathrm{Sp}(W_p)(\mathbb{Q}_p)$ as products of the form

$$g_p = m(a_p)n(b_p)k_p, \tag{1.-3}$$

with $a_p \in \mathrm{GL}_n(\mathbb{Q}_p)$ and $b_p \in \mathrm{Sym}_n(\mathbb{Q}_p)$. Throughout this paper we will denote

$$|a(g_p)| := |\det(a)|_p.$$

For the following discussion $\widetilde{P}(\mathbb{Q}_p)$ will be the inverse image of $P(\mathbb{Q}_p)$ under the exact sequence given in remark 1.5.8.

Lemma 1.5.13. *There is a unique homomorphism of the form $j : N(\mathbb{Q}_p) \to \widetilde{P}(\mathbb{Q}_p)$ lifting the natural inclusion $N(\mathbb{Q}_p) \to P(\mathbb{Q}_p)$ and that it is normalized by $P(\mathbb{Q}_p)$.*

Proof. See [MVW87, lem. p. 43]. □

Remark 1.5.14. *The above lemma provides a decomposition of the group $\widetilde{P}(\mathbb{Q}_p)$ of the form*

$$\widetilde{P}(\mathbb{Q}_p) = \widetilde{M}(\mathbb{Q}_p)N(\mathbb{Q}_p),$$

where $\widetilde{M}(\mathbb{Q}_p)$ is the inverse image in $\mathrm{Mp}(W_p)(\mathbb{Q}_p)$ of $M(\mathbb{Q}_p)$ under the exact sequence given in remark 1.5.8. The group $\widetilde{M}(\mathbb{Q}_p)$ satisfies the isomorphism

$$\widetilde{M}(\mathbb{Q}_p) \simeq \widetilde{\mathrm{GL}}_n(\mathbb{Q}_p),$$

where $\widetilde{\mathrm{GL}}_n(\mathbb{Q}_p)$ *is the twofold topological cover of* $\mathrm{GL}_n(\mathbb{Q}_p)$*: a topological group with underlying set* $\mathrm{GL}_n(\mathbb{Q}_p) \times \{\pm 1\}$ *and group law*

$$(g_1, \eta_1) \cdot (g_2, \eta_2) = \Big(g_1 g_2, \eta_1 \eta_2 \left(\det g_1, \det g_2\right)_p \Big),$$

where $(\cdot, \cdot)_p$ *is the Hilbert symbol. The determinant map lifts to a homomorphism of the form*

$$\begin{aligned} \widetilde{\det} :\ \widetilde{\mathrm{GL}}_n(\mathbb{Q}_p) &\to \widetilde{\mathrm{GL}}_1(\mathbb{Q}_p), \\ (g_p, \eta) &\to (\det(g_p), \eta). \end{aligned}$$

In [RR93, Appendix p. 365], the author defines the Weil index $\gamma_p(g_p, \psi_p)$: a function depending on $a_p \in \mathbb{Q}_p^\times$ and on an additive character ψ_p. This object gives rise to the genuine character

$$\begin{aligned} \chi_{\psi_p} :\ \widetilde{\mathrm{GL}}_1(\mathbb{Q}_p) &\to \mathbb{C}^\times, \\ (a_p, \eta) &\mapsto \eta \gamma_p(a_p, \psi_p)^{-1}. \end{aligned}$$

Definition 1.5.15. Let (V, q) be a rational quadratic space of dimension m. We define a character $\chi_{V,p} : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ by

$$\chi_{V,p}(x_p) := \Big(x_p, (-1)^{m(m-1)/2} \det(V) \Big)_p.$$

Definition 1.5.16. Let $\mathbf{M}(\mathbb{Q}_p)$ be either the group $M(\mathbb{Q}_p)$ or $\widetilde{M}(\mathbb{Q}_p)$. We define the character $\tilde{\chi}_{V,p} : \mathbf{M}(\mathbb{Q}_p) \to \mathbb{C}^\times$ by

$$\tilde{\chi}_{V,p} = \begin{cases} g_p \mapsto \chi_{V,p} \circ \det(g_p), & \text{if } \mathbf{M}(\mathbb{Q}_p) = M(\mathbb{Q}_p), \\ (p, \eta) \mapsto (\chi_{\psi_p} \circ \widetilde{\det})(g_p, \eta)(\chi_{V,p} \circ \det)(g_p), & \text{if } \mathbf{M}(\mathbb{Q}_p) = \widetilde{M}(\mathbb{Q}_p). \end{cases}$$

Remark 1.5.17. *The representation* $\tilde{\chi}_{V,p} :\ \widetilde{M}(\mathbb{Q}_p) \to \mathbb{C}^\times$ *is a genuine representation of* $\widetilde{M}(\mathbb{Q}_p)$*. In fact, every genuine representation of* $\tilde{M}(\mathbb{Q}_p)$ *is constructed in this way, see [GS12, sec. 2.4, p. 1661] for further details.*

1.5.3 Local theta correspondence

Given an irreducible admissible representation π of a non-archimedean topological group H, the local theta correspondence allows us to construct an irreducible admissible representation of certain topological group G. Throughout this subsection we will only consider the case where p is a finite place. There is an analogous construction for $(\mathfrak{g}, K^\infty)-$modules, see [How89b].

Definition 1.5.18. A *dual reductive pair* is a tuple of topological subgroups (G, G') of $\mathrm{Sp}(W_p)(\mathbb{Q}_p)$ satisfying

1. $G = C_{\mathrm{Sp}(W_p)(\mathbb{Q}_p)} G'$ and $G' = C_{\mathrm{Sp}(W_p)(\mathbb{Q}_p)} G$.
2. The actions of G and G' are completely reducible, i.e. every invariant subspace has an invariant complement.

Example 1.5.19. *Let* V_p *be a rational quadratic space. The action of the groups* $\mathrm{O}(V_p)(\mathbb{Q}_p)$ *and* $\mathrm{Sp}(W_p)(\mathbb{Q}_p)$ *on* $W \otimes_{\mathbb{Q}_p} V_p$ *defines an embedding of the form*

$$\mathrm{Sp}(W_p)(\mathbb{Q}_p) \times \mathrm{O}(V_p)(\mathbb{Q}_p) \to \mathrm{Sp}(W_p \otimes_{\mathbb{Q}_p} V_p)(\mathbb{Q}_p).$$

The tuple $(\mathrm{Sp}(W_p)(\mathbb{Q}_p), \mathrm{O}(V_p)(\mathbb{Q}_p))$ *is a dual reductive pair. From now on we will denote by* $\mathbb{W}_p := W_p \otimes_{\mathbb{Q}_p} V_p$.

Schrodinger model

Proposition 1.5.20. *Let us consider the dual reductive pair* $(\mathrm{Sp}_n(W_p)(\mathbb{Q}_p), \mathrm{O}(V_p)(\mathbb{Q}_p))$*. The homomorphism* $j_1 : \mathrm{Sp}(W_p)(\mathbb{Q}_p) \to \mathrm{Sp}(\mathbb{W}_p)(\mathbb{Q}_p)$ *lifts uniquely to an homomorphism* $\tilde{j}_1 : \mathrm{Mp}(W_p)(\mathbb{Q}_p) \to$

$\mathrm{Mp}(\mathbb{W}_p)(\mathbb{Q}_p)$. *In particular, if* $\dim V_p$ *is even,* $\tilde{j}_1$ *factors throughout* $\mathrm{Sp}(W_p)(\mathbb{Q}_p)$. *Furthermore, the homomorphism* $j_2 : \mathrm{O}(V_p)(\mathbb{Q}_p) \to \mathrm{Sp}(\mathbb{W}_p)(\mathbb{Q}_p)$ *lifts to a non unique homomorphism* $\tilde{j}_2 : \mathrm{O}(V_p)(\mathbb{Q}_p) \to \mathrm{Mp}(\mathbb{W}_p)(\mathbb{Q}_p)$.

Proof. Even though the proof of this proposition is not essential to understand the main goal of the dissertation, we give an overview of it, emphasizing the construction of the homomorphisms $\tilde{j}_1$ and stressing why the map $\tilde{j}_1$ factors throughout $\mathrm{Sp}_n(W_p)(\mathbb{Q}_p)$ depending on the dimension of V_p. We are in the following situation:

$$\begin{tikzcd}
\overline{\mathrm{Sp}(W_p)(\mathbb{Q}_p)} \arrow[r] \arrow[d, "p"] & \mathrm{Mp}(\mathbb{W}_p)(\mathbb{Q}_p) \arrow[d, "p^{top}"] \\
\mathrm{Sp}(W_p)(\mathbb{Q}_p) \arrow[r, "i"] & \mathrm{Sp}(\mathbb{W}_p)(\mathbb{Q}_p)
\end{tikzcd}$$

where p^{top} is the projection map given in remark 1.5.5, i is the embedding of the example 1.5.19 and $\overline{\mathrm{Sp}(W_p)(\mathbb{Q}_p)} := p^{top,-1}\left(i\left(\mathrm{Sp}(W_p)(\mathbb{Q}_p)\right)\right)$ with group law and topology defined via pullback. By definition the group $\overline{\mathrm{Sp}(W_p)(\mathbb{Q}_p)}$ is a twofold topological cover of $\mathrm{Sp}(W_p)(\mathbb{Q}_p)$ with underlying set $\mathrm{Sp}(W_p)(\mathbb{Q}_p)\times\mu_2$. Therefore, it can be isomorphic (as a topological group) either to $\mathrm{Mp}(W_p)(\mathbb{Q}_p)$ endowed with the multiplication $(g_1,\xi_1)\cdot(g_2,\xi_2) = (g_1g_2, \xi_1\xi_2\overline{c}(g_1,g_2))$ with $\overline{c}(g_1,g_2)$ the pullback cocycle, or to $\mathrm{Sp}(W_p)(\mathbb{Q}_p)\times\mu_2$ endowed with group law $(g_1,\xi_1)\cdot(g_2,\xi_2) = (g_1g_2,\xi_1\xi_2)$. In order to determine which kind of group is $\overline{\mathrm{Sp}(W_p)(\mathbb{Q}_p)}$, we have to compute explicitly the cocycle $\overline{c}(g_1,g_2)$. In [Swe90, lem. 2.2.2, p. 50] the author shows that $\overline{c}(g_1,g_2)$ is trivial unless $\dim V$ is odd. Hence if $\dim V$ is even, the cocycle is trivial and then the multiplication law in $\overline{\mathrm{Sp}(W_p)(\mathbb{Q}_p)}$ is given by $(g_1,\xi_1)\cdot(g_2,\xi_2) = (g_1g_2,\xi_1\xi_2)$, concluding that $\overline{\mathrm{Sp}(W_p)(\mathbb{Q}_p)} \simeq \mathrm{Sp}(W_p)(\mathbb{Q}_p)\times\mu_2$ as topological groups. Since j_1 is a homomorphism, the above discussion yields the factorization of $\tilde{j}_1$ throughout the subgroup $\mathrm{Sp}(W_p)(\mathbb{Q}_p)$. If $\dim V$ is odd, the cocycle does not vanish and hence $\overline{\mathrm{Sp}(W_p)(\mathbb{Q}_p)} \simeq \mathrm{Mp}(W_p)(\mathbb{Q}_p)$. The lift $\tilde{j}_2$ is constructed in an analogous way. □

Remark 1.5.21. *Using proposition 1.5.20 for any dual reductive pair of the form* $(\mathrm{Sp}(W_p)(\mathbb{Q}_p), \mathrm{O}(V_p)(\mathbb{Q}_p))$, *we get the following homomorphisms*

$$\mathrm{Mp}(W_p)(\mathbb{Q}_p)\times\mathrm{O}(V_p)(\mathbb{Q}_p) \xrightarrow{\tilde{j}} \mathrm{Mp}(\mathbb{W}_p)(\mathbb{Q}_p) \xrightarrow{\omega_p} \mathrm{Aut}(\mathcal{S}(V^n(\mathbb{Q}_p))),$$

where $\tilde{j} = \tilde{j}_1\times\tilde{j}_2$ *is chosen so that* $\tilde{j}_2$ *is any lift of* j_2. *Considering the pullback of the Weil representation, i.e. the map* $\tilde{j}^*(\omega_p)$, *we obtain the homomorphism*

$$\mathrm{Mp}(W_p)(\mathbb{Q}_p)\times\mathrm{O}(V_p)(\mathbb{Q}_p) \xrightarrow{\tilde{j}^*(\omega_p)} \mathrm{Aut}(\mathcal{S}(V^n(\mathbb{Q}_p))).$$

Remark 1.5.22. *From now on, we fix the lifting* $\tilde{j}_2$ *given in [Kud96, p. 39]. This choice provides a suitable formula of* $\tilde{j}_2^*(\omega_p)|_{\mathrm{O}(V_p)(\mathbb{Q}_p)}$. *We will abuse notation and denote the pullback by* $\tilde{j}^*\omega_p =: \omega_p$.

Proposition 1.5.23. Let W_p and V_p be a $p-$adic rational symplectic vector space of dimension $2n$ and a $p-$adic rational quadratic space space of dimension m respectively. The non-archimedian *Schroedinger model* of the Weil representation for the dual reductive pair $(\mathrm{Sp}(W_p)(\mathbb{Q}_p), \mathrm{O}(V_p)(\mathbb{Q}_p))$ is given by the following formulas:

- $\omega_\psi(h)\varphi(x) = \varphi(h^{-1}x)$, *if* $h\in\mathrm{O}(V_p)(\mathbb{Q}_p)$.
- $\omega_\psi\left(\left(\begin{smallmatrix} a & \\ & {}^t a^{-1}\end{smallmatrix}\right), z\right)\varphi(x) = \tilde{\chi}_{V,p}(\det(a), z)|\det(a)|^{m/2}\varphi(xa)$, *if* $a\in\mathrm{GL}_n(\mathbb{Q}_p)$,
- $\omega_\psi\left(\left(\begin{smallmatrix} 1_n & b \\ & 1_n\end{smallmatrix}\right), z\right)\varphi(x) = z\psi\left(\frac{1}{2}x^tbx\right)\varphi(x)\varphi(xa)$, *if* $b\in\mathrm{Sym}_n(\mathbb{Q}_p)$.
- $\omega_\psi\left(\left(\begin{smallmatrix} & 1_n \\ 1_n & \end{smallmatrix}\right), z\right)\varphi(x) = z\int_{\mathbb{Q}_v}\varphi(y)\psi(x^ty)dy$,

where dy is the Haar measure of $\mathbb{Q}_v$.

Proof. The formulas are deduced from [Kud96, prop. 4.3, p. 37]. □

Definition of local theta correspondence

Definition 1.5.24. Let (G, H) be a dual reductive pair so that $G \times H \hookrightarrow \mathrm{Sp}(W_p)(\mathbb{Q}_p)$ with $W_p = X_p \oplus Y_p$ a polarization. Given π_p an irreducible admissible representation of H, we consider the maximal π_p-isotypic quotient of $\mathcal{S}(Y_p)$, i.e.

$$\mathcal{S}(Y_p)/ \cap_{f \in \mathrm{Hom}_H(\mathcal{S}(Y_p),\pi_p)} \ker f.$$

The above space is a representation of $G \times H$ and hence we may express it as $\pi_p \otimes \Theta(\pi_p)$, with $\Theta(\pi_p)$ a representation of G called the *local big theta lift* of π_p.

Remark 1.5.25. *An alternative way to define the local big theta lift of π_p is*

$$\Theta(\pi_p) = (\mathcal{S}(Y_p) \otimes \tilde{\pi}_p)_H\,,$$

where $\tilde{\pi}_p$ is the contragradient of π_p, defined in 2.3.3.

Theorem 1.5.26 (Howe duality)**.** *Let $p \neq 2$. Assume that π_p is an irreducible admissible representation of H, then*

1. *Either $\Theta(\pi_p) = 0$ or $\Theta(\pi_p)$ is a non-zero admissible representation of finite length of $G(\mathbb{Q}_p)$.*

2. *If $\Theta(\pi_p) \neq 0$ there exists a unique irreducible submodule $\Theta(\pi_p)^0$ of $\Theta(\pi_p)$ so that $\theta(\pi_p) := \Theta(\pi_p)/\Theta(\pi_p)^0$ is an irreducible admissible representation. If $\Theta(\pi_p) = 0$, we define $\theta(\pi_p) := 0$.*

3. *If $\theta(\pi_1)$ and $\theta(\pi_2)$ are non-zero and isomorphic, then $\pi_1 \simeq \pi_2$.*

Proof. It is [MVW87], [GHS90]. □

To the best of the author's knowledge, the Howe duality for $p = 2$ is still unknown. Throughout this thesis we will assume it is true.

Definition 1.5.27. Let π_p be an irreducible admissible representation of H, the representation $\theta(\pi_p)$ of theorem 1.5.26 is called the *local theta correspondence* of π_p.

1.5.4 Global metaplectic group

From now on we fix

$$\psi = \otimes_p \psi_p : \; \mathbb{Q} \setminus \mathbb{A} \to \mathbb{C}^1, \tag{1.-5}$$

where ψ_p are the characters fixed in 1.5.12. For any rational vector space Y, we consider $L \subset Y$ a lattice of Y. We define

$$\varphi_p^L := \mathrm{char}(L \otimes_{\mathbb{Z}} \mathbb{Z}_p) \in \mathcal{S}(Y(\mathbb{Q}_p)).$$

Given a different lattice $M \subset Y$, the function φ_p^M satisfies

$$\varphi_p^M = \varphi_p^L, \tag{1.-5}$$

for all but finitely many finite places of $\mathbb{Q}$. We consider the following $\mathbb{C}-$vector space

$$\mathcal{S}(Y(\mathbb{A}_f)) := {\bigotimes_{p \nmid \infty}}' \mathcal{S}(Y(\mathbb{Q}_p)),$$

where the restricted product is taken with respect to the family of $p-$adic Schwartz functions φ_p^L. Due to the relation (1.5.4) between the functions φ_p^L and φ_p^M, the space $\mathcal{S}(Y(\mathbb{A}_f))$ is independent of the choice of lattice L.

Remark 1.5.28. *For any lattice L the Schwartz function φ_p^L is fixed by $\mathrm{Sp}_n(\mathbb{Z}_p)$ for all but finitely many finite places p of $\mathbb{Q}$.*

Definition 1.5.29. The space of *Schwartz functions* on $\mathbb{A}$ is defined as the following tensor product:

$$\mathcal{S}(Y(\mathbb{A})) := \mathcal{S}(Y(\mathbb{R})) \otimes \mathcal{S}(Y(\mathbb{A}_f)).$$

Definition 1.5.30. We consider the set

$$\widetilde{\mathrm{Sp}}(W)(\mathbb{A}) := \{(g, s) = (\otimes_p g_p, s) \in \mathrm{Sp}(W)(\mathbb{A}) \times U(\mathcal{S}(Y(\mathbb{A}))),\ s.t.\ s = \pm \otimes_{p\ place}\, \omega_p(g_p)\},$$

where $U(\mathcal{S}(Y(\mathbb{A})))$ is the set of unitary operators in $\mathcal{S}(Y(\mathbb{A}))$ and ω_p are the local ones defined in (1.5.1). We endow this set with the product topology as in 1.5.4 and multiplication law $(g_1, s_1) \cdot (g_2, s_2) = (g_1 g_2, s_1 s_2)$. The resulting topological group is called *global metaplectic group*.

Definition 1.5.31. The *global Weil representation* is defined by

$$\begin{aligned} \omega :\ \widetilde{\mathrm{Sp}}(W)(\mathbb{A}) &\to \mathrm{Aut}(\mathcal{S}(Y(\mathbb{A}))), \\ (g, \xi) &\mapsto \xi \cdot \otimes_{p\ place} \omega_p(g_p). \end{aligned}$$

We note that $\widetilde{\mathrm{Sp}}(W)(\mathbb{A})$ is not the restricted product of local groups. Despite this, we still have a local global relationship. Lemma 1.5.9 allows us to regard the groups K_p as subgroups of $\widetilde{\mathrm{Sp}}(W)(\mathbb{Q}_p)$, hence we define the group

$$\prod_p{}' \widetilde{\mathrm{Sp}}(W)(\mathbb{Q}_p), \tag{1.-5}$$

where the restricted product is respect to the subgroups K_p. The projection map

$$\begin{aligned} p : \prod_p{}' \widetilde{\mathrm{Sp}}(W)(\mathbb{Q}_p) &\to \widetilde{\mathrm{Sp}}(W)(\mathbb{A}), \\ \prod_p{}' (g_p, \xi_p) &\mapsto (\prod_p g_p, \prod_p \xi_p), \end{aligned}$$

is well defined because almost all components (g_p, ξ_p) of the above tensor product belongs to K_p, which implies that almost all ξ_p are equal to 1. The kernel of the previous map is $I := \{\prod_p' (id, \xi_p),\ s.t.\ \prod_p \xi_p = 1\}$, hence we have a group isomorphism of the form

$$\prod_p{}' \widetilde{\mathrm{Sp}}(W)(\mathbb{Q}_p)/I \simeq \widetilde{\mathrm{Sp}}(W)(\mathbb{A}). \tag{1.-6}$$

The previous map implies that every representation of $\prod_p' \widetilde{\mathrm{Sp}}(W)(\mathbb{Q}_p)$ trivial on I may be regarded as a representation of $\widetilde{\mathrm{Sp}}(W)(\mathbb{A})$. For example, the group $\prod_p' \widetilde{\mathrm{Sp}}(W)(\mathbb{Q}_p)$ acts on $\mathrm{Aut}(\mathcal{S}(Y(\mathbb{A})))$ by $\otimes_p \omega_p(g_p)$ and it can be checked that $\otimes_p \omega_p(g_p)|_I = \mathrm{id}$. Therefore, the local global relation is given by the following equality:

$$\otimes_p \omega_p(g_p, \xi_p) = \omega\left(p\left(\prod_p{}' (g_p, \xi_p)\right)\right). \tag{1.-6}$$

Definition 1.5.32. We consider the set

$$\mathrm{Mp}(W)(\mathbb{A}) := \mathrm{Sp}(W)(\mathbb{A}) \times \mu_2 = \{(g, \xi) = (\otimes_p g_p, \xi) \in \mathrm{Sp}(W)(\mathbb{A}) \times \mu_2\}$$

with multiplication law

$$(g_1, \xi_1) \cdot (g_2, \xi_2) = (g_1 g_2, \xi_1 \xi_2 \prod_p \tilde{c}(g_{1,p}, g_{2,p})).$$

We endow $\mathrm{Mp}(W)(\mathbb{A})$ with the pullback topology throughout the isomorphism

$$\begin{aligned} \mathrm{Mp}(W)(\mathbb{A}) &\to \widetilde{\mathrm{Sp}}(W)(\mathbb{A}), \\ (g_1, \xi_1) &\mapsto (g_1, \xi_1 \omega(g)). \end{aligned}$$

Then, the resulting topological group is isomorphic to $\widetilde{\mathrm{Sp}}(W)(\mathbb{A})$. We refer to it as the *metaplectic group*.

Remark 1.5.33. *According to [Swe90, p. 37], just a finite number of $\tilde{c}(g_{1,p}, g_{2,p})$ are equal to -1, therefore $\mathrm{Mp}(W)(\mathbb{A})$ is well defined.*

Remark 1.5.34. *The metaplectic group $\mathrm{Mp}(W)(\mathbb{A})$ fits in a short exact sequence of the form:*

$$1 \to \{\pm 1\} \to \mathrm{Mp}(W)(\mathbb{A}) \to \mathrm{Sp}(W)(\mathbb{A}) \to 1.$$

1.5.5 Adelic points of the Siegel parabolic subgroup

According to the Levi decomposition we have $P(\mathbb{A}) = M(\mathbb{A})N(\mathbb{A})$, where

$$M(\mathbb{A}) = \left\{ m(a) := \left(\begin{smallmatrix} a & \\ & {}^t a^{-1} \end{smallmatrix} \right), \ s.t.\ a \in \mathrm{GL}_n(\mathbb{A}) \right\},$$

$$N(\mathbb{A}) = \left\{ n(b) := \left(\begin{smallmatrix} 1 & b \\ & 1 \end{smallmatrix} \right), \ s.t.\ b =^t b \in \mathrm{M}_n(\mathbb{A}) = \mathrm{Sym}_n(\mathbb{A}) \right\}.$$

The maximal compact open subgroup of $\mathrm{Sp}(W)(\mathbb{A})$ is given by $K := U(n) \times \prod_{p \nmid \infty} \mathrm{Sp}(W)(\mathbb{Z}_p)$, where $U(n)$ is the unitary group of rank n. The Iwasawa decomposition yields

$$\mathrm{Sp}(W)(\mathbb{A}) = P(\mathbb{A})K,$$

According to the above equality, we express every element $g \in \mathrm{Sp}(W)(\mathbb{A})$ as

$$g = m(a)n(b)k, \tag{1.-6}$$

with $a \in \mathrm{GL}_n(\mathbb{A})$ and $b \in \mathrm{Sym}_n(\mathbb{A})$. We will use the notation

$$|a(g_p)| := |\det(a)|_A.$$

The above definitions are compatible with the ones given in 1.5.2 when we consider the restricted product $\mathrm{Sp}(W)(\mathbb{A}) = \prod_p \mathrm{Sp}(W)(\mathbb{Q}_p)$.

We denote by $\widetilde{P}(\mathbb{A})$ the inverse image in $\mathrm{Mp}(W)(\mathbb{A})$ of $P(\mathbb{A})$ under the exact sequence of remark 1.5.34. Applying lemma 1.5.13 at each place p, we obtain a factorization of the form

$$\widetilde{P}(\mathbb{A}) = \widetilde{M}(\mathbb{A})N(\mathbb{A}),$$

where $\widetilde{M}(\mathbb{A})$ is the inverse image of $M(\mathbb{A})$ under the exact sequence 1.5.8. The group $\widetilde{M}(\mathbb{A})$ satisfies the following isomorphism

$$\widetilde{M}(\mathbb{A}) \simeq \widetilde{\mathrm{GL}}_n(\mathbb{A}),$$

where $\widetilde{\mathrm{GL}}_n(\mathbb{A})$ is the twofold topological cover of $\mathrm{GL}_n(\mathbb{A})$ with underlying set $\mathrm{GL}_n(\mathbb{A}) \times \{\pm 1\}$ and multiplication law

$$(g_1, \eta_1) \cdot (g_2, \eta_2) = \left(g_1 g_2, \eta_1 \eta_2 \prod_p (\det g_{1,p}, \det g_{2,p})_p \right).$$

Analogously to (1.5.14), the determinant map gives rise to a genuine character of the form

$$\widetilde{\det} : \widetilde{\mathrm{GL}}_n(\mathbb{A}) \to \widetilde{\mathrm{GL}}_1(\mathbb{A}).$$

Furthermore, the global Weil index $\gamma(g,\psi)$ may be lifted to a character of the form

$$\begin{aligned}\chi_\psi:\ \widetilde{\mathrm{GL}}_1(\mathbb{A}) &\to \mathbb{C}^\times,\\ (g,\eta) &\mapsto \eta\gamma(g,\psi)^{-1}.\end{aligned}$$

Remark 1.5.35. *Since the group $\widetilde{M}(\mathbb{A})$ is not the restricted product of the groups $\widetilde{M}(\mathbb{Q}_p)$, the maps (1.5.14) and (1.5.2) are not compatible with the maps $\widetilde{\det}$ and χ_ψ given in this section.*

Definition 1.5.36. Let (V,q) be a rational quadratic space dimension m. We define a character $\chi_V : \mathbb{A}^\times \to \mathbb{C}^\times$ by

$$\chi_V(x) := \prod_p \chi_{V,p}(x_p).$$

Definition 1.5.37. Let $\mathbf{M}(\mathbb{A})$ be either the group $M(\mathbb{A})$ or $\widetilde{M}(\mathbb{A})$. We define the homomorphisms $\tilde{\chi}_V:\ \mathbf{M}(\mathbb{A}) \to \mathbb{C}^\times$ by

$$\tilde{\chi}_V = \begin{cases} g \mapsto \chi_V \circ \det(g), & \text{if } \mathbf{M}(\mathbb{A}) = M(\mathbb{A}),\\ (g,\eta) \mapsto (\chi_\psi \circ \widetilde{\det})(g,\eta)(\chi_V\circ\det)(g), & \text{if } \mathbf{M}(\mathbb{A}) = \widetilde{M}(\mathbb{A}).\end{cases}$$

1.5.6 Global theta correspondence

The global theta correspondence relates automorphic representations of certain pairs of groups. In this section we will show the compatibility with the global and local theta correspondence by means of the Howe duality conjecture. Throughout this section we will use the notation $\tilde{G}(\mathbb{Q}_p) := \mathrm{Mp}(W_p)(\mathbb{Q}_p)$ and $\tilde{G}(\mathbb{A}) := \mathrm{Mp}(W)(\mathbb{A})$.

Definition 1.5.38. An *algebraic dual reductive pair* is a pair of rational algebraic groups (G,H) so that for every p place of $\mathbb{Q}$, the tuples $(G(\mathbb{Q}_p), H(\mathbb{Q}_p))$ are dual reductive pairs. For simplicity, we will refer to the algebraic dual reductive pairs as dual reductive pairs.

We fix a dual reductive pair of the form $(G,H) := (\mathrm{Sp}(W), O(V))$ so that $G\times H \hookrightarrow \mathrm{Sp}(\mathbb{W})$.

Remark 1.5.39. *By remark 1.5.21 we obtain the map*

$$\omega_p:\ \tilde{G}(\mathbb{Q}_p)\times H(\mathbb{Q}_p) \to \mathrm{Aut}(\mathcal{S}(V^n(\mathbb{Q}_p))).$$

Via restricted tensor product, we may define the pullback of the global Weil representation

$$\begin{aligned}\omega:\ \tilde{G}(\mathbb{A})\times H(\mathbb{A}) &\to \mathrm{Aut}(\mathcal{S}(V^n(\mathbb{A}))),\\ (g,h) &\mapsto \otimes_p \omega_p(g_p,h_p).\end{aligned}$$

Definition 1.5.40. Given a polarization $\mathbb{W} = X\oplus Y$, we define the *theta function* on $\tilde{G}(\mathbb{A})\times H(\mathbb{A})\times \mathcal{S}(Y(\mathbb{A}))$ by

$$\theta(g,h,\varphi) = \sum_{x\in Y(\mathbb{Q})} \omega(g,h)\varphi(x).$$

Remark 1.5.41. *For the rest of this section we fix a polarization $\mathbb{W} = X\oplus Y$, where $Y = V^n$. We adopt this polarization because the formulas for the Weil representation and the description of the theta function will be suitable for our computations. Additionally, there are other contexts where other polarizations are preferred. For example, see [Mor14] where the author considers a different polarization to unfold the integral with the theta function.*

Proposition 1.5.42. *Let $\varphi\in\mathcal{S}(V^n(\mathbb{A}))$. The Schroedinger model of the representation ω is given by*

- $\omega(h)\varphi(x) = \varphi(h^{-1}x),\ \text{if } h\in H(\mathbb{A}),$
- $\omega\left(\left(\begin{smallmatrix}1 & r\\ & 1\end{smallmatrix}\right)\right)\varphi(x) = \psi_{\mathbb{A}}\left(\frac{1}{2}xb\,{}^t x\right)\varphi(x),\ \text{if } r\in \mathrm{Sym}_n(\mathbb{A}),$

- $\omega\left(\left(\begin{smallmatrix} t & \\ & {}^t t^{-1}\end{smallmatrix}\right)\right) = \tilde{\chi}_V(\det(t))|\det(t)|_{\mathbb{A}}^{m/2}\varphi(xt)$ *if* $t \in GL_n(\mathbb{A})$ *where* $\tilde{\chi}_V(\cdot)$ *is the character defined in* (1.5.36).

The action for the remaining elements of $\mathrm{Mp}(W)(\mathbb{A})$ *can be deduced from the formula given in [Kud94, p. 40].*

Proof. It follows applying [Kud94, p. 40] in each place. □

Proposition 1.5.43. *The theta function is* $G(\mathbb{Q}) \times H(\mathbb{Q})-$*invariant.*

Proof. The $H(\mathbb{Q})-$invariance follows by 1.5.42. In fact, for any $\gamma \in H(\mathbb{Q})$ we obtain

$$\theta(g,h,\varphi) = \sum_{x \in Y(\mathbb{Q})} \omega(g,\gamma h)\varphi(x) = \sum_{x \in Y(\mathbb{Q})} \omega(g,h)\varphi(\gamma^{-1}x).$$

Since $\gamma \in H(\mathbb{Q})$, we reorder the sum by

$$\sum_{x \in Y(\mathbb{Q})} \omega(g,h)\varphi(\gamma^{-1}x) = \theta(g,h,\varphi).$$

The proof for the $G(\mathbb{Q})-$invariance relies on the formulas given in 1.5.42 and the adelic Poisson summation formula. See [Wei64, thm. 6, p. 193] for the details. □

Proposition 1.5.44. *The theta function is of moderate growth.*

Proof. See [KR88, (1.1), p. 6]. □

Definition 1.5.45. Let $f \in \mathcal{A}([H])$ be an automorphic form, and $\varphi \in \mathcal{S}(V^n(\mathbb{A}))$. If the integral

$$\theta(g,f,\varphi) = \int_{[H]} \overline{f(h)}\theta(g,h,\varphi)dh,$$

converges, where dh is the Haar measure of $H(\mathbb{A})$, then the resulting function $\theta(g,f,\varphi)$ is called the *global theta lift* of f.

Remark 1.5.46. *Given a cusp form* $f \in \mathcal{A}_0([H])$*, the theta lift* $\theta(g,f,\varphi)$ *exists because* f *rapidly decreasing and* $\theta(g,h,\varphi)$ *has moderate growth by 1.5.44. Moreover, there are certain Schwartz functions* $\tilde{\varphi}$ *making* $\theta(g,h,\tilde{\varphi})$ *rapidly decreasing in the variable* h*. These kind of functions will be studied in 1.6.6 and will allow us to consider theta lifts of a wider space of automorphic forms.*

Definition 1.5.47. Let $\pi \subset \mathcal{A}_0([H])$ be a cuspidal automorphic representation of H. Then

$$\Theta^{glob}(\pi) := \{\theta(g,f,\varphi),\ f \in \pi,\ \varphi \in \mathcal{S}(V^n(\mathbb{A}))\},$$

is an automorphic representation of G called *global theta correspondence.*

We recall that in 1.4.7 we defined automorphic representations of an algebraic group G as irreducible $(\mathfrak{g}, K^\infty) \times G(\mathbb{A}_f)-$modules. From now on, in order to lighten the notation, we will refer to $(\mathfrak{g}, K^\infty) \times G(\mathbb{A}_f)-$intertwining maps as $G(\mathbb{A})-$intertwining maps.

Remark 1.5.48. *For* $\pi \in \mathcal{A}_0([H])$ *the the global theta lift defines a surjective* $H(\mathbb{A}) \times G(\mathbb{A})-$*intertwining map*

$$\theta(g,\cdot,\cdot):\ \mathcal{S}(V^n(\mathbb{A})) \times \overline{\pi} \to \Theta^{glob}(\pi),$$

where the action of $H(\mathbb{A})$ *on* $\Theta^{glob}(\pi)$ *is trivial.*

Remark 1.5.49. *Given* $f \in \mathcal{A}_0([G])$*, the integral*

$$\theta(h,f,\varphi) = \int_{[G]} \overline{f(g)}\theta(g,h,\varphi)dg,$$

defines an analogous theta correspondence from cuspidal automorphic representations $\pi' \subset \mathcal{A}_0([G])$ *to automorphic representations* $\pi \subset \mathcal{A}_0([H])$.

Proposition 1.5.50. *Given an automorphic cuspidal representation $\pi = \otimes_p \pi_p \subset \mathcal{A}_0([H])$, there is a surjective $G(\mathbb{A})-$interwinning map*

$$\otimes_p \Theta(\pi_p) \to \Theta^{glob}(\pi).$$

Proof. Remark 1.5.48 shows that the theta lift defines a surjective $H(\mathbb{A}) \times G(\mathbb{A})-$intertwining map, i.e. an element on the following group

$$\mathrm{Hom}_{G(\mathbb{A})\times H(\mathbb{A})}\left(\mathcal{S}(V^n(\mathbb{A}))\otimes \overline{\pi}, \Theta^{glob}(\pi)\right). \tag{1.-6}$$

Since π is cuspidal then $\overline{\pi} \simeq \tilde{\pi}$, and hence

$$\mathrm{Hom}_{G(\mathbb{A})\times H(\mathbb{A})}\left(\mathcal{S}(V^n(\mathbb{A}))\otimes \overline{\pi}, \Theta^{glob}(\pi)\right) \simeq \mathrm{Hom}_{G(\mathbb{A})\times H(\mathbb{A})}\left(\mathcal{S}(V^n(\mathbb{A})), \pi \otimes \Theta^{glob}(\pi)\right) \tag{1.-6}$$

Using the definition 1.5.24, the local big theta lift of a representation π_p of $H(\mathbb{Q}_p)$ is the representation $\Theta(\pi_p)$ so that the maximal $\pi-$isotypic quotient of $\mathcal{S}(V^n(\mathbb{Q}_p))$ may be expressed as $\pi_p \otimes \Theta(\pi_p)$, therefore each $p-$component of the homomorphisms (1.5.6) factors throughout

$$\mathrm{Hom}_{G(\mathbb{Q}_p)\times H(\mathbb{Q}_p)}(\mathcal{S}(V^n(\mathbb{Q}_p)), \pi_p \otimes \Theta(\pi_p)).$$

Since the map defined by the global theta lift is surjective, we obtain a surjective $G(\mathbb{A})\times H(\mathbb{A})-$intertwining map

$$\bigotimes_p \pi_p \otimes \Theta(\pi_p) \to \pi \otimes \Theta^{glob}(\pi).$$

□

Corollary 1.5.51. *Let $\pi \subset \mathcal{A}([H])$ be a cuspidal automorphic representation so that $\Theta^{glob}(\pi) \subset \mathcal{A}([G])$ is an irreducible automorphic representation, then*

$$\otimes_p \theta(\pi_p) \simeq \Theta^{glob}(\pi).$$

Proof. Proposition 1.5.50 shows that, for each place p there exists a surjective $G(\mathbb{Q}_p)-$intwetwinning map

$$t: \ \Theta(\pi_p) \to \Theta^{glob}(\pi)_p. \tag{1.-6}$$

Since $\Theta^{glob}(\pi)$ is irreducible, the local $G(\mathbb{Q}_p)-$representations $\Theta^{glob}(\pi)_p$ are irreducible and hence the kernel of the map (1.5.6) defines a subrepresentation, denoted by $\ker t$. It satisfies the following isomorphism of representations:

$$\Theta(\pi_p)/\ker t \simeq \Theta^{glob}(\pi)_p.$$

By theorem 1.5.26, there exists a unique irreducible quotient of $\Theta(\pi_p)$, the so-called local theta lift $\theta(\pi_p)$. Therefore

$$\theta(\pi_p) \simeq \Theta(\pi_p)/\ker t \simeq \Theta^{glob}(\pi)_p. \tag{1.-6}$$

Using the isomorphism (1.5.6) for each place p we obtain the isomorphism of the statement. □

Remark 1.5.52. *Let $\pi = \otimes_p \pi_p \subset \mathcal{A}([H])$ be a non cuspidal automorphic representation. One may wonder what is the relation between the local theta lifts $\theta(\pi_p)$ and the global theta lift $\Theta^{glob}(\pi)$ (if it exists, and if it does not exists, how to define an analogous theta correspondence). This question will be addressed in detail in section 1.6.6 when π is the trivial representation. For the sake of completeness, in this remark we give an overview of how these kind of questions are adressed. Let us suppose there exists a surjective $G(\mathbb{A}) \times H(\mathbb{A})-$interwinning map*

$$\omega(\alpha): \ \mathcal{S}(V^n(\mathbb{A})) \to \mathcal{S}(V^n(\mathbb{A}))_{abc},$$

where $\mathcal{S}(V^n(\mathbb{A}))_{abc} \subset \mathcal{S}(V^n(\mathbb{A}))$. Futhermore, we assume this map commutes with the Weil representation and for any $\tilde{\varphi} \in \mathcal{S}(V^n(\mathbb{A}))_{abc}$, the theta lift $\theta(g, f, \tilde{\varphi})$ converges for every $f \in \pi$. We define the representation $\Theta^{glob}_{reg}(\pi) := \{\theta(g, f, \omega(\alpha)\varphi), \ f \in \pi, \ \varphi \in \mathcal{S}(V^n(\mathbb{A}))_{abc}\}$. Only using the definition, there *exists a surjective* $G(\mathbb{A}) \times H(\mathbb{A})-$*intertwining map*

$$\pi \otimes \mathcal{S}(V^n(\mathbb{A})) \to \Theta_{reg}^{glob}(\pi). \tag{1.-6}$$

The homomorphism (1.5.52) *is similar to the maps* (1.5.6)*. Hence, depending on the structure of* $\overline{\pi}^\vee$*, we obtain analogous results to proposition 1.5.50 and corollary 1.5.51. For example, if the automorphic representation* π *is the trivial representation, it is straightforward that* $\widetilde{\pi} = \overline{\pi}$ *and hence we are able to reply all the steps in proposition 1.5.50 and corollary 1.5.51.*

For the following theorem we set (V_0, q_p) an anisotropic rational quadratic space and $V_r := V_0 \oplus \mathbb{H}^r$ isotropic quadratic spaces. We denote by $\mathrm{O}(V_r)$ their associated orthogonal groups.

Definition 1.5.53. The chain of groups

$$\mathrm{O}(V_0) \subset \mathrm{O}(V_1) \subset ...$$

is called the *Rallis tower.*

We consider dual reductive pairs of the form $(\mathrm{Sp}(W), \mathrm{O}(V_r))$ and we denote their associated theta correspondences by Θ_r^{glob}.

Theorem 1.5.54. *Let* σ *be a cuspidal automorphic representation of the group* $\mathrm{Sp}(W)$ *and let* i *be the smallest integer so that* $\Theta_i^{glob}(\sigma) \neq 0$*, then*

1. $i \leq 2n$*,*
2. $\Theta_i^{glob}(\sigma)$ *is cuspidal,*
3. $\Theta_r^{glob}(\sigma) \neq 0$ *for* $r \geq i$*.*

Proof. See [Ral84, cor. I.1.1, p. 351]. □

1.5.7 Relation between the global theta function and the classical theta functions

As stated in the previous sections we have discussed two different kinds of theta lifts: the Siegel theta lift 1.3.9 and the global theta lift 1.5.45. This section illustrates the relationship between the two correspondences. In fact, the global theta lift restricted to the Gaussian Schwartz function recovers the Siegel theta lift. For the time being, we denote by (V, q) a rational quadratic space of dimension $(2, n)$.

Remark 1.5.55. *For any lattice* $L \subset V$ *the vector space* $\mathbb{C}[L'/L]$ *is isomorphic to the* $\mathbb{C}-$*vector space generated by* S_L*, the set of characteristic functions of the coset* $L' \otimes_{\mathbb{Z}} \mathbb{A}_f / L \otimes_{\mathbb{Z}} \mathbb{A}_f$*. Then, the Weil representation defined in* (1.3.1)

$$\rho_L : \ \mathrm{Mp}_2(\mathbb{R}) \to \mathrm{End}(\mathbb{C}[L'/L]),$$

is isomorphic as a $\mathrm{Mp}_2(\mathbb{R})-$*representation to the following restriction of the Weil representation stated in 1.5.31:*

$$\omega|_{\mathrm{Mp}_2(\mathbb{R})} : \ \mathrm{Mp}_2(\mathbb{R}) \to \mathrm{Aut}(\langle S_L \rangle),$$

where $\langle S_L \rangle$ *is the* $\mathbb{C}-$*vector space generated by* S_L

Let $\mathrm{Gr}(V)$ be the connected component of the Grassmanian associated to V, see 1.2.1 for the definition. We recall that proposition 1.2.4 shows that there is an isomorphism

$$i : \ \mathrm{Gr}(V) \to \mathcal{K} = \{Z \in \mathbb{P}V(\mathbb{C}) \ s.t. \ (Z, Z) = 0, \ (Z, \overline{Z}) > 0\}.$$

Definition 1.5.56. Given $z \in \mathrm{Gr}(V)$ and $x \in V(\mathbb{R})$, we use the following notation $x_z := x_{i(z)}$ and $R(x, z) := -(x_z, x_z) = |(x, i(z))||y|^{-2}$. The *majorant* associated to z is defined by

$$(x, x)_z := (x, x) + 2R(x, z).$$

Proposition 1.5.57. *Given $\gamma \in O(V)(\mathbb{R})$, the majorant satisfies*

$$(\gamma x, \gamma x)_z = (x, x)_{\gamma^{-1} z}.$$

Proof. Given any $x, y \in V(\mathbb{R})$, we have $(\gamma x, y) = (x, \gamma^{-1} y)$. Therefore using the explicit description of $R(x, z)$ we conclude the proof. □

Definition 1.5.58. Set z_0 the element of $\mathrm{Gr}(V)$ fixed by the action of $\mathrm{SO}(2) \oplus \mathrm{SO}(n)$. The following Schwartz function:

$$\varphi_{z_0}^{\infty}(x) := e^{-\pi (x,x)_{z_0}} \in \mathcal{S}(V(\mathbb{R})).$$

is called the *Gaussian*.

Remark 1.5.59. *Due to 1.5.57, for any $h \in O(V)(\mathbb{R})$ the Gaussian satisfies*

$$\varphi_{z_0}^{\infty}(hx) = \varphi_{h^{-1} z_0}^{\infty}(x).$$

Definition 1.5.60. Given $\tau = u + iv \in \mathcal{H}$, we set $g_\tau := \left(\begin{smallmatrix} 1 & u \\ & 1 \end{smallmatrix}\right) \begin{pmatrix} v^{1/2} & \\ & v^{-1/2} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$. Since there is no danger of confusion, we will also denote by $g_\tau := (g_\tau, 1) \in \mathrm{Mp}_2(\mathbb{R})$. Furthermore, for any $z \in \mathrm{Gr}(V)$ we denote by $h_z \in SO(V)(\mathbb{R})$ the element so that $h_z^{-1} z_0 = z \in \mathrm{Gr}(L)$.

Remark 1.5.61. *Since $\mathrm{Gr}(V) \simeq SO(2) \oplus SO(n) \setminus SO(V)(\mathbb{R})$, we have $h_z \in T^{SO(V)}(\mathbb{R}) N^{SO(V)}(\mathbb{R})$.*

Proposition 1.5.62. *For any lattice $L \subset V$, we have*

$$\Theta_L(\tau, z) = \sum_{\varphi \in S_L} \theta(g_\tau, h_z, \varphi_{z_0}^{\infty} \otimes \varphi).$$

Proof. It follows by the remarks 1.5.59 and 1.5.55. □

Corollary 1.5.63. *For any $k = \frac{n}{2} - 1$ and $f \in M_{k,L}^{!}$, the singular theta lift satisfies*

$$\int^{\bullet} \langle f(\tau), \Theta_L(z, \tau) \rangle \, v d\mu(\tau) = \int^{\bullet} \sum_{\varphi \in S_L} f_\varphi(\tau) \theta(g_\tau, h_z, \varphi_{z_0}^{\infty} \otimes \varphi) v d\mu(\tau).$$

Proof. It follows directly by 1.5.62. □

1.6 Regularized Siegel-Weil formula

This section is devoted to explaining the Siegel-Weil formula. Roughly speaking, the formula relates the integral of a theta function and a Siegel Eisenstein series. It shows explicitly the most basic case of the theta correspondence, which has played an important role in the theory of automorphic representation. As an example, the Rallis inner this formula uses the Siegel-Weil formula to relate $L-$functions and the theta correspondence. More concretely, the combination of certain integral representations for the standard $L-$function, see [GPSR87], and the Siegel-Weil formula, shows in some cases a way to characterize the poles and zeroes of the $L-$function in a precise way. See [KR94, thm. 8.5, p. 77], [GQT14, thm. 11.4, p. 54] for more details. In this section we consider V a rational quadratic space of dimension m and Witt index r, i.e. $V = V_{an} \oplus \mathbb{H}^r$.

The Siegel-Weil formula had to be studied case by case depending on the ranks of the groups Sp_{2n} and $\mathrm{O}(V)$. The main reason for that was the convergence of the integral

$$\int_{[O(V)]} \theta(g, h, \varphi) dh.$$

For example, when V is an anisotropic vector space, the group $\mathrm{O}(V)(\mathbb{A})$ is compact and hence the integral is convergent. Furthermore according to [Wei65, prop. 8, p. 75], if V satisfies

$$m - r > n + 1$$

the above theta integral also converges. When the integral does not always converge, a regularization of the theta function is required. See [KR94, sec. 5.2 p. 43] and [Ich01, sec. 1, p. 204] for the definition. This process is based on a map of the form

$$\omega(\alpha): \ \mathcal{S}(V^n(\mathbb{A})) \to \mathcal{S}(V^n(\mathbb{A}))_{abc},$$

where $\mathcal{S}(V^n(\mathbb{A}))_{abc} = \{\varphi \in \mathcal{S}(V^n(\mathbb{A})),\ s.t.\ \theta(g,h,\varphi)\ absolutely\ convergent\}$. The operator $\omega(\alpha)$ is constructed using the action of an explicit element of the Hecke algebra of $\mathrm{O}(V)$ on the vector space $\mathcal{S}(V^n(\mathbb{A}))$. This machinery allows us to define a meromorphic function $\mathcal{E}(g,\varphi,s)$, see [GQT14, sec. 3.5, p. 18]. It replaces the role of $\int_{[O(V)]}\theta(g,h,\varphi)dh$ in the classical Siegel-Weil formula. This formula is studied by different methods depending on the order of the pole of the function $\mathcal{E}(s,g,\varphi)$ at $s = s_0$. In fact, this function has the following Laurent expansion

$$\mathcal{E}(s,g,\varphi) = \frac{B_{-2}(g,\varphi)}{(s-s_0)^2} + \frac{B_{-1}(g,\varphi)}{(s-s_0)} + B_0(g,\varphi) + O(s-s_0).$$

If $B_{-2}(g,\varphi) = 0$, we say that we are in the *first term range*. If $B_{-2}(g,\varphi) \neq 0$, the formula is in the *second term range*. In section 1.7 we will crucially use the second term range formula given by [GQT14]. Despite this, this section is devoted to explaining the idea of the proof for some cases of the first term range. This is because it illustrates in a simple way the nature of this formula.

We fix V a rational quadratic space of dimension m. Along this section we will denote by $H = \mathrm{O}(V)$ the orthogonal group associated to V. Let W be a rational symplectic space of dimension $2n$ and fix the basis $e_1, ..., e_n, e'_1, ..., e'_n \in W$ satisfying $\langle e_i, e_j\rangle = \left\langle e'_i, e'_j\right\rangle = 0$ and $\left\langle e_i, e'_j\right\rangle = \delta_{ij}$. Using these coordinates we have the algebraic group isomorphism

$$\mathrm{Sp}_{2n} \simeq \mathrm{Sp}(W).$$

To simplify the exposition in this subsection we denote by G the groups Sp_{2n} or Mp_{2n}, depending on the parity of m.

1.6.1 Degenerate principal series

For any $\mathbb{Q}-$algebra R, we denote

$$P(R) = \begin{cases} P(R) & when\ G = \mathrm{Sp}_{2n}, \\ \widetilde{P}(R) & when\ G = \mathrm{Mp}_{2n} \end{cases}$$

By lemma 1.5.13 we have Levi decompositions of the form $P(R) = M(R)N(R)$, where

$$M(R) = \begin{cases} M(R) & when\ G = \mathrm{Sp}_{2n}, \\ \widetilde{M}(R) & when\ G = \mathrm{Mp}_{2n} \end{cases},$$

and $N(R)$ are the $R-$points of the unipotent subgroup of the Siegel parabolic of Sp_{2n}. When $G = \mathrm{Mp}_{2n}$, the group $N(R)$ is a subgroup of G by means of the splitting

$$n \mapsto (n, 1),$$

see 1.5.13 for the details. Furthermore, lemma 1.5.9 shows that the maximal compact subgroup K_p of $\mathrm{Sp}_{2n}(\mathbb{Q}_p)$ may be regarded as a subgroup of $\mathrm{Mp}_{2n}(\mathbb{Q}_p)$. Hence, there is an Iwasawa decomposition of the form

$$G(R) = M(R)N(R)K,$$

where $K = K_p$ if $R = \mathbb{Q}_p$ or $K = \prod_p K_p$ if $R = \mathbb{A}$. We will factor each element $g \in G(R)$ according to the above decomposition:

$$g = m(a)n(b)k.$$

Local degenerate principal series

For the following discussion set $\rho_n = \frac{n+1}{2}$.

Definition 1.6.1. The local *degenerate principal series* is the representation of $G(\mathbb{Q}_p)$ given by the space of smooth functions $\Phi : G(\mathbb{Q}_p) \times \mathbb{C} \to \mathbb{C}$ satisfying

$$\Phi(m(a_p)g_p, s) = \tilde{\chi}_{V,p}(|a(g_p)|)|a(g_p)|^{s+\rho_n}\Phi(g, s),$$

where, if p is a non-archimedian place $g_p \in G(\mathbb{Q}_p)$ acts by right translation and, if p is an archimedian place $(\mathfrak{g}, K^\infty)$, it acts by right translation. This representation is denoted by $(\pi, I_n(s, \chi_V))$ or just by $I_{n,p}(s, \chi_{V,p})$ when the action is not relevant for the discussion.

Remark 1.6.2. *The local degenerate principal series representation is a particular case of the so-called induced representations. We refer the reader to section 2.3.1 for more details.*

Proposition 1.6.3. *Let $p \nmid \infty$. The local degenerate principal series $I_{n,p}(s, \chi_{V,p})$ is reducible exactly in the following situations:*

- *When $G = \mathrm{Sp}_{2n}$:*
  1. *$\chi_{V,p} = 1$ and $s = \frac{m-n+1}{2}$ with $m \in \mathbb{Z}$ even so that $0 \le m \le 2n+1$.*
  2. *$\chi_{V,p}^2 = 1$, $\chi_{V,p} \neq 1$, $s = \frac{m-n+1}{2}$ with $m \in \mathbb{Z}$ even such that $2 \le m \le 2n$.*
- *When $G = \mathrm{Mp}_{2n}$: $\chi_{V,p}^2 = 1$, $\chi_{V,p} \neq 1$ and $s = \frac{m-n+1}{2}$ with $m \in \mathbb{Z}$ odd satisfying $1 \le m \le 2n+1$.*

Proof. See [KR94, thm. 2.1, p. 18], [KR94, thm. 2.2, p. 18] and [GQT14, prop. 5.1, p. 24]. □

Along the following discussion we will denote by X_n the set of points of reducibility of $I_{n,p}(s, \chi_{V,p})$ given by proposition 1.6.3.

Definition 1.6.4. For any $s \in \mathbb{C}$ satisfying $\mathrm{Re}(s) > \frac{n+1}{2}$, the map

$$\begin{aligned} \mathrm{M}_p(s) :\ & I_n(s, \chi_V) \to I_n(-s, \chi_V^{-1}), \\ & \Phi(g, s) \mapsto \int_{N(\mathbb{Q}_p)} \Phi(\omega_l n g, s) dn, \end{aligned}$$

is a $G(\mathbb{Q}_p)-$intertwining map, where dn is the Haar measure of $N(\mathbb{Q}_p)$ and $\omega_n = \left(\begin{smallmatrix} & -I_n \\ I_n & \end{smallmatrix}\right)$ if $G = \mathrm{Sp}_{2n}$ or $\omega_n = \left(\left(\begin{smallmatrix} & -I_n \\ I_n & \end{smallmatrix}\right), 1\right)$ if $G = \mathrm{Mp}_{2n}$.

Proposition 1.6.5. *The intertwining maps $M_p(s)$ are defined by meromorphic continuation for almost all $s \in \mathbb{C}$.*

Proof. It is a particular case of [Win78, thm. 1, p. 951]. □

Global degenerate principal series

Definition 1.6.6. The *degenerate principal series* representation of $G(\mathbb{A})$ is the space of smooth functions $\Phi : G(\mathbb{A}) \times \mathbb{C} \to \mathbb{C}$ satisfying

$$\Phi(m(a)g, s) = \tilde{\chi}_V(|a(g)|)|a(g)|^{s+\rho_n}\Phi(g, s),$$

with $G(\mathbb{A})$ acting by right translation. We denote this representation by $(\pi, I_n(s, \chi_V))$ or just by $I_n(s, \chi_V)$ when the action is not relevant for the discussion.

Remark 1.6.7. *If $G = \mathrm{Sp}_{2n}$, we have $G(\mathbb{A}) = \prod' G(\mathbb{Q}_p)$. Therefore [GH22, thm. 5.7.1, p. 135] implies the following factorization*

$$I_n(s, \tilde{\chi}_V) \simeq \bigotimes_p I_{n,p}(s, \tilde{\chi}_{V,p}).$$

When $G = \mathrm{Mp}_{2n}$, the group $G(\mathbb{A})$ is not the restricted tensor product of the locally compact groups $G(\mathbb{Q}_p)$. Hence we are not able to reply the above argument for the present situation. Let us consider the tensor product representation $\left(\otimes_p \pi_p, \bigotimes_p I_{n,p}(s, \tilde{\chi}_{V,p})\right)$ of the group $\prod'_p G(\mathbb{Q}_p)$, see (1.5.4) for the definition of the group. We recall we have a projection map of the form

$$p : \prod_p{}' G(\mathbb{Q}_p) \to G(\mathbb{A}),$$

defined in (1.5.4). The map p has kernel $I = \{\prod_p (id, \xi_p), s.t. \prod_p \xi_p = 1\}$. By definition, the character $\otimes_p \tilde{\chi}_{V,p}$ is trivial on the elements of the form $\prod_p (id, \xi_p)$ with $\prod_p \xi_p = 1$. Then, the action of $\otimes_p \pi_p$ factors throughout the quotient $\prod'_p G(\mathbb{Q}_p)/I$, allowing us to regard the representation $\left(\otimes_p \pi_p, \bigotimes_p I_{n,p}(s, \tilde{\chi}_{V,p})\right)$ as a representation of the group $\prod'_p G(\mathbb{Q}_p)/I$, which, by (1.5.4), is isomorphic to $G(\mathbb{A})$. Therefore we conclude with the following isomorphism

$$\otimes_p \pi(g_p, \xi_p) = \pi\left(p\left(\prod_p (g_p, \xi_p)\right)\right),$$

obtaining the following isomorphism of $G(\mathbb{A})-$representations:

$$\bigotimes_p I_{n,p}(s, \tilde{\chi}_{V,p}) \simeq I_n(s, \tilde{\chi}_V).$$

Let us remark that for a general induced representation this local-global factorization is not always available.

Definition 1.6.8. Let s so that $\mathrm{Re}(s) > \frac{n+1}{2}$. For any $\Phi(g, s) \in I_n(s, \chi_V)$ we define the intertwining map

$$\begin{aligned} \mathrm{M}(s) :\ & I_n(s, \chi_V) \to I_n(-s, \chi_V^{-1}), \\ & \Phi(g, s) \mapsto \int_{N(\mathbb{A})} \Phi(\omega_l n g, s) dn, \end{aligned}$$

where dn is the Haar measure of $N(\mathbb{A})$ and $\omega_n = \left(\begin{smallmatrix} & -I_n \\ I_n & \end{smallmatrix}\right)$ if $G = \mathrm{Sp}_{2n}$ or $\omega_n = \left(\left(\begin{smallmatrix} & -I_n \\ I_n & \end{smallmatrix}\right), 1\right)$ if $G = \mathrm{Mp}_{2n}$. When m is even and $n + 1 < m \leq 2n$, we denote by $\mathrm{M}^*(s)$ the normalization proposed in [KR92, p. 218].

Proposition 1.6.9. *The map $\mathrm{M}(s)$ is extended for almost to all $s \in \mathbb{C}$ by meromorphic continuation.*

Proof. It is a particular case of [Art05, thm. 7.2, p. 35]. □

Remark 1.6.10. *The discussion of (1.5.4) yields the following equality:*

$$\mathrm{M}(s) = \otimes_p \mathrm{M}_p(s).$$

We will also denote $\mathrm{M}_f(s) = \otimes_{p \nmid \infty} \mathrm{M}_p(s)$.

1.6.2 Local theta correspondence of the identity

In this section we will consider the local theta correspondence of the identity from $H(\mathbb{Q}_p)$ to $G = \mathrm{Sp}_n(\mathbb{Q}_p)$ when p is a non-archimedian place.

We denote the trivial representation of $H(\mathbb{Q}_p)$ by $\mathbb{1}_V$. The local theta correspondence of the representation $\mathbb{1}_V$, for the dual reductive pair $(\mathrm{Sp}_{2n}(\mathbb{Q}_p), O(V)(\mathbb{Q}_p))$, yields the map

$$\mu_{V,p}\colon \mathcal{S}(V^n(\mathbb{Q}_p)) \to \mathbb{1}_V \otimes I_{n,p}\left(\frac{m}{2} - \frac{n+1}{2}, \tilde{\chi}_{V,p}\right),$$
$$\varphi_p \mapsto \omega(g_p)\varphi(0).$$

We recall that by definition $\mu_{V,p}\left(\mathcal{S}(V(\mathbb{Q}_p))\right) = \Theta(\mathbb{1}_{V_p})$.

Theorem 1.6.11. *The map $\mu_{V,p}$ induces an injection*

$$\Theta(\mathbb{1}_V) \to I_n\left(\frac{m}{2} - \frac{n+1}{2}, \tilde{\chi}_{V,p}\right).$$

Proof. It is in 1.5.26. $\square$

We denote by
$$R_{n,p}(V_p) := \mu_{V,p}\left(\mathcal{S}(V(\mathbb{Q}_p))\right).$$

Formally, we may regard the representation $\mathbb{1}_V$ as the Weil representation of the reductive dual pair $(O(V), \mathrm{id})$. Therefore, according to [Kud96, prop. 4.1, p. 45], the representations $R_{n,p}(V)$ are always non-trivial. The main goal of this section is to determine the irreducible constituent of the representations $R_{n,p}(V)$, which in general are reducible. In particular, by theorem 1.5.26, these representations have a unique irreducible sub-quotient, which we have denoted by $\theta(\mathbb{1}_V)$. As we saw in the proof of corollary 1.5.51, the search for these irreducible representations is essential to obtain the interplay between the local and global theta correspondence.

Remark 1.6.12. *Let V_1 and V_2 be quadratic spaces with the same dimension so that $\tilde{\chi}_{V_1,p} = \tilde{\chi}_{V_2,p}$ and they have with different Hasse invariant. There are two embeddings of the form*

$$\begin{tikzcd}
R_{n,p}(V_1) \arrow[dr] & \\
 & I_{n,p}\left(\frac{m}{2} - \frac{n+1}{2}, \tilde{\chi}_{V_1,p}\right) \\
R_{n,p}(V_2) \arrow[ur] &
\end{tikzcd}$$

For the following discussion, given a quadratic space V we will denote by V_1 and V_2 the quadratic spaces with $m = \dim V = \dim V_1 = \dim V_2$, $\tilde{\chi}_{V,p} = \tilde{\chi}_{V_1,p} = \tilde{\chi}_{V_2,p}$ and different Hasse invariants.

Proposition 1.6.13. *Let V be a quadratic space of even dimension so that $\chi_{V,p}^2 = 1$ and $\chi_{V,p} \neq 1$. For $s_0 = \frac{m-n+1}{2}$ we have*

1. *If $2 \leq m < n+1$ so that $s_0 < 0$, the representations $R_{n,p}(V_1)$ and $R_{n,p}(V_2)$ are irreducible, $R_{n,p}(V_1) \oplus R_{n,p}(V_2)$ is a submodule of $I_{n,p}(s_0, \tilde{\chi}_{V,p})$ and the quotient*
$$I_{n,p}(s, \tilde{\chi}_{V,p})/R_{n,p}(V_1) \oplus R_{n,p}(V_2),$$
is irreducible.

2. *If $m = n+1$ so that $s_0 = 0$, $R_n(V_1)$ and $R_{n,p}(V_2)$ are irreducible subrepresentations, and*
$$I_{n,p}(s, \tilde{\chi}_{V,p}) \simeq R_{n,p}(V_1) \oplus R_{n,p}(V_2).$$

3. *If $n+1 < m \leq 2n$, $R_{n,p}(V_1)$ and $R_{n,p}(V_2)$ are maximal submodules and*
$$R_{n,p}(V_1) \cap R_{n,p}(V_2),$$

is an irreducible subrepresentation.

Proof. See [KR92, p. 211]. □

Definition 1.6.14. Let us suppose V is a quadratic space of dimension $n+1 < m \leq 2n$ so that, if $m = 2n$, the space is not quaternionic. Let m_0 be the integer satisfying $m + m_0 = 2(n+1)$. The *complementary space*, denoted by V_0, is the quadratic space of dimension m_0 satisfying

$$V_0 = V_{an} \oplus \mathbb{H}^{r_0}.$$

Remark 1.6.15. *The space $R_{n,p}(V_0)$ is a subrepresentation of $I_{n,p}(-s_0, \tilde{\chi}_{V,p})$.*

Lemma 1.6.16. *Let V_1 and V_2 be quadratic spaces with the same even dimension m so that $n+1 < m \leq 2n$ and $\tilde{\chi}_{V_1,p} = \tilde{\chi}_{V_2,p}$, then*

$$\mathrm{M}^*(s_0)(R_{n,p}(V_i)) = R_{n,p}(V_{i,0}),$$
$$\mathrm{M}^*(s_0)(I_{n,p}(s_0, \tilde{\chi}_V)) = R_{n,p}(V_{1,0}) \oplus R_{n,p}(V_{2,0}).$$

Furthermore

$$\ker \mathrm{M}^*(s_0) = R_{n,p}(V_1) \cap R_{n,p}(V_2),$$

where all the above representations are $\mathrm{Sp}_{2n}(\mathbb{A})-$representations.

Proof. See [KR92, prop. 5.5, p. 240] and [KR92, prop. 5.6, p. 243]. □

Corollary 1.6.17. *Let V_1 and V_2 be quadratic spaces satisfying the hypothesis of lemma 1.6.16. Then*

$$R_{n,p}(V_i)/\left(R_{n,p}(V_1) \cap R_{n,p}(V_2)\right) \simeq R_{n,p}(V_{i,0}).$$

Proof. It follows directly by lemma 1.6.16. □

Remark 1.6.18. *If $n+1 < m \leq 2n$, hence $1 \leq \dim\ V_0 \leq n$. Therefore proposition 1.6.13 implies the irreducibility of the representation $R_{n,p}(V_0)$. The strategy used to determine certain local theta correspondences is based on the determination of the maps from our representation to representations associated to spaces satisfying the above hypothesis.*

The global theta correspondence provides analogous maps to (1.6.2)

$$\mu_{V}\colon \mathcal{S}(V^n(\mathbb{A})) \to I_n\left(\frac{m}{2} - \frac{n+1}{2}, \tilde{\chi}_{V,p}\right),$$
$$\varphi \mapsto \omega(g)\varphi(0),$$

and we denote

$$R_n(V) := \mu_V\left(\mathcal{S}(V^n(\mathbb{A}))\right).$$

Remark 1.6.19. *When $G = \mathrm{Sp}_{2n}$, according to [GH22, thm. 5.7.1, p. 135] we obtain the following isomorphism of $G(\mathbb{A})-$representations*

$$R_n(V) \to \otimes_p R_{n,p}(V).$$
$$\omega(g)\varphi(0) \mapsto \otimes \omega(g_p)\varphi_p(0).$$

For $G = \mathrm{Mp}_{2n}$, the equality (1.5.4) provides the $G(\mathbb{A})-$isomorphism

$$R_n(V) \to \otimes_p R_{n,p}(V),$$
$$\omega(g,\xi)\varphi(0) \mapsto \otimes_p \omega_p(g_p, \xi_p)\varphi_p(0),$$

with $(g,\xi) = p(\prod_p (g_p, \xi_p))$.

1.6.3 Automorphic representations associated to quadratic spaces

In the previous section we used the local theta correspondence of the identity to construct representations $R_n(V)$ of $G(\mathbb{A})$. This section is devoted to showing that these representations appear as subrepresentations of the space of automorphic forms $\mathcal{A}(G)$. We assume m even and hence $G = \mathrm{Sp}_{2n}$.

Definition 1.6.20. A representation (π, V) of $\mathrm{Sp}_{2n}(\mathbb{Q}_p)$ is *non singular in the sense of Howe* if there is a Schwartz function $f \in \mathcal{S}(\mathrm{Sym}_n(\mathbb{Q}_p))$ so that its Fourier transform $\hat{f}$ has support in the set of matrices $\beta \in \mathrm{Sym}_n(\mathbb{Q}_p)$ satisfying det $\beta \neq 0$ and f does not act by 0 in π, where the action of $f \in \mathcal{S}(\mathrm{Sym}_n(\mathbb{Q}_p))$ on V is given by

$$\pi(f)u = \int_{N(\mathbb{Q}_p)} f(b)\pi(n(b))udb,$$

with N is the Siegel parabolic of $\mathrm{Sp}_{2n}(\mathbb{Q}_p)$.

For the following discussion we consider $\mathcal{O}_V$, the space of quadratic forms of rank m that are equal to $\frac{1}{2}(x,x)$ for some $x \in V^n$ and whose components span V.

Lemma 1.6.21. *Let $\varphi_p \in \mathcal{S}(\mathrm{Sym}_n(\mathbb{Q}_p))$ be a function whose Fourier transform $\hat{\varphi}_p$ has support on the set*

$$\left(\mathcal{O}_{V_p} \cup \{\beta,\ s.t.\ \mathrm{rank}\ \beta > m\}\right) \cap \mathrm{Sym}_n(\mathbb{Z}_p).$$

Then φ_p does not act by 0 in $R_{n,p}(V)$.

Proof. In this proof we will use the following notation $w = \left(\begin{smallmatrix} & -I_n \\ I_n & \end{smallmatrix}\right)$. We consider $f_p \in R_{n,p}(V)$ satisfying $f_p = \omega(g)\alpha_p(0)$ with $\alpha_p \in \mathcal{S}(V^n(\mathbb{Q}_p))$. Applying the explicit formula of the local Weil representation given in proposition 1.5.23, we get

$$\pi(\varphi_p)\cdot v = \int_{\mathrm{Sym}_n(\mathbb{Q}_p)} \varphi_p(b) f_p(wn(b))db = \int_{\mathrm{Sym}_n(\mathbb{Q}_p)} \int_{V^n(\mathbb{Q}_p)} \psi_p\left(\mathrm{tr}\left(\frac{b}{2}(x,x)\right)\right)\alpha_p(x)\varphi_p(b)dxdb. \quad (1.\text{-}8)$$

Since φ_p is a Schwartz function it has compact support. Then we are able to change the order of the above integrals, obtaining that (1.6.3) is equal to

$$\int_{V^n(\mathbb{Q}_p)} \alpha_p(x)\hat{\varphi}_p\left(\frac{1}{2}(x,x)\right)dx$$
$$\geq \min\left(\hat{\varphi}_p|_{\left(\mathcal{O}_{V_p}\cup\{\beta,\ s.t.\ \mathrm{rank}\ \beta>m\}\right)\cap\mathrm{Sym}_n(\mathbb{Z}_p)}\right)\int_{\left(\mathcal{O}_{V_p}\cup\{\beta,\ s.t.\ \mathrm{rank}\ \beta>m\}\right)\cap\mathrm{Sym}_n(\mathbb{Z}_p)} \alpha_p(x)dx,$$

Therefore, for a suitable choice of $\alpha_p(x)$, the above integral does not vanish. □

Theorem 1.6.22. *Let V be a quadratic space of even dimension satisfying $m \leq n$, then*

$$\dim \mathrm{Hom}_{G(\mathbb{A})}\left(R_n(V), \mathcal{A}(G)\right) = 1.$$

Proof. Given $\beta \in \mathrm{Sym}_n(\mathbb{Q})$ we define a character of $N(\mathbb{A})$ of the form

$$\begin{aligned}\psi_\beta :\ & N(\mathbb{A}) \to \mathbb{C},\\ & n(b) \mapsto \psi(\mathrm{tr}(\beta b)).\end{aligned}$$

We consider the space $\mathcal{W}_\beta(R_n(V))$, consisting on functionals

$$\mu : \bigotimes_{p\nmid\infty} R_{n,p}(V) \to \mathbb{C},$$

so that $\mu(\pi(n)f) = \psi_\beta(n)\mu(f)$ for all $n \in N(\mathbb{A}_f)$ and $\mu(\pi_\infty(X)f) = d\psi_\beta(X)\mu(f)$ for all $X \in \mathrm{Lie}\ N(\mathbb{R})$. Let us denote by $\mathcal{W}^\infty(R_n(V))$ the subspace of functionals of $\mathcal{W}_\beta(R_n(V))$ having a continuous extension to

$$R_n^\infty(V) = R_{n,\infty}^\infty(V) \otimes \bigotimes_{p\nmid\infty} R_{n,p}(V)$$

where $R_{n,\infty}^\infty(V)$ is the subspace of $R_{n,\infty}(V)$ of smooth functions.

According to [KR94, prop. 2.4, p. 20] for the finite places and to [KR94, prop. 2.9, p. 23] for the infinite place, if rank $\beta > m$ or if rank $\beta = m$ and $\beta \notin \mathcal{O}_V$, we have

$$\mathcal{W}_\beta^\infty(R_n(V)) = 0. \tag{1.-8}$$

Otherwise, if $\beta \in \mathcal{O}_V$

$$\dim \mathcal{W}_\beta^\infty(R_n(V)) = 1. \tag{1.-8}$$

Given $f \in R_n(V)$ we consider its $\psi_\beta-$Fourier coefficient

$$W_{\psi_\beta}(f, g) = \int_{[N]} f(n(b)g)\psi(-\mathrm{tr}(\beta b))db.$$

It defines a functional

$$\begin{aligned} W_\beta :\ \mathcal{A}(G) &\to \mathbb{C}, \\ f &\mapsto W_{\psi_\beta}(f, \mathrm{id}), \end{aligned}$$

satisfying $W_\beta(\pi(n(b))f) = \psi(\mathrm{tr}(\beta b))W_\beta(f)$. Therefore, the map

$$R_{n,\infty}^\infty(V) \otimes \bigotimes_{p\nmid\infty} R_{n,p}(V) \to R_n(V) \xrightarrow{W_\beta(\cdot)} \mathbb{C},$$

is an element of the space $\mathcal{W}^\infty(R_n(V))$. For any $A, B \in \mathrm{Hom}_G(R_n(V), \mathcal{A}(G))$, the elements $A_\beta = W_\beta \circ A$ and $B_\beta = W_\beta \circ B$ belong to the space $\mathcal{W}_\beta^\infty(R_n(V))$. Hence, by (1.6.3), if rank $\beta > m$ or if rank $\beta = m$ and $\beta \notin \mathcal{O}_V$,

$$A_\beta = B_\beta = 0, \tag{1.-8}$$

otherwise, by (1.6.3)

$$A_\beta = c_\beta B_\beta,$$

where $c_\beta \in \mathbb{C}$. Using a change of variables we get

$$A_\beta(\pi(m(a))f) = W_\beta\left(A(\pi(m(a))f)\right) = W_{{}^t a\beta a}(A(f)) = A_{{}^t a\beta a}(f).$$

The group $\mathrm{GL}_n(\mathbb{Q})$ acts on $\mathcal{O}_V$ by

$$\beta \mapsto {}^t a\beta a,$$

for any $a \in \mathrm{GL}_n(\mathbb{Q})$. This action has one orbit. Then, every element $\beta' \in \mathcal{O}_V$ can be expressed as $\beta' = {}^t a\beta a$ for certain $a \in \mathrm{GL}_n(\mathbb{Q})$, implying $c_\beta = c_{\beta'}$ for every $\beta, \beta' \in \mathcal{O}_V$. Therefore we conclude that the constant $c := c_\beta$ is independent of β.

Let us define $D = A - cB$. Given $\varphi_p \in \mathcal{S}(N(\mathbb{Q}_p))$ a Schwartz function satisfying the hypothesis of lemma 1.6.21 and $f \in R_n(V)$, we have

$$\pi(\varphi_p) \cdot D(f)(g) = \int_{\mathrm{Sym}_n(\mathbb{Q}_p)} \varphi_p(b)D(f)(gn(b))db.$$

Since $N(\mathbb{A})$ is commutative, the automorphic form $D(f)$ has Fourier expansion

$$D(f)(g) = \sum_{\gamma\in\mathrm{Sym}_n(\mathbb{A})} W_{\psi_\gamma}(D(f), g).$$

Then

$$\int_{\mathrm{Sym}_n(\mathbb{Q}_p)} \varphi_p(b) D(f)(gn(b))db = \sum_{\gamma \in \mathrm{Sym}_n(\mathbb{A})} \int_{\mathrm{Sym}_n(\mathbb{Q}_p)} \varphi_p(b) W_{\psi_\gamma}(D(f), gn(b))db$$
$$= \sum_{\gamma \in \mathrm{Sym}_n(\mathbb{A})} \hat{\varphi}_p(g) W_{\psi_\gamma}(D(f), g).$$

By (1.6.3), the function $W_{\psi_\gamma}(D(f), g)$ vanishes if rank $\beta > m$ or if rank $\beta = m$ and $\beta \notin \mathcal{O}_V$. Otherwise, the function $\hat{\varphi}_p$ vanishes by hypothesis. Then we get

$$\sum_{\gamma \in \mathrm{Sym}_n(\mathbb{A})} \hat{\varphi}_p(g) W_{\psi_\gamma}(D(f), g) = 0, \tag{1.-8}$$

and the function φ_p acts by 0 on the image of D. Since the map D is intertwining and $R_n(V)$ irreducible, proposition 1.6.13, lemma 1.6.21 and the equation (1.6.3) imply $D = 0$, obtaining $A = cB$ and hence the statement of the theorem. □

Remark 1.6.23. *The space $\mathcal{W}_\beta(R_n(V))$ is called the space of Whittaker functionals and it plays a relevant role in the theory of automorphic representations. See section 2.4.1 for a detailed exposition of the subject.*

1.6.4 Siegel Eisenstein series

In this section we will construct the Eisenstein series involved in the Siegel-Weil formula. For the general construction of Eisenstein series see 1.4.13. Given $\Phi(g, s) \in I_n(s, \chi_V)$ and s so that $\mathrm{Re}(s) \gg 0$, we define

$$E(g, s, \Phi) = \sum_{\gamma \in P(\mathbb{Q}) \backslash \mathrm{Sp}_{2n}(\mathbb{Q})} \Phi(\gamma g, s). \tag{1.-8}$$

When $G = \mathrm{Sp}_{2n}$ it is an automorphic form on $[\mathrm{Sp}_{2n}]$. In contrast, when $G = \mathrm{Mp}_{2n}$ this function is an automorphic form on $\mathrm{Sp}_{2n}(\mathbb{Q}) \setminus \mathrm{Mp}_{2n}(\mathbb{A})$. We denote this space also by $\mathcal{A}(G)$ and we refer the reader to [Mor83] for further details about automorphic forms on metaplectic groups.

Lemma 1.6.24. *The Eisenstein series $E(g, s, \Phi)$ admits a meromorphic continuation and the following functional equation:*

$$E(g, s, \Phi) = E(g, -s, \mathrm{M}(s)\Phi).$$

Proof. See [Art05, thm. 7.2, (a), p. 35]. □

Theorem 1.6.25. *For $\mathrm{Re}(s) > 0$ the Eisenstein series $E(g, s, \Phi)$ is holomorphic except at $s \in X_n$; the points of reducibility of the local degenerate principal series $I_{n,p}(s, \chi_{V,p})$, where $E(g, s, \Phi)$ has a pole of order at most 1.*

Proof. See [KR94, thm. 1.1, p. 12] and [GQT14, prop. 6.1, p. 18]. □

Given $s_n \in X_n$, according to theorem 1.6.25, the Eisenstein series $E(g, s, \Phi)$ has the following Laurent expansion at $s = s_n$:

$$E(g, s, \Phi) = \frac{\mathrm{Res}_{s=s_n} E(g, s, \Phi)}{s - s_n} + \mathrm{CT}_{s=s_n} E(g, s, \Phi) + \mathcal{O}(s - s_n).$$

The map

$$\mathrm{Res}_{s=s_n} E(g, s, \cdot) : \ I_n(s, \chi_V) \to \mathcal{A}(G), \tag{1.-8}$$

is intertwining. The image of the above morphism is not at first an automorphic representation because $I_n(s, \chi_V)$ is not irreducible. To obtain such maps we would have to prove the factorization stated in (1.6.4): The map should factor throughout an irreducible sub-quotient of $I_n(s, \chi_V)$. Moreover, the map

$$\mathrm{CT}_{s=s_n} E(g, s, \cdot) : \ I_n(s, \chi_V) \to \mathcal{A}(G), \tag{1.-8}$$

is an intertwining map modulo the image of (1.6.4).

Theorem 1.6.26. *The residue of the Eisenstein series* $\mathrm{Res}_{s=s_n} E(g,s,\cdot)$ *factors throughout* $R_n(V_0)$ *and it is different from* 0.

Proof. Let V' be a quadratic space of dimension m and character $\tilde{\chi}_{V',p} = \tilde{\chi}_{V,p}$. Corollary 1.6.17 shows

$$R_{n,p}(V)/R_{n,p}(V) \cap R_{n,p}(V') \simeq R_{n,p}(V_0).$$

Then, the proof follows by checking the vanishing of the residue of the Eisenstein series in the subspace $R_{n,p}(V) \cap R_{n,p}(V')$. This proof is similar to the proof of theorem 1.6.22, see [KR94, thm. 4.9, p. 34] for the details. □

Classical Eisenstein series

In subsequent sections of this chapter we will consider certain Eisenstein series of complex variable. They are defined using the construction 1.6.4.

Lemma 1.6.27. *Given* $l \in \frac{1}{2}\mathbb{Z}$, *there is a unique function* $\Phi^l(g,s) \in I_1(s,\chi_V)$ *so that*

$$\Phi^l(k_\theta, s) = e^{\pi i l \theta},$$

when $k_\theta = \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right) \in \mathrm{SO}(2)$.

Proof. See [Kud03, p. 39]. □

Definition 1.6.28. The section $\Phi^l(g,s)$ of lemma 1.6.27 is called *the section of weight* l of $I_1(s,\chi_V)$.

The sections of weight l are in the image of the map

$$\lambda_\infty : \mathcal{S}(V(\mathbb{R})) \to I_{1,\infty}(s,\chi_V),$$

defined in (1.6.2). In fact, according to [Kud03, (1.28), p. 10], we get

$$\Phi^l\left(g, \frac{m}{2} - 1\right) = \lambda_\infty(\varphi_l^\infty)(g),$$

where $\varphi_l^\infty \in \mathcal{S}(V(\mathbb{R}))$ is a Schwartz function satisfying that $\omega_\infty(k_\theta)\varphi_l^\infty(x) = e^{\pi i l\theta}\varphi_l^\infty(x)$. Moreover, for any $\varphi_f \in \mathcal{S}(V(\mathbb{A}_f))$ we denote

$$E(\tau, s, l, \mu(\varphi_f)) := v^{-l/2} E(g_\tau, s, \Phi^l \otimes \lambda(\varphi_f)),$$

where $g_\tau \in G(\mathbb{R})$ is defined as in 1.5.60.

Proposition 1.6.29. *The* SL_2-*Einsenstein series satisfy the following relation:*

$$-2iv^2 \frac{\partial}{\partial \overline{\tau}} \left\{ v^{\frac{-1}{2}(l+2)} E(g_\tau, s, \Phi^{l+2} \otimes \lambda(\varphi_f)) \right\} = \frac{1}{2}(s-l-1) v^{\frac{l-1}{2}} E(g_\tau, s, \Phi^l \otimes \lambda(\varphi_f)),$$

and hence

$$-2iv^2 \frac{\partial}{\partial \overline{\tau}} \{E(\tau, s, l+2, \mu(\varphi_f))\} = \frac{1}{2}(s-l-1) E(\tau, s, l, \mu(\varphi_f)).$$

Proof. It is [Kud03, (2.15), p. 20]. □

In subsequent sections of this paper we will consider the Eisenstein series $E(\tau, s, 3/2, \mu(\varphi_f))$. According to [KY10, cor. 2.5, p. 2283], this function is holomorphic in the variable s. We denote by

$$E(\tau, s, 3/2, \mu(\varphi)) = \sum_{m \in \mathbb{Q}} A(s, m, v, \mu(\varphi)) q^m,$$

its Fourier series. Furthermore, we denote the Laurent series at $s = 1/2$ of each Fourier coefficient by

$$A(s, m, v, \mu(\varphi)) = a(m) + b(m, v, \mu(\varphi))(s - s_0) + \mathcal{O}((s - s_0)^2). \tag{1.-8}$$

1.6.5 Hecke algebra

Throughout this subsection, we will discuss the definition and main properties of the so-called Hecke algebra. Although in this section this object will be used to define the regularization of the theta function, it plays a crucial role in the theory of representations of topological groups, as described in section 2.3. In this subsection we will use concepts such as dual group, see 2.1.18, and Weyl group, see 2.1.15.

For the following definition we will denote by $\mathfrak{G}$ a topological group with maximal compact subgroup $K_{\mathfrak{G}}$.

Definition 1.6.30. The *Heke algebra* $\mathcal{H}^{\mathfrak{G}}$ is the ring of locally constant compactly supported functions $\mathfrak{G} \to \mathbb{Z}$ which are $K_{\mathfrak{G}}-$bi-invariant, endowed with multiplication law defined by convolution; given $f, g \in \mathcal{H}^{\mathfrak{G}}$

$$(f * h)(g) = \int_H f(x)h(x^{-1}g)dx,$$

where dx is the Haar measure of $\mathfrak{G}$ giving $K_{\mathfrak{G}}$ volume 1.

Set H a split reductive group defined over $\mathbb{Q}_p$. We fix a Borel subgroup $B = TN$ with maximal split torus T. This choice of Borel subgroup determines a set of positive roots denoted by Φ^+. We set $K_p = H(\mathbb{Z}_p)$ the maximal compact subgroup of $H(\mathbb{Q}_p)$. In this situation we will use the following notation $\mathcal{H}^H_p := \mathcal{H}^{H(\mathbb{Q}_p)}$.

Remark 1.6.31. *Let (π, V) be a representation of $H(\mathbb{Q}_p)$, there exists a representation*

$$\begin{aligned} \pi :\ \mathcal{H}^H_p &\to \mathrm{Aut}(V), \\ f &\mapsto \left(v \mapsto \int_{H(\mathbb{Q}_p)} f(g)\pi(g)v dg \right). \end{aligned}$$

This map allows us to regard V as a $\mathcal{H}^H_p-$module.

Proposition 1.6.32. *The group $H(\mathbb{Q}_p)$ is the disjoint union of the double cosets $K_p\mu(p)K_p$, where μ are the cocharacters of the maximal split torus of H belonging to the positive Weyl chamber of the dual group, (see 2.1.48 for the definition) i.e. $\mu \in X_\bullet(T) \cap P(\hat{\Phi}^+)$.*

Proof. See [BC80a, p. 51]. ☐

Remark 1.6.33. *The elements of the Hecke algebra are constant functions on the double cosets K_pxK_p. According to proposition 1.6.32, the characteristic functions $c_\mu := \mathrm{char}(K_p\mu(p)K_p)$ form a $\mathbb{Z}-$basis of $\mathcal{H}^H_p$.*

Proposition 1.6.34. *The Hecke algebra $\mathcal{H}^T_p$ is commutative and hence the map*

$$\begin{aligned} \mathcal{H}^T_p &\to \mathbb{Z}[X_\bullet(T)], \\ c_\mu &\mapsto [\mu], \end{aligned}$$

is an isomorphism.

Proof. The proposition follows by remark 1.6.33. ☐

Definition 1.6.35. The *Satake transform* is defined by the following ring homomorphism:

$$\begin{aligned} S :\ \mathcal{H}^H_p &\to \mathcal{H}^T_p \otimes \mathbb{Z}[p^{1/2}, p^{-1/2}], \\ f &\mapsto S(f)(t) := \delta(t)^{1/2} \int_{N(\mathbb{Q}_p)} f(tn)dn, \end{aligned}$$

where δ is the modulus character of the $B(\mathbb{Q}_p)$ and dn is the right invariant Haar measure of $N(\mathbb{Q}_p)$.

Theorem 1.6.36 (Satake isomorphism)**.** *The Satake transform is the ring isomorphism*

$$\mathcal{H}_p^H \simeq \mathbb{Z}[X_\bullet(T)]^{W(H,T)} \otimes \mathbb{Z}[p^{1/2}, p^{-1/2}].$$

Proof. By [Gro98, prop. 3.6, p. 6] we have the ring isomorphism

$$\mathcal{H}_p^H \simeq R(\hat{H}),$$

where $R(\hat{H})$ denotes ring of finite dimensional representation of the complex Lie group $\hat{H}$. To conclude, proposition 2.1.55 and remark 2.1.21 shows the isomorphism $R(\hat{H}) \simeq \mathbb{Z}[X_\bullet(T)]^{W(H,T)}$. □

Corollary 1.6.37. *The Hecke algebra $\mathcal{H}_p^H$ is commutative.*

Proof. It follows by theorem 1.6.36 and proposition 1.6.34. □

Example 1.6.38. *In this example we compute $\mathbb{Z}[X_\bullet(T)]^{W(H,T)(\mathbb{Q}_p)}$ as a ring of polynomials when $H = \mathrm{Sp}_n$ and $T = \{a_1 \times \cdots \times a_n \times a_1^{-1} \times \cdots \times a_n^{-1},\ a_i \in \mathbb{G}_m\}$. The cocharacters of the torus are $\mathbb{Z}-$linear combinations of the following maps:*

$$\begin{aligned}\hat{\alpha}_i :\ &\mathbb{G}_m \to T \simeq \mathbb{G}_m^n, \\ &c \mapsto 1 \times \overset{i}{\cdots} \times 1 \times c \times 1 \times \overset{n}{\cdots} \times 1 \times c^{-1} \times 1 \times \overset{i}{\cdots} \times 1,\end{aligned}$$

where $i < n$. The group $W(\mathrm{Sp}_n, T_n)$ acts on the set of cocharacters $\{\hat{\alpha}_i\}_{i=1}^n$ as the permutation group of degree n, denoted by $\mathcal{S}_n$. Therefore, the following identification holds:

$$\mathbb{Z}[X_\bullet(T_n)]^{W(\mathrm{Sp}_n,T)} \simeq \mathbb{Z}[\hat{\alpha}_1, ..., \hat{\alpha}_n]^{\mathcal{S}_n}. \tag{1.-8}$$

Definition 1.6.39. Fix V a quadratic space of dimension m. Let $\mathcal{H}^{\mathrm{Mp}_{2n}(\mathbb{Q}_p)}$ be the Hecke algebra of the topological group $\mathrm{Mp}_{2n}(\mathbb{Q}_p)$. We consider the following subring

$$\widetilde{\mathcal{H}}_{p,m}^{\mathrm{Mp}_{2n}} = \{\alpha \in \mathcal{H}^{\mathrm{Mp}_{2n}(\mathbb{Q}_p)},\ s.t.\ \alpha(\varepsilon g) = \varepsilon^m \alpha(g), \}$$

where $\varepsilon = (1, -1) \in \mathrm{Mp}_{2n}(\mathbb{Q}_p)$.

Lemma 1.6.40. *If $p \equiv 1$ (4) and m is odd*

$$\widetilde{\mathcal{H}}_{p,m}^{\mathrm{Mp}_{2n}} \simeq \mathcal{H}_p^{\mathrm{Sp}_{2n}}.$$

Proof. See [Ich01, p. 214]. □

To simplify the forthcoming exposition, we will always assume m, the dimension of V, being odd and the place p satisfying $p \equiv 1$ (4). Therefore, according to (1.6.40), from now on we will denote $\mathcal{H}_p^{\mathrm{Mp}_{2n}} := \mathcal{H}_p^{\mathrm{Sp}_{2n}}$.

1.6.6 Regularization map

To lighten the notation we fix V a rational quadratic space of odd dimension and we consider the algebraic group $H := O(V)$. In spite of this assumption, the following construction is done in complete generality, see [Ich01]. We fix a finite place p so that $p \equiv 1$ (4). In this section we explain the construction of Ichino (see [Ich01]) of an element $\alpha \in \mathcal{H}_p^G$ which defines a map of the form

$$\omega(\alpha) :\ \mathcal{S}(V^n(\mathbb{A})) \to \mathcal{S}(V^n(\mathbb{A}))_{abc}, \tag{1.-8}$$

where

$$\mathcal{S}(V^n(\mathbb{A}))_{abc} = \left\{\varphi \in \mathcal{S}(V(\mathbb{A})),\ s.t.\ \int_{[H]} \theta(g, h, \varphi),\ \textit{is absolutely convergent for all } g \in G(\mathbb{A})\right\}.$$

This construction allows us to relate the local theta correspondence of the identity with the regularized global theta correspondence of the identity as we explained in remark 1.5.52. When V has even dimension, there is an analogous construction for the place $p = \infty$ by Kudla and Rallis [KR94, cor. 5.1.2, p. 43]. They define an element in the center of the universal enveloping algbera of Lie G, providing an analogous map to (1.6.6). This construction is completely equivalent to the one we will describe in this subsection.

Proposition 1.6.41. *Let us choose a natural number n satisfying $r \leq n$. There exists an algebra homomorphism*

$$\theta : \mathcal{H}_p^G \to \mathcal{H}_p^H,$$

so that for all $\alpha \in \mathcal{H}_p^G$:

$$\omega(\alpha) = \omega(\theta(\alpha)) \in \mathrm{Aut}\left(\mathcal{S}(V^n(\mathbb{Q}_p))^{G(\mathbb{Z}_p)\times H(\mathbb{Z}_p)}\right).$$

Proof. The proof and definition of this map are in [Ich01, prop. 1.1, p. 206]. □

Proposition 1.6.42. *Let $T_{2n,\mathbb{Q}_p}$ be the maximal torus of $\mathrm{Sp}_{2n,\mathbb{Q}_p}$. If the Witt index of V satisfies that $r < n$, there exists a unique element*

$$\alpha_{n,r,\eta} = \sigma_{r+1} + \sum_{l=1}^{r} a_l \sigma_l \in \mathcal{H}_p^{\mathrm{Sp}_{2n}},$$

where σ_l is the $l-$symmetric polynomial in $Z[X_\bullet(T_{2n})]^{W(\mathrm{Sp}_{2n,\mathbb{Q}_p},T_{2n,\mathbb{Q}_p})}$ and $a_l \in \mathbb{C}$ such that $\theta(\alpha_{n,r,\eta}) = 0$.

Proof. See [Ich01, prop. 1.3, p. 209]. □

Lemma 1.6.43. *The element $\alpha_{n,r-1,\eta}$ satisfies the following equality:*

$$\theta(\alpha_{n,r-1,\eta}) = \eta^r \prod_{j=1}^{r} \left(Y_j - C_{m/2-n-1}\right),$$

where $\mathcal{H}_p^H = \mathbb{Z}[Y_1, ..., Y_r]^{\mathcal{S}_r} \otimes \mathbb{Z}[p^{1/2}, p^{-1/2}]$, $C_s = p^s + p^{-s}$ and $\eta = \eta_Q = \gamma_k(p, \psi_v \circ Q)^{-1} \in \{\pm 1\}$ is the Weil index defined in [Kud96, p. 12].

Proof. The proof is analogous to [KR94, lem. 5.5.4, p. 52], see [Ich01, (1.1), p. 209]. □

Remark 1.6.44. *By corollary 1.6.37 the Hecke algebra $\mathcal{H}_p^G$ is commutative. Then, for every element $\Phi(g,s) \in I_n(s, \tilde{\chi}_{V,p})$*

$$\theta(\alpha_{n,r-1,\eta}) \cdot \Phi(g,s) = c\Phi(g,s),$$

with $\cdot$ the action defined in remark 1.6.31 and $c \in \mathbb{C}$.

Example 1.6.45. *In this example we compute explicitly the element $\theta(\alpha_{n,r-1,\eta})$ when the dual reductive pair is $(\mathrm{Sp}_{2,\mathbb{Q}_p}, H_{\mathbb{Q}_p})$ with V a rational isotropic quadratic space of signature $(2,1)$. This setting will be relevant in the forthcoming exposition.*

The orthogonal group satisfies $H_{\mathbb{Q}_p} = O(V \otimes \mathbb{Q}_p)$. The dimension of the maximal torus T^H of $H_{\mathbb{Q}_p}$ is equal to 1. We fix $\mu \in X_\bullet(T^H)$ the generator of the group of cocharacters of T^H. Hence, the ring $\mathcal{H}_p^H$ is unitary and generated by $\mathrm{char}_{K_p\mu(p)K_p}(h)$ and $\mathrm{char}_{K_p}(h)$, where the latter element is the unit of the ring. By proposition 1.6.42

$$\theta(\alpha_{1,0,\eta}) = \eta\left(Y_1 - C_{-1/2}\right).$$

From the proof of [Ich01, lem. 1.4, p. 208], we deduce $\int_{H(\mathbb{Q}_v)} S^{-1}(Y_1)(h)dh = C_{1/2}$. Therefore, the Satake isomorphism given in 1.6.36 provides the following equality:

$$\mathcal{S}^{-1}(\eta Y_1 - \eta C_{-1/2})(h) = \eta C_{1/2}\mathrm{char}_{K_p\mu(p)K_p}(h) - \eta C_{-1/2}\mathrm{char}_{K_p}(h).$$

Let $\mathbb{1}_V$ be the trivial representation of $H(\mathbb{Q}_p)$, then

$$\theta(\alpha_{1,0,\eta}) \cdot \mathbb{1}_V = \eta \int_{H(\mathbb{Q}_p)} C_{1/2}\mathrm{char}_{K_p\mu(p)K_p}(h_v) - C_{-1/2}\mathrm{char}_{K_p}(h_v)dh_v = 0. \tag{1.-8}$$

The vanishing of (1.6.45) prevents us from using the first term identity of [GQT14, thm. 8.1, (i), p. 35]. In fact, this equality is the motivation to introduce the second term identity in the Siegel-Weil formula.

Let us fix a Siegel set $\mathcal{G}_H$ of $H(\mathbb{A})$ (see 1.4.1 for the definition) so that $H(\mathbb{A}) = H(\mathbb{Q})\mathcal{G}_H$.

Definition 1.6.46. Given $\varphi \in \mathcal{S}(V^n(\mathbb{A}))$ we fix a finite place p satisfying the following conditions:

- p is finite and does not divide 2.
- $p \equiv 1 \bmod 4$.
- ψ_p is of order 0.
- $q_p(\cdot)$ is unramified.
- $H(\mathbb{Q}_p) \cap \mathcal{G}_H = H(\mathbb{Z}_p)$.
- The Schwartz function φ is $\mathrm{Sp}_n(\mathbb{Z}_p) \times H(\mathbb{Z}_p)-$fixed.

Remark 1.6.47. *For any $\varphi \in \mathcal{S}(V^n(\mathbb{A}))$, there exists an infinite number of finite places where the hypothesis of 1.6.46 are satisfied.*

1.6.7 Regularized theta function

This section is devoted to introducing the replacement for $\theta(g, h, \varphi)$ in the regularized version of the Siegel-Weil formula.

Proposition 1.6.48. *The following map is an isomorphism*

$$\mathcal{S}(V^n(\mathbb{A})) \to \mathcal{S}(V^n_{an}(\mathbb{A})) \otimes \mathcal{S}(W^r(\mathbb{A})),$$
$$\varphi(x) \mapsto \hat{\varphi}(x_0, u, v) := \int_{M_{r,n}(\mathbb{A})} \varphi \begin{pmatrix} x \\ x_0 \\ u \end{pmatrix} \psi_{\mathbb{A}}(\mathrm{tr}(v^t x))dx,$$

where $w := (u, v) \in W^r(\mathbb{A})$ and $\psi_{\mathbb{A}}$ is the adelic character used to define the global Weil representation (1.5.4).

Proof. See [KR94, (5.3.2), p. 45]. □

Definition 1.6.49. The isomorphism given by 1.6.48 allows us to consider the representation

$$H(\mathbb{A}) \times G(\mathbb{A}) \to \mathrm{Aut}(\mathcal{S}(V^n_{an}(\mathbb{A})) \otimes \mathcal{S}(W^r(\mathbb{A})))$$
$$(g, h) \mapsto \left(\hat{\varphi}(x_0, u, v) \mapsto \int_{M_{r,n}(\mathbb{A})} \omega(g, h)\varphi \begin{pmatrix} x \\ x_0 \\ u \end{pmatrix} \psi_{\mathbb{A}}(\mathrm{tr}(v^t x))dx \right),$$

called the *mixed model* of the Weil representation.

Remark 1.6.50. *Using the partial Poisson summation formula of [KR94, (5.3.4), p. 45], the theta function satisfies*

$$\theta(g, h, \varphi) = \sum_{\substack{x_0 \in V_{an}(\mathbb{Q}) \\ w \in W(\mathbb{Q})}} \hat{\varphi}(x_0, w).$$

By the proof of [KR94, (5.3.6), p. 46] the sum

$$\sum_{\substack{x_0 \in V_{an}(\mathbb{Q}) \\ w \in W(\mathbb{Q}) \\ \mathrm{rank}\ w = r}} \hat{\varphi}(x_0, w),$$

is rapidly decreasing in the sense of 1.4.2.

Proposition 1.6.51. *Let V be a quadratic space so that $r > 0$, $m - r \leq n + 1$ and $r \leq n$. Given $\varphi \in \mathcal{S}(V^n(\mathbb{A}))$, we choose a finite place p satisfying the hypothesis of 1.6.46 and then*

$$\sum_{\substack{x_0 \in V_{an}(\mathbb{Q}) \\ w \in W(\mathbb{Q}) \\ \mathrm{rank}\ w = r}} \omega(\alpha_{n,r-1,\eta}) \hat{\varphi}(x_0, w) = 0,$$

and

$$\int_{[H]} \theta(g, h, \omega(\alpha)\varphi) dh,$$

is absolutely convergent for all $g \in G(\mathbb{A})$.

Proof. Since the group $G(\mathbb{Q}) \prod_{v \neq p} G(\mathbb{Q}_v)$ is dense in $G(\mathbb{A})$, we will assume $g \in K_v$ (we recall that when $G = \mathrm{Mp}_{2n}$, due to the splitting of K_v, this hypohtesis implies $g = (g', 1)$ for $g' \in \mathrm{Sp}_{2n}(\mathbb{Z}_p)$). The previous assumption implies

$$\omega(\alpha_{n,r-1,\eta})\omega(g,h)\hat{\varphi}(x_0, w) = \omega(g,h)\omega(\alpha_{n,r-1,\eta})\hat{\varphi}(x_0, w).$$

Let us define the $G(\mathbb{Q}_p)-$intertwining map

$$\mathcal{S}(V_{an}^n(\mathbb{Q}_p)) \otimes \mathcal{S}(W^r(\mathbb{Q}_p)) \to \mathcal{S}(V_{an}^n(\mathbb{Q}_p)) \otimes \mathcal{S}(W^{r-1}(\mathbb{Q}_p)),$$
$$\varphi(u, w) \mapsto \varphi_0,$$

where $\varphi_0(x_0, w_0) = \varphi\left(x_0, \left(\begin{smallmatrix} w_0 \\ 0 \end{smallmatrix}\right)\right)$. The right hand side of (1.6.7) is the mixed model of the Weil representation for the dual reductive pair $(\mathrm{Sp}_{2n}(\mathbb{Q}_p), O(V_{an} \oplus V^{r-1})(\mathbb{Q}_p))$. We denote this representation by ω_{r-1}. For any $w \in W^r(\mathbb{Q}_p)$ so that rank $w < r$, there exists $a \in \mathrm{GL}_r(\mathbb{Z}_p)$ and $w_0 \in W^{r-1}(\mathbb{Q}_p)$ satisfying $w = a\left(\begin{smallmatrix} w_0 \\ 0 \end{smallmatrix}\right)$. Hence, given $w \in W^r(\mathbb{Q}_p)$ with rank $w < r$ we get

$$\omega(\alpha_{n,r-1,\eta})\hat{\varphi}(x_0, w) = \omega(\alpha_{n,r-1,\eta})\hat{\varphi}\left(x_0, a\left(\begin{smallmatrix} w_0 \\ 0 \end{smallmatrix}\right)\right) = \omega(\alpha_{n,r-1,\eta})\hat{\varphi}\left(x_0, \left(\begin{smallmatrix} w_0 \\ 0 \end{smallmatrix}\right)\right),$$

where we have used the $\mathrm{Sp}_{2n}(\mathbb{Z}_p)-$invariance of φ_p. Moreover, since (1.6.7) is $G(\mathbb{Q}_p)-$intertwining we obtain

$$\omega(\alpha_{n,r-1,\eta})\hat{\varphi}\left(x_0, \left(\begin{smallmatrix} w_0 \\ 0 \end{smallmatrix}\right)\right) = \omega_{V^{r-1}}(\alpha_{n,r-1,\eta})\hat{\varphi}\left(x_0, \left(\begin{smallmatrix} w_0 \\ 0 \end{smallmatrix}\right)\right) = 0,$$

where the equality follows by propositions 1.6.41 and 1.6.42. The last assertion follows directly by remark 1.7.2. ☐

In order to obtain a replacement for the theta function in the present situation, we may introduce an *auxiliary Eisenstein series* (defined using the *Siegel parabolic* subgroup of H). We recall that the Levi decomposition of the Siegel parabolic Q of H is given by

$$Q = M_Q N_Q,$$

where the $\mathbb{Q}-$points of the Levi subgroup M_Q are given by

$$M_Q(\mathbb{Q}) = \left\{ m(a, h_0) = \left(\begin{smallmatrix} a & & \\ & h_0 & \\ & & (a^{-1})^t \end{smallmatrix}\right) \ s.t.\ a \in \mathrm{GL}_r(\mathbb{Q}),\ h_0 \in O(V_{an})(\mathbb{Q}) \right\},$$

and the $\mathbb{Q}-$points of the unipotent subgroup N_Q are

$$N_Q(\mathbb{Q}) = \left\{ n(c, d) = \left(\begin{smallmatrix} 1 & c & d - \frac{1}{2}(c,c) \\ & 1 & -c^t \\ & & 1 \end{smallmatrix}\right) \ s.t.\ c^t = (c_1, ..., c_r) \in V_{an}^r,\ (c, c) = ((c_i, c_j)),\ d = -{}^t d \in M_r(\mathbb{Q}) \right\}.$$

The Iwasawa decomposition provides the following equality:

$$H(\mathbb{A}) = Q(\mathbb{A})K_H,$$

where $K_H = (O(2) \oplus O(p)) \times \prod_{v \; place} H(\mathbb{Z}_v)$ is the maximal compact subgroup of $H(\mathbb{A})$. Combining the previous two decompositions we may factor every $h \in H(\mathbb{A})$ by

$$h = n(c, d)m(a, h_0)k.$$

To lighten notation we denote $|a(h)| := |\det(a)|$ and

$$\rho'_r = \frac{m - r - 1}{2}.$$

Using the previous datum we define the function

$$\Psi(h, s) := |a(h)|^{s+\rho'_r},$$

where $s \in \mathbb{C}$.

Definition 1.6.52. The *auxiliary Eisenstein series* is defined as follows:

$$E(h, s) = \sum_{\gamma \in Q(\mathbb{Q})\backslash H(\mathbb{Q})} \Psi(\gamma h, s). \tag{1.-9}$$

Remark 1.6.53. *This series is absolutely convergent when* $\mathrm{Re}(s) > \rho'_r$. *Moreover,* $E(h, s)$ *has meromorphic analytic continuation to* $\mathbb{C}$*, see [KR94, p. 47].*

Proposition 1.6.54. *Except in the case of a split binary quadratic space* V*,* $E(h, s)$ *has a simple pole at* $s = \rho'_r$ *with constant residue denoted by* k*. If* V *is the rational isotropic space of signature* $(2, 1)$ *then* $k = 2$*.*

Proof. It is [KR94, prop. 5.4.1, p. 48]. □

By 1.6.44, the Hecke algebra of $H(\mathbb{Q}_p)$ acts on $\mathrm{Ind}_{Q(\mathbb{Q}_p)}^{H(\mathbb{Q}_p)} |a(\cdot)|^{s+\rho'_r}$ by multiplication by scalars. In fact, we will denote

$$\theta(\alpha_{n,r-1,\eta}) \cdot \left(|a(\cdot)|^{s+\rho'_r}\right) = c_\alpha(s)|a(\cdot)|^{s+\rho'_r}, \tag{1.-9}$$

where $c_\alpha(s) : \mathbb{C} \to \mathbb{C}$ is a function.

Lemma 1.6.55. *Let us consider* V *an isotropic quadratic space of signature* $(2, 1)$*, then*

$$\theta(\alpha_{1,0,\eta}) \cdot \left(|a(\cdot)|^{2\rho'_1}\right) = 0,$$

i.e. $c_\alpha(\rho'_1) = 0$.

Proof. It follows by example 1.6.45 □

The replacement for the theta function in the regularized case of the Siegel-Weil formula will be the following function

$$\mathcal{E}(s, g, \varphi) := \frac{1}{c_\alpha(s)k} \int_{[H]} \theta(g, h, \omega(z)\varphi)E(h, s)dh,$$

It is meromorphic in the variable s. If the dual reductive pair is in the first term range, i.e. $r > 0$, $m \leq n + 1$ so that $m - r \leq n + 1$, then $\mathcal{E}(s, g, \varphi)$ has a pole of order at most 1 at $s = \rho'_r$. If the dual reductive pair is in the second term range, i.e. $0 < r \leq n$ and $n + 1 < m \leq n + 1 + r$, then $\mathcal{E}(s, g, \varphi)$ has a pole of order at most 2 at $s = \rho'_r$. The case of interest for section 1.7 lies in the second term range, specifically when $m = 3$, $r = 1$ and $n = 1$. In this case, proposition 1.6.54 and lemma 1.6.55 show that the function $\mathcal{E}(s,g,\varphi)$ has a pole at $s=\rho_1'$ of order at most 2. We denote the Laurent expansion of $\mathcal{E}(s,g,\varphi)$ at $s=\rho_1'$ by

$$\mathcal{E}(s,g,\varphi)=\frac{B_{-2}(g,\varphi)}{(s-\rho_1')^2}+\frac{B_{-1}(g,\varphi)}{(s-\rho_1')}+B_0(g,\varphi)+O(s-\rho_1').$$

If $B_{-2}(g,\varphi)\neq 0$, it defines an intertwining map of the form

$$B_{-2}(g,\cdot):\ R_n(V)\to\mathcal{A}(G).$$

1.6.8 Regularized Siegel-Weil formula: First term identity

In this subsection we explain the proof of the regularized Siegel-Weil formula of Kudla-Rallis given in [KR94]. We suppose $n+1<m\leq 2n$, $m-r\leq n+1$ and V isotropic.

Theorem 1.6.56. *There is a non-zero constant c so that for every $\varphi\in\mathcal{S}(V^n(\mathbb{A}))$*

$$\mathrm{Res}_{s=s_n}\ E(g,s,\Phi)=\begin{cases} B_0(g,\varphi') & \textit{if } V_0 \textit{ anisotropic},\\ B_{-1}(g,\varphi') & \textit{if } V_0 \textit{ isotropic non split},\\ B_0(g,\varphi') & \textit{if } V_0 \textit{ split binary},\end{cases}$$

where $\Phi\in I_n(s,\tilde\chi_V)$ is the section associated to φ and $\varphi'\in\mathcal{S}(V_0^n(\mathbb{A}))$ is any function whose image in $R_n(V_0)$ coincides with the image of φ under the composition of maps

$$\mathcal{S}(V^n(\mathbb{A}))\to R_n(V)\to R_n(V_0),$$

where the first map is given by μ_V and the second map is defined by the isomorphism of corollary 1.6.17.

Proof. On the one hand, according to theorem 1.6.26, the map

$$\mathrm{Res}_{s=s_n}\ E(g,s,\cdot):\ R_n(V)\to\mathcal{A}(G), \tag{1.-9}$$

is a non-zero intertwining map factorizing throughout $R_n(V_0)$, with $\dim(V_0)\leq n$. Hence, it defines an element of the group

$$\mathrm{Hom}_G\left(R_n(V_0),\mathcal{A}(G)\right),$$

On the other hand, [KR94, thm. 6.1, p. 54] shows that when V isotropic and non binary split with $m\leq n+1$, then

$$B_{-1}(g,\cdot):\ R_n(V)\to\mathcal{A}(G), \tag{1.-9}$$

is intertwining. When $m=2$ and $\tilde\chi_V=1$, the map

$$B_0(g,\cdot):\ R_n(V)\to\mathcal{A}(G), \tag{1.-9}$$

is intertwining. When V is anisotropic we are in the range of convergence of Weil and hence

$$B_0(g,\cdot):\ R_n(V)\to\mathcal{A}(G), \tag{1.-9}$$

is intertwining. The above maps define elements of

$$\mathrm{Hom}_G\left(R_n(V),\mathcal{A}(G)\right)$$

which, by theorem 1.6.22, is of dimension 1. Hence, depending on the complementary space V_0, comparing the map (1.6.8) with (1.6.8), (1.6.8) or (1.6.8), the result follows. ☐

1.6.9 Regularized Siegel-Weil formula: Second term identity

In the forthcoming exposition, the relevant Siegel-Weil formula will be

Theorem 1.6.57. *Let (V, q) be a rational quadratic isotropic space of signature $(2,1)$. Let $\varphi \in \mathcal{S}(V(\mathbb{A}))$ be a function satisfying the hypothesis of [Ich01, p. 209] for a finite place p. Then*

$$B_{-1}(g, \varphi) = A_0(g, \lambda(\varphi)) + cA_{-1}(g, \lambda(\tilde{\varphi})),$$

where $A_0(g, \lambda(\varphi)) = \mathrm{CT}_{s=1/2}\ E(g, s, \lambda(\varphi))$ and $A_{-1}(g, \lambda(\tilde{\varphi})) = \mathrm{Res}_{s=1/2} E(g, s, \lambda(\tilde{\varphi}))$, with $\lambda(\varphi)(g) := (\omega(g)\varphi)(0) \in I_n(s, \chi_V)$, $c \in \mathbb{C}$ and $\tilde{\varphi} \in \mathcal{S}(V(\mathbb{A}))$ is a non determined Schwartz function.

Proof. See [GQT14, thm. 8.1, (ii), p. 35]. □

1.7 A truncated Siegel-Weil formula and Borcherds forms

1.7.1 Introduction

This section is based on the paper [Ter22]. The integrals of the logarithm of the Borcherds forms have been related to zeta and $L-$values in a wide variety of papers [Kud03], [BBGK07], [FiMS18]. Due to the geometric nature of the Borcherds forms, those integrals have also been used to understand the arithmetic degrees of arithmetic cycles of Shimura varieties, extending the knowledge of their Chow groups, [Ehl17]. In [Kud03] the author studies the integral of the logarithm of the Borcherds forms for certain quasiprojective Shimura varieties associated to the group GSpin, obtaining an expression involving certain Fourier coefficients of Eisenstein series. One of the main tools in [Kud03] is the Siegel-Weil formula in the convergent range of Weil and the one proved in [KR88]. On account of the eventual divergence of the integral of the theta function over the modular curve, the integral of the logarithm of the Borcherds forms over the modular curve was not addressed in [Kud03]. An explicit example of the aforementioned integral is given by

$$\Delta(\tau) = e^{2\pi i\tau} \prod_{n=1}^{\infty} (1 - e^{2\pi i\tau n})^{24},$$

the so-called classical discriminant; according to [Bru04, p. 6] $\Delta(\tau)$ is the Borcherds form of the Jacobi theta function. In [Küh01, cor. 5.4, p. 21] the author shows that the integral of logarithm of the norm of $\Delta(\tau)$ diverges. This result points out the different nature between the integrals considered in [Kud03] and the one over the modular curve. A significant variation between [Kud03] and the present paper is that in order to understand our integral we have to replace the integral over the modular curve X^{mod} by the integral over the truncated modular curve, denoted by $X^{mod,\hat{T}}$.

One of the main ingredients for our proof is the Siegel-Weil formula. The classical version of it, given in [Sie35], [Sie51] and [Wei65], relates the integral of certain theta function with a special value of an Eisenstein series. Let V be a rational quadratic space of dimension m with Witt index r. For $n \geq 1$ the tuple $(\mathrm{Sp}_n, O(V))$ forms a dual reductive pair. Given $\varphi \in \mathcal{S}(V^n(\mathbb{A}))$, where $\mathcal{S}$ denotes the space of Schwartz functions, we construct a theta function $\theta(g, h, \varphi) : \mathrm{Sp}_n(\mathbb{A}) \times O(V)(\mathbb{A}) \to \mathbb{C}$. The convergence of $\int_{[O(V)]} \theta(g, h, \varphi)dh$ depends on the constants $m = \dim(V)$, r and n. When $r = 0$ or $m - r > n + 1$, we say that the datum is in the *convergent range of Weil* and by [Wei65] and [Kud86], the theta function is absolutely convergent and

$$k \int_{[O(V)]} \theta(g, h, \varphi)dh = E\left(g, \frac{m}{2} - 1, \lambda(\varphi)\right) = \sum_{\gamma \in P^{\mathrm{Sp}_n}(\mathbb{Q}) \backslash \mathrm{Sp}_n(\mathbb{Q})} \lambda(\varphi)(\gamma g),$$

where k is an explicit constant, P^{Sp_n} is the Siegel parabolic of Sp_n and

$$\lambda : \ S(V^n(\mathbb{A})) \to I_n\left(\frac{m}{2} - 1, \chi_V\right),$$

is a map where $I_n\left(\frac{m}{2} - 1, \chi_V\right)$ is the degenerated principal series representation of $\mathrm{Sp}_n(\mathbb{A})$. It is known that λ is a realization of the local theta correspondence of the identity, which has been extensively studied throughout [KR94, sec. 2. p. 17]. In order to approach the remaining cases, under certain hypothesis on

V and n, [KR94] and [Ich01] developed a regularization of the theta function. It is based on a operator

$$\omega(\alpha): \mathcal{S}(V^n(\mathbb{A})) \to \mathcal{S}(V^n(\mathbb{A}))_{abc}, \tag{1.-9}$$

where $\mathcal{S}(V^n(\mathbb{A}))_{abc} = \{\varphi \in \mathcal{S}(V^n(\mathbb{A})),\ s.t.\ \theta(g,h,\varphi)\ absolutely\ convergent\}$. The map $\omega(\alpha)$ is constructed using the action of an explicit element of the Hecke algebra of $O(V)$ on $\mathcal{S}(V^n(\mathbb{A}))$. This machinery allows us to define a meromorphic function $B(g,\varphi,s)$, [GQT14, sec. 3.5, p. 18], which replaces the role of $\int_{[O(V)]}\theta(g,h,\varphi)dh$ in the classical Siegel-Weil formula. It is known that $B(g,\varphi,s)$ has a pole at $s=\frac{m-r-1}{2}$ of order at most 2. We denote its $i-$th Laurent coefficient by $B_i(g,\varphi)$. The so-called first and second term identity of the Siegel-Weil formula relate $B_{-2}(g,\varphi)$ and $B_{-1}(g,\varphi)$ with special values of Eisenstein series (and their residues).

In this paper we consider the case $m=3$, $r=1$ and $n=1$. Geometrically it corresponds to the modular curve case. The first goal is to obtain an expression for the integral of the theta function associated to $\varphi_{z_0}^{\infty} \in \mathcal{S}(V(\mathbb{A}))$; certain Schwartz function defined using the geometry of the modular curve. It turns out that this integral does not converge, hence in order to obtain some "truncated expression" for it we use the regularized Siegel-Weil formula. One of the available theorems in this situation is given by [GQT14, thm. 8.1, (ii), p. 35]

$$\mathrm{CT}_{s=1/2}\ E(g,s,\lambda(\varphi_{z_0}^{\infty})) = B_{-1}(g,\varphi_{z_0}^{\infty}) + cA_{-1}(g,\lambda(\tilde{\varphi})), \tag{1.-9}$$

where $\mathrm{CT}_{s=1/2}\ E(g,s,\lambda(\varphi_{z_0}^{\infty}))$ and $A_{-1}(g,\lambda(\tilde{\varphi}))$ are respectively the constant and residue terms of the Laurent series at $s=1/2$ of these Eisenstein series and $c\in\mathbb{C}$. A drawback of this formula is that we can not recover information about $\int_{[O(V)]}\theta(g,h,\varphi_{z_0}^{\infty})dh$ directly. To that end we use the mixed model of the Weil representation [KR94, prop. 5.2.1, p. 44, prop. 5.3.1, p. 45]. It allows us to factor the theta function as follows:

$$\theta(g,h,\varphi_{z_0}^{\infty}) = \mathrm{Conv}(g,h,\varphi_{z_0}^{\infty}) + \mathrm{Div}(g,h,\varphi_{z_0}^{\infty}),$$

where $\int_{[O(V)]}\mathrm{Conv}(g,h,\varphi_{z_0}^{\infty})dh$ is absolutely convergent and $\int_{[O(V)]}\mathrm{Div}(g,h,\varphi_{z_0}^{\infty})dh$ diverges. Using the action of $\omega(\alpha)$ we obtain a relation between $B_{-1}(g,\varphi_{z_0}^{\infty})$ and $\int_{[O(V)]}\mathrm{Conv}(g,h,\varphi)dh$.

Theorem (1.7.8)**.** *Let $\varphi_{z_0}^{\infty}$ be the Schwartz function defined in* (1.7.2) *and let $\mathrm{Conv}(g_\tau,h,\varphi_{z_0}^{\infty})$ be the absolute convergent part of $\theta(g_\tau,h,\varphi_{z_0}^{\infty})$. Then*

$$\int_{[O(V)]}\mathrm{Conv}(g_\tau,h,\varphi_{z_0}^{\infty})dh = E(g_\tau,1/2,\lambda(\varphi_{z_0}^{\infty})) + cA_{-1}(g,\lambda(\tilde{\varphi})),$$

where $c\in\mathbb{C}$.

Let X^{mod} be the modular curve without level, fix $\hat{T}>1$ and set $X^{mod,\hat{T}} = \{z=x+iy\in X^{mod}\ s.t.\ y<\hat{T}\}$. The main body of this paper is devoted to obtaining an explicit expression of

$$\int_{X^{mod,\hat{T}}}\log||\Psi(f)(z)||_{Pet}d\mu(z),$$

where $f\in M^{!}_{-1/2,\mathbb{Z}^3}$, $\Psi(f)(z)$ is the Borcherds form of f, $||\cdot||_{Pet}$ is the Peterson norm and $d\mu$ is the hyperbolic measure. The function $\Psi(f)$ is closely related to the singular theta lift, hence the computation can be reduced to

$$\int_{X^{mod,\hat{T}}}\int^{\bullet}_{X^{mod}}\theta^{Sieg}(\tau,z)f(\tau)d\mu(\tau)d\mu(z),$$

where $\int^{\bullet}_{X^{mod}}$ is the regularization proposed in [Bor98] to ensure the convergence of the singular theta lift. Due to the behaviour of the Fourier constant term of $\theta^{Sieg}(\tau,z)$ the order of the previous two integrals can not be exchanged. With the aim of accomplishing the computation we treat separately the integrals involving the non constant and constant terms of $\theta^{Sieg}(\tau,z)$. The first one is approached following the method developed in [Kud03] that requires the truncated version of the Siegel-Weil formula stated above. The second integral is computed via an unfolding of the theta function with the integral. To ensure the convergence in this unfolding we introduce an auxiliary Eisenstein series and to conclude we apply the truncated version of the Rankin-Selberg formula developed by Zagier.

Theorem (1.7.41). *The integral $\int_{X^{mod,\hat{T}}} \log ||\Psi(f)(z)||_{Pet} d\mu(z)$ diverges as $\log(\hat{T})$. The non divergent term is an explicit combination of Fourier coefficients of Eisenstein series, gamma values, their derivatives and values of the completed Riemann zeta function. Furthermore the coefficient of the divergent term has an interpretation in terms of special values of zeta functions.*

The second subsection is devoted to stating the geometric setting of the section. In the third subsection we explain the relation between the regularized theta correspondence and the adelic theta correspondence. Moreover we describe explicitly the operator $\omega(\alpha)$. In subsection 4 we prove the truncated version of the Siegel-Weil formula. Subsection 5 is the main body of the paper, where we prove the main result. This proof is divided into two cases; the ordinary case; whose main ingredient is the truncated Siegel-Weil formula, and the limit case; approached by the unfolding of the integral with the theta series. In the final section we prove some technical computations that are used in subsection 5.

Notation

Given G an algebraic group defined over $\mathbb{Q}$, we denote by $G_{\mathbb{Q}_p} := G_{\mathbb{Q}_p} \times_{\mathrm{Spec}(\mathbb{Q})} \mathrm{Spec}(\mathbb{Q}_p)$ the base change of G to $\mathbb{Q}_p$. We use the notation $[G] = G(\mathbb{Q}) \setminus G(\mathbb{A})$. Given (V, q) a quadratic space we denote by $m := \dim V$, r its Witt index, V_{an} its maximal anisotropic subspace and $(\cdot, \cdot)$ the bilinear form associated to q. Let us denote by $O(V)$ the orthogonal algebraic group. Furthermore, given a rational symplectic space of dimension $2n$ we denote by $\mathrm{Sp}(W)$ its symplectic group, which is an algebraic group defined over $\mathbb{Q}$. There is a choice of basis for W so that $\mathrm{Sp}(W) \simeq \mathrm{Sp}_n$. Given a topological space T and a subspace S we denote by $\mathrm{char}_S(x) : T \to \{0, 1\}$ the characteristic function of S. We fix ψ_∞ the character on $\mathbb{R}$ given by $\psi_\infty(x) = e^{2\pi i x}$ and the unique characters ψ_p on $\mathbb{Q}_p$ that are additive, whose restriction to $\mathbb{Z}_p$ is trivial and they satisfy that $\psi_p(p^{-1}) = e^{\frac{-2\pi i}{p}}$.

Let $\mathcal{H}$ be the Poincaré half plane. We fix two complex variables $\tau = u + iv$ and $z = x + iy$. Given a meromorphic function $F(s)$, $s_0 \in \mathbb{C}$ and C_{s_0} a closed curve around s_0 we denote by

$$\mathrm{FT}_{s=s_0} F(s) = \int_{C_{s_0}} \frac{F(s)}{(s - s_0)^2} ds,$$

the first term in the Laurent series of $F(s)$ at $s = s_0$, by

$$\mathrm{CT}_{s=s_0} F(s) = \int_{C_{s_0}} \frac{F(s)}{(s - s_0)} ds,$$

the constant term in the Laurent series at $s = s_0$ and by

$$\mathrm{Res}_{s=s_0} F(s) = \int_{C_{s_0}} F(s) ds,$$

the residue of $F(s)$ at $s = s_0$. The Euler-Mascheroni constant is denoted by $\gamma := -\Gamma'(1)$.

This relevant constant will appear in the computations:

$$A = 2\mathrm{erf}\left(\sqrt{\frac{\pi}{2}}\right)(-4(\gamma + 1) + \log(2) + \pi \log(\pi/2)) + 2\pi \log\left(\frac{2}{\pi}\right) + \pi^{1/2}\Gamma'\left(-1/2, \frac{\pi}{2}\right),$$

where $\Gamma(a, b)$ is the incomplete Gamma function.

1.7.2 Truncated Siegel-Weil formula

Throughout this section we fix (V, q) an even rational quadratic space of signature $(2, 1)$ and Witt index 1. We denote the bilinear form associated to q by $(\cdot, \cdot)$. Let H be the algebraic group $O(V)$ and G be the metaplectic group Mp_2. We also fix the lattice $L \simeq \mathbb{Z}^3$ of V and W the rational symplectic space of dimension 2. Once we fix the basis given in subsection 1.6, we will obtain $\mathrm{Sp}(W) \simeq \mathrm{SL}_2$.

Let $\varphi^{\infty}_{z_0,\mathbb{R}} \in \mathcal{S}(V(\mathbb{R}))$ be the Gaussian associated to the quadratic space (V, q) with base point $z_0 := i \in \mathcal{H}$. Moreover we set φ_{μ_j} the characteristic functions of the two cosets of $L' \otimes_{\mathbb{Z}} \mathbb{A}_f / L \otimes_{\mathbb{Z}} \mathbb{A}_f$ given in subsection 1.2.5. Throughout this section we will consider the Schwartz functions

$$\varphi^{\infty}_{z_0,\mu_j} := \varphi^{\infty}_{z_0,\mathbb{R}} \otimes \varphi_{\mu_j} \in \mathcal{S}(V(\mathbb{A})). \tag{1.-9}$$

By the analysis done in the proof of [KR94, prop. 5.3.1, p. 45], the following integrals do not converge

$$\int_{[H]} \theta(g_\tau, h, \varphi^{\infty}_{z_0,\mu_j}) dh, \tag{1.-9}$$

where we recall that

$$g_\tau \in G(\mathbb{R}) = \left(\begin{pmatrix} v^{1/2} & uv^{1/2} \\ 0 & v^{-1/2} \end{pmatrix}, 1 \right).$$

The main goal of this section is to state an asymptotic formula for (1.7.2) i.e. we will isolate the terms of the theta function that diverge and we will compute the integral of the convergent ones. This computation is based on a manipulation of the second term identity of the Siegel-Weil formula developed in [GQT14].

Factorization of the theta function

Definition 1.7.1. The isomorphism given by 1.6.48 allows us to consider the representation

$$\begin{aligned} &H(\mathbb{A}) \times G(\mathbb{A}) \to \mathrm{Aut}(\mathcal{S}(V_{an}(\mathbb{A})) \otimes \mathcal{S}(W(\mathbb{A}))) \\ &(g, h) \mapsto \left(\hat{\varphi}(x_0, u, v) \mapsto \int_{M_{1,2}(\mathbb{A})} \omega(g, h)\varphi \begin{pmatrix} x \\ x_0 \\ u \end{pmatrix} \psi_{\mathbb{A}}(vx) dx \right), \end{aligned}$$

which is called the *mixed model* of the Weil representation.

Remark 1.7.2. *By means of the partial Poisson summation formula stated in [KR94, (5.3.4), p. 45], the theta function satisfies*

$$\theta(g, h, \varphi) = \sum_{\substack{x_0 \in V_{an}(\mathbb{Q}) \\ w \in W(\mathbb{Q})}} \hat{\varphi}(x_0, w).$$

Definition 1.7.3. Let $\varphi \in \mathcal{S}(V(\mathbb{A}))$. Given a theta function $\theta(g, h, \varphi)$ for the dual reductive pair (SL_2, H), we define the *divergent part* by

$$\mathrm{Div}(g, h, \varphi) := \sum_{x_0 \in V_{an}(\mathbb{Q}_p)} \omega(g, h)\hat{\varphi}(x_0, 0).$$

Moreover we define the *convergent part* by

$$\mathrm{Conv}(g, h, \varphi) := \sum_{\substack{x_0 \in V_{an}(\mathbb{Q}_p) \\ 0 \neq w \in W(\mathbb{Q})}} \omega(g, h)\hat{\varphi}(x_0, w).$$

Proposition 1.7.4. *The convergent part* $\mathrm{Conv}(g, h, \varphi)$ *is rapidly decreasing.*

Proof. See the proof of [KR94, prop. 5.3.1, p. 45]. □

Remark 1.7.5. *By remark 1.6.50 the previous definitions provide a well defined factorization for the theta function:*

$$\theta(g, h, \varphi) = \mathrm{Div}(g, h, \varphi) + \mathrm{Conv}(g, h, \varphi).$$

Lemma 1.7.6. *Let p be a place satisfying the hypothesis of [Ich01, p. 209] and let $\alpha := \alpha_{1,0,\eta} \in \mathcal{H}_p^{Mp_2}$ be the Hecke operator chosen in example 1.6.45. The regularized theta function satisfies*

$$\theta(g, h, \omega(\alpha)\varphi) = \mathrm{Conv}(g, h, \omega(\alpha)\varphi).$$

Proof. By [Ich01, prop. 1.5, p. 209] the function $\theta(g, h, \omega(\alpha)\varphi)$ is rapidly decreasing. Therefore, since $\mathrm{Div}(g, h, \varphi)$ diverges by the proof of [KR94, prop. 5.3.1, p. 45], the factorization given in 1.7.5 yields the result. □

Proposition 1.7.7. *The following equality holds:*

$$B_{-1}(g, h, \varphi) = \int_{[H]} \mathrm{Conv}(g, h, \varphi)dh.$$

Proof. Writing the definition of the action of the operator $\omega(\alpha)$ and applying proposition 1.7.6 to $B_{-1}(g, h, \varphi)$, we obtain

$$B_{-1}(g, h, \varphi) = \mathrm{Res}_{s=1/2}\left(\frac{1}{2c_\alpha(s)}\int_{[H]}\int_{H(\mathbb{Q}_p)} \theta(\alpha)(h_v)\mathrm{Conv}(g, hh_v, \varphi)E(h, s)dh_v dh\right).$$

By proposition 1.7.4 the function $\mathrm{Conv}(g, h, \varphi)$ is rapidly decreasing in the variable h, hence we apply a change of variables of the form $h = hh_v$, obtaining

$$B_{-1}(g, h, \varphi) = \mathrm{Res}_{s=1/2}\left(\frac{1}{2c_\alpha(s)}\int_{[H]} \mathrm{Conv}(g, h, \varphi)\int_{H(\mathbb{Q}_p)} \theta(\alpha_{1,0,\eta})(h_v)E(hh_v, s)dh_v dh\right).$$

The action of $\theta(\alpha_{1,0,\eta})$ on $E(h, s)$ factors throughout the action of this operator on $\mathrm{Ind}_{Q(\mathbb{Q}_p)}^{H(\mathbb{Q}_p)}|a(\cdot)|^{s+\rho_1'}$. We recall the reader that in (1.6.7) we defined the constant $c_\alpha(s)$ by means of the action of $\theta(\omega)$ in $\mathrm{Ind}_{Q(\mathbb{Q}_p)}^{H(\mathbb{Q}_p)}|a(\cdot)|^{s+\rho_1'}$. Therefore

$$B_{-1}(g, h, \varphi) = \mathrm{Res}_{s=1/2}\left(\frac{1}{2}\int_{[H]} \mathrm{Conv}(g, h, \varphi)E(h, s)dh\right) = \int_{[H]} \mathrm{Conv}(g, h, \varphi)dh,$$

where the last equality follows because $2 = \mathrm{Res}_{s=1/2}E(h, s)$. □

Corollary 1.7.8. *The convergent part satisfies*

$$\int_{[H]} \mathrm{Conv}(g, h, \varphi_{z_0}^\infty)dh = A_0(g, \lambda(\varphi_{z_0}^\infty)) + cA_{-1}(g, \lambda(\tilde{\varphi})),$$

where $c \in \mathbb{C}$ and $\tilde{\varphi} \in \mathcal{S}(V(\mathbb{A}))$ is a Schwartz function such that for $k_\theta = \left(\begin{smallmatrix} cos(\theta) & sin(\theta) \\ sin(\theta) & cos(\theta)\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ it satisfies that $\omega(k_\theta)\tilde{\varphi}(x) = e^{-\frac{i\theta}{2}}\tilde{\varphi}(x)$.

Proof. The equality of the statement follows by theorem 1.6.57 and proposition 1.7.7. The functions $\int_{[H]} \mathrm{Conv}(g, h, \varphi_{z_0}^\infty)dh$ and $A_0(g, \lambda(\varphi_{z_0}^\infty))$ transform as $e^{\frac{i\theta}{2}}\int_{[H]} \mathrm{Conv}(g, h, \varphi_{z_0}^\infty)dh$ and $e^{\frac{i\theta}{2}}A_0(g, \lambda(\varphi_{z_0}^\infty))$ under the action of $k_\theta = \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{smallmatrix}\right)$. The space of sections Φ with this transformation properties is one dimensional [Kud03, (4.18), p. 49]. Therefore Φ is equal to $\lambda(\tilde{\varphi})$ with $\tilde{\varphi} \in \mathcal{S}(V(\mathbb{R}))$ satisfying $\omega(k_\theta)\tilde{\varphi}(x) = e^{\frac{i\theta}{2}}\tilde{\varphi}(x)$.

□

Corollary 1.7.9. *Let $K \subset \mathrm{SL}_2(\mathbb{A}_f)$ be a compact open subgroup and let $\varphi = \varphi^\infty \otimes \varphi^L$ be a Schwartz function with $\varphi^L \in \mathcal{S}(V(\mathbb{A}_f))$ being a $K-$invariant Schwartz function. Then*

$$\int_{[H]} \mathrm{Conv}(g, h, \varphi)dh = A_0(g, \lambda(\varphi)) + cA_{-1}(g, \lambda(\tilde{\varphi})),$$

with $A_{-1}(g, \lambda(\tilde{\varphi}))$ a $K-$invariant function and $c \in \mathbb{C}$.

Proof. Since $\omega(\alpha)$ commutes with the action of the Weil representation, the regularized theta function $\theta(g, h, \omega(\alpha)\varphi)$ is right $K-$invariant. Furthermore, using lemma 1.7.6, the function $\mathrm{Conv}(g, h, \varphi)$ is $K-$invariant. Since the intertwinning map λ respects the $K-$invariance of φ, the Eisenstein series $E(g, s, \lambda(\varphi))$ is also $K-$invariant. Since corollary 1.7.8 relates the sum of three, two of them $K-$invariant, then the function $A_{-1}(g, \lambda(\tilde{\varphi}))$ is also $K-$invariant. Moreover, using the integral formula for the residue, the $K-$invariance for $E(g, s, \lambda(\tilde{\varphi}))$ follows. □

1.7.3 Integral of Borcherds forms

As in the previous section we fix (V, q) a rational isotropic quadratic space of signature $(2, 1)$ and a lattice $L \simeq \mathbb{Z}^3$ of V. The example 1.2.5 shows that the group L'/L consists of two elements. We denote them by $\mu_0 := \mathbb{Z}^3$ and $\mu_1 := \mathbb{Z} \oplus \frac{1}{2}\mathbb{Z} \oplus \mathbb{Z}$. Furthermore we will use the notation

$$f(\tau) = \sum_{\varphi_{\mu_j} \in S_L} f_{\mu_j}(\tau)\varphi_{\mu_j} = \sum_{\substack{n \in \mathbb{Z} \\ \varphi_{\mu_j} \in S_L}} c_{\mu_j}(n) q^n \varphi_{\mu_j} \in M^!_{-1/2,L},$$

for the Fourier expansion of a weakly holomorphic modular form. Let $X^{mod} := SL_2(\mathbb{Z}) \setminus \mathcal{H}$ be the modular curve without level and set $1 \leq \hat{T} \in \mathbb{R}$. The goal of this section is to compute the following integral:

$$\int_{X^{mod,\hat{T}}} \log ||\Psi(f)(z)||_{Pet} d\mu(z), \tag{1.-9}$$

where $\Psi(f)(z)$ is the Borcherds lift of f and

$$X^{mod,\hat{T}} := \{z = x + iy \in X^{mod} \; s.t. \; y < \hat{T}\} = \{x + iy \in \mathcal{H}, \; s.t. \; |x| \leq 1/2, \; |z| > 1 \; and \; y < \hat{T}\}.$$

Furthermore we denote

$$\widehat{X^{mod,\hat{T}}} := \{z = x + iy \in X^{mod}, \; y > \hat{T}\}.$$

Using the definition of the Petersson norm we deduce

$$\begin{aligned} \int_{X^{mod,\hat{T}}} \log ||\Psi(f)(z)||_{Pet} d\mu(z) &= \int_{X^{mod,\hat{T}}} \log |\Psi(f)(z) y^{c_{\mu_0}(0)/2}| d\mu(z) \\ &= \int_{X^{mod,\hat{T}}} \log |\Psi(f)(z)| d\mu(z) + \frac{c_{\mu_0}(0)}{2} \int_{X^{mod,\hat{T}}} \log |y| d\mu(z). \end{aligned}$$

Theorem [Bor98, thm. 13.3, p. 48] shows the following relation:

$$\log |\Psi(f)(z)| = -\frac{\Phi(f)(z)}{4} - \frac{c_{\mu_0}(0)}{2} \left(\log |y| + \Gamma'(1)/2 + \log \sqrt{2\pi} \right),$$

where $\Phi(f)(z)$ is the singular theta lift of f. Plugging the previous equality in (1.7.3) we find

$$\int_{X^{mod,\hat{T}}} \log ||\Psi(f)(z)||_{Pet} d\mu(z) = -\frac{1}{4} \int_{X^{mod,\hat{T}}} \Phi(f)(z) d\mu(z) + \frac{c_{\mu_0}(0) \mathrm{vol}(X^{mod,\hat{T}})}{2} \left(\Gamma'(1)/2 + \log \sqrt{2\pi} \right).$$

Therefore, our goal will be achieved by computing the following integral:

$$\begin{aligned} \int_{X^{mod,\hat{T}}} \Phi(f)(z) d\mu(z) &= \int_{X^{mod,\hat{T}}} \left(\int^{\bullet}_{X^{mod}} \left\langle f(\tau), \Theta^{Sieg}_L(\tau, z) \right\rangle d\mu(\tau) \right) d\mu(z) \\ &= \sum_{\varphi_{\mu_j} \in S_L} \int_{X^{mod,\hat{T}}} \left(\int^{\bullet}_{X^{mod}} f_{\mu_j}(\tau) \theta^{Sieg}_{\mu_j}(\tau, z) d\mu(\tau) \right) d\mu(z). \end{aligned}$$

This section is organized as follows: first of all, the truncated integral of the Siegel theta function over the modular curve will be one of the key steps to understand (1.7.3). We deduce the formula from corollary 1.7.8. More concretely, using the ideas of [Kud03, prop. 4.17, p. 44] we translate this result to the geometric setting by proving a relation between the convergent part of the integral of the theta function associated to the Gaussian over $[H]$ and the integral of the Siegel theta function over the modular curve without level. Let us denote by

$$\theta^{Sieg}_{\mu_j}(\tau,z)=\sum_{\lambda\in L+\mu_j}\theta^{Sieg}_{\mu_j}(\tau,z)_{q(\lambda)},$$

the Fourier expansion of $\theta^{Sieg}_{\mu_j}(\tau,z)$ with respect to the variable τ. The constant Fourier coefficient

$$\theta^{Sieg}_{\mu_j}(\tau,z)_0:=\sum_{\substack{\lambda\in L+\mu_j\\ q(\lambda)=0}}\theta^{Sieg}_{\mu_j}(\tau,z)_{\lambda},$$

does not have exponential decay when $v\to\infty$. Therefore, we will factor the integrals

$$\lim_{T\to\infty}\int_{\mathcal{F}^T}f_{\mu_j}(\tau)\theta^{Sieg}_{\mu_j}(\tau,z)v^{1-\sigma}d\mu(\tau),\tag{1.-11}$$

where we recall that $\mathcal{F}^T=\{\tau=u+iv\in\mathcal{H},\ s.t.\ v<T\}$, depending on the behaviour of $\theta^{Sieg}_{\mu_j}(\tau,z)_{q(\lambda)}$ with respect to the variable τ. We express (1.7.3) as the sum of two terms; the *ordinary case*:

$$\lim_{T\to\infty}\int_{\mathcal{F}^T}f_{\mu_j}(\tau)\sum_{\substack{\lambda\in L+\mu_j\\ q(\lambda)\neq 0}}\theta^{Sieg}_{\mu_j}(\tau,z)_{q(\lambda)}v^{1-\sigma}d\mu(\tau),$$

and the *limit case*:

$$\lim_{T\to\infty}\int_{\mathcal{F}^T}f_{\mu_j}(\tau)\theta^{Sieg}_{\mu_j}(\tau,z)_0v^{1-\sigma}d\mu(\tau).$$

The first integrals are studied in section 3 using the truncated Siegel-Weil formula stated in 1.7.8 and the techniques developed in [Kud03]. The second integrals are studied via the truncated unfolding of [Zag81].

Geometric version of the truncated Siegel-Weil formula.

In the previous section we computed the integral of the convergent terms of certain adelic theta function, see corollary 1.7.8. This subsection connects this result with the complex geometry point of view of Borcherds [Bor98], showing a truncated version of the classical Siegel-Weil formula for the modular curve.

Throughout this subsection we fix $K^H(\mathbb{A}_f)=\prod_{p\nmid\infty}H(\mathbb{Z}_p)$ an open compact subgroup of $H(\mathbb{A}_f)$. Let us note that $K^H(\mathbb{A}_f)$ satisfies that the lattices $\hat{\mathbb{Z}}_f^3:=\mathbb{Z}^3\otimes_{\mathbb{Z}}\hat{\mathbb{Z}}$ and $\hat{\mathbb{Z}}_f\oplus\frac{1}{2}\hat{\mathbb{Z}}_f\oplus\hat{\mathbb{Z}}_f:=(\mathbb{Z}\oplus\frac{1}{2}\mathbb{Z}\oplus\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{A}_f$ are fixed under the action of $K^H(\mathbb{A}_f)$ on $V(\mathbb{A}_f)$. Since $\mathbb{C}[L'/L]\simeq\mathcal{S}(V(\mathbb{A}_f))$ as $\mathbb{C}-$vector spaces, we denote by

$$\varphi_{\mu_0}=\mathrm{char}_{\hat{\mathbb{Z}}_f^3}\ \ and\ \ \varphi_{\mu_1}=\mathrm{char}_{\hat{\mathbb{Z}}_f\oplus\frac{1}{2}\hat{\mathbb{Z}}_f\oplus\hat{\mathbb{Z}}_f}.\tag{1.-11}$$

We set $\varphi^\infty_{z_0,\mu_0}:=\varphi^\infty_{z_0,\mathbb{R}}\otimes\varphi_{\mu_0}$ and $\varphi^\infty_{z_0,\mu_1}:=\varphi^\infty_{z_0,\mathbb{R}}\otimes\varphi_{\mu_1}$.

Lemma 1.7.10. *The Schwartz functions φ_{μ_0} and φ_{μ_1} are $K^H(\mathbb{A}_f)-$invariant.*

Proof. The group H acts on $\mathcal{S}(V(\mathbb{A}_f))$ by left translation and the group $K^H(\mathbb{A}_f)$ preserves the lattices $\hat{\mathbb{Z}}_f^3$ and $\hat{\mathbb{Z}}_f\oplus\frac{1}{2}\hat{\mathbb{Z}}_f\oplus\hat{\mathbb{Z}}_f$. □

Lemma 1.7.11. *The divergent parts $\mathrm{Div}(g,h,\varphi^\infty_{z_0,\mu_j})$ are right $(O(2)\oplus O(1))\times K^H(\mathbb{A}_f)-$invariant.*

Proof. We recall

$$\begin{aligned}\mathrm{Div}(g,h,\varphi^{\infty}_{z_0,\mu_j}) &= \sum_{x_0\in V_{an}(\mathbb{Q})} \omega(g,h)\widehat{\varphi^{\infty}_{z_0,\mu_j}}(x_0,0,0)\\ &= \sum_{x_0\in V_{an}(\mathbb{Q})} \int_{\mathbb{A}} \omega(g,h)\varphi^{\infty}_{z_0,\mu_j}\begin{pmatrix} x\\ x_0\\ 0\end{pmatrix} dx.\end{aligned}$$

Let us denote by

$$k = k_{\mathbb{R}} \times k_f \in (O(2)\oplus O(1)) \times K^H(\mathbb{A}_f).$$

It is straightforward that

$$\begin{aligned}\int_{\mathbb{A}} \omega(k)\varphi^{\infty}_{z_0,\mu_j}\begin{pmatrix} x\\ x_0\\ 0\end{pmatrix} dx &= \int_{\mathbb{R}} \omega(k_{\mathbb{R}})\varphi^{\infty}_{z_0,\mathbb{R}}\begin{pmatrix} x\\ x_0\\ 0\end{pmatrix} dx \cdot \int_{\mathbb{A}_f} \omega(k_f)\varphi_{\mu_j}\begin{pmatrix} x_f\\ x_0\\ 0\end{pmatrix} dx_f\\ &= \int_{\mathbb{A}} \varphi^{\infty}_{z_0,\mu_j}\begin{pmatrix} x\\ x_0\\ 0\end{pmatrix} dx,\end{aligned}$$

where the latter equality follows since $\varphi^{\infty}_{z_0,\mathbb{R}}$ is $O(2)\oplus O(1)-$invariant and φ_{μ_j} is $K^H(\mathbb{A}_f)-$invariant by 1.7.10. Therefore the function $\widehat{\varphi^{\infty}_{z_0,\mu_j}}$ is $(O(2)\oplus O(1))\times K^H(\mathbb{A}_f)-$invariant, implying that $\mathrm{Div}(g,h,\varphi^{\infty}_{z_0,\mu_j})$ is $(O(2)\oplus O(1)) \times K^H(\mathbb{A}_f)-$invariant. ☐

Lemma 1.7.12. *The regularized theta functions $\theta(g,h,\omega(\alpha)\varphi^{\infty}_{z_0,\mu_j})$ are right $(O(2)\oplus O(1))\times K^H(\mathbb{A}_f)-$invariant.*

Proof. Since $\omega(\alpha)$ commutes with the Weil representation the proof is analogous to lemma 1.7.11. ☐

Proposition 1.7.13. *The integral of the theta function satisfies the following equality:*

$$\int_{[H]} \theta(g_\tau,h,\omega(\alpha)\varphi^{\infty}_{z_0,\mu_j})dh = \frac{1}{2}\int_{[\mathrm{SO}(V)]} \theta(g_\tau,h,\omega(\alpha)\varphi^{\infty}_{z_0,\mu_j})dh.$$

Proof. Since the action of $\omega(\alpha)$ commutes with Weil representation, the proof follows directly by [Kud03, thm. 4.1, p. 37]. ☐

Corollary 1.7.14. *The convergent part satisfies the following equality:*

$$\int_{[H]} \mathrm{Conv}(g_\tau,h,\varphi)dh = \frac{1}{2}\int_{[\mathrm{SO}(V)]} \mathrm{Conv}(g_\tau,h,\varphi)dh.$$

Proof. Let us consider the functional

$$\begin{aligned} I' :\ \mathcal{S}(V(\mathbb{A})) &\to \mathbb{C},\\ \varphi &\mapsto \int_{[SO(V)]} \mathrm{Conv}(id,h,\varphi)dh.\end{aligned}$$

Since the group $SO(V)$ is unimodular $I' \in \mathrm{Hom}_{SO(V)(\mathbb{A})}(\mathcal{S}(V(\mathbb{A})),\mathbb{C})$. According to [Kud03, prop. 4.2, p. 37] the action of the group $C(\mathbb{A}) = O(V)(\mathbb{A})/SO(V)(\mathbb{A})$ on $\mathrm{Hom}_{SO(V)(\mathbb{A})}(\mathcal{S}(V(\mathbb{A})),\mathbb{C})$ is trivial. Hence

$$\int_{[H]} \mathrm{Conv}(g_\tau,h,\varphi)dh = \int_{C(\mathbb{Q})\backslash C(\mathbb{A})}\int_{[SO(V)]} \mathrm{Conv}(g_\tau,ch,\varphi)dhdc = \frac{1}{2}\int_{[SO(V)]} \mathrm{Conv}(g_\tau,h,\varphi)dh,$$

where we have used that $\mathrm{vol}(C(\mathbb{Q})\setminus C(\mathbb{A})) = \frac{1}{2}$. ☐

Proposition 1.7.15. *We obtain*

$$\int_{[\mathrm{SO}(V)]} \mathrm{Conv}(g_\tau, h, \varphi_{z_0,\mu_j}^\infty) dh = \frac{-\mathrm{vol}(K^H(\mathbb{A}_f))}{2} \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}} \mathrm{Conv}(g_\tau, h_1, \varphi_{z,\mu_j}^\infty) d\mu(z),$$

where $d\mu(z) = \frac{dxdy}{y^2}$.

Proof. Let us recall the notation $\tilde{\mathrm{H}} = \mathrm{GSpin}_V$. The strong approximation theorem shows that

$$\tilde{\mathrm{H}}(\mathbb{A}) = \bigcup_{t\in T} \tilde{\mathrm{H}}(\mathbb{Q})\tilde{\mathrm{H}}(\mathbb{R})^+ h_t \mathrm{K}$$

where $h_t \in \tilde{\mathrm{H}}(\mathbb{A})$ and $\tilde{\mathrm{H}}(\mathbb{R})^+$ is the connected component of the identity of $\tilde{\mathrm{H}}(\mathbb{R})$. The modular curve $\mathrm{SL}_2(\mathbb{Z}) \setminus \mathcal{H}$ has one connected component, then $T = \{1\}$. Since the functions $\varphi_{z_0,\mu_0}^\infty$ and $\varphi_{z_0,\mu_1}^\infty$ are $(O(2) \times O(1)) \times K^H(\mathbb{A}_f)-$invariant, the proof is analogous to [Kud03, prop. 4.17, p. 44]. $\square$

Corollary 1.7.16. *It holds that*

$$\int_{[\mathrm{SO}(V)]} \mathrm{Conv}(g_\tau, h, \varphi_{z_0,\mu_j}^\infty) dh = \frac{2}{\mathrm{vol}(X^{mod})} \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}} \mathrm{Conv}(g_\tau, h_1, \varphi_{z,\mu_j}^\infty) d\mu(z).$$

Proof. The result follows by proposition 1.7.15 and by [Kud03, rmk. 4.18, p. 46]. $\square$

Theorem 1.7.17. *The integral of the convergent term over the modular curve satisfies*

$$\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}} \mathrm{Conv}(g_\tau, h_1, \varphi_{z,\mu_j}^\infty) d\mu(z) = \mathrm{vol}(X^{mod}) \left[A_0(g_\tau, \lambda(\varphi_{z,\mu_j}^\infty)) + cA_{-1}(g_\tau, \lambda(\tilde{\varphi})) \right],$$

where $\tilde{\varphi} := \tilde{\varphi}_\infty \otimes \varphi^{\tilde{L}} \in \mathcal{S}(V(\mathbb{A}))$ *is a* $\prod_{p\nmid\infty} \mathrm{SL}_2(\mathbb{Z}_p)$ *Schwartz function satisfying* $\omega(k_\theta)\tilde{\varphi}(x) = e^{-\frac{i\theta}{2}}\tilde{\varphi}(x)$ *for* $k_\theta = \begin{pmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$.

Proof. Using corollaries 1.7.14 and 1.7.16 we obtain

$$\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}} \mathrm{Conv}(g_\tau, h_1, \varphi_{z,\mu_j}^\infty) d\mu(z) = \mathrm{vol}(X^{mod}) \int_{[H]} \mathrm{Conv}(g_\tau, h, \varphi_{z_0,\mu_j}^\infty) dh. \tag{1.-11}$$

Applying corollaries 1.7.8 and 1.7.9 on the right hand side of (1.7.3), the result follows. $\square$

Definition 1.7.18. We denote by

$$\vartheta(\tau, z, \mu_i) := v^{1/4}\theta(g_\tau, h_1, \varphi_{z,\mu_j}^\infty),$$

$$\mathbf{Conv}(\tau, z, \mu_j) := v^{1/4}\mathrm{Conv}(g_\tau, h_1, \varphi_{z,\mu_j}^\infty),\ \mathbf{Div}(\tau, z, \mu_j) := v^{1/4}\mathrm{Div}(g_\tau, h_1, \varphi_{z,\mu_j}^\infty).$$

Furthermore we denote their constant Fourier coefficients with respect to τ by $\vartheta(v, z, \mu_j)_0$, $\mathbf{Conv}(v, z, \mu_j)_0$ and $\mathbf{Div}(v, z, \mu_j)_0$.

Corollary 1.7.19. *Let* $\hat{T} > 1$*, the theta function satisfies the following equality:*

$$\begin{aligned}\int_{X^{mod,\hat{T}}} \vartheta(\tau, z, \mu_j) d\mu(z) =& \mathrm{vol}(X^{mod}) \left[A_0(\tau, -1/2, \mu(\varphi_{\mu_j})) + cA_{-1}(\tau, s, -1/2, \mu(\tilde{\varphi})) \right] \\ &- \int_{\widehat{X^{mod,\hat{T}}}} \mathbf{Conv}(\tau, z, \mu_j) d\mu(z) + \int_{X^{mod,\hat{T}}} \mathbf{Div}(\tau, z, \mu_j) d\mu(z).\end{aligned}$$

Proof. We factor $\vartheta(\tau, z, \mu_j)$ into its divergent and convergent parts as we did in remark 1.7.5. We obtain

$$\begin{aligned}\int_{X^{mod,\hat{T}}} \vartheta(\tau, z, \mu_j) d\mu(z) = &\int_{X^{mod,\hat{T}}} \mathbf{Conv}(\tau, z, \mu_j) d\mu(z) \\ &+ \int_{X^{mod,\hat{T}}} \mathbf{Div}(\tau, z, \mu_j) d\mu(z).\end{aligned}$$

Since $\mathbf{Conv}(\tau, z, \mu_j)$ is integrable

$$\begin{aligned}\int_{X^{mod,\hat{T}}} \mathbf{Conv}(\tau, z, \mu_j) d\mu(z) = &\int_{X^{mod}} \mathbf{Conv}(\tau, z, \mu_j) d\mu(z) \\ &- \int_{\widehat{X^{mod,\hat{T}}}} \mathbf{Conv}(\tau, z, \mu_j) d\mu(z).\end{aligned}$$

Using 1.7.17 in (1.7.3) and then plugging the result into (1.7.3) we obtain the formula. □

Corollary 1.7.20. *The constant term of the theta function satisfies the following equality:*

$$\begin{aligned}\int_{X^{mod,\hat{T}}} \vartheta(v, z, \mu_0)_0 d\mu(z) = &\mathrm{vol}(X^{mod}) \Big[A_0(v, -1/2, \mu(\varphi_{\mu_0}))_0 + cA_{-1}(v, -1/2, \mu(\varphi_{\mu_0}))_0 \Big] \\ &- \int_{\widehat{X^{mod,\hat{T}}}} \mathbf{Conv}(v, z, \mu_0)_0 d\mu(z) + \int_{X^{mod,\hat{T}}} \mathbf{Div}(v, z, \mu_0)_0 d\mu(z),\end{aligned}$$

were $A_0(v, -1/2, \mu(\varphi_{\mu_j}))_0$ and $A_{-1}(v, -1/2, \mu(\varphi_{\mu_j}))_0$ are the constant Fourier coefficients with respect to τ of $A_0(v, -1/2, \mu(\varphi_{\mu_j}))$ and $A_{-1}(v, -1/2, \mu(\varphi_{\mu_j}))$ respectively.

Proof. We obtain

$$\int_{X^{mod,\hat{T}}} \vartheta(v, z, \mu_0)_0 d\mu(z) = \int_{X^{mod,\hat{T}}} \left(\int_{-1/2}^{1/2} \vartheta(u + iv, z, \mu_0) du \right) d\mu(z).$$

Since $X^{mod,\hat{T}}$ is compact and $\vartheta(\tau, z, \mu_0)$ is continuous in both coordinates we can apply Fubini's theorem. Then

$$\int_{X^{mod,\hat{T}}} \left(\int_{-1/2}^{1/2} \vartheta(u + iv, z, \mu_j) du \right) d\mu(z) = \int_{-1/2}^{1/2} \left(\int_{X^{mod,\hat{T}}} \vartheta(u + iv, z, \mu_j) d\mu(z) \right) du.$$

We apply corollary 1.7.19, obtaining

$$\int_{-1/2}^{1/2} \left(\int_{X^{mod,\hat{T}}} \vartheta(u + iv, z, \mu_j) d\mu(z) \right) du = \mathrm{vol}(X^{mod}) \Big[A_0(v, -1/2, \mu(\varphi_{\mu_j}))_0 + cA_{-1}(v, -1/2, \mu(\varphi_{\mu_j}))_0 \Big]$$

$$- \int_{-1/2}^{1/2} \int_{\widehat{X^{mod,\hat{T}}}} \mathbf{Conv}(u + iv, z, \mu_j) d\mu(z) dv + \int_{-1/2}^{1/2} \int_{X^{mod,\hat{T}}} \mathbf{Div}(u + iv, z, \mu_j) d\mu(z) dv.$$

Using Fubini's theorem one more time

$$\begin{aligned}\int_{-1/2}^{1/2} \int_{X^{mod,\hat{T}}} \mathbf{Div}(u + iv, z, \mu_j) d\mu(z) du &= \int_{X^{mod,\hat{T}}} \left(\int_{-1/2}^{1/2} \mathbf{Div}(u + iv, z, \mu_j) du \right) d\mu(z) \\ &= \int_{X^{mod,\hat{T}}} \mathbf{Div}(v, z, \mu_j)_0 d\mu(z).\end{aligned}$$

Furthermore, analogously to (1.7.3) we obtain

$$\int_{-1/2}^{1/2}\int_{\widehat{X^{mod,\hat{T}}}}\mathbf{Conv}(u+iv,z,\mu_j)d\mu(z)du=\int_{\widehat{X^{mod,\hat{T}}}}\mathbf{Conv}(v,z,\mu_j)_0 d\mu(z). \tag{1.-15}$$

Using the equalities given in (1.7.3) and (1.7.3) on the right hand side of the equality (1.7.3), the result follows. □

Integral of the singular theta lift.

Let $f\in M^!_{1/2,L}$, the goal of this subsection is to compute the integral

$$\int_{X^{mod,\hat{T}}}\left(\int^{\bullet}_{X^{mod}}\langle f(\tau),\Theta^{Sieg}(\tau,z)\rangle d\mu(\tau)\right)d\mu(z),$$

that appears in the equality (1.7.3).

Before starting the computation we will recall the method of [Kud03]. Although it does not apply to our case, the strategy used in [Kud03] will be usefull for our goal. Let $(V^{(2,p)},q)$ be either an isotropic rational quadratic space of signature $(2,p)$ with $p\geq 3$ or an anisotropic rational quadratic space. Let us fix a lattice L of V so that $q|_L\in\mathbb{Z}$. In [Kud03] the author studies the integrals

$$\int_{X^{(2,p)}}\left(\int^{\bullet}_{X^{mod}}\langle f(\tau),\Theta^{Sieg}(\tau,z)\rangle d\mu(\tau)\right)d\mu(z),$$

where $X^{(2,p)}$ is the Shimura variety associated to the algebraic group $\mathrm{GSpin}_{V^{(2,p)}}$. More concretely, in [Kud03, sec. 3, p. 24], given $\mu\in L'(\mathbb{A}_f)/L(\mathbb{A}_f)$ the author shows

$$\begin{aligned}\int_{X^{(2,p)}}&\left(\int^{\bullet}_{X^{mod}}f_\mu(\tau)\theta^{Sieg}(\tau,z,\mu)v d\mu(\tau)\right)d\mu(z)\\&=\int_{X^{(2,p)}}\left(\int^{\bullet}_{X^{mod}}f_\mu(\tau)\vartheta(\tau,z,\mu)d\mu(\tau)\right)d\mu(z)\\&=\mathrm{CT}_{\sigma=0}\lim_{T\to\infty}\int_{\mathcal{F}^T}f_\mu(\tau)\left(\int_{X^{(2,p)}}\vartheta(\tau,z,\mu)d\mu(z)\right)v^{-\sigma}d\mu(\tau).\end{aligned}$$

Once this is achieved, the author applies the Siegel-Weil formula and unfolds the integral with the resulting Eisenstein series. Although the proof of the equality between (1.7.3) and (1.7.3) does not apply for a quadratic rational isotropic space of signature $(2,1)$, i.e. the modular curve case, we will give an overview of the proof, pointing out which parts of the proof are usefull for us and what propositions does not apply in the present case.

Proposition 1.7.21. *Let $\theta(g,h,\varphi)$ be the theta function associated to any dual reductive pair of the form $(\mathrm{Sp}_1,O(V))$ with V rational quadratic space of signature $(2,p)$. Given $\beta\in Sym_2(\mathbb{Q})$, the $\beta-$Fourier coefficient of $\theta(g,h)$ with respect to g is equal to*

$$\theta(g,h,\varphi)_\beta=\sum_{\substack{x\in\mathbb{Q}^m\\((x,x))=2\beta}}\omega(g,h)\varphi(x).$$

Let $\mu\in L'/L$, we denote by $\theta^{Sieg}_\mu(\tau,z)$ the $\mu-$component of the Siegel theta function associated to a lattice L as in definition 1.3.5. Its Fourier expansion with respect to τ is given by

$$\theta^{Sieg}_\mu(\tau,z)=\sum_{\lambda\in\mu+L}e^{-2\pi(vq(\lambda_z)-vq(\lambda_{z^\perp}))}e^{2\pi iq(\lambda)u}.$$

Furthermore

$$\left\langle f(\tau), \Theta_L^{Sieg}(\tau,z)\right\rangle = \sum_{\mu\in L'/L}\sum_{m\in\mathbb{Q}} c_\mu(-m) \sum_{\substack{\lambda\in V(\mathbb{Q})\\ Q(x)=m}} \varphi_\mu(x) e^{-2\pi v(q(\lambda_z)-q(\lambda_{z^\perp}))}.$$

Proof. We find the first assertion in [KR88, (5.1), p. 512] and the second one in [Bru02, p. 48]. □

Using the previous proposition we may factor our theta function as follows:

$$\begin{aligned}\theta_\mu^{Sieg}(\tau,z) =& C_{00}(\tau,z)_\mu^{Sieg} + C_0(\tau,z)_\mu^{Sieg} + C_1(\tau,z)_\mu^{Sieg} + C_2(\tau,z)_\mu^{Sieg}\\ =& 1 + \sum_{\substack{0\neq\lambda\in L+\mu\\ q(\lambda)=0}} e^{-2\pi(vq(\lambda_z)-vq(\lambda_{z^\perp}))}\\ &+ \sum_{\substack{\lambda\in L+\mu\\ q(\lambda)>0}} e^{-2\pi(vq(\lambda_z)-vq(\lambda_{z^\perp}))} e^{2\pi i q(\lambda)u}\\ &+ \sum_{\substack{\lambda\in L+\mu\\ q(\lambda)<0}} e^{-2\pi(vq(\lambda_z)-vq(\lambda_{z^\perp}))} e^{2\pi i q(\lambda)u},\end{aligned}$$

where the functions $C_{00}(\tau,z)_\mu^{Sieg},\ C_0(\tau,z)_\mu^{Sieg},\ C_1(\tau,z)_\mu^{Sieg},\ C_2(\tau,z)_\mu^{Sieg}$ correspond to the terms on the right hand side of the equality. Since

$$\vartheta(\tau,z,\mu) = v\theta_\mu^{Sieg}(\tau,z),$$

we may also factor $\vartheta(\tau,z,\mu)$ by

$$\begin{aligned}\vartheta(\tau,z,\mu) =& C_{00}(\tau,z,\mu) + C_0(\tau,z,\mu) + C_1(\tau,z,\mu) + C_2(\tau,z,\mu)\\ =& vC_{00}(\tau,z)_\mu^{Sieg} + vC_0(\tau,z)_\mu^{Sieg} + vC_1(\tau,z)_\mu^{Sieg} + vC_2(\tau,z)_\mu^{Sieg},\end{aligned}$$

where the functions $C_{00}(\tau,z,\mu),\ C_0(\tau,z,\mu),\ C_1(\tau,z,\mu)\ C_2(\tau,z,\mu)$ are defined by the terms of the right hand side of the above equality.

Remark 1.7.22. *In the formula (1.7.3) the terms $C_1(\tau,z,\mu)$ are the positive Fourier coefficients of $\vartheta(\tau,z,\mu)$ with respect to τ, $C_2(\tau,z,\mu)$ are the negative Fourier coefficients of $\vartheta(\tau,z,\mu)$ with respect to τ and $C_{00}(\tau,z,\mu)+C_0(\tau,z,\mu)$ is the $0-th$ Fourier coefficient of $\vartheta(\tau,z,\mu)$ with respect to τ. The motivation to split the $0-th$ Fourier coefficient into two terms is the asymptotic behaviour with respect to $\tau=u+iv$. The function*

$$C_0(\tau,z,\mu) = v\sum_{\substack{\lambda\in L+\mu\\ q(\lambda)=0\\ \lambda\neq 0}} e^{-2\pi(vq(\lambda_z)-vq(\lambda_{z^\perp}))},$$

decreases as e^{-v} when $v\to\infty$. By contrast the term $C_{00}(\tau,z,\mu)=v$ is not exponentially decreasing. In fact, the term $C_{00}(\tau,z,\mu)$ is the reason why $\mathrm{CT}_{\sigma=0}$ is needed to state the singular theta lift, see [Bor98].

In [Kud03, sec. 3, p. 24], the author factors the integral (1.7.3) according to the factorization given in

(1.7.3):

$$\begin{aligned}\mathrm{CT}_{\sigma=0}\int_{X^{(2,p)}}&\left(\lim_{T\to\infty}\int_{\mathcal{F}^T} f_\mu(\tau)\vartheta(\tau,z,\mu)v^{-\sigma}d\mu(\tau)\right)d\mu(z)\\ &=\mathrm{CT}_{\sigma=0}\int_{X^{(2,p)}}\left(\lim_{T\to\infty}\int_{\mathcal{F}^T} f_\mu(\tau)C_{00}(\tau,z,\mu)v^{-\sigma}d\mu(\tau)\right)d\mu(z)\\ &+\mathrm{CT}_{\sigma=0}\int_{X^{(2,p)}}\left(\lim_{T\to\infty}\int_{\mathcal{F}^T} f_\mu(\tau)C_{0}(\tau,z,\mu)v^{-\sigma}d\mu(\tau)\right)d\mu(z)\\ &+\mathrm{CT}_{\sigma=0}\int_{X^{(2,p)}}\left(\lim_{T\to\infty}\int_{\mathcal{F}^T} f_\mu(\tau)C_{1}(\tau,z,\mu)v^{-\sigma}d\mu(\tau)\right)d\mu(z)\\ &+\mathrm{CT}_{\sigma=0}\int_{X^{(2,p)}}\left(\lim_{T\to\infty}\int_{\mathcal{F}^T} f_\mu(\tau)C_{2}(\tau,z,\mu)v^{-\sigma}d\mu(\tau)\right)d\mu(z).\end{aligned}$$

The proof of the equality between (1.7.3) and (1.7.3) is based into two facts; the convergence of $\int_{X^{(2,p)}}\vartheta(\tau,z,\mu)d\mu(z)$ and that the following functions are holomorphic at $\sigma=0$:

$$\int_{X^{(2,p)}}\left(\lim_{T\to\infty}\int_{\mathcal{F}^T} f_\mu(\tau)C_{00}(\tau,z,\mu)v^{-\sigma}d\mu(\tau)\right)d\mu(z), \tag{1.-20}$$

$$\int_{X^{(2,p)}}\left(\lim_{T\to\infty}\int_{\mathcal{F}^T} f_\mu(\tau)C_{0}(\tau,z,\mu)v^{-\sigma}d\mu(\tau)\right)d\mu(z), \tag{1.-20}$$

$$\int_{X^{(2,p)}}\left(\lim_{T\to\infty}\int_{\mathcal{F}^T} f_\mu(\tau)C_{1}(\tau,z,\mu)v^{-\sigma}d\mu(\tau)\right)d\mu(z), \tag{1.-20}$$

$$\int_{X^{(2,p)}}\left(\lim_{T\to\infty}\int_{\mathcal{F}^T} f_\mu(\tau)C_{2}(\tau,z,\mu)v^{-\sigma}d\mu(\tau)\right)d\mu(z). \tag{1.-20}$$

The proof that the functions (1.7.3), (1.7.3) and (1.7.3) are holomorphic is in [Kud03, sec. 3, p. 24] and applies for any Shimura variety associated to a quadratic space of signature $(2,1)$. By contrast, the proof that (1.7.3) is holomorphic, see [Kud03, prop. 3.4, p. 28], does not apply for the modular curve case. As we mentioned before, in this paper we will consider the integral (1.7.3) separately and we will compute it using a different approach.

Let $T\geq 1$, we set

$$\begin{aligned}\mathcal{F}_1&=\{\tau=u+iv\in\mathcal{F}^T,\ s.t. v\leq 1\},\\ \mathcal{F}_2^T&=\{\tau=u+iv\in\mathcal{F}^T,\ s.t. v>1\},\end{aligned}$$

so that

$$\mathcal{F}^T=\mathcal{F}_1\bigsqcup\mathcal{F}_2^T. \tag{1.-20}$$

Lemma 1.7.23. *The following equality holds:*

$$\begin{aligned}\int_{X^{mod,\hat{T}}}&\left(\int^{\bullet}_{X^{mod}}\left\langle f(\tau),\Theta_L^{Sieg}(\tau,z)\right\rangle d\mu(\tau)\right)d\mu(z)\\ &=\sum_{j=0}^{1}\lim_{T\to\infty}\Big[\int_{\mathcal{F}^T} f_{\mu_j}(\tau)\left(\int_{X^{mod,\hat{T}}}\vartheta(\tau,z,\mu_j)d\mu(z)\right)d\mu(\tau)\\ &\quad-\int_{\mathcal{F}_2^T} f_{\mu_0}(\tau)\left(\int_{X^{mod,\hat{T}}}\vartheta(v,z,\mu_0)_0 d\mu(z)\right)d\mu(\tau)\Big]\\ &\quad+\int_{X^{mod,\hat{T}}}\mathrm{CT}_{\sigma=0}\lim_{T\to\infty}\int_{\mathcal{F}_2^T} f_{\mu_0}(\tau)\vartheta(v,z,\mu_0)_0 v^{-\sigma}d\mu(\tau)d\mu(z).\end{aligned}$$

Proof. Using the factorization (1.7.3) we obtain

$$\begin{aligned}\int_{X^{mod,\hat{T}}} &\mathrm{CT}_{\sigma=0}\lim_{T\to\infty}\int_{\mathcal{F}^T}\left\langle f(\tau),\Theta^{Sieg}(\tau,z)\right\rangle v^{-\sigma}d\mu(\tau)d\mu(z)\\ &=\int_{X^{mod,\hat{T}}}\mathrm{CT}_{\sigma=0}\lim_{T\to\infty}\int_{\mathcal{F}_2^T}\left\langle f(\tau),\Theta^{Sieg}(\tau,z)\right\rangle v^{-\sigma}d\mu(\tau)d\mu(z)\\ &\quad+\int_{X^{mod,\hat{T}}}\mathrm{CT}_{\sigma=0}\lim_{T\to\infty}\int_{\mathcal{F}_1}\left\langle f(\tau),\Theta^{Sieg}(\tau,z)\right\rangle v^{-\sigma}d\mu(\tau)d\mu(z).\end{aligned}$$

On the one hand, since $\mathcal{F}_1$ is compact

$$\begin{aligned}\int_{X^{mod,\hat{T}}} &\mathrm{CT}_{\sigma=0}\lim_{T\to\infty}\int_{\mathcal{F}_1}\left\langle f(\tau),\Theta^{Sieg}(\tau,z)\right\rangle v^{-\sigma}d\mu(\tau)d\mu(z)\\ &=\mathrm{CT}_{\sigma=0}\lim_{T\to\infty}\int_{\mathcal{F}_1}\int_{X^{mod,\hat{T}}}\left\langle f(\tau),\Theta^{Sieg}(\tau,z)\right\rangle v^{-\sigma}d\mu(\tau)d\mu(z).\end{aligned}$$

On the other hand, using the factorization given in (1.7.3)

$$\begin{aligned}\int_{X^{mod,\hat{T}}} &\mathrm{CT}_{\sigma=0}\lim_{T\to\infty}\int_{\mathcal{F}_2^T}\left\langle f(\tau),\Theta^{Sieg}(\tau,z)\right\rangle v^{-\sigma}d\mu(\tau)d\mu(z)\\ &=\sum_{j=0}^{1}\int_{X^{mod,\hat{T}}}\mathrm{CT}_{\sigma=0}\lim_{T\to\infty}\int_{\mathcal{F}_2^T}f_{\mu_j}(\tau)(C_1(\tau,z,\mu_j)+C_2(\tau,z,\mu_j))v^{-\sigma}d\mu(\tau)d\mu(z)\\ &\quad+\int_{X^{mod,\hat{T}}}\mathrm{CT}_{\sigma=0}\lim_{T\to\infty}\int_{\mathcal{F}_2^T}f_{\mu_j}(\tau)\left(C_{00}(\tau,z,\mu_j)+C_0(\tau,z,\mu_j)\right)v^{-\sigma}d\mu(\tau)d\mu(z).\end{aligned}$$

Using that (1.7.3) and (1.7.3) are holomorphic at $\sigma=0$, the following equality holds:

$$\begin{aligned}\int_{X^{mod,\hat{T}}} &\mathrm{CT}_{\sigma=0}\lim_{T\to\infty}\int_{\mathcal{F}_2^T}f_{\mu_j}(\tau)(C_1(\tau,z,\mu_j)+C_2(\tau,z,\mu_j))v^{-\sigma}d\mu(\tau)d\mu(z)\\ &=\lim_{T\to\infty}\int_{\mathcal{F}_2^T}f_{\mu_j}(\tau)\int_{X^{mod,\hat{T}}}(C_1(\tau,z,\mu_j)+C_2(\tau,z,\mu_j))d\mu(z)\mu(\tau).\end{aligned}$$

Moreover, since $C_1(\tau,z,\mu_j)+C_2(\tau,z,\mu_j)=\vartheta(\tau,z,\mu_j)-\vartheta(\tau,z,\mu_j)_0$ we have

$$\lim_{T\to\infty}\int_{\mathcal{F}_2^T}f_{\mu_j}(\tau)\int_{X^{mod,\hat{T}}}(C_1(\tau,z,\mu_j)+C_2(\tau,z,\mu_j))d\mu(z)\mu(\tau) \tag{1.-29}$$

$$=\lim_{T\to\infty}\left(\int_{\mathcal{F}_2^T}f_{\mu_j}(\tau)\left(\int_{X^{mod,\hat{T}}}\vartheta(\tau,z,\mu_j)d\mu(z)\right)d\mu(\tau)-\int_{\mathcal{F}_2^T}f_{\mu_j}(\tau)\left(\int_{X^{mod,\hat{T}}}\vartheta(v,z,\mu_j)_0d\mu(z)\right)d\mu(\tau)\right).$$

Since by 1.7.22 and [Fun02, (3.1), p. 295] we have

$$\vartheta(v,z,\mu_j)_0=\begin{cases}v\sum_{\substack{\lambda\in\mathbb{Z}^3\\ q(\lambda)=0}}e^{-2\pi(vq(\lambda_z)-vq(\lambda_{z^\perp}))} & if\ j=0\\ 0 & if\ j=1\end{cases},$$

plugging (1.7.3) and (1.7.3) in (1.7.3), and then (1.7.3) and (1.7.3) in (1.7.3) we obtain the result. □

In order to obtain an explicit expression for (1.7.23) we will consider separately each factor of the right hand side.

Definition 1.7.24. The integral

$$\int_{X^{mod,\hat{T}}}\mathrm{CT}_{\sigma=0}\lim_{T\to\infty}\int_{\mathcal{F}_2^T}f_{\mu_0}(\tau)\vartheta(v,z,\mu_0)_0v^{-\sigma}d\mu(\tau)d\mu(z),$$

is called the *limit case*. Furthermore we refer to the integral

$$\lim_{T\to\infty}\left(\int_{\mathcal{F}^T} f_{\mu_j}(\tau)\left(\int_{X^{mod,\hat{T}}}\vartheta(\tau,z,\mu_j)d\mu(z)\right)d\mu(\tau)-\int_{\mathcal{F}_2^T} f_{\mu_0}(\tau)\left(\int_{X^{mod,\hat{T}}}\vartheta(v,z,\mu_0)_0 d\mu(z)\right)d\mu(\tau)\right)$$

as the *ordinary case*.

Ordinary case

In this subsection we compute the ordinary case using the results of subsection 1.7.4. With the aim of simplifying the argument, we divide the computation into two terms:

$$\int_{\mathcal{F}^T} f_{\mu_j}(\tau)\left(\int_{X^{mod,\hat{T}}}\vartheta(\tau,z,\mu_j)d\mu(z)\right)d\mu(\tau),\quad \int_{\mathcal{F}_2^T} f_{\mu_0}(\tau)\left(\int_{X^{mod,\hat{T}}}\vartheta(v,z,\mu_0)_0 d\mu(z)\right)d\mu(\tau).$$

Lemma 1.7.25. *The convergent part satisfies the following estimate:*

$$\int_{\mathcal{F}_1} f_{\mu_j}(\tau)\int_{\widehat{X^{mod,\hat{T}}}}\mathbf{Conv}(\tau,z,\mu_j)d\mu(z)d\mu(\tau)\ \textit{and}\ \int_{\mathcal{F}_1} f_{\mu_0}(\tau)\int_{\widehat{X^{mod,\hat{T}}}}\mathbf{Conv}(v,z,\mu_0)_0 d\mu(z)d\mu(\tau)$$
$$=O(e^{-\hat{T}}).$$

Proof. By lemma 1.7.4 the function $\mathbf{Conv}(\tau,z,\mu_j)$ is exponentially decreasing in the variable $z=x+iy$, then

$$|\mathbf{Conv}(\tau,z,\mu_j)|\leq |F_{\mu_j}(\tau)|e^{-y},$$

where $F_{\mu_j}(\tau)$ is a continuous function. Hence

$$\left|\int_{\mathcal{F}_1} f_{\mu_j}(\tau)\int_{\widehat{X^{mod,\hat{T}}}}\mathbf{Conv}(\tau,z,\mu_j)d\mu(z)d\mu(\tau)\right|\leq e^{-\hat{T}}\int_{\mathcal{F}_1}\left|f_{\mu_j}(\tau)F_{\mu_j}(\tau)\right|d\mu(\tau).$$

Using the compactness of $\mathcal{F}_1$ we obtain the estimate of the statement. The proof for $\mathbf{Conv}(v,z,\mu_0)_0$ is completely analogous. $\square$

Proposition 1.7.26. *The following equality holds:*

$$\sum_{j=0}^{1}\int_{\mathcal{F}^T} f_{\mu_j}(\tau)\left(\int_{X^{mod,\hat{T}}}\vartheta(\tau,z,\mu_j)d\mu(z)\right)d\mu(\tau)=\sum_{j=0}^{1}2\mathrm{vol}(X^{mod})\sum_{m\geq 0}c_{\mu_j}(-m)b(m,T,\varphi_{\mu_j})$$
$$-\frac{1}{\sqrt{T}}\sum_{j=0}^{1}2c_{\mu_j}(0)\left(\log(\hat{T})+2\left(-\tanh^{-1}\left(\frac{\sqrt{7}}{4}\right)+\frac{\sqrt{7}}{4}+\frac{\log(3/4)}{2}\right)\right)+O(e^{-\hat{T}}),$$

where we recall that $b(m,T,\varphi_{\mu_j})$ is the first term at $s=1/2$ of the Laurent series of the $m-$Fourier coefficient of $E(\tau,s,3/2,\mu(\varphi_{\mu_j}))$, see (1.6.4).

Proof. We recall that corollary 1.7.19 provides

$$\int_{X^{mod,\hat{T}}}\vartheta(\tau,z,\mu(\varphi_{\mu_j}))d\mu(z)=\mathrm{vol}(X^{mod})\Big[A_0(\tau,-1/2,\mu(\varphi_{\mu_j}))$$
$$+c\mathrm{Res}_{s=1/2}\,E\left(\tau,s,1/2,\mu(\tilde{\varphi})\right)\Big]+\int_{\widehat{X^{mod,\hat{T}}}}\mathbf{Conv}(\tau,z,\mu_j)d\mu(z)$$
$$+\int_{X^{mod,\hat{T}}}\mathbf{Div}(\tau,z,\mu_j)d\mu(z).$$

Then using the above equality we factor the integral of the statement into the sum of three terms:

$$\begin{aligned}\operatorname{vol}(X^{mod}) &\left[\int_{\mathcal{F}^T} f_{\mu_j}(\tau)A_0(\tau,-1/2,\mu(\varphi_{\mu_j}))d\mu(\tau) + \int_{\mathcal{F}^T} f_{\mu_j}(\tau)A_{-1}(\tau,-1/2,\mu(\tilde{\varphi}))d\mu(\tau)\right] \\ &- \int_{\mathcal{F}^T} f_{\mu_j}(\tau)\int_{\widehat{X^{mod,\hat{T}}}} \mathbf{Conv}(\tau,z,\mu_j)d\mu(z)d\mu(\tau) \\ &+ \int_{\mathcal{F}^T} f_{\mu_j}(\tau)\int_{X^{mod,\hat{T}}} \mathbf{Div}(\tau,z,\mu_j)d\mu(z)d\mu(\tau).\end{aligned}$$

We consider each above factor separately. Using proposition 1.7.51 and lemma 1.7.53 in (1.7.3), lemma 1.7.25 in (1.7.3) and proposition 1.7.47 in (1.7.3), the result follows. □

Proposition 1.7.27. *We obtain*

$$\begin{aligned}&\int_{\mathcal{F}_2^T} f_{\mu_0}(\tau)\left(\int_{X^{mod,\hat{T}}} \vartheta(\tau,z,\mu_0)_0 d\mu(z)\right)d\mu(\tau) \\ &= c_{\mu_0}(0)\operatorname{vol}(X^{mod})\log(T) - \left(1-\frac{1}{\sqrt{T}}\right)\left[\frac{2c_{\mu_0}(0)(-12\zeta'(2)+2\gamma\pi^2+\pi^2(-\gamma-\log(8)))}{\pi^2}\right. \\ &\left.+ 2c_{\mu_0}(0)\left(\log(\hat{T}) + 2\left(-\tanh^{-1}\left(\frac{\sqrt{7}}{4}\right)+\frac{\sqrt{7}}{4}+\frac{\log(3/4)}{2}\right)\right)\right] + O(e^{-\hat{T}}).\end{aligned}$$

Proof. We recall that corollary 1.7.20 shows

$$\begin{aligned}\int_{X^{mod,\hat{T}}} \vartheta(v,z,\mu_0)_0 d\mu(z) &= \operatorname{vol}(X^{mod})\Big[A_0(v,-1/2,\mu(\varphi_{\mu_0}))_0 \\ &+ cA_{-1}(v,-1/2,\mu(\varphi_{\mu_0}))_0\Big] - \int_{\widehat{X^{mod,\hat{T}}}} \mathbf{Conv}(v,z,\mu_0)_0 d\mu(z) \\ &+ \int_{X^{mod,\hat{T}}} \mathbf{Div}(v,z,\mu_0)_0 d\mu(z).\end{aligned}$$

We proceed as in the proof of proposition 1.7.26. Using the above equality on the integral of the statement, we find

$$\begin{aligned}\int_{\mathcal{F}_2^T} f_{\mu_0}(\tau)\left(\int_{X^{mod,\hat{T}}} \vartheta(v,z,\mu_0)_0 d\mu(z)\right)d\mu(\tau) =&\operatorname{vol}(X^{mod})\Big[\int_{\mathcal{F}^T} f_{\mu_0}(\tau)A_0(v,-1/2,\mu_0)_0 d\mu(\tau) \\ &+ \int_{\mathcal{F}_2^T} f_{\mu_0}(\tau)A_{-1}(v,-1/2,\mu_0)_0 d\mu(\tau)\Big] \\ &+ \int_{\mathcal{F}_2^T} f_{\mu_0}(\tau)\int_{X^{mod,\hat{T}}} \mathbf{Div}(v,z,\mu_0)_0 d\mu(z)d\mu(\tau) \\ &- \int_{\mathcal{F}_2^T} f_{\mu_0}(\tau)\int_{\widehat{X^{mod,\hat{T}}}} \mathbf{Conv}(v,z,\mu_0)_0 d\mu(z)d\mu(\tau).\end{aligned}$$

As we did in the proof of proposition 1.7.26 we consider each factor of the above equation separately. By corollary 1.7.54 we show that (1.7.3) vanishes. Moreover using corollary 1.7.52 on the right hand side of (1.7.3), lemma 1.7.48 in (1.7.3) and lemma 1.7.25 in (1.7.3), the statement follows. □

Corollary 1.7.28. *The following equality holds:*

$$\lim_{T\to\infty}\left(\sum_{j=0}^{1}\int_{\mathcal{F}^T} f_{\mu_j}(\tau)\left(\int_{X^{mod,\hat{T}}}\vartheta(\tau,z,\mu_j)d\mu(z)\right)d\mu(\tau)-\int_{\mathcal{F}^T} f_{\mu_0}(\tau)\left(\int_{X^{mod,\hat{T}}}\vartheta(v,z,\mu_0)_0 d\mu(z)\right)d\mu(\tau)\right)$$
$$=\sum_{j=0}^{1}2\mathrm{vol}(X^{mod})\sum_{m>0}c_{\mu_j}(-m)b(m,\varphi_{\mu_j})+\frac{2c_{\mu_0}(0)(-12\zeta'(2)+2\gamma\pi^2+\pi^2(-\gamma-\log(8)))}{\pi^2}$$
$$+2c_{\mu_0}(0)\left(\log(\hat{T})+2\left(-\tanh^{-1}\left(\frac{\sqrt{7}}{4}\right)+\frac{\sqrt{7}}{4}+\frac{\log(3/4)}{2}\right)\right)+O(e^{-\hat{T}}),$$

where $b(m,\varphi_{\mu_j})=\lim_{T\to\infty}b(m,T,\varphi_{\mu_j})$.

Proof. By [KY10, thm. 6.6, p. 2303], $\lim_{T\to\infty}(2b(m,T,\varphi_{\mu_0})-\log(T))=0$. Then, using propositions 1.7.26 and 1.7.27 we conclude. □

Limit case

This section is devoted to computing the limit case 1.7.24:

$$\int_{X^{mod,\hat{T}}}\left(\mathrm{CT}_{\sigma=0}\lim_{T\to\infty}\int_{\mathcal{F}_2^T} f_{\mu_0}(\tau)\vartheta(v,z,\mu_0)_0 v^{-\sigma}d\mu(\tau)\right)d\mu(z). \tag{1.-29}$$

In [Zag81] the author developed a Rankin-Selberg method for truncated fundamental domains. Before continuing with our computation we will introduce some notation.

For every $\hat{T}\geq 1$, the subset $\mathcal{F}^{\hat{T}}=\{x+iy\in\mathcal{H},\ s.t.\ y\leq\hat{T}\}$ is a fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on

$$\mathcal{H}^{\hat{T}}=\bigcup_{\gamma\in SL_2(\mathbb{Z})}\gamma\mathcal{F}^{\hat{T}}=\{z\in\mathcal{H},\ s.t.\ \max_{\gamma\in SL_2(\mathbb{Z})}\mathrm{Im}(\gamma z)\leq\hat{T}\}.$$

According to [Zag81, (20), p. 420]

$$\mathcal{H}^{\hat{T}}=\{z\in\mathcal{H},\ s.t.\ \mathrm{Im}(z)\leq\hat{T}\}-\bigcup_{c\geq 1}\bigcup_{\substack{a\in\mathbb{Z}\\(a,c)=1}}S_{a/c},$$

where $S_{a/c}$ is the disc of radius $\frac{1}{2c^2\hat{T}}$ tangent to the real axis at a/c. Therefore, given

$$\Gamma^{\infty}=\left\{\begin{pmatrix}1&x\\0&1\end{pmatrix},\ x\in\mathbb{Z}\right\},$$

we define the following subset of the complex numbers:

$$\Gamma^{\infty}\setminus\mathcal{H}^{\hat{T}}:=\{z=x+iy\in\mathcal{H},\ |x|\leq 1/2,\ 0\leq y\leq\hat{T}\}-\bigcup_{c\geq 1}\bigcup_{\substack{a\in\mathbb{Z}\\(a,c)=1}}S_{a/c}. \tag{1.-29}$$

Definition 1.7.29. Let $s\in\mathbb{C}$ such that $\mathrm{Re}(s)>1$. The *classical Eisenstein series* considered by Zagier is defined as follows:

$$E(\tau,s)=\sum_{\gamma\in\Gamma^{\infty}\setminus\mathrm{SL}_2(\mathbb{Z})}\mathrm{Im}(\gamma z)^s.$$

Proposition 1.7.30. *The function $E(z,s)$ is holomorphic in $\mathrm{Re}(s)>1/2$ except for a pole of residue $3/\pi$ at $s=1$. Furthermore $\zeta^*(2s)E(z,s)$ is holomorphic in all $s\in\mathbb{C}$ except for $s\neq 0,1$, where $\zeta^*(s)=\pi^{-s/2}\Gamma(s/2)\zeta(s)$ is the completed Riemann zeta function.*

Proof. We refer the reader to [Zag81, p. 415, p. 416]. □

Remark 1.7.31. *The* $0-th$ *Fourier term of the modular form* $E(\tau, s)$ *is equal to*

$$v^s + \varphi(s)v^{1-s},$$

where $\varphi(s) := \frac{\zeta^*(2s-1)}{\zeta^*(2s)}$.

Proposition 1.7.32. *The following equality holds:*

$$\int_{X^{mod,\hat{T}}} E(\tau, s)\frac{dxdy}{y^2} = \hat{T}^{s-1}/(s-1) - \frac{\zeta^*(2s-1)}{\zeta^*(2s)}\hat{T}^{-s}/s,$$

where $\zeta^*(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ *is the completed Riemann zeta function.*

Proof. See [Zag81, (33), p. 426]. □

Proposition 1.7.33. *Let* A *be the constant defined in 1.7.1. The limit case satisfies the following equality:*

$$\begin{aligned}
&\int_{X^{mod,\hat{T}}} \left(\mathrm{CT}_{\sigma=0} \lim_{T\to\infty} \int_{\mathcal{F}_2^T} f_{\mu_0}(\tau)\vartheta(v, z, \mu_0)_0 v^{-\sigma} d\mu(\tau)\right) d\mu(z) \\
&= c_{\mu_0}(0)\Bigg(A\mathrm{CT}_{s=0} \int_{\Gamma^\infty\backslash\mathcal{H}^{\hat{T}}} y^s \frac{dxdy}{y^2} - 8\mathrm{erf}\left(\sqrt{\frac{\pi}{2}}\right) \mathrm{CT}_{s=0} \int_{\Gamma^\infty\backslash\mathcal{H}^{\hat{T}}} y^s \log(y)\frac{dxdy}{y^2} \\
&\quad + \mathrm{CT}_{s=0} \int_{\Gamma^\infty\backslash\mathcal{H}^{\hat{T}}} y^s \frac{dydx}{y}\Bigg),
\end{aligned}$$

Proof. We factor $\vartheta(v, z, \mu_0)_0 = C_0(v, z, \mu_0) + C_{00}(v, z, \mu_0)$, where we recall the notation $C_0(v, z, \mu_0) = v\sum_{\substack{0\neq\lambda\in\mathbb{Z}^3 \\ q(\lambda)=0}} e^{-4\pi vq(\lambda_z)}$, and $C_{00}(v, z, \mu_0) = v$. We obtain

$$\begin{aligned}
&\int_{X^{mod,\hat{T}}} \left(\mathrm{CT}_{\sigma=0} \lim_{T\to\infty} \int_{\mathcal{F}_2^T} f_{\mu_0}(\tau)_\varphi\vartheta(v, z, \mu_0)_0 v^{-\sigma} d\mu(\tau)\right) d\mu(z) \\
&= \int_{X^{mod,\hat{T}}} \left(\mathrm{CT}_{\sigma=0} \lim_{T\to\infty} \int_{\mathcal{F}_2^T} f_{\mu_0}(\tau)C_0(v, z, \mu_0)v^{-\sigma+1} d\mu(\tau)\right) d\mu(z) \\
&+ \int_{X^{mod,\hat{T}}} \left(\mathrm{CT}_{\sigma=0} \lim_{T\to\infty} \int_{\mathcal{F}_2^T} f_{\mu_0}(\tau)C_{00}(v, z, \mu_0)v^{-\sigma+1} d\mu(\tau)\right) d\mu(z).
\end{aligned}$$

By direct computation

$$\mathrm{CT}_{\sigma=0} \lim_{T\to\infty} \int_{\mathcal{F}_2^T} f_{\mu_0}(\tau)C_{00}(v, z, \mu_0)v^{-\sigma+1} d\mu(\tau) = \mathrm{CT}_{\sigma=0} \lim_{T\to\infty} \int_1^T c_{\mu_0}(0)v^{-\sigma-1}dv = 0,$$

Hence

$$\begin{aligned}
&\int_{X^{mod,\hat{T}}} \left(\mathrm{CT}_{\sigma=0} \lim_{T\to\infty} \int_{\mathcal{F}_2^T} f_{\mu_0}(\tau)\vartheta(v, z, \mu_0)_0 v^{-\sigma} d\mu(\tau)\right) d\mu(z) \\
&= c_{\mu_0}(0) \int_{X^{mod,\hat{T}}} \left(\mathrm{CT}_{\sigma=0} \lim_{T\to\infty} \int_1^T \sum_{\substack{0\neq\lambda\in\mathbb{Z}^3 \\ q(\lambda)=0}} e^{-4\pi vq(\lambda_z)}v^{-\sigma-1}dv\right) d\mu(z).
\end{aligned}$$

The isotropic elements of the quadratic space, i.e. $0 \neq \lambda \in V(\mathbb{Q})$ such that $q(\lambda) = 0$, are generated by $\mathrm{SL}_2(\mathbb{Z})$ in one orbit. Furthermore those vectors are in one to one correspondence with $\mathbb{Q}\cup\{\infty\}$ and then

$\mathrm{SL}_2(\mathbb{Z})\setminus\{0\neq\lambda\in V(\mathbb{Q})\}$ is identified with the cusp ∞ of X^{mod}. One representative of the cusp ∞ in the projective cone model, described in proposition 1.2.4, is the isotropic line $\mathbb{Q}^\times\cdot e_1 := \mathbb{Q}^\times\cdot(1,0,0)$. The stabilizer of this isotropic line is given by

$$\Gamma_\infty=\left\{\begin{pmatrix}1&0&x\\0&1&0\\0&0&1\end{pmatrix},\ x\in\mathbb{Z}\right\}\simeq\left\{\begin{pmatrix}1&x\\0&1\end{pmatrix},\ x\in\mathbb{Z}\right\},$$

where the above isomorphism is the execptional isomorphism $SO(2,1)\simeq\mathrm{SL}_2(\mathbb{R})$. Hence we obtain the following identification

$$\{0\neq\lambda\in V,\ s.t.\ q(\lambda)=0\}\simeq\Gamma_\infty\setminus\mathrm{SL}_2(\mathbb{Z})\cdot(\mathbb{Q}^\times\cdot e_1). \tag{1.-30}$$

We rewrite the inner sum of (1.7.3) as follows:

$$\sum_{\substack{0\neq\lambda\in\mathbb{Z}^3\\ q(\lambda)=0}} e^{-4\pi vq(\lambda_z)}=\sum_{\gamma\in\Gamma_\infty\setminus\mathrm{SL}_2(\mathbb{Z})}\ \sum_{x_2\in\mathbb{Z}\setminus\{0\}} e^{-4\pi vq(\gamma\cdot(x_2,0,0)_z)}.$$

By the invariance property of the Gaussian we find

$$\sum_{\gamma\in\Gamma_\infty\setminus\mathrm{SL}_2(\mathbb{Z})}\ \sum_{x_2\in\mathbb{Z}\setminus\{0\}} e^{-4\pi vq(\gamma\cdot(x_2,0,0)_z)}=\sum_{\gamma\in\Gamma_\infty\setminus\mathrm{SL}_2(\mathbb{Z})}\ \sum_{x_2\in\mathbb{Z}\setminus\{0\}} e^{-4\pi vq\left((x_2,0,0)_{\gamma^{-1}z}\right)}.$$

We would like to unfold the integral over $X^{mod,\hat{T}}$. In order to overcome the convergence problems of the unfolding we introduce the auxiliary term $\mathrm{Im}(z)^s$ with $s\in\mathbb{C}$. The integral of (1.7.3) is equal to

$$\begin{aligned}&\int_{X^{mod,\hat{T}}}\left(\mathrm{CT}_{\sigma=0}\lim_{T\to\infty}\int_1^T\sum_{\gamma\in\Gamma_\infty\setminus\mathrm{SL}_2(\mathbb{Z})}\ \sum_{x_2\in\mathbb{Z}\setminus\{0\}} e^{-4\pi vq((x_2,0,0)_{\gamma^{-1}z})}v^{-\sigma-1}\mathrm{CT}_{s=0}\mathrm{Im}(\gamma z)^s dv\right)d\mu(z)\\ &\quad=\mathrm{CT}_{s=0}\int_{X^{mod,\hat{T}}}\left(\mathrm{CT}_{\sigma=0}\lim_{T\to\infty}\int_1^T\sum_{\gamma\in\Gamma_\infty\setminus\mathrm{SL}_2(\mathbb{Z})}\ \sum_{x_2\in\mathbb{Z}\setminus\{0\}} e^{-4\pi vq((x_2,0,0)_{\gamma^{-1}z})}v^{-\sigma-1}\mathrm{Im}(\gamma z)^s dv\right)d\mu(z),\end{aligned}$$

where the equality is justified by means of Fubini's theorem. To unfold the integral (1.7.3) with the sum, we suppose that $s\in\mathbb{C}$ satisfies that $\mathrm{Re}(s)>1$ to ensure the convergence of the resulting function. After the unfolding, by means of meromorphic continuation the result will follow. Under this assumption on s, (1.7.3) is equal to

$$\mathrm{CT}_{s=0}\int_{\Gamma\infty\setminus X^{mod,\hat{T}}}\left(\mathrm{CT}_{\sigma=0}\lim_{T\to\infty}\int_1^T\sum_{x_2\in\mathbb{Z}\setminus\{0\}} e^{-4\pi vq((x_2,0,0)_z)}v^{-\sigma-1}dv\right)y^s d\mu(z), \tag{1.-31}$$

that is well defined. Using the formula given in [Fun02, (3.9), p. 296] it holds that

$$q((x_2,0,0)_z)=\frac{x_2^2}{y^2}.$$

By Poisson summation formula we obtain

$$\sum_{x_2\in\mathbb{Z}\setminus\{0\}} e^{\frac{-4\pi vx_2^2}{y^2}}=\sum_{w_1\in\mathbb{Z}}\int_{\mathbb{R}} e^{\frac{-4\pi vx_2^2}{y^2}+2\pi ix_2w_1}dx_2-1, \tag{1.-31}$$

where the -1 term in the above equation corresponds to the term $e^{\frac{-2\pi vx_2^2}{y^2}}$ evaluated at $x_2=0$. Let us divide $\mathbb{Z} = \mathbb{Z} \setminus \{0\} \cup \{0\}$, we factor the right hand side of (1.7.3) as follows:

$$\sum_{w_1 \in \mathbb{Z}} \int_{\mathbb{R}} e^{\frac{-4\pi v x_2^2}{y^2} + 2\pi i x_2 w_1} dx_2 - 1 = \sum_{w_1 \in \mathbb{Z}\setminus\{0\}} \int_{\mathbb{R}} e^{\frac{-4\pi v x_2^2}{y^2} + 2\pi i x_2 w_1} dx_2 + \int_{\mathbb{R}} e^{\frac{-4\pi v x_2^2}{y^2}} dx_2 - 1. \tag{1.-31}$$

Plugging the factorization (1.7.3) in the integral (1.7.3) we obtain

$$\begin{aligned}
&\mathrm{CT}_{s=0} \int_{\Gamma^\infty \setminus X^{mod,\hat{T}}} \left(\mathrm{CT}_{\sigma=0} \lim_{T\to\infty} \int_1^T \sum_{x_2 \in \mathbb{Z}\setminus\{0\}} e^{-4\pi v q((x_2,0,0)_z)} v^{-\sigma-1} dv \right) y^s d\mu(z) \\
&= \mathrm{CT}_{s=0} \int_{\Gamma^\infty \setminus X^{mod,\hat{T}}} \left(\mathrm{CT}_{\sigma=0} \lim_{T\to\infty} \int_1^T \int_{\mathbb{R}} e^{\frac{-4\pi v x_2^2}{y^2}} dx_2 v^{-\sigma-1} dv \right) y^s d\mu(z) \\
&+ \mathrm{CT}_{s=0} \int_{\Gamma^\infty \setminus X^{mod,\hat{T}}} \left(\mathrm{CT}_{\sigma=0} \lim_{T\to\infty} \int_1^T \sum_{w_1 \in \mathbb{Z}\setminus\{0\}} \int_{\mathbb{R}} e^{\frac{-4\pi v x_2^2}{y^2} + 2\pi i x_2 w_1} dx_2 v^{-\sigma-1} dv \right) y^s d\mu(z) \\
&+ \mathrm{CT}_{s=0} \int_{\Gamma^\infty \setminus \mathcal{H}^{\hat{T}}} \left(\mathrm{CT}_{\sigma=0} \lim_{T\to\infty} \int_1^T v^{-\sigma-1} dv \right) y^s d\mu(z).
\end{aligned}$$

The goal of this proposition is achieved by computing the sum of the above three integrals. Applying lemma 1.7.34 and the proof of lemma 1.7.39 to (1.7.3) we obtain that

$$\int_{\Gamma^\infty \setminus \mathcal{H}^{\hat{T}}} \left(\mathrm{CT}_{\sigma=0} \lim_{T\to\infty} \int_1^T \int_{\mathbb{R}} e^{\frac{-4\pi v x_2^2}{y^2}} dx_2 v^{-\sigma-1} dv \right) y^s d\mu(z),$$

is a meromorphic function in the variable $s \in \mathbb{C}$. We apply lemma 1.7.35 and the proof of lemmas 1.7.38 and 1.7.37 to (1.7.3) to obtain that

$$\int_{\Gamma^\infty \setminus \mathcal{H}^{\hat{T}}} \left(\mathrm{CT}_{\sigma=0} \lim_{T\to\infty} \int_1^T \sum_{w_1 \in \mathbb{Z}\setminus\{0\}} \int_{\mathbb{R}} e^{\frac{-4\pi v x_2^2}{y^2} + 2\pi i x_2 w_1} dx_2 v^{-\sigma-1} dv \right) y^s d\mu(z),$$

is a meromorphic function in the variable $s \in \mathbb{C}$. Applying lemma 1.7.36 to (1.7.3), the function

$$\int_{\Gamma^\infty \setminus \mathcal{H}^{\hat{T}}} \left(\mathrm{CT}_{\sigma=0} \lim_{T\to\infty} \int_1^T v^{-\sigma-1} dv \right) y^s d\mu(z),$$

is meromorphic in the variable $s \in \mathbb{C}$. Hence by meromorphic continuation in $s \in \mathbb{C}$ we obtain that the equality (1.7.3) holds for every $s \in \mathbb{C}$. To conclude we use lemma 1.7.39 in (1.7.3), lemma 1.7.35 in (1.7.3) and lemma 1.7.36 in (1.7.3). □

Lemma 1.7.34. *Let $s \in \mathbb{C}$ so that $\mathrm{Re}(s) > 1$. The integral (1.7.3) satisfies*

$$\int_{\Gamma^\infty \setminus \mathcal{H}^{\hat{T}}} \left(\mathrm{CT}_{\sigma=0} \lim_{T\to\infty} \int_1^T \int_{\mathbb{R}} e^{\frac{-4\pi v x_2^2}{y^2}} dx_2 v^{-\sigma-1} dv \right) y^s d\mu(z) = \int_{\Gamma^\infty \setminus \mathcal{H}^{\hat{T}}} y^s \frac{dy dx}{y}.$$

Proof. By direct computation

$$\int_{\mathbb{R}} e^{\frac{-4\pi v x_2^2}{y^2}} dx_2 = \frac{y}{2v^{1/2}}.$$

Therefore

$$\mathrm{CT}_{\sigma=0} \lim_{T\to\infty} \int_1^T \int_{\mathbb{R}} e^{\frac{-4\pi v x_2^2}{y^2}} dx_2 v^{-\sigma-1} dv = \mathrm{CT}_{\sigma=0} \lim_{T\to\infty} \int_1^T \frac{y}{2} v^{-\sigma-1-1/2} dv. \tag{1.-31}$$

Let us suppose that $\mathrm{Re}(\sigma) > -1/2$, then

$$\lim_{T\to\infty}\int_1^T \frac{y}{2}v^{-\sigma-1-1/2} = \frac{2}{2\sigma+1}.$$

By meromorphic continuation

$$\mathrm{CT}_{\sigma=0}\lim_{T\to\infty}\int_1^T \frac{y}{2}v^{-\sigma-1-1/2}dv = y.$$

Plugging the equality (1.7.3) into the integral of the statement the result follows. $\square$

Lemma 1.7.35. *Let A be the constant defined in 1.7.1. For $s \in \mathbb{C}$ such that* $\mathrm{Re}(s) > 1$ *the integral* (1.7.3) *satisfies the following equality:*

$$\begin{aligned}\int_{\Gamma^\infty\backslash\mathcal{H}^{\hat{T}}}&\left(\mathrm{CT}_{\sigma=0}\lim_{T\to\infty}\int_1^T\sum_{w_1\in\mathbb{Z}\backslash\{0\}}\int_{\mathbb{R}} e^{\frac{-4\pi v x_2^2}{y^2}+2\pi i x_2 w_1}dx_2 v^{-\sigma-1}dv\right)y^s d\mu(z)\\ &= A\int_{\Gamma^\infty\backslash\mathcal{H}^{\hat{T}}} y^s\frac{dxdy}{y^2} - 8\mathrm{erf}\left(\sqrt{\frac{\pi}{2}}\right)\int_{\Gamma^\infty\backslash\mathcal{H}^{\hat{T}}} y^s\log(y)\frac{dxdy}{y^2}.\end{aligned}$$

Proof. By direct computation

$$\int_{\mathbb{R}} e^{\frac{-4\pi v x_2^2}{y^2}+2\pi i x_2 w_1}dx_2 = \frac{y}{2\sqrt{v}}e^{\frac{-\pi w_1^2 y^2}{4v}}.$$

Then we obtain

$$\begin{aligned}\mathrm{CT}_{\sigma=0}&\lim_{T\to\infty}\int_1^T\sum_{w_1\in\mathbb{Z}\backslash\{0\}}\int_{\mathbb{R}} e^{\frac{-4\pi v x_2^2}{y^2}+2\pi i x_2 w_1}dx_2 v^{-\sigma-1}dv\\ &= \frac{y}{2}\left(\mathrm{CT}_{\sigma=0}\lim_{T\to\infty}\int_1^T\sum_{w_1\in\mathbb{Z}\backslash\{0\}} e^{\frac{-\pi w_1^2 y^2}{4v}}v^{-\sigma-3/2}dv\right).\end{aligned}$$

We make a change of variables of the form $2vw_1^{-2} = v$, obtaining

$$\begin{aligned}y\mathrm{CT}_{\sigma=0}&\lim_{T\to\infty}\int_1^T\sum_{w_1\in\mathbb{Z}\backslash\{0\}} e^{\frac{-\pi w_1^2 y^2}{4v}}v^{-\sigma-3/2}dv\\ &= \mathrm{CT}_{\sigma=0}2^{\sigma+3/2}\left(\sum_{w_1\in\mathbb{Z}\backslash\{0\}} w_1^{-2\sigma-1}\right)\left(y\lim_{T\to\infty}\int_1^T e^{\frac{-\pi y^2}{2v}}v^{-\sigma-3/2}dv\right).\end{aligned}$$

In order to obtain an explicit formula for (1.7.3) we have to write the Laurent series of each factor of (1.7.3). First we consider the integral in (1.7.3). Making a change of variables of the form $\frac{y^2}{v} = \frac{1}{v}$ it holds

$$y\lim_{T\to\infty}\int_1^T e^{\frac{-\pi y^2}{2v}}v^{-\sigma-3/2}dv = y^{-2\sigma}\lim_{T\to\infty}\int_1^T e^{\frac{-\pi}{2v}}v^{-\sigma-3/2}dv. \tag{1.-32}$$

The above function is holomorphic at $\sigma = 0$, then we have to consider the constant and first term of the Laurent expansion of (1.7.3). We rewrite (1.7.3) as follows:

$$\lim_{T\to\infty}\int_1^T e^{\frac{-\pi}{2v}}v^{-\sigma-3/2}dv = \left(\frac{\pi}{2}\right)^{-\sigma+1/2}\Gamma\left(-1/2+\sigma,\frac{\pi}{2}\right), \tag{1.-32}$$

where $\Gamma(\cdot,\cdot)$ is the incomplete gamma function. Then

$$\lim_{T\to\infty}\int_1^T e^{\frac{-\pi}{2v}}v^{-3/2}dv = \sqrt{2}\mathrm{erf}\left(\sqrt{\frac{\pi}{2}}\right). \tag{1.-32}$$

Furthermore the first term of the Laurent expansion of (1.7.3) satisfies

$$\begin{aligned}\mathrm{FT}_{\sigma=0}\lim_{T\to\infty}\int_1^T e^{\frac{-\pi y^2}{2v}}v^{-\sigma-3/2}dv &= \left(\frac{\pi}{2}\right)^{1/2}\left(\Gamma'\left(-1/2,\frac{\pi}{2}\right)-\Gamma\left(-1/2,\frac{\pi}{2}\right)\log\left(\frac{\pi}{2}\right)\right)\\ &=: \tilde{B},\end{aligned}$$

where $\Gamma(a,b)$ is the incomplete Gamma function. Using (1.7.3) we obtain

$$\mathrm{FT}_{\sigma=0}y^{-2\sigma}\lim_{T\to\infty}\int_1^T e^{\frac{-\pi}{2v}}v^{-\sigma-3/2}dv = \tilde{B}-2\log(y)\sqrt{2}\mathrm{erf}\left(\sqrt{\frac{\pi}{2}}\right). \tag{1.-33}$$

Now we consider the sum of the equation (1.7.3). We may observe that

$$\sum_{w_1\in\mathbb{Z}\setminus\{0\}} w_1^{-2\sigma-1} = \zeta(2\sigma+1)-1.$$

The following equalities are well known

$$\begin{aligned}\mathrm{CT}_{\sigma=0}\zeta(2\sigma+1) &= \Gamma'(1)\\ \mathrm{Res}_{\sigma=0}\zeta(2\sigma+1) &= \frac{1}{2}.\end{aligned}$$

Furthermore

$$2^{\sigma+3/2} = 2\sqrt{2}+\sqrt{2}\log(4)\sigma+\mathcal{O}(\sigma^2). \tag{1.-33}$$

Plugging the equalities (1.7.3), (1.7.3) and (1.7.3) into the function (1.7.3) we obtain

$$\begin{aligned}&y\left(\mathrm{CT}_{\sigma=0}\sum_{w_1\in\mathbb{Z}} w_2^{-\sigma-1}\lim_{T\to\infty}\int_1^T e^{\frac{-\pi y^2}{2v}}v^{-\sigma-3/2}dv\right) =\\ &= \sqrt{2}\left(\tilde{B}-2\log(y)\sqrt{2}\mathrm{erf}\left(\sqrt{\frac{\pi}{2}}\right)\right)+8(\Gamma'(1)-1)\mathrm{erf}\left(\sqrt{\frac{\pi}{2}}\right)+\log(4)\mathrm{erf}\left(\sqrt{\frac{\pi}{2}}\right).\end{aligned}$$

□

Lemma 1.7.36. *The integral of (1.7.3) vanishes, i.e.*

$$\int_{\Gamma^\infty\setminus\mathcal{H}^{\hat{T}}}\left(\mathrm{CT}_{\sigma=0}\lim_{T\to\infty}\int_1^T v^{-\sigma-1}dv\right)y^s d\mu(z) = 0.$$

Proof. It is straightforward that

$$\mathrm{CT}_{\sigma=0}\lim_{T\to\infty}\int_1^T v^{-\sigma-1}dv = 0.$$

□

Lemma 1.7.37. *We obtain the following equality*

$$\mathrm{CT}_{s=0}\int_{\Gamma^\infty\backslash\mathcal{H}^{\hat{T}}} y^s\log(y)\frac{dxdy}{y^2} = -\frac{\log(\hat{T})+1}{\hat{T}}+\gamma+\frac{3}{2}{\zeta^*}'(-1)+\zeta'(-1)\Big[\log(\pi)+2\gamma$$
$$-2\log(2\pi)\left(3\log(2\pi)+\log(2)\right)-2\log(\hat{T})\Big],$$

where we recall that γ is the Euler-Mascheroni constant.

Proof. Let us suppose that $s\in\mathbb{C}$ is satisfies $Re(s)\gg 0$

$$\frac{\partial}{\partial s}\int_{\Gamma^\infty\backslash\mathcal{H}^{\hat{T}}} y^s\frac{dxdy}{y^2} = \int_{\Gamma^\infty\backslash\mathcal{H}^{\hat{T}}} y^s\log(y)\frac{dxdy}{y^2}.$$

Using proposition 1.7.32 we obtain the following equalities

$$\int_{\Gamma^\infty\backslash\mathcal{H}^{\hat{T}}} y^s\log(y)\frac{dxdy}{y^2} = \frac{\partial}{\partial s}\int_{X^{mod,\hat{T}}} E(\tau,s)\frac{dxdy}{y^2}$$
$$= \frac{\partial}{\partial s}\left(\hat{T}^{s-1}/(s-1)-\frac{\zeta^*(2s-1)}{\zeta^*(2s)}\hat{T}^{-s}/s\right).$$

By direct computation we find that (1.7.3) satisfies the following equality

$$\frac{\partial}{\partial s}\left(\hat{T}^{s-1}/(s-1)-\frac{\zeta^*(2s-1)}{\zeta^*(2s)}\hat{T}^{-s}/s\right) = \Big(\frac{\hat{T}^{s-1}((s-1)\log(\hat{T})-1)}{(s-1)^2}$$
$$-\frac{\left({\zeta^*}'(2s-1)\zeta^*(2s)-\zeta^*(2s-1){\zeta^*}'(2s)\right)}{\zeta^*(2s)^2}\frac{\hat{T}^{-s}}{s}$$
$$+\frac{\zeta^*(2s-1)}{\zeta^*(2s)}\frac{T^{-s}(s\log(\hat{T})+1)}{s^2}\Big).$$

The function on the right hand side is meromorphic. Using meromorphic continuation we can remove the hypothesis on s. We proceed analyzing each factor of (1.7.3) separately. First we obtain

$$\mathrm{CT}_{s=0}\left(\frac{\hat{T}^{s-1}((s-1)\log(\hat{T})-1)}{(s-1)^2}\right) = -\frac{\log(\hat{T})+1}{\hat{T}}. \tag{1.-36}$$

Furthermore by direct computation

$$\mathrm{CT}_{s=0}\left(\frac{\left({\zeta^*}'(2s-1)\zeta^*(2s)-\zeta^*(2s-1){\zeta^*}'(2s)\right)}{\zeta^*(2s)^2}\frac{\hat{T}^{-s}}{s}\right) = \frac{{\zeta^*}'(-1)}{\zeta(0)}+\zeta'(-1)\Big(-\frac{\log(\pi)}{2\zeta(0)}+2\gamma$$
$$+\frac{1}{\zeta(0)}\left(-\frac{2\zeta'(0)}{\zeta(0)}+\frac{\log(\pi)}{2}\right)\frac{\zeta'(0)}{\zeta(0)^2}$$
$$-2\log(\hat{T})\Big)+2{\zeta^*}'(-1)$$

By direct computation

$$\mathrm{CT}_{s=0}\left(\frac{\zeta^*(2s-1)}{\zeta^*(2s)}\frac{T^{-s}(s\log(\hat{T})+1)}{s^2}\right) = \gamma, \tag{1.-36}$$

where γ is the Euler-Mascheroni constant. We conclude by plugging equations (1.7.3), (1.7.3) and (1.7.3) into (1.7.3). □

Lemma 1.7.38. *The following equality holds:*

$$\mathrm{CT}_{s=0}\int_{\Gamma^\infty\backslash\mathcal{H}^{\hat{T}}} y^s\frac{dxdy}{y^2}=\frac{\pi}{3}-\hat{T}^{-1}.$$

Proof. Let us suppose that $s\in\mathbb{C}$ satisfies that $\mathrm{Re}(s)\gg 0$. Using the definition 1.7.29

$$\int_{\Gamma^\infty\backslash\mathcal{H}^{\hat{T}}} y^s\frac{dxdy}{y^2}=\int_{X^{mod,\hat{T}}} E(\tau,s)\frac{dxdy}{y^2}.$$

Proceeding as in the proof of the previous lemma we use proposition 1.7.32 to obtain

$$\int_{X^{mod,\hat{T}}} E(\tau,s)\frac{dxdy}{y^2}=\left(\hat{T}^{s-1}/(s-1)-\frac{\zeta^*(2s-1)}{\zeta^*(2s)}\hat{T}^{-s}/s\right). \qquad (1.\text{-}36)$$

The right hand side is a meromorphic function on the variable s. Then by meromorphic continuation we remove the hypothesis on s. On the one hand

$$\mathrm{CT}_{s=0}\frac{\zeta^*(2s-1)}{\zeta^*(2s)}\hat{T}^{-s}/(-s)=\frac{\pi}{3}. \qquad (1.\text{-}36)$$

On the other hand

$$\mathrm{CT}_{s=0}\hat{T}^{s-1}/(s-1)=-\hat{T}^{-1}. \qquad (1.\text{-}36)$$

We conclude by plugging the equalities (1.7.3), (1.7.3) into (1.7.3). □

Lemma 1.7.39. *We obtain*

$$\mathrm{CT}_{s=0}\int_{\Gamma^\infty\backslash\mathcal{H}^{\hat{T}}} y^s\frac{dydx}{y}=\frac{3}{\pi}\left(\gamma+\log\left(\frac{\pi}{4}\right)+\frac{{\zeta^*}'(2)}{\zeta^*(2)}\right)\frac{1}{\hat{T}}-\frac{1}{2\zeta^*(2)}\frac{\log(\hat{T})+1}{\hat{T}}+\log(\hat{T}).$$

Proof. The following equality follows directly

$$\mathrm{CT}_{s=0}\int_{\Gamma^\infty\backslash\mathcal{H}^{\hat{T}}} y^s\frac{dxdy}{y}=\mathrm{CT}_{s=1}\int_{\Gamma^\infty\backslash\mathcal{H}^{\hat{T}}} y^s\frac{dxdy}{y^2}. \qquad (1.\text{-}36)$$

Let $s\in\mathbb{C}$ satisfying that $\mathrm{Re}(s)\gg 0$. Using the same argument of the proof of the previous two lemmas we use proposition 1.7.32 to rewrite the above equality as follows:

$$\int_{X^{mod,\hat{T}}} E(\tau,s)\frac{dxdy}{y^2}=\left(\hat{T}^{s-1}/(s-1)-\frac{\zeta^*(2s-1)}{\zeta^*(2s)}\hat{T}^{-s}/s\right). \qquad (1.\text{-}36)$$

The right hand side of the equality is meromorphic, applying meromorphic continuation we remove the hypothesis on s. On the one hand we obtain

$$\mathrm{CT}_{s=1}\frac{\zeta^*(2s-1)}{\zeta^*(2s)}\hat{T}^{-s}/-s=\mathrm{CT}_{s=1}\left(\frac{\zeta^*(2s-1)}{\zeta^*(2s)}\right)\frac{1}{\hat{T}}-\mathrm{Res}_{s=1}\left(\frac{\zeta^*(2s-1)}{\zeta^*(2s)}\right)\frac{\log(\hat{T})+1}{\hat{T}}. \qquad (1.\text{-}36)$$

By direct computation

$$\mathrm{CT}_{s=1}\left(\frac{\zeta^*(2s-1)}{\zeta^*(2s)}\right)=\frac{\pi^{-1/2}}{\zeta^*(2)}\left(\gamma\Gamma(1/2)+\frac{1}{2}\left(\log(\pi)\Gamma(1/2)+\Gamma'(1/2)\right)+\frac{{\zeta^*}'(2)\Gamma(1/2)}{2\zeta^*(2)}\right),$$

$$\mathrm{Res}_{s=1}\left(\frac{\zeta^*(2s-1)}{\zeta^*(2s)}\right)=\frac{\pi^{-1/2}\Gamma(1/2)}{2\zeta^*(2)}.$$

On the other hand

$$\mathrm{CT}_{s=1}\hat{T}^{s-1}/(s-1)=\log(\hat{T}). \qquad (1.\text{-}36)$$

Using the equalities (1.7.3), (1.7.3) and (1.7.3) in (1.7.3), the result follows. ☐

Corollary 1.7.40. *It holds that*

$$\begin{aligned}\int_{X^{mod,\hat{T}}} &\left(\mathrm{CT}_{\sigma=0}\lim_{T\to\infty}\int_{\mathcal{F}_2^T} f_{\mu_0}(\tau)\vartheta(v,z,\mu_0)_0 v^{-\sigma}d\mu(\tau)\right)d\mu(z) = c_{\mu_0}(0)\Bigg(A\left(\frac{\pi}{3}-\hat{T}^{-1}\right)\\ &- 8\mathrm{erf}\left(\sqrt{\frac{\pi}{2}}\right)\Bigg[-\frac{\log(\hat{T})+1}{\hat{T}}+\gamma+\frac{3}{2}\zeta^{*'}(-1)+\zeta'(-1)\Bigg(\log(\pi)+2\gamma\\ &-2\log(2\pi)\left(3\log(2\pi)+\log(2)\right)-2\log(\hat{T})\Bigg)\Bigg]\\ &+\left[\frac{3}{\pi}\left(\gamma+\log\left(\frac{\pi}{4}\right)+\frac{\zeta^{*'}(2)}{\zeta^*(2)}\right)\frac{1}{\hat{T}}-\frac{1}{2\zeta^*(2)}\frac{\log(\hat{T})+1}{\hat{T}}+\log(\hat{T})\right]\Bigg).\end{aligned}$$

Proof. The proof follows applying lemmas 1.7.37, 1.7.38 and 1.7.39 to proposition 1.7.33. ☐

Main result

The goal of this paper is to compute

$$\begin{aligned}&\int_{X^{mod,\hat{T}}}\log||\Psi(f)(z)||_{Pet}d\mu(z)\\ &\quad=-\frac{1}{4}\int_{X^{mod,\hat{T}}}\Phi(f)(z)d\mu(z)+\frac{c_{\mu_0}(0)\mathrm{vol}(X^{mod,\hat{T}})}{2}\left(\Gamma'(1)/2+\log\sqrt{2\pi}\right).\end{aligned}$$

Theorem 1.7.41 (Main result)**.** *The following equality holds:*

$$\begin{aligned}\int_{X^{mod,\hat{T}}}\log||\Psi(f)(z)||_{Pet}d\mu(z) =& -\frac{\mathrm{vol}(X^{mod})}{2}\sum_{j=0}^{1}\sum_{m\geq 0}c_{\mu_j}(-m)\kappa_{\mu_j}(m)\\ &-c_{\mu_0}(0)\log(\hat{T})\left(4\zeta'(-1)\mathrm{erf}\left(\sqrt{\frac{\pi}{2}}\right)+\frac{1}{4}\right)\\ &+O(e^{-\hat{T}})+c_{\mu_0}(0)\left(\frac{\log(\hat{T})+1}{\hat{T}}\right)\left(-2\mathrm{erf}\left(\sqrt{\frac{\pi}{2}}\right)+\frac{1}{8\zeta^*(2)}\right)\\ &-\frac{c_{\mu_0}(0)}{\hat{T}}\Bigg(\frac{1}{4\hat{T}}\left[\frac{\pi^{-1/2}}{\zeta^*(2)}\left(\gamma\Gamma(1/2)+\frac{1}{2}\left(\log(\pi)\Gamma(1/2)+\Gamma'(1/2)\right)+\frac{\zeta^{*'}(2)\Gamma(1/2)}{2\zeta^*(2)}\right)-A\right]\\ &+\frac{3}{\pi}\left(\Gamma'(1)/2+\log\sqrt{2\pi}\right)\Bigg),\end{aligned}$$

with

$$\kappa_{\mu_j}(m)=\begin{cases} b(m,\varphi_{\mu_j}) & if\ m\neq 0\\ 0 & if\ m=0\ and,\ j=1\\ C+A_0(i,-1/2,\mu(\varphi_{\mu_0}))_0-\left(\Gamma'(1)/2+1+\log\sqrt{2\pi}\right) & if\ m=0\ and,\ j=0\end{cases}$$

where

$$\begin{aligned}C=&A/2-\frac{12}{\pi}\mathrm{erf}\left(\sqrt{\frac{\pi}{2}}\right)\left[\gamma+\frac{3}{2}\zeta^{*'}(-1)+\zeta'(-1)(\log(\pi)+2\gamma-2\log(2\pi)(3\log(2\pi)+\log(2))\right]\\ &+\frac{6}{\pi}\left(-\tanh^{-1}\left(\frac{\sqrt{7}}{4}\right)+\frac{\sqrt{7}}{4}+\frac{\log(3/4)}{2}\right).\end{aligned}$$

Proof. Using the equality (1.7.3), lemma 1.7.23 and corollaries 1.7.40 and 1.7.28 the result follows. □

1.7.4 Auxiliary computations

Divergence

Let $f(\tau) = \sum_{j=0}^{1} f_{\mu_j}(\tau) \in M^!_{-1/2,L}$. The main goal of this subsection is to understand the integral

$$\sum_{j=0}^{1} \int_{\mathcal{F}^T} f_{\mu_j}(\tau) \left(\int_{X^{mod,\hat{T}}} \mathbf{Div}(\tau, z, \mu_j) d\mu(z) \right) d\mu(\tau).$$

First of all we may obtain an explicit expression for $\mathbf{Div}(\tau, z, \mu_j)$. It requires the understanding of the classical embedding

$$\mathcal{H} \hookrightarrow SO(2,1). \tag{1.-37}$$

The map (1.7.4) is given by the composition of the following two maps:

$$\begin{aligned} \mathcal{H} \to & SL_2(\mathbb{R}) \\ z = x + iy \mapsto & \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2} \end{pmatrix}. \\ SL_2(\mathbb{R}) \xrightarrow{\simeq} & SO(2,1) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto & \begin{pmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & 2cd & d^2 \end{pmatrix}. \end{aligned}$$

Then

$$\begin{aligned} \mathcal{H} \hookrightarrow & SO(2,1), \\ z = x + iy \mapsto h_z := & \begin{pmatrix} 1 & & x^2 \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} y & & \\ & 1 & \\ & & y^{-1} \end{pmatrix}. \end{aligned}$$

Throughout this subsection we will use the previous injection without referring to it. Set

$$X_0^{mod,\hat{T}} = \{z = x + iy \in X^{mod,\hat{T}},\ s.t.\ y \leq 1\},$$

$$X_1^{mod,\hat{T}} = \{z = x + iy \in X^{mod,\hat{T}},\ s.t.\ y > 1\},$$

so that

$$X^{mod,\hat{T}} = X_0^{mod,\hat{T}} \bigsqcup X_1^{mod,\hat{T}}. \tag{1.-38}$$

Lemma 1.7.42. *The divergent part satisfies*

$$\mathbf{Div}(\tau, z, \mu_j) = y\mathbf{Div}(\tau, z_0, \mu_j).$$

Proof. Using the invariance property of the Gaussian the following equality holds:

$$\mathrm{Div}(g_\tau, h_1, \varphi^\infty_{z,\mu_j}) = \sum_{x_0 \in \mathbb{Z} + \frac{j}{2}} \int_{\mathbb{R}} \omega(g_\tau, id) \varphi^\infty_{z,\mu_j} \begin{pmatrix} x_{\mathbb{R}} \\ x_0 \\ 0 \end{pmatrix} dx_{\mathbb{R}} = \sum_{x_0 \in \mathbb{Z} + \frac{j}{2}} \int_{\mathbb{R}} \omega(g_\tau, h_z) \varphi^\infty_{z_0,\mu_j} \begin{pmatrix} x_{\mathbb{R}} \\ x_0 \\ 0 \end{pmatrix} dx_{\mathbb{R}},$$

where h_z is the image of z under the map (1.7.4). Applying the Weil representation

$$\sum_{x_0 \in \mathbb{Z} + \frac{j}{2}} \int_{\mathbb{R}} \omega(g_\tau, h_z) \varphi^\infty_{z_0,\mu_j} \begin{pmatrix} x_{\mathbb{R}} \\ x_0 \\ 0 \end{pmatrix} dx_{\mathbb{R}} = \sum_{x_0 \in \mathbb{Z} + \frac{j}{2}} \int_{\mathbb{R}} \omega(g_\tau) \varphi^\infty_{z_0,\mu_j} \begin{pmatrix} y^{-1} x_{\mathbb{R}} \\ x_0 \\ 0 \end{pmatrix} dx_{\mathbb{R}}.$$

By a change of variable of the form $y^{-1}x_{\mathbb{R}} = x_{\mathbb{R}}$ we obtain

$$\sum_{x_0\in\mathbb{Z}+\frac{j}{2}} \int_{\mathbb{R}} \omega(g_\tau)\varphi^{\infty}_{z_0,\mu_j}\begin{pmatrix} y^{-1}x_{\mathbb{R}} \\ x_0 \\ 0 \end{pmatrix} dx_{\mathbb{R}} = y \sum_{x_0\in\mathbb{Z}+\frac{j}{2}} \int_{\mathbb{R}} \omega(g_\tau)\varphi^{\infty}_{z_0,\mu_j}\begin{pmatrix} x_{\mathbb{R}} \\ x_0 \\ 0 \end{pmatrix} dx_{\mathbb{R}},$$

which implies the result. $\square$

Lemma 1.7.43. *The following equality holds:*

$$\int_{X_1^{mod,\hat{T}}} \mathbf{Div}(\tau, z, \mu_j)d\mu(z) = \log(\hat{T})\mathbf{Div}(\tau, z_0, \mu_j).$$

Proof. Using lemma 1.7.42

$$\begin{aligned}\int_{X_1^{mod,\hat{T}}} \mathbf{Div}(\tau, z, \mu_j)d\mu(z) &= \int_1^{\hat{T}}\int_{-1/2}^{1/2} y\mathbf{Div}(\tau, z_0, \mu_j)\frac{dxdy}{y^2} \\ &= \mathbf{Div}(\tau, z_0, \mu_j)\int_1^{\hat{T}}\frac{1}{y}dy = \log(\hat{T})\mathbf{Div}(\tau, z_0, \mu_j).\end{aligned}$$

$\square$

Lemma 1.7.44. *We obtain*

$$\int_{X_0^{mod,\hat{T}}} \mathbf{Div}(\tau, z, \mu_j)d\mu(z) = 2\mathbf{Div}(\tau, z_0, \mu_j)\left(-\tanh^{-1}\left(\frac{\sqrt{7}}{4}\right) + \frac{\sqrt{7}}{4} + \frac{\log(3/4)}{2}\right).$$

Proof. To simplify the computation we factor $X_0^{mod,\hat{T}}$ into the following two subsets:

$$A = \{z = x + iy \in X_0^{mod,\hat{T}},\ s.t.\ x \le 0\},$$

and

$$B = \{z = x + iy \in X_0^{mod,\hat{T}},\ s.t.\ x > 0\}.$$

The truncated modular curve satisfies

$$X_0^{mod,\hat{T}} = A\bigsqcup B.$$

The proof of lemma 1.7.42 shows that $\mathbf{Div}(\tau, z, \mu_j)$ does not depend on the variable x. Then

$$\begin{aligned}\int_{X_0^{mod,\hat{T}}} \mathbf{Div}(\tau, z, \mu_j)d\mu(z) &= \int_A \mathbf{Div}(\tau, z, \mu_j)d\mu(z) + \int_B \mathbf{Div}(\tau, z, \mu_j)d\mu(z) \\ &= 2\int_A \mathbf{Div}(\tau, z, \mu_j)d\mu(z).\end{aligned}$$

We apply lemma 1.7.42 to the integral (1.7.4), then

$$\begin{aligned}\int_A \mathbf{Div}(\tau, z, \mu_j)d\mu(z) &= \int_{3/4}^1\int_{-1/2}^{-\sqrt{1-y^2}} \frac{1}{y}\mathbf{Div}(\tau, z_0, \mu_j)dxdy \\ &= \mathbf{Div}(\tau, z_0, \mu_j)\left(\int_{3/4}^1 \frac{-\sqrt{1-y^2}}{y}dy + \frac{1}{2}\int_{3/4}^1\frac{1}{y}dy\right) = \\ &= \mathbf{Div}(\tau, z_0, \mu_j)\left(-\tanh^{-1}\left(\frac{\sqrt{7}}{4}\right) + \frac{\sqrt{7}}{4} + \frac{\log(3/4)}{2}\right).\end{aligned}$$

$\square$

Proposition 1.7.45. *The following equality holds:*

$$\mathbf{Div}(\tau, z_0, \mu_0) = v^{1/2}\theta_{\mu_0}^{Jac}(\tau),$$

where $\theta_{\mu_0}^{Jac}(\tau) = \sum_{n\in\mathbb{Z}} e^{2\pi i n^2\tau}$ *is the Jacobi theta function. Furthermore*

$$\mathbf{Div}(\tau, z_0, \mu_1) = v^{1/2} \sum_{n\in\frac{1}{2}+\mathbb{Z}} e^{2\pi i n^2\tau} =: v^{1/2}\theta_{\mu_1}^{Jac}(\tau)$$

Proof. We prove the first statement, the second one follows similarly. Applying the Weil representation one obtain

$$\begin{aligned}\mathbf{Div}(\tau, z_0, \mu_0) &= v^{1/4}\sum_{x_0\in\mathbb{Z}}\int_{\mathbb{R}} \omega(g_\tau)\varphi_{z_0,\mathbb{R}}^{\infty}\begin{pmatrix} x_{\mathbb{R}} \\ x_0 \\ 0\end{pmatrix} dx_{\mathbb{R}} \\ &= v\sum_{x_0\in\mathbb{Z}}\int_{\mathbb{R}} \psi_\infty\left(2uq\left(x_{\mathbb{R}}, x_0, 0\right)\right)\varphi_{z_0,\mathbb{R}}^{\infty}\begin{pmatrix} v^{1/2}x_{\mathbb{R}} \\ v^{1/2}x_0 \\ 0\end{pmatrix} dx_{\mathbb{R}}.\end{aligned}$$

We recall that in the present case $z_0 = i$. Then

$$\psi_\infty\left(2uq\left(x_{\mathbb{R}}, x_0, 0\right)\right)\varphi_{z_0,\mathbb{R}}^{\infty}\begin{pmatrix} v^{-1/2}x_{\mathbb{R}} \\ v^{1/2}x_0 \\ 0\end{pmatrix} = e^{-\pi\left(vx_{\mathbb{R}}^2+2vx_0^2\right)+2\pi i u x_0^2}.$$

By direct computation

$$v\int_{\mathbb{R}} e^{-\pi\left(vx_{\mathbb{R}}^2+2vx_0^2\right)+2\pi i u x_0^2} dx_{\mathbb{R}} = v^{1/2}e^{2\pi i x_0^2(u+iv)}. \tag{1.-40}$$

Applying equality (1.7.4) in (1.7.4) we obtain

$$\mathbf{Div}(\tau, z_0, \mu_0) = v^{1/2}\sum_{x_0\in\mathbb{Z}} e^{2\pi i x_0^2\tau} = v^{1/2}\theta_{\mu_0}^{Jac}(\tau).$$

□

Lemma 1.7.46. *The function* $v^{-1/4}\theta_{\mu_1}^{Jac}(\tau) = v^{-1/4}\sum_{n\in\mu_1+\mathbb{Z}} e^{2\pi i n^2\tau}$ *is a non holomorphic modular form of weight* $1/2$.

Proof. According to [KY10, prop. 6.3, p. 2301]

$$\sum_{n\in\mu_1+\mathbb{Z}} e^{2\pi i n^2\tau} = E(\tau, -1/2, 1/2, \mu(\varphi_{\mu_1}))).$$

Using [Kud03, lem. 1.1, p. 11] the result holds. □

Proposition 1.7.47. *We obtain*

$$\begin{aligned}&\sum_{j=0}^{1}\int_{\mathcal{F}^T} f_{\mu_j}(\tau)\left(\int_{X^{mod,\hat{T}}} \mathbf{Div}(\tau, z, \mu_j)d\mu(z)\right)d\mu(\tau) \\ &\quad = -\frac{1}{\sqrt{T}}\sum_{j=0}^{1} 2c_{\mu_j}(0)\left(\log(\hat{T}) + 2\left(-\tanh^{-1}\left(\frac{\sqrt{7}}{4}\right) + \frac{\sqrt{7}}{4} + \frac{\log(3/4)}{2}\right)\right).\end{aligned}$$

Proof. We factor the integral of the statement according to (1.7.4)

$$\int_{\mathcal{F}^T} f_{\mu_j}(\tau)\left(\int_{X^{mod,\hat{T}}} \mathbf{Div}(\tau,z,\mu_j)d\mu(z)\right)d\mu(\tau) = \int_{\mathcal{F}^T} f_{\mu_j}(\tau)\left(\int_{X_0^{mod,\hat{T}}} \mathbf{Div}(\tau,z,\mu_j)d\mu(z)\right)d\mu(\tau)$$
$$+\int_{\mathcal{F}^T} f_{\mu_j}(\tau)\left(\int_{X_1^{mod,\hat{T}}} \mathbf{Div}(\tau,z,\mu_j)d\mu(z)\right)d\mu(\tau).$$

By lemmas 1.7.43 and 1.7.44, the function (1.7.4) is equal to

$$\left(\log(\hat{T})+2\left(-\tanh^{-1}\left(\frac{\sqrt{7}}{4}\right)+\frac{\sqrt{7}}{4}+\frac{\log(3/4)}{2}\right)\right)\int_{\mathcal{F}^T} f_{\mu_j}(\tau)\mathbf{Div}(\tau,z_0,\mu_j)d\mu(\tau). \tag{1.-41}$$

Propositions 1.7.49 and 1.7.45 imply

$$\sum_{j=0}^{1}\int_{\mathcal{F}^T} f_{\mu_j}(\tau)\mathbf{Div}(\tau,z,\mu_j)d\mu(\tau) = -\sum_{j=0}^{1}\frac{2c_{\mu_j}(0)}{\sqrt{T}}.$$

Plugging the equality (1.7.4) into the function (1.7.4) we obtain the result. □

Lemma 1.7.48. *The divergent part satisfies*

$$\int_{\mathcal{F}_2^T} f_{\mu_0}(\tau)\left(\int_{X^{mod,\hat{T}}} \mathbf{Div}(v,z,\mu_0)_0 d\mu(z)\right)d\mu(\tau)$$
$$= -2c_{\mu_0}(0)\left(1-\sqrt{T}^{-1}\right)\left(\log(\hat{T})+2\left(-\tanh^{-1}\left(\frac{\sqrt{7}}{4}\right)+\frac{\sqrt{7}}{4}+\frac{\log(3/4)}{2}\right)\right).$$

Proof. We proceed as in the proof of proposition 1.7.47. First of all we observe

$$\int_{X^{mod,\hat{T}}} \mathbf{Div}(v,z,\mu_0)_0 d\mu(z) = \int_{X^{mod,\hat{T}}}\left(\int_{-1/2}^{1/2} \mathbf{Div}(u+iv,z,\mu_0)du\right)d\mu(z). \tag{1.-41}$$

By means of Fubini's theorem it holds that (1.7.4) is equal to

$$\int_{-1/2}^{1/2}\left(\int_{X^{mod,\hat{T}}} \mathbf{Div}(u+iv,z,\mu_0)d\mu(z)\right)du, \tag{1.-41}$$

which is the Fourier constant term of the function

$$\tau\mapsto\int_{X^{mod,\hat{T}}} \mathbf{Div}(\tau,z,\mu_0)d\mu(z).$$

We factor the integral over $X^{mod,\hat{T}}$ according to (1.7.3). Applying lemmas 1.7.43 and 1.7.44 to the function (1.7.4) we obtain

$$\int_{-1/2}^{1/2}\left(\int_{X^{mod,\hat{T}}} \mathbf{Div}(u+iv,z,\mu_0)d\mu(z)\right)du$$
$$=\left(\log(\hat{T})+2\left(-\tanh^{-1}\left(\frac{\sqrt{7}}{4}\right)+\frac{\sqrt{7}}{4}+\frac{\log(3/4)}{2}\right)\right)\mathbf{Div}(v,z_0,\mu_0)_0.$$

Using proposition 1.7.45 in (1.7.4)

$$\int_{\mathcal{F}_2^T} f_{\mu_0}(\tau)\mathbf{Div}(v,z,\mu_0)_0 d\mu(\tau) = \int_{\mathcal{F}_2^T} v^{1/2} f_{\mu_0}(\tau)\theta_{\mu_0}^{Jac}(v)_0 d\mu(\tau),$$

where $\theta_{\mu_0}^{Jac}(v)_0 := \int_{-1/2}^{1/2} \theta_{\mu_0}^{Jac}(u+iv)du = 1$. Using corollary 1.7.50

$$\int_{\mathcal{F}_2^T} v^{1/2} f_{\mu_0}(\tau)\theta_{\mu_0}^{Jac}(v)_0 d\mu(\tau) = 2c_{\mu_0}(0) - \frac{2c_{\mu_0}(0)}{\sqrt{T}}$$

Lastly, we plugg the equality (1.7.4) into the function (1.7.4) to obtain the statement. □

Two integrals

This subsection is devoted to computing the integrals

$$\sum_{j=0}^{1} \int_{\mathcal{F}^T} v^{1/2} f_{\mu_j}(\tau)\theta_{\mu_j}^{Jac}(\tau)d\mu(\tau), \tag{1.-42}$$

$$\sum_{j=0}^{1} \int_{\mathcal{F}^T} f_{\mu_j}(\tau)A_0(\tau,-1/2,\mu(\varphi_{\mu_j}))d\mu(\tau), \tag{1.-42}$$

and some variants. This subsection is based on the techniques developed in [Kud03, sec. 2, p. 16].

Proposition 1.7.49. *The integral* (1.7.4) *satisfies the following equality:*

$$\sum_{j=0}^{1} \int_{\mathcal{F}^T} v^{1/2} f_{\mu_j}(\tau)\theta_{\mu_j}^{Jac}(\tau)d\mu(\tau) = \sum_{j=0}^{1} \frac{-2c_{\mu_j}(0)}{\sqrt{T}}$$

Proof. We have

$$\int_{\mathcal{F}^T} v^{1/2} f_{\mu_j}(\tau)\theta_{\mu_j}^{Jac}(\tau)d\mu(\tau) = \int_{\mathcal{F}^T} v^{-3/2} f_{\mu_j}(\tau)\theta_{\mu_j}^{Jac}(\tau)dudv. \tag{1.-42}$$

The Jacobi theta function $\theta_{\mu_j}^{Jac}(\tau)$ and $f_{\mu_j}(\tau)$ are holomorphic functions at $\tau \in \mathcal{H}$, then

$$\frac{\partial}{\partial \overline{\tau}} \theta_{\mu_j}^{Jac}(\tau) f_{\mu_j}(\tau) = 0.$$

The previous equality allow us to obtain a preimage of

$$v^{-3/2} f_{\mu_j}(\tau)\theta_{\mu_j}^{Jac}(\tau),$$

under the operator $\frac{\partial}{\partial \overline{\tau}}$, in fact by direct computation

$$\frac{2}{i}\frac{\partial}{\partial \overline{\tau}} \left\{ \frac{-2}{\sqrt{y}} f_{\mu_j}(\tau)\theta_{\mu_j}^{Jac}(\tau) \right\} = v^{-3/2} f_{\mu_j}(\tau)\theta_{\mu_j}^{Jac}(\tau). \tag{1.-42}$$

We apply Stokes theorem to (1.7.4). By (1.7.4) we obtain

$$\int_{\mathcal{F}^T} v^{-3/2} f_{\mu_j}(\tau)\theta_{\mu_j}^{Jac}(\tau)dudv = \frac{2}{i}\frac{1}{2i}\int_{\partial\mathcal{F}^T} \frac{-2}{\sqrt{v}} f_{\mu_j}(\tau)\theta_{\mu_j}^{Jac}(\tau)d\tau = \int_{\partial\mathcal{F}^T} \frac{2}{\sqrt{v}} f_{\mu_j}(\tau)\theta_{\mu_j}^{Jac}(\tau)d\tau.$$

Given $\tau \in \mathcal{H}$ such that $|\tau| = 1$, the function $\mathrm{Im}(\tau)$ is invariant under the transformation

$$\tau \mapsto -1/\tau.$$

Moreover, the function $\mathrm{Im}(\tau)$ is invariant under the transformation

$$\tau \to \tau + 1.$$

The same properties are satisfied by $\frac{1}{v^{1/4}}$. According to lemma 1.7.46 and [Kud03, (1.42), p. 13] the function

$$v^{-1/4} \sum_{j=0}^{1} f_{\mu_j}(\tau)\theta_{\mu_j}^{Jac}(\tau),$$

is invariant under $\mathrm{SL}_2(\mathbb{Z})$ and then it is invariant under the aforementioned transformations. Then

$$\frac{2}{\sqrt{v}} f_{\mu_j}(\tau)\theta_{\mu_j}^{Jac}(\tau)d\tau,$$

is invariant under $\tau \mapsto -1/\tau$ and $\tau \mapsto \tau + 1$. The above discussion implies the following equality:

$$\sum_{j=0}^{1} \int_{\partial\mathcal{F}^T} \frac{2}{\sqrt{v}} f_{\mu_j}(\tau)\theta_{\mu_j}^{Jac}(\tau)d\tau = \sum_{j=0}^{1} \left(\int_{1/2}^{-1/2} \frac{2}{\sqrt{v}} f_{\mu_j}(\tau)\theta_{\mu_j}^{Jac}(\tau)d\tau \right)_{v=T}.$$

Using the definition of the constant term of the Fourier expansion we can go further, concluding the proof:

$$\sum_{j=0}^{1} \left(\int_{1/2}^{-1/2} \frac{2}{\sqrt{v}} f_{\mu_j}(\tau)\theta_{\mu_j}^{Jac}(\tau)d\tau \right)_{v=T} = \sum_{j=0}^{1} \frac{-2c_{\mu_j}(0)}{\sqrt{T}}.$$

□

Corollary 1.7.50. *Let $f(\tau) = \sum_{j=0}^{1} f_{\mu_j}(\tau)\varphi_{\mu_j}$ be a weakly holomorphic modular form, then*

$$\sum_{j=0}^{1} \int_{\mathcal{F}_2^T} v^{1/2} f_{\mu_j}(\tau)d\mu(\tau) = \sum_{j=0}^{1} 2c_{\mu_j}(0) - \frac{2c_{\mu_j}(0)}{\sqrt{T}}.$$

Proof. The functions $f_{\mu_j}(\tau)$ are holomorphic and $\tau \mapsto \tau + 1$ invariant. Using the Stokes argument of the proof of proposition 1.7.49 we obtain the statement. □

Lemma 1.7.51. *We obtain*

$$\sum_{j=0}^{1} \int_{\mathcal{F}^T} f_{\mu_j}(\tau)A_{-1}(\tau, -1/2, \mu(\varphi_{\mu_j}))d\mu(\tau) = 2\sum_{j=0}^{1} \sum_{m\in\mathbb{Z}} c_{\mu_i}(-m)b(m, T, \mu(\varphi_{\mu_j})).$$

Proof. The proof follows as in [Kud03, p. 21]. Nevertheless for the sake of completeness we will give an overview of the proof. Proposition 1.6.29 allows us to apply Stokes theorem as in the proof of proposition 1.7.49, obtaining

$$\sum_{j=0}^{1} \int_{\mathcal{F}_2^T} v^{1/2} f_{\mu_j}(\tau)d\mu(\tau) = \frac{-2}{(s-1/2)} \int_{\partial\mathcal{F}^T} \sum_{j=0}^{1} f_{\mu_j}(\tau)E(\tau, s, 3/2, \mu(\varphi_{\mu_j}))d\tau.$$

By [Kud03, (1.42), p. 13], the function $\sum_{j=0}^{1} f_{\mu_j}(\tau)E(\tau, s, 3/2, \mu(\varphi_{\mu_j}))$ is $\mathrm{SL}_2(\mathbb{Z})-$invariant. In particular it is invariant under the transformations

$$\tau \mapsto \tau + 1,\ \tau \mapsto -1/\tau.$$

Hence

$$\sum_{j=0}^{1}\frac{-2}{(s-1/2)}\int_{\partial\mathcal{F}^T} f_{\mu_i}(\tau)E(\tau,s,3/2,\mu(\varphi_{\mu_j}))d\tau = \sum_{i=0}^{1}\frac{-2}{(s-1/2)}\int_{1/2+iT}^{-1/2+iT} f_{\mu_j}(\tau)E(\tau,s,3/2,\mu(\varphi_{\mu_j}))d\tau$$
$$= \sum_{j=0}^{1}\frac{2}{(s-1/2)}\left(f_{\mu_j}(\tau)E(\tau,s,3/2,\mu(\varphi_{\mu_j}))\right)_{0,v=T}.$$

□

Corollary 1.7.52. *The following equality holds:*

$$\int_{\mathcal{F}_2^T} f_{\mu_0}(\tau)A_0(v,-1/2,\mu(\varphi_{\mu_0}))_0 d\mu(\tau) = c_{\mu_0}(0)\Big[\log(T) - \frac{6(-12\zeta'(2)+2\gamma\pi^2+\pi^2(-\gamma-\log(8)))}{\pi^3}\left(1-\frac{1}{\sqrt{T}}\right)\Big].$$

Proof. Since we can apply [KY10, prop. 2.3, p. 2282] and [KY10, prop. 5.1, p.2293] to the present setting, the proof of [KY10, thm. 6.1, (iv) p. 2300] applies for the Eisenstein series of the statement, obtaining

$$A_0(v,-1/2,\mu(\varphi_{\mu_0})) = v + v^{1/2}\sqrt{2\pi}\,\mathrm{CT}_{s=1/2}\left(\frac{2^{-s}\Gamma(s)\zeta(2s)}{\zeta(2s+1)}\right)$$
$$= v + v^{1/2}\frac{3(-12\zeta'(2)+2\gamma\pi^2+\pi^2(-\gamma-\log(8)))}{\pi^3}.$$

Using Stokes theorem as in proposition 1.7.49, we get

$$\int_{\mathcal{F}_2^T}\sum_{j=0}^{1} f_{\mu_0}(\tau)A_0(v,-1/2,\mu_0)_0 d\mu(\tau) = c_{\mu_0}(0)\log(T) - \frac{6(-12\zeta'(2)+2\gamma\pi^2+\pi^2(-\gamma-\log(8)))}{\pi^3}\left(1-\frac{1}{\sqrt{T}}\right).$$

□

Lemma 1.7.53. *Let $\widetilde{\varphi}_f \in \mathcal{S}(V(\mathbb{A}))$ be a $\prod_{p\nmid\infty}\mathrm{SL}_2(\mathbb{Z}_p)-$invariant function, then*

$$\sum_{j=0}^{1}\int_{\mathcal{F}^T} f_{\mu_j}(\tau)A_{-1}(\tau,-1/2,\mu(\widetilde{\varphi}_f))v^{\sigma}d\mu(\tau) = 0.$$

Proof. By [Kud03, (1.42), p. 13], the function $\sum_{j=0}^{1} f_{\mu_j}(\tau)A_{-1}(\tau,-1/2,\mu(\widetilde{\varphi}_f))$ is invariant under $\tau\mapsto\tau+1$ and $\tau\mapsto -1/\tau$. Hence using lemma 1.7.48 and proposition 1.7.51 we obtain

$$\sum_{j=0}^{1}\int_{\mathcal{F}^T} f_{\mu_j}(\tau)A_{-1}(\tau,-1/2,\mu(\widetilde{\varphi}_f))v^{\sigma}d\mu(\tau) = \sum_{i=0}^{1}2\mathrm{Res}_{s=1/2}\sum_{m\in\mathbb{Z}} c_{\mu_i}(-m)\tilde{b}(m,s,T,\mu(\widetilde{\varphi}_f)).$$

By [KY10, cor. 2.5, p. 2283], the functions $\tilde{b}(m,s,T,\mu(\tilde{\varphi}))$ are holomorphic at $s=1/2$ and therefore its residue vanishes for all $m\in\mathbb{Z}$ □

Corollary 1.7.54. *Let $\widetilde{\varphi}_f \in \mathcal{S}(V(\mathbb{A}))$ be a $\prod_{p\nmid\infty}\mathrm{SL}_2(\mathbb{Z}_p)-$invariant function. Then*

$$\int_{\mathcal{F}_2^T} f_{\mu_0}(\tau)A_{-1}(v,-1/2,\mu(\widetilde{\varphi}_f))_0 v^{\sigma}d\mu(\tau) = 0.$$

Proof. Let us denote by

$$E(\tau, s, -1/2, \mu(\widetilde{\varphi}_f)) = \sum_{n\in\mathbb{Q}} E(\tau, s, -1/2, \mu(\widetilde{\varphi}_f))_n,$$

the Fourier series of $E(\tau, s, -1/2, \mu(\widetilde{\varphi}_f))$ with respect to the variable τ. It is straightforward that

$$\begin{aligned}\int_{\mathcal{F}_2^T} f_{\mu_0}(\tau) E(\tau, s, -1/2, \mu(\widetilde{\varphi}_f)) v^\sigma d\mu(\tau) &= \int_{\mathcal{F}_2^T} f_{\mu_0}(\tau) E(v, s, -1/2, \mu(\widetilde{\varphi}_f))_0 v^\sigma d\mu(\tau) \\ &\quad + \int_{\mathcal{F}_2^T} f_{\mu_0}(\tau) \sum_{n\neq 0} E(\tau, s, -1/2, \mu(\widetilde{\varphi}_f))_n v^\sigma d\mu(\tau).\end{aligned}$$

Using a similar argument to the proof of lemma 1.7.53 we have

$$\begin{aligned}0 &= \int_{\mathcal{F}_2^T} f_{\mu_0}(\tau) A_{-1}(v, -1/2, \mu(\widetilde{\varphi}_f))_0 v^\sigma d\mu(\tau) \\ &\quad + \int_{\mathcal{F}_2^T} f_{\mu_0}(\tau) \sum_{n\neq 0} A_{-1}(\tau, -1/2, \mu(\tilde{\varphi}_f))_n v^\sigma d\mu(\tau).\end{aligned}$$

The poles of the Eisenstein series are located in the constant coefficient, then

$$\int_{\mathcal{F}_2^T} f_{\mu_0}(\tau) \sum_{m\neq 0} A_{-1}(\tau, -1/2, \mu(\widetilde{\varphi}_f))_m v^\sigma d\mu(\tau) = 0.$$

□

Chapter 2

L-functions and Shalika models

In [Hec18] Hecke proposed a way to understand algebraic objects by means of complex analysis. Conjecturally you may attach a function, commonly called $L-$function, to any algebraic object. This correspondence may be done in such a way that the analytic properties of this function are related to the algebraic properties of the former object. This chapter is devoted to the study of automorphic representations. These structures have well defined $L-$functions, that in fact represent one of the cornerstones of the theory. These functions are defined by the data of a cuspidal automorphic representation π on a reductive group G and a finite dimensional representation r of the Langlands group ${}^{L}G$. Among the vast Langlands program it is conjectured that those functions encode symmetric properties of the automorphic representations and further that those $L-$functions are functorial, i.e. their values are compatible with the morphisms between Langlands groups.

As a motivation for the forthcoming exposition, in this introduction we will address the main ideas of this theory using the founding example $G = \mathrm{GL}_{2n}$, the general linear group defined over a number field F, a cuspidal automorphic representation π with trivial central character and $r = \Lambda^2$ the exterior square representation of $\mathrm{GL}_{2n}(\mathbb{C})$. In this setting, the work of [JS90] and [CKPSS01] relates the existence of a pole at $s = 1$ of the exterior square $L-$function with the weak functorial lift from the group SO_{2n+1}.

The relation between the pole of the wedge square $L-$function and the weak functorial lift is locally suggested by an explicit computation regarding the representation Λ^2. Let us suppose that the $L-$function $L(s, \pi, r)$ has a pole at $s = 1$. This assumption implies that the Satake parameter of π_p, denoted by χ_{π_p}, belongs to a subgroup $H < G(\mathbb{C})$ so that

$$\Lambda^2|_H = V \oplus \mathbb{1}, \tag{2.0}$$

with V an irreducible representation of H and $\mathbb{1}$ the trivial representation of H. In order to find the subgroup H, we set J a symplectic bilinear form of rank $2n$. It defines a $\mathrm{Sp}_{2n}(\mathbb{C})-$invariant linear map of the form

$$s: \ \Lambda^2\mathbb{C}^{2n} \to \mathbb{C}.$$

According to it, we are able to factor the vector space $\Lambda^2\mathbb{C}^{2n}$ as follows:

$$\Lambda^2\mathbb{C}^{2n} \simeq \ker(s) \oplus \mathbb{C}. \tag{2.0}$$

The group H is then characterized as the largest subgroup of G preserving the above factorization, i.e. $\ker(s)$ and $\mathbb{C}$. We will show that $H = \mathrm{Sp}_{2n}(\mathbb{C})$. Given any $e_i \wedge e_j \in \ker(s)$, an element $g \in G$ satisfies $g\,(e_i \wedge e_j) \in \ker(s)$ if and only if

$${}^{t}e_i^{t}gJge_j = {}^{t}e_iJe_j = 0.$$

The above condition can be rewritten as ${}^{t}gJg = \nu J$, with $\nu \in \mathbb{C}$. In fact, this is the definition of the group $\mathrm{GSp}_{2n}(\mathbb{C})$. But let us observe that the linear map s is just $\mathrm{Sp}_{2n}(\mathbb{C})-$invariant. Therefore the factorization (2) just follows for the group $\mathrm{Sp}_{2n}(\mathbb{C})$, concluding the search. The representation V is the irreducible $\binom{n}{2}-$dimensional representation of $\mathrm{Sp}_{2n}(\mathbb{C})$, obtainining that, if the $L-$function has a pole, then $\chi_{\pi_p} \in \mathrm{Sp}_{2n}(\mathbb{C})$ i.e. the representation π_p is a local functorial lift. We point out that the above reasoning does not imply global functoriality. The main problem of regarding the functoriality answer is to guess if there is an automorphic representation of SO_{2n+1} with Satake parameters of the form χ_{π_p} for every p with π_p unramified. This question may be solved by means of the trace formula or the converse theorem as in [CKPSS01]. In the latter paper the authors constructed a weak functorial lift from SO_{2n+1} to G (consistent with the above local lift) sending $\pi' \to \pi$. Using the factorization given in (2), this result implies the following relation between $L-$functions:

$$L(s,\pi,\Lambda^2) = L(s,\pi',V)\zeta(s). \tag{2.0}$$

From another point of view, the poles of the $L-$functions are intimately related to certain symmetries of the automorphic representations; the so-called *models*. Set $R < G$ an algebraic subgroup and $\tilde{\psi}$ a characer of R. A model of an automorphic representation is essentially a functional $F: \pi \to \mathbb{C}$ so that for any $\phi \in \pi$ the following relation holds:

$$F(\pi(r)\phi) = \tilde{\psi}(r)F(\phi),$$

with $r \in R(\mathbb{A}_F)$. In [JS90] the authors considered the *Shalika model*, which is defined with the subgroup $S = \mathrm{GL}_{2n}N_{\mathrm{GL}_{2n}}$ and the character

$$\begin{aligned} \psi_S:\ S(\mathbb{A}_F) &\to \mathbb{C}, \\ gn(x) &\mapsto \psi\left(\mathrm{Tr}(x)\right), \end{aligned}$$

where ψ is a unitary character of $\mathbb{A}_F$. By [JR96] and [AG94] this functional is unique up to constant. In fact, it is of the form

$$\mathcal{S}(\phi) = \int_{M_n(F)\backslash M_n(\mathbb{A}_F)} \int_{\mathrm{GL}_n(F)\backslash \mathrm{GL}(\mathbb{A}_F)/Z_{\mathrm{GL}_n}(\mathbb{A}_F)} \phi\left[\left(\begin{smallmatrix} I_n & X \\ & I_n \end{smallmatrix}\right)\left(\begin{smallmatrix} g & \\ & g \end{smallmatrix}\right)\right] \psi\left(\mathrm{Tr}(X)\right) dX dg, \tag{2.0}$$

where $M_n(F)$ and $M_n(\mathbb{A}_F)$ are the $n \times n$ matrices with entries in F and $\mathbb{A}_F$ respectively. We will say that the Shalika model exists if the functional S is non-zero. In order to relate the above map with certain $L-$function, the authors considered the so-called zeta integral

$$Z(\phi,s) = \int_{M_n(F)\backslash M_n(\mathbb{A}_F)} \int_{\mathrm{GL}_n(F)\backslash \mathrm{GL}(\mathbb{A}_F)/Z_{\mathrm{GL}_n}(\mathbb{A}_F)} \phi\left[\left(\begin{smallmatrix} I_n & X \\ & I_n \end{smallmatrix}\right)\left(\begin{smallmatrix} g & \\ & g \end{smallmatrix}\right)\right] \psi\left(\mathrm{Tr}(X)\right) E(g,s) dX dg \tag{2.0}$$

where $E(g,s)$ is certain Eisenstein series defined over $\mathrm{GL}_n(\mathbb{A})$ with constant residue at $s = 1$. The above integral is related (via the residue) to the Shalika functional $\mathcal{S}$. Moreover, using the series expression of the Eisenstein series, it is shown in [JS90, prop. 5, p. 203] that the integral (2) unfolds to a sum involving Whittaker functionals. These latter objects are well understood and, due to the celebrated Casselman-Shalika formula [CS80], their values can be computed explicitly in terms of the Satake parameters of π. In fact, factorizing the $L-$function via the symmetric power representation (see [JS90, sec. 2, p. 145]) and using the Weyl's character formula, Jacquet and Shalika proved in [JS90, prop. 2, p. 206] an equality (up to constant) between the zeta integral and the $L-$function. Using this connection, the existence of poles of the $L-$function at $s = 1$ is completely characterized by the vanishing of the functional (2), i.e. it is characterized in terms of the existence of a Shalika model.

We may summarize the previous discussion by stating that the following three facts are equivalent:

1. The L-function $L(s,\pi,r)$ has a pole at $s = 1$.
2. The Shalika model of π exists.
3. The cuspidal automorphic representation π is a weak functorial lift with respect to the embedding ${}^L\mathrm{SO}_{2n+1} \hookrightarrow {}^LG$.

In this section we will consider the analogous kind of questions for the group $\mathrm{GU}(2,2)$, which is intimately linked to the group GL_4. We will now replace the wedge square representation with a few adjustments. We denote the resulting map by $\Lambda_t^2 : {}^L\mathrm{GU}(2,2) \to \mathrm{GL}(\Lambda^2\mathbb{C}^4)$. In [Mor14, cor. 6.23, p. 81] it is shown that, given a cuspidal generic representation σ of GSp_4, the following equality holds:

$$L(s, \theta(\sigma), \Lambda_t^2) = L(s, \sigma, \mathrm{std})\zeta(s).$$

We note that this equality resembles the one given in (2). The result suggests that in this case, functoriality has to be replaced by the theta correspondence. Through this change we can dispense with the use of the converse theorem. Before continue we would like to point out that the appearance of the theta correspondence is not a mere coincidence. As we explained before, the wedge square $L-$function is in general linked to the Shalika model. In fact, in [Mor14] the author provided a characterization of the automorphic representations of $\mathrm{GU}(2,2)$ with Shalika model: they are the ones in the image of the theta correspondence of generic cuspidal automorphic representations of GSp_4.

To complete the above picture, [FM13] showed that the representation of $\mathrm{GU}(2,2)$ has a Shalika model if and only if the function $L(s, \theta(\sigma), \Lambda_t^2)$ has a pole. The aforementioned results are summarized by stating that the following three facts are equivalent:

1. The $L-$function $L(s, \pi, \Lambda_t^2)$ has a pole at $s = 1$.
2. The Shalika model of π exists.
3. There exists a generic cuspidal automorphic representation σ of GSp_4 so that $\theta(\sigma) = \pi$.

Along this chapter we will work in the above setting. The main contribution is the relation of a Rankin-Selberg integral of a cusp form on $\mathrm{GU}(2,2)$ with an L-function of a different cusp form on GSp_4. Given a cuspidal automorphic representation π of $\mathrm{GU}(2,2)$ we consider the zeta integral

$$I(\varphi, s) := \int_{\mathbf{H}(\mathbb{Q})Z_{\mathbf{G}}(\mathbb{A})\backslash \mathbf{H}(\mathbb{A})} E_P^*(h, s)\varphi(h)dh, \tag{2.0}$$

where $\varphi \in \pi$ and $E_P^*(h, s) = \sum_{\gamma \in P_{\mathrm{GSp}_4}(\mathbb{Q})\backslash \mathrm{GSp}_4(\mathbb{Q})} f_s(g)$ is a Siegel Eisenstein series with at most two simple poles at $s = 1, 2$. This Eisenstein series satisfies that $\mathrm{Res}_{s=2} E_P^*(g, s) \in \mathbb{C}$. We may unfold the above integral, obtaining the following equality:

$$I(\varphi, s) = \int_{\mathrm{GL}_2(\mathbb{A})N_{\mathbf{H}}(\mathbb{A})\backslash \mathbf{H}(\mathbb{A})} f_s(h)\mathcal{S}(\pi(h)\varphi)dh,$$

where $\mathcal{S}(\cdot)$ is the functional associated to the Shalika model for the group $\mathrm{GU}(2,2)$. This functional and the one defined by (2) have more in common than the name. In fact, for the primes p so that $\mathrm{GU}(2,2)(\mathbb{Q}_p) \simeq \mathrm{GL}_4(\mathbb{Q}_p) \times \mathrm{GL}_1(\mathbb{Q}_p)$, the Shalika model of $\mathrm{GU}(2,2)$ defines Shalika model for $\mathrm{GL}_4(\mathbb{Q}_p)$. The above equality implies that, if the representation π does not admit a Shalika model, the zeta integral will be equal to 0. As we explained before, the results of [Mor14] characterize the automorphic representations with Shalika model as the ones coming from generic cuspidal automorphic representations of GSp_4 via the theta correspondence. Hence, for the rest of the discussion we assume that $\pi = \theta(\sigma)$, with σ a generic cuspidal automorphic representation of GSp_4. Besides the vanishing criterion discussed above, the above equality allows us to approach the computation of $I(\varphi, s)$ via local computations. The main goal of this section is to compute the unramified components of the integral $I(\varphi, s)$.

In order to compute the local components of $I(\varphi, s)$, an explicit expression for the $p-$component $\mathcal{S}_p(\cdot)$ of the Shalika functional is required. We follow the method developed by [CS80] and generalized by [Sak06]. One of the main ingredients used in this method is the fact that the model is unique, i.e. the functional $\mathcal{S}$ is the only one satisfying

$$\mathcal{S}_p(\pi_p(s)v_0) = \tilde{\psi}_S(s)\mathcal{S}(v_0),$$

where $s \in S_{\mathrm{GU}(2,2)}(\mathbb{Q}_p)$, the Shalika subgroup of $\mathrm{GU}(2,2)$, and $\tilde{\psi}_S$ is the Shalika character in the present setting. The first result in this chapter is the proof of uniqueness for the model under certain restrictions on the representation π_p. In order to approach this problem we use ideas inspired in the classical Mackey theory of finite groups and the fact that our representation is in the image of certain theta lift. Combining the techniques of [CS80] and [Sak06] with the explicit formulas of the local theta correspondence given in [Mor14], we get the second result of this chapter; the desired explicit expression for the functional. We point out that, once again, the theta correspondence plays a crucial role in the strategy. This is because this lift is the right replacement for functoriality in the present situation.

Using the formula given in [Sak06] for the split primes, the aforementioned formula for the non split primes and the Weyl character formula we expect to achieve the following equality:

$$L(s,\sigma,\mathrm{std}) = I(\varphi,s)I(s),$$

where σ is the representation of GSp_4 so that its theta lift is equal to π and $I(s)$ is certain function associated to the non unramified places of p of π. Let us suppose that $I(s)$ is a non-zero holomorphic function. Using [Mor14] we get

$$L(s,\pi,\Lambda_t^2)\zeta^{-1}(s) = I(\varphi,s)I(s).$$

Then, taking the residue at $s=2$ at both sides, we obtain

$$\mathrm{Res}_{s=2}L(s,\pi,\Lambda_t^2) = \star\int_{[\mathrm{GSp}_4]}\varphi(h)dh,$$

where $\star$ is a non-zero constant. The right hand side of the above equality is the so-called period of an automorphic form of GSp_4. Essentially, since we are working with globally generic stuff, by [HR90, thm. 3.2.2] we have that this period is equal to 0, we refer the reader to [CT]. Hence, as a corollary we conclude that, if the representation admits a Shalika model, the function $L(s,\pi,\Lambda_t^2)$ is holomorphic at $s=2$. This result fits in the conjectural framework of the refined Ramanujan conjecture, see [Sha11, thm. 6.2, p. 21]. In fact, we recall that, by [Mor14], the representation π has a non-zero Shalika model if and only if π is generic. Under the assumption of the refined Ramanujan conjecture, this property would imply that the representation π is tempered. By an estimation based on the Satake parameters, the function $L(s,\pi,\Lambda_t^2)$ is then holomorphic at $s=2$.

This chapter is organized as follows: the first section is an introduction to the theory of reductive groups. There we will consider the different kinds of root datum associated to reductive groups, the structure of the parabolic subgroups and the construction of the Langlands dual group. The second section is devoted to presenting the main properties of the similitude groups. In the third section we explain the basic representation theory of locally compact groups, emphasizing the study of unramified representations. The section 4 is devoted to explaining what is a model of a representation. We will consider case by case relevant models for the dissertation and we will show the essential properties of each one. In addition, we explain two strategies to prove the uniqueness of a model. In the fifth section we discuss the generalization of the local and global theta correspondence for similitude groups. Furthermore, we will present the results of [Mor14], which allows us to relate the existence of models via the theta correspondence. In section 6 we explain the basic theory of the automorphic $L-$functions. We will address the relation of certain $L-$functions via the local theta correspondence. Section 7 is a joint work with Antonio Cauchi and it is the main body of the chapter. In this section we explain the original results of the chapter, where we proved the holomorphicity of the wedge square $L-$function of $\mathrm{GU}(2,2)$.

2.1 Structure of reductive groups

The results regarding the structure of reductive groups are an essential part to understand both the automorphic representations and the representations of topological groups. This section is devoted to recalling the theory required for this chapter. Throughout this section we will work with the notion of algebraic group given in [Mil18]. For a detailed discussion of the topic we refer the reader to [Mil17], [PR94], [GH22, sec. 1, p. 1] and [Bor91].

Along this section we denote by $\mathbb{G}_m$ the split algebraic torus of rank 1 defined over a field F, furthermore given a split torus of the form $\mathbb{G}_m^n$, we denote by $\alpha_i \in X^\bullet(\mathbb{G}_m^n)$ the character so that $\alpha_i((a_j)_{1\leq j\leq n}) = a_i$. Furthermore we will consider an unramified algebraic reductive group G defined over a field F of characteristic 0, with maximal torus T and Borel subgroup B. Moreover, given $g, h \in G$ we will denote by gt the image of $g \times h$ under the multiplication morphism $G \times G \to G$ of the algebraic group G.

2.1.1 Structure of split reductive groups

Unless we say the contrary in this subsection the tuple (G, T) denotes a split reductive group defined over a field F of characteristic 0 with T a maximal split torus of G.

Given X a free $\mathbb{Z}-$module, we denote by $X^\vee = \mathrm{Hom}(X, \mathbb{Z})$ its dual module. Furthermore we will consider its associated bilinear form

$$\begin{aligned} \langle \cdot, \cdot \rangle :\ & X \times X^\vee \to \mathbb{Z}, \\ & x \times y \mapsto y(x). \end{aligned}$$

Definition 2.1.1. A *root datum* is a triple $(X, R, \alpha \mapsto \alpha^\vee)$, where X is a free $\mathbb{Z}-$module of finite rank, R is a finite subset of X, called the set of *roots*, and $\alpha \mapsto \alpha^\vee$ is an injective map from R to $X^\vee$ satisfying the following conditions:

1. $\langle \alpha, \alpha^\vee \rangle = 2$ for all $\alpha \in R$.
2. Given α we consider the following automorphism of X

$$\begin{aligned} s_\alpha :\ & X \to X, \\ & x \mapsto x - \langle x, \alpha^\vee \rangle \alpha. \end{aligned}$$

   The subset R must satisfy $s_\alpha(R) \subset R$ for all $\alpha \in R$.
3. The group $W(R)$ generated by the automorphisms s_α is finite. This group is called the *Weyl group* of the root datum.

If for every $\alpha \in R$ the only multiples of it in R are $\pm\alpha$, we say that the root datum is *reduced.*

Definition 2.1.2. A *central isogeny* of root datum

$$h :\ (X, R, \alpha \mapsto \alpha^\vee) \to (Y, R', \beta \mapsto \beta^\vee),$$

is an injective morphism of $\mathbb{Z}-$modules $f : X \to Y$ with finite cokernel so that there exists a bijection $i : R' \to R$ satisfying

$$f(i(\alpha)) = \alpha,\ f^\vee(\alpha^\vee) = i(\alpha)^\vee,$$

where $f^\vee :\ Y^\vee \to X^\vee$ is the dual map of f. We say that h is an *isomorphism* if f is an isomorphism of $\mathbb{Z}-$modules.

Definition 2.1.3. The *category of reduced root datum*, denoted by **RRD**, has reduced root datums as objects and the morphisms between them are central isogenies.

Definition 2.1.4. Let $(X, R, \alpha \mapsto \alpha^\vee)$ be a root datum. We say that a subset $\Sigma \subset R$ is a *basis* for R if it is a basis for the vector space $X \otimes_{\mathbb{Z}} \mathbb{Q}$ and every root is a linear combination of elements of Σ with integer coefficients of the same sign. A subset $R^+ \subset R$ is a set of *positive roots* if for each element $\alpha \in R$, exactly one of the $\pm\alpha$ is in R^+. Moreover, a subset of positive roots $\Delta \subset R^+$ is called set of *simple roots* if the elements of Δ form a basis for R and no element of Δ can be written as the sum of two positive roots.

Using the adjoint representation:

$$\mathrm{Ad} :\ G \to \mathfrak{g} := \mathrm{Lie}(G), \tag{2.0}$$

we are able attach a reduced root datum to the tuple (G,T). Since T is a commutative subgroup, the action of T via the adjoint representation decomposes the Lie algebra of G as follows:

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha, \tag{2.0}$$

where the set Φ consists on the set of characters $\alpha \in X^\bullet(T) := \mathrm{Hom}(T, \mathbb{G}_m)$ so that $\mathfrak{g}_\alpha = \{x \in \mathfrak{g},\ s.t.\ Ad(t)x = \alpha(t)x,\ for\ all\ t \in T\}$.

Definition 2.1.5. The set Φ obtained in (2.1.1) is called the *set of roots* associated to (G,T).

Remark 2.1.6. *By [Mil15, cor. 19.19, (a), p. 341] we obtain* $\mathfrak{g}_0 = \mathrm{Lie}(T)$. *Furthermore [Mil15, thm. 22.43, (a), p. 391] shows* $\dim(\mathfrak{g}_\alpha) = 1$ *for every* $\alpha \in \Phi$.

The set of characters of T is a free $\mathbb{Z}-$module and the set of cocharacters of T, denoted by $X_\bullet(T) := \mathrm{Hom}(\mathbb{G}_m, T)$, is also a $\mathbb{Z}-$module. Given $\alpha \in X^\bullet(T)$ and $\beta \in X_\bullet(T)$

$$\alpha \circ \beta \in \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) \simeq \mathbb{Z}.$$

Then, there is a well defined bilinear form

$$\begin{aligned} \langle\cdot,\cdot\rangle : X^\bullet(T) \times X_\bullet(T) &\to \mathbb{Z}, \\ \alpha \times \beta &\mapsto \alpha \circ \beta. \end{aligned}$$

For any $\alpha \in \Phi$ we denote by s_α the reflection in $X^\bullet(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ about the hyperplane perpendicular to α. By [Mil17, thm. 22.43, (e) p. 391] there is a unique $\alpha^\vee \in X_\bullet(T)$ so that

$$s_\alpha(x) = x - \langle x, \alpha^\vee\rangle\, \alpha, \tag{2.-1}$$

for all $x \in X^\bullet(T)$. Therefore, the correspondence $\alpha \mapsto \alpha^\vee$ provided by (2.1.1) is a well defined assignment of the form $X^\bullet(T) \to X_\bullet(T)$.

Definition 2.1.7. The triple $\mathcal{R}(G,T) := (X^\bullet(T), \Phi, \alpha \mapsto \alpha^\vee)$, where the map $\alpha \to \alpha^\vee$ is given by (2.1.1) is called *root datum of* (G,T).

Proposition 2.1.8. *The triple* $\mathcal{R}(G,T)$ *is a reduced root datum. Furthermore, every reduced root datum is the root datum of a split reductive group of the form* (G,T).

Proof. The first claim follows by [Mil15, thm. 22.44, p. 392]. For the second statement see [Mil15, 22.50, p. 393] □

Remark 2.1.9. *Let* F'/F *be a field extension, then*

$$\mathcal{R}(G,T) = \mathcal{R}(G_{F'}, T_{F'}).$$

For the following lemma we recall that an element $g \in G(F)$ defines a morphism of algebraic groups of the form

$$c_g:\ G \to G, \tag{2.-1}$$

so that, for every $F-$algebra R there exists a group homomorphism

$$\begin{aligned} c_g:\ G(R) &\to G(R), \\ x &\mapsto gxg^{-1}. \end{aligned}$$

For a detailed discussion on this map we refer the reader to the definition of an algebraic group given in [GH22, sec. 1.2, p. 2] and [Mil18, sec. 2.4, p. 9].

Lemma 2.1.10. *The reduced root datum associated to* (G,T) *does not depend on the choice of* T.

Proof. By [CGP15, thm. C.2.3, p. 506], given two maximal split torus T and T' of G, there exists an element $g \in G(F)$ so that $T' = c_g(T)$, where $c_g : G \to G$ is the map defined in (2.1.1). To conclude, [Mil15, 22.48, p. 393] shows that the previous equality induces an isomorphism of root datum $\mathcal{R}(G,T) \to \mathcal{R}(G,T')$. □

Definition 2.1.11. A *central isogeny* between two connected algebraic groups G and G' is a surjective homomorphism of algebraic groups $\varphi : G \to G'$ with finite kernel contained in the centre of G. A central isogeny is an isomorphism if it is also an isomorphism of algebraic groups.

In the following definition we use the construction of the quotient of an algebraic group by a normal subgroup, we refer the reader to [Mil18, 4.3, p. 15] and [Mil18, 4.23, p. 18] to see the details.

Definition 2.1.12. We say that two isogenies $\varphi_1, \varphi_2 : G \to G'$ are *equivalent* if $\varphi_1 = t\varphi_2 t^{-1}$ with $t \in T'/Z_{G'}(F)$.

Definition 2.1.13. The *category of split reductive groups defined over* F, denoted by $\mathbf{Spl}_F$, has split reductive groups (G,T) as objects and the morphisms between the objects are the central isogenies up to equivalence.

Proposition 2.1.14. *The functor given by*

$$\mathbf{Spl}_F \to \mathbf{RRD}$$
$$(G,T) \mapsto \mathcal{R}(G,T),$$

is a contravariant equivalence of categories.

Proof. See [Mil15, 22.49, p. 393] and [Mil18, 19.58, p. 101]. □

Definition 2.1.15. Let G denote an unramified reductive group and T a torus of G. The *algebraic Weyl group* of G with respect to T is defined by the following constant group scheme:

$$W(G,T) := N_G(T)/C_G(T).$$

Lemma 2.1.16. *Let* (G,T) *be a split reductive group. The algebraic Weyl group satisfies*

$$W(G,T) = N_G(T)/T.$$

Proof. According to [Mil18, 15.14, p. 65] if T is maximal $C_G(T) = T$. □

Let G be an unramified group and T a torus of G, the $F-$points of the Weyl group $W(G,T)(F)$ act on $X^\bullet(T)$ by inner conjugation i.e. given $\alpha \in X^\bullet(T)$, $\omega \in W(G,T)(F)$ and $n \in N_G(T)(F)$ representing ω, we have

$$s \cdot \alpha(t) := \alpha(n^{-1}tn),$$

for all $t \in T$.

Proposition 2.1.17. *If* (G,T) *is split, the action of* $W(G,T)(F)$ *on* $X^\bullet(T)$ *preserves the set of roots* Φ *and permutes the sets of possible positive roots of* G *in a simply transitively way.*

Proof. The first statement is [Mil15, lem. 22.31, p. 386]. For the latter statement see [Bor91, (3), p. 14]. □

Proposition 2.1.18. *The algebraic Weyl group* $W(G,T)(F)$ *and the Weyl group* $W(\Phi)$ *associated to the root datum of* (G,T) *are isomorphic as groups.*

Proof. Let us note that the Weyl group $W(\Phi)$ depends just on the set of roots Φ, then the proposition follows by [GH22, prop. 1.8.1, p. 23]. □

Definition 2.1.19. Let G be an unramified reductive group. We denote by $\mathcal{R}(G_{\overline{F}}, T_{\overline{F}}) = (X^\bullet(T_{\overline{F}}), \Phi, \alpha \mapsto \alpha^\vee)$ the reduced root datum associated to the split reductive group $(G_{\overline{F}}, T_{\overline{F}})$. We denote by $\hat{G}$ the reductive group defined over $\mathbb{C}$ with reduced root datum of the form $(X_\bullet(T_{\overline{F}}), \Phi^\vee, \alpha^\vee \mapsto \alpha)$. The complex Lie group $\hat{G}(\mathbb{C})$ is called *the dual group* of G.

Remark 2.1.20. *Proposition 2.1.14 shows that the split reductive groups (G, T) are classified up to isomorphism by its root datum, hence the dual group of G is unique up to isomorphism.*

The dual group encodes crucial representation theoretic information of the groups $G(F)$ when is F a non-archimedian local field of characteristic 0. In section 2.3.3, we will adress this question, showing that the semisimple elements of the dual group $\hat{G}(\mathbb{C})$ classifies up to isomorphism the so-called unramified representations of the group $G(F)$. This assignment is provided by the so-called Satake isomorphism and allows us to attach local $L-$functions to this kind of representations of $G(F)$.

Remark 2.1.21. *We normalize the bilinear form (2.1.1) so that given $\alpha \in X^\bullet(T)$, $\beta \in X_\bullet(T)$ and $p \in \mathbb{Z}$, we have $\alpha(\beta(p)) = p^{\langle \alpha, \beta \rangle}$. Furthermore, using remark 2.1.9 and the bilinearity of (2.1.1), we may define the action of $W(G,T)$ on the root datum $\mathcal{R}(\hat{G}, \hat{T})$ and the action of $W(\hat{G}, \hat{T})$ on $\mathcal{R}(\hat{G}, \hat{T})$, obtaining the following group isomorphism*

$$W(G,T) \simeq W(\hat{G}, \hat{T}).$$

2.1.2 Structure of unramified groups

In contrast to the split reductive groups, the unramified non-split reductive groups over F are not classified by the root datum or similar structures. The main reason for this phenomenon is that the unramified groups are endowed with a non-trivial action of the Galois group. This action can not be realized in the linear structure of the group and must be added in order to classify these groups, for more information about the classification, we refer the reader to [Mil18, sec. 24, p. 130]. In this subsection, instead of considering the classification of non split unramified reductive groups, we will describe the algebraic structure associated to a maximal split torus. Though it does not classify the groups up to isomorphism, this construction provides enough information about their structure to approach the goals of the thesis. Throughout this subsection we fix G an unramified group defined over a field F characteristic 0 with separable closure denoted by F^{sep} and T a maximal torus of G with $T_s \subset T$ the maximal split torus of T.

Definition 2.1.22. Let V be a vector space of finite dimension defined over $\mathbb{Q}$ with $R \subset V$ a finite subset so that

1. R spans V.
2. For each α, there is a $\alpha^\vee \in V^\vee = \mathrm{Hom}(V, \mathbb{Q})$ so that $\langle \alpha, \alpha^\vee \rangle = 2$ and $\langle R, \alpha^\vee \rangle \subset \mathbb{Z}$, with

$$\langle \cdot, \cdot \rangle : \; V \times V^\vee \to \mathbb{Q},$$

   the usual bilinear form defined by duality.

The tuple (V, R) is called *root system*. We say that it is *reduced* when the only multiples of $\alpha \in R$ are $\pm\alpha$.

Throughout this section, when we discuss about subsets of R we will use the definitions given in 2.1.4.

Definition 2.1.23. Let (V, R) be a root system. The group $W(R)$ of V generated by the reflections

$$\begin{aligned} s_\alpha : \; & X \to X, \\ & x \mapsto x - \langle x, \alpha^\vee \rangle \, \alpha, \end{aligned}$$

for every $\alpha \in R$, is called the *Weyl group of* (V, R).

The adjoint representation $\mathrm{Ad} : \; G \to \mathfrak{g}$ allows us to construct a root system associated to the tuple (G, T_s). In fact, the action of Ad restricted to the maximal split torus T_s factors the Lie algebra $\mathfrak{g}$ as in (2.1.1), more concretely we have

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha, \tag{2.-1}$$

with Φ a subset of $X^\bullet(T)$.

Definition 2.1.24. The set Φ obtained in the factorization (2.1.2) is called the *relative root system associated to* (G, T_s).

Proposition 2.1.25. *The tuple* $(X^\bullet(T_s) \otimes_{\mathbb{Z}} \mathbb{Q}, \Phi)$ *forms a root system.*

Proof. See [Mil18, 23.4, (a), p. 128]. □

Remark 2.1.26. *Unlike the split case, the relative root system associated to* (G, T_s) *is not necessarily reduced. For example the group* $\mathrm{SU}(3)$ *defined in [CS80, (2), p. 217] has two simple roots that are of the form* α, 2α.

Remark 2.1.27. *As we mentioned in the introduction, the root system* (V, Φ) *is not enough to classify the unramified groups* G *over* F*. In fact, one have to define different structures that keep track of the action of* $\mathrm{Gal}(F^{sep}/F)$ *on* G*. Since this classification is not used along this thesis we do not go into details, for further information we refer the reader to [Mil17, chap. 25, p. 465].*

Proposition 2.1.28. *The algebraic Weyl group* $W(G, T_s)(F)$ *and the Weyl group* $W(\Phi)$ *associated to the relative root system of* (G, T_s) *are isomorphic as groups.*

Proof. It is [Mil18, 23.4, (c), p. 128]. □

Proposition 2.1.29. *The Weyl group satisfies*

$$W(G, T)(F) = W(G, T_s)(F) = W(G, T_s)(F^{sep}).$$

Proof. See [GH22, lem. 7.5.4, p. 175]. □

Remark 2.1.30. *Since by remark 2.1.21* $W(G, T)(F^{sep}) \simeq W(\hat{G}, \hat{T})(\mathbb{C})$*, the previous proposition shows that there is an injection*

$$W(G, T_s)(F) \to W(\hat{G}, \hat{T})(\mathbb{C}).$$

Remark 2.1.31. *The relative root datum associated to* (G, T_s) *provides a way to construct and classify the parabolic subgroups of* G*. Fix a subset of simple roots* $\Delta \subset \Phi$*. By [GH22, thm. 1.9.1, p. 30], the standard parabolic subgroups of* G *are in bijection with the subsets of* Δ*. There is an explicit construction of the parabolic subgroup attached to a subset of* Δ*. In this remark we will sketch the constructions, see [BT65, sec. 3, p. 71] and [Cas08, p. 11]. Given any subset* $\Theta \subset \Delta$ *we consider the root system generated by* Θ*, denoted by* $\langle\Theta\rangle := \mathbb{Z}\Theta \cap \Phi$*. The following parabolic group is said to be attached to* Θ

$$P^\Theta := C_G(\ker(\Theta)) \prod_{\alpha \in \Phi^+ \setminus (\langle\Theta\rangle \cap \Phi^+)} N^\alpha,$$

where $\ker(\Theta) = \cap_{\alpha\in\Theta} \ker(\alpha)$ *and* N^α *is the unimodular subgroup associated to the root* α*. We will denote by* $M^\Theta := C_G(\ker(\Theta))$ *its Levi component and by* $N^\Theta := \prod_{\alpha\in\Phi^+\setminus(\langle\Theta\rangle\cap\Phi^+)} N^\alpha$ *its unipotent subgroup. Let us note that if* $\Theta = \varnothing$*, then*

$$P^\varnothing = B,$$

with B *the Borel subgroup of* G*.*

Definition 2.1.32. Given a unipotent subgroup of the form $N^\Omega = \prod_{\alpha\in\Omega} N^\alpha$, we denote by

$$\overline{N}^\Omega = \prod_{\alpha\in\Omega} N^{-\alpha},$$

the so-called *opposite unipotent subgroup*.

Definition 2.1.33. Given $\Theta \subset \Delta$ and $P^\Theta = C_G(\ker(\Theta)) \prod_{\alpha\in\alpha\in\Phi^+\setminus(\langle\Theta\rangle\cap\Phi^+)} N^\alpha$ the parabolic associated to Θ by remark 2.1.31, we define the *opposite parabolic* of P^Θ by

$$\overline{P}^\Theta = C_G(\ker(\Theta)) \prod_{\alpha\in\alpha\in\Phi^+\setminus(\langle\Theta\rangle\cap\Phi^+)} N^{-\alpha}.$$

Proposition 2.1.34 (Bruhat decomposition). *Let F be a non-archimedean local field. Given $\Theta, \Omega \subset \Delta$, where Δ is a set of simple roots, we have*

$$G(F) = \bigsqcup_{\omega \in W_\Theta \backslash W(G,T_s)(F)/W_\Omega} P^\Theta(F)\omega P^\Sigma(F),$$

with W_Θ and W_Ω the subsets of $W(G,T_s)(F)$ generated by the reflections $\{s_\alpha,\ \alpha \in \Theta\}$ and $\{s_\alpha,\ \alpha \in \Sigma\}$ respectively.

Proof. See [Cas08, prop. 1.3.1, p. 11]. ☐

Remark 2.1.35. *If Θ is the set of roots associated to a parabolic subgroup $P^\Theta = M^\Theta N^\Theta$, by 2.1.31 we will denote W_Θ by W_{P^Θ}.*

Remark 2.1.36. *In remark 2.1.20 we showed that the assignment*

$$G \to \hat{G}$$

is well defined for the split groups. Since the unramified groups are not classified by their root system, this identification is no longer bijective. One has to construct a nice replacement for the dual group reflecting the action of $\mathrm{Gal}(F^{sep}/F)$ on G. To that end, we have to define the action of $\mathrm{Gal}(F^{sep}/F)$ on $\hat{G}(\mathbb{C})$. In this remark we will sketch the construction of this action. For a detailed discussion we refer the reader to [Mil15, p. 101] and [GH22, sec. 7.3, p. 165].

By proposition 2.1.14, the action of $\mathrm{Gal}(F^{sep}/F)$ on $G_{F^{sep}}$ allows us to regard

$$\mathrm{Gal}(F^{sep}/F) \subset \mathrm{Aut}\left(\mathcal{R}(G_{F^{sep}}, T_{F^{sep}})\right).$$

Then, just by definition, we have $\mathrm{Gal}(F^{sep}/F) \subset \mathrm{Aut}\left(\mathcal{R}(\hat{G},\hat{T})\right)$. Let us recall that the isomorphisms in the category $\mathbf{Spl}_F$ are isomorphisms of algebraic groups up to equivalence, see definition 2.1.12. Therefore, proposition 2.1.14 does not provide a unique way to consider $\mathrm{Gal}(F^{sep}/F)$ as a subset of $\mathrm{Aut}(\hat{G})$. In fact, under the equivalence of categories given in proposition 2.1.14, an element of $\mathrm{Gal}(F^{sep}/F) \subset \mathrm{Aut}\left(\mathcal{R}(\hat{G},\hat{T})\right)$ is identified with the set of conjugations by $T_{\hat{G}}/Z_{\hat{G}}(\mathbb{C})$ of certain automorphism of $\hat{G}$. To overcome the previous problem and state a well defined embedding $\mathrm{Gal}(F^{sep}/F) \subset \mathrm{Aut}(\hat{G})$, we have to fix a Borel subgroup $\hat{B} < \hat{G}$. This choice defines a based root datum $\Psi(\hat{G},\hat{T},\hat{B})$ that, stated informally, is the root datum $\mathcal{R}(\hat{G},\hat{T})$ with the extra information given by the Borel subgroup $\hat{B}$. By [GH22, prop. 7.3.3, p. 167] this choice provides a well defined map

$$\mathrm{Aut}(\Psi(\hat{G},\hat{T},\hat{B})) \to \mathrm{Aut}(\hat{G}). \tag{2.-1}$$

Therefore, we are able to regard $\mathrm{Gal}(F^{sep}/F) \subset \mathrm{Aut}(\Psi(\hat{G},\hat{T},\hat{B}))$ and then the map (2.1.36) determines the action of the Galois group on $\hat{G}$. Furthermore, by [GH22, prop. 7.3.2, p. 167] the action of the Galois group on $\hat{G}$ is independent of the choice of $\hat{B}$.

Definition 2.1.37. Let G be an unramified group, we define its associated $L-$group by

$${}^L G = \hat{G}(\mathbb{C}) \rtimes \mathrm{Gal}(F^{sep}/F),$$

where the action of $\mathrm{Gal}(F^{sep}/F)$ in $\widehat{G}(\mathbb{C})$ is given by the map (2.1.36).

2.1.3 Finite dimensional representations of split groups

The finite dimensional representations of split groups play a central role in the theory of automorphic representations. They are used to define different $L-$functions that, depending on the chosen representation, it is conjectured that their values are related to certain properties of the automorphic representation. See section 2.6.1 for a further development of the theory of $L-$functions. In this subsection we will introduce the basic theory of finite dimensional representations of split reductive groups. Furthermore, we will analyze certain representations that are relevant for the thesis. For a detailed discussion of the topic we refer the reader to [Kna86], [FH91] and [Mil18, sec. 20, p. 102]. Throughout this subsection we fix (G, T) a split reductive group defined over a characteristic 0 field F. Set a Borel subgroup B of G, determining a choice of positive roots Φ^+. Given V be a vector space defined over F, a representation of G is a natural transformation of the following group-valued functors:

$$G(R) \to \mathrm{Aut}_R\left(V \otimes_F R\right),$$

where R is any $F-$algebra. If V is finite dimensional, the previous map corresponds to a homomorphism of algebraic groups $\rho : G \to \mathrm{GL}(V)$. We denote by (ρ, V) the representation defined as above.

Definition 2.1.38. Let (ρ, V) be a finite dimensional representation. The space V factors as

$$V = \bigoplus_{\lambda \in X^{\bullet}(T)} V_\lambda \tag{2.-1}$$

with $v \in V_\lambda$ such that $\rho(t)v = \lambda(t)v$ for $t \in T(F)$. The spaces V_λ appearing in (2.1.38) are called *weight spaces* for the action of T and the elements $\lambda \in X^{\bullet}(T)$ such that $V_\lambda \neq 0$ in (2.1.38) are called *weights*.

Proposition 2.1.39. *Let (ρ, V) be a finite dimensional representation. The set of weights is invariant under the action of the Weyl group $W(G, T)$. Furthermore, the action of the Weil group respects the multiplicity of the weights.*

Proof. See [FH91, p. 201]. □

Definition 2.1.40. The lattice

$$P(\Phi^+) = \left\{\lambda \in X^{\bullet}(T) \otimes_{\mathbb{Z}} \mathbb{Q},\ s.t.\ \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z},\ s.t.\ \alpha \in \Phi^+\right\},$$

of $X^{\bullet}(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ is called the *weight lattice*.

Proposition 2.1.41. *The algebraic group G is simply connected if and only if $X^{\bullet}(T) = P(\Phi^+)$.*

Proof. It is [Mil18, 19.61, p. 102]. □

Definition 2.1.42. Let G be a connected algebraic group. A *universal covering* of G is a multiplicative isogeny $\tilde{G} \to G$ with $\tilde{G}$ smooth, connected and simply connected.

Proposition 2.1.43. *A connected group G admits a universal covering in any of the following cases:*

- *The field F is perfect and $X^{\bullet}(G) = id$.*
- *The group G is semisimple.*

Proof. See [Mil18, p. 71]. □

Remark 2.1.44. *Let G be a connected reductive group that admits a universal covering $\tilde{G} \to G$ with $\tilde{G}$ reductive. By [Mil18, 16.1, p. 68] there is a subgroup $C < Z(\tilde{G})$ so that*

$$\tilde{G}/C \simeq G.$$

For any $C_i < Z(\tilde{G})$, the groups $\tilde{G}/C_i$ are algebraic group with universal covering $\tilde{G}$. In fact, by 2.1.14 all these groups have the same root datum modulo central isogeny of root datum.

Remark 2.1.45. *By [Mil18, 20.34, p. 108] if G is simply connected, the finite dimensional representations of G are in bijection with the finite dimensional representations of $\mathfrak{g} = \mathrm{Lie}(G)$.*

Proposition 2.1.46. *If G is simply connected, the set of weights occurring in a finite dimensional representation of G belongs to the weight lattice $P(\Phi^+)$.*

Proof. The proof follows by remark 2.1.45 and [FH91, p. 200]. □

Definition 2.1.47. An element $\lambda \in P(\Phi^+)$ is a *dominant weight* if $\langle \lambda, \alpha^\vee \rangle > 0$ for all $\alpha \in \Phi^+$.

One can realize the dominant weights λ in an explicit geometric way. In fact, we may regard $\{\langle \lambda, \alpha^\vee \rangle = 0,\ \lambda \in P(\Phi^+), \alpha \in \Phi^+\}$ as the set of lines in $X^\bullet(T) \otimes_{\mathbb{Z}} \mathbb{Q}$, dividing the space into a set of pieces $\{\mathcal{C}_i\}_{i \in 2|\Phi^+|}$ called *Weyl's chambers*. Therefore, one of these chambers contains all the dominant weights associated to the choice of Φ^+.

Definition 2.1.48. The set $\{\lambda \in P(\Phi^+),\ s.t.\ \langle \lambda, \alpha^\vee \rangle > 0\ for\ all\ \alpha \in \Phi^+\}$ is called the *positive Weyl's chamber.*

Remark 2.1.49. *The Weyl group permutes the set of Weyl's chambers via the action on the roots. This action is the geometric equivalent to permute the sets of all possible positive roots of G.*

Proposition 2.1.50. *Let (ρ, V) be an irreducible finite dimensional representation of G. Then, there exists a unique one dimensional subspace V_λ of V so that*

- *It is stabilized under the action of B.*
- *The subspace V_λ is a weight space of T and its weight λ is dominant.*
- *If μ is also a weight, $\lambda - \mu = \sum_{\alpha \in \Delta} m_\alpha \alpha$ with $m_\alpha > 0$.*

Proof. We refer to [Mil18, (20.3), p. 102]. □

Definition 2.1.51. Let (ρ, V) be an irreducible finite dimensional representation of G. The weight λ obtained in 2.1.50 is called the *highest weight vector* of (ρ, V).

Theorem 2.1.52. *Let G be a simply connected algebraic group. Every dominant weight is the highest weight vector of an irreducible finite dimensional representation $V(\lambda)$ of G. Two irreducible finite dimensional representations are isomorphic if they have the same highest weight.*

Proof. It is [Mil18, p. 104]. □

Proposition 2.1.53. *Let us suppose that G has a universal covering $\tilde{G} \to G$ with $\tilde{G}$ reductive and let $\lambda \in P(\tilde{\Phi}^+)$ be a dominant weight of $\tilde{G}$. The representation $V(\lambda)$ with highest weight λ factors throughout G if and only if $\lambda \in X^\bullet(T)$.*

Proof. See [Mil18, p. 104]. □

Definition 2.1.54. The underlying set of $R(G)$, the *representation ring of* G, is the set of irreducible finite dimensional representation of G modulo isomorphism of representations. $R(G)$ is endowed with group law given by $[V], [V'] \in R(G)$, $[V] \cdot [V'] = [V \otimes V']$.

Proposition 2.1.55. *Let G be a simply connected split reductive algebraic group, the following map defines an isomorphism*

$$R(G) \to \mathbb{Z}[P(\Phi^+)]^{W(G,T)},$$
$$[V] = \bigoplus_\lambda V_\lambda \mapsto \sum_\lambda \dim(V_\lambda) e(\lambda),$$

where λ belongs to the set of weights of the representation V and $\{e(\lambda)\}_\lambda$ is a basis of $\mathbb{Z}[P(\Phi^+)]$.

Proof. It follows by proposition 2.1.39. □

Example 2.1.56. *The roots of GL_n are $\alpha_1 - \alpha_2, \ldots, \alpha_{n-1} - \alpha_n$ and their co-roots are given by $(\alpha_i - \alpha_{i+1})^\vee = \alpha_i - \alpha_{i+1}$. Therefore, the dominant weights of GL_n are the following linear combinations*

$$m_1\alpha_1 + \ldots + m_n\alpha_n,\ m_i \in \mathbb{Z},\ m_1 \geq \ldots \geq m_n.$$

Example 2.1.57. *Let us fix the set of simple roots of Sp_{2n} given by $\Delta = \{\{\alpha_i - \alpha_{i+1}\}_{i=1,\ldots,n-1} \cup \{2\alpha_n\}\}$. It is straightforward that the Weyl's chamber with respect to the previous choice of simple roots is*

$$\mathcal{W} = \{a_1\alpha_1 + \ldots + a_n\alpha_n,\ a_1 \geq \ldots \geq a_n \geq 0\}.$$

Example 2.1.58. *The 5-dimensional irreducible standard representation of* $\mathrm{Sp}_4(\mathbb{C})$ *denoted by* $(\mathrm{std}, \mathbb{C}^5)$*. Using remark 2.1.45 and [FH91, p. 245], it holds that the highest weight vector of this representation is* $\alpha_1 + \alpha_2$.

As we pointed out at the beginning of this section, the finite dimensional representations are relevant in the context of the automorphic representations because they are used to define the different $L-$functions. In fact, we have to consider representations of the Langlands dual group (or representations of the dual group in the split case). The following example is used to construct one of the $L-$functions studied in the forthcoming exposition.

Example 2.1.59. *Let* E/F *be a totally imaginary quadratic field extension. We consider the group*

$$G = (\mathrm{GL}_4(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C})) \rtimes \mathrm{Gal}(E/F),$$

where the action of the unique non-trivial element $\theta \in \mathrm{Gal}(E/F)$ *is given by*

$$\begin{aligned} \theta :\ \mathrm{GL}_4(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}) &\to \mathrm{GL}_4(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}), \\ (g, \lambda) &\mapsto (\overline{\lambda} J^{-1}\, {}^t g^{-1} J, \overline{\lambda}), \end{aligned}$$

wwhere $J = \begin{pmatrix} & J' \\ -J' & \end{pmatrix}$ *with* J' *the anti-diagonal* $n \times n$ *matrix with non-zero entries equal to* 1*. Let us denote by* Λ^2 *the wedge square representation of* $\mathrm{GL}_4(\mathbb{C})$ *on* $\mathbb{C}^6$*. By abuse of notation we also denote by*

$$\begin{aligned} \Lambda^2 :\ \mathrm{GL}_4(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}) &\to \mathrm{Aut}(\mathbb{C}^6), \\ (g, \lambda) &\mapsto \lambda \left(\Lambda^2 g\right). \end{aligned}$$

Since the representations Λ^2 *and* $\Lambda^2 \circ \theta$ *have the same highest weight, theorem 2.1.52 shows the existence of a map* $A : \mathbb{C}^6 \to \mathbb{C}^6$ *so that* $A^2 = 1$ *and*

$$\left(\Lambda^2 \circ\ \theta\right)(g, \lambda) = A^{-1} \Lambda^2(g, \lambda) A.$$

We define the representation Λ^2_t *of* G *by*

$$\Lambda^2_t(g, \lambda, 1) = \Lambda^2(g, \lambda), \quad \Lambda^2_t(1, 1, \theta) = A.$$

2.1.4 Topological groups

Let F be a non-archimedean local field, the $F-$points of G inherits the Haussdorf topology of F. To lighten the notation, throughout this subsection we will specialize to the case $F = \mathbb{Q}_p$, for a general discussion we refer the reader to [Cas08, p. 14].

For the following proposition we fix the subset

$$T_s^- = \{t \in T_s(\mathbb{Q}_p),\ s.t.\ |\alpha(a)| \leq 1,\ for\ all\ \alpha \in \Delta\}. \tag{2.-1}$$

Definition 2.1.60. Let $P = MN$ be a parabolic subgroup of G with Levi subgroup M and maximal unipotent subgroup N. Given K a compact open subgroup of $G(\mathbb{Q}_p)$, we say that K has a *Iwahori factorization* if

1. The following map is an isomorphism

$$\begin{aligned} \overline{N}_K \times M_K \times N_K &\to K, \\ \overline{n} \times m \times n &\mapsto \overline{n} m n, \end{aligned}$$

   where $\overline{N}_K := \overline{N}(\mathbb{Q}_p) \cap K$ with $\overline{N}$ the opposite unipotent of N defined in 2.1.32, $M_K := M(\mathbb{Q}_p) \cap K$ and $N_K = N(\mathbb{Q}_p) \cap K$.

2. For all $t \in T_s^-$, $t N_K t^{-1} \subset N_K$ and $t^{-1} \overline{N}_K t \subset \overline{N}_K$.

Proposition 2.1.61. *There exists a set $\{K_n\}_{n\geq 0}$ of open compact subgroups which forms a neighborhood basis of the identity so that*

- *Every K_n is a normal subgroup of K_0.*
- *If P is a parabolic group containing B, every K_n has Iwahori factorization with respect to P.*

Proof. See [Cas08, prop. 1.4.4, p. 14]. □

Definition 2.1.62. The *Iwahori subgroup* I of $G(\mathbb{Q}_p)$ is defined as the inverse image of $B(\mathbb{Z}/p\mathbb{Z})$ throughout the following projection:

$$G(\mathbb{Z}_p) \to G(\mathbb{Z}/p\mathbb{Z}).$$

Example 2.1.63. *For the group $\mathrm{GL}_n(\mathbb{Q}_p)$ with fixed Borel subgroup of upper triangular matrices, the Iwahori subgroup I is given by the subgroup of upper triangular matrices modulo p.*

Proposition 2.1.64. *The Iwahori subgroup I has Iwahori factorization given by*

$$I = (\overline{U}(\mathbb{Q}_p) \cap I)(B(\mathbb{Q}_p) \cap I),$$

where U is the maximal unipotent subgroup of G.

Proof. It follows by the definition. □

Proposition 2.1.65. *The group $G(\mathbb{Z}_p)$ satisfies the following factorization*

$$G(\mathbb{Z}_p) = \bigsqcup_{\omega \in W(G,T_s)(\mathbb{Q}_p)} I\omega I.$$

Proof. See [Cas08, p. 216]. □

2.2 Similitude groups

Throughout this section we fix $E/\mathbb{Q}$ a imaginary quadratic extension so that $E = \mathbb{Q}(\delta)$ with $\delta = \sqrt{d}$, $d \in \mathbb{Q}^\times \setminus \mathbb{Q}^{\times,2}$ and $d < 0$. Unless we say the contrary, $\mathbb{G}_m$ will be used to denote the split algebraic torus of rank 1 defined over $\mathbb{Q}$. Furthermore, given a split torus of the form $\mathbb{G}_m^n$, we denote by $\alpha_i \in X^\bullet(\mathbb{G}_m^n)$ the character so that $\alpha_i((a_j)_{1\leq j\leq n}) = a_i$. Along this section R will be any $\mathbb{Q}-$module.

2.2.1 General unitary group

Definition 2.2.1. A *hermitian space* over E is a tuple (W, S), with W a vector space of dimension n defined over E and S a non degenerate sesquilinear form. If there is no danger of confusion we will simply denote hermitian spaces by its vector space, W in the present case.

Definition 2.2.2. Let W be a hermitian space over E. The general unitary group is the algebraic group over $\mathbb{Q}$ defined by

$$\mathrm{GU}(W)(R) := \{(g, m_g) \in \mathrm{GL}(W)(R \otimes_{\mathbb{Z}} \mathcal{O}_E) \times \mathbb{G}_m(R) : \ {}^t\bar{g}Sg = m_g S\},$$

where $\bar{\cdot}$ denotes the non-trivial automorphism of order 2 of $E/\mathbb{Q}$. We denote by $\nu : \mathrm{GU}(W) \to \mathbb{G}_m$, $g \mapsto m_g$ the similitude character. Its kernel is the group $\mathrm{U}(W)$.

Remark 2.2.3. *In this chapter we will consider the general unitary group associated to an hermitian space of dimension 4. We denote by J_2 the 2×2 anti-diagonal matrix with all non-zero entries equal to 1 and $J = \begin{pmatrix} & J_2 \\ -J_2 & \end{pmatrix}$. It defines a sesquilinear form on the vector space $W = E^4$.*

Definition 2.2.4. Let us consider the vector space $W = E^4$ endowed with the sesquilinear form J. We will use the following notation:

$$\mathbf{G}(R) := \mathrm{GU}(W)(R) = \{(g, m_g) \in \mathrm{GL}_4(R \otimes_{\mathbb{Z}} \mathcal{O}_F) \times \mathbb{G}_{\mathrm{m}}(R) : \ {}^t\bar{g}Jg = m_g J\}.$$

Proposition 2.2.5. *Given p a split prime of the quadratic field extension $E/\mathbb{Q}$, the general unitary group satisfies*

$$\mathbf{G}(\mathbb{Q}_p) \simeq \mathrm{GL}_4(\mathbb{Q}_p) \times \mathrm{GL}_1(\mathbb{Q}_p).$$

Proof. Let p be a place of $\mathbb{Q}$ splitting as $p = p_1 p_2$ in E. The following map is an isomorphism

$$E \otimes_{\mathbb{Q}} \mathbb{Q}_v \simeq E_{p_1} \oplus E_{p_2},$$
$$(e \otimes r) \mapsto (er, \bar{e}r).$$

Let us observe that the complex conjugation satisfies $\overline{(g_1, g_2)} = (g_2, g_1)$. Then by (2.2.1) we obtain

$$\mathbf{G}(\mathbb{Q}_p) = \{(g, h, m_g) \in \mathrm{GL}_4(E_{p_1}) \times \mathrm{GL}_4(E_{p_1}) \times \mathbb{Q}_p^\times,\ s.t.\ ({}^t h, {}^t g)({}^t J, J)(g, h) = m_g({}^t J, J)\}. \tag{2.-2}$$

The equations defining (2.2.1) are

$${}^t h {}^t J g = m_g\, {}^t J,$$
$${}^t g J h = m_g J.$$

One equation is the transpose of the other, therefore we isolate $h = m_g J^{-1}\, {}^t g^{-1} J$. This formula implies that g a free variable i.e. we can choose any $g \in \mathrm{GL}_4(E_{p_1})$. Therefore we conclude identifying $E_{p_1} \simeq \mathbb{Q}_p$, which implies that the map

$$\mathbf{G}(\mathbb{Q}_p) \mapsto \mathrm{GL}_4(\mathbb{Q}_p) \times \mathbb{Q}_p^\times$$
$$(g, h, m_g) \mapsto (g, m_g),$$

is an isomorphism. ∎

Corollary 2.2.6. *The algebraic group $\mathbf{G}$ is an inner form of $\mathrm{GL}_4 \times \mathrm{GL}_1$.*

Proof. It follows by proposition 2.2.5. ∎

Corollary 2.2.7. *The group $\mathbf{G}$ is connected.*

Proof. By [Mil17, prop. 1.4, p. 21], the group $\mathbf{G}$ is connected if and only if $\mathbf{G} \times_{\mathrm{Spec}(\mathbb{Q})} \mathrm{Spec}(F)$ is connected for every field extension $F/\mathbb{Q}$. By proposition 2.2.5

$$\mathbf{G} \times_{\mathrm{Spec}(\mathbb{Q})} \mathrm{Spec}(\mathbb{C}) \simeq \mathrm{GL}_4 \times \mathrm{GL}_1, \tag{2.-2}$$

where GL_4 and GL_1 are algebraic groups defined over $\mathbb{C}$. The groups GL_4 and GL_1 are connected as algebraic groups, hence [Mil17, lem. 1.15, p. 21] implies that $\mathrm{GL}_4 \times \mathrm{GL}_1$ is connected. Therefore the isomorphism (2.2.1) implies that $\mathbf{G}$ is connected. ∎

Definition 2.2.8. The *special general unitary* group is the subgroup of GU given by

$$\mathbf{G}^\star(R) = \{(g, m_g) \in \mathbf{G}(R)\ :\ \det(g) = m_g^2\}.$$

Let $B_{\mathbf{G}} = T_{\mathbf{G}} U_{\mathbf{G}}$ denote the upper-triangular Borel subgroup of $\mathbf{G}$, where $T_{\mathbf{G}}$ is the maximal non split diagonal torus given by

$$T_{\mathbf{G}}(R) = \left\{ \begin{pmatrix} a &&& \\ & b && \\ && \nu\bar{b}^{-1} & \\ &&& \nu\bar{a}^{-1} \end{pmatrix}\ a, b \in \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m(R),\ \nu \in \mathbb{G}_m(R) \right\},$$

with maximal split subtorus determined by

$$T_{\mathbf{G},s}(R) = \left\{ \begin{pmatrix} a &&& \\ & b && \\ && \nu b^{-1} & \\ &&& \nu a^{-1} \end{pmatrix}\ s.t.\ a, b, \nu \in \mathbb{G}_m(R) \right\}.$$

Its maximal unipotent subgroup is the following group of matrices

$$U_{\mathbf{G}}(R) = \left\{ \begin{pmatrix} 1 & m & n & t \\ & 1 & k & \overline{n} \\ & & 1 & -\overline{m} \\ & & & 1 \end{pmatrix} \; m, n \in \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_a(R),\ t, k \in \mathbb{G}_a(R) \right\}.$$

Lemma 2.2.9. *The modulus character of $B_{\mathbf{G}}(\mathbb{Q}_p)$ is given by*

$$\begin{aligned} \delta_{B_{\mathbf{G}}} : B_{\mathbf{G}}(\mathbb{Q}_p) &\to \mathbb{C} \\ \begin{pmatrix} a & & & \\ & b & & \\ & & \nu\bar{b}^{-1} & \\ & & & \nu\bar{a}^{-1} \end{pmatrix} &\mapsto \frac{|a\bar{a}|_p^3 |b\bar{b}|_p}{|\nu|_p^4}. \end{aligned}$$

Proof. The modulus character can be computed using the formula

$$\delta_{B_{\mathbf{G}}}(b) = \det\left(\mathrm{Ad}|_{\mathrm{Lie}(U_{\mathbf{G}})}(b)\right),$$

given in [Cas08, 1.5, p. 16]. □

We fix a basis $\{\alpha_i\}_{i=0,1,2}$ of the free $\mathbb{Z}-$module $X^\bullet(T_{\mathbf{G},s})$, where

$$\begin{aligned} \alpha_i :\ T_{\mathbf{G},s}(R) &\to R^\times, \\ \begin{pmatrix} a & & & \\ & b & & \\ & & \nu b^{-1} & \\ & & & \nu a^{-1} \end{pmatrix} &\mapsto \begin{cases} \nu & if\ i = 0, \\ a & if\ i = 1, \\ b & if\ i = 2. \end{cases} \end{aligned}$$

The set of positive roots with respect to the split torus $T_{\mathbf{G},s}$ which corresponds to this choice of Borel subgroup is

$$\Phi^+_{\mathbf{G}} = \{\alpha_1 - \alpha_2, \alpha_1 + \alpha_2 - \alpha_0, 2\alpha_2 - \alpha_0, 2\alpha_1 - \alpha_0\}. \tag{2.-3}$$

The unimodular subgroups associated to each root are denoted by

$$\begin{aligned} U_{\mathbf{G}}^{\alpha_1-\alpha_2}(R) &:= \left\{ \begin{pmatrix} 1 & a & & \\ & 1 & & \\ & & 1 & -\overline{a} \\ & & & 1 \end{pmatrix},\ a \in \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_a(R) \right\}, \\ U_{\mathbf{G}}^{\alpha_1+\alpha_2-\alpha_0}(R) &:= \left\{ \begin{pmatrix} 1 & & b & \\ & 1 & & \overline{b} \\ & & 1 & \\ & & & 1 \end{pmatrix},\ b \in \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_a(R) \right\}, \\ U_{\mathbf{G}}^{2\alpha_2-\alpha_0}(R) &:= \left\{ \begin{pmatrix} 1 & & & \\ & 1 & c & \\ & & 1 & \\ & & & 1 \end{pmatrix},\ c \in \mathbb{G}_a(R) \right\}, \\ U_{\mathbf{G}}^{2\alpha_1-\alpha_0}(R) &:= \left\{ \begin{pmatrix} 1 & & & d \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix},\ d \in \mathbb{G}_a(R) \right\}. \end{aligned}$$

The Siegel parabolic subgroup $P_{\mathbf{G}}$ has Levi decomposition of the form $P_{\mathbf{G}} = M_{\mathbf{G}}N_{\mathbf{G}}$ so that

$$\begin{aligned} M_{\mathbf{G}}(R) &= \left\{ \begin{pmatrix} h & \\ & \lambda J_2^t\overline{h}^{-1}J_2 \end{pmatrix},\ s.t.\ h \in \mathrm{Res}_{E/\mathbb{Q}}\mathrm{GL}_2(R),\ \lambda \in \mathbb{G}_m(R) \right\}, \\ N_{\mathbf{G}}(R) &= \left\{ \begin{pmatrix} I_2 & X \\ & I_2 \end{pmatrix},\ s.t.\ X \in \mathrm{Herm}_2(R) \right\}, \end{aligned}$$

where $\mathrm{Herm}_2(R) := \left\{ \begin{pmatrix} \alpha & x \\ y & \overline{\alpha} \end{pmatrix},\ s.t.\ \alpha \in \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_a(R),\ x, y \in \mathbb{G}_a(R) \right\}$.

Lemma 2.2.10. *Let p be a non split prime, the Weyl group $W_{\mathbf{G}} := W(\mathbf{G}, T_s)(\mathbb{Q}_p)$ is generated by*

$$s_1 := \begin{pmatrix} 1 & & & \\ & & 1 & \\ & -1 & & \\ & & & 1 \end{pmatrix},\quad s_2 := \begin{pmatrix} & 1 & & \\ -1 & & & \\ & & & -1 \\ & & 1 & \end{pmatrix},$$

where s_1 is the reflection about $\alpha_1 + \alpha_2 - \alpha_0$ and s_2 is the reflection about $\alpha_1 - \alpha_2$.

Proof. It follows by direct computation. □

Lemma 2.2.11. *It holds that*

$$^L\mathbf{G} = \mathrm{GL}_4(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}) \rtimes \mathrm{Gal}(E/\mathbb{Q}),$$

where the action of the non-trivial element $\theta \in \mathrm{Gal}(E/\mathbb{Q})$ *in* $(g, m_g) \in \mathrm{GL}_4(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C})$ *is given by*

$$\theta(g \times m_g) = \overline{m}_g\, {}^tJ^{-1}g^{-1}J \times \overline{m}_g.$$

Proof. It is a direct application of corollary 2.2.6. □

2.2.2 General symplectic group

Definition 2.2.12. Let (W, J) be a rational symplectic space. The *general symplectic group* attached to W is the algebraic group over $\mathbb{Q}$ so that

$$\mathrm{GSp}(W)(R) = \{(g, m_g) \in \mathrm{GL}(W)(R) \times \mathbb{G}_{\mathrm{m}}(R) \ s.t.\ g^tJg = m_gJ\}.$$

We denote by $\lambda : \mathrm{GU}(W) \to \mathbb{G}_{\mathrm{m}}\ g \mapsto m_g$ the similitude character. Its kernel is the group $\mathrm{Sp}(W)$.

Remark 2.2.13. *The general symplectic group associated to certain symplectic space of dimension* 4 *will be relevant for the thesis. Let* J *be the matrix defined in remark 2.2.3, the vector space* $W = \mathbb{Q}^4$ *endowed with* J *is a symplectic space.*

Definition 2.2.14. Let W be the symplectic space described in remark 2.2.13, we define

$$\mathbf{H}(R) := \mathrm{GSp}(W)(R)\{(g, m_g) \in \mathrm{GL}_4(R) \times \mathbb{G}_{\mathrm{m}}(R) :\ {}^tgJg = m_gJ\}.$$

Remark 2.2.15. *The algebraic group* $\mathbf{H}$ *embeds into* $\mathbf{G}$ *via the map*

$$\iota : \mathbf{H} \hookrightarrow \mathbf{G},$$
$$(g, m_g) \mapsto (g, m_g),$$

induced by the inclusion $\mathrm{GL}_4 \hookrightarrow \mathrm{Res}_{\mathcal{O}_E/\mathbb{Z}}\mathrm{GL}_4$. *Furthermore the* $R-$*points satisfy that* $\mathbf{G}(R) \cap \mathrm{GL}_4(R) = \mathbf{H}(R)$.

Remark 2.2.16. *The bilinear form* $\tilde{J} = \begin{pmatrix} & I_2 \\ I_2 & \end{pmatrix}$ *is a symplectic form, then the vector space* $W = \mathbb{Q}^4$ *endowed with* $\tilde{J}$ *is a symplectic space of dimension* 4. *There exists an isomorphism of algebraic groups*

$$\mathrm{GSp}((W, J)) \simeq \mathrm{GSp}((W, \tilde{J})) \simeq \mathbf{H}.$$

The isomorphism changes the explicit description of the parabolic subgroups. Depending on our goal the choice of one of the above coordinates or the other will be more suitable.

The root datum of $\mathbf{H}$ is equal to the relative root datum of $\mathbf{G}$ with respect to the split torus $T_{\mathbf{G},s}$. Therefore the structure of the parabolic subgroups and Weyl group of $\mathbf{H}$ will be completely analogous to the one of $\mathbf{G}$. Let $B_{\mathbf{H}} = T_{\mathbf{H}}U_{\mathbf{H}}$ denote the upper-triangular Borel subgroup of $\mathbf{H}$ with $T_{\mathbf{H}}$ the diagonal torus

$$T_{\mathbf{H}}(R) = \left\{ \begin{pmatrix} a & & & \\ & b & & \\ & & \nu b^{-1} & \\ & & & \nu a^{-1} \end{pmatrix} \ s.t.\ a, b, \nu \in \mathbb{G}_m(R) \right\},$$

and maximal unipotent subgroup

$$U_{\mathbf{H}}(R) = \left\{ \begin{pmatrix} 1 & m & n & t \\ & 1 & k & n \\ & & 1 & -m \\ & & & 1 \end{pmatrix} \ s.t.\ m, n, t, k \in \mathbb{G}_a(R) \right\}. \tag{2.-4}$$

The torus $T_{\mathbf{H}}$ is split and is equal to the maximal split torus of $\mathbf{G}$.

Lemma 2.2.17. *The modulus character is given by*

$$\begin{aligned} \delta_{B_{\mathbf{H}}} :\ & B_{\mathbf{H}}(\mathbb{Q}_p) \to \mathbb{C}, \\ & \begin{pmatrix} a & & & \\ & b & & \\ & & \nu b^{-1} & \\ & & & \nu a^{-1} \end{pmatrix} \mapsto \frac{|a|_p^6 |b|_p^2}{|\nu|_p^4}. \end{aligned}$$

Proof. It is analogous to lemma 2.2.9. □

Since $T_{\mathbf{G},s} = T_{\mathbf{H}}$, we take the basis $\{\alpha_i\}_{i=0,1,2}$ of $X^\bullet(T_{\mathbf{H}})$ defined in (2.2.1). The set of positive roots of $\mathbf{H}$ which are compatible with the choice of Borel $B_{\mathbf{H}}$ is

$$\Phi_{\mathbf{H}}^+ = \{\alpha_1 - \alpha_2, \alpha_1 + \alpha_2 - \alpha_0, 2\alpha_2 - \alpha_0, 2\alpha_1 - \alpha_0\}. \tag{2.-4}$$

The associated unimodular subgroups are

$$\begin{aligned} U_{\mathbf{H}}^{\alpha_1-\alpha_2}(R) &:= \left\{ \begin{pmatrix} 1 & a & & \\ & 1 & & \\ & & 1 & -a \\ & & & 1 \end{pmatrix},\ a \in \mathbb{G}_a(R) \right\}, \\ U_{\mathbf{H}}^{\alpha_1+\alpha_2-\alpha_0}(R) &:= \left\{ \begin{pmatrix} 1 & & b & \\ & 1 & & b \\ & & 1 & \\ & & & 1 \end{pmatrix},\ b \in \mathbb{G}_a(R) \right\}, \\ U_{\mathbf{H}}^{2\alpha_2-\alpha_0}(R) &:= \left\{ \begin{pmatrix} 1 & & & \\ & 1 & c & \\ & & 1 & \\ & & & 1 \end{pmatrix}, c \in \mathbb{G}_a(R) \right\} \\ U_{\mathbf{H}}^{2\alpha_1-\alpha_0}(R) &:= \left\{ \begin{pmatrix} 1 & & & d \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix},\ d \in \mathbb{G}_a(R) \right\}. \end{aligned}$$

The Siegel parabolic $P_{\mathbf{H}}$ has Levi decomposition $P_{\mathbf{H}} = M_{\mathbf{H}} N_{\mathbf{H}}$ such that

$$\begin{aligned} M_{\mathbf{H}}(R) &= \left\{ \begin{pmatrix} h & \\ & \lambda J_2 h^{-1,t} J_2 \end{pmatrix},\ s.t.\ h \in \mathrm{GL}_2(R),\ \lambda \in \mathbb{G}_m(R) \right\}, \\ N_{\mathbf{H}}(R) &= \left\{ \begin{pmatrix} I_2 & X \\ & I_2 \end{pmatrix},\ s.t.\ X \in S(R) \right\}, \end{aligned}$$

where $S(R) = \{ \left(\begin{smallmatrix} \alpha & x \\ y & \alpha \end{smallmatrix}\right),\ s.t.\ \alpha, x, y \in \mathbb{G}_m(R)\}$.

Remark 2.2.18. *In remark 2.2.16 we pointed out that the isomorphism*

$$\mathbf{H} \simeq \mathrm{GSp}((W, \tilde{J}))$$

changes the description of the parabolic subgroups. For example, the maximal unipotent subgroup of $\mathrm{GSp}(\tilde{W})$ *is given by*

$$U_{\mathrm{GSp}(\tilde{W})}(R) = \left\{ w(a)v(A),\ s.t.\ a \in \mathbb{G}_a(R),\ \left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right) = A = {}^tA \in \mathrm{Mat}_{2\times 2}(R) \right\},$$

with $w(a) = \begin{pmatrix} 1 & a & & \\ & 1 & & \\ & & 1 & \\ & & -a & 1 \end{pmatrix}$ *and* $v(A) = \left(\begin{smallmatrix} 1_2 & A \\ & 1_2 \end{smallmatrix}\right)$.

Lemma 2.2.19. *The Weyl group* $W_{\mathbf{H}} := W(\mathbf{H}, T_{\mathbf{H}})(\mathbb{Q}_p)$ *is generated by*

$$s_1 := \begin{pmatrix} 1 & & & \\ & & 1 & \\ & -1 & & \\ & & & 1 \end{pmatrix} \quad s_2 := \begin{pmatrix} & 1 & & \\ -1 & & & \\ & & & -1 \\ & & 1 & \end{pmatrix},$$

where s_1 *the reflection about* $\alpha_1 + \alpha_2 - \alpha_0$ *and* s_2 *is the reflection about* $\alpha_1 - \alpha_2$.

Proof. Since $T_{\mathbf{H}}(R) = T_{\mathbf{G},s}(R)$ and $U_{\mathbf{H}}(R) = U_{\mathbf{G}}(R) \cap \mathrm{GL}_4(R)$ the result follows by 2.2.10. □

Lemma 2.2.20. *The general symplectic group satisfies*

$${}^L\mathbf{H} = \mathrm{GSpin}_5(\mathbb{C}).$$

Proof. See [AS06, prop. 2.9, p. 148]. □

2.2.3 General orthogonal group

Definition 2.2.21. Let (V, q) be a quadratic space with associated bilinear form $(\cdot,\cdot)$. The *general orthogonal group* is the algebraic group defined over $\mathbb{Q}$ so that

$$\mathbf{GO}(V)(R) = \{g \in \mathrm{GL}(V)(R),\ s.t.\ (gv, gv) = \nu_g(v, v)\}.$$

We denote by $\nu : \mathrm{GO}(V) \to \mathbb{G}_\mathrm{m},\ g \mapsto \nu_g$ the similitude character. Its kernel is the group $\mathrm{O}(W)$.

Definition 2.2.22. The connected component of the identity of the algebraic group $\mathbf{GO}(V)(R)$ is denoted by $\mathbf{GSO}(V)$.

Remark 2.2.23. *Throughout this chapter we will consider rational quadratic spaces of signature $(4, 2)$ and Witt index 2. For example the matrix* $S = \begin{pmatrix} &&&&&1\\ &&&&1&\\ &&2&&&\\ &&&-2d&&\\ &1&&&&\\ 1&&&&& \end{pmatrix}$ *defines a quadratic form Q on $\mathcal{V} := \mathbb{Q}^6$ which makes the tuple $(\mathcal{V}, Q)$ an isotropic quadratic space of signature $(4, 2)$ so that $V = \mathbb{H}^2 \oplus V_{an}$ where $V_{an} \simeq (E, \mathrm{N}_{E,\mathbb{Q}})$ as $\mathbb{Q}-$vector spaces.*

Definition 2.2.24. We denote by

$$\mathbf{GO}_{4,2}(R) := \mathrm{GO}_{4,2}(\mathcal{V})(R) = \{g \in \mathrm{GL}_6(R).\ s.t.\ {}^t gSg = \lambda_g S,\ \lambda_g \in R^\times\}.$$

Furthermore, its connected component of the identity is given by

$$\mathrm{GSO}_{4,2}(R) := \mathrm{GSO}(\mathcal{V})(R) = \{g \in \mathrm{GO}_{4,2}(R).\ s.t.\ \det g = \nu_g^3\}.$$

Definition 2.2.25. The general orthogonal group of signature $(2, 0)$, denoted by $\mathrm{GSO}_{2,0}$, is the algebraic group over $\mathbb{Q}$ so that

$$\mathrm{GSO}_{2,0}(F) = \left\{g \in \mathrm{GL}_2(F),\ {}^t g \begin{pmatrix} 2 & \\ & -2d \end{pmatrix} g = \nu(g) \begin{pmatrix} 2 & \\ & -2d \end{pmatrix},\ \det(g) = \nu(g)\right\}.$$

Remark 2.2.26. *Let $\mathbb{G}_m$ be the $1-$dimensional split torus defined over E. By [Mor14, p. 32] there exists a homeomorphism*

$$\mathrm{GSO}_{2,0}(F) \simeq \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m(F).$$

Furthermore, the above map identifies the determinant with the norm field $\mathrm{N}_{E/\mathbb{Q}}$.

Let us consider the Borel subgroup of $\mathrm{GSO}_{4,2}$ defined by the product $B_{\mathrm{GSO}} = T_{\mathrm{GSO}}U_{\mathrm{GSO}}$, where

$$T_{\mathrm{GSO}}(R) = \left\{ \begin{pmatrix} a &&&& \\ & b &&& \\ && x && \\ &&& \lambda b^{-1} & \\ &&&& \lambda a^{-1} \end{pmatrix},\ s.t.\ a, b \in \mathbb{G}_m(R),\ x \in \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m(R),\ \lambda = N_{E/\mathbb{Q}}(x) \right\} \tag{2.-4}$$

is a maximal torus. We alert the reader that by remark 2.2.26, the element x in the above matrix is a 2×2 matrix defined over R. The maximal unipotent subgroup U_{GSO} is given by

$$U_{\mathrm{GSO}}(R) = U_0(R)U_1(R)U_2(R)\tilde{U}(R), \tag{2.-4}$$

with

$$U_0(R)=\left\{u_0(x)=\begin{pmatrix}1 & -{}^tX_0S_1 & 0\\ & 1_4 & X_0\\ & & 1\end{pmatrix},\ s.t.\ X_0=\begin{pmatrix}x\\0\\0\\0\end{pmatrix},\ x\in\mathbb{G}_a(R)\right\},$$

$$U_1(R)=\left\{u_1(s_1,t_1)=\begin{pmatrix}1 & -{}^tX_1S_1 & -\frac{1}{2}{}^tX_1S_1X_1\\ & 1_4 & X_1\\ & & 1\end{pmatrix},\ s.t.\ X_1=\begin{pmatrix}0\\s_1\\t_1\\0\end{pmatrix},\ s_1,t_1,\ \mathbb{G}_a(\mathbb{R})\right\},$$

$$U_2(R)=\left\{u_2(s_2,t_2)=\begin{pmatrix}1 & & & & \\ & 1 & -{}^TX_2S_0 & -\frac{1}{2}{}^tX_2S_2X_2 & \\ & & 1_2 & X_2 & \\ & & & 1 & \\ & & & & 1\end{pmatrix},\ s.t.\ X_2=\begin{pmatrix}s_2\\t_2\end{pmatrix},\ s_2,t_2\in\mathbb{G}_a(R)\right\},$$

$$\tilde{U}(R)=\left\{\tilde{u}(b)=\begin{pmatrix}1 & -{}^t\tilde{X}S_1 & \\ & 1_4 & \tilde{X}\\ & & 1\end{pmatrix},\ s.t.\ \tilde{X}=\begin{pmatrix}0\\0\\0\\b\end{pmatrix},\ b\in\mathbb{G}_a(R)\right\},$$

where

$$S_0=\begin{pmatrix}2 & \\ & -2d\end{pmatrix},\ S_1=\begin{pmatrix} & & 1\\ & S_0 & \\ 1 & & \end{pmatrix},$$

and S_2 is the matrix given in 2.2.23.

2.2.4 Exceptional isomorphism

This section is devoted to explaining the exceptional isomorphism $\mathbf{PG}\simeq\mathbf{PGO}_{4,2}$. For a detailed discussion of the proof we refer the reader to [Mor14, p. 33].

Proposition 2.2.27. *The group* $\mathbf{G}$ *satisfies the following algebraic group isomorphism* $\mathbf{PG}\simeq\mathbf{PGSO}_{4,2}$

Proof. We consider the 6−dimensional vector space defined over $\mathbb{Q}$

$$\mathcal{V}:=\left\{v(x_1,x_2,x_3,x_4,x_5,x_6):=\begin{pmatrix}0 & x_1 & x_3+\delta x_4 & x_2\\ -x_1 & 0 & x_5 & -x_3+\delta x_4\\ -x_3-\delta x_4 & -x_5 & 0 & x_6\\ -x_2 & x_3-\delta x_4 & -x_6 & 0\end{pmatrix},\ s.t.\ x_i\in\mathbb{Q}\right\}.$$

The homomorphism

$$\phi:\ \mathbf{G}^\star\to\mathbf{GO}(\mathcal{V}),$$
$$g\mapsto(v\mapsto gv^tg),$$

is well defined. Let us fix the basis $e_1:=v(1,0,0,0,0,0)$, $e_2=v(0,1,0,0,0,0)$, $e_3=v(0,0,1,0,0,0)$, $e_3=v(0,0,1,0,0,0)$, $e_4=v(0,0,0,1,0,0)$,$e_5=v(0,0,0,0,1,0)$, $e_6=v(0,0,0,0,0,1)$ of $\mathcal{V}$, then the homomorphism ϕ defines an homomorphism of algebraic groups

$$\Phi:\ \mathbf{G}^\star\to\mathbf{GSO}_{4,2}.$$

Given $\alpha\in E^\times$ we set $r_\alpha=\begin{pmatrix}\alpha & & & \\ & 1 & & \\ & & \overline{\alpha}^{-1} & \\ & & & 1\end{pmatrix}$. The action of $\alpha\in E^\times$ on $v\in\mathcal{V}$ is defined as follows:

$$\alpha\cdot v=\overline{\alpha}r_\alpha vr_\alpha. \tag{2.-6}$$

Furthermore $E^\times$ also acts in $\mathbf{G}^\star$ by $\alpha \cdot g = r_\alpha g r_\alpha^{-1}$ for any $g \in \mathbf{G}^\star$. Using the above action we consider the semidirect product $\mathbf{G}^\star \rtimes E^\times$. Using (2.2.4) we define an homomorphism

$$\Gamma : \mathbf{G}^\star \rtimes E^\times \to \mathbf{PGSO}_{4,2}.$$

By [Mor14, p. 34] there exists a surjective homomorphism

$$\begin{aligned} \Xi : \mathbf{G}^\star \rtimes E^\times &\to \mathbf{PG} \\ (g,\alpha) &\mapsto g r_\alpha. \end{aligned}$$

In [Mor14, p. 34] the author claims that $\ker(\Gamma) = \ker(\Xi)$ and hence $\Gamma^* \circ (\Xi^*)^{-1} : \mathbf{PG} \simeq \mathrm{PGSO}_{4,2}$, where Γ^* is the isomorphism

$$\Gamma^* : \mathbf{G}^\star \rtimes E^\times / \ker(\Gamma) \to \mathbf{PGSO}_{4,2}, \tag{2.-6}$$

and Ξ^* is the isomorphism

$$\Xi^* : \mathbf{G}^\star \rtimes E^\times / \ker(\Xi) \to \mathbf{PG}. \tag{2.-6}$$

□

Lemma 2.2.28. *The image of* $t(a,b,\nu) := \begin{pmatrix} a & & & \\ & b & & \\ & & \nu\bar{b}^{-1} & \\ & & & \nu\bar{a}^{-1} \end{pmatrix} \in T_{\mathbf{PG}}(\mathbb{Q}_p)$ *throughout the isomorphism given by 2.2.27 is equal to* $\begin{pmatrix} |ab| & & & & \\ & \nu|a| & & & \\ & & \star & & \\ & & & \nu|b| & \\ & & & & \nu^2 \end{pmatrix} \in \mathbf{PGSO}_{4,2}(\mathbb{Q}_p)$*, where* $\star \in E_p^\times$.

Proof. This proof is divided into two steps; first we compute a representative g of $(\Xi^*)^{-1}(t(a,b,\nu))$, where we recall that Ξ is defined in (2.2.4). Having done that, we compute the image of the representative g throughout the homomorphism Γ^* defined in (2.2.4).

We factor the element $t(a,b,\nu)$ as follows:

$$t(a,b,\nu) = \begin{pmatrix} b^{-1} & & & \\ & b & & \\ & & \nu\bar{b} & \\ & & & \nu\bar{b}^{-1} \end{pmatrix} \begin{pmatrix} ab & & & \\ & 1 & & \\ & & \bar{a}^{-1}\bar{b}^{-1} & \\ & & & 1 \end{pmatrix},$$

where $\begin{pmatrix} b^{-1} & & & \\ & b & & \\ & & \nu\bar{b} & \\ & & & \nu\bar{b}^{-1} \end{pmatrix} \in \mathbf{G}^\star(\mathbb{Q}_p)$ and using the notation of the proof of 2.2.27 we denote by $r_{ab} := \begin{pmatrix} ab & & & \\ & 1 & & \\ & & \bar{a}^{-1}\bar{b}^{-1} & \\ & & & 1 \end{pmatrix}$. By direct computation we find that

$$\tilde{t}(a,b,\nu) \rtimes r_{ab} := \begin{pmatrix} b^{-1} & & & \\ & b & & \\ & & \nu\bar{a} & \\ & & & \nu\bar{a}^{-1} \end{pmatrix} \rtimes \begin{pmatrix} ab & & & \\ & 1 & & \\ & & \bar{a}^{-1}\bar{b}^{-1} & \\ & & & 1 \end{pmatrix} \in \mathbf{G}^\star(\mathbb{Q}_p) \rtimes E_p^\times,$$

is a representative of $(\Xi^*)^{-1}(t(a,b,\nu))$. Let us consider the basis $\{e_1,...,e_6\}$ of the proof of 2.2.27, then $r_{ab} \cdot e_1 = |ab|e_1$, $r_{ab} \cdot e_2 = |ab|e_2$, $r_{ab} \cdot e_5 = e_5$ and $r_{ab} \cdot e_6 = e_6$. Moreover $\tilde{t}(a,b,\nu) \cdot e_1 = e_1$, $\tilde{t}(a,b,\nu) \cdot e_2 = \frac{\nu}{\bar{b}b} e_2$, $\tilde{t}(a,b,\nu) \cdot e_5 = \bar{b}b\nu e_5$, $\tilde{t}(a,b,\nu) \cdot e_6 = \nu^2 e_6$. Comparing the action of $\mathbf{G}^\star(\mathbb{Q}_p) \rtimes E_p^\times$ on $\mathcal{V} \otimes_{\mathbb{Q}} \mathbb{Q}_p$ given by (2.2.4) and (2.2.4) with the natural action of $\mathbf{PGSO}_{4,2}(\mathbb{Q}_p)$ on $\mathcal{V} \otimes_{\mathbb{Q}} \mathbb{Q}_p$ we obtain

$$\Gamma^* \circ (\Xi^*)^{-1}(t(a,b,\nu)) = \begin{pmatrix} |ab| & & & & \\ & \nu|a| & & & \\ & & \star & & \\ & & & \nu|b| & \\ & & & & \nu^2 \end{pmatrix} \in T_{\mathbf{GSO}_{4,2}}(\mathbb{Q}_p),$$

where $\star \in E_p^\times$. □

2.3 Representations of topological groups

In this section we will denote by G a reductive algebraic group defined over $\mathbb{Q}_p$. We fix a Borel subgroup $B = TU$ with maximal torus T, maximal split torus T_s and maximal unipotent subgroup denoted by U. Furthermore we will denote $K_G := G(\mathbb{Z}_p)$.

2.3.1 Admissible representations

This subsection will be devoted to reviewing the basic definitions and theory of admissible representations. For a further discussion of the topic we refer the reader to [Cas08], [CS80] and [Ber92].

Definition 2.3.1. Let V be a complex vector space and (π, V) a representation of $G(\mathbb{Q}_p)$. If $\mathrm{Stab}_{G(\mathbb{Q}_p)}(v) = \{\pi(g)v = v,\ g \in G(\mathbb{Q}_p)\}$ is open for every $v \in V$, we say that the representation is *smooth.*

Definition 2.3.2. Let (π, V) be a smooth representation of $G(\mathbb{Q}_p)$, the *algebraic dual* representation (π^*, V^*) is defined by the vector space $V^* := \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ and the action of $G(\mathbb{Q}_p)$ via

$$\begin{aligned} \pi^*(g) :\ & V^* \to V^* \\ & v^*(\cdot) \mapsto v^*(\pi(g)\cdot) \end{aligned}$$

The algebraic dual representation of a smooth representation might not be smooth. There is a refined notion of duality in the category of smooth representations.

Definition 2.3.3. Let (π, V) a smooth representation of $G(\mathbb{Q}_p)$. The smooth dual or contragradient representation of (π, V) is denoted by $(\tilde{\pi}, \tilde{V})$. The vector space is defined by

$$\tilde{V} := \left\{ v^* \in V^*,\ s.t.\ \mathrm{Stab}_{G(\mathbb{Q}_p)}(v^*) \ is\ open \right\},$$

and the action is given by restriction $\tilde{\pi} := \pi^*|_{\tilde{V}}$.

Remark 2.3.4. *The smooth dual of a smooth representation is a smooth representation.*

Definition 2.3.5. A smooth representation (π, V) of $G(\mathbb{Q}_p)$ is *admissible* if $V^K := \{v \in V,\ s.t.\ \pi(k)v = v,\ k \in K\}$ is a finite dimensional vector space for every open compact subgroup $K < G(\mathbb{Q}_p)$.

Definition 2.3.6. A representation (π, V) of $G(\mathbb{Q}_p)$ is *unitary* if it admits a $G(\mathbb{Q}_p)-$invariant inner product in V.

Remark 2.3.7. *In certain bibliography, our notion of unitary representation is called unitarizable representation. It is also common to define unitary representation with an extra condition on V: it has to be a Hilbert space. In the present setting let (π, V) be a unitarizable representation (i.e. a unitary representation in the sense of definition 2.3.6). We can complete the vector space V with respect to the inner product, obtaining a unitary representation which is no longer smooth. Both definitions are intimately related and, depending on the goal, it is preferable to use one definition or another.*

Proposition 2.3.8. *Given a irreducible, unitary and admissible representation, its space of $G(\mathbb{Q}_p)-$invariant inner product is isomorphic to $\mathbb{C}$.*

Proof. See [Cas08, prop. 2.1.15, p. 23]. □

Proposition 2.3.9. *Let (π, V) be an admissible unitary representation. Then (π, V) is semisimple and each irreducible factor appears with finite multiplicity.*

Proof. It is [Cas08, prop. 2.1.14, p. 22]. □

Proposition 2.3.10. *Let (π, V) be a unitary representation. Then $(\tilde{\pi}, \tilde{V}) = (\overline{\pi}, \overline{V})$, where $\overline{V} = \{\overline{v},\ s.t.\ v \in V,\ and,\ \bar{\cdot}$ is the complex conjugation$\}$.*

Proof. We refer to [Ber92, rmk. 2, p. 83]. □

Definition 2.3.11. Let $H(\mathbb{Q}_p) < G(\mathbb{Q}_p)$ be a closed topological subgroup. Given a smooth representation (ξ, V) of $H(\mathbb{Q}_p)$ we define the *normalized induced* and *normalized compactly induced* representations as follows:

$$\begin{aligned}
\mathrm{Ind}_{H(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\xi &= \{f:\ G(\mathbb{Q}_p) \to V,\ s.t.\ f(hg) = \delta_{G(\mathbb{Q}_p)}^{-1/2}(h)\delta_{H(\mathbb{Q}_p)}^{1/2}(h)\xi(h)f(g),\\
&\qquad for,\ h \in H(\mathbb{Q}_p)\},\\
\mathrm{c}-\mathrm{Ind}_{H(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\xi &= \{f:\ G(\mathbb{Q}_p) \to V,\ s.t.\ f\ has\ compact\ support\ modulo\ H(\mathbb{Q}_p),\\
&\qquad and\ f(hg) = \delta_{G(\mathbb{Q}_p)}^{-1/2}(h)\delta_{H(\mathbb{Q}_p)}^{1/2}(h)\xi(h)f(g),\ for,\ h \in H(\mathbb{Q}_p)\}.
\end{aligned}$$

If the place p is clear from the context, we will denote those representations by $\mathrm{Ind}_H^G\xi$ and $\mathrm{c}-\mathrm{Ind}_H^G\xi$ respectively.

Proposition 2.3.12. *If $H(\mathbb{Q}_p) \setminus G(\mathbb{Q}_p)$ is compact and (ξ, V) is an admissible representation, then $\mathrm{Ind}_{H(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\xi$ is admissible and satisfies the following isomorphism of representations*

$$\mathrm{Ind}_{H(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\xi = \mathrm{c}-\mathrm{Ind}_{H(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\xi.$$

Proof. See [Cas08, thm. 2.4.1, (d), p. 26]. □

Proposition 2.3.13 (Frobenius reciprocity 1)**.** *Let $H(\mathbb{Q}_p) < G(\mathbb{Q}_p)$ be a closed topological group. Given (σ, W) and (ξ, V) smooth representations of $G(\mathbb{Q}_p)$ and $H(\mathbb{Q}_p)$ respectively, we have the following group isomorphisms:*

$$\begin{aligned}
\mathrm{Hom}_{G(\mathbb{Q}_p)}\left(\sigma, \mathrm{Ind}_H^G\xi\right) &\simeq \mathrm{Hom}_{H(\mathbb{Q}_p)}\left(\sigma|_{H(\mathbb{Q}_p)}, \delta_{H(\mathbb{Q}_p)}^{1/2}\delta_{G(\mathbb{Q}_p)}^{-1/2}\xi\right),\\
\mathrm{Hom}_{G(\mathbb{Q}_p)}\left(\mathrm{c}-\mathrm{Ind}_H^G\xi, \tilde{\sigma}\right) &\simeq \mathrm{Hom}_{H(\mathbb{Q}_p)}\left(\delta_{H(\mathbb{Q}_p)}^{-1/2}\delta_{G(\mathbb{Q}_p)}^{1/2}\xi, \widetilde{\sigma|_{H(\mathbb{Q}_p)}}\right).
\end{aligned}$$

Proof. It is [Cas08, thm. 2.4.1, (e), p. 26]. □

Given χ a complex character of the torus $T(\mathbb{Q}_p)$, we define a representation of $B(\mathbb{Q}_p) = T(\mathbb{Q}_p)N(\mathbb{Q}_p)$ by

$$\begin{aligned}
\hat{\chi}:\ T(\mathbb{Q}_p)N(\mathbb{Q}_p) &\to \mathbb{C}^\times\\
tn &\mapsto \chi(t).
\end{aligned}$$

Throughout this chapter we will use the following notation:

$$\mathrm{Ind}_B^G\chi := \mathrm{Ind}_B^G\hat{\chi}. \tag{2.-6}$$

The above constructions of $\mathrm{Ind}_B^G\chi$ can be seen as a functor

$$\{Smooth\ representations\ of\ T(\mathbb{Q}_p)\} \xrightarrow{\mathrm{Ind_H^G}} \{Smooth\ representations\ of\ G(\mathbb{Q}_p)\}.$$

To conclude this subsection we will introduce its adjoint, the Jacquet module.

Definition 2.3.14. Let (π, V) be a smooth representation of $G(\mathbb{Q}_p)$, $P(\mathbb{Q}_p)$ a parabolic subgroup of $G(\mathbb{Q}_p)$ and $N(\mathbb{Q}_p)$ its maximal unipotent subgroup. The *Jacquet module* of (π, V) is a representation of $P(\mathbb{Q}_p)$, denoted by $(\pi_{N(\mathbb{Q}_p)}, V_{N(\mathbb{Q}_p)})$, where

$$V_{N(\mathbb{Q}_p)} := V/\left\langle \pi(n)v - v,\ n \in N(\mathbb{Q}_p)\right\rangle,$$

and $\pi_{N(\mathbb{Q}_p)} := \pi|_{P(\mathbb{Q}_p)}$.

To simplify the notation, if the place p is clear from the context we will denote the Jacquet module associated to (π, V) just by V_N.

Proposition 2.3.15 (Frobenius reciprocity 2)**.** *Let $P(\mathbb{Q}_p) = M(\mathbb{Q}_p)N(\mathbb{Q}_p) < G(\mathbb{Q}_p)$ be the $\mathbb{Q}_p-$points of a parabolic subgroup. Given (σ, W) a smooth representation of a topological group $G(\mathbb{Q}_p)$ and (ξ, V) a smooth representation of $P(\mathbb{Q}_p)$, we have*

$$\mathrm{Hom}_{G(\mathbb{Q}_p)}\left(\sigma, \mathrm{Ind}_P^G\xi\right) \simeq \mathrm{Hom}_{P(\mathbb{Q}_p)}\left(\sigma_N, \delta_{P(\mathbb{Q}_p)}^{1/2}\delta_{G(\mathbb{Q}_p)}^{-1/2}\xi\right).$$

Proof. See [Cas08, (3.2.4), p. 2]. □

2.3.2 Induced representations as sections of a vector bundle

Let us consider $H(\mathbb{Q}_p)$ a closed subgroup of $G(\mathbb{Q}_p)$ and (ρ, V) a representation of $H(\mathbb{Q}_p)$. We define

$$G \times_H V := G \times V/\sim,$$

where $(g, v) \sim (hg, \rho(h)v)$. Consider the following vector bundle:

$$\begin{aligned} \pi : \ G(\mathbb{Q}_p) \times_H V &\to G(\mathbb{Q}_p)/H(\mathbb{Q}_p) \\ g \times v &\mapsto gH(\mathbb{Q}_p). \end{aligned}$$

The space of sections of (2.3.2) is given by

$$\Gamma(G(\mathbb{Q}_p)/H(\mathbb{Q}_p), G \times_H V) = \{f : \ G(\mathbb{Q}_p) \to V, \ s.t. \ f(hg) = \rho(h)f(g)\},$$

The previous discussion allows us to use a geometric point of view to understand the induced representations. The following proposition, that is used crucially used in this dissertation, is a result in that direction.

Proposition 2.3.16. *Let $\mathscr{V}$ be a complex vector bundle on a locally compact totally disconnected topological space X and let Z be a closed subspace. Let $\Gamma_c(X, \mathscr{V})$ denote the space of locally constant compactly supported sections of $\mathscr{V}$. Then we have the exact sequence*

$$0 \to \Gamma_c(X - Z, \mathscr{V}|_{X-Z}) \to \Gamma_c(X, \mathscr{V}) \to \Gamma_c(Z, \mathscr{V}|_Z) \to 0.$$

Proof. See [Pra90, lem. 5.1, p. 12] □

2.3.3 Unramified representations

Admissible representations with K_G-invariant subspaces play a relevant role in the theory of automorphic representations. This subsection is devoted to describing the basic properties of these representations.

Definition 2.3.17. A *model* $\mathcal{G}$ of G over $\mathbb{Z}_p$ is an affine scheme of finite type over $\mathbb{Z}_p$ of the form $\mathrm{Spec}(A)$ with $A < \mathcal{O}(G)$ a $\mathbb{Z}_p-$algebra so that $A \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \mathcal{O}(G)$.

Proposition 2.3.18. *There exists a model $\mathcal{G}$ of G over $\mathbb{Z}_p$ such that the special fiber of $\mathcal{G}$ is equal to G.*

Proof. See [GH22, thm. 2.4.1, p. 50]. □

Definition 2.3.19. A maximal compact group K is *hyperspecial* if it is of the form $\mathcal{G}(\mathbb{Z}_p)$ where $\mathcal{G}$ is a model of G over $\mathbb{Z}_p$.

Definition 2.3.20. An irreducible admissible representation (π, V) is *unramified* if for any $K < G(\mathbb{Q}_p)$ *hyperspecial* compact group, the space of $K-$fixed points satisfies that $V^K \neq 0$.

Definition 2.3.21. We define the complex valued Hecke algebra of G by $\mathcal{H}_{\mathbb{C},p}^G := \mathcal{H}_p^G \otimes_{\mathbb{Z}} \mathbb{C} = \{f \in \mathcal{C}_c(G), \ s.t \ f(kgk) = f(g), \ k \in K_G\}$.

Remark 2.3.22. *Given a unramified representation (π, V), Schur lemma implies that the complex valued Hecke algebra acts on V^{K_G} by multiplication by a constant, i.e. given $f \in \mathcal{H}^G_{\mathbb{C},p}$ and $v \in V^{K_G}$, $\pi(f)v = \mathrm{tr}(\pi(f))v$ with $\mathrm{tr}(\pi(f)) \in \mathbb{C}$. The map*

$$f \mapsto \mathrm{tr}(\pi(f)),$$

is called the Hecke character of (π, V).

Proposition 2.3.23. *The Hecke characters classify the unramified representations up to isomorphism.*

Proof. It is [GH22, prop. 7.1.1, p. 62]. □

2.3.4 Classification of unramified representations: The Satake isomorphism

Classification of unramified representations of split groups.

In this and the following subsection we state the classification of the unramified representations in terms of the dual group of G.

Proposition 2.3.24. *Let (G, T) be a split algebraic group defined over $\mathbb{Q}_p$. There exists an isomorphism of $\mathbb{C}-$algebras*

$$\mathcal{H}^G_{\mathbb{C},p} \simeq \mathbb{C}[\hat{T}]^{W(\hat{G},\hat{T})(\mathbb{C})}.$$

Proof. It follows directly by 1.6.36. □

Theorem 2.3.25 (Satake isomorphism)**.** *There is a bijection between the semisimple conjugacy classes of $\hat{G}(\mathbb{C})$ and the isomorphism classes of irreducible unramified representations of $G(\mathbb{Q}_p)$.*

Proof. By proposition 2.3.23, every unramified representation is classified up to isomorphism by its Hecke character, which is in fact an element of $\mathrm{Hom}_{\mathbb{C}}\left(\mathcal{H}^G_{\mathbb{C},p}, \mathbb{C}\right)$. Using proposition 2.3.24, we show the following group isomorphisms

$$\mathrm{Hom}_{\mathbb{C}}\left(\mathcal{H}^G_{\mathbb{C},p}, \mathbb{C}\right) \simeq \mathrm{Hom}_{\mathbb{C}}\left(\mathbb{C}[\hat{T}]^{W(\hat{G},\hat{T})(\mathbb{C})}, \mathbb{C}\right) \simeq \hat{T}(\mathbb{C})/W(\hat{G},\hat{T})(\mathbb{C}).$$

Every semisimple conjugacy class in $\hat{G}(\mathbb{C})$ intersects $\hat{T}(\mathbb{C})$, moreover two elements of $\hat{T}(\mathbb{C})$ are conjugated if and only if they are conjugated by $W(\hat{G},\hat{T})(\mathbb{C})$. □

Definition 2.3.26. Let (π, V) be an unramified representation of $G(\mathbb{Q}_p)$. The conjugacy class $\chi_\pi \in \hat{T}(\mathbb{C})/W(\hat{G},\hat{T})(\mathbb{C})$ obtained by the isomorphism of theorem 1.6.36 is called *Satake parameter* of (π, V).

Classification of unramified representations of unramified groups

One of the cornerstones of the Langlands program is the connection between representations of topological groups and Galois representations. An elementary instance of this correspondence is given by the classification of the unramified principal series. This subsection is devoted to extending corollary 2.3.25 to unramified non-split groups.

Definition 2.3.27. Let R be a $\mathbb{C}-$algebra, the group $\hat{G}(R)$ acts on $\hat{G}(R) \rtimes \mathrm{Fr}$ as follows:

$$\begin{aligned} \hat{G}(R) &\to \mathrm{Aut}(\hat{G}(R) \rtimes \mathrm{Fr}), \\ g &\mapsto \left(h \rtimes \mathrm{Fr} \mapsto gh\mathrm{Fr}(g)^{-1} \rtimes \mathrm{Fr}\right). \end{aligned}$$

Let us denote by $\sim$ the relation that defines the above action.

Theorem 2.3.28. *The complex valued Hecke algebra satisfies*

$$\mathcal{H}^G_{\mathbb{C},p} \simeq \mathbb{C}[(\hat{G} \rtimes \mathrm{Fr})/\sim].$$

Proof. It follows by 2.3.31 and [GH22, thm. 7.5.1, p. 173]. □

Theorem 2.3.29 (Satake isomorphism)**.** *There exists a bijection between the isomorphism classes of irreducible unramified representations of* $G(\mathbb{Q}_p)$ *and the set* $\left((\hat{G} \rtimes \mathrm{Fr})/\sim\right)(\mathbb{C})$.

Proof. The proof follows using proposition 2.3.23 and theorem 2.3.28 as we did in corollary 2.3.25. □

Definition 2.3.30. Let (π, V) be an unramified representation of $G(\mathbb{Q}_p)$. The conjugacy class $\chi_\pi \in \left((\hat{G} \rtimes \mathrm{Fr})/\sim\right)(\mathbb{C})$ obtained by theorem 2.3.29 is called *Satake parameter* of (π, V).

Even though theorem 2.3.29 provides a parametrization of the unramified representation in terms of the points of an algebraic group, neither the statement nor the proof of the theorem shows a sistematic way to construct such points. This difficult the task of working with $\left((\hat{G} \rtimes \mathrm{Fr})/\sim\right)(\mathbb{C})$ instead of working with representations. To conclude this subsection we will define an algebraic group isomoprhic to $(\hat{G} \rtimes \mathrm{Fr})/\sim$, that combined with the forthcoming section 2.3.10, allows us to state an explicit version of the Satake isomorphism.

Let ${}_{\mathbb{Q}_p}N$ be the constant group scheme over $\mathbb{C}$ whose $\mathbb{C}-$points are the inverse image of the surjective homomorphism

$$N_{\hat{G}}(\hat{T})(\mathbb{C}) \to W(G,T)(F) < W(G,T)(\mathbb{C}).$$

Lemma 2.3.31. *The following algebraic isomorphism of affine schemes holds:*

$$\hat{T} \rtimes \mathrm{Fr}/_{\mathbb{Q}_p}N \simeq (\hat{G} \rtimes \mathrm{Fr})/\sim .$$

Proof. We refer the reader to [GH22, (7.23) p. 173]. □

2.3.5 Unramified representations of tori

We denote by $T_K = T(\mathbb{Z}_p)$ the maximal compact subgroup of $T(\mathbb{Q}_p)$.

For the following exposition we will recall that it is known that a character $\chi \in X^\bullet(T) = \mathrm{Hom}(T, \mathbb{G}_m)$ defines a continuous homomorphism $\mathrm{Hom}(T(\mathbb{Q}_p), \mathbb{Q}_p^\times)$. In fact

$$\mathrm{Hom}(T(\mathbb{Q}_p), \mathbb{Q}_p^\times) = \mathrm{Hom}(\mathrm{Hom}(\mathrm{Spec}(\mathbb{Q}_p), T), \mathrm{Hom}(\mathrm{Spec}(\mathbb{Q}_p), \mathbb{G}_m)). \tag{2.-7}$$

Definition 2.3.32. A continuous character $T(\mathbb{Q}_p) \to \mathbb{C}^\times$ is *unramified* if it is trivial on $T(\mathbb{Q}_p) \cap T_K$.

To conclude the subsection we will examine the connection between $X^\bullet(T)$ and the unramified characters. Let us define a map

$$H_T:\ T(\mathbb{Q}_p) \to \mathrm{Hom}(X^\bullet(T), \mathbb{R}), \tag{2.-7}$$

such that $e^{\langle H_T(t), \chi\rangle} = |\chi(t)|_p$ for every $\chi \in X^\bullet(T)$.

Lemma 2.3.33. *For each* $\lambda \in X^\bullet(T) \otimes_{\mathbb{Z}} \mathbb{C}$ *the map*

$$t \mapsto e^{\langle H_T(t), \lambda\rangle},$$

is an unramified character.

Proof. See [GH22, p. 179]. □

Lemma 2.3.34. *The group of unramified characters satisfy*

$$\mathrm{Hom}(T(\mathbb{Q}_p)/T_K, \mathbb{C}^\times) \simeq \mathrm{Hom}(T_s(\mathbb{Q}_p)/T_{s,K}, \mathbb{C}^\times) \simeq X^\bullet(T_s) \otimes_{\mathbb{Z}} \mathbb{C}.$$

Proof. It is [GH22, lem. 7.6.1, p. 179]. □

Lemma 2.3.35. *Fix $E/\mathbb{Q}$ be a totally imaginary quadratic field extension with $E = \mathbb{Q}(\sqrt{d})$, p a prime number and let us denote by $\mathbb{G}_m$ the $1-$dimensional torus defined over E_p. Given ξ an unramified character of $\mathrm{Res}_{E_p/\mathbb{Q}_p}(\mathbb{G}_m)(\mathbb{Q}_p)$ there exists a unramified character $\tilde{\xi} : \mathbb{Q}_p^\times/\mathbb{Z}_p^\times \to \mathbb{C}^\times$ so that*

$$\xi = \tilde{\xi} \circ \mathrm{N}_{E_p/\mathbb{Q}_p}.$$

Proof. The spaces of unramified characters of $\mathrm{Res}_{E_p/\mathbb{Q}_p}(\mathbb{G}_m)(\mathbb{Q}_p)$ and $\mathbb{Q}_p^\times$ are both $1-$dimensional, since given an unramified character $\xi : \mathbb{Q}_p^\times \to \mathbb{C}^\times$, the map $\xi \circ \mathrm{N}_{E_p/\mathbb{Q}_p}$ is an unramified character of $\mathrm{Res}_{E_p/\mathbb{Q}_p}(\mathbb{G}_m)(\mathbb{Q}_p)$, the result follows. □

2.3.6 Unramified principal series

Theorem 1.4.9 shows that an automorphic representation is isomorphic to the restricted tensor product of infinitely many local representations, so that infinitely many of them are unramified. The classification of this kind of representations provides a way to understand the structure of the automorphic representations. From now on we will assume that p is a finite prime.

Definition 2.3.36. If ξ is an unramified character of $T(\mathbb{Q}_p)$ the representation $\mathrm{Ind}_B^G \xi$ is called *unramified principal series.*

Theorem 2.3.37. *Let π be a unramified representation. Then there exists an unramified character ξ so that π is a subrepresentation of $\mathrm{Ind}_B^G \xi$.*

Proof. It follows by [GH22, thm. 7.6.6, p. 182] and [GH22, thm. 7.6.7, p. 182] or by [Cas80, prop. 2.6, p. 396]. □

Proposition 2.3.38. *The unramified principal series satisfy the following statements:*

- *For any unramified character ξ, the representation $\mathrm{Ind}_B^G \xi$ is admissible.*
- *The contragradient satisfies $\widetilde{\mathrm{Ind}_B^G \xi} \simeq \mathrm{Ind}_B^G \xi^{-1}$.*
- *If ξ is a unitary character, the representation $\mathrm{Ind}_B^G \xi$ is unitary.*
- *For any $\omega \in W(G, T_s)$, if the representations $\mathrm{Ind}_B^G \xi$ and $\mathrm{Ind}_B^G\, {}^\omega\xi$ are irreducible then they are isomorphic as $G(\mathbb{Q}_p)-$representations.*
- *The $\mathcal{H}_{\mathbb{C},p}^G-$module $\mathrm{Ind}_B^G \xi$ has finite length.*
- *The representation $\mathrm{Ind}_B^G \xi$ admits a unique irreducible unramified sub-quotient.*

Proof. We refer the reader to [Car79, thm. 3.2, p. 136], [Car79, thm. 3.3, p. 137] and [GH22, lem. 7.6.5, p. 181]. □

2.3.7 The structure of unramified principal series

Unramified principal series are described by the combination of the central theorem 2.3.39 and Frobenius reciprocity, see proposition 2.3.15. In this subsection we will explain how those results are used to deduce certain key properties of these representations.

Theorem 2.3.39. *Let Δ be a choice of a set of simple roots of G, Ω a subset of Δ and ξ a character of $T(\mathbb{Q}_p)$. There exists a filtration*

$$0 \subseteq I_{n_l} \subseteq ... \subseteq I_0 = \mathrm{Ind}_B^G \xi,$$

by $P^\Omega(\mathbb{Q}_p)-$stable subspaces so that

$$(I_n/I_{n+1})_{N^\Omega} \simeq (I_n)_{N^\Omega}/(I_{n+1})_{N^\Omega} \simeq \bigoplus_{\substack{w \in W(G,T_s)(\mathbb{Q}_p)/W_\Omega \\ d(w)=n}} (J_\omega)_{N^\Omega},$$

where

$$(J_\omega)_{N^\Omega} \simeq \mathrm{Ind}_{\omega^{-1}B\omega\cap M^\Omega}^{M^\Omega} {}^{\omega}\xi,$$

and the group W_Ω is defined as in proposition 2.1.34.

Proof. It is [Cas08, thm. 6.3.5, p. 59] with $\Theta = \{\varnothing\}$. □

Remark 2.3.40. *If we take $\Omega = \{\varnothing\}$, theorem 2.3.39 shows that the irreducible constituents of $\left(\mathrm{Ind}_B^G\xi\right)_U$ are equal to $\delta_B^{1/2}\,{}^{\omega}\xi$, with $\omega \in W(G,T_s)(\mathbb{Q}_p)$*

Proposition 2.3.41. *Let ξ be an unramified character of $T(\mathbb{Q}_p)$. Given $\omega \in W(G,T_s)$, the semisimplifications of $\mathrm{Ind}_B^G\xi$ and $\mathrm{Ind}_B^G\,{}^{\omega}\xi$ are isomorphic.*

Proof. See [Cas08, thm. 6.3.11, p. 61]. □

Definition 2.3.42. A character $\xi : T(\mathbb{Q}_p) \to \mathbb{C}^\times$ is called *regular* if for any $w \in W(G,T_s)$, the equality $w\cdot\xi = \xi$ implies that $w = id$.

Proposition 2.3.43. *Let ξ be a unitary regular unramified character, then $\mathrm{Ind}_B^G\xi$ is irreducible.*

Proof. By proposition 2.3.38, the representation $\mathrm{Ind}_B^G\xi$ is unitary of finite length. Then, using 2.3.9 we show that the representation $\mathrm{Ind}_B^G\xi$ is a finite direct sum of irreducible representations. Therefore it is irreducible if and only if $\mathrm{End}_{G(\mathbb{Q}_p)}\left(\mathrm{Ind}_B^G\xi\right) = \mathbb{C}$. By proposition 2.3.15

$$\mathrm{End}_{G(\mathbb{Q}_p)}\left(\mathrm{Ind}_B^G\xi\right) = \mathrm{Hom}_{B(\mathbb{Q}_p)}\left(\left(\mathrm{Ind}_B^G\xi\right)_U, \xi\right).$$

Using remark 2.3.40 the irreducible constituents of $\left(\mathrm{Ind}_B^G\xi\right)_U$ are of the form $\delta_B^{1/2}\,{}^{\omega}\xi$ for $\omega \in W$ and we conclude with the equality

$$\mathrm{End}_{G(\mathbb{Q}_p)}\left(\mathrm{Ind}_B^G\xi\right) \leq \left|\mathrm{Stab}_{W(G,T_s)}\xi\right|.$$

□

Along this dissertation, the unramified representations of $\mathbf{G}$ play a key role. In the following lemma, using 2.3.43 we find a criterion for the irreducibility of these representations.

Lemma 2.3.44. *If $\xi = \xi_1 \otimes \xi_2 \otimes \xi_0$ is a unitary unramified character satisfying*

- $\xi_2|_{\mathbb{Q}_p^\times} \neq 1$,
- $\xi_1 \neq \xi_2$,
- $\xi_1|_{\mathbb{Q}_p^\times} \neq 1$ *or* $\xi_1 \neq \xi_2^{-1}$ *or* $\xi_2(b)\xi_1(\bar{b}) \neq 1$,
- $\xi_2|_{\mathbb{Q}_p^\times} \neq 1$ *or* $\xi_1 \neq \xi_2$ *or* $\xi_1(a)\xi_2(\bar{a}) \neq 1$,
- $\xi_1|_{\mathbb{Q}_p^\times} \neq \xi_2^{-1}|_{\mathbb{Q}_p^\times}$ *or* $\xi_1(a)\xi_2(\bar{a}) \neq 1$,
- $\xi_1|_{\mathbb{Q}_p^\times} \neq 1$ *or* $\xi_1(a\bar{a}) \neq 1$,
- $\xi_1|_{\mathbb{Q}_p^\times} \neq \xi_2|_{\mathbb{Q}_p^\times}^{-1}$, $\xi_1(a\bar{a}) \neq 1 \neq \xi_2(a\bar{a})$,

the representation $\mathrm{Ind}_{B_{\mathbf{G}}}^{\mathbf{G}}\xi$ is irreducible.

Proof. We take the following representatives of the Weyl group:

$$W(\mathbf{G},T_s)(\mathbb{Q}_p) = \{s_1,\ s_2,\ s_1s_2,\ s_2s_1,\ s_2s_1s_2,\ s_1s_2s_1,\ s_2s_1s_2s_1\}.$$

We fix the following notation $t(a,b,\nu) = \begin{pmatrix} a &&& \\ & b && \\ && \nu\bar{b}^{-1} & \\ &&& \nu\bar{a}^{-1}\end{pmatrix}$, and we proceed by cases

- It holds that $s_2^{-1}t(a,b,\nu)s_2 = \begin{pmatrix} a & & & \\ & \nu\overline{b}^{-1} & & \\ & & b & \\ & & & \nu\overline{a}^{-1} \end{pmatrix}$. Then $s_2 \cdot \xi = \xi$ if and only if $\xi_1(a)\xi_2(b)\xi_0(\nu) = \xi_1(a)\xi_2(\nu\overline{b}^{-1})\xi_0(\nu)$, which implies that $\xi_2|_{\mathbb{Q}_p^\times} = 1$.

- Furthermore $s_1^{-1}t(a,b,\nu)s_1 = \begin{pmatrix} b & & & \\ & a & & \\ & & \nu\overline{a}^{-1} & \\ & & & \nu\overline{b}^{-1} \end{pmatrix}$. The character ξ satisfies that $s_1 \cdot \xi = \xi$ if and only if $\xi_1(a)\xi_2(b)\xi_0(\nu) = \xi_1(b)\xi_2(a)\xi_0(\nu)$ that is equivalent to $\xi_1 = \xi_2$.

- We obtain $s_1^{-1}s_2^{-1}t(a,b,\nu)s_2s_1 = \begin{pmatrix} \nu\overline{b}^{-1} & & & \\ & a & & \\ & & \nu\overline{a}^{-1} & \\ & & & b \end{pmatrix}$. It holds that $s_2s_1 \cdot \xi = \xi$ if and only if $\xi_1(a)\xi_2(b)\xi_0(\nu) = \xi_1(\nu\overline{b}^{-1})\xi_2(a)\xi_0(\nu)$, that is equivalent to $\xi_1|_{\mathbb{Q}_p^\times} = 1$, $\xi_1 = \xi_2^{-1}$ and $\xi_2(b)\xi_1(\overline{b}) = 1$.

- It follows that $s_2^{-1}s_1^{-1}t(a,b,\nu)s_1s_2 = \begin{pmatrix} b & & & \\ & \nu\overline{a}^{-1} & & \\ & & a & \\ & & & \nu\overline{b}^{-1} \end{pmatrix}$, then $s_1s_2 \cdot \xi = \xi$ if and only if $\xi_1(a)\xi_2(b)\xi_0(\nu) = \xi_1(b)\xi_2(\nu\overline{a}^{-1})\xi_0(\nu)$, that implies $\xi_2|_{\mathbb{Q}_p^\times} = 1$, $\xi_1\xi_2^{-1} = 1$ and $\xi_1(a)\xi_2(\overline{a}) = 1$.

- Furthemore $s_2^{-1}s_1^{-1}s_2^{-1}t(a,b,\nu)s_2s_1s_2 = \begin{pmatrix} \nu\overline{b}^{-1} & & & \\ & \nu\overline{a}^{-1} & & \\ & & a & \\ & & & b \end{pmatrix}$, hence $s_1s_2s_1 \cdot \xi = \xi$ if and only if $\xi_1(a)\xi_2(b)\xi_0(\nu) = \xi_1(\nu\overline{b}^{-1})\xi_2(\nu\overline{a}^{-1})\xi_0(\nu)$. Then $s_2s_1s_2 \cdot \xi = \xi$ implies that $\xi_1|_{\mathbb{Q}_p^\times} = \xi_2^{-1}|_{\mathbb{Q}_p^\times}$ and $\xi_1(a)\xi_2(\overline{a}) = 1$.

- We obtain $s_1^{-1}s_2^{-1}s_1^{-1}t(a,b,\nu)s_1s_2s_1 = \begin{pmatrix} \nu\overline{a}^{-1} & & & \\ & b & & \\ & & \nu\overline{b}^{-1} & \\ & & & a \end{pmatrix}$ then $s_1s_2s_1 \cdot \xi = \xi$ implies that $\xi_1(a)\xi_2(b)\xi_0(\nu) = \xi_1(\nu\overline{a}^{-1})\xi_2(b)\xi_0(\nu)$ that is equivalent to $\xi_1|_{\mathbb{Q}_p^\times} = 1$ and $\xi_1(a\overline{a}) = 1$.

- Moreover $s_1^{-1}s_2^{-1}s_1^{-1}s_2^{-1}t(a,b,\nu)s_2s_1s_2s_1 = \begin{pmatrix} \nu\overline{a}^{-1} & & & \\ & \nu\overline{b}^{-1} & & \\ & & b & \\ & & & a \end{pmatrix}$. $s_2s_1s_2s_1 \cdot \xi = \xi$ is equivalent to $\xi_1(a)\xi_2(b)\xi_0(\nu) = \xi_1(\nu\overline{a}^{-1})\xi_2(\nu\overline{b}^{-1})\xi_0(\nu)$, implying $\xi_1|_{\mathbb{Q}_p^\times} = \xi_2|_{\mathbb{Q}_p^\times}^{-1}$, $\xi_1(a)\xi_1(\overline{a}) = \xi_2(a)\xi_2(\overline{a}) = 1$.

Applying 2.3.43 we obtain the result. □

2.3.8 Distributions of unramified principal series.

In the theory of automorphic representations there are certain relevant functionals that are characterized by their symmetries. Their study and computation are essential to provide integral representations for $L-$functions. This subsection will be devoted to the study of unramified principal series an harmonic analysis perspective, we refer the reader to [Mac71] for a detailed exposition. It allows us to study in a more precise way the aforementioned functionals.

Throughout this subsection we fix a regular unramified character ξ of $T(\mathbb{Q}_p)$. Let us denote by $\mathcal{C}_c^\infty(G(\mathbb{Q}_p))$ the ring of locally constant smooth functions on $G(\mathbb{Q}_p)$. We consider the representation

$$\begin{aligned} R:\ G(\mathbb{Q}_p) &\to \mathrm{Aut}\left(\mathcal{C}_c^\infty(G(\mathbb{Q}_p))\right), \\ g &\mapsto \left(f(\cdot) \mapsto f(\cdot g)\right). \end{aligned}$$

Proposition 2.3.45. *The map*

$$\begin{aligned} \mathcal{P}_\xi:\ \mathcal{C}_c^\infty(G(\mathbb{Q}_p)) &\to \mathrm{Ind}_B^G \xi, \\ f &\mapsto \int_{B(\mathbb{Q}_p)} \xi^{-1}\delta^{1/2}(b) f(bg) fb, \end{aligned}$$

is surjective and $G(\mathbb{Q}_p)-$intertwining.

Proof. See [Cas80]. □

Lemma 2.3.46. *The functions* $\phi_{K,\xi} := \mathcal{P}_\xi\left(\mathrm{char}_{G(\mathbb{Z}_p)}\right)$ *form a basis of* $\left(\mathrm{Ind}_B^G\xi\right)^{G(\mathbb{Z}_p)}$.

Proof. It follows by the Iwasawa decomposition, see [Cas80, cor. 2.2, p. 394]. □

For the following discussion we recall that proposition 2.1.65 implies

$$G(\mathbb{Z}_p) = \bigsqcup_{\omega \in W(G,T_s)(\mathbb{Q}_p)} I\omega I.$$

For each $\omega \in W(G,T)$, we consider the function $\phi_{\omega,\xi} := \mathcal{P}_\xi\left(\mathrm{char}_{I\omega I}\right)$. If there is no danger of confusion, the character ξ we will suppress the it from the notation.

Lemma 2.3.47. *The set of functions* $\{\phi_\omega\}_{\omega\in W(G,T_s)}$ *is a basis of* $\left(\mathrm{Ind}_B^G\xi\right)^I$

Proof. It follows by proposition 2.1.65. □

Proposition 2.3.48. *The canonical projection induces the isomorphism*

$$\left(\mathrm{Ind}_B^G\xi\right)^I \simeq \left(\mathrm{Ind}_B^G\xi\right)_N.$$

Proof. By [Cas08, p. 55] there exists a decreasing filtration

$$0 \subseteq I_{\omega_n} \subseteq ... \subseteq I_{\omega_0} = \mathrm{Ind}_B^G\xi,$$

labelled by the elements ω_i of the Weyl group $W(G,T_s)(\mathbb{Q}_p)$. Proposition [Cas08, lem. 1.3, (a), p. 393] shows that the element ϕ_ω obtained in lemma 2.3.47 belongs to the subspace I_ω. We denote by $J_\omega = \{f \in \mathrm{Ind}_B^G\xi,\ s.t.\ f\ \textit{has compact support and}\ supp(f) \subseteq B\omega B\}$. aWe have $\phi_\omega \in I_\omega / \bigoplus_{\substack{x>\omega \\ x\neq\omega}} J_x$. Proposition [Cas08, prop. 6.3.1, p. 56] implies the following isomorphism

$$I_\omega / \bigoplus_{\substack{x>\omega \\ x\neq\omega}} J_x \simeq J_\omega.$$

Combining theorem 2.3.39, remark 2.3.40 and [Cas08, prop. 6.3.2, p. 56] we obtain

$$\left(I_\omega / \bigoplus_{\substack{x>\omega \\ x\neq\omega}} J_x\right)_U \simeq \delta_B^{1/2}\ {}^\omega\xi. \tag{2.-7}$$

We remark that the right hand side of the above isomorphism is a 1−dimensional space isomorphic to $\mathbb{C}$. Furthermore, the map (2.3.8) is described explicitly by means of the integral

$$\phi \mapsto \int_{\omega^{-1}U(\mathbb{Q}_p)\omega\cap U(\mathbb{Q}_p)\backslash U(\mathbb{Q}_p)} \phi(\omega n)dn.$$

By [Cas08, lem. 1.3, (b), p. 393] $\int_{\omega^{-1}U(\mathbb{Q}_p)\omega\cap U(\mathbb{Q}_p)\backslash U(\mathbb{Q}_p)} \phi_\omega(\omega n)dn \neq 0$, therefore using lemma 2.3.47 we conclude the proof. □

Lemma 2.3.49. *The maps* $\phi \mapsto \int_{\omega^{-1}U(\mathbb{Q}_p)\omega\cap U(\mathbb{Q}_p)\backslash U(\mathbb{Q}_p)} \phi(\omega n)dn$ *with* $\omega \in W(G,T_s)(\mathbb{Q}_p)$ *form a basis of the dual space of* $\left(\mathrm{Ind}_B^G\xi\right)_U$.

Proof. It follows by the proof of proposition 2.3.48. □

Definition 2.3.50. The *space of distributions* is defined by

$$\mathcal{D}(G(\mathbb{Q}_p)) := \mathrm{Hom}_{\mathbb{C}}\left(\mathcal{C}_c^\infty(G(\mathbb{Q}_p)), \mathbb{C}\right).$$

We denote by $\langle\cdot,\cdot\rangle$ the pairing between $\mathcal{C}_c^\infty(G(\mathbb{Q}_p))$ and $\mathcal{D}(G(\mathbb{Q}_p))$.

The group $G(\mathbb{Q}_p)$ acts on $\mathcal{D}(G(\mathbb{Q}_p))$ by its natural action: given $T \in \mathcal{D}(G(\mathbb{Q}_p))$, the distribution $\rho(g)T$ is defined so that

$$\langle \phi, \rho(g)T\rangle = \langle \rho(g)\phi, T\rangle,$$

for every $\phi \in \mathcal{C}_c^\infty(G(\mathbb{Q}_p))$.

Remark 2.3.51. *The map $\mathcal{P}_\xi$ induces an injective map*

$$\begin{aligned}\mathcal{P}_\xi^* : \left(\mathrm{Ind}_B^G\xi\right)^* &\to \mathcal{D}(G(\mathbb{Q}_p)),\\ F &\mapsto F(\mathcal{P}_\xi(\cdot)).\end{aligned}$$

Lemma 2.3.52. *We have $G(\mathbb{Q}_p)-$intertwining isomorphism of the form*

$$\mathcal{P}_\xi^* : \left(\mathrm{Ind}_B^G\xi\right)^* \to \mathcal{D}(G)_{\xi^{-1}},$$

where

$$\mathcal{D}(G)_{\xi^{-1}} := \{T \in \mathcal{D}(G),\ \rho(b)T = \xi^{-1}\delta^{1/2}(b)T\}.$$

Proof. See [Hir99, lem. 1.2, p. 558]. □

Remark 2.3.53. *By lemma 2.3.52 there is an injective $G(\mathbb{Q}_p)-$intertwining map*

$$\widetilde{\mathrm{Ind}_B^G\xi} \to \mathcal{D}(G)_{\xi^{-1}}. \tag{2.-7}$$

Combining the isomorphism of proposition 2.3.38

$$\widetilde{\mathrm{Ind}_B^G\xi} \simeq \mathrm{Ind}_B^G\xi^{-1},$$

and the map (2.3.53),we may regard the elements of $\mathrm{Ind}_B^G\xi^{-1}$ as distributions.

2.3.9 Intertwining maps between unramified principal series.

Throughout this subsection we fix an unramified connected reductive group G defined over $\mathbb{Q}_p$ with maximal torus T so that the relative root system (V,Φ) associated to (G,T_s) is reduced in the sense of 2.1.22. These conditions are imposed to simplify the exposition: The following discussion follows in greater generality. For a general discussion of the subject we refer the reader to [Cas08], [CS80] and [Cas80]. For this subsection we fix a regular unramified character $\xi : T(\mathbb{Q}_p) \to \mathbb{C}^\times$, see 2.3.42 for the definition.

Definition 2.3.54. Given $\phi \in \mathrm{Ind}_B^G\xi$ and $\omega \in W(G,T_s)(\mathbb{Q}_p)$ we define

$$T_\omega(\phi)(g) = \int_{\omega U(\mathbb{Q}_p)\omega^{-1}\cap U(\mathbb{Q}_p)\backslash U(\mathbb{Q}_p)} \phi(\omega^{-1}ng)dn,$$

where dn is the Haar measure of $U(\mathbb{Q}_p)$.

Proposition 2.3.55. *Let $\phi \in \mathrm{Ind}_B^G\xi$, if $T_\omega(\phi)(g)$ converges for all $g \in G(\mathbb{Q}_p)$, then*

$$\begin{aligned}T_\omega : \mathrm{Ind}_B^G\xi &\to \mathrm{Ind}_B^G\, {}^\omega\xi,\\ \phi &\mapsto T_\omega(\phi)(g),\end{aligned}$$

is an intertwining map.

Proof. It follows by a change of variables. □

Proposition 2.3.56. *Set $\omega \in W(G, T_s)(\mathbb{Q}_p)$ and ξ an unramified character so that $\mathrm{Re}\,\langle \xi, \alpha^\vee \rangle > 0$ for all $\alpha \in \Phi^+$ with $\omega\alpha \notin \Phi^+$. The intertwining map T_ω converges absolutely for every $f \in \mathrm{Ind}_B^G \xi$.*

Proof. It follows by [Win78, thm. 1, p. 948], [Win78, thm. 3, p. 951] and [Key82, p. 385]. □

In order to simplify the exposition we will assume that the unramified character ξ satisfies the hypothesis of proposition 2.3.56. Since the characters of a torus varies holomorphically in $\mathbb{C}^{\mathrm{rank}(T)}$, the forthcoming results can be extended to characters outside the region of absolute convergence by a process of meromorphic continuation.

For the following discussion let M_α be the $\mathbb{Q}_p-$points of the Levi subgroup associated to a positive root α, see 2.1.31 for the details of the construction. The simply connected cover of the derived subgroup of M_α, denoted by $\tilde{M}_\alpha^{der}$, is connected, semisimple, unramified and of rank one. According to [Cas80, p. frm[o]–0] there are just two groups satisfying these hypothesis, $\mathrm{SL}_2(F)$ and $\mathrm{SU}_3(F)$ with F an unramified extension of $\mathbb{Q}_p$.

Definition 2.3.57. Let F be an unramified extension of $\mathbb{Q}_p$, $\mathcal{O}_F$ its ring of integers and $\mathfrak{p}$ its maximal ideal. Let us consider the group SL_2 defined over F. We define the following coset of $T_{\mathrm{SL}_2}(F)/T_{\mathrm{SL}_2}(\mathcal{O}_F)$

$$a = \begin{pmatrix} \mathfrak{p}-\mathfrak{p}^2 & \\ & (\mathfrak{p}-\mathfrak{p}^2)^{-1} \end{pmatrix}.$$

Let us consider the group SU_3 defined over F. We define the following coset of $T_{\mathrm{SU}_3(F)}/T_{\mathrm{SU}_3(F)}(\mathcal{O}_F)$

$$a = \begin{pmatrix} \mathfrak{p}-\mathfrak{p}^2 & & \\ & 1 & \\ & & (\mathfrak{p}-\mathfrak{p}^2)^{-1} \end{pmatrix}.$$

Given α a positive root of G, a_α is the image of a representative of a throughout the map

$$\tilde{M}_\alpha^{der} \to G(\mathbb{Q}_p).$$

Definition 2.3.58. For every root α so that $\tilde{M}_\alpha^{der} = H(F)$, we denote

$$q_\alpha = |\mathcal{O}_F/\mathfrak{p}\mathcal{O}_F|.$$

Furthermore we denote

$$q_{\alpha/2} = \begin{cases} 1 & if\ H = \mathrm{SL}_2, \\ |\mathcal{O}_F/\mathfrak{p}\mathcal{O}_F|^2 & if\ H = \mathrm{SU}_3 \end{cases}$$

Proposition 2.3.59. *Given $\alpha \in \Phi$ so that $\tilde{M}_\alpha^{der} = \mathrm{SL}_2(F)$ with F an unramified extension of $\mathbb{Q}_p$, we have*

$$q_\alpha := [I\omega_\alpha I : I],$$

where $\omega_\alpha \in W(G, T_s)(\mathbb{Q}_p)$ is the element of the Weyl group associated to the root α.

Proof. Using the alternative definition of these constants given in [Cas80, (12), p. 390], the computation follows parallel to lemmas 2.7.14 and 2.7.16. □

Definition 2.3.60. Given $\alpha \in \Phi^+$ and $\omega \in W(G, T_s)(\mathbb{Q}_p)$ we define

$$c_\alpha^G(\xi) = \frac{(1 - q_{\alpha/2}^{-1/2} q_\alpha^{-1} \xi(a_\alpha))(1 + q_{\alpha/2}^{-1/2} \xi(a_\alpha))}{(1 - q_\alpha^{-1/2} \xi(a_\alpha)^2)},$$

$$c_\omega^G(\xi) = \prod_{\substack{\alpha \in \Phi^+ \\ \omega\alpha < 0}} c_\alpha(\xi).$$

If there is no danger of confusion, we will simply denote them by $c_\alpha(\xi)$ and $c_\omega(\xi)$ respectively.

Remark 2.3.61. *If $\tilde{M}_{\alpha}^{der} = \mathrm{SL}_2(F)$, then*

$$c_{\alpha}(\xi) = \frac{1 - q_{\alpha}^{-1}\xi(a_{\alpha})}{1 - \xi(a_{\alpha})}.$$

Proposition 2.3.62. *The following equality holds:*

$$T_{\omega}\phi_{K,\xi} = c_{\omega}(\xi)\phi_{K,{}^{\omega}\xi},$$

where $c_{\omega}(\xi) = \prod_{\substack{\alpha>0 \\ \omega\alpha<0}} c_{\alpha}(\xi)$ with

Proof. See [Cas80, thm. 3.1, p. 397]. ☐

For the following discussion we will denote the adjoint map of $T_{\omega^{-1}}$ by

$$\begin{aligned} T_{\omega^{-1}}^{*} : \mathcal{D}(G)_{\xi} &\to \mathcal{D}(G)_{{}^{\omega}\xi}, \\ D &\mapsto D(T_{\omega^{-1}}(\cdot)). \end{aligned}$$

Remark 2.3.63. *Using 2.3.53 we have injective maps*

$$\mathrm{Ind}_{B}^{G}\xi \to \mathcal{D}(G)_{\xi}$$
$$\mathrm{Ind}_{B}^{G}\ {}^{\omega}\xi \to \mathcal{D}(G)_{{}^{\omega}\xi}.$$

Furthermore, the definition 2.3.3 implies

$$\mathrm{Ind}_{B}^{G}\xi \simeq \mathcal{D}(G)_{\xi} \cap \mathcal{C}_{c}^{\infty}(G(\mathbb{Q}_p)),\ \mathrm{Ind}_{B}^{G}\ {}^{\omega}\xi \simeq \mathcal{D}(G)_{{}^{\omega}\xi} \cap \mathcal{C}_{c}^{\infty}(G(\mathbb{Q}_p)).$$

Since the map $T_{\omega^{-1}}$ preserves the space of $\mathcal{C}_{c}^{\infty}-$distributions, then

$$T_{\omega^{-1}}^{*}|_{\mathcal{C}_{c}^{\infty}(G(\mathbb{Q}_p))} : \mathrm{Ind}_{B}^{G}\xi \to \mathrm{Ind}_{B}^{G}\ {}^{\omega}\xi,$$

is a bijective $G(\mathbb{Q}_p)-$intertwining map. Since T_{ω} and $T_{\omega^{-1}}^{*}|_{\mathcal{C}_{c}^{\infty}(G(\mathbb{Q}_p))}$ are intertwining maps between irreducible representations, they are proportional. In fact [Hir99, lem. 1.5, p. 561],

$$T_{\omega^{-1}}^{*}|_{\mathcal{C}_{c}^{\infty}(G(\mathbb{Q}_p))} = \frac{c_{\omega^{-1}}({}^{\omega}\xi^{-1})}{c_{\omega}(\xi)}T_{\omega}$$

Theorem 2.3.64. *Fix $\omega \in W(G, T_s)(\mathbb{Q}_p)$. Given $\alpha \in \Delta$ so that $l(\omega_{\alpha}\omega) > l(\omega)$ we obtain*

$$\begin{aligned} T_{\omega_{\alpha}}(\phi_{\omega}) &= (c_{\alpha}(\xi) - 1)\phi_{\omega} + q_{\alpha}^{-1}\phi_{\omega_{\alpha}\omega}, \\ T_{\omega_{\alpha}}(\phi_{\omega_{\alpha}\omega}) &= \phi_{\omega} + (c_{\alpha}(\xi) - q_{\alpha}^{-1})\phi_{\omega_{\alpha}\omega}. \end{aligned}$$

Proof. It is [Cas80, thm. 3.4, p. 400]. ☐

2.3.10 Satake parameter of principal series

Let π be an unramified representation of $G(\mathbb{Q}_p)$. By theorems 2.3.24 and 2.3.29, π is determined by a class of the form $\chi_{\pi} \in \hat{T}(\mathbb{C})/W(G,T)(\mathbb{C})$ or $\chi_{\pi} \in \left(\hat{G} \rtimes \mathrm{Fr}\right) / \sim (\mathbb{C})$, depending wether G is split or not. The explicit computation of such class χ_{π} is relevant for the forthcoming sections. First we show a way to compute this class when G is split and subsequently how to compute this class when G is not split.

Proposition 2.3.65. *There is a bijective correspondence*

$$\{\textit{Irreducible unramified representations}\} \to \{\textit{Unramified characters of } T(\mathbb{Q}_p)\} / \sim,$$

where $\xi_1 \sim \xi_2$ if there exists a $\omega \in W(G, T_s)(\mathbb{Q}_p)$ so that $\xi_1 = {}^{\omega}\xi_2$.

Proof. By 2.3.37, given an irreducible unramified representation π, there exists an unramified character ξ so that

$$\pi \to \mathrm{Ind}_B^G \xi,$$

is an injective map. We define the correspondence of the statement by

$$\pi \mapsto \xi. \tag{2.-7}$$

The rest of the proof is devoted to showing that the correspondence is well defined and bijective modulo the action of the Weyl group.

Let us suppose that there are two different characters ξ_1 and ξ_2 so that $\pi \mapsto \xi_1$ and $\pi \mapsto \xi_2$. Propositions 2.3.38 and 2.3.41 imply that $\xi_1 = {}^{\omega}\xi_2$, therefore the correspondence is well defined. By proposition 2.3.38, for every unramified character ξ of $T(\mathbb{Q}_p)$ the representation $\mathrm{Ind}_B^G \xi$ has a unique irreducible unramified sub-quotient, then (2.3.10) is surjective. The map (2.3.10) is injective because by proposition 2.3.38 π is the only unramified sub-quotient of $\mathrm{Ind}_B^G \xi$. $\square$

Proposition 2.3.66. *Let (G,T) be a split reductive group. There is a bijective correspondence*

$$\{\textit{Unramified characters of } T(\mathbb{Q}_p)\}/\sim \to \hat{T}(\mathbb{C})/W(G,T)(\mathbb{C}),$$
$$\xi \mapsto t_\xi.$$

Proof. By [Sha81, p. 25] the map H_T given in (2.3.5) defines an exact sequence

$$1 \to T(\mathbb{Z}_p) \to T(\mathbb{Q}_p) \xrightarrow{H_T} X_\bullet(T) \to 1,$$

Therefore there exists a $W(G,T_s)(\mathbb{Q}_p)$-equivariant group isomorphism of the form

$$\mathrm{Hom}(T(\mathbb{Q}_p)/T(\mathbb{Z}_p), \mathbb{C}^\times) \simeq \mathrm{Hom}(X_\bullet(T), \mathbb{C}^\times).$$

Taking the $W(G,T_s)(\mathbb{Q}_p)-$conjugacy classes in both sides we have

$$\mathrm{Hom}(T(\mathbb{Q}_p)/T(\mathbb{Z}_p), \mathbb{C}^\times)^{W(G,T)(\mathbb{Q}_p)} \simeq \mathrm{Hom}(X_\bullet(T), \mathbb{C}^\times)^{W(G,T)}.$$

We conclude by recalling the isomorphism $\hat{T}(\mathbb{C})/W(G,T)(\mathbb{C}) = \mathrm{Hom}(X_\bullet(T), \mathbb{C}^\times)^{W(G,T)}$. $\square$

For the next lemma we recall that there is an isomorphism

$$X^\bullet(\hat{T}_s) \to X_\bullet(T_s),$$
$$\varphi \mapsto \tilde{\varphi}.$$

Lemma 2.3.67. *Let $\xi \mapsto t_\xi$ be the corresspondence of proposition 2.3.66. The conjugacy class t_ξ is determined by*

$$\xi(\tilde{\varphi}(p)) = \varphi(t_\xi),$$

for every $\tilde{\varphi} \in X_\bullet(T)$.

Proof. See [Sha81, p. 25]. $\square$

Remark 2.3.68. *Using the above discussion, when G is split the Satake parameter of the unique unramified irreducible sub-quotient of $\mathrm{Ind}_{B_G}^G \xi$ is given by t_ξ.*

In the non split case, due to the action of the Galois group, an extra consideration is needed.

Proposition 2.3.69. *Let G be an unramified reductive group. There is a bijective correspondence*

$$\{\textit{Unramified characters of } T(\mathbb{Q}_p)\}/\sim \to \hat{T}_s(\mathbb{C})/W(G,T_s)(\mathbb{C}),$$
$$\xi \mapsto t_\xi.$$

Proof. Let us consider the following exact sequence

$$1 \to T(\mathbb{Z}_p) \to T(\mathbb{Q}_p) \xrightarrow{H_T} \Lambda(T) \to 1,$$

where $\Lambda(T) = \operatorname{im}(H_T)$. By [Sha81, p. 29] $\Lambda(T) = X_\bullet(T)^{\operatorname{Gal}(\mathbb{Q}_p^{sep}/\mathbb{Q}_p)} = X_\bullet(T_s)$. Therefore

$$\operatorname{Hom}\left(T(\mathbb{Q}_p)/T(\mathbb{Z}_p), \mathbb{C}^\times\right) = \operatorname{Hom}\left(X_\bullet(T_s), \mathbb{C}^\times\right).$$

Proceeding as in the proof of proposition 2.3.66, the statement follows. □

Lemma 2.3.70. *Let $\xi \mapsto t_\xi$ be the correspondence given in proposition 2.3.69. The element t_ξ is characterized by*

$$\xi(\tilde{\varphi}(p)) = \varphi(t_\xi),$$

for every $\tilde{\varphi} \in X_\bullet(T_s)$.

Proof. By lemma 2.3.34

$$\operatorname{Hom}\left(T(\mathbb{Q}_p)/T(\mathbb{Z}_p), \mathbb{C}^\times\right) \simeq \operatorname{Hom}\left(T_s(\mathbb{Q}_p)/T_s(\mathbb{Z}_p), \mathbb{C}^\times\right).$$

Using lemma 2.3.67 in the exact sequence

$$1 \to T_s(\mathbb{Z}_p) \to T(\mathbb{Q}_p) \xrightarrow{H_T} X_\bullet(T_s) \to 1,$$

we obtain the statement. □

Let π be an unramified representation. Proposition 2.3.69 allows us to classify unramified characters of the torus in terms of the points of

$$\hat{T}_s(\mathbb{C})/W(G,T_s)(\mathbb{C}). \tag{2.-7}$$

We may summarize the the above results with the following diagram

$$\begin{tikzcd}
\{\textit{Unramified representations}\} \arrow[r, "2.3.65"] \arrow[d, "2.3.29"] & \{\textit{Unramified characters of } T(\mathbb{Q}_p)\}/\sim \arrow[d, "2.3.69"] \\
(\hat{G} \rtimes \mathrm{Fr})/\sim(\mathbb{C}) & \hat{T}_s(\mathbb{C})/W(G,T_s)(\mathbb{C})
\end{tikzcd}$$

The determination of the Satake parameter χ_π is then equivalent to the definition of a map of the form

$$\hat{T}_s(\mathbb{C})/W(G,T_s)(\mathbb{C}) \to (\hat{G} \rtimes \mathrm{Fr})/\sim(\mathbb{C}),$$

which makes the above diagram commutative.

We recall that the injective map

$$i:\ X_\bullet(T_s) = X^\bullet(\hat{T}_s) \to X_\bullet(T) = X^\bullet(\hat{T}),$$

induces a surjective morphism

$$i^*:\ \hat{T}(\mathbb{C}) \to \hat{T}_s(\mathbb{C}).$$

Remark 2.3.71. *Since the dual group of T is equal to the dual group of $T_{\overline{\mathbb{Q}_p}}$, we may simplify the previous morphisms moving the computations to the algebraic closure, where all the groups are split. Observe that the natural inclusions define the following diagram*

$$\begin{tikzcd}
X_\bullet(T_s) \arrow[r] \arrow[d, "="] & X_\bullet(T) \arrow[r] \arrow[d, "="] & X_\bullet(T_{\overline{\mathbb{Q}_p}}) \arrow[d, "="] \\
X^\bullet(\hat{T}_s) \arrow[r] & X^\bullet(\hat{T}) \arrow[r, "="] & X^\bullet(\hat{T}_{\overline{\mathbb{Q}_p}})
\end{tikzcd}$$

The map $i:\ X^\bullet(\hat{T}_s)\to X^\bullet(\hat{T})$ is equal to the map $X_\bullet(T_s)\to X_\bullet(T_{\overline{\mathbb{Q}_p}})$. Since $T_{\overline{\mathbb{Q}_p}}$ is split, the equality $X_\bullet(T_{\overline{\mathbb{Q}_p}})=X^\bullet(\hat{T})$ and the morphism i^* are easier to determine than in the non split case.

Proposition 2.3.72. *The map*

$$\begin{aligned}\nu:\ \hat{T}(\mathbb{C})\rtimes \mathrm{Fr}&\to \hat{T}_s(\mathbb{C}),\\ t\times \mathrm{Fr}&\mapsto i^*(t),\end{aligned}$$

induces a bijection

$$\overline{\nu}:\ \hat{T}(\mathbb{C})\rtimes \mathrm{Fr}/_{\mathbb{Q}_p}N\to \hat{T}_s(\mathbb{C})/W(G,T_s)(\mathbb{C}).$$

Proof. See [BC80b, lem. 6.4, p. 47]. □

Remark 2.3.73. *The Satake parameter χ_π is equal to $\overline{\nu}^{-1}(t_\xi)$.*

Combining proposition 2.3.72 with lemma 2.3.31 we obtain a method to represent our Satake parameter in terms of the element t_ξ. To conclude this subsection we will give a some examples.

Example 2.3.74. *Let $\xi=\xi_1\otimes ...\otimes\xi_n$ be a character of $T_{\mathrm{GL}_n}(\mathbb{Q}_p)$, a representative of the Satake parameter of the unique irreducible sub-quotient of $\mathrm{Ind}_{B_{\mathrm{GL}_n}}^{\mathrm{GL}_n}\xi$ is given by*

$$\begin{pmatrix}\xi_1(p) & & \\ & \ddots & \\ & & \xi_n(p)\end{pmatrix}\in \mathrm{GL}_n(\mathbb{C}).$$

Example 2.3.75. *Let σ_p be the unique unramified sub-quotient of $\mathrm{Ind}_{B_{\mathbf{H}}(\mathbb{Q}_p)}^{\mathbf{H}(\mathbb{Q}_p)}\nu$, with $\nu=\nu_1\otimes\nu_2\otimes\nu_0$. Where those characters are defined as*

$$\nu:\begin{pmatrix}a & & & \\ & b & & \\ & & \lambda b^{-1} & \\ & & & \lambda a^{-1}\end{pmatrix}\mapsto \nu_1(a)\nu_2(b)\nu_0(\lambda).$$

Furthermore let us suppose that σ_p has central character. Therefore we can regard σ_p as a representation of $\mathbf{PH}$, which satisfies

$$^L\mathbf{PH}=\mathrm{Sp}_4(\mathbb{C}).$$

In [RS07, (2.28), p. 52] the author computes the semisimple conjugacy class attached to this representation:

$$\chi_{\sigma_p}=\begin{pmatrix}\nu_1\nu_2\nu_0(p) & & & \\ & \nu_1\nu_0(p) & & \\ & & \nu_2\nu_0)(p) & \\ & & & \nu_0(p)\end{pmatrix}\in {}^L\mathbf{PH}.$$

Example 2.3.76. *Let us consider the general unitary group $\mathbf{G}$ defined over $\mathbb{Q}_p$ with p a non split prime of $\mathbb{Q}$ over E. Let $\xi=\xi_1\otimes\xi_2\otimes\xi_0$ be an unramified character of $T_{\mathbf{G}}(\mathbb{Q}_p)$, we consider σ be the unique irreducible sub-quotient of $\mathrm{Ind}_{B_{\mathbf{G}}}^{\mathbf{G}}\xi$. Along this example we will explain how to construct the Satake parameter of σ using the ideas described in remark 2.3.71.*

Since $T_{\mathbf{G}}\simeq \mathrm{Res}_{E_p/\mathbb{Q}_p}\mathbb{G}_{m,E_p}\times\mathbb{G}_{m,\mathbb{Q}_p}$, we have $T_{\mathbf{G},\overline{\mathbb{Q}_p}}\simeq\mathbb{G}^5_{m,\overline{\mathbb{Q}_p}}$. The $\overline{\mathbb{Q}_p}-$points of the torus are realized in $\mathbf{G}(\overline{\mathbb{Q}_p})$ as follows:

$$T_{\mathbf{G}}(\overline{\mathbb{Q}_p})=\left\{\begin{pmatrix}a & & & \\ & b & & \\ & & c & \\ & & & d\end{pmatrix}\times j\in \mathrm{GL}_4(\overline{\mathbb{Q}_p})\times\mathrm{GL}_1(\overline{\mathbb{Q}_p}),\ a,b,c,d,j\in\overline{\mathbb{Q}_p}^\times\right\}.$$

Using the proof of proposition 2.2.5 we obtain that the inclusion $T_{\mathbf{G},s} \to T_{\mathbf{G},\overline{\mathbb{Q}_p}}$ is given by

$$\begin{gathered} T_{\mathbf{G},s}(\mathbb{Q}_p) \to T_{\mathbf{G}}(\overline{\mathbb{Q}_p}), \\ \begin{pmatrix} a &&& \\ & b && \\ && \mu b^{-1} & \\ &&& \mu a^{-1} \end{pmatrix} \mapsto \begin{pmatrix} a &&& \\ & b && \\ && 1 & \\ &&& 1 \end{pmatrix} \times \mu. \end{gathered}$$

at the level of points. Let us denote by $\{e_i\}_{i=0,1,2}$ the usual basis of $X^\bullet(\hat{T}_s)$ and by $\{f_i\}_{i=1,\dots,5}$ the usual basis of $X^\bullet(\hat{T}_{\mathbf{G},\overline{\mathbb{Q}_p}})$. The map (2.3.76) provides the injective map

$$X^\bullet(\hat{T}_{\mathbf{G},s}) \to X^\bullet(\hat{T}_{\mathbf{G},\overline{\mathbb{Q}_p}}),$$

defined by $e_1 \mapsto f_1$, $e_2 \mapsto f_2$ and $e_0 \mapsto f_5$, producing the surjective morphism

$$\begin{gathered} i^* : \hat{T}_{\mathbf{G}}(\mathbb{C}) \to \hat{T}_{\mathbf{G},s}(\mathbb{C}), \\ \begin{pmatrix} \alpha &&& \\ & \beta && \\ && \gamma & \\ &&& \delta \end{pmatrix} \times \mu \mapsto \begin{pmatrix} \alpha &&& \\ & \beta && \\ && \mu\beta^{-1} & \\ &&& \mu\alpha^{-1} \end{pmatrix}. \end{gathered}$$

The maximal split subtorus of $T_{\mathbf{G}}(\mathbb{Q}_p)$ satisfies

$$T_{\mathbf{G},s} \simeq \mathbb{G}^3_{m,\mathbb{Q}_p}.$$

Using lemma 2.3.69 we obtain

$$t_\xi = (\xi_1(p), \xi_2(p), \xi_0(p)) \in \mathbb{C}^{\times,3}.$$

Therefore by proposition 2.3.72, the Satake parameter of σ is given by the conjugacy class of

$$\chi_\sigma = i^{*,-1}(t_\xi) = \left(\begin{pmatrix} \xi_1(p) &&& \\ & \xi_2(p) && \\ && 1 & \\ &&& 1 \end{pmatrix}, \xi_0(p) \right) \rtimes \mathrm{Fr} \in \mathrm{GL}_4(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}) \rtimes \mathrm{Fr} = {}^L\mathbf{G},$$

modulo ${}_{\mathbb{Q}_p}N$.

2.3.11 Relation between unramified characters of similitude groups

Lemma 2.3.77. *Let $\xi = \xi_0 \otimes \xi_1 \otimes \xi_2$ be an unramified character of $T_{\mathbf{G}}(\mathbb{Q}_p)$. We denote by $\tilde{\xi} = \tau_0 \otimes \tilde{\xi}_1 \otimes \tilde{\xi}_2$ the image of ξ via the isomorphism $X^\bullet(T_{\mathbf{PG}}) \simeq X^\bullet(T_{\mathrm{PGSO}_{4,2}})$ induced by proposition 2.2.27 with $\tau_0 : E_p^\times \to \mathbb{C}$ with $\tau_0 = \tilde{\xi}_0 \circ \mathrm{N}_{E_p/\mathbb{Q}_p}$. The following relations hold:*

$$\begin{aligned} \xi_1 &= \tilde{\xi}_1 \tilde{\xi}_2 \tilde{\xi}_0 \circ \mathrm{N}_{E_p/\mathbb{Q}_p}, \\ \xi_2 &= \tilde{\xi}_1 \tilde{\xi}_0 \circ \mathrm{N}_{E_p/\mathbb{Q}_p}, \\ \xi_0 &= \tilde{\xi}_2|_{\mathbb{Q}_p} \tilde{\xi}_0^2. \end{aligned}$$

Proof. We consider $t(a,b,\nu) \in T_{\mathbf{G}}(\mathbb{Q}_p)$ as an element in $T_{\mathbf{PG}}(\mathbb{Q}_p)$. Furthermore we regard $\Gamma^* \circ (\Xi^*)^{-1}(t(a,b,\nu)) \in T_{\mathbf{GSO}_{4,2}}(\mathbb{Q}_p)$. The characters ξ and $\tilde{\xi}$ must satisfy

$$\xi(t(a,b,\nu)) = \tilde{\xi}\left(\Gamma^* \circ (\Xi^*)^{-1}(t(a,b,\nu))\right).$$

Using lemma 2.2.28 we obtain

$$\xi_1(a)\xi_2(b)\xi_0(\nu) = \tilde{\xi}_1(|a|)\tilde{\xi}_1(|b|)\tilde{\xi}_2(|a|)\tilde{\xi}_2(\nu)\tilde{\xi}_0(|a|)\tilde{\xi}_0(|b|)\tilde{\xi}_0(\nu)^2.$$

Let us recall that $\mathrm{N}_{E_p/\mathbb{Q}_p}(\alpha) = \alpha\overline{\alpha}$. Regrouping the terms we obtain the statement. ∎

2.4 Models of automorphic representations

In this section we will denote by G any reductive algebraic group defined over $\mathbb{Q}$. Further we use F to denote any $\mathbb{Q}-$algebra. We set π a representation of the topological group $G(F)$. For the forthcoming exposition $R(F) < G(F)$ will be closed subgroup and $\tilde{\psi}$ a character of $R(F)$.

Definition 2.4.1. A *model* of a representation π with respect to $(R(F), \tilde{\psi})$ is a non-zero $G(F)-$intertwining map of the form

$$i: \ \pi \to \mathrm{Ind}_{R(F)}^{G(F)} \tilde{\psi}.$$

Examples of the above construction are given by Whittaker models [JS83], Shalika models [FM13] and Whittaker-orthogonal models [BFG92].

The existence of a model for a representation π can be seen as an underlying symmetry. In fact, it is equivalent to the existence of a non-zero functional

$$\begin{aligned} F: \ \pi &\to \mathbb{C} \\ v &\mapsto i(v)(1), \end{aligned}$$

so that for any $r \in R(F)$, $F(\pi(r)v) = \tilde{\psi}(r)F(v)$. Although the definition using the above functional is less algebraic, it appears in a natural way on the integral representations of certain $L-$functions.

Definition 2.4.2. The *multiplicity* of π with respect to the model defined by $(R(F), \tilde{\psi})$ is the following constant:

$$m_\pi^R := \dim \mathrm{Hom}_{G(F)} \left(\pi, \mathrm{Ind}_{R(F)}^{G(F)} \tilde{\psi} \right).$$

Informally, the existence of a model is equivalent to the existence of m_π^R non-zero solutions of certain functional equation. They have been used to characterize representations. For example, in [Sak06, p. 21], the author proved that the unramified representations of the form $\mathrm{Ind}_{B_{\mathrm{GL}_{2n}}(\mathbb{Q}_p)}^{\mathrm{GL}_{2n}(\mathbb{Q}_p)} \xi$ have a Shalika model if and only if the character ξ is regular. Moreover, the search of compatibility between models have been used to relate values of $L-$functions, see for example [JNQ08], [Mor14], [FM13].

This section will be devoted to describing two significant models for the thesis, the *Whittaker models* and the *Shalika models*. For the forthcoming exposition we fix a unitary character $\psi: \ \mathbb{Q} \setminus \mathbb{A} \to \mathbb{C}^1$ so that $\psi = \otimes_p \psi_p$ with $\psi_p: \ \mathbb{Q}_p \to \mathbb{C}^\times$ unramified characters , i.e. $\psi_p|_{\mathbb{Z}_p} = 1$.

2.4.1 Whittaker model

In this subsection we will denote by K the maximal compact subgroup of $G(\mathbb{A})$. We fix $B = TU$ a Borel subgroup of G with maximal torus T and maximal unipotent subgroup U. By [GH22, thm. 1.9.1, p. 30] the choice of a Borel subgroup is equivalent to the choice of a set of simple roots Δ. We set $\{X_\alpha, \ \alpha \in \Delta\}$ a choice of root vectors in $\mathrm{Lie}(U)$ and we consider the map

$$\phi: \ U \to \prod_{\alpha \in \Delta} \mathbb{G}_a,$$

that sends $\exp(x_\alpha X_\alpha)$ to $x_\alpha \in \mathbb{Q}$. Further let us consider

$$\begin{aligned} \Omega: \ \prod_{\alpha \in \Delta} \mathbb{G}_a &\to \mathbb{G}_a, \\ (x_\alpha)_{\alpha \in \Delta} &\mapsto \sum_{\alpha \in \Delta} x_\alpha. \end{aligned}$$

Suppose that the group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ fixes the set $\{X_\alpha\}_{\alpha \in \Delta}$, which implies that the map $\phi \circ \Omega: \ U \to \mathbb{G}_a$ is defined over $\mathbb{Q}$. For every $\mathbb{Q}-$algebra F we will also denote by $\phi \circ \Omega$ the map induced by $\phi \circ \Omega$ on $U(F)$.

Definition 2.4.3. The characters $\chi_p : U(\mathbb{Q}_p) \to \mathbb{C}^\times$ of the form $\chi_p = \phi \circ \Omega \circ \psi_p$, with $\psi_p : \mathbb{Q}_p \to \mathbb{C}^\times$ non-trivial, are called *locally generic characters*. The characters $\chi : U(\mathbb{A}) \to \mathbb{C}^\times$ of the form $\chi = \phi \circ \Omega \circ \psi$, with $\psi : \mathbb{Q} \setminus \mathbb{A} \to \mathbb{C}^\times$ non-trivial, are called *globally generic characters*.

Remark 2.4.4. *The locally and globally generic characters are characters defined over $U(\mathbb{Q}_p)$ and $U(\mathbb{A})$ respectively, that are not trivial on the root spaces associated to the set of simple roots Δ.*

Definition 2.4.5. Let (π_p, V_p) be a representation of the topological group $G(\mathbb{Q}_p)$. Given a locally generic character χ_p of $U(\mathbb{Q}_p)$, we say that π_p is *generic* if there exists a non-zero functional $W_{\chi_p} \in \tilde{V}_p$ so that

$$W_{\chi_p}(\pi_p(u)v) = \chi_p(u) W_\chi(v),$$

for every $v \in V_p$ and $u \in U(\mathbb{Q}_p)$. When (π_p, V_p) is an induced representation of the form $\mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\xi$, we will denote this functional by $W_{\chi,\xi}$.

Remark 2.4.6. *As we shown in (2.4), the existence of a functional of the form 2.4.5 is equivalent to the existence of a $G(\mathbb{Q}_p)-$intertwining map:*

$$\begin{aligned} F_{\chi_p} :\ \pi_p &\to \mathrm{Ind}_{U(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\chi_p, \\ v &\to W_{\chi_p}(\pi(\cdot)v). \end{aligned}$$

When π_p is an induced representation of the form $\mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\xi$ we will denote this intertwining map by $F_{\chi_p,\xi}$. Let us note that the functional-intertwining map correspondence can be described algebraically via [CS80, prop. 1.1, p. 209] and [CS80, prop. 1.3, p. 209] taking $V' = \chi_p$.

Proposition 2.4.7. *For any locally generic character χ_p and any unramified character ξ of $T(\mathbb{Q}_p)$, the space of functionals $W_{\chi_p,\xi} \in \widetilde{\mathrm{Ind}_B^G}\xi$ so that $W_{\chi,\xi}(\pi_p(u)v) = \chi_p(u)W_\chi(v)$, has dimension at most one, i.e.*

$$\dim \mathrm{Hom}_G\left(\mathrm{Ind}_B^G\xi, \mathrm{Ind}_U^G\chi_p\right) \leq 1.$$

Proof. This result follows by the combination of two propositions, the first one the following modification of the Frobenius reciprocity theorem [CS80, prop. 1.3, p. 209]:

$$\mathrm{Hom}_G\left(\mathrm{Ind}_B^G\xi, \mathrm{Ind}_U^G\chi_p\right) = \mathrm{Hom}_{\mathbb{C}}\left(\left(\mathrm{Ind}_B^G\xi\right)_{\chi_p,U}, \mathbb{C}\right),$$

where $\left(\mathrm{Ind}_B^G\xi\right)_{\chi_p,U}$ is the so-called twisted Jacquet module (see [CS80, p. 2]). The result is the equality between $\left(\mathrm{Ind}_B^G\xi\right)_{\psi_p,U}$ and ξ given in [CS80, cor. 1.7, p. 213]. □

Definition 2.4.8. Given a globally generic character χ of $[U]$, a cuspidal automorphic representation (π, V) of G is *globally generic* if the Fourier coefficient associated to the character χ does not vanish for at least some $\varphi \in \pi$, i.e.

$$W_\chi(\varphi, g) = \int_{[U]} \varphi(ug)\overline{\chi(u)}du \neq 0,$$

for some $\varphi \in \pi$.

Remark 2.4.9. *Analogously to the local case, a cuspidal automorphic representation π is globally generic if and only if there exists a $(\mathfrak{g}, K) \times G(\mathbb{A}_f)-$intertwining map of the form*

$$\begin{aligned} \pi &\to \mathrm{Ind}_U^G\chi \\ \varphi &\mapsto W_\chi(\varphi, \cdot). \end{aligned}$$

Remark 2.4.10. *The cuspidal automorphic representations of GL_n are always generic, see [Bum97, thm. 3.5.5, p. 329] for the proof when $n = 2$.*

Definition 2.4.11. Let $\{\chi_p\}_p$ be a set of locally generic characters for every place p. We say that an automorphic representation π is *locally generic* if for every p the representation π_p is generic with respect to χ_p.

Remark 2.4.12. *Set $\chi = \otimes_p \chi_p$ a globally generic character. If an automorphic representation π is globally generic with respect to χ, it is a locally generic representation with respect to the set of characters $\{\chi_p\}_p$. The converse of this statement is more subtle and in fact, it is only proved for the group GL_n in [Sha74]. The general case is still conjectural, in [AS06, conj. 2.10, p. 8] the author studied cases where locally generic implies globally generic under the assumption of the conjectures of Arthur and Clozel; (see [Clo07, conj. 2A] and [Sha11, conj. 6.4, p. 21]). The reliance of this result on Arthur's conjectures shows the relevant role of the models to classify automorphic representations.*

2.4.2 Shalika model for the group GL_{2n}.

For simplicity, let us consider the group GL_{2n} defined over $\mathbb{Q}$. We fix B_{GL} the upper diagonal Borel subgroup of GL_{2n}, with T_{GL} and U_{GL} the maximal torus and the maximal unipotent subgroup of GL_n, satisfying $B_{\mathrm{GL}} = T_{\mathrm{GL}} U_{\mathrm{GL}}$.

Definition 2.4.13. The *Shalika subgroup* of $S_{\mathrm{GL}_{2n}} < \mathrm{GL}_{2n}$ is the algebraic subgroup so that

$$S_{\mathrm{GL}_{2n}}(F) = \left\{ \left(\begin{smallmatrix} g & \\ & g \end{smallmatrix} \right) \left(\begin{smallmatrix} I_n & X \\ & I_n \end{smallmatrix} \right),\ g \in \mathrm{GL}_n(F),\ X \in \mathrm{M}_n(F) \right\},$$

for any $\mathbb{Q}-$algebra F. If there is no danger of confusion, we will suppress the subscript $2n$ from the notation.

Remark 2.4.14. *The Shalika subgroup of GL_{2n} is isomorphic to the algebraic group $\mathrm{GL}_n N_{\mathrm{GL}_{2n}}$, where $N_{\mathrm{GL}_{2n}}$ is the Siegel parabolic of GL_{2n}. We will factor each element $s \in S_{\mathrm{GL}}$ according to the previous isomorphism by*

$$s = gn,$$

where $g \in \mathrm{GL}_n$ and $n \in N_{\mathrm{GL}_{2n}}$.

Definition 2.4.15. The *local Shalika character* of GL_{2n} is defined by

$$\begin{aligned} \chi_{p,\mathrm{GL}_{2n}} :\ S_{\mathrm{GL}}(\mathbb{Q}_p) &\to \mathbb{C}^1, \\ gn &\mapsto \psi_p\left(\mathrm{tr}\left(X\right)\right), \end{aligned}$$

where $X \in M_n(\mathbb{Q}_p)$ so that $n = \left(\begin{smallmatrix} I_n & X \\ & I_n \end{smallmatrix} \right)$. We suppress the subindex $2n$ from the notation unless it is not clear.

Definition 2.4.16. A representation (π_p, V_p) of $\mathrm{GL}_{2n}(\mathbb{Q}_p)$ *has a Shalika model* if there exists a non-zero functional $\mathcal{S}_{\mathrm{GL}_{2n},p} \in \tilde{V}_p$, called *local Shalika functional*, satisfying

$$\mathcal{S}_{\mathrm{GL}_{2n},p}(\pi_p(s)v) = \chi_{p,\mathrm{GL}_{2n}}(s)\mathcal{S}_{\mathrm{GL}_{2n},p}(v),$$

for any $v \in V_p$ and $s \in S_{\mathrm{GL}_{2n}}(\mathbb{Q}_p)$. If the subindex n is clear from the context we will suppress it from the notation. When (π_p, V_p) is the unique unramified sub-quotient of a representation of the form $\mathrm{Ind}_{B_{\mathrm{GL}}}^{\mathrm{GL}_{2n}(\mathbb{Q}_p)} \xi$, we denote the above functional by $\mathcal{S}_{\mathrm{GL},\xi}$.

Theorem 2.4.17 (Multiplicity one). *For any representation (π_p, V_p) of $\mathrm{GL}_{2n}(\mathbb{Q}_p)$, we have*

$$\dim \mathrm{Hom}_{\mathrm{GL}_{2n}(\mathbb{Q}_p)}\left(\pi_p, \mathrm{Ind}_{S_{\mathrm{GL}}}^{\mathrm{GL}_{2n}} \chi_{p,\mathrm{GL}}\right) \leq 1.$$

Proof. It is [JR96, prop. 6.1, p. 117]. □

Theorem 2.4.18. *Let ξ be an unramified character of $T_{\mathrm{GL}_{2n}}(\mathbb{Q}_p)$. The representation $\mathrm{Ind}_{B_{\mathrm{GL}_{2n}}}^{\mathrm{GL}_{2n}} \xi$ has a non-zero Shalika model if and only if*

- *The representation $\mathrm{Ind}_{B_{\mathrm{GL}_{2n}}}^{\mathrm{GL}_{2n}} \xi$ is reducible with the spherical vector generating a proper $\mathrm{GL}_n(\mathbb{Q}_p)-$subspace.*

- *The character ξ is regular.*
- *The character ξ is also a character of the torus of Sp_{2n} embedded in GL_{2n}.*

Proof. See [Sak06, p. 21] and [Sak06, p. 2]. $\square$

2.4.3 Shalika model for the group G.

Definition 2.4.19. The *Shalika subgroup* of $\mathbf{G}$ is the algebraic subgroup $S < \mathbf{G}$ so that for any $\mathbb{Q}$-algebra F

$$S(F) = \left\{ \begin{pmatrix} a & b & & \\ c & d & & \\ & & a & -b \\ & & -c & d \end{pmatrix} \begin{pmatrix} 1 & & \alpha & x \\ & 1 & y & \overline{\alpha} \\ & & 1 & \\ & & & 1 \end{pmatrix} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(F), x, y \in \mathbb{G}_a(F),\ \alpha \in \mathrm{Res}_{E/F}\mathbb{G}_a(F) \right\}.$$

For any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ it holds that $\begin{pmatrix} a & -b \\ -c & d \end{pmatrix} = J_2^t g^{-1} J_2$, where the matrix J_2 the 2×2 anti-diagonal matrix with all non-zero entries equal to 1.

Remark 2.4.20. *The Shalika subgroup S is isomorphic to $\mathrm{GL}_2 N_{\mathbf{G}}$ where*

$$N_{\mathbf{G}} = \prod_{\alpha \in \{\alpha_1+\alpha_2-\alpha_0, 2\alpha_2-\alpha_0, 2\alpha_1-\alpha_0\}} N_\alpha,$$

is the unipotent subgroup of the Siegel parabolic $P_{\mathbf{G}}$ of $\mathbf{G}$.

Definition 2.4.21. The *local Shalika character* of $\mathbf{G}$ is defined by

$$\chi_{\delta,p} :\ S(\mathbb{Q}_p) \to \mathbb{C}^1,$$
$$\left(\begin{pmatrix} g & \\ & J_2{}^t g^{-1} J_2 \end{pmatrix} \begin{pmatrix} 1 & & \alpha & x \\ & 1 & y & \overline{\alpha} \\ & & 1 & \\ & & & 1 \end{pmatrix} \right) \mapsto \psi_p(\delta\alpha - \delta\overline{\alpha}),$$

where we recall that δ is the element so that $E = \mathbb{Q}(\delta)$.

Definition 2.4.22. A representation (π_p, V_p) of $\mathbf{G}(\mathbb{Q}_p)$ *has a Shalika model* if there exists a non-zero functional $\mathcal{S}_p \in \tilde{V}_p$, called *local Shalika functional*, satisfying

$$\mathcal{S}_p(\pi_p(s)v) = \chi_{\delta,p}(s)\mathcal{S}_p(v),$$

with $s \in S(\mathbb{Q}_p)$ and $v \in \tilde{V}_p$.

Definition 2.4.23. The *global Shalika character* is defined by $\chi_\delta := \otimes_p \chi_{\delta,p} : S(\mathbb{Q}) \setminus S(\mathbb{A}) \to \mathbb{C}^1$.

Definition 2.4.24. A cuspidal automorphic representation (π, V) of $\mathbf{G}$ *has a Shalika model* if there exists a non-zero functional $\mathcal{S}_\delta$, called *Shalika functional*, so that

$$\mathcal{S}_\delta(\pi(s)v) = \chi_\delta(s)\mathcal{S}_\delta(v),$$

for any $v \in V$ and $s \in S(\mathbb{A})$.

Remark 2.4.25. *For any cuspidal automorphic representation (π, V) of $\mathbf{G}$ we consider the functional*

$$S_\delta(\varphi) = \int_{\mathbb{A}^\times \mathrm{GL}_2(\mathbb{Q})\setminus \mathrm{GL}_2(\mathbb{A})} \int_{[N_{\mathbf{G}}]} \varphi(nh)\chi_\delta(nh)dndh,$$

where dh and dn are the Haar measures of $\mathrm{GL}_2(\mathbb{A})$ and $N^{Sieg}(\mathbb{A})$ respectively, and the embedding

$$\left(\mathbb{A}^\times \mathrm{GL}_2(\mathbb{Q}) \setminus \mathrm{GL}_2(\mathbb{A})\right)[N_{\mathbf{G}}] \hookrightarrow \mathbf{G}(\mathbb{A}),$$

is given via the structure of the Shalika subgroup described in remark 2.4.20. If there is a cusp form $\varphi \in V$ satisfying $S_\delta(\varphi) \neq 0$, the functional S_δ is an explicit example of non-zero Shalika functional of (π, V).

Remark 2.4.26. *The existence of a global Shalika functional is intimately related to the existence of poles at $s = 1$ of the wedge square $L-$functions, see [FM13, thm. 4.1, p. 4136]. Furthermore, the existence of such functionals can be related to the theta correspondence, see [Mor14, thm. 3.6, p. 43]. Since these both results are essential to understand the main goal of this chapter, we will discuss them in detail in the forthcoming sections.*

Remark 2.4.27. *Due to the isomorphism 2.2.27, the Shalika models for the groups* GL_4 *and* $\mathbf{G}$ *are intimately related. We will adress this question in subsection 2.7.2.*

2.4.4 Shalika model for the group $\mathrm{GSO}_{4,2}$.

Definition 2.4.28. The *Shalika subgroup* of $\mathrm{GSO}_{4,2}$ is the algebraic group $S_O < \mathrm{GSO}_{4,2}$, so that

$$S_O(F) = R(F)N_O(F),$$

for any $\mathbb{Q}-$algebra F, where

$$R(F) = \left\{ \begin{pmatrix} 1 & & \\ & r & \\ & & 1 \end{pmatrix}, \ s.t. \ r \in \mathrm{SO}_{3,1}(F), \ re_2 = e_2 \right\},$$

and $N_O = U_0 U_1 \tilde{U}$ is the unipotent subgroup of the Siegel parabolic. The reader can find the definition of this unipotent subgroup in (2.2.3), (2.2.3) and (2.2.3).

Definition 2.4.29. The *Shalika character* ψ_S is defined by

$$\psi_S(ru_0(x)u_1(y,z)\tilde{u}(w)) := \psi(2dz).$$

Remark 2.4.30. *The Shalika character factorizes as follows:*

$$\psi_S = \otimes_p \psi_{S,p},$$

where $\psi_{S,p}(ru_0(x)u_1(y,z)\tilde{u}(w)) = \psi_p(2dz_p)$ *and* $z = \prod_p' z_p \in \mathbb{A}$.

Definition 2.4.31. A representation (π_p, V_p) of $\mathrm{GSO}_{4,2}(\mathbb{Q}_p)$ *has a Shalika model* if there exists a non-zero functional $\mathcal{S}_{O,p} \in \tilde{V}_p$ so that

$$\mathcal{S}_{O,p}(\pi_p(s)v) = \psi_{S,p}(s)\mathcal{S}_{O,p}(v),$$

for $v \in V_p$ and $s \in S_O(\mathbb{Q}_p)$.

Definition 2.4.32. A cuspidal automorphic representation (π, V) of $\mathbf{G}$ *has a Shalika model* if there exists a non-zero functional $\mathcal{S}_O \in \tilde{V}$, called *Shalika functional*, so that

$$\mathcal{S}_O(\pi(s)v) = \psi_S(s)\mathcal{S}_O(v),$$

for $v \in V$ and $s \in S_O(\mathbb{Q}_p)$.

Remark 2.4.33. *Let* (π, V) *be a cuspidal automorphic representation of* $\mathrm{GSO}_{4,2}$. *If the functional*

$$\mathrm{Sh}(f) := \int_{[S]} f(g)\psi_S(g)dg,$$

is non-zero for some $f \in V$, $\mathrm{Sh}(\cdot)$ *is a non-zero Shalika functional.*

2.4.5 Uniqueness of models

Results concerning multiplicities of representations with respect to models have been widely used to study the structure of automorphic representation. In particular, multiplicity one results are essential to obtain certain integral representation of $L-$functions. In this section we explain two different methods to understand the multiplicity of a representation. The first one is based on the search of an involution. Even though it is completely general, it is only usefull when the multiplicity is one. The second method is based on Mackey theory, and allows us to understand the multiplicity for unramified representations.

Let (π_p, V_p) be a smooth irreducible representation of the group $G(\mathbb{Q}_p)$. Fix $R(\mathbb{Q}_p) < G(\mathbb{Q}_p)$ a closed subgroup and $\psi_p : R(\mathbb{Q}_p) \to \mathbb{C}$ a character. There are several strategies to prove the uniqueness of a model, i.e. the inequality

$$\dim \operatorname{Hom}_G \left(\pi_p, \operatorname{Ind}_R^G \psi_p\right) \leq 1.$$

One of the most fruitful ideas is based on the search of an involution which fixes the group $R(\mathbb{Q}_p)$ and the functional F associated to the model by (2.4). We will sketch this method stressing out why such involution is needed. For a complete discussion of this method we refer the reader to [Sou87] and [Bum97, sec. 4.4, p. 492], where particular cases are treated.

Given $l \in \tilde{V}_p$, we consider the space

$$W = \left\{\varphi * l,\ \varphi \in \mathcal{H}_{p,\mathbb{C}}^G\right\},$$

where $*$ is the action of the Hecke algebra on the representation. The space W is a subspace of $\tilde{V}_p$ and we denote by ρ be the action by right translations of the group $G(\mathbb{Q}_p)$ on $\mathcal{H}_p^G$. Since

$$\rho(g)\varphi * l = \pi_p(g)(\varphi * l),$$

the space W is an invariant subspace of V_p. Since V_p is irreducible, then

$$W \simeq V_p.$$

Let us suppose that $r_1, r_2 \in \tilde{V}_p$ are two functionals associated to the model $\operatorname{Ind}_R^G \psi_p$ in the sense of (2.4). We define a map

$$\begin{aligned} V_p &\to V_p, \\ \varphi * r_2 &\mapsto \varphi * r_1. \end{aligned}$$

It is surjective and intertwining. Let us suppose that the previous map were also injective, then using the Schur's lemma there would exist a constant $C \in \mathbb{C}$ so that

$$\varphi * r_1 = C\varphi * r_2.$$

Concluding that $r_1 = Cr_2$ and obtaining the multiplicity one result. Therefore, the main goal of this method has been reduced to prove the injectivity of the map (2.4.5), i.e. if $\varphi * r_1 = 0$ then $\varphi * r_2 = 0$. From now on, let us suppose that $\varphi * r_1 = 0$. We consider the following distribution:

$$T(\varphi) = r_2(\varphi * r_1).$$

It is straightforward that $T(\varphi) = 0$.

Lemma 2.4.34. *The following equality holds:*

$$\varphi * l = \int_{G(\mathbb{Q}_p)} \varphi(g)\pi_p(g^{-1})\hat{l}dg,$$

where $\hat{l}$ is the image of l under the map $V \hookrightarrow \hat{V}$ defined by the Riesz representation theorem.

Proof. See [Sou87, (2.9), p. 273]. ∎

Let $g_0 \in G(\mathbb{Q}_p)$ and denote by $\langle \cdot, \cdot \rangle : V \times \tilde{V} \to \mathbb{C}$ the natural bilinear form between a representation and its smooth dual. Using the lemma 2.4.34 we obtain

$$
\begin{aligned}
0 = T(\varphi) &= r_2(\rho(g_0)\varphi * r_1) = r_2\left(\int_{G(\mathbb{Q}_p)} \rho(g_0)\varphi(g)\pi_p(g^{-1})\hat{r}_1 dg\right) \\
&= \int_{G(\mathbb{Q}_p)} \rho(g_0)\varphi(g)r_2(\pi_p(g^{-1})\hat{r}_1)dg = \int_{G(\mathbb{Q}_p)} \rho(g_0)\varphi(g)\left\langle r_2, \pi_p(g^{-1})\hat{r}_1\right\rangle dg = \int_{G(\mathbb{Q}_p)} \rho(g_0)\varphi(g)\left\langle \pi_p(g)r_2, \hat{r}_1\right\rangle dg \\
&= \int_{G(\mathbb{Q}_p)} \varphi(g)\left\langle \pi_p(gg_0^{-1})r_2, \hat{r}_1\right\rangle dg,
\end{aligned}
$$

where the last equality follows by a change of variables. Let us suppose that there exists an involution τ i.e. a map $\tau: G(\mathbb{Q}_p) \to G(\mathbb{Q}_p)$ so that $(ab)^\tau = b^\tau a^\tau$, satisfying

$$
R^\tau(\mathbb{Q}_p) = R(\mathbb{Q}_p) \text{ and } r_2(\varphi^\tau * r_1) = r_2(\varphi * r_1). \tag{2.-11}
$$

The existence of such involution would allow us to change the order of the multiplication gg_0^{-1}, obtaining

$$
\int_{G(\mathbb{Q}_p)} \varphi(g)\left\langle \pi_p(gg_0^{-1})r_2, \hat{r}_1\right\rangle dg = \int_{G(\mathbb{Q}_p)} \varphi(g)\left\langle \pi_p(g_0^{-1})\pi_p(g)r_2, \hat{r}_1\right\rangle dg = \int_{G(\mathbb{Q}_p)} \varphi(g)r_1(\pi_p(g_0^{-1})\pi_p(g)r_2)dg
$$

$$
= r_1\left(\pi_p(g_0^{-1})\int_{G(\mathbb{Q}_p)} \varphi(g)\pi_p(g)r_2 dg\right).
$$

Since for any $g_0 \in G(\mathbb{Q}_p)$

$$
0 = T(\varphi) = r_1\left(\pi_p(g_0^{-1})\int_{G(\mathbb{Q}_p)} \varphi(g)\pi_p(g)r_2 dg\right),
$$

and $r_1 \neq 0$, we obtain

$$
\int_{G(\mathbb{Q}_p)} \varphi(g)\pi_p(g)r_2 dg = 0,
$$

implying that $\varphi * r_2 = 0$.

Therefore any smooth representation (π_p, V_p) has multiplicity one with respect to the model given by the tuple $(R(\mathbb{Q}_p), \psi_p)$ if there exists an involution satisfying the hypothesis (2.4.5). Although this method has been very fruitful in many cases and it is completely general, it presents some disadvantages. For example it does not provide any information when the multiplicity is not one and it may happen that the multiplicity of certain family of representations varies according to certain invariant. Since throughout the previous discussion we have not used any property of the representation (π_p, V_p), we are not able to relate any invariant with the multiplicity of the representation using the above method.

One of the main results of this chapter is a criterion which determines the multiplicity of an unramified representation of $\mathbf{G}(\mathbb{Q}_p)$ with respect to the local Shalika model. The multiplicity one of the local Shalika model for the group $\mathbf{G}(\mathbb{Q}_p)$ was not clear at the beginning of the project (for example in [CG21, thm. 4.3, p. 12] the authors gave a formula for the multiplicity of discrete series representation with respect to Shalika models for unitary groups, showing that it is not always one). In order to approach this computation, we used the so-called Mackey theory. Althought in subsection 2.7.3 we will explain this method in complete detail, to conclude this subsection we sketch it, stressing out its main idea.

For simplicity let us take $\pi_p = \mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\xi$ an irreducible unramified principal series. The multiplicity of π_p with respect to the model defined by $(R(\mathbb{Q}_p), \psi_p)$ is equal to

$$
\mathrm{Hom}_G\left(\mathrm{Ind}_B^G\xi, \mathrm{Ind}_R^G\psi_p\right). \tag{2.-12}
$$

An initial idea to understand the above vector space is to apply proposition 2.3.13. Due to the lack of relation between the groups $B(\mathbb{Q}_p)$ and $R(\mathbb{Q}_p)$, the resulting vector space is just as difficult to compute as (2.4.5). In order to understand the group (2.4.5) we use the line bundle description of the unramified principal series presented in 2.3.2. This point of view allows us to apply the exact sequence of proposition 2.3.16 to our representation π_p, obtaining that the multiplicity of the representation π_p with respect to the model is described by the smaller groups

$$\mathrm{Hom}_{R(\mathbb{Q}_p)}\left(\text{c-Ind}_{\mathrm{Stab}_{R(\mathbb{Q}_p)}w_i}^{R(\mathbb{Q}_p)}\xi_{w_i},\psi_p\right), \tag{2.-12}$$

where $w_i \in G(\mathbb{Q}_p)$ so that $R(\mathbb{Q}_p)\setminus G(\mathbb{Q}_p)/B(\mathbb{Q}_p)=\bigsqcup_{i\in I} w_i$ and ξ_{ω_i} are certain twists of the character ξ. We compute the dimension of the vector spaces (2.4.5) via proposition 2.3.13. Applying an inductive argument to the exact sequence obtained in 2.3.16, we may obtain a criterion on ξ explaining the multiplicity of the representation.

2.4.6 Casselman-Shalika formula for Whittaker models

One of the main reasons why models of representations are considered in the theory of automorphic forms is because they are closely connected to functional equations and integral representations of certain $L-$functions. In fact, their associated functionals appear in a wide variety of computations using Rankin-Selberg integrals or Godement-Jacquet integrals. The combination of an explicit formula for the functional, usually called *Casselman-Shalika formula for the model*, and the Weyl's character formula, connects the aforementioned integrals with the $L-$functions. Case by case, this kind of formulas have been usefull tools to understand the Langlands $L-$functions, leading mathematicians to study different methods to compute Casselman-Shalika formulas: [Shi76], [CS80] or [Sak06]. In this subsection we will introduce the basic theory to understand the proof of [CS80], whose ideas have been used to state a diverse range of Casselman-Shalika formulas.

Let G be a connected unramified reductive group defined over $\mathbb{Q}_p$ and $K=G(\mathbb{Z}_p)$. We fix a Borel subgroup $B=TU$, defining a set of positive roots Φ^+. Given ξ an unramified character of $T(\mathbb{Q}_p)$, we consider the representation $\left(\pi,\mathrm{Ind}_B^G\xi\right)$. The main goal of this subsection is to find a formula for

$$W_{\chi,\xi}(\pi(t)\phi_{K,\xi}),$$

where $W_{\chi,\xi}(\cdot)$ is the functional defined in 2.4.5, χ is a locally generic character of $U(\mathbb{Q}_p)$ satisfying $\chi|_{U(\mathbb{Z}_p)}=1$, $\phi_{K,\xi}=\mathcal{P}_\xi(\mathrm{char}_K)$ is the element generating $\left(\mathrm{Ind}_B^G\xi\right)^K$ by lemma 2.3.46 and $t\in T(\mathbb{Q}_p)$.

We recall that by proposition 2.4.7 the Whittaker functional $W_{\chi,\xi}(\pi(t)\phi_K)$ is unique up to constant. The following proposition will define an explicit choice of such Whittaker functional.

Proposition 2.4.35. *Fix a representative $x\in N_G(T_s)(\mathbb{Q}_p)$ for $\omega_l\in W(G,T_s)(\mathbb{Q}_p)$, the longest element of the Weyl group. The functional*

$$\Omega_{\chi,\xi}(\phi):=\int_{U(\mathbb{Q}_p)}\phi(xu)\chi^{-1}(u)du,$$

is defined for the sections $\phi\in\mathrm{Ind}_B^G\xi$ with support on $B(\mathbb{Q}_p)\omega_l B(\mathbb{Q}_p)$. This functional extends uniquely to a Whittaker functional on the whole space $\mathrm{Ind}_B^G\xi$.

Proof. It is [CS80, cor. 1.8, p. 213]. □

From now on we will denote by $W_{\chi,\xi}(\cdot)$ the Whittaker functional determined by proposition 2.4.35.

Lemma 2.4.36. *Let T_s^- be the subset of $T_s(\mathbb{Q}_p)$ defined in (2.1.4). Given $t\in T_s(\mathbb{Q}_p)$ so that $t\notin T_s^-$*

$$W_{\chi,\xi}(\pi(t)\phi_{K,\xi})=0.$$

Proof. The proof follows using the functional equation of $W_{\chi,\xi}(\cdot)$ given in 2.4.5. See [CS80, lem. 5.1, p. 224] for further details. □

Despite having reduced our computation to a smaller family of elements, we still have to compute the value of the functional given in 2.4.35 on a continuous family of vectors of the form $\pi(t)\phi_{K,\xi}$. Moreover, we do not have an explicit formula for the Whittaker functional evaluated in every element of $\mathrm{Ind}_B^G\xi$. The main novelty of [CS80] is the definition of a finite basis $\{f_\omega\}_{\omega\in W(G,T_s)(\mathbb{Q}_p)}$, which reduces the computation of $W_{\chi,\xi}(\pi(t)\phi_{K,\xi})$ to the computation of $W_{\chi,\xi}(\phi_\omega)$.

Definition 2.4.37. We denote by $f_\omega \in \mathrm{Ind}_B^G\xi$ the element dual to the operator defined by

$$\begin{aligned}\mathrm{Ind}_B^G\xi &\to \mathbb{C},\\ \phi &\mapsto T_\omega(\phi)(1),\end{aligned}$$

where T_ω is the map defined in 2.3.54. The set $\{f_\omega\}_{\omega\in W(G,T_s)(\mathbb{Q}_p)}$ is called *Casselman basis.*

Proposition 2.4.38. *The Casselman basis $\{f_\omega\}_{\omega\in W(G,T_s)(\mathbb{Q}_p)}$ is a basis of $\left(\mathrm{Ind}_B^G\xi\right)^I$, where we recall that I is the Iwahori subgroup of $G(\mathbb{Q}_p)$ defined in 2.1.62.*

Proof. Lemma 2.3.49 shows that the maps $\phi \mapsto T_\omega(\phi)(1)$ form a basis of the dual space of $\left(\mathrm{Ind}^G\xi\right)_N$. Proposition 2.3.48 implies the result. □

Remark 2.4.39. *Since*

$$\phi_{K,\xi} \in \left(\mathrm{Ind}_B^G\xi\right)^I,$$

using proposition 2.4.38, we are able to express $\phi_{K,\xi}$ as a combination of the elements $\{f_\omega\}_{\omega\in W(G,T_s)(\mathbb{Q}_p)}$. In fact, applying proposition 2.3.62, we obtain

$$\phi_{K,\xi} = \sum_{\omega\in W(G,T_s)(\mathbb{Q}_p)} c_\omega(\xi) f_\omega,$$

where we recall that $c_\omega(\xi)$ is the constant defined in 2.3.60.

Definition 2.4.40. Given $U_0 < U(\mathbb{Q}_p)$ an open compact subgroup of $U(\mathbb{Q}_p)$ and $\phi \in \mathrm{Ind}_B^G\xi$, we define

$$\mathcal{P}_{\chi,U_0}(\phi)(g) := (\mathrm{meas}(U_0))^{-1}\int_{U_0}\phi(gu)\chi^{-1}(u)du,$$

where $\mathrm{meas}(U_0) = \int_{U_0} du$.

Lemma 2.4.41. *For any open compact subgroup $\tilde{K}$ of $G(\mathbb{Q}_p)$, there is a compact open subgroup $U_0 < U(\mathbb{Q}_p)$ so that for every unramified character ξ and $\phi \in \left(\mathrm{Ind}_B^G\xi\right)^{\tilde{K}}$, the function $\mathcal{P}_{\chi,U_0}(\phi)$ has support on $B(\mathbb{Q}_p)\omega_l B(\mathbb{Q}_p)$.*

Proof. See [CS80, lem. 2.2, p. 214]. □

Let $\tilde{K}$ be any open compact subgroup of $G(\mathbb{Q}_p)$, the above lemma allows us to extend the formula given in proposition 2.4.35 for every $\phi \in \left(\mathrm{Ind}_B^G\xi\right)^{\tilde{K}}$, removing the condition on the support. More concretely

$$W_{\chi,\xi}\left(\mathcal{P}_{\chi,U_0}(\phi)\right) = (\mathrm{meas}(U_0))^{-1}\int_{U(\mathbb{Q}_p)}\int_{U_0}\phi(xuu')\chi^{-1}(u')\chi^{-1}(u)du'du.$$

Applying a change of variables we obtain

$$W_{\chi,\xi}\left(\mathcal{P}_{\chi,U_0}(\phi)\right) = W_{\chi,\xi}(\phi), \tag{2.-12}$$

obtaining an explicit formula for $W_{\chi,\xi}(\phi)$ when $\phi \in \left(\mathrm{Ind}_B^G \xi\right)^{\tilde{K}}$. We recall that proposition 2.3.69 implies that the set of unramified characters of $T(\mathbb{Q}_p)$ is isomorphic to $\mathbb{C}^{\times,\mathrm{rank}(T)}$. Given any function $f \in \mathcal{C}_c^\infty(G(\mathbb{Q}_p))$ and a compact open subgroup $\tilde{K}$ with the property that f is $\tilde{K}-$invariant, the formula(2.4.6) allows us to show that the function

$$\begin{aligned} W_{\chi,\xi}(\mathcal{P}_\xi(f)) :\ \mathbb{C}^{\times,\mathrm{rank}(T)} &\to \mathbb{C}, \\ \xi &\mapsto W_{\chi,\xi}(\mathcal{P}_\xi(f)), \end{aligned}$$

is a holomorphic function on ξ, see [CS80, prop. 2.1, p. 214] for more details.

Proposition 2.4.42. *We obtain*

$$W_\chi(\pi(t)\phi_{K,\xi}) = \sum_{\omega \in W(G,T_s)(\mathbb{Q}_p)} c_\omega(\xi)\ {}^\omega\xi \delta_B^{1/2}(t) W_\psi(f_\omega).$$

Proof. By [CS80, p. 228]

$$\mathcal{P}_{\chi,N_0}(\pi(t)\phi_{K,\xi}) = \sum_{\omega \in W(G,T_s)(\mathbb{Q}_p)} c_\omega(\xi)\ {}^\omega\xi \delta^{1/2}(t) f_\omega. \tag{2.-12}$$

Using the equation (2.4.6) on (2.4.6), the result follows. ☐

The computation of $W_{\chi,\xi}(f_\omega)$ using the formula of proposition 2.4.35 is too involved for the elements of the form $\omega \neq \omega_l$. We will use the aforementioned formula to compute the constant $W_{\chi,\xi}(f_{\omega_l})$ and then, by means of a functional equation we will show the final formula.

Lemma 2.4.43. *The following equality holds:*

$$W_{\chi,\xi}(f_{\omega_l}) = 1.$$

Proof. See [Cas80, prop. 3.7, p. 402]. ☐

Definition 2.4.44. Given $\omega \in W(G,T_s)(\mathbb{Q}_p)$, we introduce the following constants:

$$\begin{aligned} \rho_\alpha(\xi) &= (1 - q_{\alpha/2}^{-1/2} q_\alpha^{-1} \xi(a_\alpha))(1 + q_{\alpha/2}^{-1/2} \xi(a_\alpha)), \\ \rho_\omega(\xi) &= \prod_{\substack{\alpha>0 \\ \omega\alpha<0}} \rho_\alpha(\xi). \end{aligned}$$

Lemma 2.4.45. *For any $\omega \in W(G,T_s)(\mathbb{Q}_p)$ so that $\rho_\omega(\xi) \neq 0$,*

$$W_{\chi,{}^\omega\xi}(\cdot) = (\rho_\omega(\xi^{-1})/\rho_\omega(\xi)) W_{\chi,\xi}(\cdot).$$

Proof. Let

$$F_{\chi,\xi} :\ \mathrm{Ind}_B^G \xi \to \mathrm{Ind}_U^G \chi,$$

be the map associated to the functional $W_{\chi,\xi}(\cdot)$ by remark 2.4.6. It is defined by

$$F_{\chi,\xi}(\phi_{K,\xi})(t) = W_{\chi,\xi}(\pi(t)\phi_{K,\xi}).$$

The map $F_{\chi,\xi}$ is unique by 2.4.7, then there exists a non-zero constant $C \in \mathbb{C}$ so that

$$F_{\chi,{}^\omega\xi}(T_\omega \phi_{K,\xi}) = C F_{\chi,\xi}(\phi_{K,\xi}). \tag{2.-12}$$

In [CS80, prop. 4.3, p. 223], the author shows that $C = c_\omega(\xi)\rho_\omega(\xi^{-1})/\rho_\omega(\xi)$. Furthermore, by proposition 2.3.62, we have

$$F_{\chi,{}^\omega\xi}(T_\omega \phi_{K,\xi}) = c_\omega(\xi) F_{\chi,{}^\omega\xi}(\phi_{K,{}^\omega\xi}). \tag{2.-12}$$

Plugging (2.4.6) in (2.4.6) we conclude. ☐

Proposition 2.4.46. *The function $W_{\chi,\xi}(\pi(t)\phi_K)/\rho_{\omega_l}(\xi)$ is $W(G,T_s)(\mathbb{Q}_p)-$invariant as a function on ξ.*

Proof. Lemma 2.4.45 reduces the statement to the proof of the following equality:

$$\frac{\rho_{\omega_l}({}^{\omega}\xi)}{\rho_{\omega_l}(\xi)} = \frac{\rho_{\omega}(\xi^{-1})}{\rho_{\omega}(\xi)},$$

for every $\omega \in W(G,T)(\mathbb{Q}_p)$. See [CS80, cor. 5.3, p. 225] for the details of the computation. □

Theorem 2.4.47. *For $t \in T_s^-$*

$$W_{\chi,\xi}(\pi(t)\phi_{K,\xi}) = \rho_{\omega_l}(\xi)\prod_{\alpha\in\Phi^+}\frac{1}{1-\xi(a_\alpha)^{-1}}\sum_{\omega\in W(G,T_s)(\mathbb{Q}_p)}(-1)^{l(\omega)}\prod_{\substack{\beta\in\Phi^+\\ \omega\alpha<0}}\xi(a_\alpha)\ {}^{\omega}\xi\delta^{1/2}(t).$$

Proof. Propositions 2.4.42 and 2.4.43 show the equality

$$W_{\chi,\xi}(\pi(t)\phi_K)/\rho_{\omega_l}(\xi) = \sum_{\substack{\omega\in W(G,T_s)(\mathbb{Q}_p)\\ \omega\neq\omega_l}}\rho_{\omega_l}(\xi)^{-1}c_{\omega}(\xi)\ {}^{\omega}\xi\delta_B^{1/2}(t)W_{\chi,\xi}(f_{\omega,\xi}) + \rho_{\omega_l}(\xi)^{-1}c_{\omega_l}(\xi)\ {}^{\omega_l}\xi\delta_B^{1/2}(t). \tag{2.-12}$$

By direct computation

$$\rho_{\omega_l}(\xi)^{-1}c_{\omega_l}(\xi) = \prod_{\alpha>0}\frac{1}{1-\xi(a_\alpha)} = \prod_{\alpha>0}\frac{1}{1-\ {}^{\omega_l\omega_l}\xi(a_\alpha)}.$$

Since ω_l sends all the positive roots to the negative roots, it hold that ${}^{\omega_l\omega_l}\xi(a_\alpha) ={}^{\omega_l}\xi(a_\alpha)^{-1}$, obtaining

$$\rho_{\omega_l}(\xi)^{-1}c_{\omega_l}(\xi)\ {}^{\omega_l}\xi\delta_B^{1/2}(t) = \prod_{\alpha>0}\frac{1}{1-\ {}^{\omega_l}\xi(a_\alpha)^{-1}}\ {}^{\omega_l}\xi\delta_B^{1/2}(t). \tag{2.-12}$$

We have chosen ξ regular, then the set $\{{}^{\omega}\xi\}_{\omega\in W(G,T_s)(\mathbb{Q}_p)}$ is a linear independent set of characters. Using proposition 2.4.46, lemma 2.4.43 and the equation (2.4.6) recursively, we obtain

$$W_{\chi,\xi}(\pi(t)\phi_K)/\rho_{\omega_l}(\xi) = \sum_{\omega\in W(G,T_s)(\mathbb{Q}_p)}\prod_{\alpha>0}\frac{1}{1-\ {}^{\omega}\xi(a_\alpha)^{-1}}\ {}^{\omega}\xi\delta_B^{1/2}(t).$$

By a formal manipulation the result follows, see [CS80, p. 226-227] for the details. □

2.4.7 Casselman Shalika formula for the Shalika model of GL_{2n}.

In this subsection we explain the main result of [Sak06], an explicit formula for the Shalika model of GL_{2n}. Even thought the proofs of [Sak06] and [CS80] follow a similar philosophy, their techniques differ in certain aspects.

We consider the group GL_{2n} defined over $\mathbb{Q}_p$ and $K = \mathrm{GL}_{2n}(\mathbb{Z}_p)$. We fix the upper triangular Borel $B_{\mathrm{GL}} = T_{\mathrm{GL}}U_{\mathrm{GL}}$. This choice of Borel determines a set of positive roots that we will denote by Φ^+_{GL}. In this subsection we fix an unramified character of $T_{\mathrm{GL}}(\mathbb{Q}_p)$ of the form:

$$\xi = \xi_1 \otimes ... \otimes \xi_n \otimes \xi_n^{-1} \otimes ... \otimes \xi_1^{-1}. \tag{2.-12}$$

We consider the representation $\mathrm{Ind}_{B_{\mathrm{GL}}}^{\mathrm{GL}_{2n}}\xi$. The main goal of this subsection is to obtain an explicit formula for

$$\Lambda_{\mathrm{GL},\xi}\left(\pi(g)\phi_K\right),$$

where $\phi_K = \mathcal{P}_{\xi}\left(\mathrm{char}_K\right) = \delta_B^{-1/2}\xi$ and $\Lambda_{\mathrm{GL},\xi}$ is an explicit choice of the Shalika functional defined in 2.4.16. Let us recall that we assume that the character ξ is also a character of the torus of Sp_{2n} because of proposition 2.4.18 if the character ξ is not of this form. the representation does not admit a Shalika functional. Furthermore, by theorem 2.4.17, the representation $\mathrm{Ind}_{B_{\mathrm{GL}}}^{\mathrm{GL}_{2n}}\xi$ has at most multiplicity one with respect to the Shalika model. Therefore the functional $\Lambda_{\mathrm{GL},\xi}$ is unique up to constant. For the time being, we do not choose a particular Shalika functional.

Remark 2.4.48. *Since the element ϕ_K is $K-$invariant and the Shalika functonal satisfies the functional equation of 2.4.16, we just need to compute the value of $\Lambda_{\mathrm{GL},\xi}(\pi(g)\phi_K)$ in representatives of the double quotient $S_{\mathrm{GL}}(\mathbb{Q}_p) \setminus \mathrm{GL}_{2n}(\mathbb{Q}_p)/\mathrm{GL}_2(\mathbb{Z}_p)$. Using the Iwasawa and Cartan decompositions we obtain that this double quotient has representatives of the form*

$$g_\lambda = \begin{pmatrix} p^\lambda I_n & \\ & I_n \end{pmatrix},$$

where $p^\lambda = \begin{pmatrix} p^{\lambda_1} & & & \\ & p^{\lambda_2} & & \\ & & \ddots & \\ & & & p^{\lambda_n} \end{pmatrix}$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Furthermore, in [Sak06, p. 12] it is proved that the Shalika functional vanishes when $\lambda_i \leq 0$, then from now on we restrict our computations to g_λ with $\lambda_i > 0$. We recall that 2.1.56 shows that the vector defined by λ is a dominant weight of GL_n.

By lemma 2.3.52 there is a distribution $\Delta_{\mathrm{GL}_{2n},\xi} \in \mathcal{D}(\mathrm{GL}_{2n}(\mathbb{Q}_p))_{\xi^{-1}}$ so that for all $f \in \mathcal{C}_c^\infty(\mathrm{GL}_{2n}(\mathbb{Q}_p))$

$$\Lambda_{\mathrm{GL},\xi}(\mathcal{P}_\xi(f)) = \int_{\mathrm{GL}_{2n}(\mathbb{Q}_p)} \Delta_{\mathrm{GL}_{2n},\xi}(x) f(x)dx = \Delta_{\mathrm{GL}_{2n},\xi}(f)$$

If there is no confusion with the subindex will supress $2n$ from the notation. We recall that since $\mathcal{P}_\xi$ is a $\mathrm{GL}_{2n}(\mathbb{Q}_p)-$intertwining map, we have

$$\Lambda_{\mathrm{GL},\xi}\left(\pi(g)\phi_K\right) = \Delta_{\mathrm{GL}_{2n},\xi}(R_g \mathrm{char}_K),$$

where R_g is the right translation by $\mathrm{GL}_{2n}(\mathbb{Q}_p)$ on $\mathcal{C}_c^\infty(\mathrm{GL}_{2n}(\mathbb{Q}_p))$. Using that $\Delta_{\mathrm{GL},\xi} \in \mathcal{D}(\mathrm{GL}_{2n}(\mathbb{Q}_p))_{\xi^{-1}}$ and the fact that it is the distribution associated to the Shalika functional $\Lambda_{\mathrm{GL},\xi}$, it satisfies

$$\Delta_{\mathrm{GL},\xi}(bgs) = \xi^{-1}\delta^{1/2}(b)\chi_{p,\mathrm{GL}}(s)\Delta_{\mathrm{GL},\xi}(g), \tag{2.-12}$$

for any $s \in S_{\mathrm{GL}}(\mathbb{Q}_p)$ and $b \in B_{\mathrm{GL}}(\mathbb{Q}_p)$. This functional equation implies that the distribution Δ_{GL} is determined by their values on representatives of the coset

$$B_{\mathrm{GL}}(\mathbb{Q}_p) \setminus \mathrm{GL}_{2n}(\mathbb{Q}_p)/S_{\mathrm{GL}}(\mathbb{Q}_p). \tag{2.-12}$$

Let

$$\beta := \left(\begin{smallmatrix} & I_n \\ J_n & \end{smallmatrix}\right),$$

with J_n the $n \times n$ antidiagonal matrix and all entries different from 0 equal to 1. In [Sak06, lem. 3.1, p. 8] the author shows that the double quotient (2.4.7) has a Zariski open coset in $\mathrm{GL}_{2n}(\mathbb{Q}_p)$

$$B_{\mathrm{GL}}(\mathbb{Q}_p)\beta S_{\mathrm{GL}}(\mathbb{Q}_p).$$

In order to simplify the forthcoming computations we modify the Shalika functional.

Definition 2.4.49. Let $H := \beta S_{\mathrm{GL}}(\mathbb{Q}_p)\beta^{-1}$. We denote

$$\begin{aligned} \chi_{p,H} : \ & H \to \mathbb{C}^1, \\ & h \mapsto \chi_{p,\mathrm{GL}}(\beta^{-1}h\beta). \end{aligned}$$

Lemma 2.4.50. *The map*

$$\begin{aligned} i : \ & \mathrm{Ind}_{S_{\mathrm{GL}}}^{\mathrm{GL}}\chi_{p,\mathrm{GL}} \to \mathrm{Ind}_H^{\mathrm{GL}}\chi_{p,H}, \\ & f(\cdot) \mapsto f(\beta^{-1}\cdot), \end{aligned}$$

is a bijective $\mathrm{GL}_{2n}(\mathbb{Q}_p)-$*intertwining map.*

Proof. It follows directly. □

Definition 2.4.51. Let us consider the map

$$\begin{aligned} F^S_{\mathrm{GL},\xi} :\ &\mathrm{Ind}^{\mathrm{GL}}_{B_{\mathrm{GL}}}\xi \to \mathrm{Ind}^{\mathrm{GL}}_{S_{\mathrm{GL}}}\chi_{p,\mathrm{GL}} \\ &\phi(\cdot) \mapsto \Lambda_{\mathrm{GL},\xi}(\pi(\cdot)\phi). \end{aligned}$$

We denote by $\Lambda_{H,\xi} := i \circ F^S_{\mathrm{GL},\xi}$ its associated functional and by $\Delta_{H,\xi}$ its associated distribution.

It is straightforward

$$\Lambda_{\mathrm{GL},\xi}\left(\pi(g_\lambda)\phi_K\right) = \Lambda_{H,\xi}\left(\pi(g_{-\lambda})\phi_K\right).$$

Proposition 2.4.52. *Given a regular character* ξ *and an element* $\phi \in \mathrm{Ind}^{\mathrm{GL}_{2n}}_{B_{\mathrm{GL}}}\xi$ *with* $\mathrm{supp}(\phi) \subset PH$*, the integral*

$$\int_{B_{\mathrm{GL}}(\mathbb{Q}_p)\cap H\backslash H} \phi(h)\chi_{p,H}(h)^{-1}dh,$$

converges and defines a Shalika functional (after the change of coordinates given by 2.4.51). Moreover, it extends uniquely to a Shalika functional.

Proof. It is a combination of [Sak06, cor. 5.5, p. 13], [Sak06, prop. 7.1, p. 15] and [Sak06, prop. 7.2, p. 16] under the assumption that ξ is regular. If ξ were not regular, theorem 2.4.18 would imply that the only Shalika functional is equal to 0. □

From now on, we fix the Shalika functional given by proposition 2.4.52 and we denote it by $\Lambda_{H,\xi}$. Let us note that this step of the proof is analogous to the proposition 2.4.35 in the case of the Casselman-Shalika formula for Whittaker models.

Definition 2.4.53. We consider the following projection

$$\begin{aligned} \mathcal{P}_I :\ &\mathcal{D}(\mathrm{GL}_{2n}(\mathbb{Q}_p))_{\xi^{-1}} \to \left(\mathrm{Ind}^{\mathrm{GL}_{2n}}_{B_{\mathrm{GL}}}\xi^{-1}\right)^{I_{\mathrm{GL}}}, \\ &D(\cdot) \mapsto \int_{I^{\mathrm{GL}}} D(\cdot h)dh, \end{aligned}$$

where I_{GL} is the Iwahori subgroup of $\mathrm{GL}_{2n}(\mathbb{Q}_p)$ defined in 2.1.62 and db is the Haar measure of the Iwahori subgroup giving I_{GL} volume 1. We point out that remark 2.3.53 is used implicitly in this definition.

The projection map $\mathcal{P}_I$ plays an analogous role to the projection $\mathcal{P}_U$ of 2.4.40. Since the adjoint operator of R_g satisfies $R_g^* = R_{g^{-1}}$, we have

$$\Lambda_{H,\xi}(\pi(g)\phi_K) = \Delta_{H,\xi}(R_g\mathrm{char}_K) = R_{g^{-1}}\Delta_{H,\xi}(\mathrm{char}_K).$$

Considering char_K as a distribution it is straightforward that $\mathcal{P}_H\left(\mathrm{char}_K\right) = \mathrm{char}_K$ and hence

$$R_{g^{-1}}\Delta_{H,\xi}(\mathrm{char}_K) = R_{g^{-1}}\Delta_{H,\xi}(\mathcal{P}_I\left(\mathrm{char}_K\right)).$$

The adjoint operator of $\mathcal{P}_I$ satisfies $\mathcal{P}_I^* = \mathcal{P}_I$ and hence

$$R_{g^{-1}}\Delta_{H,\xi}(\mathcal{P}_I\left(\mathrm{char}_K\right)) = \mathcal{P}_I\left(R_{g^{-1}}\Delta_{H,\xi}(\mathrm{char}_K)\right).$$

Proposition 2.4.54. *The following equality holds:*

$$\Lambda_{H,\xi}(\pi(g)\phi_K) = Q^{-1}\sum_{\omega\in W(\mathrm{GL}_{2n},T_{\mathrm{GL}})(\mathbb{Q}_p)}\prod_{\substack{\alpha\in\Phi^+\\ \omega\alpha>0}} c_\alpha(\xi)T^*_{\omega^{-1}}\Delta_{H,{}^\omega\xi}(R_g\mathrm{char}_B),$$

where $Q = \frac{\mathrm{meas}(I\omega_l I)}{\mathrm{meas}(K)}$.

Proof. Let $\{f_{\omega,\xi^{-1}}\}_{\omega\in W(\mathrm{GL}_{2n},T_{\mathrm{GL}})(\mathbb{Q}_p)}$ be the Casselman basis defined in 2.4.37 for the representation $\mathrm{Ind}_{B_{\mathrm{GL}}}^{\mathrm{GL}_{2n}}\xi^{-1}$. By definition 2.4.53

$$\mathcal{P}_I(R_{g^{-1}}\Delta_{H,\xi})\in\left(\mathrm{Ind}_{B_{\mathrm{GL}}}^{\mathrm{GL}_{2n}}\xi^{-1}\right)^{I_{\mathrm{GL}}}.$$

Proposition 2.3.47 implies that there exists constants $a_\omega(g)\in\mathbb{C}$ so that

$$\mathcal{P}_I(R_{g^{-1}}\Delta_{H,\xi})=\sum_{\omega\in W(\mathrm{GL}_{2n},T_{\mathrm{GL}})(\mathbb{Q}_p)}a_\omega(g)f_{\omega,\xi^{-1}}. \tag{2.-12}$$

For any $\omega\in W(\mathrm{GL}_{2n},T_{\mathrm{GL}})(\mathbb{Q}_p)$ we apply the operator

$$T_\omega:\ \mathrm{Ind}_{B_{\mathrm{GL}}}^{\mathrm{GL}_{2n}}\xi^{-1}\to\mathrm{Ind}_{B_{\mathrm{GL}}}^{\mathrm{GL}_{2n}}\ {}^\omega\xi^{-1},$$

in both sides of (2.4.7). We evaluate the resulting function at 1, obtaining

$$T_\omega\mathcal{P}_I(R_{g^{-1}}\Delta_{H,\xi})(1)=a_\omega(g). \tag{2.-12}$$

Be aware that $R_{g^{-1}}\Delta_{H,\xi}\in\mathcal{D}(\mathrm{GL}_{2n}(\mathbb{Q}_p))_{\xi^{-1}}$. Then, applying remark 2.3.63 in (2.4.7) (with ξ^{-1} instead of ξ because our intertwining map T_ω has domain $\mathrm{Ind}_{B_{\mathrm{GL}}}^{\mathrm{GL}_{2n}}\xi^{-1}$ instead of $\mathrm{Ind}_{B_{\mathrm{GL}}}^{\mathrm{GL}_{2n}}\xi$), we obtain

$$\frac{c_\omega(\xi^{-1})}{c_{\omega^{-1}}({}^\omega\xi)}\mathcal{P}_I(T^*_{\omega^{-1}}R_{g^{-1}}\Delta_{H,\xi})(1)=a_\omega(g).$$

Furthermore, by definition

$$\mathcal{P}_I(T^*_{\omega^{-1}}R_{g^{-1}}\Delta_{H,\xi})(1)=T^*_{\omega^{-1}}\Delta_{H,\xi}(R_g\mathrm{char}_{I_{\mathrm{GL}}}),$$

and then

$$a_\omega(g)=\frac{c_\omega(\xi^{-1})}{c_{\omega^{-1}}({}^\omega\xi)}T^*_{\omega^{-1}}\Delta_{H,\xi}(R_g\mathrm{char}_{I_{\mathrm{GL}}}).$$

Using remark 2.3.53, we consider $f_{\omega,\xi^{-1}}$ as a distribution of $\mathrm{Ind}_B^{\mathrm{GL}}\xi$. In the proof of [Cas80, thm. 4.2, p. 403] it is shown the following equality

$$f_\omega(\phi_K)=Q^{-1}\frac{c_{\omega_l}({}^\omega\xi)}{c_\omega(\xi^{-1})}.$$

By a formal manipulation of the constants $c_\omega(\xi)$, we obtain the result. See [Sak06, p. 11] for further details. □

Proposition 2.4.54 is analogous to the proposition 2.4.42, it reduces the computation of the Shalika functional to the computation of

$$T^*_{\omega^{-1}}\Delta_{H,{}^\omega\xi}(R_g\mathrm{char}_{I_{\mathrm{GL}}}).$$

For the following proposition we denote by T_{Sp} the maximal torus of the algebraic group Sp_{2n} defined over $\mathbb{Q}_p$. We fix the upper triangular Borel subgroup B_{Sp} of Sp_{2n} determining a set of positive roots Φ^+_{Sp}. Furthermore we will denote the set of positive short roots by Φ^{S+}_{Sp}, and we regard $W(\mathrm{Sp}_{2n},T_{\mathrm{Sp}})$ as a subgroup of $W(\mathrm{GL}_{2n},T_{\mathrm{GL}})$.

Proposition 2.4.55. *If $\omega\notin W(\mathrm{Sp}_{2n},T_{\mathrm{Sp}})(\mathbb{Q}_p)$ the distribution $T^*_{\omega^{-1}}\Delta_{H,{}^\omega\xi}$ is supported away from $B_{\mathrm{GL}}(\mathbb{Q}_p)H$.*

Proof. The proof of this result is in [Sak06, 5.2, p. 12]. For the sake of completeness we will give a formal sketch of the proof, considering $T^*_{\omega^{-1}}\Delta_{H,{}^\omega\xi}$ as a function instead of a distribution. Using the functional equation of the Shalika functional we have

$$T^*_{\omega^{-1}}\Delta_{H,{}^\omega\xi}(bgh)={}^\omega\xi^{-1}\delta^{1/2}_{B_{\mathrm{GL}}}(b)\chi_{p,H}(h)^{-1}T^*_{\omega^{-1}}\Delta_{H,{}^\omega\xi}(g).$$

Given $x \in B_{\mathrm{GL}}(\mathbb{Q}_p) \cap H$, we obtain

$$\chi_{p,H}(x)^{-1} T^*_{\omega^{-1}} \Delta_{H,{}^\omega\xi}(1) =^\omega \xi^{-1} \delta^{1/2}_{B_{\mathrm{GL}}}(x) T^*_{\omega^{-1}} \Delta_{H,{}^\omega\xi}(1),$$

and hence, if $T^*_{\omega^{-1}} \Delta_{H,{}^\omega\xi}(1) \neq 0$

$$\chi_{p,H}(x)^{-1} =^\omega \xi^{-1} \delta^{1/2}_{B_{\mathrm{GL}}}(x),$$

The group $B_{\mathrm{GL}}(\mathbb{Q}_p) \cap H$ consists on matrices of the form

$$\begin{pmatrix} \alpha_1 & & & & & \\ & \ddots & & & & \\ & & \alpha_n & & & \\ & & & \alpha_n & & \\ & & & & \ddots & \\ & & & & & \alpha_1 \end{pmatrix}.$$

On the one hand, $\chi_{p,H}(x)^{-1} = 1$. On the other hand, by the choice of ξ in (2.4.7)

$${}^\omega\xi^{-1} \delta^{1/2}_{B_{\mathrm{GL}}}(x) = 1,$$

if and only if $\omega \in W(\mathrm{Sp}_{2n}, T_{\mathrm{Sp}})(\mathbb{Q}_p)$. If ${}^\omega\xi^{-1} \delta^{1/2}_{B_{\mathrm{GL}}}(x) = 1$ is non-trivial, that would imply

$$T^*_{\omega^{-1}} \Delta_{H,{}^\omega\xi}(1) = 0.$$

□

Proposition 2.4.56. *Let $g_\lambda \in \mathrm{GL}_{2n}(\mathbb{Q}_p)$ with $\lambda = (\lambda_1, ..., \lambda_n)$ all entries $\lambda_i > 0$. Then*

$$a_{\mathrm{id}}(g_\lambda) = \xi^{-1} \delta^{1/2}(g_{-\lambda}).$$

Proof. See [Sak06, prop. 6.1, p. 14]. □

Proposition 2.4.57. *For $\omega \in W(\mathrm{Sp}_{2n}, T_{\mathrm{Sp}})(\mathbb{Q}_p)$*

$$\frac{a_\omega(g_{-\lambda})}{a_\omega(g_0)} =^\omega \xi^{-1} \delta^{1/2}(g_\lambda).$$

Proof. The operator $T_{\omega^{-1}}$ is a $\mathrm{GL}_{2n}(\mathbb{Q}_p)$–intertwining map. Then $T_{\omega^{-1}}(\Lambda_{H,\xi})$ is a Shalika functional of $\mathrm{Ind}^{\mathrm{GL}_{2n}}_{B_{\mathrm{GL}}} {}^\omega\xi$. By theorem 2.4.17, the space of Shalika functionals is one dimensional, then there exists a non-zero constant c so that

$$T^*_{\omega^{-1}}(\Delta_{H,\xi}) = c \Delta_{H,{}^\omega\xi}.$$

By proposition 2.4.56

$$\Delta_{\tilde{S},{}^\omega\xi}(R_{g_{-\lambda}} \phi_B) = {}^\omega\xi^{-1} \delta^{1/2}(g_{-\lambda}). \tag{2.-12}$$

Then

$$\frac{a_\omega(g_{-\lambda})}{a_\omega(g_0)} = \frac{c\, {}^\omega\xi^{-1} \delta^{1/2}(g_{-\lambda})}{c\, \xi^{-1} \delta^{1/2}(I_{2n})} = \frac{c\, {}^\omega\xi^{-1} \delta^{1/2}(g_{-\lambda})}{c} = {}^\omega\xi^{-1} \delta^{1/2}(g_{-\lambda}).$$

□

Proposition 2.4.57 is the replacement for proposition 2.4.46 in the present setting.

Theorem 2.4.58. *The following equality holds:*

$$\Lambda_{H,\xi}(\pi(g)\phi_K) = Q^{-1} \prod_{\alpha \in \Phi^+_{\mathrm{GL}}} c_\alpha(\xi) \sum_{\omega \in W(\mathrm{GL}_{2n}, T_{\mathrm{GL}})(\mathbb{Q}_p)} (-1)^{l(\omega)} \prod_{\substack{\alpha \in \Phi^+_{\mathrm{Sp}} \\ \omega\alpha > 0}} \xi(a_\alpha) \prod_{\substack{\alpha \in \Phi^{S+}_{\mathrm{Sp}} \\ \omega\alpha < 0}} \frac{1 - p^{-1}\xi(a_\alpha)}{1 - p^{-1}\xi(a_\alpha)} \, {}^\omega\xi^{-1} \delta^{1/2}(g_\lambda).$$

Proof. It is [Sak06, (17), p. 7]. It is based on proposition 2.4.57 and the explicit description of the functional given in proposition 2.4.52. □

2.5 Theta correspondence for similitude groups

In this section we recall the construction of the local and global theta correspondence for similitude groups. Throughout this section we fix V a quadratic space of even dimension m, W a symplectic space of dimension $2n$ and we form $\mathbb{W} = V \otimes_{\mathbb{Q}} W$, a the symplectic space, where we fix a polarization of the form $\mathbb{W} = X \oplus Y$. We will use the following notation $W_p := W \otimes_{\mathbb{Q}} \mathbb{Q}_p$, $V_p := V \otimes_{\mathbb{Q}} \mathbb{Q}_p$ and $\mathbb{W}_p := \mathbb{W} \otimes_{\mathbb{Q}} \mathbb{Q}_p$.

The main difference between the theta correspondence for isometry groups and the theta correspondence for similitude groups is that there is no embedding of $\mathrm{GO}(V) \times \mathrm{GSp}(W)$ in any metaplectic group and we are not able to use directly the construction given in section 1.5. In order to define a replacement for the Weil representation, we restrict the setting to a subgroup denoted by $\mathrm{GSp}^+(W) \times \mathrm{GO}(V)$, where the group $\mathrm{GSp}^+(W)$ depends on the group $\mathrm{GSO}(V)$. At first, it seems that this construction does not give us information about the automorphic representations of $\mathrm{GSp}(W)$. But in fact, following Morimoto [Mor14, thm. 3.6, p. 45], given an automorphic representation π of $\mathrm{GSp}^+(W)$ we are able to construct an automorphic representation of $\mathrm{GSp}(W)$ and then realize the correspondence in terms of the group $\mathrm{GSp}(W)$.

2.5.1 Replacement of the Weil representation

Let p be a place of $\mathbb{Q}$. Throughout this section we fix the character $\psi_p : \mathbb{Q}_p \to \mathbb{C}^1$ defined in 1.5.1. Since V has even dimension, remark 1.5.21 shows that the Weil representation can be realized as follows:

$$\omega_p : \mathrm{Sp}(W_p)(\mathbb{Q}_p) \times \mathrm{O}(V_p)(\mathbb{Q}_p) \to \mathrm{Aut}(\mathcal{S}(Y(\mathbb{Q}_p))).$$

Due to the lack of an embedding of the form

$$\mathrm{GSp}(W_p)(\mathbb{Q}_p) \times \mathrm{GO}(V_p)(\mathbb{Q}_p) \to \mathrm{Sp}(\mathbb{W}_p)(\mathbb{Q}_p),$$

we have to define a replacement of the Weil representation. It must be constructed in a way such that the resulting correspondence satisfies a Howe duality property as in 1.5.26. Let us consider the group

$$\tilde{R} := \{(g,h) \in \mathrm{GSp}(W_p)(\mathbb{A}) \times \mathrm{GO}(V_p)(\mathbb{A}),\ s.t.\ \lambda(g)\nu(h) = 1\}.$$

By [Rob96, p. 14], the group $\mathrm{GO}(V_p)(\mathbb{Q}_p)$ acts naturally by left translations on $\mathcal{S}(Y(\mathbb{Q}_p))$. Furthermore, according to [GT11, p. 1847], the Weil representation extends naturally to the full group $\tilde{R}$ by

$$\omega_p(g,h)\varphi(x) = |\lambda(h)|^{-\frac{\dim V \dim W}{8}} \omega_p(g_1, 1)\varphi(h^{-1}x), \tag{2.-12}$$

where

$$g_1 := g \begin{pmatrix} \lambda(g)^{-1} I_n & \\ & I_n \end{pmatrix} \in \mathrm{Sp}(W_p)(\mathbb{Q}_p).$$

Although the Weil representation is defined for the group $\tilde{R}$, we can not define a theta correspondence yet because we are not able to factor $\tilde{R}$ as a product of two different topological groups. Let us consider the group

$$\mathrm{GSp}^+(W_p)(\mathbb{Q}_p) = \{g \in \mathrm{GSp}(W_p)(\mathbb{Q}_p),\ s.t.\ \lambda(g) = \nu(h)\ for\ some\ g \in \mathrm{GO}(V_p)(\mathbb{Q}_p)\}.$$

It is straightforward that $\tilde{R} \subset \mathrm{GSp}^+(W_p)(\mathbb{Q}_p) \times \mathrm{GO}(V_p)(\mathbb{Q}_p)$.

Definition 2.5.1. Let $\Omega_p = \mathrm{c} - \mathrm{Ind}_{\tilde{R}}^{\mathrm{GSp}^+(W_p)\times\mathrm{GO}(V_p)} \omega_p$,the *induced Weil representation* is the map of the form

$$\Omega_p : \mathrm{GSp}^+(W_p)(\mathbb{Q}_p) \times \mathrm{GO}(V_p)(\mathbb{Q}_p) \to \mathrm{Aut}\left(\Omega_p\right),$$

defined by the formula (2.5.1) and the induction defined in 2.3.11.

For further information about this representation we refer the reader to [Rob96, p. 17] (in [Rob96] this representation is denoted by Ω^+). It is shown in [Rob96, sec. 4, p. 18] and [Rob96, sec. 5, p. 24] that the representation Ω_p inherits crucial properties of the representation ω_p via Frobenius reciprocity, allowing us to define a correspondence between the groups $(\mathrm{GSp}^+(W_p)(\mathbb{Q}_p), \mathrm{GO}(V_p)(\mathbb{Q}_p))$ which is in fact intimately related to the classical theta correspondence between $(\mathrm{Sp}(W_p)(\mathbb{Q}_p), \mathrm{O}(V_p)(\mathbb{Q}_p))$.

2.5.2 Local theta correspondence

Let π_p be an admissible representation of $\mathrm{GO}(V_p)(\mathbb{Q}_p)$. In order to construct the local theta correspondence we proceed as in 1.5.24. The maximal π_p-isotypic quotient of Ω_p is of the form

$$\pi_p \otimes \widetilde{\Theta}(\pi_p), \tag{2.-12}$$

where $\widetilde{\Theta}(\pi_p)$ is an admissible representation of $\mathrm{GSp}^+(W_p)(\mathbb{Q}_p)$.

Definition 2.5.2. Let π_p be an admissible representation of $\mathrm{GO}(V_p)(\mathbb{Q}_p)$. The $\mathrm{GSp}^+(W_p)(\mathbb{Q}_p)-$representation $\widetilde{\Theta}(\pi_p)$ obtained in (2.5.2) is called the *local big theta lift* of π_p. There is a unique maximal semisimple quotient of $\widetilde{\Theta}(\pi_p)$, denoted by $\widetilde{\theta}(\pi_p)$ and called the *local theta lift.*

Remark 2.5.3. *As in the theta correspondence for isometry groups. Given a irreducible representation σ_p of $\mathrm{GSp}^+(W_p)(\mathbb{Q}_p)$, the constructions of the big theta lift $\widetilde{\Theta}(\sigma_p)$ and the theta lift $\widetilde{\theta}(\sigma_p)$ are completely analogous to the definition 2.5.2.*

Theorem 2.5.4 (Howe duality)**.** *Given a irreducible smooth admissible representation π_p of $\mathrm{GO}(V_p)(\mathbb{Q}_p)$ so that $\pi_p|_{\mathrm{O}(V_p)(\mathbb{Q}_p)}$ is multiplicity free, then*

1. *The representation $\widetilde{\Theta}(\pi_p)$ is admissible of finite length.*
2. *The representation $\widetilde{\theta}(\pi_p)$ is irreducible.*
3. *If $\widetilde{\theta}(\pi_1)$ and $\widetilde{\theta}(\pi_2)$ are non-zero and isomorphic, then $\pi_1 \simeq \pi_2$.*

The analogous theorem also follows for representations of $\mathrm{GSp}^+(W_p)(\mathbb{Q}_p)$.

Proof. The proof is based on the transfer of the Howe duality for isometry groups to the similitude groups. The first assertion follows by [GT11, lem. 2.2, (iii), p. 1848]. The second and third assertions are given by [Rob96, thm. 5.2, p. 24] under the assumption of the Howe duality conjecture. □

Lemma 2.5.5. *Suppose that π_p is an irreducible representation of $\mathrm{GO}(V_p)(\mathbb{Q}_p)$ and take τ_p a constituent of the $\mathrm{O}(V_p)-$representation $\pi_p|_{\mathrm{O}(V_p)(\mathbb{Q}_p)}$. Then $\theta(\tau_p) \neq 0$ implies $\widetilde{\theta}(\pi_p) \neq 0$. The result for representations of $\mathrm{GSp}^+(W_p)(\mathbb{Q}_p)$ is completely analogous.*

Proof. See [GT11, lem. 2.2, (i), p. 1848]. □

Lemma 2.5.6. *Let π_1 and π_2 be irreducible admissible representations of $\mathrm{GO}(V_p)(\mathbb{Q}_p)$ and $\mathrm{GSp}^+(W_p)(\mathbb{Q}_p)$ respectively. Suppose*

$$\mathrm{Hom}_R(\Omega_p, \pi_1 \otimes \pi_2) \neq 0,$$

then $\pi_1|_{\mathrm{Sp}(W_p)(\mathbb{Q}_p)}$ and $\pi_2|_{\mathrm{O}(V_p)(\mathbb{Q}_p)}$ are semisimple and multiplicity free.

Proof. It follows by Frobenius reciprocity, see [GT11, lem. 2.2, (i), p. 1848]. □

Lemma 2.5.7. *Let π_1 and π_2 be irreducible admissible representations of $\mathrm{GO}(V_p)(\mathbb{Q}_p)$ and $\mathrm{GSp}^+(W_p)(\mathbb{Q}_p)$ respectively. Suppose*

$$\mathrm{Hom}_R(\Omega, \pi_2 \otimes \pi_1) \neq 0,$$

and that, for each constituent τ_1 *of* $\pi_1|_{\mathrm{O}(V_p)(\mathbb{Q}_p)}$*, the representation* $\theta(\tau_1)$ *is irreducible and the theta correspondence is injective in* $\pi_1|_{\mathrm{O}(V_p)(\mathbb{Q}_p)}$*. Then, there is a uniquely determined bijection*

$$f:\ \{irr\ summands\ of\ \pi_1|_{\mathrm{O}(V_p)(\mathbb{Q}_p)}\} \to \{irr\ summands\ of\ \pi_2|_{\mathrm{Sp}(W_p)(\mathbb{Q}_p)}\},$$

so that, for any τ_i *in* $\pi_i|_{\mathrm{O}(V_p)(\mathbb{Q}_p)}$*, then* $\tau_2 = f(\tau_1)$ *if and only if* $\mathrm{Hom}_{\mathrm{O}(V_p)\times\mathrm{Sp}(W_p)}(\omega_p, \tau_1 \otimes \tau_2) \neq 0$.

Proof. It is [GT11, lem. 2.2, p. 1848]. □

Remark 2.5.8. *According to theorem 1.5.26, if the representation* $\pi_1|_{\mathrm{O}(V_p)(\mathbb{Q}_p)}$ *is multiplicity free, the second assumption in lemma 2.5.7 is always true (under the assumption of the Howe duality conjecture). We refer the reader to [AP06, prop. 11.1, p. 20], where an explicit example of a multiplicity one result for the restriction is given.*

Lemma 2.5.9. *Set* π_p *an admissible representation of* $\mathrm{GO}(V_p)(\mathbb{Q}_p)$ *so that the* $\mathrm{O}(V_p)(\mathbb{Q}_p)-$*representation* $\pi_p|_{\mathrm{O}(V_p)(\mathbb{Q}_p)}$ *is semisimple. We denote its factorization into irreducibles by* $\pi_p|_{\mathrm{O}(V_p)(\mathbb{Q}_p)} \simeq \bigoplus_i \tau_{p,i}$*. Then*

$$\widetilde{\Theta}(\pi_p)|_{\mathrm{Sp}(W_p)(\mathbb{Q}_p)} \simeq \bigoplus_i \Theta(\tau_{p,i}).$$

The representation $\widetilde{\Theta}(\pi_p)$ *is admissible and of finite length. Moreover, if* $\Theta(\tau_{p,i}) = \theta(\tau_{p,i})$ *for all* i*, then* $\widetilde{\Theta}(\pi_p) = \widetilde{\theta}(\pi_p)$*. The result for* $\mathrm{GSp}^+(W_p)(\mathbb{Q}_p)-$*representations is completely analogous.*

Proof. See [GT11, lem. 2.2, (iii), p. 1848]. □

The following proposition specializes to the case of interest in this dissertation. This result is not in general true, even though it is proved for other combination of groups as in [GT11, lem. 3.1, p. 1851].

Proposition 2.5.10. *Let us suppose that* W *is a symplectic space of dimension* 4*,* V *is a quadratic space of signature* $(2,4)$ *and*

$$\mathrm{Hom}_{\mathrm{GSp}^+(W_p)\times\mathrm{GO}(V_p)}(\Omega, \pi_1 \otimes \pi_2) \neq 0,$$

where π_1 *(respectively* π_2*) is an irreducible representation of* $\mathrm{GSp}^+(W_p)(\mathbb{Q}_p)$ *(respectively of* $\mathrm{GO}(V_p)(\mathbb{Q}_p)$*). Then, the* $\mathrm{GSO}(V_p)(\mathbb{Q}_p)-$*representation* $\pi_2|_{\mathrm{GSO}(V_p)(\mathbb{Q}_p)}$ *is irreducible. Furthermore, in the present case the theorem 2.5.4 holds without the Howe duality conjecture hypothesis.*

Proof. It is [Mor14, lem. 6.3, p. 68] and [Mor14, cor. 6.22, p. 81]. □

2.5.3 Global theta lift

Throughout this section we denote by W the symplectic space of dimension 4 endowed with the symplectic form J defined in 2.2.14. Furthermore, we fix $\mathcal{V}$ the quadratic space of signature $(2,4)$ and Witt index 2 defined in 2.2.21. This subsection will be devoted to describing an analogous global theta correspondence for similitude groups in a compatible way with the local theta correspondence addressed in section 2.5.2.

Fix a polarization of the form $\mathbb{W} = W \otimes_{\mathbb{Q}} \mathcal{V} = X \oplus Y$ and consider the character $\psi:\ \mathbb{Q}\setminus\mathbb{A} \to \mathbb{C}^1$ defined in (1.5.4). Throughout this subsection we will denote by $(\omega, \mathcal{S}(Y(\mathbb{A})))$ the Schrodinger model of the Weil representation of $\mathrm{Mp}(\mathbb{W})(\mathbb{A})$, see 1.5.31 for the details. We proceed in a similar way to the previous subsection. Let us consider the homomorphism

$$i:\ \mathrm{GSp}(W)(\mathbb{A}) \times \mathrm{GSO}(\mathcal{V})(\mathbb{A}) \to \mathrm{GSp}(\mathbb{W})(\mathbb{A}). \tag{2.-12}$$

It is straightforward that $\lambda(i(g,h)) = \lambda(g)\nu(h)^{-1}$. Therefore, the map (2.5.3) restricted to the subgroup

$$R := \{(g,h) \in \mathrm{GSp}(W)(\mathbb{A}) \times \mathrm{GSO}(\mathcal{V})(\mathbb{A}),\ s.t.\ \lambda(g) = \nu(h)\},$$

has image in $\mathrm{Sp}(\mathbb{W})(\mathbb{A})$. The previous proposition discussion allows us to consider the pullback of ω to the subgroup R.

Definition 2.5.11. Given $\varphi \in \mathcal{S}(Y(\mathbb{A}))$, we define the *similitude theta function* by

$$\theta(g,h,\varphi) = \sum_{z\in Y(\mathbb{Q})} \omega(g,h)\varphi(z),$$

with $(g,h) \in R$.

As in the local case, since R is not the product of two topological groups we are not able to state the theta correspondence yet. In fact, we consider a group between R and $\mathrm{GSp}(W)(\mathbb{A}) \times \mathrm{GSO}(\mathcal{V})(\mathbb{A})$, denoted by $\mathrm{GSp}^+(W)(\mathbb{A}) \times \mathrm{GSO}(\mathcal{V})(\mathbb{A})$, with

$$\mathrm{GSp}^+(W)(\mathbb{A}) := \{g \in \mathrm{GSp}(W)(\mathbb{A}),\ s.t.\ \lambda(g) = \nu(h)\ for\ some\ h \in \mathrm{GSO}(\mathcal{V})(\mathbb{A})\}.$$

Further we define $\mathrm{GSp}^+(W)(\mathbb{Q}) := \mathrm{GSp}^+(W)(\mathbb{A}) \cap \mathrm{GSp}(W)(\mathbb{Q})$. At first, this group has two disadvantages; the first one is that the theta kernel is not defined over $\mathrm{GSp}^+(W)(\mathbb{A}) \times \mathrm{GSO}(\mathcal{V})(\mathbb{A})$. We will approach this difficulty by translating our theta function by certain element of $\mathrm{GSp}^+(W)(\mathbb{A})$. Secondly, the group $\mathrm{GSp}^+(W)(\mathbb{A})$ depends both on the symplectic vector space W and on the quadratic space $\mathcal{V}$. To get rid of the dependence of $\mathcal{V}$, and then obtain results for the group $\mathrm{GSp}(W)(\mathbb{A})$, we will show that there is an explicit interplay between the representations of $\mathrm{GSp}(W)(\mathbb{A})$ and the representations of $\mathrm{GSp}^+(W)(\mathbb{A})$ given by an "induction and restriction" argument.

Remark 2.5.12. *We recall that the Bruhat decomposition 2.1.34 in the present setting shows the following factorization:*

$$\mathrm{GSO}(\mathcal{V})(\mathbb{A}) = \prod_p \bigsqcup_{w_{i,p}\in W_{\mathrm{GSO}(\mathcal{V})_{\mathbb{Q}_p}}} B_{\mathrm{GSO}(\mathcal{V})}(\mathbb{Q}_p) w_{i,p} B_{\mathrm{GSO}(\mathcal{V})}(\mathbb{Q}_p).$$

Since the similitude character of the elements of the Weyl group is equal to 1*, the similitude character of each element of* $\mathrm{GSO}(\mathcal{V})(\mathbb{A})$ *is determined by the similitude character of* $B_{\mathrm{GSO}(\mathcal{V})}(\mathbb{A})$*. Given* $b = tn \in B_{\mathrm{GSO}(\mathcal{V})}(\mathbb{A})$*, it is straightforward that* $\nu(b) = \nu(t)$ *and by (2.2.3),* $\nu(t) \in \mathrm{N}_{E/\mathbb{Q}}(\mathbb{A}_E^\times)$*. The previous discussion gives an alternative description of* $\mathrm{GSp}^+(W)(\mathbb{A})$*:*

$$\mathrm{GSp}^+(W)(\mathbb{A}) = \{g \in \mathrm{GSp}(W)(\mathbb{A}),\ s.t.\ \lambda(g) \in \mathrm{N}_{E/\mathbb{Q}}(\mathbb{A}_E^\times)\}.$$

Definition 2.5.13. Throughout this section we say that a smooth moderate growth function f defined on $\mathrm{GSp}^+(W)(\mathbb{A})$ is an *automorphic form* if it is

- $\mathrm{GSp}^+(W)(\mathbb{Q})-$invariant,
- $K-$finite,
- $Z(U(\mathrm{Lie}(\mathrm{GSp}^+(W)(\mathbb{R}))))-$finite

where K is a maximal compact subgroup of $\mathrm{GSp}(W)(\mathbb{A})$ and $U(\mathrm{Lie}(\mathrm{GSp}^+(W)(\mathbb{R})))$ is the universal envelopping algebra of $\mathrm{Lie}(\mathrm{GSp}^+(W)(\mathbb{R}))$. We say that an automorphic form f on $\mathrm{GSp}^+(W)(\mathbb{A})$ is a *cusp form* if its restriction to $\mathrm{Sp}(W)(\mathbb{A})$ is a cusp form.

Definition 2.5.14. Let π be a cuspidal automorphic representation of $\mathrm{GSp}^+(W)$. Given $\varphi \in \pi$ we define $\varphi^* :\ \mathrm{GSp}(W)(\mathbb{A}) \to \mathbb{C}$ by

$$\varphi^*(g) = \begin{cases} \varphi(g), & if\ g \in \mathrm{GSp}^+(W)(\mathbb{Q})\mathrm{GSp}^+(W)(\mathbb{A}), \\ 0 & o.w. \end{cases}$$

The function φ^* is an automorphic form of $\mathrm{GSp}(W)$. We denote by π^* the automorphic representation generated by φ^*.

Definition 2.5.15. Let f be a cusp form on $\mathrm{GSp}^+(W)(\mathbb{A})$ and $\varphi \in \mathcal{S}(Y(\mathbb{A}))$. The *theta lift* of f is defined by

$$\theta(h,f,\varphi) = \int_{[\mathrm{Sp}(W)]} \theta(g_1 g, h, \varphi)\overline{f(g_1 g)}dg_1,$$

where $g \in \mathrm{GSp}^+(W)(\mathbb{A})$ is chosen so that $\lambda(g) = \nu(h)$.

Proposition 2.5.16. *Given f a cusp form on* $\mathrm{GSp}^+(W)(\mathbb{A})$ *and* $\varphi \in \mathcal{S}(Y(\mathbb{A}))$*, the theta lift* $\theta(h, f, \varphi)$ *is an automorphic form on* $\mathrm{GSO}(\mathcal{V})$*.*

Definition 2.5.17. Let f' be an automorphic cusp form on $\mathrm{GSO}(\mathcal{V})$ and $\varphi \in \mathcal{S}(Y(\mathbb{A}))$. The *theta lift* of f' is defined by

$$\theta(g, f', \varphi) = \int_{[\mathrm{SO}(\mathcal{V})]} \theta(g, h_1 h, \varphi)\overline{f(h_1 h)} dh_1,$$

where h is chosen such that $\nu(h) = \lambda(g)$.

Proposition 2.5.18. *Given f' a cusp form on* $\mathrm{GSO}(\mathcal{V})(\mathbb{A})$ *and* $\varphi \in \mathcal{S}(Y(\mathbb{A}))$*, the theta lift* $\theta(g, f', \varphi)$ *is an automorphic form on* $\mathrm{GSp}^+(W)(\mathbb{A})$*.*

Definition 2.5.19. Given π an automorphic cuspidal representation of $\mathrm{GSp}^+(W)$ the *global theta correspondence* of π is defined by

$$\widetilde{\Theta}^{glob}(\pi) := \{\theta(g, f, \varphi),\ f \in \pi, \varphi \in \mathcal{S}(Y(\mathbb{A}))\}.$$

For an automorphic representation σ of $\mathrm{GSO}(\mathcal{V})$, its global theta lift $\widetilde{\Theta}^{glob}(\sigma)$ is defined in an analogous way.

Proposition 2.5.20. *Given cuspidal automorphic representation π of* $\mathrm{GSp}^+(W)$*, we have*

$$\widetilde{\Theta}^{glob}(\pi) \simeq \bigotimes_p \widetilde{\theta}(\pi_p)|_{\mathrm{GSO}(\mathcal{V})(\mathbb{Q}_p)}.$$

This result also follows for the cuspidal automorphic representations of $\mathrm{GSO}(\mathcal{V})$*.*

Proof. The definitions of the local and global theta correspondence for isometry and similitude groups are analogous. Using theorem 2.5.4 we can adapt the proof of proposition 1.5.50 for the present case, obtaining a surjective $\mathrm{GSO}(\mathcal{V})(\mathbb{A})-$intertwining map

$$\otimes_p \widetilde{\Theta}(\pi_p) \to \widetilde{\Theta}^{glob}(\pi_p).$$

We conclude using the second statement of theorem 2.5.4 and proposition 2.5.10 as in the proof of corollary 1.5.51. □

For the following theorem we recall that given $(\mathcal{V}_0, q_p)$ an anisotropic rational quadratic space, we may construct the following quadratic space $\mathcal{V}_r := \mathcal{V}_0 \oplus \mathbb{H}^r$ of $r \geq 1$. We attach the similitude orthogonal group $\mathrm{GSO}(\mathcal{V}_r)$ to the above space.

Definition 2.5.21. The chain of groups

$$\mathrm{GSO}(\mathcal{V}_0) \subset \mathrm{GSO}(\mathcal{V}_1) \subset ...$$

is called the *Rallis tower*.

We consider the dual reductive pairs of the form $(\mathrm{GSp}(W), \mathrm{GSO}(\mathcal{V}_r))$ and we denote its associated theta correspondence by Θ_r^{glob}.

Theorem 2.5.22. *Let σ be a cuspidal automorphic representation of the group* $\mathrm{GSp}(W)$ *and let i be the smallest integer such that* $\widetilde{\Theta}_i^{glob}(\sigma) \neq 0$*, then*

1. $i \leq 2n$*,*
2. $\widetilde{\Theta}_i^{glob}(\sigma)$ *is cuspidal,*
3. $\widetilde{\Theta}_r^{glob}(\sigma) \neq 0$ *for* $r \geq i$*.*

Proof. It follows by proposition 2.5.5, theorem 2.5.4 and theorem 1.5.54. □

2.5.4 Theta correspondence and relation between models

In this subsection we explain the results of [Mor14]. The theta correspondence can be used to realize an equivalence between the Shalika model for the group $\mathbf{G}$ and the Whittaker model for the group $\mathbf{H}$.

Throughout this section we will consider the theta correspondence for the pair $(\mathbf{H}, \mathrm{GSO}_{4,2})$. When the input are automorphic representations with trivial central character, proposition 2.2.27 allows us to extend the correspondence for the pair of groups $(\mathbf{H}, \mathbf{G})$. Let us fix the character $\psi : \mathbb{Q} \setminus \mathbb{A} \to \mathbb{C}^1$ given in (1.5.4). Further, let us denote by U the maximal unipotent subgroup of $\mathrm{GSO}_{4,2}$ given in (2.2.3).

Definition 2.5.23. We consider the character

$$\psi_U : U(\mathbb{A}) \longrightarrow \mathbb{C}^1,$$
$$u = u_0(x)u_1(s_1,t_1)u_2(s_2,t_2)\tilde{u}(b) \mapsto \psi(2dt_2 + b).$$

Remark 2.5.24. *The character ψ_U is globally generic in the sense of 2.4.3. The pair (U, ψ_U) defines a Whittaker model which will be studied in this section.*

Given a cuspidal automorphic form φ of $\mathrm{GSO}_{4,2}$, we consider the following Whittaker functional

$$W(\varphi) = \int_{[U]} \varphi(u)\psi_U^{-1}(u)du.$$

Definition 2.5.25. Let $U_{\mathbf{H}}$ be the maximal unipotent subgroup of $\mathbf{H}$ defined in (2.2.2). We consider the character

$$\psi_{\mathbf{H}} : U_{\mathbf{H}}(\mathbb{A}) \to \mathbb{C}^1,$$
$$n = w(a)v(A) \mapsto \psi(-a - da_{22}),$$

where we recall that a_{22} is certain entry of the matrix A defined in (2.2.18).

For any cuspidal automorphic form f of $\mathbf{H}^+$ we consider the Whittaker functional associated to the character $\psi_{\mathbf{H}}$, denoted by

$$W'(f) = \int_{[U_{\mathbf{H}}]} \varphi(u)\psi_{\mathbf{H}}^{-1}(u)du.$$

Definition 2.5.26. A cuspidal automorphic representation σ of $\mathbf{H}^+$ is *generic with respect to* $\psi_{\mathbf{H}}$ if there is some $f \in \sigma$ such that $W'(f) \neq 0$.

Proposition 2.5.27. *Let σ be a cuspidal automorphic representation of $\mathbf{H}^+$. The representation σ is generic with respect to ψ_U if and only if $\widetilde{\Theta}^{glob}(\sigma)$ is generic with respect to $\psi_{\mathbf{H}}$.*

Proof. In [Mor14, prop. 3.3, p. 40], given $f \in \sigma$ and $\varphi \in \mathcal{S}(Y(\mathbb{A}))$ the author was able to compute $W(\theta(h, f, \varphi))$ in therms of the Whittaker model of f. More concretely, via an unfolding it is shown that $W(\theta(h, f, \varphi))$ can be expressed in terms of $W'(\tilde{f})$, where $\tilde{f} \in \sigma$ is certain $\mathbf{H}^+(\mathbb{A})-$translate of f. Following [GS03, p. 2718] the result follows. ☐

Proposition 2.5.28. *Let π be a cuspidal automorphic representation of $\mathrm{GSO}_{4,2}$. The representation π has a Shalika model if and only if $\widetilde{\Theta}^{glob}(\pi)$ is a $\mathbf{H}^+-$automorphic representation generic with respect to $\psi_{\mathbf{H}}$.*

Proof. Given aby $f \in \pi$ and $\varphi \in \mathcal{S}(Y(\mathbb{A}))$, [Mor14, prop. 3.4, p. 42] expresses $W'(\theta(g, f, \varphi))$ in terms of the constant $\mathrm{Sh}(f)$ defined in remark 2.4.33. ☐

2.5.5 Appendix: Restriction of admissible representations

The previous theta correspondence may be used to relate representations of $\mathbf{H}$ and $\mathrm{GSO}_{4,2}$. Despite this, the construction is developed for representations of the group $\mathbf{H}^+$ and hence we relate representations of $\mathbf{H}$ via the theta correspondence of their restrictions to $\mathbf{H}^+$. Then, it is natural to ask how are the admissible representations of $\mathbf{H}$ when we restrict them to $\mathbf{H}^+$. In this subsection we recall a result of [GK82] which allows us to approach this problem.

Lemma 2.5.29. *Let π be an irreducible admissible representation of a topological group $\mathfrak{G}$, $\mathfrak{H}$ an open normal subgroup of $\mathfrak{G}$ so that $\mathfrak{G}/\mathfrak{H}$ is finite and abelian. Then*

1. *$\pi|_{\mathfrak{H}}$ is the finite direct sum of irreducible admissible representations of $\mathfrak{H}$.*

2. *When the irreducible constituents of $\pi|_{\mathfrak{H}}$ are grouped according to their equivalence classes as*

$$\pi|_{\mathfrak{H}} \simeq \bigoplus_{i=1}^{M} m_i \pi_i,$$

with π_i irreducible and inequivalent, the integers m_i are all equal.

3. *The subgroup*

$$\mathfrak{G}_{\pi_1} = \{g \in \mathfrak{G},\ s.t.\ g \cdot \pi_1 \simeq \pi_1\},$$

has the property so that $\mathfrak{G}/\mathfrak{G}_{\pi_1}$ permutes the classes of the π_i simply transitively.

4. *The set*

$$X_{\mathfrak{H}}(\pi) := \{\nu : \ \mathfrak{G} \to \mathbb{C},\ homomorphism\ s.t.\ |\nu|_{\mathfrak{H}} = 1,\ and\ \pi \otimes \nu \simeq \pi\}$$

has $m^2 M$ elements, where m is the common m obtained in the second result above.

5. *Given π an admissible representation so that the representations in $\pi|_{\mathfrak{H}}$ have multiplicity one, then*

$$\mathfrak{G}_{\pi_1} = \{g \in \mathfrak{G},\ s.t.\ \nu(g) = 1\ for\ all\ \nu \in X_{\mathfrak{H}}(\pi)\}.$$

Proof. It is [GK82, lem. 2.1, p. 105] and [GK82, cor. 2.2, p. 107]. □

Theorem 2.5.30. *Given an admissible irreducible representation π of $\mathbf{H}$, the representations $\pi|_{\mathrm{Sp}_4}$ appear with multiplicity one.*

Proof. It is a particular case of [AP06, thm. 1.4, p. 4]. □

Corollary 2.5.31. *The restriction of $\mathrm{Ind}_{B_{\mathbf{H}}}^{\mathbf{H}}\chi$ to $\mathbf{H}^+(\mathbb{Q}_p)$ is irreducible if and only if*

$$\mathrm{Ind}_{B_{\mathbf{H}}}^{\mathbf{H}}\chi \otimes \chi_{E_p/\mathbb{Q}_p} \not\simeq \mathrm{Ind}_{B_{\mathbf{H}}}^{\mathbf{H}}\chi.$$

Proof. By 2.5.30 and (4) of lemma 2.5.29, the restriction of $\mathrm{Ind}_{B_{\mathbf{H}}}^{\mathbf{H}}\chi$ to $\mathbf{H}^+(\mathbb{Q}_p)$ is irreducible if and only if

$$X_{\mathbf{H}^+(\mathbb{Q}_p)}(\pi) = \{\mathrm{id}\}.$$

The trivial characters on $\mathbf{H}^+(\mathbb{Q}_p)$ define characters of $\mathbf{H}(\mathbb{Q}_p)/\mathbf{H}^+(\mathbb{Q}_p)$, which, according to remark 2.5.12, is a group isomorphic to $\mathbb{Q}_p^\times/\mathrm{im}(\mathrm{N}_{E_p/\mathbb{Q}_p}|_{E_p^\times})$. By [Mil13, thm. 1.1, (b) p. 10] this quotient is isomorphic to $\mathrm{Gal}(E_p/\mathbb{Q}_p) = \{\pm 1\}$. Therefore, there are only two characters of this form, which are $\chi_{E_p/\mathbb{Q}_p}$ and the trivial character. □

Corollary 2.5.32. *The restriction of $\mathrm{Ind}_{B_{\mathbf{H}}}^{\mathbf{H}}\chi$ to $\mathbf{H}^+(\mathbb{Q}_p)$ is irreducible if and only if*

$$\chi_{E_p/\mathbb{Q}_p} \notin \{\chi_1, \chi_2, \chi_1\chi_2, \chi_1\chi_2^{-1}\},$$

where $\chi = \chi_1 \otimes \chi_2 \otimes \chi_0$.

Proof. Using the corollary 2.5.31 and [GT11, (H3), p. 4, prop. 2.2, p. 4], the proposition follows determining whether $\mathrm{Ind}_{B_{\mathbf{H}}}^{\mathbf{H}}\chi$ is isomorphic to $\mathrm{Ind}_{B_{\mathbf{H}}}^{\mathbf{H}}\chi \otimes \chi_{E_p/\mathbb{Q}_p}$ at the level of Satake parameters. □

2.6 L functions and models

The $L-$functions are analytic objects associated to algebraic objects, which conjecturally encode information about the algebraic object in their analytic properties. More concretely it is conjectured (and in many cases proved) that the analytic nature of an L-function (such as holomorphicity, poles or equalities between $L-$functions), encode relevant algebraic information (for example self-duality of a representation, the existence of embeddings in other representation or even algebraic geometry properties of Shimura varieties). In the present setting, the relevant $L-$functions are the so-called Langlands $L-$functions. In this dissertation we just consider the unramified part of this function, which is defined via the Satake isomorphism. In this section we will adress some examples where the $L-$functions and the models of automorphic representations are intimately linked.

2.6.1 Automorphic $L-$functions

In this subsection we will explain the construction of the Langlands $L-$function. There is a wide variety of conjectures relating the behaviour of the $L-$functions with the properties of the representations, we refer the reader to [KR94, sec. 7, p. 61], [Bum97, sec. 3.8, 3.10, p. 368, 385], [JS83] to see some known examples of this phenomena.

Definition 2.6.1. Let π be an automorphic representation, S the set containing the archimedian places and the finite places p where π_p is ramified, furthermore let $r:\ {}^L G \to \mathrm{GL}_m$ be a finite dimensional representation. The *partial Langlands $L-$function* is defined by

$$L^S(s,\pi_p,r) = \prod_{p\notin S} L_p(s,\pi_p,r),$$

where $L_p(s,\pi,r) = \det(I - r(\chi_{\pi_p})p^{-s})^{-1}$ is the so-called *Euler factor*, with $\chi_{\pi_p} \in\ {}^L G$ the Satake parameter associated to π_p defined in 2.3.26 and 2.3.30.

Proposition 2.6.2. *There exists a real number s_0 depending on G, π and r, so that $L^S(s,\pi,r)$ converges when $\mathrm{Re}(s) > s_0$.*

Proof. See [Lan71]. □

Remark 2.6.3. *Let r be a representation of ${}^L G$ and $\mu_1, ..., \mu_d$ the eigenvalues of $r(\chi_{\pi_p})$. The coefficient of p^{-ks} of $\det(I - r(\chi_{\pi_p})p^{-s})^{-1}$ is equal to the $k-$th elementary symmetric polynomial in the variables $\mu_1, ..., \mu_d$. It coincides with the trace of $\mathrm{Sym}^k r(\chi_{\pi_p})$ and hence we express*

$$L_p(s,\pi_p,r) = \sum_{k=0}^{\infty} \mathrm{tr}\left(\chi_{\pi_p}|\mathrm{Sym}^k r\right) p^{-ks}.$$

Among the vast Langlands program it is conjctured that the partial Langlands $L-$functions have meromorphic continuation and satisfy a functional equation. In order to approach these problems, a wide variety of methods have been developed. Two of the most fruitfull methods are the Godement-Jacquet integrals (see [GPSR87]) and the Rankin-Selberg integrals (see [Bum97, sec. 3.8, p. 368]). The latter one has been used in 2.7 to study the residue of certain $L-$function.

Example 2.6.4. *The Riemann zeta function is the most basic example of Langlands $L-$function. An automorphic representation π of $\mathrm{GL}_1(\mathbb{A})$ is given by a Hecke character $\chi:\ \mathrm{GL}_1(\mathbb{A}) \to \mathbb{C}$. In the present case, we consider the automorphic representation π determined by the Hecke character $\chi(g) = \bigotimes_p |g_p|_p$. Furthermore, we take the standard representation of the form $\mathrm{std}:\ {}^L\mathrm{GL}_1 = \mathrm{GL}_1(\mathbb{C}) \to \mathrm{GL}_1$. Therefore, it is straightforward*

$$L(s,\pi,\mathrm{std}) = \prod_{p\nmid\infty} \zeta_p(s) = \zeta(s),$$

with $\zeta^p(s) = (1-p^{-s})^{-1}$ the classical Euler factor of the Riemann zeta function. Furthermore, given a set of non-archimedian places places A so that the non archimedian places minus A are denoted by S, we *will use the following notation*

$$\zeta^S(s) := \prod_{p\notin S} \zeta_p(s).$$

Example 2.6.5. *The Satake parameter of the unique unramified sub-quotient* π *of* $\mathrm{Ind}_{B_{\mathrm{Sp}_4}(\mathbb{Q}_p)}^{\mathrm{Sp}_4(\mathbb{Q}_p)}\xi$*, with* $\xi = \xi_1 \otimes \xi_2$*, is given by*

$$\chi_\pi = \begin{pmatrix} \xi(p) & & & & \\ & \xi_2(p) & & & \\ & & 1 & & \\ & & & \xi_2(p)^{-1} & \\ & & & & \xi_1(p)^{-1} \end{pmatrix} \in^L \mathrm{Sp}_4 = \mathrm{SO}_5(\mathbb{C}).$$

Let us consider the standard representation

$$\mathrm{std}:\ {}^L\mathrm{Sp}_4 \to \mathrm{GL}(\mathbb{C}^5).$$

By direct computation, the Euler factor satisfies

$$L_p(s,\pi,\mathrm{std}) = (1-p^{-s})^{-1}\prod_{i=1}^{2}(1-\xi_i(p)p^{-s})^{-1}(1-\xi_i(p)^{-1}p^{-s})^{-1}. \tag{2.-13}$$

Example 2.6.6. *Let us consider* π *the unique unramified sub-quotient of* $\mathrm{Ind}_{B_{\mathbf{H}}(\mathbb{Q}_p)}^{\mathbf{H}(\mathbb{Q}_p)}\xi$*, with* $\xi = \xi_1\otimes\xi_2\otimes\xi_0$*. Further, we suppose that* π *has trivial central character, which implies* $\chi_\pi \in \mathrm{Spin}_5(\mathbb{C}) \simeq \mathrm{Sp}_4(\mathbb{C})$*. In the present case, the so-called standard representation is defined by*

$$\mathrm{std}:\ \mathrm{GSpin}_5(\mathbb{C}) \to \mathrm{GL}(\mathbb{C}^5).$$

A direct computation shows

$$L_p(s,\pi,\mathrm{std}) = (1-p^{-s})^{-1}\prod_{i=1}^{2}(1-\xi_i(p)p^{-s})^{-1}(1-\xi_i(p)^{-1}p^{-s})^{-1}.$$

Let us observe that according to (2.6.5)*, it coincides with the partial standard* $L-$*function of the unique irreducible sub-quotient of* $\mathrm{Ind}_{B_{\mathrm{Sp}_4}(\mathbb{Q}_p)}^{\mathrm{Sp}_4(\mathbb{Q}_p)}\xi_1\otimes\xi_2$*.*

Remark 2.6.7. *The poles and values of the* $L-$*functions of* Sp_{2n} *(and twists of them) are related via the Siegel-Weil formula to the non vanishing of theta lifts, see [PSR88], [KR94], [KRS92].*

Example 2.6.8. *With the notation of the above example, for any character* η *of* $\mathbb{C}^\times$*, the Euler factor of the twisted standard* $L-$*function of* GSp_4 *has the following form:*

$$L_p(s,\pi,\mathrm{std}\otimes\eta) = (1-\eta(p)p^{-s})^{-1}\prod_{i=1}^{2}(1-\eta(p)\xi_i(p)p^{-s})^{-1}(1-\eta(p)\xi_i(p)^{-1}p^{-s})^{-1}.$$

Proposition 2.6.9. *For any generic cuspidal automorphic representation* σ *of* $\mathbf{H}$ *its partial twisted standard* $L-$*function satisfies*

$$L^S(1,\sigma,\mathrm{std}\otimes\chi_{E/\mathbb{Q}}) \neq 0.$$

Proof. See [Sha81, thm. 5.1, p. 351]. □

Example 2.6.10. Set p an inert prime of $E/\mathbb{Q}$ and let π be the unique unramified sub-quotient of $\mathrm{Ind}_{B_{\mathbf{G}}(\mathbb{Q}_p)}^{\mathbf{G}(\mathbb{Q}_p)}\xi$, where $\xi = \xi_1\otimes\xi_2\otimes\xi_0$ is any unramified character of $T_{\mathbf{G}}(\mathbb{Q}_p)$. We denote by

$$\Lambda_t:\ \mathrm{GL}_4(\mathbb{C})\times\mathrm{GL}_1(\mathbb{C})\rtimes\mathrm{Gal}(E_p/\mathbb{Q}_p) =^L \mathbf{G} \to \mathrm{GL}(\Lambda^2\mathbb{C}^4),$$

the representation given in 2.1.59. Using [KK08, p. 12] we obtain

$$\Lambda_t^2(\mathrm{diag}(a,b,1,1),\lambda,\theta) = \lambda\mathrm{diag}\left(\left(\begin{smallmatrix} ab & \\ & 1\end{smallmatrix}\right),\left(\begin{smallmatrix} & a \\ b & \end{smallmatrix}\right),\left(\begin{smallmatrix} a & \\ & b\end{smallmatrix}\right)\right).$$

Therefore

$$L_p(s,\pi,\Lambda^2_t) = (1-\xi_0(p)p^{-s})^{-1}(1-\xi_0(p)\xi_1(p)\xi_2(p)p^{-s})^{-1} \cdot (1-\xi_0(p)\alpha p^{-s})^{-1}(1-\xi_0(p)\beta p^{-s})^{-1}\prod_{i=1}^{2}(1-\xi_0(p)\xi_i(p)p^{-s})^{-1},$$

where α and β are the roots of the polynomial $X^2-\xi_1(p)\xi_2(p)$.

2.6.2 Local theta correspondence and $L-$functions

In this subsection we address the results of [Mor14], where the $L-$functions 2.6.10 and 2.6.8 are related via the local theta correspondence defined in 2.5.2.

In this subsection we will use the followung notation: $B_{\mathbf{H}^+}(\mathbb{Q}_p) := B_{\mathbf{H}}(\mathbb{Q}_p)\cap \mathbf{H}^+(\mathbb{Q}_p)$, $T_{\mathbf{H}^+}(\mathbb{Q}_p) := T_{\mathbf{H}}(\mathbb{Q}_p)\cap \mathbf{H}^+(\mathbb{Q}_p)$, $N_{\mathbf{H}^+}(\mathbb{Q}_p) := N_{\mathbf{H}}(\mathbb{Q}_p)\cap \mathbf{H}^+(\mathbb{Q}_p)$ so that

$$B_{\mathbf{H}^+}(\mathbb{Q}_p) = T_{\mathbf{H}^+}(\mathbb{Q}_p)N_{\mathbf{H}^+}(\mathbb{Q}_p). \tag{2.-13}$$

By remark 2.5.12 we have $T_{\mathbf{H}^+}(\mathbb{Q}_p)\simeq \mathbb{Q}_p^{\times,2}\times \mathrm{Im}\left(\mathrm{N}_{E_p/\mathbb{Q}_p}\right)$. Given any character $\chi_0:\ \mathbb{Q}_p^\times\to\mathbb{C}$ we fix the following notation:

$$\chi_{0,+} := \chi_0|_{\mathrm{Im}\left(\mathrm{N}_{E_p/\mathbb{Q}_p}\right)}.$$

Given χ_1,χ_2 characters of $\mathbb{Q}_p^\times$ and χ_0 a character of $E_p^\times$, we consider the normalized induced representation of the form

$$\mathrm{Ind}_{B_{\mathbf{H}^+}(\mathbb{Q}_p)}^{\mathbf{H}^+(\mathbb{Q}_p)}\left(\chi_1\otimes\chi_2\otimes\chi_{0,+}\right),$$

where we recall that for $b\in B_{\mathbf{H}^+}(\mathbb{Q}_p)$, $\chi_1\otimes\chi_2\otimes\chi_{0,+}(b)=\chi_1\otimes\chi_2\otimes\chi_{0,+}(t)$.

Lemma 2.6.11. *Let π^+ be an unramified representation of $\mathbf{H}^+(\mathbb{Q}_p)$ so that*

$$\pi^+\hookrightarrow \mathrm{Ind}_{B_{\mathbf{H}^+}}^{\mathbf{H}^+}(\xi_1\otimes\xi_2\otimes\xi_{0,+}),$$

with ξ_1,ξ_2,ξ_0 be unitary unramified characters of $\mathbb{Q}_p^\times$, then

$$\pi^+\simeq \mathrm{Ind}_{B_{\mathbf{H}^+}}^{\mathbf{H}^+}(\xi_1\otimes\xi_2\otimes\xi_{0,+}).$$

Proof. Since ξ_1,ξ_2,ξ_0 are unitary unramified characters. We can apply [Mor14, lem. 4.2, (3), (d), p. 50] with $s_1=s_2=0$ in its notation. □

Theorem 2.6.12. *Let π^+ be an unramified representation of $\mathbf{H}^+(\mathbb{Q}_p)$ satisfying the hypothesis of the lemma 2.6.11, then*

$$\tilde\theta\left(\pi^+\right) = \mathrm{J}_{B_{\mathrm{GSO}}}^{\mathrm{GSO}}\left(\xi_1^{-1}\chi_{E_p/\mathbb{Q}_p}\otimes\xi_2\chi_{E_p/\mathbb{Q}_p}\otimes(\xi_1\xi_2\xi_0)\circ\mathrm{N}_{E_p/\mathbb{Q}_p}\right),$$

where the right hand side representation is the unique irreducible sub-quotient of

$$\mathrm{Ind}_{B_{\mathrm{GSO}}}^{\mathrm{GSO}}\left(\xi_1^{-1}\chi_{E_p/\mathbb{Q}_p}\otimes\xi_2\chi_{E_p/\mathbb{Q}_p}\otimes(\xi_1\xi_2\xi_0)\circ\mathrm{N}_{E_p/\mathbb{Q}_p}\right).$$

Proof. It is [Mor14, thm. 6. 21, (3), (d), p. 75].

□

Remark 2.6.13. *Given an unramified character $\xi:\mathbb{Q}_p^\times\to\mathbb{C}^\times$, the function $\xi\chi_{E_p/\mathbb{Q}_p}$ is an unramified character. In fact, $\chi_{E_p/\mathbb{Q}_p}$ is a continuous map of the form $\mathbb{Q}_p^\times\to\{\pm1\}$, where the right hand side is endowed with the discrete topology. Therefore the product*

$$\xi\chi_{E_p/\mathbb{Q}_p}:\ \mathbb{Q}_p^\times\to\mathbb{C}^\times,$$

is continuous with the usual topology of $\mathbb{C}^\times$. *Moreover by definition both characters* ξ *and* $\chi|_{E_p/\mathbb{Q}_p}$ *are identically 1 in* $\mathbb{Z}_p^\times$. *The previous discussion implies that* $\mathrm{Ind}_{B_{\mathrm{GSO}}}^{\mathrm{GSO}}\left(\xi_1^{-1}\chi_{E_p/\mathbb{Q}_p}\otimes\xi_2\chi_{E_p/\mathbb{Q}_p}\otimes(\xi_1\xi_2\xi_0)\circ \mathrm{N}_{E_p/\mathbb{Q}_p}\right)$ *is an unramified representation.*

Theorem 2.6.14. *Let* σ *be an unramified representation of* $\mathrm{GSO}_{4,2}(\mathbb{Q}_p)$, *then*

$$L_p(s,\sigma,\Lambda_t) = L_p(s,\pi,\mathrm{std}\otimes\chi_{E/\mathbb{Q}})\zeta_p(s),$$

where π *is an unramified representation of* $\mathbf{H}$ *so that* $\theta(\sigma)\subset\pi|_{\mathbf{H}^+}$.

Proof. The proof follows from comparing the Euler factors at $p\notin S$, which is done in [Mor14, cor. 6.23, p. 81]. For the sake of completeness we show the proof of the theorem. Suppose that p inert in E and that π is an unramified irreducible sub-quotient of $\mathrm{Ind}_{B_{\mathbf{G}}}^{\mathbf{G}}\xi$. If $\pi=\tilde{\theta}(\sigma^+)$, with σ an unramified generic principal series $\mathrm{Ind}_{B_{\mathbf{H}}}^{\mathbf{H}}\chi$, then by [Mor14, thm. 6.21, (3), p. 73] we have that

$$\xi_1 = \chi_0\circ N_{E_p/\mathbb{Q}_p},\ \xi_2 = (\chi_2\chi_0)\circ N_{E_p/\mathbb{Q}_p},\ \xi_0 = \chi_1\chi_{E_p/\mathbb{Q}_p}.$$

Therefore, the Frobenius conjugacy class $A_\pi = (\mathrm{diag}(\chi_1^{-1}\chi_2^{-1}(p),\chi_1^{-1}\chi_2(p),1,1),\chi_1(p)\chi_{E_p/\mathbb{Q}_p}(p),\tau)$. As in example 2.6.10, we thus have that $\wedge_t^2(A_{\Pi_p}) = \chi_{E_p/\mathbb{Q}_p}(p)\cdot\mathrm{diag}(\chi_1(p),\chi_2^{-1}(p),\chi_2(p),\chi_1^{-1}(p),-1,1)$, hence

$$\begin{aligned}L(s,\Pi_p,\wedge_t^2) &= [(1+\chi_{E_p/\mathbb{Q}_p}(p)p^{-s})(1-\chi_{E_p/\mathbb{Q}_p}(p)p^{-s})\prod_{i=1}^{2}(1-\chi_{E_p/\mathbb{Q}_p}\chi_i(p)p^{-s})(1-\chi_{E_p/\mathbb{Q}_p}\chi_i^{-1}(p)p^{-s})]^{-1}\\ &= (1-p^{-s})^{-1}L(s,\sigma_p,\mathrm{std}\otimes\chi_{E_p/\mathbb{Q}_p}),\end{aligned}$$

as $\chi_{E_p/\mathbb{Q}_p}(p)=-1$. The proof in the split case is analogous. □

2.6.3 Shalika model and $L-$functions

In this section we explain the relation between the poles of the function $L^S(s,\pi,\Lambda_t)$, defined in 2.6.10, and the existence of a Shalika model for a cuspidal automorphic representation π. Furthermore, we will explain key applications of this result; the equivalence between Shalika models and Whittaker models via the theta correspondence and an integral expression for the $L-$function 2.6.8 which, combined with the aforementioned relation, proves analytic continuation and functional equations for both $L-$functions.

Definition 2.6.15. Given any Schwartz function $\Phi\in\mathcal{S}(\mathbb{A}^2)$ we consider the function $f_\Phi(h,s):\ \mathrm{GL}_2(\mathbb{A})\times\mathbb{C}\to\mathbb{C}$ defined by

$$f_\Phi(h,s) := |\det h|^s\int_{\mathbb{A}^\times}\Phi((0,t)h)|t|^{2s}d^\times t.$$

It satisfies that $f_\Phi(\cdot,s)\in\mathrm{Ind}_{B_{\mathrm{GL}_2}(\mathbb{A})}^{\mathrm{GL}_2(\mathbb{A})}\chi^s$ where

$$\begin{aligned}\chi^s:\ &B_{\mathrm{GL}_2}(\mathbb{A})\to\mathbb{C},\\ &b=\left(\begin{smallmatrix}\alpha & \beta\\ & \gamma\end{smallmatrix}\right)\mapsto|\alpha|_{\mathbb{A}}^s|\gamma|_{\mathbb{A}}^{-s}.\end{aligned}$$

Throughout this subsection we consider the Eisenstein series defined by the following expression

$$E(h,f_\Phi,s) := \sum_{\gamma\in B_{\mathrm{GL}_2}(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{Q})} f_\Phi(\gamma h,s).$$

Proposition 2.6.16. *Let* $f_\Phi(h,s)$ *be a section of the form 2.6.15, the sum* $\sum_{\gamma\in B_{\mathrm{GL}_2}(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{Q})}f_\Phi(\gamma h,s)$ *converges for all* s *with* $\mathrm{Re}(s)>1$. *Furthermore the Eisenstein series* $E(h,f_\Phi,s)$ *has meromorphic continuation to* $\mathbb{C}$.

Proof. See [Jac72, prop. 19.3, p. 118]. □

Let us recall that the Shalika subgroup S of $\mathbf{G}$, defined in 2.4.19, is isomorphic to $\mathrm{GL}_2 N_{\mathbf{G}}$ as algebraic groups. We denote by

$$\begin{aligned}\mathrm{GL}_2(\mathbb{Q}_p) &\to S(\mathbb{Q}_p),\\ h &\mapsto h^*,\end{aligned}$$

the embedding of GL_2 into the GL_2 part of the Shalika subgroup. Given a cuspidal automorphic representation π of $\mathrm{GU}(2,2)$ and $\varphi \in \pi$, we consider the *global zeta integral*

$$Z(s,\varphi,f_\Phi) = \int_{\mathbb{A}^\times \mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A})} \int_{[N_{\mathbf{G}}]} \varphi(nh^*)\psi(n)E(h,f_\Phi,s)dndh.$$

Proposition 2.6.17. *The function $Z(s,\varphi,f_\Phi)$ has a simple pole at $s=1$ if and only if the Shalika functional defined in (2.4.25) satisfies $S_\delta(\varphi) \neq 0$.*

Proof. It is [FM13, prop. 2.1, p. 4129]. □

Theorem 2.6.18. *Let π be a cuspidal automorphic representation and $\varphi \in \pi$. The global zeta integral has an Euler product expansion i.e.*

$$Z\left(s,\varphi,f_\Phi\right) = \prod_p Z\left(s,\pi_p,f_\Phi\right),$$

so that

$$\prod_{p\notin S} Z\left(s,\pi_p,f_{\mathrm{char}_{\mathcal{O}\oplus\mathcal{O}}}\right) = L^S\left(s,\pi,\Lambda_t\right),$$

where S is the set of primes so that the representations $\{\pi_p\}_{s\notin S}$ are ramified.

Proof. The proof is in [FM13, se. 2, 3, p. 4129]. It is based on the unfolding of the global zeta integral with the Eisenstein series. They express the zeta integral in terms of the Whittaker model of π and hence, using the Casselman-Shalika formula of [CS80] the result follows. □

Theorem 2.6.19. *The partial $L-$function $L^S(s,\pi,\Lambda_t)$ extends to a meromorphic function in $s \in \mathbb{C}$. It has a pole at $s=1$ if and only if π has a Shalika model.*

Proof. See [FM13, thm. 4.1, p. 4136]. □

Theorem 2.6.20. *Let π be a cuspidal automorphic representation of $\mathbf{G}$ with trivial central character. The representation π has a Shalika model if and only if there exists a generic representation of $\mathbf{H}$, denoted by σ, so that $\pi = \widetilde{\Theta}^{glob}(\sigma^+)$ with σ^+ a subrepresentation of $\sigma|_{\mathbf{H}^+}$.*

Proof. The proof of this result is given in [Mor14, thm. 3.6, p. 44]. For the sake of completeness we give a sketch of the proof.

We start assuming that π has a Shalika model. By proposition 2.5.28, the representation $\widetilde{\Theta}^{glob}(\pi)$ is generic with respect to the character $\psi_{\mathbf{H}}$. We assume that $\widetilde{\Theta}^{glob}(\pi)$ is not cuspidal to obtain a contradiction. Let us recall that $\mathbf{H} \simeq \mathrm{GSO}(3,2)$ and $\mathrm{GSp}_2 \simeq \mathrm{GSO}(2,1)$. We define the group GSp_2^+ in an analogous way to $\mathbf{H}^+$. Since $\widetilde{\Theta}^{glob}(\pi)$ is non cuspidal, theorem 2.5.22 implies that the theta lift $\widetilde{\Theta}^{glob}(\pi)$ to GSp_2^+ is non-zero and hence by 2.5.4 that there exists an automorphic representation ω of GSp_2^+ so that $\widetilde{\Theta}^{glob}(\omega) = \pi$. In [Mor14, lem. 3.8, p. 44] the author uses the fact that π has a Shalika model and $\widetilde{\Theta}^{glob}(\omega) = \pi$ to obtain that the theta lift of ω to $\mathrm{GSO}_{3,1}$ is non-zero. Since the group $\mathrm{GSO}_{3,1}$ is the element below $\mathrm{GSO}_{4,2}$ in the Rallis tower, theorem 2.5.22 implies that π is non cuspidal, obtaining the desired contradiction. The previous discussion implies that $\widetilde{\Theta}^{glob}(\pi)$ is a cuspidal automorphic representation of $\mathbf{H}^+$ and then we are able to define $\widetilde{\Theta}^{glob}(\pi)^*$, the cuspidal automorphic representation of $\mathbf{H}$ defined in 2.5.14. From the definition of $\widetilde{\Theta}^{glob}(\pi)^*$, there exists an irreducible constituent σ of $\widetilde{\Theta}^{glob}(\pi)^*|_{\mathbf{H}^+}$ so that $\pi = \widetilde{\Theta}^{glob}(\sigma)$.

Now let us suppose that there exists σ a generic representation of $\mathbf{H}$ so that $\pi = \widetilde{\Theta}^{glob}(\sigma^+)$ with σ^+ a subrepresentation of $\sigma|_{\mathbf{H}^+}$. Using 2.6.12 we have

$$L^S(s,\pi,\Lambda_t) = L^S(s,\sigma,\mathrm{std}\otimes\chi_{E/\mathbb{Q}})\zeta^S(s),$$

where the above functions are defined in 2.1.59, 2.6.8 and 2.6.4. If σ is generic by 2.6.9, the $L-$function $L^S(s,\sigma,\mathrm{std}\otimes\chi_{E/\mathbb{Q}})$ does not vanish at $s=1$. Since $\zeta^S(s)$ has a pole at $s=1$ we conclude that $L^S(s,\pi,\Lambda_t)$ has a pole at $s=1$. Therefore, plugging together theorem 2.6.19 and proposition 2.6.8 we conclude that π has a non-zero Shalika model. □

Theorem 2.6.21. *Given a cuspidal automorphic representation π of $\mathbf{G}$ the following three facts are equivalent*

1. *The L-function $L(s,\pi,\Lambda_t)$ has a pole at $s=1$.*
2. *π has a non-zero Shalika model.*
3. *$\pi = \widetilde{\Theta}^{glob}(\sigma^+)$, with σ^+ a subrepresentation of $\sigma|_{\mathbf{H}^+}$ where σ is a generic cuspidal automorphic representation of $\mathbf{H}$.*

Proof. Plugging together theorems 2.6.19 and 2.6.20 we obtain the result. □

2.7 Zeta integrals and Shalika models for $\mathrm{GU}_{2,2}$

This part of the thesis mainly based on [CT], a work with Antonio Cauchi. The main contribution of this section consists of giving (partially) a new instance of the curious phenomenon where a Rankin-Selberg integral of a cusp form on one group is used to represent an L-function of a different cusp form on another group. The principal ingredient to obtain this equality is the determination of an explicit formula for the Shalika functional. Therefore, one of the main focus of this section is on the proof of a local criterion that implies the multiplicity one result of the Shalika model. The previous result allows us to approach the computation of an explicit formula for the Shalika functional using harmonic analysis on p-adic reductive groups. In the future we expect to relate this formula to the standard $L-$function of $\mathbf{H}$. Along the theta correspondence between similitude groups will play a crucial role (both local and global). Then, in order to lighten the notation we will denote those correspondences by θ and Θ^{glob} as in the classical case. Furthermore, given an unramified reductive group G and a character ξ of the $\mathbb{Q}_p-$points of the maximal torus of G, we will use the following notation $I_G(\xi) := \mathrm{Ind}_{B_G}^G\xi$.

Throughout this section we will consider the zeta integral

$$I(\varphi,s) := \int_{\mathbf{H}(\mathbb{Q})Z_{\mathbf{H}}(\mathbb{A})\backslash\mathbf{H}(\mathbb{A})} E_P^*(s,g)\varphi(g)dg,$$

where $E_P^*(s,g)$ is the normalised degenerate Siegel Eisenstein series for $\mathbf{H}$ and φ is a cusp form in the space of a cuspidal representation π of $\mathbf{G}(\mathbb{A})$ with trivial central character. This integral is analogous to the integral for the group $\mathrm{SO}_{3,3}$ studied in [BFG92] and can be regarded as the degenerate companion of the integral for $\mathrm{SO}_{4,2}$ studied in [Sug85] and [Pol18].

The integral $I(\varphi,s)$ unfolds to the Shalika period of φ. Thus, by the characterization given by theorem 2.6.21, it is identically zero if π does not come from $\mathbf{H}$. From now on we will assume we are not in this situation. Due to the nature of the resulting integral after the unfolding, there is an Euler product expression for the zeta integral, more concretely

$$I(\varphi,s) = \prod_p I_p(v_0,s) := \prod_p \int_{\mathrm{GL}_2(\mathbb{Q}_p)N_{\mathbf{H}}(\mathbb{Q}_p)\backslash\mathbf{H}(\mathbb{Q}_p)} f_{p,s}(h)\mathcal{S}_p(h\cdot v_0)dh$$

We approach the computation of the zeta integral from local to global. Firstly, we will describe explicitly the functional $\mathcal{S}_p(\cdot)$. Then, using the Weyl's character formula and the equality 2.6.3, we will relate the resulting expression with the standard $L-$function of $\mathbf{H}$.

The first (and main) step of the proof is the computation of the Shalika functional. We have to divide our calculation depending on the prime p. In fact, according to proposition 2.2.5, when p is a split prime of $\mathbb{Q}$ over E we get

$$\mathbf{G}(\mathbb{Q}_p) \simeq \mathrm{GL}_4(\mathbb{Q}_p) \times \mathrm{GL}_1(\mathbb{Q}_p).$$

In this situation, the Shalika functional of $\mathbf{G}(\mathbb{Q}_p)$ is intimately linked to the Shalika functional of $\mathrm{GL}_4(\mathbb{Q}_p)$. Hence, it is unique and we may adapt the main result of [Sak06] to describe explicitly $\mathcal{S}_p(h \cdot v_0)$. The previous discussion allows us to obtain the following result.

Theorem (2.7.43)**.** *For* $\mathrm{Re}(s)$ *big enough, we have*

$$I_p(v_0, s) = L(s, \sigma_p, \mathrm{std}). \tag{2.-13}$$

The main body of this section is devoted to understanding the Shalika functional when p is inert over E. In this case, there is no Casselman-Shalika formula available. Moreover, the multiplicity of the model is not clear, see [CG21]. The strategy used in this section to approach this computation is based on [CS80] and [Sak06]. This theory relies on the multiplicity one property of the model. Therefore, the first step of the proof is the determination of a multiplicity one result for the model. In accordance with the global situation we assume every representation π of $\mathbf{G}$ in the image of the theta correspondence with the group $\mathbf{H}^+$. These are the kind of representations that will appear when we consider the above zeta integral. If $\pi = \theta(\sigma^+)$ with σ^+ a constituent of the restriction of the generic representation $\mathrm{Ind}_{B_{\mathbf{H}}}^{\mathbf{H}} \chi$ to $\mathbf{H}^+(\mathbb{Q}_p)$, we will distinguish the following two cases:

- When $|\cdot|^{\pm 1}, \chi_{E_p/\mathbb{Q}_p} \notin \{\chi_1, \chi_2, \chi_1\chi_2, \chi_1\chi_2^{-1}\}$, we will say that π is in the *non-degenerate case*.
- When $\chi_{E_p/\mathbb{Q}_p} \in \{\chi_1, \chi_2, \chi_1\chi_2, \chi_1\chi_2^{-1}\}$ and $|\cdot|^{\pm 1} \notin \{\chi_1, \chi_2, \chi_1\chi_2, \chi_1\chi_2^{-1}\}$, we will say that π is in the *degenerate case*.

The above classification is related to the reducibility of $\mathrm{Ind}_{B_{\mathbf{H}}}^{\mathbf{H}} \chi|_{\mathbf{H}^+(\mathbb{Q}_p)}$ and allows us to exclude cases in which the multiplicity is not yet known to be 1. The following theorem follows from applying Mackey theory to the present situation.

Theorem (2.7.1)**.** *If* $\pi = \theta(\sigma^+)$ *is a representation in the non-degenerate case satisfying the hypothesis of lemma 2.7.10, there exists a unique Shalika functional on* π *(up to constant).*

From now on, we will assume every representation of $\mathbf{G}(\mathbb{Q}_p)$ is in the non-degenerate case. We will approach the degenerate case in future projects. Using the ideas of [CS80] and [Sak06] we get the following result.

Theorem. *We obtain the following expression:*

$$\mathcal{S}_{\tilde{S},p}\left(\pi(g_{-\lambda})v_0\right) = \frac{p^{-2n}\prod_{\alpha \in \Phi_{\mathbf{H}}^+} c_\alpha(\xi)}{Qe^{-\check{\rho}}\prod_{\check{\alpha} \in \Phi_{\mathrm{Sp}_4}^{\vee,+,S}}(1 - p^{-1}e^{\check{\alpha}})(g_{\tilde{\xi}})} \mathcal{A}\left(e^{\check{\rho}+n(\check{\alpha}_1+\check{\alpha}_2)} \prod_{\check{\alpha} \in \Phi_{\mathrm{Sp}_4}^{\vee,+,S}} (1 - p^{-1}e^{-\check{\alpha}})\right)(g_{\tilde{\xi}}).$$

Proceeding as in the split case, we get the following theorem:

Theorem. *For* $\mathrm{Re}(s)$ *big enough and* p *a non split prime we have*

$$I_p(v_0, s) = L_p(s, \sigma_p, \mathrm{std} \otimes \chi_{E_p/\mathbb{Q}_p}).$$

The main theorem can be derived from the previous two expressions for the $L-$function:

Theorem. *Let* π *be an irreducible cuspidal automorphic representation of* $\mathbf{G}(\mathbb{A})$ *with trivial central character. Suppose that* $\pi = \widetilde{\Theta}^{glob}(\sigma^+)$*, with* σ^+ *a subrepresentation of* $\sigma|_{\mathbf{H}^+}$ *where* σ *is a generic cuspidal automorphic representation of* $\mathbf{H}$ *with trivial central character. We have*

$$I(\varphi, s) = I_S(s)L^S(s, \sigma, \mathrm{std} \otimes \chi_{E/\mathbb{Q}}),$$

where S is a finite set of primes containing the ramified primes for π, σ, and $E/\mathbb{Q}$, and $I_S(s)$ is the integral over the places in S and infinity.

This section is organized as follows: The first subsection is devoted to explaining the ideas that motivated this section but are still unknown. In the second subsection we obtain a Casselman-Shalika formula for the Shalika model of $\mathbf{G}(\mathbb{Q}_p)$ when p is split. Due to the structure of $\mathbf{G}$, it is easily deduced from the one treated in subsection 2.4.7. For the non split primes, we first prove in the third subsection the uniqueness of the model using Mackey theory. Then, in the fourth subsection, using the methods of [Sak06] and [CS80] we obtain the formula for the functional. In subsections 5 and we prove the equality between the local zeta integrals and the local $L-$factors for the split primes using the previous results. To conclude in section 6 we obtain the vanishing of the zeta integral with the Klingen Eisentein series and the relation between the wedge square $L-$function and some period integral.

2.7.1 Future directions

Throughout the introduction, we emphasized that the main motivation for this section was the computation of the standard $L-$function of $\mathbf{H}$ using an integral derived from $\mathbf{G}$. We did indeed achieve equality for the unramified primes. The purpose of this section is to discuss the unfinished results and how we plan to obtain them.

The two points that remain to be proved are:

- The multiplicity one property for the so-called degenerate case.
- The non vanishing of the ramified places.

Throughout this section, we will assume the automorphic representations with all their constituents in the non-degenerate case. As a matter of fact, we intend to exclude the degenerate case by showing that, if an automorphic representation belongs to the non degenerate case, then it cannot belong to the cuspidal spectrum.

Our second expectation is that the function $I_S(s)$ obtained in our main theorem will not vanish.

Theorem 2.7.1. *Let π be an irreducible cuspidal automorphic representation of $\mathbf{G}(\mathbb{A})$ with trivial central character. Suppose that $\pi = \widetilde{\Theta}^{glob}(\sigma^+)$, with σ^+ a subrepresentation of $\sigma|_{\mathbf{H}^+}$ where σ is a generic cuspidal automorphic representation of $\mathbf{H}$ with trivial central character. We have*

$$I(\varphi, s) = I_S(s)L^S(s, \sigma, \mathrm{std} \otimes \chi_{E/\mathbb{Q}}),$$

where S is a finite set of primes containing the ramified primes for π, σ, and $E/\mathbb{Q}$, and $I_S(s) \neq 0$ is the integral over the places in S and infinity.

2.7.2 A Casselman-Shalika formula for the Shalika model of $\mathrm{GU}_{2,2}$: The split case

In this subsection we will consider the split places, where we may deduce the result by a simple modification of 2.4.7. Let π be a unramified representation of $\mathbf{G}(\mathbb{Q}_p)$ with trivial central character and let p be split over E. In what follows we describe how the Shalika model of π defines a Shalika model for GL_4.

According to proposition 2.2.27, when p is split there is a group isomorphism

$$\mathbf{G}(\mathbb{Q}_p) \simeq \mathbf{GL}_4(\mathbb{Q}_p) \times \mathbf{GL}_1(\mathbb{Q}_p).$$

Furthermore, taking quotient at both sides we get $\mathbf{PG}(\mathbb{Q}_p) \simeq \mathbf{PGL}_4(\mathbb{Q}_p)$. Since π has trivial central character by hypothesis, we may consider π as an irreducible unramified representation of $\mathbf{PGL}_4(\mathbb{Q}_p)$. We conveniently combine the isomorphism 2.2.27 with conjugation by the diagonal matrix $w = (\mathrm{diag}(1, 1, -1, 1), -1) \in$

$\mathrm{GL}_4(\mathbb{Q}_p) \times \mathbb{Q}_p^\times$. This has the effect of identifying the subgroup

$$\mathrm{GL}_2(\mathbb{Q}_p) \simeq \left\{ \begin{pmatrix} g & \\ & \det(g) I_2' {}^t g^{-1} I_2' \end{pmatrix} \right\} < \mathbf{G}(\mathbb{Q}_p),$$

with $\{(\left(\begin{smallmatrix} g & \\ & g \end{smallmatrix}\right), \det(g))\} \times \{1\} < \mathrm{GL}_4(\mathbb{Q}_p) \times \mathrm{GL}_1(\mathbb{Q}_p)$ and the subgroup $N_{\mathbf{G}}(\mathbb{Q}_p) < \mathbf{G}(\mathbb{Q}_p)$ with

$$N_{\mathrm{GL}_4}(\mathbb{Q}_p) = \{u(X) = \left(\begin{smallmatrix} I & X \\ & I \end{smallmatrix}\right), X \in M_{2\times 2}(\mathbb{Q}_p)\} < \mathrm{GL}_4(\mathbb{Q}_p) \times \mathrm{GL}_1(\mathbb{Q}_p).$$

Furthermore, under the previous identification, the Shalika character of $\mathbf{G}$, denoted by $\chi_{\delta,p}$, corresponds to the character

$$\begin{aligned} \chi_{p,\mathrm{GL}_4} :\ N_{\mathrm{GL}_4}(\mathbb{Q}_p) &\to \mathbb{C}^\times \\ u(X) &\mapsto \psi_p(\mathrm{tr}(X)), \end{aligned}$$

the so-called Shalika character for the group GL_4, see 2.4.15. Moreover, according to the isomorphism $\mathbf{PG}(\mathbb{Q}_p) \simeq \mathbf{PGL}_4(\mathbb{Q}_p)$, the irreducible unramified representation π of $\mathbf{PG}(\mathbb{Q}_p)$ gets identified with

$$\tilde{\pi} = \mathrm{Ind}_{B_{\mathrm{GL}}}^{\mathrm{GL}_4}(\chi),$$

where B_{GL} is the Borel subgroup of GL_4 and $\chi = (\chi_1, \chi_2, \chi_3, \chi_4)$ is an unramified character of the diagonal torus of $\mathrm{GL}_4(\mathbb{Q}_p)$ satisfying $\prod_{i=1}^4 \chi_i = \mathrm{id}$ (i.e. the representation $\tilde{\pi}$ has trivial central character). From now on, we assume the character χ of the form $\chi = \chi_1 \otimes \chi_2 \otimes \chi_2^{-1} \otimes \chi_1^{-1}$. According to theorem 2.4.18 this hypothesis ensures the existence of a Shalika model for $\tilde{\pi}$. This assumption is consistent with the global setting. In fact, if π is a constituent of an automorphic representation in the image of a generic cuspidal representation of $\mathbf{H}$, the global Shalika model must exists, implying the existence of a local Shalika functional. By the previous discussion, the existence of the model is equivalent to the fact that the character of $\tilde{\pi}$ is of the form $\chi = \chi_1 \otimes \chi_2 \otimes \chi_2^{-1} \otimes \chi_1^{-1}$.

Before continue we address the relation between the structures of Sp_4 and GL_4. We let Γ, resp. W, denote the Weyl group of Sp_4, resp. GL_4; we also let Φ_{GL_4}, Φ_{Sp_4} denote the root systems for GL_4 and Sp_4. The embedding $\mathrm{Sp}_4 \hookrightarrow \mathrm{GL}_4$ identifies Γ as a subgroup of the Weyl group of GL_4 and induces a surjection

$$\Phi_{\mathrm{GL}_4} \to \Phi_{\mathrm{Sp}_4},$$

which is one-to-one onto the set of long roots $\Phi_{\mathrm{Sp}_4}^L$ and two-to-one on the set of short roots $\Phi_{\mathrm{Sp}_4}^S$. For each $\alpha \in \Phi_{\mathrm{Sp}_4}$, we denote by α, $\tilde{\alpha}$ the (possibly equal) elements in the pre-image of it.

Lemma 2.7.2. *The functional $\mathcal{S}_p : \pi \to \mathbb{C}$ defines a functional $\mathcal{S}_{\mathrm{GL}_n,p} : \tilde{\pi} \to \mathbb{C}$ so that for all $v \in \tilde{\pi}$ we have*

$$\mathcal{S}_{\mathrm{GL}_n,p}(\left(\begin{smallmatrix} g & \\ & g \end{smallmatrix}\right) u(X) \cdot v) = \psi_p(\mathrm{tr}(X)) \mathcal{S}_{\mathrm{GL}_n,p}(v),\ \forall g \in \mathrm{GL}_2(\mathbb{Q}_p), \forall X \in M_{2\times 2}(\mathbb{Q}_p).$$

Proof. It follows by the previous identifications. □

Remark 2.7.3. *The functional $\mathcal{S}_{\mathrm{GL}_n,p}$ obtained in the previous lemma is a Shalika functional for GL_4, see section 2.4.2 for more details.*

We recall that theorem 2.4.17 shows the uniqueness of the Shalika model for GL_n unconditionally. We therefore normalize the functional $\mathcal{S}_p$ according to the explicit expression given in theorem 2.4.58. Let α be a root, we denote by $\check{\alpha}$ its co-root. In this section we will use the following notation $e^{\check{\alpha}}(\xi) := \xi(a_\alpha)$.

Definition 2.7.4. *We define*

$$\Lambda_p^{\mathrm{split}}(g) := \mathcal{S}_{\mathrm{GL}_4,p}(g \cdot v_0),$$

where v_0 is the spherical vector of π such that $v_0(1) = 1$ (with v_0 seen as an element of $\tilde{\pi}$), $\mathcal{S}_{\mathrm{GL}_4,p}(\cdot)$ is the Shalika functional such that

$$\mathcal{S}_{\mathrm{GL}_4,p}(v_0) = \frac{\mathcal{Q}}{1+p^{-1}} \cdot \frac{e^{-\check{\rho}} \prod_{\beta \in \Phi_{\mathrm{GL}_4}^+} (1 - e^{-\check{\beta}})}{\prod_{\alpha \in \Phi_{\mathrm{Sp}_4}^+} (1 - p^{-1} e^{-\check{\alpha}}) \mathcal{A}(e^{\check{\rho}})} (\chi_{\tilde{\pi}}),$$

where $\mathcal{Q} = \sum_{w\in W}(I_{\mathrm{GL}_4} w I_{\mathrm{GL}_4} : I_{\mathrm{GL}_4})^{-1}$, $\check{\rho} = \frac{1}{2}\sum_{\alpha\in\Phi^+_{\mathrm{Sp}_4}} \check{\alpha}$, *and* $\mathcal{A}$ *denotes the alternator*

$$\mathcal{A}(\cdot) = \sum_{w\in W_{\mathbf{H}}} (-1)^{\ell(w)} w(\cdot).$$

Remark 2.7.5. *As it is explained in the proof of [Sak06, thm. 2.1, p. 21], the factor* $\prod_{\alpha\in\Phi^+_{\mathrm{Sp}_4}}(1-p^{-1}e^{-\check{\alpha}})(\chi_{\tilde{\pi}})$ *appearing in the denominator of the formula above is non-zero as long as* $\tilde{\pi}$ *is irreducible.*

The formula given in 2.4.58 will be used to compute a $L-$function for the group Sp_4. Then, in order to relate the formula of 2.4.7, first we have to write the constants attached to GL_4 in terms of constants attached to Sp_4. More concretely, on the one hand, for $\alpha \in \Phi^{\mathrm{GL}_4}$ so that $\alpha\in\Phi^{\mathrm{Sp}_4}$, trivially $a_\alpha \in \mathrm{Sp}_4(\mathbb{Q}_p)$. On the other hand, by assumption ξ is a character of the torus of $\mathrm{Sp}_4(\mathbb{C})$, see remark (2.7.3). Then, regarding ξ as a character of T_{Sp_4} instead of T_{GL_4} we get

$$\xi(a_\alpha) = e^{\check{\alpha}}(\widehat{\chi_{\tilde{\pi}}}),$$

where $\widehat{\chi_{\tilde{\pi}}} \in \mathrm{SO}_5(\mathbb{C})$ is the semisimple conjugacy class attached to the representation $\mathrm{Ind}^{\mathrm{Sp}_4}_{B_{\mathrm{Sp}_4}}\xi$. To lighten the notation from now on we will simply denote $\widehat{\chi_{\tilde{\pi}}}$ by $\chi_{\tilde{\pi}}$.

Theorem 2.7.6. *For all* $n\geq 0$ *we have*

$$\Lambda\left(\begin{pmatrix} p^n I & \\ & I\end{pmatrix}\right) = \frac{p^{-2n}}{1+p^{-1}}\cdot\mathcal{A}\Big(e^{\check{\rho}+n(\check{\alpha}_1+\check{\alpha}_2)}\cdot\prod_{\alpha\in\Phi^{S+}_{\mathrm{Sp}_4}}(1-p^{-1}e^{-\check{\alpha}})\Big)(\chi_{\tilde{\pi}})\left(\mathcal{A}\left(e^{\check{\rho}}\right)(\chi_{\tilde{\pi}})\right)^{-1}.$$

Proof. It follows from a combination of the previous discussion and theorem 2.4.58. □

2.7.3 Multiplicity of the Shalika model for $\mathrm{GU}_{2,2}$: The non split case

There is no formula available in the bibliography for the Shalika functional when the place p is not split. In order to find it using harmonic analysis for reductive p-adic groups, we need to establish a multiplicity one result for the model. The forthcoming exposition is based on Mackey theory, i.e. in proposition 2.3.16.

In accordance with the global situation we will assume every representation of $\mathbf{G}(\mathbb{Q}_p)$ with trivial central character and in the image of the theta correspondence with $\mathbf{H}^+(\mathbb{Q}_p)$.

Orbits computations

The first step of this proof consists on the computation of representatives of the double quotient $S(\mathbb{Q}_p)\backslash\mathbf{G}(\mathbb{Q}_p)/B_{\mathbf{G}}(\mathbb{Q}_p)$ when p is a inert prime in E. We denote by $E_p = \mathbb{Q}_p(\delta)$ the corresponding quadratic extension of $\mathbb{Q}_p$. Let $P_{\mathbf{G}} \simeq M_{\mathbf{G}}N_{\mathbf{G}}$ denote the Siegel parabolic of $\mathbf{G}$ and recall that $S(\mathbb{Q}_p)\hookrightarrow P_{\mathbf{G}}(\mathbb{Q}_p)$ induced by the inclusion $\mathrm{GL}_2(\mathbb{Q}_p)\hookrightarrow\mathrm{GL}_2(E_p)\times\mathrm{GL}_1(\mathbb{Q}_p)$, $g\mapsto(g,\det(g))$.

Lemma 2.7.7. *We have the following decomposition:*

$$\mathbf{G}(\mathbb{Q}_p) = \bigsqcup_{w\in W_\delta}\bigsqcup_{\tilde{w}\in {}^MW_{\mathbf{H}}} S(\mathbb{Q}_p)w\tilde{w}B_{\mathbf{G}}(\mathbb{Q}_p),$$

where $W_\delta = \left\{id, w_\delta = \begin{pmatrix}\frac{1}{\delta} & 1 & & \\ & & \frac{1}{-\bar{\delta}} & 1\end{pmatrix}\right\}$ *and* ${}^MW_{\mathbf{H}} := \{id, s_2, s_2s_1, s_2s_1s_2\}$ *the set of minimal length representatives (Kostant representatives) of* $W_{M_{\mathbf{H}}}\backslash W_{\mathbf{H}}$ *with* s_1 *and* s_2 *defined as in lemma 2.2.10.*

Proof. Using the Bruhat decomposition 2.1.34 we have

$$\mathbf{G}(\mathbb{Q}_p) = \bigsqcup_{\tilde{w}\in\tilde{W}} P_{\mathbf{G}}(\mathbb{Q}_p)\tilde{w}B_{\mathbf{G}}(\mathbb{Q}_p),$$

where $\tilde{W}$ is the set of Konstant representatives of $W_{M_{\mathbf{G}}}\backslash W_{\mathbf{G}}$, with $W_{M_{\mathbf{G}}}$ the Weyl group associated to the Levi of the Siegel subgroup $P_{\mathbf{G}}$. According to (2.2.1) and (2.2.2), the root datum of $\mathbf{H}$ equals to the relative root datum of $\mathbf{G}$, then $W_{M_{\mathbf{G}}}\backslash W_{\mathbf{G}} = W_{M_{\mathbf{H}}}\backslash W_{\mathbf{H}}$. Therefore the disjoint union above is indexed by the elements in the set of Kostant representatives for the quotient $W_{M_{\mathbf{H}}}\backslash W_{\mathbf{H}}$. Thus any $g \in \mathbf{G}(\mathbb{Q}_p)$ can be expressed as $m \cdot n \cdot \tilde{w} \cdot b$ for a unique $\tilde{w} \in {}^M W_{\mathbf{H}}$ and some $b \in B_{\mathbf{G}}(\mathbb{Q}_p)$. We use the following notation $m \cdot n = \left(\begin{smallmatrix} h & \\ & \nu J {}^t\bar{h}^{-1} J \end{smallmatrix}\right)\left(\begin{smallmatrix} I & X \\ & I \end{smallmatrix}\right) \in P_{\mathbf{G}}(\mathbb{Q}_p)$.

Write m as the product of $m'\lambda_\nu = \left(\begin{smallmatrix} h & \\ & J {}^t\bar{h}^{-1} J \end{smallmatrix}\right)\left(\begin{smallmatrix} I & \\ & \nu I \end{smallmatrix}\right)$, then we have

$$g = m'\lambda_\nu n\tilde{w}b = m'\lambda_\nu n\lambda_\nu^{-1}\lambda_\nu\tilde{w}b = m'n'\lambda_\nu\tilde{w}b = m'n'\tilde{w}\lambda'_\nu b \in \mathrm{GL}_2(E_p)N_{\mathbf{G}}(\mathbb{Q}_p)\tilde{w}B_{\mathbf{G}}(\mathbb{Q}_p),$$

where we have denoted $n' = \lambda_\nu n \lambda_\nu^{-1} \in N_{\mathbf{G}}(\mathbb{Q}_p)$ and $\lambda'_\nu = \tilde{w}\lambda_\nu\tilde{w}^{-1} \in T_{\mathbf{G}}(\mathbb{Q}_p)$. The previous discussion implies

$$\begin{aligned} S(\mathbb{Q}_p)\backslash\mathbf{G}(\mathbb{Q}_p)/B_{\mathbf{G}}(\mathbb{Q}_p) &= \bigsqcup_{\tilde{w}\in {}^M W_{\mathbf{H}}} S(\mathbb{Q}_p)\backslash P_{\mathbf{G}}(\mathbb{Q}_p)\tilde{w}B_{\mathbf{G}}(\mathbb{Q}_p)/B_{\mathbf{G}}(\mathbb{Q}_p) \\ &= \bigsqcup_{\tilde{w}\in {}^M W_{\mathbf{H}}} \mathrm{GL}_2(\mathbb{Q}_p)\backslash\mathrm{GL}_2(E_p)N_{\mathbf{G}}(\mathbb{Q}_p)\tilde{w}B_{\mathbf{G}}(\mathbb{Q}_p)/B_{\mathbf{G}}(\mathbb{Q}_p). \end{aligned}$$

Finally, recall that $\mathrm{GL}_2(\mathbb{Q}_p)$ acts on $\mathbb{P}^1_{E_p} = \mathrm{GL}_2(E_p)/B_{\mathrm{GL}_2}(E_p)$ with two orbits, one closed and one open with generators given by the set

$$W_\delta = \left\{ id, w_\delta = \begin{pmatrix} 1 & & & \\ \delta & 1 & & \\ & & 1 & \\ & & -\bar{\delta} & 1 \end{pmatrix} \right\},$$

see [Gro20, p. 18]. Since any $\tilde{w}$ is defined by the condition $\tilde{w}^{-1}\Phi^+_{M_{\mathbf{G}}} \subset \Phi^+_{\mathbf{G}}$, i.e. $\tilde{w}^{-1}B_{\mathrm{GL}_2}(E_p)\tilde{w} \subset B_{\mathbf{G}}(\mathbb{Q}_p)$, we use this fact to deduce

$$\mathbf{G}(\mathbb{Q}_p) = \bigsqcup_{w\in W_\delta}\bigsqcup_{\tilde{w}\in {}^M W_{\mathbf{H}}} S(\mathbb{Q}_p)w\tilde{w}B_{\mathbf{G}}(\mathbb{Q}_p),$$

which completes the proof. $\square$

Following Lemma 2.7.7, we now calculate the stabiliser of each orbit for the action of $S(\mathbb{Q}_p)$ on the flag $\mathbf{G}(\mathbb{Q}_p)/B_{\mathbf{G}}(\mathbb{Q}_p)$. Before doing so, we denote by T_δ the maximal non-split torus of $\mathrm{GL}_2(\mathbb{Q}_p)$ given by

$$T_\delta := \{\left(\begin{smallmatrix} a & b \\ b\delta^2 & a \end{smallmatrix}\right) : (a,b) \in \mathbb{Q}_p^2 \smallsetminus \{(0,0)\}\},$$

which we see inside $S(\mathbb{Q}_p)$ via the usual embedding $\mathrm{GL}_2(\mathbb{Q}_p) \hookrightarrow S(\mathbb{Q}_p)$.

Proposition 2.7.8. *The group $S(\mathbb{Q}_p)$ acts on the flag $\mathbf{G}(\mathbb{Q}_p)/B_{\mathbf{G}}(\mathbb{Q}_p)$ with an open orbit $\mathcal{O}_{w_\delta s_2 s_1 s_2}$ and seven closed ones. All possible cases are listed below:*

- $\mathcal{O}_{\mathrm{id}} = \mathrm{Stab}_{\mathrm{id}}\backslash S(\mathbb{Q}_p)$, *with* $\mathrm{Stab}_{\mathrm{id}} = B_{\mathrm{GL}_2}(\mathbb{Q}_p)N_{\mathbf{G}}(\mathbb{Q}_p)$.
- $\mathcal{O}_{s_2} = \mathrm{Stab}_{s_2}\backslash S(\mathbb{Q}_p)$, *with*
$$\mathrm{Stab}_{s_2} = B_{\mathrm{GL}_2}(\mathbb{Q}_p)\left\{\begin{pmatrix} 1 & & \alpha & x \\ & 1 & & \bar{\alpha} \\ & & 1 & \\ & & & 1 \end{pmatrix} : x \in \mathbb{Q}_p, \alpha \in E_p\right\}.$$
- $\mathcal{O}_{s_2 s_1} = \mathrm{Stab}_{s_2 s_1}\backslash S(\mathbb{Q}_p)$, *with*
$$\mathrm{Stab}_{s_2 s_1} = B_{\mathrm{GL}_2}(\mathbb{Q}_p)\left\{\begin{pmatrix} 1 & & & x \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} : x \in \mathbb{Q}_p\right\}.$$
- $\mathcal{O}_{s_2 s_1 s_2} = \mathrm{Stab}_{s_2 s_1 s_2}\backslash S(\mathbb{Q}_p)$, *with*
$$\mathrm{Stab}_{s_2 s_1 s_2} = B_{\mathrm{GL}_2}(\mathbb{Q}_p).$$

- $\mathcal{O}_{w_\delta} = \mathrm{Stab}_{w_\delta} \backslash S(\mathbb{Q}_p)$, *with*
$$\mathrm{Stab}_{w_\delta} = T_\delta N_{\mathbf{G}}(\mathbb{Q}_p).$$

- $\mathcal{O}_{w_\delta s_2} = \mathrm{Stab}_{w_\delta s_2} \backslash S(\mathbb{Q}_p)$, *with*
$$\mathrm{Stab}_{w_\delta s_2} = T_\delta \left\{ \begin{pmatrix} 1 & & \alpha & x \\ & 1 & \delta\alpha + \bar{\delta}\bar{\alpha} - \delta\bar{\delta}x & \bar{\alpha} \\ & & 1 & \\ & & & 1 \end{pmatrix} : x \in \mathbb{Q}_p, \alpha \in E_p \right\}.$$

- $\mathcal{O}_{w_\delta s_2 s_1} = \mathrm{Stab}_{w_\delta s_2 s_1} \backslash S(\mathbb{Q}_p)$, *with*
$$\mathrm{Stab}_{w_\delta s_2 s_1} = T_\delta \left\{ \begin{pmatrix} 1 & & \bar{\delta}x & x \\ & 1 & \delta\bar{\delta}x & \delta x \\ & & 1 & \\ & & & 1 \end{pmatrix} : x \in \mathbb{Q}_p \right\}.$$

- $\mathcal{O}_{w_\delta s_2 s_1 s_2} = \mathrm{Stab}_{w_\delta s_2 s_1 s_2} \backslash S(\mathbb{Q}_p)$, *with*
$$\mathrm{Stab}_{w_\delta s_2 s_1 s_2} = T_\delta.$$

Proof. By Lemma 2.7.7, a set of representatives for the $S(\mathbb{Q}_p)$-orbits of $\mathbf{G}(\mathbb{Q}_p)/B_{\mathbf{G}}(\mathbb{Q}_p)$ is given by $\{w\tilde{w}\}$ with w and $\tilde{w}$ varying in W_δ and ${}^M W_{\mathbf{H}}$ respectively. It is straightforward that the stabilizer of each element v of this set is given by
$$\mathrm{Stab}_v := S(\mathbb{Q}_p) \cap v B_{\mathbf{G}}(\mathbb{Q}_p) v^{-1}.$$
We start by computing the stabiliser of each element $\tilde{w} \in {}^M W_{\mathbf{H}}$. Since $\Phi^+_{M_{\mathbf{G}}} \subset \tilde{w}\Phi^+$, we have
$$\mathrm{Stab}_{\tilde{w}} = B_{\mathrm{GL}_2}(\mathbb{Q}_p)(N_{\mathbf{G}} \cap \tilde{w} U_{\mathbf{G}} \tilde{w}^{-1})(\mathbb{Q}_p),$$
where we recall that $U_{\mathbf{G}}$ denotes the unipotent radical of the upper triangular Borel parabolic of $\mathbf{G}$ and B_{GL_2} embeds into S via the map $b \mapsto \begin{pmatrix} b & \\ & \det(b) J {}^t b^{-1} J \end{pmatrix}$. Hence the following equality is straightforward:
$$\mathrm{Stab}_{\mathrm{id}} = B_{\mathrm{GL}_2}(\mathbb{Q}_p) N_{\mathbf{G}}(\mathbb{Q}_p),$$
Since s_2 sends the Sp_4-positive roots $\Phi^+_{\mathrm{Sp}_4} = \{\alpha_1 - \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1, 2\alpha_2\}$ to $\{\alpha_1 + \alpha_2, \alpha_1 - \alpha_2, 2\alpha_1, -2\alpha_2\}$, then
$$\mathrm{Stab}_{s_2} = B_{\mathrm{GL}_2}(\mathbb{Q}_p) \left\{ \begin{pmatrix} 1 & & \alpha & x \\ & 1 & & \bar{\alpha} \\ & & 1 & \\ & & & 1 \end{pmatrix} : x \in \mathbb{Q}_p, \alpha \in E_p \right\}.$$
Similarly, $s_2 s_1$ and resp. $s_2 s_1 s_2$, sends $\Phi^+_{\mathrm{Sp}_4}$ to $\{-\alpha_1 - \alpha_2, \alpha_1 - \alpha_2, -2\alpha_2, 2\alpha_1\}$, resp. to $\{\alpha_1 - \alpha_2, -\alpha_1 - \alpha_2, -2\alpha_2, -2\alpha_1\}$, hence
$$\mathrm{Stab}_{s_2 s_1} = B_{\mathrm{GL}_2}(\mathbb{Q}_p) \left\{ \begin{pmatrix} 1 & & & x \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} : x \in \mathbb{Q}_p \right\},$$
$$\mathrm{Stab}_{s_2 s_1 s_2} = B_{\mathrm{GL}_2}(\mathbb{Q}_p)$$
We are now left with calculating the stabiliser of the orbits associated to the elements $w_\delta \tilde{w}$. Conjugating by w_δ^{-1} we obtain
$$w_\delta^{-1} \mathrm{Stab}_{w_\delta \tilde{w}} w_\delta = w_\delta^{-1} S(\mathbb{Q}_p) w_\delta \cap \tilde{w} B_{\mathbf{G}}(\mathbb{Q}_p) \tilde{w}^{-1}.$$
Given $g = mn = \begin{pmatrix} a & b & & \\ c & d & & \\ & & a & -b \\ & & -c & d \end{pmatrix} \begin{pmatrix} 1 & & \alpha & x \\ & 1 & y & \bar{\alpha} \\ & & 1 & \\ & & & 1 \end{pmatrix} \in S(\mathbb{Q}_p)$ we get
$$w_\delta^{-1} g w_\delta = w_\delta^{-1} m w_\delta w_\delta^{-1} n w_\delta = \begin{pmatrix} a+b\delta & b & & \\ c' & d-b\delta & & \\ & & a+b\bar{\delta} & -b \\ & & -\bar{c}' & d-\bar{\delta}b \end{pmatrix} \begin{pmatrix} 1 & & \alpha-\bar{\delta}x & x \\ & 1 & y' & \bar{\alpha}-\delta x \\ & & 1 & \\ & & & 1 \end{pmatrix},$$
with $c' = c - b\delta^2 + \delta(d-a)$ and $y' = y + \delta\bar{\delta}x - \delta\alpha - \bar{\delta}\bar{\alpha}$. Using again that $\Phi^+_{M_{\mathbf{G}}} \subset \tilde{w}\Phi^+$, $w_\delta^{-1} m w_\delta \in$

$\tilde{w}B_{\mathbf{G}}(\mathbb{Q}_p)\tilde{w}^{-1}$ if $c' = 0$, i.e.

$$c = b\delta^2 + \delta(a-d).$$

Since $c \in \mathbb{Q}_p$, this further implies that $a - d = 0$, hence m equals to $\begin{pmatrix} a & b & & \\ b\delta^2 & a & & \\ & & a & -b \\ & & -b\delta^2 & a \end{pmatrix}$. When $\tilde{w} = \mathrm{id}$, $w_\delta^{-1} n w_\delta \in B_{\mathbf{G}}(\mathbb{Q}_p)$, which implies

$$\mathrm{Stab}_{w_\delta} = \left\{ \begin{pmatrix} a & b & & \\ b\delta^2 & a & & \\ & & a & -b \\ & & -b\delta^2 & a \end{pmatrix} : \begin{pmatrix} a & b \\ b\delta^2 & a \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q}_\ell) \right\} U_{\mathbf{G}}(\mathbb{Q}_\ell) = T_\delta N_{\mathbf{G}}(\mathbb{Q}_\ell).$$

When $\tilde{w} = s_2$, $N_{\mathbf{G}}(\mathbb{Q}_p) \cap \tilde{w}U_{\mathbf{G}}(\mathbb{Q}_p)\tilde{w}^{-1} = \begin{pmatrix} 1 & & \star & \star \\ & 1 & & \star \\ & & 1 & \\ & & & 1 \end{pmatrix}$, hence $w_\delta^{-1} u w_\delta \in \tilde{w}U_{\mathbf{G}}\tilde{w}^{-1}$ if $y' = 0$. Thus

$$\mathrm{Stab}_{w_\delta s_2} = T_\delta \left\{ \begin{pmatrix} 1 & & \alpha & x \\ & 1 & \delta\alpha + \bar{\delta}\bar{\alpha} - \delta\bar{\delta}x & \bar{\alpha} \\ & & 1 & \\ & & & 1 \end{pmatrix} \right\}.$$

Finally, when $\tilde{w} = s_2 s_1$, resp. $\tilde{w} = s_2 s_1 s_2$, $U_{\mathbf{G}}(\mathbb{Q}_p) \cap \tilde{w}N_{\mathbf{G}}(\mathbb{Q}_p)\tilde{w}^{-1}$ equals to $\begin{pmatrix} 1 & & & \star \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$, resp. I_4. From this, one deduces the last two cases. $\square$

Following the same ordering as in Proposition 2.7.8, we denote by $\{w_i\}_{i\in I}$, with $I = \{1, ..., 7\}$, the set of representatives of the closed orbits, each with stabiliser Stab_i. Moreover, we let w_{op} be the representative $w_\delta s_2 s_1 s_2$ of the open orbit with stabiliser T_δ.

Mackey theory

Now we are in position to apply proposition 2.3.16. In fact, let $X = \mathbf{G}(\mathbb{Q}_p)$, $Y = B_{\mathbf{G}}(\mathbb{Q}_p)$ and $\mathscr{V}$ the vector bundle associated to the representation $I_{\mathbf{G}}(\xi) := \mathrm{Ind}_{B_{\mathbf{G}}}^{\mathbf{G}}\xi$ via 2.3.2. If $Z = X/Y - \mathcal{O}_{w_{\mathrm{op}}}$, proposition 2.3.16 yields an exact sequence of $S(\mathbb{Q}_p)$-modules, with $S(\mathbb{Q}_p)$ acting on the right

$$0 \to \text{c-Ind}_{T_\delta}^{S}\xi_{\mathrm{op}} \to \pi := \mathrm{Ind}_{B_{\mathbf{G}}}^{\mathbf{G}}\xi \to \Gamma_c\left(\bigsqcup_{i\in I}\mathcal{O}_i, \mathscr{V}|_{\bigsqcup_{i\in I}\mathcal{O}_i}\right) \to 0, \tag{2.-13}$$

with ξ_{op} the representation of T_δ given by

$$\xi_{\mathrm{op}}(g) = \delta_{B_{\mathbf{G}}}^{1/2}(w_{\mathrm{op}}^{-1} g w_{\mathrm{op}})\delta_{T_\delta}^{-1/2}(g)\xi(w_{\mathrm{op}}^{-1} g w_{\mathrm{op}}) = \delta_{B_{\mathbf{G}}}^{1/2}\xi(w_{\mathrm{op}}^{-1} g w_{\mathrm{op}})$$

where the latter equality follows since T_δ is unimodular. Notice that the restriction of $\mathscr{V}$ to each orbit $\mathcal{O}_i$ can be identified with $\text{c-Ind}_{\mathrm{Stab}_i}^{S(\mathbb{Q}_p)}\xi_{w_i}$, where

$$\xi_{w_i}(g) := \delta_{B_{\mathbf{G}}}^{1/2}(w_i^{-1} g w_i)\delta_{Stab_{w_i}}^{-1/2}(g)\xi(w_i^{-1} g w_i).$$

In the present setting we get

$$\Gamma_c\left(\bigsqcup_{i\in I}\mathcal{O}_i, \mathscr{V}|_{\bigsqcup_{i\in I}\mathcal{O}_i}\right) = \Gamma_c\left(\bigsqcup_{i\in I}\mathrm{Stab}_i \setminus S(\mathbb{Q}_p)w_i, \mathscr{V}|_{\bigsqcup_{i\in I}\mathrm{Stab}_i\setminus S(\mathbb{Q}_p)w_i}\right).$$

In order to lighten the notation we will denote $\mathscr{V}|_J := \mathscr{V}|_{\bigsqcup_{i\in J}\mathrm{Stab}_i\setminus S(\mathbb{Q}_p)w_i}$ for any subset J of I. Let $\chi_{\delta,p}$ be the character on $S(\mathbb{Q}_p)$ defined in 2.4.21. Applying the functor $\mathrm{Hom}_S(-,\chi_\eta) := \mathrm{Hom}_{S(\mathbb{Q}_p)}(-,\chi_\eta)$ to the exact sequence (2.7.3), we obtain

$$0 \to \mathrm{Hom}_S\left(\Gamma_c\left(\bigsqcup_{i\in I}\mathcal{O}_i, \mathscr{V}|_{\bigsqcup_{i\in I}\mathcal{O}_i}\right), \chi_{\delta,p}\right) \to \mathrm{Hom}_S\left(\pi|_{S(\mathbb{Q}_p)}, \chi_{\delta,p}\right) \to \mathrm{Hom}_S\left(\text{c-Ind}_{T_\delta}^S \xi_{\mathrm{op}}, \chi_{\delta,p}\right) \to$$

$$\to \mathrm{Ext}^1_S\left(\Gamma_c\left(\bigsqcup_{i\in I}\mathcal{O}_i, \mathscr{V}|_{\bigsqcup_{i\in I}\mathcal{O}_i}\right), \chi_{\delta,p}\right) \to \mathrm{Ext}^1_S\left(\pi|_{S(\mathbb{Q}_p)}, \chi_{\delta,p}\right) \to \mathrm{Ext}^1_S\left(\text{c-Ind}_{T_\delta}^S \xi_{\mathrm{op}}, \chi_{\delta,p}\right) \to \ldots$$

We use this exact sequence to study the space $\mathrm{Hom}_S(\pi|_S, \chi_{\delta,p})$. To do so, we study the contributions of the open and closed orbits separately.

Lemma 2.7.9. *The space* $\mathrm{Hom}_S\left(c\text{-Ind}_{T_\delta}^S \xi_{\mathrm{op}}, \chi_{\delta,p}\right)$ *is trivial unless* $\xi_1(\alpha)\xi_2(\bar\alpha)\xi_0(\alpha\bar\alpha) = 1$, $\forall \alpha \in E_p^\times$, *in which case*

$$\dim \mathrm{Hom}_S\left(c\text{-Ind}_{T_\delta}^S \xi_{\mathrm{op}}, \chi_{\delta,p}\right) = 1.$$

Proof. By Frobenius reciprocity

$$\mathrm{Hom}_S\left(\text{c-Ind}_{T_\delta}^S \xi_{\mathrm{op}}, \chi_{\delta,p}\right) \simeq \mathrm{Hom}_{T_\delta}\left(\delta_S^{1/2}\xi_{\mathrm{op}}, \mathbf{1}\right).$$

We now explicit $\delta_S^{1/2}\xi_{\mathrm{op}}(g) = \delta_S^{1/2}(g)\delta_{B_{\mathbf{G}}}^{1/2}\xi(w_{\mathrm{op}}^{-1} g w_{\mathrm{op}})$. Firstly, recall the notation

$$s_2 = \begin{pmatrix} 1 &&& \\ && 1 & \\ & -1 && \\ &&& 1 \end{pmatrix},\ s_1 = \begin{pmatrix} & 1 && \\ -1 &&& \\ &&& -1 \\ && 1 & \end{pmatrix},\ w_\delta = \begin{pmatrix} 1 &&& \\ \delta & 1 && \\ && 1 & \\ && -\bar\delta & 1 \end{pmatrix}.$$

Hence $w_{\mathrm{op}} = \begin{pmatrix} && 1 & \\ && \delta & -1 \\ 1 &&& \\ -\bar\delta & -1 && \end{pmatrix}$ and, if $g = \begin{pmatrix} a & b && \\ b\delta^2 & a && \\ && a & -b \\ && -b\delta^2 & a \end{pmatrix} \in T_\delta$, we have $w_{\mathrm{op}}^{-1} g w_{\mathrm{op}} = \begin{pmatrix} a-b\delta & b && \\ & a+b\delta && \\ && a+b\delta & -b \\ &&& a-b\delta \end{pmatrix}$.

Since $\delta_S(g) = \delta_{B_{\mathbf{G}}}(w_{\mathrm{op}}^{-1} g w_{\mathrm{op}}) = 1$, the space $\mathrm{Hom}_{T_\delta}\left(\delta_S^{1/2}\xi_{\mathrm{op}}, \mathbf{1}\right)$ is trivial unless $\xi(w_{\mathrm{op}}^{-1} g w_{\mathrm{op}}) = 1$ for all $g \in T_\delta$, in which case it is 1 dimensional. This condition translates to asking that $\xi_1(\alpha)\xi_2(\bar\alpha)\xi_0(\alpha\bar\alpha) = 1$ for all $\alpha \in E_p^\times$. $\square$

Recall that π has trivial central character, i.e. $\xi_1(\alpha)\xi_2(\alpha)\xi_0(\alpha\bar\alpha) = 1$ $\forall \alpha \in E_p^\times$.

Lemma 2.7.10. *We have the following:*

- $\mathrm{Hom}_S(c\text{-Ind}_{\mathrm{Stab}_i}^S \xi_{w_i}, \chi_\eta)$ *is trivial if* $i = 1, 4, 5, 6, 7$;
- $\dim \mathrm{Hom}_S(c\text{-Ind}_{\mathrm{Stab}_2}^S \xi_{w_2}, \chi_\eta) \leq 1$ *and it equals to one if and only if*

$$\xi_1(a)\xi_2(a)\xi_0(ad) = 1,\ \forall a, d \in \mathbb{Q}_p^\times.$$

  The latter is equivalent to asking that $\xi_0 = id$ *and* $\xi_1 = \xi_2^{-1}$.
- $\dim \mathrm{Hom}_S(c\text{-Ind}_{\mathrm{Stab}_3}^S \xi_{w_3}, \chi_\eta) \leq 1$ *and it equals to one if and only if*

$$\xi_1(a)\xi_2(d)\xi_0(ad) = 1,\ \forall a, d \in \mathbb{Q}_p^\times.$$

  This is equivalent to asking that $\xi_1|_{\mathbb{Q}_p} = \xi_0^{-1}$ *and* $\xi_2(\alpha) = \xi_1(\bar\alpha)$.

Proof. By Frobenius reciprocity

$$\mathrm{Hom}_S(\text{c-Ind}_{\mathrm{Stab}_i}^S \xi_{w_i}, \chi_{\delta,p}) \simeq \mathrm{Hom}_{\mathrm{Stab}_i}\left(\delta_S^{1/2}\delta_{\mathrm{Stab}_i}^{-1/2}\xi_{w_i}, \chi_{\delta,p}|_{Stab_i}\right) = \mathrm{Hom}_{\mathrm{Stab}_i}\left(\delta_{Stab_i}^{-1/2}\xi_{w_i}, \chi_{\delta,p}|_{\mathrm{Stab}_i}\right).$$

In the case where $w_i \in \{id, s_2, w_\delta, w_\delta s_2, w_\delta s_2 s_1\}$, this space is trivial since $\chi_{\delta,p}|_{\mathrm{Stab}_i}$ is not trivial while $\delta_{\mathrm{Stab}_i}^{-1/2}\xi_{w_i}$ is trivial on the unipotent part of Stab_i. We are left with examining the cases of $w_2 = s_2 s_1$ and $w_3 = s_2 s_1 s_2$. Let us start with w_2. In this case

$$\mathrm{Stab}_2 = B_{\mathrm{GL}_2}(\mathbb{Q}_p)\left\{\begin{pmatrix} 1 &&& x \\ & 1 && \\ && 1 & \\ &&& 1 \end{pmatrix} : x \in \mathbb{Q}_p\right\},\ \delta_{\mathrm{Stab}_2}\left(\begin{pmatrix} a & b && \\ & d && \\ && a & -b \\ &&& d \end{pmatrix}\begin{pmatrix} 1 &&& x \\ & 1 && \\ && 1 & \\ &&& 1 \end{pmatrix}\right) = \frac{|a|^2}{|d|^2},$$

hence, if $g = \begin{pmatrix} a & b & & \\ & d & & \\ & & a & -b \\ & & & d \end{pmatrix} \begin{pmatrix} 1 & & & x \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$, we have

$$\begin{aligned} \delta_{\mathrm{Stab}_2}^{-1/2} \xi_{w_2}(g) &= \delta_{\mathrm{Stab}_2}^{-1}(g) \delta_{B_{\mathbf{G}}}^{1/2} \xi(w_2^{-1} g w_2) \\ &= \tfrac{|d|^2}{|a|^2} \cdot \delta_{B_{\mathbf{G}}}^{1/2} \xi \left(\begin{pmatrix} a & & -b & \\ & a & ax & -b \\ & & d & \\ & & & d \end{pmatrix} \right) \\ &= \tfrac{|d|^2}{|a|^2} \cdot \tfrac{|a|^2}{|d|^2} \xi_1(a) \xi_2(a) \xi_0(ad) \\ &= \xi_1(a)\xi_2(a)\xi_0(ad). \end{aligned}$$

Therefore $\mathrm{Hom}_{\mathrm{Stab}_2}\left(\delta_{\mathrm{Stab}_2}^{-1/2}\xi_{w_2}, \chi_\eta|_{\mathrm{Stab}_2}\right)$ is trivial unless $\xi_1(d)\xi_2(d)\xi_0(ad) = 1 \ \forall a, d \in \mathbb{Q}_p^\times$, in which case it is one-dimensional. Similarly, $\mathrm{Stab}_3 = B_{\mathrm{GL}_2}(\mathbb{Q}_p)$ and $\delta_{\mathrm{Stab}_3} \begin{pmatrix} a & b & & \\ & d & & \\ & & a & -b \\ & & & d \end{pmatrix} = \frac{|a|}{|d|}$ which implies the following equalities:

$$\begin{aligned} \delta_{\mathrm{Stab}_3}^{-1/2} \xi_{w_3}(g) &= \delta_{\mathrm{Stab}_3}^{-1}(g) \delta_{B_{\mathbf{G}}}^{1/2} \xi(w_3^{-1} g w_3) \\ &= \tfrac{|d|}{|a|} \cdot \delta_{B_{\mathbf{G}}}^{1/2} \xi \begin{pmatrix} a & b & & \\ & d & & \\ & & a & -b \\ & & & d \end{pmatrix} \\ &= \tfrac{|d|}{|a|} \cdot \tfrac{|a|}{|d|} \xi_1(a)\xi_2(d)\xi_0(ad) \\ &= \xi_1(a)\xi_2(d)\xi_0(ad). \end{aligned}$$

Hence $\mathrm{Hom}_{\mathrm{Stab}_3}\left(\delta_{\mathrm{Stab}_3}^{-1/2}\xi_{w_3}, \chi_\eta|_{\mathrm{Stab}_3}\right)$ is trivial unless $\xi_1(a)\xi_2(d)\xi_0(ad) = 1 \ \forall a, d \in \mathbb{Q}_p^\times$, in which case it is one-dimensional. $\square$

Role of the theta correspondence

We now finalize the calculations of 2.7.3 and discuss the uniqueness of the Shalika model for unramified representations of $\mathbf{PG}(\mathbb{Q}_p)$. After applying the exceptional isomorphism $j : \mathbf{PG}(\mathbb{Q}_p) \simeq \mathrm{PGSO}_{4,2}(\mathbb{Q}_p)$ of Lemma 2.2.27, these unramified representations are assumed to be in the image of the local theta correspondence for $(\mathbf{H}^+, \mathrm{GSO}_{4,2})$ of [Mor14]. Before continue the discussion about the multiplicity one, we recall the following lemma, which allows us to divide the proof into two cases:

Lemma 2.7.11. *Let $\xi' = \xi_1' \otimes \xi_2' \otimes \xi_0'$ be a character of $T_{\mathbf{PG}'}(\mathbb{Q}_p)$ defined by*

$$\xi' : \begin{pmatrix} a & & & & \\ & b & & & \\ & & x & & \\ & & & \nu b^{-1} & \\ & & & & \nu a^{-1} \end{pmatrix} \mapsto \xi_1'(a)\xi_2'(b)\xi_0'(x), \ \forall a, b \in \mathbb{Q}_p^\times, x \in E_p^\times.$$

The character $\xi := \xi' \circ j$ of $T_{\mathbf{PG}}(\mathbb{Q}_p)$ is given by $\begin{pmatrix} a & & & \\ & b & & \\ & & \nu\bar{b}^{-1} & \\ & & & \nu\bar{a}^{-1} \end{pmatrix} \mapsto \xi_1(a)\xi_2(b)\xi_0(\nu)$, where

$$\begin{aligned} \xi_1(a) &= (\xi_1'\xi_2' \circ \mathrm{N}_{E_p/\mathbb{Q}_p})(a)\xi_0'(\bar{a}), \\ \xi_2(b) &= \xi_1' \circ \mathrm{N}_{E_p/\mathbb{Q}_p}(b)\xi_0'(b), \\ \xi_0(\nu) &= \xi_2'\xi_0'(\nu). \end{aligned}$$

Proof. According to lemma 2.2.27, we have

$$j : \begin{pmatrix} a & & & \\ & b & & \\ & & \nu\bar{b}^{-1} & \\ & & & \nu\bar{a}^{-1} \end{pmatrix} \mapsto \begin{pmatrix} \mathrm{N}_{E_p/\mathbb{Q}_p}(ab) & & & & \\ & \nu\mathrm{N}_{E_p/\mathbb{Q}_p}(a) & & & \\ & & \nu\bar{a}b & & \\ & & & \nu\mathrm{N}_{E_p/\mathbb{Q}_p}(b) & \\ & & & & \nu^2 \end{pmatrix},$$

with $a, b \in E_p^\times$ and $\nu \in \mathbb{Q}_p^\times$. Thus, $\xi := \xi' \circ j$ is explicitly given by

$$\xi : \begin{pmatrix} a & & & \\ & b & & \\ & & \nu\bar{b}^{-1} & \\ & & & \nu\bar{a}^{-1} \end{pmatrix} \mapsto \xi_1'(\mathrm{N}_{E_p/\mathbb{Q}_p}(ab))\xi_2'(\nu \mathrm{N}_{E_p/\mathbb{Q}_p}(a))\xi_0'(\nu\bar{a}b).$$

Regrouping the terms of the latter as

$$\xi_1'(\mathrm{N}_{E_p/\mathbb{Q}_p}(a))\xi_2'(\mathrm{N}_{E_p/\mathbb{Q}_p}(a))\xi_0'(\bar{a})\xi_1'(\mathrm{N}_{E_p/\mathbb{Q}_p}(b))\xi_0'(b)\xi_2'(\nu)\xi_0'(\nu),$$

we get the result. □

We now turn our attention to local unramified generic representations in the image of the local theta correspondence studied by [Mor14]. By [Mor14, thm. 6.21, p. 73], the theta lift $\tilde{\theta}(\sigma^+)$ of an irreducible representation σ^+ of $\mathbf{H}^+(\mathbb{Q}_p)$ to $\mathrm{GSO}_{4,2}(\mathbb{Q}_p)$ has a unique irreducible quotient $\theta(\sigma^+)$ if it is non-zero. Moreover, note that an irreducible generic unramified representation σ of $\mathbf{PH}(\mathbb{Q}_p)$ is necessarily a principal series $I_{\mathbf{H}}(\chi) := \mathrm{Ind}_{B_{\mathbf{H}}(\mathbb{Q}_p)}^{\mathbf{H}(\mathbb{Q}_p)}\chi$, for a unique $W_{\mathbf{H}}$-orbit of characters $\chi = \chi_1 \otimes \chi_2 \otimes \chi_0$, where χ_1, χ_2, χ_0 are unramified smooth characters such that $\chi_1\chi_2\chi_0^2 = 1$ and $|\cdot|^{\pm 1} \notin \{\chi_1, \chi_2, \chi_1\chi_2, \chi_1\chi_2^{-1}\}$ (see [RS07, Tables A.1 & A.13]). We recall that, by corollary 2.5.32, the restriction $I_{\mathbf{H}^+}(\chi)$ of $I_{\mathbf{H}}(\chi)$ to $\mathbf{H}^+$ is irreducible if and only if $\chi_{E_p/\mathbb{Q}_p} \in \{\chi_1, \chi_2, \chi_1\chi_2, \chi_1\chi_2^{-1}\}$. To study the multiplicity of local Shalika models, it is thus natural to distinguish between the following two cases.

The non-degenerate case

We say we are in the *non-degenerate case* when π is the local theta lift of $I_{\mathbf{H}^+}(\chi)$ from $\mathbf{H}^+(\mathbb{Q}_p)$ so that

$$\begin{aligned}&\xi = (\chi_0 \circ N_{E_p/\mathbb{Q}_p}, \chi_2\chi_0 \circ N_{E_p/\mathbb{Q}_p}, \chi_1\chi_{E_p/\mathbb{Q}_p}),\\ &|\cdot|^{\pm 1}, \chi_{E_p/\mathbb{Q}_p} \notin \{\chi_1, \chi_2, \chi_1\chi_2, \chi_1\chi_2^{-1}\}.\end{aligned}$$

Lemma 2.7.12. *Let π be an irreducible generic unramified representation of $\mathbf{G}(\mathbb{Q}_p)$ with trivial central character. Let us assume π is in the non-degenerate case. Then $\pi = I_{\mathbf{G}}(\xi)$ such that $\xi_1(\alpha)\xi_2(\bar{\alpha})\xi_0(\alpha\bar{\alpha}) = 1$, $\forall \alpha \in E_p^\times$. Moreover, we have*

$$\mathrm{Hom}_{S(\mathbb{Q}_p)}(c\text{-}\mathrm{Ind}_{Stab_i}^{S(\mathbb{Q}_p)}\xi_{w_i}, \chi_{\delta,p}) = 0, \quad \textit{for every } 1 \le i \le 7.$$

Proof. Since $\chi_{E_p/\mathbb{Q}_p} \notin \{\chi_1, \chi_2, \chi_1\chi_2, \chi_1\chi_2^{-1}\}$, we can identify $\sigma^+ = I_{\mathbf{H}^+}(\chi)$. The theta lift of σ^+ is calculated explicitly in theorem 2.6.12. Then $\theta(\sigma^+)$ is the unique irreducible unramified quotient of

$$\mathrm{Ind}_{B_{\mathrm{GSO}_{4,2}}}^{\mathrm{GSO}_{4,2}}\left(\chi_1^{-1}\chi_{E_p/\mathbb{Q}_p} \otimes \chi_2^{-1}\chi_{E_p/\mathbb{Q}_p} \otimes (\chi_1\chi_2\chi_0) \circ N_{E_p/\mathbb{Q}_p}\right).$$

By [Mor14, sec. 5.2.2, p. 53], this principal series is irreducible and thus isomorphic to $\theta(\sigma^+)$. Using the exceptional isomorphism $j : \mathbf{PG}(\mathbb{Q}_p) \simeq \mathbf{PG}'(\mathbb{Q}_p)$ and lemma 2.7.11, we thus have $\Pi = I_{\mathbf{G}}(\xi)$, with

$$\begin{aligned}\xi_1 &= \chi_0 \circ N_{E_p/\mathbb{Q}_p},\\ \xi_2 &= (\chi_2\chi_0) \circ N_{E_p/\mathbb{Q}_p},\\ \xi_0 &= \chi_1\chi_{E_p/\mathbb{Q}_p}.\end{aligned}$$

We now examine the conditions of lemma 2.7.9 and lemma 2.7.10. Firstly, notice that the condition of Lemma 2.7.9 is automatically satisfied; indeed, for any $\alpha \in E_p^\times$, we have

$$\xi_1(\alpha)\xi_2(\bar{\alpha})\xi_0(\alpha\bar{\alpha}) = (\chi_1\chi_2\chi_0^2\chi_{E_p/\mathbb{Q}_p})(\alpha\bar{\alpha}) = 1,$$

because $I_{\mathbf{H}}(\chi)$ has trivial central character and $\chi_{E_p/\mathbb{Q}_p}$ is trivial on $N_{E_p/\mathbb{Q}_p}(E_p^\times)$. Now, notice that the conditions $\xi_0 = \mathrm{id}$, resp. $\xi_1|_{\mathbb{Q}_p^\times} = \xi_0^{-1}$, imply that χ_1, resp. χ_2, equal to $\chi_{E_p/\mathbb{Q}_p}$, which would yield a contradiction. Therefore, by Lemma 2.7.10, we get

$$\mathrm{Hom}_{S(\mathbb{Q}_p)}(\text{c-Ind}_{Stab_i}^{S(\mathbb{Q}_p)}\xi_{w_i},\chi_{\delta,p})=0,\quad \text{for every } 1\leq i\leq 7.$$

□

We can finally prove the uniqueness of the Shalika functionals.

Theorem 2.7.1 (Multiplicity one)**.** *Let $\pi=\theta(\sigma^+)$ be as in Lemma 2.7.12. Then π admits up to constant multiple a unique Shalika functional.*

Proof. By lemma 2.7.12, π is an irreducible unramified principal series of the form $I_{\mathbf{G}}(\xi)$, with $\xi_1=\chi_0\circ N_{E_p/\mathbb{Q}_p}$, $\xi_2=(\chi_2\chi_0)\circ N_{E_p/\mathbb{Q}_p}$, $\xi_0=\chi_1\chi_{E_p/\mathbb{Q}_p}$, for certain unramified smooth characters χ_1,χ_2,χ_0 of $\mathbb{Q}_p^\times$. Recall that we have the long exact sequence (2.7.3)

$$0\to \mathrm{Hom}_{S(\mathbb{Q}_p)}\left(\Gamma_c\left(\bigsqcup_{i\in I}\mathcal{O}_i,\mathscr{V}_{|\bigsqcup_{i\in I}\mathcal{O}_i}\right),\chi_{\delta,p}\right)\to \mathrm{Hom}_{S(\mathbb{Q}_p)}\left(I_{\mathbf{G}}(\xi),\chi_{\delta,p}\right)\to \mathrm{Hom}_{S(\mathbb{Q}_p)}\left(\text{c-Ind}_{T_\delta}^{S(\mathbb{Q}_p)}\xi_{\mathrm{op}},\chi_{\delta,p}\right)\to$$

$$\to \mathrm{Ext}^1_{S(\mathbb{Q}_p)}\left(\Gamma_c\left(\bigsqcup_{i\in I}\mathcal{O}_i,\mathscr{V}_{|\bigsqcup_{i\in I}\mathcal{O}_i}\right),\chi_{\delta,p}\right)\to \mathrm{Ext}^1_{S(\mathbb{Q}_p)}\left(I_{\mathbf{G}}(\xi),\chi_{\delta,p}\right)\to \mathrm{Ext}^1_{S(\mathbb{Q}_p)}\left(\text{c-Ind}_{T_\delta}^{S(\mathbb{Q}_p)}\xi_{\mathrm{op}},\chi_{\delta,p}\right)\to \dots$$

By lemma 2.7.10, we have

$$\mathrm{Hom}_{S(\mathbb{Q}_p)}(\text{c-Ind}_{Stab_i}^{S(\mathbb{Q}_p)}\xi_{w_i},\chi_{\delta,p})=0,\quad \text{for every } 1\leq i\leq 7.$$

Thus, we can apply lemma 2.7.13 to get

$$\mathrm{Hom}_{S(\mathbb{Q}_p)}\left(\Gamma_c\left(\bigsqcup_{i\in I}\mathcal{O}_i,\mathscr{V}_{|\bigsqcup_{i\in I}\mathcal{O}_i}\right),\chi_{\delta,p}\right)=\mathrm{Ext}^1_{S(\mathbb{Q}_p)}\left(\Gamma_c\left(\bigsqcup_{i\in I}\mathcal{O}_i,\mathscr{V}_{|\bigsqcup_{i\in I}\mathcal{O}_i}\right),\chi_{\delta,p}\right)=0.$$

Therefore, the short exact sequence gives an isomorphism

$$\mathrm{Hom}_{S(\mathbb{Q}_p)}\left(I_{\mathbf{G}}(\xi),\chi_{\delta,p}\right)\simeq \mathrm{Hom}_{S(\mathbb{Q}_p)}\left(\text{c-Ind}_{T_\delta}^{S(\mathbb{Q}_p)}\xi_{\mathrm{op}},\chi_{\delta,p}\right).$$

By lemmas 2.7.12 and 2.7.9, the latter is exactly of dimension one. □

Lemma 2.7.13. *Let ξ be such that $\mathrm{Hom}_{S(\mathbb{Q}_p)}(\text{c-Ind}_{Stab_i}^{S(\mathbb{Q}_p)}\xi_{w_i},\chi_{\delta,p})=0$ for every $1\leq i\leq 7$, where we recall that $\{w_i\}_{i\in I}$ is the set of representatives of the closed orbits of $S(\mathbb{Q}_p)\backslash\mathbf{G}(\mathbb{Q}_p)/B_{\mathbf{G}}(\mathbb{Q}_p)$. Then*

$$\mathrm{Hom}_{S(\mathbb{Q}_p)}\left(\Gamma_c\left(\bigsqcup_{i\in I}\mathcal{O}_i,\mathscr{V}_{|\bigsqcup_{i\in I}\mathcal{O}_i}\right),\chi_{\delta,p}\right)=\mathrm{Ext}^1_{S(\mathbb{Q}_p)}\left(\Gamma_c\left(\bigsqcup_{i\in I}\mathcal{O}_i,\mathscr{V}_{|\bigsqcup_{i\in I}\mathcal{O}_i}\right),\chi_{\delta,p}\right)=0.$$

Proof. We will prove the statement by applying Proposition 2.3.16 inductively on the number of closed orbits.

Firstly, let $X=\mathcal{O}_1\bigsqcup\mathcal{O}_2$, $Z=\mathcal{O}_1$ and let $\mathscr{V}|_X$ be the restriction to X of the vector bundle $\mathscr{V}$ associated to the representation $I_{\mathbf{G}}(\xi)$. By Proposition 2.3.16, we have the exact sequence

$$0\to\text{c-Ind}^S_{\mathrm{Stab}_1}\xi_{w_1}\to\Gamma_c\left(X,\mathscr{V}|_X\right)\to\text{c-Ind}^S_{\mathrm{Stab}_2}\xi_{w_2}\to 0.$$

We apply the functor $\mathrm{Hom}_{S(\mathbb{Q}_p)}\left(\cdot,\chi_{\delta,p}\right)$ to the previous exact sequence, obtaining

$$0\to \mathrm{Hom}_{S(\mathbb{Q}_p)}\left(\text{c-Ind}^S_{\mathrm{Stab}_2}\xi_{w_2},\chi_{\delta,p}\right)\to \mathrm{Hom}_{S(\mathbb{Q}_p)}\left(\Gamma_c\left(X,\mathscr{V}|_X\right),\chi_{\delta,p}\right)\to \mathrm{Hom}_{S(\mathbb{Q}_p)}\left(\text{c-Ind}^S_{\mathrm{Stab}_1}\xi_{w_1},\chi_{\delta,p}\right)\to\dots \tag{2.-13}$$

By hypothesis, $\mathrm{Hom}_{S(\mathbb{Q}_p)}(\text{c-Ind}_{Stab_i}^{S(\mathbb{Q}_p)}\xi_{w_i},\chi_{\delta,p})=0$ for $i=1,2$, which shows the equality

$$\mathrm{Hom}_{S(\mathbb{Q}_p)}\left(\Gamma_c\left(X,\mathscr{V}|_X\right),\chi_{\delta,p}\right)=0.$$

We now apply an inductive argument on the number of closed orbits; given $J = \{1, ..., j\} \subset I$ so that $j+1 \in I$. Let us suppose

$$\mathrm{Hom}_{S(\mathbb{Q}_p)}\left(\Gamma_c\left(\bigsqcup_{i\in J}\mathcal{O}_i, \mathscr{V}|_J\right), \chi_{\delta,p}\right) = 0, \tag{2.-13}$$

where $\mathscr{V}|_J$ denotes the restriction of $\mathscr{V}$ to $\bigsqcup_{i\in J}\mathcal{O}_i$. Applying Proposition 2.3.16 with $X = \bigsqcup_{i\in J\cup\{j+1\}}\mathcal{O}_i$, $Z = \mathcal{O}_{j+1}$ and with the line bundle $\mathscr{V}|_{J\cup\{j+1\}}$, we get an analogous exact sequence to (2.7.3). Using the inductive hypothesis (2.7.3) and our hypothesis on the vanishing of $\mathrm{Hom}_{S(\mathbb{Q}_p)}(\text{c-Ind}_{Stab_{j+1}}^{S(\mathbb{Q}_p)}\xi_{w_{j+1}}, \chi_{\delta,p})$, we obtain

$$\mathrm{Hom}_{S(\mathbb{Q}_p)}\left(\Gamma_c\left(\bigsqcup_{i\in J\cup\{j+1\}}\mathcal{O}_i, \mathscr{V}|_{J\cup\{j+1\}}\right), \chi_{\delta,p}\right) = 0.$$

By induction, this proves the following equality:

$$\mathrm{Hom}_{S(\mathbb{Q}_p)}\left(\Gamma_c\left(\bigsqcup_{i\in I}\mathcal{O}_i, \mathscr{V}_{|\bigsqcup_{i\in I}\mathcal{O}_i}\right), \chi_{\delta,p}\right) = 0.$$

The vanishing of the Ext^1 is proved similarly. Firstly, notice that $\mathrm{Hom}_{S(\mathbb{Q}_p)}(\text{c-Ind}_{Stab_i}^{S(\mathbb{Q}_p)}\xi_{w_i}, \chi_{\delta,p}) = 0$ if and only if $\mathrm{Ext}^1_{S(\mathbb{Q}_p)}(\text{c-Ind}_{Stab_i}^{S(\mathbb{Q}_p)}\xi_{w_i}, \chi_{\delta,p}) = 0$. This fact is proved in the exact same way as for [Pra90, prop 5.9 p. 17]. Then, the exact sequence (2.7.3) becomes

$$0 \to \mathrm{Ext}^1_{S(\mathbb{Q}_p)}(\text{c-Ind}_{Stab_2}^{S(\mathbb{Q}_p)}\xi_{w_2}, \chi_{\delta,p}) \to \mathrm{Ext}^1_{S(\mathbb{Q}_p)}\left(\Gamma_c\left(X, \mathscr{V}|_X\right), \chi_{\delta,p}\right) \to \mathrm{Ext}^1_{S(\mathbb{Q}_p)}\left(\text{c-Ind}_{\mathrm{Stab}_1}^{S}\xi_1, \chi_{\delta,p}\right) \to ...,$$

thus, by hypothesis, $\mathrm{Ext}^1_{S(\mathbb{Q}_p)}\left(\Gamma_c\left(X, \mathscr{V}|_X\right), \chi_{\delta,p}\right) = 0$. Using the same inductive argument as above we obtain the result. □

The degenerate case

We are in the degenerate case when $\chi_{E_p/\mathbb{Q}_p} \in \{\chi_1, \chi_2, \chi_1\chi_2, \chi_1\chi_2^{-1}\}$, i.e. $I_{\mathbf{H}}(\chi) \otimes \chi_{E_p/\mathbb{Q}_p} \simeq I_{\mathbf{H}}(\chi)$. At present we have not proved the multiplicity one property in this situation. We expect to exclude this situation by proving that this would imply the existence of a pole of order two for the standard $L-$function of $\mathbf{H}$.

2.7.4 A Casselman-Shalika formula for the Shalika model of $\mathrm{GU}_{2,2}$: The non split case

Along all this section we will assume every representation π of $G(\mathbb{Q}_p)$ (with p inert), unramified, with trivial central character and in the non-degenerate case. Furthermore, we will assume the existence of the Shalika model of π. The last assumption is consistent with the global picture. In fact, any automorphic representation which is in the image of the theta correspondence has a non-zero Shalika model by theorem 2.6.21.

Sketch of the proof

The strategy used to determine the formula will be analogous to the one described in subsection 2.4.7. For the sake of clarity, we will discuss here an sketch of the forthcoming subsections, emphasizing the differences with the proof of [Sak06].

The main goal of this subsection is to show an explicit expression of $\mathcal{S}_p\left(\begin{pmatrix} p^n I_2 & \\ & I_2\end{pmatrix} v_0\right)$. Firstly, in order to take advantage of the structure of the locally compact group $\mathbf{G}(\mathbb{Q}_p)$, we rather work with the following translation of the Shalika functional:

$$\tilde{\mathcal{S}}_p(g \cdot v_0) := \mathcal{S}_p((w_{op}g) \cdot v_0). \tag{2.-13}$$

As in [Sak06, sec. 4, p. 9], using the work of Casselman [CS80] and Hironaka [Hir99], we may write the above twist by

$$\tilde{\mathcal{S}}_p(g_{-\lambda}\cdot v_0) = Q^{-1}\sum_{\omega\in W_{\mathbf{H}}}\prod_{\substack{\alpha\in\Phi^+_{\mathbf{H}}\\ \omega\alpha>0}} c_\alpha(\xi)T^*_{\omega^{-1}}\Delta_{\tilde{S},{}^\omega\xi}(R_{g_{-\lambda}}\mathrm{char}_I). \tag{2.-13}$$

The previous formula reduces the computation to the constants

$$T^*_{\omega^{-1}}\Delta_{\tilde{S},{}^\omega\xi}(R_{g_{-\lambda}}\mathrm{char}_I).$$

It is necessary to remove this dependence of $T^*_{\omega^{-1}}\Delta_{\tilde{S},{}^\omega\xi}$ on $g_{-\lambda}$. This step is called *the first reduction.* Using the multiplicity one result obtained in the previous subsection (see theorem 2.7.1), we show that (2.7.4) is equal to

$$Q^{-1}\prod_{\omega\in W_{\mathbf{H}}}\sum_{\substack{\alpha\in\Phi^+_{\mathbf{H}}\\ \omega\alpha>0}} c_\alpha(\xi)({}^\omega\xi^{-1}\delta^{1/2}_{B_{\mathbf{G}}})(g_\lambda)T^*_{\omega^{-1}}\Delta_{\tilde{S},{}^\omega\xi}(\mathrm{char}_I). \tag{2.-13}$$

By 2.2.19, the elements $\omega_{\alpha_1-\alpha_2}$ and $\omega_{2\alpha_2-\alpha_0}$ generate Weyl group. Therefore, once again by the multiplicity one result, the computation will conclude by determining the following constants:

$$T^*_{\omega^{-1}_{\alpha_1-\alpha_2}}\Delta_{\tilde{S},{}^{\omega_{\alpha_1-\alpha_2}}\xi}(\mathrm{char}_I) = c_{\alpha_1-\alpha_2}({}^{\omega_{\alpha_1-\alpha_2}}\xi)-1+p^{-2}\Delta_{\tilde{S},{}^{\omega_{\alpha_1-\alpha_2}}\xi}(\mathrm{char}_{I\omega_{\alpha_1-\alpha_2}I,{}^{\omega_{\alpha_1-\alpha_2}}\xi}),$$
$$T^*_{\omega^{-1}_{2\alpha_2-\alpha_0}}\Delta_{\tilde{S},{}^{\omega_{2\alpha_2-\alpha_0}}\xi}(\mathrm{char}_I) = c_{2\alpha_2-\alpha_0}({}^{\omega_{2\alpha_2-\alpha_0}}\xi)-1+p^{-1}\Delta_{\tilde{S},{}^{\omega_{2\alpha_2-\alpha_0}}\xi}(\mathrm{char}_{I\omega_{2\alpha_2-\alpha_0}I,{}^{\omega_{2\alpha_2-\alpha_0}}\xi}),$$

On first thought, the only indeterminacy above are the constants

$$\Delta_{\tilde{S},{}^\omega\xi}(\mathrm{char}_{I\omega I,{}^\omega\xi}),$$

when $\omega=\omega_{\alpha_1-\alpha_2}$ or $\omega_{2\alpha_2-\alpha_0}$. Hence, in order to conclude the computation we have to obtain a formula for the distribution $\Delta_{\tilde{S},{}^\omega\xi}$. This step is called *the second reduction.* Due to the non-split structure of the group $\mathbf{G}$, the computation differs from that in 2.4.7. The absolute convergence of the intertwining operators T_ω is the main ingredient for this part. The *third reduction* consists of presenting a formula based on the recently determined formula for the above constants. In the *fourth reduction* we will use the previous results to show an explicit formula for the Shalika functional.

Throughout this subsection we fix a non split prime p of $\mathbb{Q}$ over E and $\pi=\mathrm{Ind}^{\mathbf{G}}_{B_{\mathbf{G}}}\xi$ an unramified representation of $\mathbf{G}(\mathbb{Q}_p)$ with trivial central character and in the non-degenerate case. Further, we suppose π lies in the image of the local theta correspondence from non-degenerate representations of $\mathbf{H}^+$. For the following exposition we recall that the relative root datum of $\mathbf{G}\times_{\mathbb{Q}}\mathbb{Q}_p$ with respect to $T_{\mathbf{G},s}$ is equal to $\Phi_{\mathbf{H}}$, the reduced root datum of the group $\mathbf{H}$.

Preliminary step: structure theory of G

This subsection is devoted to computing constants needed to give an explicit expression for $c_\alpha(\xi)$ in the cases of interests for the dissertation. We recommend skipping this section for expert readers.

For the following discussion we will denote by $\omega_1,\omega_2\in W(\mathbf{G},T_{\mathbf{G},s})(\mathbb{Q}_p)$ the Weyl elements associated to the roots $\alpha_1-\alpha_2$ and $2\alpha_2-\alpha_0$. We recall that these two elements generate the group $W(\mathbf{G},T_{\mathbf{G},s})(\mathbb{Q}_p)$. Furthermore, we fix s_1 and s_2 two representatives of ω_1 and ω_2 in $N_G(T)(\mathbb{Q}_p)$. An explicit choice of representatives may be found in lemma 2.2.10.

Lemma 2.7.14. *We have*

$$[I\omega_2 I : I] = p.$$

Proof. Let us note that there exists a group isomorphism

$$U^{2\alpha_2-\alpha_0}_{\mathbf{G}}(\mathbb{Z}_p)\simeq\mathbb{Z}_p.$$

By 2.1.64, the Iwahori subgroup I has Iwahori factorization of the form

$$I = B_{\mathbf{G}}(\mathbb{Z}_p)\overline{U_{\mathbf{G}}}(p\mathbb{Z}_p).$$

Therefore

$$I\omega_2 I/I \simeq B_{\mathbf{G}}(\mathbb{Z}_p)\overline{U_{\mathbf{G}}}(p\mathbb{Z}_p)\omega_2/B_{\mathbf{G}}(\mathbb{Z}_p)\overline{U_{\mathbf{G}}}(p\mathbb{Z}_p). \tag{2.-13}$$

By direct computation (2.7.4) is equal to

$$\omega_2 B_{\mathbf{G}}^{\widehat{2\alpha_2-\alpha_0}}(\mathbb{Z}_p)\overline{U_{\mathbf{G}}}^{2\alpha_2-\alpha_0}(\mathbb{Z}_p)\overline{U_{\mathbf{G}}}^{\widehat{2\alpha_2-\alpha_0}}(p\mathbb{Z}_p)U_{\mathbf{G}}^{2\alpha_2-\alpha_0}(p\mathbb{Z}_p)/B_{\mathbf{G}}(\mathbb{Z}_p)\overline{U_{\mathbf{G}}}(p\mathbb{Z}_p). \tag{2.-13}$$

Since $U_{\mathbf{G}}^{2\alpha_2-\alpha_0}(p\mathbb{Z}_p) \subset B_{\mathbf{G}}(\mathbb{Z}_p)$ and $\overline{U_{\mathbf{G}}}^{\widehat{2\alpha_2-\alpha_0}}(p\mathbb{Z}_p) \subset \overline{U_{\mathbf{G}}}(p\mathbb{Z}_p)$ we have (2.7.4) equal to

$$\omega_2 B_{\mathbf{G}}^{\widehat{2\alpha_2-\alpha_0}}(\mathbb{Z}_p)\overline{U_{\mathbf{G}}}^{2\alpha_2-\alpha_0}(\mathbb{Z}_p)/B_{\mathbf{G}}(\mathbb{Z}_p)\overline{U_{\mathbf{G}}}(p\mathbb{Z}_p).$$

Using $B_{\mathbf{G}}^{\widehat{2\alpha_2-\alpha_0}}(\mathbb{Z}_p) = N_G(\mathbb{Z}_p)T_G(\mathbb{Z}_p)$ and $\overline{U_{\mathbf{G}}}^{2\alpha_2-\alpha_0}(\mathbb{Z}_p) \in M_{\mathbf{G}}(\mathbb{Z}_p)$ the above quotient is isomorphic to

$$\omega_2\overline{U_{\mathbf{G}}}^{2\alpha_2-\alpha_0}(\mathbb{Z}_p)/B_{\mathbf{G}}(\mathbb{Z}_p)\overline{U_{\mathbf{G}}}(p\mathbb{Z}_p). \tag{2.-13}$$

Equation (2.7.4) implies

$$\overline{U_{\mathbf{G}}}^{2\alpha_2-\alpha_0}(\mathbb{Z}_p)/\overline{U_{\mathbf{G}}}^{2\alpha_2-\alpha_0}(p\mathbb{Z}_p) \simeq \mathbb{F}_p.$$

Therefore, there are p cosets of the form

$$\omega_2\overline{U_{\mathbf{G}}}^{2\alpha_2-\alpha_0}(j)/B_{\mathbf{G}}(\mathbb{Z}_p)\overline{U_{\mathbf{G}}}(p\mathbb{Z}_p),$$

with $j \in \mathbb{F}_p$ and

$$\overline{U_{\mathbf{G}}}^{2\alpha_2-\alpha_0}(j) := \begin{pmatrix} 1 & & & \\ & 1 & & \\ & j & 1 & \\ & & & 1 \end{pmatrix}.$$

□

Corollary 2.7.15. *We obtain $q_{2\alpha_2-\alpha_0} = p$.*

Proof. It follows combining lemma 2.7.14 and the proof of [Cas80, thm. 3.4, p. 14]. □

Lemma 2.7.16. *We obtain*

$$[I\omega_1 I : I] = p^2$$

Proof. Proceeding as in the proof of lemma 2.7.14, we find

$$\begin{aligned} I\omega_1 I &\simeq \omega_1 B_{\mathbf{G}}^{\widehat{\alpha_1-\alpha_2}}(\mathbb{Z}_p)\overline{U_{\mathbf{G}}}^{\alpha_1-\alpha_2}(\mathbb{Z}_p)\overline{U_{\mathbf{G}}}^{\widehat{\alpha_1-\alpha_2}}(p\mathbb{Z}_p)U_{\mathbf{G}}^{\alpha_1-\alpha_2}(p\mathbb{Z}_p)/B_{\mathbf{G}}(\mathbb{Z}_p)\overline{U_{\mathbf{G}}}(p\mathbb{Z}_p) \\ &\simeq \omega_1\overline{U_{\mathbf{G}}}^{\alpha_1-\alpha_2}(\mathbb{Z}_p)/B_{\mathbf{G}}(\mathbb{Z}_p)\overline{U_{\mathbf{G}}}(p\mathbb{Z}_p). \end{aligned}$$

We recall the group isomorphism

$$U_{\mathbf{G}}^{\alpha_1-\alpha_2}(\mathbb{Z}_p) \simeq \mathcal{O}_{E_p}.$$

Therefore, since

$$\overline{U_{\mathbf{G}}}^{\alpha_1-\alpha_2}(\mathbb{Z}_p)/\overline{U_{\mathbf{G}}}^{\alpha_1-\alpha_2}(p\mathbb{Z}_p) \simeq \mathbb{F}_p \oplus \mathbb{F}_p,$$

we proceed as in the proof of lemma 2.7.14 to conclude. □

Corollary 2.7.17. *It holds that $q_{\alpha_1-\alpha_2} = p^2$.*

Proof. It follows combining lemma 2.7.16 and the proof of [Cas80, thm. 3.4, p. 14]. □

Lemma 2.7.18. *We have $a_{\alpha_1-\alpha_2} = \begin{pmatrix} p & & & \\ & p^{-1} & & \\ & & p & \\ & & & p^{-1} \end{pmatrix}$ and $a_{2\alpha_2-\alpha_0} = \begin{pmatrix} 1 & & & \\ & p & & \\ & & p^{-1} & \\ & & & 1 \end{pmatrix}$.*

Proof. The $\mathbb{Q}_p-$points of the Levi subgroup associated to $\alpha_1 - \alpha_2$ is isomorphic to $\mathrm{GL}_2(E_p) \times \mathrm{GL}_1(\mathbb{Q}_p)$. It is known that $(\mathrm{GL}_2(E_p) \times \mathrm{GL}_1(\mathbb{Q}_p))^{der} = \mathrm{SL}_2(E_p)$, which is simply connected. According to definition 2.3.57, the element $a_{\alpha_1-\alpha_2}$ is the image of a representative of the following coset

$$a = \begin{pmatrix} (p)-(p)^2 & \\ & ((p)-(p)^2)^{-1} \end{pmatrix}$$

throughout the map $\mathrm{SL}_2(E_p) \to \mathbf{G}(\mathbb{Q}_p)$. By direct computation the first equality follows. For the root $2\alpha_2-\alpha_0$, the simply connected cover of the $\mathbb{Q}_p-$points of its associated Levi subgroup is equal to $\mathrm{SL}_2(\mathbb{Q}_p)$. Proceeding as above we conclude. □

Remark 2.7.19. *Using lemma 2.7.18 and [CS80, p. 19] we may show alternative proofs for lemmas 2.7.14 and 2.7.16.*

Preliminary step: Twist of the Shalika functional

The main goal of this subsection is to justify and explain why the twist of the Shalika character may benefit us. Furthermore we will show the equality (2.7.4).

Let w_{op} be the representative of the open orbit of $S(\mathbb{Q}_p) \setminus \mathbf{G}(\mathbb{Q}_p)/B_{\mathbf{G}}(\mathbb{Q}_p)$ (see 2.7.8). We define the group

$$\tilde{S}(\mathbb{Q}_p) := w_{op}S(\mathbb{Q}_p)w_{op}^{-1} = \omega_\delta \mathrm{GL}_2(\mathbb{Q}_p)\overline{N}_{\mathbf{G}}(\mathbb{Q}_p)\omega_\delta^{-1},$$

Furthermore, we set the character

$$\begin{aligned} \chi_{\tilde{S},p}: \ \tilde{S}(\mathbb{Q}_p) &\to \mathbb{C}^\times, \\ s &\mapsto \chi_{\delta,p}(w_{op}^{-1} s w_{op}), \end{aligned}$$

and the map

$$\begin{aligned} \mathrm{Ind}_S^{\mathbf{G}} \chi_{\delta,p} &\to \mathrm{Ind}_{\tilde{S}}^{\mathbf{G}} \chi_{\tilde{S},p}, \\ f &\mapsto f_{\tilde{S}}(g) := f(w_{op}^{-1} g). \end{aligned}$$

It determines an isomorphism of representations. Hence, the composition of (2.7.4) with the map $\mathrm{Ind}_{B_{\mathbf{G}}}^{\mathbf{G}} \xi \to \mathrm{Ind}_S^{\mathbf{G}} \chi_{\delta,p}$ given by the existence of the Shalika model, results in a $\mathbf{G}(\mathbb{Q}_p)-$intertwining map of the form

$$\mathrm{Ind}_{B_{\mathbf{G}}}^{\mathbf{G}} \xi \to \mathrm{Ind}_{\tilde{S}}^{\mathbf{G}} \chi_{\tilde{S},p}.$$

The above representation may be seen as a different realization of the Shalika model of $\mathbf{G}$. Its associated Shalika functional is given by

$$\mathcal{S}_{\tilde{S},p}(\pi(g)v_0) := \mathcal{S}_p(\pi(w_{op}^{-1}g)v_0) = \mathcal{S}_p(\pi(w_{op}^{-1} g w_{op})v_0), \tag{2.-15}$$

where the last equality follows because $w_{op} \in \mathbf{G}(\mathbb{Z}_p)$. Furthermore, using the map P_ξ defined in 2.3.45, there exists an associated distribution to $\mathcal{S}_{\tilde{S},p}$ which we will denote by

$$\Delta_{\tilde{S},\xi} \in \mathcal{D}(G(\mathbb{Q}_p))_{\xi^{-1}}.$$

The main computation of this subsection may be rewritten as follows:

$$\mathcal{S}_p\left(\pi\begin{pmatrix} p^\lambda & & & \\ & p^\lambda & & \\ & & 1 & \\ & & & 1 \end{pmatrix} v_0\right) = \mathcal{S}_{\tilde{S},p}\left(\pi\begin{pmatrix} p^{-\lambda} & & & \\ & p^{-\lambda} & & \\ & & 1 & \\ & & & 1 \end{pmatrix} v_0\right),$$

where $\lambda \geq 0$. From now on let us denote $g_\lambda := \begin{pmatrix} p^\lambda & & & \\ & p^\lambda & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$. Using the work of [CS80] and [Hir99] as we did in section 2.4.7, we obtain

$$\mathcal{S}_{\tilde{S},p}\left(\pi(g_{-\lambda})v_0\right) = Q^{-1} \sum_{\omega \in W^{\mathbf{H}}} \frac{c_{\omega_l}({}^{\omega}\xi)}{c_{\omega^{-1}}({}^{\omega}\xi)} T^*_{\omega^{-1}} \Delta_{\tilde{S},{}^{\omega}\xi}(R_{g_{-\lambda}} \mathrm{char}_I), \tag{2.-15}$$

where I is the Iwahori subgroup of $\mathbf{G}(\mathbb{Q}_p)$, $R_{g_{-\lambda}}$ is the right translation by $\mathbf{G}(\mathbb{Q}_p)$ on $\mathcal{C}_c^{\infty}(\mathbf{G}(\mathbb{Q}_p))$ and $c_\omega(\xi)$ and $c_\alpha(\xi)$ are the constants associated to the group $\mathbf{G}$ defined in propositions 2.3.62 and 2.3.60 respectively. From now on, we will use the following notation :

$$a_\omega(g_{-\lambda}) := \frac{c_{\omega_l}({}^{\omega}\xi)}{c_{\omega^{-1}}({}^{\omega}\xi)} T^*_{\omega^{-1}} \Delta_{\tilde{S},{}^{\omega}\xi}(R_{g_{-\lambda}} \mathrm{char}_I).$$

To conclude this subsection, we will simplify the constants $\frac{c_{\omega_l}({}^{\omega}\xi)}{c_{\omega^{-1}}({}^{\omega}\xi)}$. In fact, by direct computation

$$\frac{c_{\omega_l}({}^{\omega}\xi)}{c_{\omega^{-1}}({}^{\omega}\xi)} = \frac{\prod_{\substack{\alpha>0 \\ \omega_l\alpha<0}} c_\alpha({}^{\omega}\xi)}{\prod_{\substack{\alpha>0 \\ \omega^{-1}\alpha<0}} c_\alpha({}^{\omega}\xi)} = \prod_{\substack{\alpha>0 \\ \omega^{-1}\alpha>0}} c_\alpha({}^{\omega}\xi).$$

Furthermore, since

$$c_\alpha({}^{\omega}\xi) = \frac{1 - q_\alpha^{-1}\, {}^{\omega}\xi(a_\alpha)}{1 -{}^{\omega}\, \xi(a_\alpha)} = \frac{1 - q_\alpha^{-1}\xi(a_{\omega^{-1}\alpha})}{1 - \xi(a_{\omega^{-1}\alpha})} = c_{\omega^{-1}\alpha}(\xi),$$

we obtain

$$\prod_{\substack{\alpha>0 \\ \omega^{-1}\alpha>0}} c_{\omega^{-1}\alpha}(\xi) = \prod_{\substack{\alpha>0 \\ \omega\alpha>0}} c_\alpha(\xi).$$

The previous discussion allows us to rewrite the formula (2.7.4) by

$$\mathcal{S}_{\tilde{S},p}\left(\pi(g_{-\lambda})v_0\right) = Q^{-1} \sum_{\omega \in W^{\mathbf{H}}} \prod_{\substack{\alpha \in \Phi^+_{\mathbf{H}} \\ \omega\alpha>0}} c_\alpha(\xi) T^*_{\omega^{-1}} \Delta_{\tilde{S},{}^{\omega}\xi}(R_{g_{-\lambda}} \mathrm{char}_I). \tag{2.-15}$$

First reduction: Dependence on λ

The main goal of this reduction is the proof of the formula (2.7.4).

Lemma 2.7.20. *The Iwahori subgroup I satisfies the following property*

$$I g_\lambda^{-1} \subset B_{\mathbf{G}}(\mathbb{Q}_p)\tilde{S}(\mathbb{Q}_p).$$

Proof. The Iwahori factorization provides $I = B_{\mathbf{G}}(\mathbb{Z}_p)\overline{U}_1$ where $\overline{U}_1 := \left(\prod_{\alpha>0} U^\alpha_{\mathbf{G}}((p))\right)^t$. First of all, it is straightforward that $Ig_\lambda^{-1} = B_{\mathbf{G}}(\mathbb{Z}_p)\overline{U}_1 g_\lambda^{-1}$. Since $B_{\mathbf{G}}(\mathbb{Z}_p)$ is trivially a subgroup of $B_{\mathbf{G}}(\mathbb{Q}_p)$, we reduce the proof of the statement to show the inclusion $\overline{U}_1 g_\lambda^{-1} \subset \tilde{S}(\mathbb{Q}_p)$. Since $g_{-\lambda} \subset B_{\mathbf{G}}(\mathbb{Q}_p)$, this is equivalent to show that $g_\lambda \overline{U}_1 g_\lambda^{-1} \subset B_{\mathbf{G}}(\mathbb{Q}_p)\tilde{S}(\mathbb{Q}_p)$. Moreover, by direct computation we get $g_\lambda \overline{U}_1 g_\lambda^{-1} \subset \overline{U}_1$. Therefore we conclude this proof showing that $\overline{U}_1 \subset B_{\mathbf{G}}(\mathbb{Q}_p)\tilde{S}(\mathbb{Q}_p)$.

Let us write an element of $\overline{U}_1$ as $I + u_1$, where u_1 has all their entries in (p). Proposition 2.7.8 implies that the algebraic groups $B_{\mathbf{G}}$ and $\tilde{S}$ are complementary and hence their Lie alebras. Therefore we may find h_1, an element of the Lie algebra of $\tilde{S}$ with coefficients in $\mathbb{Z}_p$, so that

$$b_1 = u_1 + h_1, \tag{2.-15}$$

with b_1 an upper triangular matrix. Then

$$(I + u_1)(I + h_1) = I + u_1 + h_1 + u_1 h_1,$$

where $u_1 + h_1$ is upper triangular by (2.7.4). We repeat this process recursively, obtaining an element $h = I + h_1 + h_2 + ... \in \tilde{S}(\mathbb{Z}_p)$, (where the convergence of this sum is provided by the fact that h_i has coefficients in p^j, with $j \geq 0$), so that

$$(I + u_1)h \in B_{\mathbf{G}}(\mathbb{Z}_p),$$

obtaining the result. □

Proposition 2.7.21. *For any $\omega \in W^{\mathbf{H}}$ and $\phi \in \mathrm{Ind}_{B_{\mathbf{G}}}^{\mathbf{G}} {}^{\omega}\xi$ with $\mathrm{supp}(\phi) \subset B_{\mathbf{G}}(\mathbb{Q}_p)\tilde{S}(\mathbb{Q}_p)$ we have*

$$\mathcal{S}_{\tilde{S},p}(\phi) = \int_{B_{\mathbf{G}}(\mathbb{Q}_p)\cap\tilde{S}(\mathbb{Q}_p)\backslash\tilde{S}(\mathbb{Q}_p)} \phi(h)\chi_{\tilde{S}}(h)dh.$$

Proof. Since we have assumed the representation $\mathrm{Ind}_{B_{\mathbf{G}}}^{\mathbf{G}}\xi$ with trivial central character, the proof follows parallel to the proof of 2.4.52 given in [Sak06, cor. 5.5, p. 13]. □

Let $f \in \mathcal{C}_c^{\infty}(\mathbf{G}(\mathbb{Q}_p))$ be a function so that $\phi = P_{\xi}(f) \in \mathrm{Ind}_B^{\mathbf{G}}\xi$ has support contained in $B_{\mathbf{G}}(\mathbb{Q}_p)\tilde{S}(\mathbb{Q}_p)$. Combining the definition 2.3.45 with the formula 2.7.21, we get

$$\Delta_{\tilde{S},\xi}(f) = \int_{B(\mathbb{Q}_p)\tilde{S}(\mathbb{Q}_p)} \xi^{-1}\delta_B^{1/2}(b(x))f(x)\chi_{\tilde{S}}^{-1}(\tilde{s}(x))dx, \tag{2.-15}$$

where dx is the Haar measure of $\mathbf{G}(\mathbb{Q}_p)$ and the constants $b(x)$, $\tilde{s}(x)$ are the $B_{\mathbf{G}}(\mathbb{Q}_p)$ and $\tilde{S}(\mathbb{Q}_p)$ parts of x respectively. Since the group $B_{\mathbf{G}}(\mathbb{Q}_p)\tilde{S}(\mathbb{Q}_p)$ is dense in $\mathbf{G}(\mathbb{Q}_p)$, almost every element of $\mathbf{G}(\mathbb{Q}_p)$ may be expressed in the above form and hence the equality makes sense. From now on, we normalize the measure dx in such a way $\mathrm{vol}(I) = 1$. We fix our Shalika functional as the one defined by the distribution (2.7.4), i.e. fixing the normalization

$$\mathcal{S}_{\tilde{S},p}(P_{\xi}(\mathrm{char}_I)) = 1. \tag{2.-15}$$

Proposition 2.7.22. *The Shalika functional satisfies*

$$\Delta_{\tilde{S},\xi}(R_{g_{-\lambda}}\mathrm{char}_I) = \xi^{-1}\delta^{1/2}(g_{\lambda}) = a_1(g_{-\lambda}).$$

Hence, for any $\omega \in W^{\mathbf{H}}$

$$\frac{a_{\omega}(g_{-\lambda})}{a_{\omega}(g_0)} = {}^{\omega}\xi^{-1}\delta^{1/2}(g_{\lambda}).$$

Proof. The first statement follows parallel to proposition 2.4.56.

Since the operator $T_{\omega^{-1}}$ is $\mathbf{G}(\mathbb{Q}_p)-$equivariant, the functional $T^*_{\omega^{-1}}(\Delta_{\tilde{S},\xi})$ is a Shalika functional on $\mathrm{Ind}_B^{\mathbf{G}} {}^{\omega}\xi$. According to the assumption at the beginning of the subsection and theorem 2.7.12, the Shalika functional is unique (up to constant). Therefore, there exists a constant $c \in \mathbb{C}$ satisfying

$$T^*_{\omega^{-1}}(\Delta_{\tilde{S},\xi}) = c\Delta_{\tilde{S},{}^{\omega}\xi}.$$

Using the first statement of this proposition, we get

$$\Delta_{\tilde{S},{}^{\omega}\xi}(R_{g_{-\lambda}}\phi_B) = {}^{\omega}\xi^{-1}\delta^{1/2}(g_{-\lambda}). \tag{2.-15}$$

Therefore

$$\frac{a_{\omega}(g_{-\lambda})}{a_{\omega}(g_0)} = \frac{c\,{}^{\omega}\xi^{-1}\delta^{1/2}(g_{-\lambda})}{c} = {}^{\omega}\xi^{-1}\delta^{1/2}(g_{-\lambda}).$$

□

The above proposition allows us to simplify the formula (2.7.4), obtaining the *first reduction*:

$$\mathcal{S}_{\tilde{S},p}(\pi(g_{-\lambda})v_0) = Q^{-1} \prod_{\omega\in W^{\mathbf{H}}} \sum_{\substack{\alpha\in\Phi^+_{\mathbf{H}} \\ \omega\alpha>0}} c_{\alpha}(\xi)^{\omega}\xi^{-1}\delta_B^{1/2}(g_{\lambda})T^*_{\omega^{-1}}\Delta_{\tilde{S},{}^{\omega}\xi}(\mathrm{char}_I). \tag{2.-15}$$

Second reduction: Convergence of the Shalika functional on all of its domain

The main goal of this subsection is the proof of the convergence of the formula (2.7.21) for all the elements of π, where π is an unramified representation satisfying certain local constraints. Moreover, this reduction concludes with the meromorphic continuation of thios formula to all kind of unramified representations.

We first consider the following auxiliary calculation. Recall that we have denoted by T_δ the maximal non-split torus of $\mathrm{GL}_2(\mathbb{Q}_p)$ which is equal to the stabilizer of the open orbit w_{op} (see proposition 2.7.8)

Lemma 2.7.23. *We have an isomorphism*

$$T_\delta \setminus \mathrm{GL}_2(\mathbb{Q}_p) \simeq T_1(\mathbb{Q}_p)\overline{N}_{\mathrm{GL}_2}(\mathbb{Q}_p),$$

where $T_1(\mathbb{Q}_p)\overline{N}_{\mathrm{GL}_2}(\mathbb{Q}_p) = \left\{ t(y)\bar{n}(x) = \left(\begin{smallmatrix} y & \\ & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & \\ x & 1\end{smallmatrix}\right),\ y \in \mathbb{Q}_p^\times,\ x \in \mathbb{Q}_p \right\}$.

Proof. Recall that the group $\mathrm{GL}_2(\mathbb{Q}_p)$ acts on $\mathbb{P}^1_{E_p} \simeq \mathrm{GL}_2(E_p)/B_{\mathrm{GL}_2}(E_p)$ with two orbits, generated by the elements $(1:0)$ and $(1:\delta)$. Every matrix $g = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in T_\delta \setminus \mathrm{GL}_2(\mathbb{Q}_p)$ acts on $(1:\delta)$ by

$$\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \cdot (1:\delta) = (a + b\delta : c + d\delta).$$

Denote by O_δ the corresponding orbit. It consists of elements $(m:n)$ such that $mn \neq 0$ and $mn^{-1} \notin \mathbb{Q}_p$. Since $T_\delta \setminus \mathrm{GL}_2(\mathbb{Q}_p)$ acts transitively on O_δ, we have $T_\delta \setminus \mathrm{GL}_2(\mathbb{Q}_p) \simeq O_\delta$ as topological spaces. On the other hand, the subgroup $T_1(\mathbb{Q}_p)\overline{N}_{\mathrm{GL}_2}(\mathbb{Q}_p) = \{\left(\begin{smallmatrix} y & \\ & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & \\ x & 1\end{smallmatrix}\right)\} \subset \mathrm{GL}_2(\mathbb{Q}_p)$ acts on O_δ with one orbit, with representative $(1:\delta)$. Concretely, if $\left(\begin{smallmatrix} y & \\ x & 1\end{smallmatrix}\right) \in T_1(\mathbb{Q}_p)\overline{N}_{\mathrm{GL}_2}(\mathbb{Q}_p)T_1(\mathbb{Q}_p)$, we have

$$\left(\begin{smallmatrix} y & \\ x & 1\end{smallmatrix}\right)(1:\delta) = (y : x+\delta),$$

which implies that the stabilizer of $(1:\delta)$ is just the identity matrix; hence $T_1(\mathbb{Q}_p)\overline{N}_{\mathrm{GL}_2}(\mathbb{Q}_p) \simeq O_\delta$ as topological spaces. $\square$

We now can study the convergence of the period integral of Lemma 2.7.21 for all $\phi \in I_{\mathbf{G}}(\xi)$. Since ξ is an unramified character, we can write $\xi_1 = |\cdot|_p^{z_1} \circ \mathrm{N}_{E_p/\mathbb{Q}_p}$, $\xi_2 = |\cdot|_p^{z_2} \circ \mathrm{N}_{E_p/\mathbb{Q}_p}$, and $\xi_0 = |\cdot|_p^{z_0}$, where z_1, z_2, z_0 are complex numbers such that $|\cdot|_p^{\sum 2z_i} = 1$. In particular, $\mathrm{Re}(z_0) = -\mathrm{Re}(z_1) - \mathrm{Re}(z_2)$.

Proposition 2.7.24. *Suppose that* $\mathrm{Re}(z_1) > \mathrm{Re}(z_2) + 1 > 1$. *The period integral of Lemma 2.7.21 converges absolutely for every* $\phi \in I_{\mathbf{G}}(\xi)$ *and thus represents a Shalika functional.*

Proof. As $I_{\mathbf{G}}(\xi)$ is unramified, it suffices to prove the proposition when ϕ is equal to the spherical vector ϕ_0, which we therefore assume from now on. In order to simplify the calculation, after conjugating by w_{op} (using (2.7.4)), the convergence of the integral for $\tilde{S}$ follows from the one over the Shalika group S given by

$$\begin{aligned} \mathcal{S}_{\tilde{S},p}(\phi_0) &:= \int_{T_\delta \setminus S(\mathbb{Q}_p)} \phi_0(w_{\mathrm{op}}^{-1}s)\chi_{\delta,p}^{-1}(s)ds \\ &= \int_{\mathbb{Q}_p^\times}\int_{\mathbb{Q}_p}\int_{N_{\mathbf{G}}(\mathbb{Q}_p)} \phi_0(\omega^{-1}w_\delta^{-1}t(y)\bar{n}(x)n)\chi_{\delta,p}^{-1}(n)|y|_p^{-1}dndxd^\times y, \end{aligned}$$

where, for the second equality, we have used the isomorphism of Lemma 2.7.23, and denoted $\omega := s_2s_1s_2$; moreover, we have used the fact that the right Haar measure ds corresponds to the right Haar measure on $T_1(\mathbb{Q}_p)\overline{N}_{\mathrm{GL}_2}(\mathbb{Q}_p)$, which is $|y|_p^{-1}dxd^\times y$ (up to a volume factor coming from the normalization of ds chosen in 2.7.4).

Notice that $w_\delta^{-1}t(y)\bar{n}(x) = \iota\left(\begin{smallmatrix} y & \\ x-y\delta & 1\end{smallmatrix}\right) = t(y)\iota\left(\begin{smallmatrix} 1 & \\ x-y\delta & 1\end{smallmatrix}\right)$ where $\iota : \mathrm{GL}_2(E_p) \hookrightarrow P_{\mathbf{G}}(\mathbb{Q}_p)$, so that the integral becomes equal to

$$\begin{aligned}\mathcal{S}_{\tilde{S},p}(\phi_0) &= \int_{\mathbb{Q}_p^\times}\int_{\mathbb{Q}_p}\int_{N_{\mathbf{G}}(\mathbb{Q}_p)} \phi_0(\omega^{-1}t(y)\iota\left(\begin{smallmatrix}1 & \\ x-y\delta & 1\end{smallmatrix}\right)n)\chi_{\delta,p}^{-1}(n)|y|_p^{-1}dndxd^\times y \\ &= \int_{\mathbb{Q}_p^\times}\int_{\mathbb{Q}_p}\int_{N_{\mathbf{G}}(\mathbb{Q}_p)} \phi_0(t(y)\omega^{-1}\iota\left(\begin{smallmatrix}1 & \\ x-y\delta & 1\end{smallmatrix}\right)n)\chi_{\delta,p}^{-1}(n)|y|_p^{-1}dndxd^\times y \\ &= \int_{\mathbb{Q}_p^\times}\int_{\mathbb{Q}_p}\int_{N_{\mathbf{G}}(\mathbb{Q}_p)} \phi_0(\omega^{-1}\iota\left(\begin{smallmatrix}1 & \\ x-y\delta & 1\end{smallmatrix}\right)n)\chi_{\delta,p}^{-1}(n)(\xi_1\xi_0)(y)dndxd^\times y,\end{aligned}$$

where in the last equality we have used that $\phi_0(t(y)g) = \xi\delta_{B_{\mathbf{G}}}^{1/2}(t(y))\phi_0(g)$. As ϕ_0 is fixed by the right translation action of $\mathbf{G}(\mathbb{Z}_p)$ and since conjugation by the Weyl element s_1 yields $s_1\iota\left(\begin{smallmatrix}1 & \\ x-y\delta & 1\end{smallmatrix}\right)s_1 = \iota\left(\begin{smallmatrix}1 & x-y\delta \\ & 1\end{smallmatrix}\right)$, the integral simplifies further as

$$\int_{\mathbb{Q}_p^\times}\int_{\mathbb{Q}_p}\int_{N_{\mathbf{G}}(\mathbb{Q}_p)} \phi_0(\omega_0^{-1}\iota\left(\begin{smallmatrix}1 & x-y\delta \\ & 1\end{smallmatrix}\right)n)\chi_{\delta,p}^{-1}(n)(\xi_1\xi_0)(y)dndxd^\times y,$$

where we have denoted $\omega_0 := s_2s_1s_2s_1$, which is a representative of the longest Weyl element of $W_{\mathbf{G}}$. Since $\xi_1 = |\cdot|_p^{z_1}\circ \mathrm{N}_{E_p/\mathbb{Q}_p}$, $\xi_2 = |\cdot|_p^{z_2}\circ \mathrm{N}_{E_p/\mathbb{Q}_p}$, and $\xi_0 = |\cdot|_p^{z_0}$, we have that $(\xi_1\xi_0)(y) = |y|_p^{2z_1+z_0}$ and the integral above is dominated absolutely by

$$\int_{\mathbb{Q}_p^\times}\int_{\mathbb{Q}_p}\int_{N_{\mathbf{G}}(\mathbb{Q}_p)} \left|\phi_0(\omega_0^{-1}\iota\left(\begin{smallmatrix}1 & x-y\delta \\ & 1\end{smallmatrix}\right)n)\right| |y|_p^{\mathrm{Re}(z_1)-\mathrm{Re}(z_2)}dndxd^\times y,$$

where we have used $|\chi_{\delta,p}^{-1}(n)| = 1$. Furthermore, since $\{0\}\in\mathbb{Q}_p$ is a measure 0 set of $\mathbb{Q}_p$ and $\phi\left(\omega_0\iota\left(\begin{smallmatrix}1 & x+y\delta \\ & 1\end{smallmatrix}\right)n\right)$ is a continuous function in all $y\in\mathbb{Q}_p$, if $\mathrm{Re}(z_1) > \mathrm{Re}(z_2)+1$ the latter extends to

$$\int_{\mathbb{Q}_p}\int_{\mathbb{Q}_p}\int_{N_{\mathbf{G}}(\mathbb{Q}_p)} \left|\phi_0(\omega_0^{-1}\iota\left(\begin{smallmatrix}1 & x-y\delta \\ & 1\end{smallmatrix}\right)n)\right| |y|_p^{\mathrm{Re}(z_1)-\mathrm{Re}(z_2)-1}dndxdy.$$

We now show that the latter integral converges. We write it as the sum of two integrals by splitting the domain of x, y into $\{x,y : x-\delta y\in\mathcal{O}_p\}\cup\{x,y : x-\delta y\notin\mathcal{O}_p\}$ and analyze each separately. The first integral equals to

$$\begin{aligned}\iint_{\mathcal{O}_p}\cdots dxdy &= \sum_{n\geq 0} p^{-n(\mathrm{Re}(z_1)-\mathrm{Re}(z_2)-1)}\int_{\Omega_n}\int_{\mathbb{Z}_p}\int_{N_{\mathbf{G}}(\mathbb{Q}_p)}\left|\phi_0(\omega_0^{-1}\iota\left(\begin{smallmatrix}1 & x-y\delta \\ & 1\end{smallmatrix}\right)n)\right| dndxdy \\ &\leq \sum_{n\geq 0} p^{-n(\mathrm{Re}(z_1)-\mathrm{Re}(z_2)-1)}\cdot\int_{U_{\mathbf{G}}(\mathbb{Q}_p)}\left|\phi_0(\omega_0^{-1}u)\right| du,\end{aligned}$$

where we denoted $\Omega_n = \{y : |y|_p = p^{-n}\}$. The latter converges because $\mathrm{Re}(z_1) > \mathrm{Re}(z_2)+1$ and because the intertwining operator

$$T_{\omega_0}(\phi_0)(1) = \int_{U_{\mathbf{G}}(\mathbb{Q}_p)}\phi_0(\omega_0^{-1}u)du$$

converges absolutely as long as $\mathrm{Re}(z_1) > \mathrm{Re}(z_2) > 0$ (see [Sha81, (2.3)]). The second integral equals to

$$\begin{aligned}&= \iint_{\{x-\delta y\notin\mathcal{O}_p\}}\int_{N_{\mathbf{G}}(\mathbb{Q}_p)}\left|\phi_0(\omega_0^{-1}\iota\left(\begin{smallmatrix}1 & x-y\delta \\ & 1\end{smallmatrix}\right)n)\right| |y|_p^{\mathrm{Re}(z_1)-\mathrm{Re}(z_2)-1}dndxdy \\ &= \iint_{\{x-\delta y\notin\mathcal{O}_p\}}\int_{N_{\mathbf{G}}(\mathbb{Q}_p)}\left|\phi_0(\omega_0^{-1}\iota\begin{pmatrix}x-y\delta & \\ 1 & (x-y\delta)^{-1}\end{pmatrix}\iota\begin{pmatrix}(x-y\delta)^{-1} & 1 \\ -1 & \end{pmatrix}n)\right| |y|_p^{\mathrm{Re}(z_1)-\mathrm{Re}(z_2)-1}dndxdy \\ &= \iint_{\{x-\delta y\notin\mathcal{O}_p\}}\int_{N_{\mathbf{G}}(\mathbb{Q}_p)}\left|\phi_0(\iota\begin{pmatrix}(x-y\delta)^{-1} & -1 \\ & x-y\delta\end{pmatrix}\omega_0^{-1}n)\right| |y|_p^{\mathrm{Re}(z_1)-\mathrm{Re}(z_2)-1}dndxdy \\ &= \iint_{\{x-\delta y\notin\mathcal{O}_p\}}\int_{N_{\mathbf{G}}(\mathbb{Q}_p)}\left|\phi_0(\omega_0^{-1}n)\right| |y^{-1}x^2-y\delta^2|_p^{1-\mathrm{Re}(z_1)+\mathrm{Re}(z_2)}dndxdy,\end{aligned}$$

where for the third equality we have made the change of variable $n \mapsto \iota\left(\begin{smallmatrix} (x-y\delta)^{-1} & 1 \\ & -1 \end{smallmatrix}\right) n \iota\left(\begin{smallmatrix} (x-y\delta)^{-1} & 1 \\ & -1 \end{smallmatrix}\right)^{-1}$ and we have used that $\left(\begin{smallmatrix} (x-y\delta)^{-1} & 1 \\ & -1 \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathcal{O}_p)$; for the fourth, $\phi_0(\iota\left(\begin{smallmatrix} (x-y\delta)^{-1} & -1 \\ & x-y\delta \end{smallmatrix}\right) g) = \xi_1^{-1}\xi_2(x-y\delta)|x^2-\delta^2y^2|_p\phi_0(g)$. Notice that

$$|y^{-1}x^2 - y\delta^2|_p = \begin{cases} |y|_p & \text{if } v_p(y) \leq v_p(x) \\ |x^2y^{-1}|_p & \text{if } v_p(y) > v_p(x). \end{cases}$$

In particular, $|y^{-1}x^2 - y\delta^2|_p < 1$ because $x - \delta y \notin \mathcal{O}_p$. Therefore

$$(2.7.4) \leq \sum_{n \geq 1} p^{-n(\mathrm{Re}(z_1)-\mathrm{Re}(z_2)-1)} \cdot \int_{N_{\mathbf{G}}(\mathbb{Q}_p)} \left|\phi_0(\omega^{-1}n)\right| dn,$$

and the latter converges as above because of our hypotheses. □

This section concludes with a meromorphic continuation of the functional above. The principal tool for solving this problem is Bernstein's theorem. We will closely follow [GPSR87, (12.2), p. 126] to state this theorem.

For the following discussion we fix Y a vector space defined over a field K, $Y^* = \mathrm{Hom}_K(Y, K)$ its dual vector space and D an algebraic variety defined over K with ring of regular functions $K[D]$ over K.

Definition 2.7.25. A *system of equations* Ξ in Y^* is a collection of pairs $\{(y_\nu, \lambda_\nu)\}_{\nu \in R}$, with $y_\nu \in Y$, $\lambda_\nu \in K$ and R some set. A *solution* of Ξ is an element $l \in Y^*$ satisfying $l(y_\nu) = \lambda_\nu$ for all $\nu \in R$.

Definition 2.7.26. A function $f : D \to Y$ is *regular* on D if it belongs to $Y \otimes_K K[D]$. In other words, if it can be expressed as a finite sum

$$\sum_i y_i \varphi_i,$$

where $y_i \in Y$ and $\varphi_i \in K[D]$.

Definition 2.7.27. Let $\{\Xi_d\}$ be a family of systems of equations in Y^* parametrized by $d \in D$. $\{\Xi_d\}$ is *polynomial* in d if all the systems Ξ_d are indexed by the same set $R = \{\nu\}$ and, (before specializing at any $d \in D$) given any ν, $x_{\nu,d} \in Y \otimes_K K[D]$ is a regular function and $\lambda_{v,d} \in K[D]$ (in other words, when we evaluate $x_{\nu,d}$ and $\lambda_{\nu,d}$ at certain $d_0 \in D$, we will obtain elements of Y and K respectively).

Let $L = \mathrm{Frac}\,(K[D])$. We will denote $Y_L := Y \otimes_K L$ and $Y_L^* = \mathrm{Hom}_L(Y_L, L)$. We may regard a system of equations over Y as a system of equations over Y_L.

Theorem 2.7.28. *Fix a polynomial family $\{\Xi_d\}$ with $K = \mathbb{C}$, D an irreducible variety over $\mathbb{C}$ and Y of countable dimension over $\mathbb{C}$. Suppose further that there is a subset $\Omega \subset D$, non empty and open (with the Hausdorf topology), so that for each $d \in \Omega$, the system Ξ_d has a unique (rational) solution $l_d \in Y_L^*$ on some subset $D' \subset D$, whose complement is a countable union of hypersurfaces $l(d)$, then l_d is the unique (rational) solution of Ξ_d.*

Proof. It is in [BL19, Appendix A, p. 47]. □

This result is specially useful to obtain meromorphic continuations for expressions satisfying certain symmetry conditions. The following corollary is a concrete example of the application of this result.

Corollary 2.7.29. *Let D be the set of unramified characters of $\mathbf{PG}(\mathbb{Q}_p)$ of the form (2.7.3), for which (2.7.3) holds. There exists a unique non-zero Shalika functional on $I_{\mathbf{G}}(\xi)$, which satisfies (2.7.4), for almost all $\xi \in D$. Moreover, if $\{\phi_\xi\}_{\xi \in D}$ is a rational family of functions $\phi_\xi \in I_{\mathbf{G}}(\xi)$, then the period integral $\mathcal{S}_{\tilde{S},p}(\phi_\xi)$ of (2.7.21) is a rational function on $\xi \in D$.*

Proof. We apply theorem 2.7.28 to the system of equations given by

- D is the complex variety defined by the unramified characters of $T_{\mathbf{G},s}(\mathbb{Q}_p)$.

- The rational functions of $I_{\mathbf{G}}(\xi)$ are $X \otimes_{\mathbb{C}} \mathbb{C}(D)$, where X is the space of locally constant functions on $\mathbf{G}(\mathbb{Z}_p)$ that are left-invariant under $B_{\mathbf{G}}(\mathbb{Z}_p)$.
- $\mathcal{S}_{\tilde{S},p}$ is a Shalika functional on $I_{\mathbf{G}}(\xi)$, for all $\xi \in D$,
- $\mathcal{S}_{\tilde{S},p}$ satisfies (2.7.4).

Indeed, by Proposition 2.7.24, there exists a unique normalized Shalika functional $\mathcal{S}_{\tilde{S},p}$, given by the period integral of Lemma 2.7.21, for $\xi \in D$ which satisfies that $\mathrm{Re}(z_1) > \mathrm{Re}(z_2)+1$ and $\mathrm{Re}(z_2) > 0$. The latter conditions define an open subset of D - here we are identifying D with the variety $\mathbb{C}^2$ by sending $\xi \mapsto (z_1, z_2)$. We can thus apply Bernstein's theorem to deduce that there exists a unique non-zero Shalika functional $\mathcal{S}_{\tilde{S},p}$, satisfying (2.7.4) for almost all $\xi \in D$. This has the further property that, if $\{\phi_\xi\}_{\xi\in D}$ is a rational family of functions $\phi_\xi \in I_{\mathbf{G}}(\xi)$, then $\mathcal{S}_{\tilde{S},p}(\phi_\xi)$ is a rational function of $\xi \in D$. This concludes the proof. □

Third reduction: expliciting the right hand side of (2.7.4)

The main results of this reduction are the propositions 2.7.32 and 2.7.34. They determine the constants $T^*_{\omega^{-1}}\Delta_{\tilde{S},{}^{\omega}\xi}(\mathrm{char}_I)$ for the Weyl elements associated to the roots $2\alpha_2-\alpha_0$ and $\alpha_1-\alpha_2$, i.e. the generators of the Weyl group.

For $\alpha \in \Phi^+_{\mathbf{G}}$, denote by $U^{\alpha}_{\mathbf{G}} \subset U_{\mathbf{G}}$ the one parameter unipotent subgroup associated to it. Explicitly, if α equals $2\alpha_2-\alpha_0$, resp. $\alpha_1-\alpha_2$, we let

$$x_{2\alpha_2-\alpha_0} : \mathbb{G}_{\mathrm{a}} \to U^{2\alpha_2-\alpha_0}_{\mathbf{G}},\ y \mapsto \begin{pmatrix} 1 &&& \\ & 1 & y & \\ && 1 & \\ &&& 1 \end{pmatrix},$$

$$x_{\alpha_1-\alpha_2} : \mathrm{Res}_{\mathcal{O}_p/\mathbb{Z}_p}\mathbb{G}_{\mathrm{a}} \to U^{\alpha_1-\alpha_2}_{\mathbf{G}},\ y \mapsto \begin{pmatrix} 1 & y && \\ & 1 && \\ && 1 & -\bar{y} \\ &&& 1 \end{pmatrix}.$$

We have the following decomposition.

Lemma 2.7.30.

1. *The element $s_2x_{2\alpha_2-\alpha_0}(y) \in s_2N^{2\alpha_2-\alpha_0}(\mathbb{Z}_p)$ factors as*

$$\begin{pmatrix} 1 &&& \\ && 1 & \\ & -1 & -y & \\ &&& 1 \end{pmatrix} = \begin{pmatrix} 1 &&& \\ & -y^{-1} & 1 & \\ && -y & \\ &&& 1 \end{pmatrix}\begin{pmatrix} 1 &&& \\ & 1 && \\ & y^{-1} & 1 & \\ &&& 1 \end{pmatrix} \in B_{\mathbf{G}}(\mathbb{Q}_p)\tilde{S}(\mathbb{Q}_p),$$

if $y \neq 0$.

2. *The element $s_1x_{\alpha_1-\alpha_2}(y) \in s_1N^{\alpha_1-\alpha_2}(\mathbb{Z}_p)$ factors as*

$$\begin{pmatrix} & 1 && \\ 1 & y && \\ &&& 1 \\ && 1 & -\overline{y} \end{pmatrix} = \begin{pmatrix} -(y\overline{y}+b_y)^{-1} & \overline{y}(y\overline{y}+b_y)^{-1} && \\ & 1 && \\ && (y\overline{y}+b_y)^{-1} & y(y\overline{y}+b_y)^{-1} \\ &&& -1 \end{pmatrix}\begin{pmatrix} \overline{y} & -b_y && \\ 1 & y && \\ && y & b_y \\ && -1 & \overline{y} \end{pmatrix} \in B_{\mathbf{G}}(\mathbb{Q}_p)\tilde{S}(\mathbb{Q}_p),$$

where $y = a_y + \delta b_y$, if $y\overline{y} + b_y \neq 0$.

Proof. It follows from a direct matrix calculation. □

For the following discussion we denote by dy the additive Haar measures of E_p. Since the homomorphism $\mathcal{O}_{E_p} \simeq \mathbb{Z}_p^2$ preserves the topologies of both sides, we have

$$dy = C dadb,$$

where da and db are the additive Haar measures of $\mathbb{Q}_p$ giving $\mathbb{Z}_p$ measure 1. Since

$$1 = \int_{\mathcal{O}_{E_p}} dy = C\int_{\mathbb{Z}_p^2} dadb = C,$$

we have $dy = dadb$. To compute the ratio of $T^*_{\omega^{-1}}\Delta_{\tilde{S},{}^{\omega}\xi}$ by $\Delta_{\tilde{S},\xi}$, it is enough to evaluate it at the element $\mathbf{1}_{\mathrm{Iw}}$. Recall that, thanks to our normalization (2.7.4), $\Delta_{\tilde{S},\xi}(\mathbf{1}_{\mathrm{Iw}}) = 1$.

Finally, recall that, since our principal series is in the image of the theta lift the character ξ is of the form (2.7.3):

$$\xi = (\chi_0 \circ N_{E_p/\mathbb{Q}_p}, \chi_2\chi_0 \circ N_{E_p/\mathbb{Q}_p}, \chi_1\chi_{E_p/\mathbb{Q}_p}),$$

for some regular unramified character χ of $T_{\mathbf{H}}(\mathbb{Q}_p)$. In the following discussion, we will use the following equalities derived from theta correspondence:

$$\xi(a_{2\alpha_2-\alpha_0}) = \chi^{-1}(a_{\alpha_1-\alpha_2}),\ \ \xi(a_{\alpha_1-\alpha_2}) = \chi^{-2}(a_{2\alpha_2-\alpha_0}).$$

Lemma 2.7.31. *Suppose that ξ satifies the hypothesis of Proposition 2.7.24; then*

$$\int_{s_2N^{2\alpha_2-\alpha_0}(\mathbb{Z}_p)} (\xi^{-1}\delta^{1/2}_{B_{\mathbf{G}}})(b(x))\chi^{-1}_{\tilde{S}}(\tilde{s}(x))dx = 1 - p^{-1} - p^{-1}\xi(a_{2\alpha_2-\alpha_0}).$$

Proof. Given $x = s_2x_{2\alpha_2-\alpha_0}(y) \in s_2U^{2\alpha_2-\alpha_0}_{\mathbf{G}}(\mathbb{Z}_p)$, the first point of lemma 2.7.30 shows that, if $y \neq 0$,

$$(\xi^{-1}\delta^{1/2}_{B_{\mathbf{G}}})(b(x))\chi^{-1}_{\tilde{S}}(\tilde{s}(x)) = |y|^{-1}_p\xi_2(y)\chi^{-1}_{\delta,p}(w_{\mathrm{op}}\tilde{s}(x)w^{-1}_{\mathrm{op}}) = |y|^{2z_2-1}_p\psi^{-1}_p(2\delta^2y^{-1}).$$

Since $N^{2\alpha_2-\alpha_0}(\mathbb{Z}_p) \simeq \mathbb{Z}_p$, the integral, away from $\{0\}$, reduces to

$$\int_{s_2N^{2\alpha_2-\alpha_0}(\mathbb{Z}_p)} (\xi^{-1}\delta^{1/2}_{B_{\mathbf{G}}})(b(x))\chi^{-1}_{\tilde{S}}(\tilde{s}(x))dx = \int_{\mathbb{Z}_p} |y|^{2z_2-1}_p\psi^{-1}_p(2\delta^2y^{-1})dy,$$

where we recall that dy is the additive Haar measure of $\mathbb{Z}_p$. The previous integral is equal to

$$\sum_{j=0}^{\infty} p^{-j(2z_2-1)}\int_{(p)^j-(p)^{j+1}} \psi^{-1}_p(-2\delta^2y^{-1})dy. \tag{2.-18}$$

Since the conductor of the character $y \mapsto \psi^{-1}_p(-2\delta^2y^{-1})$ is $\mathbb{Z}_p$, the only two integrals of (2.7.4) that contribute to the sum are the ones with $j = 0$ and $j = 1$. They are equal to $1 - p^{-1}$ and $-p^{-2}$ respectively. Thus, the integral is equal to $1 - p^{-1} - p^{-(2z_2+1)} = 1 - p^{-1} - p^{-1}\xi(a_{2\alpha_2-\alpha_0})$, where for the latter equality we have conveniently used the matrix $a_{2\alpha_2-\alpha_0}$. □

Proposition 2.7.32. *We obtain the following equality:*

$$T^*_{\omega_2^{-1}}\Delta_{\tilde{S},{}^{\omega_2}\xi}(\mathrm{char}_I) = -\chi^{-1}(a_{\alpha_1-\alpha_2})c_{2\alpha_2-\alpha_0}(\xi),$$

where we recall that $c_{\alpha_1-\alpha_2}(\xi)$ is the constant defined in 2.3.60.

Proof. Due to the resemblance with [Sak06, prop. 8.1, p. 17], the result is completely analogous to the aforementioned one. Despite this, for the sake of completeness we will show the proof of this result. Using theorem 2.3.64 and corollary 2.7.15 we obtain

$$T_{\omega_2^{-1}}\left(P_{{}^{\omega_2}\xi}\left(\mathrm{char}_{I,{}^{\omega_2}\xi}\right)\right) = (c_{2\alpha_2-\alpha_0}({}^{\omega_2}\xi) - 1)P_{{}^{\omega_2}\xi}\left(\mathrm{char}_{I,{}^{\omega_2}\xi}\right) + p^{-1}P_{{}^{\omega_2}\xi}\left(\mathrm{char}_{Is_2I,{}^{\omega_2}\xi}\right).$$

Using the first statement of proposition 2.7.22 and equation (2.7.4) we obtain

$$T^*_{\omega_2^{-1}}\Delta_{\tilde{S},{}^{\omega_2}\xi}(\mathrm{char}_I) = c_{2\alpha_2-\alpha_0}({}^{\omega_2}\xi) - 1 + p^{-1}\Delta_{\tilde{S},{}^{\omega_2}\xi}(\mathrm{char}_{Is_2I,{}^{\omega_2}\xi}),$$

reducing our computation to $\Delta_{\tilde{S},{}^{\omega_2}\xi}(\mathrm{char}_{Is_2I,{}^{\omega_2}\xi})$.

If we express every element of Is_2I as a product of the form $B(\mathbb{Q}_p)\tilde{S}(\mathbb{Q}_p)$, the formula (2.7.4) provide an explicit expression for $\Delta_{\tilde{S},{}^{\omega_2}\xi}(\mathrm{char}_{Is_2I,{}^{\omega_2}\xi})$. Using the Iwahori factorization of I we find that $Is_2I = B_{\mathbf{G}}(\mathbb{Z}_p)\overline{U_{\mathbf{G}}}(p\mathbb{Z}_p)s_2B_{\mathbf{G}}(\mathbb{Z}_p)\overline{U_{\mathbf{G}}}(p\mathbb{Z}_p)$. Considering the unimodular subgroup attached to the root $2\alpha_2 - \alpha_0$

we get

$$\begin{aligned} &B_{\mathbf{G}}(\mathbb{Z}_p)\overline{U_{\mathbf{G}}}^{2\alpha_2-\alpha_0}(p\mathbb{Z}_p)\overline{U_{\mathbf{G}}}^{\widehat{2\alpha_2-\alpha_0}}(p\mathbb{Z}_p)s_2B_{\mathbf{G}}^{2\alpha_2-\alpha_0}(\mathbb{Z}_p)B_{\mathbf{G}}^{\widehat{2\alpha_2-\alpha_0}}(\mathbb{Z}_p)\overline{U_{\mathbf{G}}}^{2\alpha_2-\alpha_0}(p\mathbb{Z}_p)\overline{U_{\mathbf{G}}}^{\widehat{2\alpha_2-\alpha_0}}(p\mathbb{Z}_p) \\ &\qquad = B_{\mathbf{G}}(\mathbb{Z}_p)\overline{U_{\mathbf{G}}}^{2\alpha_2-\alpha_0}(p\mathbb{Z}_p)s_2U_{\mathbf{G}}^{2\alpha_2-\alpha_0}(\mathbb{Z}_p)\overline{U_{\mathbf{G}}}^{2\alpha_2-\alpha_0}(p\mathbb{Z}_p)\overline{U_{\mathbf{G}}}^{\widehat{2\alpha_2-\alpha_0}}(p\mathbb{Z}_p). \end{aligned}$$

Since $s_2^{-1}\overline{U_{\mathbf{G}}}^{2\alpha_2-\alpha_0}(p\mathbb{Z}_p)s_2 = U_{\mathbf{G}}^{2\alpha_2-\alpha_0}(p\mathbb{Z}_p)$, the group (2.7.4) is equal to

$$B_{\mathbf{G}}(\mathbb{Z}_p)s_2U_{\mathbf{G}}^{2\alpha_2-\alpha_0}(\mathbb{Z}_p)\overline{U_{\mathbf{G}}}^{\widehat{2\alpha_2-\alpha_0}}(p\mathbb{Z}_p).$$

The previous expression discussion allows us to rewrite the coset Is_2I as $B_{\mathbf{G}}(\mathbb{Z}_p)s_2U_{\mathbf{G}}^{2\alpha_2-\alpha_0}(\mathbb{Z}_p)\overline{U_{\mathbf{G}}}^{\widehat{2\alpha_2-\alpha_0}}(p\mathbb{Z}_p)$ in a measure preserving way except for the constant $[Is_2I : I]$ that is equal to p by lemma 2.7.14. Moreover using lemma 2.7.20

$$\begin{aligned} B_{\mathbf{G}}(\mathbb{Z}_p)s_2U_{\mathbf{G}}^{2\alpha_2-\alpha_0}(\mathbb{Z}_p)\overline{U_{\mathbf{G}}}^{\widehat{2\alpha_2-\alpha_0}}(p\mathbb{Z}_p) &\subset B_{\mathbf{G}}(\mathbb{Z}_p)s_2U_{\mathbf{G}}^{2\alpha_2-\alpha_0}(\mathbb{Z}_p)B_{\mathbf{G}}^{\widehat{2\alpha_2-\alpha_0}}(\mathbb{Z}_p)\tilde{S}(\mathbb{Z}_p) \\ &= B_{\mathbf{G}}(\mathbb{Z}_p)s_2U_{\mathbf{G}}^{2\alpha_2-\alpha_0}(\mathbb{Z}_p)\tilde{S}(\mathbb{Z}_p). \end{aligned}$$

By the formula (2.7.4) and the left and right $\mathbf{G}(\mathbb{Z}_p)-$invariance of $\xi^{-1}\delta_B^{1/2}(b(x))\chi_{\tilde{S}}^{-1}(\tilde{s}(x))$ we obtain

$$\Delta_{\tilde{S},\omega_2\xi}(1_{Is_2I}) = \int_{s_2U_{\mathbf{G}}^{2\alpha_2-\alpha_0}(\mathbb{Z}_p)} \xi^{-1}\delta_B^{1/2}(b(x))f(x)\chi_{\tilde{S}}^{-1}(\tilde{s}(x))dx, \tag{2.-19}$$

where the Haar measure dx is normalized such that $\mathrm{vol}(B_{\mathbf{G}}(\mathbb{Z}_p)) = \mathrm{vol}(\tilde{S}(\mathbb{Z}_p)) = 1$. Applying lemma 2.7.31 to equation (2.7.4) we get

$$T^*_{\omega_2^{-1}}\Delta_{\tilde{S},\omega_2\xi}(\mathrm{char}_I) = -\xi(a_{2\alpha_2-\alpha_0})c_{2\alpha_2-\alpha_0}(\xi) = -\chi^{-1}(a_{\alpha_1-\alpha_2})c_{2\alpha_2-\alpha_0}(\xi),$$

where we have used (2.7.3) to identify $\xi(a_{2\alpha_2-\alpha_0}) = \chi^{-1}(a_{\alpha_1-\alpha_2})$. This proves the result for the set of characters which satisfy the hypothesis of Proposition 2.7.24. By Bernstein's theorem, 2.7.28, the section $T_{\omega_2^{-1}}(P_{\omega_2\xi}(\mathrm{char}_I))$ is a rational function on the character variety of $T_{\mathbf{G},s}$. Then we apply Corollary 2.7.29 to deduce that $S_{\tilde{S},p}\left(T_{\omega_2^{-1}}(P_{\omega_2\xi}(\mathrm{char}_I))\right)$ is a rational function on the character variety and hence we extend the expression to every ξ. □

Lemma 2.7.33. *Suppose that ξ satisfies the hypothesis of Proposition 2.7.24. We have*

$$\int_{s_1U_{\mathbf{G}}^{\alpha_1-\alpha_2}(\mathbb{Z}_p)} \xi^{-1}\delta_{B_{\mathbf{G}}}^{1/2}(b(x))\chi_{\tilde{S}}^{-1}(\tilde{s}(x))dx = -\frac{1}{p^2}\left[1+(p-1)\frac{1-p\chi(a_{2\alpha_2-\alpha_0})}{1+\chi(a_{2\alpha_2-\alpha_0})}\right].$$

Proof. Given $x = s_1x_{\alpha_1-\alpha_2}(y) \in s_1U_{\mathbf{G}}^{\alpha_1-\alpha_2}(\mathbb{Z}_p)$, with $y = a_y + b_y\delta$ such that $y\overline{y}+b_y \neq 0$, lemma 2.7.30(2) shows

$$(\xi^{-1}\delta_{B_{\mathbf{G}}}^{1/2})(b(x))\chi_{\tilde{S}}^{-1}(\tilde{s}(x)) = |y\overline{y}+b_y|_p^{-1}(\xi_1\xi_0)(y\overline{y}+b_y).$$

Since ξ is unramified, we can write

$$(\xi_1\xi_0)(y\overline{y}+b_y) = |y\overline{y}+b_y|_p^{2z_1+z_0},$$

for some complex numbers z_1, z_0. Let us assume for now that z_0 and z_1 satisfy $2\mathrm{Re}(z_1)+\mathrm{Re}(z_0)-1>0$; we first prove the formula in this region and then extend it to other values of z_0 and z_1 by meromorphic continuation. Fixing an isomorphism $\mathcal{O}_p \simeq \mathbb{Z}_p^2$ of $\mathbb{Z}_p$-modules, for which the topology on $\mathcal{O}_p$ corresponds to the product topology on $\mathbb{Z}_p^2$, then we can decompose

$$\mathcal{O}_p = (p)\times(p)\bigsqcup\mathbb{Z}_p^\times\times\mathbb{Z}_p^\times\bigsqcup\mathbb{Z}_p^\times\times(p)\bigsqcup(p)\times\mathbb{Z}_p^\times =: \bigsqcup_{i=1}^4 A_i.$$

Using this, we write the integral away from the elements $y = a_y + b_y\delta$ such that $a_y^2 - \delta^2 b_y^2 + b_y \neq 0$, as

$$\int_{s_1 N^{\alpha_1-\alpha_2}(\mathbb{Z}_p)} \xi^{-1}\delta_{B_{\mathbf{G}}}^{1/2}(b(x))\chi_{\tilde{S}}^{-1}(\tilde{s}(x))dx = \sum_i \int_{A_i} \left|a_y^2 - \delta^2 b_y^2 + b_y\right|_p^{2z_1+z_0-1} da_y db_y, \tag{2.-19}$$

where da_y, db_y are additive Haar measures of $\mathbb{Z}_p$. We now solve each of the integrals appearing in the right hand side separately.

We may factor the ideal $(p) = \bigsqcup_{i\geq 1}(p)^i - (p)^{i+1}$. We thus get

$$\int_{(p)\times(p)} \left|a_y^2 - \delta^2 b_y^2 + b_y\right|_p^{2z_1+z_0-1} da_y db_y = \sum_{i,j\geq 1}\int_{(p)^i-(p)^{i+1}}\int_{(p)^j-(p)^{j+1}} \left|a_y^2 - \delta^2 b_y^2 + b_y\right|_p^{2z_1+z_0-1} db_y da_y.$$

By direct computation we get

$$\left|a_y^2 - \delta^2 b_y^2 + b_y\right|_p |_{(p)^i-(p)^{i+1}\times(p)^j-(p)^{j+1}} = \begin{cases} p^{-j} & if\ j<2i \\ p^{-2i} & if\ j>2i \end{cases}$$

Writing $\mathrm{vol}_{i,j} = \mathrm{vol}((p)^i - (p)^{i+1})\cdot \mathrm{vol}((p)^j - (p)^{j+1})$, then the integral is equal to

$$\sum_{i\geq 1}\sum_{j=1}^{j=2i-1} \mathrm{vol}_{i,j}\cdot p^{-j(2z_1+z_0-1)} + \sum_{i\geq 1}\sum_{j=2i+1} \mathrm{vol}_{i,j}\cdot p^{-2i(2z_1+z_0-1)}$$
$$+\sum_{i\geq 1}\int_{(p)^i-(p)^{i+1}}\int_{(p)^{2i}-(p)^{2i+1}} \left|a_y^2 - \delta^2 b_y^2 + b_y\right|_p^{2z_1+z_0-1} db_y da_y.$$

First of all, we compute (2.7.4) explicitly: since $\mathrm{vol}((p)^i - (p)^{i+1}) = p^{-i}(1-p^{-1})$, (2.7.4) is equal to

$$(1-p^{-1})^2\left(\sum_{i\geq 1}\sum_{j=1}^{2i-1} p^{-j(2z_1+z_0)-i} + \sum_{i\geq 1}\sum_{j\geq 2i+1} p^{-2i(2z_1+z_0-\frac{1}{2})-j}\right).$$

We analyze each sum separately. By direct computation,

$$\sum_{j=1}^{2i-1} p^{-j(2z_1+z_0)-i} = p^{-i}\sum_{j=1}^{2i-1} p^{-j(2z_1+z_0)} = -p^{-i}\frac{1-p^{2z_1+z_0-2i(2z_1+z_0)}}{1-p^{2z_1+z_0}} = \frac{p^{2z_1+z_0-i(2(2z_1+z_0)+1)} - p^{-i}}{1-p^{2z_1+z_0}}.$$

Therefore

$$\begin{aligned}\sum_{i\geq 1}\sum_{j=1}^{2i-1} p^{-j(2z_1+z_0)-i} &= \frac{1}{1-p^{2z_1+z_0}}\left(p^{2z_1+z_0}\sum_{i\geq 1}p^{-i(2(2z_1+z_0)+1)} - \sum_{i\geq 1}p^{-i}\right)\\ &= \frac{1}{1-p^{2z_1+z_0}}\left(\frac{p^{2z_1+z_0}}{p^{2(2z_1+z_0)+1}-1} - \frac{1}{p-1}\right)\\ &= \frac{p^{2z_1+z_0+1}+1}{(p-1)(p^{2(2z_1+z_0)+1}-1)},\end{aligned}$$

as long as $\mathrm{Re}(2z_1+z_0) > -1/2$. Regarding the second sum, we get

$$\sum_{j\geq 2i+1} p^{-2i(2z_1+z_0-\frac{1}{2})-j} = p^{-2i(2z_1+z_0-\frac{1}{2})}\sum_{j\geq 2i+1} p^{-j} = p^{-2i(2z_1+z_0-\frac{1}{2})}\frac{p^{-2i}}{p-1} = \frac{p^{-2i(2z_1+z_0+\frac{1}{2})}}{p-1}.$$

Therefore

$$\sum_{i\geq 1}\sum_{j\geq 2i+1} p^{-2i(2z_1+z_0-\frac{1}{2})-j} = \frac{1}{p-1}\sum_{i\geq 1} p^{-2i(2z_1+z_0+\frac{1}{2})} = \frac{1}{(p-1)(p^{2(2z_1+z_0)+1}-1)}.$$

Hence, the sum of (2.7.4) is equal to

$$(p-1)\frac{p^{2z_1+z_0-1}+2p^{-2}}{p^{2(2z_1+z_0)+1}-1}.$$

We now evaluate (2.7.4). Writing any element $\alpha \in (p)^\bullet - (p)^{\bullet+1}$ as $\alpha = p^\bullet\tilde{\alpha}$, with $\tilde{\alpha}\in\mathbb{Z}_p^\times$, we may write it as

$$\sum_{i\geq 1} p^{-2i(2z_1+z_0-1)}p^{-3i}\int_{\mathbb{Z}_p^\times\times\mathbb{Z}_p^\times}\left|\tilde{a}_y^2-\delta^2p^{2i}\tilde{b}_y^2+\tilde{b}_y\right|_p^{2z_1+z_0-1}d\tilde{a}_yd\tilde{b}_y, \tag{2.-19}$$

where the volume factor comes from the Jacobian of the change of variables $(\tilde{a}_y,\tilde{b}_y)=(p^{-i}a_y,p^{-2i}b_y)$, which is exactly

$$\tfrac{\mathrm{vol}(p^i\mathbb{Z}_p-p^{i+1}\mathbb{Z}_p)\cdot\mathrm{vol}(p^{2i}\mathbb{Z}_p-p^{2i+1}\mathbb{Z}_p)}{\mathrm{vol}(\mathbb{Z}_p^\times)^2}=p^{-3i}.$$

By doing the variables change $(\alpha_y,\beta_y)=(\tilde{a}_y\tilde{b}_y^{-1},\tilde{b}_y^{-1}-\delta^2p^{2i})$, which has trivial Jacobian, then

$$\begin{aligned}
(2.7.4) &= \sum_{i\geq 1}p^{-i(2(2z_1+z_0)+1)}\int_{\mathbb{Z}_p^\times\times\mathbb{Z}_p^\times}\left|\alpha_y^2+\beta_y\right|_p^{2z_1+z_0-1}d\alpha_yd\beta_y\\
&= \sum_{i\geq 1}p^{-i(2(2z_1+z_0)+1)}(1-p^{-1})\int_{\mathbb{Z}_p^\times}|1+u|_p^{2z_1+z_0-1}du\\
&= \sum_{i\geq 1}p^{-i(2(2z_1+z_0)+1)}(1-p^{-1})(\tfrac{p-2}{p}+\tfrac{1-p^{-1}}{p^{2z_1+z_0}-1})\\
&= (1-p^{-1})\tfrac{1}{p^{2(2z_1+z_0)+1}-1}(\tfrac{p-2}{p}+\tfrac{1-p^{-1}}{p^{2z_1+z_0}-1}),
\end{aligned}$$

where for the second equality we did the change of variables $u=\beta_y/\alpha_y^2$, for the third we have divide the integral into the one over $1+u\notin p\mathbb{Z}_p$ and $1+u\in p\mathbb{Z}_p$, and, for the convergence of the geometric series, we have used that $\mathrm{Re}(2z_1+z_0)>1$.

We now calculate the other three terms of (2.7.4). Firstly, the integral over $\mathbb{Z}_p^\times\times(p)$ is immediately equal to $\mathrm{vol}(\mathbb{Z}_p^\times)\mathrm{vol}((p))=\frac{p-1}{p^2}$, while the integral over $(p)\times\mathbb{Z}_p^\times$ equals

$$\begin{aligned}
\int_{(p)\times\mathbb{Z}_p^\times}|a_y^2-\delta^2b_y^2+b_y|_p^{2z_1+z_0-1}da_ydb_y &= \int_{(p)\times\mathbb{Z}_p^\times}|a_y^2b_y^{-2}-\delta^2+b_y^{-1}|_p^{2z_1+z_0-1}da_ydb_y\\
&= \int_{(p)\times\mathbb{Z}_p^\times}|a_y^2-\delta^2+b_y^{-1}|_p^{2z_1+z_0-1}da_ydb_y\\
&= \mathrm{vol}((p))\int_{\mathbb{Z}_p^\times}|u-\delta^2|_p^{2z_1+z_0-1}du\\
&= p^{-1}(\tfrac{p-2}{p}+\tfrac{1-p^{-1}}{p^{2z_1+z_0}-1}),
\end{aligned}$$

where for the second, resp. third, equality we have used the change of variable $a_y=a_yb_y^{-1}$, resp. $u=b_y^{-1}+a_y^2$.

Doing the analogous changes of variables as above, we may write

$$\int_{\mathbb{Z}_p^\times\times\mathbb{Z}_p^\times}|a_y^2-\delta^2b_y^2+b_y|_p^{2z_1+z_0-1}da_ydb_y=\int_{\mathbb{Z}_p^\times\times\mathbb{Z}_p^\times}|a_y^2-\delta^2+b_y|_p^{2z_1+z_0-1}da_ydb_y.$$

Furthermore, since p is inert, δ^2 is not a square modulo p, hence $a_y^2-\delta^2\in\mathbb{Z}_p^\times$. This fact (and a change of variable $u=b_y(a_y^2-\delta^2)^{-1}$) let us write the integral as

$$\int_{\mathbb{Z}_p^\times\times\mathbb{Z}_p^\times}|a_y^2-\delta^2+b_y|_p^{2z_1+z_0-1}da_ydb_y=\mathrm{vol}(\mathbb{Z}_p^\times)\int_{\mathbb{Z}_p^\times}|1+u|_p^{2z_1+z_0-1}du$$
$$=(1-p^{-1})(\tfrac{p-2}{p}+\tfrac{1-p^{-1}}{p^{2z_1+z_0-1}}).$$

Hence, the contributions of the last three terms of (2.7.4) add up to $\frac{p-1}{p^2}+\frac{p-2}{p}+\frac{1-p^{-1}}{p^{2z_1+z_0-1}}$. Summing this with (2.7.4) and (2.7.4) gives

$$\begin{aligned}
&=\frac{1}{p^2}\left[p^2-p-1+\tfrac{p^2-p}{p^{2z_1+z_0}-1}+\tfrac{(p-1)(p^{2z_1+z_0+1}+2)}{p^{2(2z_1+z_0)+1}-1}+\tfrac{p-1}{p^{2(2z_1+z_0)+1}-1}(p-2+\tfrac{p-1}{p^{2z_1+z_0}-1})\right]\\
&=\frac{1}{p^2}\left[-1+(p-1)\frac{p^{2z_1+z_0+1}+1}{p^{2z_1+z_0}-1}\right]\\
&=-\frac{1}{p^2}\left[1+(p-1)\frac{1-p\chi(a_{2\alpha_2-\alpha_0})}{1+\chi(a_{2\alpha_2-\alpha_0})}\right],
\end{aligned}$$

where, by (2.7.3) and the fact that $\chi_{E_p/\mathbb{Q}_p}(p)=-1$, in the last equality we have used that $-\chi(a_{2\alpha_2-\alpha_0})=p^{2z_1+z_0}$. ☐

Proposition 2.7.34. *We obtain the following equality:*

$$T^*_{\omega_1^{-1}}\Delta_{\tilde{S},{}^{\omega_1}\xi}(\mathrm{char}_I)=-\chi^{-1}(a_{2\alpha_2-\alpha_0})c_{\alpha_1-\alpha_2}(\xi)\frac{1+p^{-1}\chi(a_{2\alpha_2-\alpha_0})}{1+p^{-1}\chi^{-1}(a_{2\alpha_2-\alpha_0})}.$$

Proof. We prove the formula following the same strategy as in Proposition 2.7.32. In particular, by proposition 2.3.64 and corollary 2.7.17, we have

$$T_{\omega_1^{-1}}(P_{^{\omega_1}\xi}(\mathrm{char}_I))=(c_{\alpha_1-\alpha_2}(^{\omega_1}\xi)-1)P_{^{\omega_1}\xi}(\mathrm{char}_I)+p^{-2}P_{^{\omega_1}\xi}(\mathrm{char}_{Is_1I}).$$

Applying $\Delta_{\tilde{S},\xi}$ to (2.7.4), we get

$$T^*_{s_1^{-1}}\Delta_{\tilde{S},{}^{\omega_1}\xi}(\mathrm{char}_I)=c_{\alpha_1-\alpha_2}(^{\omega_1}\xi)-1+p^{-2}\Delta_{\tilde{S},\xi}(\mathrm{char}_{Is_1I}).$$

To calculate $\Delta_{\tilde{S},\xi}(\mathrm{char}_{Is_1I})$ explicitly we use two facts. Firstly, we express every element of Is_1I as a product of the form $B_{\mathbf{G}}(\mathbb{Q}_p)\tilde{S}(\mathbb{Q}_p)$. Secondly, we initially assume that ξ satisfies the hypothesis of proposition 2.7.24 to use formula (2.7.4) for the Shalika distribution and then extend the result for all ξ by using corollary 2.7.29.

In the exact same way as in Proposition 2.7.32, we get that $Is_1I\subset B_{\mathbf{G}}(\mathbb{Z}_p)s_1N^\alpha(\mathbb{Z}_p)\tilde{S}(\mathbb{Z}_p)$. Thus, if ξ satisfies the hypothesis of proposition 2.7.24, we get

$$\Delta_{\tilde{S},\xi}(\mathrm{char}_I)=p^2\cdot\int_{s_1U_{\mathbf{G}}^{\alpha_1-\alpha_2}(\mathbb{Z}_p)}\xi^{-1}\delta_{B_{\mathbf{G}}}^{1/2}(b(x))\chi_{\tilde{S}}^{-1}(\tilde{s}(x))dx,\qquad(2.\text{-}22)$$

where the Haar measure dx is normalized such that $\mathrm{vol}(B_{\mathbf{G}}(\mathbb{Z}_p))=\mathrm{vol}(\tilde{S}(\mathbb{Z}_p))=1$, and p^2 comes from the volume factor $[Is_1I:I]$. By lemma 2.7.33, we get

$$\Delta_{\tilde{S},\xi}(\mathrm{char}_{Is_1I})=-1+(1-p)\frac{1-p\chi(a_{2\alpha_2-\alpha_0})}{1+\chi(a_{2\alpha_2-\alpha_0})}=-1+p(p-1)\frac{1-p^{-1}\chi^{-1}(a_{2\alpha_2-\alpha_0})}{1+\chi^{-1}(a_{2\alpha_2-\alpha_0})}.$$

Hence, (2.7.4) becomes

$$T^*_{\omega_1^{-1}}\Delta_{\tilde{S},{}^{\omega_1}\xi}(\mathrm{char}_I)=c_{\alpha_1-\alpha_2}(^{\omega_1}\xi)-1-p^{-2}+p^{-1}(p-1)\frac{1-p^{-1}\chi^{-1}(a_{2\alpha_2-\alpha_0})}{1+\chi^{-1}(a_{2\alpha_2-\alpha_0})}.$$

Since
$$c_{\alpha_1-\alpha_2}({}^{\omega_1}\xi)-1-p^{-2}=\frac{1-p^{-2}\chi^2(a_{2\alpha_2-\alpha_0})}{1-\chi^2(a_{2\alpha_2-\alpha_0})}-1-p^{-2}=-\frac{1-p^{-2}\chi^{-2}(a_{2\alpha_2-\alpha_0})}{1-\chi^{-2}(a_{2\alpha_2-\alpha_0})},$$
we thus have
$$\begin{aligned}T^*_{\omega_1^{-1}}\Delta_{\tilde{S},{}^{\omega_1}\xi}(\mathrm{char}_I)&=-\frac{1-p^{-2}\chi^{-2}(a_{2\alpha_2-\alpha_0})}{1-\chi^{-2}(a_{2\alpha_2-\alpha_0})}+p^{-1}(p-1)\frac{1-p^{-1}\chi^{-1}(a_{2\alpha_2-\alpha_0})}{1+\chi^{-1}(a_{2\alpha_2-\alpha_0})}\\&=\frac{1-p^{-1}\chi^{-1}(a_{2\alpha_2-\alpha_0})}{1+\chi^{-1}(a_{2\alpha_2-\alpha_0})}\left[\frac{1+p^{-1}\chi^{-1}(a_{2\alpha_2-\alpha_0})}{\chi^{-1}(a_{2\alpha_2-\alpha_0})-1}+1-p^{-1}\right]\\&=-\frac{1-p^{-1}\chi^{-1}(a_{2\alpha_2-\alpha_0})}{1+\chi^{-1}(a_{2\alpha_2-\alpha_0})}\cdot\frac{p^{-1}+\chi^{-1}(a_{2\alpha_2-\alpha_0})}{1-\chi^{-1}(a_{2\alpha_2-\alpha_0})}\\&=-\chi^{-1}(a_{2\alpha_2-\alpha_0})\frac{1-p^{-2}\chi^{-2}(a_{2\alpha_2-\alpha_0})}{1-\chi^{-2}(a_{2\alpha_2-\alpha_0})}\cdot\frac{1+p^{-1}\chi(a_{2\alpha_2-\alpha_0})}{1+p^{-1}\chi^{-1}(a_{2\alpha_2-\alpha_0})}.\end{aligned}$$

Since $c_{\alpha_1-\alpha_2}(\xi)=\frac{1-p^{-2}\chi^{-2}(a_{2\alpha_2-\alpha_0})}{1-\chi^{-2}(a_{2\alpha_2-\alpha_0})}$, we get the desired formula. □

Fourth reduction: The Casselman-Shalika formula

Before starting the proof of the final theorem, we will recall a simple formula involving products of covariant expressions. If k_α is an expression which is covariant with α (in the sense that $\omega k_\alpha=k_{\omega^{-1}\alpha}$), we have
$$\frac{\omega^{-1}\prod_{\alpha>0}k_\alpha}{\prod_{\alpha>0}k_\alpha}=\frac{\prod_{\substack{\alpha>0\\ \omega\alpha<0}}k_{-\alpha}}{\prod_{\substack{\alpha>0\\ \omega\alpha<0}}k_\alpha}. \tag{2.-22}$$

Lemma 2.7.35. *We get*
$$\prod_{\alpha\in\Phi^+_{\mathbf{H}}}\chi^{-1}(a_\alpha)=e^{\rho-\omega^{-1}\rho}(g_{\chi^{-1}}).$$

Proof. Taking $k_\alpha=\xi(a_\alpha)$ in (2.7.4), we get
$$\frac{\prod_{\substack{\alpha\in\Phi^+_{\mathbf{H}}\\ \omega\alpha<0}}\chi^{-1}(a_\alpha)}{\prod_{\substack{\alpha\in\Phi^+_{\mathbf{H}}\\ \omega\alpha<0}}\chi^{-1}(a_{-\alpha})}=\frac{\prod_{\alpha\in\Phi^+_{\mathbf{H}}}\chi^{-1}(a_\alpha)}{\omega^{-1}\prod_{\alpha\in\Phi^+_{\mathbf{H}}}\chi^{-1}(a_\alpha)}.$$

Isolating the term we want to compute, we obtain
$$\prod_{\substack{\alpha\in\Phi^+_{\mathbf{H}}\\ \omega\alpha<0}}\chi^{-1}(a_\alpha)^2=\prod_{\alpha\in\Phi^+_{\mathbf{H}}}\chi^{-1}(a_{\alpha-\omega^{-1}\alpha}).$$

Hence, using the fact that χ is a character, we get
$$\prod_{\substack{\alpha\in\Phi^+_{\mathbf{H}}\\ \omega\alpha<0}}\chi^{-1}(a_\alpha)=\chi(a_{\rho-\omega^{-1}\rho}),$$
where we recall that $\rho=\frac{1}{2}\sum_{\alpha\in\Phi^+_{\mathbf{H}}}\alpha$. Furthermore, expressing the result in the dual group notation we conclude. □

Lemma 2.7.36. *We obtain*
$$\prod_{\substack{\alpha\in\Phi^{+,l}_{\mathbf{H}}\\ \omega\alpha<0}}\frac{1+p^{-1}\chi(a_\alpha)}{1+p^{-1}\chi^{-1}(a_\alpha)}=\frac{\prod_{\alpha\in\Phi^{+,l}_{\mathbf{H}}}1+p^{-1}e^{\omega^{-1}\alpha}}{\prod_{\alpha\in\Phi^{+,l}_{\mathbf{H}}}1+p^{-1}e^{\alpha}}\left(g_{\chi^{-1}}\right).$$

Proof. Taking $k_\alpha = 1 + p^{-1}\chi(a_{-\alpha})$ and $\Phi^{+,l}$ as the set of roots in (2.7.4), we get the result directly. □

For the following theorem let us denote by $\tilde{\xi}$, the restriction of ξ to the split torus.

Theorem 2.7.37. *We obtain the following expression*

$$\mathcal{S}_{\tilde{S},p}(g_{-\lambda}v_0) = \frac{p^{-2n}\prod_{\alpha\in\Phi^+_{\mathbf{H}}} c_\alpha(\xi)}{Qe^{-\check{\rho}}\prod_{\check{\alpha}\in\Phi^{\vee,+,S}_{\mathrm{Sp}_4}}(1-p^{-1}e^{\check{\alpha}})(g_{\tilde{\xi}})}\mathcal{A}\left(e^{\check{\rho}+n(\check{\alpha}_1+\check{\alpha}_2)}\prod_{\check{\alpha}\in\Phi^{\vee,+,S}_{\mathrm{Sp}_4}}(1-p^{-1}e^{-\check{\alpha}})\right)(g_{\tilde{\xi}}).$$

Proof. Let us start by recalling the formula obtained in (2.7.4),

$$\mathcal{S}_{\tilde{S},p}\left(\pi(g_{-\lambda})v_0\right) = Q^{-1}\sum_{\omega\in W^{\mathbf{H}}}\prod_{\substack{\alpha\in\Phi^+_{\mathbf{H}}\\ \omega\alpha>0}} c_\alpha(\xi)^\omega\xi^{-1}\delta_B^{1/2}(g_\lambda)T^*_{\omega^{-1}}\Delta_{\tilde{S},{}^\omega\xi}(\mathrm{char}_I).$$

As we explained in 1.5.42, the group $W^{\mathbf{H}}$ is generated by ω_1 and ω_2, with representatives s_1 and s_2. We will proceed then by induction. By 2.7.32 and 2.7.34 we get

$$\begin{aligned}
T^*_{\omega_1^{-1}}\Delta_{\tilde{S},{}^{\omega_1}\xi}(\mathrm{char}_I) &= -\chi^{-1}(a_{2\alpha_2-\alpha_0})c_{\alpha_1-\alpha_2}(\xi)\frac{1+p^{-1}\chi(a_{2\alpha_2-\alpha_0})}{1+p^{-1}\chi^{-1}(a_{2\alpha_2-\alpha_0})}\\
&= (-1)^{\ell(\omega_2)}\prod_{\substack{\alpha\in\Phi^+\\ \omega_2\alpha<0}}\chi^{-1}(a_\alpha)\prod_{\substack{\alpha\in\Phi^+\\ \omega_1\alpha<0}}c_\alpha(\xi)\prod_{\substack{\alpha\in\Phi^{+,l}\\ \omega_2\alpha<0}}\frac{1+p^{-1}\chi(a_{2\alpha_2-\alpha_0})}{1+p^{-1}\chi^{-1}(a_{2\alpha_2-\alpha_0})},
\end{aligned}$$

$$\begin{aligned}
T^*_{\omega_2^{-1}}\Delta_{\tilde{S},{}^{\omega_2}\xi}(\mathrm{char}_I) &= -\chi^{-1}(a_{\alpha_1-\alpha_2})c_{2\alpha_2-\alpha_0}(\xi)\\
&= (-1)^{\ell(\omega_1)}\prod_{\substack{\alpha\in\Phi^+\\ \omega_1\alpha<0}}\chi^{-1}(a_\alpha)\prod_{\substack{\alpha\in\Phi^+\\ \omega_2\alpha<0}}c_\alpha(\xi)\prod_{\substack{\alpha\in\Phi^{+,l}\\ \omega_1\alpha<0}}\frac{1+p^{-1}\chi(a_{2\alpha_2-\alpha_0})}{1+p^{-1}\chi^{-1}(a_{2\alpha_2-\alpha_0})}.
\end{aligned}$$

Our inductive hypothesis is that, for every $\omega\in W_{\mathbf{H}}$, then

$$T^*_{\omega^{-1}}\Delta_{\tilde{S},{}^\omega\xi}(\mathrm{char}_I) = (-1)^{\ell(\iota(\omega))}\prod_{\substack{\alpha\in\Phi^+\\ \iota(\omega)\alpha<0}}\chi^{-1}(a_\alpha)\prod_{\substack{\alpha\in\Phi^+\\ \omega\alpha<0}}c_\alpha(\xi)\prod_{\substack{\alpha\in\Phi^{+,l}\\ \iota(\omega)\alpha<0}}\frac{1+p^{-1}\chi(a_{2\alpha_2-\alpha_0})}{1+p^{-1}\chi^{-1}(a_{2\alpha_2-\alpha_0})},$$

where $\iota: W^{\mathbf{H}}\to W^{\mathbf{H}}$ is the map which sends ω_1 to ω_2 and viceversa. We point out that $\ell(\iota(\omega)) = \ell(\omega)$. We will prove that $T^*_{\omega^{-1}\omega_1^{-1}}\Delta_{\tilde{S},{}^{\omega_1\omega}\xi}(\mathrm{char}_I)$ satisfies the expression of the statement. The case $\omega_2\omega$ is completely analogous. First of all we may write the above expression as $T^*_{\omega^{-1}}\left(T^*_{\omega_1^{-1}}\Delta_{\tilde{S},{}^{\omega_1\omega}\xi}(\mathrm{char}_I)\right)$. Then, proposition 2.7.34 yields

$$T^*_{\omega^{-1}}\left(T^*_{\omega_1^{-1}}\Delta_{\tilde{S},{}^{\omega_1\omega}\xi}(\mathrm{char}_I)\right) = -\omega\cdot\chi^{-1}(a_{2\alpha_2-\alpha_0})c_{\alpha_1-\alpha_2}({}^\omega\xi)\frac{1+p^{-1}\omega\cdot\chi(a_{2\alpha_2-\alpha_0})}{1+p^{-1}\omega\cdot\chi^{-1}(a_{2\alpha_2-\alpha_0})}T^*_{\omega^{-1}}\Delta_{\tilde{S},{}^\omega\xi}, \quad (2.\text{-}22)$$

where ω acts on the character χ via the theta correspondence, i.e.

$$\omega\cdot\chi^{-1}(a_{2\alpha_2-\alpha_0}) := {}^\omega\xi^{1/2}(a_{\alpha_1-\alpha_2}) = \xi^{1/2}(a_{\omega^{-1}\cdot(\alpha_1-\alpha_2)}).$$

Using the inductive hypothesis we get that

$$(2.7.4) = (-1)^{\ell(\iota(\omega))} \prod_{\substack{\alpha\in\Phi^+\\ \iota(\omega)\alpha<0}} \chi^{-1}(a_\alpha) \prod_{\substack{\alpha\in\Phi^+\\ \omega\alpha<0}} c_\alpha(\xi) \prod_{\substack{\alpha\in\Phi^{+,l}\\ \iota(\omega)\alpha<0}} \frac{1+p^{-1}\chi(a_{2\alpha_2-\alpha_0})}{1+p^{-1}\chi^{-1}(a_{2\alpha_2-\alpha_0})}$$
$$\cdot -\omega\cdot\chi^{-1}(a_{2\alpha_2-\alpha_0})c_{\alpha_1-\alpha_2}({}^{\omega}\xi)\frac{1+p^{-1}\omega\cdot\chi(a_{2\alpha_2-\alpha_0})}{1+p^{-1}\omega\cdot\chi^{-1}(a_{2\alpha_2-\alpha_0})}.$$

First of all we observe $-(-1)^{\ell(\iota(\omega))} = (-1)^{\ell(\iota(\omega_1\omega))}$. The $c_\alpha(\xi)$ constants satisfy

$$c_{\alpha_1-\alpha_2}({}^{\omega}\xi)\prod_{\substack{\alpha\in\Phi^+\\ \omega\alpha<0}} c_\alpha(\xi) = c_{\omega^{-1}\cdot(\alpha_1-\alpha_2)}(\xi)\prod_{\substack{\alpha\in\Phi^+\\ \omega\alpha<0}} c_\alpha(\xi) = \prod_{\substack{\alpha\in\Phi^+\\ \omega_1\omega\alpha<0}} c_\alpha(\xi).$$

In fact,

$$\begin{aligned}\{\alpha\in\Phi^+,\ s.t.\ \omega_1\omega\alpha<0\} &= \{\alpha\in\Phi^+,\ s.t.\ \omega\alpha<0\}\cup\{\alpha\in\Phi^+,\ s.t.\ \omega\alpha=\alpha_1-\alpha_2\}\\ &= \{\alpha\in\Phi^+,\ s.t.\ \omega\alpha<0\}\cup\{\omega^{-1}\cdot(\alpha_1-\alpha_2)\}.\end{aligned}$$

As have been used consistently throughout this subsection, the constants $\xi(a_\alpha)$ and $\chi(a_\alpha)$ are related by the Theta correspondence. At the level of roots, it has the effect of the map $t:\ \Phi_{\mathbf{H}}\to\Phi_{\mathbf{H}}$, which sends $\alpha_1-\alpha_2$ to $2\alpha_2-\alpha_0$ and $2\alpha_2-\alpha_0$ to $\alpha_1-\alpha_2$. The previous map between root systems induces the morphism ι between the Weyl groups of both sides. Hence, we get that $t(\omega\alpha)=\iota(\omega)t(\alpha)$, which yields the following equality:

$$\omega\cdot\chi^{-1}(a_{2\alpha_2-\alpha_0}) = \chi^{-1}(a_{\iota(\omega)^{-1}(2\alpha_2-\alpha_0)}).$$

Replying the argument we used to deal with the $c_\alpha(\xi)$ constants, we get

$$\omega\cdot\chi^{-1}(a_{2\alpha_2-\alpha_0})\prod_{\substack{\alpha\in\Phi^+\\ \iota(\omega)\alpha<0}}\chi^{-1}(a_\alpha) = \prod_{\substack{\alpha\in\Phi^+\\ \iota(\omega_1\omega)\alpha<0}}\chi^{-1}(a_\alpha),$$
$$\frac{1+p^{-1}\omega\cdot\chi(a_{2\alpha_2-\alpha_0})}{1+p^{-1}\omega\cdot\chi^{-1}(a_{2\alpha_2-\alpha_0})}\prod_{\substack{\alpha\in\Phi^{+,l}\\ \iota(\omega)\alpha<0}}\frac{1+p^{-1}\chi(a_{2\alpha_2-\alpha_0})}{1+p^{-1}\chi^{-1}(a_{2\alpha_2-\alpha_0})} = \prod_{\substack{\alpha\in\Phi^{+,l}\\ \iota(\omega_1\omega)\alpha<0}}\frac{1+p^{-1}\chi(a_{2\alpha_2-\alpha_0})}{1+p^{-1}\chi^{-1}(a_{2\alpha_2-\alpha_0})}.$$

Therefore by induction we prove the equality (2.7.4).

Furthermore note that, since $g_\lambda = \begin{pmatrix} p^nI & \\ & I\end{pmatrix}$, we can write $({}^{\omega}\xi^{-1})(g_\lambda) = (-1)^n\cdot{}^{\iota(\omega)}\chi(a_{n(2\alpha_1-\alpha_0)})$; hence, after reordering the sum, the above expression is equal to

$$Q^{-1}\delta_{B_{\mathbf{G}}}^{1/2}(g_\lambda)\prod_{\alpha\in\Phi^+_{\mathbf{H}}}c_\alpha(\xi)\sum_{\omega\in W^{\mathbf{H}}}(-1)^{\ell(\omega)+n}({}^{\omega}\chi(a_{n(2\alpha_1-\alpha_0)}))\prod_{\substack{\alpha\in\Phi^+_{\mathbf{H}}\\ \omega\alpha<0}}\chi(a_{-\alpha})\prod_{\substack{\alpha\in\Phi^{+,l}_{\mathbf{H}}\\ \omega\alpha<0}}\frac{1+p^{-1}\chi(a_\alpha)}{1+p^{-1}\chi(a_{-\alpha})}.$$

Using lemmas 2.7.35 and 2.7.36 and writing $\omega^{-1}=\omega\omega_0$, the formula becomes

$$Q^{-1}\delta_{B_{\mathbf{G}}}^{1/2}(g_\lambda)\prod_{\alpha\in\Phi^+_{\mathbf{H}}}c_\alpha(\xi)\sum_{\omega\in W^{\mathbf{H}}}(-1)^{\ell(\omega)+n}\chi^{-1}(a_{\rho+\omega(\rho+n(2\alpha_1-\alpha_0))})\frac{\omega\left(\prod_{\alpha\in\Phi^{+,l}_{\mathbf{H}}}(1+p^{-1}\chi(a_\alpha))\right)}{\prod_{\alpha\in\Phi^{+,l}_{\mathbf{H}}}(1+p^{-1}\chi(a_{-\alpha}))},$$

where we have written ${}^{\omega}\chi(a_{n(2\alpha_1-\alpha_0)}) = \chi^{-1}(a_{n\omega(2\alpha_1-\alpha_0)})$. To further simplify the formula, we can express it all in terms of the Satake paramaters of $I_{\mathbf{H}}(\tilde{\xi})$. To do so, first observe that via the isomorphism between the dual group of $\mathbf{PH}$ and $\mathrm{Sp}_4(\mathbb{C})$, the set of positive roots of $\mathbf{PH}$ is isomorphic to set of positive co-roots of Sp_4 via the map (explicited in the proof of [RS07, Lemma 2.3.1])

$$\{\alpha_1-\alpha_2, \alpha_1+\alpha_2-\alpha_0, 2\alpha_1-\alpha_0, 2\alpha_2-\alpha_0\} \to \{\check{\alpha}_2, \check{\alpha}_1, \check{\alpha}_1+\check{\alpha}_2, \check{\alpha}_1-\check{\alpha}_2\}.$$

Using this and the fact that $g_{\tilde{\xi}} = \chi_{E_p/\mathbb{Q}_p}(p)\mathrm{diag}(\chi_1^{-1}(p), \chi_2^{-1}(p), \chi_2(p), \chi_1(p))$, the expression reads as

$$\frac{\delta_{B_{\mathbf{G}}}^{1/2}(g_\lambda)\prod_{\alpha\in\Phi_{\mathbf{H}}^+} c_\alpha(\xi)}{Qe^{-\check{\rho}}\prod_{\check{\alpha}\in\Phi_{\mathrm{Sp}_4}^{\vee,+,S}}(1-p^{-1}e^{\check{\alpha}})(g_{\tilde{\xi}})}\mathcal{A}\left(e^{\check{\rho}+n(\check{\alpha}_1+\check{\alpha}_2)}\prod_{\check{\alpha}\in\Phi_{\mathrm{Sp}_4}^{\vee,+,S}}(1-p^{-1}e^{-\check{\alpha}})\right)(g_{\tilde{\xi}}),$$

where we used the convenient notation of $\xi(a_\alpha) = e^{\check{\alpha}}(g_{\tilde{\xi}})$ and the definition of the alternator $\mathcal{A}(\cdot) = \sum_\omega(-1)^{\ell(\omega)}\omega(\cdot)$. As $\delta_{B_{\mathbf{G}}}^{1/2}(g_\lambda) = p^{-2n}$, we get the formula. □

Remark 2.7.38. *If $I_{\mathbf{G}}(\xi)$ is irreducible, applying the previous we get the vanishing of the Shalika functional.*

2.7.5 The zeta integral

Siegel Eisenstein series on $\mathbf{H}$

We define the Siegel Eisenstein series associated to $\mathbf{H}$ and describe some of their properties. We follow [KR94, sec. 1] closely, with the only caveat of moving to $s = 1/2$ the center of the functional equation. Recall that in (2.2.2) and (2.2.2) we introduced the Siegel parabolic $P_{\mathbf{H}}$ of $\mathbf{H}$ with Levi deomposition $M_{\mathbf{H}}N_{\mathbf{H}}$ where

$$M_P(R) = \left\{m = \begin{pmatrix} g & \\ & \mu I_2'{}^tg^{-1}I_2' \end{pmatrix},\ g\in\mathrm{GL}_2, \mu\in\mathrm{GL}_1\right\},$$

for every $\mathbb{Q}-$algebra R. Denote by δ_P the modulus character of P given by

$$\delta_{P_{\mathbf{H}}} : \begin{pmatrix} g & \star \\ & \mu I_2'{}^tg^{-1}I_2' \end{pmatrix} \mapsto \left|\frac{\det(g)}{\mu}\right|^3.$$

We denote by $I_P(s)$ the degenerate principal series representation of $\mathbf{H}(\mathbb{A})$ consisting of smooth functions f_s on $\mathbf{H}(\mathbb{A})$ so that

$$f_s(nmg) = \delta_{P_{\mathbf{H}}}^{\frac{1}{3}(s+1)}(m)f_s(g),\ \forall n\in N_{\mathbf{H}}(\mathbb{A}), \forall m\in M_{\mathbf{H}}(\mathbb{A}).$$

Let us recall that as in remark 1.6.7, the representation $I_P(s)$ factorizes as follows

$$I_P(s) \simeq \bigotimes_p I_{P(\mathbb{Q}_p)}(s),$$

where $I_{P(\mathbb{Q}_p)}(s)$ is the representation of $\mathbf{H}(\mathbb{Q}_p)$ consisting on smooth functions $f_{s,p}$ on $\mathbf{H}(\mathbb{Q}_p)$ so that

$$f_{s,p}(nmg) = \delta_{P_{\mathbf{H},\mathbb{Q}_p}}^{\frac{1}{3}(s+1)}(m)f_{s,p}(g),\ \forall n\in N_{\mathbf{H}}(\mathbb{Q}_p), \forall m\in M_{\mathbf{H}}(\mathbb{A}).$$

Given a standard holomorphic section

$$f_s \in I_P(s),$$

we define the degenerate Eisenstein series

$$E_P(g,s,f_s) = \sum_{\gamma\in P_{\mathbf{H}}(\mathbb{Q})\backslash\mathbf{H}(\mathbb{Q})} f_s(\gamma g),$$

which is absolutely convergent in the half-plane $\mathrm{Re}(s) > 2$ and admits analytic continuation to a meromorphic function on $\mathbb{C}$ which satisfies a functional equation (cf. [KR94, (1.5), p. 11])

$$E_P(g, s, f_s) = E_P(g, 1-s, M(s)f_s),$$

where $M(s)$ is an intertwining operator defined in 1.6.8. At a finite prime p, let $f^0_{p,s}$ be the function in $I_{P(\mathbb{Q}_p)}(s)$ such that $f^0_{p,s}(k) = 1$ for all $k \in \mathbf{H}(\mathbb{Z}_p)$. We assume that f_s is a pure tensor and denote by $f_{p,s}$ its p-adic component, which equals $f^0_{p,s}$ almost everywhere. Let S be the finite set of places containing ∞ and all the primes p such that $f_{p,s} \neq f^0_{p,s}$. Following [KR94, (1.11), p. 12], we then define the normalised Siegel Eisenstein series to be equal to

$$E^*_P(g, s) := \zeta^S(s+1)\zeta^S(2s)E_P(g, s, f_s). \tag{2.-22}$$

It follows from [KR94, thm 1.1, p. 12] and [KR94, thm, 4.12, p. 39] that $E^*_P(g, s)$ has at most simple poles at the points $s = 1, 2$.

The zeta integral and Shalika periods

Let π be a cuspidal automorphic representation of $\mathbf{G}$ with trivial central character. Let S be a finite set of places containing ∞ and all the ramified primes for π. We take a cusp form $\varphi \in \pi$. Recall that by remark 2.2.15 we have the embedding

$$\mathbf{H} = \mathrm{GSp}_4 \hookrightarrow \mathbf{G}.$$

We consider the following integral:

$$I(\varphi, s) := \int_{\mathbf{H}(\mathbb{Q})Z_{\mathbf{H}}(\mathbb{A})\backslash \mathbf{H}(\mathbb{A})} E^*_P(h, s)\varphi(h)dh, \tag{2.-22}$$

where $E^*_P(g, s)$, defined in defined in (2.7.5), is the normalised Siegel Eisenstein series for $\mathbf{H}$ associated to a normalised standard holomorphic section $f_s = \otimes'_p f_{p,s} \in I_P(s)$, i.e. such that

$$f_{p,s} = \zeta_p(s+1)\zeta_p(2s)f^0_{p,s}, \ \forall p \notin S.$$

Note that this integral converges absolutely as the restriction to $\mathbf{H}(\mathbb{A})$ of φ is rapidly decreasing.

Proposition 2.7.39. *The integral $I(\varphi, s)$ unfolds to*

$$\int_{\mathrm{GL}_2(\mathbb{A})N_{\mathbf{H}}(\mathbb{A})\backslash \mathbf{H}(\mathbb{A})} f_s(h)\mathcal{S}_\delta(\pi(h)\varphi)dh,$$

where $\mathcal{S}_\delta$ is the functional defined in remark 2.4.25.

Proof. We start by unfolding the Eisenstein series to get

$$\int_{P_{\mathbf{H}}(\mathbb{Q})Z_{\mathbf{G}}(\mathbb{A})\backslash \mathbf{H}(\mathbb{A})} f_s(h)\varphi(h)dh. \tag{2.-22}$$

Collapse the integral over the unipotent radical $N_{\mathbf{H}}$ obtaining that (2.7.5) is equal to

$$\begin{aligned}\int_{M_{\mathbf{H}}(\mathbb{Q})N_{\mathbf{H}}(\mathbb{A})Z_{\mathbf{G}}(\mathbb{A})\backslash \mathbf{H}(\mathbb{A})} \int_{N_{\mathbf{H}}(\mathbb{Q})\backslash N_{\mathbf{H}}(\mathbb{A})} f_s(nh)\varphi(nh)dndh \\ = \int_{M_{\mathbf{H}}(\mathbb{Q})N_{\mathbf{H}}(\mathbb{A})Z_{\mathbf{G}}(\mathbb{A})\backslash \mathbf{H}(\mathbb{A})} f_s(h) \int_{N_{\mathbf{H}}(\mathbb{Q})\backslash N_{\mathbf{H}}(\mathbb{A})} \varphi(nh)dndh,\end{aligned}$$

where the equality follows from the definition of $f_s \in I_P(s)$. Let us denote unipotent radical $N_{\mathbf{G}}$ of the Siegel parabolic subgroup of $\mathbf{G}$, which is commutative. Using the Fourier expansion of the cusp form φ

over $[N_{\mathbf{G}}/N_{\mathbf{H}}]$, see 1.4.11 for the details, we obtain that the above integral equals

$$\int_{M_{\mathbf{H}}(\mathbb{Q})N_{\mathbf{H}}(\mathbb{A})Z_{\mathbf{G}}(\mathbb{A})\backslash\mathbf{H}(\mathbb{A})} f_s(h)\sum_{\chi}\varphi_{\chi}(h)dh, \tag{2.-22}$$

where the sum runs over the characters $\chi : [N_{\mathbf{G}}/N_{\mathbf{H}}] \to \mathbb{C}^{\times}$ and where we have denoted

$$\varphi_{\chi}(h) = \int_{N_{\mathbf{G}}(\mathbb{Q})\backslash N_{\mathbf{G}}(\mathbb{A})} \varphi(n'h)\chi^{-1}(n')dn'.$$

Recall that any character $\chi : [N_{\mathbf{G}}] \to \mathbb{C}^{\times}$ is of the form

$$n' \mapsto \psi\left(\mathrm{tr}(A \cdot \left(\begin{smallmatrix} \alpha & x \\ y & \bar{\alpha}\end{smallmatrix}\right))\right),$$

where $A \in M_{2\times 2}(E)$ satisfies $\overline{A}^t = I_2' A I_2'$, ψ is a non-trivial additive character on $\mathbb{Q}\backslash\mathbb{A}$ and where we have used the notation from (2.2.1) i.e.

$$n' = \begin{pmatrix} 1 & & \alpha & x \\ & 1 & y & \bar{\alpha} \\ & & 1 & \\ & & & 1\end{pmatrix}, \quad \text{with } x, y \in \mathbb{A}, \alpha \in \mathbb{A}_E,$$

If $A = (a_{i,j})$, the associated character χ is trivial on $N_{\mathbf{H}}(\mathbb{A})$ if $a_{1,1} + a_{2,2} = 0$ and $a_{1,2} = a_{2,1} = 0$. Therefore any character $\chi : [N_{\mathbf{G}}/N_{\mathbf{H}}] \to \mathbb{C}^{\times}$ can be described as follows by

$$n' \mapsto \psi(\mathrm{Tr}(\left(\begin{smallmatrix} \delta & \\ & -\delta\end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} \alpha & x \\ y & \bar{\alpha}\end{smallmatrix}\right))) = \psi(\delta\alpha - \delta\bar{\alpha}),$$

with $\delta \in E$ such that $\bar{\delta} = -\delta$. To emphasize its dependence on δ, we denote this character by χ_{δ}.

As $M_{\mathbf{H}}$ acts on $N_{\mathbf{G}}$ by conjugation and this action preserves $N_{\mathbf{H}}$, $M_{\mathbf{H}}(\mathbb{Q})$ acts on the space of characters of $[N_{\mathbf{G}}/N_{\mathbf{H}}]$. Indeed, if $m = \left(\begin{smallmatrix} g & \\ & \mu I_2' {}^t g^{-1} I_2'\end{smallmatrix}\right) \in M_{\mathbf{H}}(\mathbb{Q})$,

$$\chi_{\delta}(mn'm^{-1}) = \chi_{\delta'}(n'), \quad \text{with } \delta' = \mu^{-1}\det(g)\eta.$$

Then the group $M_{\mathbf{H}}(\mathbb{Q})$ acts on the space of characters of $[N_{\mathbf{G}}/N_{\mathbf{H}}]$ with two orbits, the trivial one and an open one associated to any χ_{δ} with $\delta \neq 0$. Using this action, we can write the integral (2.7.5) as

$$\int_{M_{\mathbf{H}}(\mathbb{Q})N_{\mathbf{H}}(\mathbb{A})Z_{\mathbf{G}}(\mathbb{A})\backslash\mathbf{H}(\mathbb{A})} f_s(h)\int_{[N_{\mathbf{G}}]}\varphi(n'h)dn'dh + \int_{M_{\delta}(\mathbb{Q})N_{\mathbf{H}}(\mathbb{A})Z_{\mathbf{G}}(\mathbb{A})\backslash\mathbf{H}(\mathbb{A})} f_s(h)\varphi_{\chi_{\delta}}(h)dh,$$

where $M_{\delta}(\mathbb{Q})$ denotes the stabiliser in $M_{\mathbf{H}}(\mathbb{Q})$ of χ_{δ}. Precisely, $m \in M_{\delta}(\mathbb{Q})$ if $\mu = \det(g)$ and hence we can fix an isomorphism

$$\begin{aligned}\mathrm{GL}_2(\mathbb{Q}) &\to M_{\delta}(\mathbb{Q}) \\ g &\mapsto m(g, \det(g)),\end{aligned}$$

where $m(g, \det(g)) = \left(\begin{smallmatrix} g & \\ & \det(g) I_2' {}^t g^{-1} I_2'\end{smallmatrix}\right)$. As the first integral of (2.7.5) vanishes because of cuspidality of φ along $[N_{\mathbf{G}}]$,

$$I(\varphi, s) = \int_{\mathrm{GL}_2(\mathbb{Q})N_{\mathbf{H}}(\mathbb{A})Z_{\mathbf{G}}(\mathbb{A})\backslash\mathbf{H}(\mathbb{A})} f_s(h)\varphi_{\chi_{\delta}}(h)dh.$$

Collapse the integral over $[\mathrm{GL}_2]$, to get

$$\begin{aligned}
I(\varphi, s) &= \int_{\mathrm{GL}_2(\mathbb{Q})N_{\mathbf{H}}(\mathbb{A})Z_{\mathbf{G}}\backslash \mathbf{H}(\mathbb{A})} f_s(h)\varphi_{\chi_\delta}(h)dh \\
&= \int_{\mathrm{GL}_2(\mathbb{A})N_{\mathbf{H}}(\mathbb{A})\backslash \mathbf{H}(\mathbb{A})} \int_{\mathrm{GL}_2(\mathbb{Q})Z_{\mathbf{G}}(\mathbb{A})\backslash \mathrm{GL}_2(\mathbb{A})} f_s(m(g,\det(g))h)\varphi_{\chi_\delta}(m(g,\det(g))h)dgdh \\
&= \int_{\mathrm{GL}_2(\mathbb{A})N_{\mathbf{H}}(\mathbb{A})\backslash \mathbf{H}(\mathbb{A})} f_s(h) \int_{\mathrm{GL}_2(\mathbb{Q})Z_{\mathbf{G}}(\mathbb{A})\backslash \mathrm{GL}_2(\mathbb{A})} \varphi_{\chi_\delta}(gh)dgdh \\
&= \int_{\mathrm{GL}_2(\mathbb{A})N_{\mathbf{H}}(\mathbb{A})\backslash \mathbf{H}(\mathbb{A})} f_s(h)\mathcal{S}_\delta(\pi(h)\varphi)dh,
\end{aligned}$$

where we have used that $\delta_{P_{\mathbf{H}}}(m(g,\det(g))) = \left|\frac{\det(g)}{\det(g)}\right|^3 = 1$. □

Corollary 2.7.40. *The integral $I(\varphi, s)$ is identically zero unless one of the following equivalent conditions hold*

1. *π is (globally) generic and $L^S(s, \pi, \Lambda_t)$ has a pole at $s = 1$,*
2. *$\pi = \widetilde{\Theta}^{glob}(\sigma^+)$, where σ^+ is an irreducible cuspidal automorphic representation of $\mathbf{H}^+$ appearing in $\sigma|_{\mathbf{H}^+}$, with σ a generic cuspidal automorphic representation of $\mathbf{H}$ with trivial central character.*

Proof. It follows from the combination of proposition 2.7.39 and theorems 2.6.19 and 2.6.20. □

In order to relate $I(\varphi, s)$ with the partial standard L-function, we study the local integral

$$I_p(v, s) = \int_{\mathrm{GL}_2(\mathbb{Q}_p)N_{\mathbf{H}}(\mathbb{Q}_p)\backslash \mathbf{H}(\mathbb{Q}_p)} f_{p,s}(h)\mathcal{S}_p(h\cdot v)dh,$$

where $v \in \pi_p$ and $\mathcal{S}_p : \pi_p \to \mathbb{C}$ is a local Shalika functional defined in 2.4.22, i.e. a functional with the property

$$\mathcal{S}_p(\pi_p(gu)v) = \chi_{\delta,p}(u)\mathcal{S}_p(v),\ \forall g \in \mathrm{GL}_2(\mathbb{Q}_), \forall u \in N_{\mathbf{G}}(\mathbb{Q}_p).$$

Suppose that p is a finite place of $\mathbb{Q}$ so that π_p is unramified. Let v_0 be a spherical vector in π_p, then, using the Iwasawa decomposition $\mathbf{H}(\mathbb{Q}_p) = P_{\mathbf{H}}(\mathbb{Q}_p)\mathbf{H}(\mathbb{Z}_p)$ and the fact that $f_{p,s}$ and v_0 are $\mathbf{H}(\mathbb{Z}_p)$-invariants, we get

$$I_p(v_0, s) = \int_{\mathrm{GL}_2(\mathbb{Q}_p)N_{\mathbf{H}}(\mathbb{Q}_p)\backslash P_{\mathbf{H}}(\mathbb{Q}_p)} \delta^{-1}_{P_{\mathbf{H}}}(p)f_{p,s}(p)\mathcal{S}_p(p\cdot v)dp.$$

As $P_{\mathbf{H}}(\mathbb{Q}_p) = M_{\mathbf{H}}(\mathbb{Q}_p)N_{\mathbf{H}}(\mathbb{Q}_p)$ and the multiplier identifies $\mathrm{GL}_2(\mathbb{Q}_p)\backslash M_{\mathbf{H}}(\mathbb{Q}_p) \simeq \mathbb{Q}_p^\times$, we have

$$\begin{aligned}
I_p(v_0, s) &= \int_{\mathbb{Q}_p^\times} \delta^{-1}_{P_{\mathbf{H}}}(\left(\begin{smallmatrix} I & \\ & \mu I\end{smallmatrix}\right))f_{p,s}(\left(\begin{smallmatrix} I & \\ & \mu I\end{smallmatrix}\right))\mathcal{S}_p(\left(\begin{smallmatrix} I & \\ & \mu I\end{smallmatrix}\right)\cdot v_0)d^\times\mu \\
&= f_{p,s}(1)\int_{\mathbb{Q}_p^\times} |\mu|^{2-s}\mathcal{S}_p(\left(\begin{smallmatrix} I & \\ & \mu I\end{smallmatrix}\right)\cdot v_0)d^\times\mu.
\end{aligned}$$

Lemma 2.7.41. *If $\mathcal{S}_p(\left(\begin{smallmatrix} I & \\ & \mu I\end{smallmatrix}\right)\cdot v_0) \neq 0$, then $|\mu|_p \geq 1$.*

Proof. As v_0 is spherical, we have for $u \in N_{\mathbf{G}}(\mathbb{Z}_p)$

$$\begin{aligned}
\mathcal{S}_p\left(\left(\begin{smallmatrix} I & \\ & \mu I\end{smallmatrix}\right)\cdot v_0\right) &= \mathcal{S}_p\left(\left(\left(\begin{smallmatrix} I & \\ & \mu I\end{smallmatrix}\right)u\right)\cdot v_0\right) \\
&= \mathcal{S}_p\left(\left(\left(\begin{smallmatrix} I & \\ & \mu I\end{smallmatrix}\right)u\left(\begin{smallmatrix} I & \\ & \mu^{-1} I\end{smallmatrix}\right)\left(\begin{smallmatrix} I & \\ & \mu I\end{smallmatrix}\right)\right)\cdot v_0\right) \\
&= \chi_\eta\left(\left(\begin{smallmatrix} I & \\ & \mu I\end{smallmatrix}\right)u\left(\begin{smallmatrix} I & \\ & \mu^{-1} I\end{smallmatrix}\right)\right)\mathcal{S}_p\left(\left(\begin{smallmatrix} I & \\ & \mu I\end{smallmatrix}\right)\cdot v_0\right).
\end{aligned}$$

Hence, since by hypothesis $\mathcal{S}_p\left(\left(\begin{smallmatrix} I & \\ & \mu I\end{smallmatrix}\right)\cdot v_0\right)\neq 0$, we deduce the equality

$$\chi_\delta\left(\left(\begin{smallmatrix} I & \\ & \mu I\end{smallmatrix}\right)u\left(\begin{smallmatrix} I & \\ & \mu^{-1} I\end{smallmatrix}\right)\right)=1,$$

since δ is a $p-$adic integer. This implies that $|\mu|\geq 1$. $\square$

By Lemma 2.7.41, we can further write the integral as

$$\begin{aligned} I_p(v_0,s) &= f_{p,s}(1)\sum_{n\geq 0} p^{n(2-s)}\mathcal{S}_p\left(\left(\begin{smallmatrix} I & \\ & p^{-n} I\end{smallmatrix}\right)\cdot v_0\right) \\ &= f_{p,s}(1)\sum_{n\geq 0} p^{n(2-s)}\mathcal{S}_p\left(\left(\begin{smallmatrix} ^{n}I & \\ & I\end{smallmatrix}\right)\cdot v_0\right), \end{aligned}$$

where in the latter we have used that the central character of π is assumed to be trivial. From now on, the following two subsections will be devoted to finding an explicit expression for $\mathcal{S}_p\left(\left(\begin{smallmatrix} p^n I & \\ & I\end{smallmatrix}\right)\cdot v_0\right)$ and then relate $I_p(v_0,s)$ with the $L-$function $L_p(s,\pi_p,\Lambda_t)$. Since by proposition 2.2.5 the group $\mathbf{G}(\mathbb{Q}_p)$ has a different behaviour depending if p is split or non split, we divide the computation of our functional.

2.7.6 The split prime: Unramified computation

In what follows, we explicitly calculate $I_p(v_0,s)$ when p is an unramified prime for π which splits in E.

Factorization of the standard L-function

We recall that, according to theorem 2.6.14, we get

$$L_p(s,\pi_p,\Lambda_t^2)=L_p(s,\sigma_p,\mathrm{std})(1-p^{-s})^{-1},$$

where σ_p is an irreducible unramified representation of $\mathbf{H}(\mathbb{Q}_p)$ with trivial central character such that $\theta(\pi_p)\subset\sigma_p$ and $L(s,\sigma_p,\mathrm{std})$ denotes the Euler factor associated to the 5-dimensional representation $\rho_{1,1}$ of $\mathrm{Sp}_4(\mathbb{C})={}^L\mathbf{PH}$ of highest weight e_1+e_2. Notice that the Satake parameter associated to σ_p equals to the Satake parameter of π_p and thus is equal to $\chi_{\tilde\pi}$, see 2.3.74. We recall the reader that $\tilde\pi_p$ is the unramified representation of $\mathrm{GL}_4(\mathbb{Q}_p)$ defined by π_p, see lemma 2.7.2 for further details. By remark 2.6.3, the Euler factor of the standard $L-$function satisfies the following equality

$$L_p(s,\sigma_p,\mathrm{std})=\sum_{k=0}^{\infty}\mathrm{tr}\left(\chi_{\tilde\pi}|\mathrm{Sym}^k\rho_{1,1}\right)p^{-ks},$$

where $\mathrm{tr}\left(\chi_{\tilde\pi}|\mathrm{Sym}^k\rho_{1,1}\right)$ is the character associated to the representation $\mathrm{Sym}^k\rho_{1,1}$ of $\mathrm{Sp}_4(\mathbb{C})$, evaluated at $\chi_{\tilde\pi}$. In order to give a more explicit formula for the L-function, we have to decompose it into irreducible factors the representation $\mathrm{Sym}^k\rho_{1,1}$ for every $k\geq 0$.

Lemma 2.7.42. *We have that*

$$\mathrm{Sym}^k(\rho_{1,1})=\bigoplus_{i=0}^{\lfloor k/2\rfloor}\rho_{k-2i,k-2i},$$

where $\rho_{k-2i,k-2i}$ is the irreducible representation of highest weight $(k-2i)(\alpha_1+\alpha_2)$.

Proof. The symplectic form defining Sp_4 defines a surjection

$$\mathrm{Sym}^2\rho_{1,1}\to\mathbb{C},$$

which induces a surjection, for all $k\geq 2$,

$$\mathrm{Sym}^k(\rho_{1,1})\to\mathrm{Sym}^{k-2}(\rho_{1,1}),$$

with kernel the representation $\rho_{k,k}$ of highest weight $k\alpha_1 + k\alpha_2$ (see [FH91, ex. 16.11, p. 250]). Therefore

$$\mathrm{Sym}^k(\rho_{1,1}) = \rho_{k,k} \oplus \mathrm{Sym}^{k-2}(\rho_{1,1}).$$

Using this formula recursively, we obtain the desired factorization

$$\mathrm{Sym}^k(\rho_{1,1}) = \bigoplus_{i=0}^{\lfloor k/2 \rfloor} \rho_{k-2i,k-2i}.$$

□

Lemma 2.7.42 implies the equality

$$L_p(s, \sigma_p, \mathrm{std}) = \sum_{k=0}^{\infty} \sum_{i=0}^{\lfloor k/2 \rfloor} \mathrm{tr}\left(A_{\chi_{\tilde{\pi}}} | \rho_{k-2i,k-2i}\right) p^{-ks}.$$

The Weyl's character formula, stated in [FH91, thm. 24.2, p. 400], provides an equality between the above $L-$function and

$$\sum_{k=0}^{\infty} \sum_{i=0}^{\lfloor k/2 \rfloor} \frac{\mathcal{A}\left(e^{\check{\rho}+(k-2i,k-2i)}\right)}{\mathcal{A}\left(e^{\check{\rho}}\right)} (\chi_{\tilde{\pi}}) p^{-ks}, \tag{2.-22}$$

where we denote $(k-2i, k-2i) := (k-2i)(\alpha_1 + \alpha_2)$.

The unramified computation at split primes

Theorem 2.7.43. *For* $\mathrm{Re}(s)$ *big enough, we have*

$$I_p(v_0, s) = L_p(s, \sigma_p, \mathrm{std}).$$

Proof. Substituting the formula of Theorem 2.7.6 in the integral, we have

$$\begin{aligned} I_p(v_0, s) &= f_{p,s}(1) \sum_{n \geq 0} p^{n(2-s)} \mathcal{S}_p(\left(\begin{smallmatrix} p^n I & \\ & I \end{smallmatrix}\right) \cdot v_0) \\ &= f_{p,s}(1)(1+p^{-1})^{-1} \sum_{n \geq 0} p^{-ns} \mathcal{A}\Big(e^{\check{\rho}+n(\check{\alpha}_1+\check{\alpha}_2)} \cdot \prod_{\alpha \in \Phi^{S+}_{\mathrm{Sp}_4}} (1 - p^{-1} e^{-\check{\alpha}})\Big)(\chi_{\tilde{\pi}}) (\mathcal{A}\left(e^{\check{\rho}}\right)(\chi_{\tilde{\pi}}))^{-1}, \end{aligned}$$

where we recall that $\chi_{\tilde{\pi}}$ is equal to the Satake parameter of the representation σ_p. We now examine the alternator sum appearing above. First, recall that $\Phi^{S+}_{\mathrm{Sp}_4}$ consists of the roots $\alpha_1 - \alpha_2$ and $\alpha_1 + \alpha_2$. Futher, we have $\check{\rho} = 2\check{\alpha}_1 + \check{\alpha}_2$. Then, the term $\mathcal{A}(e^{\check{\rho}+n(\check{\alpha}_1+\check{\alpha}_2)} \cdot \prod_{\alpha \in \Phi^{S+}_{\mathrm{Sp}_4}} (1 - p^{-1} e^{-\check{\alpha}}))$ equals

$$\mathcal{A}(e^{(n+2)\check{\alpha}_1+(n+1)\check{\alpha}_2} - p^{-1} e^{(n+1)\check{\alpha}_1+n\check{\alpha}_2} - p^{-1} e^{(n+1)\check{\alpha}_1+(n+2)\check{\alpha}_2} + p^{-2} e^{n\check{\alpha}_1+(n+1)\check{\alpha}_2}).$$

Now, let $s_1 \in \Gamma$ be the simple Weyl element associated to the root $\alpha_1 - \alpha_2$; it acts on the alternator as multiplication by -1, i.e.

$$\mathcal{A}(e^{s_1 \cdot \mu}) = -\mathcal{A}(e^{\mu}). \tag{2.-22}$$

Applying it to (2.7.6), it rearranges the sum as

$$(1+p^{-1})\mathcal{A}(e^{(n+2)\check{e}_1+(n+1)\check{e}_2} - p^{-1} e^{(n+1)\check{e}_1+n\check{e}_2}).$$

Substituting (2.7.6) in the integral and using the definition of $\check{\rho}$, yields

$$I_p(v_0, s) = f_{p,s}(1) \sum_{n \geq 0} p^{-ns} \frac{\mathcal{A}(e^{\check{\rho}+n\check{e}_1+n\check{e}_2} - p^{-1} e^{\check{\rho}+(n-1)\check{e}_1+(n-1)\check{e}_2})}{\mathcal{A}\left(e^{\check{\rho}}\right)} (\chi_{\tilde{\pi}}).$$

For every $n \geq -1$, denote the term $a_n := \frac{\mathcal{A}(e^{\check{\rho}+n\check{e}_1+n\check{e}_2})}{\mathcal{A}(e^{\check{\rho}})}(A_{\tilde{\pi}_p})$; notice that for $n=-1$ we have that $a_{-1} = \frac{\mathcal{A}(e^{\check{e}_1})}{\mathcal{A}(e^{\check{\rho}})}(A_{\tilde{\pi}_p})$. Let $s_2 \in \Gamma$ be the Weyl element associated to the root $2e_2$, it acts on the alternator as in (2.7.6). Since $\check{e}_1$ is invariant under s_2 we conclude that $a_{-1}=0$.

Recall that

$$f_{p,s}(1) = \zeta_p(s+1)\zeta_p(2s).$$

Using the Cauchy product formula, we have

$$\begin{aligned} I_p(v_0,s) &= \zeta_p(s+1)\sum_{n\geq 0} p^{-2ns} \cdot \sum_{n\geq 0} p^{-ns}(a_n - p^{-1}a_{n-1}) \\ &= \zeta_p(s+1)\sum_{n\geq 0}\Big(\sum_{k=0}^{\lfloor n/2\rfloor}(a_{n-2k}-p^{-1}a_{n-1-2k})\Big)p^{-ns} \\ &= \zeta_p(s+1)\sum_{n\geq 0}\Big(\sum_{k=0}^{\lfloor n/2\rfloor}a_{n-2k} - p^{-1}\sum_{k=0}^{\lfloor (n-1)/2\rfloor}a_{n-1-2k}\Big)p^{-ns}, \end{aligned}$$

where for the latter equality we have used that $a_{-1}=0$. This expression is terribly close to the one of (2.7.6). Indeed, a simple manipulation of the series above gives that

$$\begin{aligned} I_p(v_0,s) &= \zeta_p(s+1)\left(\sum_{n\geq 0}p^{-ns}\sum_{k=0}^{\lfloor n/2\rfloor}a_{n-2k} - p^{-1}\sum_{n\geq 0}p^{-ns}\sum_{k=0}^{\lfloor (n-1)/2\rfloor}a_{n-1-2k}\right) \\ &= \zeta_p(s+1)\left(\sum_{n\geq 0}p^{-ns}\sum_{k=0}^{\lfloor n/2\rfloor}a_{n-2k} - p^{-1-s}\sum_{n\geq 0}p^{-(n-1)s}\sum_{k=0}^{\lfloor (n-1)/2\rfloor}a_{n-1-2k}\right) \\ &= \zeta_p(s+1)(1-p^{-1-s})\sum_{n\geq 0}p^{-ns}\Big(\sum_{k=0}^{\lfloor n/2\rfloor}a_{n-2k}\Big) \\ &= \sum_{n\geq 0}\sum_{k=0}^{\lfloor n/2\rfloor}\frac{\mathcal{A}\left(e^{\check{\rho}+(n-2k,n-2k)}\right)}{\mathcal{A}\left(e^{\check{\rho}}\right)}(\chi_{\tilde{\pi}})p^{-ns} \\ &= L_p(s,\sigma_p,\mathrm{std}), \end{aligned}$$

where the last equality follows from (2.7.6). □

2.7.7 The inert prime: Unramified computation

The exterior square $L-$function

In this section we fix $\pi_p = I_{\mathbf{G}}(\xi)$ and $\sigma_p = I_{\mathbf{H}}(\chi)$, both with trivial central character and so that $\theta(\pi_p) \subset \sigma_p$. We denote by $\tilde{\xi}$ the restriction of the ξ to the torus of $\mathbf{H}(\mathbb{Q}_p)$, and we also consider the representation $\tilde{\sigma}_p = I_{\mathbf{H}}(\tilde{\xi})$. The relevant $L-$function in this subsection will be 2.6.6

$$L_p(s,\tilde{\sigma}_p,\mathrm{std}),$$

see 2.6.6 for the definition. Lemma 2.7.42 implies the following equality

$$L_p(s,\tilde{\sigma}_p,\mathrm{std}) = \sum_{k=0}^{\infty}\sum_{i=0}^{\lfloor k/2\rfloor}\mathrm{tr}\left(\chi_{\tilde{\sigma}_p}|\rho_{k-2i,k-2i}\right)p^{-ks}.$$

and further, the Weyl's character formula (see [FH91, thm. 24.2, p. 400]) conclude with

$$L_p(s,\tilde{\sigma}_p,\mathrm{std}) = \sum_{k=0}^{\infty}\sum_{i=0}^{\lfloor k/2\rfloor} \frac{\mathcal{A}\left(e^{\check{\rho}+(k-2i,k-2i)}\right)}{\mathcal{A}\left(e^{\check{\rho}}\right)}(\chi_{\tilde{\sigma}_p})p^{-ks},$$

where we have denoted $(k-2i,k-2i)=(k-2i)(\alpha_1+\alpha_2)$. The above $L-$function will appear naturally applying the Casselman-Shalika formula developed in subsection 2.7.4 when we compute $I_p(v_0,s)$ with p inert. Furthermore, this $L-$function is consistent with the computation at split primes, since it is related to both $L_p(s,\sigma_p,\mathrm{std}\otimes\chi_{E_p/\mathbb{Q}_p})$ and $L_p(s,\pi_p,\Lambda_t)$. The theta correspondence is used to realize such connection. In fact, recall that the characters ξ and χ are related by

$$\begin{aligned}\xi_1 &= \chi_0\circ N_{E_p/\mathbb{Q}_p},\\ \xi_2 &= (\chi_2\chi_0)\circ N_{E_p/\mathbb{Q}_p},\\ \xi_0 &= \chi_1\chi_{E_p/\mathbb{Q}_p},\end{aligned}$$

and then, using the example 2.3.75, we see that the Satake parameter of $\tilde{\sigma}_p$ (we recall that it is a conjugacy class modulo $W_{\mathbf{H}}$) is equal to

$$\chi_{\tilde{\sigma}_p} = \begin{pmatrix}\chi_1\chi_{E_p/\mathbb{Q}_p}(p) &&& \\ & \chi_2\chi_{E_p/\mathbb{Q}_p}(p) && \\ && \chi_2^{-1}\chi_{E_p/\mathbb{Q}_p} & \\ &&& \chi_1^{-1}\chi_{E_p/\mathbb{Q}_p}\end{pmatrix} = \chi_{E_p/\mathbb{Q}_p}\chi_{\sigma_p}$$

Hence, we get

$$L_p(s,\tilde{\sigma}_p,\mathrm{std}) = L_p(s,\sigma_p,\mathrm{std}\otimes\chi_{E_p/\mathbb{Q}_p}), \tag{2.-22}$$

which, together with theorem 2.6.14, yields

$$L_p(s,\pi_p,\Lambda_t^2) = L_p(s,\tilde{\sigma}_p,\mathrm{std})(1-p^{-s})^{-1}.$$

The unramified computation at inert primes

Theorem 2.7.44. *For* $\mathrm{Re}(s)$ *big enough, we have*

$$I_p(v_0,s) = L_p(s,\sigma_p,\mathrm{std}\otimes\chi_{E_p/\mathbb{Q}_p}).$$

Proof. Recall that $I_{\mathbf{G}}(\xi)$ is irreducible; this implies, by [Cas80, Proposition 3.5],

$$C := \frac{\prod_{\alpha\in\Phi^+_{\mathbf{H}}} c_\alpha(\xi)}{e^{-\check{\rho}}\prod_{\check{\alpha}\in\Phi^{\vee,+,s}_{\mathrm{Sp}_4}}(1-p^{-1}e^{\check{\alpha}})(g_{\check{\xi}})}$$

is invertible. We then normalize the Shalika functional by multiplying our formula by $\frac{C^{-1}Q}{1+p^{-1}}$. Using theorem 2.7.37, the proof follows parallel to theorem 2.7.43. To conclude we use the equality (2.7.7). □

2.7.8 Location of poles

In what follows, we locate the possible poles of the zeta integral and relate the period introduced in the introduction to the residue of the standard L-function at $s=2$.

The Klingen Eisenstein series

Let Q denote the Klingen parabolic of $\mathbf{H}$ with Levi deomposition M_QN_Q where

$$M_Q = \{\begin{pmatrix}a && \\ & g & \\ && d\end{pmatrix},\ g\in\mathrm{GL}_2, a,d\in\mathrm{GL}_1\ :\ ad=\det(g)\}\simeq \mathrm{GL}_2\times\mathrm{GL}_1,$$

and $N_Q \simeq \mathbb{G}_a^3$.

Denote by δ_Q the modulus character of Q, given by

$$\delta_Q : \begin{pmatrix} a & & \\ & g & \\ & & d \end{pmatrix} \mapsto \left|\tfrac{a}{d}\right|^2.$$

Let $I_Q(s)$ be the degenerate principal series representation of $\mathbf{H}(\mathbb{A})$ consisting of smooth functions f'_s on $\mathbf{H}(\mathbb{A})$ such that

$$f'_s(nmg) = \delta_P^{\frac{1}{2}(s+1/2)}(m) f'_s(g),\ \forall n \in N_Q(\mathbb{A}), \forall m \in M_Q(\mathbb{A}).$$

Given a smooth section

$$f'_s \in I_Q(s),$$

we define the degenerate Eisenstein series

$$\mathrm{Eis}_Q(g,s) = \sum_{\gamma \in Q(\mathbb{Q})\backslash \mathbf{H}(\mathbb{Q})} f'_s(\gamma g).$$

In the study of the poles of our integral, we employ two facts regarding this Eisenstein series. The first one is a result of Ikeda and regards a first term identity between the Siegel and Klingen Eisenstein series (cf. [Ike96, Proposition 1.8]).

Proposition 2.7.45. *There exists a section $f'_s \in I_Q(s)$ such that $\mathrm{Eis}_Q(g,s)$ is holomorphic at $s = 1/2$ and*

$$\mathrm{Res}_{s=1} E^*_P(g,s) = \frac{\mathrm{Res}_{s=1}(\zeta^S(s))\zeta^S(3)}{2\zeta^S(2)} \mathrm{Eis}_Q(g, 1/2).$$

The second fact needed concerns the vanishing of the following integral:

$$J(\varphi, f'_s) := \int_{\mathbf{H}(\mathbb{Q})Z_{\mathbf{H}}(\mathbb{A})\backslash \mathbf{H}(\mathbb{A})} \mathrm{Eis}_Q(h,s)\varphi(h)dh, \tag{2.-22}$$

where φ is a cusp form in π and $\mathrm{Eis}_Q(g,s)$ is the Klingen Eisenstein series for $\mathbf{H}$. We have the following.

Proposition 2.7.46. *The integral $J(\varphi, f'_s)$ is identically zero.*

Proof. We start by unfolding the Eisenstein series. For big enough $\mathrm{Re}(s)$, we get

$$\int_{Q(\mathbb{Q})Z_{\mathbf{H}}(\mathbb{A})\backslash \mathbf{H}(\mathbb{A})} f'_s(h)\varphi(h)dh.$$

Collapse the sum over the unipotent radical N_Q of Q to get

$$\int_{M_Q(\mathbb{Q})N_Q(\mathbb{A})Z_{\mathbf{H}}(\mathbb{A})\backslash \mathbf{H}(\mathbb{A})} f'_s(h) \int_{N_Q(\mathbb{Q})\backslash N_Q(\mathbb{A})} \varphi(nh)dndh.$$

We now Fourier expand over $[N''/N_Q]$, with N'' being the unipotent radical of the Klingen parabolic of $\mathbf{G}$. The integral equals

$$\int_{M_Q(\mathbb{Q})N_Q(\mathbb{A})Z_{\mathbf{H}}(\mathbb{A})\backslash \mathbf{H}(\mathbb{A})} f'_s(h) \sum_{\chi} \varphi_\chi(h)dh,$$

where the sum runs over the characters $\chi : [N''/N_Q] \to \mathbb{C}^\times$ and where we have denoted

$$\varphi_\chi(h) = \int_{N''(\mathbb{Q})\backslash N''(\mathbb{A})} \varphi(n''h)\chi^{-1}(n'')dn''.$$

Any character $\chi : [N''/N_Q] \to \mathbb{C}^\times$ can be described as follows. Recall that any character $\chi : [N''/[N'', N'']] \to \mathbb{C}^\times$ is of the form $n'' \mapsto \psi(\mathrm{Tr}_{E/\mathbb{Q}}(\alpha x + \beta y))$, where $\alpha, \beta \in E$, ψ is a non-trivial additive character on

$\mathbb{Q}\backslash\mathbb{A}$, and where we have written

$$n'' \equiv \begin{pmatrix} 1 & x & y & 0 \\ & 1 & & \bar{y} \\ & & 1 & -\bar{x} \\ & & & 1 \end{pmatrix}, \quad \text{mod } [N'', N''] \ \text{ with } x, y \in \mathbb{A}_E.$$

Such a character χ is trivial on $N_Q(\mathbb{A})$ if $\alpha + \bar{\alpha} = \beta + \bar{\beta} = 0$.

We deduce that any character $\chi : [N''/N_Q] \to \mathbb{C}^\times$ is of the form

$$n'' \mapsto \psi(\mathrm{Tr}_{E/\mathbb{Q}}(\alpha x + \beta y)),$$

with $\alpha, \beta \in E$ such that $\bar{\alpha} = -\alpha$ and $\bar{\beta} = -\beta$. To remark the dependence on these data, denote such a character by $\chi_{\alpha,\beta}$.

As M_Q acts on N'' by conjugation and this action preserves N_Q, $M_Q(\mathbb{Q})$ acts on the space of characters of $[N''/N_Q]$. Indeed, if $m = \begin{pmatrix} a & & \\ & g & \\ & & d \end{pmatrix} \in M_Q(\mathbb{Q})$ and $g^{-1} = (\tilde{g}_{i,j})$,

$$\begin{aligned} \chi_{\alpha,\beta}(mn'm^{-1}) &= \psi(a \cdot \mathrm{Tr}_{E/\mathbb{Q}}(\alpha(\tilde{g}_{1,1}x + \tilde{g}_{2,1}y) + \beta(\tilde{g}_{1,2}x + \tilde{g}_{2,2}y))) \\ &= \psi(\mathrm{Tr}_{E/\mathbb{Q}}(a(\alpha\tilde{g}_{1,1} + \beta\tilde{g}_{1,2})x + a(\alpha\tilde{g}_{2,1} + \tilde{g}_{2,2})y)) \\ &= \chi_{\alpha',\beta'}(n'), \end{aligned}$$

with $\alpha' = a(\alpha\tilde{g}_{1,1} + \beta\tilde{g}_{1,2})$ and $\beta' = a(\alpha\tilde{g}_{2,1} + \tilde{g}_{2,2})$. Then, $M_Q(\mathbb{Q})$ acts on the space of characters of $[N''/N_Q]$ with two orbits, the trivial one and an open one. A representative of the open orbit is given by the character $\chi_{\alpha,0}$ with $\bar{\alpha} = -\alpha \neq 0$. Using this action, we can write the integral as

$$\int_{M_Q(\mathbb{Q})N_Q(\mathbb{A})Z_{\mathbf{H}}(\mathbb{A})\backslash\mathbf{H}(\mathbb{A})} f_s'(h) \int_{[N'']} \varphi(n''h)dn''dh + \int_{M_\alpha(\mathbb{Q})N_Q(\mathbb{A})Z_{\mathbf{H}}(\mathbb{A})\backslash\mathbf{H}(\mathbb{A})} f_s'(h)\varphi_{\chi_{\alpha,0}}(h)dh,$$

where $M_\alpha(\mathbb{Q})$ denotes the stabiliser in $M_Q(\mathbb{Q})$ of $\chi_{\alpha,0}$. From the explicit description above, notice that

$$M_\alpha(\mathbb{Q}) = \left\{ \begin{pmatrix} a & & & \\ & a & b & \\ & & d & \\ & & & d \end{pmatrix} \right\} = L_\alpha(\mathbb{Q})N_\alpha(\mathbb{Q}),$$

where L_α (resp. N_α) denotes its Levi part (resp. unipotent part).

The first integral of (2.7.5) vanishes because of cuspidality of φ along $[N'']$, hence

$$J(\varphi, f_s') = \int_{M_\alpha(\mathbb{Q})N_Q(\mathbb{A})Z_{\mathbf{H}}(\mathbb{A})\backslash\mathbf{H}(\mathbb{A})} f_s'(h)\varphi_{\chi_{\alpha,0}}(h)dh.$$

Collapse the sum over $[N_\alpha]$ to get

$$J(\varphi, f_s') = \int_{L_\alpha(\mathbb{Q})N_\alpha(\mathbb{A})N_Q(\mathbb{A})Z_{\mathbf{H}}\backslash\mathbf{H}(\mathbb{A})} f_s'(h) \int_{[N_\alpha]} \varphi_{\chi_{\alpha,0}}(nh)dndh.$$

Notice that the inner integral $\int_{[N_\alpha]} \varphi_{\chi_{\alpha,0}}(nh)dn$ equals the Fourier coefficient of φ with respect to the unipotent radical of the Borel subgroup of $\mathbf{G}$ and the choice of a character on it which is degenerate (as it's only supported on the entry (1,2)). As a consequence, $\int_{[N_\alpha]} \varphi_{\chi_{\alpha,0}}(nh)dn$ contains as an inner integral the period of φ over the unipotent parabolic of the Siegel parabolic of $\mathbf{G}$, which vanishes by cuspidality of φ. This completes the proof. □

Periods and residues

Theorem 2.7.47. *The integral $I(\varphi, s)$ is holomorphic for all s except two. In particular,* $\mathrm{Res}_{s=2} I(\varphi, s) = \mathcal{P}_{\mathbf{H}}(\varphi)$, *where*

$$\mathcal{P}_{\mathbf{H}}(\varphi) = \int_{\mathbf{H}(\mathbb{Q})Z_{\mathbf{H}}(\mathbb{A})\backslash \mathbf{H}(\mathbb{A})} \varphi(h)dh.$$

Proof. Recall that the normalised Eisenstein series $E_P^*(g, s)$ achieves poles at $s = 1, 2$ of order at most one, thus $I(\varphi, s)$ may have a pole at these two values.

At $s = 1$, the residue of the integral equals to

$$\begin{aligned}
\mathrm{Res}_{s=1} I(\varphi, s) &= \mathrm{Res}_{s=1} \int_{\mathbf{H}(\mathbb{Q})Z_{\mathbf{H}}(\mathbb{A})\backslash \mathbf{H}(\mathbb{A})} E_P^*(h, s)\varphi(h)dh \\
&= \int_{\mathbf{H}(\mathbb{Q})Z_{\mathbf{H}}(\mathbb{A})\backslash \mathbf{H}(\mathbb{A})} \mathrm{Res}_{s=1} E_P^*(h, s)\varphi(h)dh \\
&= C \cdot \int_{\mathbf{H}(\mathbb{Q})Z_{\mathbf{H}}(\mathbb{A})\backslash \mathbf{H}(\mathbb{A})} \mathrm{Eis}_Q(g, 1/2)\varphi(h)dh \\
&= C \cdot J(\varphi, f'_{1/2}),
\end{aligned}$$

where the third equality follows from Proposition 2.7.45 and $C = \frac{\mathrm{Res}_{s=1}(\zeta^S(s))\zeta^S(3)}{2\zeta^S(2)}$. The latter integral vanishes by Proposition 2.7.46, hence $I(\varphi, s)$ is holomorphic at $s = 1$.

At $s = 2$, the picture is different. Indeed by [KR94, prop. 5.4.1, p. 48], there exists $f_s \in I_P(s)$ such that $\mathrm{Res}_{s=2} E_P^*(g, s) = 1$ therefore

$$\begin{aligned}
\mathrm{Res}_{s=2} I(\varphi, s) &= \int_{\mathbf{H}(\mathbb{Q})Z_{\mathbf{H}}(\mathbb{A})\backslash \mathbf{H}(\mathbb{A})} \mathrm{Res}_{s=2} E_P^*(h, s)\varphi(h)dh \\
&= \int_{\mathbf{H}(\mathbb{Q})Z_{\mathbf{H}}(\mathbb{A})\backslash \mathbf{H}(\mathbb{A})} \varphi(h)dh.
\end{aligned}$$

Hence, $I(\varphi, s)$ achieves a pole at $s = 2$ if and only if $\mathcal{P}_{\mathbf{H}}(\varphi) \neq 0$. □

Theorem 2.7.48. *Let π be an irreducible cuspidal automorphic representation of $\mathbf{G}(\mathbb{A})$ with trivial central character. Suppose that $\pi = \widetilde{\Theta}^{glob}(\sigma^+)$, with σ^+ a subrepresentation of $\sigma|_{\mathbf{H}^+}$ where σ is a generic cuspidal automorphic representation of $\mathbf{H}$ with trivial central character. We have*

$$I(\varphi, s) = I_S(s) L^S(s, \sigma, \mathrm{std} \otimes \chi_{E/\mathbb{Q}}),$$

where S is a finite set of primes containing the ramified primes for π, σ, and $E/\mathbb{Q}$, and $I_S(s)$ is the integral over the places in S and infinity.

Proof. It follows by theorems 2.7.43 and 2.7.44. □

For the next corollary we assume $I_S(s) \neq 0$.

Corollary 2.7.49. *The wedge square $L-$function $L(s, \pi, \Lambda_t)$ is holomorphic at $s = 2$.*

Proof. By essentially [CT], $\mathcal{P}_{\mathbf{H}}(\varphi) = 0$ and hence by the previous theorem and theorems 2.7.43 and 2.7.44 the $L-$function is holomorphic at $s = 2$. □

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