Degenerate invariant tori in KAM theory

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Degenerate Invariant Tori in KAM Theory

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Contents

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Abstract

The project consists in studying bifurcations of invariant tori for one-dimensional dynamical systems under external quasi–periodic forcing. The (extended) phase space is a bundle whose base is a torus of dimension d, and the real-line is the fiber. The systems themselves are bundle maps over translations on the torus with d frequencies.
The methodology involves KAM theory, bifurcation theory, modifying term techniques and translated curve theorems (in the spirit of Moser, Rüßmann, Herman, Delshams and Ortega).
The goal of the project is to obtain rigorous results in an a posteriori format for the existence of families of translated tori, and establishing a methodology for studying the bifurcations of invariant tori. The a posteriori format is suitable for numerical and rigorous computations.

Introduction

The aim of this work is to understand the general dynamics of quasi–periodic forced skew–products in which the phase space is a bundle whose base is the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and the real line is the fiber:

$$\begin{array}{rccc} \Psi: & \mathbb{T}\times\mathbb{R} & \longrightarrow & \mathbb{T}\times\mathbb{R} \\ & (\theta, x) & \longmapsto & \Psi(\theta, x) = (\theta+\omega, f(\theta, x)) \end{array},$$

where $\mathcal{R}_\omega(\theta) = \theta + \omega$ is an ergodic rigid rotation with a Diophantine frequency $\omega \in \mathcal{DC}(\gamma, \nu)$ and f is analytic.

Due to the irrationality of the frequency ω there are no fixed points nor periodic orbits. Thus, the first part of the work is devoted to obtain sufficient conditions to find invariant tori, proving the existence and the analyticity of these invariant curves[1] by means of an iterative procedure whose methodology involves KAM theory[2]. In addition to the aforementioned non-resonance condition, certain non-degeneracy conditions are necessary to guarantee the constructability of the procedure and their convergence. In the context of analytic functions, formal solutions to the invariance equation, expressed by means of their Fourier series expansions, can be find. The convergence of these Fourier series leads to the problem of small divisors and cohomological equations. The constructability of the formal Fourier series satisfying the cohomological equation is feasible whenever the frequency is an irrational number. In spite os this fact, there is a set of Fourier coefficients with small denominators whose effect can lead to the non-convergence of the series or, in the case of being convergent, its regularity may not be guaranteed. These problem was resolved by the founders of KAM theory. There are different ways to prove the existence of invariant curves in these kind of skew–products (Hermann, De la Llave, Haro, Fontich, and others), even in the context of non–analytic functions (e.g. R. Ortega, Invariant curves for skew product diffeomorphisms, Milano September 1999). Regarding cohomological equations, Rüßmann ([49],[51],[50],[48]) provided sharp estimates for the solutions to these kind of difference equations, demanding the Diophantine character of the frequency. We give in Chapter 1 a detailed version of their estimates, adapted to the framework that concerns us (see Theorem 1.20).

In the spirit of Delshams and De la Llave ([16]), we adopt then the so–called translated graph method, also adapted to our framework. This method consists essentially of fixing an average $p \in \mathbb{R}$ (in addition to the frequency ω, which is also fixed a priori) and finding p–average invariant translated curves following a Newton–like iterative procedure.

In Chapter 2, before dealing with the translated graph method and the KAM procedure, we

[1]Briefly speaking, curves of the form $\kappa : \mathbb{T} \longrightarrow \mathbb{R}$ such that the invariance equation, $f(\theta, \kappa(\theta)) = \kappa(\theta + \omega)$ is satisfied.

[2]The acronym KAM stands for Kolmogorov [39] (1954), Arnold [3] (1961) and Moser [43] (1962), the founders of the theory. Other recent contributions: [14], [15] , [46], [7], [24], [27].

discuss the concept of reducibility of a skew-product, which is essential in the construction of the process and we show, as it is well known, that every non–singular one–dimensional linear quasi–periodic skew–product is reducible (see Theorem 2.6). This fact, and the explicit expression of the Floquet transformation that relates the reducibility with the Lyapunov exponents by means of the cohomological operator will be used later to simplify some computations and obtain important dynamical properties. Nevertheless, the first attempt to build the KAM iterative procedure elapses simultaneously combining two procedures. On the one hand, the invariance of the translated curve and, on the other hand, the reducibility of the skew–product without using the aforementioned explicit expression of the reducibility function (Floquet transformation).
Henceforth, our challenge consists, briefly speaking, in proving the following result:

> *If we have a good enough approximation of a translated invariant curve, then under certain non–degeneracy and non–resonance conditions, there exists a true invariant translated curve nearby.*

In Section 2.4 the whole process is performed, starting with the non–degeneracy conditions needed for one step, the corresponding estimates (Lemma 2.16), the iterative lemma (Lemma 2.18), and finally, the KAM theorem (Theorem 2.19), in which the convergence and the analyticity of the solutions are proved. This is one of the most important results obtained in this work.
Chapter 3 is devoted to showing the translated graph method with a slightly different approach. In this case the reducibility of the skew–product is taken for granted, and we use, at every step of the process, the expression of the Floquet transformation obtained in Theorem 2.6. Consequently, the part of the process described in Section 2.4 corresponding to the reducibility is avoided. Moreover, the average of the reducibility function does not need to be the same value at each step. In fact, it can be chosen freely at every step. This allows to reduce the obtained error estimates. Another difference that is taken in account here is that the spatial partial derivative of the function which describes the skew–product is assumed to be bounded from below (in modulus). This restriction allows to assure that the Lyapunov exponents obtained at every step are globally bounded. Under these conditions, we can obtain sharp explicit estimates for one step and for the corrections generated along the whole iterative procedure (Lemma 3.5 and Lemma 3.9).
These expressions can be significantly simplified by assuming some a priori conditions that do not detract from the generality of the approach. For instance, the first guess of the procedure can be a p–average curve where p is the value previously fixed, and then all the average errors given by $e_n(p) = <\kappa_n> - p$ $(n \in \mathbb{N})$ vanish. Moreover, since the average of the n–th Floquet transformation $c_{n,0} = \int_{\mathbb{T}} c_n(\theta)d\theta$ can be freely chosen at every step, we opt to take its value such that

$$\alpha_n = \frac{c_{n,0}}{\displaystyle\int_{\mathbb{T}} e^{\Re_1\left(\log\left(\frac{\partial f}{\partial x}(\theta,\kappa_n(\theta)) - \Lambda_n\right)\right)} d\theta} = 1 \; (n \in \mathbb{N}).$$

This is what we do in Section 3.6, obtaining as a result a narrow version of the aforementioned estimates. With these new bounds, we are in a position to state another version of the KAM theorem (see Theorem 3.15).

It is worth mentioning that there are examples of skew product systems without invariant curves. In [2] the authors (Alsedà et al.) construct an example in a domain $\Omega \subset \mathbb{S}^1 \times \mathbb{R}$ limited by an upper and a lower circle that are permuted by the map and do not have any invariant curve in its interior.

Delshams and Ortega [17] provided the existence of translated curves for quasi-periodically forced maps which was established, under very mild regularity hypotheses, for rotation numbers of constant type. Among the translated curves, the invariant curves are characterized as the solutions of an scalar bifurcation equation, from which their existence, stability as well as bifurcation can be easily described.

Once the existence of invariant curves has been determined, under the non–degeneracy conditions required by the translated graph method described above, the next objective consists in the establishment of a methodology to study the local bifurcation theory of one parametric families of skew–products as they were considered previously. The approach starts with a family of invariant translated curves $\{(\kappa(\cdot;\mu,p),\tau(\mu,p))\}_{(\mu,p)\in\mathcal{I}\times\mathbb{R}}$, i.e. solutions of a system of the form

$$\left\{\begin{array}{rcl} f(\theta,\kappa(\theta;\mu,p);\mu)-\kappa(\theta+\omega;\mu,p)+\tau(\mu,p) & = & 0\,, \\ <\kappa(\cdot\,;\mu,p)> & = & p\,. \end{array}\right. \qquad (\theta\in\mathbb{T};\mu\in\mathcal{I},\ p\in\mathbb{R})$$

In this scenario we are mostly interested in the study of qualitative geometric properties of the family of invariant curves, namely, those invariant translated curves whose translation parameter is equal to zero, i.e.

$$\tau(\mu,p)=0.$$

Here, we have two parameters: on the one hand, the bifurcation parameter, $\mu\in\mathcal{I}\subseteq\mathbb{R}$; and, on the other hand, the average parameter, $p\in\mathbb{R}$.
The implicit function theorem (IFT) provides the appropriate framework for this study, through sufficient conditions that allow information to be obtained from one of the parameters as a function of the other. This is the content of Chapter 4.
The approach focuses specifically on the study of several concrete types of bifurcations: saddle–node or fold bifurcations (Section 4.2), transcritical and pitchfork bifurcations (Section 4.3), and period–doubling or flip bifurcations (Section 4.4).

Another objective of this thesis is to implement numerical procedures that allow validating the theoretical results, as well as reinforcing the numerical results obtained by applying them to specific examples. To achieve these objectives, it is necessary to implement programs that allow the reproduction of the KAM procedures described above. Among others, the study of the numerical representation of functions defined by their Fourier coefficients is required, for which the most efficient tool is the Discrete Fourier Transform (DFT) and its inverse (IDFT).
Chapter 5, Chapter 6 and Appendix II are devoted to this purpose.

General summary of the thesis

Chapter 1.

This chapter is dedicated to introducing cohomological equations and the spaces of real analytic functions, the problem of small divisors and how the Diophantine condition affects the convergence and analyticity of the solutions to these cohomological equations. Some properties of these functions are discussed. Among them their Fourier expansions, Cauchy's inequality, the exponential decay of their Fourier coefficients and the uniform convergence. Next, there is a brief presentation of the Diophantine condition which leads to the important lemma of small divisors. The most important result of this chapter, due to Rüßmann [51], is stated in Theorem 1.20, showing what we call Rüßmann estimates for the solutions to the cohomological equation. The complete proof provided here is adapted, in all of its terms, to the one dimensional frame. The chapter concludes with the definition of the cohomological operator and a detailed description of its properties. Among them, it is remarkable that Proposition 1.26 shows two slightly different ways to estimate cohomological operator corrections, one of them sharper than the one obtained by applying the Rüßmann estimates twice. Some references related to this chapter are [49], [51], [50], [10], [11], [14], [31], and [18].

Chapter 2.

In this chapter we face up to one of the main objectives of this work. Our challenge is to design a KAM procedure to demonstrate the existence of invariant curves for one–dimensional quasi–periodic skew–products under certain non–degeneracy conditions. We will use the translated graph method for the very particular frame in which the base is the torus, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, and the fiber is the real line, $\mathbb{R}$, giving sufficient conditions for which the Newton–like method to be designed converges quadratically, and thus formulate them in a posteriori format. The challenge, on the one hand, is to fertilize the land for the creation of a methodology for the study and classification of bifurcations of invariant curves related to perturbations of this kind of skew–products, and on the other hand, implement numerical methods of representation. We will employ all the tools which were described in the corresponding sections (as the invariance equation, topological and linear conjugacy of skew–products, small denominators and cohomological equations,...)[3] plus new ones (as linearization of a skew–product, reducibility, the translated graph method itself, and KAM theory).
Some references for this chapter are [24], [22], [10]

[3] See **Appendix** I. and **Chapter** 1.

Chapter 3.

This chapter is devoted to showing the translated graph method with a slightly different approach. In this case the reducibility of the skew–product is taken for granted, and we use, at every step of the process, the expression of the Floquet transformation obtained in Theorem 2.6. Consequently, the part of the process described in Section 2.4 corresponding to the reducibility is avoided. Moreover, the average of the reducibility function does not need to be the same value at each step. In fact, it can be chosen freely at every step. This allows to reduce the obtained error estimates. Another difference that is taken in account here is that the spatial partial derivative of the function which describes the skew–product is assumed to be bounded from below (in modulus). This restriction allows to assure that the Lyapunov exponents obtained at every step are globally bounded. Under these conditions, we can obtain sharp explicit estimates for one step and for the corrections generated along the whole iterative procedure (Lemma 3.5 and Lemma 3.9).
These expressions can be significantly simplified by assuming some a priori conditions that do not detract from the generality of the approach. For instance, the first guess of the procedure can be a p–average curve where p is the value previously fixed, and then all the average errors given by $e_n(p) =< \kappa_n > -p$ $(n \in \mathbb{N})$ vanish. Moreover, since the average of the n–th Floquet transformation $c_{n,0} = \int_{\mathbb{T}} c_n(\theta)d\theta$ can be freely chosen at every step, we opt to take its value such that

$$\alpha_n = \frac{c_{n,0}}{\displaystyle\int_{\mathbb{T}} e^{\Re_1\left(\log\left(\frac{\partial f}{\partial x}(\theta,\kappa_n(\theta))-\Lambda_n\right)\right)} d\theta} = 1 \; (n \in \mathbb{N}).$$

This is what we do in Section 3.6, obtaining as a result a narrow version of the aforementioned estimates. With these new bounds, we are in a position to state another version of the KAM theorem (see Theorem 3.15).

Section 3.1 reproduces the translation graph method with the reducibility of the skew–product taken for granted.
Section 3.2 speaks about the details on the non–degeneracy condition in one step of the KAM procedure.
Section 3.3 gives the expresion of the invariance error produced in one step.
Section 3.4 and Section 3.5 show the explicit form of the estimates and correction estimates obtained for one step in the most general and sharp way.
In Section 3.6 we come out with some reductions to obtain explicit estimates with simpler expressions although less precise.
Section 3.7 shows a new version of the KAM theorem for skew–products bases on the estimates obtained in the above section.
Some references for this chpther: [36], [10], [24], [22].

Chapter 4.

This chapter establishes a methodology to study the local bifurcation theory of one parametric families of skew–products as they were considered in previous chapters. The approach starts with a family of invariant translated curves $\{(\kappa(\cdot;\mu,p),\tau(\mu,p))\}_{(\mu,p)\in\mathcal{I}\times\mathbb{R}}$, i.e. solutions of a system of the form

$$\left\{ \begin{array}{rcl} f(\theta,\kappa(\theta;\mu,p);\mu) - \kappa(\theta+\omega;\mu,p) + \tau(\mu,p) & = & 0 \, , \\ < \kappa(\cdot\,;\mu,p) > & = & p \, . \end{array} \right. \qquad (\theta \in \mathbb{T};\mu \in \mathcal{I},\ p \in \mathbb{R})$$

In this scenario we are mostly interested in the study of qualitative geometric properties of the family of invariant curves, namely, those invariant translated curves whose translation parameter is equal to zero, i.e.

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Here, we have two parameters: on the one hand, the bifurcation parameter, $\mu \in \mathcal{I} \subseteq \mathbb{R}$; and, on the other hand, the average parameter, $p \in \mathbb{R}$.

The implicit function theorem (IFT) provides the appropriate framework for this study, through sufficient conditions that allow information to be obtained from one of the parameters as a function of the other.

This approach focuses specifically on the study of several concrete types of bifurcations: saddle–node or fold bifurcations (Section 4.2), Trancritical and pitchfork bifurcations (Section 4.3), and period–doubling or flip bifurcations (Section 4.4).

Chapter 5.

The fundamental objective of this chapter is the introduction of all those concepts necessary for the numerical implementation of the procedures described in the previous chapters[4].

Section 5.2 is devoted to introduce the discrete Fourier transform (DFT) and its inverse (IDFT), definitions and some of those properties which will be used later on in the computations. These tools constitute an efficient way to compute functions given by their Fourier series expansion on the torus. In Section 5.3 we introduce a method to compute numerically the Fourier coefficients of a function by means of the DFT, providing moreover an estimate of the error made in the aforementioned approximation. With a finite collection of Fourier coefficients it is possible to reconstruct, by means of the convolution with the Dirichlet kernel, the partial sums of the Fourier series. This is explained in Section 5.4. Moreover, there is an efficient way to reconstruct functions from their Fourier coefficients employing the inverse discrete Fourier transform (IDFT). Once we have solved the problem of the numerical implementation for Fourier series and Fourier coefficients, we are in a position to solve cohomological equations and, as a particular case, to compute the Floquet transformation of a given curve, which is necessary in the reducibility process of skew–products. These aspects will be dealt with in the last section of the chapter, Section 5.6.

Chapter 6.

In this section, the model presented will serve as a support to develop all the algorithms built in previous sections and chapters, as well as their subsequent numerical implementation. This model, with slight differences, was presented by Tobias H. Jäger in 2003 [34] and has been deeply analyzed by Àngel Jorba, Francisco Javier Muñoz–Almaraz, and Joan Carles Tatjer in 2018 [36]. Here we expose an extended version complexified with the aim of adapting the model to the previous exposition.

[4] Mainly, cohomological equations and the derived computation of the Floquet transformation of a curve. Subject to these procedures everything is related to the numerical implementation of Fourier series and the Fourier coefficients of a function.

Appendix I.

This appendix lays the foundations in a more general context of the background in which the entire thesis is focused. It is devoted to introduce the notion of skew–product and some general definitions related with this smattering. In particular, the concept of invariant section. First, there is a description of fiber and vector bundles, bundle maps and vector bundle maps, and the concept of cross sections over fiber bundles and vector bundles. For a more complete account on these topics, we refer the reader to [55], [32], [47], [1], and [33].
In this context, it is introduced the definition of skew–product dynamical system, giving rise to the concept of invariant section which justifies what is called invariance equation, and is the basis to understand what are invariant tori in skew–products. Additionally, the concept of invertibility in skew-products is described and, above all, topological conjugacy and linear conjugacy of skew-products.

Appendix II.

Based on the numerical aspects developed in Chapter 5 this appendix shows some of the algorithms which are implemented in Matlab® programming environment (R2022b).

- Orbits and the error function;
- Complex Fourier series estimates by means of the Discrete Fourier Transform (DFT);
- The cohomological operator; and
- The KAM step.

Appendix III.

This appendix provides an important matrix lemma (Lemma III.1) which is often used to obtain several estimates needed in the KAM procedure (see Lemma 2.18).

Chapter 1

Cohomological equation and Rüßmann estimates

The first part of this introductory chapter (**Section 1.1**) is devoted to introducing the spaces of real analytic periodic functions defined in a complex strip that will appear throughout the upcoming exposition. This implies the fact of precisely defining topological aspects and other features of these spaces.
Before facing up the main problem (I.9), we still need some more background.

1.1 Analytic periodic functions

Definition 1.1 Complex strip

We define the complex strip of width $\varrho > 0$ as the set

$$\mathbb{T}_\varrho = \{z \in \mathbb{C}/\mathbb{Z} : |\operatorname{Im}(z)| \leq \varrho\} .$$

◇

Definition 1.2 Analytic periodic functions

Let $\varrho > 0$. We define the set of analytic periodic functions *on the complex strip $\mathbb{T}_\varrho$ as:*

$$\begin{aligned} \mathcal{A}_\varrho \quad = \quad & \{u : \mathbb{T}_\varrho \subseteq \mathbb{C}/\mathbb{Z} \to \mathbb{C} \mid u \textit{ is analytic in } \operatorname{Int}(\mathbb{T}_\varrho), \\ & \textit{and continuously extendable to } \partial\mathbb{T}_\varrho\} . \end{aligned}$$

Moreover, we say that $u \in \mathcal{A}_\varrho$ is real analytic on $\mathbb{T}_\varrho$ if it takes real values for real arguments.

◇

Remark 1.3

We identify functions defined on $\mathbb{T}_\varrho$,

$$u : \quad \mathbb{T}_\varrho \subseteq \mathbb{C}/\mathbb{Z} \quad \longrightarrow \quad \mathbb{C}$$

with 1–periodic functions defined on the complex strip $S_\varrho = \{z \in \mathbb{C} : |\operatorname{Im}(z)| \leq \varrho\}$,

$$u : \quad S_\varrho \subseteq \mathbb{C} \quad \longrightarrow \quad \mathbb{C} .$$

Thus, we say that $u \in \mathcal{A}_\varrho$ is holomorphic (analytic) in an open set $U \subseteq \mathbb{T}_\varrho$ if the corresponding 1–periodic function defined on S_ϱ is holomorphic (analytic) in an open set $V \subseteq S_\varrho$, such that

$U = \{z + \mathbb{Z} \in \mathbb{T}_\varrho : z \in V \subseteq S_\varrho\} = V/\mathbb{Z}$. *Notice that* $\mathbb{T}_\varrho$ *is a topological space endowed with the topology inherited from the usual topology of* $\mathbb{C}$ *and* $\mathcal{A}_\varrho$ *is a* $\mathbb{C}$*–vector space. In short, we can write:* $\mathcal{A}_\varrho = \mathcal{H}(\mathrm{Int}(\mathbb{T}_\varrho)) \cap \mathcal{C}(\mathbb{T}_\varrho)$.

Definition 1.4 Pre–Hilbert structure of the space of analytic periodic functions

We define the inner product for analytic periodic functions:

$$\begin{array}{rccl} <\cdot,\cdot>: & \mathcal{A}_\varrho \times \mathcal{A}_\varrho & \longrightarrow & \mathbb{C} \\ & (u,v) & \longmapsto & <u,v>:= \displaystyle\int_{\mathbb{T}} u(\theta)\overline{v(\theta)}\, d\theta \,. \end{array}$$

This inner product endows $\mathcal{A}_\varrho$ *with a structure of a pre–Hilbert space. Furthermore,*

$$\{e^{2\pi k z i} : k \in \mathbb{Z}\}$$

is an orthonormal set in $\mathcal{A}_\varrho$.

Notice that $\mathcal{A}_\varrho$ *is a subspace of the usual Hilbert space* $L^2(\mathbb{T}_\varrho) := \{u : \mathbb{T}_\varrho \to \mathbb{C} \mid \int_{\mathbb{T}} |u(\theta)|^2 d\theta < \infty\}$, *and the inner product defined above in* $\mathcal{A}_\varrho$ *is the restriction of the inner product*

$$\begin{array}{rccl} <\cdot,\cdot>: & L^2(\mathbb{T}_\varrho) \times L^2(\mathbb{T}_\varrho) & \longrightarrow & \mathbb{C} \\ & (u,v) & \longmapsto & <u,v>:= \displaystyle\int_{\mathbb{T}} u(\theta)\overline{v(\theta)}\, d\theta \end{array}$$

to the complex strip $\mathcal{A}_\varrho$.

The L^2*–norm in* $\mathcal{A}_\varrho$ *is, accordingly, defined by:*

$$\|u\| = <u,u>^{\frac{1}{2}} = \left(\int_{\mathbb{T}} |u(\theta)|^2\, dz\right)^{\frac{1}{2}} \,,$$

whereas the supremum norm or uniform norm is defined as

$$\|u\|_\varrho = \sup_{z\in\mathbb{T}_\varrho} |u(z)| \,.$$

If $u \in \mathcal{A}_\varrho$ *for some* $\varrho > 0$, *the Fourier coefficients of* u *are defined as:*

$$\widehat{u}_k := \int_{\mathbb{T}} u(\theta) e^{-2\pi k\theta i} d\theta \quad (k \in \mathbb{Z}), \tag{1.1}$$

and the Fourier expansion of u *is the formal series*

$$\mathfrak{F}u(z) := \sum_{k\in\mathbb{Z}} \widehat{u}_k e^{2\pi k z i} \quad (z \in \mathbb{T}_\varrho). \tag{1.2}$$

◇

Remark 1.5

$(L^2(\mathbb{T}_\varrho), \|\cdot\|_\varrho)$ *is a Banach space. The uniform norm satisfies the inequality:*

$$\sum_{k\in\mathbb{Z}} |\widehat{u}_k|^2 \le \|u\|_\varrho^2, \ \forall u \in \mathcal{A}_\varrho. \tag{1.3}$$

Indeed, by Parseval's Theorem,

$$\sum_{k\in\mathbb{Z}} |\widehat{u}_k|^2 = \|u\|^2 = \int_{\mathbb{T}} |u(\theta)|^2\, d\theta \le \sup_{\theta\in\mathbb{T}} |u(\theta)|^2 \int_{\mathbb{T}} d\theta \le \sup_{z\in\mathbb{T}_\varrho} |u(z)|^2 = \|u\|_\varrho^2 \,.$$

Moreover, $(\mathcal{C}^0(\mathbb{T}_\varrho), \|\cdot\|_\varrho)$ *is also a Banach space, since it is a closed subspace of* $(L^2(\mathbb{T}_\varrho), \|\cdot\|_\varrho)$*.*
$(\mathcal{A}_\varrho), \|\cdot\|_\varrho)$ *is a Fréchet space.*
Finally, as a consequence of Morera's theorem, uniform limits on compact sets of analytic functions in an open set are analytic.

Remark 1.6

Define

$$\mathcal{A}_{\varrho,0} = \left\{ u \in \mathcal{A}_\varrho : < u >= \int_{\mathbb{T}} u(\theta)\, d\theta = 0 \right\} ,$$

that is, the subspace of A_ϱ *of that analytic functions over the complex strip* $\mathbb{T}_\varrho$ *with zero–average.*
Notice that $\mathcal{A}_{\varrho,0}$ *is a closed subspace of* $\mathcal{A}_\varrho$ *and thus it is also a Fréchet space.*
In what follows, if $u \in \mathcal{A}_\varrho$*, for some* $\varrho > 0$*, we will write:*

$$u = u_0 + \widetilde{u} ,$$

being u_0 *the average of* u *and* $\widetilde{u} \in \mathcal{A}_{\varrho,0}$ *a zero-average function, namely:*

$$u_0 =< u >:= \int_{\mathbb{T}} u(\theta)\, d\theta$$

and

$$< \widetilde{u} >= \int_{\mathbb{T}} \widetilde{u}(\theta)\, d\theta = 0 .$$

Lemma 1.7 (Cauchy inequality[1]) Fourier coefficients of an analytic 1–periodic function *If $u \in \mathcal{A}_\varrho$ is an analytic 1–periodic function defined on the complex strip $\mathbb{T}_\varrho$ for some $\varrho > 0$, then the Fourier coefficients of u,*

$$\widehat{u}_k := \int_{\mathbb{T}} u(\theta) e^{-2\pi k\theta i} d\theta \quad (k \in \mathbb{Z}), \tag{1.4}$$

satisfy the following estimate:

$$|\widehat{u}_k| \le e^{-2\pi|k|\varrho} \|u\|_\varrho \ , \ \forall k \in \mathbb{Z}, \tag{1.5}$$

where $\|u\|_\varrho = \sup_{z\in\mathbb{T}_\varrho} |u(z)|$. That is, the Fourier coefficients of an analytic 1–periodic function decay exponentially.

Proof.
Let $\delta \in (0, \varrho)$ be the height of the rectangle $R_\delta = [0, 1] \times [0, \delta] \subset \mathrm{Int}(S_\varrho)$.
Since $u \in \mathcal{A}_\varrho$ and $R_\delta \subset \mathrm{Int}(S_\varrho)$, then $u \in \mathcal{H}(R_\delta)$. Therefore, for every $k \in \mathbb{Z}$, the function

$$v_k(z) := u(z) e^{-2\pi k z i} \quad (z \in S_\varrho) \ ,$$

is holomorphic in $\mathcal{S}_\delta$ (and also 1−periodic). By Cauchy's integral theorem the contour integral of v_k along the boundary of R_δ vanishes:

$$\oint_{\partial R_\delta} v_k(z) dz = 0 \ .$$

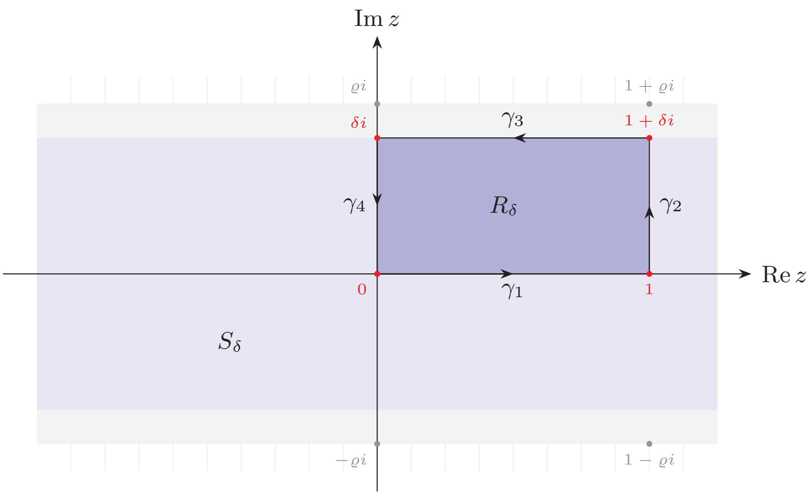

Figure 1.1: Upper integration path.

The boundary of the rectangle R_δ, counterclockwise oriented, is the juxtaposition of four line segments,

$$\partial R_\delta = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 \ ,$$

[1] cf. [14] A tutorial on KAM theory. Smooth Ergodic Theory and its Applications.

that can be parameterized, respectively, by

$$\begin{array}{rcccl} \gamma_1 : & [0,1] & \longrightarrow & \mathbb{C} & , \\ & t & \longmapsto & \gamma_1(t) = t & \end{array} \quad \begin{array}{rcccl} \gamma_2 : & [0,1] & \longrightarrow & \mathbb{C} & , \\ & t & \longmapsto & \gamma_2(t) = 1 + t\delta i & \end{array}$$

$$\begin{array}{rcccl} \gamma_3 : & [0,1] & \longrightarrow & \mathbb{C} & , \\ & t & \longmapsto & \gamma_3(t) = 1 - t + \delta i & \end{array} \quad \begin{array}{rcccl} \gamma_4 : & [0,1] & \longrightarrow & \mathbb{C} & . \\ & t & \longmapsto & \gamma_4(t) = (1-t)\delta i & \end{array}$$

Accordingly,

$$\begin{aligned} 0 = \oint_{\partial R_\delta} v_k(z)dz &= \sum_{j=1}^{4} \int_0^1 v_k(\gamma_j(t))\gamma_j'(t)dt \\ &= \int_0^1 u(t)e^{-2\pi k t i}dt + \int_0^1 u(1+t\delta i)e^{-2\pi k(1+t\delta i)i}\delta i dt \\ &+ \int_0^1 u(1-t+\delta i)e^{-2\pi k(1-t+\delta i)i}(-1)dt + \int_0^1 u((1-t)\delta i)e^{-2\pi k(1-t)\delta i^2}(-\delta i)dt \\ &= \int_0^1 u(t)e^{-2\pi k t i}dt + \delta i \int_0^1 u(1+t\delta i)e^{2\pi k t\delta}dt \\ &- e^{2\pi k\delta}\int_0^1 u(1-t+\delta i)e^{-2\pi k(1-t)i}dt - \delta i\int_0^1 u((1-t)\delta i)e^{2\pi k(1-t)\delta}dt \ , \end{aligned}$$

where we have used Euler's formula, $e^{2\pi i} = 1$. Now we show that the second integral and the fourth one are equal. On the one hand, since u is $1-$periodic,

$$\int_0^1 u(1+t\delta i)e^{2\pi k t\delta}dt = \int_0^1 u(t\delta i)e^{2\pi k t\delta}dt \ .$$

On the other hand, by means of the change of variable $\theta = 1-t$, we have

$$\int_0^1 u((1-t)\delta i)e^{2\pi k(1-t)\delta}dt = \int_0^1 u(\theta\delta i)e^{2\pi k\theta\delta}d\theta \ .$$

Thence,

$$0 = \int_0^1 u(t)e^{-2\pi k t i}dt - e^{2\pi k\delta}\int_0^1 u(1-t+\delta i)e^{-2\pi k(1-t)i}dt \ .$$

Equivalently,

$$\widehat{u}_k = e^{2\pi k\delta}\int_0^1 u(1-t+\delta i)e^{-2\pi k(1-t)i}dt \ , \forall k \in \mathbb{Z} \ ,$$

or

$$\widehat{u}_k = e^{2\pi k\delta}\int_0^1 u(t+\delta i)e^{-2\pi k t i}dt \ , \forall k \in \mathbb{Z} \quad (0 < \delta < \varrho) \ . \tag{1.6}$$

In particular, if $k < 0$,

$$\begin{aligned} |\widehat{u}_k| &= e^{-2\pi|k|\delta}\left|\int_0^1 u(t+\delta i)e^{-2\pi k(1-t)i}dt\right| \le e^{-2\pi|k|\delta}\int_0^1 |u(t+\delta i)|dt \le e^{-2\pi|k|\delta}\sup_{z\in R_\delta}|u(z)| \\ &\le e^{-2\pi|k|\delta}\sup_{z\in S_\delta}|u(z)| \le e^{-2\pi|k|\delta}\sup_{z\in S_\varrho}|u(z)| = e^{-2\pi|k|\delta}\sup_{z\in \mathbb{T}_\varrho}|u(z)| = e^{-2\pi|k|\delta}\ \|u\|_\varrho \ . \end{aligned}$$

Summarizing,

$$|\widehat{u}_k| \le e^{-2\pi|k|\delta}\ \|u\|_\varrho, \ \forall\delta \in (0,\varrho) \text{ and } \forall k \in \mathbb{Z}, \ k < 0 \ . \tag{1.7}$$

In like manner, we consider now the rectangle $R'_\delta = [0,1] \times [-\delta, 0] \subset \mathrm{Int}(\mathbb{T}_\varrho)$.

Since $u \in \mathcal{A}_\varrho$ and $R'_\delta \subset \mathrm{Int}(S_\varrho)$, then $u \in \mathcal{H}(R_\delta)$. Again, by Cauchy's integral theorem the contour integral of v_k along the boundary of R'_δ vanishes:

$$\oint_{\partial R'_\delta} v_k(z)dz = 0 \, .$$

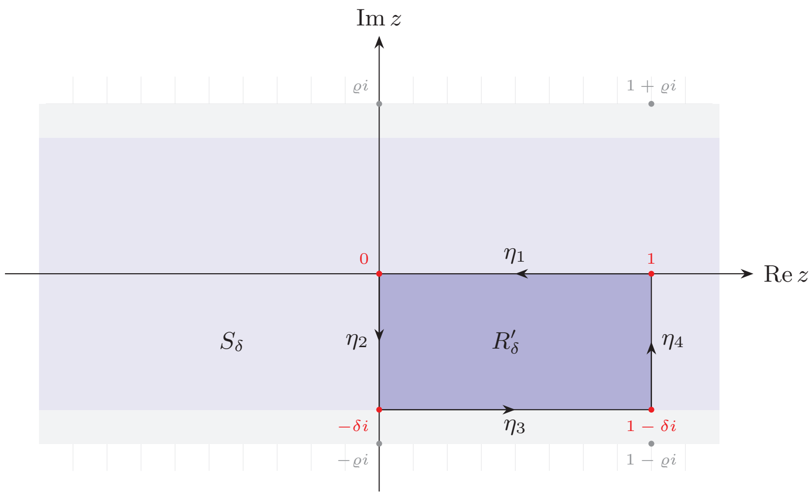

FIGURE 1.2: Lower integration path.

The boundary of the rectangle R'_δ, counterclockwise oriented, is the juxtaposition of four line segments,

$$\partial R'_\delta = \eta_1 + \eta_2 + \eta_3 + \eta_4 \, ,$$

that can be parameterized, respectively, by

$$\begin{array}{rccl} \eta_1 : & [0,1] & \longrightarrow & \mathbb{C} \\ & t & \longmapsto & \eta_1(t) = 1-t \end{array} , \quad \begin{array}{rccl} \eta_2 : & [0,1] & \longrightarrow & \mathbb{C} \\ & t & \longmapsto & \eta_2(t) = -\delta t i \end{array} ,$$
$$\begin{array}{rccl} \eta_3 : & [0,1] & \longrightarrow & \mathbb{C} \\ & t & \longmapsto & \eta_3(t) = t-\delta i \end{array} , \quad \begin{array}{rccl} \eta_4 : & [0,1] & \longrightarrow & \mathbb{C} \\ & t & \longmapsto & \eta_4(t) = 1-(1-t)\delta i \end{array} .$$

It follows that:

$$\begin{aligned} 0 = \oint_{\partial R_\delta} v_k(z)dz &= \sum_{j=1}^{4} \int_0^1 v_k(\eta_j(t))\eta_j'(t)dt \\ &= \int_0^1 u(1-t)e^{-2\pi k(1-t)i}(-1)dt + \int_0^1 u(-\delta ti)e^{-2\pi k(-\delta ti)i}(-\delta i)dt \\ &+ \int_0^1 u(t-\delta i)e^{-2\pi k(t-\delta i)i}dt + \int_0^1 u(1-(1-t)\delta i)e^{-2\pi k(1-(1-t)\delta i)i}\delta i dt \\ &= -\int_0^1 u(1-t)e^{-2\pi k(1-t)i}dt - \delta i \int_0^1 u(-\delta ti)e^{-2\pi k\delta t}dt \\ &+ e^{-2\pi k\delta} \int_0^1 u(t-\delta i)e^{-2\pi kti}dt + \delta i \int_0^1 u(1-\delta ti)e^{-2\pi k\delta t}dt \end{aligned}$$

where we have used again Euler's formula, $e^{2\pi i} = 1$. Now, the second integral and the fourth one are equal by the 1−periodicity of u. On the other hand, by means of the change of variable $\theta = 1-t$, we have

$$\int_0^1 u(1-t)e^{-2\pi k(1-t)i}dt = \int_0^1 u(\theta)e^{-2\pi k\theta i}d\theta \, .$$

Thence,

$$0 = -\int_0^1 u(t)e^{-2\pi kti}dt + e^{-2\pi k\delta}\int_0^1 u(t-\delta i)e^{-2\pi kti}dt\ .$$

Equivalently,

$$\widehat{u}_k = e^{-2\pi k\delta}\int_0^1 u(t-\delta i)e^{-2\pi kti}dt\ ,\forall k\in\mathbb{Z}\quad (0<\delta<\varrho)\ . \tag{1.8}$$

In particular, if $k > 0$,

$$\begin{aligned}
|\widehat{u}_k| &= e^{-2\pi|k|\delta}\left|\int_0^1 u(t-\delta i)e^{-2\pi kti}dt\right| \le e^{-2\pi|k|\delta}\int_0^1 |u(t-\delta i)|dt \\
&\le e^{-2\pi|k|\delta}\sup_{z\in R_\delta}|u(z)| \le e^{-2\pi|k|\delta}\sup_{z\in S_\delta}|u(z)| \\
&\le e^{-2\pi|k|\delta}\sup_{z\in S_\varrho}|u(z)| = e^{-2\pi|k|\delta}\sup_{z\in \mathbb{T}_\varrho}|u(z)| = e^{-2\pi|k|\delta}\ \|u\|_\varrho\ .
\end{aligned}$$

Summarizing,

$$|\widehat{u}_k| \le e^{-2\pi|k|\delta}\ \|u\|_\varrho,\ \forall\delta\in(0,\varrho)\text{ and }\forall k\in\mathbb{Z},\ k>0\ . \tag{1.9}$$

Finally, joining (1.7) and (1.9) we have:

$$|\widehat{u}_k| \le e^{-2\pi|k|\delta}\ \|u\|_\varrho,\ \forall\delta\in(0,\varrho)\text{ and }\forall k\in\mathbb{Z}. \tag{1.10}$$

It follows, from (1.10), that:

$$\forall k\in\mathbb{Z},\ |\widehat{u}_k| \le \inf\{e^{-2\pi|k|\delta}\ \|u\|_\varrho : 0<\delta<\varrho\} = e^{-2\pi|k|\varrho}\ \|u\|_\varrho\ ,$$

which is the estimate (1.5) that we wanted to prove. □

Lemma 1.8 [2]

Let $u\in\mathcal{A}_\varrho$ for some $\varrho>0$ and assume that there exist a positive constant $M>0$ such that $|u(z)| \le M,\ \forall z\in\mathbb{T}_\varrho$, i.e. $\|u\|_\varrho = \sup_{z\in\mathbb{T}_\varrho}|u(z)| < \infty$.

Then:

$$\sum_{k\in\mathbb{Z}}|\widehat{u}_k|^2 e^{4\pi\varrho|k|} \le 2\|u\|_\varrho^2\ , \tag{1.11}$$

where $\widehat{u}_k = \int_0^1 u(\theta)e^{-2\pi k\theta i}d\theta$ $(k\in\mathbb{Z})$ are the Fourier coefficients of u.

Proof. Let $s\in(-\varrho,\varrho)$ and define

$$\begin{array}{cccc}\varphi_s: & \mathcal{S}_{\varrho-|s|}\subseteq\mathbb{C} & \longrightarrow & \mathbb{C} \\ & z & \longmapsto & \varphi_s(z) := u(z+si)\end{array}\ .$$

Notice that whenever $z\in\mathcal{S}_{\varrho-|s|}$,

$$|\operatorname{Im}(z+si)| = |\operatorname{Im}(z)+s| \le |\operatorname{Im}(z)| + |s| \le \varrho - |s| + |s| = \varrho\ .$$

[2] cf. [49] **Lemma 2.1** p. 605 . The proof is essentially the same, but here it has been adapted to the one dimensional frame.

Hence, $z + si \in \mathcal{S}_\varrho$ and φ_s is well defined. Moreover, since $u \in \mathcal{A}_\varrho$ then $\varphi_s \in \mathcal{A}_{\varrho-|s|}$.
The Fourier coefficients of φ_s are given by:

$$\begin{aligned}\widehat{\varphi_{s}}_k &= \int_0^1 \varphi_s(\theta)e^{-2\pi k\theta i}d\theta = \int_0^1 u(\theta+si)e^{-2\pi k\theta i}d\theta \\ &= e^{-2\pi ks}\int_0^1 u(\theta+si)e^{-2\pi k(\theta+si)i}d\theta\ .\end{aligned}$$

Let us define, for each $k \in \mathbb{Z}$, the function

$$\begin{aligned}\Phi_k :\ (-\varrho,\varrho)\subseteq\mathbb{R} &\longrightarrow \mathbb{C} \\ s &\longmapsto \Phi_k(s) := e^{2\pi ks}\widehat{\varphi_{s}}_k = \int_0^1 u(\theta+si)e^{-2\pi k(\theta+si)i}d\theta\ .\end{aligned} \tag{1.12}$$

We claim that Φ_k is a constant function. Indeed, its derivative vanishes everywhere. Since u is analytic, we can differentiate under the the integral sign, that is, applying the Leibnitz integral rule:

$$\begin{aligned}\Phi_k'(s) &= \frac{d}{ds}\int_0^1 u(\theta+si)e^{-2\pi k(\theta+si)i}d\theta = \int_0^1 \frac{\partial}{\partial s}\left(u(\theta+si)e^{-2\pi k(\theta+si)i}\right)d\theta \\ &= \int_0^1\left(u'(\theta+si)ie^{-2\pi k(\theta+si)i} + u(\theta+si)e^{-2\pi k(\theta+si)i}2\pi k\right)d\theta \\ &= 2\pi k\int_0^1 u(\theta+si)e^{-2\pi k(\theta+si)i}d\theta + i\int_0^1 u'(\theta+si)e^{-2\pi k(\theta+si)i}d\theta\end{aligned}$$

Now, we integrate by parts the last term:

$$\begin{aligned}\int_0^1 u'(\theta+si)e^{-2\pi k(\theta+si)i}d\theta &= u(\theta+si)e^{-2\pi k(\theta+si)i}\Big]_{\theta=0}^{\theta=1} + 2\pi ki\int_0^1 u(\theta+si)e^{-2\pi k(\theta+si)i}d\theta \\ &= u(1+si)e^{-2\pi k(1+si)i} - u(si)e^{2\pi ks} \\ &\quad + 2\pi ki\int_0^1 u(\theta+si)e^{-2\pi k(\theta+si)i}d\theta\end{aligned}$$

and hence,

$$\begin{aligned}\Phi_k'(s) &= i\left(u(1+si)e^{-2\pi k(1+si)i} - u(si)e^{2\pi ks}\right) \\ &= ie^{2\pi ks}\left(u(1+si) - u(si)\right) = 0,\end{aligned}$$

since u is $1-$periodic.
It follows that $\Phi_k(s) = \Phi_k(0) = \int_0^1 u(\theta)e^{-2\pi k\theta i}d\theta = \hat{u}_k$, $\forall s \in (-\varrho,\varrho)$ and $k \in \mathbb{Z}$. Thus:

$$\widehat{\varphi_{s}}_k = \hat{u}_k e^{-2\pi ks}$$

and

$$\sum_{k\in\mathbb{Z}}|\widehat{\varphi_{s}}_k|^2 = \sum_{k\in\mathbb{Z}}|\hat{u}_k|^2 e^{-4\pi ks}. \tag{1.13}$$

Furthermore, Bessel's inequality gives, for the Fourier coefficients of φ_s:

$$\sum_{k\in\mathbb{Z}}|\widehat{\varphi_{s}}_k|^2 \le \int_0^1 |u(\theta+si)|^2 d\theta \le \left(\sup_{z\in S_\varrho}|u(z)|\right)^2 = \|u\|_\varrho^2\ . \tag{1.14}$$

Equations (1.13) and (1.14) together imply:

$$\sum_{k\in\mathbb{Z}} |\widehat{u}_k|^2 \, e^{-4\pi ks} \leq \|u\|_\varrho^2 \, , \forall s \in (-\varrho, \varrho) \, . \tag{1.15}$$

Thus, for every $s \in (-\varrho, \varrho)$, we can write:

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} |\widehat{u}_k|^2 \, e^{4\pi|k|s} &= \sum_{\substack{k\in\mathbb{Z}\\ k<0}} |\widehat{u}_k|^2 \, e^{4\pi|k|s} + \sum_{\substack{k\in\mathbb{Z}\\ k\geq 0}} |\widehat{u}_k|^2 \, e^{4\pi|k|s} \\
&= \sum_{\substack{k\in\mathbb{Z}\\ k<0}} |\widehat{u}_k|^2 \, e^{-4\pi ks} + \sum_{\substack{k\in\mathbb{Z}\\ k\geq 0}} |\widehat{u}_k|^2 \, e^{-4\pi k(-s)} \\
&\leq \sum_{k\in\mathbb{Z}} |\widehat{u}_k|^2 \, e^{-4\pi ks} + \sum_{k\in\mathbb{Z}} |\widehat{u}_k|^2 \, e^{-4\pi k(-s)} \, .
\end{aligned}$$

Since $s \in (-\varrho, \varrho)$, then $-s \in (-\varrho, \varrho)$ too, and we can apply (1.15) to both of the latter sums:

$$\sum_{k\in\mathbb{Z}} |\widehat{u}_k|^2 \, e^{4\pi|k|s} \leq 2\|u\|_\varrho^2 \, , \; \forall s \in (-\varrho, \varrho).$$

To end the proof we take limits as $s \to \varrho$ and we have:

$$\sum_{k\in\mathbb{Z}} |\widehat{u}_k|^2 \, e^{4\pi|k|\varrho} \leq 2\|u\|_\varrho^2 \, ,$$

as we wanted to prove. □

Lemma 1.9 Uniform convergence

Let $u \in A_\varrho$ for some $\varrho > 0$. Then u can be expanded in its Fourier series:

$$u(z) = \sum_{k\in\mathbb{Z}} \widehat{u}_k \, e^{2\pi kzi}, \; \forall z \in \mathbb{T}_\varrho \, ,$$

where

$$\widehat{u}_k := \int_{\mathbb{T}} u(\theta) e^{-2\pi k\theta i}, \; \forall k \in \mathbb{Z} \, .$$

The series is absolutely and uniformly convergent in every complex strip of the form:

$$\mathbb{T}_{\varrho-\delta} = \{z \in \mathbb{C}/\mathbb{Z} : |\operatorname{Im}(z)| \leq \varrho - \delta\} \text{ with } 0 < \delta < \varrho \, .$$

Furthermore, if $0 < \delta < \varrho$, then we have the following estimate:

$$|u(z)| \leq \sqrt{2}\sqrt{\frac{e^{4\pi\delta} + 1}{e^{4\pi\delta} - 1}} \|u\|_\varrho \, , \forall z \in \mathbb{T}_{\varrho-\delta} \, .$$

Proof. [3] Since u is continuous in $\mathbb{T}_\varrho$ and 1–periodic, then

$$\|u\|_{\varrho-\delta} = \sup_{z\in\mathbb{T}_{\varrho-\delta}} |u(z)| < +\infty, \; \forall \, \delta \in (0, \varrho) \, .$$

[3] cf. **Lemma 2.4** p. 610 On optimal estimates for the solutions of linear partial differential equations of first order with constant coefficients on the torus [49].

Let us denote the Fourier series of u by

$$\mathfrak{F}u(z)=\sum_{k\in\mathbb{Z}}\widehat{u}_k\, e^{2\pi kzi},\ \forall z\in\mathbb{T}_\varrho\,. \tag{1.16}$$

This series converges absolutely and uniformly in every complex strip $\mathbb{T}_{\varrho-\delta}$, with $0<\delta<\varrho$, and consequently the series converges uniformly in every compact subset of the strip $\mathbb{T}_\varrho$. Therefore, $\mathfrak{F}u$ is analytic in $\mathrm{Int}(\mathbb{T}_\varrho)$ and it is well defined[4] in $\mathbb{T}_\varrho$.
Next, we would like to find an estimate for the Fourier series.
$\forall z\in\mathbb{T}_{\varrho-\delta}$, we have:

$$|\mathfrak{F}u(z)|=\left|\sum_{k\in\mathbb{Z}}\widehat{u}_k\, e^{2\pi kzi}\right|\leq\sum_{k\in\mathbb{Z}}|\widehat{u}_k|\,\left|e^{2\pi kzi}\right|\,.$$

Since $z\in\mathbb{T}_{\varrho-\delta}$, we may write $z=x+yi$, with $|y|\leq\varrho-\delta$. So,

$$|e^{2\pi kzi}|=e^{-2\pi ky}\leq e^{2\pi|k|(\varrho-\delta)},\ \forall k\in\mathbb{Z}.$$

It follows that:

$$|\mathfrak{F}u(z)|\leq\sum_{k\in\mathbb{Z}}|\widehat{u}_k|\, e^{2\pi|k|(\varrho-\delta)}=\sum_{k\in\mathbb{Z}}|\widehat{u}_k|\, e^{2\pi|k|\varrho}\, e^{-2\pi|k|\delta}\,.$$

Applying now the Cauchy-Schwartz inequality to the latter sum, we get:

$$|\mathfrak{F}u(z)|\leq\left(\sum_{k\in\mathbb{Z}}|\widehat{u}_k|^2\, e^{4\pi|k|\varrho}\right)^{\frac{1}{2}}\left(\sum_{k\in\mathbb{Z}}e^{-4\pi|k|\delta}\right)^{\frac{1}{2}}\,.$$

The first factor of this product can be estimated by means of **Lemma 1.8**. Thus:

$$|\mathfrak{F}u(z)|\leq\left(2\|u\|_\varrho^2\right)^{\frac{1}{2}}\left(\sum_{k\in\mathbb{Z}}e^{-4\pi|k|\delta}\right)^{\frac{1}{2}}=\sqrt{2}\|u\|_\varrho\left(\sum_{k\in\mathbb{Z}}e^{-4\pi|k|\delta}\right)^{\frac{1}{2}}\,.$$

Notice that the geometric series $\sum_{k=1}^{\infty}\left(e^{-4\pi\delta}\right)^k$ is convergent since $\left|e^{-4\pi\delta}\right|<1\,,\forall\delta>0$.
Hence, we can compute directly the series:

$$\begin{aligned}\sum_{k\in\mathbb{Z}}e^{-4\pi|k|\delta} &= \sum_{\substack{k\in\mathbb{Z}\\ k<0}}e^{-4\pi|k|\delta}+1+\sum_{\substack{k\in\mathbb{Z}\\ k>0}}e^{-4\pi|k|\delta}\\ &= 1+2\sum_{k=1}^{\infty}e^{-4\pi k\delta}=1+2\sum_{k=1}^{\infty}\left(e^{-4\pi\delta}\right)^k\\ &= 1+2\frac{e^{-4\pi\delta}}{1-e^{-4\pi\delta}}=\frac{e^{4\pi\delta}+1}{e^{4\pi\delta}-1}\,.\end{aligned}$$

It follows that:

$$|\mathfrak{F}u(z)|\leq\sqrt{2}\sqrt{\frac{e^{4\pi\delta}+1}{e^{4\pi\delta}-1}}\|u\|_\varrho\,,\forall z\in\mathbb{T}_{\varrho-\delta}\,.$$

[4] cf. **Theorem 9.12.1** *Foundations of Modern Analysis* [18]

To finish the proof we show that $u(z) = \mathfrak{F}u(z)$, $\forall z \in \mathbb{T}_\varrho$.
On one side, $u(x) = \mathfrak{F}u(x)$, $\forall x \in \mathbb{R}$, since u and $\mathfrak{F}u$ have the same Fourier coefficients and the set of exponentials $\{e^{2\pi kxi} : k \in \mathbb{Z}\}$ is complete[5]. On the other hand, by analytical continuation[6] we obtain as a consequence the desired result. □

Lemma 1.10 Cauchy estimates

Let $\varrho > 0$, $v : \mathbb{T}_\varrho \longrightarrow \mathbb{C}$, and $m \in \mathbb{N}$. If $v \in \mathcal{A}_\varrho$, then for every $\delta \in (0, \frac{1}{m}\varrho)$, $\frac{d^m v}{dz^m} \in \mathcal{A}_{\varrho - m\delta}$ and

$$\|\frac{d^m v}{dz^m}\|_{\varrho - m\delta} \leq m!\delta^{-m}\|v\|_\varrho \, . \tag{1.17}$$

Proof. Let $\delta^* = m\delta$. Then $0 < \delta^* < \varrho$ and for any $z \in \mathbb{T}_{\varrho-\delta^*}$, $\overline{\mathbb{D}}(z,\delta) \subset \operatorname{Int}(\mathbb{T}_{\varrho-\delta^*})$. Since $v \in \mathcal{A}_\varrho$, then v is holomorphic in a neighborhood of the closed disk $\overline{\mathbb{D}}(z,\delta)$. By the Cauchy integral formula,

$$\forall z \in \mathbb{T}_{\varrho-\delta^*}, \; \frac{d^m v}{dz^m}(z) = \frac{m!}{2\pi i}\int_{\partial\mathbb{D}(z,\delta)} \frac{v(\zeta)}{(\zeta - z)^{m+1}} d\zeta \, .$$

Taking modulus on both sides, we have

$$\begin{aligned}
\forall z \in \mathbb{T}_{\varrho-\delta^*}, \; \left|\frac{d^m v}{dz^m}(z)\right| &= \frac{m!}{2\pi}\left|\int_{\partial\mathbb{D}(z,\delta)} \frac{v(\zeta)}{(\zeta - z)^{m+1}} d\zeta\right| \\
&\leq \frac{m!}{2\pi}\int_{\partial\mathbb{D}(z,\delta)} \frac{|v(\zeta)|}{|\zeta - z|^{m+1}} |d\zeta| \leq \frac{m!}{2\pi} \sup_{\zeta \in \partial\mathbb{D}(z,\delta)} |v(\zeta)| \int_{\partial\mathbb{D}(z,\delta)} \frac{1}{|\zeta - z|^{m+1}} |d\zeta| \\
&= \frac{m!}{2\pi} \sup_{\zeta \in \mathbb{T}_\varrho} |v(\zeta)| \frac{1}{\delta^{m+1}} 2\pi\delta = m!\delta^{-m}\|v_\varrho\| \, .
\end{aligned}$$

□

1.2 Liouville numbers and the Diophantine condition

The following bi-decomposition of irrational numbers, in Liouville numbers and Diophantine numbers, plays a special role in the dynamics of a skew–product and the related *KAM* process that we will build to look for invariant tori, as will be brought out later. For this reason we refer now the main concepts which are needed to know about the arithmetics of these real numbers.
Recall first that a real number $\omega \in \mathbb{R}$ is said to be *algebraic* if there exists a polynomial $P(z) = a_r z^r + a_{r-1}z^{r-1} + \cdots + a_1 z + a_0 \in \mathbb{Q}[z]$, with rational coefficients, $a_i \in \mathbb{Q}$, such that $P(\omega) = 0$. The set of all algebraic numbers is a subfield of $\mathbb{R}$. Indeed, it is the algebraic closure of $\mathbb{Q}$, which is denoted by $\overline{\mathbb{Q}}$, that is, the minimal subfield of $\mathbb{R}$ which contains all the real roots of polynomials with coefficients in $\mathbb{Q}$. The complement of algebraic numbers with respect to $\mathbb{R}$ is the set of *transcendental* numbers, $\mathbb{R} \setminus \overline{\mathbb{Q}}$.

[5]cf. **Chapter II. §4.**, *Methods of Mathematical Physics* [31]

[6]cf. **§(9.4.4)**, p.203, *Foundations of Modern Analysis* [18]

Remark 1.11

For every algebraic number $\omega \in \overline{\mathbb{Q}}$*, there is a polynomial* $Q \in \mathbb{Q}[z]$ *of least degree, denoted by* $Q = \mathrm{Irr}(\omega, \mathbb{Q})$*, for which* ω *is a root.* Q *is also known as the irreducible polynomial of* ω *over the field of the rational numbers. Namely:*

(i) $Q(z) = \alpha_r z^r + \alpha_{r-1} z^{r-1} + \cdots + \alpha_1 z + \alpha_0 \in \mathbb{Q}[z]$*, i.e.* $\alpha_j = \frac{b_j}{c_j}$*, with* $b_j, c_j \in \mathbb{Z}$*,* $c_j \neq 0$ $(j = 0, 1, \dots, r)$*;*

(ii) $Q(\omega) = 0$*;*

(iii) Q *is irreducible, that is, if* $Q(z) = Q_1(z) Q_2(z)$*, with* $Q_1, Q_2 \in \mathbb{Q}[z]$*, then either* Q_1 *is constant or* Q_2 *is constant. In other words* Q *cannot be factorized in* $\mathbb{Q}[z]$ *as a product of non–trivial polynomials.*

By (i), Q *can be expressed in the following way:*

$$Q(z) = \frac{1}{\prod_{j=0}^{r} c_j} \left(a_r z^r + a_{r-1} z^{r-1} + \cdots + a_1 z + a_0 \right),$$

with $a_j = b_j \prod_{\substack{i=0 \\ i \neq j}}^{r} c_i \in \mathbb{Z}$*.*

Let us call $P(z) = a_r z^r + a_{r-1} z^{r-1} + \cdots + a_1 z + a_0 \in \mathbb{Z}[z]$*. Thus,* $Q(z) = \frac{1}{c} P(z)$*, with* $c = \prod_{j=0}^{r} c_j$*. Observe that the roots of* Q *and the roots of* P *are the same. Furthermore, by Gauss Lemma,* Q *is irreducible in* $\mathbb{Q}[z]$ *if and only if* P *is irreducible in* $\mathbb{Z}[z]$*.*

Definition 1.12 Liouville numbers

It is said that a real number $\omega \in \mathbb{R}$ *is a Liouville number if for any* $\nu \geq 1$ *there is a positive integer* $k \in \mathbb{N}$*,* $k \geq 2$*, and* $l \in \mathbb{Z}$ *such that*

$$0 < |k\omega - l| \leq \frac{1}{k^\nu}.$$

◇

Lemma 1.13 (Liouville, 1844)

For any algebraic irrational number $\omega \in \overline{\mathbb{Q}}$*, there are constants* $\gamma > 0$ *and* $\nu \geq 1$ *such that:*

$$|k\omega - l| \geq \frac{\gamma}{|k|^\nu}, \ \forall k \in \mathbb{Z}, k \neq 0, l \in \mathbb{Z}. \tag{1.18}$$

Remark 1.14

This lemma means that every irrational Liouville number is transcendental.

Proof. Let $Q = \mathrm{Irr}(\omega, \mathbb{Q})$ be the irreducible polynomial of ω over $\mathbb{Q}$. This polynomial can be written as
$Q(z) = \alpha_r z^r + \alpha_{r-1} z^{r-1} + \cdots + \alpha_1 z + \alpha_0 \in \mathbb{Q}[z]$, with $\alpha_j = \frac{b_j}{c_j}$, with $b_j, c_j \in \mathbb{Z}$, $c_j \neq 0$ $(j = 0, 1, \dots, r)$.

Define $P(z) = a_r z^r + a_{r-1}z^{r-1} + \cdots + a_1 z + a_0 \in \mathbb{Z}[z]$, with $a_j = b_j \prod_{\substack{i=0 \\ i \neq j}}^{r} c_i \in \mathbb{Z}$.

Thus, $Q(z) = \frac{1}{c}P(z)$, with $c = \prod_{j=0}^{r} c_j$. The irreducibility properties of Q and the definition of P lead us to the following properties for P:

(i) $P(\omega) = 0$;

(ii) P is irreducible in $\mathbb{Z}[z]$, that is, if $P(z) = P_1(z)P_2(z)$, with $P_1, P_2 \in \mathbb{Z}[z]$, then either P_1 is constant or P_2 is constant, i.e. P cannot be factorized in $\mathbb{Z}[z]$ as a product of non–trivial polynomials.

(iii) For any $k, l \in \mathbb{Z}$, with $k \neq 0$,

$$k^r P\left(\frac{l}{k}\right) = a_r l^r + a_{r-1}kl^{r-1} + \cdots + a_1 k^{r-1} l + a_0 \in \mathbb{Z}.$$

(iv) $P\left(\frac{l}{k}\right) \neq 0$, $\forall k, l \in \mathbb{Z}$, $k \neq 0$. Otherwise, ω and $\frac{l}{k}$ would be distinct roots of P, which is not possible due to the irreducibility.

On one side, the degree of the polynomial P is $r = \deg(\mathrm{Irr}(\mathbb{Q}, \omega)) \geq 2$. Otherwise, if $r = 1$, since $P(\omega) = 0$, ω would be rational, against our assumption. We take now $\nu = r - 1 \geq 1$.
On the other hand we can express P centered at ω, namely

$$P(z) = \sum_{j=0}^{r} \widetilde{a}_j (z - \omega)^j .$$

Observe that $\widetilde{a}_j = \frac{1}{j!}P^{(j)}(\omega)$, $\forall j = 0, 1, \ldots, r$. Furthermore, since Q is the irreducible polynomial of ω over $\mathbb{Q}$, then $P(\omega) = \widetilde{a}_0 = 0$ and $P'(\omega) = \widetilde{a}_1 \neq 0$. Otherwise P would have ω as a multiple root, which is not possible because of the irreducibility. Now, if we call $M = \sum_{j=1}^{r} |\widetilde{a}_j|$, then $M > 0$, since $\widetilde{a}_1 \neq 0$. So, we can define $\gamma = \min\left\{1, \frac{1}{M}\right\} > 0$. Notice that the constant γ depends only on ω.
To finish the proof, we check that with these definitions for γ and ν the inequalities (1.18) hold. Let $k \in \mathbb{Z} \setminus \{0\}$ and $l \in \mathbb{Z}$. If $|k\omega - l| \geq 1$, $|k\omega - l| \geq \gamma \geq \frac{\gamma}{|k|^\nu}$. If $|k\omega - l| < 1$, then $|k\omega - l|^j \leq 1$, $\forall j = 1, \ldots r$, and we can write:

$$\begin{aligned} \left|P\left(\frac{l}{k}\right)\right| &= \left|\sum_{j=1}^{r} \widetilde{a}_j \left(\frac{l}{k} - \omega\right)^j\right| \leq \sum_{j=1}^{r} |\widetilde{a}_j| \left|\frac{l}{k} - \omega\right|^j = \sum_{j=1}^{r} |\widetilde{a}_j| \frac{|\omega k - l|^j}{|k|^j} \\ &= \frac{|\omega k - l|}{|k|} \sum_{j=1}^{r} |\widetilde{a}_j| \frac{|\omega k - l|^{j-1}}{|k|^{j-1}} \leq \frac{|\omega k - l|}{|k|} \sum_{j=1}^{r} |\widetilde{a}_j| = \frac{|\omega k - l|}{|k|} \cdot M . \end{aligned}$$

Therefore,

$$\left|k^r P\left(\frac{l}{k}\right)\right| \leq |\omega k - l| \, |k|^{r-1} \cdot M = |\omega k - l| \, |k|^\nu \cdot M .$$

Since $k^r P\left(\frac{l}{k}\right) \in \mathbb{Z}$ and $P\left(\frac{l}{k}\right) \neq 0$, we have:

$$1 \leq |\omega k - l| \, |k|^\nu \cdot M .$$

It follows that:

$$|\omega k - l| \geq \frac{1}{M}|k|^{-\nu} \geq \gamma |k|^{-\nu} \ ,$$

which is the inequality (1.18) that we wanted to prove. □

Definition 1.15 Diophantine condition

It is said that $\omega \in \mathbb{T}$ *satisfies the Diophantine condition if*

$$|\omega k - l| \geq \gamma |k|^{-\nu} \ , \ \forall k \in \mathbb{Z} \setminus \{0\}, l \in \mathbb{Z} \tag{1.19}$$

for some suitable constants $\gamma > 0$ *and* $\nu \geq 1$*. In this case, we say that* ω *is Diophantine of class* ν *and constant* γ*.*
The set of all Diophantine frequencies[7] $\omega \in \mathbb{T}$ *of class* ν *and constant* γ *will be denoted by* $\mathcal{DC}(\gamma, \nu)$*.*
◇

Remark 1.16

Given an irrational algebraic number $\omega \in \overline{\mathbb{Q}}$*, Liouville's* ***Lemma 1.13*** *assures that* ω *is Diophantine. Namely,* $\omega \in \mathcal{DC}(\gamma, \nu)$ *for some* $\gamma > 0$ *and* $\nu \geq 1$*. Our proof of the lemma brings out a way to obtain* γ *and* ν *explicitly.*
For instance, the irreducible polynomial of the golden ratio $\omega = \frac{1+\sqrt{5}}{2}$ *is* $P(z) = z^2 - z - 1$*. So, we can take* $\nu = \deg(\mathrm{Irr}(\omega, \mathbb{Q})) - 1 = 1$*, and* $\gamma = \min\{1, \frac{1}{M}\}$*, with* $M = |P'(\omega)| + |P''(\omega)| = 2 + \sqrt{5}$*. Thus,* $\gamma = \frac{1}{M} = -2 + \sqrt{5}$*, and we have* $\omega = \frac{1+\sqrt{5}}{2} \in \mathcal{DC}(-2 + \sqrt{5}, 1)$*.*
In general, the set of the so–called Diophantine numbers of class $\nu \geq 1$ *is*

$$\mathcal{DC}(\nu) = \bigcup_{\gamma > 0} \mathcal{DC}(\gamma, \nu)$$

and the set of all Diophantine numbers is defined as

$$\mathcal{DC}(\infty) = \bigcup_{\nu \geq 1} \mathcal{DC}(\nu) \ .$$

In particular, $\mathcal{DC}(1)$ *is known as the class of bad approximable numbers. Of course, every quadratic irrational number belongs to this class, for instance, the golden ratio.*
With these notations, we can mention the following properties that we are not going to prove:

(i) $\forall \gamma > 0$ *and* $\nu \geq 1$*,* $\mathcal{DC}(\gamma, \nu)$ *is a set of full Lebesgue measure in* $\mathbb{R}$*.*

(ii) $\mathcal{DC}(\nu)$ *has zero Lebesgue measure but it is everywhere dense.*

(iii) If $1 < \nu_1 < \nu_2$ *then:*
$$\mathcal{DC}(1) \subsetneq \mathcal{DC}(\nu_1) \subsetneq \mathcal{DC}(\nu_2) \subsetneq \mathcal{DC}(\infty) \ .$$

---

[7] With this notation we mean implicitly that $\gamma > 0$ and $\nu \geq 1$ are given.

1.3 Small denominators

In order to help proving the following **Lemma 1.19** which concerns some arithmetical properties relative to the so–called small denominators, we define now some auxiliary real functions.

Definition 1.17 Some auxiliary functions

(*i*) $d : \mathbb{R} \longrightarrow \mathbb{R}$ *the* $1-$*periodic function, i.e.* $d(x+1) = d(x)$, $\forall x \in \mathbb{R}$, *given by* $d(x) = x$ *on the interval* $(-\frac{1}{2}, \frac{1}{2}]$.

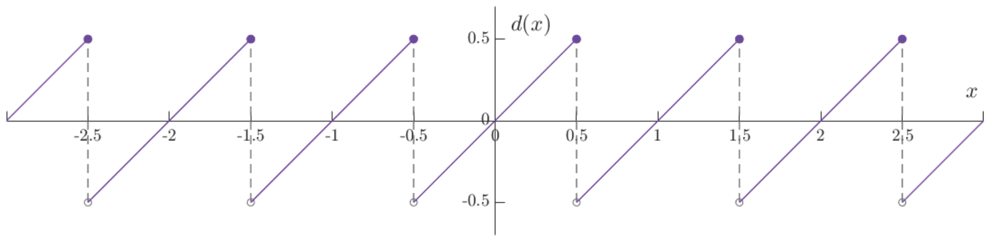

FIGURE 1.3: $d(x) = x$, $\forall x \in (-\frac{1}{2}, \frac{1}{2}]$ and $d(x+1) = d(x)$, $\forall x \in \mathbb{R}$

(*ii*) $l : \mathbb{R} \longrightarrow \mathbb{R}$ *given by* $l(x) = x - d(x)$, $\forall x \in \mathbb{R}$.

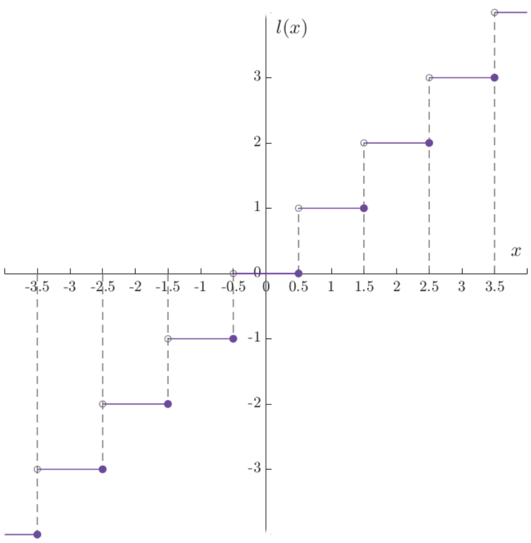

FIGURE 1.4: $l(x) = x - d(x)$, $\forall x \in \mathbb{R}$

(iii) $D : \mathbb{R} \longrightarrow \mathbb{R}$ *defined as* $D(x) = |d(x)|$.

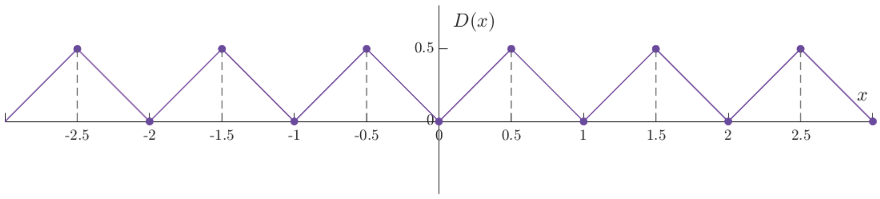

FIGURE 1.5: $D(x) = |d(x)|,\ \forall x \in \mathbb{R}$

◇

REMARK 1.18

Observe that, from the above definitions, the following properties become obvious:

(1) $d(x+l) = d(x) = x,\ \forall x \in (-\frac{1}{2}, \frac{1}{2}]$ *and* $l \in \mathbb{Z}$.

(2) $d(x) = 0 \Longleftrightarrow x \in \mathbb{Z}$.

(3) $l(x) \in \mathbb{Z},\ \forall x \in \mathbb{R}$.

(4) $D(-x) = D(x),\ \forall x \in \mathbb{R}$.

(5) $0 \leq D(x) \leq \frac{1}{2},\ \forall x \in \mathbb{R}$.

(6) $D(x) = 0 \Longleftrightarrow x \in \mathbb{Z}$.

(7) $D(x) = \frac{1}{2} \Longleftrightarrow x = \frac{2k+1}{2}$ *for some* $k \in \mathbb{Z}$.

(8) $\forall x \in \mathbb{R},\ D(x) = |d(x)| = |x - l(x)| = \min_{l \in \mathbb{Z}} |x - l|$ *which is the minimum distance from* x *to an integer number.*

Lemma 1.19 Small denominators

Let $\omega \in \mathbb{R}\backslash\mathbb{Q}$ *be any irrational number. Lets call*

$$D_k = \min_{l \in \mathbb{Z}} |\omega k - l|,\ k \in \mathbb{Z}$$

and

$$D_n^* = \min_{1 \leq k \leq n} D_k,\ n \in \mathbb{N}\ .$$

Then:

(a) *For every* $\lambda > 0$,

$$|e^{2\pi k\omega i} - \lambda| \geq 2(1+\lambda)D_k,\ \forall k \in \mathbb{Z}. \tag{1.20}$$

(b)

$$\sum_{k=1}^{n} \frac{1}{D_k^{2m}} \leq 2\zeta(2m)\frac{1}{(D_n^*)^{2m}}\ ,\ \forall n \in \mathbb{N}\ ,\ m > \frac{1}{2}\ , \tag{1.21}$$

where $\zeta(z) = \sum_{k=1}^{\infty} \frac{1}{k^z}$ *is the Riemann zeta function.*

(*c*) *If, besides, ω is Diophantine, i.e. $\omega \in \mathcal{DC}(\gamma,\nu)$, then for every $\lambda > 0$,*

$$|e^{2\pi k\omega i} - \lambda| \geq 2(1+\lambda)\gamma|k|^{-\nu}, \forall k \in \mathbb{Z}\setminus\{0\}, \tag{1.22}$$

and

(*d*)

$$\sum_{k=1}^{n} \frac{1}{D_k^{2m}} \leq 2\zeta(2m)\gamma^{-2m}n^{2m\nu}\ ,\ \forall n \in \mathbb{N}\ ,\ m > \frac{1}{2}\ . \tag{1.23}$$

Proof. Once $\omega \in \mathbb{R}\setminus\mathbb{Q}$ is fixed, we define the following sequences:

$$\begin{aligned} d_k &= d(k\omega),\ k \in \mathbb{Z}\ , \\ l_k &= l(k\omega) = k\omega - d_k,\ k \in \mathbb{Z}\ ,\ \text{and} \\ D_k &= D(k\omega) = |d_k| = |k\omega - l_k| = \inf_{l\in\mathbb{Z}} |k\omega - l|,\ k \in \mathbb{Z}\ . \end{aligned}$$

Due to the irrationality of ω and the remarked properties of the auxiliary functions which were described in **Definition 1.17**, $\forall k \in \mathbb{Z}\ ,\ k\omega \in \mathbb{R}\setminus\mathbb{Q},\ -\frac{1}{2} < d_k < \frac{1}{2},\ d_k \neq 0$, and $0 < D_k < \frac{1}{2}$. Notice also that $D_{-k} = D_k = D_{|k|}, \forall k \in \mathbb{Z}$ since D is an even function.

(a) On one side,

$$|e^{2\pi k\omega i} - \lambda|^2 = (1-\lambda)^2\cos^2(\pi k\omega) + (1+\lambda)^2\sin^2(\pi k\omega),\ \forall k \in \mathbb{Z}. \tag{1.24}$$

Indeed,

$$\begin{aligned} |e^{2\pi k\omega i} - \lambda|^2 &= |\cos(2\pi k\omega) + i\sin(2\pi k\omega) - \lambda|^2 = (-\lambda + \cos(2\pi k\omega))^2 + \sin^2(2\pi k\omega) \\ &= \lambda^2 - 2\lambda\cos(2\pi k\omega) + \cos^2(2\pi k\omega) + \sin^2(2\pi k\omega) \\ &= 1 - 2\lambda\cos(2\pi k\omega) + \lambda^2 = 1 - 2\lambda(\cos^2(\pi k\omega) - \sin^2(\pi k\omega)) + \lambda^2 \\ &= 1 - 2\lambda(1 - 2\sin^2(\pi k\omega)) + \lambda^2 = (1-\lambda)^2 + 4\lambda\sin^2(\pi k\omega) \\ &= (1-\lambda)^2(\cos^2(\pi k\omega) + \sin^2(\pi k\omega))) + 4\lambda\sin^2(\pi k\omega) \\ &= (1-\lambda)^2\cos^2(\pi k\omega) + ((1-\lambda)^2 + 4\lambda)\sin^2(\pi k\omega) \\ &= (1-\lambda)^2\cos^2(\pi k\omega) + (1+\lambda)^2\sin^2(\pi k\omega)\ . \end{aligned}$$

Thus, from (1.24), it follows that:

$$\begin{aligned} |e^{2\pi k\omega i} - \lambda|^2 &\geq (1+\lambda)^2\sin^2(\pi k\omega) = (1+\lambda)^2\sin^2(\pi k\omega - \pi l_k) \\ &= (1+\lambda)^2\sin^2(\pi(k\omega - l_k)) = (1+\lambda)^2\sin^2(\pi d_k)\ , \end{aligned} \tag{1.25}$$

for every $k \in \mathbb{Z}$, since the square of the sinus is a π–periodic function and $l_k \in \mathbb{Z}$.

Furthermore, since $-\frac{1}{2} < d_k < \frac{1}{2},\ d_k \neq 0$ then the following estimate

$$\frac{2}{\pi} < \frac{\sin x}{x} < 1,\ \forall x \in (-\frac{\pi}{2}, \frac{\pi}{2}),\ x \neq 0,$$

can be applied to πd_k.

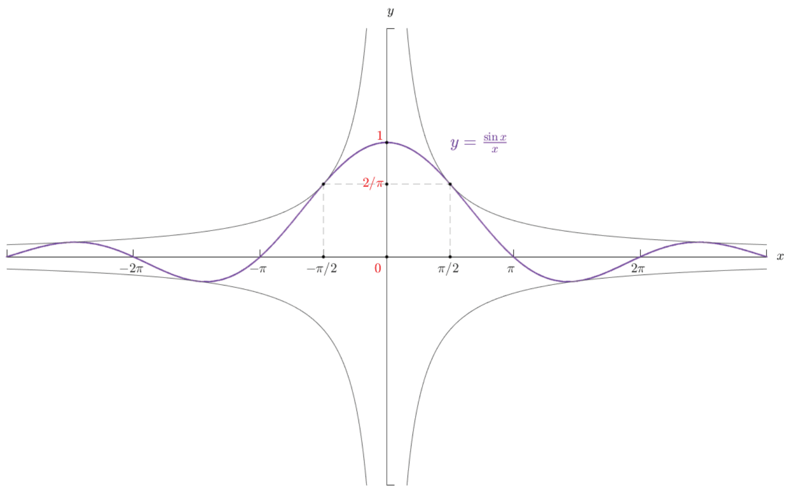

FIGURE 1.6: $\frac{2}{\pi} < \frac{\sin x}{x} < 1,\ \forall x \in (-\frac{\pi}{2}, \frac{\pi}{2}),\ x \neq 0.$

So, we get: $\forall k \in \mathbb{Z},\ \frac{2}{\pi} < \frac{\sin(\pi d_k)}{\pi d_k} < 1$, or equivalently:

$$\forall k \in \mathbb{Z},\ 2 < \frac{\sin(\pi d_k)}{d_k} < \pi. \tag{1.26}$$

From (1.25) and (1.26) we have, finally:

$$|e^{2\pi k\omega i} - \lambda|^2 \quad \geq \quad (1+\lambda)^2 4d_k^2 = (1+\lambda)^2 4D_k^2\ , \forall k \in \mathbb{Z}.$$

Therefore,

$$|e^{2\pi k\omega i} - \lambda| \geq 2(1+\lambda)D_k,\ \forall k \in \mathbb{Z},$$

and (1.20) is proved.

(b) [8] Given $n \in \mathbb{N}$ let us consider the set

$$\{d_1, d_2, \ldots, d_n\}\ .$$

These numbers are all different from each other in view of the irrationality of ω.

Indeed, if $d_j = j\omega - l_j = k\omega - l_k = d_k$ for some $j \neq k$, then $\omega = \frac{l_j - l_k}{j-k} \in \mathbb{Q}$ which is a contradiction.

Since all of these numbers are different from zero, we can assume that p of them are negative and the remainder $q = n - p$ are positive. Therefore, we can choose a permutation $\sigma \in S_n$ in order to sort the set $\{d_1, d_2, \ldots, d_n\}$ so that

$$-\frac{1}{2} < d_{\sigma(1)} < d_{\sigma(2)} < \cdots < d_{\sigma(p)} < 0 < d_{\sigma(p+1)} < d_{\sigma(p+2)} < \cdots < d_{\sigma(p+q)} < \frac{1}{2}\ ,$$

[8] cf. [51] Lemma 2.1. p. 36 On Optimal Estimates for the Solutions of Linear Differences Equations on the Circle. See also [50] Note on sums containing small denominators.

where $p + q = n$. For the sake of clarity we denote

$$\alpha_j = d_{\sigma(j)}, \; j = 1, \ldots, p \tag{1.27}$$
$$\beta_j = d_{\sigma(p+j)}, \; j = 1, \ldots, q \, . \tag{1.28}$$

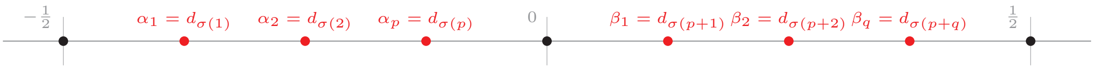

FIGURE 1.7: Sorting denominators $\{d_1, d_2, \ldots, d_n\}$.

Then, we can write for every $j = 2, \ldots, q$,

$$\beta_j = \beta_1 + \sum_{i=2}^{j} (\beta_i - \beta_{i-1})$$

Taking in account now the distribution of the positive values we have, for $i = 2, \ldots, j$:

$$\begin{aligned}
0 < \beta_i - \beta_{i-1} &= |\beta_i - \beta_{i-1}| \\
&= |d_{\sigma(p+i)} - d_{\sigma(p+i-1)}| \\
&= |D_{\sigma(p+i)} - D_{\sigma(p+i-1)}| \\
&= |(\omega\sigma(p+i) - l_{\sigma(p+i)}) - (\omega\sigma(p+i-1) - l_{\sigma(p+i-1)})| \\
&= |\omega(\sigma(p+i) - \sigma(p+i-1)) - (l_{\sigma(p+i)} - l_{\sigma(p+i-1)})| \\
&\geq \min_{l \in \mathbb{Z}} |\omega(\sigma(p+i) - \sigma(p+i-1)) - l| \\
&= \min_{l \in \mathbb{Z}} |\omega|\sigma(p+i) - \sigma(p+i-1)| - l| \\
&= D_{|\sigma(p+i) - \sigma(p+i-1)|} \\
&\geq \min_{1 \leq k \leq n} D_k = D_n^* \, ,
\end{aligned}$$

since $|\sigma(p+i) - \sigma(p+i-1)| \in \{1, 2, \ldots, n-1\}$.

Thus:

$$\beta_j = \beta_1 + \sum_{i=2}^{j} (\beta_i - \beta_{i-1}) \geq \beta_1 + (j-1) D_n^*, \; \forall j = 2, \ldots, q \, .$$

Furthermore, for $j = 1$ we have:

$$\begin{aligned}
\beta_1 &= d_{\sigma(p+1)} = |d_{\sigma(p+1)}| = D_{\sigma(p+1)} \\
&= \min_{l \in \mathbb{Z}} |\omega\sigma(p+1) - l| \\
&\geq \min_{1 \leq k \leq n} D_k = D_n^* \, ,
\end{aligned}$$

since $\sigma(p+1) \in \{1, 2, \ldots, n\}$.

Hence,

$$\beta_j \geq j D_n^*, \; \forall j = 1, \ldots, q \, . \tag{1.29}$$

On the other hand, we consider now the negative values, α_j , $j = 1, \ldots, p$, and argue with them in a similar way. For every $j = 1, \ldots, p-1$ we have,

$$\alpha_j = \sum_{i=j}^{p-1} (\alpha_i - \alpha_{i+1}) + \alpha_p$$

Now we have, for $i = j, \ldots, p-1$:

$$\begin{aligned} 0 > \alpha_i - \alpha_{i+1} &= d_{\sigma(i)} - d_{\sigma(i+1)} \\ &= -|d_{\sigma(i)} - d_{\sigma(i+1)}| \\ &= -|(\omega\sigma(i) - l_{\sigma(i)}) - (\omega\sigma(i+1) - l_{\sigma(i+1)})| \\ &= -|\omega(\sigma(i) - \sigma(i+1)) - (l_{\sigma(i)} - l_{\sigma(i+1)})| \\ &\leq -\min_{l\in\mathbb{Z}} |\omega(\sigma(i) - \sigma(i+1)) - l| \\ &= -D_{\sigma(i)-\sigma(i+1)} \\ &= -D_{|\sigma(i)-\sigma(i+1)|} \\ &\leq -\min_{1\leq k\leq n} D_k = -D_n^* \ , \end{aligned}$$

since $|\sigma(i) - \sigma(i+1)| \in \{1, \ldots, n-1\}$. Thence,

$$\alpha_j = \sum_{i=j}^{p-1}(\alpha_i - \alpha_{i+1}) + \alpha_p \leq -(p-j)D_n^* + \alpha_p \ .$$

Moreover,

$$\begin{aligned} \alpha_p &= d_{\sigma(p)} = -|d_{\sigma(p)}| = -D_{\sigma(p)} \\ &= -\min_{l\in\mathbb{Z}} |\omega\sigma(p) - l| \\ &\leq -\min_{1\leq k\leq n} D_k = -D_n^* \ , \end{aligned}$$

since $\sigma(p) \in \{1, 2, \ldots, n\}$ and then,

$$\alpha_j \leq -(p-j+1)D_n^*, \ \forall j = 1, \ldots, p \ . \tag{1.30}$$

To end the proof of this part we write:

$$\begin{aligned} \sum_{k=1}^{n} \frac{1}{D_k^{2m}} &= \sum_{k=1}^{n} \frac{1}{{D_{\sigma(k)}}^{2m}} = \sum_{k=1}^{n} \frac{1}{{d_{\sigma(k)}}^{2m}} \\ &= \sum_{j=1}^{p} \frac{1}{{d_{\sigma(j)}}^{2m}} + \sum_{j=p+1}^{n} \frac{1}{{d_{\sigma(j)}}^{2m}} \\ &= \sum_{j=1}^{p} \frac{1}{\alpha_j^{2m}} + \sum_{j=1}^{q} \frac{1}{\beta_j^{2m}} \ . \end{aligned} \tag{1.31}$$

From (1.29) and (1.30) we have also:

$$\frac{1}{\alpha_j^{2m}} \leq \frac{1}{(p-j+1)^{2m}} \frac{1}{{D_n^*}^{2m}}, \quad \forall j = 1, \ldots, p \ , \tag{1.32}$$

$$\frac{1}{\beta_j^{2m}} \leq \frac{1}{j^{2m}} \frac{1}{{D_n^*}^{2m}}, \quad \forall j = 1, \ldots, q \ . \tag{1.33}$$

Finally, from (1.31), (1.32), and (1.33) we conclude,

$$\begin{aligned} \sum_{k=1}^{n} \frac{1}{D_k^{2m}} &\leq \left(\sum_{j=1}^{p} \frac{1}{(p-j+1)^{2m}} + \sum_{j=1}^{q} \frac{1}{j^{2m}} \right) \frac{1}{{D_n^*}^{2m}} \\ &= \left(\sum_{j=1}^{p} \frac{1}{j^{2m}} + \sum_{j=1}^{q} \frac{1}{j^{2m}} \right) \frac{1}{{D_n^*}^{2m}} \\ &\leq 2\sum_{k=1}^{n} \frac{1}{k^{2m}} \frac{1}{{D_n^*}^{2m}} \leq 2\sum_{k=1}^{\infty} \frac{1}{k^{2m}} \frac{1}{{D_n^*}^{2m}} = 2\zeta(2m) \frac{1}{{D_n^*}^{2m}} \ , \end{aligned}$$

as we wanted to prove[9]. Notice that the series $\zeta(2m) = \sum_{k=1}^{\infty} \frac{1}{k^{2m}}$ is convergent since $m > \frac{1}{2}$.

(c) It follows directly from part (a) and the Diophantine condition of ω.
$\omega \in \mathcal{DC}(\gamma, \nu) \Rightarrow |\omega k - l| \geq \gamma |k|^{-\nu}$, $\forall k \in \mathbb{Z} \setminus \{0\}, l \in \mathbb{Z} \Rightarrow \forall k \in \mathbb{Z} \setminus \{0\}, D_k = \min_{l \in \mathbb{Z}} |\omega k - l| \geq \gamma |k|^{-\nu} \Rightarrow |e^{2\pi k \omega} - \lambda| \geq 2(1+\lambda)\gamma |k|^{-\nu}$.

(d) The last part follows immediately from (b) and the Diophantine condition of ω.
Indeed, if $\omega \in \mathcal{DC}(\gamma, \nu)$ then $|\omega k - l| \geq \gamma |k|^{-\nu}$, $\forall k \in \mathbb{Z} \setminus \{0\}$ and $l \in \mathbb{Z}$.
Thus, $\forall n \in \mathbb{N}$, $D_n^* = \min_{1 \leq k \leq n} D_k = \min_{1 \leq k \leq n} \min_{l \in \mathbb{Z}} |\omega k - l| \geq \min_{1 \leq k \leq n} \gamma |k|^{-\nu} = \gamma n^{-\nu}$ and hence, by (b):

$$\sum_{k=1}^{n} \frac{1}{D_k^{\,2m}} \leq 2\zeta(2m) \frac{1}{D_n^{*\,2m}} \leq 2\zeta(2m)\gamma^{-2m} n^{2m\nu} \ , \ \forall n \in \mathbb{N} \ , \ m > \frac{1}{2} \ .$$

□

[9] Recall that the Riemann zeta function is defined by $\zeta(z) = \sum_{k=1}^{\infty} \frac{1}{k^z}$. In particular, $\zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$ and $\zeta(4) = \sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}$.

1.4 Cohomological equation

Theorem 1.20 Rüßmann estimates

Let $\lambda \in (a, \frac{1}{a})$ *for some* $a \in (0,1)$ *and* $\omega \in \mathcal{DC}(\gamma,\nu)$ *be a Diophantine frequency*[10] *satisfying the Diophantine condition* (1.19) *for some constants* $\gamma \in (0,+\infty)$ *and* $\nu \in [1,+\infty)$.
Let us consider the so–called cohomological equation*:*

$$u(\theta+\omega) - \lambda u(\theta) = v(\theta)\ ,\ \theta \in \mathbb{T} \tag{1.34}$$

for some function u*, where* $v \in \mathcal{A}_\varrho$ *is given, with* $\varrho > 0$ *and* $\|v\|_\varrho < \infty$*. Assume moreover that* v *is a zero-average function, i.e.* $< v >= \int_{\mathbb{T}} v(\theta)d\theta = 0$*. Then:*

(*a*) *There is one and only one solution to* (1.34), $u \in \mathcal{H}(\mathrm{Int}(\mathbb{T}_\varrho))$*, with zero average. This solution* u *can be expanded in Fourier series*

$$u(z) = \sum_{k\in\mathbb{Z}} \widehat{u}_k e^{2\pi k z i}, \quad z \in \mathrm{Int}(\mathbb{T}_\varrho),$$

where $\widehat{u}_0 = 0$ *and*

$$\widehat{u}_k = \frac{\widehat{v}_k}{e^{2\pi k\omega i} - \lambda},\ \forall k \in \mathbb{Z}\setminus\{0\}. \tag{1.35}$$

(*b*) *The series is absolutely and uniformly convergent in every strip* $\mathbb{T}_{\varrho-\delta}$*, with* $0 < \delta < \varrho$.

(*c*) *The solution* u *holds the following Rüßmann estimate:*

$$\|u\|_{\varrho-\delta} \le \mathfrak{C}_R\, \gamma^{-1}\delta^{-\nu}\|v\|_\varrho, \quad \forall\delta \in (0,\varrho). \tag{1.36}$$

where the constant $\mathfrak{C}_R = \mathfrak{C}_R(a,\nu)$ *is independent of* γ*, uniform in* λ*, and it is given by*

$$\mathfrak{C}_R = \frac{1}{1+a}\frac{\sqrt{2\zeta(2)\Gamma(2\nu+1)}}{(4\pi)^\nu} = \frac{1}{1+a}\frac{\pi}{\sqrt{3}}\frac{\sqrt{\Gamma(2\nu+1)}}{(4\pi)^\nu}\ . \tag{1.37}$$

Note:

$\zeta(z) = \sum_{k=1}^{\infty}\frac{1}{k^z} \quad (\mathrm{Re}(z) > 1)$*, is the Riemann zeta function.*

$\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx \quad (\mathrm{Re}(z) >\ 0)$*, is the gamma function.*

Proof.

(a) Assuming that such a solution u exist, notice that, by Lemma 1.9, both functions u and v can be expandable in Fourier series since they are both analytic in a complex strip. If we consider the Fourier expansions of u and v,

$$v(\theta) = \sum_{k\in\mathbb{Z}} \widehat{v}_k e^{2\pi k\theta i} \tag{1.38}$$

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[10] See **Definition 1.15**.

which is given, and

$$u(\theta) = \sum_{k\in\mathbb{Z}} \widehat{u}_k e^{2\pi k\theta i} \tag{1.39}$$

which is unknown, then for every $\theta \in \mathbb{T}$, we have:

$u(\theta+\omega) - \lambda u(\theta) = v(\theta) \;\Leftrightarrow \sum_{k\in\mathbb{Z}} \widehat{u}_k e^{2\pi k\omega i} e^{2\pi k\theta i} - \lambda \sum_{k\in\mathbb{Z}} \widehat{u}_k e^{2\pi k\theta i} = \sum_{k\in\mathbb{Z}} \widehat{v}_k e^{2\pi k\theta i} \;\Leftrightarrow$

$\widehat{u}_k(e^{2\pi k\omega i} - \lambda) = \widehat{v}_k\ ,\ \forall k \in \mathbb{Z}$.

Since $\lambda \in \mathbb{R}$ and $\omega \in \mathbb{R}\setminus\mathbb{Q}$ then $\forall k \in \mathbb{Z}\setminus\{0\}, e^{2\pi k\omega i} - \lambda \neq 0$.

It turns out that the following formal solution is obtained. Let us define the series:

$$\mathfrak{R}_\lambda v(z) = \sum_{k\in\mathbb{Z}} \widehat{u}_k e^{2\pi kzi}\ (z \in \mathbb{T}_{\varrho-\delta})\ , \tag{1.40}$$

where $\widehat{u}_0 = 0$ and

$$\widehat{u}_k = \frac{\widehat{v}_k}{e^{2\pi k\omega i} - \lambda},\ \forall k \in \mathbb{Z}\setminus\{0\}\ .$$

In the next part of this proof we show that this series actually belongs to $\mathcal{A}_{\varrho-\delta}$ and it is, therefore, the unique solution to the cohomological equation (1.34) with zero–average.

(b) In spite of the fact that the coefficients $\widehat{u}_k$ are well defined, one cannot assure the regularity of the solution if the irrationality of ω is the only property required. As we show here, a sufficient condition for that regularity is the Diophantine character of the frequency.

If we take $s \in (-\varrho, \varrho)$ and $z \in \mathbb{T}_{\varrho-|s|}$, we can write $z = x + yi$ and then

$$\left|e^{2\pi kzi}\right| = \left|e^{2\pi k(x+yi)i}\right| = \left|e^{2\pi kxi}e^{-2\pi ky}\right| = e^{-2\pi ky} \leq e^{2\pi|k|(\varrho-|s|)},\ \forall z \in \mathbb{T}_{\varrho-|s|}.$$

It follows that:

$$\begin{aligned}
|\mathfrak{R}_\lambda v(z)| &= \left|\sum_{k\in\mathbb{Z}} \widehat{u}_k e^{2\pi kzi}\right| \leq \sum_{k\in\mathbb{Z}} \left|\widehat{u}_k e^{2\pi kzi}\right| = \sum_{k\in\mathbb{Z}\setminus\{0\}} \left|\widehat{u}_k e^{2\pi kzi}\right| \\
&= \sum_{k\in\mathbb{Z}\setminus\{0\}} \left|\frac{\widehat{v}_k}{e^{2\pi k\omega i} - \lambda} e^{2\pi kzi}\right| = \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{|\widehat{v}_k|}{|e^{2\pi k\omega i} - \lambda|} \left|e^{2\pi kzi}\right| \\
&\leq \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{|\widehat{v}_k|}{|e^{2\pi k\omega i} - \lambda|} e^{2\pi|k|(\varrho-|s|)} = \sum_{k\in\mathbb{Z}\setminus\{0\}} |\widehat{v}_k| e^{2\pi|k|\varrho} \frac{e^{-2\pi|k||s|}}{|e^{2\pi k\omega i} - \lambda|}
\end{aligned}$$

Using now the Cauchy–Schwartz's inequality we obtain the following estimate:

$$\begin{aligned}
|\mathfrak{R}_\lambda v(z)| &\leq \left(\sum_{k\in\mathbb{Z}\setminus\{0\}} |\widehat{v}_k|^2 e^{4\pi|k|\varrho}\right)^{\frac{1}{2}} \left(\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{e^{-4\pi|k||s|}}{|e^{2\pi k\omega i} - \lambda|^2}\right)^{\frac{1}{2}} \\
&\leq \left(2\|v\|_\varrho^2\right)^{\frac{1}{2}} \left(\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{e^{-4\pi|k||s|}}{|e^{2\pi k\omega i} - \lambda|^2}\right)^{\frac{1}{2}} = \left(\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{2e^{-4\pi|k||s|}}{|e^{2\pi k\omega i} - \lambda|^2}\right)^{\frac{1}{2}} \|v\|_\varrho \\
&= \sqrt{\Phi(s)}\ \|v\|_\varrho\ ,
\end{aligned}$$

where the first factor has been bounded by means of **Lemma 1.8.** and

$$\begin{aligned}
\Phi :\ (-\varrho, \varrho) &\longrightarrow \mathbb{R} \\
s &\longmapsto \Phi(s) := \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{2e^{-4\pi|k||s|}}{|e^{2\pi k\omega i} - \lambda|^2}
\end{aligned} \tag{1.41}$$

As long as $\Phi(s)$ is finite, all of the above inequalities hold. So, we are going to prove now that $\Phi(s) < \infty$, $\forall s \in (-\varrho, \varrho)$.

First of all, we express the denominators in the same way as in (1.24), namely:

$$|e^{2\pi k\omega i} - \lambda|^2 = (1-\lambda)^2 \cos^2(\pi k\omega) + (1+\lambda)^2 \sin^2(\pi k\omega) \ . \tag{1.42}$$

Thus, we get:

$$\Phi(s) = \sum_{n=1}^{\infty} \frac{4}{(1-\lambda)^2 \cos^2(\pi n\omega) + (1+\lambda)^2 \sin^2(\pi n\omega)} e^{-4\pi n|s|} \ , \ \forall s \in (-\varrho, \varrho) \ . \tag{1.43}$$

Let us call now

$$c_n = \frac{4}{(1-\lambda)^2 \cos^2(\pi n\omega) + (1+\lambda)^2 \sin^2(\pi n\omega)} \ , \ n \in \mathbb{N} \tag{1.44}$$

and

$$C_n = \sum_{k=1}^{n} c_k \ , \ n \in \mathbb{N} \ . \tag{1.45}$$

We also denote

$$\begin{array}{llll} \chi: & [1,+\infty) \times (-\varrho, \varrho) & \longrightarrow & \mathbb{R} \\ & (t,s) & \longmapsto & \chi(t,s) := e^{-4\pi t|s|} \end{array} .$$

Thus, $\Phi(s) = \sum_{n=1}^{\infty} c_n \chi(n,s)$, $\forall s \in (-\varrho, \varrho)$.

As we have seen, if $s \in (-\varrho, \varrho)$ then:

$$\forall z \in \mathbb{T}_{\varrho-|s|}, \ |\mathfrak{R}_\lambda v(z)| \le \sqrt{\Phi(s)} \ \|v\|_\varrho \tag{1.46}$$

and we want to obtain an estimate of the square root of Φ.

First, we prove that the function Φ can be expressed in the following way[11]:

$$\Phi(s) = \sum_{n=1}^{\infty} C_n(\chi(n,s) - \chi(n+1,s)) \ , \ \forall s \in (-\varrho, \varrho) \ . \tag{1.47}$$

$$\Phi(s) = \sum_{n=1}^{\infty} c_n \chi(n,s) = \lim_{n\to\infty} \sum_{k=1}^{n} c_k \chi(k,s) = \lim_{n\to\infty} \left(c_1 \chi(1,s) + \sum_{k=2}^{n} (C_k - C_{k-1}) \chi(k,s) \right)$$
$$= \lim_{n\to\infty} \left(\sum_{k=1}^{n} C_k \chi(k,s) - \sum_{k=2}^{n} C_{k-1} \chi(k,s) \right)$$
$$= \lim_{n\to\infty} \left(C_n \chi(n,s) + \sum_{k=1}^{n-1} C_k \chi(k,s) - \sum_{k=1}^{n-1} C_k \chi(k+1,s) \right)$$
$$= \lim_{n\to\infty} \left(C_n \chi(n,s) + \sum_{k=1}^{n-1} C_k (\chi(k,s) - \chi(k+1,s)) \right) .$$

[11] For this purpose we use the technique based on Abel's summation formula.

Now, by means of **Lemma 1.19**, we can see that

$$C_n \leq \frac{2\zeta(2)}{(1+\lambda)^2}\gamma^{-2}n^{2\nu} \ , \tag{1.48}$$

and hence $\lim_{n\to\infty} C_n\chi(n,s) = 0$.

Thus:

$\Phi(s) = \lim_{n\to\infty}\sum_{k=1}^{n-1} C_k(\chi(k,s)-\chi(k+1,s)) = \sum_{n=1}^{\infty} C_n(\chi(n,s)-\chi(n+1,s))$, $\forall s \in (-\varrho,\varrho)$, and this is the expression (1.47) that we wanted to prove.

Furthermore, we deduce finally the following estimate:

$$\Phi(s) \leq \frac{2\zeta(2)}{(1+\lambda)^2}\gamma^{-2}\sum_{n=1}^{\infty} n^{2\nu}(\chi(n,s)-\chi(n+1,s)) \ . \tag{1.49}$$

To finish the proof of the finiteness of Φ we can estimate the latter sum by an integral:

$$\sum_{n=1}^{\infty} n^{2\nu}(\chi(n,s)-\chi(n+1,s)) = \sum_{n=1}^{\infty} n^{2\nu}\int_n^{n+1} -\frac{\partial\chi}{\partial t}(t,s)dt = \sum_{n=1}^{\infty} -\int_n^{n+1} n^{2\nu}\frac{\partial\chi}{\partial t}(t,s)dt$$
$$\leq \sum_{n=1}^{\infty} -\int_n^{n+1} t^{2\nu}\frac{\partial\chi}{\partial t}(t,s)dt = \int_1^{\infty} -t^{2\nu}\frac{\partial\chi}{\partial t}(t,s)dt \leq \int_0^{\infty} -t^{2\nu}\frac{\partial\chi}{\partial t}(t,s)dt.$$

The last integral is related with the gamma function:

$$\begin{array}{rccl} \Gamma : & \{z\in\mathbb{C} : \mathrm{Re}(z)>0\} & \longrightarrow & \mathbb{C} \\ & z & \longmapsto & \Gamma(z) = \int_0^{\infty} x^{z-1}e^{-x}dx \ . \end{array}$$

Indeed,

$$\int_0^{\infty} -t^{2\nu}\frac{\partial\chi}{\partial t}(t,s)dt = \int_0^{\infty} 4\pi|s|t^{2\nu}e^{-4\pi|s|t}dt.$$

With the change of variable $x = 4\pi|s|t$ we have:

$$\int_0^{\infty} -t^{2\nu}\frac{\partial\chi}{\partial t}(t,s)dt = (4\pi|s|)^{-2\nu}\int_0^{\infty} x^{2\nu}e^{-x}dx = (4\pi|s|)^{-2\nu}\Gamma(2\nu+1).$$

The convergence of the integral that defines the gamma function leads us to the finiteness of Φ. In fact, from (1.49), we get:

$$\Phi(s) \leq \frac{2\zeta(2)}{(1+\lambda)^2}\gamma^{-2}(4\pi|s|)^{-2\nu}\Gamma(2\nu+1) \ . \tag{1.50}$$

Since $\Phi(s) < +\infty$, $\forall s \in (-\varrho,\varrho)$ then the series given by $\mathfrak{R}_\lambda v$ in (1.40) converges absolutely and uniformly in every compact subset of $\mathrm{Int}(\mathbb{T}_\varrho)$ and hence represents an analytic function in this open strip, which satisfies obviously the cohomological equation (1.34). By analytic continuation $\mathfrak{R}_\lambda v$, analytic a priori in $\mathrm{Int}(\mathbb{T}_{\varrho-\delta})$ can be extended continuously to the boundary of this strip $\partial\mathbb{T}_{\varrho-\delta}$. Hence, $\mathfrak{R}_\lambda v \in \mathcal{A}_{\varrho-\delta}$, for every $\delta \in (0,\varrho)$.

(c) Taking in account (1.50) and getting back to (1.46) we finally obtain the desired estimate, that is, for any $0<\delta<\varrho$:

$$\forall z \in \mathbb{T}_{\varrho-\delta},\ |\mathfrak{R}_\lambda v(z)| \leq \sqrt{\Phi(\delta)}\ \|v\|_\varrho \leq \frac{1}{1+\lambda}\frac{\sqrt{2\zeta(2)\Gamma(2\nu+1)}}{(4\pi)^\nu}\gamma^{-1}\delta^{-\nu}\|v\|_\varrho \ . \tag{1.51}$$

Since $0 < a < \lambda$ and calling[12] $\mathfrak{C}_R = \mathfrak{C}_R(a,\nu) = \frac{1}{1+a}\frac{\sqrt{2\zeta(2)\Gamma(2\nu+1)}}{(4\pi)^\nu} = \frac{1}{1+a}\frac{\pi}{\sqrt{3}}\frac{\sqrt{\Gamma(2\nu+1)}}{(4\pi)^\nu}$ we have:

$$\forall z \in \mathbb{T}_{\varrho-\delta},\ |\mathfrak{R}_\lambda v(z)| \le \sqrt{\Phi(\delta)}\ \|v\|_\varrho \le \mathfrak{C}_R \gamma^{-1}\delta^{-\nu}\|v\|_\varrho\ , \tag{1.52}$$

which leads to the desired Rüßmann estimate (1.36).

□

REMARK 1.21

*In what follows we will denote by $u = \mathfrak{R}_\lambda v$ the unique solution to a cohomological equation (1.34) obtained under the same conditions as in **Theorem 1.20**[13].*
In spite of the fact that $\mathfrak{R}_\lambda v \in \mathcal{A}_{\varrho-\delta}$, $\forall\delta \in (0,\varrho)$, one cannot assure that $\mathfrak{R}_\lambda v \in \mathcal{A}_\varrho$ unless that it is bounded in $\mathrm{Int}(\mathbb{T}_\varrho)$ and hence extendable by continuity to the boundary of the strip.

Corollary 1.22

Let $\varrho \in (0,1/2)$, $\gamma > 0$, $\nu \ge 1$, $\lambda > 0$, and $\omega \in \mathcal{DC}(\gamma,\nu)$.
If $v \in A_\varrho$, then

$$\exists m = m(\gamma,\nu) \in \mathbb{N}\ ,\ \textit{such that}\ \forall\delta \in (0,\varrho),\ \|\mathfrak{R}_\lambda \widetilde{v}\|_{\varrho-\delta} \le \delta^{-m}\|\widetilde{v}\|_\varrho \le 2\delta^{-m}\|v\|_\varrho\ . \tag{1.53}$$

REMARK 1.23

More specifically, we can take $m \in \mathbb{N}$ such that

$$m \ge \nu + \frac{\log\left(\frac{\pi\sqrt{\Gamma(2\nu+1)}\gamma^{-1}}{2\sqrt{3}(4\pi)^\nu}\right)}{\log 2}\ .$$

Proof. Let $a \in (0,1)$ such that $a < \lambda < \frac{1}{a}$. By **Theorem 1.20** we know that there is a unique solution $u = \mathfrak{R}_\lambda \widetilde{v}$ to the cohomological equation

$$u(\theta) - \lambda u(\theta+\omega) = \widetilde{v}(\theta),\ \theta \in \mathbb{T},$$

and u is extendable analytically to the complex strip $\mathbb{T}_{\varrho-\delta}$, $\forall\delta \in (0,\varrho)$, being $\|u\|_{\varrho-\delta} \le \mathfrak{C}_R\gamma^{-1}\delta^{-\nu}\|\widetilde{v}\|_\varrho$.
Let us call $C(\gamma,\nu) = (a+1)\mathfrak{C}_R\gamma^{-1} = \frac{\pi\sqrt{\Gamma(2\nu+1)}\gamma^{-1}}{\sqrt{3}(4\pi)^\nu}$.
Since $a \in (0,1)$, $\mathfrak{C}_R \le (a+1)\mathfrak{C}_R$. Therefore, $\mathfrak{C}_R\gamma^{-1} \le C(\gamma,\nu)$. It follows that

$$\forall\delta \in (0,\varrho),\ \|\mathfrak{R}_\lambda v\|_{\varrho-\delta} \le C(\gamma,\nu)\delta^{-\nu}\|v\|_\varrho.$$

We need to find $m \in \mathbb{N}$ sufficiently large so that $C(\gamma,\nu)\delta^{-\nu} \le \delta^{-m}$, i.e. $\delta^{m-\nu} \le \frac{1}{C(\gamma,\nu)}$.

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[12]Recall that $\zeta(2) = \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$.

[13]See also **Definition 1.24**.

Since we have taken $\varrho < 1/2$, then $\delta^{m-\nu} < \frac{1}{2^{m-\nu}}$. Hence, if $\frac{1}{2^{m-\nu}} \leq \frac{1}{C(\gamma,\nu)}$, then the condition holds. So, we must solve the unknown $m \in \mathbb{N}$ of the latter inequality. Thus,

$$m \geq \nu + \frac{\log C(\gamma,\nu)}{\log 2} .$$

With this value of $m \in \mathbb{N}$ we have, finally:

$$\forall \delta \in (0,\varrho), \; \|\mathfrak{R}_\lambda v\|_{\varrho-\delta} \leq \delta^{-m} \|\widetilde{v}\|_\varrho ,$$

and $\|\widetilde{v}\|_\varrho \leq 2\|v\|_\varrho$. □

EXAMPLE

If $\omega = \frac{1+\sqrt{5}}{2}$*, then* $\omega \in \mathcal{DC}(\gamma,\nu)$*, with* $\gamma = -2+\sqrt{5}$ *and* $\nu = 1$*.*
Thus, $C(\gamma,\nu) = \frac{\sqrt{6}}{12}(2+\sqrt{(5)}) \approx 0.864683755051501...$
With these values we obtain,

$$m \geq 1 + \frac{\log(C(\gamma,\nu))}{\log 2} = 0.790244490531274...$$

For instance, we can take $m = 1$*.*
If we choose $0 < \varrho < \frac{1}{2}$*, it turns out that*

$$\|\mathfrak{R}_\lambda \widetilde{v}\|_{\varrho-\delta} \leq \delta^{-1} \|\widetilde{v}\|_\varrho \, , \forall \delta \in (0,\varrho).$$

1.5 The cohomological operator

According to **Theorem 1.20** we can define an specific operator related with the solutions of cohomological equations.

Definition 1.24 Cohomological operator

Given $\varrho > 0$ and $\omega \in \mathcal{DC}(\gamma,\nu)$ with $\gamma > 0$ and $\nu \geq 1$, we define for each $\lambda > 0$ and $\delta \in (0,\varrho)$, the so–called cohomological operator *$\mathfrak{R}_\lambda$ over the space of analytic 1–periodic functions with zero–average $\mathcal{A}_{\varrho,0}$ (endowed with the uniform convergence topology) by:*

$$\begin{array}{rcccccl} \mathfrak{R}_\lambda : & \mathcal{A}_{\varrho,0} & \longrightarrow & \mathcal{A}_{\varrho-\delta,0} & & & \\ & v & \longmapsto & \mathfrak{R}_\lambda v : & \mathbb{T}_{\varrho-\delta} & \longrightarrow & \mathbb{C} \\ & & & & z & \longmapsto & \mathfrak{R}_\lambda v(z) = \displaystyle\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{\widehat{v}_k}{e^{2\pi k\omega i}-\lambda} e^{2\pi k z i} . \end{array} \tag{1.54}$$

◇

Proposition 1.25 Properties of the cohomological operator

The cohomological operator $\mathfrak{R}_\lambda$ (see **Definition 1.24***) holds the following properties:*

(i) *$\mathfrak{R}_\lambda$ is well defined and for each $v \in \mathcal{A}_{\varrho,0}$, $\mathfrak{R}_\lambda v$ is the unique solution in $\mathcal{A}_{\varrho-\delta,0}$ for any $\delta \in (0,\varrho)$ to the cohomological equation:*

$$u(\theta+\omega) - \lambda u(\theta) = v(\theta),\ \forall \theta \in \mathbb{T}. \tag{1.55}$$

(ii) *$\forall \delta \in (0,\varrho)$, $\mathfrak{R}_\lambda \in \mathcal{L}(\mathcal{A}_{\varrho,0}, \mathcal{A}_{\varrho-\delta,0})$, i.e. it is a continuous linear operator.*

(iii) *Let*

$$\begin{array}{rccl} \mathfrak{L}_\lambda : & \mathcal{A}_{\varrho,0} & \longrightarrow & \mathcal{A}_{\varrho,0} \\ & u & \longmapsto & \mathfrak{L}_\lambda u = u \circ \mathcal{R}_\omega - \lambda u \end{array} \tag{1.56}$$

where $\mathcal{R}_\omega(\theta) = \theta + \omega$ is the ergodic rigid rotation with the Diophantine frequency $\omega \in \mathcal{DC}(\gamma,\nu)$.

Then, the compositions $\mathfrak{R}_\lambda \circ \mathfrak{L}_\lambda : \mathcal{A}_{\varrho,0} \to \mathcal{A}_{\varrho-\delta,0}$ and $\mathfrak{L}_\lambda \circ \mathfrak{R}_\lambda : \mathcal{A}_{\varrho,0} \to \mathcal{A}_{\varrho-\delta,0}$ are well defined. Moreover:

$$(\mathfrak{R}_\lambda \circ \mathfrak{L}_\lambda)u = u|_{\mathbb{T}_{\varrho-\delta}},\ \forall u \in \mathcal{A}_{\varrho,0}\ , \tag{1.57}$$

$$(\mathfrak{L}_\lambda \circ \mathfrak{R}_\lambda)v = v|_{\mathbb{T}_{\varrho-\delta}},\ \forall v \in \mathcal{A}_{\varrho,0}\ . \tag{1.58}$$

(iv) *If $v \in \mathcal{A}_{\varrho,0}$ is real for real values, so is $\mathfrak{R}_\lambda v$.*

Proof.

(i) This can be seen at once from **Theorem 1.20**.

(ii) The linearity of the cohomological operator $\mathfrak{R}_\lambda$ is obvious from the linearity of the Fourier coefficients. The continuity follows from (1.36).

(iii) Let $u \in \mathcal{A}_{\varrho,0}$. Then, $\mathfrak{L}_\lambda u \in \mathcal{A}_{\varrho,0}$ since $\mathfrak{L}_\lambda u = u \circ \mathcal{R}_\omega - \lambda u$ is analytic in $\mathbb{T}_\varrho$ and also has zero average, by the 1–periodicity of u. Therefore, $(\mathfrak{R}_\lambda \circ \mathfrak{L}_\lambda)u \in \mathcal{A}_{\varrho-\delta,0}$. Let us call $u^* = (\mathfrak{R}_\lambda \circ \mathfrak{L}_\lambda)u$. We want to show that $u^* = u|_{\mathbb{T}_{\varrho-\delta}}$.

u^* is, by the definition of $\mathfrak{R}_\lambda$ the unique solution in $\mathbb{T}_{\varrho-\delta}$ to the cohomological equation

$$w(\theta+\omega) - \lambda w(\theta) = \mathfrak{L}_\lambda u(\theta),$$

but

$$u(\theta+\omega) - \lambda u(\theta) = \mathfrak{L}_\lambda u(\theta),\ \forall\theta \in \mathbb{T}_\varrho.$$

In particular,

$$u(\theta+\omega) - \lambda u(\theta) = \mathfrak{L}_\lambda u(\theta),\ \forall\theta \in \mathbb{T}_{\varrho-\delta}.$$

Therefore, $u^*(\theta) = u(\theta),\ \forall\theta \in \mathbb{T}_{\varrho-\delta}$, i.e. $u^* = u|_{\mathbb{T}_{\varrho-\delta}}$.

For the second part, let $v \in \mathcal{A}_{\varrho,0}$. Then $\mathfrak{R}_\lambda v \in \mathcal{A}_{\varrho-\delta,0}$ and $(\mathfrak{L}_\lambda \circ \mathfrak{R}_\lambda)v \in \mathcal{A}_{\varrho-\delta,0}$ since $(\mathfrak{L}_\lambda \circ \mathfrak{R}_\lambda)v = \mathfrak{L}_\lambda(\mathfrak{R}_\lambda v) = \mathfrak{R}_\lambda v(\theta+\omega) - \lambda\mathfrak{R}_\lambda v(\theta)$ and by the linearity of the average:

$< (\mathfrak{L}_\lambda \circ \mathfrak{R}_\lambda)v >=< \mathfrak{R}_\lambda(v \circ \mathcal{R}_\omega) > -\lambda < \mathfrak{R}_\lambda v >= 0$, inasmuch as $< v >= 0$.

Let us call $v^* = (\mathfrak{L}_\lambda \circ \mathfrak{R}_\lambda)v$. We want to show that $v^* = v|_{\mathbb{T}_{\varrho-\delta}}$.

We argue $\forall\theta \in \mathbb{T}_{\varrho-\delta}$:

$v^*(\theta) = (\mathfrak{L}_\lambda \circ \mathfrak{R}_\lambda)v(\theta) = \mathfrak{R}_\lambda v(\theta+\omega) - \lambda\mathfrak{R}_\lambda v(\theta)$. By the definition of $\mathfrak{R}_\lambda$, $\mathfrak{R}_\lambda v \in \mathcal{A}_{\varrho-\delta}$ is the unique solution to the cohomological equation $u(\theta+\omega) - \lambda u(\theta) = v(\theta)$ in $\mathbb{T}_{\varrho-\delta}$.

Therefore, $\forall\theta \in \mathbb{T}_{\varrho-\delta},\ v^*(\theta) = \mathfrak{R}_\lambda v(\theta+\omega) - \lambda\mathfrak{R}_\lambda v(\theta) = v(\theta)$, i.e. $v^* = v|_{\mathbb{T}_{\varrho-\delta}}$.

(*iv*) Assume that $\forall z = x + yi \in \mathbb{T}_\varrho$, with $y = 0$, $\mathrm{Im}(v(z)) = 0$. Then,

(a)

$$\begin{aligned}\overline{\widehat{v}_k} &= \overline{\int_{\mathbb{T}} v(\theta)e^{-2\pi k\theta i}\,d\theta} = \int_{\mathbb{T}} \overline{v(\theta)}\,\overline{e^{-2\pi k\theta i}}\,d\theta = \int_{\mathbb{T}} v(\theta)e^{2\pi k\theta i}\,d\theta \\ &= \int_{\mathbb{T}} v(\theta)e^{-2\pi(-k)\theta i}\,d\theta = \widehat{v}_{-k}, \quad \forall k \in \mathbb{Z}\setminus\{0\}.\end{aligned}$$

(b) $\overline{e^{2\pi k\omega i} - \lambda} = \overline{e^{2\pi k\omega i}} - \overline{\lambda} = e^{-2\pi k\omega i} - \lambda = e^{2\pi(-k)\omega i} - \lambda$, since λ is real.

(c) $\overline{e^{2\pi kzi}} = e^{-2\pi kzi} = e^{2\pi(-k)zi}$.

From these three facts we get:

$$\begin{aligned}\overline{\mathfrak{R}_\lambda v(z)} &= \overline{\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{\widehat{v}_k}{e^{2\pi k\omega i} - \lambda}e^{2\pi kzi}} = \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{\overline{\widehat{v}_k}}{\overline{e^{2\pi k\omega i} - \lambda}}\overline{e^{2\pi kzi}} \\ &= \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{\widehat{v}_{-k}}{e^{2\pi(-k)\omega i} - \lambda}e^{2\pi(-k)zi} = \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{\widehat{v}_k}{e^{2\pi k\omega i} - \lambda}e^{2\pi kzi} = \mathfrak{R}_\lambda v(z)\ .\end{aligned}$$

In other words, if z is real, $\mathfrak{R}_\lambda v(z)$ is real.

□

Next proposition shows two slightly different ways to estimate some cohomological operator corrections.

Proposition 1.26 Cohomological operator correction estimates

Let $\lambda, \bar{\lambda} \in [a, \frac{1}{a}]$ for some $a \in (0,1)$ and $\omega \in \mathcal{DC}(\gamma,\nu)$ be a Diophantine frequency[14] *satisfying the Diophantine condition (1.19) for some constants $\gamma \in (0,+\infty)$ and $\nu \in [1,+\infty)$.*
Given $\varrho > 0$, let us consider the cohomological operator[15]*:*

$$\begin{array}{rcccccl} \mathfrak{R}_\lambda : & \mathcal{A}_{\varrho,0} & \longrightarrow & \mathcal{A}_{\varrho-\delta,0} & & & \\ & v & \longmapsto & \mathfrak{R}_\lambda v : & \mathbb{T}_{\varrho-\delta} & \longrightarrow & \mathbb{C} \\ & & & & z & \longmapsto & \mathfrak{R}_\lambda v(z) = \displaystyle\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{\widehat{v}_k}{e^{2\pi k\omega i}-\lambda} e^{2\pi k z i} , \end{array} \tag{1.59}$$

where the series is absolutely and uniformly convergent in every strip $\mathbb{T}_{\varrho-\delta}$, with $0 < \delta < \varrho$.
Let us denote $\Delta\lambda = \bar{\lambda} - \lambda$ and $\Delta\mathfrak{R}_\lambda = \mathfrak{R}_{\bar{\lambda}} - \mathfrak{R}_\lambda$.
Then, the following properties hold, for every $v \in \mathcal{A}_{\varrho,0}$:

(*a*) $\Delta\mathfrak{R}_\lambda v(z) = \Delta\lambda\, \mathfrak{R}_{\bar{\lambda}}\mathfrak{R}_\lambda v(z)\ , \forall z \in \mathbb{T}_{\varrho-2\delta}, \quad \forall \delta \in (0, \frac{1}{2}\varrho)$.

(*b*) $\|\Delta\mathfrak{R}_\lambda v\|_{\varrho-2\delta} \leq |\Delta\lambda| \mathfrak{C}_R^2\, \gamma^{-2}\delta^{-2\nu}\, \|v\|_\varrho, \quad \forall \delta \in (0, \frac{1}{2}\varrho)$,

where $\mathfrak{C}_R = \frac{1}{1+a}\frac{\sqrt{2\zeta(2)}\sqrt{\Gamma(2\nu+1)}}{(4\pi)^\nu}$ *is the Rüßmann constant.*

(*c*) $\|\Delta\mathfrak{R}_\lambda v\|_{\varrho-\delta} \leq |\Delta\lambda|\, \mathfrak{C}_R^*\, \gamma^{-2}\delta^{-2\nu}\, \|v\|_\varrho, \quad \forall \delta \in (0, \varrho)$,

where $\mathfrak{C}_R^* = \frac{1}{2}\frac{1}{(1+a)^2}\frac{\sqrt{2\zeta(4)}\sqrt{\Gamma(4\nu+1)}}{(4\pi)^{2\nu}}$.

is independent of γ and uniform in λ and $\bar{\lambda}$.

Proof.

(a) Let $\delta \in (0, \frac{1}{2}\varrho)$ and $v \in \mathcal{A}_{\varrho,0}$. Then, $\Delta\mathfrak{R}_\lambda = \mathfrak{R}_{\bar{\lambda}} - \mathfrak{R}_\lambda \in \mathcal{A}_{\varrho-\delta,0}$. Now, take $\varrho^* = \varrho - \delta$ and $u = \mathfrak{R}_\lambda v \in \mathcal{A}_{\varrho^*,0}$. Applying (1.57) we obtain:

$$(\mathfrak{R}_{\bar{\lambda}} \circ \mathfrak{L}_{\bar{\lambda}})u = u|_{\mathbb{T}_{\varrho^*-\delta}}\ ,$$

that is,

$$(\mathfrak{R}_{\bar{\lambda}} \circ \mathfrak{L}_{\bar{\lambda}})\Delta\mathfrak{R}_\lambda v(z) = \Delta\mathfrak{R}_\lambda v(z), \quad \forall z \in \mathbb{T}_{\varrho-2\delta}, \tag{1.60}$$

since $\varrho^* - \delta = \varrho - 2\delta$. Moreover, developing the first term of (1.60) and using the definition of the left cohomological operator $\mathfrak{L}_{\bar{\lambda}}$, we can write:

$$\begin{aligned} (\mathfrak{R}_{\bar{\lambda}} \circ \mathfrak{L}_{\bar{\lambda}})\Delta\mathfrak{R}_\lambda v(z) &= \mathfrak{R}_{\bar{\lambda}}(\mathfrak{L}_{\bar{\lambda}}\mathfrak{R}_{\bar{\lambda}} v(z) - \mathfrak{L}_{\bar{\lambda}}\mathfrak{R}_\lambda v(z)) \\ &= \mathfrak{R}_{\bar{\lambda}}(v(z) - (\mathfrak{R}_\lambda v(z+\omega) - \bar{\lambda}\mathfrak{R}_\lambda v(z))) \\ &= \mathfrak{R}_{\bar{\lambda}}(v(z) - (\mathfrak{R}_\lambda v(z+\omega) - (\lambda+\Delta\lambda)\mathfrak{R}_\lambda v(z))) \\ &= \mathfrak{R}_{\bar{\lambda}}(v(z) - (\mathfrak{R}_\lambda v(z+\omega) - \lambda\mathfrak{R}_\lambda v(z)) + \Delta\lambda\mathfrak{R}_\lambda v(z)) \\ &= \mathfrak{R}_{\bar{\lambda}}(v(z) - v(z) + \Delta\lambda\mathfrak{R}_\lambda v(z)) \\ &= \mathfrak{R}_{\bar{\lambda}}(\Delta\lambda\mathfrak{R}_\lambda v(z)) = \Delta\lambda\mathfrak{R}_{\bar{\lambda}}\mathfrak{R}_\lambda v(z)\ . \end{aligned}$$

---

[14]See **Definition 1.15**.

[15]See **Definition 1.24**.

Notice that, since $v \in \mathcal{A}_{\varrho,0} \subseteq \mathcal{A}_{\varrho^*,0}$, then by (1.58),

$$\mathfrak{L}_{\bar{\lambda}}\mathfrak{R}_{\bar{\lambda}}v(z) = v(z),\ \forall z \in \mathbb{T}_{\varrho^*-\delta}.$$

Summarizing,

$$\Delta\mathfrak{R}_\lambda v(z) = \Delta\lambda\,\mathfrak{R}_{\bar{\lambda}}\mathfrak{R}_\lambda v(z), \quad \forall z \in \mathbb{T}_{\varrho-2\delta}, \tag{1.61}$$

as we wanted to prove.

(b) Let $\delta \in (0, \frac{1}{2}\varrho)$, $\varrho^* = \varrho - \delta$, and $v \in A_{\varrho,0}$. Then, by part (a) and applying **Theorem 1.20** (Rüßmann estimates) twice, we have:

$$\begin{aligned}
\|\Delta\mathfrak{R}_\lambda v\|_{\varrho^*-\delta} &= \|\Delta\lambda\,\mathfrak{R}_{\bar{\lambda}}\mathfrak{R}_\lambda\|_{\varrho^*-\delta} \\
&= |\Delta\lambda|\,\|\mathfrak{R}_{\bar{\lambda}}\mathfrak{R}_\lambda\|_{\varrho^*-\delta} \\
&\leq |\Delta\lambda|\,\mathfrak{C}_R\gamma^{-1}\delta^{-\nu}\|\mathfrak{R}_\lambda v\|_{\varrho^*} \\
&= |\Delta\lambda|\,\mathfrak{C}_R\gamma^{-1}\delta^{-\nu}\|\mathfrak{R}_\lambda v\|_{\varrho-\delta} \\
&\leq |\Delta\lambda|\,\mathfrak{C}_R\gamma^{-1}\delta^{-\nu}\mathfrak{C}_R\gamma^{-1}\delta^{-\nu}\|v\|_\varrho \\
&= |\Delta\lambda|\,\mathfrak{C}_R^2\gamma^{-2}\delta^{-2\nu}\|v\|_\varrho\ ,
\end{aligned} \tag{1.62}$$

i.e.

$$\|\Delta\mathfrak{R}_\lambda v\|_{\varrho-2\delta} \leq |\Delta\lambda|\,\mathfrak{C}_R^2\gamma^{-2}\delta^{-2\nu}\|v\|_\varrho, \quad \forall\delta \in (0, \frac{1}{2}\varrho). \tag{1.63}$$

(c) The proof of this part is carried out, in some sense, in a parallel way to the one of the previous **Theorem 1.20** (Rüßmann estimates). First of all, we know from the above theorem that

$$\mathfrak{R}_\lambda v(\theta) = \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{\hat{v}_k}{e^{2\pi k\omega i} - \lambda} e^{2\pi k\theta i}\ ,\ \theta \in \mathbb{T}_{\varrho-\delta}$$

and

$$\mathfrak{R}_{\bar{\lambda}} v(\theta) = \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{\hat{v}_k}{e^{2\pi k\omega i} - \bar{\lambda}} e^{2\pi k\theta i}\ ,\ \theta \in \mathbb{T}_{\varrho-\delta}$$

are the unique zero–average solutions to the cohomological equations

$$u(\theta + \omega) - \lambda u(\theta) = v(\theta)$$

and

$$u(\theta + \omega) - \bar{\lambda} u(\theta) = v(\theta)\ ,$$

respectively.

It follows that the difference of the cohomological operators applied to a given zero–average function $v \in \mathcal{A}_{\varrho,0}$ is given by:

$$\begin{aligned}
\Delta\mathfrak{R}_\lambda v(\theta) &= \mathfrak{R}_{\bar{\lambda}}v(\theta) - \mathfrak{R}_\lambda v(\theta) \\
&= \sum_{k\in\mathbb{Z}\setminus\{0\}} \left(\frac{1}{e^{2\pi k\omega i} - \bar{\lambda}} - \frac{1}{e^{2\pi k\omega i} - \lambda}\right)\hat{v}_k\, e^{2\pi k\theta i} \\
&= \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{\bar{\lambda} - \lambda}{(e^{2\pi k\omega i} - \bar{\lambda})(e^{2\pi k\omega i} - \lambda)}\,\hat{v}_k\, e^{2\pi k\theta i} \\
&= \Delta\lambda \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{\hat{v}_k}{(e^{2\pi k\omega i} - \bar{\lambda})(e^{2\pi k\omega i} - \lambda)}\, e^{2\pi k\theta i}\ .
\end{aligned} \tag{1.64}$$

Notice that (1.64) fits with (1.61) obtained previously in part (a).

Again, if we take $s \in (-\varrho, \varrho)$ and $\theta \in \mathbb{T}_{\varrho-|s|}$, we can write $\theta = x + yi$ and then

$$\left|e^{2\pi k\theta i}\right| = \left|e^{2\pi k(x+yi)i}\right| = \left|e^{2\pi kxi}e^{-2\pi ky}\right| = e^{-2\pi ky} \leq e^{2\pi|k|(\varrho-|s|)}, \ \forall\theta \in \mathbb{T}_{\varrho-|s|}.$$

Therefore,

$$|\Delta\mathfrak{R}_\lambda v(\theta)| \leq |\Delta\lambda| \sum_{k\in\mathbb{Z}\setminus\{0\}} |\hat{v}_k|\, e^{2\pi|k|\varrho} \frac{e^{-2\pi|k|\,|s|}}{|e^{2\pi k\omega i} - \bar{\lambda}|\,|e^{2\pi k\omega i} - \lambda|}. \tag{1.65}$$

By the Cauchy-Schwartz's inequality and applying also **Lemma 1.8**, we have:

$$\begin{aligned} |\Delta\mathfrak{R}_\lambda v(\theta)| &\leq |\Delta\lambda| \left(\sum_{k\in\mathbb{Z}\setminus\{0\}} |\hat{v}_k|^2\, e^{4\pi|k|\varrho}\right)^{\frac{1}{2}} \left(\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{e^{-4\pi|k||s|}}{|e^{2\pi k\omega i} - \bar{\lambda}|^2\,|e^{2\pi k\omega i} - \lambda|^2}\right)^{\frac{1}{2}} \\ &\leq |\Delta\lambda| \left(2\|v\|_\varrho^2\right)^{\frac{1}{2}} \left(2\sum_{n=1}^{\infty} \frac{e^{-4\pi n|s|}}{|e^{2\pi n\omega i} - \bar{\lambda}|^2\,|e^{2\pi n\omega i} - \lambda|^2}\right)^{\frac{1}{2}} \\ &= 2|\Delta\lambda| \left(\sum_{n=1}^{\infty} c_n\chi(n,s)\right)^{\frac{1}{2}} \|v\|_\varrho\ , \end{aligned} \tag{1.66}$$

where

$$c_n = \frac{1}{|e^{2\pi n\omega i} - \bar{\lambda}|^2\,|e^{2\pi n\omega i} - \lambda|^2}\ ,\ n \in \mathbb{N}$$

and

$$\begin{aligned} \chi: \quad (0,+\infty)\times(-\varrho,\varrho) &\longrightarrow \mathbb{R} \\ (t,s) &\longmapsto \chi(t,s) := e^{-4\pi t|s|} \end{aligned}\ .$$

Let us denote $\Upsilon(s) = \sum_{n=1}^{\infty} c_n\chi(n,s),\ s \in (-\varrho,\varrho)$. Thus, (1.66) is written as

$$|\Delta\mathfrak{R}_\lambda v(\theta)| \leq 2|\Delta\lambda|\sqrt{\Upsilon(s)}\|v\|_\varrho\ ,\ \forall\theta \in \mathbb{T}_{\varrho-|s|}\ ,\ s \in (-\varrho,\varrho)\ . \tag{1.67}$$

Our aim now is to obtain an estimate for $\Upsilon(s)$.

Let $(C_n)_{n\in\mathbb{N}}$ be the sequence given by

$$C_n = \sum_{k=1}^{n} c_k\ ,\ n \in \mathbb{N}\ . \tag{1.68}$$

By the argument of the Abel's summation formula[16], we can write:

$$\Upsilon(s) = \sum_{n=1}^{\infty} c_n\chi(n,s) = \sum_{n=1}^{\infty} C_n(\chi(n,s) - \chi(n+1,s))\ , \tag{1.69}$$

whenever $\lim_{n\to\infty} C_n\chi(n,s) = 0$.

[16]See **Theorem 1.20**, part (c), (1.47) .

Notice that

$$\begin{aligned} c_n &= \frac{1}{|e^{2\pi n\omega i}-\bar{\lambda}|^2\,|e^{2\pi n\omega i}-\lambda|^2} \\ &= \frac{1}{(1-\lambda)^2\cos^2(\pi n\omega)+(1+\lambda)^2\sin^2(\pi n\omega)} \\ &\qquad \cdot\frac{1}{(1-\bar{\lambda})^2\cos^2(\pi n\omega)+(1+\bar{\lambda})^2\sin^2(\pi n\omega)} \\ &\leq \frac{1}{(1+\bar{\lambda})^2(1+\lambda)^2}\,\frac{1}{\sin^4(\pi n\omega)}\,. \end{aligned} \tag{1.70}$$

Therefore,

$$\begin{aligned} C_n &= \sum_{k=1}^{n} c_k = \sum_{k=1}^{n}\frac{1}{|e^{2\pi k\omega i}-\bar{\lambda}|^2\,|e^{2\pi k\omega i}-\lambda|^2} \\ &\leq \frac{1}{(1+\bar{\lambda})^2(1+\lambda)^2}\sum_{k=1}^{n}\frac{1}{\sin^4(\pi k\omega)}\,. \end{aligned}$$

As it was seen in **Theorem 1.19**, $\sin^2(\pi k\omega)=\sin^2(\pi d_k)$. Then, from (1.26) we have:

$$\sum_{k=1}^{n}\frac{1}{\sin^4(\pi k\omega)} = \sum_{k=1}^{n}\frac{1}{\sin^4(\pi d_k)} \leq \sum_{k=1}^{n}\frac{1}{4^2 d_k^4} = \sum_{k=1}^{n}\frac{1}{4^2 D_k^4} = \frac{1}{4^2}\sum_{k=1}^{n}\frac{1}{D_k^4} \tag{1.71}$$

and hence

$$C_n \leq \frac{1}{(1+\bar{\lambda})^2(1+\lambda)^2}\,\frac{1}{4^2}\sum_{k=1}^{n}\frac{1}{D_k^4}\,. \tag{1.72}$$

We now make the following claim[17]:

$$\sum_{k=1}^{n}\frac{1}{D_k^4} \leq 2\zeta(4)\gamma^{-4}n^{4\nu}\ ,\ \forall n\in\mathbb{N}\,. \tag{1.73}$$

Joining (1.72), and (1.73),

$$C_n \leq \frac{1}{(1+\bar{\lambda})^2(1+\lambda)^2}\,\frac{1}{4^2}2\zeta(4)\gamma^{-4}n^{4\nu}\,. \tag{1.74}$$

This inequality implies that

$$\lim_{n\to\infty} C_n\chi(n,s) \leq \frac{1}{(1+\bar{\lambda})^2(1+\lambda)^2}\,\frac{1}{4^2}2\zeta(4)\gamma^{-4}\lim_{n\to\infty} n^{4\nu}e^{-4\pi n|s|} = 0,$$

which proves the veracity of (1.69).

Moreover,

$$\Upsilon(s) \leq \frac{1}{4^2}\,\frac{1}{(1+\bar{\lambda})^2(1+\lambda)^2}2\,\zeta(4)\gamma^{-4}\sum_{n=1}^{\infty} n^{4\nu}(\chi(n,s)-\chi(n+1,s))\,. \tag{1.75}$$

[17] In fact, this inequality is a particular case of (1.23) for $m=2$, see **Lemma 1.19 Small denominators**.

From (1.67) and (1.75) we obtain: $\forall\theta \in \mathbb{T}_{\varrho-|s|}$,

$$\begin{aligned} |\Delta\mathfrak{R}_\lambda v(\theta)| &\leq 2|\Delta\lambda|\sqrt{\frac{1}{4^2}\,\frac{2\,\zeta(4)\gamma^{-4}}{(1+\bar\lambda)^2(1+\lambda)^2}\sum_{n=1}^{\infty} n^{4\nu}(\chi(n,s)-\chi(n+1,s))}\,\|v\|_\varrho \\ &= \frac{1}{2}\frac{|\Delta\lambda|}{(1+\bar\lambda)(1+\lambda)}\sqrt{2\zeta(4)}\gamma^{-2}\sqrt{\sum_{n=1}^{\infty} n^{4\nu}(\chi(n,s)-\chi(n+1,s))}\,\|v\|_\varrho. \end{aligned} \tag{1.76}$$

It only remains to estimate the series:

$$\begin{aligned} \sum_{n=1}^{\infty} n^{4\nu}(\chi(n,s)-\chi(n+1,s)) &= \sum_{n=1}^{\infty} n^{4\nu}\int_n^{n+1} -\frac{\partial\chi}{\partial t}(t,s)dt \\ =\sum_{n=1}^{\infty} -\int_n^{n+1} n^{4\nu}\frac{\partial\chi}{\partial t}(t,s)dt &\leq \sum_{n=1}^{\infty} -\int_n^{n+1} t^{4\nu}\frac{\partial\chi}{\partial t}(t,s)dt \\ =\int_1^{\infty} -t^{4\nu}\frac{\partial\chi}{\partial t}(t,s)dt &\leq \int_0^{\infty} -t^{4\nu}\frac{\partial\chi}{\partial t}(t,s)dt\ . \end{aligned}$$

The last integral is related with the gamma function:

$$\begin{aligned} \Gamma:\quad \{z\in\mathbb{C}:\mathrm{Re}(z)>0\} &\longrightarrow \mathbb{C} \\ z &\longmapsto \Gamma(z)=\int_0^{\infty} x^{z-1}e^{-x}dx\ . \end{aligned}$$

Indeed, $\displaystyle\int_0^{\infty} -t^{4\nu}\frac{\partial\chi}{\partial t}(t,s)dt = \int_0^{\infty} 4\pi|s|t^{4\nu}e^{-4\pi|s|t}dt,$

and with the change of variable $x = 4\pi|s|t$ we have:

$$\int_0^{\infty} -t^{4\nu}\frac{\partial\chi}{\partial t}(t,s)dt = (4\pi|s|)^{-4\nu}\int_0^{\infty} x^{4\nu}e^{-x}dx = (4\pi|s|)^{-4\nu}\Gamma(4\nu+1)\ .$$

Thus,

$$\sum_{n=1}^{\infty} n^{4\nu}(\chi(n,s)-\chi(n+1,s)) \leq (4\pi|s|)^{-4\nu}\Gamma(4\nu+1)\ . \tag{1.77}$$

Finally, introducing (1.77) in (1.76) we get:

$$\begin{aligned} |\Delta\mathfrak{R}_\lambda v(\theta)| &\leq \frac{1}{2}\frac{|\Delta\lambda|}{(1+\bar\lambda)(1+\lambda)}\sqrt{2\zeta(4)}\gamma^{-2}\sqrt{(4\pi|s|)^{-4\nu}\Gamma(4\nu+1)}\,\|v\|_\varrho \\ &= \frac{1}{2}\frac{|\Delta\lambda|}{(1+\bar\lambda)(1+\lambda)}\frac{\sqrt{2\zeta(4)}\sqrt{\Gamma(4\nu+1)}}{(4\pi)^{2\nu}}\gamma^{-2}|s|^{-2\nu}\,\|v\|_\varrho\ . \end{aligned} \tag{1.78}$$

In particular, if $\delta \in (0,\varrho)$ and taking limits in (1.78) as $s \to \delta$,

$$|\Delta\mathfrak{R}_\lambda v(\theta)| \leq \frac{1}{2}\frac{|\Delta\lambda|}{(1+\bar\lambda)(1+\lambda)}\frac{\sqrt{2\zeta(4)}\sqrt{\Gamma(4\nu+1)}}{(4\pi)^{2\nu}}\gamma^{-2}\delta^{-2\nu}\,\|v\|_\varrho,$$

$$\forall\theta\in(0,\varrho-\delta),\delta\in(0,\varrho). \tag{1.79}$$

Since $\lambda,\bar\lambda \in (a,\frac{1}{a})$, $\frac{1}{(1+\bar\lambda)(1+\lambda)} \leq \frac{1}{(1+a)^2}$ and we may write: $\forall\,\theta\in(0,\varrho-\delta),\delta\in(0,\varrho)$,

$$|\Delta\mathfrak{R}_\lambda v(\theta)| \leq \frac{1}{2}|\Delta\lambda|\frac{1}{(1+a)^2}\frac{\sqrt{2\zeta(4)}\sqrt{\Gamma(4\nu+1)}}{(4\pi)^{2\nu}}\gamma^{-2}\delta^{-2\nu}\,\|v\|_\varrho\ , \tag{1.80}$$

Denoting

$$\mathfrak{C}_R^* = \mathfrak{C}_R^*(a,\nu) := \frac{1}{2}\frac{1}{(1+a)^2}\frac{\sqrt{2\zeta(4)}\sqrt{\Gamma(4\nu+1)}}{(4\pi)^{2\nu}}\ , \tag{1.81}$$

we obtain finally:

$$|\Delta\mathfrak{R}_\lambda v(\theta)| \le |\Delta\lambda|\ \mathfrak{C}_R^*\,\gamma^{-2}\delta^{-2\nu}\ \|v\|_\varrho, \quad \forall\theta\in(0,\varrho-\delta),\ \delta\in(0,\varrho), \tag{1.82}$$

or equivalently,

$$\|\Delta\mathfrak{R}_\lambda v\|_{\varrho-\delta} \le |\Delta\lambda|\ \mathfrak{C}_R^*\,\gamma^{-2}\delta^{-2\nu}\ \|v\|_\varrho, \quad \delta\in(0,\varrho). \tag{1.83}$$

□

Remark 1.27

We should make a comparison between the constants $\mathfrak{C}_R$ and $\mathfrak{C}_R^*$ in order to determine whether estimate (1.63) *is sharper than estimate* (1.83).
To do this we must make the comparison in the same complex strip.
*Let $\delta\in(0,\frac{1}{2}\varrho)$. On the one hand, we have by **Proposition 1.26** part* (*b*),

$$\|\Delta\mathfrak{R}_\lambda v\|_{\varrho-2\delta} \le |\Delta\lambda|\mathfrak{C}_R^2\,\gamma^{-2}\delta^{-2\nu}\ \|v\|_\varrho. \tag{1.84}$$

*On the other hand, since $2\delta\in(0,\varrho)$, then by **Proposition 1.26** part* (*c*),

$$\|\Delta\mathfrak{R}_\lambda v\|_{\varrho-2\delta} \le |\Delta\lambda|\mathfrak{C}_R^*\,2^{-2\nu}\,\gamma^{-2}\delta^{-2\nu}\ \|v\|_\varrho. \tag{1.85}$$

Thus, the comparison that we must carry out must be made between $\mathfrak{C}_R^2$ and $\mathfrak{C}_R^*\,2^{-2\nu}$.

$$\begin{aligned}\frac{\mathfrak{C}_R^2}{\mathfrak{C}_R^*\,2^{-2\nu}} &= \frac{\left(\frac{1}{1+a}\frac{\sqrt{2\zeta(2)\Gamma(2\nu+1)}}{(4\pi)^\nu}\right)^2}{\frac{1}{2}\frac{1}{(1+a)^2}\frac{\sqrt{2\zeta(4)\Gamma(4\nu+1)}}{(4\pi)^{2\nu}}}2^{2\nu} = \sqrt{\frac{8\zeta(2)^2\Gamma(2\nu+1)^2\,2^{4\nu}}{\zeta(4)\Gamma(4\nu+1)}}\\ &= \sqrt{\frac{8\left(\frac{\pi^2}{6}\right)^2\Gamma(2\nu+1)^2 2^{4\nu}}{\frac{\pi^4}{90}\Gamma(4\nu+1)}} = \sqrt{\frac{5\cdot 2^{4\nu+2}\cdot\Gamma(2\nu+1)^2}{\Gamma(4\nu+1)}}\\ &= \sqrt{5\cdot\frac{(2\nu\Gamma(2\nu))^2 2^{4\nu}2^2}{4\nu\Gamma(4\nu)}} = \sqrt{20\nu\cdot 2^{4\nu}\frac{\Gamma(2\nu)^2}{\Gamma(4\nu)}} \quad (\nu\ge 1).\end{aligned} \tag{1.86}$$

We have used the property: $\Gamma(z+1)=z\Gamma(z)$, for $\mathrm{Re}(z)>0$.
Moreover, a good approximation for the gamma function is given by Stirling's formula:

$$\Gamma(z) = \sqrt{\frac{2\pi}{z}}\left(\frac{z}{e}\right)^z\left(1+\mathrm{O}\left(\frac{1}{z}\right)\right) \quad (\mathrm{Re}(z)>0). \tag{1.87}$$

From (1.86)*and* (1.87) *we obtain:*

$$\begin{aligned}\frac{\mathfrak{C}_R^2}{\mathfrak{C}_R^*\,2^{-2\nu}} &= \sqrt{20\nu\cdot 2^{4\nu}\frac{\Gamma(2\nu)^2}{\Gamma(4\nu)}} \simeq \sqrt{\frac{20\nu\frac{2\pi}{2\nu}\left(\frac{2\nu}{e}\right)^{4\nu}2^{4\nu}}{\sqrt{\frac{2\pi}{4\nu}}\left(\frac{4\nu}{e}\right)^{4\nu}}}\\ &= \sqrt{20\sqrt{2\pi\nu}} = (800\pi\nu)^{\frac{1}{4}} \quad (\nu\ge 1).\end{aligned} \tag{1.88}$$

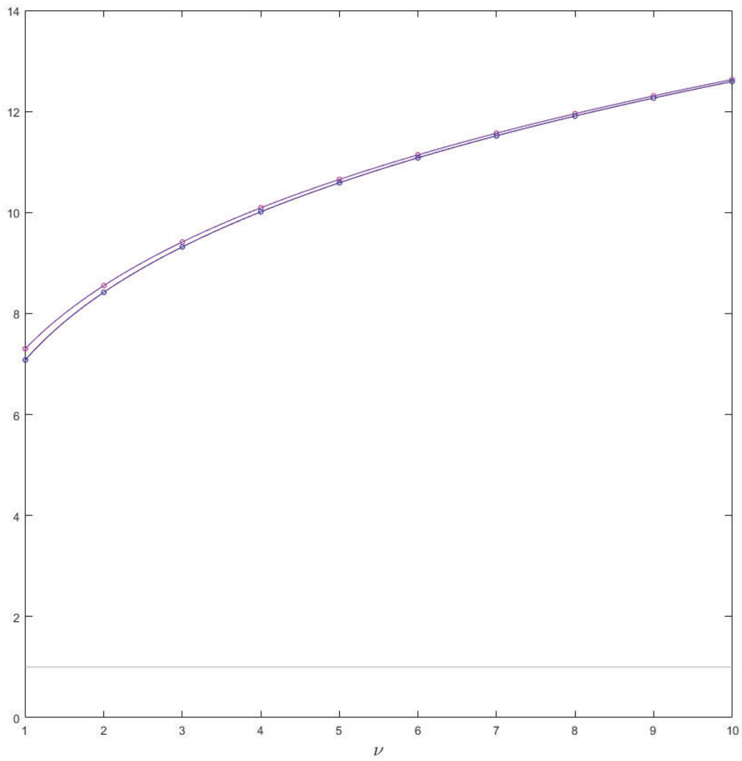

FIGURE 1.8: $\frac{\mathfrak{C}_R^2}{\mathfrak{C}_R^*\, 2^{-2\nu}} = \sqrt{20\nu \cdot 2^{4\nu}\frac{\Gamma(2\nu)^2}{\Gamma(4\nu)}} \simeq (800\pi\nu)^{\frac{1}{4}}$

ν	$\frac{\mathfrak{C}_R^2}{\mathfrak{C}_R^*\, 2^{-2\nu}} = \sqrt{20\nu \cdot 2^{4\nu}\frac{\Gamma(2\nu)^2}{\Gamma(4\nu)}}$	$\frac{\mathfrak{C}_R^2}{\mathfrak{C}_R^*\, 2^{-2\nu}} \simeq (800\pi\nu)^{\frac{1}{4}}$
1	7.3029674334022152	7.0804354027573764
2	8.5523597411975807	8.4201041582762297
3	9.4158381813839949	9.3183770339577983
4	10.0917315582136773	10.0132477740860892
5	10.6540732633396313	10.5877204500288240
6	11.1393301229580857	11.0814802690605649
7	11.5683888457418096	11.5168702727403698
8	11.9544238169449581	11.9078254972283375
9	12.3063245811731967	12.2636738572851822
10	12.6303967775344343	12.5909924908340898

```matlab
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Author   :     Pello Garcia, Juan
% Subject  :     PhD
% Date     :     January 2022
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% File     :     RussmannRatioTable.m
% Input    :     'filename_table.tex', text file in which the results will be
%                written in LaTeX code
%                numax , maximun value of the nu variable to be considered
% Output   :     LaTeX file (.tex) with the code of a table (tabular) with
%                three columns:
%                nu, C_r^2/(C_r^* 2^(-2 nu)), and the Stirling approximation
%                (800 pi nu)^(1/4)
% Synopsis :     Comparisson of Rüssmann constants
% Syntax   :     RussmannRatioTable('filename_table.tex',numax)
% Example  :     RussmannRatioTable('RussmannRatio.tex',10)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function RussmannRatioTable(filename,numax)

fichero = fopen(filename,'w');  % Opening the output file
                                % Writing the header of the table
fprintf(fichero,'\\begin{center}\n');
fprintf(fichero,'\\begin{tabular}{|c|r|r|}\n');
fprintf(fichero,'\\hline\\cline{1-3}&&\\\\ \n');
fprintf(fichero,['$\\nu$ & $\\frac{\\mathfrak{C}_R^2}{\\mathfrak{C}_R^*\\: ' ...
' 2^{-2\\nu}}=\\sqrt{20\\nu\\cdot 2^{4\\nu}\\frac{\\Gamma(2\\nu)^2}' ...
    '{\\Gamma(4\\nu)}}$ & $\\frac{\\mathfrak{C}_R^2}{\\mathfrak{C}_R^*\\:' ...
    ' 2^{-2\\nu}}\\simeq(800\\pi\\nu)^{\\frac{1}{4}}$ \\\\\n']);
fprintf(fichero,'&&\\\\ \\cline{1-3}&&\\\\\n');

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
format long

for k=1:numax
    fprintf(fichero,' $%d$ & $%.16f$ & $%.16f$ \\\\[2mm]\n',...
        k,RussmannRatio(k),RussmannRatioStirling(k));
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf(fichero,'\\hline\n');   % Writing the foot of the table
fprintf(fichero,'\\end{tabular}\n');
fprintf(fichero,'\\end{center}\n');
fclose(fichero);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function mu=RussmannRatio(t)

h=@(nu)sqrt(20*nu*2.^(4*nu).*gamma(2*nu).^2./gamma(4*nu));
mu=h(t);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function mu=RussmannRatioStirling(t)

format long
h=@(nu)(800*pi*nu).^(1/4);
mu=h(t);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

Corollary 1.28

Let $\varrho > 0$, $m \in \mathbb{N}$, $a \in (0,1)$, $\lambda \in [a, \frac{1}{a}]$, $\omega \in \mathcal{D}(\gamma, \nu)$, *and* $\delta \in (0, \frac{1}{m+1}\varrho)$.
The following estimates hold, $\forall v \in \mathcal{A}_{\varrho - m\delta, 0}$:

(*a*)

$$|\mathfrak{R}_\lambda v(\theta)| \leq \mathfrak{C}_R\, \gamma^{-1}\delta^{-\nu}\|v\|_{\varrho - m\delta}, \quad \forall \theta \in \mathbb{T}_{\varrho-(m+1)\delta}. \tag{1.89}$$

(*b*)

$$|\mathfrak{R}_\lambda v(\theta)| \leq (m+1)^\nu \mathfrak{C}_R\, \gamma^{-1}\varrho^{-\nu}\|v\|_{\varrho - m\delta}, \quad \forall \theta \in \mathbb{T}_{\frac{m}{m+1}\varrho - m\delta}. \tag{1.90}$$

(*c*)

$$|\mathfrak{R}_\lambda v(\theta)| \leq (m+1)^\nu \mathfrak{C}_R\, \gamma^{-1}\varrho^{-\nu}\|v\|_{\frac{1}{m+1}\varrho}, \quad \forall \theta \in \mathbb{T}. \tag{1.91}$$

Proof.

(a) First, fix any $\delta \in (0, \frac{1}{m+1}\varrho)$ and call $\varrho^* = \varrho - m\delta$. According with **Theorem 1.20** we know that for every $\delta^* \in (0, \varrho^*)$,

$$|\mathfrak{R}_\lambda v(\theta)| \leq \mathfrak{C}_R\, \gamma^{-1}(\delta^*)^{-\nu}\|v\|_{\varrho^*}, \quad \forall \theta \in \mathbb{T}_{\varrho^* - \delta^*}. \tag{1.92}$$

Notice that due to our choice we have $0 < \delta < \frac{1}{m+1}\varrho < \varrho^* = \varrho - m\delta < \varrho$.

Therefore, if we choose $\delta^* = \delta$ in (1.92), we obtain

$$|\mathfrak{R}_\lambda v(\theta)| \leq \mathfrak{C}_R\, \gamma^{-1}\delta^{-\nu}\|v\|_{\varrho - m\delta}, \quad \forall \theta \in \mathbb{T}_{\varrho-(m+1)\delta}. \tag{1.93}$$

(b) If we choose $\delta^* = \frac{1}{m+1}\varrho$ in (1.92), we obtain

$$|\mathfrak{R}_\lambda v(\theta)| \leq (m+1)^\nu \mathfrak{C}_R\, \gamma^{-1}\varrho^{-\nu}\|v\|_{\varrho - m\delta}, \quad \forall \theta \in \mathbb{T}_{\frac{m}{m+1}\varrho - m\delta}. \tag{1.94}$$

(c) Taking limits in (a) or (b) as $\delta \to \frac{1}{m+1}\varrho^-$ we get straightforward the desired result:

$$|\mathfrak{R}_\lambda v(\theta)| \leq (m+1)^\nu \mathfrak{C}_R\, \gamma^{-1}\varrho^{-\nu}\|v\|_{\frac{1}{m+1}\varrho}, \quad \forall \theta \in \mathbb{T}. \tag{1.95}$$

□

Chapter 2

Invariant curves in 1–D skew–products

In this chapter we face up to the initial objective of this work. Our aim is to design a KAM procedure to demonstrate the existence of invariant curves for one–dimensional quasi–periodic skew–products under certain non–degeneracy conditions. We will use the translated graph method for the very particular frame in which the base is the torus, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, and the fiber is the real line, $\mathbb{R}$, giving sufficient conditions for which the Newton–like method to be designed converges quadratically, and thus formulate them in a posteriori format. The challenge, on the one hand, is to fertilize the land for the creation of a methodology for the study and classification of bifurcations of invariant curves related to perturbations of this kind of skew–products, and on the other hand, implement numerical methods of representation. We will employ all the tools which were described in the corresponding sections (as the invariance equation, topological and linear conjugacy of skew–products, small denominators and cohomological equations,...)[1] plus new ones (as linearization of a skew–product, reducibility, the translated graph method itself, and KAM theory). We start describing in detail the setting with the explicit conditions that we assume for granted henceforth.
Consider a quasi–periodic skew–product of the form:

$$\begin{array}{rccc} \psi: & \mathbb{T}\times\mathbb{R} & \longrightarrow & \mathbb{T}\times\mathbb{R} \\ & (\theta,x) & \longmapsto & \psi(\theta,x) = (\theta+\omega, f(\theta,x))\ , \end{array} \tag{2.1}$$

where the frequency is given, $\omega \in \mathcal{DC}(\gamma,\nu)$, and it is Diophantine with constant $\gamma > 0$ and class $\nu \geq 1$, and the function $f : \mathbb{T}\times\mathbb{R} \longrightarrow \mathbb{R}$ is real analytic in $\mathbb{T}\times\mathbb{R}$.
In other words, we can think of f as the restriction to $\mathbb{T}\times\mathbb{R}$ of a function

$$f : \mathbb{T}_\varrho \times U \subseteq \mathbb{C} \longrightarrow \mathbb{C}$$

which we denote in the same manner, taking real values for real arguments and such that for every $\kappa \in \mathcal{A}_\varrho$ the composition $f \circ (\mathrm{I}_{\mathbb{T}_\varrho} \times \kappa) \in \mathcal{A}_\varrho$ is also a real analytic function on the same strip $\mathbb{T}_\varrho$.
Notice that for every $x \in \mathbb{R}$ the restriction to the one dimensional torus

$$\begin{array}{rccc} f(\,\cdot\,,x): & \mathbb{T} & \longrightarrow & \mathbb{R} \\ & \theta & \longmapsto & f(\theta,x) \end{array}$$

is a real analytic function, i.e. $f(\,\cdot\,,x) \in \mathcal{A}_\varrho, \forall x \in \mathbb{R}$.

---

[1] See **Appendix** I. and **Chapter** 1.

Once we have established this starting point, we want to find sufficient conditions in order to prove the existence of invariant curves for this kind of skew–products, namely, functions $\kappa : \mathbb{T} \longrightarrow \mathbb{R}$ such that the invariance equation,

$$f(\theta, \kappa(\theta)) = \kappa(\theta + \omega) \, , \, \forall \theta \in \mathbb{T} \tag{2.2}$$

holds[2].
Henceforth, our challenge consists, briefly speaking, in proving the following result:
If we have a good enough approximation of an invariant curve with frequency ω, then, under certain non–degeneracy and non–resonance conditions, there exists a true invariant curve nearby. After finding such conditions, we also analyze the regularity and local uniqueness of these invariant curves.

Remark 2.1

Let us assume the specified properties for f.

- *If $\sigma = \mathrm{I} \times \kappa \in \Gamma(\mathbb{T}, \mathbb{R})$ is a cross section with $\kappa \in \mathcal{A}_\varrho$, then $f \circ \sigma \in \mathcal{A}_\varrho$ as well.*
- *If*
$$\begin{array}{rccl} E: & \mathbb{T}_\varrho & \longrightarrow & \mathbb{C} \\ & \theta & \longmapsto & E(\theta) = f(\theta, \kappa(\theta)) - \kappa(\theta + \omega) \end{array}$$
is the so–called error function *related to ψ w.r.t. κ and $\kappa \in \mathcal{A}_\varrho$, then $E \in \mathcal{A}_\varrho$ too.*

[2] Recall that this is equivalent to the existence of a cross section $\sigma \in \Gamma(\mathbb{T} \times \mathbb{R})$ such that $\psi \circ \sigma = \sigma \circ \mathcal{R}_\omega$ (see **Definition I.15** and **Proposition I.16**). We know that, under these conditions, the invariant section is of the form $\sigma = \mathrm{I}_{\mathbb{T}} \times \kappa$.

2.1 Linearization of a skew–product

Before going on to describe the translated graph method, we dedicate this section to providing the specific definition of linearization of a skew–product.
Let us consider a skew–product of the form

$$\begin{array}{rccl} \psi: & \mathbb{T}^d \times \mathbb{R}^n & \longrightarrow & \mathbb{T}^d \times \mathbb{R}^n \\ & (\theta, x) & \longmapsto & \psi(\theta, x) = (\varphi(\theta), f(\theta, x)) \ , \end{array} \tag{2.3}$$

where $\varphi : \mathbb{T}^d \longrightarrow \mathbb{T}^d$ is a diffeomorphism and the function

$$f: \ \mathbb{T}^d \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

satisfies:

(i) $f \in \mathcal{C}^0(\mathbb{T}^d, \mathbb{R}^n)$;

(ii) If we consider the fiber maps

$$\begin{array}{rccl} f_\theta: & \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ & x & \longmapsto & f_\theta(x) = f(\theta, x) \end{array} \quad (\theta \in \mathbb{T}^d) \ ,$$

then $f_\theta \in \mathcal{C}^k(\mathbb{R}^n)$ $(k \geq 1)$, $\forall \theta \in \mathbb{T}^d$.

Fixing some $\theta \in \mathbb{T}^d$ and applying the Taylor expansion of $\psi_\theta = \psi(\theta, \cdot)$ about $x_0 \in \mathbb{R}^n$, there exists $r > 0$ such that for every $h \in B(x_0, r)$,

$$\psi_\theta(x_0 + h) = \psi_\theta(x_0) + D\psi_\theta(x_0)h + o(\|h\|^2).$$

Equivalently,

$$\begin{aligned} \psi(\theta, x_0 + h) &= (\varphi(\theta), f(\theta, x_0)) + (0, D_x f(\theta, x_0)h) + O(\|(0,h)\|^2_{\mathbb{T}^n \times \mathbb{R}^d}) \\ &= (\varphi(\theta), f(\theta, x_0) + D_x f(\theta, x_0)h) + O(\|(0,h)\|^2_{\mathbb{T}^d \times \mathbb{R}^n}) \ . \end{aligned}$$

This fact gives rise to the following definition.

Definition 2.2 Linearization of a skew–product

We call linearization *of the skew–product* (2.3) *at a point* $x_0 \in \mathbb{R}^n$ *to the linear skew–product given by*

$$\begin{array}{rccl} D\psi(x_0): & \mathbb{T}^d \times \mathbb{R}^n & \longrightarrow & \mathbb{T}^d \times \mathbb{R}^n \\ & (\theta, x) & \longmapsto & D\psi(x_0)(\theta, x) = (\varphi(\theta), D_x f(\theta, x_0)x) \ , \end{array} \tag{2.4}$$

This definition can be extended to a map parameterized by $\kappa : \mathbb{T}^d \longrightarrow \mathbb{R}^n$ *in a way that the linearization of* (2.3) *about* κ *is the linear skew–product*

$$\begin{array}{rccl} D\psi(\kappa): & \mathbb{T}^d \times \mathbb{R}^n & \longrightarrow & \mathbb{T}^d \times \mathbb{R}^n \\ & (\theta, x) & \longmapsto & D\psi(\kappa)(\theta, x) = (\varphi(\theta), D_x f(\theta, \kappa(\theta))x) \ . \end{array} \tag{2.5}$$

◇

Thereby, the dynamics of ψ at a point x_0 (resp. about a map $\kappa(\theta)$), is determined by the linear skew–product given by its linearization, $D\psi(x_0)$ (resp. $D\psi(\kappa)$).

2.2 Reducibility and the Lyapunov exponent

We set aside for a while the translated graph method in order to define, in this section, the reducibility property for linear skew–products in general, give a characterization, and thereafter we return to it. Furthermore, we prove here that one-dimensional non–singular quasi–periodic skew–products are always reducible[3].

Definition 2.3 Reducibility for linear skew–products

Let (φ, ψ) be a linear skew–product defined over the trivial vector bundle[4] $(E = B \times F, B, \pi, F)$, with

$$\begin{array}{rccc} \psi: & B \times F & \longrightarrow & B \times F \\ & (\theta, x) & \longmapsto & \psi(\theta, x) = (\varphi(\theta), m(\theta)x) \end{array} \tag{2.6}$$

where φ is a homeomorphism of B and $m : B \longrightarrow \mathcal{M}_n(\mathbb{K})$ is continuous.
It is said that (φ, ψ) is reducible if it is linearly conjugate to another linear skew–product $(\widetilde{\varphi}, \widetilde{\psi})$ of the form:

$$\begin{array}{rccc} \widetilde{\psi}: & B \times F & \longrightarrow & B \times F \\ & (\theta, x) & \longmapsto & \widetilde{\psi}(\theta, x) = (\widetilde{\varphi}(\theta), \lambda x) \end{array} \tag{2.7}$$

where $\lambda \in \mathcal{M}_n(\mathbb{K})$ is independent of θ, namely it is a constant $n \times n-$dimensional matrix, being $n = \dim_{\mathbb{K}}(F)$.

◇

Remark 2.4

A linear quasi–periodic skew–product of the form

$$\psi(\theta, x) = (\theta + \omega, m(\theta)x), \ (\theta, x) \in \mathbb{T}^d \times \mathbb{R}^n$$

is invertible if and only if $\det(m(\theta)) \neq 0, \ \forall \theta \in \mathbb{T}^d$.
This is an immediate consequence of **Proposition I.19**. *Furthermore, in this case the inverse is given by:*

$$\psi^{-1}(\theta, x) = (\theta - \omega, m(\theta - \omega)^{-1}x), \ (\theta, x) \in \mathbb{T}^d \times \mathbb{R}^n.$$

In the one dimensional case, $n = 1$, the condition for the invertibility becomes into

$$m(\theta) \neq 0, \ \forall \theta \in \mathbb{T}^d.$$

Such an invertible skew–product will be referred from now on as a non–singular skew–product. Notice that, by continuity, $m(\theta) > 0, \ \forall \theta \in \mathbb{T}^d$ or $m(\theta) < 0, \ \forall \theta \in \mathbb{T}^d$. Henceforth, we may assume that $m(\theta) > 0, \ \forall \theta \in \mathbb{T}$ without mentioning it. This assumption will not entail any loss of generality in what has to do with our challenges.

---

[3] This property will allow us to make a change of variable in the system (3.5) that will lead us to obtain solutions by means of cohomological equations.

[4] Here we assume that F is a topological finite dimensional $\mathbb{K}-$vector space with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, and $n = \dim_{\mathbb{K}}(F) < \infty$. The base space B is a topological space.

Definition 2.5 Lyapunov exponent

Let us consider a linear quasi–periodic skew–product over the real line defined by

$$\begin{array}{rccc} \psi: & \mathbb{T}\times\mathbb{R} & \longrightarrow & \mathbb{T}\times\mathbb{R} \\ & (\theta,x) & \longmapsto & \psi(\theta,x)=(\theta+\omega,m(\theta)x) \end{array} \tag{2.8}$$

where $m:\mathbb{T}\longrightarrow\mathbb{R}$ *is continuous and positive, and* $\omega\in\mathcal{DC}(\gamma,\nu)$ *is Diophantine*[5].
The Lyapunov exponent of the linear skew–product (2.8) *is defined*[6] *as*

$$\Lambda=\int_{\mathbb{T}}\log(m(\theta))\,d\theta\ . \tag{2.9}$$

We say that ψ *is a non–singular quasi–periodic skew–product if its Lyapunov exponent is finite.*

◇

Theorem 2.6 Reducibility of one-dimensional skew–products

Every non–singular one–dimensional linear quasi–periodic skew–product is reducible.

Proof. Let (φ,ψ) be a linear quasi–periodic skew–product defined on the cylinder $\mathbb{T}\times\mathbb{R}$, that is, $\varphi=\mathcal{R}_\omega$ be an ergodic rigid rotation with a Diophantine frequency, $\omega\in\mathcal{DC}(\gamma,\nu)$, and

$$\begin{array}{rccc} \psi: & \mathbb{T}\times\mathbb{R} & \longrightarrow & \mathbb{T}\times\mathbb{R} \\ & (\theta,x) & \longmapsto & \psi(\theta,x)=(\theta+\omega,m(\theta)x) \end{array}\ ,$$

with $m:\mathbb{T}\longrightarrow\mathbb{R}$ a non–vanishing continuous function.
We have to prove that there exist another linear skew–product (h,H) which is invertible and conjugates (φ,ψ) to the quasi–periodic skew–product $(\widetilde{\varphi},\widetilde{\psi})$ given by:

$$\begin{array}{rccc} \widetilde{\psi}: & \mathbb{T}\times\mathbb{R} & \longrightarrow & \mathbb{T}\times\mathbb{R} \\ & (\theta,x) & \longmapsto & \widetilde{\psi}(\theta,x)=(\theta+\widetilde{\omega},\widetilde{m}(\theta)x) \end{array}\ ,$$

and such that $\widetilde{m}(\theta)=\lambda$ is a real constant, independent of θ. We are going to show that (h,H) can be chosen also to be quasi–periodic, that is, $h(\theta)=\theta+\nu$, and

$$\begin{array}{rccc} H: & \mathbb{T}\times\mathbb{R} & \longrightarrow & \mathbb{T}\times\mathbb{R} \\ & (\theta,x) & \longmapsto & H(\theta,x)=(\theta+\nu,c(\theta)x) \end{array}\ ,$$

with $c:\mathbb{T}\longrightarrow\mathbb{R}$ a non–vanishing continuous function, since H must be invertible[7].

According to **Proposition I.27**, and translating terms to this case, $(\mathcal{R}_\omega,\psi)$ and $(\mathcal{R}_{\widetilde{\omega}},\widetilde{\psi})$ are linearly conjugate, by means of $(\mathcal{R}_\nu,H)$ if and only if the following conditions hold:

---

[5]See **Definition 1.15**

[6]In fact, the Lyapunov exponent is defined as $\limsup_{n\to\infty}\frac{1}{n}\log\prod_{j=0}^{n-1}m(\theta+j\omega)$, but this lim sup, when $m(\theta)$ never vanishes, does not depend on θ and coincides with the value taken in our definition.This is a consequence of the unique ergodicity of irrational rotations on $\mathbb{T}$ and the Birkhoff Ergodic Theorem. cf. [19], [38].

[7]In fact, we can chose $\nu=0$ and $c(\theta)>0,\forall\theta\in\mathbb{T}$.

(i) $c(\theta) \neq 0, \ \forall \theta \in \mathbb{T}$;

(ii) $\mathcal{R}_{\widetilde{\omega}} = {\mathcal{R}_{\nu}}^{-1} \circ \mathcal{R}_{\omega} \circ \mathcal{R}_{\nu}$, i.e. $\mathcal{R}_{\omega}$ and $\mathcal{R}_{\widetilde{\omega}}$ are topologically conjugate in $\mathbb{T}$, by means of $\mathcal{R}_{\nu}$;

(iii) $m(\theta+\nu)c(\theta) = c(\theta+\widetilde{\omega})\lambda, \ \forall \theta \in \mathbb{T}$.

Since ${\mathcal{R}_{\nu}}^{-1} = \mathcal{R}_{-\nu}$, condition (ii) holds if and only if $\widetilde{\omega} = \omega$, no matter the value of ν.
Then, condition (iii) writes

$$m(\theta+\nu)c(\theta) = c(\theta+\omega)\lambda, \ \forall \theta \in \mathbb{T}. \tag{2.10}$$

which is equivalent to

$$m(\theta)c(\theta-\nu) = c(\theta-\nu+\omega)\lambda, \ \forall \theta \in \mathbb{T}. \tag{2.11}$$

Taking natural logarithms in this equation we have:

$$\log(m(\theta)) + \log(c(\theta-\nu)) = \log(c(\theta-\nu+\omega)) + \log(\lambda), \ \forall \theta \in \mathbb{T}. \tag{2.12}$$

Choose $\lambda = e^{\Lambda}$, where Λ is the Lyapunov exponent of the linear skew–product $(\mathcal{R}_{\omega}, \psi)$, that is, $\Lambda = \displaystyle\int_{\mathbb{T}} \log(m(\theta))\, d\theta$, and define the following functions:

$$\begin{aligned} u(\theta) &= \log(c(\theta-\nu)), & (2.13)\\ v(\theta) &= \log(m(\theta)) - \Lambda. & (2.14) \end{aligned}$$

Notice that v is a zero–average function which is known. Indeed,

$$\begin{aligned} <v> &= \int_{\mathbb{T}} v(\theta)\, d\theta = \int_{\mathbb{T}} (\log(m(\theta)) - \Lambda)\, d\theta = \int_{\mathbb{T}} \log(m(\theta))\, d\theta - \Lambda \int_{\mathbb{T}} d\theta \\ &= \int_{\mathbb{T}} \log(m(\theta))\, d\theta - \Lambda = 0 \,. \end{aligned}$$

Moreover, (2.10) is equivalent, by means of (2.11), (2.12), (2.13), and (2.14), to the small denominators equation:

$$u(\theta+\omega) - u(\theta) = v(\theta), \ \theta \in \mathbb{T} \,. \tag{2.15}$$

As long as the frequency ω is Diophantine, that is, it satisfies the Diophantine condition (1.19), the small denominators equation (2.15) has only one solution with zero average, which can be denoted by $u(\theta) = \mathfrak{R}_1 v(\theta)$. Namely, according to **Theorem 1.20**, u is a 1–periodic function which is defined by its Fourier expansion:

$$u(\theta) = \sum_{k \in \mathbb{Z}\setminus\{0\}} \widehat{u}_k\, e^{2\pi k\theta i} \quad (\theta \in \mathbb{T}) \,, \tag{2.16}$$

where

$$\widehat{u}_k = \frac{\widehat{v}_k}{e^{2\pi k\omega i} - 1}, \ \forall k \in \mathbb{Z} \setminus \{0\}. \tag{2.17}$$

Here, $\widehat{v}_k$ represents the Fourier coefficients of the function v defined in (2.14), that is:

$$\begin{aligned} \widehat{v}_k &= \int_{\mathbb{T}} v(\theta) e^{-2\pi k\theta i}\, d\theta = \int_{\mathbb{T}} (\log(m(\theta)) - \Lambda) e^{-2\pi k\theta i}\, d\theta \\ &= \int_{\mathbb{T}} \log(m(\theta)) e^{-2\pi k\theta i}\, d\theta - \Lambda \int_{\mathbb{T}} e^{-2\pi k\theta i}\, d\theta \\ &= \int_{\mathbb{T}} \log(m(\theta)) e^{-2\pi k\theta i}\, d\theta \,, \end{aligned}$$

since $\int_{\mathbb{T}} e^{-2\pi k\theta i}\, d\theta = 0,\ \forall k \in \mathbb{Z}\setminus\{0\}$.
Thus,

$$u(\theta) = \mathfrak{R}_1 v(\theta) = \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{T}} \log(m(\vartheta))e^{-2\pi k\vartheta i}\, d\vartheta}{e^{2\pi k\omega i}-1}\, e^{2\pi k\theta i} \quad (\theta\in\mathbb{T}) \ , \tag{2.18}$$

Finally, we have, from (2.13):

$$c(\theta) = e^{u(\theta+\nu)} \ , \tag{2.19}$$

where u is given by (2.18) and $\nu \in \mathbb{T}$ can be chosen freely. To end the proof, we define:

$$H(\theta,x) = (\theta+\nu, c(\theta)x), \quad (\theta,x)\in\mathbb{T}\times\mathbb{R}, \tag{2.20}$$

and we only need to check that $\widetilde{\psi} = H^{-1}\circ\psi\circ H$.
Indeed, from (2.10) and the fact that $H^{-1}(\theta,x) = (\theta-\nu, c(\theta-\nu)^{-1}x)$, $\forall(\theta,x)\in\mathbb{T}\times\mathbb{R}$ we have:

$$\begin{aligned}
(H^{-1}\circ\psi\circ H)(\theta,x) &= H^{-1}(\psi(H(\theta,x))) = H^{-1}(\psi(\theta+\nu, c(\theta)x)) \\
&= H^{-1}(\theta+\nu+\omega, m(\theta+\nu)c(\theta)x) \\
&= ((\theta+\nu+\omega)-\nu, c((\theta+\nu+\omega)-\nu)^{-1}m(\theta+\nu)c(\theta)x) \\
&= (\theta+\omega, c(\theta+\omega)^{-1}m(\theta+\nu)c(\theta)x) = (\theta+\omega,\lambda x) \\
&= \widetilde{\psi}(\theta,x) \ , \ \forall(\theta,x)\in\mathbb{T}\times\mathbb{R},
\end{aligned}$$

and the proof is complete. □

Remark 2.7

In the above exposition, we have proved the reducibility of a one dimensional non–singular linear quasi–periodic skew–product. Furthermore, we have seen that the frequency $\nu\in\mathbb{T}$ can be chosen freely. In practical cases we will take usually $\nu = 0$. With all, we can restate the results obtained summarizing them in the following way:
Given a one dimensional non–singular linear quasi–periodic skew–product,

$$\begin{array}{rccc}
\psi: & \mathbb{T}\times\mathbb{R} & \longrightarrow & \mathbb{T}\times\mathbb{R} \\
& (\theta,x) & \longmapsto & \psi(\theta,x) = (\theta+\omega, m(\theta)x)
\end{array} \ ,$$

where $\omega\in\mathcal{DC}(\gamma,\nu)$ is Diophantine, if $m:\mathbb{T}\longrightarrow\mathbb{R}$ is a continuous positive function, there exist a positive constant $\lambda > 0$ which is called reducibility constant *or* Lyapunov multiplier *and a so–called* Floquet transformation *$c:\mathbb{T}\longrightarrow\mathbb{R}$, which is continuous and positive, such that the skew–product ψ is linearly conjugate to the one of the form:*

$$\begin{array}{rccc}
\widetilde{\psi}: & \mathbb{T}\times\mathbb{R} & \longrightarrow & \mathbb{T}\times\mathbb{R} \\
& (\theta,x) & \longmapsto & \widetilde{\psi}(\theta,x) = (\theta+\omega, \lambda x)
\end{array} \ ,$$

where $\lambda = c(\theta+\omega)^{-1}m(\theta)c(\theta)$ does not depend on θ.
More precisely:

$$\lambda = e^{\Lambda} \ , \ \textit{with } \Lambda = \int_{\mathbb{T}} \log(m(\theta))\, d\theta \tag{2.21}$$

the Lyapunov exponent of ψ and

$$c(\theta) = e^{u(\theta)} \tag{2.22}$$

with $u(\theta) = \mathfrak{R}_1(\log(m(\theta))-\Lambda)$, that is:

$$u(\theta) = \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{T}} \log(m(\vartheta))e^{-2\pi k\vartheta i}\, d\vartheta}{e^{2\pi k\omega i}-1}\, e^{2\pi k\theta i} \quad (\theta\in\mathbb{T}) \ . \tag{2.23}$$

Under these conditions, if $H(\theta, x) = (\theta, c(\theta)x)$, $\forall(\theta, x) \in \mathbb{T} \times \mathbb{R}$, *then* $\widetilde{\psi} = H^{-1} \circ \psi \circ H$.

Under the same conditions as in **Theorem 2.6** we have the following result.

Corollary 2.8 Normalization

Given any $c_0 > 0$, *there exist a reducibility constant* $\lambda > 0$ *and a Floquet transformation* $c : \mathbb{T} \longrightarrow \mathbb{R}$ (*continuous and positive*) *such that:*

(*i*) $m(\theta)c(\theta) = c(\theta + \omega)\lambda,\ \theta \in \mathbb{T}$.

(*ii*) $< c >= c_0$.

Proof. Take $\lambda = e^{\Lambda}$, with $\Lambda = \displaystyle\int_{\mathbb{T}} \log(m(\theta))\, d\theta$ the Lyapunov exponent of the linear skew–product. Now, define:

$$c(\theta) = \alpha\, e^{\mathfrak{R}_1 v(\theta)}, \quad \theta \in \mathbb{T}. \tag{2.24}$$

where $\alpha = \dfrac{c_0}{\displaystyle\int_{\mathbb{T}} e^{\mathfrak{R}_1 v(\vartheta)} d\vartheta}$, $v(\theta) = \log(m(\theta)) - \Lambda$, $\theta \in \mathbb{T}$, and $\mathfrak{R}_1 v$ is the unique solution with zero average to the cohomological equation:

$$u(\theta + \omega) - u(\theta) = v(\theta),\ \theta \in \mathbb{T}.$$

(i) Let $u(\theta) = \log(c(\theta)) = u_0 + \widetilde{u}(\theta)$, $\theta \in \mathbb{T}$, where $u_0 =< u >$ and $< \widetilde{u} >= 0$. Then:

$$\begin{aligned}
\widetilde{u}(\theta + \omega) - \widetilde{u}(\theta) &= (u(\theta + \omega) - u_0) - (u(\theta) - u_0) = u(\theta + \omega) - u(\theta) \\
&= \log(c(\theta + \omega)) - \log(c(\theta)) \\
&= \log\left(\alpha\, e^{\mathfrak{R}_1 v(\theta+\omega)}\right) - \log\left(\alpha\, e^{\mathfrak{R}_1 v(\theta)}\right) \\
&= \log\left(e^{\mathfrak{R}_1 v(\theta+\omega)}\right) - \log\left(e^{\mathfrak{R}_1 v(\theta)}\right) \\
&= \mathfrak{R}_1 v(\theta + \omega) - \mathfrak{R}_1 v(\theta) = v(\theta)\ .
\end{aligned}$$

Therefore,
$v(\theta) = \widetilde{u}(\theta + \omega) - \widetilde{u}(\theta) \Rightarrow (v(\theta) + \Lambda) + (\widetilde{u}(\theta) + u_0) = (\widetilde{u}(\theta + \omega) + u_0) + \Lambda$
$\Rightarrow \log(m(\theta)) + \log(c(\theta)) = \log(c(\theta + \omega)) + \log(\lambda)$
$\Rightarrow m(\theta)c(\theta) = c(\theta + \omega)\lambda,\ \theta \in \mathbb{T}$.

(ii) $< c >= \displaystyle\int_{\mathbb{T}} c(\theta) d\theta = \int_{\mathbb{T}} \alpha e^{\mathfrak{R}_1 v(\theta)} d\theta = \alpha \int_{\mathbb{T}} e^{\mathfrak{R}_1 v(\theta)} d\theta = c_0$.

□

2.3 The translated graph method

Definition 2.9 Translated curves

Given a quasi–periodic skew–product as (2.1), which is of the form $\psi = \mathcal{R}_\omega \times f$*, and a cross section* $\sigma = \mathrm{I}_{\mathbb{T}} \times \kappa \in \Gamma(\mathbb{T} \times \mathbb{R})$*, we say that* $\mathfrak{T} = \sigma(\mathbb{T}) = \{(\theta, \kappa(\theta) : \theta \in \mathbb{T})\}$ *(which is the graph of* κ*) is a translated curve w.r.t.* ψ *if there exists a real number* $\tau \in \mathbb{R}$*, which is called translation number, such that:*

$$\psi(\mathfrak{T}) = \mathfrak{T} + \tau\mathfrak{I}\ , \tag{2.25}$$

where $\mathfrak{I} = \{(0, 1)\} \subseteq \mathbb{T} \times \mathbb{R}$.

◇

Definition 2.10 Error function

Given a quasi–periodic skew–product as (2.1), $\psi = \mathcal{R}_\omega \times f$*, the error function related to* ψ *is defined by:*

$$\begin{array}{rrcl} E_\psi : & \Gamma(\mathbb{T} \times \mathbb{R}) & \longrightarrow & \mathcal{C}(\mathbb{T}, \mathbb{T} \times \mathbb{R}) \\ & \sigma = \mathrm{I}_{\mathbb{T}} \times \kappa & \longmapsto & E_\psi(\sigma) = \psi \circ \sigma - \sigma \circ \mathcal{R}_\omega \end{array} \tag{2.26}$$

◇

Remark 2.11

The error function E_ψ*, when applied to a cross section* σ*, is a measure of how far* σ *is from being an invariant section with respect to the skew–product* ψ*, as will become clear in the following proposition..*

Proposition 2.12 Translated curves

Let $\psi = \mathcal{R}_\omega \times f$ *be a quasi–periodic skew–product (2.1),* $\sigma = \mathrm{I}_{\mathbb{T}} \times \kappa \in \Gamma(\mathbb{T} \times \mathbb{R})$ *be a cross section, and* $\tau \in \mathbb{R}$*. Then, the following statements are equivalent:*

(i) $\mathfrak{T} = \sigma(\mathbb{T})$ *is* ψ_τ*–invariant, that is,* $\psi_\tau(\mathfrak{T}) = \mathfrak{T}$*, where* ψ_τ *is the translated quasi–periodic skew–product* $\psi_\tau = \psi + \tau\mathfrak{I}$ *with translation number* τ *(i.e.* $\psi_\tau(\theta, x) = \psi(\theta, x) + (0, \tau)$*,* $\forall(\theta, x) \in \mathbb{T} \times \mathbb{R}$*).*

(ii) $E_{\psi_\tau}(\sigma) = 0$*, where* $E_{\psi_\tau}(\sigma) = \psi_\tau \circ \sigma - \sigma \circ R_\omega$ *is the error function w.r.t the translated quasi–periodic skew–product* ψ_τ*.*

(iii) $f(\theta, \kappa(\theta)) - \kappa(\theta + \omega) + \tau = 0,\ \forall\theta \in \mathbb{T}$.

Proof. First of all, observe that:

$$\begin{aligned} \forall\theta \in \mathbb{T},\ E_{\psi_\tau}(\sigma)(\theta) &= (\psi_\tau \circ \sigma)(\theta) - (\sigma \circ \mathcal{R}_\omega)(\theta) = \psi_\tau(\sigma(\theta)) - \sigma(\mathcal{R}_\omega)(\theta) \\ &= \psi_\tau(\theta, \kappa(\theta)) - \sigma(\theta + \omega) = (\theta + \omega, f(\theta, \kappa(\theta)) + \tau) - (\theta + \omega, \kappa(\theta + \omega)) \\ &= (0, f(\theta, \kappa(\theta)) - \kappa(\theta + \omega) + \tau)\ . \end{aligned}$$

Therefore, $E_{\psi_\tau}(\sigma) = 0 \Leftrightarrow E_{\psi_\tau}(\sigma)(\theta) = 0,\ \forall\theta \in \mathbb{T} \Leftrightarrow f(\theta, \kappa(\theta)) - \kappa(\theta + \omega) + \tau = 0, \forall\theta \in \mathbb{T}$. And (ii) $\Leftrightarrow$ (iii) is proved.
On the other hand, if $\mathfrak{T} = \sigma(\mathbb{T}) = \{(\theta, \kappa(\theta)) : \theta \in \mathbb{T}\}$ is ψ_τ–invariant, then $\psi_\tau(\sigma(\mathbb{T})) = \sigma(\mathbb{T})$. Namely,

$$\begin{aligned} &\forall\theta \in \mathbb{T},\ \exists\,\overline{\theta} \in \mathbb{T} \quad \text{such that} \quad \psi_\tau(\sigma(\theta)) = \sigma(\overline{\theta}),\ \text{and} \\ &\forall\overline{\theta} \in \mathbb{T},\ \exists\,\theta \in \mathbb{T} \quad \text{such that} \quad \psi_\tau(\sigma(\theta)) = \sigma(\overline{\theta}). \end{aligned}$$

Since $\sigma(\overline{\theta}) = (\overline{\theta}, \kappa(\overline{\theta}))$ and $\psi_\tau(\sigma(\theta)) = \psi_\tau(\theta, \kappa(\theta)) = (\theta + \omega, f(\theta, \kappa(\theta)) + \tau)$, we have: $(\overline{\theta}, \kappa(\overline{\theta})) = (\theta + \omega, f(\theta, \kappa(\theta)) + \tau)$ and hence, $\overline{\theta} = \theta + \omega$ and $\kappa(\overline{\theta}) = f(\theta, \kappa(\theta)) + \tau$, that is, $f(\theta, \kappa(\theta)) + \tau = \kappa(\theta + \omega)$ or, $f(\theta, \kappa(\theta)) + \tau - \kappa(\theta + \omega) + \tau = 0$. This proves (i) $\Rightarrow$ (iii).
We end the proof showing (iii) $\Rightarrow$ (i). Assume that (iii) holds. Then, given $\theta \in \mathbb{T}$ we can take $\overline{\theta} = \theta + \omega$ and then: $\sigma(\overline{\theta}) = (\overline{\theta}, \kappa(\overline{\theta})) = (\theta + \omega, \kappa(\theta + \omega)) = (\theta + \omega, f(\theta, \kappa(\theta)) + \tau) = \psi_\tau(\theta, \kappa(\theta)) = \psi_\tau(\sigma(\theta))$.
In the same manner, if $\overline{\theta} \in \mathbb{T}$ we can take $\theta = \overline{\theta} - \omega$, and hence: $\sigma(\overline{\theta}) = \sigma(\theta + \omega) = \psi_\tau(\sigma(\theta))$. $\square$

Definition 2.13 Family of invariant translated curves

Given a quasi–periodic skew–product as (2.1)*, of the form* $\psi = \mathcal{R}_\omega \times f$*, we define the family of invariant translated curves related to* ψ *as the collection*

$$\{(\kappa^{(p)}, \tau^{(p)})\}_{p \in \mathbb{R}}$$

where $\kappa^{(p)} \in \mathcal{A}_\varrho$ *is an invariant translated curve w.r.t* ψ *with translation number* $\tau^{(p)} \in \mathbb{R}$*, i.e.*

$$\left\{ \begin{array}{rcl} f(\theta, \kappa^{(p)}(\theta)) - \kappa^{(p)}(\theta + \omega) + \tau^{(p)} & = & 0 \quad , \forall \theta \in \mathbb{T} \\ < \kappa^{(p)} > & = & p \end{array} \right. \tag{2.27}$$

Another way to refer to this family is to consider the following functions:

$$\begin{array}{rccl} \kappa : & \mathbb{T}_\varrho \times \mathbb{R} & \longrightarrow & \mathbb{C} \\ & (\theta; p) & \longmapsto & \kappa(\theta; p) = \kappa^{(p)}(\theta) \end{array}$$

and

$$\begin{array}{rccl} \tau : & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & p & \longmapsto & \tau(p) = \tau^{(p)} \end{array} .$$

Thus, we may write equations (2.27) *as*

$$\left\{ \begin{array}{rcl} f(\theta, \kappa(\theta; p)) - \kappa(\theta + \omega; p) + \tau(p) & = & 0 \quad (\theta \in \mathbb{T}) \\ < \kappa(\cdot\,; p) > & = & p \end{array} \right. \tag{2.28}$$

$\diamond$

Remark 2.14

Among all the invariant translated curves we are mostly interested in that ones whose translation number is $\tau = 0$*, because those are, obviously, the invariant curves of the original skew–product. This is, in essence, the objective of the so called* translated graph method*. Next, we give a complete and detailed description of the modus operandi.*

2.4 KAM procedure

The objective of this section is to obtain invariant translated curves by simultaneously combining two procedures. On the one hand the invariance of the translated curve and, on the other hand, the reducibility of the skew–product.
Given a skew–product

$$\begin{array}{rccl} \Psi: & \mathbb{T}\times\mathbb{R} & \longrightarrow & \mathbb{T}\times\mathbb{R} \\ & (\theta,x) & \longmapsto & \Psi((\theta,x)=(\theta+\omega,f(\theta,x)) \end{array}$$

and $p \in \mathbb{R}$, we look for $\kappa : \mathbb{T} \longrightarrow \mathbb{R}$ and $\tau \in \mathbb{R}$ such that

$$\left\{ \begin{array}{rcl} f(\theta,\kappa(\theta))-\kappa(\theta+\omega)+\tau & = & 0 \\ <\kappa> & = & p \end{array} \right. \tag{2.29}$$

In an iteration procedure, we assume that we have an approximated solution such that

$$f(\theta,\kappa(\theta))-\kappa(\theta+\omega)+\tau = e^i(\theta) \tag{2.30}$$

$$\frac{\partial f}{\partial x}(\theta,\kappa(\theta))\, c(\theta)-c(\theta+\omega)\lambda = e^r(\theta) \tag{2.31}$$

$$<\kappa> = p \tag{2.32}$$

$$<c> = c_0 \tag{2.33}$$

where c_0 can be freely chosen a priori. We may take, for simplicity, $c_0 = 1$.
We have introduced new conditions that deal with the reducibility.
Here $e^i(\theta)$ and $e^r(\theta)$ are error functions measuring how far $\kappa(\theta)$ is from being an invariant translated curve with translation parameter τ, and what extent the function $c(\theta)$ is a reducibility function with $\lambda > 0$ as the corresponding reducibility constant, respectively. That is, we assume the linearized skew–product be reduced up to an error $e^r(\theta)$.
In these equations, $p \in \mathbb{R}$, and $c_0 > 0$ are fixed, $\kappa(\theta)$, $c(\theta)$, τ, and λ will change through the iterative process:

$$\left\{ \begin{array}{rcll} \kappa(\theta) & \longmapsto & \kappa_1(\theta) & = \kappa(\theta)+c(\theta)\xi^i(\theta) \\ c(\theta) & \longmapsto & c_1(\theta) & = c(\theta)+c(\theta)\xi^r(\theta) \\ \tau & \longmapsto & \tau_1 & = \tau+\Delta\tau \\ \lambda & \longmapsto & \lambda_1 & = \lambda+\Delta\lambda \end{array} \right.$$

with specially chosen $\xi^i(\theta)$, $\xi^r(\theta)$, $\Delta\tau$, and $\Delta\lambda$.
Next, we look for the corrections $\xi^i(\theta)$, $\xi^r(\theta)$, $\Delta\tau$, and $\Delta\lambda$ such that the new objects κ_1, c_1, τ_1, and λ_1 are better approximations, essentially the new error is the square of the previous one.

We use here these notations:

$$\begin{array}{rclrcl} c & = & c(\theta) & c_+ & = & c(\theta+\omega) \\ \xi^i_+ & = & \xi^i(\theta+\omega) & \xi^r_+ & = & \xi^r(\theta+\omega) \\ \eta^i & = & c_+^{-1}e^i & \eta^r & = & c_+^{-1}e^r \end{array}$$

Starting from eqs. (2.30) and (2.32), that is,

$$\left\{ \begin{array}{rcl} f(\theta,\kappa(\theta)) - \kappa(\theta+\omega) + \tau &=& e^i(\theta) \\ <\kappa> &=& p \end{array} \right. \tag{2.34}$$

the error of the invariance condition after one step, with $\kappa_1(\theta) = \kappa(\theta) + c(\theta)\xi^i(\theta)$ and $\tau_1 = \tau + \Delta\tau$, is:

$$\begin{aligned} e_1^i(\theta) &= f(\theta,\kappa_1(\theta)) - \kappa_1(\theta+\omega) + \tau_1 \\ &= f(\theta,\kappa(\theta)+c(\theta)\xi^i(\theta)) - (\kappa(\theta+\omega)+c(\theta+\omega)\xi^i(\theta+\omega)) + \tau + \Delta\tau \\ &= f(\theta,\kappa(\theta)) + \frac{\partial f}{\partial x}(\theta,\kappa(\theta))c(\theta)\xi^i(\theta) + \int_0^1 (1-t)\frac{\partial^2 f}{\partial x^2}(\theta,\kappa(\theta)+tc(\theta)\xi^i(\theta))\left(c(\theta)\xi^i(\theta)\right)^2 dt \\ &\quad -\kappa(\theta+\omega) + \tau - c(\theta+\omega)\xi^i(\theta+\omega) + \Delta\tau \\ &= e^i(\theta) + \left(\frac{\partial f}{\partial x}(\theta,\kappa(\theta))c(\theta) - c(\theta+\omega)\lambda\right)\xi^i(\theta) + c(\theta+\omega)\lambda\xi^i(\theta) \\ &\quad -c(\theta+\omega)\xi^i(\theta+\omega) + \Delta\tau + R^i(\theta) \\ &= e^i(\theta) + e^r(\theta)\xi^i(\theta) - c(\theta+\omega)(\xi^i(\theta+\omega) - \lambda\xi^i(\theta)) + \Delta\tau + R^i(\theta)\ , \end{aligned} \tag{2.35}$$

where

$$R^i(\theta) = \int_0^1 (1-t)\frac{\partial^2 f}{\partial x^2}(\theta,\kappa(\theta)+tc(\theta)\xi^i(\theta))dt\left(c(\theta)\xi^i(\theta)\right)^2\ . \tag{2.36}$$

In order to reduce the error $\xi^i(\theta)$ and the remainder $R^i(\theta)$ as much as possible, we ask

$$c(\theta+\omega)(\xi^i(\theta+\omega) - \lambda\xi^i(\theta)) = e^i(\theta) + \Delta\tau\ . \tag{2.37}$$

Notice that, in such a case,

$$e_1^i(\theta) = e^r(\theta)\xi^i(\theta) + R^i(\theta). \tag{2.38}$$

Dividing (2.37) by c_+,

$$\xi^i(\theta+\omega) - \lambda\xi^i(\theta) = \frac{e^i(\theta)}{c(\theta+\omega)} + \frac{1}{c(\theta+\omega)}\Delta\tau = \eta^i(\theta) + c_+^{-1}(\theta)\Delta\tau. \tag{2.39}$$

Taking averages on both sides,

$$\overline{\xi^i} - \lambda\overline{\xi^i} = \overline{\eta^i} + <c_+^{-1}>\Delta\tau = \overline{\eta^i} + <c^{-1}>\Delta\tau$$

that is,

$$(1-\lambda)\overline{\xi^i} - <c^{-1}>\Delta\tau = <c_+^{-1}e^i>\ . \tag{2.40}$$

Eq. (2.39) can also be written as

$$\overline{\xi^i_+} + \widetilde{\xi^i} - \lambda(\overline{\xi^i} + \widetilde{\xi^i}) = \overline{\eta^i} + \widetilde{\eta^i} + (\overline{c_+^{-1}} + \widetilde{c_+^{-1}})\Delta\tau\ .$$

Since $\overline{\xi^i_+} = \overline{\xi^i}$ and $\overline{c_+^{-1}} = \overline{c^{-1}}$,

$$(1-\lambda)\overline{\xi^i} + \widetilde{\xi^i_+} - \lambda\widetilde{\xi^i} = \overline{\eta^i} + \widetilde{\eta^i} + <c^{-1}>\Delta\tau + \widetilde{c_+^{-1}}\Delta\tau.$$

Subtracting (2.40) from the latter,

$$\widetilde{\xi^i_+} - \lambda\widetilde{\xi^i} = \widetilde{\eta^i} + \widetilde{c_+^{-1}}\Delta\tau,$$

that means,

$$\widetilde{\xi^i} = \mathfrak{R}_\lambda(\widetilde{\eta^i} + \widetilde{c_+^{-1}}\Delta\tau) = \mathfrak{R}_\lambda\widetilde{\eta^i} + \Delta\tau\,\mathfrak{R}_\lambda\widetilde{c_+^{-1}}, \tag{2.41}$$

where $\mathfrak{R}_\lambda$ is the cohomological operator[8].
Additionally, from (2.32), $p =< \kappa_1 >=< \kappa + c\xi^i >=< \kappa > + < c\xi^i >= p+ < c\xi^i >$, which implies that $< c\xi^i >= 0$. Therefore, by (2.41),
$0 =< c\xi^i >=< c(\overline{\xi^i} + \widetilde{\xi^i}) >=< c > \overline{\xi^i}+ < c\,\widetilde{\xi^i} >=< c > \overline{\xi^i}+ < c\,\mathfrak{R}_\lambda\widetilde{\eta^i} > +\Delta\tau < c\,\mathfrak{R}_\lambda\widetilde{c_+^{-1}} >$, or

$$< c > \overline{\xi^i}+ < c\,\mathfrak{R}_\lambda\widetilde{c_+^{-1}} > \Delta\tau = - < c\,\mathfrak{R}_\lambda\widetilde{\eta^i} > \tag{2.42}$$

Equations (2.40) and (2.42) can be written as the linear system,

$$\begin{pmatrix} 1-\lambda & - < c^{-1} > \\ < c > & < c\,\mathfrak{R}_\lambda\widetilde{c_+^{-1}} > \end{pmatrix} \begin{pmatrix} \overline{\xi^i} \\ \Delta\tau \end{pmatrix} = \begin{pmatrix} < c_+^{-1} e^i > \\ - < c\,\mathfrak{R}_\lambda\widetilde{c_+^{-1} e^i} > \end{pmatrix} \tag{2.43}$$

Denoting $b = \mathfrak{R}_\lambda\widetilde{c_+^{-1}}$ and $D = \begin{pmatrix} 1-\lambda & - < c^{-1} > \\ < c > & < c\,b > \end{pmatrix}$, eq. (2.43) can be written as

$$D \begin{pmatrix} \overline{\xi^i} \\ \Delta\tau \end{pmatrix} = \begin{pmatrix} < c_+^{-1} e^i > \\ - < c\,\mathfrak{R}_\lambda\widetilde{c_+^{-1} e^i} > \end{pmatrix} . \tag{2.44}$$

Next we deal with eqs. (2.31) and (2.33), that is, the error of the reducibility condition after one step. From

$$\left\{ \begin{array}{rcl} \frac{\partial f}{\partial x}(\theta, \kappa(\theta))c(\theta) - c(\theta+\omega)\lambda & = & e^r(\theta) \\ < c > & = & c_0 \ , \end{array} \right. \tag{2.45}$$

with $c_1(\theta) = c(\theta)(1 + \xi^r(\theta))$ and $\lambda_1 = \lambda + \Delta\lambda$, we write:

$$\begin{aligned} e_1^r(\theta) &= \frac{\partial f}{\partial x}(\theta, \kappa_1(\theta))c_1(\theta) - c_1(\theta+\omega)\lambda_1 \\ &= \frac{\partial f}{\partial x}(\theta, \kappa(\theta) + c(\theta)\xi^i(\theta))c(\theta)(1+\xi^r(\theta)) - c(\theta+\omega)(1+\xi^r(\theta+\omega))(\lambda+\Delta\lambda) \\ &= \left(\frac{\partial f}{\partial x}(\theta, \kappa(\theta)) + \int_0^1 \frac{\partial^2 f}{\partial x^2}(\theta, \kappa(\theta) + tc(\theta)\xi^i(\theta))\, c(\theta)\xi^i(\theta)dt\right) c(\theta)(1+\xi^r(\theta)) \\ &\quad -c(\theta+\omega)(1+\xi^r(\theta+\omega))(\lambda+\Delta\lambda) \\ &= \frac{\partial f}{\partial x}(\theta, \kappa(\theta))c(\theta) - c(\theta+\omega)\lambda + \int_0^1 \frac{\partial^2 f}{\partial x^2}(\theta, \kappa(\theta) + tc(\theta)\xi^i(\theta))dt\, c(\theta)^2\xi^i(\theta)(1+\xi^r(\theta)) \\ &\quad +\frac{\partial f}{\partial x}(\theta, \kappa(\theta))c(\theta)\xi^r(\theta) - c(\theta+\omega)\Delta\lambda - c(\theta+\omega)\xi^r(\theta+\omega)(\lambda+\Delta\lambda) \\ &= e^r(\theta) + \left(\frac{\partial f}{\partial x}(\theta, \kappa(\theta))c(\theta) - c(\theta+\omega)\lambda\right)\xi^r(\theta) + c(\theta+\omega)\lambda\xi^r(\theta) \\ &\quad -c(\theta+\omega)\Delta\lambda - c(\theta+\omega)\xi^r(\theta+\omega)(\lambda+\Delta\lambda) + R^r(\theta) \\ &= e^r(\theta) + e^r(\theta)\xi^r(\theta) - c(\theta+\omega)(\xi^r(\theta+\omega) - \xi^r(\theta))\lambda \\ &\quad -c(\theta+\omega)(1+\xi^r(\theta+\omega))\Delta\lambda + R_1^i(\theta)(1+\xi^r(\theta)) \ , \end{aligned} \tag{2.46}$$

where $R^r(\theta) = R_1^i(\theta)(1+\xi^r(\theta))$ and $R_1^i(\theta) = \displaystyle\int_0^1 \frac{\partial^2 f}{\partial x^2}(\theta, \kappa(\theta) + tc(\theta)\xi^i(\theta))dt\, c(\theta)^2\xi^i(\theta)$.
In order to reduce the error $e_1^r(\theta)$ as much as possible, we ask

$$e^r(\theta) + R_1^i(\theta) - c(\theta+\omega)(\xi^r(\theta+\omega) - \xi^r(\theta))\lambda - c(\theta+\omega)\Delta\lambda = 0. \tag{2.47}$$

---

[8]c.f. **Definition 1.24**.

Notice that, in such a case, $e_1^r(\theta) = (e^r(\theta) + R_1^i(\theta))\xi^r(\theta) - c(\theta+\omega)\xi^r(\theta+\omega)\Delta\lambda$.
Dividing eq. (2.47) by c_+,

$$(\xi^r(\theta+\omega) - \xi^r(\theta))\lambda + \Delta\lambda = \frac{e^r(\theta)}{c(\theta+\omega)} + \frac{R_1^i(\theta)}{c(\theta+\omega)} \,. \tag{2.48}$$

Taking averages,

$$(\overline{\xi_+^r} - \overline{\xi^r})\lambda + \Delta\lambda =< \frac{e^r}{c_+} + \frac{R_1^i}{c_+} > .$$

Since $\overline{\xi_+^r} = \overline{\xi^r}$, we have

$$\Delta\lambda =< c_+^{-1} e^r > + < c_+^{-1} R_1^i > \,. \tag{2.49}$$

In eq. (2.48) each function can be decomposed in the sum of the average plus the oscillatory part:

$$((\overline{\xi_+^r} + \widetilde{\xi_+^r}) - (\overline{\xi^r} + \widetilde{\xi^r}))\lambda + \Delta\lambda =< c_+^{-1} e^r > + \widetilde{c_+^{-1} e^r} + < c_+^{-1} R_1^i > + \widetilde{c_+^{-1} R_1^i} \,,$$

i.e.

$$((\overline{\xi_+^r} - \overline{\xi^r})\lambda + (\widetilde{\xi_+^r} - \widetilde{\xi^r}))\lambda + \Delta\lambda =< c_+^{-1} e^r > + \widetilde{c_+^{-1} e^r} + < c_+^{-1} R_1^i > + \widetilde{c_+^{-1} R_1^i} \,.$$

Since $\overline{\xi_+^r} = \overline{\xi^r}$ and subtracting eq.(2.49), we have

$$(\widetilde{\xi_+^r} - \widetilde{\xi^r})\lambda = \widetilde{c_+^{-1} e^r} + \widetilde{c_+^{-1} R_1^i}. \tag{2.50}$$

This means that

$$\lambda \widetilde{\xi^r} = \mathfrak{R}_1(\widetilde{c_+^{-1} e^r} + \widetilde{c_+^{-1} R_1^i}) \,, \tag{2.51}$$

where $\mathfrak{R}_\lambda$ is the cohomological operator.
Finally, by eq.(2.33), $c_0 =< c_1 >=< c(1+\xi^r) >=< c > + < c\xi^r >= c_0 + < c\xi^r >$, i.e. $< c\xi^r >= 0$.
Therefore, $0 =< c\xi^r >=< c(\overline{\xi^r} + \widetilde{\xi^r}) >=< c > \overline{\xi^r} + < c\,\widetilde{\xi^r} >$, and

$$\overline{\xi^r} = -\frac{1}{< c >} < c\,\widetilde{\xi^r} > .$$

From eq. (2.51) we get

$$\lambda \overline{\xi^r} = -\frac{1}{< c >} < c\,\mathfrak{R}_1(\widetilde{c_+^{-1} e^r} + \widetilde{c_+^{-1} R_1^i}) > . \tag{2.52}$$

Summary

The *invariance system*

$$\left\{ \begin{array}{rcl} f(\theta,\kappa(\theta)) - \kappa(\theta+\omega) + \tau & = & e^i(\theta) \\ <\kappa> & = & p \end{array} \right. \tag{2.53}$$

with $\kappa_1(\theta) = \kappa(\theta) + c(\theta)\xi^i(\theta)$ and $\tau_1 = \tau + \Delta\tau$, produces

$$D\begin{pmatrix} \overline{\xi^i} \\ \Delta\tau \end{pmatrix} = \begin{pmatrix} <c_+^{-1}e^i> \\ -<c\,\mathfrak{R}_\lambda \widetilde{c_+^{-1}e^i}> \end{pmatrix}$$

$$\widetilde{\xi^i} = \mathfrak{R}_\lambda(\widetilde{c_+^{-1}e^i} + \widetilde{c_+^{-1}}\Delta\tau), \text{ and}$$

$$\xi^i = \widetilde{\xi^i} + \overline{\xi^i}\ ,$$

where $D = \begin{pmatrix} 1-\lambda & -<c^{-1}> \\ <c> & <c\,b> \end{pmatrix}$ and $b = \mathfrak{R}_\lambda\widetilde{c_+^{-1}}$.

The *reducibility system*

$$\left\{ \begin{array}{rcl} \frac{\partial f}{\partial x}(\theta,\kappa(\theta))c(\theta) - c(\theta+\omega)\lambda & = & e^r(\theta) \\ <c> & = & c_0 \end{array} \right. \tag{2.54}$$

with $c_1(\theta) = c(\theta)(1+\xi^r(\theta))$ and $\lambda_1 = \lambda + \Delta\lambda$, produces

$$\lambda\overline{\xi^r} = -\tfrac{1}{<c>} < c\,\mathfrak{R}_1(\widetilde{c_+^{-1}e^r} + \widetilde{c_+^{-1}R_1^i}) >,$$

$$\lambda\widetilde{\xi^r} = \mathfrak{R}_1(\widetilde{c_+^{-1}e^r} + \widetilde{c_+^{-1}R_1^i}),$$

$$\xi^r = \widetilde{\xi^r} + \overline{\xi^r}\ , \text{ and}$$

$$\Delta\lambda = <c_+^{-1}e^r> + <c_+^{-1}R_1^i>\ ,$$

where $R_1^i(\theta) = \displaystyle\int_0^1 \frac{\partial^2 f}{\partial x^2}(\theta, \kappa(\theta + tc(\theta)\xi^i(\theta))dt\, c(\theta)^2\xi^i(\theta)$.

Notice that $\kappa_1(\theta), \tau_1, c_1(\theta)$, and λ_1 are uniquely determined whenever the non-degeneracy conditions $\det(D) \neq 0$ and $\lambda \neq 0$ are satisfied.
Moreover, if the domain of analyticity of κ and c is $\mathbb{T}_\varrho$, i.e. $\kappa, c \in \mathcal{A}_\varrho$, then $\kappa_1 \in \mathcal{A}_{\varrho-\delta}$ and $c_1 \in \mathcal{A}_{\varrho-2\delta}$, due to the clipping in the analyticity domain that occurs each time the cohomological operator is applied. Notice that $\xi^i \in \mathcal{A}_{\varrho-\delta}$ but $\xi^r \in \mathcal{A}_{\varrho-2\delta}$, since the integral remainder $R_1^i \in A_{\varrho-\delta}$.

Once we have $\widetilde{\xi^i}$, $\widetilde{\xi^r}$, $<\xi^i>$, $<\xi^r>$, $\Delta\tau$, and $\Delta\lambda$, we can compute the new errors, e_1^i and e_1^r:

$$
\begin{aligned}
e_1^i(\theta) &= e^r(\theta)\xi^i(\theta) + R^i(\theta) && (2.55)\\
e_1^r(\theta) &= e^r(\theta)\xi^r(\theta) + R^r(\theta) \; - c(\theta+\omega)\xi^r(\theta+\omega)\Delta\lambda, && (2.56)
\end{aligned}
$$

where

$$
\begin{aligned}
R^i(\theta) &= \int_0^1 (1-t)\frac{\partial^2 f}{\partial x^2}(\theta,\kappa(\theta)+tc(\theta)\xi^i(\theta))dt\;\left(c(\theta)\xi^i(\theta)\right)^2 \quad \text{and} && (2.57)\\
R^r(\theta) &= \int_0^1 \frac{\partial^2 f}{\partial x^2}(\theta,\kappa(\theta)+tc(\theta)\xi^i(\theta))dt\; c(\theta)^2\xi^i(\theta)\xi^r(\theta). && (2.58)
\end{aligned}
$$

Remark 2.15

If we call

$$R_1^i(\theta) = \int_0^1 \frac{\partial^2 f}{\partial x^2}(\theta,\kappa(\theta)+tc(\theta)\xi^i(\theta))dt\; c(\theta)^2\,\xi^i(\theta)$$

and

$$R_2^i(\theta) = \int_0^1 t\frac{\partial^2 f}{\partial x^2}(\theta,\kappa(\theta)+tc(\theta)\xi^i(\theta))dt\; c(\theta)^2\,\xi^i(\theta),$$

then $R^i(\theta) = (R_1^i(\theta) - R_2^i(\theta))\xi^i(\theta)$ *and* $R^r(\theta) = R_1^i(\theta)\xi^r(\theta)$. *Therefore,*

$$
\begin{aligned}
e_1^i(\theta) &= (e^r(\theta) + R_1^i(\theta) - R_2^i(\theta))\xi^i(\theta),\; \theta\in\mathbb{T}_{\varrho-\delta} && (2.59)\\
e_1^r(\theta) &= (e^r(\theta) + R_1^i(\theta))\xi^r(\theta) - \Delta\lambda c(\theta+\omega)\xi^r(\theta+\omega),\; \theta\in\mathbb{T}_{\varrho-2\delta}\,. && (2.60)
\end{aligned}
$$

Next, we compute the estimates for one step. Recall from eq.(2.44) that if the non–degeneracy condition $\det(D)\neq 0$ holds, then

$$\begin{pmatrix} \overline{\xi^i} \\ \Delta\tau \end{pmatrix} = D^{-1}\begin{pmatrix} <c_+^{-1}e^i> \\ -<c\,\mathfrak{R}_\lambda \widetilde{c_+^{-1}e^i}> \end{pmatrix}$$

To bound the solution $\begin{pmatrix} \overline{\xi^i} \\ \Delta\tau \end{pmatrix}$ we use the ∞–norm in $\mathbb{C}^2$, $\|z\|_\infty = \max\{|z_1|,|z_2|\}$ and the matrix norm[9]

$$\|D\| = \max_j \sum_i |d_{ij}|, \quad \text{if } D=(d_{ij}),$$

that is, the maximum 1–norm of the columns of D.

[9]This matrix norm is compatible with the ∞–norm in $\mathbb{C}^2$, that is, $\|Dz\|_\infty \le \|D\|\,\|z\|_\infty$, $\forall z=(z_1,z_2)\in\mathbb{C}^2$

Assume that there exists $\kappa^* : \mathbb{T}_{\varrho_0} \longrightarrow \mathbb{C}$ such that if

$$\Omega = \Omega_{\varrho_0, r_0} = \{(\theta, z) \in \mathbb{C}/\mathbb{Z} \times \mathbb{C} : |\operatorname{Im} \theta| \leq \varrho_0,\ |z - \kappa^*(\theta)| \leq r_0\}\ ,$$

the map f is defined in Ω_{ϱ_0, r_0} and we have

$$\|f\|_\Omega < C_f \tag{2.61}$$

$$\|\frac{\partial f}{\partial z}\|_\Omega < C_{\partial_z f} \tag{2.62}$$

$$\|\frac{\partial^2 f}{\partial z^2}\|_\Omega < C_{\partial_{zz} f} \tag{2.63}$$

We consider the following condition numbers:

$$\begin{array}{ll} \|c\|_\varrho < \sigma_c & \|c^{-1}\|_\varrho < \sigma_{c^{-1}} \\ \|D^{-1}\| < \sigma_{D^{-1}} & a < |\lambda| < \frac{1}{a}, \text{ with } a \in (0,1) \\ \|\kappa - \kappa^*\|_\varrho < r_0 & \\ \|\mathfrak{R}_\lambda \widetilde{c_+^{-1}}\|_{\varrho-\delta} < \sigma_b & (b = \mathfrak{R}_\lambda \widetilde{c_+^{-1}} \in \mathcal{A}_{\varrho-\delta} \text{ with } 0 < \delta < \varrho) \end{array}$$

We expect in the iterative procedure, something like

$$\begin{array}{ll} \|c_n\|_{\varrho-2n\delta} < \sigma_c & \|c_n^{-1}\|_{\varrho-2n\delta} < \sigma_{c^{-1}} \\ \|D_n^{-1}\| < \sigma_{D^{-1}} & a < |\lambda_n| < \frac{1}{a}, \text{ with } a \in (0,1) \\ \|\kappa_n - \kappa^*\|_{\varrho-2n\delta} < r_0 & \\ \|\mathfrak{R}_{\lambda_n} \widetilde{c_{n,+}^{-1}}\|_{\varrho-3n\delta} < \sigma_b & \end{array}$$

The following lemma provides a number of estimates obtained in one step of the KAM procedure whenever certain non-degeneracy conditions hold.

Lemma 2.16 Estimates for one step

Let $\varepsilon = \max\left\{\|e^r\|_\varrho, \dfrac{1}{\gamma\,\delta^\nu}\|e^i\|_\varrho\right\}$, *where* $0 < \delta < \frac{1}{2}\varrho$. *Assume that* δ *is sufficiently small so that* $\gamma\delta^\nu < 1$. *If the non–degeneracy conditions* $\det(D) \neq 0$ *and* $\lambda \neq 0$ *are satisfied, then the following estimates hold:*

(*i*) $\max\{|\overline{\xi^i}|, |\Delta\tau|\} \leq \sigma_s\, \frac{1}{\gamma\,\delta^\nu}\|e^i\|_\varrho$, *with* $\sigma_s = \sigma_{D^{-1}}\sigma_{c^{-1}} \max\{1, 2\sigma_c \mathfrak{C}_R\}$;

(*ii*) $\|\widetilde{\xi^i}\|_{\varrho-\delta} \leq \sigma_{\widetilde{\xi^i}}\, \frac{1}{\gamma\,\delta^\nu}\|e^i\|_\varrho$, *with* $\sigma_{\widetilde{\xi_i}} = 2\mathfrak{C}_R\sigma_{c^{-1}} + \sigma_s\sigma_b$;

(*iii*) $\|\xi^i\|_{\varrho-\delta} \leq \sigma_{\xi^i}\, \frac{1}{\gamma\,\delta^\nu}\|e^i\|_\varrho$, *with* $\sigma_{\xi^i} = \sigma_{\widetilde{\xi^i}} + \sigma_s$;

(*iv*) $|\overline{\xi^r}| \leq \sigma_{\overline{\xi^r}}\, \frac{1}{\gamma\,\delta^\nu}\varepsilon$, *with* $\sigma_{\overline{\xi^r}} = \frac{2}{a}\sigma_c\sigma_{c^{-1}}\mathfrak{C}_R(1 + \sigma_c^2\sigma_{\xi^i}C_{\partial_{zz}f})$;

(*v*) $\|\widetilde{\xi^r}\|_{\varrho-2\delta} \leq \sigma_{\widetilde{\xi^r}}\, \frac{1}{\gamma\,\delta^\nu}\varepsilon$, *with* $\sigma_{\widetilde{\xi^r}} = \frac{2}{a}\sigma_{c^{-1}}\mathfrak{C}_R(1 + \sigma_c^2\sigma_{\xi^i}C_{\partial_{zz}f})$;

(*vi*) $\|\xi^r\|_{\varrho-2\delta} \leq \sigma_{\xi^r}\, \frac{1}{\gamma\,\delta^\nu}\varepsilon$, *with* $\sigma_{\xi^r} = (1+\sigma_c)\sigma_{\widetilde{\xi^r}}$;

(*vii*) $\|\Delta c\|_{\varrho-2\delta} \leq \sigma_{\Delta c}\, \frac{1}{\gamma\,\delta^\nu}\varepsilon$, *with* $\sigma_{\Delta c} = \sigma_c\sigma_{\xi^r}$;

(*viii*) $|\Delta\lambda| \leq \sigma_{\Delta\lambda}\varepsilon$, *with* $\sigma_{\Delta\lambda} = \sigma_{c^{-1}}(1 + \sigma_c^2\sigma_{\xi^i}C_{\partial_{zz}f})$.

Proof.

(i) Since $\begin{pmatrix} \overline{\xi^i} \\ \Delta\tau \end{pmatrix} = D^{-1} \begin{pmatrix} < c_+^{-1} e^i > \\ - < c\, \mathfrak{R}_\lambda \widetilde{c_+^{-1} e^i} > \end{pmatrix}$, then

$$\begin{aligned}
\max\{|\overline{\xi^i}|, |\Delta\tau|\} &\leq \sigma_{D^{-1}} \max\left\{ | < c_+^{-1} e^i > |, | < c\, \mathfrak{R}_\lambda \widetilde{c_+^{-1} e^i} > | \right\} \\
&\leq \sigma_{D^{-1}} \max\left\{ \|c_+^{-1} e^i\|_\varrho, \|c\, \mathfrak{R}_\lambda \widetilde{c_+^{-1} e^i}\|_{\varrho-\delta} \right\} \\
&\leq \sigma_{D^{-1}} \max\left\{ \sigma_{c^{-1}} \|e^i\|_\varrho, \sigma_c \|\mathfrak{R}_\lambda \widetilde{c_+^{-1} e^i}\|_{\varrho-\delta} \right\} \\
&\leq \sigma_{D^{-1}} \max\left\{ \sigma_{c^{-1}} \|e^i\|_\varrho, \sigma_c \frac{1}{\gamma\, \delta^\nu}\, \mathfrak{C}_R\, \|\widetilde{c_+^{-1} e^i}\|_\varrho \right\} \\
&\leq \sigma_{D^{-1}} \max\left\{ \sigma_{c^{-1}} \|e^i\|_\varrho, 2\sigma_c \frac{1}{\gamma\, \delta^\nu}\, \mathfrak{C}_R\, \sigma_{c^{-1}} \|e^i\|_\varrho \right\} \\
&= \sigma_{D^{-1}} \sigma_{c^{-1}} \max\{\gamma\, \delta^\nu, 2\sigma_c \mathfrak{C}_R\} \frac{1}{\gamma\, \delta^\nu}\, \|e^i\|_\varrho \\
&\leq \sigma_{D^{-1}} \sigma_{c^{-1}} \max\{1, 2\sigma_c \mathfrak{C}_R\} \frac{1}{\gamma\, \delta^\nu}\, \|e^i\|_\varrho = \sigma_s \frac{1}{\gamma\, \delta^\nu}\, \|e^i\|_\varrho\,,
\end{aligned}$$

where $\sigma_s = \sigma_{D^{-1}} \sigma_{c^{-1}} \max\{1, 2\sigma_c \mathfrak{C}_R\}$.

(ii) Since $\widetilde{\xi^i} = \mathfrak{R}_\lambda(\widetilde{c_+^{-1} e^i} + \Delta\tau\, \widetilde{c_+^{-1}})$, then
$\|\widetilde{\xi^i}\|_{\varrho-\delta} \leq \mathfrak{C}_R \frac{1}{\gamma\,\delta^\nu} 2\sigma_{c^{-1}} \|e^i\|_\varrho + |\Delta\tau|\ \|\mathfrak{R}_\lambda \widetilde{c_+^{-1}}\|_{\varrho-\delta} \leq \mathfrak{C}_R \frac{1}{\gamma\,\delta^\nu} 2\sigma_{c^{-1}} \|e^i\|_\varrho + \sigma_s \frac{1}{\gamma\,\delta^\nu} \|e^i\|_\varrho\, \sigma_b$
$\leq (2\mathfrak{C}_R \sigma_{c^{-1}} + \sigma_s \sigma_b) \frac{1}{\gamma\,\delta^\nu} \|e^i\|_\varrho = \sigma_{\widetilde{\xi^i}}\ \frac{1}{\gamma\,\delta^\nu} \|e^i\|_\varrho$, where $\sigma_{\widetilde{\xi^i}} = 2\mathfrak{C}_R \sigma_{c^{-1}} + \sigma_s \sigma_b$.

(iii) $\xi^i = \widetilde{\xi^i} + \overline{\xi^i}$. Therefore, $\|\xi^i\|_{\varrho-\delta} \leq \|\widetilde{\xi^i}\|_{\varrho-\delta} + |\overline{\xi^i}| \leq \sigma_{\widetilde{\xi^i}}\ \frac{1}{\gamma\,\delta^\nu} \|e^i\|_\varrho + \sigma_s\ \frac{1}{\gamma\,\delta^\nu} \|e^i\|_\varrho$
$\leq (\sigma_{\widetilde{\xi^i}} + \sigma_s)\ \frac{1}{\gamma\,\delta^\nu} \|e^i\|_\varrho = \sigma_{\xi^i}\ \frac{1}{\gamma\,\delta^\nu} \|e^i\|_\varrho$, where $\sigma_{\xi^i} = \sigma_{\widetilde{\xi^i}} + \sigma_s$.

(iv) $\overline{\xi^r} = -\frac{1}{\lambda <c>} < c\mathfrak{R}_1(\widetilde{c_+^{-1} e^r} + \widetilde{c_+^{-1} R_1^i}) >$. Hence, $|\overline{\xi^r}| \leq \frac{1}{|\lambda|\,|<c>|} \sigma_c \frac{1}{\gamma\,\delta^\nu} \mathfrak{C}_R 2\sigma_{c^{-1}} (\|e^r\|_\varrho + \|R_1^i\|_{\varrho-\delta})$.
Now we bound the integral remainder,
$R_1^i(\theta) = \displaystyle\int_0^1 \frac{\partial^2 f}{\partial x^2}(\theta, \kappa(\theta) + t\, c(\theta)\xi^i(\theta))dt\, c(\theta)^2 \xi^i(\theta)\ ,\ \theta \in \mathbb{T}_{\varrho-\delta}$.
$|R_1^i(\theta)| \leq \displaystyle\int_0^1 |\frac{\partial^2 f}{\partial x^2}(\theta, \kappa(\theta) + tc(\theta)\xi^i(\theta))| dt\ \|c\|_\varrho^2\ \|\xi^i\|_{\varrho-\delta} < C_{\partial_{zz} f} \sigma_c^2 \sigma_{\xi^i}\ \frac{1}{\gamma\,\delta^\nu} \|e^i\|_\varrho$.
Thus,
$|\overline{\xi^r}| \leq \frac{1}{|\lambda|} \frac{1}{\gamma\,\delta^\nu} \mathfrak{C}_R\, 2\sigma_c\, \sigma_{c^{-1}} (\|e^r\|_\varrho + C_{\partial_{zz} f} \sigma_c^2 \sigma_{\xi^i}\ \frac{1}{\gamma\,\delta^\nu} \|e^i\|_\varrho) \leq \frac{2}{a} \mathfrak{C}_R\, \sigma_c \sigma_{c^{-1}} (1 + C_{\partial_{zz} f} \sigma_c^2 \sigma_{\xi^i})\ \frac{1}{\gamma\,\delta^\nu}\ \varepsilon$
$= \sigma_{\overline{\xi^r}}\ \frac{1}{\gamma\,\delta^\nu}\ \varepsilon$, where $\sigma_{\overline{\xi^r}} = \frac{2}{a} \mathfrak{C}_R\, \sigma_c \sigma_{c^{-1}} (1 + C_{\partial_{zz} f} \sigma_c^2 \sigma_{\xi^i})$.

(v) Since $\widetilde{\xi^r} = \frac{1}{\lambda} \mathfrak{R}_1(\widetilde{c_+^{-1} e^r} + \widetilde{c_+^{-1} R_1^i})$, then
$\|\widetilde{\xi^r}\|_{\varrho-2\delta} \leq \frac{1}{|\lambda|} \mathfrak{C}_R \frac{1}{\gamma\,\delta^\nu}\ 2\sigma_{c^{-1}} (\|e^r\|_\varrho + C_{\partial_{zz} f} \sigma_c^2 \sigma_{\xi^i} \frac{1}{\gamma\,\delta^\nu} \|e^i\|_\varrho) \leq \frac{2}{a} \mathfrak{C}_R\, \sigma_{c^{-1}} (1 + C_{\partial_{zz} f} \sigma_c^2 \sigma_{\xi^i}) \frac{1}{\gamma\,\delta^\nu}\ \varepsilon$
$= \sigma_{\widetilde{\xi^r}}\ \frac{1}{\gamma\,\delta^\nu}\ \varepsilon$, where $\sigma_{\widetilde{\xi^r}} = \frac{2}{a} \mathfrak{C}_R\, \sigma_{c^{-1}} (1 + C_{\partial_{zz} f} \sigma_c^2 \sigma_{\xi^i})$. Notice that $\sigma_{\overline{\xi^r}} = \sigma_{\widetilde{\xi^r}}\ \sigma_c$.

(vi) $\xi^r = \widetilde{\xi^r} + \overline{\xi^r}$. Then, $\|\xi^r\|_{\varrho-2\delta} \leq \|\widetilde{\xi^r}\|_{\varrho-2\delta} + |\overline{\xi^r}| \leq (\sigma_{\widetilde{\xi^r}} + \sigma_{\overline{\xi^r}})\ \frac{1}{\gamma\,\delta^\nu}\ \varepsilon = \sigma_{\xi^r}\ \frac{1}{\gamma\,\delta^\nu}\ \varepsilon$, where $\sigma_{\xi^r} = \sigma_{\widetilde{\xi^r}} + \sigma_{\overline{\xi^r}} = \sigma_{\widetilde{\xi^r}} (1 + \sigma_c)$.

(vii) $c_1 = c(1+\xi^r) = c + c\xi^r = c + \Delta c$. Therefore, $\|\Delta c\|_{\varrho-2\delta} = \| c\xi^r\|_{\varrho-2\delta} \le \|c\|_\varrho \, \|\xi^r\|_{\varrho-2\delta} \le \sigma_c \, \sigma_{\xi^r} \frac{1}{\gamma\,\delta^\nu} \, \|e^r\|_\varrho \le \sigma_{\Delta c} \frac{1}{\gamma\,\delta^\nu} \, \|e^r\|_\varrho$, where $\sigma_{\Delta c} = \sigma_c \, \sigma_{\xi^r}$.

(viii) $\Delta\lambda = < c_+^{-1} e^r > + < c_+^{-1} R_1^i >$. Hence, $|\Delta\lambda| \le | < c_+^{-1} e^r > | + | < c_+^{-1} R_1^i > |$
$\le \sigma_{c^{-1}} (\|e^r\|_\varrho + \|R_1^i\|_{\varrho-\delta}) \le \sigma_{c^{-1}} (\|e^r\|_\varrho + C_{\partial_{zz}f} \sigma_c^2 \sigma_{\xi^i} \frac{1}{\gamma\,\delta^\nu} \|e^i\|_\varrho)$
$\le \sigma_{c^{-1}} (1 + C_{\partial_{zz}f} \sigma_c^2 \sigma_{\xi^i}) \, \varepsilon = \sigma_{\Delta\lambda} \, \varepsilon$, where $\sigma_{\Delta\lambda} = \sigma_{c^{-1}} (1 + C_{\partial_{zz}f} \sigma_c^2 \sigma_{\xi^i})$.

□

Corollary 2.17

Under the same conditions as in the previous ***Lemma 2.16****,* $\exists \, Q_{e^i}, \, Q_{e^r} > 0$*, such that*

(a) $\|e_1^i\|_{\varrho-\delta} < Q_{e^i} \, \varepsilon^2$;

(b) $\|e_1^r\|_{\varrho-2\delta} < Q_{e^r} \, \frac{1}{\gamma\,\delta^\nu} \, \varepsilon^2$.

Proof.
(a) $e_1^i(\theta) = (e^r(\theta) + R_1^i(\theta) - R_2^i(\theta))\xi^i(\theta)$, $\forall \theta \in \mathbb{T}_{\varrho-\delta}$.

$$\|e_1^i\|_{\varrho-\delta} \le (\|e^r\|_\varrho + \|R_1^i - R_2^i\|_{\varrho-\delta})\|\xi^i\|_{\varrho-\delta} \le (\|e^r\|_\varrho + \frac{1}{2} C_{\partial_{zz}f} \|c\|_\varrho^2 \|\xi^i\|_{\varrho-\delta})\|\xi^i\|_{\varrho-\delta}.$$

Since $\|\xi^i\|_{\varrho-\delta} \le \sigma_{\xi^i} \frac{1}{\gamma\delta^\nu} \|e^i\|_\varrho$, then

$$\begin{aligned} \|e_1^i\|_{\varrho-\delta} &\le (\|e^r\|_\varrho + \frac{1}{2} C_{\partial_{zz}f} \|c\|_\varrho^2 \sigma_{\xi^i} \frac{1}{\gamma\delta^\nu} \|e^i\|_\varrho) \sigma_{\xi^i} \frac{1}{\gamma\delta^\nu} \|e^i\|_\varrho \\ \le &= (\varepsilon + \frac{1}{2} C_{\partial_{zz}f} \sigma_c^2 \sigma_{\xi^i} \varepsilon) \sigma_{\xi^i} \varepsilon = \sigma_{\xi^i} (1 + \frac{1}{2} C_{\partial_{zz}f} \sigma_{\xi^i}) \, \varepsilon^2 = Q_{e^i} \, \varepsilon^2 \, , \end{aligned}$$

where $Q_{e^i} = \sigma_{\xi^i} (1 + \frac{1}{2} C_{\partial_{zz}f} \sigma_{\xi^i})$.

(b) $e_1^r(\theta) = (\mathrm{e}^r(\theta) + R_1^i(\theta))\xi^r(\theta) - \Delta\lambda \, c(\theta+\omega)\xi^r(\theta+\omega)$, $\forall \theta \in \mathbb{T}_{\varrho-2\delta}$.

$$\begin{aligned} \|e_1^r\|_{\varrho-2\delta} &\le (\|e^r\|_\varrho + \|R_1^i\|_{\varrho-\delta}) \|\xi^r\|_{\varrho-2\delta} + |\Delta\lambda| \, \|c\|_\varrho \, \|\xi^r\|_{\varrho-2\delta} \\ &\le (\|e^r\|_\varrho + C_{\partial_{zz}f} \sigma_c^2 \, \|\xi^i\|_{\varrho-\delta} + |\Delta\lambda| \, \sigma_c) \|\xi^r\|_{\varrho-2\delta} \\ &\le (\varepsilon + C_{\partial_{zz}f} \sigma_c^2 \sigma_{\xi^i} \varepsilon + \sigma_{\Delta\lambda} \varepsilon) \sigma_{\xi^r} \frac{1}{\gamma\,\delta^\nu} \varepsilon = \sigma_{\xi^r} (1 + C_{\partial_{zz}f} \sigma_c^2 \sigma_{\xi^i} + \sigma_{\Delta\lambda}) \frac{1}{\gamma\,\delta^\nu} \varepsilon^2 \\ &= Q_{e^r} \, \frac{1}{\gamma\,\delta^\nu} \varepsilon^2, \end{aligned}$$

where $Q_{e^r} = \sigma_{\xi^r} (1 + C_{\partial_{zz}f} \sigma_c^2 \sigma_{\xi^i} + \sigma_{\Delta\lambda})$.

□

In the following lemma, all the conditions of the previous **Lemma 2.16** are assumed.

Lemma 2.18 Iterative step

Let $0<\delta<\frac{1}{2}\varrho$ *and* $\varepsilon=\max\{\|e^r\|_\varrho, \frac{1}{\gamma\,\delta^\nu}\|e^i\|_\varrho\}$. *If the following conditions hold:*

$$(H1)\ \ \frac{\sigma_{\Delta\kappa}}{r_0-\|\kappa-\kappa^*\|_\varrho}\,\varepsilon<1$$

$$(H2)\ \ \max\left\{\frac{\sigma_{\Delta c}}{\sigma_c-\|c\|_\varrho}, \frac{\sigma_{\Delta c^{-1}}}{\sigma_{c^{-1}}-\|c^{-1}\|_\varrho}\right\}\frac{1}{\gamma\,\delta^\nu}\,\varepsilon<1$$

$$(H3)\ \ \max\left\{\frac{1}{\frac{1}{a}-|\lambda|}, \frac{1}{|\lambda|-a}\right\}\sigma_{\Delta\lambda}\,\varepsilon<1$$

$$(H4)\ \ \max\left\{\frac{\sigma_{\Delta b}}{\sigma_b-\|b\|_{\varrho-\delta}}, \frac{\sigma^2_{D^{-1}}\,\sigma_{\Delta D}}{\sigma_{D^{-1}}-\|D^{-1}\|}\right\}\frac{1}{(\gamma\,\delta^\nu)^2}\,\varepsilon<1\ ,$$

then

(*i*) $<\kappa_1>=p$ *and* $<c_1>=1$;

(*ii*) $\|\kappa_1-\kappa^*\|_{\varrho-\delta}<r_0$, *i.e.* $\kappa_1(\mathbb{T}_\varrho)\subseteq\Omega_{\varrho,r_0}$ *and* $\|\Delta\kappa\|_{\varrho-\delta}<\sigma_{\Delta\kappa}\,\varepsilon$;

(*iii*) $|\Delta\tau|<\sigma_{\Delta\tau}\,\varepsilon$ *and* $|\Delta\lambda|<\sigma_{\Delta\lambda}\,\varepsilon$;

(*iv*) $\|c_1\|_{\varrho-2\delta}<\sigma_c$ *and* $\|c_1^{-1}\|_{\varrho-2\delta}<\sigma_{c^{-1}}$. *Moreover,*

$$\|\Delta c\|_{\varrho-2\delta}<\sigma_{\Delta c}\,\frac{1}{\gamma\,\delta^\nu}\,\varepsilon \text{ and } \|\Delta c^{-1}\|_{\varrho-2\delta}<\sigma_{\Delta c^{-1}}\,\frac{1}{\gamma\,\delta^\nu}\,\varepsilon;$$

(*v*) $a<|\lambda_1|<\dfrac{1}{a}$;

(*vi*) $\|b_1\|_{\varrho-3\delta}<\sigma_b$ *and* $\|\Delta b\|_{\varrho-3\delta}<\sigma_{\Delta b}\,\dfrac{1}{(\gamma\,\delta^\nu)^2}\,\varepsilon$ (*whenever* $0<\delta<\frac{1}{3}\varrho$) ;

(*vii*) $\|D_1^{-1}\|<\sigma_{D^{-1}}$, $\|\Delta D\|<\sigma_{\Delta D}\,\dfrac{1}{(\gamma\,\delta^\nu)^2}\,\varepsilon$ *and* $\|\Delta D^{-1}\|<\sigma_{\Delta D^{-1}}\,\dfrac{1}{(\gamma\,\delta^\nu)^2}\,\varepsilon$,

where

$$\begin{array}{lcl}
\sigma_{\Delta\kappa} & = & \sigma_c\,\sigma_{\xi^i}\\
\sigma_{\Delta c} & = & \sigma_c\,\sigma_{\xi^r}\\
\sigma_{\Delta c^{-1}} & = & \sigma^2_{c^{-1}}\,\sigma_c\,\sigma_{\xi^r}\\
\sigma_{\Delta\lambda} & = & \sigma_{c^{-1}}(1+\sigma_c^2\sigma_{\xi^i}C_{\partial_{zz}f})\\
\sigma_{\Delta b} & = & 2\mathfrak{C}_R\sigma_{c^{-1}}(\sigma^2_{c^{-1}}\,\sigma_c\,\sigma_{\xi^r}+\sigma_{\Delta\lambda}\mathfrak{C}_R)\\
\sigma_{\Delta D} & = & \max\{\sigma_{\Delta\lambda},\sigma_{\Delta c^{-1}}+\sigma_c\sigma_{\Delta b}\sigma_{\Delta c}\sigma_b\}\\
\sigma_{\Delta D^{-1}} & = & \sigma^2_{D^{-1}}\sigma_{\Delta D}\\
\sigma_{\Delta\tau} & = & \sigma_{D^{-1}}\,\sigma_{c^{-1}}\max\{1,2\sigma_c\mathfrak{C}_R\}\ .
\end{array}$$

Additionally, $\exists\,Q_{e^i},\,Q_{e^r}>0$ *such that* $\|e_1^i\|_{\varrho-\delta}<Q_{e^i}\,\varepsilon^2$ *and* $\|e_1^r\|_{\varrho-2\delta}<Q_{e^r}\,\frac{1}{\gamma\delta^\nu}\,\varepsilon^2$.

Proof.

(i) Recall that $\kappa_1(\theta),\tau_1,c_1(\theta)$, and λ_1 are uniquely determined whenever the non-degeneracy conditions $\det(D)\neq 0$ and $\lambda\neq 0$ are satisfied. In such a case, $<\kappa_1>=p$ and $<c_1>=1$ by construction. Moreover, $\kappa_1\in\mathcal{A}_{\varrho-\delta}$ for every $\delta\in(0,\varrho)$, and $c_1\in\mathcal{A}_{\varrho-2\delta}$ for every $\delta\in(0,\frac{1}{2}\varrho)$. We can also say that $k_1,c_1\in\mathcal{A}_{\varrho-2\delta}$, for every $\delta\in(0,\frac{1}{2}\varrho)$.

(ii) $\Delta\kappa = \kappa_1 - \kappa = c\,\xi^i \Rightarrow \|\Delta\kappa\|_{\varrho-\delta} \le \|c\|_\varrho \|\xi^i\|_{\varrho-\delta} < \sigma_c\sigma_{\xi^i} \frac{1}{\gamma\,\delta^\nu} \|e^i\|_\varrho$. This means that $\|\Delta\kappa\|_{\varrho-\delta} \le \sigma_{\Delta\kappa} \frac{1}{\gamma\,\delta^\nu} \|e^i\|_\varrho < \sigma_{\Delta\kappa}\,\varepsilon$, with $\sigma_{\Delta\kappa} = \sigma_c\sigma_{\xi^i}$. Moreover,

$$\begin{aligned} \frac{\|\Delta\kappa\|_{\varrho-\delta}}{r_0 - \|\kappa-\kappa^*\|_\varrho} &\le \frac{\sigma_c\sigma_{\xi^i}}{r_0 - \|\kappa-\kappa^*\|_\varrho} \cdot \frac{1}{\gamma\,\delta^\nu} \|e^i\|_\varrho = \frac{\sigma_{\Delta\kappa}}{r_0 - \|\kappa-\kappa^*\|_\varrho} \cdot \frac{1}{\gamma\,\delta^\nu} \|e^i\|_\varrho \\ &= \frac{\sigma_{\Delta\kappa}}{r_0 - \|\kappa-\kappa^*\|_\varrho} \cdot \varepsilon < 1 \qquad (\text{ by hypothesis (H1) }) \\ &\Rightarrow \|\Delta\kappa\|_{\varrho-\delta} < r_0 - \|\kappa-\kappa^*\|_{\varrho-\delta}. \end{aligned}$$

Thus,

$$\|\kappa_1-\kappa^*\|_{\varrho-\delta} = \|\kappa+\Delta\kappa-\kappa^*\|_{\varrho-\delta} \le \|\kappa-\kappa^*\|_{\varrho-\delta} + \|\Delta\kappa\|_{\varrho-\delta} \le \|\kappa-\kappa^*\|_{\varrho-\delta} + r_0 - \|\kappa-\kappa^*\|_{\varrho-\delta} = r_0.$$

(iii) Consequence of **Lemma 2.16**, parts (i) and (viii).

(iv) On the one hand,

$$\begin{aligned} \|\Delta c\|_{\varrho-2\delta} &= \|c\xi^r\|_{\varrho-2\delta} \le \|c\|_\varrho \|\xi^r\|_{\varrho-2\delta} \le \sigma_c\sigma_{\xi^r} \frac{1}{\gamma\,\delta^\nu} \|e^r\|_\varrho \\ &= \sigma_{\Delta c} \frac{1}{\gamma\,\delta^\nu} \|e^r\|_\varrho \le \sigma_{\Delta c} \frac{1}{\gamma\,\delta^\nu} \varepsilon, \end{aligned}$$

where $\sigma_{\Delta c} = \sigma_c\,\sigma_{\xi^r}$. Then,

$$\begin{aligned} \|c_1\|_{\varrho-2\delta} = \|c+\Delta c\|_{\varrho-2\delta} &\le \|c\|_\varrho + \|\Delta c\|_{\varrho-2\delta} \le \|c\|_\varrho + \sigma_{\Delta c} \frac{1}{\gamma\,\delta^\nu} \varepsilon < \qquad (\text{ by hypothesis (H2) }) \\ &< \|c\|_\varrho + \sigma_c - \|c\|_\varrho = \sigma_c\,. \end{aligned}$$

On the other hand, if $c \neq 0$, $\|c^{-1}\|_\varrho < \sigma_{c^{-1}}$, $c_1 = c + \Delta c$, and we take $\sigma_{\Delta c^{-1}} = \sigma^2_{c^{-1}}\sigma_{\Delta c} = \sigma^2_{c^{-1}}\sigma_c\sigma_{\xi^r}$, then using hypothesis (H2) we have $\dfrac{\sigma^2_{c^{-1}} \|\Delta c\|_{\varrho-2\delta}}{\sigma_{c^{-1}} - \|c^{-1}\|_\varrho} < \dfrac{\sigma_{\Delta c^{-1}}}{\sigma_{c^{-1}} - \|c^{-1}\|_\varrho} \dfrac{1}{\gamma\,\delta^\nu}\,\varepsilon < 1$. Applying now **Lemma III.1**, we get

$$\|c_1^{-1}\|_{\varrho-2\delta} < \sigma_{c^{-1}}.$$

Moreover, by the same lemma,

$$\begin{aligned} \|\Delta c^{-1}\|_{\varrho-2\delta} = \|c_1^{-1} - c^{-1}\|_{\varrho-2\delta} &\le \sigma^2_{c^{-1}} \|\Delta c\|_{\varrho-2\delta} < \sigma^2_{c^{-1}}\sigma_c\sigma_{\xi^r} \frac{1}{\gamma\,\delta^\mu} \|e^r\|_\varrho \\ &= \sigma_{\Delta c^{-1}} \frac{1}{\gamma\,\delta^\nu} \|e^r\|_\varrho \le \sigma_{\Delta c^{-1}} \frac{1}{\gamma\,\delta^\nu} \varepsilon\,, \end{aligned}$$

with $\sigma_{\Delta c^{-1}} = \sigma^2_{c^{-1}}\sigma_c\sigma_{\xi^r}$.

(v)

$$\begin{aligned} ||\lambda_1| - |\lambda|| &\le |\lambda_1 - \lambda| = |\Delta\lambda| < \sigma_{\Delta\lambda}\,\varepsilon \qquad (\text{ by hypothesis (H3) }) \\ &\le \frac{1}{\max\left\{\frac{1}{\frac{1}{a}-|\lambda|}, \frac{1}{|\lambda|-a}\right\}} = \min\left\{\frac{1}{a} - |\lambda|, |\lambda| - a\right\}. \end{aligned}$$

Therefore, $a < |\lambda_1| < \frac{1}{a}$.

(vi) First, we bound Δb.

Notice that,

$$\begin{aligned}
\Delta b(\theta) &= b_1(\theta) - b(\theta) = \mathfrak{R}_{\lambda_1}\widetilde{c_{1,+}^{-1}}(\theta) - \mathfrak{R}_{\lambda}\widetilde{c_{+}^{-1}}(\theta) \\
&= \mathfrak{R}_{\lambda_1}\widetilde{c_{1,+}^{-1}}(\theta) - \mathfrak{R}_{\lambda_1}\widetilde{c_{+}^{-1}}(\theta) + \mathfrak{R}_{\lambda_1}\widetilde{c_{+}^{-1}}(\theta) - \mathfrak{R}_{\lambda}\widetilde{c_{+}^{-1}}(\theta) \\
&= \mathfrak{R}_{\lambda_1}(\widetilde{c_{1,+}^{-1}} - \widetilde{c_{+}^{-1}})(\theta) + (\mathfrak{R}_{\lambda_1} - \mathfrak{R}_{\lambda})\widetilde{c_{+}^{-1}}(\theta) \\
&= \mathfrak{R}_{\lambda_1}(\Delta\widetilde{c_{+}^{-1}})(\theta) + (\Delta\mathfrak{R}_{\lambda})\widetilde{c_{+}^{-1}}(\theta), \ \forall\theta \in \mathbb{T}_{\varrho-3\delta}.
\end{aligned}$$

By **Proposition 1.26**, part (a),

$$\Delta\mathfrak{R}_{\lambda}\widetilde{c_{+}^{-1}}(\theta) = \Delta\lambda\, \mathfrak{R}_{\lambda_1}\mathfrak{R}_{\lambda}\widetilde{c_{+}^{-1}}(\theta), \ \forall\theta \in \mathbb{T}_{\varrho-2\delta}.$$

Thus,

$$\Delta b(\theta) = \mathfrak{R}_{\lambda_1}(\Delta\widetilde{c_{+}^{-1}})(\theta) + \Delta\lambda\, \mathfrak{R}_{\lambda_1}\mathfrak{R}_{\lambda}\widetilde{c_{+}^{-1}}(\theta), \ \forall\theta \in \mathbb{T}_{\varrho-3\delta}.$$

It follows that

$$\begin{aligned}
\|\Delta b\|_{\varrho-3\delta} &\le \mathfrak{C}_R\,\frac{1}{\gamma\,\delta^{\nu}}\,\|\Delta\widetilde{c_{+}^{-1}}\|_{\varrho-2\delta} + |\Delta\lambda|\,\mathfrak{C}_R\,\frac{1}{\gamma\,\delta^{\nu}}\,\mathfrak{C}_R\,\frac{1}{\gamma\,\delta^{\nu}}\,\|\widetilde{c_{+}^{-1}}\|_{\varrho} \\
&\le 2\mathfrak{C}_R\,\frac{1}{\gamma\,\delta^{\nu}}\left(\|\Delta c^{-1}\|_{\varrho-2\delta} + |\Delta\lambda|\mathfrak{C}_R\,\frac{1}{\gamma\,\delta^{\nu}}\|c^{-1}\|_{\varrho}\right) \\
&\le 2\mathfrak{C}_R\,\frac{1}{\gamma\,\delta^{\nu}}\left(\sigma_{\Delta c^{-1}}\frac{1}{\gamma\,\delta^{\nu}}\varepsilon + \sigma_{\Delta\lambda}\,\varepsilon\mathfrak{C}_R\,\frac{1}{\gamma\,\delta^{\nu}}\sigma_{c^{-1}}\right) \\
&= 2\mathfrak{C}_R(\sigma_{\Delta c^{-1}} + \mathfrak{C}_R\sigma_{\Delta\lambda}\sigma_{c^{-1}})\frac{1}{(\gamma\,\delta^{\nu})^2}\,\varepsilon = \sigma_{\Delta b}\,\frac{1}{(\gamma\,\delta^{\nu})^2}\,\varepsilon,
\end{aligned}$$

where $\sigma_{\Delta b} = 2\mathfrak{C}_R(\sigma_{\Delta c^{-1}} + \mathfrak{C}_R\sigma_{\Delta\lambda}\sigma_{c^{-1}}) = 2\mathfrak{C}_R\sigma_{c^{-1}}(\sigma_{c^{-1}}\sigma_c\sigma_{\xi^r} + \mathfrak{C}_R\sigma_{\Delta\lambda})$.

$$\begin{aligned}
\|b_1\|_{\varrho-3\delta} &= \|b + \Delta b\|_{\varrho-3\delta} \le \|b\|_{\varrho-\delta} + \|\Delta b\|_{\varrho-3\delta} \qquad (\text{ by hypothesis (H4) }) \\
&< \|b\|_{\varrho-\delta} + \sigma_{\Delta b}\,\frac{1}{(\gamma\,\delta^{\nu})^2}\,\varepsilon < \|b\|_{\varrho-\delta} + \sigma_b - \|b\|_{\varrho-\delta} = \sigma_b.
\end{aligned}$$

(vii) Next, we bound ΔD.

$$\Delta D = D_1 - D = \begin{pmatrix} 1-\lambda_1 & -< c_1^{-1} > \\ < c_1 > & < c_1 b_1 > \end{pmatrix} - \begin{pmatrix} 1-\lambda & -< c^{-1} > \\ < c > & < cb > \end{pmatrix} = \begin{pmatrix} -\Delta\lambda & -< \Delta c^{-1} > \\ 0 & < \Delta(c\,b) > \end{pmatrix}.$$

Thus,

$$\begin{aligned}
\|\Delta D\| &= \|D_1 - D\| = \max\left\{|\Delta\lambda|, |< \Delta c^{-1} >| + |< \Delta(c\,b) >|\right\} \\
&\le \max\left\{|\Delta\lambda|, \|\Delta c^{-1}\|_{\varrho-2\delta} + \|\Delta(cb)\|_{\varrho-3\delta}\right\}
\end{aligned}$$

$$\begin{aligned}
|< \Delta(c\,b) >| &= |< c_1\,b_1 > - < c\,b >| = |< c_1\,(b_1 - b) + (c_1 - c)\,b >| \\
&\le \|c_1\Delta b\|_{\varrho-3\delta} + \|\Delta c\,b\|_{\varrho-2\delta} \le \|c_1\|_{\varrho-2\delta}\,\|\Delta b\|_{\varrho-3\delta} + \|\Delta c\|_{\varrho-2\delta}\,\|b\|_{\varrho-\delta} \\
&= \sigma_c\,\sigma_{\Delta b}\,\frac{1}{(\gamma\,\delta^{\nu})^2}\,\varepsilon + \sigma_{\Delta c}\,\frac{1}{\gamma\,\delta^{\nu}}\,\varepsilon\,\sigma_b = (\sigma_c\sigma_{\Delta b}\,\gamma\delta^{\nu} + \sigma_{\delta c}\sigma b)\,\frac{1}{(\gamma\,\Delta^{\nu})^2}\,\varepsilon \\
&\le (\sigma_c\sigma_{\Delta b} + \sigma_{\Delta c}\sigma b)\,\frac{1}{(\gamma\,\delta^{\nu})^2}\,\varepsilon = \sigma_{\Delta(c\,b)}\,\frac{1}{(\gamma\,\delta^{\nu})^2}\,\varepsilon,
\end{aligned}$$

where $\sigma_{\Delta(c\,b)} = \sigma_c \sigma_{\Delta b} + \sigma_{\Delta c}\sigma_b$.

Thus,

$$\begin{aligned} \|\Delta D\| &\leq \max\left\{\sigma_{\Delta\lambda}, \sigma_{\Delta c^{-1}} \frac{1}{\gamma\,\Delta^{\nu}} + \sigma_{\Delta(c\,b)} \frac{1}{(\gamma\,\delta^{\nu})^2}\,\varepsilon\right\} \\ &< \max\left\{\sigma_{\Delta\lambda}, \sigma_{\Delta c^{-1}} + \sigma_{\Delta(c\,b)}\right\} \frac{1}{(\gamma\,\delta^{\nu})^2}\,\varepsilon = \sigma_{\Delta D} \frac{1}{(\gamma\,\delta^{\nu})^2}\,\varepsilon, \end{aligned}$$

where $\sigma_{\Delta D} = \max\left\{\sigma_{\Delta\lambda}, \sigma_{\Delta c^{-1}} + \sigma_{\Delta(c\,b)}\right\}$.

As a consequence of this bound, we can write

$$\frac{\sigma^2_{D^{-1}}\,\|\Delta D\|}{\sigma_{D^{-1}} - \|D^{-1}\|} \leq \frac{\sigma^2_{D^{-1}}\,\sigma_{\Delta D}}{\sigma_{D^{-1}} - \|D^{-1}\|}\,\frac{1}{(\gamma\,\delta^{\nu})^2}\,\varepsilon < 1 \quad (\text{ by hypothesis (H4)}).$$

Now, we apply again **Lemma III.1**:

$$\frac{\sigma^2_{D^{-1}}\,\|\Delta D\|}{\sigma_{D^{-1}} - \|D^{-1}\|} < 1 \Rightarrow \begin{cases} \|D_1^{-1}\| < \sigma_{D^{-1}} \\ \|\Delta D^{-1}\| = \|D_1^{-1} - D^{-1}\| < \sigma^2_{D^{-1}}\,\|\Delta D\| \end{cases}.$$

Moreover, since $\|\Delta D\| < \sigma_{\Delta D}\,\frac{1}{(\gamma\,\delta^{\nu})^2}\,\varepsilon$, then

$$\|\Delta D^{-1}\| < \sigma^2_{D^{-1}}\,\|\Delta D\| < \sigma^2_{D^{-1}}\sigma_{\Delta D}\,\frac{1}{(\gamma\,\delta^{\nu})^2}\,\varepsilon = \sigma_{\Delta D^{-1}}\,\frac{1}{(\gamma\,\delta^{\nu})^2}\,\varepsilon,$$

where $\sigma_{\Delta D^{-1}} = \sigma^2_{D^{-1}}\sigma_{\Delta D}$.

$\square$

Theorem 2.19 (KAM Theorem)

Let $\Psi = \mathcal{R}_\omega \times f$ be a quasi–periodic skew-product

$$\begin{array}{rccc} \Psi: & \mathbb{T}\times\mathbb{R} & \longrightarrow & \mathbb{T}\times\mathbb{R} \\ & (\theta, x) & \longmapsto & \Psi(\theta,x) = (\theta+\omega, f(\theta,x)) \end{array}$$

where the frequency $\omega \in \mathcal{DC}(\gamma,\nu)$ is Diophantine and $f : \mathbb{T}\times\mathbb{R} \longrightarrow \mathbb{R}$ is a real analytic function. Assume that there is a complex extension of f,

$$f : \mathbb{T}_\varrho \times U \longrightarrow \mathbb{C},$$

where $\varrho > 0$ and $U \subseteq \mathbb{C}$ is an open connected set such that exist $\kappa^* : \mathbb{T}_{\varrho_0} \longrightarrow \mathbb{C}$, $\kappa^* \in \mathcal{A}_{\varrho_0}$, with $0 < \varrho_0 < \varrho$, and $r_0 > 0$ satisfying the following properties:

If $\Omega = \Omega_{\varrho_0,r_0} := \{(\theta,z) \in \mathbb{T}_{\varrho_0}\times\mathbb{C} : |z-\kappa^*(\theta)| \le r_0\}$, then

(a) $\Omega_{\varrho_0,r_0} \subseteq \mathbb{T}_\varrho \times U$,

(b) $\left\|\dfrac{\partial f}{\partial z}\right\|_\Omega := \sup\limits_{(\theta,z)\in\Omega}\left|\dfrac{\partial f}{\partial z}(\theta,z)\right| < C_{\partial_z f}$, *and*

(c) $\left\|\dfrac{\partial^2 f}{\partial z^2}\right\|_\Omega := \sup\limits_{(\theta,z)\in\Omega}\left|\dfrac{\partial^2 f}{\partial z^2}(\theta,z)\right| < C_{\partial_{zz} f}$.

Let $p \in \mathbb{R}$ be a fixed average.
Assume that we have $\kappa_0, c_0 \in \mathcal{A}_{\varrho_0}$, and $\tau_0, \lambda_0 \in \mathbb{R}$ satisfying the following conditions:

(i) $<\kappa_0> = p$;

(ii) $<c_0> = 1$.

(iii) $\|\kappa_0 - \kappa^*\|_{\varrho_0} \le r_0$, *i.e.* $\{(\theta,\kappa_0(\theta)) : \theta \in \mathbb{T}_{\varrho_0}\} \subseteq \mathbb{T}_{\varrho_0}\times U$;

(iv) $\det(D_0) \neq 0$, *where*

$$D_0 = \begin{pmatrix} 1-\lambda_0 & -<\widetilde{c_0^{-1}>} \\ <c_0> & <c_0\mathfrak{R}_{\lambda_0} c_{0,+}^{-1}> \end{pmatrix}, \text{ and}$$

(v) $\lambda_0 \neq 0$.

Define,

$$\begin{array}{rcl} e_0^i(\theta) & = & f(\theta,\kappa_0(\theta)) - \kappa_0(\theta+\omega) + \tau_0,\ \theta \in \mathbb{T}_{\varrho_0} \\ e_0^r(\theta) & = & \frac{\partial f}{\partial z}(\theta,\kappa_0(\theta))c_0(\theta) - c_0(\theta+\omega)\lambda_0,\ \theta\in\mathbb{T}_{\varrho_0} \\ \varepsilon_0 & = & \max\left\{\|e_0^r\|_{\varrho_0}, \frac{1}{\gamma\,\delta_0^\nu}\|e_0^i\|_{\varrho_0}\right\},\ \text{where } \delta_0 = \min\{\frac{1}{4}\varrho_0, \gamma^{\frac{1}{\nu}}\}, \\ Q_{\varepsilon_0} & = & \max\{Q_{e_0^r}, \alpha^\nu\, Q_{e_0^i}\},\ \text{and} \\ \Sigma_s(\mu,\alpha) & = & \displaystyle\sum_{j=0}^{\infty} \alpha^{s\nu j}\mu^{2^j-1} \quad (s=-1,0,1),\ \text{where } \alpha > 1 \text{ and } \mu \in (0,1). \end{array}$$

Assume that ε_0 satisfy the following smallness condition:

$$\frac{Q_{\varepsilon_0}\,\varepsilon_0}{\gamma\,\delta_1^\nu} \le \mu < 1,$$

where $\delta_1 = \frac{\delta_0}{\alpha}$.

Under these conditions, if moreover $\kappa_0(\theta), c_0(\theta), \tau_0$, and λ_0 satisfy the following hypothesis:

$$(H.I) \quad \frac{\sigma_{\Delta\kappa_0}\Sigma_{-1}}{r_0 - \|\kappa_0 - \kappa^*\|_{\varrho_0}}\, \varepsilon_0 < 1$$

$$(H.II) \quad \max\left\{\frac{1}{\frac{1}{a} - |\lambda_0|}, \frac{1}{|\lambda_0| - a}\right\} \sigma_{\Delta\lambda_0}\, \Sigma_{-1}\varepsilon_0 < 1$$

$$(H.III) \quad \frac{\sigma_{\Delta c_0}\Sigma_0}{\sigma_{c_0} - \|c_0\|_{\varrho_0}}\, \frac{1}{\gamma\, \delta_0^{\nu}}\, \varepsilon_0 < 1$$

$$(H.IV) \quad \frac{\sigma_{\Delta c_0^{-1}}\Sigma_0}{\sigma_{c_0^{-1}} - \|c_0^{-1}\|_{\varrho_0}}\, \frac{1}{\gamma\, \delta_0^{\nu}}\, \varepsilon_0 < 1$$

$$(H.V) \quad \frac{\sigma_{\Delta D_0^{-1}}\Sigma_1}{\sigma_{D_0^{-1}} - \|D_0^{-1}\|}\, \frac{1}{(\gamma\, \delta_0^{\nu})^2}\varepsilon_0 < 1$$

$$(H.VI) \quad \frac{\sigma_{\Delta b_0}\Sigma_1}{\sigma_{b_0} - \|b_0\|_{\varrho_0 - \delta_0}}\, \frac{1}{(\gamma\, \delta_0^{\nu})^2}\varepsilon_0 < 1$$

then $\exists\, \kappa \in \mathcal{A}_{\varrho_\infty}$ *(for some* $0 < \varrho_\infty < \varrho_0$*) and* $\tau \in \mathbb{R}$ *such that*

$$\begin{cases} f(\theta, \kappa(\theta)) - \kappa(\theta + \omega) + \tau & = \quad 0 \\ \qquad\qquad < \kappa > & = \quad p \end{cases}$$

i.e. κ *is an analytic invariant translated curve of the skew–product* Ψ *with translation parameter* τ*. In particular, if* $\tau = 0$ *then* κ *is an invariant curve.*

Additionally, $\exists\, c \in \mathcal{A}_{\varrho_\infty}$ *and* $\lambda \in \mathbb{R} \setminus \{0\}$ *such that*

$$\begin{cases} \frac{\partial f}{\partial z}(\theta, \kappa(\theta))\, c(\theta) - c(\theta + \omega)\lambda & = \quad 0 \\ \qquad\qquad < c > & = \quad 1 \end{cases}$$

Moreover, κ, c*, and* c^{-1} *satisfy the desired bounds.*

Proof. The proof consist in the generation of a sequence of objects $\{\kappa_j, \tau_j, c_j, \lambda_j\}$ $(j \in \mathbb{N})$ with $\kappa_j, c_j : \mathbb{T}_{\varrho_j} \longrightarrow \mathbb{C}$, $\tau_j, \lambda_j \in \mathbb{R}$, which produce errors $\{e_j^i, e_j^r\}$, $e_j^i, e_j^r : \mathbb{T}_{\varrho_j} \longrightarrow \mathbb{C}$ and

$$\varepsilon_j = \max\left\{\|e_j^r\|_{\varrho_j}, \frac{1}{\gamma\, \delta_j^{\nu}}\|e_j^i\|_{\varrho_j}\right\}.$$

When the guess functions κ_j, c_j are analytic in the complex strip $\mathbb{T}_{\varrho_j}$, the new ones κ_{j+1}, c_{j+1} will turn out to be analytic in a reduced strip $\mathbb{T}_{\varrho_{j+1}}$, with $\varrho_{j+1} = \varrho_j - 2\delta_j$, for some $\delta_j \in (0, \frac{1}{2}\varrho_j)$. This fact is a consequence of the required application of the cohomological operator at each step of the procedure. So, during the procedure and in the limit, we want each curve to be analytic in a strip $\mathbb{T}_{\varrho_\infty}$, with $\varrho_\infty > 0$. This can be achieved with an adequate choice of the δ_j's, so that $\varrho_\infty = \lim_{j\to\infty} \varrho_j$ with $\varrho_\infty < \varrho_j$, $\forall j \in \mathbb{N}$.

The choice of the deltas

$\delta_0 = \min\{\frac{1}{4}\varrho_0, \gamma^{-\frac{1}{\nu}}\}$	$\delta_j = \frac{\delta_0}{\alpha^j}$, $j = 0, 1, \dots$
$0 < \varrho_0 < \varrho$	$\varrho_{j+1} = \varrho_j - 2\delta_j$, $j = 0, 1, \dots$
$\varrho_\infty = \varrho_0 - \frac{2\alpha}{\alpha-1}\delta_0 \in (0, \varrho_0)$	$\alpha = \frac{\varrho_0 - \varrho_\infty}{\varrho_1 - \varrho_\infty} \in (1, \infty)$.

Observe that, with this choice, the following properties hold:

- $(\varrho_j)_{j\in\mathbb{N}}$ and $(\delta_j)_{j\in\mathbb{N}}$ are strictly decreasing sequences with $0 < 2\delta_j < \varrho_j$, $\forall j \in \mathbb{N}$;
- $\delta_j = \dfrac{\delta_0}{\alpha^j}$, $\forall j \in \mathbb{N}$;
- $\varrho_j = \varrho_0 - 2\displaystyle\sum_{i=0}^{j-1} \delta_i \longrightarrow \varrho_\infty$.

Next, we bound the errors produced at each step of the procedure.

$$\begin{aligned}
\varepsilon_j &= \max\left\{\|e_j^r\|_{\varrho_j}, \frac{1}{\gamma\delta_j^\nu}\|e_j^i\|_{\varrho_j}\right\} \leq \max\left\{Q_{e_{j-1}^r}\frac{1}{\gamma\,\delta_{j-1}^\nu}\varepsilon_{j-1}^2, \frac{1}{\gamma\,\delta_j^\nu}Q_{e_{j-1}^i}\varepsilon_{j-1}^2\right\} \\
&= \max\left\{Q_{e_{j-1}^r}\frac{1}{\delta_{j-1}^\nu}, \frac{1}{\delta_j^\nu}Q_{e_{j-1}^i}\right\}\frac{1}{\gamma}\varepsilon_{j-1}^2 = \max\left\{Q_{e_{j-1}^r}\frac{1}{\delta_{j-1}^\nu}, \frac{1}{\left(\frac{\delta_{j-1}}{\alpha}\right)^\nu}Q_{e_{j-1}^i}\right\}\frac{1}{\gamma}\varepsilon_{j-1}^2 \\
&= \max\left\{Q_{e_{j-1}^r}, Q_{e_{j-1}^i}\alpha^\nu\right\}\frac{1}{\gamma\,\delta_{j-1}^\nu}\,\varepsilon_{j-1}^2 = Q_{\varepsilon_{j-1}}\,\frac{1}{\gamma\,\delta_{j-1}^\nu}\,\varepsilon_{j-1}^2\,,
\end{aligned}$$

where $Q_{\varepsilon_j} := \max\left\{Q_{e_j^r}, Q_{e_j^i}\,\alpha^\nu\right\}$ $\quad(j = 0, 1, \dots)$.
It follows that

$$\begin{aligned}
\varepsilon_j &\leq Q_{\varepsilon_{j-1}}\,\frac{1}{\gamma\,\delta_{j-1}^\nu}\,\varepsilon_{j-1}^2 \leq Q_{\varepsilon_{j-1}}\,\frac{1}{\gamma\,\delta_{j-1}^\nu}\left(Q_{\varepsilon_{j-2}}\,\frac{1}{\gamma\,\delta_{j-2}^\nu}\,\varepsilon_{j-2}^2\right)^2 \\
&= Q_{\varepsilon_{j-1}}\,Q_{\varepsilon_{j-2}}^2\frac{1}{\gamma\,\gamma^2}\cdot\frac{1}{\left(\delta_{j-1}\,\delta_{j-2}^2\right)^\nu}\cdot\varepsilon_{j-2}^{2^2} \leq \dots \\
&\leq Q_{\varepsilon_{j-1}}^{2^0}Q_{\varepsilon_{j-2}}^{2^1}\dots Q_{\varepsilon_1}^{2^{j-2}}Q_{\varepsilon_0}^{2^{j-1}}\frac{1}{\gamma\,\gamma^2\dots\gamma^{2^{j-1}}}\cdot\frac{1}{\left(\delta_{j-1}\delta_{j-2}^2\dots\delta_0^{2^{j-1}}\right)^\nu}\cdot\varepsilon_0^{2^j} \\
&\leq Q_{\varepsilon_0}^{2^0+2^1+\dots+2^{j-1}}\,\frac{1}{\gamma^{2^0+2^1+\dots+2^{j-1}}}\cdot\left(\frac{\delta_0^{2^j-1}}{\alpha^{2^j-j-1}}\right)^{-\nu}\varepsilon_0^{2^j} \\
&= Q_{\varepsilon_0}^{2^j-1}\,\frac{1}{\gamma^{2^j-1}}\left(\frac{\delta_0^{2^j-1}}{\alpha^{2^j-1}}\right)^{-\nu}\frac{1}{\alpha^{\nu j}}\,\varepsilon_0^{2^j} \\
&= \left(Q_{\varepsilon_0}\,\frac{1}{\gamma}\left(\frac{\delta_0}{\alpha}\right)^{-\nu}\varepsilon_0\right)^{2^j-1}\frac{1}{\alpha^{\nu j}}\cdot\varepsilon_0 = \left(\frac{Q_{\varepsilon_0}\,\varepsilon_0}{\gamma\,\delta_1^\nu}\right)^{2^j-1}\frac{1}{\alpha^{\nu j}}\,\varepsilon_0.
\end{aligned} \tag{2.64}$$

Moreover, by assumption $\mu \in \left[\dfrac{Q_{\varepsilon_0}\,\varepsilon_0}{\gamma\,\delta_1^{\nu}}, 1\right)$. Therefore,

$$\varepsilon_j \le \mu^{2^j-1}\frac{1}{\alpha^{\nu j}}\,\varepsilon_0,\ \forall j\in\mathbb{N}. \tag{2.65}$$

Remark 2.20

$$\delta_{j-1}\delta_{j-2}^2\ldots\delta_{j_0}^{2^{j-1}} = \frac{\delta_0^{2^j-1}}{\alpha^{2^j-j-1}}.$$

Proof.

$$\begin{aligned}\delta_{j-1}\delta_{j-2}^2\ldots\delta_{j_0}^{2^{j-1}} &= \left(\frac{\delta_0}{\alpha^{j-1}}\right)^{2^0}\cdot\left(\frac{\delta_0}{\alpha^{j-2}}\right)^{2^1}\cdot\dots\cdot\left(\frac{\delta_0}{\alpha^{0}}\right)^{2^{j-1}}\\ &= \frac{\delta_0^{1+2+\dots+2^{j-1}}}{\alpha^{(j-1)\cdot 2^0+(j-2)\cdot 2^1+\dots+1\cdot 2^{j-2}}}.\end{aligned}$$

If we call, $S_j=\sum_{i=0}^{j-1}2^i$ and $T_j=\sum_{i=0}^{j-1} i\,2^i$, then

$(j-1)\cdot 2^0+(j-2)\cdot 2^1+\dots+1\cdot 2^{j-2}=\sum_{i=0}^{j-1}(j-1-i)2^i=(j-1)S_j-T_j.$

On the one hand, $S_j=\frac{1-2^j}{1-2}=2^j-1$.

On the other hand, $-T_j=T_j-2T_j=2^1+2^2+\dots+2^{j-1}-(j-1)2^j=\frac{2-2^j}{1-2}-(j-1)2^j$, and $T_j=2-2^j+(j-1)2^j=2-2^j(1-(j-1))=2-2^j(1-(j-1))$. Therefore, $(j-1)\cdot 2^0+(j-2)\cdot 2^1+\dots+1\cdot 2^{j-2}=(j-1)S_j-T_j=(j-1)(2^j-1)-(2-2^j(1-(j-1)))=2^j-j-1$. It follows from (2.64), that

$$\delta_{j-1}\delta_{j-2}^2\ldots\delta_{j_0}^{2^{j-1}} = \frac{\delta_0^{2^j-1}}{\alpha^{2^j-j-1}}.$$

□

Notice that, from (2.65) we also have

$$\sum_{i=0}^{j-1}\varepsilon_i \le \sum_{i=0}^{j-1}\mu^{2^i-1}\frac{1}{\alpha^{\nu i}}\varepsilon_0 \le \sum_{j=0}^{\infty}\mu^{2^i-1}\frac{1}{\alpha^{\nu i}}\varepsilon_0 = \Sigma_{-1}\,\varepsilon_0. \tag{2.66}$$

This means that the series $\sum_{j=0}^{\infty}\varepsilon_j$ is convergent as well and hence,

$$\lim_{j\to\infty}\varepsilon_j = 0\,. \tag{2.67}$$

Next, we check the following conditions:

(I) $\|\kappa_j - \kappa^*\|_{\varrho_j} < r_0$, $\forall j = 0, 1, \ldots$, and hence $\kappa_j(\mathbb{T}_{\varrho_j}) \subseteq \Omega_{\varrho_0, r_0}$, $\forall j = 0, 1, \ldots$.

(II) $a < |\lambda_j| < \frac{1}{a}$, $\forall j = 0, 1, \ldots$.

(III) $\|c_j\|_{\varrho_j} < \sigma_{c_0}$, $\forall j = 0, 1, \ldots$.

(IV) $\|c_j^{-1}\|_{\varrho_j} < \sigma_{c_0^{-1}}$, $\forall j = 0, 1, \ldots$.

(V) $\|D_j^{-1}\| < \sigma_{D_0^{-1}}$, $\forall j = 0, 1, \ldots$.

(VI) $\|b_j\|_{\varrho_{j+1}} < \sigma_{b_0}$, $\forall j = 0, 1, \ldots$.

(I) $\|\kappa_j - \kappa^*\|_{\varrho_j} < r_0$, $\forall j = 0, 1, \ldots$, and hence $\kappa_j(\mathbb{T}_{\varrho_j}) \subseteq \Omega_{\varrho_0, r_0}$, $\forall j = 0, 1, \ldots$.

$$\begin{aligned}
\|\kappa_j - \kappa^*\|_{\varrho_j} &\le \|\kappa_0 - \kappa^*\|_{\varrho_0} + \sum_{i=0}^{j-1} \|\kappa_{i+1} - \kappa_i\|_{\varrho_{i+1}} = \|\kappa_0 - \kappa^*\|_{\varrho_0} + \sum_{i=0}^{j-1} \|\Delta\kappa_i\|_{\varrho_{i+1}} \\
&\le \|\kappa_0 - \kappa^*\|_{\varrho_0} + \sum_{i=0}^{j-1} \sigma_{\Delta\kappa_0}\, \varepsilon_i \le \|\kappa_0 - \kappa^*\|_{\varrho_0} + \sigma_{\Delta\kappa_0} \sum_{i=0}^{j-1} \varepsilon_i \\
&\le \|\kappa_0 - \kappa^*\|_{\varrho_0} + \sigma_{\Delta\kappa_0} \sum_{i=0}^{j-1} \frac{\mu^{2^i - 1}}{\alpha^{\nu i}} \varepsilon_0 < \|\kappa_0 - \kappa^*\|_{\varrho_0} + \sigma_{\Delta\kappa_0} \sum_{i=0}^{\infty} \frac{\mu^{2^i - 1}}{\alpha^{\nu i}} \varepsilon_0 \\
&= \|\kappa_0 - \kappa^*\|_{\varrho_0} + \sigma_{\Delta\kappa_0} \varepsilon_0 \Sigma_{-1} \le \|\kappa_0 - \kappa^*\|_{\varrho_0} + r_0 - \|\kappa_0 - \kappa^*\|_{\varrho_0} = r_0.
\end{aligned}$$

(II) $a < |\lambda_j| < \frac{1}{a}$, $\forall j = 0, 1, \ldots$.

$$\begin{aligned}
|\lambda_j| &\le |\lambda_0| + \sum_{i=0}^{j-1} |\Delta\lambda_i| \le |\lambda_0| + \sum_{i=0}^{j-1} \sigma_{\Delta\lambda_i}\, \varepsilon_i \\
&\le |\lambda_0| + \sigma_{\Delta\lambda_0} \sum_{i=0}^{j-1} \varepsilon_i < |\lambda_0| + \sigma_{\Delta\lambda_0} \sum_{i=0}^{j-1} \varepsilon_i = |\lambda_0| + \sigma_{\Delta\lambda_0} \sum_{i=0}^{\infty} \varepsilon_i \\
&= |\lambda_0| + \sigma_{\Delta\lambda_0} \Sigma_{-1} \varepsilon_0 < |\lambda_0| + \frac{1}{a} - |\lambda_0| = \frac{1}{a}.
\end{aligned}$$

Since (H.I) implies that $\sigma_{\Delta\lambda_0} \Sigma_{-1} \varepsilon_0 < \dfrac{1}{\max\left\{\frac{1}{\frac{1}{a} - |\lambda_0|}, \frac{1}{|\lambda_0| - a}\right\}} = \min\left\{\dfrac{1}{a} - |\lambda_0|, |\lambda_0| - a\right\}$, and therefore $\sigma_{\Delta\lambda_0} \Sigma_{-1} \varepsilon_0 < \frac{1}{a} - |\lambda_0|$ and $\sigma_{\Delta\lambda_0} \Sigma_{-1} \varepsilon_0 < |\lambda_0| - a$.

On the other hand,

$$||\lambda_j| - |\lambda_0|| \le |\lambda_j - \lambda_0| \le \sum_{i=0}^{j-1} |\Delta\lambda_i| \le \sigma_{\Delta\lambda_0} \Sigma_{-1} \varepsilon_0 < |\lambda_0| - a.$$

Hence, $a < |\lambda_j|$.

(III) $\|c_j\|_{\varrho_j} < \sigma_{c_0}$, $\forall j = 0, 1, \ldots$.

$$\begin{aligned}
\|c_j\|_{\varrho_j} &\le \|c_0\|_{\varrho_0} + \sum_{i=0}^{j-1} \|c_{i+1} - c_i\|_{\varrho_{i+1}} = \|c_0\|_{\varrho_0} + \sum_{i=0}^{j-1} \|\Delta c_i\|_{\varrho_{i+1}} \\
&\le \|c_0\|_{\varrho_0} + \sum_{i=0}^{j-1} \sigma_{\Delta c_i} \frac{1}{\gamma\, \delta_i^\nu}\, \varepsilon_i \le \|c_0\|_{\varrho_0} + \sigma_{\Delta c_0} \sum_{i=0}^{j-1} \frac{1}{\gamma\, \delta_i^\nu}\, \varepsilon_i \\
&\le \|c_0\|_{\varrho_0} + \sigma_{\Delta c_0} \sum_{i=0}^{j-1} \frac{1}{\gamma\, \delta_i^\nu}\, \mu^{2^i-1} \frac{1}{\alpha^{\nu i}} \varepsilon_0 \le \|c_0\|_{\varrho_0} + \sigma_{\Delta c_0} \sum_{i=0}^{j-1} \mu^{2^i-1}\, \frac{1}{\gamma\, \delta_0^\nu} \varepsilon_0 \\
&< \|c_0\|_{\varrho_0} + \sigma_{\Delta c_0} \sum_{i=0}^{\infty} \mu^{2^i-1}\, \frac{1}{\gamma\, \delta_0^\nu} \varepsilon_0 = \|c_0\|_{\varrho_0} + \sigma_{\Delta c_0} \Sigma_0\, \frac{1}{\gamma\, \delta_0^\nu} \varepsilon_0 \\
&\le \|c_0\|_{\varrho_0} + \sigma_{c_0} - \|c_0\|_{\varrho_0} = \sigma_{c_0}.
\end{aligned}$$

Notice that $\delta_i\, \alpha^i = \delta_0$. The last inequality is due to (H.III).

(IV) $\|c_j^{-1}\|_{\varrho_j} < \sigma_{c_0^{-1}}$, $\forall j = 0, 1, \ldots$.

In a completely analogous way,

$$\begin{aligned}
\|c_j^{-1}\|_{\varrho_j} &\le \|c_0^{-1}\|_{\varrho_0} + \sum_{i=0}^{j-1} \|c_{i+1}^{-1} - c_i^{-1}\|_{\varrho_{i+1}} = \|c_0^{-1}\|_{\varrho_0} + \sum_{i=0}^{j-1} \|\Delta c_i^{-1}\|_{\varrho_{i+1}} \\
&\le \|c_0^{-1}\|_{\varrho_0} + \sum_{i=0}^{j-1} \sigma_{\Delta c_i^{-1}} \frac{1}{\gamma\, \delta_i^\nu}\, \varepsilon_i \le \|c_0^{-1}\|_{\varrho_0} + \sigma_{\Delta c_0^{-1}} \sum_{i=0}^{j-1} \frac{1}{\gamma\, \delta_i^\nu}\, \varepsilon_i \\
&\le \|c_0^{-1}\|_{\varrho_0} + \sigma_{\Delta c_0^{-1}} \sum_{i=0}^{j-1} \frac{1}{\gamma\, \delta_i^\nu}\, \mu^{2^i-1} \frac{1}{\alpha^{\nu i}} \varepsilon_0 \le \|c_0^{-1}\|_{\varrho_0} + \sigma_{\Delta c_0^{-1}} \sum_{i=0}^{j-1} \mu^{2^i-1}\, \frac{1}{\gamma\, \delta_0^\nu} \varepsilon_0 \\
&< \|c_0^{-1}\|_{\varrho_0} + \sigma_{\Delta c_0^{-1}} \sum_{i=0}^{\infty} \mu^{2^i-1}\, \frac{1}{\gamma\, \delta_0^\nu} \varepsilon_0 = \|c_0^{-1}\|_{\varrho_0} + \sigma_{\Delta c_0^{-1}} \Sigma_0\, \frac{1}{\gamma\, \delta_0^\nu} \varepsilon_0 \\
&\le \|c_0^{-1}\|_{\varrho_0} + \sigma_{c_0^{-1}} - \|c_0^{-1}\|_{\varrho_0} = \sigma_{c_0^{-1}}.
\end{aligned}$$

Notice that $\delta_i\, \alpha^i = \delta_0$. The last inequality is due to (H.IV).

(V) $\|D_j^{-1}\| < \sigma_{D_0^{-1}}$, $\forall j = 0, 1, \ldots$.

$$\begin{aligned}
\|D_j^{-1}\| &\le \|D_0^{-1}\| + \sum_{i=0}^{j-1} \|D_{i+1}^{-1} - D_i^{-1}\| = \|D_0^{-1}\| + \sum_{i=0}^{j-1} \|\Delta D_i^{-1}\| \\
&\le \|D_0^{-1}\| + \sum_{i=0}^{j-1} \sigma_{\Delta D_i^{-1}} \frac{1}{(\gamma\, \delta_i^\nu)^2}\, \varepsilon_i \le \|D_0^{-1}\| + \sigma_{\Delta D_0^{-1}} \sum_{i=0}^{j-1} \frac{1}{(\gamma\, \delta_i^\nu)^2}\, \varepsilon_i \\
&\le \|D_0^{-1}\| + \sigma_{\Delta D_0^{-1}} \sum_{i=0}^{j-1} \frac{1}{(\gamma\, \delta_i^\nu)^2}\, \mu^{2^i-1} \frac{1}{\alpha^{\nu i}} \varepsilon_0 \le \|D_0^{-1}\| + \sigma_{\Delta D_0^{-1}} \sum_{i=0}^{j-1} \mu^{2^i-1}\, \frac{\alpha^{\nu i}}{(\gamma\, \delta_0^\nu)^2} \varepsilon_0 \\
&< \|D_0^{-1}\| + \sigma_{\Delta D_0^{-1}} \frac{1}{(\gamma\, \delta_0^\nu)^2} \sum_{i=0}^{\infty} \alpha^{\nu i} \mu^{2^i-1} \varepsilon_0 = \|D_0^{-1}\| + \sigma_{\Delta D_0^{-1}} \Sigma_1\, \frac{1}{(\gamma\, \delta_0^\nu)^2} \varepsilon_0 \\
&\le \|D_0^{-1}\| + \sigma_{D_0^{-1}} - \|D_0^{-1}\| = \sigma_{D_0^{-1}}.
\end{aligned}$$

Notice that $\delta_i\, \alpha^i = \delta_0$. The last inequality is due to (H.V).

(VI) $\|b_j\|_{\varrho_{j+1}} < \sigma_{b_0}, \forall j = 0, 1, \ldots$.

$$\begin{aligned}
\|b_j\|_{\varrho_{j+1}} &\leq \|b_0\|_{\varrho_1} + \sum_{i=0}^{j-1} \|b_{i+1} - b_i\|_{\varrho_{i+2}} = \|b_0\|_{\varrho_1} + \sum_{i=0}^{j-1} \|\Delta b_i\|_{\varrho_{i+2}} \\
&\leq \|b_0\|_{\varrho_1} + \sum_{i=0}^{j-1} \sigma_{\Delta b_i} \frac{1}{(\gamma\, \delta_i^\nu)^2}\, \varepsilon_i \leq \|b_0\|_{\varrho_1} + \sigma_{\Delta b_0} \sum_{i=0}^{j-1} \frac{1}{(\gamma\, \delta_i^\nu)^2}\, \varepsilon_i \\
&\leq \|b_0\|_{\varrho_1} + \sigma_{\Delta b_0} \sum_{i=0}^{j-1} \frac{1}{\gamma\, \delta_i^\nu}\, \mu^{2^i-1} \frac{1}{(\gamma\, \delta_i^\nu)^2} \varepsilon_0 \leq \|b_0\|_{\varrho_1} + \sigma_{\Delta c_0} \sum_{i=0}^{j-1} \mu^{2^i-1}\, \frac{\alpha^{\nu i}}{(\gamma\, \delta_0^\nu)^2} \varepsilon_0 \\
&< \|b_0\|_{\varrho_1} + \sigma_{\Delta c_0} \frac{1}{(\gamma\, \delta_0^\nu)^2} \sum_{i=0}^{\infty} \alpha^{\nu i} \mu^{2^i-1} \varepsilon_0 = \|b_0\|_{\varrho_1} + \sigma_{\Delta b_0} \Sigma_1 \frac{1}{(\gamma\, \delta_0^\nu)^2} \varepsilon_0 \\
&\leq \|b_0\|_{\varrho_1} + \sigma_{b_0} - \|b_0\|_{\varrho_1} = \sigma_{b_0}.
\end{aligned}$$

Again we have used that $\delta_i\, \alpha^i = \delta_0$ and the last inequality in this case is due to (H.VI).

Finally, we prove the convergence of the generated sequences.
Notice that $\varrho_\infty < \varrho_j$, $\forall j = 0, 1, \ldots$ and therefore $\kappa_j, c_j \in \mathcal{A}_{\varrho_\infty}$, $\forall j = 0, 1, \ldots$. By the uniform convergence of the series Σ_{-1} and Σ_0, we can see that $(\kappa_j)_{j\in\mathbb{N}}$, $(c_j)_{j\in\mathbb{N}}$ are Cauchy sequences in $\mathcal{A}_{\varrho_\infty}$ and $(\tau_j)_{j\in\mathbb{N}}$, $(\lambda_j)_{j\in\mathbb{N}}$ are Cauchy sequences in $\mathbb{R}$.

Since $\kappa_j = \kappa_0 + \sum_{i=0}^{j-1} \Delta\kappa_i$, then, $\forall j, l \in \mathbb{N}$, with $l > j$, we have

$$\begin{aligned}
\|\kappa_l - \kappa_j\|_{\varrho_\infty} &= \left\| \sum_{i=0}^{l-1} \Delta\kappa_i - \sum_{i=0}^{j-1} \Delta\kappa_i \right\|_{\varrho_\infty} \leq \sum_{i=j}^{l-1} \|\Delta\kappa_i\|_{\varrho_\infty} \\
&\leq \sum_{i=j}^{l-1} \sigma_{c_0} \sigma_{\xi_0^i} \varepsilon_i = \sigma_{c_0} \sigma_{\xi_0^i} \sum_{i=j}^{l-1} \varepsilon_i \\
&< \sigma_{c_0} \sigma_{\xi_0^i} \sum_{i=j}^{\infty} \varepsilon_i \;\to\; 0 \text{ as } j \to \infty,
\end{aligned}$$

Indeed, $(\kappa_j)_{j\in\mathbb{N}}$ is Cauchy in $\mathcal{A}_{\varrho_\infty}$, which is a Banach space. Therefore, $\exists \kappa \in \mathcal{A}_{\varrho_\infty}$, such that $\lim_{j\to\infty} \kappa_j = \kappa$.

Similarly, $\tau_j = \tau_0 + \sum_{i=0}^{j-1} \Delta\tau_i$ and hence, $|\tau_l - \tau_j| = \left| \sum_{i=j}^{l-1} \Delta\tau_i \right| \leq \sum_{i=j}^{l-1} |\Delta\tau_i| \leq \sigma_{\Delta\tau_0} \sum_{i=j}^{l-1} \varepsilon_i$ $< \sigma_{\Delta\tau_0} \sum_{i=j}^{\infty} \varepsilon_i \;\to\; 0$ as $j \to \infty$. Consequently, $\exists \tau \in \mathbb{R}$ such that $\lim_{j\to\infty} \tau_j = \tau$.

In the same manner, since $c_j = c_0 + \sum_{i=0}^{j-1} \Delta c_i$, then, $\forall j, l \in \mathbb{N}$, with $l > j$, we have

$$\begin{aligned}
\|c_l - c_j\|_{\varrho_\infty} &= \left\| \sum_{i=0}^{l-1} \Delta c_i - \sum_{i=0}^{j-1} \Delta c_i \right\|_{\varrho_\infty} \leq \sum_{i=j}^{l-1} \|\Delta c_i\|_{\varrho_\infty} \\
&\leq \sum_{i=j}^{l-1} \sigma_{\Delta c_i} \frac{1}{\gamma\, \delta_i^\nu} \varepsilon_i < \sigma_{\Delta c_0} \sum_{i=j}^{\infty} \frac{1}{\gamma\, \delta_i^\nu} \varepsilon_i \leq \sigma_{\Delta c_0} \frac{1}{\gamma\, \delta_0^\nu} \sum_{i=j}^{\infty} \mu^{2^i-1} \varepsilon_0 \;\to\; 0 \text{ as } j \to \infty,
\end{aligned}$$

since $\Sigma_0 = \sum_{j=0}^{\infty} \mu^{2^j - 1}$ is uniformly convergent.
Thus, $(c_j)_{j\in\mathbb{N}}$ is Cauchy in $\mathcal{A}_{\varrho_\infty}$ and therefore, $\exists c \in \mathcal{A}_{\varrho_\infty}$, such that $\lim_{j\to\infty} c_j = c$.
Finally, $\lambda_j = \lambda_0 + \sum_{i=0}^{j-1} \Delta\lambda_i$ and hence, $|\lambda_l - \lambda_j| = \left|\sum_{i=j}^{l-1} \Delta\lambda_i\right| \leq \sum_{i=j}^{l-1} |\Delta\lambda_i| \leq \sigma_{\Delta\lambda_0} \sum_{i=j}^{l-1} \varepsilon_i$ $< \sigma_{\Delta\lambda_0} \sum_{i=j}^{\infty} \varepsilon_i \;\to\; 0$ as $j \to \infty$. Consequently, $\exists \lambda \in \mathbb{R}$ such that $\lim_{j\to\infty} \lambda_j = \lambda$. Moreover, $\lambda \neq 0$, since $a < |\lambda_j| < \frac{1}{a}$, $\forall j \in \mathbb{N}$.
Furthermore,
$\varepsilon_j = \max\left\{ \|e_j^r\|_{\varrho_j}, \dfrac{1}{\gamma\, \|e_j^i\|_{\varrho_j}} \right\} \to 0$ as $j \to \infty$ implies that $\|e_j^r\|_{\varrho_j} \to 0$ and $\|e_j^i\|_{\varrho_j} \to 0$ as $j \to \infty$.
Taking limits as $j \to \infty$ in

$$\begin{aligned} f(\theta, \kappa_j(\theta)) - \kappa_j(\theta + \omega) + \tau_j &= e_j^i(\theta)\ , \theta \in \mathbb{T}_{\varrho_\infty} \text{ and} \\ \frac{\partial f}{\partial z}(\theta, \kappa_j(\theta)) c_j(\theta) + c_j(\theta + \omega)\lambda_j &= e_j^r(\theta)\ , \theta \in \mathbb{T}_{\varrho_\infty} \end{aligned}$$

we obtain

$$\begin{aligned} f(\theta, \kappa(\theta)) - \kappa(\theta + \omega) + \tau &= 0\ , \theta \in \mathbb{T}_{\varrho_\infty} \text{ and} \\ \frac{\partial f}{\partial z}(\theta, \kappa(\theta)) c(\theta) + c(\theta + \omega)\lambda &= 0\ , \theta \in \mathbb{T}_{\varrho_\infty}\ . \end{aligned}$$

So, $\kappa \in \mathcal{A}_{\varrho_\infty}$ is an invariant translated curve of the skew–product with translation parameter $\tau \in \mathbb{R}$ and $c \in \mathcal{A}_{\varrho_\infty}$ is a reducibility function for κ with reducibility constant $\lambda \neq 0$.

Additionally, if we call $v(\theta) = \log\left|\frac{\partial f}{\partial z}(\theta, \kappa(\theta))\right|$ and $u(\theta) = \log|c(\theta)|$ and taking in account that

$$\left|\frac{\partial f}{\partial z}(\theta, \kappa(\theta))\right|\ |c(\theta)| = |c(\theta + \omega)|\ |\lambda|,$$

we obtain, taking logarithms,

$$u(\theta + \omega) - u(\theta) = v(\theta) - \log|\lambda|.$$

Taking averages,

$$0 = \int_{\mathbb{T}} v(\theta) d\theta - \log|\lambda|.$$

Therefore, $|\lambda| = e^{\int_{\mathbb{T}} \log\left|\frac{\partial f}{\partial z}(\theta,\kappa(\theta))\right| d\theta}$.
If $\dfrac{\partial f}{\partial z}(\theta, \kappa(\theta)) > 0$, $\forall\theta \in \mathbb{T}$, then $|\lambda| = e^{\Lambda}$, with $\Lambda = \int_{\mathbb{T}} \log\left(\dfrac{\partial f}{\partial z}(\theta, \kappa(\theta))\right) d\theta$, the Lyapunov exponent of κ. □

Chapter 3

The translated graph method

3.1 The KAM procedure: one step of the Newton-like method

The key point of the method for finding invariant translated curves is to restrict the search to those whose average is a fixed number. More explicitly: Given an average $p \in \mathbb{R}$ our aim is to find, according to **Proposition 2.12**, solutions $(\kappa(\theta), \tau)$ to the system:

$$\left\{ \begin{array}{rcl} f(\theta, \kappa(\theta)) - \kappa(\theta+\omega) + \tau & = & 0 \\ < \kappa > & = & p \end{array} \right. \quad (\theta \in \mathbb{T}) \tag{3.1}$$

where $f : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is the fiber map of a given skew–product of the form described in (2.1), with the frequency $\omega \in \mathcal{DC}(\gamma, \nu)$ being Diophantine, whereas $\kappa : \mathbb{T} \to \mathbb{R}$ and the translation number $\tau \in \mathbb{R}$ are unknown.
We are going to show the constructability, under certain non-degeneracy conditions, of a sequence

$$\{(\kappa_n(\theta), \tau_n)\}_{n \in \mathbb{N}} \tag{3.2}$$

which converges to such a solution, i.e. $\kappa_n(\theta) \longrightarrow \kappa(\theta) \quad (\theta \in \mathbb{T})$ and $\tau_n \longrightarrow \tau$ as $n \to \infty$.
Let us proceed by recurrence, that is, assume that we are given an approximation $(\kappa_n(\theta), \tau_n)$, and we want to build a new one, $(\kappa_{n+1}(\theta), \tau_{n+1})$ with

$$\left\{ \begin{array}{rcl} \kappa_{n+1}(\theta) & = & \kappa_n(\theta) + \Delta\kappa_n(\theta) \\ \tau_{n+1} & = & \tau_n + \Delta\tau_n \end{array} \right. . \tag{3.3}$$

We denote the *invariance error* for the n^{th}–terms by:

$$\left\{ \begin{array}{rcl} E_n(\theta) & = & f(\theta, \kappa_n(\theta)) - \kappa_n(\theta+\omega) + \tau_n \\ e_n(p) & = & < \kappa_n > - p \end{array} \right. \tag{3.4}$$

The new approximation (3.3) is obtained by solving the system:

$$\left\{ \begin{array}{rcl} m_n(\theta)\Delta\kappa_n(\theta) - \Delta\kappa_n(\theta+\omega) + \Delta\tau_n & = & -E_n(\theta) \\ < \Delta\kappa_n > & = & -e_n(p) \end{array} \right. \tag{3.5}$$

where[1]

$$m_n(\theta) = D_x f(\theta, \kappa_n(\theta)) \ , \ \theta \in \mathbb{T} \quad , \tag{3.6}$$

[1] From now on we assume that $D_x f(\theta, x) > 0$, $\forall (\theta, x) \in \mathbb{T} \times \mathbb{R}$. Accordingly, $m_n(\theta) > 0$, $\forall \theta \in \mathbb{T}$.

and the unknowns are $\Delta\kappa_n(\theta)$ and $\Delta\tau_n$.
The motivation for this approach lies in the following argument:
The first equation of (3.5) comes from the first order approximation of the new error, by means of the Taylor expansion of f with respect to the second variable at $(\theta,\kappa(\theta))$. Namely,

$$\begin{array}{rcl} E_{n+1}(\theta) & = & f(\theta,\kappa_{n+1}(\theta)) - \kappa_{n+1}(\theta+\omega) + \tau_{n+1} \\ & = & f(\theta,\kappa_n(\theta)+\Delta\kappa_n(\theta)) - (\kappa_n(\theta+\omega)+\Delta\kappa_n(\theta+\omega)) + \tau_n + \Delta\tau_n \\ & = & f(\theta,\kappa_n(\theta)) + D_x f(\theta,\kappa_n(\theta))\Delta\kappa_n(\theta) + \mathrm{O}\left(\Delta\kappa_n(\theta)^2\right) \\ & & \quad -\kappa_n(\theta+\omega) - \Delta\kappa_n(\theta+\omega) + \tau_n + \Delta\tau_n \\ & = & E_n(\theta) + m_n(\theta)\Delta\kappa_n(\theta) - \Delta\kappa_n(\theta+\omega) + \Delta\tau_n + \mathrm{O}\left(\Delta\kappa_n(\theta)^2\right) \end{array}$$

So that, when (3.5) holds, then:

$$E_{n+1}(\theta) = \mathrm{O}\left(\Delta\kappa_n(\theta)^2\right) \ ,$$

and the order of convergence is quadratic[2].
On the other hand, by the linearity of the average,

$$\begin{array}{rcl} e_{n+1}(p) & = & <\kappa_{n+1}> - p \\ & = & <\kappa_n + \Delta\kappa_n> - p \\ & = & <\kappa_n> + <\Delta\kappa_n> - p \\ & = & e_n(p) + <\Delta\kappa_n> \ . \end{array}$$

Hence, $e_{n+1}(p) = 0$ whenever the second condition of (3.5) holds.
Observe that, in such a case, $<\kappa_{n+1}> = p$.

3.2 The non–degeneracy condition

Let us now return to the problem of finding translated invariant curves, i.e. solutions of the system (3.1), where we left off. More precisely, we need to investigate, first, sufficient conditions under which the system (3.5), scilicet,

$$\left\{ \begin{array}{rcl} m_n(\theta)\Delta\kappa_n(\theta) - \Delta\kappa_n(\theta+\omega) + \Delta\tau_n & = & -E_n(\theta) \\ <\Delta\kappa_n> & = & -e_n(p) \ , \end{array} \right. \tag{3.7}$$

is solvable.
According to **Theorem 2.6** and **Corollary 2.8**, the reducibility of the linear quasi–periodic skew–product

$$\begin{array}{rccl} D\psi(\kappa_n): & \mathbb{T}\times\mathbb{R} & \longrightarrow & \mathbb{T}\times\mathbb{R} \\ & (\theta,x) & \longmapsto & D\psi(\kappa_n)(\theta,x) = (\theta+\omega, m_n(\theta)x) \ , \end{array} \tag{3.8}$$

where $m_n(\theta) = D_x f(\theta,\kappa_n(\theta))$, which is the linearization[3] of ψ_τ about κ_n, allows us finding a positive reducibility constant $\lambda_n > 0$ and a Floquet transformation $c_n : \mathbb{T} \longrightarrow \mathbb{R}$, such that for a given $c_{n,0} > 0$ (average which can be chosen freely a priori):

$$\lambda_n = c_n(\theta+\omega)^{-1} m_n(\theta) c_n(\theta) = e^{\Lambda_n} \ , \tag{3.9}$$

[2] We provide more details about this fact later on. See **Proposition 3.4**.

[3] See **Definition 2.2**.

with

$$\Lambda_n = \int_{\mathbb{T}} \log(m_n(\theta))d\theta \ , \tag{3.10}$$

and

$$c_n(\theta) = \frac{c_{n,0}}{\displaystyle\int_{\mathbb{T}} e^{\Re_1(\log(m_n(\theta))-\Lambda_n)}d\theta} \cdot e^{\Re_1(\log(m_n(\theta))-\Lambda_n)} \ . \tag{3.11}$$

Now, we put forward the following change of variable:

$$\Delta\kappa_n(\theta) = c_n(\theta)\varsigma_n(\theta) \ . \tag{3.12}$$

In this way, the first equation of (3.7) is transformed into the following one:

$$m_n(\theta)c_n(\theta)\varsigma_n(\theta) - c_n(\theta+\omega)\varsigma_n(\theta+\omega) + \Delta\tau_n = -E_n(\theta) \ . \tag{3.13}$$

Since $m_n(\theta)c_n(\theta) = c_n(\theta+\omega)\lambda_n$, we have

$$\varsigma_n(\theta+\omega) - \lambda_n\varsigma_n(\theta) = \frac{E_n(\theta)}{c_n(\theta+\omega)} + \frac{1}{c_n(\theta+\omega)}\Delta\tau_n$$

or, in other words,

$$\varsigma_n(\theta+\omega) - \lambda\varsigma_n(\theta) = \xi_n(\theta) + \eta_n(\theta)\Delta\tau_n \ , \tag{3.14}$$

where we have defined the functions:

$$\xi_n(\theta) = \frac{E_n(\theta)}{c_n(\theta+\omega)} \ , \text{ and} \tag{3.15}$$

$$\eta_n(\theta) = \frac{1}{c_n(\theta+\omega)} \ , \tag{3.16}$$

being, both of them, known.
Summarizing, the main problem, given by the equations (3.5), is converted now into the following:

$$\left\{ \begin{array}{rcl} \varsigma_n(\theta+\omega) - \lambda\varsigma_n(\theta) & = & \xi_n(\theta) + \eta_n(\theta)\Delta\tau_n \quad (\theta \in \mathbb{T}) \\ < c_n\varsigma_n > & = & -e_n(p) \ . \end{array} \right. \tag{3.17}$$

Here, the unknowns are the curve $\varsigma_n(\theta)$ and the deviation of the translation parameter $\Delta\tau_n$.
Recall the notation pointed out at Remark 1.6. In this case, we write:

$$\varsigma_n = \varsigma_{n,0} + \widetilde{\varsigma}_n, \text{ with } \varsigma_{n,0} =< \varsigma_n >= \int_{\mathbb{T}} \varsigma_n(\theta)\, d\theta \ , \tag{3.18}$$

$$\xi_n = \xi_{n,0} + \widetilde{\xi}_n, \text{ with } \xi_{n,0} =< \xi_n >= \int_{\mathbb{T}} \xi_n(\theta)\, d\theta \ , \tag{3.19}$$

$$\eta_n = \eta_{n,0} + \widetilde{\eta}_n, \text{ with } \eta_{n,0} =< \eta_n >= \int_{\mathbb{T}} \eta_n(\theta)\, d\theta \ . \tag{3.20}$$

Thus, the first equation of (3.17) can be written as:

$$\varsigma_{n,0} + \widetilde{\varsigma}_n(\theta+\omega) - \lambda_n(\varsigma_{n,0} + \widetilde{\varsigma}_n(\theta)) = \xi_{n,0} + \widetilde{\xi}_n(\theta) + (\eta_{n,0} + \widetilde{\eta}_n(\theta))\Delta\tau_n \ . \tag{3.21}$$

Taking the average on both sides of (3.21) we have:

$$(1-\lambda_n)\varsigma_{n,0} - \eta_{n,0}\Delta\tau_n = \xi_{n,0} \ . \tag{3.22}$$

Subtracting (3.22) from (3.21) we also have:

$$\widetilde{\varsigma}_n(\theta+\omega) - \lambda_n\widetilde{\varsigma}_n(\theta) = \widetilde{\xi}_n(\theta) + \widetilde{\eta}_n(\theta)\Delta\tau_n \ . \tag{3.23}$$

Observe that the average of the right hand side of this equation is zero.
Consequently, it turns out that $\widetilde{\varsigma}_n$ is, in accordance with **Theorem 1.20**, the unique solution to the cohomological equation (3.23) with zero average, that is, $\widetilde{\varsigma}_n(\theta) = \mathfrak{R}_{\lambda_n}(\widetilde{\xi}_n(\theta) + \widetilde{\eta}_n(\theta)\Delta\tau_n)$, i.e.

$$\widetilde{\varsigma}_n(\theta) = \mathfrak{R}_{\lambda_n}(\widetilde{\xi}_n(\theta)) + \mathfrak{R}_{\lambda_n}(\widetilde{\eta}_n(\theta))\Delta\tau_n \quad , \tag{3.24}$$

since $\mathfrak{R}_{\lambda_n}$ is linear.
Additionally, from the second equation of (3.17), we obtain:

$$\begin{aligned} < c_n\varsigma_n > &= < (c_{n,0} + \widetilde{c}_n)(\varsigma_{n,0} + \widetilde{\varsigma}_n) >=< c_{n,0}\varsigma_{n,0} + c_{n,0}\widetilde{\varsigma}_n + \varsigma_{n,0}\widetilde{c}_n + \widetilde{c}_n\widetilde{\varsigma}_n > \\ &= c_{n,0}\varsigma_{n,0}+ < \widetilde{c}_n\widetilde{\varsigma}_n >, \text{ since } < \widetilde{\varsigma}_n >=< \widetilde{c}_n >= 0. \end{aligned}$$

Hence,

$$c_{n,0}\varsigma_{n,0}+ < \widetilde{c}_n\widetilde{\varsigma}_n >= -e_n(p) \, . \tag{3.25}$$

Now, from (3.24) and (3.25) we have:

$$\begin{aligned} -e_n(p) &= c_{n,0}\varsigma_{n,0}+ < \widetilde{c}_n\widetilde{\varsigma}_n > \\ &= c_{n,0}\varsigma_{n,0}+ < \widetilde{c}_n(\mathfrak{R}_{\lambda_n}(\widetilde{\xi}_n) + \mathfrak{R}_{\lambda_n}(\widetilde{\eta}_n)\Delta\tau_n) > \\ &= c_{n,0}\varsigma_{n,0}+ < \widetilde{c}_n\mathfrak{R}_{\lambda_n}(\widetilde{\xi}_n) > + < \widetilde{c}_n\mathfrak{R}_{\lambda_n}(\widetilde{\eta}_n) > \Delta\tau_n \, , \end{aligned}$$

that is,

$$c_{n,0}\varsigma_{n,0}+ < \widetilde{c}_n\mathfrak{R}_{\lambda_n}(\widetilde{\eta}_n) > \Delta\tau_n = -e_n(p)- < \widetilde{c}_n\mathfrak{R}_{\lambda_n}(\widetilde{\xi}_n) > . \tag{3.26}$$

Equations (3.22) and (3.26) joined together, provide us with the following linear system:

$$\begin{pmatrix} 1-\lambda_n & -\eta_{n,0} \\ c_{n,0} & < \widetilde{c}_n\mathfrak{R}_{\lambda_n}(\widetilde{\eta}_n) > \end{pmatrix} \begin{pmatrix} \varsigma_{n,0} \\ \Delta\tau_n \end{pmatrix} = \begin{pmatrix} \xi_{n,0} \\ -e_n(p)- < \widetilde{c}_n\mathfrak{R}_{\lambda_n}(\widetilde{\xi}_n) > \end{pmatrix}. \tag{3.27}$$

This linear system is the key point in the construction of one step in this process.
Let us denote

$$\Omega_n = \begin{pmatrix} 1-\lambda_n & -\eta_{n,0} \\ c_{n,0} & < \widetilde{c}_n\mathfrak{R}_{\lambda_n}(\widetilde{\eta}_n) > \end{pmatrix} \tag{3.28}$$

and

$$b_n = \begin{pmatrix} \xi_{n,0} \\ -e_n(p)- < \widetilde{c}_n\mathfrak{R}_{\lambda_n}(\widetilde{\xi}_n) > \end{pmatrix} . \tag{3.29}$$

The system (3.27) is then expressed as:

$$\Omega_n \begin{pmatrix} \varsigma_{n,0} \\ \Delta\tau_n \end{pmatrix} = b_n. \tag{3.30}$$

Notice that (3.30) has a unique solution if and only if the following non–degeneracy condition is satisfied:

$$\det(\Omega_n) = \det \begin{pmatrix} 1-\lambda_n & -\eta_{n,0} \\ c_{n,0} & < \widetilde{c}_n\mathfrak{R}_{\lambda_n}(\widetilde{\eta}_n) > \end{pmatrix} \neq 0 \quad . \tag{3.31}$$

Equivalently, we may write (3.31) as:

$$< (1-\lambda_n)\widetilde{c}_n\mathfrak{R}_{\lambda_n}(\widetilde{\eta}_n) + c_{n,0}\eta_{n,0} >\neq 0 \quad . \tag{3.32}$$

Remark 3.1

The non–degeneracy condition (3.31) is independent of the chosen value for the average $c_{n,0}$ of the Floquet transformation. Henceforth, for the sake of simplicity we may take $c_{n,0} = 1$. However, we can also consider other options, as we will see below.

All the previous arguments are fully resumed in the following statement.

Lemma 3.2 Constructability of one step of the KAM process

Let $\psi = \mathcal{R}_\omega \times f$ be a quasi–periodic skew–product defined under the same conditions as in (2.1). Let $(\kappa_n(\theta), \tau_n)$ be an approximation of a solution to the system (3.1), and $E_n(\theta)$ and $e_n(p)$ the corresponding invariance errors given by (3.4).
Consider the following constants and functions:

$$\begin{aligned}
m_n(\theta) &= \frac{\partial f}{\partial x}(\theta, \kappa_n(\theta)),\ \theta \in \mathbb{T}_\varrho\ , \\
\Lambda_n &= \int_{\mathbb{T}} \log(m_n(\theta))\, d\theta,\ \text{the Lyapunov exponent}\ , \\
\lambda_n &= e^{\Lambda_n},\ \text{the Lyapunov multiplier}\ , \\
v_n(\theta) &= \log(m_n(\theta)) - \Lambda_n\ ,\ \forall\theta \in \mathbb{T}_\varrho\ , \\
u_n(\theta) &= u_{n,0} + \widetilde{u}_n(\theta), \theta \in \mathbb{T}_{\varrho-\delta},\ \delta \in (0, \varrho)\ \ \text{with} \\
\widetilde{u}_n(\theta) &= \mathfrak{R}_1 v_n(\theta)\ ,\ \forall\theta \in \mathbb{T}_{\varrho-\delta}\ , \\
u_{n,0} &= \log \alpha_n\ ,\ \text{with} \\
\alpha_n &= \frac{c_{n,0}}{\displaystyle\int_{\mathbb{T}} e^{\widetilde{u}_n(\theta)} d\theta}\ \ \text{and } c_{n,0} > 0 \text{ freely chosen}\ , \\
c_n(\theta) &= e^{u_n(\theta)} = \alpha_n e^{\widetilde{u}_n(\theta)}\ ,\ \forall\theta \in \mathbb{T}_{\varrho-\delta}\ ,\ \text{the Floquet transformation}, \\
\xi_n(\theta) &= \frac{E_n(\theta)}{c_n(\theta+\omega)}\ ,\ \forall\theta \in \mathbb{T}_{\varrho-\delta}\ , \\
\eta_n(\theta) &= \frac{1}{c_n(\theta+\omega)}\ ,\ \forall\theta \in \mathbb{T}_{\varrho-\delta}\ .
\end{aligned}$$

Finally, define:

$$\Omega_n = \begin{pmatrix} 1-\lambda_n & -\eta_{n,0} \\ c_{n,0} & < \widetilde{c}_n \mathfrak{R}_{\lambda_n} \widetilde{\eta}_n > \end{pmatrix} \ \text{and} \tag{3.33}$$

$$b_n = \begin{pmatrix} \xi_{n,0} \\ -e_n(p) - < \widetilde{c}_n \mathfrak{R}_{\lambda_n} \widetilde{\xi}_n > \end{pmatrix} . \tag{3.34}$$

If the non–degeneracy condition

$$\det(\Omega_n) = < (1-\lambda_n)\widetilde{c}_n \mathfrak{R}_{\lambda_n} \widetilde{\eta}_n + c_{n,0}\eta_{n,0} > \neq 0 \tag{3.35}$$

holds, then there exist a new approximation $(\kappa_{n+1}(\theta), \tau_{n+1})$ for the system (3.1) of the form:

$$\kappa_{n+1}(\theta) = \kappa_n(\theta) + \Delta\kappa_n(\theta)\ ,\ \forall\theta \in \mathbb{T} \tag{3.36}$$

$$\tau_{n+1} = \tau_n + \Delta\tau_n\ , \tag{3.37}$$

satisfying equations (3.5).

Furthermore, this new approximation can be obtained explicitly, solving the linear system:

$$\Omega_n(\varsigma_{n,0}, \Delta\tau_n)^\top = b_n\ . \tag{3.38}$$

Namely, on the one hand

$$\Delta\kappa_n(\theta) = c_n(\theta)\varsigma_n(\theta)\ ,\ \forall\theta \in \mathbb{T}\ , \tag{3.39}$$

where

$$\varsigma_n(\theta) = \varsigma_{n,0} + \widetilde{\varsigma}_n(\theta)\ ,\ \forall\theta \in \mathbb{T}\ , \tag{3.40}$$

$$\varsigma_{n,0} = \frac{< \xi_{n,0}\widetilde{c}_n \mathfrak{R}_{\lambda_n}\widetilde{\eta}_n + \eta_{n,0}(-e_n(p) - \widetilde{c}_n \mathfrak{R}_{\lambda_n}\widetilde{\xi}_n) >}{\det(\Omega_n)}\ ,\ \text{and} \tag{3.41}$$

$$\widetilde{\varsigma}_n(\theta) = \mathfrak{R}_{\lambda_n}(\widetilde{\xi}_n(\theta) + \widetilde{\eta}_n(\theta)\Delta\tau_n)\ ,\ \forall\theta \in \mathbb{T}. \tag{3.42}$$

and, on the other hand

$$\Delta\tau_n = \frac{< (1-\lambda_n)(-e_n(p) - \widetilde{c}_n \mathfrak{R}_{\lambda_n}\widetilde{\xi}_n) - c_{n,0}\xi_{n,0} >}{\det(\Omega_n)} , \tag{3.43}$$

Remark 3.3

We should point out that, in the described construction process, the domain of some functions involved is broader than the one–dimensional torus. More specifically, if $\varrho > 0$ *and* $v_n \in \mathcal{A}_\varrho$*, then we can take* $\delta \in (0, \frac{\varrho}{2})$ *so that* $c_n = \alpha_n e^{\mathfrak{R}_1 v_n} \in \mathcal{A}_{\varrho-\delta}$*, since* $\mathfrak{R}_1 v_n \in \mathcal{A}_{\varrho-\delta}$ *as it was seen in* ***Theorem 1.20****. Moreover,* $\eta_n = \frac{1}{c_n \circ \mathcal{R}_\omega} \in \mathcal{A}_{\varrho-\delta}$ *and* $\xi_n = \frac{E_n}{c_n \circ \mathcal{R}_\omega} \in \mathcal{A}_{\varrho-\delta}$*, too. Calling now* $\varrho^* = \varrho - \delta$ *we can say that* $\widetilde{\eta}_n \in \mathcal{A}_{\varrho^*}$*, and* $\delta \in (0, \varrho^*)$*. Again by* ***Theorem 1.20****,* $\mathfrak{R}_{\lambda_n}\widetilde{\eta}_n \in \mathcal{A}_{\varrho^*-\delta} = \mathcal{A}_{\varrho-2\delta}$*. By the same argument,* $\mathfrak{R}_{\lambda_n}\widetilde{\xi}_n \in \mathcal{A}_{\varrho^*-\delta} = \mathcal{A}_{\varrho-2\delta}$*. Moreover, since* $\widetilde{\varsigma} = \mathfrak{R}_{\lambda_n}(\widetilde{\xi}_n + \widetilde{\eta}_n \Delta\tau_n)$*, then* $\widetilde{\varsigma} \in \mathcal{A}_{\varrho-2\delta}$*, too. It follows that* $\Delta\kappa_n = c_n\varsigma_n = c_n(\varsigma_{n,0} + \widetilde{\varsigma}_n) \in \mathcal{A}_{\varrho-2\delta}$*. Finally, we can conclude that*

$$\kappa_{n+1} = \kappa_n + \Delta\kappa_n \in \mathcal{A}_{\varrho-2\delta},\ \forall\delta \in (0, \frac{1}{2}\varrho). \tag{3.44}$$

3.3 Error estimates

In this section we show a number of estimates regarding the control of some geometric properties of an approximately invariant curve.
First, we come back to the expressions of the invariance error (3.4). For the sake of simplicity, we write these expressions as follows:

$$E(\theta) = f(\theta, \kappa(\theta)) - \kappa(\theta+\omega) + \tau \tag{3.45}$$

$$e_p = <\kappa> - p \tag{3.46}$$

For the new approximation the error is given by:

$$\overline{E}(\theta) = f(\theta, \overline{\kappa}(\theta)) - \overline{\kappa}(\theta+\omega) + \overline{\tau} \tag{3.47}$$

$$\overline{e}_p = <\overline{\kappa}> - p \tag{3.48}$$

Our goal is to express $\overline{E}(\theta)$ and $\overline{e}_p$ in terms of $\Delta\kappa(\theta)$ and $\Delta\tau$.

Proposition 3.4 Invariance error in the KAM iterative step

(a)

$$\overline{m}(\theta) = m(\theta) + \int_0^1 \frac{\partial^2 f}{\partial x^2}(\theta, \kappa(\theta) + s\Delta\kappa(\theta))ds\ \Delta\kappa(\theta)\ ; \tag{3.49}$$

(b)

$$\overline{E}(\theta) = \int_0^1 \frac{\partial^2 f}{\partial x^2}(\theta, \kappa(\theta) + s\Delta\kappa(\theta))(1-s)ds\ \Delta\kappa(\theta)^2 \quad . \tag{3.50}$$

Proof. To demonstrate parts (a) and (b) we use a technique based upon the first-order Taylor expansion with integral remainder.

(a)

$$\begin{aligned}\overline{m}(\theta) - m(\theta) &= \frac{\partial f}{\partial x}(\theta, \kappa(\theta) + \Delta\kappa(\theta)) - \frac{\partial f}{\partial x}(\theta, \kappa(\theta)) \\ &= \int_{\kappa(\theta)}^{\overline{\kappa}(\theta)} \frac{\partial^2 f}{\partial x^2}(\theta, x)dx\ .\end{aligned}$$

Apply now the following change of variable $x = h(s)$ with

$$\begin{array}{rccl} h: & [0,1] & \longrightarrow & [\kappa(\theta), \overline{\kappa}(\theta)] \\ & s & \longmapsto & h(s) = \kappa(\theta) + s\Delta\kappa(\theta) \end{array} \tag{3.51}$$

and hence

$$\begin{aligned} \overline{m}(\theta) - m(\theta) &= \int_0^1 \frac{\partial^2 f}{\partial x^2}(\theta, \kappa(\theta) + s\Delta\kappa(\theta))\, \Delta\kappa(\theta) ds \\ &= \int_0^1 \frac{\partial^2 f}{\partial x^2}(\theta, \kappa(\theta) + s\Delta\kappa(\theta)) ds\, \Delta\kappa(\theta)\,. \end{aligned}$$

(b) First we write $f(\theta, \overline{\kappa}(\theta)) = f(\theta, \kappa(\theta)) + \int_{\kappa(\theta)}^{\overline{\kappa}(\theta)} \frac{\partial f}{\partial x}(\theta, x) dx$. Now, we integrate by parts, taking:

$$\begin{aligned} u &= \frac{\partial f}{\partial x}(\theta, x) \\ dv &= dx \\ du &= \frac{\partial^2 f}{\partial x^2}(\theta, x) dx \\ v &= x - \kappa(\theta) \end{aligned}$$

Therefore,

$$\begin{aligned} f(\theta, \overline{\kappa}(\theta)) &= f(\theta, \kappa(\theta)) + \left[\frac{\partial f}{\partial x}(\theta, x)(x - \kappa(\theta))\right]_{x=\kappa(\theta)}^{x=\overline{\kappa}(\theta)} - \int_{\kappa(\theta)}^{\overline{\kappa}(\theta)} \frac{\partial^2 f}{\partial x^2}(\theta, x)\,(x - \kappa(\theta))\, dx \\ &= f(\theta, \kappa(\theta)) + \frac{\partial f}{\partial x}(\theta, \overline{\kappa}(\theta))\Delta\kappa(\theta) - \int_{\kappa(\theta)}^{\overline{\kappa}(\theta)} \frac{\partial^2 f}{\partial x^2}(\theta, x)\,(x - \kappa(\theta))\, dx \\ &= f(\theta, \kappa(\theta)) + \overline{m}(\theta)\Delta\kappa(\theta) - \int_{\kappa(\theta)}^{\overline{\kappa}(\theta)} \frac{\partial^2 f}{\partial x^2}(\theta, x)\,(x - \kappa(\theta))\, dx \quad . \end{aligned}$$

For the integral we apply again the change (3.51), that is,

$$\begin{aligned} \int_{\kappa(\theta)}^{\overline{\kappa}(\theta)} \frac{\partial^2 f}{\partial x^2}(\theta, x)\,(x - \kappa(\theta))\, dx &= \int_0^1 \frac{\partial^2 f}{\partial x^2}(\theta, \kappa(\theta) + s\Delta\kappa(\theta))\; s\; \Delta\kappa(\theta)^2\, ds \\ &= \int_0^1 \frac{\partial^2 f}{\partial x^2}(\theta, \kappa(\theta) + s\Delta\kappa(\theta))\; s\, ds\; \Delta\kappa(\theta)^2 \quad . \end{aligned}$$

It follows that:

$$f(\theta, \overline{\kappa}(\theta)) = f(\theta, \kappa(\theta)) + \overline{m}(\theta)\Delta\kappa(\theta) - \int_0^1 \frac{\partial^2 f}{\partial x^2}(\theta, \kappa(\theta) + s\Delta\kappa(\theta))\; s\, ds\; \Delta\kappa(\theta)^2\,. \tag{3.52}$$

Finally, with (3.47) and (3.52), we can compute the new invariance error:

$$\begin{aligned} \overline{E}(\theta) &= f(\theta, \overline{\kappa}(\theta)) - \overline{\kappa}(\theta + \omega) + \overline{\tau} \\ &= f(\theta, \kappa(\theta)) + \overline{m}(\theta)\Delta\kappa(\theta) - \int_0^1 \frac{\partial^2 f}{\partial x^2}(\theta, \kappa(\theta) + s\Delta\kappa(\theta))\; s\, ds\; \Delta\kappa(\theta)^2 \\ &\quad -(\kappa(\theta + \omega) + \Delta\kappa(\theta + \omega)) + \tau + \Delta\tau \\ &= f(\theta, \kappa(\theta)) - \kappa(\theta + \omega) + \tau \\ &+ \overline{m}(\theta)\Delta\kappa(\theta) - \Delta\kappa(\theta + \omega)) + \Delta\tau - \int_0^1 \frac{\partial^2 f}{\partial x^2}(\theta, \kappa(\theta) + s\Delta\kappa(\theta))\; s\, ds\; \Delta\kappa(\theta)^2 \\ &= E(\theta) + \overline{m}(\theta)\Delta\kappa(\theta) - E(\theta) - m(\theta)\Delta\kappa(\theta) - \int_0^1 \frac{\partial^2 f}{\partial x^2}(\theta, \kappa(\theta) + s\Delta\kappa(\theta))\; s\, ds\; \Delta\kappa(\theta)^2 \\ &= (\overline{m}(\theta) - m(\theta))\Delta\kappa(\theta) - \int_0^1 \frac{\partial^2 f}{\partial x^2}(\theta, \kappa(\theta) + s\Delta\kappa(\theta))\; s\, ds\; \Delta\kappa(\theta)^2\,, \end{aligned}$$

where we have used (3.5). Now for the first part we apply (3.49):

$$\begin{aligned}\overline{E}(\theta) &= \int_0^1 \frac{\partial^2 f}{\partial x^2}(\theta, \kappa(\theta) + s\Delta\kappa(\theta))\, ds\, \Delta\kappa(\theta)^2 - \int_0^1 \frac{\partial^2 f}{\partial x^2}(\theta, \kappa(\theta) + s\Delta\kappa(\theta))\; s\; \Delta\kappa(\theta))^2\, ds \\ &= \int_0^1 \frac{\partial^2 f}{\partial x^2}(\theta, \kappa(\theta) + s\Delta\kappa(\theta))\,(1-s)\, ds\, \Delta\kappa(\theta)^2 \;,\end{aligned}$$

and (3.50) is proved.

□

3.4 KAM procedure estimates

Next we show a number of estimates satisfied under certain sufficient conditions and lead to the convergence of the KAM process described before.

Lemma 3.5 Estimates

Let $\psi = \mathcal{R}_\omega \times f$ be a quasi–periodic skew–product defined under the same conditions as in (2.1), with the frequency $\omega \in \mathcal{DC}(\gamma, \nu)$ being Diophantine.
Let $(\kappa_n(\theta), \tau_n)$ be an approximation of a solution to the system (3.1), and $E_n(\theta)$ and $e_n(p)$ the corresponding invariance errors given by (3.4). Assume that the following conditions are fulfilled:

(a) *The new approximation $(\kappa_{n+1}(\theta), \tau_{n+1})$ is constructible in the sense of* **Lemma 3.2**.

Moreover, assume for now that there is a (global) positive constant σ_D such that for every real analytic curve, $\kappa_n : \mathbb{T}_\varrho \to \mathbb{C}$, $0 < \sigma_D \le |\det(\Omega_n)|$.

(b) $0 < K_1^* = \inf_{(\theta,z)\in\mathbb{T}_\varrho\times\mathbb{C}} \left|\frac{\partial f}{\partial x}(\theta, z)\right| < \sup_{(\theta,z)\in\mathbb{T}_\varrho\times\mathbb{C}} \left|\frac{\partial f}{\partial x}(\theta, z)\right| = K_1 < \infty$

and for real arguments $\frac{\partial f}{\partial x}(\theta, x) > 0, \forall (\theta, x) \in \mathbb{T} \times \mathbb{R}$.

(c) $\sup_{(\theta,z)\in\mathbb{T}_\varrho\times\mathbb{C}} \left|\frac{\partial^2 f}{\partial x^2}(\theta, z)\right| \le K_2 < \infty.$

(d) $\exists\, \alpha \in (0, \pi)$ *such that* $\left|\mathrm{Arg}\, \frac{\partial f}{\partial x}(\theta, z)\right| \le \alpha,\ \forall (\theta, z) \in \mathbb{T}_\varrho \times \mathbb{C}$.

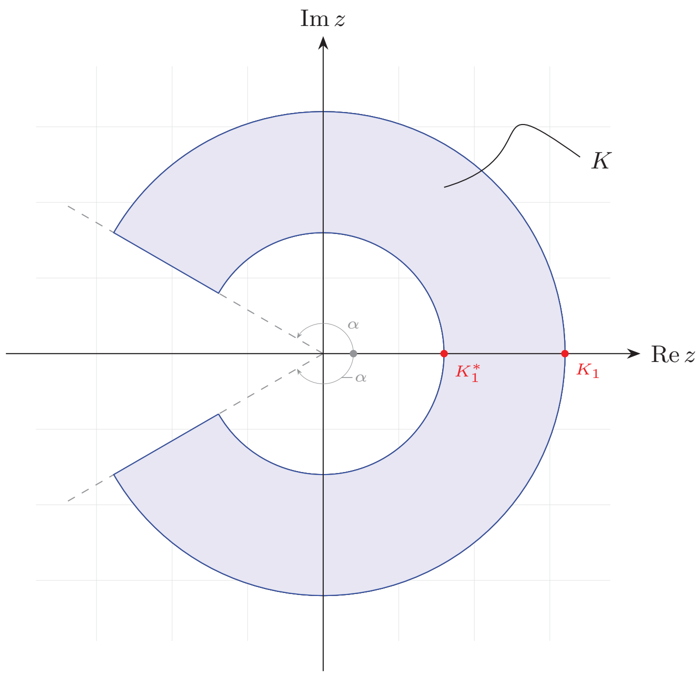

FIGURE 3.1: $K = \{z = re^{\theta i} : K_1^* \le r \le K_1,\ |\mathrm{Arg}(z)| \le \alpha\}$ with $\alpha \in (0, \pi)$. K is a compact annulus sector containing the image of the derivative, i.e., $\frac{\partial f}{\partial x}(\theta, z) \in K,\ \forall (\theta, z) \in \mathbb{T}_\varrho \times \mathbb{C}$.

Let $a = \min\{K_1^*, K_1^{-1}, \frac{\alpha}{\pi}\}$ *and* $A = \sqrt{\max\{|\log K_1^*|, |\log K_1|\}^2 + \alpha^2}$.
Call $C_\delta = \dfrac{1}{a}\exp\left(\dfrac{A\,\mathfrak{C}_R}{\gamma\,\delta^\nu}\right)$, *for any* $\delta \in (0,\varrho)$, *where* $\mathfrak{C}_R = \frac{1}{1+a}\frac{\pi}{\sqrt{3}}\frac{\sqrt{\Gamma(2\nu+1)}}{(4\pi)^\nu}$ *is the Rüßmann constant*[4].

Then, the following estimates hold:

(i) $a \leq \lambda_n \leq \frac{1}{a}$, *with* $a \in (0,1)$.

(ii) $|\Delta m_n(\theta)| \leq K_2 |\Delta\kappa_n(\theta)|$, $\forall \theta \in \mathbb{T}_{\varrho-2\delta}$, $\delta \in (0, \frac{1}{2}\varrho)$.

(iii) $|E_{n+1}(\theta)| \leq \frac{1}{2}K_2\,|\Delta\kappa_n(\theta)|^2$, $\forall\theta \in \mathbb{T}_{\varrho-2\delta}$, $\delta \in (0,\frac{1}{2}\varrho)$.

(iv) $\|v_n\|_\varrho \leq 2A$ *and* $\|\mathfrak{R}_1 v_n\|_{\varrho-\delta} \leq 2\log(a\,C_\delta)$, $\delta \in (0,\varrho)$.
In particular, $|\mathfrak{R}_1 v_n(\theta)| \leq 2\log(aC_\varrho)$, $\forall\theta \in \mathbb{T}$.

(v) *For any* $\delta \in (0,\varrho)$, $|c_n(\theta)| \leq \alpha_n a^2 C_\delta^2$, $\forall\theta\in\mathbb{T}_{\varrho-\delta}$ *and* $\frac{1}{|c_n(\theta)|} \leq \frac{1}{\alpha_n}a^2C_\delta^2$, $\forall\theta\in\mathbb{T}_{\varrho-\delta}$.
In particular, $|c_n(\theta)| \leq \alpha_n a^2 C_\varrho^2$, $\forall\theta\in\mathbb{T}$ *and* $\frac{1}{|c_n(\theta)|} \leq \frac{1}{\alpha_n}a^2C_\varrho^2$, $\forall\theta\in\mathbb{T}$.
Moreover, $\alpha_n \leq c_{n,0}\,a^2C_\varrho^2$ *and* $\frac{1}{\alpha_n} \leq \frac{1}{c_{n,0}}\,a^2C_\varrho^2$.
Consequently, for any $\delta\in(0,\varrho)$, $\max\{\frac{1}{c_{n,0}}\|c_n\|_{\varrho-\delta}, c_{n,0}\|c_n^{-1}\|_{\varrho-\delta}\} \leq a^4C_\varrho^2C_\delta^2$.
Additionally, if we choose $\alpha_n = 1$, *then we have the sharper estimate:*
$\max\{\|c_n\|_{\varrho-\delta}, \|c_n^{-1}\|_{\varrho-\delta}\} \leq a^2C_\delta^2$, $\delta\in(0,\varrho)$.

(vi) $\|\eta_n\|_{\varrho-\delta} \leq \frac{1}{\alpha_n}C_\delta^2$, $\forall\delta\in(0,\varrho)$.
Moreover, for any $c_{n,0} > 0$ *we have* $\|\eta_n\|_{\varrho-\delta} \leq \frac{1}{c_{n,0}}a^2C_\varrho^2C_\delta^2$, $\delta\in(0,\varrho)$.
In particular, $\forall\theta\in\mathbb{T}, |\eta_n(\theta)| \leq \frac{1}{\alpha_n}C_\varrho^2 \leq \frac{1}{c_{n,0}}a^2C_\varrho^4$.
Besides this, if $\alpha_n = 1$, *then* $\|\eta_n\|_{\varrho-\delta} \leq C_\delta^2$, $\delta\in(0,\varrho)$.

(vii) $\|\xi_n\|_{\varrho-\delta} \leq \|\eta_n\|_{\varrho-\delta}\|E_n\|_\varrho \leq \frac{1}{\alpha_n}C_\delta^2\|E_n\|_\varrho$, $\delta\in(0,\varrho)$.
Consequently, for any $c_{n,0} > 0$, $\|\xi_n\|_{\varrho-\delta} \leq \frac{1}{c_{n,0}}a^2C_\varrho^2C_\delta^2\|E_n\|_\varrho$, $\delta\in(0,\varrho)$.
Moreover, if $\alpha_n = 1$, *then* $\|\xi_n\|_{\varrho-\delta} \leq \|\eta_n\|_{\varrho-\delta}\|E_n\|_\varrho \leq C_\delta^2\|E_n\|_\varrho$, $\delta\in(0,\varrho)$.
In particular, $\forall\theta\in\mathbb{T}, |\xi_n(\theta)| \leq \frac{1}{\alpha_n}C_\varrho^2\|E_n\|_\varrho \leq \frac{1}{c_{n,0}}a^2C_\varrho^4\|E_n\|_\varrho$.

(viii) $\|\widetilde{\eta}_n\|_{\varrho-\delta} \leq \|\eta_n\|_{\varrho-\delta} + \eta_{n,0} \leq \frac{1}{\alpha_n}(C_\varrho^2 + C_\delta^2)$, $\delta\in(0,\varrho)$.
Thus, for any $c_{n,0} > 0$, $\|\widetilde{\eta}_n\|_{\varrho-\delta} \leq \frac{1}{c_{n,0}}a^2C_\varrho^2(C_\varrho^2 + C_\delta^2)$, $\delta\in(0,\varrho)$.
If $\alpha_n = 1$, $\|\widetilde{\eta}_n\|_{\varrho-\delta} \leq C_\varrho^2 + C_\delta^2$, $\delta\in(0,\varrho)$.
In particular, $\forall\theta\in\mathbb{T}, |\widetilde{\eta}_n(\theta)| \leq \frac{2}{\alpha_n}C_\varrho^2 \leq \frac{2}{c_{n,0}}a^2C_\varrho^4$.

[4] c.f. **Theorem 1.20**.

(ix) $\|\widetilde{\xi}_n\|_{\varrho-\delta} \leq \|\xi_n\|_{\varrho-\delta} + |\xi_{n,0}| \leq \frac{1}{\alpha_n}(C_\varrho^2 + C_\delta^2)\|E_n\|_\varrho,\ \delta \in (0,\varrho)$.

Thus, for any $c_{n,0} > 0$, $\|\widetilde{\xi}_n\|_{\varrho-\delta} \leq \frac{1}{c_{n,0}} a^2 C_\varrho^2 (C_\varrho^2 + C_\delta^2)\|E_n\|_\varrho,\ \delta \in (0,\varrho)$.

If $\alpha_n = 1$, $\|\widetilde{\xi}_n\|_{\varrho-\delta} \leq (C_\varrho^2 + C_\delta^2)\|E_n\|_\varrho,\ \delta \in (0,\varrho)$.

In particular, $\forall \theta \in \mathbb{T}, |\widetilde{\xi}_n(\theta)| \leq \frac{2}{\alpha_n} C_\varrho^2 \|E_n\|_\varrho \leq \frac{2}{c_{n,0}} a^2 C_\varrho^4 \|E_n\|_\varrho$.

(x) $\|\mathfrak{R}_{\lambda_n}\widetilde{\eta}_n\|_{\varrho-2\delta} \leq \frac{1}{\alpha_n}\frac{1}{A}\log(aC_\delta)(C_\varrho^2 + C_\delta^2),\ \delta \in (0,\frac{1}{2}\varrho)$.

For any $c_{n,0} > 0$, $\|\mathfrak{R}_{\lambda_n}\widetilde{\eta}_n\|_{\varrho-2\delta} \leq \frac{1}{c_{n,0}}\frac{a^2}{A}\log(aC_\delta) C_\varrho^2 (C_\varrho^2 + C_\delta^2),\ \delta \in (0,\frac{1}{2}\varrho)$.

In the case where $\alpha_n = 1$, $\|\mathfrak{R}_{\lambda_n}\widetilde{\eta}_n\|_{\varrho-2\delta} \leq \frac{1}{A}\log(aC_\delta)(C_\varrho^2 + C_\delta^2),\ \delta \in (0,\frac{1}{2}\varrho)$.

In particular, $\forall \theta \in \mathbb{T}, |\mathfrak{R}_{\lambda_n}\widetilde{\eta}_n(\theta)| \leq \frac{1}{\alpha_n}\frac{1}{A}\log(aC_{\frac{1}{2}\varrho})(C_\varrho^2 + C_{\frac{1}{2}\varrho}^2)$.

(xi) $\|\mathfrak{R}_{\lambda_n}\widetilde{\xi}_n\|_{\varrho-2\delta} \leq \frac{1}{\alpha_n}\frac{1}{A}\log(aC_\delta)(C_\varrho^2 + C_\delta^2)\|E_n\|_\varrho,\ \delta \in (0,\frac{1}{2}\varrho)$.

For any $c_{n,0} > 0$, $\|\mathfrak{R}_{\lambda_n}\widetilde{\xi}_n\|_{\varrho-2\delta} \leq \frac{1}{c_{n,0}}\frac{a^2}{A}\log(aC_\delta) C_\varrho^2 (C_\varrho^2 + C_\delta^2)\|E_n\|_\varrho,\ \delta \in (0,\frac{1}{2}\varrho)$.

In the case where $\alpha_n = 1$, $\|\mathfrak{R}_{\lambda_n}\widetilde{\xi}_n\|_{\varrho-2\delta} \leq \frac{1}{A}\log(aC_\delta)(C_\varrho^2 + C_\delta^2)\|E_n\|_\varrho,\ \delta \in (0,\frac{1}{2}\varrho)$.

In particular, $\forall \theta \in \mathbb{T}, |\mathfrak{R}_{\lambda_n}\widetilde{\xi}_n(\theta)| \leq \frac{1}{\alpha_n}\frac{1}{A}\log(aC_{\frac{1}{2}\varrho})(C_\varrho^2 + C_{\frac{1}{2}\varrho}^2)\|E_n\|_\varrho$.

(xii) $\eta_{n,0} \leq \frac{1}{\alpha_n} C_\varrho^2$.

For any $c_{n,0} > 0$, $a^2 \leq c_{n,0}\,\eta_{n,0} \leq a^2 C_\varrho^4$.

If $\alpha_n = 1$, *then we also have* $\eta_{n,0} \leq C_\varrho^2$.

(xiii) $|\xi_{n,0}| \leq \eta_{n,0}\|E_n\|_\varrho \leq \frac{1}{\alpha_n} C_\varrho^2 \|E_n\|_\varrho$.

For any $c_{n,0} > 0$, $|\xi_{n,0}| \leq \eta_{n,0}\|E_n\|_\varrho \leq \frac{1}{c_{n,0}} a^2 C_\varrho^4 \|E_n\|_\varrho$.

If $\alpha_n = 1$, *then* $|\xi_{n,0}| \leq \eta_{n,0}\|E_n\|_\varrho \leq C_\varrho^2 \|E_n\|_\varrho$.

(xiv) $| < \widetilde{c}_n \mathfrak{R}_{\lambda_n}\widetilde{\eta}_n > | \leq 2c_{n,0} \sup_{\theta\in\mathbb{T}} |\mathfrak{R}_{\lambda_n}\widetilde{\eta}_n| \leq \dfrac{2a^2}{A}\log(aC_{\frac{1}{2}\varrho}) C_\varrho^2 (C_\varrho^2 + C_{\frac{1}{2}\varrho}^2)$.

(xv) $| < \widetilde{c}_n \mathfrak{R}_{\lambda_n}\widetilde{\xi}_n > | \leq 2c_{n,0} \sup_{\theta\in\mathbb{T}} |\mathfrak{R}_{\lambda_n}\widetilde{\xi}_n| \leq \dfrac{2a^2}{A}\log(aC_{\frac{1}{2}\varrho}) C_\varrho^2 (C_\varrho^2 + C_{\frac{1}{2}\varrho}^2)\|E_n\|_\varrho$.

(xvi) $|\varsigma_{n,0}| \leq \frac{1}{|\det(\Omega_n)|}\frac{1}{\alpha_n} C_\varrho^2 \left(\frac{4a^2}{A}\log(aC_{\frac{1}{2}\varrho} C_\varrho^2 (C_\varrho^2 + C_{\frac{1}{2}\varrho}^2)\|E_n\|_\varrho + |e_n(p)|\right)$.

Furthermore, for any $c_{n,0} > 0$, $|\varsigma_{n,0}| \leq \frac{1}{\sigma_D}\frac{1}{c_{n,0}}(P_1(C_\varrho, C_{\frac{1}{2}\varrho})\|E_n\|_\varrho + Q_1(C_\varrho)|e_n(p)|)$, *where*

$$\begin{aligned} P_1(s,t) &= \frac{4a^4}{A} s^6 (s^2 + t^2)\log(at), \\ Q_1(s) &= a^2 s^4. \end{aligned}$$

If $\alpha_n = 1$, *then* $|\varsigma_{n,0}| \leq \frac{1}{\sigma_D}(P_1^*(C_\varrho, C_{\frac{1}{2}\varrho})\|E_n\|_\varrho + Q_1^*(C_\varrho)|e_n(p)|)$, *where*

$$\begin{aligned} P_1^*(s,t) &= \frac{4a^2}{A} s^4 (s^2 + t^2)\log(at) \\ Q_1^*(s) &= s^2. \end{aligned}$$

(xvii) $|\triangle\tau_n| \le \frac{1}{\sigma_D}(P_2(C_\varrho, C_{\frac{1}{2}\varrho})\|E_n\|_\varrho + Q_2|e_n(p)|)$, *where*

$$\begin{aligned} P_2(s,t) &= a^2 s^2 \left(1 + \frac{2(1-a)}{aA}(s^2+t^2)\log(at)\right), \\ Q_2 &= \frac{1-a}{a}. \end{aligned}$$

(xviii) $|\det(\Omega_n)| \le P_2(C_\varrho, C_{\frac{1}{2}\varrho})$.

(xix) *For any* $\delta \in (0, \frac{1}{2}\varrho)$,

$\|\widetilde{\varsigma}_n\|_{\varrho-2\delta} \le \frac{1}{|\det(\Omega_n)|}\frac{1}{\alpha_n}\frac{1}{A}\log(aC_\delta)(C_\varrho^2 + C_\delta^2)\left(2P_2(C_\varrho, C_{\frac{1}{2}\varrho})\|E_n\|_\varrho + Q_2|e_n(p)|\right)$.

Furthermore, for any $c_{n,0} > 0$,

$\|\widetilde{\varsigma}_n\|_{\varrho-2\delta} \le \frac{1}{\sigma_D}\frac{1}{c_{n,0}}\left(P_3(C_\varrho, C_{\frac{1}{2}\varrho}, C_\delta)\|E_n\|_\varrho + Q_3(C_\varrho, C_\delta)|e_n(p)|\right)$, *where*

$$\begin{aligned} P_3(s,t,u) &= \frac{2a^4}{A}s^4(s^2+u^2)\log(au)\left(1 + \frac{2(1-a)}{aA}(s^2+t^2)\log(at)\right), \\ Q_3(s,u) &= \frac{a(1-a)}{A}s^2(s^2+u^2)\log(au). \end{aligned}$$

If $\alpha_n = 1$, *then* $\|\widetilde{\varsigma}_n\|_{\varrho-2\delta} \le \frac{1}{\sigma_D}\left(P_3^*(C_\varrho, C_{\frac{1}{2}\varrho}, C_\delta)\|E_n\|_\varrho + Q_3^*(C_\varrho, C_\delta)|e_n(p)|\right)$, *with*

$$\begin{aligned} P_3^*(s,t,u) &= \frac{2a^2}{A}s^2(s^2+ut^2)\log(au)\left(1 + \frac{2(1-a)}{aA}(s^2+t^2)\log(at)\right), \\ Q_3^*(s,u) &= \frac{1-a}{aA}(s^2+u^2)\log(au). \end{aligned}$$

(xx) *For any* $\delta \in (0, \frac{1}{2}\varrho)$, $\|\Delta\kappa_n(\theta)\|_{\varrho-2\delta} \le \frac{1}{\sigma_D}(P(C_\varrho, C_{\frac{1}{2}\varrho}, C_\delta)\|E_n\|_\varrho + Q(C_\varrho, C_\delta)|e_n(p)|)$,

where

$$\begin{aligned} P(s,t,u) &= a^2u^2(P_1^*(s,t) + P_3^*(s,t,u)) \\ &= \frac{2a^4}{A}(s^2+t^2)u^2\log(at)\Bigg(2s^4 \\ &\quad + \left(1 + \frac{2(1-a)}{aA}(s^2+t^2)\log(at)\right)\log(au)\Bigg), \\ Q(s,u) &= a^2u^2(Q_1^*(s) + Q_3^*(s,u)) \\ &= a^2s^2u^2 + \frac{1-a}{aA}(s^2+u^2)\log(au). \end{aligned}$$

In particular,

$$|\Delta\kappa_n(\theta)| \le \frac{1}{\sigma_D}(P^*(C_\varrho, C_{\frac{1}{2}\varrho})\|E_n\|_\varrho + Q^*(C_\varrho, C_{\frac{1}{2}\varrho})|e_n(p)|), \ \forall\theta \in \mathbb{T},$$

where

$$\begin{aligned} P^*(s,t) = P(s,t,t) &= a^2t^2(P_1^*(s,t) + P_3^*(s,t,t)) \\ &= \frac{2a^4}{A}(s^2+t^2)t^2\log(at)\Bigg(2s^4 \\ &\quad + \left(1 + \frac{2(1-a)}{aA}(s^2+t^2)\log(at)\right)\log(at)\Bigg), \\ Q^*(s,t) = Q(s,t)) &= a^2t^2(Q_1^*(s) + Q_3^*(s,t)) \\ &= a^2s^2t^2 + \frac{1-a}{aA}(s^2+t^2)\log(at). \end{aligned}$$

Proof.

First of all, notice that since $a = \min\{K_1^*, K_1^{-1}, \frac{\alpha}{\pi}\}$ and $A = \sqrt{\max\{|\log K_1^*|, |\log K_1|\}^2 + \alpha^2}$, then

$$a \leq K_1^* < K_1 \leq \tfrac{1}{a} , \tag{3.53}$$

$$a \leq \tfrac{1}{K_1} < \tfrac{1}{K_1^*} \leq \tfrac{1}{a} , \tag{3.54}$$

$$\text{and } \frac{1}{A} = \frac{1}{\sqrt{\max\{|\log K_1^*|, |\log K_1|\}^2 + \alpha^2}} \leq \tfrac{1}{\alpha} \leq \tfrac{1}{\pi a} . \tag{3.55}$$

Additionally, define

$$I_n = \int_{\mathbb{T}} e^{\Re_1 v_n(\theta)} d\theta \int_{\mathbb{T}} e^{-\Re_1 v_n(\theta)} d\theta . \tag{3.56}$$

By Cauchy-Schwartz's inequality,

$$\begin{aligned} 1 = \int_{\mathbb{T}} d\theta &= \int_{\mathbb{T}} e^{\frac{1}{2}\Re_1 v_n(\theta)} \, e^{-\frac{1}{2}\Re_1 v_n(\theta)} d\theta \\ &\leq \left(\int_{\mathbb{T}} \left(e^{\frac{1}{2}\Re_1 v_n(\theta)} \right)^2 d\theta \right)^{\frac{1}{2}} \left(\int_{\mathbb{T}} \left(e^{-\frac{1}{2}\Re_1 v_n(\theta)} \right)^2 d\theta \right)^{\frac{1}{2}} \\ &= \left(\int_{\mathbb{T}} e^{\Re_1 v_n(\theta)} d\theta \int_{\mathbb{T}} e^{-\Re_1 v_n(\theta)} d\theta \right)^{\frac{1}{2}} = I_n^{\frac{1}{2}} . \end{aligned}$$

Hence,

$$I_n \geq 1 . \tag{3.57}$$

We will use these inequalities many times throughout the proof.

(*i*) The Lyapunov multiplier is the exponential of the Lyapunov exponent, that is

$$\lambda_n = e^{\Lambda_n} = \exp\left(\int_{\mathbb{T}} \log(m_n(\theta)) \, d\theta \right) ,$$

where $m_n(\theta) = \frac{\partial f}{\partial x}(\theta, \kappa_n(\theta))$. Thus,

$$\begin{aligned} 0 < a \leq K_1^* \leq |m_n(\theta)| \leq K_1 \leq \frac{1}{a} \quad &\Rightarrow \quad \text{(taking logarithms)} \\ \log(K_1^*) \leq \log|m_n(\theta)| \leq \log(K_1) \quad &\Rightarrow \quad \text{(integrating over the torus)} \\ \log(K_1^*) \leq \int_{\mathbb{T}} \log|m_n(\theta)| \, d\theta \leq \log(K_1) \quad &\Rightarrow \quad \text{(taking exponentials)} \\ a \leq K_1^* \leq e^{\Lambda_n} = \lambda_n \leq K_1 \leq \frac{1}{a} . \end{aligned} \tag{3.58}$$

(*ii*) Let $\delta \in (0, \frac{1}{2}\varrho)$. Notice that by **Theorem 1.20** and according to Remark 3.3, we know that the domain of κ_{n+1} is the complex strip $\mathbb{T}_{\varrho-2\delta}$. More exactly, the new approximation is analytic in that domain, i.e. $\kappa_{n+1} \in \mathcal{A}_{\varrho-2\delta}$, $\forall \delta \in (0, \frac{1}{2}\varrho)$.

Since $\lambda_n \in [a, \frac{1}{a}]$, then by **Proposition 3.4** (part (a)), we can write:

$$m_{n+1}(\theta) = m_n(\theta) + \int_0^1 \frac{\partial^2 f}{\partial x^2}(\theta, \kappa_n(\theta) + s\Delta\kappa_n(\theta)) \, ds \, \Delta\kappa_n(\theta) , \quad \forall \theta \in T_{\varrho-2\delta}.$$

It follows that

$$\begin{aligned} |\Delta m_n(\theta)| &= |m_{n+1}(\theta) - m_n(\theta)| = \left| \int_0^1 \frac{\partial^2 f}{\partial x^2}(\theta, \kappa_n(\theta) + s\Delta\kappa_n(\theta))\, ds\, \Delta\kappa_n(\theta) \right| \\ &\leq \int_0^1 \left| \frac{\partial^2 f}{\partial x^2}(\theta, \kappa_n(\theta) + s\Delta\kappa_n(\theta)) \right| ds\, |\Delta\kappa_n(\theta)| \\ &\leq \sup_{s\in[0,1]} \left| \frac{\partial^2 f}{\partial x^2}(\theta, \kappa_n(\theta) + s\Delta\kappa_n(\theta)) \right| \cdot |\Delta\kappa_n(\theta)| \\ &\leq \sup_{z\in\mathbb{C}} \left| \frac{\partial^2 f}{\partial x^2}(\theta, z) \right| \cdot |\Delta\kappa_n(\theta)| \leq K_2 |\Delta\kappa_n(\theta)| \, . \end{aligned} \tag{3.59}$$

Hence,

$$|\Delta m_n(\theta)| \leq K_2 |\Delta\kappa_n(\theta)|, \ \forall\theta \in \mathbb{T}_{\varrho-2\delta} \, . \tag{3.60}$$

In particular,

$$|\Delta m_n(\theta)| \leq K_2 \sup_{\theta\in\mathbb{T}} |\Delta\kappa_n(\theta)|, \ \forall\theta \in \mathbb{T} \, . \tag{3.61}$$

(iii) By **Proposition 3.4** (part (b)), for any $\delta \in (0, \frac{1}{2}\varrho)$,

$$E_{n+1}(\theta) = \int_0^1 \frac{\partial^2 f}{\partial x^2}(\theta, \kappa_n(\theta) + s\Delta\kappa_n(\theta))(1-s)\, ds \cdot \Delta\kappa_n(\theta)^2 \, , \forall\theta \in \mathbb{T}_{\varrho-2\delta}.$$

Then,

$$\begin{aligned} |E_{n+1}(\theta)| &\leq \int_0^1 \left| \frac{\partial^2 f}{\partial x^2}(\theta, \kappa_n(\theta) + s\Delta\kappa_n(\theta))(1-s) \right| ds \cdot |\Delta\kappa_n(\theta)|^2 \\ &\leq \sup_{s\in[0,1]} \left| \frac{\partial^2 f}{\partial x^2}(\theta, \kappa_n(\theta) + s\Delta\kappa_n(\theta)) \right| \int_0^1 (1-s)\, ds \cdot |\Delta\kappa_n(\theta)|^2 \\ &\leq \sup_{z\in\mathbb{C}} \left| \frac{\partial^2 f}{\partial x^2}(\theta, z) \right| \int_0^1 (1-s)\, ds \cdot |\Delta\kappa_n(\theta)|^2 \\ &\leq K_2 \int_0^1 (1-s)\, ds \cdot |\Delta\kappa_n(\theta)|^2 = \frac{1}{2} K_2 |\Delta\kappa_n(\theta)|^2 \, , \forall\theta \in \mathbb{T}_{\varrho-2\delta} \, . \end{aligned} \tag{3.62}$$

(iv) $v_n(\theta) = \log(m_n(\theta)) - \Lambda_n$, $\forall\theta \in \mathbb{T}_\varrho$.

$$\begin{aligned} |v_n(\theta)| &\leq |\log(m_n(\theta))| + |\Lambda_n| = |\log(m_n(\theta))| + \left| \int_{\mathbb{T}} \log(m_n(\theta))\, d\theta \right| \\ &\leq |\log(m_n(\theta))| + \int_{\mathbb{T}} |\log(m_n(\theta))|\, d\theta \\ &\leq 2 \sup_{\theta\in\mathbb{T}_\varrho} |\log(m_n(\theta))| = 2 \sup_{\theta\in\mathbb{T}_\varrho} \sqrt{(\log|m_n(\theta)|)^2 + (\mathrm{Arg}(m_n(\theta)))^2} \\ &\leq 2\sqrt{\max\{|\log(K_1)|, |\log(K_1^*)|\}^2 + (\mathrm{Arg}(m_n(\theta)))^2} \\ &\leq 2\sqrt{\max\{|\log(K_1)|, |\log(K_1^*)|\}^2 + \alpha^2} = 2A \, . \end{aligned} \tag{3.63}$$

Additionally, by **Theorem 1.20**,

$$\begin{aligned} |\mathfrak{R}_1 v_n(\theta)| &\leq \frac{\mathfrak{C}_R}{\gamma\, \delta^\nu} \|v_n\|_\varrho = \frac{1}{A} \cdot \frac{A\, \mathfrak{C}_R}{\gamma\, \delta^\nu} \|v_n\|_\varrho \\ &\leq \frac{1}{A} \log(a\, C_\delta) \|v_n\|_\varrho \leq \frac{1}{A} \log(a\, C_\delta) 2A \\ &= 2\log(a\, C_\delta), \ \forall\theta \in \mathbb{T}_{\varrho-\delta} \, , \delta \in (0, \varrho) \, . \end{aligned} \tag{3.64}$$

In particular[5], for every $\varrho > 0$, if we restrict ourselves only to real values of the argument, we have the following bound:

$$|\mathfrak{R}_1 v_n(\theta)| \leq \frac{2A\,\mathfrak{C}_R}{\gamma\,\varrho^\nu} = 2\log(aC_\varrho), \quad \forall\theta \in \mathbb{T}, \tag{3.65}$$

where $C_\varrho = \frac{1}{a}\exp\left(\frac{A\,\mathfrak{C}_R}{\gamma\,\varrho^\nu}\right)$.

(*v*) The Floquet transformation, for any $\delta \in (0, \varrho)$, is of the form[6]:

$c_n(\theta) = \alpha_n e^{\mathfrak{R}_1 v_n(\theta)}$, $\forall\theta \in \mathbb{T}_{\varrho-\delta}$, where $\alpha_n = \dfrac{c_{n,0}}{\displaystyle\int_{\mathbb{T}} e^{\mathfrak{R}_1 v_n(\theta)}\,d\theta}$ and $c_{n,0} > 0$ can be chosen freely.

Thus,

$$\begin{aligned} |c_n(\theta)| &= \alpha_n|e^{\mathfrak{R}_1 v_n(\theta)}| = \alpha_n e^{\mathrm{Re}(\mathfrak{R}_1 v_n(\theta))} \\ &\leq \alpha_n e^{|\mathfrak{R}_1 v_n(\theta|} \leq \alpha_n e^{2\log(aC_\delta)} = \alpha_n a^2 C_\delta^2\,. \end{aligned} \tag{3.66}$$

Regardless of the choice we make of the average, α_n can be estimated in the following way, using (3.56) and (3.57) together with (3.65):

$$\begin{aligned} \alpha_n &= \frac{c_{n,0}}{\displaystyle\int_{\mathbb{T}} e^{\mathfrak{R}_1 v_n(\theta)}\,d\theta} = \frac{c_{n,0}}{I_n}\int_{\mathbb{T}} e^{-\mathfrak{R}_1 v_n(\theta)}\,d\theta \\ &\leq c_{n,0}\int_{\mathbb{T}} e^{-\mathfrak{R}_1 v_n(\theta)}\,d\theta \leq c_{n,0}\sup_{\theta\in\mathbb{T}}|e^{-\mathfrak{R}_1 v_n(\theta)}| \\ &\leq c_{n,0}\sup_{\theta\in\mathbb{T}} e^{|\mathfrak{R}_1 v_n(\theta)|} = c_{n,0}\exp(\sup_{\theta\in\mathbb{T}}|\mathfrak{R}_1 v_n(\theta)|) \\ &\leq c_{n,0}\,e^{2\log(aC_\varrho)} = c_{n,0}\,a^2 C_\varrho^2\,. \end{aligned} \tag{3.67}$$

Thus,

$$|c_n(\theta)| \leq c_{n,0}\,a^4 C_\varrho^2 C_\delta^2\,, \forall\theta \in \mathbb{T}_{\varrho-\delta}\,. \tag{3.68}$$

Notice that if we choose $c_{n,0} = \displaystyle\int_{\mathbb{T}} e^{\mathfrak{R}_1 v_n(\theta)}\,d\theta$, i.e. $\alpha_n = 1$, then

$$|c_n(\theta)| \leq a^2 C_\delta^2\,, \forall\theta \in \mathbb{T}_{\varrho-\delta}\,. \tag{3.69}$$

In a similar way, we can write:

$$\begin{aligned} \frac{1}{|c_n(\theta)|} &= \frac{1}{\alpha_n|e^{\mathfrak{R}_1 v_n(\theta)}|} = \frac{1}{\alpha_n e^{\mathrm{Re}(\mathfrak{R}_1 v_n(\theta))}} = \frac{1}{\alpha_n}e^{-\mathrm{Re}(\mathfrak{R}_1 v_n(\theta))} \\ &\leq \frac{1}{\alpha_n}e^{|\mathfrak{R}_1 v_n(\theta)|} \leq \frac{1}{\alpha_n}e^{2\log(aC_\delta)} = \frac{1}{\alpha_n}a^2 C_\delta^2\,. \end{aligned} \tag{3.70}$$

[5] Notice that when $\delta \to \varrho$, $\mathbb{T}_{\varrho-\delta} \to \mathbb{T}_0 = \mathbb{T}$.

[6] See **Corollary 2.8** .

$$\begin{aligned}\frac{1}{\alpha_n} &= \frac{\displaystyle\int_{\mathbb{T}} e^{\mathfrak{R}_1 v_n(\theta)}\,d\theta}{c_{n,0}} \le \frac{1}{c_{n,0}} \sup_{\theta\in\mathbb{T}} |e^{\mathfrak{R}_1 v_n(\theta)}| \\ &\le \frac{1}{c_{n,0}}\, e^{2\log(aC_\varrho)} = \frac{1}{c_{n,0}}\, a^2 C_\varrho^2 \,. \end{aligned} \tag{3.71}$$

Thus,

$$\frac{1}{|c_n(\theta)|} \le \frac{1}{c_{n,0}}\, a^4 C_\varrho^2 C_\delta^2 \,, \forall\theta\in\mathbb{T}_{\varrho-\delta} \,. \tag{3.72}$$

In the same manner, notice that if we choose $c_{n,0} = \displaystyle\int_{\mathbb{T}} e^{\mathfrak{R}_1 v_n(\theta)}\,d\theta$, i.e. $\alpha_n = 1$, then

$$\frac{1}{|c_n(\theta)|} \le a^2 C_\delta^2 \,, \forall\theta\in\mathbb{T}_{\varrho-\delta} \,. \tag{3.73}$$

(vi) $\eta_n(\theta) = \dfrac{1}{c_n(\theta+\omega)} = \dfrac{1}{e^{u_n(\theta+\omega)}}$, $\forall\theta\in\mathbb{T}_{\varrho-\delta}$, where $u_n(\theta) = \log(\alpha_n) + \mathfrak{R}_1 v_n(\theta)$.

Taking in account that u_n satisfies the cohomological equation[7] $u_n(\theta+\omega) - u_n(\theta) = v_n(\theta)$, we can write:

$$\begin{aligned}\eta_n(\theta) &= e^{-u_n(\theta+\omega)} = e^{-(u_n(\theta)+v_n(\theta))} = e^{-u_n(\theta)} e^{-v_n(\theta)} \\ &= e^{-(\log\alpha_n + \mathfrak{R}_1 v_n(\theta)} e^{-(\log(m_n(\theta)) - \Lambda_n)} \\ &= \frac{1}{\alpha_n} e^{-\mathfrak{R}_1 v_n(\theta)} e^{\Lambda_n} e^{-\log(m_n(\theta))} \\ &= \frac{1}{\alpha_n} \lambda_n \frac{1}{e^{\log(m_n(\theta))}} e^{-\mathfrak{R}_1 v_n(\theta)} \\ &= \frac{1}{\alpha_n} \lambda_n \frac{1}{e^{\log(|m_n(\theta)|) + i\operatorname{Arg}(m_n(\theta))}} e^{-\mathfrak{R}_1 v_n(\theta)} \\ &= \frac{1}{\alpha_n} \frac{\lambda_n}{|m_n(\theta)|} e^{-i\operatorname{Arg}(m_n(\theta))} e^{-\mathfrak{R}_1 v_n(\theta)}, \ \forall\theta\in\mathbb{T}_{\varrho-\delta} \,. \end{aligned} \tag{3.74}$$

Thanks to (3.74) we are in a position to estimate $\|\eta_n\|_{\varrho-\delta}$, $\|\widetilde{\eta}_n\|_{\varrho-\delta}$, $\|\mathfrak{R}_{\lambda_n}\widetilde{\eta}_n\|_{\varrho-2\delta}$, and $\eta_{n,0}$.

$$\begin{aligned}|\eta_n(\theta)| &= \frac{1}{\alpha_n}\frac{\lambda_n}{|m_n(\theta)|} \left| e^{-\mathfrak{R}_1 v_n(\theta)} \right| \\ &= \frac{1}{\alpha_n}\frac{\lambda_n}{|m_n(\theta)|} e^{\operatorname{Re}(-\mathfrak{R}_1 v_n(\theta))} \\ &\le \frac{1}{\alpha_n}\frac{\lambda_n}{|m_n(\theta)|} e^{|-\mathfrak{R}_1 v_n(\theta)|} = \frac{1}{\alpha_n}\frac{\lambda_n}{|m_n(\theta)|} e^{|\mathfrak{R}_1 v_n(\theta)|} \ \text{(by (3.64))} \\ &\le \frac{1}{\alpha_n}\frac{\lambda_n}{|m_n(\theta)|} e^{2\log(aC_\delta)} = \frac{1}{\alpha_n}\frac{\lambda_n}{|m_n(\theta)|} a^2 C_\delta^2 \ \text{(by (3.58))} \\ &\le \frac{1}{\alpha_n}\frac{1}{a^2} a^2 C_\delta^2 = \frac{1}{\alpha_n} C_\delta^2 \,, \forall\theta\in\mathbb{T}_{\varrho-\delta} \,. \end{aligned} \tag{3.75}$$

[7] See **Corollary 2.8**.

In particular,

$$|\eta_n(\theta)| \le \frac{1}{\alpha_n} C_\varrho^2, \quad \forall \theta \in \mathbb{T}. \tag{3.76}$$

As a consequence of (3.75) we can highlight the following estimates:
On one side, if $\alpha_n = 1$, then

$$|\eta_n(\theta)| \le C_\delta^2 \ , \forall \theta \in \mathbb{T}_{\varrho-\delta} \tag{3.77}$$

or, equivalently

$$\|\eta_n\|_{\varrho-\delta} \le C_\delta^2 \ , \forall \delta \in (0, \varrho) \ . \tag{3.78}$$

And, on the other side, without this assumption, we obtain from (3.71) and (3.75),

$$|\eta_n(\theta)| \le \frac{1}{c_{n,0}} a^2 C_\varrho^2 C_\delta^2 \ , \forall \theta \in \mathbb{T}_{\varrho-\delta} \ , \tag{3.79}$$

namely,

$$\|\eta_n\|_{\varrho-\delta} \le \frac{1}{c_{n,0}} a^2 C_\varrho^2 C_\delta^2 \ , \forall \delta \in (0, \varrho) \ . \tag{3.80}$$

(*vii*) By definition, $\xi_n(\theta) = \dfrac{E_n(\theta)}{c_n(\theta+\omega)} = \eta_n(\theta) E_n(\theta)$.

Therefore, by (3.75),

$$\begin{aligned} |\xi_n(\theta)| &= |\eta_n(\theta)||E_n(\theta)| \\ &\le \frac{1}{\alpha_n} C_\delta^2 |E_n(\theta)| \le \frac{1}{\alpha_n} C_\delta^2 \sup_{\theta \in \mathbb{T}_{\varrho-\delta}} |E_n(\theta)| \\ &= \frac{1}{\alpha_n} C_\delta^2 \sup_{\theta \in T_\varrho} |E_n(\theta)| = \frac{1}{\alpha_n} C_\delta^2 \|E_n\|_\varrho \ , \ \forall \theta \in \mathbb{T}_{\varrho-\delta} \ . \end{aligned} \tag{3.81}$$

Clearly, if $\alpha_n = 1$,

$$|\xi_n(\theta)| \le C_\delta^2 \|E_n\|_\varrho \ , \ \forall \theta \in \mathbb{T}_{\varrho-\delta} \ . \tag{3.82}$$

In general, from (3.71) and (3.81), we obtain

$$|\xi_n(\theta)| \le \frac{1}{c_{n,0}} a^2 C_\varrho^2 C_\delta^2 \|E_n\|_\varrho \ , \ \forall \theta \in \mathbb{T}_{\varrho-\delta} \ . \tag{3.83}$$

(*viii*) Since $\widetilde{\eta}_n(\theta) = \eta_n(\theta) - \eta_{n,0}$, then

$$\begin{aligned} |\widetilde{\eta}_n(\theta)| &\le |\eta_n(\theta)| + |\eta_{n,0}| \le \|\eta_n\|_{\varrho-\delta} + \left| \int_{\mathbb{T}} \eta_n(\theta)\, d\theta \right| \\ &\le \|\eta_n\|_{\varrho-\delta} + \int_{\mathbb{T}} |\eta_n(\theta)|\, d\theta = \|\eta_n\|_{\varrho-\delta} + \int_{\mathbb{T}} \eta_n(\theta)\, d\theta \\ &= \|\eta_n\|_{\varrho-\delta} + \eta_{n,0} \le \frac{1}{\alpha_n} C_\delta^2 + \frac{1}{\alpha_n} C_\varrho^2 = \frac{1}{\alpha_n}(C_\varrho^2 + C_\delta^2) \ , \ \forall \theta \in \mathbb{T}_{\varrho-\delta} \ . \end{aligned} \tag{3.84}$$

Now, from (3.80) and (3.84) follows

$$\|\widetilde{\eta}_n\|_{\varrho-\delta} \le \frac{1}{c_{n,0}} a^2 C_\varrho^2 (C_\varrho^2 + C_\delta^2) \ , \ \forall \delta \in (0, \varrho) \ . \tag{3.85}$$

If $\alpha_n = 1$, then from (3.77) and (3.84), we have

$$\|\widetilde{\eta}_n\|_{\varrho-\delta} \le C_\varrho^2 + C_\delta^2 \ , \ \forall \delta \in (0, \varrho) \ . \tag{3.86}$$

(ix) On the one hand, by the definition of ξ_n and applying (vii) and (xiii), we can write:

$$\begin{aligned}\|\widetilde{\xi}_n\|_{\varrho-\delta} &= \sup_{\theta\in\mathbb{T}_{\varrho-\delta}} |\widetilde{\xi}_n(\theta)| = \sup_{\theta\in\mathbb{T}_{\varrho-\delta}} |\xi_n(\theta)-\xi_{n,0}| \\ &\leq \sup_{\theta\in\mathbb{T}_{\varrho-\delta}} (|\xi_n(\theta)|+|\xi_{n,0}|) \leq \sup_{\theta\in\mathbb{T}_{\varrho-\delta}} |\xi_n(\theta)|+|\xi_{n,0}| \\ &= \|\xi_n\|_{\varrho-\delta}+|\xi_{n,0}| \leq \|\eta_n\|_{\varrho-\delta}\|E_n\|_\varrho+\eta_{n,0}\|E_n\|_\varrho \\ &= (\|\eta_n\|_{\varrho-\delta}+\eta_{n,0})\,\|E_n\|_\varrho \leq \frac{1}{\alpha_n}(C_\varrho^2+C_\delta^2)\|E_n\|_\varrho\ ,\ \forall\theta\in\mathbb{T}_{\varrho-\delta}\ , \end{aligned} \tag{3.87}$$

where we have used (vi) and (xii).

Again, for any $c_{n,0}>0$ we have, from (3.87) and (v):

$$\|\widetilde{\xi}_n\|_{\varrho-\delta} \leq \frac{1}{c_{n,0}} a^2 C_\varrho^2 (C_\varrho^2+C_\delta^2)\|E_n\|_\varrho\ ,\ \forall\delta\in(0,\varrho)\ . \tag{3.88}$$

On the other hand, if $\alpha_n=1$, then from (3.87),

$$\|\widetilde{\xi}_n\|_{\varrho-\delta} \leq \left(C_\varrho^2+C_\delta^2\right)\|E_n\|_\varrho\ ,\ \forall\delta\in(0,\varrho)\ . \tag{3.89}$$

(x) Since $<\widetilde{\eta}_n>=0$, then by Rüßmann estimates (**Theorem 1.20**) and applying (viii),

$$\begin{aligned}|\mathfrak{R}_{\lambda_n}\widetilde{\eta}_n(\theta)| &\leq \|\mathfrak{R}_{\lambda_n}\widetilde{\eta}_n\|_{\varrho-2\delta} \leq \frac{\mathfrak{C}_R}{\gamma\delta^\nu}\|\widetilde{\eta}_n\|_{\varrho-\delta} \\ &= \frac{1}{A}\log(aC_\delta)\|\widetilde{\eta}_n\|_{\varrho-\delta} \leq \frac{1}{A}\log(aC_\delta)(\|\eta_n\|_{\varrho-\delta}+\eta_{n,0}) \\ &\leq \frac{1}{A}\log(aC_\delta)\frac{1}{\alpha_n}(C_\varrho^2+C_\delta^2)\ ,\ \forall\theta\in\mathbb{T}_{\varrho-2\delta}\ . \end{aligned} \tag{3.90}$$

If $\alpha_n=1$, then by (3.90),

$$\|\mathfrak{R}_{\lambda_n}\widetilde{\eta}_n\|_{\varrho-2\delta} \leq \frac{1}{A}\log(aC_\delta)(C_\varrho^2+C_\delta^2)\ ,\ \forall\delta\in(0,\frac{1}{2}\varrho)\ . \tag{3.91}$$

In any case, from (3.90) and (3.80), we have

$$\|\mathfrak{R}_{\lambda_n}\widetilde{\eta}_n\|_{\varrho-2\delta} \leq \frac{1}{c_{n,0}}\frac{a^2}{A}\log(aC_\delta)C_\varrho^2(C_\varrho^2+C_\delta^2)\ ,\ \forall\delta\in(0,\frac{1}{2}\varrho)\ , \tag{3.92}$$

and, in particular, taking limits[8] in (3.90) as $\delta\to\frac{1}{2}\varrho$,

$$\forall\theta\in\mathbb{T},\ |\mathfrak{R}_{\lambda_n}\widetilde{\eta}_n(\theta)| \leq \frac{1}{\alpha_n}\frac{1}{A}\log(aC_{\frac{1}{2}\varrho})(C_\varrho^2+C_{\frac{1}{2}\varrho}^2). \tag{3.93}$$

Since $C_\varrho < C_{\frac{1}{2}\varrho}$, we can also write:

$$\forall\theta\in\mathbb{T},\ |\mathfrak{R}_{\lambda_n}\widetilde{\eta}_n(\theta)| \leq \frac{1}{\alpha_n}\frac{2}{A}\log(aC_{\frac{1}{2}\varrho})C_{\frac{1}{2}\varrho}^2. \tag{3.94}$$

[8] According to **Corollary 1.28**, with $m=1$,

(*xi*) Again, by Rüßmann estimates (**Theorem 1.20**) and (viii),

$$\begin{aligned} |\mathfrak{R}_{\lambda_n}\widetilde{\xi}_n(\theta)| &\leq \|\mathfrak{R}_{\lambda_n}\widetilde{\xi}_n\|_{\varrho-2\delta} \leq \frac{\mathfrak{C}_R}{\gamma\,\delta^\nu}\|\widetilde{\xi}_n\|_{\varrho-\delta} \\ &= \frac{1}{A}\log(aC_\delta)\|\widetilde{\xi}_n\|_{\varrho-\delta} \leq \frac{1}{A}\log(aC_\delta)(\|\eta_n\|_\varrho + \eta_{n,0})\|E_n\|_\varrho \\ &\leq \frac{1}{A}\log(aC_\delta)\frac{1}{\alpha_n}(C_\varrho^2 + C_\delta^2)\|E_n\|_\varrho\ ,\ \forall\theta\in\mathbb{T}_{\varrho-2\delta}\ . \end{aligned} \tag{3.95}$$

In particular, taking limits in (3.95) as $\delta \to \frac{1}{2}\varrho$,

$$\forall\theta\in\mathbb{T},\ |\mathfrak{R}_{\lambda_n}\widetilde{\xi}_n(\theta)| \leq \frac{1}{\alpha_n}\frac{1}{A}\log(aC_{\frac{1}{2}\varrho})(C_\varrho^2 + C_{\frac{1}{2}\varrho}^2)\|E_n\|_\varrho. \tag{3.96}$$

If $\alpha_n = 1$, then by (3.95),

$$\|\mathfrak{R}_{\lambda_n}\widetilde{\xi}_n\|_{\varrho-2\delta} \leq \frac{1}{A}\log(aC_\delta)(C_\varrho^2 + C_\delta^2)\|E_n\|_\varrho\ ,\ \forall\delta\in(0,\frac{1}{2}\varrho)\ . \tag{3.97}$$

In particular,

$$\forall\theta\in\mathbb{T},\ \ |\mathfrak{R}_{\lambda_n}\widetilde{\xi}_n(\theta)| \leq \frac{1}{A}\log(aC_{\frac{1}{2}\varrho})(C_\varrho^2 + C_{\frac{1}{2}\varrho}^2)\|E_n\|_\varrho\ . \tag{3.98}$$

Finally, for any $c_{n,0} > 0$, from (3.95) and (3.71), we have

$$\|\mathfrak{R}_{\lambda_n}\widetilde{\xi}_n\|_{\varrho-2\delta} \leq \frac{1}{c_{n,0}}\frac{a^2}{A}\log(aC_\delta)C_\varrho^2(C_\varrho^2 + C_\delta^2)\|E_n\|_\varrho\ ,\forall\delta\in(0,\frac{1}{2}\varrho)\ , \tag{3.99}$$

and, in particular,

$$\forall\theta\in\mathbb{T},\ |\mathfrak{R}_{\lambda_n}\widetilde{\xi}_n(\theta)| \leq \frac{1}{c_{n,0}}\frac{a^2}{A}\log(aC_{\frac{1}{2}\varrho})C_\varrho^2(C_\varrho^2 + C_{\frac{1}{2}\varrho}^2)\|E_n\|_\varrho. \tag{3.100}$$

(*xii*) By (3.74) we can write the average of η_n as

$$\begin{aligned} \eta_{n,0} &= <\eta_n> = \int_{\mathbb{T}}\eta_n(\theta)\,d\theta = \int_{\mathbb{T}}\frac{1}{\alpha_n}\frac{\lambda_n}{|m_n(\theta)|}e^{-i\,\mathrm{Arg}(m_n(\theta))}e^{-\mathfrak{R}_1 v_n(\theta)}\,d\theta \\ &= \frac{\lambda_n}{\alpha_n}\int_{\mathbb{T}}\frac{1}{|m_n(\theta)|}e^{-\mathfrak{R}_1 v_n(\theta)}\,d\theta\ . \end{aligned} \tag{3.101}$$

Since $\frac{1}{K_1} \leq \frac{1}{|m_n(\theta)|} \leq \frac{1}{K_1^*}$, it follows that

$$\frac{\lambda_n}{\alpha_n}\frac{1}{K_1}\int_{\mathbb{T}}e^{-\mathfrak{R}_1 v_n(\theta)}\,d\theta \leq \eta_{n,0} \leq \frac{\lambda_n}{\alpha_n}\frac{1}{K_1^*}\int_{\mathbb{T}}e^{-\mathfrak{R}_1 v_n(\theta)}\,d\theta\ . \tag{3.102}$$

Now, by (3.54) and (3.58),

$$\frac{1}{\alpha_n}a^2\int_{\mathbb{T}}e^{-\mathfrak{R}_1 v_n(\theta)}\,d\theta \leq \eta_{n,0} \leq \frac{1}{\alpha_n}\frac{1}{a^2}\int_{\mathbb{T}}e^{-\mathfrak{R}_1 v_n(\theta)}\,d\theta\ . \tag{3.103}$$

Since $\alpha_n = \dfrac{c_{n,0}}{\int_{\mathbb{T}}e^{\mathfrak{R}_1 v_n(\theta)}\,d\theta}$, then from the former inequality of (3.103),

$$\frac{1}{\alpha_n}a^2\int_{\mathbb{T}}e^{-\mathfrak{R}_1 v_n(\theta)}\,d\theta = \frac{1}{c_{n,0}}a^2\int_{\mathbb{T}}e^{\mathfrak{R}_1 v_n(\theta)}\,d\theta\int_{\mathbb{T}}e^{-\mathfrak{R}_1 v_n(\theta)}\,d\theta = \frac{1}{c_{n,0}}a^2 I_n \leq \eta_{n,0} \tag{3.104}$$

We also know that $I_n \geq 1$ by (3.57). Therefore:

$$a^2 \leq c_{n,0}\eta_{n,0} \ . \tag{3.105}$$

On the other hand, from the latter inequality of (3.103) and applying (3.71) we have,

$$\eta_{n,0} \leq \frac{1}{\alpha_n}\frac{1}{a^2}\int_{\mathbb{T}} e^{-\mathfrak{R}_1 v_n(\theta)}\, d\theta \leq \frac{1}{c_{n,0}}a^2 C_\varrho^2 \frac{1}{a^2}\int_{\mathbb{T}} e^{-\mathfrak{R}_1 v_n(\theta)}\, d\theta = \frac{1}{c_{n,0}}C_\varrho^2 \int_{\mathbb{T}} e^{-\mathfrak{R}_1 v_n(\theta)}\, d\theta \ . \tag{3.106}$$

Finally, by means of (3.64), we can write:

$$\begin{aligned}
\int_{\mathbb{T}} e^{-\mathfrak{R}_1 v_n(\theta)}\, d\theta &\leq \sup_{\theta\in\mathbb{T}} \left| e^{-\mathfrak{R}_1 v_n(\theta)} \right| = \sup_{\theta\in\mathbb{T}} e^{\mathrm{Re}(-\mathfrak{R}_1 v_n(\theta))} \\
&\leq \sup_{\theta\in\mathbb{T}} e^{|-\mathfrak{R}_1 v_n(\theta)|} = \sup_{\theta\in\mathbb{T}} e^{|\mathfrak{R}_1 v_n(\theta)|} \leq e^{2\log(aC_\varrho)} \\
&= a^2 C_\varrho^2 \ .
\end{aligned} \tag{3.107}$$

Joining (3.106) and (3.107),

$$\eta_{n,0} \leq \frac{1}{c_{n,0}} C_\varrho^2\, a^2 C_\varrho^2 = \frac{1}{c_{n,0}} a^2 C_\varrho^4 \ , \tag{3.108}$$

i.e.

$$c_{n,0}\eta_{n,0} \leq a^2 C_\varrho^4 \ . \tag{3.109}$$

Additionally, when $\alpha_n = 1$, from (3.103) and (3.107) we have:

$$\eta_{n,0} \leq \frac{1}{a^2}\int_{\mathbb{T}} e^{-\mathfrak{R}_1 v_n(\theta)} d\theta \leq \frac{1}{a^2} a^2 C_\varrho^2 = C_\varrho^2 \ . \tag{3.110}$$

(*xiii*)

$$\begin{aligned}
|\xi_{n,0}| = |<\xi_n>| &= \left| \int_{\mathbb{T}} \xi_n(\theta)\, d\theta \right| \leq \int_{\mathbb{T}} |\xi_n(\theta)|\, d\theta \\
&= \int_{\mathbb{T}} |\eta_n(\theta) E_n(\theta)|\, d\theta = \int_{\mathbb{T}} \eta_n(\theta) |E_n(\theta)|\, d\theta \\
&\leq \int_{\mathbb{T}} \eta_n(\theta)\, d\theta \ \sup_{\theta\in\mathbb{T}} |E_n(\theta)| \\
&= \eta_{n,0} \sup_{\theta\in\mathbb{T}} |E_n(\theta)| \leq \eta_{n,0} \sup_{\theta\in\mathbb{T}_\varrho} |E_n(\theta)| \\
&= \eta_{n,0} \|E_n\|_\varrho \leq \frac{1}{\alpha_n} C_\varrho^2 \|E_n\|_\varrho \ .
\end{aligned} \tag{3.111}$$

Thus, from (3.111) and (xii), we have:

For any $c_{n,0} > 0$,

$$|\xi_{n,0}| \leq \frac{1}{c_{n,0}} a^2 C_\varrho^4 \|E_n\|_\varrho \ . \tag{3.112}$$

If $\alpha_n = 1$, then

$$|\xi_{n,0}| \leq C_\varrho^2 \|E_n\|_\varrho \ . \tag{3.113}$$

(*xiv*) For this part and the next one, recall again that for every $\theta \in \mathbb{T}$, $\theta \in \mathbb{R}$ and hence $\mathfrak{R}_1 v_n(\theta) \in \mathbb{R}$. Thus $c_n(\theta) = \alpha_n e^{\mathfrak{R}_1 v_n(\theta)} \in \mathbb{R}^+$. Therefore,

$$\begin{aligned}
\int_{\mathbb{T}} |\widetilde{c}_n(\theta)|\, d\theta &= \int_{\mathbb{T}} |c_n(\theta) - c_{n,0}|\, d\theta \leq \int_{\mathbb{T}} (|c_n(\theta)| + c_{n,0})\, d\theta \\
&= \int_{\mathbb{T}} |c_n(\theta)|\, d\theta + c_{n,0} = \int_{\mathbb{T}} c_n(\theta)\, d\theta + c_{n,0} = 2c_{n,0} \ .
\end{aligned} \tag{3.114}$$

From (3.93) and (3.114), we can develop the following estimate:

$$
\begin{aligned}
|<\widetilde{c}_n \mathfrak{R}_{\lambda_n} \widetilde{\eta}_n>| &= \left|\int_{\mathbb{T}} \widetilde{c}_n(\theta) \mathfrak{R}_{\lambda_n} \widetilde{\eta}_n(\theta)\, d\theta\right| \leq \int_{\mathbb{T}} |\widetilde{c}_n(\theta)|\, |\mathfrak{R}_{\lambda_n} \widetilde{\eta}_n(\theta)|\, d\theta \\
&\leq \int_{\mathbb{T}} |\widetilde{c}_n(\theta)|\, d\theta \sup_{\theta\in\mathbb{T}} |\mathfrak{R}_{\lambda_n} \widetilde{\eta}_n(\theta)| \leq 2c_{n,0} \sup_{\theta\in\mathbb{T}} |\mathfrak{R}_{\lambda_n} \widetilde{\eta}_n(\theta)| \\
&\leq 2c_{n,0} \frac{1}{\alpha_n} \frac{1}{A} \log(aC_{\frac{1}{2}\varrho})(C_\varrho^2 + C_{\frac{1}{2}\varrho}^2) \\
&\leq 2c_{n,0} \frac{1}{c_{n,0}} C_\varrho^2 \frac{1}{A} \log(aC_{\frac{1}{2}\varrho})(C_\varrho^2 + C_{\frac{1}{2}\varrho}^2) \\
&= \frac{2a^2}{A} \log(aC_{\frac{1}{2}\varrho}) C_\varrho^2 (C_\varrho^2 + C_{\frac{1}{2}\varrho}^2)\,.
\end{aligned}
\tag{3.115}
$$

(*xv*) In a similar way, from (3.100) and (3.114), we can develop the following estimate as well:

$$
\begin{aligned}
|<\widetilde{c}_n \mathfrak{R}_{\lambda_n} \widetilde{\xi}_n>| &= \left|\int_{\mathbb{T}} \widetilde{c}_n(\theta) \mathfrak{R}_{\lambda_n} \widetilde{\xi}_n(\theta)\, d\theta\right| \leq \int_{\mathbb{T}} |\widetilde{c}_n(\theta)|\, |\mathfrak{R}_{\lambda_n} \widetilde{\xi}_n(\theta)|\, d\theta \\
&\leq \int_{\mathbb{T}} |\widetilde{c}_n(\theta)|\, d\theta \sup_{\theta\in\mathbb{T}} |\mathfrak{R}_{\lambda_n} \widetilde{\xi}_n(\theta)| \leq 2c_{n,0} \sup_{\theta\in\mathbb{T}} |\mathfrak{R}_{\lambda_n} \widetilde{\xi}_n(\theta)| \\
&\leq 2c_{n,0} \frac{1}{\alpha_n} \frac{1}{A} \log(aC_{\frac{1}{2}\varrho})(C_\varrho^2 + C_{\frac{1}{2}\varrho}^2)\|E_n\|_\varrho \\
&\leq 2c_{n,0} \frac{1}{c_{n,0}} C_\varrho^2 \frac{1}{A} \log(aC_{\frac{1}{2}\varrho})(C_\varrho^2 + C_{\frac{1}{2}\varrho}^2)\|E_n\|_\varrho \\
&= \frac{2a^2}{A} \log(aC_{\frac{1}{2}\varrho}) C_\varrho^2 (C_\varrho^2 + C_{\frac{1}{2}\varrho}^2)\|E_n\|_\varrho\,.
\end{aligned}
\tag{3.116}
$$

Notice that this estimate (3.116) and the one before (3.115) do not depend on the value assigned to $c_{n,0}$.

(*xvi*) By (3.41), $\varsigma_{n,0} = \dfrac{1}{\det(\Omega_n)} < \xi_{n,0} \widetilde{c}_n \mathfrak{R}_{\lambda_n} \widetilde{\eta}_n + \eta_{n,0}(-e_n(p) - \widetilde{c}_n \mathfrak{R}_{\lambda_n} \widetilde{\xi}_n) >$.

Therefore, by (3.111), (3.115), (3.116), and at last (3.108), we have:

$$
\begin{aligned}
|\varsigma_{n,0}| &= \frac{1}{|\det(\Omega_n)|} \Big(|\xi_{n,0}|\, |<\widetilde{c}_n \mathfrak{R}_{\lambda_n} \widetilde{\eta}_n>| + |\eta_{n,0}|\, |<\widetilde{c}_n \mathfrak{R}_{\lambda_n} \widetilde{\xi}_n>| + |\eta_{n,0}|\, |e_n(p)| \Big) \\
&\leq \frac{1}{|\det(\Omega_n)|} \Big(\eta_{n,0} \|E_n\|_\varrho \cdot \frac{2a^2}{A} \log(aC_{\frac{1}{2}\varrho}) C_\varrho^2 (C_\varrho^2 + C_{\frac{1}{2}\varrho}^2) \\
&\qquad + \eta_{n,0} \cdot \frac{2a^2}{A} \log(aC_{\frac{1}{2}\varrho}) C_\varrho^2 (C_\varrho^2 + C_{\frac{1}{2}\varrho}^2)\|E_n\|_\varrho + \eta_{n,0} |e_n(p)| \Big) \\
&= \frac{1}{|\det(\Omega_n)|} \eta_{n,0} \left(\frac{4a^2}{A} \log(aC_{\frac{1}{2}\varrho}) C_\varrho^2 (C_\varrho^2 + C_{\frac{1}{2}\varrho}^2)\|E_n\|_\varrho + |e_n(p)| \right) \\
&\leq \frac{1}{|\det(\Omega_n)|} \frac{1}{\alpha_n} C_\varrho^2 \left(\frac{4a^2}{A} \log(aC_{\frac{1}{2}\varrho}) C_\varrho^2 (C_\varrho^2 + C_{\frac{1}{2}\varrho}^2)\|E_n\|_\varrho + |e_n(p)| \right)\,,
\end{aligned}
\tag{3.117}
$$

Now, for any $c_{n,0} > 0$ we have

$$
\begin{aligned}
|\varsigma_{n,0}| &\leq \frac{1}{|\det(\Omega_n)|} \frac{1}{c_{n,0}} a^2 C_\varrho^4 \left(\frac{4a^2}{A} \log(aC_{\frac{1}{2}\varrho}) C_\varrho^2 (C_\varrho^2 + C_{\frac{1}{2}\varrho}^2)\|E_n\|_\varrho + |e_n(p)| \right) \\
&\quad \frac{1}{\sigma_D} \frac{1}{c_{n,0}} \Big(P_1(C_\varrho, C_{\frac{1}{2}\varrho})\|E_n\|_\varrho + Q_1(C_\varrho)|e_n(p)| \Big)\,,
\end{aligned}
\tag{3.118}
$$

where

$$\begin{aligned} P_1(s,t) &= \frac{4a^4}{A}s^6(s^2+t^2)\log(at), &(3.119)\\ Q_1(s) &= a^2s^4. &(3.120)\end{aligned}$$

Moreover, in case where $\alpha = 1$ we have $\eta_{n,0} \leq C_\varrho^2$. Therefore,

$$\begin{aligned} |\varsigma_{n,0}| &\leq \frac{1}{|\det(\Omega_n)|}C_\varrho^2\left(\frac{4a^2}{A}\log(aC_{\frac{1}{2}\varrho})C_\varrho^2(C_\varrho^2+C_{\frac{1}{2}\varrho}^2)\|E_n\|_\varrho + |e_n(p)|\right)\\ &\quad \frac{1}{\sigma_D}\left(P_1^*(C_\varrho, C_{\frac{1}{2}\varrho})\|E_n\|_\varrho + Q_1^*(C_\varrho)|e_n(p)|\right), &(3.121)\end{aligned}$$

where

$$\begin{aligned} P_1^*(s,t) &= \frac{4a^2}{A}s^4(s^2+t^2)\log(at), &(3.122)\\ Q_1^*(s) &= s^2. &(3.123)\end{aligned}$$

(*xvii*) As we saw in (3.43), the translation correction is of the form

$$\Delta\tau_n = \frac{1}{\det(\Omega_n)}\left(< (1-\lambda_n)(-e_n(p) - \widetilde{c}_n\mathfrak{R}_{\lambda_n}\widetilde{\xi}_n) - c_{n,0}\xi_{n,0} >\right) \tag{3.124}$$

Hence,

$$|\Delta\tau_n| \leq \frac{1}{|\det(\Omega_n)|}\left(|1-\lambda_n|\left(|e_n(p)| + |<\widetilde{c}_n\mathfrak{R}_{\lambda_n}\widetilde{\xi}_n)>|\right) + c_{n,0}|\xi_{n,0}|\right) \tag{3.125}$$

Taking in account now parts (xiii) and (xv), we have

$$\begin{aligned} |\Delta\tau_n| &\leq \frac{1}{|\det(\Omega_n)|}\left(|1-\lambda_n|\left(|e_n(p)| + \frac{2a^2}{A}\log(aC_{\frac{1}{2}\varrho})C_\varrho^2(C_\varrho^2+C_{\frac{1}{2}\varrho}^2)\|E_n\|_\varrho\right)\right.\\ &\quad \left. + c_{n,0}\eta_{n,0}\|E_n\|_\varrho\right) &(3.126)\end{aligned}$$

Moreover, by part (xii),

$$\begin{aligned} |\Delta\tau_n| &\leq \frac{1}{|\det(\Omega_n)|}\left(|1-\lambda_n|\left(|e_n(p)| + \frac{2a^2}{A}\log(aC_{\frac{1}{2}\varrho})C_\varrho^2(C_\varrho^2+C_{\frac{1}{2}\varrho}^2)\|E_n\|_\varrho\right)\right.\\ &\quad \left. + a^2C_\varrho^4\|E_n\|_\varrho\right) &(3.127)\end{aligned}$$

Notice that $|1-\lambda_n| \leq \frac{1-a}{a}$. Indeed, by (i), $a \leq \lambda_n \leq \frac{1}{a}$, so if $a \leq \lambda_n \leq 1$, then $|1-\lambda_n| \leq 1-a$, and if $1 \leq \lambda_n \leq \frac{1}{a}$, then $|1-\lambda_n| \leq \frac{1}{a} - 1 = \frac{1-a}{a}$. Thence, $|\lambda_n - 1| \leq \max\{1-a, \frac{1-a}{a}\}$. But $a \in (0,1)$, so $1-a \leq \frac{1-a}{a}$. Therefore,

$$|\lambda_n - 1| \leq \frac{1-a}{a}\ . \tag{3.128}$$

It follows, from (3.127) and (3.128), that

$$\begin{aligned} |\Delta\tau_n| &\leq \frac{1}{|\det(\Omega_n)|}\left(\frac{1-a}{a}\left(|e_n(p)| + \frac{2a^2}{A}\log(aC_{\frac{1}{2}\varrho})C_\varrho^2(C_\varrho^2+C_{\frac{1}{2}\varrho}^2)\|E_n\|_\varrho\right)\right.\\ &\quad \left. + a^2C_\varrho^4\|E_n\|_\varrho\right) &(3.129)\end{aligned}$$

Rearranging terms, we have finally:

$$\begin{aligned} |\Delta\tau_n| &\leq \frac{1}{|\det(\Omega_n)|}\left(a^2C_\varrho^2\left(1+\frac{2(1-a)}{aA}(C_\varrho^2+C_{\frac{1}{2}\varrho}^2)\log(aC_{\frac{1}{2}\varrho})\right)\|E_n\|_\varrho \right. \\ &\qquad \left. +\frac{1-a}{a}|e_n(p)|\right) \\ &= \frac{1}{\sigma_D}\left(P_2(C_\varrho,C_{\frac{1}{2}\varrho})\|E_n\|_\varrho+Q_2|e_n(p)|\right) , \end{aligned} \tag{3.130}$$

where

$$P_2(s,t) = a^2s^2\left(1+\frac{2(1-a)}{aA}(s^2+t^2)\log(at)\right), \tag{3.131}$$

$$Q_2 = \frac{1-a}{a}. \tag{3.132}$$

$(xviii)$ From (3.33) we get the determinant,

$$\det(\Omega_n) = (1-\lambda_n) < \widetilde{c}_n\mathfrak{R}_{\lambda_n}\widetilde{\eta}_n > +c_{n,0}\eta_{n,0} , \tag{3.133}$$

whose value can be therefore estimated, according to (3.128), (3.115), and (3.109), as

$$\begin{aligned} |\det(\Omega_n)| &\leq |1-\lambda_n|\,|<\widetilde{c}_n\mathfrak{R}_{\lambda_n}\widetilde{\eta}_n>|+c_{n,0}\eta_{n,0} \\ &\leq \frac{1-a}{a}\frac{2a^2}{A}\log(aC_{\frac{1}{2}\varrho})C_\varrho^2(C_\varrho^2+C_{\frac{1}{2}\varrho}^2)+a^2C_\varrho^4 \\ &= a^2C_\varrho^4\left(1+\frac{2(1-a)}{aA}(C_\varrho^2+C_{\frac{1}{2}\varrho}^2)\ \log(aC_{\frac{1}{2}\varrho})\right) = P_2(C_\varrho,C_{\frac{1}{2}\varrho}) . \end{aligned} \tag{3.134}$$

(xix) According to (3.42),

$$\widetilde{\varsigma}_n(\theta) = \mathfrak{R}_{\lambda_n}(\widetilde{\xi}_n(\theta)+\widetilde{\eta}_n(\theta)\Delta\tau_n) , \ \forall\theta\in\mathbb{T} .$$

In fact, for any $\delta\in(0,\frac{1}{2}\varrho)$, $\widetilde{\varsigma}_n\in\mathcal{A}_{\varrho-2\delta}$. From **Theorem 1.20** (Rüßmann estimates) we obtain:

$$\|\widetilde{\varsigma}_n\|_{\varrho-2\delta} \leq \frac{1}{A}\log(aC_\delta)\ \|\widetilde{\xi}_n+\widetilde{\eta}_n\ \Delta\tau_n\|_{\varrho-\delta} . \tag{3.135}$$

By the triangular inequality,

$$\|\widetilde{\varsigma}_n\|_{\varrho-2\delta} \leq \frac{1}{A}\log(aC_\delta)\ \left(\|\widetilde{\xi}_n\|_{\varrho-\delta}+\|\widetilde{\eta}_n\|_{\varrho-\delta}\,|\Delta\tau_n|\right) .$$

Notice that, by (3.87), $\|\widetilde{\xi}_n\|_{\varrho-\delta}\leq(\|\eta_n\|_{\varrho-\delta}+\eta_{n,0})\|E_n\|_\varrho$, and from (3.84), we have also $\|\widetilde{\eta}_n\|_{\varrho-\delta}\leq\|\eta_n\|_{\varrho-\delta}+\eta_{n,0}$. Therefore,

$$\begin{aligned} \|\widetilde{\varsigma}_n\|_{\varrho-2\delta} &\leq \frac{1}{A}\log(aC_\delta)\left((\|\eta_n\|_{\varrho-\delta}+\eta_{n,0})\|E_n\|_\varrho+(\|\eta_n\|_{\varrho-\delta}+\eta_{n,0})\,|\Delta\tau_n|\right) \\ &= \frac{1}{A}\log(aC_\delta)(\|\eta_n\|_{\varrho-\delta}+\eta_{n,0})\left(\|E_n\|_\varrho+|\Delta\tau_n|\right) . \end{aligned} \tag{3.136}$$

Now, by the estimate of the translation parameter correction (3.130) and the estimate of the determinant (3.134), we have:

$$\begin{aligned}
\|\widetilde{\varsigma}_n\|_{\varrho-2\delta} &\leq \frac{1}{A}\log(aC_\delta)(\|\eta_n\|_{\varrho-\delta} \\
&\qquad +\eta_{n,0})\left(\|E_n\|_\varrho + \frac{1}{|\det(\Omega_n)|}\left(P_2(C_\varrho, C_{\frac{1}{2}\varrho})\|E_n\|_\varrho + Q_2|e_n(p)|\right)\right) \\
&= \frac{1}{|\det(\Omega_n)|}\frac{1}{A}\log(aC_\delta)(\|\eta_n\|_{\varrho-\delta} \\
&\qquad +\eta_{n,0})\left((|\det(\Omega_n)| + P_2(C_\varrho, C_{\frac{1}{2}\varrho})\|E_n\|_\varrho + Q_2|e_n(p)|\right) \\
&\leq \frac{1}{|\det(\Omega_n)|}\frac{1}{A}\log(aC_\delta)(\|\eta_n\|_{\varrho-\delta} \\
&\qquad +\eta_{n,0})\left(2P_2(C_\varrho, C_{\frac{1}{2}\varrho})\|E_n\|_\varrho + Q_2(C_\varrho)|e_n(p)|\right) . \qquad (3.137)
\end{aligned}$$

Finally, from (vi) and (xii), $\|\eta_n\|_{\varrho-\delta} + \eta_{n,0} \leq \frac{1}{\alpha_n}(C_\varrho^2 + C_\delta^2)$. Therefore,

$$\|\widetilde{\varsigma}_n\|_{\varrho-2\delta} \leq \frac{1}{|\det(\Omega_n)|}\frac{1}{\alpha_n}\frac{1}{A}\log(aC_\delta)(C_\varrho^2 + C_\delta^2)\left(2P_2(C_\varrho, C_{\frac{1}{2}\varrho})\|E_n\|_\varrho + Q_2(C_\varrho)|e_n(p)|\right) . \qquad (3.138)$$

Clearly, if $\alpha_n = 1$, then from (3.138) we obtain:

$$\begin{aligned}
\|\widetilde{\varsigma}_n\|_{\varrho-2\delta} &\leq \frac{1}{|\det(\Omega_n)|}\frac{1}{A}\log(aC_\delta)(C_\varrho^2 + C_\delta^2)\left(2P_2(C_\varrho, C_{\frac{1}{2}\varrho})\|E_n\|_\varrho + Q_2(C_\varrho)|e_n(p)|\right) \\
&= \frac{1}{|\det(\Omega_n)|}\left(P_3^*(C_\varrho, C_{\frac{1}{2}\varrho}, C_\delta)\|E_n\|_\varrho + Q_3^*(C_\varrho, C_\delta)|e_n(p)|\right) \\
&\leq \frac{1}{\sigma_D}\left(P_3^*(C_\varrho, C_{\frac{1}{2}\varrho}, C_\delta)\|E_n\|_\varrho + Q_3^*(C_\varrho, C_\delta)|e_n(p)|\right), \qquad (3.139)
\end{aligned}$$

with

$$\begin{aligned}
P_3^*(s,t,u) &= \frac{2a^2}{A}s^2(s^2+u^2)\log(au)\left(1 + \frac{2(1-a)}{aA}(s^2+t^2)\log(at)\right), \qquad (3.140) \\
Q_3^*(s,u) &= \frac{1-a}{aA}(s^2+u^2)\log(au) . \qquad (3.141)
\end{aligned}$$

In general, for any $c_{n,0} > 0$, from (3.138) and applying (v), we get the estimate:

$$\begin{aligned}
\|\widetilde{\varsigma}_n\|_{\varrho-2\delta} &\leq \frac{1}{|\det(\Omega_n)|}\frac{1}{c_{n,0}}\frac{a^2}{A}\log(aC_\delta)C_\varrho^2(C_\varrho^2 + C_\delta^2) \\
&\qquad \cdot (2P_2(C_\varrho)\|E_n\|_\varrho + Q_2(C_\varrho)|e_n(p)|) \\
&= \frac{1}{|\det(\Omega_n)|}\frac{1}{c_{n,0}}\left(P_3(C_\varrho, C_{\frac{1}{2}\varrho}, C_\delta)\|E_n\|_\varrho + Q_3(C_\varrho, C_\delta)|e_n(p)|\right) \\
&\leq \frac{1}{\sigma_D}\frac{1}{c_{n,0}}\left(P_3(C_\varrho, C_{\frac{1}{2}\varrho}, C_\delta)\|E_n\|_\varrho + Q_3(C_\varrho, C_\delta)|e_n(p)|\right) , \qquad (3.142)
\end{aligned}$$

with

$$\begin{aligned}
P_3(s,t,u) &= \frac{2a^4}{A}s^4(s^2+u^2)\log(au)\left(1 + \frac{2(1-a)}{aA}(s^2+t^2)\log(at)\right), \qquad (3.143) \\
Q_3(s,u) &= \frac{a(1-a)}{A}s^2(s^2+u^2)\log(au). \qquad (3.144)
\end{aligned}$$

(*xx*) Let $\delta \in (0, \frac{1}{2}\varrho)$. From (3.12) and (3.18), the correction of the curve κ_n is given by

$$\Delta\kappa_n(\theta) = c_n(\theta)\varsigma_n(\theta) = c_n(\theta)(\varsigma_{n,0} + \widetilde{\varsigma}_n(\theta)),\ \forall\theta \in \mathbb{T}_{\varrho-2\delta}.$$

Therefore, by the previous parts (v) (3.66), (xvi), and (xix) of this lemma, we have:

$$\begin{aligned}
\|\Delta\kappa_n\|_{\varrho-2\delta} &\leq \|c_n\|_{\varrho-\delta}(\|\widetilde{\varsigma}_n\|_{\varrho-2\delta} + |\varsigma_{n,0}|)\\
&\leq \alpha_n a^2 C_\delta^2 \left(\frac{1}{\sigma_D}\frac{1}{\alpha_n}(P_1^*(C_\varrho, C_{\frac{1}{2}\varrho})\|E_n\|_\varrho + Q_1^*(C_\varrho))\right.\\
&+ \left.\frac{1}{\sigma_D}\frac{1}{\alpha_n}(P_3^*(C_\varrho, C_{\frac{1}{2}\varrho}, C_\delta)\|E_n\|_\varrho + Q_3^*(C_\varrho, C_\delta))\right)\\
&= \frac{1}{\sigma_D}a^2 C_\delta^2\left(\left(P_1^*(C_\varrho, C_{\frac{1}{2}\varrho}) + P_3^*(C_\varrho, C_{\frac{1}{2}\varrho}, C_\delta)\right)\|E_n\|_\varrho\right.\\
&+ \left.(Q_1^*(C_\varrho) + Q_3^*(C_\varrho, C_\delta))\,|e_n(p)|\right)\\
&= \frac{1}{\sigma_D}\left(P(C_\varrho, C_{\frac{1}{2}\varrho}, C_\delta)\|E_n\|_\varrho + Q(C_\varrho, C_\delta)|e_n(p)|\right),
\end{aligned} \tag{3.145}$$

where

$$\begin{aligned}
P(s,t,u) &= a^2u^2(P_1^*(s,t) + P_3^*(s,t,u))\\
&= \frac{2a^4}{A}(s^2+t^2)u^2\log(at)\left(2s^4\right.\\
&+ \left.\left(1 + \frac{2(1-a)}{aA}(s^2+t^2)\log(at)\right)\log(au)\right),
\end{aligned} \tag{3.146}$$

$$\begin{aligned}
Q(s,u) &= a^2u^2(Q_1^*(s) + Q_3^*(s,u))\\
&= a^2s^2u^2 + \frac{1-a}{aA}(s^2+u^2)\log(au).
\end{aligned} \tag{3.147}$$

In particular,

$$|\Delta\kappa_n(\theta)| \leq \frac{1}{\sigma_D}(P^*(C_\varrho, C_{\frac{1}{2}\varrho})\|E_n\|_\varrho + Q^*(C_\varrho, C_{\frac{1}{2}\varrho})|e_n(p)|),\ \forall\theta\in\mathbb{T}, \tag{3.148}$$

where

$$\begin{aligned}
P^*(s,t) = P(s,t,t) &= a^2t^2(P_1^*(s,t) + P_3^*(s,t,t))\\
&= \frac{2a^4}{A}(s^2+t^2)t^2\log(at)\left(2s^4\right.\\
&+ \left.\left(1 + \frac{2(1-a)}{aA}(s^2+t^2)\log(at)\right)\log(at)\right),
\end{aligned} \tag{3.149}$$

$$\begin{aligned}
Q^*(s,t) = Q(s,t)) &= a^2t^2(Q_1^*(s) + Q_3^*(s,t))\\
&= a^2s^2t^2 + \frac{1-a}{aA}(s^2+t^2)\log(at).
\end{aligned} \tag{3.150}$$

□

Remark 3.6

Notice that estimates (3.130) and (3.145) obtained for $|\Delta\tau_n|$ and $\|\Delta\kappa_n\|_{\varrho-2\delta}$, respectively, do not depend on the value assigned to $c_{n,0}$.

Corollary

*Under the same conditions as in **Lemma 3.5** we have the following estimate* $\forall \delta \in (0, \frac{1}{2}\varrho)$*:*

$$|E_{n+1}(\theta)| \leq \frac{1}{2}K_2\frac{1}{\sigma_D^2}\left(P(C_\varrho, C_{\frac{1}{2}\varrho}, C_\delta)\|E_n\|_\varrho + Q(C_\varrho, C_\delta)|e_n(p)|\right)^2, \ \forall\theta \in \mathbb{T}_{\varrho-2\delta}. \tag{3.151}$$

In particular,

$$|E_{n+1}(\theta)| \leq \frac{1}{2}K_2\frac{1}{\sigma_D^2}\left(P^*(C_\varrho, C_{\frac{1}{2}\varrho})\|E_n\|_\varrho + Q^*(C_\varrho, C_{\frac{1}{2}\varrho})|e_n(p)|\right)^2, \ \forall\theta \in \mathbb{T}. \tag{3.152}$$

Proof. It is a consequence of parts (iii) and (xx) of the previous lemma. □

Remark 3.8

$$|E_{n+1}(\theta)| \leq \frac{1}{\sigma_D^2}R(C_\varrho, C_{\frac{1}{2}\varrho}, C_\delta)(\|E_n\|_\varrho + |e_n(p)|)^2, \ \forall\theta \in T_{\varrho-2\delta}. \tag{3.153}$$

with

$$R(C_\varrho, C_{\frac{1}{2}\varrho}, C_\delta) = \frac{1}{2}K_2 \max\left\{P(C_\varrho, C_{\frac{1}{2}\varrho}, C_\delta), Q(C_\varrho, C_\delta)\right\}^2 \tag{3.154}$$

3.5 KAM correction estimates

Lemma 3.9 Correction estimates

*Under the same hypothesis as in the previous **Lemma 3.5** the corresponding corrections satisfy the following estimates:*

(i) $|\Delta m_n(\theta)| \leq K_2 |\Delta\kappa_n(\theta)|$, $\forall\theta \in \mathbb{T}_{\varrho-2\delta}$, $\delta \in (0, \frac{1}{2}\varrho)$.

(ii) $|E_{n+1}(\theta)| \leq \frac{1}{2}K_2 |\Delta\kappa_n(\theta)|^2$, $\forall\theta \in \mathbb{T}_{\varrho-2\delta}$, $\delta \in (0, \frac{1}{2}\varrho)$.

From now on we will consider the following additional hypothesis that will be useful in concluding many of the new correction estimates:

$$\exists\, r \in (0,1) \text{ such that } |\Delta m_n(\theta)| \leq r|m_n(\theta)|,\ \forall\theta \in \mathbb{T}_{\varrho-2\delta},\ \delta \in (0, \frac{1}{2}\varrho). \tag{H}$$

(iii) *For any* $\delta \in (0, \dfrac{1}{2}\varrho)$, $\left|\log\left(1 + \frac{\Delta m_n(\theta)}{m_n(\theta)}\right)\right| \leq \frac{1}{r}\log\frac{1}{1-r}\frac{|\Delta m_n(\theta)|}{|m_n(\theta)|}$, $\forall\theta \in \mathbb{T}_{\varrho-2\delta}$.

(iv) *Whenever* (H) *holds, the correction of the Lyapunov multiplier can be estimated by* $|\Delta\lambda_n| \leq L \sup_{\theta\in\mathbb{T}} |\Delta\kappa_n(\theta)|$, *where* $L = K_2\, \frac{1}{a^2}\, \frac{1}{1-r}$.

(v) *For any* $\delta \in (0, \frac{1}{2}\varrho)$ *the correction of the function* v_n *can be estimated by* $|\Delta v_n(\theta)| \leq L^* \|\Delta\kappa_n\|_{\varrho-2\delta}$, $\forall\theta \in \mathbb{T}_{\varrho-2\delta}$, *where* $L^* = \frac{K_2}{a}\,\frac{2}{r}\log\frac{1}{1-r}$.
In particular, $|\Delta v_n(\theta)| \leq L^* \sup_{\theta\in\mathbb{T}} |\kappa_n(\theta)|$, $\forall\theta \in \mathbb{T}$.

(vi) *Let* $\delta \in (0, \frac{1}{3}\varrho)$. *If* $\alpha_{n+1} = \alpha_n$, $\Delta u_n(\theta) = \mathfrak{R}_1 \Delta v_n(\theta)$, $\forall\theta \in \mathbb{T}_{\varrho-3\delta}$ *and*

$$|\Delta u_n(\theta)| \leq \frac{1}{A}\log(aC_\delta)\|\Delta v_n\|_{\varrho-2\delta}\ ,\ \forall\theta \in \mathbb{T}_{\varrho-3\delta}\ .$$

Additionally, whenever (H) *holds,* $|\Delta u_n(\theta)| \leq \frac{L^*}{A}\log(aC_\delta)\|\Delta\kappa_n\|_{\varrho-2\delta}$, $\forall\theta \in \mathbb{T}_{\varrho-3\delta}$.
In particular, $\forall\theta \in \mathbb{T}$, $|\Delta u_n(\theta)| \leq 3^\nu \dfrac{L^*}{A}\log(aC_\varrho) \sup_{\theta\in\mathbb{T}_{\frac{1}{3}\varrho}} |\Delta\kappa_n(\theta)|$.

(vii) *For any* $\delta \in (0, \frac{1}{3}\varrho)$, *the correction of the Floquet transformation* Δc_n *is given by:*

$$\Delta c_n(\theta) = c_n(\theta)\Delta u_n(\theta)\int_0^1 e^{t\Delta u_n(\theta)}dt,\ \forall\theta \in \mathbb{T}_{\varrho-3\delta}.$$

If $\alpha_{n+1} = \alpha_n$, Δc_n *can be estimated by:*

$$|\Delta c_n(\theta)| \leq \alpha_n \frac{a^2}{4A} C_\delta^2 (a^4 C_\delta^4 - 1)\|\Delta v_n\|_{\varrho-2\delta}\ ,\ \forall\theta \in \mathbb{T}_{\varrho-3\delta}.$$

Additionally, whenever (H) *holds,*

$$|\Delta c_n(\theta)| \leq\ \alpha_n \frac{a^2 L^*}{4A} C_\delta^2 (a^4 C_\delta^4 - 1)\|\Delta\kappa_n\|_{\varrho-2\delta}\ ,\ \forall\theta \in \mathbb{T}_{\varrho-3\delta}.$$

In particular, $\forall\theta \in \mathbb{T}$,

$$|\Delta c_n(\theta)| \leq\ \alpha_n \frac{a^2 L^*}{4A} C_{\frac{1}{3}\varrho}^2 (a^4 C_{\frac{1}{3}\varrho}^4 - 1)\|\Delta\kappa_n\|_{\frac{1}{3}\varrho}.$$

(viii) *Let* $\delta \in (0, \frac{1}{3}\varrho)$*. The correction of the function* η_n *is given by*

$$\Delta\eta_n(\theta) = -\eta_n(\theta)\,\Delta u_n(\theta+\omega)\int_0^1 e^{-t\Delta u_n(\theta+\omega)}dt,\ \forall\theta \in \mathbb{T}_{\varrho-3\delta}.$$

Whenever $\alpha_{n+1} = \alpha_n$ *and* (H) *holds,*

$$|\Delta\eta_n(\theta)| \le \frac{1}{\alpha_n}\frac{L^*}{4A}C_\delta^2(e^{4A}a^4C_\delta^4 - 1)\|\Delta\kappa_n\|_{\varrho-2\delta}\ ,\ \forall\theta \in \mathbb{T}_{\varrho-3\delta}\ .$$

In particular, $\forall\theta \in \mathbb{T}$*,*

$$|\Delta\eta_n(\theta)| \le \frac{1}{\alpha_n}\frac{L^*}{4A}C_{\frac{1}{3}\varrho}^2(e^{4A}a^4C_{\frac{1}{3}\varrho}^4 - 1)\|\Delta\kappa_n\|_{\frac{1}{3}\varrho}\ .$$

(ix) *For any* $\delta \in (0, \frac{1}{4}\varrho)$*,* $|\mathfrak{R}_{\lambda_{n+1}}\Delta\widetilde{\eta}_n(\theta)| \le \frac{1}{A}\log(aC_\delta)\|\Delta\widetilde{\eta}_n\|_{\varrho-3\delta}$*,* $\forall\theta\in\mathbb{T}_{\varrho-3\delta}$*. Additionally, whenever* $\alpha_{n+1} = \alpha_n$ *and* (H) *holds:*

$$\begin{aligned}|\mathfrak{R}_{\lambda_{n+1}}\Delta\widetilde{\eta}_n(\theta)| \quad &\le \quad \frac{1}{\alpha_n}\frac{L^*}{4A^2}\log(aC_\delta)\Big(C_\delta^2(e^{4A}a^4C_\delta^4 - 1)\|\Delta\kappa_n\|_{\varrho-2\delta} \\ &\quad + C_{\frac{1}{3}\varrho}^2(e^{4A}a^4C_{\frac{1}{3}\varrho}^4 - 1)\|\Delta\kappa_n\|_{\frac{1}{3}\varrho}\Big)\ ,\ \forall\theta\in\mathbb{T}_{\varrho-4\delta}.\end{aligned}$$

In particular, $\forall\theta \in \mathbb{T}$*,*

$$\begin{aligned}|\mathfrak{R}_{\lambda_{n+1}}\Delta\widetilde{\eta}_n(\theta)| \quad &\le \quad \frac{1}{\alpha_n}\frac{L^*}{4A^2}\log(aC_{\frac{1}{4}\varrho})\Big(C_{\frac{1}{4}\varrho}^2(e^{4A}a^4C_{\frac{1}{4}\varrho}^4 - 1)\|\Delta\kappa_n\|_{\frac{1}{2}\varrho} \\ &\quad + C_{\frac{1}{3}\varrho}^2(e^{4A}a^4C_{\frac{1}{3}\varrho}^4 - 1)\|\Delta\kappa_n\|_{\frac{1}{3}\varrho}\Big)\ .\end{aligned}$$

(x) *The correction of the translation parameter* τ_n *can be estimated by*

$$|\triangle\tau_n| \le \frac{1}{\sigma_D}(P_2(C_\varrho, C_{\frac{1}{2}\varrho})\|E_n\|_\varrho + Q_2|e_n(p)|),$$

where

$$P_2(s,t) \quad = \quad a^2s^2\left(1 + \frac{2(1-a)}{aA}(s^2+t^2)\log(at)\right), \tag{3.155}$$

$$Q_2 \quad = \quad \frac{1-a}{a}. \tag{3.156}$$

(xi) *The correction of the curve* κ_n *may be estimated for any* $\delta \in (0, \frac{1}{2}\varrho)$*, by*

$$\|\Delta\kappa_n(\theta)\|_{\varrho-2\delta} \le \frac{1}{\sigma_D}(P(C_\varrho, C_{\frac{1}{2}\varrho}, C_\delta)\|E_n\|_\varrho + Q(C_\varrho, C_\delta)|e_n(p)|),$$

where

$$\begin{aligned}P(s,t,u) \quad &= \quad a^2u^2(P_1^*(s,t) + P_3^*(s,t,u)) \\ &= \quad \frac{2a^4}{A}(s^2+t^2)u^2\log(at)\left(2s^4 + \left(1 + \frac{2(1-a)}{aA}(s^2+t^2)\log(at)\right)\log(au)\right), \\ Q(s,u) \quad &= \quad a^2u^2(Q_1^*(s) + Q_3^*(s,u)) \\ &= \quad a^2s^2u^2 + \frac{1-a}{aA}(s^2+u^2)\log(au).\end{aligned} \tag{3.157}$$

In particular, $|\Delta\kappa_n(\theta)| \leq \frac{1}{\sigma_D}(P^*(C_\varrho, C_{\frac{1}{2}\varrho})\|E_n\|_\varrho + Q^*(C_\varrho, C_{\frac{1}{2}\varrho})|e_n(p)|)$, $\forall\theta\in\mathbb{T}$, *where*

$$\begin{aligned} P^*(s,t) &= P(s,t,t) \\ &= \frac{2a^4}{A}(s^2+t^2)t^2\log(at)\left(2s^4+\left(1+\frac{2(1-a)}{aA}(s^2+t^2)\log(at)\right)\log(at)\right), \qquad (3.158) \\ Q^*(s,t) &= Q(s,t) \\ &= a^2s^2t^2+\frac{1-a}{aA}(s^2+t^2)\log(at). \qquad (3.159) \end{aligned}$$

(xii) *The correction of the cohomological operator may be estimated by:*

$$|(\Delta\mathfrak{R}_{\lambda_n})\widetilde{\eta}_n(\theta)| \leq |\Delta\lambda_n|\,\mathfrak{C}_R^*\,\gamma^{-2}\delta^{-2\nu}\,\|\widetilde{\eta}_n\|_{\varrho-\delta}\ ,\ \forall\theta\in\mathbb{T}_{\varrho-2\delta}\ ,\ \delta\in(0,\frac{1}{2}\varrho)\ ,$$

where

$$\mathfrak{C}_R^* = \mathfrak{C}_R^*(a,\nu) = \frac{1}{2}\frac{1}{(1+a)^2}\sqrt{2\zeta(4)}\frac{\sqrt{\Gamma(4\nu+1)}}{(4\pi)^{2\nu}}$$

is independent of γ.

$(xiii)$ *Let* $\delta\in(0,\frac{1}{3}\varrho)$. *The correction of the cohomological operator may be also estimated in this alternative way:*

$$|(\Delta\mathfrak{R}_{\lambda_n})\widetilde{\eta}_n(\theta)| \leq |\Delta\lambda_n|\,\frac{1}{A^2}(\log(aC_\delta))^2\|\widetilde{\eta}_n\|_{\varrho-\delta},\ \forall\theta\in\mathbb{T}_{\varrho-3\delta},$$

from which is derived the following

$$|(\Delta\mathfrak{R}_{\lambda_n})\widetilde{\eta}_n(\theta)| \leq \frac{1}{\alpha_n}\frac{L}{A^2}(C_\varrho^2+C_\delta^2)(\log(aC_\delta))^2\sup_{\theta\in\mathbb{T}}|\Delta\kappa_n(\theta)|,\ \forall\theta\in\mathbb{T}_{\varrho-3\delta}.$$

(xiv) *Whenever* $\alpha_{n+1}=\alpha_n$ *and* (H) *holds:*

$$|\Delta c_{n,0}| \leq \alpha_n\frac{a^2L^*}{4A}C_{\frac{1}{3}\varrho}^2(a^4C_{\frac{1}{3}\varrho}^4-1)\|\Delta\kappa_n\|_{\frac{1}{3}\varrho}.$$

(xv) *Whenever* $\alpha_{n+1}=\alpha_n$ *and* (H) *holds:*

$$|\Delta\eta_{n,0}| \leq \alpha_n\frac{L^*}{4A}C_{\frac{1}{3}\varrho}^2(a^4C_{\frac{1}{3}\varrho}^4-1)\|\Delta\kappa_n\|_{\frac{1}{3}\varrho}.$$

Proof.

(i) See **Lemma 3.5**, part (ii).

(ii) See **Lemma 3.5**, part (iii).

Before proving next parts, consider the following remark about complex logarithms.

Remark 3.10

Some local bounds of the complex logarithm.

(a) *If* $0<r<1$, *then* $\forall z\in\overline{\mathbb{D}}(0,r)$:

$$(1-R)|z| \leq |\log(1+z)| \leq (1+R)|z|\ , \qquad (3.160)$$

where $R=\frac{1}{2}\frac{r}{1-r}$.

(*b*) *If* $0 < r < 1$*, then* $\forall z \in \overline{\mathbb{D}}(0,r)$*:*

$$|\log(1+z)| \leq R|z| \,. \tag{3.161}$$

where $R = \frac{1}{r}\log\frac{1}{1-r}$.

Proof. The function

$$\log(1+z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} z^{n+1} \,,\; z \in \mathbb{D}(0,1)$$

is analytic in the unit open disk, and the series is absolute and uniformly convergent over compact sets. In particular, the convergence of the series is uniform over the compact closed disk $\overline{\mathbb{D}}(0,r) \subset \mathbb{D}(0,1)$.

(a) For any $z \in \overline{\mathbb{D}}(0,r) \setminus \{0\}$ we can write:

$$1 - \frac{\log(1+z)}{z} = 1 - \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} z^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1} z^n \,.$$

By the triangular inequality,

$$\begin{aligned} \left|1 - \frac{\log(1+z)}{z}\right| &\leq \sum_{n=1}^{\infty} \frac{1}{n+1}|z|^n \leq \sum_{n=1}^{\infty} \frac{1}{2}|z|^n \\ &= \frac{1}{2}\sum_{n=1}^{\infty}|z|^n = \frac{1}{2}\frac{|z|}{1-|z|} \leq \frac{1}{2}\frac{r}{1-r} = R \,. \end{aligned}$$

It follows that

$$(1-R)|z| \leq |\log(1+z)| \leq (1+R)|z| \;, \forall z \in \overline{\mathbb{D}}(0,r) \,,$$

as we stated.

(b) On the other hand, for any $z \in \overline{\mathbb{D}}(0,r) \setminus \{0\}$ we can also write:

$$\begin{aligned} \left|\frac{\log(1+z)}{z}\right| &\leq \sum_{n=0}^{\infty} \frac{1}{n+1}|z|^n \leq \sum_{n=0}^{\infty} \frac{1}{n+1} r^n \\ &= -\frac{1}{r}\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}(-r)^{n+1} \\ &= -\frac{1}{r}\log(1-r) = \frac{1}{r}\log\frac{1}{1-r} = R \,. \end{aligned}$$

It follows that

$$|\log(1+z)| \leq R\,|z| \;, \forall z \in \overline{\mathbb{D}}(0,r) \,,$$

as we wanted to prove.

As far as the upper bound of the logarithm module is concerned, the latter option (3.161) is sharper than the former (3.160), since $\frac{1}{r}\log\frac{1}{1-r} < 1 + \frac{1}{2}\frac{r}{1-r}$, $\forall r \in (0,1)$. □

(*iii*) Let $\delta \in (0, \frac{1}{2}\varrho)$. Assuming (H) holds, we may apply (3.161) to $z = \frac{\Delta m_n(\theta)}{m_n(\theta)}$ obtaining

$$\left|\log\left(1 + \frac{\Delta m_n(\theta)}{m_n(\theta)}\right)\right| \leq \frac{1}{r}\log\frac{1}{1-r}\,\frac{|\Delta m_n(\theta)|}{|m_n(\theta)|},\; \forall \theta \in \mathbb{T}_{\varrho-2\delta}. \tag{3.162}$$

(*iv*) The Lyapunov multiplier of κ_n is $\lambda_n = e^{\Lambda_n}$, with $\Lambda_n = \int_{\mathbb{T}} \log(m_n(\theta))\, d\theta$, where $m_n(\theta) = \frac{\partial f}{\partial x}(\theta, \kappa_n(\theta))$, $\theta \in \mathbb{T}_\varrho$. In like manner, the Lyapunov multiplier of κ_{n+1} is $\lambda_{n+1} = e^{\Lambda_{n+1}}$, with $\Lambda_{n+1} = \int_{\mathbb{T}} \log(m_{n+1}(\theta))\, d\theta$, where $m_{n+1}(\theta) = \frac{\partial f}{\partial x}(\theta, \kappa_{n+1}(\theta))$, $\theta \in \mathbb{T}_{\varrho-2\delta}$ (see Remark 3.3).

Thus, $\dfrac{\lambda_{n+1}}{\lambda_n} = \dfrac{e^{\Lambda_{n+1}}}{e^{\Lambda_n}} = e^{\Lambda_{n+1}-\Lambda_n} = e^{\Delta\Lambda_n}$. So, we can write:

$\Delta\lambda_n = \lambda_{n+1} - \lambda_n = \lambda_n \left(\frac{\lambda_{n+1}}{\lambda_n} - 1\right) = \lambda_n(e^{\Delta\Lambda_n} - 1)$.

Our claim now is that

$$|\Delta\lambda| \leq \lambda_n \left(e^{|\Delta\Lambda_n|} - 1\right). \tag{3.163}$$

Indeed, if $\Delta\lambda_n \geq 0$, equality holds. Notice that $\lambda_n > 0$.

On the other hand, if $\Delta\lambda_n < 0$,

$|\Delta\lambda_n| = -\Delta\lambda_n = -\lambda_n(e^{\Delta\Lambda_n} - 1) = -\lambda_n(e^{-|\Delta\Lambda_n|} - 1) = \lambda_n \frac{e^{|\Delta\Lambda_n|}-1}{e^{|\Delta\Lambda_n|}} \leq \lambda_n(e^{|\Delta\Lambda_n|} - 1)$ since $\frac{1}{e^{|\Delta\Lambda_n|}} < 1$.

Due to the identity[9]:

$$e^z - 1 = z \int_0^1 e^{tz}\, dt\ ,\ z \in \mathbb{C}\ , \tag{3.164}$$

with $z = \Delta\Lambda_n$, the modulus of the correction of the Lyapunov multiplier can be estimated by

$$|\Delta\lambda_n| \quad \leq \quad \lambda_n \left(e^{|\Delta\Lambda_n|} - 1\right) \leq \lambda_n |\Delta\Lambda_n| \int_0^1 e^{t|\Delta\Lambda_n|} dt\ . \tag{3.165}$$

Next, we estimate the Lyapunov exponent correction:

$$\begin{aligned} \Delta\Lambda_n = \Lambda_{n+1} - \Lambda_n &= \int_{\mathbb{T}} \log(m_{n+1}(\theta))\, d\theta - \int_{\mathbb{T}} \log(m_n(\theta))\, d\theta \\ &= \int_{\mathbb{T}} \left(\log(m_{n+1}(\theta)) - \log(m_n(\theta))\right)\, d\theta \\ &= \int_{\mathbb{T}} \log\left(\frac{m_{n+1}(\theta)}{m_n(\theta)}\right)\, d\theta = \int_{\mathbb{T}} \log\left(\frac{m_n(\theta) + \Delta m_n(\theta)}{m_n(\theta)}\right)\, d\theta \\ &= \int_{\mathbb{T}} \log\left(1 + \frac{\Delta m_n(\theta)}{m_n\theta}\right)\, d\theta. \end{aligned} \tag{3.166}$$

It follows that

$$|\Delta\Lambda_n| \quad = \quad \left|\int_{\mathbb{T}} \log\left(1 + \frac{\Delta m_n(\theta)}{m_n\theta}\right)\, d\theta\right| \leq \int_{\mathbb{T}} \left|\log\left(1 + \frac{\Delta m_n(\theta)}{m_n\theta}\right)\right|\, d\theta\ . \tag{3.167}$$

9 $e^z - 1 = \left[e^{tz}\right]_{t=0}^{t=1} = \int_0^1 \frac{de^{tz}}{dt}\, dt = \int_0^1 ze^{tz}\, dt = z\int_0^1 e^{tz}\, dt.$

Now, on the one hand, from (3.162) and (3.59) we obtain

$$\begin{aligned}
|\Delta\Lambda_n| &= \int_{\mathbb{T}} \left|\log\left(1+\frac{\Delta m_n(\theta)}{m_n\theta}\right)\right| \, d\theta \leq \int_{\mathbb{T}} \frac{1}{r}\log\frac{1}{1-r}\frac{|\Delta m_n(\theta)|}{|m_n(\theta)|} d\theta \\
&= \frac{1}{r}\log\frac{1}{1-r}\int_{\mathbb{T}}\frac{|\Delta m_n(\theta)|}{|m_n(\theta)|}d\theta \leq \frac{1}{r}\log\frac{1}{1-r}\int_{\mathbb{T}}\frac{|\Delta m_n(\theta)|}{K_1^*}d\theta \\
&= \frac{1}{K_1^*}\frac{1}{r}\log\frac{1}{1-r}\int_{\mathbb{T}}|\Delta m_n(\theta)|d\theta \leq \frac{1}{a}\frac{1}{r}\log\frac{1}{1-r}\int_{\mathbb{T}}|\Delta m_n(\theta)|d\theta \\
&\leq \frac{1}{a}\frac{1}{r}\log\frac{1}{1-r}\int_{\mathbb{T}} K_2|\Delta\kappa_n(\theta)|d\theta \leq K_2\frac{1}{a}\frac{1}{r}\log\frac{1}{1-r}\sup_{\theta\in\mathbb{T}}|\Delta\kappa_n(\theta)| \, . \qquad (3.168)
\end{aligned}$$

On the other hand, by hypothesis (H) we can also write

$$\begin{aligned}
|\Delta\Lambda_n| &= \int_{\mathbb{T}} \left|\log\left(1+\frac{\Delta m_n(\theta)}{m_n\theta}\right)\right| \, d\theta \leq \int_{\mathbb{T}} \frac{1}{r}\log\frac{1}{1-r}\frac{|\Delta m_n(\theta)|}{|m_n(\theta)|} d\theta \\
&= \frac{1}{r}\log\frac{1}{1-r}\int_{\mathbb{T}}\frac{|\Delta m_n(\theta)|}{|m_n(\theta)|}d\theta \leq \frac{1}{r}\log\frac{1}{1-r}r = \log\frac{1}{1-r} \, , \qquad (3.169)
\end{aligned}$$

and then

$$\begin{aligned}
\int_0^1 e^{t|\Delta\Lambda_n|}dt &\leq \int_0^1 e^{t\log\frac{1}{1-r}}dt = \frac{1}{\log\frac{1}{1-r}}\left[e^{t\log\frac{1}{1-r}}\right]_{t=0}^{t=1} \\
&= \frac{1}{\log\frac{1}{1-r}}\left(e^{\log\frac{1}{1-r}}-1\right) = \frac{1}{\log\frac{1}{1-r}}\left(\frac{1}{1-r}-1\right) \\
&= \frac{1}{\log\frac{1}{1-r}}\frac{r}{1-r}. \qquad (3.170)
\end{aligned}$$

Since $0 < a \leq \lambda_n \leq \frac{1}{a}$, then joining together (3.165), (3.168), and (3.170) we can finally write:

$$\begin{aligned}
|\Delta\lambda_n| &\leq \lambda_n\,|\Delta\Lambda_n|\int_0^1 e^{t|\Delta\Lambda_n|}dt \\
&\leq \frac{1}{a}\frac{K_2}{a}\,\frac{1}{r}\log\frac{1}{1-r}\sup_{\theta\in\mathbb{T}}|\Delta\kappa_n(\theta)|\frac{1}{\log\frac{1}{1-r}}\frac{r}{1-r} \\
&= \frac{K_2}{a^2}\frac{1}{1-r}\sup_{\theta\in\mathbb{T}}|\Delta\kappa_n(\theta)| = L\sup_{\theta\in\mathbb{T}}|\Delta\kappa_n(\theta)|, \qquad (3.171)
\end{aligned}$$

with $L = \frac{K_2}{a^2}\frac{1}{1-r}$.

(*v*) Given $\delta \in (0, \frac{1}{2}\varrho)$, the correction of the function v_n can be written as

$$\begin{aligned}
\Delta v_n(\theta) &= v_{n+1}(\theta) - v_n(\theta) = \log(m_{n+1}(\theta)) - \Lambda_{n+1} - (\log(m_n(\theta)) - \Lambda_n) \\
&= \log\left(\frac{m_{n+1}(\theta)}{m_n(\theta)}\right) - (\Lambda_{n+1} - \Lambda_n) \\
&= \log\left(\frac{m_n(\theta)+\Delta m_n(\theta)}{m_n(\theta)}\right) - \left(\int_{\mathbb{T}}\log(m_{n+1}(\theta))\,d\theta - \int_{\mathbb{T}}\log(m_n(\theta))\,d\theta\right) \\
&= \log\left(1+\frac{\Delta m_n(\theta)}{m_n(\theta)}\right) - \int_{\mathbb{T}}\log\left(1+\frac{\Delta m_n(\theta)}{m_n(\theta)}\right)\,d\theta, \quad \forall\theta\in\mathbb{T}_{\varrho-2\delta}\,. \qquad (3.172)
\end{aligned}$$

Taking moduli on both sides and applying the triangular inequality, we can write:

$$\begin{aligned}|\Delta v_n(\theta)| &\leq \left|\log\left(1+\frac{\Delta m_n(\theta)}{m_n(\theta)}\right)\right| + \left|\int_{\mathbb{T}} \log\left(1+\frac{\Delta m_n(\theta)}{m_n(\theta)}\right)\, d\theta\right| \\ &\leq \left|\log\left(1+\frac{\Delta m_n(\theta)}{m_n(\theta)}\right)\right| + \int_{\mathbb{T}} \left|\log\left(1+\frac{\Delta m_n(\theta)}{m_n(\theta)}\right)\right| \, d\theta \\ &\leq 2 \sup_{\theta\in\mathbb{T}_{\varrho-2\delta}} \left|\log\left(1+\frac{\Delta m_n(\theta)}{m_n(\theta)}\right)\right| \ . \end{aligned} \tag{3.173}$$

As to obtain an estimate of (3.173) we may apply local bounds of the complex logarithm. Whenever (H) holds, i.e. $|\Delta m_n(\theta)| \leq r|m_n(\theta)|$, as a consequence of (3.173), and by means of (3.162), we have:

$$\begin{aligned}|\Delta v_n(\theta)| &\leq 2 \sup_{\theta\in\mathbb{T}_{\varrho-2\delta}} \frac{1}{r}\log\frac{1}{1-r}\left|\frac{\Delta m_n(\theta)}{m_n(\theta)}\right| = \frac{2}{r}\log\frac{1}{1-r} \sup_{\theta\in\mathbb{T}_{\varrho-2\delta}} \frac{|\Delta m_n(\theta)|}{|m_n(\theta)|} \\ &\leq \frac{2}{r}\log\frac{1}{1-r}\frac{1}{K_1^*} \sup_{\theta\in\mathbb{T}_{\varrho-2\delta}} |\Delta m_n(\theta)| \leq \frac{2}{a}\frac{1}{r}\log\frac{1}{1-r} \sup_{\theta\in\mathbb{T}_{\varrho-2\delta}} K_2|\Delta\kappa_n(\theta)| \\ &= K_2\frac{2}{a}\frac{1}{r}\log\frac{1}{1-r} \sup_{\theta\in\mathbb{T}_{\varrho-2\delta}} |\Delta\kappa_n(\theta)| = K_2\frac{2}{a}\frac{1}{r}\log\frac{1}{1-r}\|\Delta\kappa_n\|_{\varrho-2\delta} \\ &= L^*\|\Delta\kappa_n\|_{\varrho-2\delta} \ , \ \forall\theta\in\mathbb{T}_{\varrho-2\delta} \ , \ \delta\in(0,\frac{1}{2}\varrho) \ , \end{aligned} \tag{3.174}$$

where $L^* = K_2\frac{2}{a}\frac{1}{r}\log\frac{1}{1-r}$.

On the other hand, restricting θ to the real torus $\mathbb{T}$ in (3.172), we can obtain with similar arguments:

$$|\Delta v_n(\theta)| \leq 2\sup_{\theta\in\mathbb{T}}\left|\log\left(1+\frac{\Delta m_n(\theta)}{m_n(\theta)}\right)\right| \leq L^* \sup_{\theta\in\mathbb{T}} |\Delta\kappa_n(\theta)| \ , \forall\theta\in\mathbb{T}.$$

(vi) $u_n(\theta) = \mathfrak{R}_1 v_n(\theta) + \log(\alpha_n)$. Thus,

$$\begin{aligned}\Delta u_n(\theta) &= u_{n+1}(\theta) - u_n(\theta) = \log(\alpha_{n+1}) + \mathfrak{R}_1 v_{n+1}(\theta) - (\log(\alpha_n) + \mathfrak{R}_1 v_n(\theta)) \\ &= \log\frac{\alpha_{n+1}}{\alpha_n} + \mathfrak{R}_1(v_{n+1}(\theta) - v_n(\theta)) = \log\frac{\alpha_{n+1}}{\alpha_n} + \mathfrak{R}_1\Delta v_n(\theta) \ . \end{aligned} \tag{3.175}$$

Whenever $\alpha_{n+1} = \alpha_n$ we have

$$\Delta u_n(\theta) = \mathfrak{R}_1\Delta v_n(\theta) \ , \ \forall\theta\in\mathbb{T}_{\varrho-3\delta} \ . \tag{3.176}$$

Additionally, in such a case, we can apply **Corollary 1.28** to Δv_n (with $m = 2$), thus obtaining the following estimates:

$$|\Delta u_n(\theta)| = |\mathfrak{R}_1\Delta v_n(\theta)| \leq \frac{1}{A}\log(aC_\delta)\|\Delta v_n\|_{\varrho-2\delta}, \quad \forall\theta\in\mathbb{T}_{\varrho-3\delta}. \tag{3.177}$$

In particular,

$$|\Delta u_n(\theta)| \leq \frac{L^*}{A}\log(aC_{\frac{1}{3}\varrho})\|\Delta\kappa_n\|_{\frac{1}{3}\varrho} \leq \frac{3^\nu L^*}{A}\log(aC_\varrho)\|\Delta\kappa_n\|_{\frac{1}{3}\varrho}, \quad \forall\theta\in\mathbb{T}. \tag{3.178}$$

(vii) Let $\delta\in(0,\frac{1}{3}\varrho)$.

The correction of the Floquet transformation can be written as

$$
\begin{aligned}
\Delta c_n(\theta) &= c_{n+1}(\theta) - c_n(\theta) = c_n(\theta)\left(\frac{c_{n+1}(\theta)}{c_n(\theta)} - 1\right) \\
&= c_n(\theta)\left(\frac{e^{u_{n+1}(\theta)}}{e^{u_n(\theta)}} - 1\right) = c_n(\theta)\left(e^{u_{n+1}(\theta)-u_n(\theta)} - 1\right) \\
&= c_n(\theta)\left(e^{\Delta u_n(\theta)} - 1\right) = c_n(\theta)\Delta u_n(\theta)\int_0^1 e^{t\Delta u_n(\theta)}\, dt \ . \qquad (3.179)
\end{aligned}
$$

The last step is due to the identity[10]:

$$
e^z - 1 = z\int_0^1 e^{tz}\, dt \ , \ z \in \mathbb{C} \ , \qquad (3.180)
$$

with $z = \Delta u_n(\theta)$.

By (3.176), whenever $\alpha_{n+1} = \alpha_n$, $\Delta u_n(\theta) = \mathfrak{R}_1 \Delta v_n(\theta)$, $\forall \theta \in \mathbb{T}_{\varrho-3\delta}$. Thus, in such a case

$$
\Delta c_n(\theta) = c_n(\theta)\mathfrak{R}_1\Delta v_n(\theta)\int_0^1 e^{t\mathfrak{R}_1\Delta v_n(\theta)}\, dt, \quad \forall \theta \in \mathbb{T}_{\varrho-3\delta}. \qquad (3.181)
$$

Notice that, applying **Lemma 3.5**, part (iv), we may narrow down the norm of Δv_n: $\|\Delta v_n\|_{\varrho-2\delta} = \|\ v_{n+1} - v_n\|_{\varrho-2\delta} \leq \|v_{n+1}\|_{\varrho-2\delta} + \|v_n\|_{\varrho-2\delta} \leq 2A + 2A = 4A.$

Thus, taking moduli on both sides of (3.181), and taking in account (3.177), we can write:

$$
\begin{aligned}
|\Delta c_n(\theta)| &= |c_n(\theta)||\mathfrak{R}_1\Delta v_n(\theta)|\left|\int_0^1 e^{t\mathfrak{R}_1\Delta v_n(\theta)}\, dt\right| \\
&\leq |c_n(\theta)||\mathfrak{R}_1\Delta v_n(\theta)|\int_0^1 \left|e^{t\mathfrak{R}_1\Delta v_n(\theta)}\right| dt \\
&\leq |c_n(\theta)||\mathfrak{R}_1\Delta v_n(\theta)|\int_0^1 e^{t|\mathfrak{R}_1\Delta v_n(\theta)|}\, dt \\
&\leq |c_n(\theta)|\frac{1}{A}\log(aC_\delta)\|\Delta v_n\|_{\varrho-2\delta}\int_0^1 e^{t\frac{1}{A}\log(aC_\delta)\|\Delta v_n\|_{\varrho-2\delta}}\, dt \\
&\leq |c_n(\theta)|\frac{1}{A}\log(aC_\delta)\|\Delta v_n\|_{\varrho-2\delta}\int_0^1 e^{t\frac{1}{A}\log(aC_\delta)4A}\, dt \\
&= |c_n(\theta)|\|\Delta v_n\|_{\varrho-2\delta}\frac{1}{4A}\int_0^1 4\log(aC_\delta)e^{4t\log(aC_\delta)}\, dt \\
&= |c_n(\theta)|\|\Delta v_n\|_{\varrho-2\delta}\frac{1}{4A}\left[e^{4t\log(aC_\delta)}\right]_{t=0}^{t=1} \\
&= |c_n(\theta)|\|\Delta v_n\|_{\varrho-2\delta}\frac{1}{4A}\left(e^{4\log(aC_\delta)} - 1\right) \\
&= |c_n(\theta)|\|\Delta v_n\|_{\varrho-2\delta}\frac{1}{4A}\left(a^4C_\delta^4 - 1\right) \ . \qquad (3.182)
\end{aligned}
$$

Now, we can use the estimates obtained for the Floquet transformation (3.66) and, whenever

10 $e^z - 1 = \left[e^{tz}\right]_{t=0}^{t=1} = \int_0^1 \frac{de^{tz}}{dt}\, dt = \int_0^1 ze^{tz}\, dt = z\int_0^1 e^{tz}\, dt.$

(H) holds, the estimate of the correction of the function v_n (3.174).

$$\begin{aligned} |\Delta c_n(\theta)| &\leq \alpha_n a^2 C_\delta^2 L^* \|\Delta\kappa_n\|_{\varrho-2\delta} \frac{1}{4A}(a^4 C_\delta^4 - 1) \\ &= \alpha_n \frac{a^2 L^*}{4A} C_\delta^2 (a^4 C_\delta^4 - 1) \|\Delta\kappa_n\|_{\varrho-2\delta}, \quad \forall\theta \in \mathbb{T}_{\varrho-3\delta}. \end{aligned} \tag{3.183}$$

In particular, taking limits as $\delta \to \frac{1}{3}\varrho$, according with **Corollary 1.28**, we obtain,

$$\forall\theta \in \mathbb{T}, \quad |\Delta c_n(\theta)| \leq \alpha_n \frac{a^2 L^*}{4A} C_{\frac{1}{3}\varrho}^2 (a^4 C_{\frac{1}{3}\varrho}^4 - 1) \|\Delta\kappa_n\|_{\frac{1}{3}\varrho}. \tag{3.184}$$

(viii) The correction of the function η_n may be written as

$$\begin{aligned} \Delta\eta_n(\theta) &= \eta_{n+1}(\theta) - \eta_n(\theta) = \eta_n(\theta)\left(\frac{\eta_{n+1}(\theta)}{\eta_n(\theta)} - 1\right) \\ &= \eta_n(\theta)\left(\frac{\frac{1}{c_{n+1}(\theta+\omega)}}{\frac{1}{c_n(\theta+\omega)}}\right) = \eta_n(\theta)\left(\frac{c_n(\theta+\omega)}{c_{n+1}(\theta+\omega)}\right) \\ &= \eta_n(\theta)\left(\frac{e^{u_n(\theta+\omega)}}{e^{u_{n+1}(\theta+\omega)}} - 1\right) = \eta_n(\theta)\left(e^{-(u_{n+1}(\theta+\omega)-u_n(\theta+\omega))} - 1\right) \\ &= \eta_n(\theta)\left(e^{-\Delta u_n(\theta+\omega)} - 1\right) = -\eta_n(\theta)\Delta u_n(\theta+\omega)\int_0^1 e^{-t\Delta u_n(\theta+\omega)}\,dt \,, \end{aligned} \tag{3.185}$$

where the last step is due again to the identity (3.180).

On the one hand, recall that u_n satisfies the cohomological equation[11]

$$u_n(\theta+\omega) - u_n(\theta) = v_n(\theta)\ .$$

And, in the same manner,

$$u_{n+1}(\theta+\omega) - u_{n+1}(\theta) = v_{n+1}(\theta)\ .$$

Thus,

$$\begin{aligned} \Delta u_n(\theta+\omega) &= u_{n+1}(\theta+\omega) - u_n(\theta+\omega) = (u_{n+1}(\theta) + v_{n+1}(\theta)) - (u_n(\theta) + v_n(\theta)) \\ &= (u_{n+1}(\theta) - u_n(\theta)) + (v_{n+1}(\theta) - v_n(\theta)) \\ &= \Delta u_n(\theta) + \Delta v_n(\theta)\ . \end{aligned}$$

On the other hand, whenever $\alpha_{n+1} = \alpha_n$,

$$\Delta u_n(\theta) = \mathfrak{R}_1 \Delta v_n(\theta)\ ,\ \forall\theta \in \mathbb{T}_{\varrho-3\delta} \tag{3.186}$$

so, $\Delta\eta_n(\theta+\omega) = \mathfrak{R}_1\Delta v_n(\theta) + \Delta v_n(\theta)$ and

$$\begin{aligned} |\Delta\eta(\theta+\omega)| &\leq |\mathfrak{R}_1 \Delta v_n(\theta)| + |\Delta v_n(\theta)| \\ &\leq \frac{1}{A}\log(aC_\delta)\|\Delta v_n\|_{\varrho-2\delta} + \|\Delta v_n\|_{\varrho-2\delta} \\ &= (1 + \frac{1}{A}\log(aC_\delta))\|\Delta v_n\|_{\varrho-2\delta}\ . \end{aligned} \tag{3.187}$$

---

[11] $u_n(\theta+\omega) - u_n(\theta) = \log c_n(\theta+\omega) - \log c_n(\theta) = \log \frac{c_n(\theta+\omega)}{c_n(\theta)} = \log \frac{m_n(\theta)c_n(\theta)/\lambda_n}{c_n(\theta)} = \log \frac{m_n(\theta)}{\lambda_n} = \log m_n(\theta) - \log\lambda_n = \log m_n(\theta) - \Lambda_n = v_n(\theta)$. See more details in **Corollary 2.8**.

From (3.185) and (3.187) we obtain:

$$\begin{aligned} |\Delta\eta_n(\theta)| &= |\eta_n(\theta)|\,|\Delta u_n(\theta+\omega)| \left|\int_0^1 e^{-t\Delta u_n(\theta+\omega)}dt\right| \\ &\le |\eta_n(\theta)|\,|\Delta u_n(\theta+\omega)| \int_0^1 \left|e^{-t\Delta u_n(\theta+\omega)}\right| dt \\ &\le |\eta_n(\theta)|(1+\frac{1}{A}\log(aC_\delta))\|\Delta v_n\|_{\varrho-2\delta}\int_0^1 e^{\mathrm{Re}(-t\Delta u_n(\theta+\omega))}dt\,. \end{aligned} \tag{3.188}$$

The integral can be estimated as follows:

$$\begin{aligned} \int_0^1 e^{\mathrm{Re}(-t\Delta u_n(\theta+\omega))}dt &\le \int_0^1 e^{|-t\Delta u_n(\theta+\omega)|}dt = \int_0^1 e^{t|\Delta u_n(\theta+\omega)|}dt \\ &\le \int_0^1 e^{t(1+\frac{1}{A}\log(aC_\delta))\|\Delta v_n\|_{\varrho-2\delta}}dt \\ &\le \int_0^1 e^{t(1+\frac{1}{A}\log(aC_\delta))4A}dt = \int_0^1 e^{4t(A+\log(aC_\delta))}dt\,, \end{aligned} \tag{3.189}$$

where we have used the fact that, by (3.63),
$\|\Delta v_n\|_{\varrho-2\delta} = \|v_{n+1}-v_n\|_{\varrho-2\delta} \le \|v_{n+1}\|_{\varrho-2\delta} + \|v_n\|_{\varrho-2\delta} \le 2A+2A = 4A.$
Finally, from (3.188) and (3.189) we get:

$$\begin{aligned} |\Delta\eta_n(\theta)| &\le |\eta_n(\theta)|(1+\frac{1}{A}\log(aC_\delta))\|\Delta v_n\|_{\varrho-2\delta}\int_0^1 e^{\mathrm{Re}(-t\Delta u_n(\theta+\omega))}dt \\ &\le |\eta_n(\theta)|(1+\frac{1}{A}\log(aC_\delta))\|\Delta v_n\|_{\varrho-2\delta}\int_0^1 e^{4t(A+\log(aC_\delta))}dt \\ &= |\eta_n(\theta)|\|\Delta v_n\|_{\varrho-2\delta}\frac{1}{4A}\int_0^1 (4(A+\log(aC_\delta)))e^{4t(A+\log(aC_\delta))}dt \\ &= |\eta_n(\theta)|\|\Delta v_n\|_{\varrho-2\delta}\frac{1}{4A}\left[e^{4t(A+\log(aC_\delta))}-1\right]_{t=0}^{t=1} \\ &= |\eta_n(\theta)|\|\Delta v_n\|_{\varrho-2\delta}\frac{1}{4A}\left(e^{4(A+\log(aC_\delta))}-1\right) \\ &= |\eta_n(\theta)|\|\Delta v_n\|_{\varrho-2\delta}\frac{1}{4A}\left(e^{4A}a^4C_\delta^4-1\right)\,. \end{aligned} \tag{3.190}$$

Now, it only remains to consider the estimates obtained previously for η_n and Δv_n in each of the respective cases.

Namely, from (3.174) we know that whenever (H) holds,

$$|\Delta v_n(\theta)| \le L^*\|\Delta\kappa_n\|_{\varrho-2\delta}\,,\ \forall\theta\in T_{\varrho-2\delta}$$

and hence,

$$|\Delta\eta_n(\theta)| \le |\eta_n(\theta)|\, L^*\, \|\Delta\kappa_n\|_{\varrho-3\delta}\frac{1}{4A}(e^{4A}a^4C_\delta^4-1). \tag{3.191}$$

Regarding the estimate of η_n, we will use the previous **Lemma 3.5**, part (vi). More specifically, from (3.75) we have

$$|\eta_n(\theta)| \le \frac{1}{\alpha_n}C_\delta^2, \quad \forall\theta\in\mathbb{T},$$

and consequently

$$|\Delta\eta_n(\theta)| \le \frac{1}{\alpha_n}\frac{L^*}{4A}C_\delta^2(e^{4A}a^4C_\delta^4-1)\|\Delta\kappa_n\|_{\varrho-2\delta}\,,\ \forall\theta\in\mathbb{T}_{\varrho-3\delta}\,. \tag{3.192}$$

In particular, taking limits as $\delta \to \frac{1}{3}\varrho$, we obtain $\forall\theta \in \mathbb{T}$,

$$|\Delta\eta_n(\theta)| \leq \frac{1}{\alpha_n}\frac{L^*}{4A}C^2_{\frac{1}{3}\varrho}(e^{4A}a^4C^4_{\frac{1}{3}\varrho} - 1)\|\kappa_n\|_{\frac{1}{3}\varrho} \, . \tag{3.193}$$

(ix) $\Delta\widetilde{\eta}_n$ is defined in $\mathbb{T}_{\varrho-3\delta}$, with $\delta \in (0, \frac{1}{3}\varrho)$. Therefore, by **Theorem 1.20**, $\mathfrak{R}_{\lambda_{n+1}}\Delta\widetilde{\eta}_n$ is defined and analytic in $\mathbb{T}_{\varrho-4\delta}$ for any $\delta \in (0, \frac{1}{4}\varrho)$, and the following estimate holds:

$$|\mathfrak{R}_{\lambda_{n+1}}\Delta\widetilde{\eta}_n(\theta)| \leq \frac{1}{A}\log(aC_\delta)\|\Delta\widetilde{\eta}_n\|_{\varrho-3\delta},\ \forall\theta \in \mathbb{T}_{\varrho-4\delta}\ ,\ \delta \in (0, \frac{1}{4}\varrho)\ . \tag{3.194}$$

On the other hand, $\|\Delta\widetilde{\eta}_n\|_{\varrho-3\delta} \leq \|\eta_n\|_{\varrho-3\delta} + \sup_{\theta\in\mathbb{T}}|\Delta\eta_n(\theta)|$.

Indeed, $\Delta\widetilde{\eta}_n(\theta) = \widetilde{\eta}_{n+1}(\theta) - \widetilde{\eta}_n(\theta) = (\eta_{n+1}(\theta) - \eta_{n+1,0}) - (\eta_n(\theta) - \eta_{n,0})$
$= (\eta_{n+1}(\theta) - \eta_n(\theta)) - (\eta_{n+1,0} - \eta_{n,0}) = \Delta\eta_n(\theta) - \Delta\eta_{n,0}$.

Thus, $|\Delta\widetilde{\eta}_n(\theta)| \leq |\Delta\eta_n(\theta)| + |\Delta\eta_{n,0}| = |\Delta\eta_n(\theta)| + \left|\int_{\mathbb{T}}\Delta\eta_n(\theta)\,d\theta\right| \leq |\Delta\eta_n(\theta)| + \int_{\mathbb{T}}|\Delta\eta_n(\theta)|\,d\theta$
$\leq |\Delta\eta_n(\theta)| + \sup_{\theta\in\mathbb{T}}|\Delta\eta_n(\theta)|$.

From (3.192) and (3.193) we have, whenever $\alpha_{n+1} = \alpha_n$ and (H) holds,

$$\begin{aligned}
|\Delta\widetilde{\eta}_n(\theta)| &= \frac{1}{\alpha_n}\frac{L^*}{4A}C^2_\delta(e^{4A}a^4C^4_\delta - 1)\|\Delta\kappa_n\|_{\varrho-2\delta} \\
&+ \frac{1}{\alpha_n}\frac{L^*}{4A}C^2_{\frac{1}{3}\varrho}(e^{4A}a^4C^4_{\frac{1}{3}\varrho} - 1)\|\kappa_n\|_{\frac{1}{3}\varrho} \\
&= \frac{1}{\alpha_n}\frac{L^*}{4A}\Big(C^2_\delta(e^{4A}a^4C^4_\delta - 1)\|\Delta\kappa_n\|_{\varrho-2\delta} \\
&+ C^2_{\frac{1}{3}\varrho}(e^{4A}a^4C^4_{\frac{1}{3}\varrho} - 1)\|\kappa_n\|_{\frac{1}{3}\varrho}\Big)
\end{aligned} \tag{3.195}$$

The statements of this part follow straightforward from (3.194) and (3.195).

$$\begin{aligned}
|\mathfrak{R}_{\lambda_{n+1}}\Delta\widetilde{\eta}_n(\theta)| \leq\ & \frac{1}{\alpha_n}\frac{L^*}{4A^2}\log(aC_\delta)\Big(C^2_\delta(e^{4A}a^4C^4_\delta - 1)\|\kappa_n\|_{\varrho-2\delta} \\
& + C^2_{\frac{1}{3}\varrho}(e^{4A}a^4C^4_{\frac{1}{3}\varrho} - 1)\|\kappa_n\|_{\frac{1}{3}\varrho}\Big)\ ,\ \forall\theta \in \mathbb{T}_{\varrho-4\delta}.
\end{aligned} \tag{3.196}$$

In particular, $\forall\theta \in \mathbb{T}$,

$$\begin{aligned}
|\mathfrak{R}_{\lambda_{n+1}}\Delta\widetilde{\eta}_n(\theta)| \leq\ & \frac{1}{\alpha_n}\frac{L^*}{4A^2}\log(aC_{\frac{1}{4}\varrho})\Big(C^2_{\frac{1}{4}\varrho}(e^{4A}a^4C_{\frac{1}{4}\varrho})^4 - 1)\|\kappa_n\|_{\frac{1}{2}\varrho} \\
& + C^2_{\frac{1}{3}\varrho}(e^{4A}a^4C^4_{\frac{1}{3}\varrho} - 1)\|\kappa_n\|_{\frac{1}{3}\varrho}\Big)\,.
\end{aligned} \tag{3.197}$$

Remark 3.11

Since $C_\varrho < C_{\frac{1}{2}\varrho} < C_{\frac{1}{3}\varrho} < C_{\frac{1}{4}\varrho}$ and $\|\Delta\kappa_n\|_{\frac{1}{3}\varrho} \leq \|\Delta\kappa_n\|_{\frac{1}{2}\varrho}$, we could also have written

$$|\mathfrak{R}_{\lambda_{n+1}}\Delta\widetilde{\eta}_n(\theta)| \leq \frac{1}{\alpha_n}\frac{L^*}{2A^2}\log(aC_{\frac{1}{4}\varrho})C^2_{\frac{1}{3}\varrho}(e^{4A}a^4C^4_{\frac{1}{4}\varrho} - 1)\|\kappa_n\|_{\frac{1}{2}\varrho},\quad \forall\theta \in \mathbb{T}. \tag{3.198}$$

(x) See **Lemma 3.5**, part $(xvii)$.

(xi) See **Lemma 3.5**, part (xx).

(xii) On the one hand, $<\widetilde{\eta}_n>=0$, $\widetilde{\eta}_n \in \mathcal{A}_{\varrho-\delta,0}$.

Moreover, $(\Delta\mathfrak{R}_{\lambda_n})\widetilde{\eta}_n(\theta) = \mathfrak{R}_{\lambda_{n+1}}\widetilde{\eta}_n(\theta) - \mathfrak{R}_{\lambda_n}\widetilde{\eta}_n(\theta),\ \forall\theta \in \mathbb{T}_{\varrho-2\delta}$.

On the other hand, by **Lemma 3.5**, $\lambda_n, \lambda_{n+1} \in [a, \frac{1}{a}]$, so **Proposition 1.26** part (c) is applicable and we have:

$$|\Delta\mathfrak{R}_{\lambda_n}\widetilde{\eta}_n(\theta)| \leq \mathfrak{C}_R^*\,\gamma^{-2}\delta^{-2\nu}\,|\Delta\lambda_n|\,\|\widetilde{\eta}_n\|_{\varrho-\delta}\ ,\ \forall\theta \in \mathbb{T}_{\varrho-2\delta}\ ,\ \delta \in (0, \frac{1}{2}\varrho)\ . \tag{3.199}$$

$(xiii)$ Let $\delta \in (0, \frac{1}{3}\varrho)$.

By **Lemma 3.5**, part (i), $\lambda_n, \lambda_{n+1} \in [a, \frac{1}{a}]$ and we can apply **Proposition 1.26**, parts (a) and (b). So, we have on one side:

$$\Delta\mathfrak{R}_{\lambda_n}\widetilde{\eta}_n(\theta) = \Delta\lambda_n\,\mathfrak{R}_{\lambda_{n+1}}\mathfrak{R}_{\lambda_n}\widetilde{\eta}_n(\theta)\ ,\forall\theta \in \mathbb{T}_{\varrho-3\delta}. \tag{3.200}$$

On the other side,

$$\begin{aligned} |\Delta\mathfrak{R}_{\lambda_n}\widetilde{\eta}_n(\theta)| &\leq |\Delta\lambda_n|\mathfrak{C}_R^2\gamma^{-2}\delta^{-2\nu}\|\widetilde{\eta}_n\|_{\varrho-\delta} \\ &= |\Delta\lambda_n|\frac{1}{A^2}(\log(aC_\delta))^2\|\widetilde{\eta}_n\|_{\varrho-\delta}\ ,\forall\theta \in \mathbb{T}_{\varrho-3\delta}. \end{aligned} \tag{3.201}$$

Moreover, we can use the estimate obtained for $\Delta\lambda_n$ in (3.171) and the corresponding for $\widetilde{\eta}_n$ in (3.84). Thus, we finally have:

$$|(\Delta\mathfrak{R}_{\lambda_n})\widetilde{\eta}_n(\theta)| \leq \frac{1}{\alpha_n}\frac{L}{A^2}(C_\varrho^2 + C_\delta^2)(\log(aC_\delta))^2 \sup_{\theta\in\mathbb{T}}|\Delta\kappa_n(\theta)|,\ \forall\theta \in \mathbb{T}_{\varrho-3\delta}. \tag{3.202}$$

(xiv) First of all, notice that the correction of the Floquet transformation average can be written as

$$\begin{aligned} \Delta c_{n,0} &= c_{n+1,0} - c_{n,0} = \int_{\mathbb{T}} c_{n+1}(\theta)d\theta - \int_{\mathbb{T}} c_n(\theta)d\theta \\ &= \int_{\mathbb{T}} (c_{n+1}(\theta) - c_n(\theta))d\theta = \int_{\mathbb{T}} \Delta c_n(\theta)d\theta\ . \end{aligned}$$

Thus,

$$|\Delta c_{n,0}| \leq \int_{\mathbb{T}} |\Delta c_n(\theta)|d\theta \leq \sup_{\theta\in\mathbb{T}}|\Delta c_n(\theta)|. \tag{3.203}$$

According to (3.184) we obtain

$$|\Delta c_{n,0}| \leq \alpha_n\frac{a^2L^*}{4A}C_{\frac{1}{3}\varrho}^2(a^4C_{\frac{1}{3}\varrho}^4 - 1)\|\Delta\kappa_n\|_{\frac{1}{3}\varrho} \tag{3.204}$$

whenever $\alpha_{n+1} = \alpha_n$ and (H) holds, so the statement of this part is proved.

(xv) In like manner as the part before, notice that the correction of the function η_n can be written as

$$\begin{aligned} \Delta\eta_{n,0} &= \eta_{n+1,0} - \eta_{n,0} = \int_{\mathbb{T}} \eta_{n+1}(\theta)d\theta - \int_{\mathbb{T}} \eta_n(\theta)d\theta \\ &= \int_{\mathbb{T}} (\eta_{n+1}(\theta) - \eta_n(\theta))d\theta = \int_{\mathbb{T}} \Delta\eta_n(\theta)d\theta\ . \end{aligned}$$

Thus,

$$|\eta_{n,0}| \leq \int_{\mathbb{T}} |\eta_n(\theta)|d\theta \leq \sup_{\theta\in\mathbb{T}}|\eta_n(\theta)|. \tag{3.205}$$

According to (3.193) we obtain

$$|\eta_{n,0}| \leq \alpha_n \frac{L^*}{4A} C^2_{\frac{1}{3}\varrho}(a^4 C^4_{\frac{1}{3}\varrho} - 1)\|\Delta\kappa_n\|_{\frac{1}{3}\varrho} \tag{3.206}$$

whenever $\alpha_{n+1} = \alpha_n$ and (H) holds.

□

Remark 3.12

To state the KAM theorem, in a posteriori format, we need to control the size of the deltas at every step of the process in such a way that the non–degeneracy condition holds along the whole process. Moreover, we need to impose, in addition to the hypotheses of the previous lemma, a new one that guarantees the convergence based on the error estimate that we have found here.

The following proposition allows to find sufficient conditions for the constructability.

Proposition 3.13 Determinant correction estimate

Under the same conditions as in ***Lemma 3.5*** *and* ***Lemma 3.9*** *there is a real function* $G : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *such that:*

$$|\Delta \det(\Omega_n)| \leq G(C_{\frac{1}{4}\varrho})\|\Delta\kappa_n\|_{\frac{1}{2}\varrho}. \tag{3.207}$$

Proof.

$$\begin{aligned}
\Delta \det(\Omega_n) &= \det(\Omega_{n+1}) - \det(\Omega_n) \\
&= < (1-\lambda_{n+1})\widetilde{c}_{n+1}\mathfrak{R}_{\lambda_{n+1}}\widetilde{\eta}_{n+1} + c_{n+1,0}\eta_{n+1,0} - ((1-\lambda_n)\widetilde{c}_n\mathfrak{R}_{\lambda_n}\widetilde{\eta}_n + c_{n,0}\eta_{n,0}) > \\
&= < (1-\lambda_n)\widetilde{c}_n\Delta\mathfrak{R}_{\lambda_n}\widetilde{\eta}_n + (1-\lambda_n)\widetilde{c}_n\mathfrak{R}_{\lambda_{n+1}}\Delta\widetilde{\eta}_n + \\
&+ (1-\lambda_{n+1})\Delta\widetilde{c}_n\mathfrak{R}_{\lambda_{n+1}}\widetilde{\eta}_{n+1} + c_{n,0}\Delta\eta_{n,0} + \Delta c_{n,0}\eta_{n,0} + \Delta c_{n,0}\Delta\eta_{n,0} > .
\end{aligned}$$

Thus,

$$\begin{aligned}
|\Delta \det(\Omega_n)| &\leq |1-\lambda_n| \int_{\mathbb{T}} |\widetilde{c}_n(\theta)|\, |\Delta\mathfrak{R}_{\lambda_n}\widetilde{\eta}_n(\theta)| d\theta \\
&+ |1-\lambda_n| \int_{\mathbb{T}} |\widetilde{c}_n(\theta)|\, |\mathfrak{R}_{\lambda_{n+1}}\Delta\widetilde{\eta}_n(\theta)| d\theta \\
&+ |1-\lambda_{n+1}| \int_{\mathbb{T}} |\Delta\widetilde{c}_n(\theta)|\, |\mathfrak{R}_{\lambda_{n+1}}\widetilde{\eta}_{n+1}(\theta)| d\theta \\
&+ |c_{n,0}\Delta\eta_{n,0} + \Delta c_{n,0}\eta_{n,0} + \Delta c_{n,0}\Delta\eta_{n,0}|.
\end{aligned}$$

□

3.6 The iterative step of the KAM procedure

Lemma 3.14 KAM step

Let $\Psi = \mathcal{R}_\omega \times f$ be the skew–product

$$\begin{array}{rccl} \Psi: & \mathbb{T}_\varrho \times \mathcal{U} & \longrightarrow & \mathbb{T}_\varrho \times \mathbb{C} \\ & (\theta, z) & \longmapsto & \Psi(\theta, z) = (\theta + \omega, f(\theta, z)) \,, \end{array} \tag{3.208}$$

where $\omega \in \mathcal{DC}(\gamma, \nu)$ is Diophantine and $0 < \varrho < \frac{1}{2}$.
Assume that f is a real analytic function on the spatial component satisfying the following global conditions:

(a) $\exists K_1, K_1^* > 0, K_1^* \leq \left|\frac{\partial f}{\partial z}(\theta, z)\right| \leq K_1, \ \forall (\theta, z) \in \mathbb{T}_\varrho \times \mathcal{U}$;

(b) $\exists K_2 > 0, \left|\frac{\partial^2 f}{\partial z^2}(\theta, z)\right| \leq K_2, \ \forall (\theta, z) \in \mathbb{T}_\varrho \times \mathcal{U}$;

(c) $\exists \alpha \in (0, \pi), \left|\mathrm{Arg}(\frac{\partial f}{\partial z}(\theta, z))\right| \leq \alpha, \ \forall (\theta, z) \in \mathbb{T}_\varrho \times \mathcal{U}$.

Let $p \in \mathbb{R}$ be a given average and suppose that there exists $(\kappa_n(\theta), \tau_n) \in \mathcal{A}_\varrho \times \mathbb{R}$ such that:

(i) $f(\theta, \kappa_n(\theta)) \in \mathcal{U}, \ \forall \theta \in \mathbb{T}_\varrho$;

(ii) $|\det(\Omega_n)| > 0$;

(iii) $< \kappa_n >= p$.

Then, there exist $(\kappa_{n+1}(\theta), \tau_{n+1}) \in \mathcal{A}_{\varrho - 2\delta} \times \mathbb{R}$ for any $\delta \in (0, \frac{1}{2}\varrho)$ such that:

$$\left\{ \begin{array}{rcl} m_n(\theta)\Delta\kappa_n(\theta) - \Delta\kappa(\theta + \omega) + \Delta\tau_n & = & -E_n(\theta) \\ < \Delta\kappa_n > & = & 0 \,, \end{array} \right. \tag{3.209}$$

where

$$\left\{ \begin{array}{rcl} \kappa_{n+1}(\theta) & = & \kappa_n(\theta) + \Delta\kappa_n(\theta) \\ \tau_{n+1} & = & \tau_n + \Delta\tau_n \end{array} \right.$$

Here E_n denotes the error function $E_n(\theta) = f(\theta, \kappa_n(\theta)) - \kappa_n(\theta + \omega) + \tau_n$ and $m_n(\theta) = \frac{\partial f}{\partial z}(\theta, \kappa_n(\theta))$, for any $\theta \in \mathbb{T}_\varrho$.

Additionally, denoting $v_n(\theta) = \log(m_n(\theta))$ and $\widetilde{v}_n(\theta) = v_n(\theta) - \Lambda_n$, with $\Lambda_n = \displaystyle\int_{\mathbb{T}} \log(m_n(\theta))d\theta$, the Lyapunov exponent of the curve κ_n, and assuming that

(iv) $\exists r \in (0, 1)$ *such that* $\left|\frac{\Delta m_n(\theta)}{m_n(\theta)}\right| \leq r, \forall \theta \in \mathbb{T}_{\varrho - 2\delta}$,

the following estimates hold for some $m = m(\gamma, \nu) > 0$:

(I) $\|\Delta\kappa_n\|_{\varrho - 2\delta} \leq \frac{1}{|\det(\Omega_n)|} \cdot \frac{8}{a}\delta^{-2m} e^{8A\delta^{-m}} \, \|E_n\|_\varrho, \ \forall \delta \in (0, \frac{1}{2}\varrho)$,

(II) $\|E_{n+1}\|_{\varrho - 2\delta} \leq \frac{1}{2}K_2 \|\Delta\kappa_n\|^2_{\varrho - 2\delta}, \ \forall \delta \in (0, \frac{1}{2}\varrho)$,

(III) $|\Delta\tau_n| \leq \frac{1}{|\det(\Omega_n)|} \cdot \frac{2}{a}\delta^{-m} e^{4A\delta^{-m}} \, \|E_n\|_\varrho, \ \forall \delta \in (0, \frac{1}{2}\varrho)$, *and*

(IV) $|\Delta \det(\Omega_n)| \leq \frac{1}{d}\frac{10}{a^2}\delta^{-m} e^{2(2A+B)\delta^{-m}} \|\Delta\kappa_n\|_{\varrho - 2\delta}, \ \forall \delta \in (0, \frac{1}{4}\varrho)$.

Proof. Under the conditions of the statement, all the estimates found in **Lemma 3.5** and **Lemma 3.9** hold. Here we consider the particular case where $\alpha_n = 1$, that is,

$$c_{n,0} = \int_{\mathbb{T}} e^{\mathfrak{R}_1\left(\log\left(\frac{\partial f}{\partial x}(\theta,\kappa_n(\theta))\right)-\Lambda_n\right)d\theta},$$

which is feasible because $c_{n,0}$ can be freely chosen. Moreover, since we are assuming now that $<\kappa_n>= p$, then $e_n(p) = 0$, and this term disappears in those estimates where it was present before. Additionally, we will apply **Corollary 1.22** which assures that

$$\exists m = m(\gamma,\nu) \in \mathbb{N} \text{ , such that } \forall \delta \in (0,\varrho),\ \|\mathfrak{R}_\lambda \widetilde{v}\|_{\varrho-\delta} \le \delta^{-m}\|\widetilde{v}\|_\varrho \le 2\delta^{-m}\|v\|_\varrho \ ,$$

for any $v \in \mathcal{A}_\varrho$. Recall that $m = m(\gamma,\nu)$ depends only on ω.
Taking in account these facts and following the same scheme as in **Lemma 3.5** we obtain the following estimates:

(i) $a \le \lambda_n \le \frac{1}{a}$, where $a = \min\{K_1^*, K_1^{-1}, \frac{\alpha}{2\pi}\} \in (0,1)$;

(ii) $\|\Delta m_n\|_{\varrho-2\delta} \le K_2\|\Delta K_n\|_{\varrho-2\delta}$, $\delta \in (0, \frac{1}{2}\varrho)$;

(iii) $\|E_{n+1}\|_{\varrho-2\delta} \le \frac{1}{2}K_2\|\Delta K_n\|^2_{\varrho-2\delta}$, $\delta \in (0,\frac{1}{2}\varrho)$;

(iv) $\|\widetilde{v_n}\|_\varrho \le 2A$, where $A = (\max\{|\log K_1^*|, |\log K_1|\}^2 + \alpha^2)^{\frac{1}{2}}$;
$\|\mathfrak{R}_1\widetilde{v_n}\|_{\varrho-\delta} \le \delta^{-m}\|\widetilde{v_n}\|_\varrho$, $\delta \in (0,\frac{1}{2}\varrho)$;

(v) $\max\{\|c_n\|_{\varrho-\delta}, \|\frac{1}{c_n}\|_{\varrho-\delta}\} \le e^{\delta^{-m}\|\widetilde{v_n}\|_\varrho}$, $\delta \in (0,\varrho)$;

(vi) $\|\eta_n\|_{\varrho-\delta} \le e^{\delta^{-m}\|\widetilde{v_n}\|_\varrho}$, $\delta \in (0,\varrho)$;

(vii) $\|\xi_n\|_{\varrho-\delta} \le e^{\delta^{-m}\|\widetilde{v_n}\|_\varrho}\|E_n\|_\varrho$, $\delta \in (0,\varrho)$;

(viii) $\|\widetilde{\eta_n}\|_{\varrho-\delta} \le 2\|\eta_n\|_{\varrho-\delta} \le 2e^{\delta^{-m}\|\widetilde{v_n}\|_\varrho}$, $\delta \in (0,\varrho)$;

(ix) $\|\widetilde{\xi_n}\|_{\varrho-\delta} \le 2\|\xi_n\|_{\varrho-\delta} \le 2e^{\delta^{-m}\|\widetilde{v_n}\|_\varrho}\|E_n\|_\varrho$, $\delta \in (0,\varrho)$;

(x) $\|\mathfrak{R}_{\lambda_n}\widetilde{\eta_n}\|_{\varrho-2\delta} \le \delta^{-m}\|\widetilde{\eta_n}\|_{\varrho-\delta} \le 2\delta^{-m}e^{\delta^{-m}\|\widetilde{v_n}\|_\varrho}$, $\delta \in (0,\frac{1}{2}\varrho)$;

(xi) $\|\mathfrak{R}_{\lambda_n}\widetilde{\xi_n}\|_{\varrho-2\delta} \le \delta^{-m}\|\widetilde{\xi_n}\|_{\varrho-\delta} \le 2\delta^{-m}e^{\delta^{-m}\|\widetilde{v_n}\|_\varrho}\|E_n\|_\varrho$, $\delta \in (0,\frac{1}{2}\varrho)$;

(xii) $a^2 \le \frac{K_1^*}{K_1} \le c_{n,0}\eta_{n,0} \le \frac{K_1}{K_1^*}I_n \le \frac{1}{a^2}I_n$, with $1 \le I_n = \int_{\mathbb{T}} e^{\mathfrak{R}_1\widetilde{v_n}}d\theta \int_{\mathbb{T}} e^{-\mathfrak{R}_1\widetilde{v_n}}d\theta \le e^{2\delta^{-m}\|\widetilde{v_n}\|_\varrho}$, and whenever $\alpha_n = 1$, $\eta_{n,0} \le e^{\delta^{-m}\|\widetilde{v_n}\|_\varrho}$, $\delta \in (0,\varrho)$;

(xiii) $|\xi_{n,0}| \le \eta_{n,0}\|E_n\|_\varrho$;

(xiv) $| < c_n\mathfrak{R}_{\lambda_n}\widetilde{\eta_n} > | \le 2c_{n,0}\delta^{-m}e^{\delta^{-m}\|\widetilde{v_n}\|_\varrho}$, $\delta \in (0,\frac{1}{2}\varrho)$;

(xv) $| < c_n\mathfrak{R}_{\lambda_n}\widetilde{\xi_n} > | \le 2c_{n,0}\delta^{-m}e^{\delta^{-m}\|\widetilde{v_n}\|_\varrho}\|E_n\|_\varrho$, $\delta \in (0,\frac{1}{2}\varrho)$;

(xvi) $|\varsigma_{n,0}| \le \frac{1}{|\det(\Omega_n)|}4\delta^{-m}e^{2\delta^{-m}\|\widetilde{v_n}\|_\varrho}\|E_n\|_\varrho$, $\delta \in (0,\frac{1}{2}\varrho)$;

(xvii) $|\Delta\tau_n| \le \frac{1}{|\det(\Omega_n)|}\frac{2}{a}\delta^{-m}e^{2\delta^{-m}\|\widetilde{v_n}\|_\varrho}$, $\delta \in (0,\frac{1}{2}\varrho)$;

(xviii) $|\det(\Omega_n)| \le \frac{2}{a}\delta^{-m}e^{2\delta^{-m}\|\widetilde{v_n}\|_\varrho}$, $\delta \in (0,\frac{1}{2}\varrho)$;

(xix) $\|\widetilde{\varsigma_n}\|_{\varrho-2\delta} \le \frac{1}{|\det(\Omega_n)|}\frac{8}{a}\delta^{-2m}e^{3\delta^{-m}\|\widetilde{v_n}\|_\varrho}\|E_n\|_\varrho$, $\delta \in (0,\frac{1}{2}\varrho)$;

(xx) $\|\Delta\kappa_n\|_{\varrho-2\delta} \leq \frac{1}{|\det(\Omega_n)|}\frac{8}{a}\delta^{-2m}e^{3\delta^{-m}\|\widetilde{v_n}\|_\varrho}\|E_n\|_\varrho,\ \delta \in (0, \frac{1}{2}\varrho)$.

Next, we follow the same scheme of **Lemma 3.9** to obtain the correction estimates. Notice first that if $\left|\frac{\Delta m_n(\theta)}{m_n(\theta)}\right| \leq r < 1$, then $\left|\log\left(1+\frac{\Delta m_n(\theta)}{m_n(\theta)}\right)\right| \leq \frac{1}{r}\log\frac{1}{1-r}\left|\frac{\Delta m_n(\theta)}{m_n(\theta)}\right| \leq \log\frac{1}{1-r} =: B$. After the corresponding computations, we have:

(1) $|\Delta\lambda_n| \leq L\|\Delta\kappa_n\|_{\varrho-2\delta}$, with $L = \frac{K_1K_2}{K_1^*}\cdot\frac{1}{1-r}$;

(2) $|\Delta\widetilde{v_n}(\theta)| \leq 2B$ and $|\Delta\widetilde{v_n}(\theta)| \leq L_*\|\Delta\kappa_n\|_{\varrho-2\delta}$, with $L_* := \frac{K_2}{K_1^*}\frac{2}{r}\log\frac{1}{1-r} = \frac{2B}{d}$;

(3) $|\mathfrak{R}_{\lambda_n}\Delta\widetilde{v_n}(\theta)| \leq L_*\delta^{-m}\|\Delta\kappa_n\|_{\varrho-2\delta},\ \theta \in \mathbb{T}_{\varrho-3\delta},\ \delta \in (0, \frac{1}{3}\varrho)$;

(4) $|\Delta c_n(\theta)| \leq \frac{1}{d}e^{2(A+B)\delta^{-m}}\|\Delta\kappa_n\|_{\varrho-2\delta},\ \theta \in \mathbb{T}_{\varrho-3\delta},\ \delta \in (0, \frac{1}{3}\varrho)$;

(5) $|\Delta\eta_n(\theta)| \leq \|\Delta\eta_n\|_{\varrho-3\delta} \leq \frac{1}{d}e^{2(A+B)\delta^{-m}}\|\Delta\kappa_n\|_{\varrho-2\delta},\ \theta \in \mathbb{T}_{\varrho-3\delta},\ \delta \in (0, \frac{1}{3}\varrho)$;

(6) $\|\mathfrak{R}_{\lambda_{n+1}}\Delta\widetilde{\eta_n}\|_{\varrho-4\delta} \leq \|\Delta\widetilde{\eta_n}\|_{\varrho-3\delta} \leq \frac{1}{d}\delta^{-m}e^{2(A+B)\delta^{-m}}\|\Delta\kappa_n\|_{\varrho-2\delta},\ \theta \in \mathbb{T}_{\varrho-4\delta},\ \delta \in (0, \frac{1}{4}\varrho)$;

(7) $\Delta\mathfrak{R}_{\lambda_n}\widetilde{\eta_n}(\theta) = \Delta\lambda_n\mathfrak{R}_{\lambda_{n+1}}\mathfrak{R}_{\lambda_n}\widetilde{\eta_n}(\theta),\ \forall\theta \in \mathbb{T}_{\varrho-4\delta},\ \delta \in (0, \frac{1}{4}\varrho)$, and
$\|\Delta\mathfrak{R}_{\lambda_n}\widetilde{\eta_n}\|_{\varrho-4\delta} \leq |\Delta\lambda_n|\|\mathfrak{R}_{\lambda_{n+1}}\mathfrak{R}_{\lambda_n}\widetilde{\eta_n}\|_{\varrho-4\delta} \leq |\Delta\lambda_n|\delta^{-2m}\|\widetilde{\eta_n}\|_{\varrho-2\delta} \leq L\delta^{-2m}e^{2A\delta^{-m}}\|\Delta\kappa_n\|_{\varrho-2\delta}$,
$\forall\theta \in \mathbb{T}_{\varrho-4\delta},\ \delta \in (0, \frac{1}{4}\varrho)$;

(8) $|\Delta c_{n,0}| \leq \|\Delta c_n\|_{\varrho-3\delta} \leq \frac{1}{d}e^{2(A+B)\delta^{-m}}\|\Delta\kappa_n\|_{\varrho-2\delta},\ \delta \in (0, \frac{1}{3}\varrho)$;

(9) $|\Delta\eta_{n,0}| \leq \|\Delta\eta_n\|_{\varrho-3\delta} \leq \frac{1}{d}e^{2(A+B)\delta^{-m}}\|\Delta\kappa_n\|_{\varrho-2\delta},\ \delta \in (0, \frac{1}{3}\varrho)$.

Summary of constants

$$\begin{aligned}
a &= \min\{K_1^*, K_1^{-1}, \frac{\alpha}{2\pi}\} \in (0,1);\\
A &= (\max\{|\log K_1^*|, |\log K_1|\}^2 + \alpha^2)^{\frac{1}{2}};\\
B &= \log\frac{1}{1-r};\\
L &= \frac{K_1K_2}{K_1^*}\cdot\frac{1}{1-r};\\
L_* &= \frac{K_2}{K_1^*}\frac{2}{r}\log\frac{1}{1-r} = \frac{2B}{d};\\
d &= \frac{K_1^*}{K_2}r.
\end{aligned}$$

With all the estimates obtained above and after some substitutions and computations we get:

(I) $\|\Delta\kappa_n\|_{\varrho-2\delta} \leq \frac{1}{|\det(\Omega_n)|}\cdot\frac{8}{a}\delta^{-2m}e^{8A\delta^{-m}}\ \|E_n\|_\varrho,\ \forall\delta \in (0, \frac{1}{2}\varrho)$,

(II) $\|E_{n+1}\|_{\varrho-2\delta} \leq \frac{1}{2}K_2\|\Delta\kappa_n\|^2_{\varrho-2\delta},\ \forall\delta \in (0, \frac{1}{2}\varrho)$,

(III) $|\Delta\tau_n| \leq \frac{1}{|\det(\Omega_n)|}\cdot\frac{2}{a}\delta^{-m}e^{4A\delta^{-m}}\ \|E_n\|_\varrho,\ \forall\delta \in (0, \frac{1}{2}\varrho)$, and

(IV) $|\Delta\det(\Omega_n)| \leq \frac{1}{d}\frac{10}{a^2}\delta^{-m}e^{2(2A+B)\delta^{-m}}\|\Delta\kappa_n\|_{\varrho-2\delta},\ \forall\delta \in (0, \frac{1}{4}\varrho)$.

It is worth to mention that

$$\begin{array}{rcl} \Delta\det(\Omega_n) & = & \det(\Omega_{n+1}) - \det(\Omega_n) \\ & = & <(1-\lambda_{n+1})\widetilde{c}_{n+1}\mathfrak{R}_{\lambda_{n+1}}\widetilde{\eta}_{n+1} + c_{n+1,0}\eta_{n+1,0} - ((1-\lambda_n)\widetilde{c}_n\mathfrak{R}_{\lambda_n}\widetilde{\eta}_n + c_{n,0}\eta_{n,0})> \\ & = & (1-(\lambda_{n+1}\lambda_n))(\Delta\lambda_n < c_n\mathfrak{R}_{\lambda_{n+1}}\mathfrak{R}_{\lambda_n}\widetilde{\eta_n}> + < c_{n+1}\mathfrak{R}_{\lambda_{n+1}}\Delta\widetilde{\eta_n}>) \\ & + & c_{n,0}\Delta\eta_{n,0} + \Delta c_{n,0}\eta_{n+1,0}. \end{array}$$

Therefore,

$$\begin{array}{rcl} |\Delta\det(\Omega_n)| & \leq & |1-(\lambda_{n+1}\lambda_n)|(|\Delta\lambda_n|| < c_n\mathfrak{R}_{\lambda_{n+1}}\mathfrak{R}_{\lambda_n}\widetilde{\eta_n}>| + | < c_{n+1}\mathfrak{R}_{\lambda_{n+1}}\Delta\widetilde{\eta_n}>|) \\ & + & c_{n,0}|\Delta\eta_{n,0}| + |\Delta c_{n,0}|\eta_{n+1,0}. \end{array}$$

Now, we have

- $|1-(\lambda_{n+1}\lambda_n)| \leq \frac{2}{a}$;
- $|\Delta\lambda_n| \leq L\|\Delta\kappa_n\|_{\varrho-2\delta}$;
- $| < c_n\mathfrak{R}_{\lambda_{n+1}}\mathfrak{R}_{\lambda_n}\widetilde{\eta_n}>| \leq 2c_{n,0}\delta^{-2m}e^{2A\delta^{-m}} \leq 2\delta^{-2m}e^{4A\delta^{-m}}$;
- $| < c_{n+1}\mathfrak{R}_{\lambda_{n+1}}\Delta\widetilde{\eta_n}>| \leq \frac{2}{d}c_{n+1,0}\delta^{-m}e^{2(A+B)\delta^{-m}}\|\Delta\kappa_n\|_{\varrho-2\delta} \leq \frac{2}{d}\delta^{-m}e^{2(2A+B)\delta^{-m}}\|\Delta\kappa_n\|_{\varrho-2\delta}$;
- $c_{n,0}|\Delta\eta_{n,0}| + |\Delta c_{n,0}|\eta_{n+1,0} \leq \frac{2}{d}e^{2(2A+B)\delta^{-m}}\|\Delta\kappa_n\|_{\varrho-2\delta}$.

It follows that $|\Delta\det(\Omega_n)| \leq \frac{1}{d}\frac{10}{a^2}\delta^{-m}e^{2(2A+B)\delta^{-m}}\|\Delta\kappa_n\|_{\varrho-2\delta}$, $\forall\delta\in(0,\frac{1}{4}\varrho)$.

$\square$

3.7 The KAM theorem

Thanks to **Lemma 3.14** we are in a position now to state a new version of a KAM theorem, which determines sufficient conditions for the KAM procedure convergence and the analyticity of the invariant translated curves found[12].

Theorem 3.15 KAM

Let $\Psi = \mathcal{R}_\omega \times f$ *be a quasi–periodic skew-product*

$$\begin{array}{rccc} \Psi: & \mathbb{T}\times\mathbb{R} & \longrightarrow & \mathbb{T}\times\mathbb{R} \\ & (\theta,x) & \longmapsto & \Psi(\theta,x) = (\theta+\omega, f(\theta,x)) \end{array}$$

where the frequency $\omega \in \mathcal{DC}(\gamma,\nu)$ *is Diophantine and* $f : \mathbb{T}\times\mathbb{R} \longrightarrow \mathbb{R}$ *is a real analytic function. Assume that there is a complex extension of* f*,*

$$f : \mathbb{T}_\varrho \times \mathcal{U} \longrightarrow \mathbb{C},$$

where $\varrho > 0$ *and* $\mathcal{U} \subseteq \mathbb{C}$ *is an open connected set such that exist* $\kappa^* : \mathbb{T}_{\varrho_0} \longrightarrow \mathbb{C}$*,* $\kappa^* \in \mathcal{A}_{\varrho_0}$*, with* $0 < \varrho_0 < \varrho$*, and* $r_0 > 0$ *satisfying:*

$$\text{If } \Upsilon = \Upsilon_{\varrho_0, r_0} := \{(\theta,z) \in \mathbb{T}_{\varrho_0} \times \mathbb{C} : |z - \kappa^*(\theta)| \le r_0\}, \text{ then } \Upsilon_{\varrho_0,r_0} \subseteq \mathbb{T}_\varrho \times \mathcal{U}.$$

Assume, additionally, that f *satisfies the following global conditions:*

(a) $\exists K_1, K_1^* > 0, K_1^* \le \left|\frac{\partial f}{\partial z}(\theta,z)\right| \le K_1,\ \forall(\theta,z) \in \mathbb{T}_\varrho \times \mathcal{U}$;

(b) $\exists K_2 > 0, \left|\frac{\partial^2 f}{\partial z^2}(\theta,z)\right| \le K_2,\ \forall(\theta,z) \in \mathbb{T}_\varrho \times \mathcal{U}$;

(c) $\exists \alpha \in (0,\pi), \left|\mathrm{Arg}(\frac{\partial f}{\partial z}(\theta,z))\right| \le \alpha,\ \forall(\theta,z) \in \mathbb{T}_\varrho \times \mathcal{U}$.

Let $p \in \mathbb{R}$ *be a fixed average. Then,* $\exists\, \varepsilon = \varepsilon(K_1, K_1^*, K_2, \alpha, \gamma, \nu) > 0$*, i.e. depending only on global constants of the skew–product, such that if* $\kappa_0 \in \mathcal{A}_{\varrho_0}$ *is a an analytic curve and* $\tau_0(p) \in \mathbb{R}$ *a real number satisfying:*

(i) $f(\theta, \kappa_0(\theta)) \in \mathcal{U}$, $\forall\theta \in \mathbb{T}_{\varrho_0}$;

(ii) $< \kappa_0 >= p$;

(iii) $\det(\Omega_0) \neq 0$; *where*

$$\Omega_0 = \begin{pmatrix} 1-\lambda_0 & -<\eta_0> \\ <c_0> & <c_0 \mathfrak{R}_{\lambda_0} \widetilde{\eta}_0> \end{pmatrix} \tag{3.210}$$

[12] In [36] Jorba, Tatjer and Muñoz–Almaraz give a proof of a KAM theorem for affine skew–products with certain kind of additional symmetries.

with

$$\begin{aligned}
\lambda_0 &= e^{\Lambda_0} \text{ the Lyapunov multiplier of } \kappa_0 \,, \\
\Lambda_0 &= \int_{\mathbb{T}} \log\left(\frac{\partial f}{\partial x}(\theta, \kappa_0(\theta))\right) d\theta \text{ the Lyapunov exponent of } \kappa_0 \,, \\
c_0(\theta) &= e^{\Re_1\left(\log\left(\frac{\partial f}{\partial x}(\theta,\kappa_0(\theta))\right)-\Lambda_0\right)} \,, \theta \in \mathbb{T}_{\varrho_0} \,, \text{ and} \\
\eta_0(\theta) &= \frac{1}{c_0(\theta+\omega)} \,, \theta \in \mathbb{T}_{\varrho_0} \,,
\end{aligned}$$

and

(iv) $\|E_0\|_{\varrho_0} < D_0\, \varepsilon$, *with* $D_0 = d_0\, \mathrm{e}^{-2\beta\delta_0^{-m}\Sigma}$,

where E_0 *denotes the error function*

$$E_0(\theta) = f(\theta, \kappa_0(\theta)) - \kappa_0(\theta+\omega) + \tau_0 \,, \theta \in \mathbb{T}_{\varrho_0} \,,$$

and

$$\begin{aligned}
\delta_0 &= \tfrac{6}{\pi^2}\delta, \text{ with } \delta < \tfrac{1}{4}\varrho_0 \,, \\
\Sigma &= \left(\frac{\pi^2}{6}\right)^m \sum_{k=0}^{\infty} (k+1)^{2m} 2^{-k} \,.
\end{aligned}$$

then $\exists\, \kappa \in \mathcal{A}_{\rho_0/2}$ *and* $\tau(p) \in \mathbb{R}$ *such that*

$$\left\{ \begin{aligned} f(\theta, \kappa(\theta)) - \kappa(\theta+\omega) + \tau(p) &= 0 \\ < \kappa > &= p \end{aligned} \right. \,. \tag{3.211}$$

This means that κ *is an analytic invariant translated curve of the skew–product* Ψ *with translation number* $\tau(p)$.
In the case where the translation parameter $\tau(p)$ *is zero,* κ *is an invariant curve.*

Chapter 4

Bifurcation theory: local analysis and stability

In this chapter the framework to be considered consists of a one–parameter family of quasi–periodic skew–products $\{\psi_\mu\}_{\mu\in\mathcal{I}}$, where $\mathcal{I} \subseteq \mathbb{R}$ is an open interval of the real line, and for each $\mu \in \mathcal{I}$, $\psi_\mu = \mathcal{R}_\omega \times f_\mu$, and $f_\mu : \mathbb{T}_\varrho \times \mathbb{C} \longrightarrow \mathbb{C}$ satisfies the properties described in (2.1).
This means that, for any $\mu \in \mathcal{I}$, there is in the family a discrete dynamical system of the form:

$$\begin{array}{rccl} \psi_\mu : & \mathbb{T}_\varrho \times \mathbb{C} & \longrightarrow & \mathbb{T}_\varrho \times \mathbb{C} \\ & (\theta, z) & \longmapsto & \psi_\mu(\theta, z) = (\theta + \omega, f_\mu(\theta, z)) \end{array} .$$

If we define

$$\begin{array}{rccl} f : & \mathbb{T}_\varrho \times \mathbb{C} \times \mathcal{I} & \longrightarrow & \mathbb{C} \\ & (\theta, z; \mu) & \longmapsto & f(\theta, z; \mu) = f_\mu(\theta, z) \end{array} , \tag{4.1}$$

we can refer to the one–parameter family of skew–products by only mentioning the function f.
In what follows, we assume that this family of skew–products represented by (4.1) is depending analytically on the parameter μ, denoting this fact by saying that $f \in \mathcal{C}^\omega(\mathbb{T}_\varrho \times \mathbb{C} \times \mathcal{I}, \mathbb{C})$.

Definition 4.1 Family of invariant translated curves of a one–parameter family of skew–products

Given a one–parameter family of quasi–periodic skew–products $\{\psi_\mu\}_{\mu\in\mathcal{I}}$ of the form $\psi_\mu = \mathcal{R}_\omega \times f_\mu$, we define the family of invariant translated curves associated to $\{\psi_\mu\}_{\mu\in\mathcal{I}}$ as the collection

$$\left\{ \{(\kappa_\mu^{(p)}, \tau_\mu^{(p)})\}_{p\in\mathbb{R}} \right\}_{\mu\in\mathcal{I}},$$

where $\kappa_\mu^{(p)} \in \mathcal{A}_\varrho$ is an invariant translated curve w.r.t ψ_μ with translation number[1] $\tau_\mu^{(p)} \in \mathbb{R}$, i.e.

$$\left\{ \begin{array}{rcl} f(\theta, \kappa_\mu^{(p)}(\theta)) - \kappa_\mu^{(p)}(\theta + \omega) + \tau_\mu^{(p)} & = & 0, \\ < \kappa_\mu^{(p)} > & = & p \end{array} \right. \quad (\theta \in \mathbb{T}; \mu \in \mathcal{I},\ p \in \mathbb{R}) \tag{4.2}$$

Another way to refer to this family is to consider the following functions:

$$\begin{array}{rccl} \kappa : & \mathbb{T}_\varrho \times \mathcal{I} \times \mathbb{R} & \longrightarrow & \mathbb{C} \\ & (\theta; \mu, p) & \longmapsto & \kappa(\theta; \mu, p) = \kappa_\mu^{(p)}(\theta) \end{array}$$

---

[1] See **Definition 2.13**.

and

$$\begin{array}{rccl} \tau: & \mathcal{I}\times\mathbb{R} & \longrightarrow & \mathbb{R} \\ & (\mu,p) & \longmapsto & \tau_\mu(p)=\tau_\mu^{(p)} \end{array} .$$

Thus, we may write equations (4.2) *as*

$$\left\{ \begin{array}{rcl} f(\theta,\kappa(\theta;\mu,p);\mu)-\kappa(\theta+\omega;\mu,p)+\tau(\mu,p) & = & 0\,, \\ <\kappa(\cdot\,;\mu,p)> & = & p\,. \end{array} \right. \quad (\theta\in\mathbb{T};\mu\in\mathcal{I},\ p\in\mathbb{R}) \tag{4.3}$$

◇

In this scenario we are mostly interested in the study of qualitative geometric properties of the family of invariant curves, namely, those invariant translated curves whose translation parameter is equal to zero, i.e.

$$\tau(\mu,p)=0. \tag{4.4}$$

This is the so–called *bifurcation equation* of the one–parameter family of skew–products. The equation leads to the points of the parameter space where the local behavior of the invariant translated curves changes.
Thus, the objective is to study geometric properties of the level curve $\tau(\mu,p)=0$ or, the so–called *bifurcation diagram*,

$$\mathcal{B}_\tau := \{(\mu,p)\in\mathcal{I}\times R : \tau(\mu,p)=0\}\,.$$

4.1 Local bifurcation theory of invariant curves in 1–D quasi–periodic skew–products

On the context described above, we are going to establish a methodology to study the theory of bifurcations that concerns us. The implicit function theorem (IFT) provides the appropriate framework for this study, through sufficient conditions that allow information to be obtained from one of the parameters as a function of the other. Therefore, the starting point is made up by the equations of the invariant translated curves (4.3).

In the KAM prodedure applied to (4.3), all the objects involved are considered to be depending, in addition to their own variables, on the parameters μ and p in an open set of the complex plane, $\mathcal{I}\times\mathbb{R}\subseteq\mathcal{U}\times\mathcal{V}\subseteq\mathbb{C}\times\mathbb{C}$. Using bounds of the objects delimiting for all the values of the parameters, the whole process converges in the considered domain in the same way that has been indicated in the corresponding KAM theorem stated previously (**Theorem 2.19**). Since we have analytic functions over open domains of the complex plane, the limit is analytic in all of its variables, including the parameters.

With this objective, we start by considering that the analyticity of the functions involved in (4.3) allows us to take derivatives of any order w.r.t. both parameters. As we will see, we may obtain these derivatives of any order under a unique non–degeneracy condition. As a consequence, the conditions that we need to apply the IFT to the bifurcation equation may be related to the equations of the invariant translated curves and their correspondent derivatives. In short, this is the common link between the type of dynamics of invariant curves and the type of root of the bifurcation equation (4.4).

Thus, we start with the equations:

$$
\begin{aligned}
f(\theta,\kappa(\theta;\mu,p);\mu)-\kappa(\theta+\omega;\mu,p)+\tau(\mu,p) &= 0\,,\ \forall\theta\in\mathbb{T} \qquad (4.5)\\
<\kappa(\cdot\,;\mu,p)> &= p\,. \qquad (4.6)
\end{aligned}
$$

Taking derivatives, first, in (4.5) and (4.6) with respect to the average parameter p, we obtain:

$$
\begin{aligned}
\frac{\partial f}{\partial x}(\theta,\kappa(\theta;\mu,p);\mu)\frac{\partial\kappa}{\partial p}(\theta;\mu,p)-\frac{\partial\kappa}{\partial p}(\theta+\omega;\mu,p)+\frac{\partial\tau}{\partial p}(\mu,p) &= 0 \quad ,\ \forall\theta\in\mathbb{T} \qquad (4.7)\\
<\frac{\partial\kappa}{\partial p}(\cdot\,;\mu,p)> &= 1. \qquad (4.8)
\end{aligned}
$$

Indeed, $\frac{\partial}{\partial p} <\kappa(\cdot\,;\mu,p)> = \frac{\partial}{\partial p}\int_{\mathbb{T}}\kappa(\theta;\mu,p)d\theta = \int_{\mathbb{T}}\frac{\partial\kappa}{\partial p}(\theta;\mu,p)d\theta = <\frac{\partial\kappa}{\partial p}(\cdot\,;\mu,p)>$, where we have applied derivation under the integral sign, which is feasible because of the analyticity of the integrand.

On the other hand, if we take derivatives in (4.5) and (4.6) with respect to the bifurcation parameter μ, we obtain:

$$
\begin{aligned}
&\frac{\partial f}{\partial x}(\theta,\kappa(\theta;\mu,p);\mu)\frac{\partial\kappa}{\partial\mu}(\theta;\mu,p)-\frac{\partial\kappa}{\partial\mu}(\theta+\omega;\mu,p)\\
&\qquad +\frac{\partial f}{\partial\mu}(\theta,\kappa(\theta;\mu,p);\mu)+\frac{\partial\tau}{\partial\mu}(\mu,p) = 0 \quad ,\ \forall\theta\in\mathbb{T} \qquad (4.9)\\
&<\frac{\partial\kappa}{\partial\mu}(\cdot\,;\mu,p)> = 0. \qquad (4.10)
\end{aligned}
$$

Now, we are in a position to take derivatives again in (4.7), (4.8), (4.9), and (4.10).

This process can be generalized to obtain all the corresponding equations at once. For this purpose we will use the following notation: For $i,j=0,1,2,\ldots$

$$
\begin{aligned}
\kappa^{ij}(\theta;\mu,p) &= \frac{\partial^{i+j}\kappa}{\partial\mu^i\partial p^j}(\theta;\mu,p) \qquad (4.11)\\
\tau^{ij}(\mu,p) &= \frac{\partial^{i+j}\tau}{\partial\mu^i\partial p^j}(\mu,p). \qquad (4.12)
\end{aligned}
$$

Additionally, to state the general result, we call[2]:

$$
\begin{aligned}
m(\theta;\mu,p) &= \frac{\partial f}{\partial x}(\theta,\kappa(\theta;\mu,p);\mu) \qquad (4.13)\\
n(\theta;\mu,p) &= \frac{\partial f}{\partial\mu}(\theta,\kappa(\theta;\mu,p);\mu) \qquad (4.14)\\
c(\theta;\mu,p) &= \alpha(\mu,p)e^{\Re_1 v(\theta;\mu,p)} \qquad (4.15)\\
\alpha(\mu,p) &= \frac{c_0(\mu,p)}{\displaystyle\int_{\mathbb{T}} e^{\Re_1 v(\theta;\mu,p)}d\theta},\ \text{with } c_0(\mu,p)>0 \text{ freely chosen}, \qquad (4.16)\\
v(\theta;\mu,p) &= \log(m(\theta;\mu,p))-\Lambda(\mu,p) \qquad (4.17)\\
\Lambda(\mu,p) &= \int_{\mathbb{T}}\log(m(\theta;\mu,p))d\theta \qquad (4.18)\\
\lambda(\mu,p) &= e^{\Lambda(\mu,p)} \qquad (4.19)\\
\eta(\theta;\mu,p) &= \frac{1}{c(\theta+\omega;\mu,p)}. \qquad (4.20)
\end{aligned}
$$

[2] By analogy with **Lemma 3.2**

Theorem 4.2 Derivatives of the invariant translated curves

Let $(\kappa(\cdot\,;\mu,p),\tau(\mu,p))$ be an invariant translated curve for some $(\mu,p)\in\mathcal{I}\times\mathbb{R}$. Then:

(*a*) *For $i,j=0,1,2,\dots$ and whenever $i+j>0$,*

$$\begin{aligned} m(\theta;\mu,p)\kappa^{ij}(\theta;\mu,p)-\kappa^{ij}(\theta+\omega;\mu,p)+\zeta^{ij}(\theta;\mu,p)+\tau^{ij}(\mu,p) &= 0\,,\ \forall\theta\in\mathbb{T} && (4.21)\\ <\kappa^{ij}(\cdot\,;\mu,p)> &= \varepsilon^{ij}\,. && (4.22)\end{aligned}$$

where

$$\varepsilon^{ij}=\begin{cases}1 & ,\quad (i,j)=(0,1)\\ 0 & ,\quad \textit{otherwise}\end{cases}\ , \tag{4.23}$$

and the functions ζ^{ij} are given by

$$\zeta^{10}(\theta;\mu,p) = n(\theta;\mu,p)=\frac{\partial f}{\partial\mu}(\theta,\kappa(\theta;\mu,p);\mu), \tag{4.24}$$

$$\zeta^{01}(\theta;\mu,p) = 0 \tag{4.25}$$

$$\zeta^{ij}(\theta;\mu,p) = \begin{cases}\frac{\partial m}{\partial p}(\theta;\mu,p)\kappa^{i,j-1}(\theta;\mu,p)+\frac{\partial\zeta^{i,j-1}}{\partial p}(\theta;\mu,p) & ,\quad 0<i\le j\\ \frac{\partial m}{\partial\mu}(\theta;\mu,p)\kappa^{i-1,j}(\theta;\mu,p)+\frac{\partial\zeta^{i-1,j}}{\partial\mu}(\theta;\mu,p) & ,\quad 0<j\le i\end{cases}. \tag{4.26}$$

In particular, for lower indices,

$$\zeta^{ij}(\theta;\mu,p)=\begin{cases}0 & ,\quad (i,j)=(0,1)\\ n(\theta;\mu,p) & ,\quad (i,j)=(1,0)\\ \frac{\partial m}{\partial\mu}(\theta;\mu,p)\kappa^{01}(\theta;\mu,p) & ,\quad (i,j)=(1,1)\\ \frac{\partial m}{\partial p}(\theta;\mu,p)\kappa^{01}(\theta;\mu,p) & ,\quad (i,j)=(0,2)\\ \frac{\partial m}{\partial\mu}(\theta;\mu,p)\kappa^{10}(\theta;\mu,p)+\frac{\partial n}{\partial\mu}(\theta;\mu,p) & ,\quad (i,j)=(2,0)\end{cases}. \tag{4.27}$$

(*b*) *Under the change of variable*

$$\kappa^{ij}(\theta;\mu,p)=c(\theta;\mu,p)\chi^{ij}(\theta;\mu,p) \tag{4.28}$$

equations (4.21) and (4.22) take the form:

$$\begin{aligned}\chi^{ij}(\theta+\omega;\mu,p)-\lambda(\mu,p)\chi^{ij}(\theta;\mu,p) &= \xi^{ij}(\theta;\mu,p)+\eta(\theta;\mu,p)\tau^{ij}(\mu,p)\,,\ \forall\theta\in\mathbb{T} && (4.29)\\ <c(\cdot\,;\mu,p)\chi^{ij}(\cdot\,;\mu,p)> &= \varepsilon^{ij}\,, && (4.30)\end{aligned}$$

where

$$\xi^{ij}(\theta;\mu,p)=\eta(\theta;\mu,p)\zeta^{ij}(\theta;\mu,p). \tag{4.31}$$

In what follows, we consider the usual decomposition of a function as the sum of its average plus the oscillating part. More specifically:

$$\begin{aligned}\chi^{ij}(\theta;\mu,p) &= \chi_0^{ij}(\mu,p)+\widetilde{\chi^{ij}}(\theta;\mu,p)\\ \eta(\theta;\mu,p) &= \eta_0(\mu,p)+\widetilde{\eta}(\theta;\mu,p)\\ \xi^{ij}(\theta;\mu,p) &= \xi_0^{ij}(\mu,p)+\widetilde{\xi^{ij}}(\theta;\mu,p)\\ \zeta^{ij}(\theta;\mu,p) &= \zeta_0^{ij}(\mu,p)+\widetilde{\zeta^{ij}}(\theta;\mu,p)\end{aligned}$$

with $<\widetilde{\chi^{ij}}(\cdot\,;\mu,p)>=<\widetilde{\eta^{ij}}(\cdot\,;\mu,p)>=<\widetilde{\xi^{ij}}(\cdot\,;\mu,p)>=<\widetilde{\zeta^{ij}}(\cdot\,;\mu,p)>=0$.

(*c*) *Let* $\Omega(\mu,p)$ *be the matrix*

$$\Omega(\mu,p) = \begin{pmatrix} 1-\lambda(\mu,p) & -\eta_0(\mu,p) \\ c_0(\mu,p) & <\widetilde{c}\,\mathfrak{R}_{\lambda(\mu,p)}\widetilde{\eta}(\cdot\,;\mu,p)> \end{pmatrix} \tag{4.32}$$

and

$$b^{ij}(\mu,p) = \left(\xi_0^{ij}(\mu,p)\,,\,\varepsilon^{ij} - <\widetilde{c}\,\mathfrak{R}_{\lambda(\mu,p)}\widetilde{\xi^{ij}}(\cdot\,;\mu,p)>\right)^\top . \tag{4.33}$$

If the non–degeneracy condition $\det(\Omega(\mu,p)) \neq 0$ *holds, then the system* (4.29)–(4.30) *has a unique solution* $\left(\chi^{ij}(\theta;\mu,p), \tau^{ij}(\mu,p)\right)$*:*

$$\chi^{ij}(\theta;\mu,p) = \chi_0^{ij}(\mu,p) + \mathfrak{R}_{\lambda(\mu,p)}\widetilde{\eta}(\theta;\mu,p)\,\tau^{ij}(\mu,p) + \mathfrak{R}_{\lambda(\mu,p)}\widetilde{\xi^{ij}}(\theta;\mu,p), \tag{4.34}$$

where $\chi_0^{ij}(\mu,p), \tau^{ij}(\mu,p)$ *are the unique solutions to the linear system*

$$\Omega(\mu,p)\begin{pmatrix} \chi_0^{ij}(\mu,p) \\ \tau^{ij}(\mu,p) \end{pmatrix} = b^{ij}(\mu,p)\,, \tag{4.35}$$

namely,

$$\chi_0^{ij}(\mu,p) = \frac{\xi_0^{ij}(\mu,p) <\widetilde{c}\,\mathfrak{R}_{\lambda(\mu,p)}\widetilde{\eta}(\cdot\,;\mu,p)> + (\varepsilon^{ij} - <\widetilde{c}\,\mathfrak{R}_{\lambda(\mu,p)}\widetilde{\xi^{ij}}(\cdot\,;\mu,p)>)\eta_0(\mu,p)}{(1-\lambda(\mu,p)) <\widetilde{c}\,\mathfrak{R}_{\lambda(\mu,p)}\widetilde{\eta}(\cdot\,;\mu,p)> + c_0(\mu,p)\eta_0(\mu,p)} \tag{4.36}$$

$$\tau^{ij}(\mu,p) = \frac{(1-\lambda(\mu,p))(\varepsilon^{ij} - <\widetilde{c}\,\mathfrak{R}_{\lambda(\mu,p)}\widetilde{\xi^{ij}}(\cdot\,;\mu,p)>) - c_0(\mu,p)\xi_0^{ij}(\mu,p)}{(1-\lambda(\mu,p)) <\widetilde{c}\,\mathfrak{R}_{\lambda(\mu,p)}\widetilde{\eta}(\cdot\,;\mu,p)> + c_0(\mu,p)\eta_0(\mu,p)} \tag{4.37}$$

Proof.

(a) This part can be proved by induction.

Starting from eqs. (4.5) and (4.6), taking derivatives w.r.t. p we obtain eqs. (4.7) and (4.8), which can be expressed as

$$\begin{aligned} m(\theta;\mu,p)\kappa^{01}(\theta;\mu,p) - \kappa^{01}(\theta+\omega;\mu,p) + \zeta^{01}(\theta;\mu,p) + \tau^{01}(\mu,p) &= 0\,,\ \forall\theta\in\mathbb{T} &(4.38)\\ <\kappa^{01}(\cdot\,;\mu,p)> &= \varepsilon^{01}\,, &(4.39)\end{aligned}$$

where $\varepsilon^{01} = 1$ and $\zeta^{01}(\theta;\mu,p) = 0$.

Similarly, starting from eqs. (4.5) and (4.6), taking derivatives w.r.t. μ we obtain eqs. (4.9) and (4.10), which can be expressed as

$$\begin{aligned} m(\theta;\mu,p)\kappa^{10}(\theta;\mu,p) - \kappa^{10}(\theta+\omega;\mu,p) + \zeta^{10}(\theta;\mu,p) + \tau^{10}(\mu,p) &= 0\,,\ \forall\theta\in\mathbb{T} &(4.40)\\ <\kappa^{10}(\cdot\,;\mu,p)> &= \varepsilon^{10}\,, &(4.41)\end{aligned}$$

where $\varepsilon^{10} = 0$ and $\zeta^{10}(\theta;\mu,p) = n(\theta;\mu,p) = \frac{\partial f}{\partial\mu}(\theta,\kappa(\theta;\mu,p);\mu)$.

Assume now that for every $k = 0,1,\ldots i$ and $l = 0,1,\ldots j$, with $k+l > 1$ we have

$$\begin{aligned} m(\theta;\mu,p)\kappa^{kl}(\theta;\mu,p) - \kappa^{kl}(\theta+\omega;\mu,p) + \zeta^{kl}(\theta;\mu,p) + \tau^{kl}(\mu,p) &= 0\,,\ \forall\theta\in\mathbb{T} &(4.42)\\ <\kappa^{kl}(\cdot\,;\mu,p)> &= \varepsilon^{kl}\,. &(4.43)\end{aligned}$$

Then, taking derivatives again w.r.t.μ in these equations for $k = i$ and $l = j$, we have

$$\begin{aligned} &\frac{\partial m}{\partial\mu}(\theta;\mu,p)\kappa^{ij}(\theta;\mu,p) + m(\theta;\mu,p)\kappa^{i+1,j}(\theta;\mu,p)\\ &-\kappa^{i+1,j}(\theta+\omega;\mu,p) + \frac{\partial\zeta^{ij}}{\partial\mu}(\theta;\mu,p) + \tau^{i+1,j}(\mu,p) \quad = \\ &m(\theta;\mu,p)\kappa^{i+1,j}(\theta;\mu,p) - \kappa^{i+1,j}(\theta+\omega;\mu,p)\\ &+\zeta^{i+1,j}(\theta;\mu,p) + \tau^{i+1,j}(\mu,p) \quad = \quad 0\,,\ \forall\theta\in\mathbb{T} &(4.44)\\ &<\kappa^{i+1,j}(\cdot\,;\mu,p)> \quad = \quad 0\,, &(4.45)\end{aligned}$$

where

$$\zeta^{i+1,j}(\theta;\mu,p) = \frac{\partial m}{\partial \mu}(\theta;\mu,p)\kappa^{ij}(\theta;\mu,p) + \frac{\partial \zeta^{ij}}{\partial \mu}(\theta;\mu,p). \tag{4.46}$$

In the same way, taking derivatives w.r.t. p in eqs. (4.42) and (4.43) for $k = i$ and $l = j$, we have

$$\begin{aligned} \frac{\partial m}{\partial p}(\theta;\mu,p)\kappa^{ij}(\theta;\mu,p) + m(\theta;\mu,p)\kappa^{i,j+1}(\theta;\mu,p) & \\ -\kappa^{i,j+1}(\theta+\omega;\mu,p) + \frac{\partial \zeta^{ij}}{\partial p}(\theta;\mu,p) + \tau^{i,j+1}(\mu,p) &= \\ m(\theta;\mu,p)\kappa^{i+1,j}(\theta;\mu,p) - \kappa^{i+1,j}(\theta+\omega;\mu,p) & \\ +\zeta^{i,j+1}(\theta;\mu,p) + \tau^{i,j+1}(\mu,p) &= 0\,,\ \forall\theta\in\mathbb{T} \qquad (4.47) \\ < \kappa^{i,j+1}(\cdot\,;\mu,p) > &= 0\,, \qquad (4.48) \end{aligned}$$

where

$$\zeta^{i,j+1}(\theta;\mu,p) = \frac{\partial m}{\partial p}(\theta;\mu,p)\kappa^{ij}(\theta;\mu,p) + \frac{\partial \zeta^{ij}}{\partial p}(\theta;\mu,p). \tag{4.49}$$

(b) Under the change of variable

$$\kappa^{ij}(\theta;\mu,p) = c(\theta;\mu,p)\chi^{ij}(\theta;\mu,p) \tag{4.50}$$

eq. (4.21) can be written as

$$\begin{aligned} m(\theta;\mu,p)c(\theta;\mu,p)\chi^{ij}(\theta;\mu,p) - c(\theta+\omega;\mu,p)\chi^{ij}(\theta+\omega;\mu,p) & \\ +\zeta^{ij}(\theta;\mu,p) + \tau^{ij}(\theta;\mu,p) &= 0. \end{aligned}$$

By construction, c is a Floquet transformation (cf. (4.15)), that is,

$$m(\theta;\mu,p)c(\theta;\mu,p) = \lambda(\mu,p)c(\theta+\omega;\mu,p),$$

and hence

$$c(\theta+\omega;\mu,p)\left(\chi^{ij}(\theta+\omega;\mu,p) - \lambda(\mu,p)\chi^{ij}(\theta;\mu,p)\right) = \zeta^{ij}(\theta;\mu,p) + \tau^{ij}(\mu,p).$$

Calling

$$\begin{aligned} \eta(\theta;\mu,p) &= \frac{1}{c(\theta+\omega;\mu,p)} \text{ and} \\ \xi^{ij}(\theta;\mu,p) &= \eta(\theta;\mu,p)\zeta^{ij}(\theta;\mu,p)\,, \end{aligned}$$

we have

$$\chi^{ij}(\theta+\omega;\mu,p) - \lambda(\mu,p)\chi^{ij}(\theta;\mu,p) = \xi^{ij}(\theta;\mu,p) + \eta(\theta;\mu,p)\tau^{ij}(\mu,p). \tag{4.51}$$

Thus, equations (4.21) and (4.22) take the form:

$$\begin{aligned} \chi^{ij}(\theta+\omega;\mu,p) - \lambda(\mu,p)\chi^{ij}(\theta;\mu,p) &= \xi^{ij}(\theta;\mu,p) + \eta(\theta;\mu,p)\tau^{ij}(\mu,p)\,,\ \forall\theta\in\mathbb{T} \qquad (4.52) \\ < c(\cdot\,;\mu,p)\chi^{ij}(\cdot\,;\mu,p) > &= \varepsilon^{ij}\,. \qquad (4.53) \end{aligned}$$

(c) With the usual decomposition of a function as the sum of its average plus the oscillating part,

$$\begin{aligned} \chi^{ij}(\theta;\mu,p) &= \chi_0^{ij}(\mu,p) + \widetilde{\chi^{ij}}(\theta;\mu,p) \\ \eta(\theta;\mu,p) &= \eta_0(\mu,p) + \widetilde{\eta}(\theta;\mu,p) \\ \xi^{ij}(\theta;\mu,p) &= \xi_0^{ij}(\mu,p) + \widetilde{\xi^{ij}}(\theta;\mu,p) \\ \zeta^{ij}(\theta;\mu,p) &= \zeta_0^{ij}(\mu,p) + \widetilde{\zeta^{ij}}(\theta;\mu,p) \end{aligned}$$

eq. (4.51) is written as

$$\left(\chi_0^{ij}(\mu,p)+\widetilde{\chi^{ij}}(\theta+\omega;\mu,p)\right)-\lambda(\mu,p)\left(\chi_0^{ij}(\mu,p)+\widetilde{\chi^{ij}}(\theta;\mu,p)\right)$$
$$=\xi_0^{ij}(\mu,p)+\widetilde{\xi^{ij}}(\theta;\mu,p)+\left(\eta_0(\mu,p)+\widetilde{\eta}(\theta;\mu,p)\right)\tau^{ij}(\mu,p). \tag{4.54}$$

Taking averages in eq. (4.54), we have

$$\chi_0^{ij}(\mu,p)-\lambda(\mu,p)\chi_0^{ij}(\mu,p)=\xi_0^{ij}(\mu,p)+\eta_0(\mu,p)\tau^{ij}(\mu,p), \tag{4.55}$$

and subtracting eq. (4.55) from (4.54),

$$\widetilde{\chi^{ij}}(\theta+\omega;\mu,p)-\lambda(\mu,p)\widetilde{\chi^{ij}}(\theta;\mu,p)=\widetilde{\xi^{ij}}(\theta;\mu,p)+\widetilde{\eta}(\theta;\mu,p)\tau^{ij}(\mu,p). \tag{4.56}$$

This is a cohomological equation. Therefore, there is a unique zero–average solution which is written as

$$\widetilde{\chi^{ij}}(\theta;\mu,p)=\mathfrak{R}_{\lambda(\mu,p)}\widetilde{\xi^{ij}}(\theta;\mu,p)+\mathfrak{R}_{\lambda(\mu,p)}\widetilde{\eta}(\theta;\mu,p)\,\tau^{ij}(\mu,p), \tag{4.57}$$

where $\mathfrak{R}$ is the cohomological operator.

On the other hand,

$$\begin{aligned}\kappa^{ij}(\theta;\mu,p) &= c(\theta;\mu,p)\chi^{ij}(\theta;\mu,p)=\left(c_0(\mu,p)+\widetilde{c}(\theta;\mu,p)\right)\cdot\left(\chi_0(\mu,p)+\widetilde{\chi^{ij}}(\theta;\mu,p)\right)\\ &= c_0(\mu,p)\chi_0^{ij}(\mu,p)+c_0(\mu,p)\widetilde{\chi^{ij}}(\theta;\mu,p)+\chi_0(\mu,p)\widetilde{c}(\theta;\mu,p)+\widetilde{c}(\theta;\mu,p)\widetilde{\chi^{ij}}(\theta;\mu,p).\end{aligned}$$

Now, taking averages on both sides,

$$\begin{aligned}<\kappa^{ij}(\theta;\mu,p)> &= c_0(\mu,p)\chi_0^{ij}(\mu,p)+<\widetilde{c}\widetilde{\chi^{ij}}(\theta;\mu,p)>\\ &= c_0(\mu,p)\chi_0^{ij}(\mu,p)+<\widetilde{c}\mathfrak{R}_{\lambda(\mu,p)}\widetilde{\xi^{ij}}(\theta;\mu,p)>+<\widetilde{c}\mathfrak{R}_{\lambda(\mu,p)}\widetilde{\eta}(\theta;\mu,p)>\ \tau^{ij}(\mu,p)\\ &= \varepsilon^{ij}.\end{aligned} \tag{4.58}$$

Finally, eqs. (4.55) and (4.58) can be written as

$$(1-\lambda(\mu,p))\chi_0^{ij}(\mu,p)-\eta_0(\mu,p)\tau^{ij}(\mu,p) = \xi_0^{ij}(\mu,p)\,, \tag{4.59}$$
$$c_0(\mu,p)\chi_0(\mu,p)+<\widetilde{c}\mathfrak{R}_{\lambda(\mu,p)}\widetilde{\eta}(\theta;\mu,p)>\ \tau^{ij}(\mu,p) = \varepsilon^{ij}-<\widetilde{c}\mathfrak{R}_{\lambda(\mu,p)}\widetilde{\xi^{ij}}(\theta;\mu,p)>, \tag{4.60}$$

which is the linear system

$$\Omega(\mu,p)\begin{pmatrix}\chi_0^{ij}(\mu,p)\\ \tau^{ij}(\mu,p)\end{pmatrix}=b^{ij}(\mu,p)\,, \tag{4.61}$$

where

$$\Omega(\mu,p)=\begin{pmatrix}1-\lambda(\mu,p) & -\eta_0(\mu,p)\\ c_0(\mu,p) & <\widetilde{c}\,\mathfrak{R}_{\lambda(\mu,p)}\widetilde{\eta}(\cdot\,;\mu,p)>\end{pmatrix} \tag{4.62}$$

and

$$b^{ij}(\mu,p)=\left(\xi_0^{ij}(\mu,p)\,,\ \varepsilon^{ij}-<\widetilde{c}\,\mathfrak{R}_{\lambda(\mu,p)}\widetilde{\xi^{ij}}(\cdot\,;\mu,p)>\right)^{\top}. \tag{4.63}$$

If the non–degeneracy condition $\det(\Omega(\mu,p))\neq 0$ holds, then the system (4.59) has a unique solution $\left(\chi_0^{ij}(\theta;\mu,p),\tau^{ij}(\mu,p)\right)$ and then

$$\chi^{ij}(\theta;\mu,p)=\chi_0^{ij}(\mu,p)+\mathfrak{R}_{\lambda(\mu,p)}\widetilde{\eta}(\theta;\mu,p)\,\tau^{ij}(\mu,p)+\mathfrak{R}_{\lambda(\mu,p)}\widetilde{\xi^{ij}}(\theta;\mu,p). \tag{4.64}$$

With all, this part is proved.

□

Remark 4.3

$$
\begin{aligned}
\frac{\partial^{i+j}\kappa}{\partial\mu^i\partial p^j}(\theta;\mu,p) &= \kappa^{ij}(\theta;\mu,p) = c(\theta;\mu,p)\chi^{ij}(\theta;\mu,p)\\
&= \left(c_0(\mu,p)+\widetilde{c}(\theta;\mu,p)\right)\left(\chi_0^{ij}(\mu,p)+\widetilde{\chi^{ij}}(\theta;\mu,p)\right)\\
&= \left(c_0(\mu,p)+\widetilde{c}(\theta;\mu,p)\right)\Big(\chi_0^{ij}(\mu,p)+\mathfrak{R}_{\lambda(\mu,p)}\widetilde{\eta}(\theta;\mu,p)\,\tau^{ij}(\mu,p)\\
&\qquad\qquad + \mathfrak{R}_{\lambda(\mu,p)}\widetilde{\xi^{ij}}(\theta;\mu,p)\Big)\\
&= \frac{1}{\det(\Omega(\mu,p))}\left(c_0(\mu,p)+\widetilde{c}(\theta;\mu,p)\right)\Big(<\widetilde{c}\,\mathfrak{R}_{\lambda(\mu,p)}\widetilde{\eta}(\cdot\,;\mu,p)>\Big(\xi_0^{ij}(\mu,p)\\
&\qquad\qquad +(1-\lambda(\mu,p))\mathfrak{R}_{\lambda(\mu,p)}\widetilde{\xi^{ij}}(\theta;\mu,p)\Big)\\
&\qquad + \left(\varepsilon^{ij}-<\widetilde{c}\,\mathfrak{R}_{\lambda(\mu,p)}\widetilde{\xi^{ij}}(\cdot\,;\mu,p)>\right)(\eta_0(\mu,p)\\
&\qquad\qquad +(1-\lambda(\mu,p))\mathfrak{R}_{\lambda(\mu,p)}\widetilde{\eta}(\theta;\mu,p))\\
&\quad + c_0(\mu,p)\left(\eta_0(\mu,p)\mathfrak{R}_{\lambda(\mu,p)}\widetilde{\xi^{ij}}(\theta;\mu,p)-\xi_0^{ij}(\mu,p)\mathfrak{R}_{\lambda(\mu,p)}\widetilde{\eta}(\theta;\mu,p)\right)\Big). \qquad (4.65)
\end{aligned}
$$

Corollary 4.4

Let $(\kappa(\cdot\,;\mu,p),\tau(\mu,p))$ be an invariant translated curve for some $(\mu,p)\in\mathcal{I}\times\mathbb{R}$. Then:

$$\frac{\partial\tau}{\partial p}(\mu,p)=0 \Longleftrightarrow \lambda(\mu,p)=1\,.$$

Moreover, in general,

$$\frac{\partial\kappa}{\partial p}(\theta;\mu,p)=c(\theta;\mu,p)\left(\chi_0^{01}(\mu,p)+\mathfrak{R}_{\lambda(\mu,p)}\widetilde{\eta}(\cdot\,;\mu,p)\,\frac{\partial\tau}{\partial p}(\mu,p)\right)\,.$$

In the particular case where $\lambda(\mu,p)=1$,

$$
\begin{aligned}
\det(\Omega(\mu,p)) &= c_0(\mu,p)\eta_0(\mu,p)\geq a^2>0 \text{ and}\\
\frac{\partial\kappa}{\partial p}(\theta;\mu,p) &= \frac{1}{c_0(\mu,p)}c(\theta;\mu,p)=\frac{e^{\mathfrak{R}_1 v(\theta;\mu,p)}}{\displaystyle\int_{\mathbb{T}} e^{\mathfrak{R}_1 v(\theta;\mu,p)}d\theta}\,.
\end{aligned}
$$

Proof. According to **Theorem 4.2**, the linear system (4.35) in the case where $i=0, j=1$ has the form:

$$\begin{pmatrix} 1-\lambda(\mu,p) & -\eta_0(\mu,p)\\ c_0(\mu,p) & <\widetilde{c}\,\mathfrak{R}_{\lambda(\mu,p)}\widetilde{\eta}(\cdot\,;\mu,p)> \end{pmatrix}\begin{pmatrix}\chi_0^{01}(\mu,p)\\ \frac{\partial\tau}{\partial p}(\mu,p)\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix} \qquad (4.66)$$

($\Rightarrow$) If $\frac{\partial\tau}{\partial p}(\mu,p)=0$, then $(1-\lambda(\mu,p))\chi_0^{01}(\mu,p)=0$ and $c_0(\mu,p)\chi_0^{01}(\mu,p)=1$.

Thus, $\chi_0^{01}(\mu,p)=\frac{1}{c_0(\mu,p)}>0$ and $1-\lambda(\mu,p)=0$.

Moreover, whenever $\lambda(\mu,p)=1$, $\det(\Omega(\mu,p))=c_0(\mu,p)\eta_0(\mu,p)$.

In the same way as in the previous **Lemma 3.5**, part (xii), $c_0(\mu,p)\eta_0(\mu,p)\geq a^2>0$.

($\Leftarrow$) If $\lambda(\mu,p)=1$, from (4.66), we have:

$$\begin{pmatrix} 0 & -\eta_0(\mu,p)\\ c_0(\mu,p) & <\widetilde{c}\,\mathfrak{R}_{\lambda(\mu,p)}\widetilde{\eta}(\cdot\,;\mu,p)> \end{pmatrix}\begin{pmatrix}\chi_0^{01}(\mu,p)\\ \frac{\partial\tau}{\partial p}(\mu,p)\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix} \qquad (4.67)$$

Thus, $-\eta_0(\mu,p)\frac{\partial\tau}{\partial p}(\mu,p)=0 \Longrightarrow \frac{\partial\tau}{\partial p}(\mu,p)=0$.

Additionally, from (4.11), (4.28), and (4.34), we have for $i = 0, j = 1$:

$$\begin{aligned}\frac{\partial \kappa}{\partial p}(\theta;\mu,p) &= c(\theta;\mu,p)\chi^{01}(\theta;\mu,p) \\ &= c(\theta;\mu,p)\left(\chi_0^{01}(\mu,p) + \mathfrak{R}_{\lambda(\mu,p)}\widetilde{\eta}(\theta;\mu,p)\,\frac{\partial \tau}{\partial p}(\mu,p) + \mathfrak{R}_{\lambda(\mu,p)}\widetilde{\xi^{01}}(\theta;\mu,p)\right) \\ &= c(\theta;\mu,p)\left(\chi_0^{01}(\mu,p) + \mathfrak{R}_{\lambda(\mu,p)}\widetilde{\eta}(\cdot\,;\mu,p)\,\frac{\partial \tau}{\partial p}(\mu,p)\right)\,. \end{aligned} \tag{4.68}$$

since $\zeta^{01}(\theta;\mu,p) = 0 \Rightarrow \xi^{01}(\theta;\mu,p) = \eta(\theta;\mu,p)\zeta^{01}(\theta;\mu,p) = 0 \Rightarrow \widetilde{\xi^{01}}(\theta;\mu,p) = 0$
$\Rightarrow \mathfrak{R}_{\lambda(\mu,p)}\widetilde{\xi^{01}}(\theta;\mu,p) = 0$.

Thus, in the particular case where $\lambda(\mu,p) = 1$:

$$\frac{\partial \tau}{\partial p}(\mu,p) = 0, \quad \chi_0^{01}(\mu,p) = \frac{1}{c_0(\mu,p)}, \quad \Lambda(\mu,p) = 0, \quad v(\theta;\mu,p) = \log(m(\theta;\mu,p)),$$

$$\det(\Omega(\mu,p)) = c_0(\mu,p)\eta_0(\mu,p) \geq a^2 > 0 \text{ and} \tag{4.69}$$

$$\frac{\partial \kappa}{\partial p}(\theta;\mu,p) = \frac{1}{c_0(\mu,p)}c(\theta;\mu,p) = \frac{\alpha(\mu,p)e^{\mathfrak{R}_1 v(\theta;\mu,p)}}{\displaystyle\int_{\mathbb{T}}\alpha(\mu,p)e^{\mathfrak{R}_1 v(\theta;\mu,p)}d\theta} = \frac{e^{\mathfrak{R}_1 v(\theta;\mu,p)}}{\displaystyle\int_{\mathbb{T}}e^{\mathfrak{R}_1 v(\theta;\mu,p)}d\theta}\,. \tag{4.70}$$

Notice that $\forall\theta \in \mathbb{T}$, $\frac{\partial\kappa}{\partial p}(\theta;\mu,p) > 0$ and $< \frac{\partial\kappa}{\partial p}(\cdot\,;\mu,p) >= 1$. These facts do not depend on the chosen value for $c_0(\mu,p)$. □

Corollary 4.5

Let $(\kappa(\cdot\,;\mu,p),\tau(\mu,p)) \in \mathcal{A}_\varrho$ be an invariant translated curve for some $(\mu,p) \in \mathcal{I}\times\mathbb{R}$ such that $\lambda(\mu,p) = 1$, or equivalently $\Lambda(\mu,p) = \displaystyle\int_{\mathbb{T}}\log\left(\frac{\partial f}{\partial x}(\theta,\kappa(\theta;\mu,p))\right)d\theta = 0$. Then:

$$\frac{\partial^{i+j}\tau}{\partial\mu^i\partial p^j}(\mu,p) = -\frac{< \eta(\cdot;\mu,p)\zeta^{ij}(\cdot;\mu,p) >}{< \eta(\cdot;\mu,p) >}\,,\ \forall i,j = 0,1,2,\ldots,\quad i+j > 1\,, \tag{4.71}$$

where $\eta(\theta;\mu,p) = \dfrac{1}{c(\theta+\omega;\mu,p)}$, $\forall\,\theta \in \mathbb{T}_\varrho$, and the functions ζ^{ij} are given by eqs. (4.24), (4.25), and (4.26).

Proof. It is a consequence of (4.37) with $\lambda(\mu,p) = 1$, which gives

$$\tau^{ij}(\mu,p) = -\frac{\xi_0^{ij}(\mu,p)}{\eta_0(\mu,p)}\,. \tag{4.72}$$

Since $\xi^{ij}(\theta;\mu,p) = \eta(\theta;\mu,p)\zeta^{ij}(\theta;\mu,p)$, then

$$\tau^{ij}(\mu,p) = \frac{\partial^{i+j}\tau}{\partial\mu^i\partial p^j}(\mu,p) = -\frac{< \eta(\cdot;\mu,p)\zeta^{ij}(\cdot;\mu,p) >}{< \eta(\cdot;\mu,p) >}\,,\ \forall i,j = 0,1,2,\ldots,\quad i+j > 1\,. \tag{4.73}$$

□

4.2 Saddle–node or fold bifurcation

If $(\mu_0, p_0) \in \mathcal{B}_\tau$ and $\dfrac{\partial \tau}{\partial \mu}(\mu_0, p_0) \neq 0$, then by the implicit function theorem, there exits $\mathcal{I}_0, \mathcal{J}_0$ open intervals, with $\mathcal{I}_0 \subseteq \mathcal{I}$, $\mathcal{J}_0 \subseteq \mathbb{R}$ and $\mu_0 \in \mathcal{I}_0$, $p_0 \in \mathcal{J}_0$, and there exists a function $\overline{\mu} : \mathcal{I}_0 \longrightarrow \mathcal{J}_0$ with the same regularity as τ, such that

$$\{(\mu, p) \in \mathcal{I}_0 \times \mathcal{J}_0 : \; \tau(\mu, p) = 0\} = \{(\overline{\mu}(p), p) : \; p \in \mathcal{J}_0\}\,.$$

Thus, $\tau(\mu_0, p_0) = 0$ and $\tau(\overline{\mu}(p), p) = 0$, $\forall p \in \mathcal{J}_0$.
Taking derivatives,

$$\frac{\partial \tau}{\partial \mu}(\overline{\mu}(p), p) \cdot \overline{\mu}'(p) + \frac{\partial \tau}{\partial p}(\overline{\mu}(p), p) = 0\ ,\ \forall p \in \mathcal{J}_0. \tag{4.74}$$

In particular, for $p = p_0$,

$$\frac{\partial \tau}{\partial \mu}(\mu_0, p_0) \cdot \overline{\mu}'(p_0) + \frac{\partial \tau}{\partial p}(\mu_0, p_0) = 0\,.$$

Therefore,

$$\overline{\mu}'(p_0) = -\frac{\frac{\partial \tau}{\partial p}(\mu_0, p_0)}{\frac{\partial \tau}{\partial \mu}(\mu_0, p_0)}. \tag{4.75}$$

Taking derivatives again in (4.74),

$$\begin{aligned}\left(\frac{\partial^2 \tau}{\partial \mu^2}(\overline{\mu}(p), p) \cdot \overline{\mu}'(p) + \frac{\partial^2 \tau}{\partial \mu \partial p}(\overline{\mu}(p), p)\right)\overline{\mu}'(p) + \frac{\partial \tau}{\partial \mu}(\overline{\mu}(p), p) \cdot \overline{\mu}''(p)\\ + \frac{\partial^2 \tau}{\partial \mu \partial p}(\overline{\mu}(p), p) \cdot \overline{\mu}'(p) + \frac{\partial^2 \tau}{\partial p^2}(\overline{\mu}(p), p) = 0, \quad \forall p \in \mathcal{J}_0.\end{aligned} \tag{4.76}$$

Particularizing again for $p = p_0$,

$$\begin{aligned}\left(\frac{\partial^2 \tau}{\partial \mu^2}(\mu_0, p_0) \cdot \overline{\mu}'(p_0) + \frac{\partial^2 \tau}{\partial \mu \partial p}(\mu_0, p_0)\right)\overline{\mu}'(p_0) + \frac{\partial \tau}{\partial \mu}(\mu_0, p_0) \cdot \overline{\mu}''(p_0)\\ + \frac{\partial^2 \tau}{\partial p \partial \mu}(\mu_0, p_0) \cdot \overline{\mu}'(p_0) + \frac{\partial^2 \tau}{\partial p^2}(\mu_0, p_0) = 0.\end{aligned} \tag{4.77}$$

Whenever $\dfrac{\partial \tau}{\partial p}(\mu_0, p_0) = 0$, we obtain:

$$\begin{aligned}\overline{\mu}(p_0) &= \mu_0\,,\\ \overline{\mu}'(p_0) &= 0\,, \text{and}\\ \overline{\mu}''(p_0) &= -\frac{\frac{\partial^2 \tau}{\partial p^2}(\mu_0,p_0)}{\frac{\partial \tau}{\partial \mu}(\mu_0,p_0)}.\end{aligned} \tag{4.78}$$

Assume, additionally, that $\dfrac{\partial^2 \tau}{\partial p^2}(\mu_0, p_0) \neq 0$, and define

$$\Psi(p) = \frac{\partial \tau}{\partial p}(\overline{\mu}(p), p),\ p \in \mathcal{J}_0.$$

Then,

$$\Psi(p_0) = \frac{\partial \tau}{\partial p}(\mu_0, p_0) = 0 \tag{4.79}$$

$$\begin{aligned}\Psi'(p_0) &= \frac{d}{dp}\left(\frac{\partial \tau}{\partial p}(\overline{\mu}(p), p)\right)_{p=p_0}\\ &= \left(\frac{\partial^2 \tau}{\partial \mu \partial p}(\overline{\mu}(p), p) \cdot \overline{\mu}'(p) + \frac{\partial^2 \tau}{\partial p^2}(\overline{\mu}(p), p)\right)_{p=p_0}\\ &= \frac{\partial^2 \tau}{\partial \mu \partial p}(\mu_0, p_0) \cdot \overline{\mu}'(p_0) + \frac{\partial^2 \tau}{\partial p^2}(\mu_0, p_0) = \frac{\partial^2 \tau}{\partial p^2}(\mu_0, p_0) \neq 0.\end{aligned} \tag{4.80}$$

It follows that, $\exists\delta > 0$, such that $\forall p \in (p_0 - \delta, p_0 + \delta) \subseteq \mathcal{J}_0$,

$$\Psi(p) = \Psi(p_0) + \Psi'(p_0)(p - p_0) + O(p - p_0)^2,$$

that is,

$$\frac{\partial \tau}{\partial p}(\overline{\mu}(p), p) = \frac{\partial^2 \tau}{\partial p^2}(\mu_0, p_0)(p - p_0) + O(p - p_0)^2 \;, \forall p \in (p_0 - \delta, p_0 + \delta) \subseteq \mathcal{J}_0. \tag{4.81}$$

We say that τ has a *saddle–node bifurcation* or *fold bifurcation* at (μ_0, p_0) if

SN(a). $\tau(\mu_0, p_0) = 0$;

SN(b). $\frac{\partial \tau}{\partial p}(\mu_0, p_0) = 0$;

SN(c). $\frac{\partial \tau}{\partial \mu}(\mu_0, p_0) \neq 0$; and

SN(d). $\frac{\partial^2 \tau}{\partial p^2}(\mu_0, p_0) \neq 0$.

Theorem 4.6 Saddle-node or fold bifurcation

Let $(\mu_0, p_0) \in \mathcal{I} \times \mathbb{R}$ such that:

(a) $\kappa(\cdot; \mu_0, p_0) \in \mathcal{A}_\varrho$ *is an invariant curve;*

(b) $\lambda(\mu_0, p_0) = 1$ *or, equivalently,* $\Lambda(\mu_0, p_0) = 0$*;*

(c) $\displaystyle\int_{\mathbb{T}} \eta(\theta; \mu_0, p_0) n(\theta; \mu_0, p_0) d\theta \neq 0$*;*

(d) $\displaystyle\int_{\mathbb{T}} \eta(\theta; \mu_0, p_0) \frac{\partial m}{\partial p}(\theta; \mu_0, p_0) \frac{\partial \kappa}{\partial p}(\theta; \mu_0, p_0) d\theta \neq 0.$

Then, τ has a saddle–node bifurcation or fold bifurcation at (μ_0, p_0), that is,

(i) $(\mu_0, p_0) \in \mathcal{B}_\tau$, *i.e.* $\tau(\mu_0, p_0) = 0$*;*

(ii) $\dfrac{\partial \tau}{\partial p}(\mu_0, p_0) = 0$*;*

(iii) $\dfrac{\partial \tau}{\partial \mu}(\mu_0, p_0) \neq 0$*; and*

(iv) $\dfrac{\partial^2 \tau}{\partial p^2}(\mu_0, p_0) \neq 0.$

Additionally, $\exists\, \mathcal{I}_0 \times \mathcal{J}_0 \subseteq \mathcal{I} \times \mathbb{R}$ ($\mathcal{I}_0, \mathcal{J}_0$ open intervals) and $\overline{\mu} : \mathcal{J}_0 \longrightarrow \mathcal{I}_0$, $\overline{\mu} \in \mathcal{C}^\infty$, such that:

$$\{(\mu, p) \in \mathcal{I}_0 \times \mathcal{J}_0 :\ \tau(\mu, p) = 0\} = \{(\overline{\mu}(p), p) :\ p \in \mathcal{J}_0\}.$$

Moreover,

$$\begin{aligned} \overline{\mu}(p_0) &= \mu_0 \\ \overline{\mu}'(p_0) &= 0 \\ \overline{\mu}''(p_0) &= -\frac{\frac{\partial^2 \tau}{\partial p^2}(\mu_0, p_0)}{\frac{\partial \tau}{\partial \mu}(\mu_0, p_0)} \neq 0. \end{aligned} \tag{4.82}$$

and for some $\delta > 0$,

$$\frac{\partial \tau}{\partial p}(\overline{\mu}(p), p) = \frac{\partial^2 \tau}{\partial p^2}(\mu_0, p_0)(p - p_0) + O(p - p_0)^2,\ \forall p \in (p_0 - \delta, p_0 + \delta) \subseteq \mathcal{J}_0. \tag{4.83}$$

Consequently, there are four cases depending on the sign of $\frac{\partial \tau}{\partial \mu}(\mu_0, p_0)$ and $\frac{\partial^2 \tau}{\partial p^2}(\mu_0, p_0)$, corresponding to four respective bifurcation diagrams.

Proof. (i) If $\kappa(\cdot; \mu_0, p_0) \in \mathcal{A}_\varrho$ is an invariant curve, then the translation parameter is zero, i.e. $\tau(\mu_0, p_0) = 0$ and $(\mu_0, p_0) \in \mathcal{B}_\tau$.

(ii) Since $\lambda(\mu_0, p_0) = 1$, then by **Corollary 4.4**, $\dfrac{\partial \tau}{\partial p}(\mu_0, p_0) = 0$;

(iii) By **Corollary 4.5**

$$\begin{aligned} \frac{\partial \tau}{\partial \mu}(\mu_0, p_0) &= \tau^{10}(\mu_0, p_0) = -\frac{\xi_0^{10}(\mu_0, p_0)}{\eta_0(\mu_0, p_0)} \\ &= -\frac{< \eta(\cdot; \mu_0, p_0)\zeta^{10}(\cdot; \mu_0, p_0) >}{\eta(\cdot; \mu_0, p_0)} = -\frac{< \eta(\cdot; \mu_0, p_0) n(\cdot; \mu_0, p_0) >}{\eta(\cdot; \mu_0, p_0)}. \end{aligned}$$

Thus, $\dfrac{\partial \tau}{\partial \mu}(\mu_0, p_0) \neq 0 \Leftrightarrow < \eta(\cdot; \mu_0, p_0) n(\cdot; \mu_0, p_0) > \neq 0 \Leftrightarrow \displaystyle\int_{\mathbb{T}} \eta(\theta; \mu_0, p_0) n(\theta; \mu_0, p_0) d\theta \neq 0.$

(iv) By **Corollary 4.5**

$$\begin{aligned}\frac{\partial^2\tau}{\partial p^2}(\mu_0,p_0) &= \tau^{02}(\mu_0,p_0) = -\frac{\xi_0^{02}(\mu_0,p_0)}{\eta_0(\mu_0,p_0)} \\ &= -\frac{< \eta(\cdot;\mu_0,p_0)\zeta^{02}(\cdot;\mu_0,p_0) >}{\eta(\cdot;\mu_0,p_0)} = -\frac{< \eta(\cdot;\mu_0,p_0)\frac{\partial m}{\partial p}(\cdot;\mu_0,p_0)\frac{\partial \kappa}{\partial p}(\cdot;\mu_0,p_0) >}{\eta(\cdot;\mu_0,p_0)}.\end{aligned}$$

Thus,

$$\frac{\partial^2\tau}{\partial p^2}(\mu_0,p_0) \neq 0 \Leftrightarrow < \eta(\cdot;\mu_0,p_0)n(\cdot;\mu_0,p_0) > \neq 0 \Leftrightarrow \int_{\mathbb{T}} \eta(\theta;\mu_0,p_0)\frac{\partial m}{\partial p}(\theta;\mu_0,p_0)\frac{\partial \kappa}{\partial p}(\theta;\mu_0,p_0)d\theta \neq 0.$$

The remainder of the theorem is already proved just before the statement. □

4.3 Transcritical and Pitchfork bifurcations

Let $(\mu_0, p_0) \in \mathcal{B}_\tau$ be a critical point of τ, that is:

$$\begin{cases} \frac{\partial \tau}{\partial \mu}(\mu_0, p_0) &= 0 \\ \frac{\partial \tau}{\partial p}(\mu_0, p_0) &= 0\,. \end{cases} \tag{4.84}$$

Assume that (μ_0, p_0) is a non–degenerate critical point, i.e. $\det(\operatorname{Hess}\tau)(\mu_0, p_0) \neq 0$, namely:

$$\det \begin{pmatrix} \frac{\partial^2 \tau}{\partial \mu^2}(\mu_0, p_0) & \frac{\partial^2 \tau}{\partial \mu \partial p}(\mu_0, p_0) \\ \frac{\partial^2 \tau}{\partial \mu \partial p}(\mu_0, p_0) & \frac{\partial^2 \tau}{\partial p^2}(\mu_0, p_0) \end{pmatrix} \neq 0. \tag{4.85}$$

Morse's theorem assures that there exist a local diffeomorphism

$$\begin{array}{rccc} h: & \Lambda_1 \times \Lambda_2 & \longrightarrow & \mathcal{I}_0 \times \mathcal{J}_0 \\ & (y_1, y_2) & \longmapsto & (\mu, p)\,, \end{array}$$

with $(0,0) \in \Lambda_1 \times \Lambda_2 \subseteq \mathbb{R}^2$, Λ_1, Λ_2 open intervals, such that $h(0,0) = (\mu_0, p_0)$ and

(a) If $\frac{\partial^2 \tau}{\partial \mu^2}(\mu_0, p_0) > 0$ and $\det(\operatorname{Hess}\tau)(\mu_0, p_0) > 0$, then

$$(\tau \circ h)(y_1, y_2) = y_1^2 + y_2^2\,,$$

and τ has a local minimum at (μ_0, p_0).

(b) If $\frac{\partial^2 \tau}{\partial \mu^2}(\mu_0, p_0) < 0$ and $\det(\operatorname{Hess}\tau)(\mu_0, p_0) > 0$, then

$$(\tau \circ h)(y_1, y_2) = -(y_1^2 + y_2^2)\,,$$

and τ has a local maximum at (μ_0, p_0).

(c) If $\det(\operatorname{Hess}\tau)(\mu_0, p_0) < 0$, then

$$(\tau \circ h)(y_1, y_2) = y_1\, y_2\,,$$

and τ has a local saddle point at (μ_0, p_0).

In the first two cases the point $(\mu_0, p_0) \in \mathcal{B}_\tau$ is isolated. In the last case, $\mathcal{B}_\tau$ is a product of two curves in a neighborhood of (μ_0, p_0):
Notice that

$$\det(\operatorname{Hess}\tau)(\mu_0, p_0) < 0 \Longleftrightarrow \left. \left(\frac{\partial^2 \tau}{\partial \mu^2} \frac{\partial^2 \tau}{\partial p^2} - \left(\frac{\partial^2 \tau}{\partial \mu \partial p} \right)^2 \right) \right|_{(\mu_0, p_0)} < 0. \tag{4.86}$$

Since h is a diffeomorphism,

$$\exists y_1, y_2 : \mathcal{I}_0 \times \mathcal{J}_0 \longrightarrow \mathbb{R}\,,\ y_1, y_2 \in \mathcal{C}^\infty \text{ such that } \forall (\mu, p) \in \mathcal{I}_0 \times \mathcal{J}_0, \tau(\mu, p) = y_1(\mu, p) \cdot y_2(\mu, p).$$

Moreover,

$$y_1(\mu_0, p_0) = y_2(\mu_0, p_0) = 0\,, \tag{4.87}$$

and

$$\det \begin{pmatrix} \frac{\partial y_1}{\partial \mu} & \frac{\partial y_1}{\partial p} \\ \frac{\partial y_2}{\partial \mu} & \frac{\partial y_2}{\partial p} \end{pmatrix}_{(\mu_0, p_0)} \neq 0. \tag{4.88}$$

Thus,

$$\mathcal{B}_\tau \cap (\mathcal{I}_0 \times \mathcal{J}_0) = \mathcal{B}^1 \cup \mathcal{B}^2\,, \tag{4.89}$$

where

$$\mathcal{B}^1 = \{(\mu, p) \in \mathcal{I}_0 \times \mathcal{J}_0 : y_1(\mu, p) = 0\} \text{ and} \tag{4.90}$$
$$\mathcal{B}^2 = \{(\mu, p) \in \mathcal{I}_0 \times \mathcal{J}_0 : y_2(\mu, p) = 0\}. \tag{4.91}$$

From (4.88) we know that, at least one of the partial derivatives, $\frac{\partial y_1}{\partial p}$ or $\frac{\partial y_2}{\partial p}$ is different from zero at (μ_0, p_0).
If $\frac{\partial y_1}{\partial p}(\mu_0, p_0) \cdot \frac{\partial y_2}{\partial p}(\mu_0, p_0) \neq 0$ we say that τ has a *transcritical bifurcation* at (μ_0, p_0).
If $\frac{\partial y_1}{\partial p}(\mu_0, p_0) \cdot \frac{\partial y_2}{\partial p}(\mu_0, p_0) = 0$ we say that τ has a *pitchfork bifurcation* at (μ_0, p_0).

To be more explicit, transcritical bifurcations are characterized, in terms of y_1 and y_2 by the following conditions:

TB1 $y_1(\mu_0, p_0) = y_2(\mu_0, p_0) = 0$;

TB2 $\det \left(\begin{array}{cc} \frac{\partial y_1}{\partial \mu} & \frac{\partial y_1}{\partial p} \\ \frac{\partial y_2}{\partial \mu} & \frac{\partial y_2}{\partial p} \end{array} \right) \Bigg|_{(\mu_0, p_0)} \neq 0$;

TB3 $\frac{\partial y_1}{\partial p}(\mu_0, p_0) \cdot \frac{\partial y_2}{\partial p}(\mu_0, p_0) \neq 0$,

while pitchfork bifurcations are characterized by

PB1 $y_1(\mu_0, p_0) = y_2(\mu_0, p_0) = 0$;

PB2 $\det \left(\begin{array}{cc} \frac{\partial y_1}{\partial \mu} & \frac{\partial y_1}{\partial p} \\ \frac{\partial y_2}{\partial \mu} & \frac{\partial y_2}{\partial p} \end{array} \right) \Bigg|_{(\mu_0, p_0)} \neq 0$;

PB3 $\frac{\partial y_1}{\partial p}(\mu_0, p_0) \neq 0$ and $\frac{\partial y_2}{\partial p}(\mu_0, p_0) = 0$ or
$\frac{\partial y_1}{\partial p}(\mu_0, p_0) = 0$ and $\frac{\partial y_2}{\partial p}(\mu_0, p_0) \neq 0$.

Next, we want to express these conditions over y_1, y_2 into conditions over τ, i.e. μ, p.

(i)

$$\tau(\mu, p) = y_1(\mu, p) \cdot y_2(\mu, p). \tag{4.92}$$
$$\tau(\mu_0, p_0) = 0. \tag{4.93}$$

(ii)

$$\frac{\partial \tau}{\partial \mu}(\mu, p) = \frac{\partial y_1}{\partial \mu}(\mu, p) y_2(\mu, p) + y_1(\mu, p) \frac{\partial y_2}{\partial \mu}(\mu, p). \tag{4.94}$$
$$\frac{\partial \tau}{\partial \mu}(\mu_0, p_0) = 0. \tag{4.95}$$

(iii)

$$\frac{\partial \tau}{\partial p}(\mu, p) = \frac{\partial y_1}{\partial p}(\mu, p) y_2(\mu, p) + y_1(\mu, p) \frac{\partial y_2}{\partial p}(\mu, p). \tag{4.96}$$
$$\frac{\partial \tau}{\partial p}(\mu_0, p_0) = 0. \tag{4.97}$$

(iv)

$$\frac{\partial^2 \tau}{\partial \mu^2}(\mu, p) = \frac{\partial^2 y_1}{\partial \mu^2}(\mu, p) y_2(\mu, p) + 2 \frac{\partial y_1}{\partial \mu}(\mu, p) \frac{\partial y_2}{\partial \mu}(\mu, p) + y_1(\mu, p) \frac{\partial^2 y_2}{\partial \mu^2}(\mu, p). \tag{4.98}$$

$$\frac{\partial^2 \tau}{\partial \mu^2}(\mu_0, p_0) = 2 \frac{\partial y_1}{\partial \mu}(\mu_0, p_0) \frac{\partial y_2}{\partial \mu}(\mu_0, p_0). \tag{4.99}$$

(v)

$$\frac{\partial^2 \tau}{\partial p^2}(\mu,p) = \frac{\partial^2 y_1}{\partial p^2}(\mu,p)y_2(\mu,p) + 2\frac{\partial y_1}{\partial p}(\mu,p)\frac{\partial y_2}{\partial p}(\mu,p) + y_1(\mu,p)\frac{\partial^2 y_2}{\partial p^2}(\mu,p). \tag{4.100}$$

$$\frac{\partial^2 \tau}{\partial p^2}(\mu_0,p_0) = 2\frac{\partial y_1}{\partial p}(\mu_0,p_0)\frac{\partial y_2}{\partial p}(\mu_0,p_0). \tag{4.101}$$

(vi)

$$\frac{\partial^2 \tau}{\partial \mu \partial p}(\mu,p) = \frac{\partial^2 y_1}{\partial \mu \partial p}(\mu,p)y_2(\mu,p) + \frac{\partial y_1}{\partial \mu}(\mu,p)\frac{\partial y_2}{\partial p}(\mu,p) + \frac{\partial y_1}{\partial p}(\mu,p)\frac{\partial y_2}{\partial \mu}(\mu,p) + y_1(\mu,p)\frac{\partial^2 y_2}{\partial \mu \partial p}(\mu,p). \tag{4.102}$$

$$\frac{\partial^2 \tau}{\partial \mu \partial p}(\mu_0,p_0) = \frac{\partial y_1}{\partial \mu}(\mu_0,p_0)\frac{\partial y_2}{\partial p}(\mu_0,p_0) + \frac{\partial y_1}{\partial p}(\mu_0,p_0)\frac{\partial y_2}{\partial \mu}(\mu_0,p_0). \tag{4.103}$$

(vii)

$$\det(\mathrm{Hess}\,\tau)(\mu,p) = \det\begin{pmatrix} \frac{\partial^2 \tau}{\partial \mu^2}(\mu,p) & \frac{\partial^\tau}{\partial \mu \partial p}(\mu,p) \\ \frac{\partial^\tau}{\partial \mu \partial p}(\mu,p) & \frac{\partial^2 \tau}{\partial p^2}(\mu,p) \end{pmatrix} \tag{4.104}$$

$$\begin{aligned}
\det(\mathrm{Hess}\,\tau)(\mu_0,p_0) &= \det\begin{pmatrix} \frac{\partial^2 \tau}{\partial \mu^2}(\mu_0,p_0) & \frac{\partial^\tau}{\partial \mu \partial p}(\mu_0,p_0) \\ \frac{\partial^\tau}{\partial \mu \partial p}(\mu_0,p_0) & \frac{\partial^2 \tau}{\partial p^2}(\mu_0,p_0) \end{pmatrix} \\
&= \left.\begin{pmatrix} 2\frac{\partial y_1}{\partial \mu}\frac{\partial y_2}{\partial \mu} & \frac{\partial y_1}{\partial \mu}\frac{\partial y_2}{\partial p} + \frac{\partial y_1}{\partial p}\frac{\partial y_2}{\partial \mu} \\ \frac{\partial y_1}{\partial \mu}\frac{\partial y_2}{\partial p} + \frac{\partial y_1}{\partial p}\frac{\partial y_2}{\partial \mu} & 2\frac{\partial y_1}{\partial p}\frac{\partial y_2}{\partial p} \end{pmatrix}\right|_{(\mu_0,p_0)} \\
&= 4\frac{\partial y_1}{\partial \mu}\frac{\partial y_1}{\partial p}\frac{\partial y_2}{\partial \mu}\frac{\partial y_2}{\partial p} - \left.\left(\frac{\partial y_1}{\partial \mu}\frac{\partial y_2}{\partial p} + \frac{\partial y_1}{\partial p}\frac{\partial y_2}{\partial \mu}\right)^2\right|_{(\mu_0,p_0)} \\
&= -\left.\left(\frac{\partial y_1}{\partial \mu}\frac{\partial y_2}{\partial p} - \frac{\partial y_1}{\partial p}\frac{\partial y_2}{\partial \mu}\right)^2\right|_{(\mu_0,p_0)} \\
&= -\left(\det\left.\begin{pmatrix} \frac{\partial y_1}{\partial \mu} & \frac{\partial y_1}{\partial p} \\ \frac{\partial y_2}{\partial \mu} & \frac{\partial y_2}{\partial p} \end{pmatrix}\right|_{(\mu_0,p_0)}\right)^2 < 0
\end{aligned} \tag{4.105}$$

(viii)

$$\begin{aligned}
\frac{\partial^3 \tau}{\partial p^3}(\mu,p) &= \frac{\partial^3 y_1}{\partial p^3}y_2 + \frac{\partial^2 y_1}{\partial p^2}\frac{\partial y_2}{\partial p} + 2\left(\frac{\partial^2 y_1}{\partial p^2}\frac{\partial y_2}{\partial p} + \frac{\partial y_1}{\partial p}\frac{\partial^2 y_2}{\partial p^2}\right) + \frac{\partial y_1}{\partial p}\frac{\partial^2 y_2}{\partial p^2} + y_1\frac{\partial y_2^3}{\partial p^3} \\
&= \frac{\partial^3 y_1}{\partial p^3}y_2 + 3\frac{\partial^2 y_1}{\partial p^2}\frac{\partial y_2}{\partial p} + 3\frac{\partial y_1}{\partial p}\frac{\partial^2 y_2}{\partial p^2} + y_1\frac{\partial y_2^3}{\partial p^3}\,.
\end{aligned} \tag{4.106}$$

$$\frac{\partial^3 \tau}{\partial p^3}(\mu_0,p_0) = 3\frac{\partial^2 y_1}{\partial p^2}\frac{\partial y_2}{\partial p} + 3\left.\frac{\partial y_1}{\partial p}\frac{\partial^2 y_2}{\partial p^2}\right|_{(\mu_0,p_0)}. \tag{4.107}$$

Accordingly, transcritical bifurcations are characterized, in terms of τ and the parameters μ, p by:

T(a). $\tau(\mu_0, p_0) = 0$;

T(b). $\frac{\partial \tau}{\partial \mu}(\mu_0, p_0) = 0$;

T(c). $\frac{\partial \tau}{\partial p}(\mu_0, p_0) = 0$;

T(d). $\frac{\partial^2 \tau}{\partial p^2}(\mu_0, p_0) \neq 0$;

T(e). $\det(\text{Hess}\,\tau)(\mu_0, p_0) < 0$,

while pitchfork bifurcations by:

P(a). $\tau(\mu_0, p_0) = 0$;

P(b). $\frac{\partial \tau}{\partial \mu}(\mu_0, p_0) = 0$;

P(c). $\frac{\partial \tau}{\partial p}(\mu_0, p_0) = 0$;

P(d). $\frac{\partial^2 \tau}{\partial p^2}(\mu_0, p_0) = 0$;

P(e). $\frac{\partial^2 \tau}{\partial \mu \partial p}(\mu_0, p_0) \neq 0$;

P(f). $\frac{\partial^3 \tau}{\partial p^3}(\mu_0, p_0) \neq 0$;

Remark 4.7

Notice that, in this case, $\det(\text{Hess}\,\tau)(\mu_0, p_0) < 0$ *as well, since*

$$\det(\text{Hess}\,\tau)(\mu_0, p_0) = -\left(\det\left(\begin{array}{cc} \frac{\partial y_1}{\partial \mu} & \frac{\partial y_1}{\partial p} \\ \frac{\partial y_2}{\partial \mu} & \frac{\partial y_2}{\partial p} \end{array}\right)\Bigg|_{(\mu_0, p_0)}\right)^2 = -\left(\frac{\partial^2 \tau}{\partial \mu \partial p}(\mu_0, p_0)\right)^2,$$

due to the conditions P(d). *and* P(e).

Next we carry out the local analysis and stability of both bifurcations.

In the case of the transcritical bifurcation, by the implicit function theorem,

$$\exists\, \overline{p}_1, \overline{p}_2 : \mathcal{I}_0 \longrightarrow \mathcal{J}_0,\ \overline{p}_1, \overline{p}_2 \in \mathcal{C}^\infty,$$

such that

$$\{(\mu, p) \in \mathcal{I}_0 \times \mathcal{J}_0 : \tau(\mu, p) = 0\} = \{(\mu, \overline{p}_1(\mu)) : \mu \in \mathcal{I}_0\} \cup \{(\mu, \overline{p}_2(\mu)) : \mu \in \mathcal{I}_0\}.$$

Hence, $\tau(\mu, \overline{p}_i(\mu)) = 0$, $\forall \mu \in \mathcal{I}_0$. In particular, for $\mu = \mu_0, \tau(\mu_0, p_0) = 0$. Taking derivatives,

$$\frac{\partial \tau}{\partial \mu}(\mu, \overline{p}_i(\mu)) + \frac{\partial \tau}{\partial p}(\mu, \overline{p}_i(\mu)) \cdot \overline{p}_i'(\mu) = 0,\ \forall \mu \in \mathcal{I}_0 \quad (i = 1, 2)\,. \tag{4.108}$$

Taking derivatives again,

$$\begin{aligned} 0 &= \frac{\partial^2 \tau}{\partial \mu^2}(\mu, \overline{p}_i(\mu)) + \frac{\partial^2 \tau}{\partial \mu \partial p}(\mu, \overline{p}_i(\mu)) \cdot \overline{p}_i'(\mu) + \left(\frac{\partial^2 \tau}{\partial \mu \partial p}(\mu, \overline{p}_i(\mu)) + \frac{\partial^2 \tau}{\partial p^2}(\mu, \overline{p}_i(\mu)) \cdot \overline{p}_i'(\mu)\right) \cdot \overline{p}_i'(\mu) \\ &+ \frac{\partial \tau}{\partial p}(\mu, \overline{p}_i(\mu)) \cdot \overline{p}_i''(\mu),\ \forall \mu \in \mathcal{I}_0. \end{aligned} \tag{4.109}$$

Particularizing (4.109) for $\mu = \mu_0$,

$$\begin{aligned} 0 &= \frac{\partial^2 \tau}{\partial \mu^2}(\mu_0, p_0) + \frac{\partial^2 \tau}{\partial \mu \partial p}(\mu_0, p_0) \cdot \overline{p}_i'(\mu_0) + \left(\frac{\partial^2 \tau}{\partial \mu \partial p}(\mu_0, p_0) + \frac{\partial^2 \tau}{\partial p^2}(\mu_0, p_0) \cdot \overline{p}_i'(\mu_0) \right) \cdot \overline{p}_i'(\mu_0) \\ &+ \frac{\partial \tau}{\partial p}(\mu_0, p_0) \cdot \overline{p}_i''(\mu_0). \end{aligned} \quad (4.110)$$

Since $\frac{\partial \tau}{\partial p}(\mu_0, p_0) = 0$, we obtain

$$\frac{\partial^2 \tau}{\partial p^2}(\mu_0, p_0)(\overline{p}_i'(\mu_0))^2 + 2\frac{\partial^2 \tau}{\partial \mu \partial p}(\mu_0, p_0)\overline{p}_i'(\mu_0) + \frac{\partial^2 \tau}{\partial \mu^2}(\mu_0, p_0) = 0 \, . \quad (4.111)$$

This is a quadratic equation with leading coefficient $\frac{\partial^2 \tau}{\partial p^2}(\mu_0, p_0) \neq 0$ and positive discriminant:

$$\begin{aligned} \Delta &= \left(2\frac{\partial^2 \tau}{\partial \mu \partial p}(\mu_0, p_0) \right)^2 - 4\frac{\partial^2 \tau}{\partial p^2}(\mu_0, p_0)\frac{\partial^2 \tau}{\partial \mu^2}(\mu_0, p_0) \\ &= -4\left(\frac{\partial^2 \tau}{\partial p^2}(\mu_0, p_0)\frac{\partial^2 \tau}{\partial \mu^2}(\mu_0, p_0) - \left(\frac{\partial^2 \tau}{\partial \mu \partial p}(\mu_0, p_0) \right)^2 \right) \\ &= -4\det(\mathrm{Hess}\,\tau)(\mu_0, p_0) > 0. \end{aligned}$$

Accordingly, (4.111) has two different real solutions:

$$\overline{p}_i'(\mu_0) = \frac{-\frac{\partial^2 \tau}{\partial \mu \partial p}(\mu_0, p_0) \pm \sqrt{-\det(\mathrm{Hess}\,\tau)(\mu_0, p_0)}}{\frac{\partial^2 \tau}{\partial p^2}(\mu_0, p_0)} \quad (i = 1, 2) \, . \quad (4.112)$$

If we call

$$\Phi_i(\mu) = \frac{\partial \tau}{\partial p}(\mu, \overline{p}_i(\mu)), \ \mu \in \mathcal{I}_0 \quad (i = 1, 2),$$

then $\exists \delta > 0$, such that

$$\Phi_i(\mu) = \Phi_i(\mu_0) + \Phi_i'(\mu_0)(\mu - \mu_0) + O(\mu - \mu_0)^2, \ \forall \mu \in (\mu_0 - \delta, \mu_0 + \delta) \subseteq \mathcal{I}_0,$$

and we have

$$\begin{aligned} \Phi_i(\mu_0) &= \frac{\partial \tau}{\partial p}(\mu_0, p_0) = 0 \text{ and} \\ \Phi_i'(\mu_0) &= \frac{d}{d\mu}\left(\frac{\partial \tau}{\partial p}(\mu, \overline{p}_i(\mu)) \right)\Big|_{\mu=\mu_0} = \left(\frac{\partial^2 \tau}{\partial \mu \partial p}(\mu, \overline{p}_i(\mu)) + \frac{\partial^2 \tau}{\partial p^2}(\mu, \overline{p}_i(\mu)) \cdot \overline{p}_i'(\mu) \right)\Big|_{\mu=\mu_0} \\ &= \frac{\partial^2 \tau}{\partial \mu \partial p}(\mu_0, p_0) + \frac{\partial^2 \tau}{\partial p^2}(\mu_0, p_0) \cdot \overline{p}_i'(\mu_0) = \pm\sqrt{-\det(\mathrm{Hess}\,\tau)(\mu_0, p_0)}. \end{aligned}$$

Therefore,

$$\frac{\partial \tau}{\partial p}(\mu, \overline{p}_i(\mu)) = \pm\sqrt{-\det(\mathrm{Hess}\,\tau)(\mu_0, p_0)}(\mu - \mu_0) + O(\mu - \mu_0)^2, \forall \mu \in (\mu_0 - \delta, \mu_0 + \delta) \subseteq \mathcal{I}_0, \quad (i = 1, 2). \quad (4.113)$$

In the case of the pitchfork bifurcation we have two possibilities, since $\frac{\partial y_1}{\partial p}(\mu_0, p_0) \cdot \frac{\partial y_2}{\partial p}(\mu_0, p_0) = 0$:

Case 1. $\frac{\partial y_1}{\partial p}(\mu_0, p_0) \neq 0$ and $\frac{\partial y_2}{\partial p}(\mu_0, p_0) = 0$;

Case 2. $\frac{\partial y_1}{\partial p}(\mu_0, p_0) = 0$ and $\frac{\partial y_2}{\partial p}(\mu_0, p_0) \neq 0$.

In the first one, we have additionally $\frac{\partial y_2}{\partial \mu}(\mu_0, p_0) \neq 0$, since
$\det \begin{pmatrix} \frac{\partial y_1}{\partial \mu}(\mu_0, p_0) & \frac{\partial y_1}{\partial p}(\mu_0, p_0) \\ \frac{\partial y_2}{\partial \mu}(\mu_0, p_0) & \frac{\partial y_2}{\partial p}(\mu_0, p_0) \end{pmatrix} = \det \begin{pmatrix} \frac{\partial y_1}{\partial \mu}(\mu_0, p_0) & \frac{\partial y_1}{\partial p}(\mu_0, p_0) \\ \frac{\partial y_2}{\partial \mu}(\mu_0, p_0) & 0 \end{pmatrix} = -\frac{\partial y_1}{\partial p}(\mu_0, p_0) \cdot \frac{\partial y_2}{\partial \mu}(\mu_0, p_0) \neq 0.$
In the second one, we have additionally $\frac{\partial y_1}{\partial \mu}(\mu_0, p_0) \neq 0$ as well, since
$\det \begin{pmatrix} \frac{\partial y_1}{\partial \mu}(\mu_0, p_0) & \frac{\partial y_1}{\partial p}(\mu_0, p_0) \\ \frac{\partial y_2}{\partial \mu}(\mu_0, p_0) & \frac{\partial y_2}{\partial p}(\mu_0, p_0) \end{pmatrix} = \det \begin{pmatrix} \frac{\partial y_1}{\partial \mu}(\mu_0, p_0) & 0 \\ \frac{\partial y_2}{\partial \mu}(\mu_0, p_0) & \frac{\partial y_2}{\partial p}(\mu_0, p_0) \end{pmatrix} = \frac{\partial y_1}{\partial \mu}(\mu_0, p_0) \cdot \frac{\partial y_2}{\partial p}(\mu_0, p_0) \neq 0.$

We then analyze **Case** 1., that is:

(a) $y_1(\mu_0, p_0) = 0$ and $\frac{\partial y_1}{\partial p}(\mu_0, p_0) \neq 0$ and

(b) $y_2(\mu_0, p_0) = 0$ and $\frac{\partial y_2}{\partial \mu}(\mu_0, p_0) \neq 0$.

It follows, by the implicit function theorem,

$$\exists\, \overline{\mu} : \mathcal{J}_0 \longrightarrow \mathcal{I}_0 \text{ and } \overline{p} : \mathcal{I}_0 \longrightarrow \mathcal{J}_0 \;, \overline{\mu}, \overline{p} \in \mathcal{C}^\infty \text{ such that}$$

$$\{(\mu, p) \in \mathcal{I}_0 \times \mathcal{J}_0 : y_1(\mu, p) = 0\} = \{(\mu, \overline{p}(\mu)) : \mu \in \mathcal{I}_0\},$$

and

$$\{(\mu, p) \in \mathcal{I}_0 \times \mathcal{J}_0 : y_2(\mu, p) = 0\} = \{(\overline{\mu}(p), p) : p \in \mathcal{J}_0\}.$$

Equivalently,

$$\{(\mu, p) \in \mathcal{I}_0 \times \mathcal{J}_0 : \tau(\mu, p) = 0\} = \{(\mu, \overline{p}(\mu)) : \mu \in \mathcal{I}_0\} \cup \{(\overline{\mu}(p), p) : p \in \mathcal{J}_0\}.$$

Thus, on the one hand:
$y_1(\mu, \overline{p}(\mu)) = 0$, $\forall \mu \in \mathcal{I}_0$. In particular, for $\mu = \mu_0, y_1(\mu_0, p_0) = 0$.
Taking derivatives,

$$\frac{\partial y_1}{\partial \mu}(\mu, \overline{p}(\mu)) + \frac{\partial y_1}{\partial p}(\mu, \overline{p}(\mu)) \cdot \overline{p}'(\mu) = 0, \; \forall \mu \in \mathcal{I}_0 \,. \tag{4.114}$$

Particularizing for $\mu = \mu_0$,

$$\frac{\partial y_1}{\partial \mu}(\mu_0, p_0) + \frac{\partial y_1}{\partial p}(\mu_0, p_0) \cdot \overline{p}'(\mu_0) = 0, \; \forall \mu \in \mathcal{I}_0 \,. \tag{4.115}$$

Since $\frac{\partial y_1}{\partial p}(\mu_0, p_0) \neq 0$,

$$\overline{p}'(\mu_0) = -\frac{\frac{\partial y_1}{\partial \mu}(\mu_0, p_0)}{\frac{\partial y_1}{\partial p}(\mu_0, p_0)} \tag{4.116}$$

Moreover,

$$\begin{aligned} \frac{\frac{\partial^2 \tau}{\partial \mu^2}(\mu_0, p_0)}{\frac{\partial^2 \tau}{\partial \mu \partial p}(\mu_0, p_0)} &= \frac{2\frac{\partial y_1}{\partial \mu}(\mu_0, p_0)\frac{\partial y_2}{\partial \mu}(\mu_0, p_0)}{\frac{\partial y_1}{\partial \mu}(\mu_0, p_0)\frac{\partial y_2}{\partial p}(\mu_0, p_0) + \frac{\partial y_1}{\partial p}(\mu_0, p_0)\frac{\partial y_2}{\partial \mu}(\mu_0, p_0)} \\ &= 2\frac{\frac{\partial y_1}{\partial \mu}(\mu_0, p_0)}{\frac{\partial y_1}{\partial p}(\mu_0, p_0)} = -2\overline{p}'(\mu_0). \end{aligned}$$

Hence,

$$\overline{p}'(\mu_0) = -\frac{1}{2}\frac{\frac{\partial^2\tau}{\partial\mu^2}(\mu_0,p_0)}{\frac{\partial^2\tau}{\partial\mu\partial p}(\mu_0,p_0)}. \tag{4.117}$$

On the other hand:

$y_2(\overline{\mu}(p),p)=0$, $\forall p\in\mathcal{J}_0$. In particular, for $p=p_0, y_2(\mu_0,p_0)=0$.
Taking derivatives,

$$\frac{\partial y_2}{\partial\mu}(\overline{\mu}(p),p)\cdot\overline{\mu}'(p)+\frac{\partial y_2}{\partial p}(\overline{\mu}(p),p)=0,\ \forall p\in\mathcal{J}_0. \tag{4.118}$$

In particular, for $p=p_0$,

$$\frac{\partial y_2}{\partial\mu}(\mu_0,p_0)\cdot\overline{\mu}'(p_0)+\frac{\partial y_2}{\partial p}(\mu_0,p_0)=0. \tag{4.119}$$

Since $\frac{\partial y_2}{\partial\mu}(\mu_0,p_0)\neq 0$,

$$\overline{\mu}'(p_0) = -\frac{\frac{\partial y_2}{\partial p}(\mu_0,p_0)}{\frac{\partial y_2}{\partial\mu}(\mu_0,p_0)}=0. \tag{4.120}$$

Taking derivatives again,

$$\begin{aligned} 0 &= \left(\frac{\partial^2 y_2}{\partial\mu^2}(\overline{\mu}(p),p)\cdot\overline{\mu}'(p)+\frac{\partial^2 y_2}{\partial\mu\partial p}(\overline{\mu}(p),p)\right)\cdot\overline{\mu}'(p)\\ &+ \frac{\partial y_2}{\partial\mu}(\overline{\mu}(p),p)\cdot\overline{\mu}''(p)+\frac{\partial^2 y_2}{\partial\mu\partial p}(\overline{\mu}(p),p)\cdot\overline{\mu}'(p)+\frac{\partial^2 y_2}{\partial p^2}(\overline{\mu}(p),p),\ \forall p\in\mathcal{J}_0. \end{aligned} \tag{4.121}$$

In particular, for $p=p_0$,

$$\begin{aligned} 0 &= \left(\frac{\partial^2 y_2}{\partial\mu^2}(\mu_0,p_0)\cdot\overline{\mu}'(p_0)+\frac{\partial^2 y_2}{\partial\mu\partial p}(\mu_0,p_0)\right)\cdot\overline{\mu}'(p_0)\\ &+ \frac{\partial y_2}{\partial\mu}(\mu_0,p_0)\cdot\overline{\mu}''(p_0)+\frac{\partial^2 y_2}{\partial\mu\partial p}(\mu_0,p_0)\cdot\overline{\mu}'(p_0)+\frac{\partial^2 y_2}{\partial p^2}(\mu_0,p_0). \end{aligned} \tag{4.122}$$

Since $\overline{\mu}'(p_0)=0$ and $\frac{\partial y_2}{\partial\mu}(\mu_0,p_0)\neq 0$, we have

$$\overline{\mu}''(p_0) = -\frac{\frac{\partial^2 y_2}{\partial p^2}(\mu_0,p_0)}{\frac{\partial y_2}{\partial\mu}(\mu_0,p_0)}. \tag{4.123}$$

Furthermore,

$$\begin{aligned} \frac{\frac{\partial^3\tau}{\partial p^3}(\mu_0,p_0)}{\frac{\partial^2\tau}{\partial\mu\partial p}(\mu_0,p_0)} &= \frac{3\frac{\partial^2 y_1}{\partial p^2}(\mu_0,p_0)\frac{\partial y_2}{\partial p}(\mu_0,p_0)+3\frac{\partial y_1}{\partial p}(\mu_0,p_0)\frac{\partial^2 y_2}{\partial p^2}(\mu_0,p_0)}{\frac{\partial y_1}{\partial\mu}(\mu_0,p_0)\frac{\partial y_2}{\partial p}(\mu_0,p_0)+\frac{\partial y_1}{\partial p}(\mu_0,p_0)\frac{\partial y_2}{\partial\mu}(\mu_0,p_0)}\\ &= 3\frac{\frac{\partial^2 y_2}{\partial p^2}(\mu_0,p_0)}{\frac{\partial y_2}{\partial\mu}(\mu_0,p_0)} = -3\overline{\mu}''(p_0), \end{aligned}$$

since $\frac{\partial y_2}{\partial p}(\mu_0,p_0)=0$, $\frac{\partial y_1}{\partial p}(\mu_0,p_0)\neq 0$, and $\frac{\partial y_2}{\partial\mu}(\mu_0,p_0)=0$.
Thus,

$$\overline{\mu}''(p_0) = -\frac{1}{3}\frac{\frac{\partial^3\tau}{\partial p^3}(\mu_0,p_0)}{\frac{\partial^2\tau}{\partial\mu\partial p}(\mu_0,p_0)}. \tag{4.124}$$

Summarizing,

$\overline{p}(\mu_0) = p_0$	$\overline{\mu}(p_0) = \mu_0$
$\overline{p}'(\mu_0) = -\frac{1}{2}\frac{\frac{\partial^2\tau}{\partial\mu^2}(\mu_0,p_0)}{\frac{\partial^2\tau}{\partial\mu\partial p}(\mu_0,p_0)} \neq 0$	$\overline{\mu}'(p_0) = 0$
	$\overline{\mu}''(p_0) = -\frac{1}{3}\frac{\frac{\partial^3\tau}{\partial p^3}(\mu_0,p_0)}{\frac{\partial^2\tau}{\partial\mu\partial p}(\mu_0,p_0)} \neq 0$

(4.125)

Next, in order to determine the stability of the bifurcation points, we define the functions:

$$\begin{aligned} \Phi(\mu) &= \frac{\partial\tau}{\partial p}(\mu,\overline{p}(\mu))\ ,\ \mu\in\mathcal{I}_0 \text{ and} \\ \Psi(p) &= \frac{\partial\tau}{\partial p}(\overline{\mu}(p),p)\ ,\ p\in\mathcal{J}_0. \end{aligned} \tag{4.126}$$

On the one hand,

$$\Phi(\mu_0) = \frac{\partial\tau}{\partial p}(\mu_0,p_0), \tag{4.127}$$

$$\begin{aligned} \Phi'(\mu_0) &= \frac{d}{d\mu}\left(\frac{\partial\tau}{\partial p}(\mu,\overline{p}(\mu))\right)_{\mu=\mu_0} = \left(\frac{\partial^2\tau}{\partial\mu\partial p}(\mu,\overline{p}(\mu)) + \frac{\partial^2\tau}{\partial p^2}(\mu,\overline{p}(\mu))\cdot\overline{p}'(\mu)\right)_{\mu=\mu_0} \\ &= \frac{\partial^2\tau}{\partial\mu\partial p}(\mu_0,p_0) + \frac{\partial^2\tau}{\partial p^2}(\mu_0,p_0)\cdot\overline{p}'(\mu_0) = \frac{\partial^2\tau}{\partial\mu\partial p}(\mu_0,p_0), \end{aligned} \tag{4.128}$$

since $\frac{\partial^2\tau}{\partial p^2}(\mu_0,p_0) = 0$.
On the other hand:

$$\Psi(p_0) = \frac{\partial\tau}{\partial p}(\mu_0,p_0) = 0, \tag{4.129}$$

$$\begin{aligned} \Psi'(p_0) &= \frac{d}{dp}\left(\frac{\partial\tau}{\partial p}(\overline{\mu}(p),p)\right)_{p=p_0} = \left(\frac{\partial^2\tau}{\partial\mu\partial p}(\overline{\mu}(p),p)\cdot\overline{\mu}'(p) + \frac{\partial^2\tau}{\partial p^2}(\overline{\mu}(p),p)\right)_{p=p_0} \\ &= \frac{\partial^2\tau}{\partial\mu\partial p}(\mu_0,p_0)\cdot\overline{\mu}'(p_0) + \frac{\partial^2\tau}{\partial p^2}(\mu_0,p_0) = 0, \end{aligned} \tag{4.130}$$

since $\overline{\mu}'(p_0) = 0$ and $\frac{\partial^2\tau}{\partial p^2}(\mu_0,p_0) = 0$.

$$\begin{aligned} \Psi''(p_0) &= \frac{d}{dp}\left(\frac{\partial^2\tau}{\partial\mu\partial p}(\overline{\mu}(p),p)\cdot\overline{\mu}'(p) + \frac{\partial^2\tau}{\partial p^2}(\overline{\mu}(p),p)\right)_{p=p_0} \\ &= \left(\left(\frac{\partial^3\tau}{\partial\mu^2\partial p}(\overline{\mu}(p),p)\cdot\overline{\mu}'(p) + \frac{\partial^3\tau}{\partial\mu\partial p^2}(\overline{\mu}(p),p)\right)\cdot\overline{\mu}'(p)\right. \\ &+ \left.\frac{\partial^2\tau}{\partial\mu\partial p}(\overline{\mu}(p),p)\cdot\overline{\mu}''(p) + \frac{\partial^3\tau}{\partial\mu\partial p^2}(\overline{\mu}(p),p)\cdot\overline{\mu_0}'(p) + \frac{\partial^3\tau}{\partial p^3}(\overline{\mu}(p),p)\right)_{p=p_0} \\ &= \left(\frac{\partial^3\tau}{\partial\mu^2\partial p}(\mu_0,p_0)\cdot\overline{\mu_0}'(p_0) + \frac{\partial^3\tau}{\partial\mu\partial p^2}(\mu_0,p_0)\right)\cdot\overline{\mu}'(p_0) \\ &+ \frac{\partial^2\tau}{\partial\mu\partial p}(\mu_0,p_0)\cdot\overline{\mu}''(p_0) + \frac{\partial^3\tau}{\partial\mu\partial p^2}(\mu_0,p_0)\cdot\overline{\mu_0}'(p_0) + \frac{\partial^3\tau}{\partial p^3}(\mu_0,p_0) \\ &= \frac{\partial^2\tau}{\partial\mu\partial p}(\mu_0,p_0)\cdot\overline{\mu}''(p_0) + \frac{\partial^3\tau}{\partial p^3}(\mu_0,p_0) \\ &= \frac{\partial^2\tau}{\partial\mu\partial p}(\mu_0,p_0)\cdot\left(-\frac{1}{3}\frac{\frac{\partial^3\tau}{\partial p^3}(\mu_0,p_0)}{\frac{\partial^2\tau}{\partial\mu\partial p}(\mu_0,p_0)}\right) + \frac{\partial^3\tau}{\partial p^3}(\mu_0,p_0) = \frac{2}{3}\frac{\partial^3\tau}{\partial p^3}(\mu_0,p_0). \end{aligned} \tag{4.131}$$

Summarizing,

$$\begin{array}{|lcl|lcl|} \hline \Phi(\mu_0) & = & 0 & \Psi(p_0) & = & 0 \\ \Phi'(\mu_0) & = & \frac{\partial \tau^2}{\partial \mu \partial p}(\mu_0, p_0) \neq 0 & \Psi'(p_0) & = & 0 \\ & & & \Psi''(p_0) & = & \frac{2}{3}\frac{\partial^3 \tau}{\partial p^3}(\mu_0, p_0) \neq 0 \\ \hline \end{array} \tag{4.132}$$

It follows that, $\exists \delta > 0$ such that

$$\Phi(\mu) = \Phi(\mu_0) + \Phi'(\mu_0)(\mu - \mu_0) + O(\mu - \mu_0)^2 \ , \ \forall \mu \in (\mu_0 - \delta, \mu_0 + \delta) \subseteq \mathcal{I}_0 \tag{4.133}$$

and

$$\Psi(p) = \Psi(p_0) + \Phi'(p_0)(p - p_0) + \frac{1}{2}\Phi''(p_0)(p - p_0)^2 + O(p - p_0)^3 \ , \ \forall p \in (p_0 - \delta, p_0 + \delta) \subseteq \mathcal{J}_0. \tag{4.134}$$

Namely,

$$\frac{\partial \tau}{\partial p}(\mu, \overline{p}(\mu)) = \frac{\partial \tau^2}{\partial \mu \partial p}(\mu_0, p_0)(\mu - \mu_0) + O(\mu - \mu_0)^2 \ , \ \forall \mu \in (\mu_0 - \delta, \mu_0 + \delta) \subseteq \mathcal{I}_0 \tag{4.135}$$

and

$$\frac{\partial \tau}{\partial p}(\overline{\mu}(p), p) = \frac{1}{3}\frac{\partial^3 \tau}{\partial p^3}(\mu_0, p_0)(p - p_0)^2 + O(p - p_0)^3 \ , \ \forall p \in (p_0 - \delta, p_0 + \delta) \subseteq \mathcal{J}_0. \tag{4.136}$$

Case 2. is obtained from **Case** 1. exchanging the roles of y_1 and y_2.

Theorem 4.8 Transcritical bifurcations

Let $(\mu_0, p_0) \in \mathcal{I} \times \mathbb{R}$ *such that:*

(*a*) $\kappa(\cdot;\mu_0,p_0) \in \mathcal{A}_\varrho$ *is an invariant curve;*

(*b*) $\lambda(\mu_0,p_0) = 1$ *or, equivalently,* $\Lambda(\mu_0,p_0) = 0$*;*

(*c*) $\displaystyle\int_{\mathbb{T}} \eta(\theta;\mu_0,p_0) n(\theta;\mu_0,p_0) d\theta \neq 0$*;*

(*d*) $\displaystyle\int_{\mathbb{T}} \eta(\theta;\mu_0,p_0) \frac{\partial m}{\partial p}(\theta;\mu_0,p_0) \frac{\partial \kappa}{\partial p}(\theta;\mu_0,p_0) d\theta \neq 0.$

(*e*) $$\int_{\mathbb{T}} \eta(\theta;\mu_0,p_0) \left(\frac{\partial m}{\partial \mu}(\theta;\mu_0,p_0) \frac{\partial \kappa}{\partial \mu}(\theta;\mu_0,p_0) + \frac{\partial n}{\partial \mu}(\theta;\mu_0,p_0) \right) d\theta \cdot \int_{\mathbb{T}} \eta(\theta;\mu_0,p_0) \frac{\partial m}{\partial p}(\theta;\mu_0,p_0) \frac{\partial \kappa}{\partial p}(\theta;\mu_0,p_0) d\theta$$
$$< \left(\int_{\mathbb{T}} \eta(\theta;\mu_0,p_0) \frac{\partial m}{\partial p}(\theta;\mu_0,p_0) \frac{\partial \kappa}{\partial p}(\theta;\mu_0,p_0) d\theta \right)^2.$$

Then, τ *has a transcritical bifurcation at* (μ_0, p_0)*, that is,*

(*i*) $(\mu_0,p_0) \in \mathcal{B}_\tau$*, i.e.* $\tau(\mu_0,p_0) = 0$*;*

(*ii*) $\dfrac{\partial \tau}{\partial \mu}(\mu_0,p_0) = 0$*;*

(*iii*) $\dfrac{\partial \tau}{\partial p}(\mu_0,p_0) = 0$*;*

(*iv*) $\dfrac{\partial^2 \tau}{\partial p^2}(\mu_0,p_0) \neq 0$*; and*

(*v*) $\det(\mathrm{Hess}\,\tau)(\mu_0,p_0) < 0.$

Additionally, $\exists\, \mathcal{I}_0 \times \mathcal{J}_0 \subseteq \mathcal{I} \times \mathbb{R}$ *(*$\mathcal{I}_0, \mathcal{J}_0$ *open intervals) and* $\exists\, \overline{p}_1, \overline{p}_2 : \mathcal{I}_0 \longrightarrow \mathcal{J}_0, \overline{p}_1, \overline{p}_2 \in \mathcal{C}^\infty$*, such that:*

$$\{(\mu,p) \in \mathcal{I}_0 \times \mathcal{J}_0 : \tau(\mu,p) = 0\} = \{(\mu,\overline{p}_1(\mu)) : \mu \in \mathcal{I}_0\} \cup \{(\mu,\overline{p}_2(\mu)) : \mu \in \mathcal{I}_0\}.$$

Moreover,

$$\begin{aligned} \overline{p}_i(\mu_0) &= p_0 \\ \overline{p}_i'(\mu_0) &= \frac{-\frac{\partial^2 \tau}{\partial \mu \partial p}(\mu_0,p_0) \pm \sqrt{-\det(\mathrm{Hess}\,\tau)(\mu_0,p_0)}}{\frac{\partial^2 \tau}{\partial p^2}(\mu_0,p_0)} \quad (i=1,2)\,. \end{aligned} \tag{4.137}$$

and for some $\delta > 0$*,*

$$\frac{\partial \tau}{\partial p}(\mu,\overline{p}_i(\mu)) = \pm\sqrt{-\det(\mathrm{Hess}\,\tau)(\mu_0,p_0)}(\mu-\mu_0) + O(\mu-\mu_0)^2, \forall \mu \in (\mu_0-\delta,\mu_0+\delta) \subseteq \mathcal{I}_0, \quad (i=1,2). \tag{4.138}$$

Proof. It only remains to prove that conditions $(a)-(e)$ are equivalent to conditions $(i)-(v)$ since the remainder of the statement is already proved above.

(i) If $\kappa(\cdot;\mu_0,p_0) \in \mathcal{A}_\varrho$ is an invariant curve, then the translation parameter is zero, i.e. $\tau(\mu_0,p_0) = 0$ and $(\mu_0,p_0) \in \mathcal{B}_\tau$.

(ii) Since $\lambda(\mu_0,p_0) = 1$, then by **Corollary 4.4**, $\dfrac{\partial \tau}{\partial p}(\mu_0,p_0) = 0$;

(iii) By **Corollary 4.5**

$$\begin{aligned}\frac{\partial\tau}{\partial\mu}(\mu_0,p_0) &= \tau^{10}(\mu_0,p_0) = -\frac{\xi_0^{10}(\mu_0,p_0)}{\eta_0(\mu_0,p_0)}\\ &= -\frac{<\eta(\cdot;\mu_0,p_0)\zeta^{10}(\cdot;\mu_0,p_0)>}{<\eta(\cdot;\mu_0,p_0)>} = -\frac{<\eta(\cdot;\mu_0,p_0)n(\cdot;\mu_0,p_0)>}{<\eta(\cdot;\mu_0,p_0)>}.\end{aligned}$$

Thus, $\frac{\partial\tau}{\partial\mu}(\mu_0,p_0)\neq 0 \Leftrightarrow <\eta(\cdot;\mu_0,p_0)n(\cdot;\mu_0,p_0)>= 0 \Leftrightarrow \int_{\mathbb{T}}\eta(\theta;\mu_0,p_0)n(\theta;\mu_0,p_0)d\theta\neq 0.$

(iv) By **Corollary 4.5**

$$\begin{aligned}\frac{\partial^2\tau}{\partial\mu\partial p}(\mu_0,p_0) &= \tau^{11}(\mu_0,p_0) = -\frac{\xi_0^{11}(\mu_0,p_0)}{\eta_0(\mu_0,p_0)}\\ &= -\frac{<\eta(\cdot;\mu_0,p_0)\zeta^{11}(\cdot;\mu_0,p_0)>}{<\eta(\cdot;\mu_0,p_0)>} = -\frac{<\eta(\cdot;\mu_0,p_0)\frac{\partial m}{\partial\mu}(\cdot;\mu_0,p_0)\frac{\partial\kappa}{\partial p}(\cdot;\mu_0,p_0)>}{<\eta(\cdot;\mu_0,p_0)>}.\end{aligned}$$

Thus,
$\frac{\partial^2\tau}{\partial\mu\partial p}(\mu_0,p_0)\neq 0 \Leftrightarrow <\eta(\cdot;\mu_0,p_0)n(\cdot;\mu_0,p_0)>\neq 0 \Leftrightarrow \int_{\mathbb{T}}\eta(\theta;\mu_0,p_0)\frac{\partial m}{\partial\mu}(\theta;\mu_0,p_0)\frac{\partial\kappa}{\partial p}(\theta;\mu_0,p_0)d\theta\neq 0.$

(v) $\det(\mathrm{Hess}\,\tau)(\mu_0,p_0) = \frac{\partial^2\tau}{\partial\mu^2}(\mu_0,p_0)\frac{\partial^2\tau}{\partial p^2}(\mu_0,p_0) - \left(\frac{\partial^2\tau}{\partial\mu\partial p}(\mu_0,p_0)\right)^2.$

Applying again **Corollary 4.5**,

$$\begin{aligned}\tfrac{\partial^2\tau}{\partial\mu^2}(\mu_0,p_0) &= \tau^{20}(\mu_0,p_0) = -\tfrac{\xi_0^{20}(\mu_0,p_0)}{\eta_0(\mu_0,p_0)}\\ \tfrac{\partial^2\tau}{\partial p^2}(\mu_0,p_0) &= \tau^{02}(\mu_0,p_0) = -\tfrac{\xi_0^{02}(\mu_0,p_0)}{\eta_0(\mu_0,p_0)}\\ \tfrac{\partial^2\tau}{\partial\mu\partial p}(\mu_0,p_0) &= \tau^{11}(\mu_0,p_0) = -\tfrac{\xi_0^{11}(\mu_0,p_0)}{\eta_0(\mu_0,p_0)}\end{aligned}$$

Therefore,
$\det(\mathrm{Hess}\,\tau)(\mu_0,p_0) = \frac{1}{(\eta_0(\mu_0,p_0))^2}\left(\xi_0^{20}(\mu_0,p_0)\xi_0^{02}(\mu_0,p_0) - \left(\xi_0^{11}(\mu_0,p_0)\right)^2\right).$

Thus,
$\det(\mathrm{Hess}\,\tau)(\mu_0,p_0) < 0 \Leftrightarrow \xi_0^{20}(\mu_0,p_0)\xi_0^{02}(\mu_0,p_0) < \left(\xi_0^{11}(\mu_0,p_0)\right)^2.$
and the latter is equivalent to

$$\int_{\mathbb{T}}\eta(\theta;\mu_0,p_0)\left(\frac{\partial m}{\partial\mu}(\theta;\mu_0,p_0)\frac{\partial\kappa}{\partial\mu}(\theta;\mu_0,p_0)+\frac{\partial n}{\partial\mu}(\theta;\mu_0,p_0)\right)d\theta\cdot\int_{\mathbb{T}}\eta(\theta;\mu_0,p_0)\frac{\partial m}{\partial p}(\theta;\mu_0,p_0)\frac{\partial\kappa}{\partial p}(\theta;\mu_0,p_0)d\theta$$
$$< \left(\int_{\mathbb{T}}\eta(\theta;\mu_0,p_0)\frac{\partial m}{\partial p}(\theta;\mu_0,p_0)\frac{\partial\kappa}{\partial p}(\theta;\mu_0,p_0)d\theta\right)^2.$$

□

Theorem 4.9 Pitchfork bifurcations

Let $(\mu_0, p_0) \in \mathcal{I} \times \mathbb{R}$ *such that:*

(a) $\kappa(\cdot; \mu_0, p_0) \in \mathcal{A}_\varrho$ *is an invariant curve;*

(b) $\lambda(\mu_0, p_0) = 1$ *or, equivalently,* $\Lambda(\mu_0, p_0) = 0$*;*

(c) $\int_{\mathbb{T}} \eta(\theta; \mu_0, p_0) n(\theta; \mu_0, p_0) d\theta = 0;$

(d) $\int_{\mathbb{T}} \eta(\theta; \mu_0, p_0) \frac{\partial m}{\partial p}(\theta; \mu_0, p_0) \frac{\partial \kappa}{\partial p}(\theta; \mu_0, p_0) d\theta = 0.$

(e) $\int_{\mathbb{T}} \eta(\theta; \mu_0, p_0) \frac{\partial m}{\partial \mu}(\theta; \mu_0, p_0) \frac{\partial \kappa}{\partial p}(\theta; \mu_0, p_0) d\theta \neq 0.$

(f) $\int_{\mathbb{T}} \eta(\theta; \mu_0, p_0) \left(2\frac{\partial m}{\partial p}(\theta; \mu_0, p_0) \frac{\partial^2 \kappa}{\partial p^2}(\theta; \mu_0, p_0) + \frac{\partial^2 m}{\partial p^2}(\theta; \mu_0, p_0) \frac{\partial \kappa}{\partial p}(\theta; \mu_0, p_0) \right) d\theta \neq 0.$

Then, τ *has a* pitchfork bifurcation *at* (μ_0, p_0)*, that is,*

(i) $\tau(\mu_0, p_0) = 0;$

(ii) $\frac{\partial \tau}{\partial p}(\mu_0, p_0) = 0;$

(iii) $\frac{\partial \tau}{\partial \mu}(\mu_0, p_0) = 0;$

(iv) $\frac{\partial^2 \tau}{\partial p^2}(\mu_0, p_0) = 0;$

(v) $\frac{\partial^2 \tau}{\partial \mu \partial p}(\mu_0, p_0) \neq 0;$

(vi) $\frac{\partial^3 \tau}{\partial p^3}(\mu_0, p_0) \neq 0;$

Additionally, $\exists\, \mathcal{I}_0 \times \mathcal{J}_0 \subseteq \mathcal{I} \times \mathbb{R}$ $(\mathcal{I}_0, \mathcal{J}_0$ *open intervals) and*

$$\exists\, \overline{\mu} : \mathcal{J}_0 \longrightarrow \mathcal{I}_0 \text{ and } \overline{p} : \mathcal{I}_0 \longrightarrow \mathcal{J}_0\ , \overline{\mu}, \overline{p} \in \mathcal{C}^\infty \text{ such that}$$

$$\{(\mu, p) \in \mathcal{I}_0 \times \mathcal{J}_0 : y_1(\mu, p) = 0\} = \{(\mu, \overline{p}(\mu)) : \mu \in \mathcal{I}_0\},$$

and

$$\{(\mu, p) \in \mathcal{I}_0 \times \mathcal{J}_0 : y_2(\mu, p) = 0\} = \{(\overline{\mu}(p), p) : p \in \mathcal{J}_0\}.$$

Equivalently,

$$\{(\mu, p) \in \mathcal{I}_0 \times \mathcal{J}_0 : \tau(\mu, p) = 0\} = \{(\mu, \overline{p}(\mu)) : \mu \in \mathcal{I}_0\} \cup \{(\overline{\mu}(p), p) : p \in \mathcal{J}_0\}.$$

Moreover,

$$\begin{array}{rcl} \overline{p}(\mu_0) & = & p_0 \\ \overline{p}'(\mu_0) & = & -\frac{1}{2} \frac{\frac{\partial^2 \tau}{\partial \mu^2}(\mu_0, p_0)}{\frac{\partial^2 \tau}{\partial \mu \partial p}(\mu_0, p_0)} \neq 0 \\ \overline{\mu}(p_0) & = & \mu_0 \\ \overline{\mu}'(p_0) & = & 0 \\ \overline{\mu}''(p_0) & = & -\frac{1}{3} \frac{\frac{\partial^3 \tau}{\partial p^3}(\mu_0, p_0)}{\frac{\partial^2 \tau}{\partial \mu \partial p}(\mu_0, p_0)} \neq 0 \end{array} . \tag{4.139}$$

and for some $\delta > 0$,

$$\frac{\partial \tau}{\partial p}(\mu, \overline{p}(\mu)) = \frac{\partial \tau^2}{\partial \mu \partial p}(\mu_0, p_0)(\mu - \mu_0) + O(\mu - \mu_0)^2 \ , \ \forall \mu \in (\mu_0 - \delta, \mu_0 + \delta) \subseteq \mathcal{I}_0 \tag{4.140}$$

and

$$\frac{\partial \tau}{\partial p}(\overline{\mu}(p), p) = \frac{1}{3}\frac{\partial^3 \tau}{\partial p^3}(\mu_0, p_0)(p - p_0)^2 + O(p - p_0)^3 \ , \ \forall p \in (p_0 - \delta, p_0 + \delta) \subseteq \mathcal{J}_0. \tag{4.141}$$

Proof. It only remains to prove that conditions $(a) - (f)$ are equivalent to conditions $(i) - (vi)$ since the remainder of the statement is already proved above.

(i) If $\kappa(\cdot; \mu_0, p_0) \in \mathcal{A}_\varrho$ is an invariant curve, then the translation parameter is zero, i.e. $\tau(\mu_0, p_0) = 0$ and $(\mu_0, p_0) \in \mathcal{B}_\tau$.

(ii) Since $\lambda(\mu_0, p_0) = 1$, then by **Corollary 4.4**, $\dfrac{\partial \tau}{\partial p}(\mu_0, p_0) = 0$;

(iii) By **Corollary 4.5**

$$\begin{aligned}\frac{\partial \tau}{\partial \mu}(\mu_0, p_0) &= \tau^{10}(\mu_0, p_0) = -\frac{\xi_0^{10}(\mu_0, p_0)}{\eta_0(\mu_0, p_0)} \\ &= -\frac{< \eta(\cdot; \mu_0, p_0)\zeta^{10}(\cdot; \mu_0, p_0) >}{< \eta(\cdot; \mu_0, p_0) >} = -\frac{< \eta(\cdot; \mu_0, p_0) n(\cdot; \mu_0, p_0) >}{< \eta(\cdot; \mu_0, p_0) >}.\end{aligned}$$

Thus, $\dfrac{\partial \tau}{\partial \mu}(\mu_0, p_0) = 0 \Leftrightarrow < \eta(\cdot; \mu_0, p_0) n(\cdot; \mu_0, p_0) >= 0 \Leftrightarrow \displaystyle\int_{\mathbb{T}} \eta(\theta; \mu_0, p_0) n(\theta; \mu_0, p_0) d\theta = 0.$

(iv) By **Corollary 4.5**

$$\begin{aligned}\frac{\partial^2 \tau}{\partial p^2}(\mu_0, p_0) &= \tau^{02}(\mu_0, p_0) = -\frac{\xi_0^{02}(\mu_0, p_0)}{\eta_0(\mu_0, p_0)} \\ &= -\frac{< \eta(\cdot; \mu_0, p_0)\zeta^{02}(\cdot; \mu_0, p_0) >}{< \eta(\cdot; \mu_0, p_0) >} = -\frac{< \eta(\cdot; \mu_0, p_0)\frac{\partial m}{\partial p}(\cdot; \mu_0, p_0)\frac{\partial \kappa}{\partial p}(\cdot; \mu_0, p_0) >}{< \eta(\cdot; \mu_0, p_0) >}.\end{aligned}$$

Thus,

$$\begin{aligned}\frac{\partial^2 \tau}{\partial p^2}(\mu_0, p_0) = 0 \quad &\Leftrightarrow \quad < \eta(\cdot; \mu_0, p_0)\frac{\partial m}{\partial p}(\cdot; \mu_0, p_0)\frac{\partial \kappa}{\partial p}(\cdot; \mu_0, p_0) >= 0 \\ &\Leftrightarrow \quad \int_{\mathbb{T}} \eta(\theta; \mu_0, p_0)\frac{\partial m}{\partial p}(\theta; \mu_0, p_0)\frac{\partial \kappa}{\partial p}(\theta; \mu_0, p_0) d\theta = 0 \,.\end{aligned}$$

(v) By **Corollary 4.5**

$$\begin{aligned}\frac{\partial^2 \tau}{\partial \mu \partial p}(\mu_0, p_0) &= \tau^{11}(\mu_0, p_0) = -\frac{\xi_0^{11}(\mu_0, p_0)}{\eta_0(\mu_0, p_0)} \\ &= -\frac{< \eta(\cdot; \mu_0, p_0)\zeta^{11}(\cdot; \mu_0, p_0) >}{< \eta(\cdot; \mu_0, p_0) >} = -\frac{< \eta(\cdot; \mu_0, p_0)\frac{\partial m}{\partial \mu}(\cdot; \mu_0, p_0)\frac{\partial \kappa}{\partial p}(\cdot; \mu_0, p_0) >}{< \eta(\cdot; \mu_0, p_0) >}.\end{aligned}$$

Thus,

$$\begin{aligned}\frac{\partial^2 \tau}{\partial \mu \partial p}(\mu_0, p_0) \neq 0 \quad &\Leftrightarrow \quad < \eta(\cdot; \mu_0, p_0)\frac{\partial m}{\partial \mu}(\cdot; \mu_0, p_0)\frac{\partial \kappa}{\partial p}(\cdot; \mu_0, p_0) >\neq 0 \\ &\Leftrightarrow \quad \int_{\mathbb{T}} \eta(\theta; \mu_0, p_0)\frac{\partial m}{\partial \mu}(\theta; \mu_0, p_0)\frac{\partial \kappa}{\partial p}(\theta; \mu_0, p_0) d\theta \neq 0 \,.\end{aligned}$$

(vi) By **Corollary 4.5**

$$\begin{aligned}\frac{\partial^3\tau}{\partial p^3}(\mu_0,p_0) &= \tau^{03}(\mu_0,p_0) = -\frac{\xi_0^{03}(\mu_0,p_0)}{\eta_0(\mu_0,p_0)}\\ &= -\frac{<\eta(\cdot;\mu_0,p_0)\zeta^{03}(\cdot;\mu_0,p_0)>}{<\eta(\cdot;\mu_0,p_0)>}\\ &= -\frac{<\eta(\cdot;\mu_0,p_0)\left(2\frac{\partial m}{\partial p}(\cdot;\mu_0,p_0)\frac{\partial^2\kappa}{\partial p^2}(\cdot;\mu_0,p_0)+\frac{\partial^2 m}{\partial p^2}(\cdot;\mu_0,p_0)\frac{\partial\kappa}{\partial p}(\cdot;\mu_0,p_0)\right)>}{<\eta(\cdot;\mu_0,p_0)>}.\end{aligned}$$

Thus,

$$\begin{aligned}0 &\neq \frac{\partial^3\tau}{\partial p^3}(\mu_0,p_0)\\ \Leftrightarrow 0 &\neq <\eta(\cdot;\mu_0,p_0)\left(2\frac{\partial m}{\partial p}(\cdot;\mu_0,p_0)\frac{\partial^2\kappa}{\partial p^2}(\cdot;\mu_0,p_0)+\frac{\partial^2 m}{\partial p^2}(\cdot;\mu_0,p_0)\frac{\partial\kappa}{\partial p}(\cdot;\mu_0,p_0)\right)>\\ \Leftrightarrow 0 &\neq \int_{\mathbb{T}}\eta(\theta;\mu_0,p_0)\left(2\frac{\partial m}{\partial p}(\theta;\mu_0,p_0)\frac{\partial^2\kappa}{\partial p^2}(\theta;\mu_0,p_0)+\frac{\partial^2 m}{\partial p^2}(\theta;\mu_0,p_0)\frac{\partial\kappa}{\partial p}(\theta;\mu_0,p_0)\right)d\theta.\end{aligned}$$

□

4.4 Period–doubling or flip bifurcations

Remark 4.10

The bifurcation diagram can be expressed in the following way:

$$\mathcal{B}_\tau = \{(\mu, p) \in \mathcal{I} \times \mathbb{R} : \tau(\mu, p) = 0\} = \{(\mu, p) \in \mathcal{I} \times \mathbb{R} : \upsilon(\mu, p) = p\},$$

where $\upsilon(\mu, p) = \tau(\mu, p) + p$.
Thus, the bifurcation points (μ_0, p_0) *are fixed points of the function* $\upsilon(\mu_0, \cdot)$*, i.e.*

$$(\mu_0, p_0) \in \mathcal{B}_\tau \Longleftrightarrow p_0 \in \operatorname{Fix} \tau(\mu_0, \cdot),$$

where $\operatorname{Fix} \tau(\mu_0, \cdot) := \{p \in \mathbb{R} : \upsilon(\mu_0, p) = p\}$*, for a given* $\mu_0 \in \mathcal{I}$.
Then, different kind of bifurcation points can be described in terms of υ*. For instance, pitchfork bifurcations are characterized by the following conditions:*

P(a). $\upsilon(\mu_0, p_0) = p_0$;

P(b). $\frac{\partial \upsilon}{\partial \mu}(\mu_0, p_0) = 0$;

P(c). $\frac{\partial \upsilon}{\partial p}(\mu_0, p_0) = 1$;

P(d). $\frac{\partial^2 \upsilon}{\partial p^2}(\mu_0, p_0) = 0$;

P(e). $\frac{\partial^2 \upsilon}{\partial \mu \partial p}(\mu_0, p_0) \neq 0$;

P(f). $\frac{\partial^3 \upsilon}{\partial p^3}(\mu_0, p_0) \neq 0$;

Let us consider a bifurcation point $(\mu_0, p_0) \in \mathcal{B}_\tau$ such that $\dfrac{\partial \upsilon}{\partial p}(\mu_0, p_0) = -1$, that is, (μ_0, p_0) is a continuable point w.r.t μ, or p_0 is a non–hyperbolic fixed point of $\upsilon(\mu_0, \cdot)$.
By the IFT, $\exists \mathcal{I}_0 \subseteq \mathcal{I}$, $\mathcal{J}_0 \subseteq \mathbb{R}$, open intervals such that $(\mu_0, p_0) \in \mathcal{I}_0 \times \mathcal{J}_0$, and $\exists\, \overline{p} : \mathcal{I}_0 \longrightarrow \mathcal{J}_0$, $\overline{p} \in \mathcal{C}^\infty(\mathcal{I}_0)$, such that $\overline{p}(\mu_0) = p_0$ and

$$\{(\mu, p) \in \mathcal{I}_0 \times \mathcal{J}_0 : \upsilon(\mu, p) = p\} = \{(\mu, \overline{p}(\mu) : \mu \in \mathcal{I}_0\}.$$

Thus,

$$\upsilon(\mu, \overline{p}(\mu) = \overline{p}(\mu), \ \forall \mu \in \mathcal{I}_0. \tag{4.142}$$

In particular, $\upsilon(\mu_0, p_0) = p_0$. Taking derivatives in (4.142),

$$\frac{\partial \upsilon}{\partial \mu}(\mu, \overline{p}(\mu)) + \frac{\partial \upsilon}{\partial p}(\mu, \overline{p}(\mu)) \cdot \overline{p}'(\mu) = \overline{p}'(\mu), \ \forall \mu \in \mathcal{I}_0. \tag{4.143}$$

In particular, for $\mu = \mu_0$,

$$\frac{\partial \upsilon}{\partial \mu}(\mu_0, p_0) + \frac{\partial \upsilon}{\partial p}(\mu_0, p_0) \cdot \overline{p}'(\mu_0) = \overline{p}'(\mu_0). \tag{4.144}$$

Since $\frac{\partial \upsilon}{\partial p}(\mu_0, p_0) = -1$, then

$$\overline{p}'(\mu_0) = \frac{1}{2}\frac{\partial \upsilon}{\partial \mu}(\mu_0, p_0). \tag{4.145}$$

Let $\Phi(\mu) = \frac{\partial \upsilon}{\partial p}(\mu, \overline{p}(\mu))$, $\mu \in \mathcal{I}_0$. Then,

$$\Phi(\mu_0) = \frac{\partial \upsilon}{\partial p}(\mu_0, p_0) = -1. \tag{4.146}$$

Moreover,

$$\begin{aligned}\Phi'(\mu_0) &= \frac{d}{d\mu}\left(\frac{\partial \upsilon}{\partial p}(\mu,\overline{p}(\mu))\right)\Big|_{\mu=\mu_0} = \left(\frac{\partial^2 \upsilon}{\partial\mu\partial p}(\mu,\overline{p}(\mu)) + \frac{\partial^2 \upsilon}{\partial p^2}(\mu,\overline{p}(\mu))\cdot \overline{p}'(\mu)\right)_{\mu=\mu_0} \\ &= \frac{\partial^2 \upsilon}{\partial\mu\partial p}(\mu_0,p_0) + \frac{\partial^2 \upsilon}{\partial p^2}(\mu_0,p_0)\cdot \overline{p}'(\mu_0) \\ &= \frac{1}{2}\frac{\partial^2 \upsilon}{\partial p^2}(\mu_0,p_0)\frac{\partial \upsilon}{\partial \mu}(\mu_0,p_0) + \frac{\partial^2 \upsilon}{\partial\mu\partial p}(\mu_0,p_0).\end{aligned} \tag{4.147}$$

If we assume that $\frac{1}{2}\frac{\partial^2 \upsilon}{\partial p^2}(\mu_0,p_0)\frac{\partial \upsilon}{\partial \mu}(\mu_0,p_0) + \frac{\partial^2 \upsilon}{\partial\mu\partial p}(\mu_0,p_0) \neq 0$, then $\exists \delta > 0$ such that $\forall \mu \in (\mu_0-\delta,\mu_0+\delta) \subseteq \mathcal{I}_0,\ \Phi(\mu) = \Psi(\mu_0) + \Psi'(\mu_0)(\mu-\mu_0) + O(\mu-\mu_0)^2$, that is,

$$\frac{\partial \upsilon}{\partial p}(\mu,\overline{p}(\mu)) = -1 + \left(\frac{1}{2}\frac{\partial^2 \upsilon}{\partial p^2}(\mu_0,p_0)\frac{\partial \upsilon}{\partial \mu}(\mu_0,p_0) + \frac{\partial^2 \upsilon}{\partial\mu\partial p}(\mu_0,p_0)\right)(\mu-\mu_0) + O(\mu-\mu_0)^2. \tag{4.148}$$

Consider now the function given by the second iterate,

$$\upsilon^2(\mu,p) = \upsilon(\mu,\upsilon(\mu,p))\ ,(\mu,p) \in \mathcal{I}_0^* \times \mathcal{J}_0, \tag{4.149}$$

where $\mathcal{I}_0^* = \{\mu \in \mathcal{I}_0 : \upsilon(\mu,p) \in \mathcal{J}_0,\ \forall p \in \mathcal{I}_0\}$.
Notice that $\upsilon^2(\mu_0,p_0) = \upsilon(\mu_0,\upsilon(\mu_0,p_0)) = \upsilon(\mu_0,p_0) = p_0$.
Differentiating (4.149),

$$\frac{\partial \upsilon^2}{\partial \mu}(\mu,p) = \frac{\partial \upsilon}{\partial \mu}(\mu,\upsilon(\mu,p)) + \frac{\partial \upsilon}{\partial p}(\mu,\upsilon(\mu,p))\cdot\frac{\partial \upsilon}{\partial \mu}(\mu,p) \tag{4.150}$$

$$\frac{\partial \upsilon^2}{\partial p}(\mu,p) = \frac{\partial \upsilon}{\partial p}(\mu,\upsilon(\mu,p))\cdot\frac{\partial \upsilon}{\partial p}(\mu,p). \tag{4.151}$$

In particular, for $\mu = \mu_0$ and $p = \overline{p}(\mu_0) = p_0$,

$$\begin{aligned}\frac{\partial \upsilon^2}{\partial \mu}(\mu_0,p_0) &= \frac{\partial \upsilon}{\partial \mu}(\mu_0,p_0) + \frac{\partial \upsilon}{\partial p}(\mu_0,p_0)\cdot\frac{\partial \upsilon}{\partial \mu}(\mu_0,p_0) \\ &= \frac{\partial \upsilon}{\partial \mu}(\mu_0,p_0)\left(1 + \frac{\partial \upsilon}{\partial p}(\mu_0,p_0)\right) = 0\end{aligned} \tag{4.152}$$

$$\frac{\partial \upsilon^2}{\partial p}(\mu_0,p_0) = \frac{\partial \upsilon}{\partial p}(\mu_0,p_0)\cdot\frac{\partial \upsilon}{\partial p}(\mu,p) = (-1)^2 = 1. \tag{4.153}$$

Derivating (4.150) w.r.t μ,

$$\begin{aligned}\frac{\partial^2 \upsilon^2}{\partial \mu^2}(\mu,p) &= \frac{\partial^2 \upsilon}{\partial \mu^2}(\mu,\upsilon(\mu,p)) + \frac{\partial^2 \upsilon}{\partial\mu\partial p}(\mu,\upsilon(\mu,p))\cdot\frac{\partial \upsilon}{\partial \mu}(\mu,p) \\ &+ \left(\frac{\partial^2 \upsilon}{\partial\mu\partial p}(\mu,\upsilon(\mu,p)) + \frac{\partial^2 \upsilon}{\partial p^2}(\mu,\upsilon(\mu,p))\cdot\frac{\partial \upsilon}{\partial \mu}(\mu,p)\right)\cdot\frac{\partial \upsilon}{\partial \mu}(\mu,p) \\ &+ \frac{\partial \upsilon}{\partial p}(\mu,p)\cdot\frac{\partial^2 \upsilon}{\partial \mu^2}(\mu,p)\ ,\forall(\mu,p) \in \mathcal{I}_0^* \times \mathcal{J}_0.\end{aligned}$$

In particular, for $\mu = \mu_0$ and $p = \overline{p}(\mu_0) = p_0$,

$$\begin{aligned}\frac{\partial^2 \upsilon^2}{\partial \mu^2}(\mu_0, p_0) &= \frac{\partial^2 \upsilon}{\partial \mu^2}(\mu_0, p_0) + \frac{\partial^2 \upsilon}{\partial \mu \partial p}(\mu_0, p_0) \cdot \frac{\partial \upsilon}{\partial \mu}(\mu_0, p_0) \\ &+ \left(\frac{\partial^2 \upsilon}{\partial \mu \partial p}(\mu_0, p_0) + \frac{\partial^2 \upsilon}{\partial p^2}(\mu_0, p_0) \cdot \frac{\partial \upsilon}{\partial \mu}(\mu_0, p_0)\right) \cdot \frac{\partial \upsilon}{\partial \mu}(\mu_0, p_0) \\ &+ \frac{\partial \upsilon}{\partial p}(\mu_0, p_0) \cdot \frac{\partial^2 \upsilon}{\partial \mu^2}(\mu_0, p_0) \\ &= 2\left(\frac{1}{2}\frac{\partial \upsilon^2}{\partial p^2}(\mu_0, p_0)\frac{\partial \upsilon}{\partial \mu}(\mu_0, p_0) + \frac{\partial^2 \upsilon}{\partial \mu \partial p}(\mu_0, p_0)\right) \cdot \frac{\partial \upsilon}{\partial \mu}(\mu_0, p_0) \\ &= 2\Phi'(\mu_0) \cdot \frac{\partial \upsilon}{\partial \mu}(\mu_0, p_0). \qquad (4.154)\end{aligned}$$

Derivating (4.151) w.r.t p,

$$\begin{aligned}\frac{\partial^2 \upsilon^2}{\partial p^2}(\mu, p) &= \frac{\partial^2 \upsilon}{\partial p^2}(\mu, \upsilon(\mu, p)) \left(\frac{\partial \upsilon}{\partial p}(\mu, p)\right)^2 \\ &+ \frac{\partial \upsilon}{\partial p}(\mu, \upsilon(\mu, p)) \cdot \frac{\partial^2 \upsilon}{\partial p^2}(\mu, p) \ , \forall (\mu, p) \in \mathcal{I}_0^* \times \mathcal{J}_0. \qquad (4.155)\end{aligned}$$

In particular, for $\mu = \mu_0$ and $p = \overline{p}(\mu_0) = p_0$,

$$\begin{aligned}\frac{\partial^2 \upsilon^2}{\partial p^2}(\mu_0, p_0) &= \frac{\partial^2 \upsilon}{\partial p^2}(\mu_0, p_0) \left(\frac{\partial \upsilon}{\partial p}(\mu_0, p_0)\right)^2 + \frac{\partial \upsilon}{\partial p}(\mu_0, p_0) \cdot \frac{\partial^2 \upsilon}{\partial p^2}(\mu_0, p_0) \\ &= \frac{\partial^2}{\partial p^2}(\mu_0, p_0)(-1)^2 - \frac{\partial^2 \upsilon}{\partial p^2}(\mu_0, p_0) = 0. \qquad (4.156)\end{aligned}$$

Derivating (4.151) w.r.t μ,

$$\begin{aligned}\frac{\partial^2 \upsilon^2}{\partial \mu \partial p}(\mu, p) &= \left(\frac{\partial^2 \upsilon}{\partial \mu \partial p}(\mu, \upsilon(\mu, p)) + \frac{\partial^2 \upsilon}{\partial p^2}(\mu, \upsilon(\mu, p)) \cdot \frac{\partial \upsilon}{\partial \mu}(\mu, p)\right) \cdot \frac{\partial \upsilon}{\partial p}(\mu, p) \\ &+ \frac{\partial \upsilon}{\partial p}(\mu, \upsilon(\mu, p)) \cdot \frac{\partial^2 \upsilon}{\partial \mu \partial p}(\mu, p) \ , \forall (\mu, p) \in \mathcal{I}_0^* \times \mathcal{J}_0. \qquad (4.157)\end{aligned}$$

In particular, for $\mu = \mu_0$ and $p = \overline{p}(\mu_0) = p_0$,

$$\begin{aligned}\frac{\partial^2 \upsilon^2}{\partial \mu \partial p}(\mu_0, p_0) &= \left(\frac{\partial^2 \upsilon}{\partial \mu \partial p}(\mu_0, p_0) + \frac{\partial^2 \upsilon}{\partial p^2}(\mu_0, p_0) \cdot \frac{\partial \upsilon}{\partial \mu}(\mu_0, p_0)\right) \cdot \frac{\partial \upsilon}{\partial p}(\mu_0, p_0) \\ &+ \frac{\partial \upsilon}{\partial p}(\mu_0, p_0) \cdot \frac{\partial^2 \upsilon}{\partial \mu \partial p}(\mu_0, p_0) \qquad (4.158) \\ &= -2\left(\frac{1}{2}\frac{\partial \upsilon^2}{\partial p^2}(\mu_0, p_0)\frac{\partial \upsilon}{\partial \mu}(\mu_0, p_0) + \frac{\partial^2 \upsilon}{\partial \mu \partial p}(\mu_0, p_0)\right) = -2\Phi'(\mu_0) \neq 0. \qquad (4.159)\end{aligned}$$

Finally, taking derivatives in (4.155) w.r.t. p,

$$\begin{aligned}\frac{\partial^3 \upsilon^2}{\partial p^3}(\mu, p) &= \frac{\partial^3 \upsilon}{\partial p^3}(\mu, \upsilon(\mu, p)) \left(\frac{\partial \upsilon}{\partial p}(\mu, p)\right)^3 \\ &+ 3\frac{\partial^2 \upsilon}{\partial p^2}(\mu, \upsilon(\mu, p))\frac{\partial \upsilon}{\partial p}(\mu, p)\frac{\partial^2 \upsilon}{\partial p^2}(\mu, p) \qquad (4.160) \\ &+ \frac{\partial \upsilon}{\partial p}(\mu, \upsilon(\mu, p))\frac{\partial^3 \upsilon}{\partial p^3}(\mu, p) \ , \forall (\mu, p) \in \mathcal{I}_0^* \times \mathcal{J}_0. \qquad (4.161)\end{aligned}$$

Particularizing for $\mu = \mu_0$ and $p = \overline{p}(\mu_0) = p_0$,

$$\begin{aligned}
\frac{\partial^3 \upsilon^2}{\partial p^3}(\mu_0, p_0) &= \frac{\partial^3 \upsilon}{\partial p^3}(\mu_0, p_0)\left(\frac{\partial \upsilon}{\partial p}(\mu_0, p_0)\right)^3 \\
&+ 3\frac{\partial^2 \upsilon}{\partial p^2}(\mu_0, p_0)\frac{\partial \upsilon}{\partial p}(\mu_0, p_0)\frac{\partial^2 \upsilon}{\partial p^2}(\mu_0, p_0) \\
&+ \frac{\partial \upsilon}{\partial p}(\mu_0, p_0)\frac{\partial^3 \upsilon}{\partial p^3}(\mu_0, p_0) \\
&= -2\frac{\partial^3 \upsilon}{\partial p^3}(\mu_0, p_0) - 3\left(\frac{\partial^2 \upsilon}{\partial p^2}(\mu_0, p_0)\right)^2 = 2S_p\upsilon(\mu_0, p_0),
\end{aligned} \tag{4.162}$$

where

$$\begin{aligned}
S_p\upsilon(\mu_0, p_0) &:= \frac{\frac{\partial^3 \upsilon}{\partial p^3}(\mu_0, p_0)}{\frac{\partial \upsilon}{\partial p}(\mu_0, p_0)} - \frac{3}{2}\left(\frac{\frac{\partial^2 \upsilon}{\partial p^2}(\mu_0, p_0)}{\frac{\partial \upsilon}{\partial p}(\mu_0, p_0)}\right)^2 \\
&= \frac{1}{2}\left(-2\frac{\partial^3 \upsilon}{\partial p^3}(\mu_0, p_0) - 3\left(\frac{\partial^2 \upsilon}{\partial p^2}(\mu_0, p_0)\right)^2\right) \\
&= \frac{1}{2}\frac{\partial^3 \upsilon^2}{\partial p^3}(\mu_0, p_0)
\end{aligned} \tag{4.163}$$

is the *Schwarzian derivative* of $\upsilon(\mu_0, \cdot)$ at p_0.

Chapter 5

Numerical aspects related to Fourier series, discrete Fourier transform (DFT), and cohomological equations

5.1 Introduction

The fundamental objective of this chapter is the introduction of all those concepts necessary for the numerical implementation of the procedures described in the previous chapters. Mainly, cohomological equations and the derived computation of the Floquet transformation of a curve. Subject to these procedures is everything related to the numerical implementation of Fourier series and the Fourier coefficients of a function.

Section 5.2 is devoted to introduce the discrete Fourier transform (DFT) and its inverse (IDFT), definitions and some of those properties which will be used later on in the computations. These tools constitute an efficient way to compute functions given by their Fourier series expansion on the torus. In Section 5.3 we introduce a method to compute numerically the Fourier coefficients of a function by means of the DFT, providing moreover an estimate of the error made in the aforementioned approximation. With a finite collection of Fourier coefficients it is possible to reconstruct, by means of the convolution with the Dirichlet kernel, the partial sums of the Fourier series. This is explained in Section 5.4. Moreover, there is an efficient way to reconstruct functions from their Fourier coefficients employing the inverse discrete Fourier transform (IDFT). Once we have solved the problem of the numerical implementation for Fourier series and Fourier coefficients, we are in a position to solve cohomological equations and, as a particular case, to compute the Floquet transformation of a given curve, which is necessary in the reducibility process of skew–products. These aspects will be dealt with in the last section of the chapter, Section 5.6.

5.2 The discrete Fourier Transform (DFT) and its inverse (IDFT)

This section is devoted to providing definitions and fundamental properties of the Discrete Fourier Transform (DFT) and its inverse (IDFT). It is not intended to make an exhaustive study of digital signal processing (DSP), but only of some tools necessary for implementing those procedures involved in the numerical simulation of the KAM algorithm designed in previous chapters and the corresponding error control[1], as will be shown in the following sections[2].

Given $N \in \mathbb{N}$ define the set[3] $\mathbb{Z}_N = \{1, 2, \ldots, N\}$ and let

$$l^2(\mathbb{Z}_N) = \{x = (x_k)_{k\in\mathbb{Z}} : x_k \in \mathbb{C}, x_{k+N} = x_k, \ \forall k \in \mathbb{Z}\}.$$

that is, the set of all the both–sided N–periodic sequences of complex numbers.
One element $x = (x_k)_{k\in\mathbb{Z}}$ of $l^2(\mathbb{Z}_N)$ can be represented by the terms corresponding to one period. We will consider here the convention of denoting by $x = [x(1), x(2), \ldots, x(N)]$ the elements of $l^2(\mathbb{Z}_N)$. This space has a natural structure of a $\mathbb{C}$–vector space. Moreover, it is a Hilbert space with the inner product

$$\begin{array}{rccc} < \cdot, \cdot >: & l^2(\mathbb{Z}_N) \times l^2(\mathbb{Z}_N) & \longrightarrow & \mathbb{C} \\ & (x = [x(1), x(2), \ldots, x(N)], y = [y(1), y(2), \ldots, y(N)]) & \longmapsto & < x, y > := \sum_{k=1}^{N} x(k)\overline{y(k)} \,. \end{array}$$

One orthonormal basis is $\mathcal{E} = \{e_1, \ldots, e_N\}$ with $e_j(k) = \delta_{jk}$, $\forall j, k \in \mathbb{Z}_N$, and the dimension of the space is $\dim_{\mathbb{C}}(l^2(\mathbb{Z}_N)) = N$, hence it is isomorphic to $\mathbb{C}^N$.

Definition 5.1 DFT, IDFT

Given $N \in \mathbb{N}$, the Discrete Fourier Transform (DFT) is defined as the linear operator

$$\begin{array}{rccc} \mathfrak{F}_N : & l^2(\mathbb{Z}_N) & \longrightarrow & l^2(\mathbb{Z}_N) \\ & x = [x(1), x(2), \ldots, x(N)] & \longmapsto & \mathfrak{F}_N x := X = [X(1), X(2), \ldots, X(N)] \end{array}$$

where

$$X(k+1) = \sum_{n=0}^{N-1} x(n+1) W_N^{kn} \,, \ \forall k = 0, 1, \ldots, N-1 \tag{5.1}$$

and $W_N = e^{-\frac{2\pi}{N} i}$. This operator is invertible and the inverse is the so–called Inverse Discrete Fourier Transform (IDFT) *which, in fact, is defined by*

$$\begin{array}{rccc} \mathfrak{F}_N^{-1} : & l^2(\mathbb{Z}_N) & \longrightarrow & l^2(\mathbb{Z}_N) \\ & X = [X(1), X(2), \ldots, X(N)] & \longmapsto & \mathfrak{F}_N^{-1} X := x = [x(1), x(2), \ldots, x(N)] \end{array}$$

with

$$x(n+1) = \frac{1}{N} \sum_{k=0}^{N-1} X(k+1) W_N^{-kn} \,, \ \forall n = 0, 1, \ldots, N-1 \,. \tag{5.2}$$

◇

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[1] For extended documentation and more details the reader is referred to the voluminous literature about the DFT e.g. [6, 13, 20, 56, 28]. .

[2] In this way, it is about justifying the programming making exposed in **Appendix II**, which are implemented in Matlab® programming environment (R2022b).

[3] This set can be considered as a representative of the cyclic Abelian group of order N, typically represented as $\mathbb{Z}/_N\mathbb{Z} = \{0, 1, \ldots, N-1\} = \{1, 2, \ldots, N\}$. In our case, the operation of the group is derived from the usual sum of integers modulus N and the zero element is $0 = N$.

Remark 5.2

The factor $W_N = e^{-\frac{2\pi}{N}i}$ *is an* N*–th root of the unity, that is,* $W_N^N = 1$ *. Moreover, given any* $m \in \mathbb{Z}$*,*

$$1 + W_N^m + W_N^{2m} + \cdots + W_N^{(N-1)m} = \begin{cases} N & , \quad \text{if } m = \dot{N} \\ 0 & , \quad \text{otherwise} \end{cases}$$

Notice that $\mathcal{G} = \{W_N, W_N^2, \ldots, W_N^{N-1}, W_N^N = 1\}$ *is an Abelian cyclic group and* $\mathcal{G} \cong \mathbb{Z}_N$*.*
Thus, the above definitions are consistent with the fact that the sequences involved are N*–periodic. Indeed,*

$$\begin{aligned} X(k+N+1) &= \sum_{n=0}^{N-1} x(n+1) W_N^{(k+N)n} = \sum_{n=0}^{N-1} x(n+1) W_N^{kn} W_N^{Nn} \\ &= \sum_{n=0}^{N-1} x(n+1) W_N^{kn} = X(k+1), \ \forall k \in \mathbb{Z}. \end{aligned}$$

In the same manner,

$$\begin{aligned} x(n+N+1) &= \frac{1}{N} \sum_{k=0}^{N-1} X(k+1) W_N^{-(n+N)k} = \frac{1}{N} \sum_{k=0}^{N-1} X(k+1) W_N^{-nk} W_N^{-Nk} \\ &= \frac{1}{N} \sum_{k=0}^{N-1} X(k+1) W_N^{-nk} = x(n+1), \ \forall n \in \mathbb{Z}. \end{aligned}$$

Notice that $\mathfrak{F}_N^{-1}$ is effectively the inverse operator of $\mathfrak{F}_N$ and this fact justifies the notation. Indeed, for any $x = [x(1), x(2), \ldots, x(N)] \in l^2(\mathbb{Z}_N)$ we have:

$$\begin{aligned} \mathfrak{F}_N^{-1}(\mathfrak{F}_N x)(n+1) &= \frac{1}{N} \sum_{k=0}^{N-1} (\mathfrak{F}_N x)(n+1) W_N^{-nk} = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{m=0}^{N-1} x(m+1) W_N^{mk} W_N^{-nk} \\ &= \frac{1}{N} \sum_{m=0}^{N-1} x(m+1) \sum_{k=0}^{N-1} W_N^{k(m-n)} = \frac{1}{N} \sum_{m=0}^{N-1} x(m+1) N \delta_{mn} \\ &= \frac{1}{N} x(n+1) N = x(n+1), \ \forall n \in 0, 1, \ldots, N-1. \end{aligned}$$

Therefore, $\mathfrak{F}_N^{-1} \mathfrak{F}_N x = x, \forall x \in l^2(\mathbb{Z}_N)$. Analogously, $\mathfrak{F}_N \mathfrak{F}_N^{-1} X = X, \forall X \in l^2(\mathbb{Z}_N)$.

Before stating some properties of the DFT we need to define some linear operators.

Definition 5.3 Conjugation, shift, reversal, and rotation operators

The conjugation operator produces the complex conjugation of all the terms of the sequence.

$$\begin{array}{rccl} \mathfrak{C}_N : & l^2(\mathbb{Z}_N) & \longrightarrow & l^2(\mathbb{Z}_N) \\ & x = [x(1), x(2), \ldots, x(N)] & \longmapsto & \mathfrak{C}_N x = \bar{x} = [\overline{x(1)}, \overline{x(2)}, \ldots, \overline{x(N)}] \,. \end{array} \tag{5.3}$$

The shift operator displaces the terms of the sequence one position to the right.

$$\begin{array}{rccl} \mathfrak{S}_N : & l^2(\mathbb{Z}_N) & \longrightarrow & l^2(\mathbb{Z}_N) \\ & x = [x(1), x(2), \ldots, x(N)] & \longmapsto & \mathfrak{S}_N x = [x(N), x(1), \ldots, x(N-1)] \,. \end{array} \tag{5.4}$$

The reversal operator reverses the order of the elements in a period.

$$\begin{array}{rccl} \mathfrak{I}_N : & l^2(\mathbb{Z}_N) & \longrightarrow & l^2(\mathbb{Z}_N) \\ & x = [x(1), x(2), \ldots, x(N)] & \longmapsto & \mathfrak{I}_N x = [x(N), x(N-1), \ldots, x(1)] \,. \end{array} \tag{5.5}$$

The rotation operator rotates each element $x(n+1)$ by multiplication by the factor W_N^n.

$$\begin{array}{llll} \mathfrak{R}_N : & l^2(\mathbb{Z}_N) & \longrightarrow & l^2(\mathbb{Z}_N) \\ & x = [x(1), x(2), \ldots, x(N)] & \longmapsto & \mathfrak{R}_N x = [W_N^0 x(1), W_N^1 x(2), \ldots, W_N^{N-1} x(N)] \,. \end{array} \tag{5.6}$$

◇

REMARK 5.4

Notice that $\mathfrak{C}_N^2 = \mathrm{I}_N$, $\mathfrak{S}_N^N = \mathrm{I}_N$, $\mathfrak{I}_N^2 = \mathrm{I}_N$, and $\mathfrak{R}_N^N = \mathrm{I}_N$, where I_N denotes here the identity operator.
Moreover, $\mathfrak{S}_N = \mathfrak{F}_N^{-1}\mathfrak{R}_N\mathfrak{F}_N$, i.e. $\mathfrak{S}_N$ is the conjugate of $\mathfrak{R}_N$ by means of the DFT $\mathfrak{F}_N$.

Proof. Let $x = [x(1), x(2), \ldots, x(N)] \in l^2(\mathbb{Z}_N)$, $X = \mathfrak{F}_N x = [X(1), X(2), \ldots, X(N)]$, $y = \mathfrak{S}_N x = [y(1), y(2), \ldots, y(N)]$, $Y = \mathfrak{F}_N y = [Y(1), Y(2), \ldots, Y(N)]$, and $Z = \mathfrak{R}_N X = [Z(1), Z(2), \ldots, Z(N)]$. Then, $Z = \mathfrak{R}_N\mathfrak{F}_N x = [W_N^0 X(1), W_N^1 X(2), \ldots, W_N^{N-1} X(N)]$. Thus,

$$\begin{aligned} Z(k+1) &= W_N^k X(k+1) = W_N^k \sum_{n=0}^{N-1} x(n+1) W_N^{kn} = \sum_{n=0}^{N-1} x(n+1) W_N^{k(n+1)} \\ &= \sum_{n=0}^{N-1} x(n) W_N^{kn} = \sum_{n=0}^{N-1} y(n+1) W_N^{kn} = Y(k+1), \ \forall k = 0, 1, \ldots, N-1. \end{aligned}$$

Therefore, $Z = Y$ and this implies that $\mathfrak{R}_N X = \mathfrak{F}_N y \Rightarrow \mathfrak{R}_N\mathfrak{F}_N x = \mathfrak{F}_N\mathfrak{S}_N x$, $\forall x \in l^2(Z_N) \Rightarrow \mathfrak{R}_N\mathfrak{F}_N = \mathfrak{F}_N\mathfrak{S}_N \Rightarrow \mathfrak{S}_N = \mathfrak{F}_N^{-1}\mathfrak{R}_N\mathfrak{F}_N$. Moreover, $\mathfrak{S}_N^m = (\mathfrak{F}_N^{-1}\mathfrak{R}_N\mathfrak{F}_N)^m = \mathfrak{F}_N^{-1}\mathfrak{R}_N^m\mathfrak{F}_N$, $\forall m \in \mathbb{Z}$.

□

For the sake of clarity, one may omit the subindex N in the notation of the operators, at least when there is no ambiguity.

Next we state a series of fundamental properties of the DFT that can be demonstrated by applying the definitions.

Proposition 5.5 Properties of the DFT

(*a*) *Linearity* $\forall x, y \in l^2(\mathbb{Z}_N)$, $\alpha, \beta \in \mathbb{C}, \mathfrak{F}_N(\alpha x + \beta y) = \alpha\mathfrak{F}_N x + \beta\mathfrak{F}_N y$.

(*b*) *Orthogonality*

$\{e_k = [W_N^{0k}, W_N^{1k}, \ldots, W_N^{(N-1)k}] : k = 0, 1, \ldots, N-1\}$ *is an orthogonal basis of* $l^2(\mathbb{Z}_N)$.

(*c*) *Plancherel theorem and Parseval's identity*

(Plancherel) *Let* $x = [x(1), x(2), \ldots, x(N)] \in l^2(\mathbb{Z}_N)$, $X = \mathfrak{F}_N x = [X(1), X(2), \ldots, X(N)]$, $y = [y(1), y(2), \ldots, y(N)] \in l^2(\mathbb{Z}_N)$, *and* $Y = \mathfrak{F}_N y = [Y(1), Y(2), \ldots, Y(N)]$. *Then,*

$$\sum_{n=0}^{N-1} x(n+1)\overline{y}(n+1) = \frac{1}{N}\sum_{k=0}^{N-1} X(k+1)\overline{Y}(k+1),$$

that is, $< x, y > = \frac{1}{N} < \mathfrak{F}_N x, \mathfrak{F}_N y >$, $\forall x, y \in l^2(\mathbb{Z}_N)$.

(Parseval) *Let* $x = [x(1), x(2), \ldots, x(N)] \in l^2(\mathbb{Z}_N)$. *Then,*

$$\sum_{n=0}^{N-1} |x(n+1)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k+1)|^2 \,,$$

that is,

$$\|x\|_{l^2(\mathbb{Z}_N)} = \frac{1}{\sqrt{N}} \|\mathfrak{F}_N x\|_{l^2(\mathbb{Z}_N)}.$$

(*d*) *Shift theorem.*

Let $x, y \in l^2(\mathbb{Z}_N)$, $X = \mathfrak{F}_N x$, $Y = \mathfrak{F}_N y$, *and* $m \in \{0, 1, \ldots, N-1\}$.

Time shifting *If* $y(n+1) = x(n+1-m)$, $\forall n = 0, 1, \ldots, N-1$ *(i.e.* $y = \mathfrak{S}_N^m x$*), then* $Y(k+1) = X(k+1)W_N^{mk}$, $\forall k = 0, 1, \ldots, N-1$ *(i.e.* $Y = \mathfrak{R}_N^m X$*).*

Frequency shifting *If* $y(n+1) = x(n+1)W_N^{-mn}$, $\forall n = 0, 1, \ldots, N-1$ *(i.e.* $x = \mathfrak{R}_N^m y$*), then* $Y(k+1) = X(k+1-m)$, $\forall k = 0, 1, \ldots, N-1$ *(i.e.* $Y = \mathfrak{S}_N^m X$*).*

(*e*) *Time and frequency reversal*

Time reversal $\forall x \in l^2(\mathbb{Z}_N)$, $\mathfrak{F}_N \mathfrak{I}_N x = \mathfrak{S}_N \mathfrak{I}_N \mathfrak{F}_N \mathfrak{S}_N x$.

Frequency reversal $\forall x \in l^2(\mathbb{Z}_N)$, $\mathfrak{I}_N \mathfrak{F}_N x = \mathfrak{F}_N \mathfrak{R}_N \mathfrak{S}_N \mathfrak{I}_N x$.

(*f*) *Conjugacy*

$\mathfrak{F}_N^{-1} = \frac{1}{N} \mathfrak{C}_N \mathfrak{F}_N \mathfrak{C}_N$.

(*g*) *Complex conjugate*

Time conjugate $\mathfrak{F}_N \mathfrak{C}_N = \mathfrak{C}_N \mathfrak{S}_N \mathfrak{I}_N \mathfrak{F}_N$.

Frequency conjugate $\mathfrak{C}_N \mathfrak{F}_N = \mathfrak{F}_N \mathfrak{C}_N \mathfrak{S}_N \mathfrak{I}_N$.

Proof.

(a) Linearity The linearity of the DFT is immediate from the definition.

Matrix representation of the DFT

Let $\mathcal{E} = \{e_{j+1}\}_{j=0}^{N-1}$ be the canonical basis of $l^2(\mathbb{Z}_N)$.

The DFT of the basis elements is given by $\widehat{e}_{j+1} = \mathfrak{F}_N e_{j+1}$, $j = 0, 1, \ldots, N$, where

$\widehat{e}_{j+1}(k+1) = \sum_{n=0}^{N-1} e_{j+1}(n+1) W_N^{kn} = W_N^{jk}$, $\forall j, k = 0, 1, \ldots, N$. Thus,

$$\widehat{e}_{j+1} = [1, W_N^j, W_N^{2j}, \ldots, W_N^{Nj}, W_N^{(N+1)j}], \ \forall j = 0, 1, \ldots, N.$$

The consequent matrix representation of these identities has the form

$$\begin{pmatrix} \widehat{e}_1 \\ \widehat{e}_2 \\ \vdots \\ \widehat{e}_N \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & W_N & W_N^2 & \ldots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \ldots & W_N^{2(N-1)j} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \ldots & W_N^{(N-1)(N-1)} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_N \end{pmatrix} . \tag{5.7}$$

The square matrix $A(W_N) \in GL_N(\mathbb{C})$ which appears in the above representation is useful to prove some important properties[4] of the DFT.

In a first sight, one can see that the matrix is symmetric. Some terms of the matrix can be often reduced using the fact that $W_N^{jN} = 1$, $\forall j \in \mathbb{Z}$. Despite this, if N is a prime number this reduction is not feasible. On the other hand note that $A(W_N)A(\overline{W_N}) = NI_N$, where I_N is the identity matrix, so $\mathfrak{F}_N$ is, indeed, invertible and the matrix of the IDFT, $\mathfrak{F}_N^{-1}$, w.r.t the canonical basis is $A(W_N)^{-1} = \frac{1}{N}A(\overline{W_N})$.

(b) Orthogonality

$$< e_k, e_l >= \sum_{n=0}^{N-1} W_N^{nk}\overline{W_N}^{nl} = \sum_{n=0}^{N-1} W_N^{n(k-l)} = N\delta_{kl}.$$

(c) Plancherel theorem and Parseval's identity

Plancherel

$$\begin{aligned}
< x, y > &= \sum_{n=0}^{N-1} x(n+1)\overline{y(n+1)} = \sum_{n=0}^{N-1}\left(\frac{1}{N}\sum_{k=0}^{N-1} X(k+1)W_N^{kn}\frac{1}{N}\sum_{l=0}^{N-1}\overline{Y(k+1)}W_N^{-ln}\right)\\
&= \frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} X(k+1)\overline{Y(k+1)}W_N^{n(k-l)}\\
&= \frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} X(k+1)\overline{Y(k+1)}N\delta_{k(n-l)}\\
&= \frac{1}{N}\sum_{k=0}^{N-1} X(k+1)\overline{Y(k+1)} = \frac{1}{N} < \mathfrak{F}_N x, \mathfrak{F}_N y > \ .
\end{aligned}$$

Parseval This identity is a particular case of the latter, when $y = x$.

(d) Shift theorem

Let $x, y \in l^2(\mathbb{Z}_N)$, $X = \mathfrak{F}_N x$, $Y = \mathfrak{F}_N y$, and $m \in \{0, 1, \ldots, N-1\}$.

Time shifting If $y(n+1) = x(n+1-m)$, $\forall n = 0, 1, \ldots, N-1$ (i.e. $y = \mathfrak{S}_N^m x$), then

$$\begin{aligned}
Y(k+1) &= \sum_{n=0}^{N-1} y(n+1)W_N^{kn} = \sum_{n=0}^{N-1} x(n+1-m)W_N^{kn} \qquad \text{(making } \nu = n-m\text{)}\\
&= \sum_{\nu=-m}^{N-1-m} x(\nu+1)W_N^{k(\nu+m)} = \sum_{\nu=-m}^{N-1-m} x(\nu+1)W_N^{\nu k}W_N^{mk}\\
&\qquad \text{(by the } N\text{–periodicity)}\\
&= \sum_{\nu=0}^{N-1} x(\nu+1)W_N^{\nu k}W_N^{mk} = X(k+1)W_N^{mk},\ \forall k = 0, 1, \ldots, N-1.
\end{aligned}$$

[4]For more information about the eigenstructure of DFT matrices see [12].

Frequency shifting If $y(n+1) = x(n+1)W_N^{-mn}$, $\forall n = 0, 1, \ldots, N-1$, then

$$\begin{aligned} Y(k+1) &= \sum_{n=0}^{N-1} y(n+1)W_N^{kn} = \sum_{n=0}^{N-1} x(n+1)W_N^{-nm}W_N^{kn} \\ &= \sum_{n=0}^{N-1} x(n+1)W_N^{n(k-m)} \quad (\text{calling } \bar{k} = k-m) \\ &= \sum_{n=0}^{N-1} y(n+1)W_N^{\bar{k}n} = X(\bar{k}+1) = X(k-m+1),\ \forall k = 0, 1, \ldots, N-1. \end{aligned}$$

(e) Time and frequency reversal

Let $x = [x(1), x(2), \ldots, x(N)] \in l^2(\mathbb{Z}_N)$, $X = \mathfrak{F}_N x = [X(1), X(2), \ldots, X(N)]$, $y = \mathfrak{I}_N x = [y(1), y(2), \ldots, y(N)]$, and $Y = \mathfrak{F}_N y = [Y(1), Y(2), \ldots, Y(N)]$. Then,

$$\begin{aligned} Y(k+1) &= \sum_{n=0}^{N-1} y(n+1)W_N^{nk} = \sum_{n=0}^{N-1} x(N-n)W_N^{nk} \ (\text{inverting the order of the sum}) \\ &= \sum_{n=0}^{N-1} x(n+1)W_N^{(N-1-n)k} = \sum_{n=0}^{N-1} x(n+1)W_N^{N(n-k)}W_N^{N-k} \\ &= W_N^{N-k}\sum_{n=0}^{N-1} x(n+1)W_N^{n(n-k)} = W_N^{N-k}X(N-k-1), \forall k = 0, 1, \ldots, N-1\,. \end{aligned}$$

(f) Conjugacy

Let $x = [x(1), x(2), \ldots, x(N)] \in l^2(\mathbb{Z}_N)$, $X = [X(1), X(2), \ldots, X(N)] = \mathfrak{F}_N x$,
$y = [y(1), y(2), \ldots, y(N)] = \mathfrak{C}_N X = [\overline{X(1)}, \overline{X(2)}, \ldots, \overline{X(N)}]$,
$Y = [Y(1), Y(2), \ldots, Y(N)] = \mathfrak{F}_N y$, and
$z = [z(1), z(2), \ldots, z(N)] = \mathfrak{C}_N Y = [\overline{Y(1)}, \overline{Y(2)}, \ldots, \overline{Y(N)}]$.
Then, $\forall n = 0, 1, \ldots, N-1$,

$$\begin{aligned} z(n+1) &= \overline{Y(n+1)} = \overline{\sum_{l=0}^{N-1} y(l+1)W_N^{ln}} = \overline{\sum_{l=0}^{N-1} \overline{X(l+1)}W_N^{ln}} \\ &= \overline{\sum_{l=0}^{N-1}\overline{\sum_{k=0}^{N-1} x(k+1)W_N^{kl}}W_N^{ln}} = \sum_{l=0}^{N-1}\sum_{k=0}^{N-1} x(k+1)W_N^{kl}W_N^{-ln} \\ &= \sum_{k=0}^{N-1} x(k+1)\sum_{l=0}^{N-1} W_N^{l(k-n)} = \sum_{k=0}^{N-1} x(k+1)N\delta_{kn} = Nx(n+1). \end{aligned}$$

Therefore, $x(n+1) = \frac{1}{N}z(n+1)$, $\forall n = 0, 1, \ldots, N$, and then
$x = \frac{1}{N}z = \frac{1}{N}\mathfrak{C}_N Y = \frac{1}{N}\mathfrak{C}_N\mathfrak{F}_N y = \frac{1}{N}\mathfrak{C}_N\mathfrak{F}_N\mathfrak{C}_N X = \frac{1}{N}\mathfrak{C}_N\mathfrak{F}_N\mathfrak{C}_N\mathfrak{F}_N x$, that is, $\mathrm{I}_N = \frac{1}{N}\mathfrak{C}_N\mathfrak{F}_N\mathfrak{C}_N\mathfrak{F}_N$ or equivalently, $\mathfrak{F}_N^{-1} = \frac{1}{N}\mathfrak{C}_N\mathfrak{F}_N\mathfrak{C}_N$.

This property allows to compute the IDFT using the direct DFT.

(g) Complex conjugate

□

Proposition 5.6 Properties of the DFT operators

(i) $\mathfrak{C}_N^2 = \mathrm{I}_N$;

(ii) $\mathfrak{I}_N^2 = \mathrm{I}_N$;

(iii) $\mathfrak{S}_N^N = \mathrm{I}_N$;

(iv) $\mathfrak{R}_N^N = \mathrm{I}_N$;
Let $x = [x(1), x(2), \ldots, x(N)] \in l^2(\mathbb{Z}_N)$, $X = \mathfrak{F}_N x = [X(1), X(2), \ldots, X(N)]$, $y = \mathfrak{I}_N x = [y(1), y(2), \ldots, y(N)]$, *and* $Y = \mathfrak{F}_N y = [Y(1), Y(2), \ldots, Y(N)]$. *Then,*

(v) ***Time conjugation*** $y = \mathfrak{C}_N x \Rightarrow Y = \mathfrak{C}_N \mathfrak{S}_N \mathfrak{I}_N X$, *i.e.* $\mathfrak{F}_N \mathfrak{C}_N = \mathfrak{C}_N \mathfrak{S}_N \mathfrak{I}_N \mathfrak{F}_N$;

(vi) ***Frequency conjugation*** $Y = \mathfrak{C}_N X \Rightarrow y = \mathfrak{S}_N \mathfrak{I}_N \mathfrak{C}_N x$, *i.e.* $\mathfrak{C}_N \mathfrak{F}_N = \mathfrak{F}_N \mathfrak{C}_N \mathfrak{S}_N \mathfrak{I}_N$;

(vii) ***Time reversal*** $y = \mathfrak{I}_N x \Rightarrow Y = \mathfrak{S}_N \mathfrak{I}_N \mathfrak{R}_N X$, *i.e.* $\mathfrak{F}_N \mathfrak{I}_N = \mathfrak{S}_N \mathfrak{I}_N \mathfrak{R}_N \mathfrak{F}_N$;

(viii) ***Frequency reversal*** $Y = \mathfrak{I}_N X \Rightarrow y = \mathfrak{R}_N \mathfrak{S}_N \mathfrak{I}_N x$, *i.e.* $\mathfrak{I}_N \mathfrak{F}_N = \mathfrak{F}_N \mathfrak{R}_N \mathfrak{S}_N \mathfrak{I}_N$;

(ix) ***Time shifting*** $y = \mathfrak{S}_N x \Rightarrow Y = \mathfrak{R}_N X$, *i.e.* $\mathfrak{F}_N \mathfrak{S}_N = \mathfrak{R}_N \mathfrak{F}_N$;

(x) ***Frequency shifting*** $y = \mathfrak{S}_N X \Rightarrow y = \mathfrak{R}_N^{-1} x$,*i.e.* $\mathfrak{S}_N \mathfrak{F}_N = \mathfrak{F}_N \mathfrak{R}_N^{-1}$.

Definition 5.7 Periodic Discrete Convolution (PDC)
The periodic discrete convolution is defined as the following inner operation in $l^2(\mathbb{Z}_N)$.

$$\begin{array}{lccl} \star : & l^2(\mathbb{Z}_N) \times l^2(\mathbb{Z}_N) & \longrightarrow & l^2(\mathbb{Z}_N) \\ & (x = [x(1), x(2), \ldots, x(N)], y = [y(1), y(2), \ldots, y(N)]) & \longmapsto & z = x \star y \end{array}$$

where

$$(x \star y)(n+1) := \sum_{m=0}^{N-1} x(m+1) y(n-m+1) \ , \ \forall n = 0, 1, \ldots, N-1 \ . \tag{5.8}$$

◇

Theorem 5.8 (PDC)
Let $x, y \in l^2(\mathbb{Z}_N)$. *Then*

$$\mathfrak{F}_N(x \star y)(k+1) = \mathfrak{F}_N x(k+1) \mathfrak{F}_N y(k+1), \ \forall k = 0, 1, \ldots N-1 \ .$$

Proof.
Let $X = [X(1), X(2), \ldots, X(N)] = \mathfrak{F}_N x$, $Y = [Y(1), Y(2), \ldots, Y(N)] = \mathfrak{F}_N y$, and $Z = [Z(1), Z(2), \ldots, Z(N)] = \mathfrak{F}_N(x \star y)$. Then:
$X(k+1)Y(k+1) = \sum_{m=0}^{N-1} x(m+1) W_N^{kn} \sum_{l=0}^{N-1} y(l+1) W_N^{kl} = \sum_{m=0}^{N-1} \sum_{l=0}^{N-1} x(m+1) y(l+1) W_N^{k(m+l)}$.
Now, taking $n = m + l$, we get:

$$\begin{aligned}
X(k+1)Y(k+1) &= \sum_{m=0}^{N-1}\sum_{n=m}^{N+m-1} x(m+1)y(n-m+1)W_N^{kn} \text{ (by the } N-\text{periodicity)}\\
&= \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} x(m+1)y(n-m+1)W_N^{kn}\\
&= \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} x(m+1)y(n-m+1)W_N^{kn}\\
&= \sum_{n=0}^{N-1} z(n+1)W_N^{kn} = Z(k+1),\ \forall k = 0,1,\ldots,N-1.
\end{aligned}$$

□

Remark 5.9

If $X, Y \in l^2(\mathbb{Z}_N)$,

$$(\mathfrak{F}_N^{-1}X \star \mathfrak{F}_N^{-1}Y)(n+1) = X(n+1)Y(n+1)\ ,\ \forall n = 0,1,\ldots,N-1.$$

This property is obtained directly from the PDC theorem taking $x = \mathfrak{F}_N^{-1}X$ and $y = \mathfrak{F}_N^{-1}Y$.

5.3 Fourier coefficients and the DFT approximation: error estimates

Lemma 5.10 Approximation of the Fourier coefficients

Let $u \in \mathcal{C}^1(\mathbb{T}, \mathbb{R})$ and $\widehat{u}_k = \int_{\mathbb{T}} u(\theta) e^{-2\pi k \theta i} d\theta,\ k \in \mathbb{Z}$ the Fourier coefficients of u. If we choose a partition $(\theta_0, \theta_1, \dots, \theta_N)$ on the torus $\mathbb{T}$, i.e. $0 = \theta_0 < \theta_1 < \cdots < \theta_{N-1} < \theta_N = 1$, then, for every $n = 0, 1, \dots, N-1$ there exists $\xi_n \in [\theta_n, \theta_{n+1}]$ such that the error in the approximation of the Fourier coefficient $\widehat{u}_k$ by means of

$$\widehat{u}_k^* = \sum_{n=0}^{N-1} (\theta_{n+1} - \theta_n) u(\theta_n) e^{-2\pi k \theta_n i},\ k \in \mathbb{Z} \tag{5.9}$$

is given by

$$E_k = \widehat{u}_k - \widehat{u}_k^* = \frac{1}{2} \sum_{n=0}^{N-1} w_k'(\xi_n)(\theta_{n+1} - \theta_n)^2\ , \tag{5.10}$$

where $w_k(\theta) = u(\theta) e^{-2\pi k \theta i},\ \forall \theta \in \mathbb{T}$.

Proof. First of all, the k–th Fourier coefficient of u can be expressed in the following way:

$$\widehat{u}_k = \int_{\mathbb{T}} u(\theta) e^{-2\pi k \theta i} d\theta = \sum_{n=0}^{N-1} \int_{\theta_n}^{\theta_{n+1}} u(\theta) e^{-2\pi k \theta i} d\theta = \sum_{n=0}^{N-1} \int_{\theta_n}^{\theta_{n+1}} w_k(\theta) d\theta\ , \tag{5.11}$$

where we have called $w_k(\theta) = u(\theta) e^{-2\pi k \theta i}$. Now, each of the integrals in the above sum can be approximated by means of any method among the extensive family of methods that are based on approximating the function to integrate. The simplest of all this methods is the so–called *left rectangle rule*:

$$\int_{\theta_n}^{\theta_{n+1}} w_k(\theta) d\theta \approx (\theta_{n+1} - \theta_n) w_k(\theta_n),\ n = 0, 1, \dots, N-1. \tag{5.12}$$

Set

$$\widehat{u}_k^* = \sum_{n=0}^{N-1} (\theta_{n+1} - \theta_n) w_k(\theta_n) \tag{5.13}$$

the approximation of $\widehat{u}_k$.
The error in the approximation is

$$E_k = \widehat{u}_k - \widehat{u}_k^* = \sum_{n=0}^{N-1} \left(\int_{\theta_n}^{\theta_{n+1}} w_k(\theta) d\theta - (\theta_{n+1} - \theta_n) w_k(\theta_n) \right). \tag{5.14}$$

Now, we denote $I_{n,k} = \int_{\theta_n}^{\theta_{n+1}} w_k(\theta) d\theta$ and $I_{n,k}^* = (\theta_{n+1} - \theta_n) w_k(\theta_n)$.
Thus,

$$E_k = \sum_{n=0}^{N-1} (I_{n,k} - I_{n,k}^*) = \sum_{n=0}^{N-1} \int_{\theta_n}^{\theta_{n+1}} (w_k(\theta) - w_k(\theta_n)) d\theta. \tag{5.15}$$

Since $u \in \mathcal{C}^1(\mathbb{T}, \mathbb{R})$, then $w_k \in \mathcal{C}^1(\mathbb{T}, \mathbb{R})$, too. By the fundamental theorem of calculus, we can write

$$w_k(\theta) = w_k(\theta_n) + \int_{\theta_n}^{\theta} w_k'(t) dt,\ \forall \theta \in \mathbb{T}. \tag{5.16}$$

It follows that

$$I_{n,k} - I_{n,k}^* = \int_{\theta_n}^{\theta_{n+1}} \int_{\theta_n}^{\theta} w_k'(t) dt d\theta. \tag{5.17}$$

Let $D_n = \{(\theta, t) \in \mathbb{R}^2 : \theta_n \leq \theta \leq \theta_{n+1}, \theta_n \leq t \leq \theta\}$.
One can write $D_n = \{(\theta, t) \in \mathbb{R}^2 : \theta_n \leq t \leq \theta_{n+1}, t \leq \theta \leq \theta_{n+1}\}$.
Consequently, by Fubini's theorem, the order of integration can be changed and the result is

$$I_{n,k} - I^*_{n,k} = \int_{\theta_n}^{\theta_{n+1}} \int_t^{\theta_{n+1}} w'_k(t) d\theta dt = \int_{\theta_n}^{\theta_{n+1}} w'_k(t)(\theta_{n+1} - t)dt. \tag{5.18}$$

Now, we come with the following change of variable in the integral:

$$\begin{array}{ccc} [-1, 1] & \longrightarrow & [\theta_n, \theta_{n+1}] \\ s & \longmapsto & t = \frac{\theta_n + \theta_{n+1}}{2} + \frac{\theta_{n+1} - \theta_n}{2} s \ , \end{array}$$

obtaining, after calling $h_n = \theta_{n+1} - \theta_n$,

$$\begin{aligned} E_k &= \sum_{n=0}^{N-1} (I_{n,k} - I^*_{n,k}) \\ &= \sum_{n=0}^{N-1} \int_{-1}^{1} w'_k(\theta_n + \frac{h_n}{2}(1+s))\frac{h_n}{2}(1-s)\frac{h_n}{2})ds \\ &= \sum_{n=0}^{N-1} \frac{h_n^2}{4} \int_{-1}^{1} w'_k(\theta_n + \frac{h_n}{2}(1+s))(1-s)ds \ . \end{aligned}$$

By the mean value theorem for integrals, $\forall n = 0, 1, \ldots, N$ there exist $\xi_n \in [\theta_n, \theta_{n+1}]$, such that

$$\int_{-1}^{1} w'_k(\theta_n + \frac{h_n}{2}(1+s))(1-s)ds = w'_k(\xi_n) \int_{-1}^{1} (1-s)ds = 2w'_k(\xi_n).$$

Therefore,

$$E_k = \frac{1}{2} \sum_{n=0}^{N-1} w'_k(\xi_n) h_n^2 = \frac{1}{2} \sum_{n=0}^{N-1} w'_k(\xi_n)(\theta_{n+1} - \theta_n)^2, \text{ with } \xi_n \in [\theta_n, \theta_{n+1}], \ \forall n = 0, 1, \ldots N-1. \tag{5.19}$$

□

Corollary 5.11 Approximation of the Fourier coefficients by means of the DFT

Taking $\theta_n = \frac{n}{N}$, $\forall n = 0, 1, \ldots, N-1, N$ *in the above lemma, and calling*

$$x = [x(1), x(2), \ldots, x(N)]$$

with $x(n+1) = u(\theta_n)$ *and the* DFT *of* x,

$$X = [X(1), X(2), \ldots, X(N)] = \mathfrak{F}x$$

i.e. $X(k+1) = \sum_{n=0}^{N-1} x(n+1) W_N^{-nk}$, $\forall k = 0, 1, \ldots, N-1$ *where* $W_N = e^{-\frac{2\pi}{N}i}$, *then the error in the approximation of* $\widehat{u}_k$ *by*

$$\widehat{u}^*_k = \frac{1}{N} X(k+1), \ \forall k = 0, 1, \ldots, N-1, \tag{5.20}$$

is given, for some $\xi_n \in [\frac{n}{N}, \frac{n+1}{N}]$, $n = 0, 1, \ldots, N-1$, *by*

$$E_k = \widehat{u}_k - \widehat{u}^*_k = \frac{1}{2N^2} \sum_{n=0}^{N-1} w'_k(\xi_n) \ , \tag{5.21}$$

where $w_k(\theta) = u(\theta) e^{-2\pi k \theta i}$, $\forall \theta \in \mathbb{T}$.

Proof. □

EXAMPLE

Take the function

$$\begin{array}{rccl} u: & \mathbb{T} & \longrightarrow & \mathbb{R} \\ & \theta & \longmapsto & u(\theta) = \left\{ \begin{array}{rcl} 1 & , & 0 < \theta < \frac{1}{2} \\ 0 & , & \theta = 0, \frac{1}{2}, 0 \\ -1 & , & \frac{1}{2} < \theta < 1 \end{array} \right. , \end{array}$$

whose Fourier coefficients are given by

$$\begin{array}{rcl} \widehat{u}_0 & = & 0 \\ \widehat{u}_k & = & -\dfrac{1-(-1)^k}{\pi k}, \ k \in \mathbb{Z} \setminus \{0\}. \end{array}$$

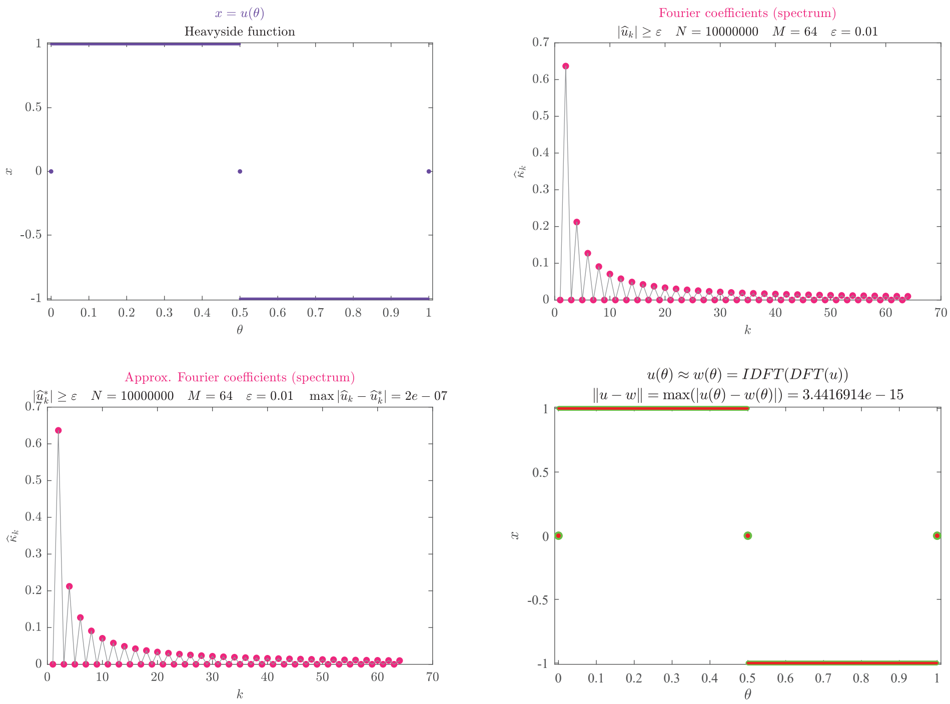


FIGURE 5.1: Heavyside function
$x = u(\theta)$(purple) (top left);
Fourier coefficients of $u(\theta)$, $\widehat{u}_k$ (magenta) (top right);
Approximate Fourier coefficients of $u(\theta)$, $\widehat{u}^*_k$ (magenta) (bottom left);
u versus $w = \mathfrak{F}_N^{-1}\mathfrak{F}_N u$ (bottom right).

EXAMPLE

Take the function

$$\begin{array}{rccl} u: & \mathbb{T} & \longrightarrow & \mathbb{R} \\ & \theta & \longmapsto & u(\theta) = \frac{1}{2} - |\theta - \frac{1}{2}| \ , \end{array}$$

whose Fourier coefficients are given by

$$\widehat{u}_0 = \frac{1}{4}$$
$$\widehat{u}_k = -\frac{1-(-1)^k}{2\pi^2 k^2},\ k \in \mathbb{Z}\setminus\{0\}.$$

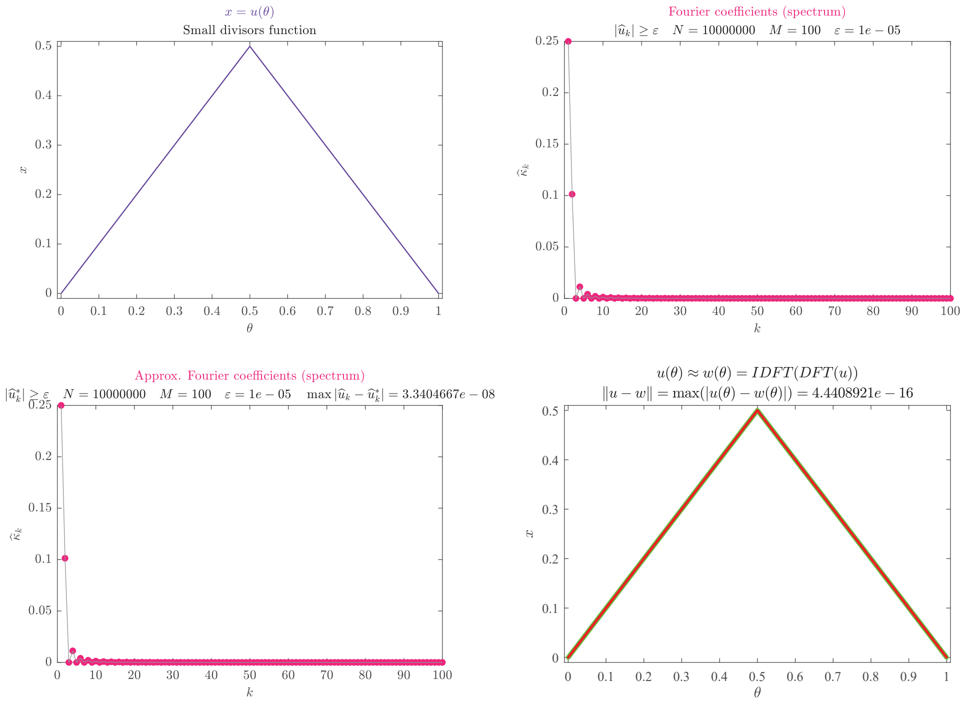


FIGURE 5.2: Small denominators function
$x = u(\theta)$(purple) (top left);
Fourier coefficients of $u(\theta)$, $\widehat{u}_k$ (magenta) (top right);
Approximate Fourier coefficients of $u(\theta)$, $\widehat{u}^*_k$ (magenta) (bottom left);
u versus $w = \mathfrak{F}_N^{-1}\mathfrak{F}_N u$ (bottom right).

5.4 Fourier series approximation: the Dirichlet Kernel

The objective of this section is to obtain numerical approximations of functions expressed by their Fourier series expansion on the torus, i.e.

$$u(\theta) = \sum_{k\in\mathbb{Z}} \widehat{u}_k e^{2\pi k\theta i}, \ \theta \in \mathbb{T}\,. \tag{5.22}$$

We assume here that $u \in \mathcal{A}_\varrho$ for some $\varrho > 0$ and u takes real values for real arguments, i.e. $\overline{u(\theta)} = u(\theta)$, $\forall\theta \in \mathbb{T}$.

The Fourier coefficients of u are defined by

$$\widehat{u}_k = \int_{\mathbb{T}} u(\theta) e^{-2\pi k\theta i} d\theta, \ k \in \mathbb{Z}. \tag{5.23}$$

Since u takes real values for real arguments, $\widehat{u}_{-k} = \overline{\widehat{u}_k}$, $\forall k \in \mathbb{Z}$.

Indeed, $\widehat{u}_{-k} = \int_{\mathbb{T}} u(\theta)e^{-2\pi(-k)\theta i}d\theta = \int_{\mathbb{T}} u(\theta)\overline{e^{-2\pi k\theta i}}d\theta = \overline{\int_{\mathbb{T}} u(\theta)e^{-2\pi k\theta i}d\theta} = \overline{\widehat{u}_k}$.

Thanks to this property we can express u in the following way:

$$\begin{aligned} u(\theta) &= \sum_{k=1}^{\infty} \widehat{u}_{-k} e^{2\pi(-k)\theta i} + \widehat{u}_0 + \sum_{k=1}^{\infty} \widehat{u}_k e^{2\pi k\theta i} \\ &= \sum_{k=1}^{\infty} \overline{\widehat{u}_k e^{2\pi k\theta i}} + \widehat{u}_0 + \sum_{k=1}^{\infty} \widehat{u}_k e^{2\pi k\theta i} \\ &= \widehat{u}_0 + 2\,\mathrm{Re}\left(\sum_{k=1}^{\infty} \widehat{u}_k e^{2\pi k\theta i}\right), \quad \theta \in \mathbb{T}. \end{aligned} \tag{5.24}$$

Definition 5.12 Dirichlet Kernel

Given $M \in \mathbb{N}$, the Dirichlet Kernel of order M is the function:

$$\begin{aligned} D_M :\ & \mathbb{T} \longrightarrow \mathbb{R} \\ & \theta \longmapsto D_M(\theta) := \sum_{k=-M}^{M} e^{2\pi k\theta i}\,. \end{aligned} \tag{5.25}$$

◇

Proposition

$D_M(\theta) = 1 + 2\sum_{k=1}^{M} \cos(2\pi k\theta) = \dfrac{\sin((2M+1)\pi\theta)}{\sin(\pi\theta)}$, $\forall\theta \in \mathbb{T}$.

Proof. □

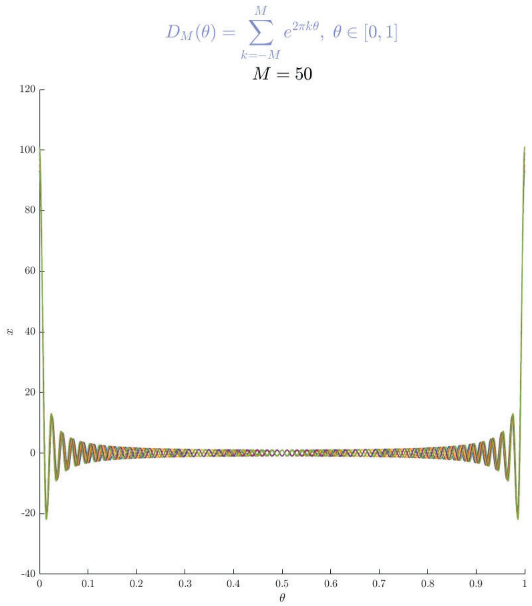


FIGURE 5.3: Dirichlet Kernel $D_M(\theta) = \sum_{k=-M}^{M} e^{2\pi k\theta i},\ \theta \in \mathbb{T}$.

Definition 5.14 Convolution

The convolution between two functions $u, v : \mathbb{T} \longrightarrow \mathbb{C}$ is the function $u * v$ defined by

$$\begin{aligned} u * v :\ \ & \mathbb{T} \longrightarrow \mathbb{C} \\ & \theta \longmapsto (u*v)(\theta) := \int_{\mathbb{T}} u(\eta)v(\theta-\eta)d\eta\ . \end{aligned} \tag{5.26}$$

◇

Proposition

The convolution of a real analytic function $u : \mathbb{T} \longrightarrow \mathbb{R}$ with the $M-$th order Dirichlet Kernel D_M is the $M-$th partial sum approximation of the Fourier series expansion of u.

More explicitly, if $u(\theta) = \sum_{k\in\mathbb{Z}} \widehat{u}_k e^{2k\pi\theta i}$ and $u_M(\theta) = \sum_{k=-M}^{M} \widehat{u}_k e^{2k\pi\theta i}$, then:

$$u_M = u * D_M\ . \tag{5.27}$$

5.5 Fourier series approximation by means of the IDFT

This section is devoted to providing the numerical procedures necessary to approximate a function defined by its Fourier series expansion approximated from a finite list of its Fourier coefficients. It must be taken into account that what is sought is to obtain the evaluation of the function in a finite collection of points, in each of which the series must be approximated by a finite partial sum.

As seen in the previous **Section 5.4**, functions given by their Fourier series expansion on the torus, i.e.

$$u(\theta) = \sum_{k\in\mathbb{Z}} \widehat{u}_k e^{2\pi k\theta i},\ \theta \in \mathbb{T}\,. \tag{5.28}$$

with Fourier coefficients

$$\widehat{u}_k = \int_{\mathbb{T}} u(\theta) e^{-2\pi k\theta i} d\theta,\ k \in \mathbb{Z}. \tag{5.29}$$

can be expressed as

$$u(\theta) = \widehat{u}_0 + 2\,\mathrm{Re}\left(\sum_{k=1}^{\infty} \widehat{u}_k e^{2\pi k\theta i}\right), \quad \theta \in \mathbb{T}. \tag{5.30}$$

Let

$$u_M(\theta) = \widehat{u}_0 + 2\,\mathrm{Re}\left(\sum_{k=1}^{M} \widehat{u}_k e^{2\pi k\theta i}\right), \quad \theta \in \mathbb{T} \tag{5.31}$$

be the M–th partial sum of (5.30).

Given $\widehat{u}_0, \widehat{u}_1, \ldots, \widehat{u}_M$, we want to compute $u_M(\theta)$ by evaluation in a mesh (equidistibuted) $(t_0, t_1, \ldots, t_{N-1}, t_N)$, with $N = 2M$ and $t_n = \frac{n}{N}$, $\forall n = 0, 1, \ldots, N$. Notice that the partition has $N+1$ terminal points. Nonetheless, $u(t_M) = u(t_0)$, since $e^{2\pi i} = 1$. Therefore, it is enough to evaluate u_M in $(t_0, t_1, \ldots, t_{N-1})$.

Let $w = [w(1), w(2), \ldots, w(N)] \in l^2(\mathbb{Z}_N)$ with

$$w(n+1) = u_M(t_n) = \widehat{u}_0 + 2\,\mathrm{Re}\left(\sum_{k=1}^{M} \widehat{u}_k e^{2\pi k t_n i}\right) = \widehat{u}_0 + 2\,\mathrm{Re}\left(\sum_{k=1}^{M} \widehat{u}_k W_N^{-kn}\right),\ \forall n = 0, 1, \ldots, N-1\,,$$

where $W_N = e^{-\frac{2\pi}{N} i}$.

We will distinguish two cases[5]:

(i) $n = 2m$, $m = 0, 1, \ldots, M-1$.

$$\begin{aligned}
w(2m+1) &= \widehat{u}_0 + 2\,\mathrm{Re}\left(\sum_{k=1}^{M} \widehat{u}_k W_N^{-2km}\right) = \widehat{u}_0 + 2\,\mathrm{Re}\left(\sum_{k=1}^{M} \widehat{u}_k W_M^{-km}\right) \\
&= \widehat{u}_0 + 2\,\mathrm{Re}\left(\sum_{k=0}^{M-1} \widehat{u}_{k+1} W_M^{-km} W_M^{-m}\right) = \widehat{u}_0 + 2\,\mathrm{Re}\left(W_M^{-m} \sum_{k=0}^{M-1} \widehat{u}_{k+1} W_M^{-km}\right) \\
&= \widehat{u}_0 + 2\,\mathrm{Re}\left(W_M^{-m} \sum_{k=0}^{M-1} X(k+1) W_M^{-km}\right) \\
&= \widehat{u}_0 + 2\,\mathrm{Re}\left(M W_M^{-m} x(m+1)\right) = \widehat{u}_0 + 2M\,\mathrm{Re}\left(W_M^{-m} x(m+1)\right)\,,
\end{aligned}$$

where $x = \mathfrak{F}_M^{-1} X$, $X = [X(1), X(2), \ldots, X(M)]$ and $X(k+1) = \widehat{u}_{k+1}$, $\forall k = 0, 1, \ldots, M-1$.

[5]Notice that $W_M = W_N^2$.

(ii) $n = 2m+1$, $m = 0, 1, \ldots, M-1$.

$$\begin{aligned}
w(2m+2) &= \widehat{u}_0 + 2\,\mathrm{Re}\left(\sum_{k=1}^{M} \widehat{u}_k W_N^{-k(2m+1)}\right) = \widehat{u}_0 + 2\,\mathrm{Re}\left(\sum_{k=1}^{M} \widehat{u}_k W_M^{-2km} W_N^{-k}\right) \\
&= \widehat{u}_0 + 2\,\mathrm{Re}\left(\sum_{k=1}^{M} \widehat{u}_{k+1} W_M^{-km} W_M^{-k}\right) \\
&= \widehat{u}_0 + 2\,\mathrm{Re}\left(\sum_{k=0}^{M-1} \widehat{u}_{k+1} W_M^{-(k+1)m} W_N^{-(k+1)}\right) \\
&= \widehat{u}_0 + 2\,\mathrm{Re}\left(W_M^{-m} \sum_{k=0}^{M-1} \widehat{u}_{k+1} W_N^{-(k+1)} W_M^{-km}\right) \\
&= \widehat{u}_0 + 2\,\mathrm{Re}\left(W_M^{-m} \sum_{k=0}^{M-1} Y(k+1) W_M^{-km}\right) \\
&= \widehat{u}_0 + 2\,\mathrm{Re}\left(M(W_M^{-m} y(m+1)\right) = \widehat{u}_0 + 2M\,\mathrm{Re}\left(W_M^{-m} y(m+1)\right)\ ,
\end{aligned}$$

where

$y = \mathfrak{F}_M^{-1} Y$, $Y = [Y(1), Y(2), \ldots, Y(M)]$ and $Y(k+1) = \widehat{u}_{k+1} W_N^{-(k+1)}$, $\forall k = 0, 1, \ldots, M-1$.

Summarizing,

$$w(2m+1) = \widehat{u}_0 + 2M\,\mathrm{Re}\left(W_M^{-m} x(m+1)\right),\ \forall m = 0, 1, \ldots, M-1, \tag{5.32}$$

$$w(2m+2) = \widehat{u}_0 + 2M\,\mathrm{Re}\left(W_M^{-m} y(m+1)\right),\ \forall m = 0, 1, \ldots, M-1. \tag{5.33}$$

with

$$\begin{aligned}
x &= \mathfrak{F}_M^{-1} X,\ X(k+1) = \widehat{u}_{k+1},\ \forall k = 0, 1, \ldots, M-1 \\
y &= \mathfrak{F}_M^{-1} Y,\ Y(k+1) = \widehat{u}_{k+1} W_N^{-(k+1)},\ \forall k = 0, 1, \ldots, M-1
\end{aligned}$$

Now, by the shift theorem (**Proposition 5.5** part (d) frequency shifting, $m = 1$)

$$\mathfrak{F}_M[W_M^0 x(1), W_M^{-1} x(2), \ldots, W_M^{-(M-1)} x(M)] = [X(M), X(1), \ldots, X(M-1)] = \mathfrak{S}_M X\ . \tag{5.34}$$

On the other hand, by the same argument,

$$\mathfrak{F}_M[W_M^0 y(1), W_M^{-1} y(2), \ldots, W_M^{-(M-1)} y(M)] = [Y(M), Y(1), \ldots, Y(M-1)] = \mathfrak{S}_M Y\ . \tag{5.35}$$

Consequently, relations (5.32) and (5.33) can be written as

$$[w(1), w(3), \ldots, w(2M-1)] = \widehat{u}_0 + 2M\,\mathrm{Re}\left(\mathfrak{F}_M^{-1} \mathfrak{S}_M X\right) \tag{5.36}$$

$$[w(2), w(4), \ldots, w(2M)] = \widehat{u}_0 + 2M\,\mathrm{Re}\left(\mathfrak{F}_M^{-1} \mathfrak{S}_M Y\right)\ , \tag{5.37}$$

with

$$\begin{aligned}
X &= [X(1), X(2), \ldots X(M)] = [\widehat{u}_1, \widehat{u}_2, \ldots, \widehat{u}_M] = \mathfrak{F}_M x \\
Y &= [Y(1), Y(2), \ldots Y(M)] = [\widehat{u}_1 W_N^{-1}, \widehat{u}_2 W_N^{-2}, \ldots, \widehat{u}_M W_N^{-M}] = \mathfrak{F}_M y\ .
\end{aligned}$$

Finally, in spite of the fact that X is known and Y can be therefore computed directly, we go further providing a relationship between X and Y through the DFT that will turn out to be more efficient as far as numerical computation is concerned.

Let $Z = [Z(1), Z(2), \ldots, Z(N)] = [0 \cdot W_N^0, \widehat{u}_1 \cdot W_N^{-1}, \ldots, \widehat{u}_M \cdot W_N^{-M}, 0 \cdot W_N^{-(M+1)}, \ldots, 0 \cdot W_N^{-(N-1)}] = [0, Y, 0_{M-1}] = \mathfrak{S}_N[Y, 0_M] \in l^2(\mathbb{Z}_N)$, and $T = \mathfrak{S}_N[X, 0_M] = [0, X, 0_M] = [0, \widehat{v}_1, \ldots, \widehat{v}_M, \underbrace{0, \ldots, 0}_{M-1_{\text{zeros}}}]$.

Thus, $\mathfrak{R}_N^{-1} T = [0 \cdot W_N^0, \widehat{v}_1 \cdot W_N^{-1}, \ldots, \widehat{v}_M \cdot W_N^{-M}, 0, \ldots, 0] = [0, Y, 0_{M-1}] = \mathfrak{S}_N[Y, 0_M] = Z$.
Hence, $T = \mathfrak{R}_N Z$ and By the shift theorem (**Proposition 5.5** part (d) time shifting, $m = 1$) $\mathfrak{F}_N^{-1} T = \mathfrak{S}_N \mathfrak{F}_N^{-1} Z$. Therefore, $Z = \mathfrak{F}_N \mathfrak{S}_N^{-1} \mathfrak{F}_N^{-1} T = \mathfrak{F}_N \mathfrak{S}_N^{-1} \mathfrak{F}_N^{-1} \mathfrak{S}_N[X, 0_M]$.
On the other hand, $Z = \mathfrak{S}_N[Y, 0_M]$, so

$$[Y, 0_M] = \mathfrak{F}_N \mathfrak{S}_N^{-1} \mathfrak{F}_N^{-1} T = \mathfrak{F}_N \mathfrak{S}_N^{-1} \mathfrak{F}_N^{-1} \mathfrak{S}_N[X, 0_M].$$

Then, Y is performed by the M first components of the latter vector, which does nor require the products $[\widehat{u}_1 \cdot W_N^{-1}, \ldots, \widehat{u}_M \cdot W_N^{-M}] =: Y$ to be computed directly, being the time of computation and the corresponding error reduced[6].

[6] This algorithm has been implemented in a Matlab® function named `IDFTAPPROX.m` which is showed in **Appendix II.2**

5.6 Cohomological equations and the Floquet transformation

See **Appendix II.3**.

5.7 The KAM procedure

The algorithm followed to implement the KAM procedure ws described in 2.4 and 3.2 with the corresponding links to that parts which are necessary to reproduce the whole process (See II.4). The following tables show the results obtained in one particular example which is detailed in greater extension in **Chapter 6**.
Let $\varrho > 0$, $a, b \in \mathbb{R}^+$, $\omega \in \mathcal{DC}(\gamma, \nu)$, and $\mathcal{U} = \mathbb{C} \setminus \{-\frac{1}{a}i, \frac{1}{a}i\}$.
The exapmple is defined by:

$$\begin{array}{rccl} \psi : & \mathbb{T}_\varrho \times \mathcal{U} & \longrightarrow & \mathbb{T}_\varrho \times \mathbb{C} \\ & (\theta, z) & \longmapsto & \psi(\theta, z) := (\theta + \omega, \arctan(az) + b\sin(2\pi\theta)) \end{array} \quad . \tag{5.38}$$

As usual, we denote by $f(\theta, z)$ the second component of the skew–product, that is,

$$\begin{array}{rccl} f : & \mathbb{T}_\varrho \times \mathcal{U} & \longrightarrow & \mathbb{C} \\ & (\theta, z) & \longmapsto & f(\theta, z) := \arctan(az) + b\sin(2\pi\theta) \end{array} \quad . \tag{5.39}$$

$\diamond$

The initial guess is a curve κ_0 obtained from the orbit of a point (θ_0, x_0) (forward or backward) subjected to a previous smoothing process. N is the number of points calculated for the orbit and N_0 is the number of them discarded. The average of the curve is p. Λ_0 and λ_0 represent the Lyapunov exponent and the Lyapunov multiplier of the curve κ_0, respectively. τ_0 is the initial guess for the translation parameter and τ_n is the translation parameter obtained after n iterations. Finally, $\|E_n\|$ is the invariance error after n iterations and $e_n(p)$ is the average error. This value should be zero when the average of κ_0 is equal to p. The difference is produced because of the propagation of computational errors.

Stable invariant curves

n	$< \kappa_n >$	Λ_n	λ_n	$\|E_n\| = \sup_{\theta\in\mathbb{T}} \|E_n(\theta)\|$	$\|e_n(p)\| = \|< \kappa_n > -p\|$
0	1.115300308834	−1.42097756798	0.241477840472	$4.93328710727e-06$	0
1	1.115300308834	−1.42097771529	0.241477804901	$8.19850516867e-09$	0
2	1.115300308834	−1.42097771529	0.241477804901	$9.31448094501e-11$	0

Parameters
$a = 6.8$
$b = 1.62$
$\omega = \frac{1+\sqrt{5}}{2} \approx 1.618033988749895$
Initial proposal $(\kappa_0,\ \tau_0)$
$\theta_0 = 0$
$x_0 = 1$
$N = 10000000$
$N_0 = 128$
Initial characteristic features
$p = 1.115300308339138$
$\Lambda_0 = -1.420977567979511$
$\lambda_0 = 0.2414778404719825$
$\det(\Omega_0) = 1.996578052326427$
$\tau_0 = 0$
$\|E_0\| = \max_{\theta\in\mathbb{T}}(|E_0(\theta)|) = 4.933287107267148e-06$

$|e_0(p)| = | < \kappa_0 > -p| = 0$

Number of KAM iterations: $n = 2$

Final characteristic features

$\Lambda_n = -1.420977715285497$

$\lambda_n = 0.2414778049008537$

$\tau_n = -3.900965064576669e - 08$

$\det(\Omega_n) = 1.996577812178111$

$\|E_n\| = \max_{\theta \in \mathbb{T}}(|E(\theta)|) = 9.314480945012101e - 11$

$|e_n(p)| = | < \kappa_n > -p| = 0$

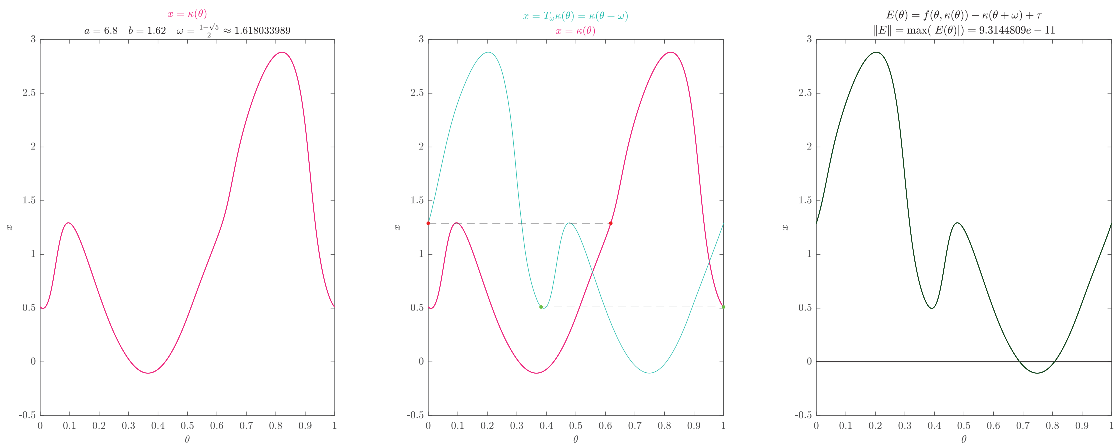


FIGURE 5.4: Curve obtained after n iterations
$x = \kappa_n(\theta)$(magenta) (left);
Translated curve $x = \kappa_n(\theta + \omega)$ (light green) (center);
$x = f(\theta, \kappa_n(\theta))$ (dark green) and the error function $E_n(\theta)$ (right).

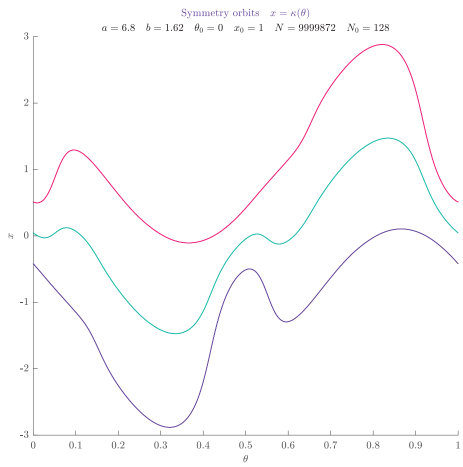


FIGURE 5.5: Symmetry curves $\theta_0 = 0$, $x_0 = 1$ with $N = 10^7$.
$x = \kappa_n(\theta)$ (magenta)
$x = \gamma_n(\theta) = -\kappa_n(\theta + \frac{1}{2})$ (purple)
$x = \eta_n(\theta) = \frac{1}{2}(\kappa_n(\theta) + \gamma_n(\theta))$ (light green)

n	$< \kappa_n >$	Λ_n	λ_n	$\|E_n\| = \sup_{\theta\in\mathbb{T}} \|E_n(\theta)\|$	$\|e_n(p)\| = \| < \kappa_n > -p\|$
0	1.11530030834	-1.42097756798	0.241477840472	$3.13447635114e-06$	0
1	1.11530030834	-1.42097754127	0.241477846923	$1.04472829751e-08$	$4.4408920985e-16$
2	1.11530030834	-1.42097754127	0.241477846923	$1.25859706492e-10$	$4.4408920985e-16$

Parameters

$a = 6.8$

$b = 1.62$

$\omega = \frac{1+\sqrt{5}}{2} \approx 1.618033988749895$

Initial proposal $(\kappa_0,\ \tau_0)$

$\theta_0 = 0$

$x_0 = 1$

$N = 10000000$

$N_0 = 128$

Initial characteristic features

$p = 1.115300308339138$

$\Lambda_0 = -1.420977567979511$

$\lambda_0 = 0.2414778404719825$

$\det(\Omega_0) = 1.996578052326427$

$\tau_0 = 0$

$\|E_0\| = \max_{\theta\in\mathbb{T}}(|E_0(\theta)|) = 3.134476351140947e-06$

$|e_0(p)| = | < \kappa_0 > -p| = 0$

Number of KAM iterations: $n = 2$

Final characteristic features

$\Lambda_n = -1.420977541265186$

$\lambda_n = 0.2414778469229001$

$\tau_n = -5.25003669794802e-08$

$\det(\Omega_n) = 1.996578138176017$

$\|E_n\| = \max_{\theta\in\mathbb{T}}(|E(\theta)|) = 1.258597064916809e-10$

$|e_n(p)| = | < \kappa_n > -p| = 4.440892098500626e-16$

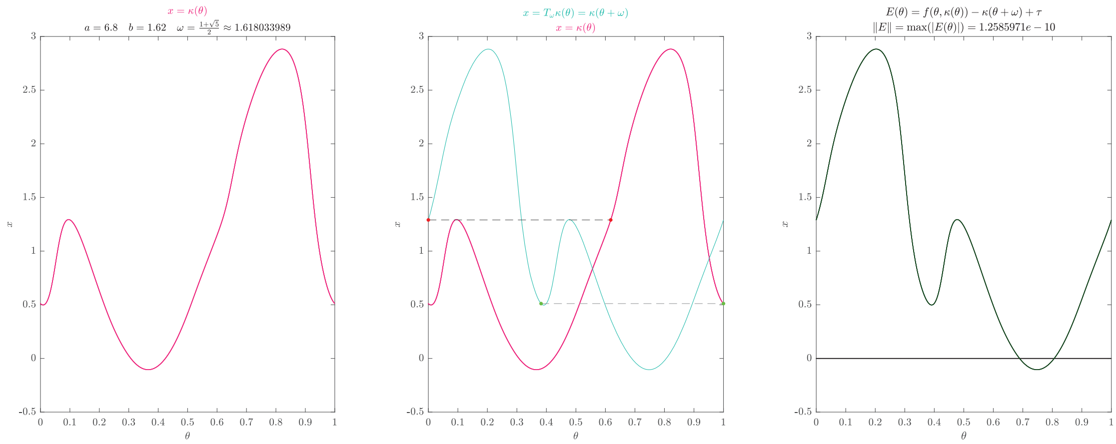


FIGURE 5.6: Curve obtained after n iterations
$x = \kappa_n(\theta)$(magenta) (left);
Translated curve $x = \kappa_n(\theta + \omega)$ (light green) (center);
$x = f(\theta, \kappa_n(\theta))$ (dark green) and the error function $E_n(\theta)$ (right).

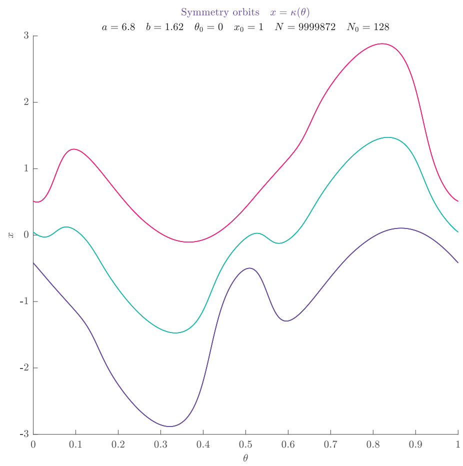


FIGURE 5.7: Symmetry curves $\theta_0 = 0\,,\ x_0 = 1$ with $N = 10^7$.
$x = \kappa_n(\theta)$ (magenta)
$x = \gamma_n(\theta) = -\kappa_n(\theta + \frac{1}{2})$ (purple)
$x = \eta_n(\theta) = \frac{1}{2}(\kappa_n(\theta) + \gamma_n(\theta))$ (light green)

n	$< \kappa_n >$	Λ_n	λ_n	$\|E_n\| = \sup_{\theta\in\mathbb{T}} \|E_n(\theta)\|$	$\|e_n(p)\| = \| < \kappa_n > -p\|$
0	-1.11530055032	-1.42097805145	0.241477723726	$3.05875921325e-06$	0
1	-1.11530055032	-1.42097795532	0.241477746939	$3.34431080571e-08$	$4.4408920985e-16$
2	-1.11530055032	-1.42097795532	0.241477746939	$4.57780300568e-11$	$1.99840144433e-15$

Parameters
$a = 6.8$
$b = 1.62$
$\omega = \frac{1+\sqrt{5}}{2} \approx 1.618033988749895$
Initial proposal $(\kappa_0,\ \tau_0)$
$\theta_0 = 0$

$x_0 = -1$

$N = 10000000$

$N_0 = 128$

Initial characteristic features

$p = -1.115300550317159$

$\Lambda_0 = -1.420978051445764$

$\lambda_0 = 0.2414777237256241$

$\det(\Omega_0) = 1.996577577281046$

$\tau_0 = 0$

$\|E_0\| = \max_{\theta \in \mathbb{T}}(|E_0(\theta)|) = 3.058759213248052e - 06$

$|e_0(p)| = | < \kappa_0 > -p| = 0$

Number of KAM iterations: $n = 2$

Final characteristic features

$\Lambda_n = -1.420977955316531$

$\lambda_n = 0.2414777469386935$

$\tau_n = -5.744774538529812e - 08$

$\det(\Omega_n) = 1.996578028748249$

$\|E_n\| = \max_{\theta \in \mathbb{T}}(|E(\theta)|) = 4.577803005676686e - 11$

$|e_n(p)| = | < \kappa_n > -p| = 1.998401444325282e - 15$

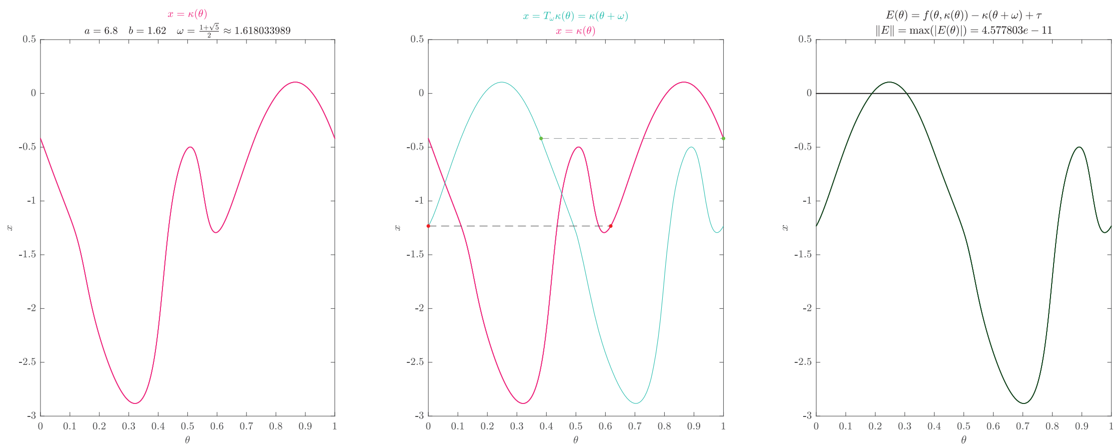


FIGURE 5.8: Curve obtained after n iterations
$x = \kappa_n(\theta)$(magenta) (left);
Translated curve $x = \kappa_n(\theta + \omega)$ (light green) (center);
$x = f(\theta, \kappa_n(\theta))$ (dark green) and the error function $E_n(\theta)$ (right).

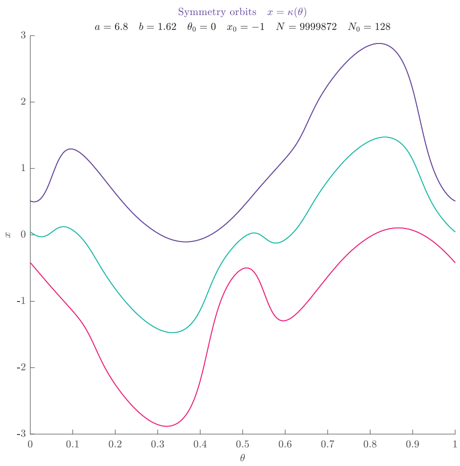


FIGURE 5.9: Symmetry curves $\theta_0 = 0\,,\ x_0 = 1$ with $N = 10^7$.
$x = \kappa_n(\theta)$ (magenta)
$x = \gamma_n(\theta) = -\kappa_n(\theta + \frac{1}{2})$ (purple)
$x = \eta_n(\theta) = \frac{1}{2}(\kappa_n(\theta) + \gamma_n(\theta))$ (light green)

n	$< \kappa_n >$	Λ_n	λ_n	$\|E_n\| = \sup_{\theta\in\mathbb{T}} \|E_n(\theta)\|$	$\|e_n(p)\| = \|< \kappa_n > -p\|$
0	-1.11530055032	-1.42097805145	0.241477723726	$1.03987595645e-06$	0
1	-1.11530055032	-1.42097787817	0.241477765567	$3.69311680115e-08$	$8.881784197e-16$
2	-1.11530055032	-1.42097787817	0.241477765567	$3.58276588012e-09$	$2.44249065418e-15$

Parameters
$a = 6.8$
$b = 1.62$
$\omega = \frac{1+\sqrt{5}}{2} \approx 1.618033988749895$
Initial proposal $(\kappa_0,\ \tau_0)$
$\theta_0 = 0$
$x_0 = -1$
$N = 10000000$
$N_0 = 128$
Initial characteristic features
$p = -1.115300550317159$
$\Lambda_0 = -1.420978051445764$
$\lambda_0 = 0.2414777237256241$
$\det(\Omega_0) = 1.996577577281046$
$\tau_0 = 0$
$\|E_0\| = \max_{\theta\in\mathbb{T}}(|E_0(\theta)|) = 1.039875956454495e-06$
$|e_0(p)| = |< \kappa_0 > -p| = 0$
Number of KAM iterations: $n = 2$
Final characteristic features
$\Lambda_n = -1.42097787817254$
$\lambda_n = 0.2414777655672513$
$\tau_n = -1.402031841840645e-07$
$\det(\Omega_n) = 1.996577368507178$
$\|E_n\| = \max_{\theta\in\mathbb{T}}(|E(\theta)|) = 3.582765880120126e-09$
$|e_n(p)| = |< \kappa_n > -p| = 2.442490654175344e-15$

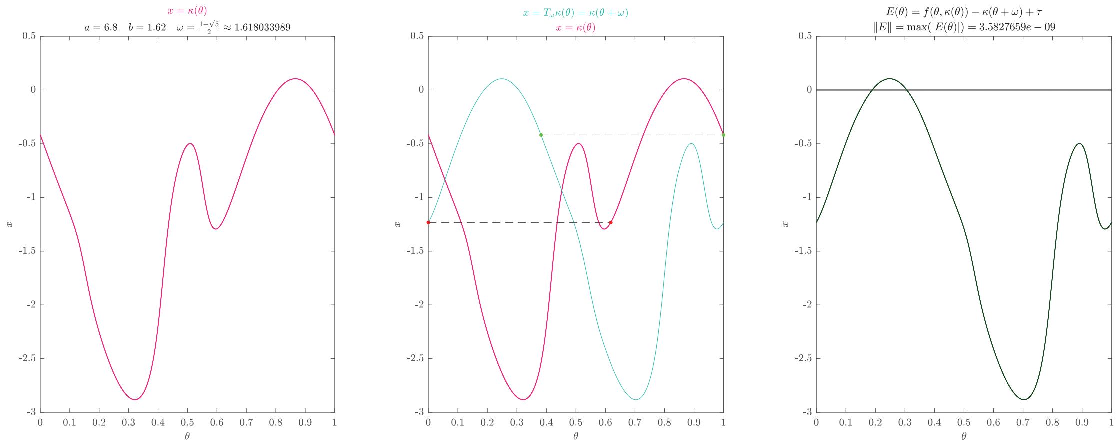


FIGURE 5.10: Curve obtained after n iterations
$x = \kappa_n(\theta)$(magenta) (left);
Translated curve $x = \kappa_n(\theta + \omega)$ (light green) (center);
$x = f(\theta, \kappa_n(\theta))$ (dark green) and the error function $E_n(\theta)$ (right).

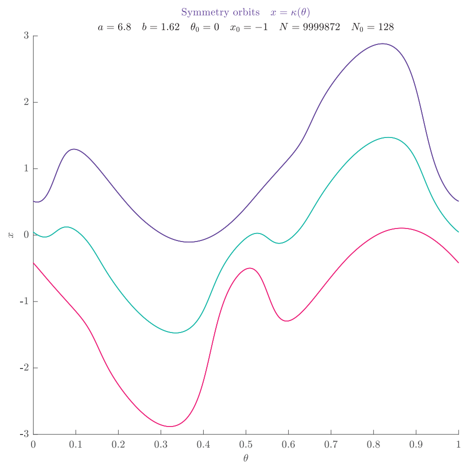


FIGURE 5.11: Symmetry curves $\theta_0 = 0\,,\ x_0 = 1$ with $N = 10^7$.
$x = \kappa_n(\theta)$ (magenta)
$x = \gamma_n(\theta) = -\kappa_n(\theta + \frac{1}{2})$ (purple)
$x = \eta_n(\theta) = \frac{1}{2}(\kappa_n(\theta) + \gamma_n(\theta))$ (light green)

The following example shows an unstable invariant curve with spline interpolation for the seed.

n	$< \kappa_n >$	Λ_n	λ_n	$\|E_n\| = \sup_{\theta\in\mathbb{T}} \|E_n(\theta)\|$	$\|e_n(p)\| = \| < \kappa_n > -p\|$
0	$-3.29574701041e-08$	0.751526426994	2.12023393012	$3.1482287252e-06$	0
1	$-3.29574703253e-08$	0.751526498008	2.12023408069	$2.8583364213e-09$	$2.21191940532e-16$
2	$-3.29574704534e-08$	0.751526498008	2.12023408069	$6.66282955708e-12$	$3.49250437992e-16$

Parameters
$a = 6.8$
$b = 1.62$
$\omega = \frac{1+\sqrt{5}}{2} \approx 1.618033988749895$
Initial proposal $(\kappa_0,\ \tau_0)$

$\theta_0 = 0$

$x_0 = 0$

$N = 10000000$

$N_0 = 128$

Initial characteristic features

$p = -3.295747010411697e - 08$

$\Lambda_0 = 0.7515264269943018$

$\lambda_0 = 2.120233930123497$

$\det(\Omega_0) = 1.983649446320672$

$\tau_0 = 0$

$\|E_0\| = \max_{\theta \in \mathbb{T}}(|E_0(\theta)|) = 3.148228725199931e - 06$

$|e_0(p)| = | < \kappa_0 > -p| = 0$

Number of KAM iterations: $n = 2$

Final characteristic features

$\Lambda_n = 0.7515264980077757$

$\lambda_n = 2.120234080688679$

$\tau_n = -6.547116112843343e - 08$

$\det(\Omega_n) = 1.98364969603699$

$\|E_n\| = \max_{\theta \in \mathbb{T}}(|E(\theta)|) = 6.6628295570820 55e - 12$

$|e_n(p)| = | < \kappa_n > -p| = 3.492504379917958e - 16$

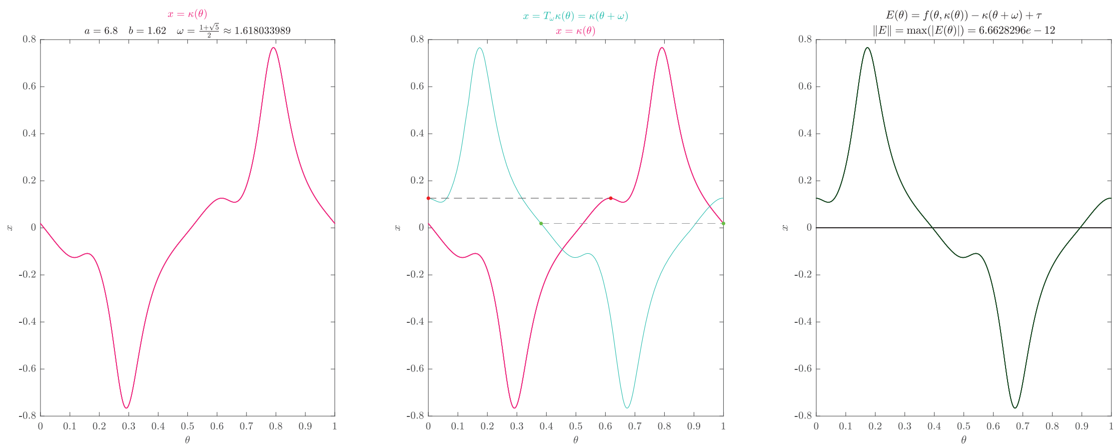


FIGURE 5.12: Curve obtained after n iterations
$x = \kappa_n(\theta)$(magenta) (left);
Translated curve $x = \kappa_n(\theta + \omega)$ (light green) (center);
$x = f(\theta, \kappa_n(\theta))$ (dark green) and the error function $E_n(\theta)$ (right).

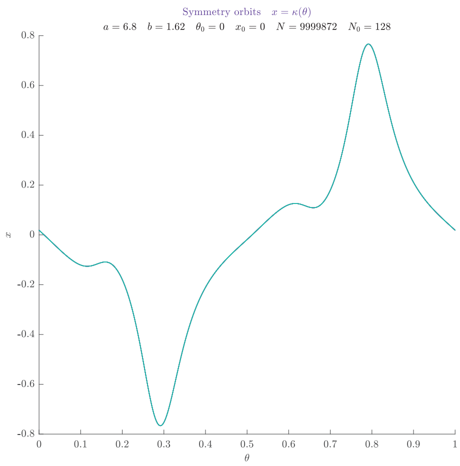


FIGURE 5.13: Symmetry curves $\theta_0 = 0\,,\ x_0 = 1$ with $N = 10^7$.
$x = \kappa_n(\theta)$ (magenta)
$x = \gamma_n(\theta) = -\kappa_n(\theta + \frac{1}{2})$ (purple)
$x = \eta_n(\theta) = \frac{1}{2}(\kappa_n(\theta) + \gamma_n(\theta))$ (light green)

n	$< \kappa_n >$	Λ_n	λ_n	$\|E_n\| = \sup_{\theta\in\mathbb{T}} \|E_n(\theta)\|$	$\|e_n(p)\| = \|< \kappa_n > -p\|$
0	$1.86330473196e-09$	0.75152651734	2.12023412168	$5.8619805865e-07$	0
1	$1.8633049648e-09$	0.751526498008	2.12023408069	$5.1409186815e-12$	$2.32833624225e-16$
2	$1.86330520927e-09$	0.751526498008	2.12023408069	$7.01423272764e-12$	$4.77308929248e-16$

Parameters
$a = 6.8$
$b = 1.62$
$\omega = \frac{1+\sqrt{5}}{2} \approx 1.618033988749895$
Initial proposal $(\kappa_0,\ \tau_0)$
$\theta_0 = 0$
$x_0 = 0$
$N = 10000000$
$N_0 = 128$
Initial characteristic features
$p = 1.863304731962491e-09$
$\Lambda_0 = 0.7515265173397319$
$\lambda_0 = 2.120234121676952$
$\det(\Omega_0) = 1.983649244512972$
$\tau_0 = 0$
$\|E_0\| = \max_{\theta\in\mathbb{T}}(|E_0(\theta)|) = 5.861980586496784e-07$
$|e_0(p)| = |<\kappa_0> -p| = 0$
Number of KAM iterations: $n = 2$
Final characteristic features
$\Lambda_n = 0.7515264980077548$
$\lambda_n = 2.120234080688635$
$\tau_n = -8.513563132896642e-08$
$\det(\Omega_n) = 1.983649696036903$
$\|E_n\| = \max_{\theta\in\mathbb{T}}(|E(\theta)|) = 7.014232727641702e-12$
$|e_n(p)| = |<\kappa_n> -p| = 4.773089292475703e-16$

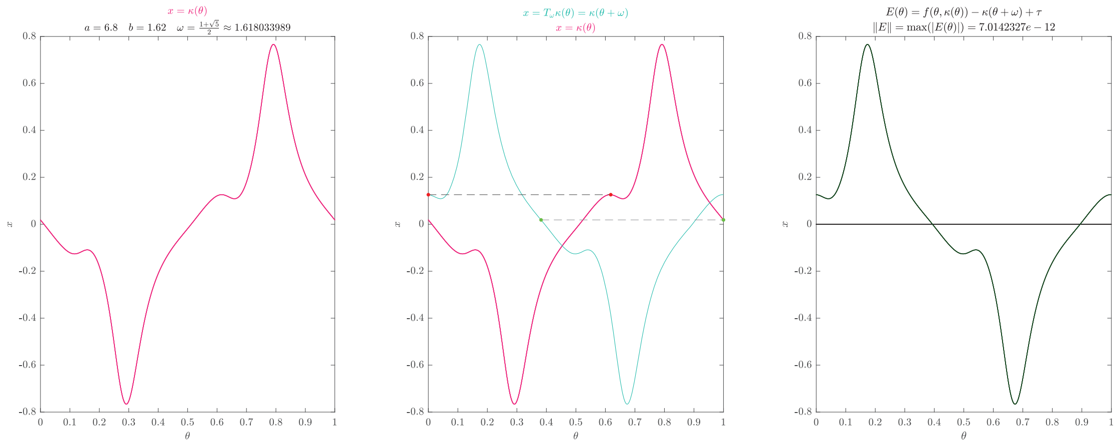


FIGURE 5.14: Curve obtained after n iterations
$x = \kappa_n(\theta)$(magenta) (left);
Translated curve $x = \kappa_n(\theta + \omega)$ (light green) (center);
$x = f(\theta, \kappa_n(\theta))$ (dark green) and the error function $E_n(\theta)$ (right).

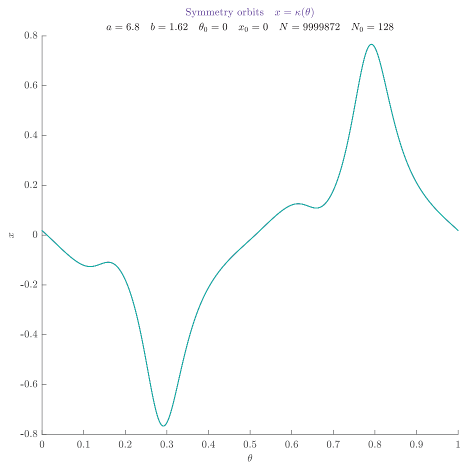


FIGURE 5.15: Symmetry curves $\theta_0 = 0\,,\ x_0 = 1$ with $N = 10^7$.
$x = \kappa_n(\theta)$ (magenta)
$x = \gamma_n(\theta) = -\kappa_n(\theta + \frac{1}{2})$ (purple)
$x = \eta_n(\theta) = \frac{1}{2}(\kappa_n(\theta) + \gamma_n(\theta))$ (light green)
Notice that the three curves are, in fact, the same since this curve is self–symmetric.

n	$<\kappa_n>$	Λ_n	λ_n	$\|E_n\| = \sup_{\theta \in \mathbb{T}} \|E_n(\theta)\|$	$\|e_n(p)\| = \|<\kappa_n> -p\|$
0	0.590406740555	-0.509019427517	0.601084697043	$6.07732360267e-05$	0

Parameters
$a = 6.8$
$b = 1.82$
$\omega = \frac{1+\sqrt{5}}{2} \approx 1.618033988749895$
Initial proposal $(\kappa_0,\ \tau_0)$
$\theta_0 = 0$

$x_0 = 1$

$N = 10000000$

$N_0 = 1024$

Initial characteristic features

$p = 0.5904067405545206$

$\Lambda_0 = -0.5090194275165796$

$\lambda_0 = 0.6010846970426238$

$\det(\Omega_0) = 46.1951214269993$

$\tau_0 = 0$

$\|E_0\| = \max_{\theta \in \mathbb{T}}(|E_0(\theta)|) = 6.077323602671214e - 05$

$|e_0(p)| = |<\kappa_0> -p| = 0$

Number of KAM iterations: $n = 0$

Final characteristic features

$\Lambda_n = -0.5090194275165796$

$\lambda_n = 0.6010846970426238$

$\tau_n = -7.324208908281058e - 08$

$\det(\Omega_n) = 46.1951214269993$

$\|E_n\| = \max_{\theta \in \mathbb{T}}(|E(\theta)|) = 6.077323602671214e - 05$

$|e_n(p)| = |<\kappa_n> -p| = 0$

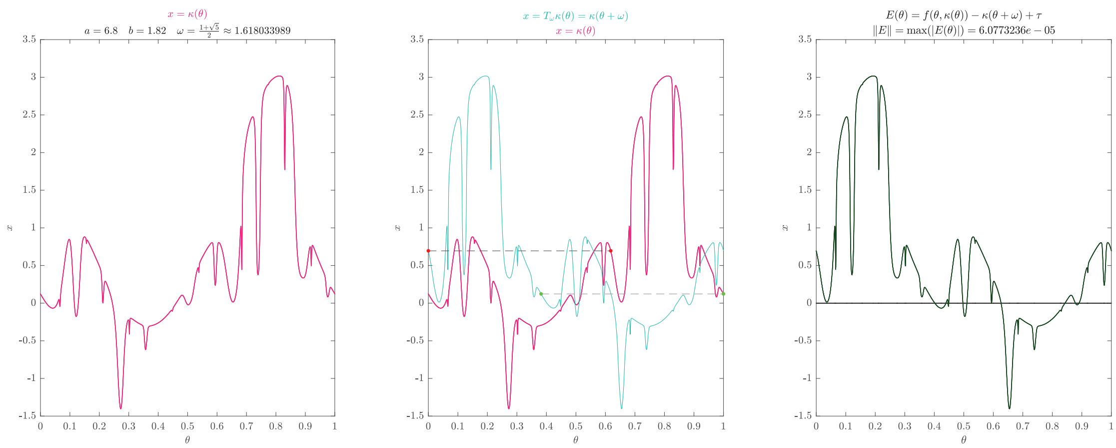


FIGURE 5.16: Curve obtained after n iterations
$x = \kappa_n(\theta)$(magenta) (left);
Translated curve $x = \kappa_n(\theta + \omega)$ (light green) (center);
$x = f(\theta, \kappa_n(\theta))$ (dark green) and the error function $E_n(\theta)$ (right).

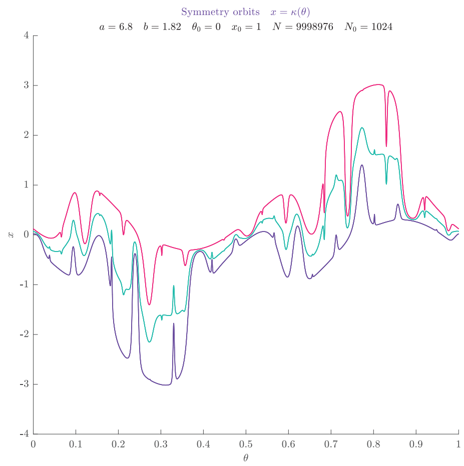


FIGURE 5.17: Symmetry curves $\theta_0 = 0$, $x_0 = 1$ with $N = 10^7$.
$x = \kappa_n(\theta)$ (magenta)
$x = \gamma_n(\theta) = -\kappa_n(\theta + \frac{1}{2})$ (purple)
$x = \eta_n(\theta) = \frac{1}{2}(\kappa_n(\theta) + \gamma_n(\theta))$ (light green)

n	$< \kappa_n >$	Λ_n	λ_n	$\|E_n\| = \sup_{\theta\in\mathbb{T}} \|E_n(\theta)\|$	$\|e_n(p)\| = \| < \kappa_n > -p\|$
0	-0.590406726398	-0.509019376445	0.601084727741	$3.98600345086e-05$	0

Parameters
$a = 6.8$
$b = 1.82$
$\omega = \frac{1+\sqrt{5}}{2} \approx 1.618033988749895$
Initial proposal $(\kappa_0,\ \tau_0)$
$\theta_0 = 0$
$x_0 = -1$
$N = 10000000$
$N_0 = 1024$
Initial characteristic features
$p = -0.5904067263981806$
$\Lambda_0 = -0.5090193764446215$
$\lambda_0 = 0.6010847277741197$
$\det(\Omega_0) = 46.19558497466932$
$\tau_0 = 0$
$\|E_0\| = \max_{\theta\in\mathbb{T}}(|E_0(\theta)|) = 3.986003450862086e-05$
$|e_0(p)| = |<\kappa_0> -p| = 0$
Number of KAM iterations: $n = 0$
Final characteristic features
$\Lambda_n = -0.5090193764446215$
$\lambda_n = 0.601084727741197$
$\tau_n = -1.358955629788376e-07$
$\det(\Omega_n) = 46.19558497466932$
$\|E_n\| = \max_{\theta\in\mathbb{T}}(|E(\theta)|) = 3.986003450862086e-05$
$|e_n(p)| = |<\kappa_n> -p| = 0$

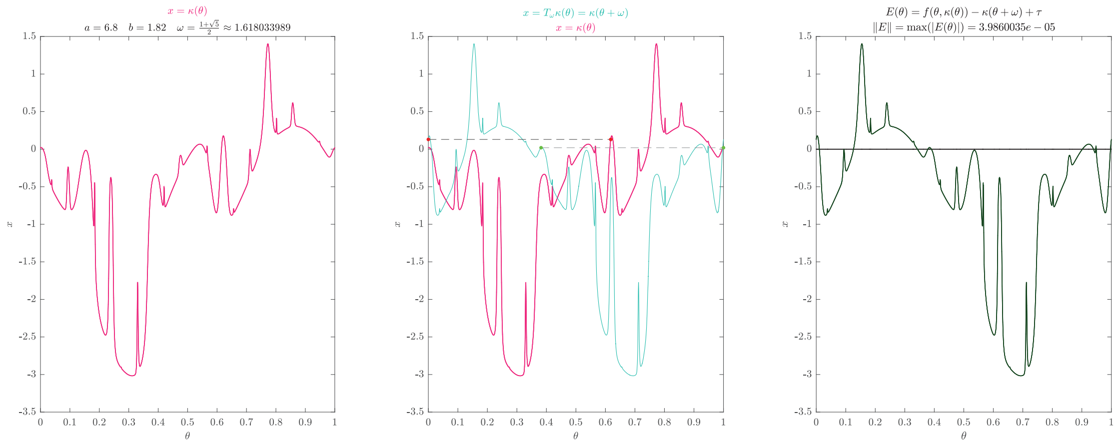


FIGURE 5.18: Curve obtained after n iterations
$x = \kappa_n(\theta)$(magenta) (left);
Translated curve $x = \kappa_n(\theta + \omega)$ (light green) (center);
$x = f(\theta, \kappa_n(\theta))$ (dark green) and the error function $E_n(\theta)$ (right).

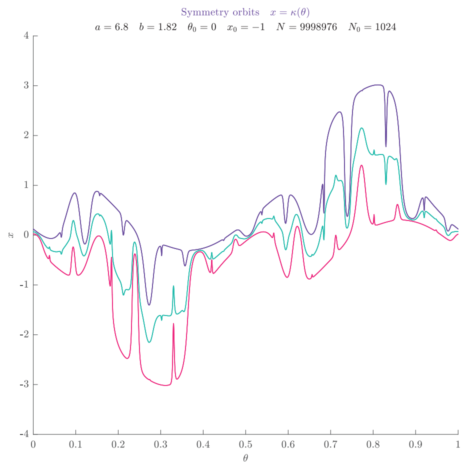


FIGURE 5.19: Symmetry curves $\theta_0 = 0\,,\ x_0 = 1$ with $N = 10^7$.
$x = \kappa_n(\theta)$ (magenta)
$x = \gamma_n(\theta) = -\kappa_n(\theta + \frac{1}{2})$ (purple)
$x = \eta_n(\theta) = \frac{1}{2}(\kappa_n(\theta) + \gamma_n(\theta))$ (light green)

n	$< \kappa_n >$	Λ_n	λ_n	$\|E_n\| = \sup_{\theta \in \mathbb{T}} \|E_n(\theta)\|$	$\|e_n(p)\| = \|< \kappa_n > -p\|$
0	$3.13072670441e-09$	0.193406444853	1.21337586368	$1.87069028474e-05$	0
1	$3.13072665784e-09$	0.193403136375	1.21337184926	$7.72329917504e-09$	$4.65708975576e-17$
2	$3.13072696055e-09$	0.193403136048	1.21337184887	$1.84614323239e-08$	$2.56139936774e-16$

Parameters
$a = 6.8$
$b = 1.82$
$\omega = \frac{1+\sqrt{5}}{2} \approx 1.618033988749895$
Initial proposal $(\kappa_0,\ \tau_0)$
$\theta_0 = 0$

$x_0 = 0$

$N = 10000000$

$N_0 = 1024$

Initial characteristic features

$p = 3.130726704412596e - 09$

$\Lambda_0 = 0.1934064448532904$

$\lambda_0 = 1.213375863684317$

$\det(\Omega_0) = 19.49054356302629$

$\tau_0 = 0$

$\|E_0\| = \max\limits_{\theta \in \mathbb{T}}(|E_0(\theta)|) = 1.870690284744292e - 05$

$|e_0(p)| = | < \kappa_0 > -p| = 0$

Number of KAM iterations: $n = 2$

Final characteristic features

$\Lambda_n = 0.1934031360480563$

$\lambda_n = 1.213371848866551$

$\tau_n = -1.05779693622118e - 07$

$\det(\Omega_n) = 19.49275527026337$

$\|E_n\| = \max\limits_{\theta \in \mathbb{T}}(|E(\theta)|) = 1.846143232392429e - 08$

$|e_n(p)| = | < \kappa_n > -p| = 2.561399367735473e - 16$

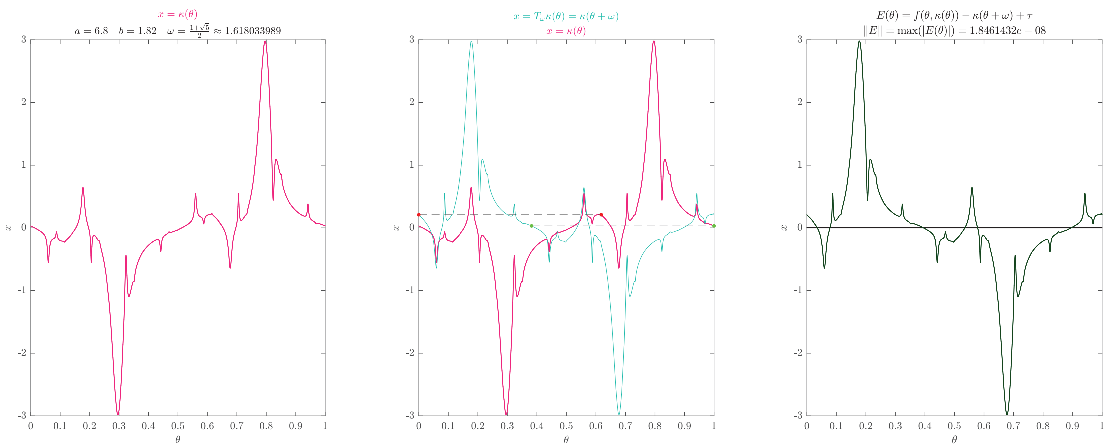


FIGURE 5.20: Curve obtained after n iterations
$x = \kappa_n(\theta)$(magenta) (left);
Translated curve $x = \kappa_n(\theta + \omega)$ (light green) (center);
$x = f(\theta, \kappa_n(\theta))$ (dark green) and the error function $E_n(\theta)$ (right).

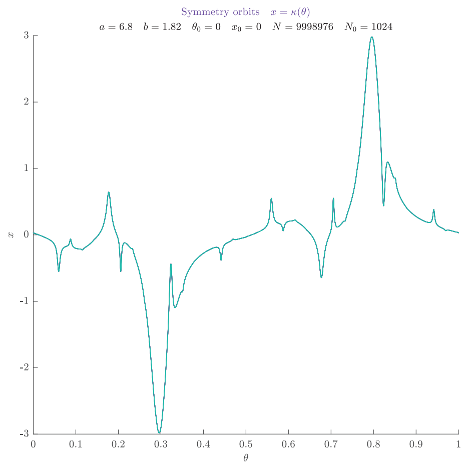


FIGURE 5.21: Symmetry curves $\theta_0 = 0\,,\ x_0 = 1$ with $N = 10^7$.
$x = \kappa_n(\theta)$ (magenta)
$x = \gamma_n(\theta) = -\kappa_n(\theta + \frac{1}{2})$ (purple)
$x = \eta_n(\theta) = \frac{1}{2}(\kappa_n(\theta) + \gamma_n(\theta))$ (light green)

n	$< \kappa_n >$	Λ_n	λ_n	$\|E_n\| = \sup_{\theta\in\mathbb{T}} \|E_n(\theta)\|$	$\|e_n(p)\| = \| < \kappa_n > -p\|$
0	$8.05162793094e-09$	-0.407063725217	0.665601773333	0.00070520531343	0

Parameters
$a = 6.8$
$b = 1.84$
$\omega = \frac{1+\sqrt{5}}{2} \approx 1.618033988749895$
Initial proposal $(\kappa_0,\ \tau_0)$
$\theta_0 = 0$
$x_0 = 1$
$N = 10000000$
$N_0 = 128$
Initial characteristic features
$p = 8.051627930935152e-09$
$\Lambda_0 = -0.4070637252170804$
$\lambda_0 = 0.6656017733325256$
$\det(\Omega_0) = 23.31026525875345$
$\tau_0 = 0$
$\|E_0\| = \max_{\theta\in\mathbb{T}}(|E_0(\theta)|) = 0.0007052053134297687$
$|e_0(p)| = |< \kappa_0 > -p| = 0$
Number of KAM iterations: $n = 0$
Final characteristic features
$\Lambda_n = -0.4070637252170804$
$\lambda_n = 0.6656017733325256$
$\tau_n = -1.041867063577868e-07$
$\det(\Omega_n) = 23.31026525875345$
$\|E_n\| = \max_{\theta\in\mathbb{T}}(|E(\theta)|) = 0.0007052053134297687$
$|e_n(p)| = |< \kappa_n > -p| = 0$

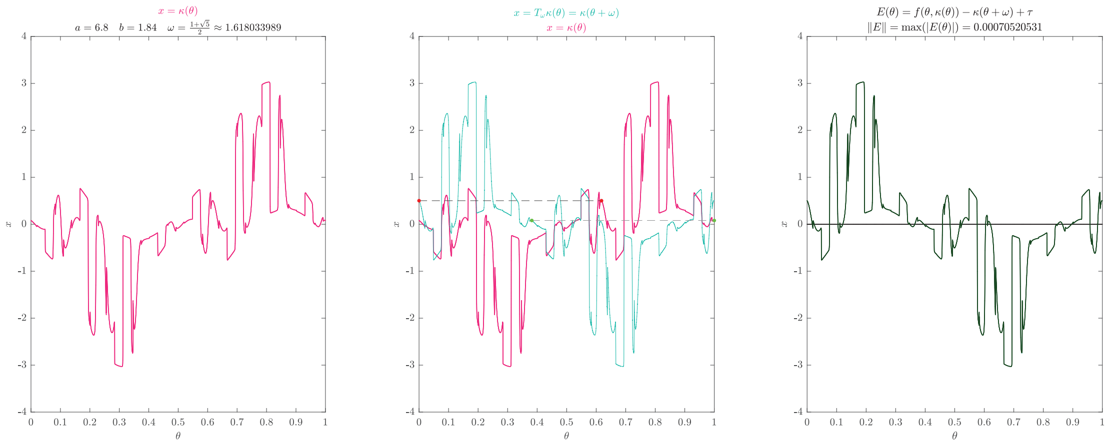


FIGURE 5.22: Curve obtained after n iterations
$x = \kappa_n(\theta)$(magenta) (left);
Translated curve $x = \kappa_n(\theta + \omega)$ (light green) (center);
$x = f(\theta, \kappa_n(\theta))$ (dark green) and the error function $E_n(\theta)$ (right).

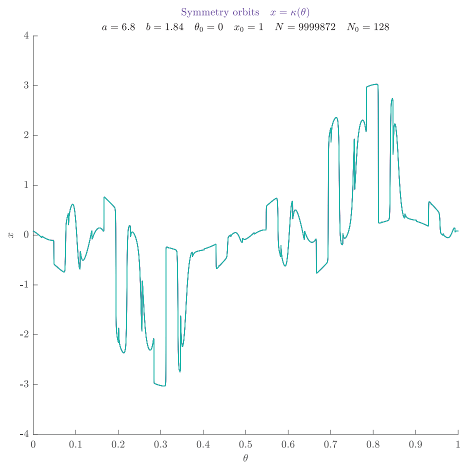


FIGURE 5.23: Symmetry curves $\theta_0 = 0\,,\ x_0 = 1$ with $N = 10^7$.
$x = \kappa_n(\theta)$ (magenta)
$x = \gamma_n(\theta) = -\kappa_n(\theta + \frac{1}{2})$ (purple)
$x = \eta_n(\theta) = \frac{1}{2}(\kappa_n(\theta) + \gamma_n(\theta))$ (light green)

n	$< \kappa_n >$	Λ_n	λ_n	$\|E_n\| = \sup_{\theta\in\mathbb{T}} \|E_n(\theta)\|$	$\|e_n(p)\| = \| < \kappa_n > -p\|$
0	1.11530031925	−1.42097748324	0.241477860935	$2.81484447884e-06$	0
1	1.11530031925	−1.42097749529	0.241477858024	$2.04363060521e-08$	$4.4408920985e-16$
2	1.11530031925	−1.42097749529	0.241477858024	$1.22414088392e-09$	0

Parameters
$a = 6.8$
$b = 1.62$
$\omega = \frac{1+\sqrt{5}}{2} \approx 1.618033988749895$
Initial proposal $(\kappa_0,\ \tau_0)$
$\theta_0 = 0$

$x_0 = 1$
$N = 10000000$
$N_0 = 256$
Initial characteristic features
$p = 1.115300319245528$
$\Lambda_0 = -1.420977483238893$
$\lambda_0 = 0.2414778609349648$
$\det(\Omega_0) = 1.996578066171518$
$\tau_0 = -9.182297182144148e - 08$
$\|E_0\| = \max_{\theta \in \mathbb{T}}(|E_0(\theta)|) = 2.814844478837841e - 06$
$|e_0(p)| = |<\kappa_0> - p| = 0$
Number of KAM iterations: $n = 2$
Final characteristic features
$\Lambda_n = -1.4209774952928$
$\lambda_n = 0.2414778580242132$
$\tau_n = -9.182294006889637e - 08$
$\det(\Omega_n) = 1.996578275066772$
$\|E_n\| = \max_{\theta \in \mathbb{T}}(|E(\theta)|) = 1.224140883923697e - 09$
$|e_n(p)| = |<\kappa_n> - p| = 0$

n	$<\kappa_n>$	Λ_n	λ_n	$\|E_n\| = \sup_{\theta\in\mathbb{T}} \|E_n(\theta)\|$	$\|e_n(p)\| = \|<\kappa_n> - p\|$
0	1.11530031925	−1.42097748324	0.241477860935	$2.81484447884e - 06$	0
1	1.11530031925	−1.42097749529	0.241477858024	$2.04363060521e - 08$	$4.4408920985e - 16$
2	1.11530031925	−1.42097749529	0.241477858024	$1.22414088392e - 09$	0
3	1.11530031925	−1.42097749529	0.241477858024	$4.93978905902e - 11$	$1.7763568394e - 15$
4	1.11530031925	−1.42097749529	0.241477858024	$3.66341938488e - 11$	$3.77475828373e - 15$

Parameters
$a = 6.8$
$b = 1.62$
$\omega = \frac{1+\sqrt{5}}{2} \approx 1.618033988749895$
Initial proposal $(\kappa_0,\ \tau_0)$
$\theta_0 = 0$
$x_0 = 1$
$N = 10000000$
$N_0 = 256$
Initial characteristic features
$p = 1.115300319245528$
$\Lambda_0 = -1.420977483238893$
$\lambda_0 = 0.2414778609349648$
$\det(\Omega_0) = 1.996578066171518$
$\tau_0 = 0$
$\|E_0\| = \max_{\theta \in \mathbb{T}}(|E_0(\theta)|) = 2.814844478837841e - 06$
$|e_0(p)| = |<\kappa_0> - p| = 0$
Number of KAM iterations: $n = 4$
Final characteristic features
$\Lambda_n = -1.420977495292836$
$\lambda_n = 0.2414778580242045$

$\tau_n = -9.182293798437145e-08$

$\det(\Omega_n) = 1.996578274678293$

$\|E_n\| = \max_{\theta\in\mathbb{T}}(|E(\theta)|) = 3.663419384884819e-11$

$|e_n(p)| = |<\kappa_n> -p| = 3.774758283725532e-15$

n	$<\kappa_n>$	Λ_n	λ_n	$\|E_n\| = \sup_{\theta\in\mathbb{T}}\|E_n(\theta)\|$	$\|e_n(p)\| = \|<\kappa_n> -p\|$
0	1.11530031341	-1.42097761065	0.241477830169	$1.81826142032e-06$	0
1	1.11530031341	-1.42097772454	0.241477802666	$3.31766090047e-12$	$6.66133814775e-16$
2	1.11530031341	-1.42097772454	0.241477802666	$3.66339903764e-13$	$1.33226762955e-15$
3	1.11530031341	-1.42097772454	0.241477802666	$5.22675299156e-13$	$2.44249065418e-15$
4	1.11530031341	-1.42097772454	0.241477802666	$1.46968662239e-12$	$5.10702591328e-15$

Parameters

$a = 6.8$

$b = 1.62$

$\omega = \frac{1+\sqrt{5}}{2} \approx 1.618033988749895$

Initial proposal $(\kappa_0,\ \tau_0)$

$\theta_0 = 0$

$x_0 = 1$

$N = 10000000$

$N_0 = 256$

Initial characteristic features

$p = 1.115300313413207$

$\Lambda_0 = -1.420977610646873$

$\lambda_0 = 0.2414778301687604$

$\det(\Omega_0) = 1.996578017355469$

$\tau_0 = 0$

$\|E_0\| = \max_{\theta\in\mathbb{T}}(|E_0(\theta)|) = 1.818261420316603e-06$

$|e_0(p)| = |<\kappa_0> -p| = 0$

Number of KAM iterations: $n = 4$

Final characteristic features

$\Lambda_n = -1.420977724538793$

$\lambda_n = 0.2414778026663881$

$\tau_n = -3.708213172918934e-08$

$\det(\Omega_n) = 1.99657780883491$

$\|E_n\| = \max_{\theta\in\mathbb{T}}(|E(\theta)|) = 1.469686622392079e-12$

$|e_n(p)| = |<\kappa_n> -p| = 5.10702591327572e-15$

n	$<\kappa_n>$	Λ_n	λ_n	$\|E_n\| = \sup_{\theta\in\mathbb{T}} |E_n(\theta)|$	$|e_n(p)| = |<\kappa_n> - p|$
0	$-2.68897208283e-09$	0.751526320597	2.12023370454	$9.03460874557e-07$	0
1	$-2.68897197805e-09$	0.751526428125	2.12023393252	$4.22203411101e-09$	$1.04776472175e-16$
2	$-2.68897189656e-09$	0.751526428125	2.12023393252	$5.87811104465e-12$	$1.86269283314e-16$
3	$-2.68897167536e-09$	0.751526428125	2.12023393252	$1.17963579637e-12$	$4.07464057354e-16$
4	$-2.68897127954e-09$	0.751526428125	2.12023393252	$2.931912132e-13$	$8.03286284604e-16$
5	$-2.68897049954e-09$	0.751526428125	2.12023393252	$4.14899322049e-13$	$1.583288909e-15$
6	$-2.68896885804e-09$	0.751526428125	2.12023393252	$2.32362458953e-13$	$3.22478696935e-15$
7	$-2.68896557505e-09$	0.751526428125	2.12023393252	$2.09432034022e-13$	$6.50778308963e-15$
8	$-2.68895917204e-09$	0.751526428125	2.12023393252	$1.59154098468e-13$	$1.29107897071e-14$
9	$-2.68894636603e-09$	0.751526428125	2.12023393252	$1.32320667309e-13$	$2.5716802942e-14$

Parameters

$a = 6.8$

$b = 1.62$

$\omega = \frac{1+\sqrt{5}}{2} \approx 1.618033988749895$

Initial proposal $(\kappa_0,\ \tau_0)$

$\theta_0 = 0$

$x_0 = 0$

$N = 10000000$

$N_0 = 256$

Initial characteristic features

$p = -2.688972082828913e-09$

$\Lambda_0 = 0.7515263205973447$

$\lambda_0 = 2.120233704537071$

$\det(\Omega_0) = 1.98364937809758$

$\tau_0 = 0$

$\|E_0\| = \max_{\theta\in\mathbb{T}}(|E_0(\theta)|) = 9.034608745572825e-07$

$|e_0(p)| = |<\kappa_0> - p| = 0$

Number of KAM iterations: $n = 9$

Final characteristic features

$\Lambda_n = 0.7515264281245346$

$\lambda_n = 2.120233932519855$

$\tau_n = -1.085811362433058e-08$

$\det(\Omega_n) = 1.983648720495918$

$\|E_n\| = \max_{\theta\in\mathbb{T}}(|E(\theta)|) = 1.323206673085968e-13$

$|e_n(p)| = |<\kappa_n> - p| = 2.571680294197911e-14$

n	$<\kappa_n>$	Λ_n	λ_n	$\|E_n\| = \sup_{\theta\in\mathbb{T}} \|E_n(\theta)\|$	$\|e_n(p)\| = \|<\kappa_n> -p\|$
0	$1.86332048016e-09$	0.751526517341	2.12023412168	$5.86419266924e-07$	0
1	$1.86332055001e-09$	0.751526498009	2.12023408069	$1.13568007919e-12$	$6.98509810361e-17$
2	$1.86332058494e-09$	0.751526498009	2.12023408069	$5.70862191743e-14$	$1.04776471761e-16$
3	$1.86332073628e-09$	0.751526498009	2.12023408069	$7.40865459657e-14$	$2.56120264764e-16$
4	$1.86332100404e-09$	0.751526498009	2.12023408069	$2.88680339999e-14$	$5.23882359632e-16$
5	$1.86332158613e-09$	0.751526498009	2.12023408069	$3.07942539434e-14$	$1.1059738702e-15$
6	$1.86332250584e-09$	0.751526498009	2.12023408069	$2.06818218507e-14$	$2.02567845708e-15$
7	$1.8633245548e-09$	0.751526498009	2.12023408069	$1.79229880944e-14$	$4.07464057478e-15$
8	$1.86332868765e-09$	0.751526498009	2.12023408069	$1.45149622014e-14$	$8.20749030049e-15$
9	$1.86333679036e-09$	0.751526498009	2.12023408069	$1.23151604097e-14$	$1.63102041292e-14$

Parameters

$a = 6.8$

$b = 1.62$

$\omega = \frac{1+\sqrt{5}}{2} \approx 1.618033988749895$

Initial proposal $(\kappa_0,\ \tau_0)$

$\theta_0 = 0$

$x_0 = 0$

$N = 10000000$

$N_0 = 256$

Initial characteristic features

$p = 1.863320480159815e-09$

$\Lambda_0 = 0.7515265173412089$

$\lambda_0 = 2.120234121680084$

$\det(\Omega_0) = 1.98364924451342$

$\tau_0 = 0$

$\|E_0\| = \max_{\theta\in\mathbb{T}}(|E_0(\theta)|) = 5.86419266923599e-07$

$|e_0(p)| = |<\kappa_0> -p| = 0$

Number of KAM iterations: $n = 9$

Final characteristic features

$\Lambda_n = 0.7515264980089791$

$\lambda_n = 2.120234080691231$

$\tau_n = -8.513673455284271e-08$

$\det(\Omega_n) = 1.98364969604328$

$\|E_n\| = \max_{\theta\in\mathbb{T}}(|E(\theta)|) = 1.231516040971472e-14$

$|e_n(p)| = |<\kappa_n> -p| = 1.631020412922066e-14$

n	$<\kappa_n>$	Λ_n	λ_n	$\|E_n\| = \sup_{\theta\in\mathbb{T}} \|E_n(\theta)\|$	$\|e_n(p)\| = \| <\kappa_n> -p\|$
0	$1.86332048016e-09$	0.751526517341	2.12023412168	$5.86419266924e-07$	0
1	$1.86332055001e-09$	0.751526498009	2.12023408069	$1.13568007919e-12$	$6.98509810361e-17$
2	$1.86332058494e-09$	0.751526498009	2.12023408069	$5.70862191743e-14$	$1.04776471761e-16$
3	$1.86332073628e-09$	0.751526498009	2.12023408069	$7.40865459657e-14$	$2.56120264764e-16$
4	$1.86332100404e-09$	0.751526498009	2.12023408069	$2.88680339999e-14$	$5.23882359632e-16$
5	$1.86332158613e-09$	0.751526498009	2.12023408069	$3.07942539434e-14$	$1.1059738702e-15$
6	$1.86332250584e-09$	0.751526498009	2.12023408069	$2.06818218507e-14$	$2.02567845708e-15$
7	$1.8633245548e-09$	0.751526498009	2.12023408069	$1.79229880944e-14$	$4.07464057478e-15$
8	$1.86332868765e-09$	0.751526498009	2.12023408069	$1.45149622014e-14$	$8.20749030049e-15$
9	$1.86333679036e-09$	0.751526498009	2.12023408069	$1.23151604097e-14$	$1.63102041292e-14$
10	$1.86335306564e-09$	0.751526498009	2.12023408069	$1.07708782497e-14$	$3.25854827677e-14$
11	$1.86338565113e-09$	0.751526498009	2.12023408069	$8.96764415492e-15$	$6.51709655354e-14$
12	$1.86345084537e-09$	0.751526498009	2.12023408069	$7.85407353697e-15$	$1.30365214731e-13$
13	$1.86358136193e-09$	0.751526498009	2.12023408069	$6.61832414709e-15$	$2.60881773255e-13$
14	$1.86384222042e-09$	0.751526498009	2.12023408069	$6.15525422591e-15$	$5.2174026285e-13$

Parameters

$a = 6.8$

$b = 1.62$

$\omega = \frac{1+\sqrt{5}}{2} \approx 1.618033988749895$

Initial proposal $(\kappa_0,\ \tau_0)$

$\theta_0 = 0$

$x_0 = 0$

$N = 10000000$

$N_0 = 256$

Initial characteristic features

$p = 1.863320480159815e-09$

$\Lambda_0 = 0.7515265173412089$

$\lambda_0 = 2.120234121680084$

$\det(\Omega_0) = 1.98364924451342$

$\tau_0 = 0$

$\|E_0\| = \max_{\theta\in\mathbb{T}}(|E_0(\theta)|) = 5.86419266923599e-07$

$|e_0(p)| = |<\kappa_0> -p| = 0$

Number of KAM iterations: $n = 14$

Final characteristic features

$\Lambda_n = 0.7515264980089793$

$\lambda_n = 2.120234080691231$

$\tau_n = -8.513730535354296e-08$

$\det(\Omega_n) = 1.983649696043277$

$\|E_n\| = \max_{\theta\in\mathbb{T}}(|E(\theta)|) = 6.15525422591442e-15$

$|e_n(p)| = |<\kappa_n> -p| = 5.217402628500924e-13$

n	$<\kappa_n>$	Λ_n	λ_n	$\|E_n\| = \sup_{\theta\in\mathbb{T}} \|E_n(\theta)\|$	$\|e_n(p)\| = \|<\kappa_n> - p\|$
0	$1.86332048016e-09$	0.751526517341	2.12023412168	$5.86419266924e-07$	0
1	$1.86332055001e-09$	0.751526498009	2.12023408069	$1.13568007919e-12$	$6.98509810361e-17$
2	$1.86332058494e-09$	0.751526498009	2.12023408069	$5.70862191743e-14$	$1.04776471761e-16$
3	$1.86332073628e-09$	0.751526498009	2.12023408069	$7.40865459657e-14$	$2.56120264764e-16$
4	$1.86332100404e-09$	0.751526498009	2.12023408069	$2.88680339999e-14$	$5.23882359632e-16$
5	$1.86332158613e-09$	0.751526498009	2.12023408069	$3.07942539434e-14$	$1.1059738702e-15$
6	$1.86332250584e-09$	0.751526498009	2.12023408069	$2.06818218507e-14$	$2.02567845708e-15$
7	$1.8633245548e-09$	0.751526498009	2.12023408069	$1.79229880944e-14$	$4.07464057478e-15$
8	$1.86332868765e-09$	0.751526498009	2.12023408069	$1.45149622014e-14$	$8.20749030049e-15$
9	$1.86333679036e-09$	0.751526498009	2.12023408069	$1.23151604097e-14$	$1.63102041292e-14$
10	$1.86335306564e-09$	0.751526498009	2.12023408069	$1.07708782497e-14$	$3.25854827677e-14$
11	$1.86338565113e-09$	0.751526498009	2.12023408069	$8.96764415492e-15$	$6.51709655354e-14$
12	$1.86345084537e-09$	0.751526498009	2.12023408069	$7.85407353697e-15$	$1.30365214731e-13$
13	$1.86358136193e-09$	0.751526498009	2.12023408069	$6.61832414709e-15$	$2.60881773255e-13$
14	$1.86384222042e-09$	0.751526498009	2.12023408069	$6.15525422591e-15$	$5.2174026285e-13$

Parameters

$a = 6.8$

$b = 1.62$

$\omega = \frac{1+\sqrt{5}}{2} \approx 1.618033988749895$

Initial proposal $(\kappa_0,\ \tau_0)$

$\theta_0 = 0$

$x_0 = 0$

$N = 10000000$

$N_0 = 256$

Initial characteristic features

$p = 1.863320480159815e-09$

$\Lambda_0 = 0.7515265173412089$

$\lambda_0 = 2.120234121680084$

$\det(\Omega_0) = 1.98364924451342$

$\tau_0 = 0$

$\|E_0\| = \max_{\theta\in\mathbb{T}}(|E_0(\theta)|) = 5.86419266923599e-07$

$|e_0(p)| = |<\kappa_0> - p| = 0$

Number of KAM iterations: $n = 14$

Final characteristic features

$\Lambda_n = 0.7515264980089793$

$\lambda_n = 2.120234080691231$

$\tau_n = -8.513730535354296e-08$

$\det(\Omega_n) = 1.983649696043277$

$\|E_n\| = \max_{\theta\in\mathbb{T}}(|E(\theta)|) = 6.15525422591442e-15$

$|e_n(p)| = |<\kappa_n> - p| = 5.217402628500924e-13$

n	$<\kappa_n>$	Λ_n	λ_n	$\|E_n\| = \sup_{\theta\in\mathbb{T}} \|E_n(\theta)\|$	$\|e_n(p)\| = \| <\kappa_n> -p\|$
0	1.11530031341	−1.42097761065	0.241477830169	$1.81826142032e-06$	0
1	1.11530031341	−1.42097772454	0.241477802666	$3.31766090047e-12$	$6.66133814775e-16$
2	1.11530031341	−1.42097772454	0.241477802666	$3.66339903764e-13$	$1.33226762955e-15$
3	1.11530031341	−1.42097772454	0.241477802666	$5.22675299156e-13$	$2.44249065418e-15$
4	1.11530031341	−1.42097772454	0.241477802666	$1.46968662239e-12$	$5.10702591328e-15$

$a = 6.8$
$b = 1.62$
$N = 10000000$
$N_0 = 256$
$\omega = \frac{1+\sqrt{5}}{2} \approx 1.618033988749895$
$p = 1.115300313413207$
$n = 4$
$\Lambda_n = -1.420977724538793$
$\lambda_n = 0.2414778026663881$
$\tau_n = -3.708213172918934e-08$
$\|E_0\| = \sup_{\theta\in\mathbb{T}}(|E_0(\theta)|) = 1.818261420316603e-06$
$\|E_n\| = \sup_{\theta\in\mathbb{T}}(|E(\theta)|) = 1.469686622392079e-12$
$\det(\Omega_0) = 1.996578017355469$
$\det(\Omega_n) = 1.99657780883491$

n	$<\kappa_n>$	Λ_n	λ_n	$\|E_n\| = \sup_{\theta\in\mathbb{T}} \|E_n(\theta)\|$	$\|e_n(p)\| = \| <\kappa_n> -p\|$
0	−1.11530030417	−1.42097757442	0.241477838916	$1.14711105148e-06$	0
1	−1.11530030417	−1.42097750652	0.241477855312	$2.44644605755e-12$	$1.11022302463e-15$
2	−1.11530030417	−1.42097750652	0.241477855312	$7.24616551907e-13$	$1.7763568394e-15$
3	−1.11530030417	−1.42097750652	0.241477855312	$2.0928732574e-12$	$3.99680288865e-15$
4	−1.11530030417	−1.42097750652	0.241477855312	$6.02923071497e-12$	$7.1054273576e-15$

$a = 6.8$
$b = 1.62$
$N = 10000000$
$N_0 = 256$
$\omega = \frac{1+\sqrt{5}}{2} \approx 1.618033988749895$
$p = -1.115300304170598$
$n = 4$
$\Lambda_n = -1.420977506523903$
$\lambda_n = 0.2414778553121505$
$\tau_n = 3.606618546312675e-08$
$\|E_0\| = \max_{\theta\in\mathbb{T}}(|E_0(\theta)|) = 1.147111051480465e-06$
$\|E_n\| = \max_{\theta\in\mathbb{T}}(|E(\theta)|) = 6.029230714966026e-12$
$\det(\Omega_0) = 1.996577861978446$
$\det(\Omega_n) = 1.996578190831171$

n	$<\kappa_n>$	Λ_n	λ_n	$\|E_n\| = \sup_{\theta\in\mathbb{T}} \|E_n(\theta)\|$	$\|e_n(p)\| = \|<\kappa_n> -p\|$
0	$-2.68897208283e-09$	0.751526320597	2.12023370454	$9.03460874557e-07$	0
1	$-2.68897197805e-09$	0.751526428125	2.12023393252	$4.22203411101e-09$	$1.04776472175e-16$
2	$-2.68897189656e-09$	0.751526428125	2.12023393252	$5.87811104465e-12$	$1.86269283314e-16$
3	$-2.68897167536e-09$	0.751526428125	2.12023393252	$1.17963579637e-12$	$4.07464057354e-16$
4	$-2.68897127954e-09$	0.751526428125	2.12023393252	$2.931912132e-13$	$8.03286284604e-16$

$a = 6.8$

$b = 1.62$

$N = 10000000$

$N_0 = 256$

$\omega = \frac{1+\sqrt{5}}{2} \approx 1.618033988749895$

$p = -2.688972082828913e-09$

$n = 4$

$\Lambda_n = 0.7515264281245347$

$\lambda_n = 2.120233932519855$

$\tau_n = -1.08580854217983e-08$

$\|E_0\| = \max_{\theta\in\mathbb{T}}(|E_0(\theta)|) = 9.034608745572825e-07$

$\|E_n\| = \max_{\theta\in\mathbb{T}}(|E(\theta)|) = 2.931912132003164e-13$

$\det(\Omega_0) = 1.98364937809758$

$\det(\Omega_n) = 1.983648720495924$

n	$<\kappa_n>$	Λ_n	λ_n	$\|E_n\| = \sup_{\theta\in\mathbb{T}} \|E_n(\theta)\|$	$\|e_n(p)\| = \|<\kappa_n> -p\|$
0	1.11530031925	-1.42097748324	0.241477860935	$2.81484447884e-06$	0
1	1.11530031925	-1.42097749529	0.241477858024	$2.04363060521e-08$	$4.4408920985e-16$
2	1.11530031925	-1.42097749529	0.241477858024	$1.22414088392e-09$	0
3	1.11530031925	-1.42097749529	0.241477858024	$4.93978905902e-11$	$1.7763568394e-15$
4	1.11530031925	-1.42097749529	0.241477858024	$3.66341938488e-11$	$3.77475828373e-15$

n	$<\kappa_n>$	Λ_n	λ_n	$\|E_n\| = \sup_{\theta\in\mathbb{T}} \|E_n(\theta)\|$	$\|e_n(p)\| = \|<\kappa_n> -p\|$
0	1.11530031341	-1.42097761065	0.241477830169	$1.81826142032e-06$	0
1	1.11530031341	-1.42097772454	0.241477802666	$3.31766090047e-12$	$6.66133814775e-16$
2	1.11530031341	-1.42097772454	0.241477802666	$3.66339903764e-13$	$1.33226762955e-15$
3	1.11530031341	-1.42097772454	0.241477802666	$5.22675299156e-13$	$2.44249065418e-15$
4	1.11530031341	-1.42097772454	0.241477802666	$1.46968662239e-12$	$5.10702591328e-15$

n	$< \kappa_n >$	Λ_n	λ_n	$\|E_n\| = \sup_{\theta\in\mathbb{T}} \|E_n(\theta)\|$	$\|e_n(p)\| = \| < \kappa_n > -p\|$
0	-1.11530030417	-1.42097757442	0.241477838916	$1.14711105148e-06$	0
1	-1.11530030417	-1.42097750652	0.241477855312	$2.44644605755e-12$	$1.11022302463e-15$
2	-1.11530030417	-1.42097750652	0.241477855312	$7.24616551907e-13$	$1.7763568394e-15$
3	-1.11530030417	-1.42097750652	0.241477855312	$2.0928732574e-12$	$3.99680288865e-15$
4	-1.11530030417	-1.42097750652	0.241477855312	$6.02923071497e-12$	$7.1054273576e-15$

n	$< \kappa_n >$	Λ_n	λ_n	$\|E_n\| = \sup_{\theta\in\mathbb{T}} \|E_n(\theta)\|$	$\|e_n(p)\| = \| < \kappa_n > -p\|$
0	$-3.29574701041e-08$	0.751526426994	2.12023393012	$1.00495609567e-06$	0
1	$-3.29574702787e-08$	0.75152651564	2.12023411807	$2.59163795774e-09$	$1.74625218996e-16$
2	$-3.29574703602e-08$	0.75152651564	2.12023411807	$1.28089691143e-11$	$2.56116988302e-16$
3	$-3.29574705814e-08$	0.75152651564	2.12023411807	$4.61497552674e-12$	$4.77308928834e-16$
4	$-3.29574710471e-08$	0.75152651564	2.12023411807	$7.10349142546e-12$	$9.42976177284e-16$

Parameters
$a = 6.8$
$b = 1.62$
$\omega = \frac{1+\sqrt{5}}{2} \approx 1.618033988749895$
Initial proposal $(\kappa_0,\ \tau_0)$
$\theta_0 = 0$
$x_0 = 1$
$N = 10000000$
$N_0 = 128$
Initial characteristic features
$p = -3.295747010411697e-08$
$\Lambda_0 = 0.7515264269943018$
$\lambda_0 = 2.120233930123497$
$\det(\Omega_0) = 1.983649446320672$
$\tau_0 = 0$
$\|E_0\| = \max_{\theta\in\mathbb{T}}(|E_0(\theta)|) = 1.004956095673748e-06$
$|e_0(p)| = | < \kappa_0 > -p| = 0$
Number of KAM iterations: $n = 4$
Final characteristic features
$\Lambda_n = 0.7515265156396997$
$\lambda_n = 2.120234118072486$
$\tau_n = -1.19051621416144e-09$
$\det(\Omega_n) = 1.983649153752742$
$\|E_n\| = \max_{\theta\in\mathbb{T}}(|E(\theta)|) = 7.103491425456302e-12$
$|e_n(p)| = | < \kappa_n > -p| = 9.429761772838928e-16$

Unstable invariant curve

n	$<\kappa_n>$	Λ_n	λ_n	$\|E_n\| = \sup_{\theta\in\mathbb{T}} \|E_n(\theta)\|$	$\|e_n(p)\| = \|<\kappa_n> - p\|$
0	$-2.68897208283e-09$	0.751526320597	2.12023370454	$9.03460874557e-07$	0
1	$-2.68897197805e-09$	0.751526428125	2.12023393252	$4.22203411101e-09$	$1.04776472175e-16$
2	$-2.68897189656e-09$	0.751526428125	2.12023393252	$5.87811104465e-12$	$1.86269283314e-16$
3	$-2.68897167536e-09$	0.751526428125	2.12023393252	$1.17963579637e-12$	$4.07464057354e-16$
4	$-2.68897127954e-09$	0.751526428125	2.12023393252	$2.931912132e-13$	$8.03286284604e-16$
5	$-2.68897049954e-09$	0.751526428125	2.12023393252	$4.14899322049e-13$	$1.583288909e-15$
6	$-2.68896885804e-09$	0.751526428125	2.12023393252	$2.32362458953e-13$	$3.22478696935e-15$
7	$-2.68896557505e-09$	0.751526428125	2.12023393252	$2.09432034022e-13$	$6.50778308963e-15$
8	$-2.68895917204e-09$	0.751526428125	2.12023393252	$1.59154098468e-13$	$1.29107897071e-14$
9	$-2.68894636603e-09$	0.751526428125	2.12023393252	$1.32320667309e-13$	$2.5716802942e-14$
10	$-2.68892068415e-09$	0.751526428125	2.12023393252	$1.09474100512e-13$	$5.13986803928e-14$
11	$-2.68886907592e-09$	0.751526428125	2.12023393252	$9.10874111954e-14$	$1.0300691373e-13$
12	$-2.68876623199e-09$	0.751526428125	2.12023393252	$7.65636248248e-14$	$2.05850841836e-13$
13	$-2.68856047428e-09$	0.751526428125	2.12023393252	$6.47806102933e-14$	$4.1160854903e-13$
14	$-2.68814870275e-09$	0.751526428125	2.12023393252	$5.43810413504e-14$	$8.23380083684e-13$
15	$-2.68732534595e-09$	0.751526428125	2.12023393252	$4.61384775104e-14$	$1.64673688371e-12$
16	$-2.6856786207e-09$	0.751526428125	2.12023393252	$3.89824875914e-14$	$3.29346212558e-12$
17	$-2.68238520515e-09$	0.751526428125	2.12023393252	$3.29066115874e-14$	$6.58687768384e-12$
18	$-2.67579825761e-09$	0.751526428125	2.12023393252	$2.79883448273e-14$	$1.31738252187e-11$
19	$-2.66262438582e-09$	0.751526428125	2.12023393252	$2.38898097364e-14$	$2.63476970047e-11$

n	$<\kappa_n>$	Λ_n	λ_n	$\|E_n\| = \sup_{\theta\in\mathbb{T}} \|E_n(\theta)\|$	$\|e_n(p)\| = \|<\kappa_n> - p\|$
0	$1.86332048016e-09$	0.751526517341	2.12023412168	$5.86419266924e-07$	0
1	$1.86332055001e-09$	0.751526498009	2.12023408069	$1.13568007919e-12$	$6.98509810361e-17$
2	$1.86332058494e-09$	0.751526498009	2.12023408069	$5.70862191743e-14$	$1.04776471761e-16$
3	$1.86332073628e-09$	0.751526498009	2.12023408069	$7.40865459657e-14$	$2.56120264764e-16$
4	$1.86332100404e-09$	0.751526498009	2.12023408069	$2.88680339999e-14$	$5.23882359632e-16$
5	$1.86332158613e-09$	0.751526498009	2.12023408069	$3.07942539434e-14$	$1.1059738702e-15$
6	$1.86332250584e-09$	0.751526498009	2.12023408069	$2.06818218507e-14$	$2.02567845708e-15$
7	$1.8633245548e-09$	0.751526498009	2.12023408069	$1.79229880944e-14$	$4.07464057478e-15$
8	$1.86332868765e-09$	0.751526498009	2.12023408069	$1.45149622014e-14$	$8.20749030049e-15$
9	$1.86333679036e-09$	0.751526498009	2.12023408069	$1.23151604097e-14$	$1.63102041292e-14$
10	$1.86335306564e-09$	0.751526498009	2.12023408069	$1.07708782497e-14$	$3.25854827677e-14$
11	$1.86338565113e-09$	0.751526498009	2.12023408069	$8.96764415492e-15$	$6.51709655354e-14$
12	$1.86345084537e-09$	0.751526498009	2.12023408069	$7.85407353697e-15$	$1.30365214731e-13$
13	$1.86358136193e-09$	0.751526498009	2.12023408069	$6.61832414709e-15$	$2.60881773255e-13$
14	$1.86384222042e-09$	0.751526498009	2.12023408069	$6.15525422591e-15$	$5.2174026285e-13$
15	$1.86436396069e-09$	0.751526498009	2.12023408069	$5.25480641486e-15$	$1.0434805257e-12$
16	$1.86540748778e-09$	0.751526498009	2.12023408069	$4.90313839925e-15$	$2.08700761872e-12$
17	$1.86749444883e-09$	0.751526498009	2.12023408069	$4.12317228992e-15$	$4.17396867012e-12$
18	$1.87166831272e-09$	0.751526498009	2.12023408069	$3.63126669377e-15$	$8.34783256377e-12$
19	$1.880016122e-09$	0.751526498009	2.12023408069	$3.48808445612e-15$	$1.66956418439e-11$

n	$< \kappa_n >$	Λ_n	λ_n	$\|E_n\| = \sup_{\theta\in\mathbb{T}} \|E_n(\theta)\|$	$\|e_n(p)\| = \| < \kappa_n > -p\|$
0	$1.86332048016e-09$	0.751526517341	2.12023412168	$5.86419266924e-07$	0
1	$1.86332055001e-09$	0.751526498009	2.12023408069	$1.13568007919e-12$	$6.98509810361e-17$
2	$1.86332058494e-09$	0.751526498009	2.12023408069	$5.70862191743e-14$	$1.04776471761e-16$
3	$1.86332073628e-09$	0.751526498009	2.12023408069	$7.40865459657e-14$	$2.56120264764e-16$
4	$1.86332100404e-09$	0.751526498009	2.12023408069	$2.88680339999e-14$	$5.23882359632e-16$
5	$1.86332158613e-09$	0.751526498009	2.12023408069	$3.07942539434e-14$	$1.1059738702e-15$
6	$1.86332250584e-09$	0.751526498009	2.12023408069	$2.06818218507e-14$	$2.02567845708e-15$
7	$1.8633245548e-09$	0.751526498009	2.12023408069	$1.79229880944e-14$	$4.07464057478e-15$
8	$1.86332868765e-09$	0.751526498009	2.12023408069	$1.45149622014e-14$	$8.20749030049e-15$
9	$1.86333679036e-09$	0.751526498009	2.12023408069	$1.23151604097e-14$	$1.63102041292e-14$
10	$1.86335306564e-09$	0.751526498009	2.12023408069	$1.07708782497e-14$	$3.25854827677e-14$
11	$1.86338565113e-09$	0.751526498009	2.12023408069	$8.96764415492e-15$	$6.51709655354e-14$
12	$1.86345084537e-09$	0.751526498009	2.12023408069	$7.85407353697e-15$	$1.30365214731e-13$
13	$1.86358136193e-09$	0.751526498009	2.12023408069	$6.61832414709e-15$	$2.60881773255e-13$
14	$1.86384222042e-09$	0.751526498009	2.12023408069	$6.15525422591e-15$	$5.2174026285e-13$
15	$1.86436396069e-09$	0.751526498009	2.12023408069	$5.25480641486e-15$	$1.0434805257e-12$
16	$1.86540748778e-09$	0.751526498009	2.12023408069	$4.90313839925e-15$	$2.08700761872e-12$
17	$1.86749444883e-09$	0.751526498009	2.12023408069	$4.12317228992e-15$	$4.17396867012e-12$
18	$1.87166831272e-09$	0.751526498009	2.12023408069	$3.63126669377e-15$	$8.34783256377e-12$
19	$1.880016122e-09$	0.751526498009	2.12023408069	$3.48808445612e-15$	$1.66956418439e-11$

n	$< \kappa_n >$	Λ_n	λ_n	$\|E_n\| = \sup_{\theta\in\mathbb{T}} \|E_n(\theta)\|$	$\|e_n(p)\| = \| < \kappa_n > -p\|$
0	$4.76471550321e-10$	1.7811320816	5.93657330125	$2.31443690966e-08$	0
1	$4.76471591067e-10$	1.78113208167	5.93657330164	$6.44256392132e-15$	$4.07464057767e-17$
2	$4.76471628903e-10$	1.78113208167	5.93657330164	$4.93920067974e-15$	$7.85823539242e-17$
3	$4.7647170312e-10$	1.78113208167	5.93657330164	$3.79540287878e-15$	$1.52799021559e-16$
4	$4.76471832635e-10$	1.78113208167	5.93657330164	$2.94401836263e-15$	$2.82314382704e-16$

n	$< \kappa_n >$	Λ_n	λ_n	$\|E_n\| = \sup_{\theta\in\mathbb{T}} \|E_n(\theta)\|$	$\|e_n(p)\| = \| < \kappa_n > -p\|$
0	$3.98768775472e-07$	-3.01060856251	0.0492616908183	$2.30120842322e-05$	0
1	$3.98768775361e-07$	-3.01060791127	0.0492617228997	$1.9509617846e-10$	$1.11130828559e-16$
2	$3.98768775361e-07$	-3.01060791127	0.0492617228996	$9.75242654723e-10$	$1.11130828559e-16$
3	$3.98768775027e-07$	-3.01060791127	0.0492617228996	$7.6948926864e-09$	$4.44523314236e-16$
4	$3.98768774138e-07$	-3.01060791127	0.0492617228996	$6.05788270823e-08$	$1.33356994271e-15$

n	$<\kappa_n>$	Λ_n	λ_n	$\|E_n\| = \sup_{\theta\in\mathbb{T}} \|E_n(\theta)\|$	$\|e_n(p)\| = \| <\kappa_n> -p\|$
0	$1.67096936354e-07$	-3.01060857585	0.0492616901613	$9.936411417e-06$	0
1	$1.6709693654e-07$	-3.01060830296	0.0492617036044	$2.81870667586e-11$	$1.86269268425e-16$
2	$1.67096936633e-07$	-3.01060830296	0.0492617036044	$2.11359268164e-10$	$2.79403902638e-16$
3	$1.67096937099e-07$	-3.01060830296	0.0492617036044	$2.3440134395e-09$	$7.45077126641e-16$
4	$1.67096937006e-07$	-3.01060830296	0.0492617036044	$2.50229554579e-08$	$6.51942492428e-16$

Chapter 6

Numerical aspects related to some examples

6.1 Jäger's model

In this section, the model presented will serve as a support to develop all the algorithms built in previous sections and chapters, as well as their subsequent numerical implementation[1].

Let us define the extended quasi–periodic skew–product:

Definition 6.1 Jäger's model
Let $\varrho > 0$*,* $a, b \in \mathbb{R}^+$*,* $\omega \in \mathcal{DC}(\gamma, \nu)$*, and* $\mathcal{U} = \mathbb{C} \setminus \{-\frac{1}{a}i, \frac{1}{a}i\}$*.*

$$\begin{array}{rccl} \psi: & \mathbb{T}_\varrho \times \mathcal{U} & \longrightarrow & \mathbb{T}_\varrho \times \mathbb{C} \\ & (\theta, z) & \longmapsto & \psi(\theta, z) := (\theta + \omega, \arctan(az) + b\sin(2\pi\theta)) \end{array} . \tag{6.1}$$

As usual, we denote by $f(\theta, z)$ *the second component of the skew–product, that is,*

$$\begin{array}{rccl} f: & \mathbb{T}_\varrho \times \mathcal{U} & \longrightarrow & \mathbb{C} \\ & (\theta, z) & \longmapsto & f(\theta, z) := \arctan(az) + b\sin(2\pi\theta) \end{array} . \tag{6.2}$$

◇

Remark 6.2

ψ is invertible and

$$\begin{array}{rccl} \psi^{-1}: & \mathbb{T}_\varrho \times \mathcal{V} & \longrightarrow & \mathbb{T}_\varrho \times \mathcal{U} \\ & (\theta, z) & \longmapsto & \psi^{-1}(\theta, z) := (\theta - \omega, g(\theta, z)) \end{array} , \tag{6.3}$$

where $g(\theta, z) = \frac{1}{a}\tan(z - b\sin(2\pi(\theta - \omega)))$*,* $\forall (\theta, z) \in \mathbb{T}_\varrho \times \mathcal{V}$*.*

[1] This model, with slight differences, was presented by Tobias H. Jäger in 2003 [34] and has been deeply analyzed by Àngel Jorba, Francisco Javier Muñoz–Almaraz, and Joan Carles Tatjer in 2018 [36]. Here we expose an extended version complexified with the aim of adapting the model to the previous exposition.

We take **Lemma 3.5** as a starting point to certify the performance of the algorithms described for obtaining invariant curves. And we will do it without taking into account some specific symmetry properties of this particular example that, on the other hand, may be used for other specific purposes. Thus, the study of this skew–product does not detract from the generality of the example. We will see, first of all, that the hypotheses for the KAM procedure are satisfied. In this sense we remark the following properties of the skew–product (6.1).

Proposition 6.3

Let $\psi = R_\omega \times f$ be the skew–product (6.1). We consider the closed disk $\overline{\mathbb{D}}(0,R)$, with $0 < R < \frac{1}{a}$. Then, the following properties hold:

(*i*) *The spatial derivative of the function f (6.2) is*

$$\frac{\partial f}{\partial z}(\theta, z) = \frac{a}{1+a^2z^2},\ \forall(\theta,z) \in \mathbb{T}_\varrho \times \mathcal{U}. \tag{6.4}$$

and $\frac{\partial f}{\partial z}\left(\overline{\mathbb{D}}(0,R)\right) = \overline{\mathbb{D}}(z_0, R^*)$, with $z_0 = \frac{a}{1-a^4R^4}$ and $R^* = \frac{a^3R^2}{1-a^4R^4}$.
Moreover, there are positive constants K_1 and K_1^* such that

$$0 < K_1^* := \frac{a}{1+a^2R^2} \le \left|\frac{\partial f}{\partial z}(\theta,z)\right| \le \frac{a}{1-a^2R^2} := K_1,\ \forall(\theta,z) \in \mathbb{T}_\varrho \times \overline{\mathbb{D}}(0,R). \tag{6.5}$$

(*ii*) *The second spatial derivative of the function f is*

$$\frac{\partial^2 f}{\partial z^2}(\theta,z) = -\frac{2a^3z}{(1+a^2z^2)^2},\ \forall(\theta,z) \in \mathbb{T}_\varrho \times \mathcal{U}. \tag{6.6}$$

Moreover, there is a positive constant K_2 such that

$$\left|\frac{\partial^2 f}{\partial z^2}(\theta,z)\right| \le \frac{2a^3R}{(1-a^2R^2)^2} := K_2,\ \forall(\theta,z) \in \mathbb{T}_\varrho \times \overline{\mathbb{D}}(0,R). \tag{6.7}$$

(*iii*) *$\exists \alpha \in (0,\pi)$ such that*

$$\left|\operatorname{Arg}\frac{\partial f}{\partial z}(\theta,z)\right| \le \alpha,\ \forall(\theta,z) \in \mathbb{T}_\varrho \times \overline{\mathbb{D}}(0,R)\ . \tag{6.8}$$

More specifically, $\alpha = \arctan\dfrac{a^2R^2}{\sqrt{1-a^4R^4}} \in (0,\pi/2)$.

Proof.

(i) The first partial derivative of f with respect to the spatial coordinate is

$$\frac{\partial f}{\partial z}(\theta,z) = \frac{a}{1+(az)^2},\ \forall z \in \mathcal{U} = \mathbb{C}\setminus\{-\frac{1}{a}i, \frac{1}{a}i\}.$$

The image of the closed disk

$$\overline{\mathbb{D}}(0,R) = \{z = \frac{r}{a}e^{ti} : (r,t) \in [0,aR]\times(-\pi,\pi]\}$$

can be written as

$$\frac{\partial f}{\partial z}\left(\overline{\mathbb{D}}(0,R)\right) = \{\frac{a}{1+(az)^2} : z = \frac{1}{a}re^{ti} : (r,t) \in [0,aR]\times(-\pi,\pi]\}.$$

First we show that, given any $r \in [0, aR]$, the image of the circle $\partial\mathbb{D}(0, \frac{1}{a}r)$ under $\frac{\partial f}{\partial z}$ is the circle $\partial\mathbb{D}(c_r, R_r)$, with $c_r = \frac{a}{1-r^4}$ and $R_r = \frac{ar^2}{1-r^4}$.

On the one hand, if $z = \frac{1}{a}re^{ti}$, with $(r,t) \in [0, aR] \times (-\pi, \pi]$, then

$$\begin{aligned}\frac{\partial f}{\partial z}(\theta, z) &= \frac{a}{1+(az)^2} = \frac{a}{1+r^2e^{2ti}} \\ &= \frac{a}{1+r^2e^{2ti}}\frac{1+r^2e^{-2ti}}{1+r^2e^{-2ti}} = \frac{a}{|1+r^2e^{2ti}|^2}(1+r^2e^{-2ti}).\end{aligned}$$

By Euler's formula, $e^{2ti} = \cos(2t) + i\sin(2t)$ and $e^{-2ti} = \cos(2t) - i\sin(2t)$. Therefore,

$$|1+r^2e^{2ti}|^2 = (1+r^2\cos(2t))^2 + r^2\sin^2(2t) = 1 + 2r^2\cos(2t) + r^4 \;.$$

It follows that, $\frac{\partial f}{\partial z}(\theta, z) = u(r,t) + iv(r,t)$ with

$$\begin{cases} u(r,t) &= \frac{a}{1+2r^2\cos(2t)+r^4}(1+r^2\cos(2t)) \\ v(r,t) &= \frac{a}{1+2r^2\cos(2t)+r^4}(-r^2\sin(2t)) \end{cases} \quad (r,t) \in [0, aR] \times (-\pi, \pi] \;. \tag{6.9}$$

On the other hand, for each $r \in [0, aR]$, the distance from $\frac{\partial f}{\partial z}(\theta, \frac{1}{a}re^{ti})$ to the real point c_r is

$$\begin{aligned}\left|\frac{\partial f}{\partial z}(\theta, \frac{1}{a}re^{ti}) - c_r\right| &= \left|\frac{a}{1+r^2e^{2ti}} - \frac{a}{1-r^4}\right| \\ &= \left|\frac{1}{1-r^4}\frac{ar^2}{1+r^2e^{2ti}}(r^2+e^{2ti})\right| \\ &= \frac{ar^2}{1-r^4}\frac{|r^2+e^{2ti}|}{|1+r^2e^{2ti}|} \;.\end{aligned}$$

Since $|r^2 + e^{2ti}| = |1 + r^2e^{2ti}| = (1 + 2r^2\cos(2t) + r^4)^{\frac{1}{2}}$, we have

$$\left|\frac{\partial f}{\partial z}(\theta, \frac{1}{a}re^{ti}) - c_r\right| = \frac{ar^2}{1-r^4} = R_r \;.$$

This proves that $\frac{\partial f}{\partial z}\left(\partial\mathbb{D}(0, \frac{1}{a}r)\right) = \partial\mathbb{D}(c_r, R_r)$, $\forall r \in [0, aR]$. These non-concentric circles are all centered in the real line, and the corresponding closed disks $\overline{\mathbb{D}}(c_r, R_r)$ constitute an increasing family of sets with respect to r. In particular,

$$K_1^* = c_{aR} - R_{aR} = \frac{a}{1+a^2R^2} \le c_r - R_r = \frac{a}{1+r^2} \le \frac{a}{1-r^2} = c_r + R_r \le \frac{a}{1-a^2R^2} = c_{aR} + R_{aR} = K_1.$$

Thus, we may write finally:

$$\begin{aligned}\frac{\partial f}{\partial z}(\overline{\mathbb{D}}(0, R)) &= \frac{\partial f}{\partial z}\left(\bigcup_{r\in[0,aR]} \partial\mathbb{D}(0, \frac{1}{a}r)\right) = \bigcup_{r\in[0,aR]} \frac{\partial f}{\partial z}(\partial\mathbb{D}(0, \frac{1}{a}r)) \\ &= \bigcup_{r\in[0,aR]} \partial\mathbb{D}(c_r, R_r) = \bigcup_{r\in[0,aR]} \overline{\mathbb{D}}(c_r, R_r) = \lim_{r\to aR} \overline{\mathbb{D}}(c_r, R_r) \\ &= \overline{\mathbb{D}}(c_{aR}, R_{aR}) = \overline{\mathbb{D}}(z_0, R^*).\end{aligned}$$

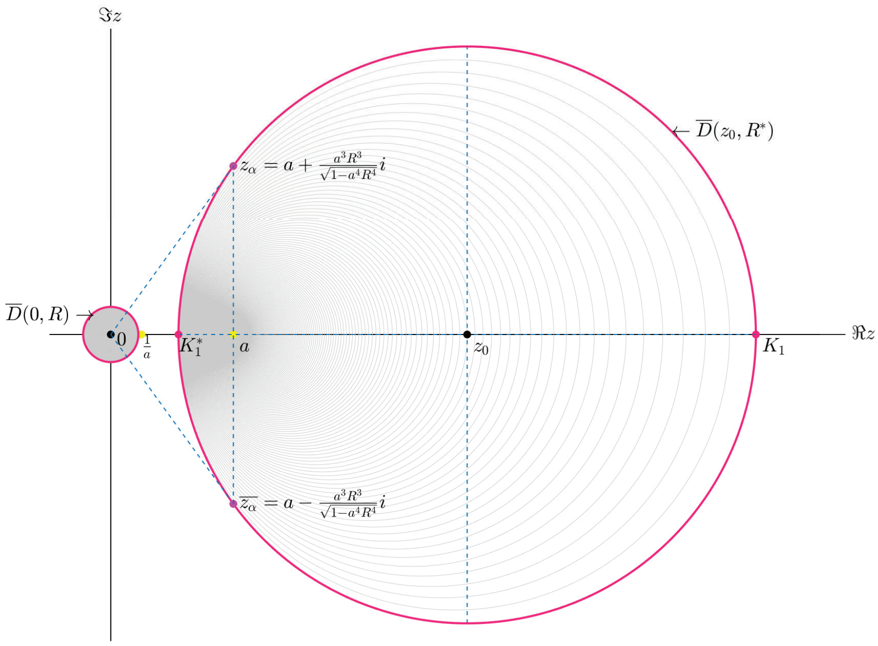


FIGURE 6.1: Image of the closed disk $\overline{\mathbb{D}}(0,R)$ under $\frac{\partial f}{\partial z}$

(ii) The second partial derivative of f with respect to the spatial coordinate is

$$\frac{\partial^2 f}{\partial z^2}(\theta, z) = -\frac{2a^3 z}{(1+(az)^2)^2}, \ \forall z \in \mathcal{U} = \mathbb{C} \setminus \{-\frac{1}{a}i, \frac{1}{a}i\}.$$

The image of the closed disk

$$\overline{\mathbb{D}}(0,R) = \{z = \frac{r}{a}e^{ti} : (r,t) \in [0, aR] \times (-\pi, \pi]\}$$

can be written as

$$\frac{\partial f}{\partial z}\left(\overline{\mathbb{D}}(0,R)\right) = \left\{-\frac{2a^3 z}{(1+(az)^2)^2} : z = \frac{1}{a}re^{ti} : (r,t) \in [0, aR] \times (-\pi, \pi]\right\}.$$

Let $z = x + yi = \frac{1}{a}re^{ti} \in \overline{\mathbb{D}}(0,R)$, i.e. $(r,t) \in [0, aR] \times (-\pi, \pi]$. Then,

$$\begin{aligned}
\frac{\partial^2 f}{\partial z^2}(\theta, z) &= \frac{\partial^2 f}{\partial z^2}(\theta, \frac{1}{a}re^{ti}) = -\frac{2a^3 \frac{1}{a}re^{ti}}{(1+(re^{ti})^2)^2} = -\frac{2a^2 re^{ti}}{(1+r^2e^{2ti})^2}\frac{(1+r^2e^{-2ti})^2}{(1+r^2e^{-2ti})^2} \\
&= -2a^2 r\frac{e^{ti}(1+r^2e^{-2ti})^2}{|1+r^2e^{2ti}|^4} = \frac{-2a^2 r}{|1+r^2e^{2ti}|^4}e^{ti}(1+2r^2e^{-2ti}+r^4e^{-4ti}) \\
&= \frac{-2a^3 r}{|1+r^2e^{2ti}|^4}(e^{ti}+2r^2e^{-ti}+r^4e^{-3ti}).
\end{aligned}$$

By means of Euler's formula, complex exponentials can be written as follows:

$$\begin{aligned}
e^{ti}+2r^2e^{-ti}+r^4e^{-3ti} &= \cos t + i\sin t + 2r^2(\cos(-t)+i\sin(-t)) + r^4(\cos(-3t)+i\sin(-3t)) \\
&= \cos t + 2r^2\cos t + r^4\cos(3t) + i(\sin t - 2r^2\sin t - r^4\sin(3t)) \\
&= (1+2r^2)\cos t + r^4\cos(3t) + i((1-2r^2)\sin t - r^4\sin(3t)),
\end{aligned}$$

and

$$\begin{aligned} |1+r^2e^{2ti}|^2 &= (1+r^2\cos(2t))^2 + r^4\sin^2(2t) = 1+2r^2\cos(2t)+r^4\cos^2(2t)+r^4\sin^2(2t) \\ &= 1+2r^2\cos(2t)+r^4 \ . \end{aligned}$$

It follows that

$$\frac{\partial^2 f}{\partial z^2}(\theta,z) = \frac{\partial^2 f}{\partial z^2}(\theta,\frac{1}{a}re^{ti}) = u(r,t)+iv(r,t) \ ,$$

with

$$\left\{ \begin{aligned} u(r,t) &= -\tfrac{2a^2r}{(1+2r^2\cos(2t)+r^4)^2}((1+2r^2)\cos t + r^4\cos(3t)) \\ v(r,t) &= -\tfrac{2a^2r}{(1+2r^2\cos(2t)+r^4)^2}((1-2r^2)\sin t - r^4\sin(3t)) \end{aligned} \right. \qquad (r,t)\in[0,aR]\times(-\pi,\pi] \ . \tag{6.10}$$

For each $r\in[0,aR]$ these equations represent a closed curve. All the curves of the family together constitute the image of the closed disk $\overline{\mathbb{D}}(0,R)$, and they are all enclosed inside the curve corresponding to $r=aR$. (See FIGURE 6.2).

Consequently, they are also enclosed inside the close disk $\overline{\mathbb{D}}(0,K_2)$, where K_2 is the maximum distance to the origin. In other words,

$$\begin{aligned} \left|\frac{\partial^2 f}{\partial z^2}(\theta,z)\right| &= \frac{2a^3|z|}{|1+a^2z^2|^2}, \ (\theta,z)\in\mathbb{T}_\varrho\times\mathcal{U} \\ \left|\frac{\partial^2 f}{\partial z^2}(\theta,\frac{1}{a}re^{ti})\right| &= \frac{2a^2r}{1+2r^2\cos(2t)+r^4}, \ (r,t)\in[0,aR]\times(-\pi,\pi] \end{aligned}$$

Moreover,

$$\frac{2a^2r}{1+2r^2\cos(2t)+r^4} \le \frac{2a^2r}{1-2r^2+r^4} = \frac{2a^2r}{(1-r^2)^2} \le \frac{2a^3R}{(1-a^2R^2)^2} =: K_2 \ .$$

Thus,

$$\left|\frac{\partial^2 f}{\partial z^2}(\theta,z)\right| \le \frac{2a^3R}{(1-a^2R^2)^2} =: K_2, \ \forall(\theta,z)\in\mathbb{T}_\varrho\times\overline{\mathbb{D}}(0,R). \tag{6.11}$$

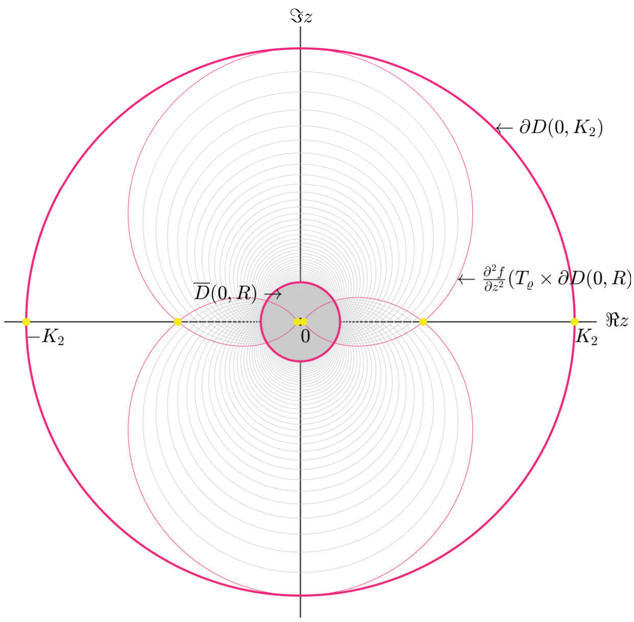


FIGURE 6.2: Image of the closed disk $\overline{\mathbb{D}}(0,R)$ under $\frac{\partial^2 f}{\partial z^2}$

Remark 6.4

Curves given by (6.10) are symmetric with respect to the axes, and there are some points lying on the real axis:

$$\operatorname{Im}\left(\frac{\partial^2 f}{\partial z^2}(\theta, \frac{1}{a}re^{ti})\right) = 0 \Longleftrightarrow \begin{cases} r = 0 & , \quad u(0,t) = 0 \\ t = 0 & , \quad u(r,t) = -\frac{a^2 r}{(1+r^2)^2} \\ t = \pi & , \quad u(r,\pi) = \frac{a^2 r}{(1+r^2)^2} \\ t = \arccos\left(-\frac{1-r^2}{2r^2}\right) & , \quad u(r,t) = \frac{2a^2 r^3}{(1-r^2)(1-r^4)} \\ t = \arccos\left(\frac{1-r^2}{2r^2}\right) & , \quad u(r,t) = -\frac{2a^2 r^3}{(1-r^2)(1-r^4)} \end{cases}.$$

(iii) The image of the closed disk $\overline{\mathbb{D}}(0,R)$ under $\frac{\partial f}{\partial z}$ is the closed disk $\overline{\mathbb{D}}(z_0,R^*)$, which is centered at $z_0 = \frac{a}{1-a^4R^4}$ placed on the real line and whose radius $R^* = \frac{a^3R^2}{a-a^4R^4}$ is small enough so that the disc does not reach to cut the imaginary axis (See Figure 6.1). Therefore, the maximum principal argument of the disk is the angle $\alpha \in (0, \frac{\pi}{2})$ between the positive real semiaxis and the tangent line from the origin to the circumference $\partial\mathbb{D}(z_0,R^*)$. By means of geometric arguments it can be shown that $\alpha = \arctan \frac{R^*}{\sqrt{z_0^2 - R^{*2}}}$.

Let $z_\alpha = x_\alpha + iy_\alpha$ be the tangent point. z_α may be written as

$$z_\alpha = z_0 + R^* e^{(\frac{\pi}{2}+\alpha)i} = z_0 - R^* \sin\alpha + R^* i \cos\alpha.$$

Thus,

$$\begin{cases} x_\alpha = z_0 - R^* \sin\alpha \ , \\ y_\alpha = R^* \cos\alpha \ . \end{cases}$$

Since triangles $0z_\alpha x_\alpha$ and $0z_\alpha z_0$ are right–angled triangles we have:

$$\tan\alpha = \frac{y_\alpha}{x_\alpha} = \frac{R^*}{\sqrt{z_0^2 - R^{*2}}} = \frac{a^3R^3/(1-a^4R^4)}{a/(1-a^4R^4)} = \frac{a^2R^2}{\sqrt{1-a^4R^4}}.$$

Moreover,

$$\begin{aligned} \cos\alpha &= \frac{1}{\sqrt{1+\tan^2\alpha}} = \sqrt{1-a^4R^4} \\ \sin\alpha &= \tan\alpha\cos\alpha = a^2R^2 \ . \end{aligned}$$

It follows that

$$\begin{cases} x_\alpha = z_0 - R^* \sin\alpha = a \\ y_\alpha = R^* \cos\alpha = \frac{a^3R^2}{\sqrt{1-a^4R^4}} \ , \end{cases}$$

$\alpha = \arctan \frac{a^2R^2}{\sqrt{1-a^4R^4}}$, and $\left|\operatorname{Arg}\left(\frac{\partial f}{\partial z}(\theta,z)\right)\right| \leq \alpha,\ \forall(\theta,z) \in \mathbb{T}_\varrho \times \overline{\mathbb{D}}(0,R)$.

□

There are some important symmetry additional properties in this example. Some of the numerical results obtained can be better understood if these properties are taken into account, to which we allude in the following statement.

Proposition 6.5 Symmetry properties

Let $\psi = R_\omega \times f$ be the skew–product (6.1). Then, the following properties hold:

(i) *ψ is invariant under conjugation by the symmetry*

$$\begin{array}{rrcl} S: & \mathbb{T}_\varrho \times \mathcal{U} & \longrightarrow & \mathbb{T}_\varrho \times \mathcal{U} \\ & (\theta, z) & \longmapsto & S(\theta, z) := (\theta + \frac{1}{2}, -z)\,, \end{array} \tag{6.12}$$

namely, $S \circ \psi = \psi \circ S$.

(ii) *If $\kappa : \mathbb{T} \longrightarrow \mathbb{R}$ is an invariant curve for ψ, i.e.*

$$f(\theta, \kappa(\theta)) = \kappa(\theta + \omega),\ \forall \theta \in \mathbb{T},$$

then the symmetric curve

$$\begin{array}{rrcl} \gamma: & \mathbb{T} & \longrightarrow & \mathbb{R} \\ & \theta & \longmapsto & \gamma(\theta) := -\kappa(\theta + \frac{1}{2})\,, \end{array}$$

is also an invariant curve for ψ, i.e.

$$f(\theta, \gamma(\theta)) = \gamma(\theta + \omega),\ \forall \theta \in \mathbb{T},$$

(iii) *If $\kappa : \mathbb{T} \longrightarrow \mathbb{R}$ is a curve (not necessary invariant for ψ) whose Lyapunov exponent is*

$$\Lambda(\kappa) = \int_{\mathbb{T}} \log\left(\frac{\partial f}{\partial z}(\theta, \kappa(\theta))\right) d\theta\ ,$$

then the symmetric curve $\gamma(\theta)) = -\kappa(\theta + \frac{1}{2})$, $\forall \theta \in \mathbb{T}$, has the same Lyapunov exponent, i.e.

$$\Lambda(\gamma) = \int_{\mathbb{T}} \log\left(\frac{\partial f}{\partial z}(\theta, \gamma(\theta))\right) d\theta\ = \Lambda(\kappa).$$

(iv) *If $\kappa : \mathbb{T} \longrightarrow \mathbb{R}$ is a curve and $\gamma(\theta)) = -\kappa(\theta + \frac{1}{2})$, $\forall \theta \in \mathbb{T}$, is the corresponding symmetric, then their Fourier coefficients are related by*

$$\widehat{\gamma}_k = \int_{\mathbb{T}} \gamma(\theta) e^{-2\pi k\theta i} d\theta = (-1)^{k+1} \int_{\mathbb{T}} \kappa(\theta) e^{-2\pi k\theta i} d\theta = (-1)^{k+1} \widehat{\kappa}_k,\ \forall k \in \mathbb{Z}.$$

If κ is self–symmetric, all its even Fourier coefficients vanish. In particular, $<\kappa>= \widehat{k}_0 = 0$.

(v) *Let $\kappa : \mathbb{T} \longrightarrow \mathbb{R}$ be a curve (not necessary invariant for ψ) and define the matrix*

$$\Omega(\kappa) = \begin{pmatrix} 1 - \lambda(\kappa) & -\eta_0(\kappa) \\ c_0(\kappa) & <\widetilde{c}(\kappa) \mathfrak{R}_{\lambda(\kappa)} \widetilde{\eta}(\kappa)> \end{pmatrix},$$

where $\lambda(\kappa)$ is the Lyapunov multiplier of the curve, $c(\kappa)$ is the corresponding Floquet transformation, and $\eta(\kappa)(\theta) = \frac{1}{c(\kappa)(\theta+\omega)}$ $(\theta \in \mathbb{T})$, then the symmetric curve $\gamma(\theta) = -\kappa(\theta + \frac{1}{2})$, $\forall \theta \in \mathbb{T}$, has the same determinant, i.e,

$$\det \Omega(\gamma) = \det \Omega(\kappa).$$

In fact $\Omega(\gamma) = \Omega(\kappa)$.

Proof.

(i) Let $(\theta, z) \in T_\varrho \times \mathcal{U}$. Then,
$(S \circ \psi)(\theta, z) = S(\psi(\theta, z)) = S(\theta + \omega, f(\theta, z)) = (\theta + \frac{1}{2} + \omega, -f(\theta, z))$.
$(\psi \circ S)(\theta, z) = \psi(S(\theta, z)) = \psi(\theta + \frac{1}{2}, -z) = (\theta + \frac{1}{2} + \omega, f(\theta + \frac{1}{2}, -z)) = (\theta + \frac{1}{2} + \omega, (f \circ S)(\theta, z))$.
Since $f(\theta, z) = \arctan(az) + b\sin(2\pi\theta)$, then $(f \circ S)(\theta, z) = f(\theta + \frac{1}{2}, -z) = \arctan(-az) + b\sin(2\pi(\theta + \frac{1}{2})) = -\arctan(az) + b\sin(2\pi\theta + \pi) = -\arctan(az) - b\sin(2\pi\theta) = -f(\theta, z)$.
Thus, $f \circ S = -f$ and therefore $S \circ \psi = \psi \circ S$.

Remark 6.6

Notice that, in general

$$S \circ \psi = \psi \circ S \Longleftrightarrow f \circ S = -f.$$

Furthermore, differentiating $f \circ S = -f$ we obtain $\frac{\partial f}{\partial z} \circ S = \frac{\partial f}{\partial z}$. Moreover, inductively we may also write,

$$\frac{\partial^k f}{\partial z^k} \circ S = (-1)^{k+1} \frac{\partial^k f}{\partial z^k},\ \forall k = 0, 1, 2, \dots\ . \tag{6.13}$$

(ii) Assume that κ is invariant. Then:
$\gamma(\theta + \omega) = -\kappa(\theta + \omega + \frac{1}{2}) = -f(\theta + \frac{1}{2}, \kappa(\theta + \frac{1}{2})) = -f(\theta + \frac{1}{2}, -\gamma(\theta)) = -(f \circ S)(\theta, \gamma(\theta)) = f(\theta, \gamma(\theta))$. Therefore, γ is invariant.

(iii) Let $m_\kappa(\theta) = \frac{\partial f}{\partial z}(\theta, \kappa(\theta))$, $\theta \in \mathbb{T}_\varrho$ and $m_\gamma(\theta) = \frac{\partial f}{\partial z}(\theta, \gamma(\theta))$, $\theta \in \mathbb{T}_\varrho$. Then, applying de definition of γ and taking into account (6.13), we have:
$m_\gamma(\theta) = \frac{\partial f}{\partial z}(\theta, \gamma(\theta)) = \left(\frac{\partial f}{\partial z} \circ S\right)(\theta, \gamma(\theta)) = \frac{\partial f}{\partial z}(\theta + \frac{1}{2}, -\gamma(\theta)) = \frac{\partial f}{\partial z}(\theta + \frac{1}{2}, \kappa(\theta + \frac{1}{2})) = m_\kappa(\theta + \frac{1}{2})$, $\forall \theta \in \mathbb{T}_\varrho$. It follows that $\Lambda(\kappa) = \int_{\mathbb{T}} \log m_\gamma(\theta) d\theta = \int_{\mathbb{T}} \log m_\kappa(\theta + \frac{1}{2}) d\theta$.
Taking into account the 1–periodicity of the integrand and applying the change of variable $\zeta = \theta + \frac{1}{2}$, we get $\Lambda(\gamma) = \int_{\mathbb{T}} \log m_\kappa(\zeta) d\zeta = \Lambda(\kappa)$.

(iv) Since $e^{\pi k i} = (-1)^k$, $\forall k \in \mathbb{Z}$ and applying the change of variable $\zeta = \theta + \frac{1}{2}$, the Fourier coefficients of γ can be written as

$$\begin{aligned} \widehat{\gamma}_k &= \int_{\mathbb{T}} \gamma(\theta) e^{-2\pi k\theta i} d\theta = \int_{\mathbb{T}} -\kappa(\theta + \frac{1}{2}) e^{-2\pi k\theta i} d\theta = -e^{\pi k i} \int_{\mathbb{T}} \kappa(\theta + \frac{1}{2}) e^{-2\pi k(\theta + \frac{1}{2})i} d\theta \\ &= (-1)^{k+1} \int_{\mathbb{T}} \kappa(\zeta) e^{-2\pi k\zeta i} d\zeta = (-1)^{k+1} \widehat{\kappa}_k,\ \forall k \in \mathbb{Z}\ . \end{aligned}$$

If κ is self–symmetric, $\gamma = \kappa$ and then $\widehat{\kappa}_k = (-1)^{k+1} \widehat{\kappa}_k$. Therefore, if κ is even, $\widehat{\kappa}_k = 0$. In particular, $\widehat{\kappa}_0 =< \kappa >= 0$.

(v) This property can be proved following the same scheme as in the above properties.

□

Appendix I. Fiber bundles, bundle maps and invariant curves in skew–products

This appendix lays the foundations in a more general context of the background in which the entire thesis is focused. It is devoted to introduce the notion of skew–product and some general definitions related with this smattering. In particular, the concept of invariant section. For a more complete account on the topic, we refer the reader to [55], [32], [47], [1]. Let us start with some definitions and basic properties.

I.1 Fiber bundles

Definition I.1 Fiber bundle

Formally, a fiber bundle is a structure (E, B, p, F) *where*

(i) E *is a topological space called* bundle space *or* total space*;*

(ii) B *is a topological space, usually connected, called* base space*;*

(iii) $p : E \longrightarrow B$ *is a continuous surjective map of* E *onto* B *called the* bundle projection*; and*

(iv) F *is a topological space called the* fiber*,*

satisfying the following local triviality condition*:*

$$\forall\theta \in B, \exists U_\theta \in \tau(B), \text{ namely an open set, and exists a fiber preserving homeomorphism}$$

$$\varphi_{U_\theta} : p^{-1}(U_\theta) \longrightarrow U_\theta \times F$$

such that the diagram

$$\begin{array}{ccc} p^{-1}(U_\theta) \subseteq E & \xrightarrow{\varphi_{U_\theta}} & U_\theta \times F \subseteq B \times F \\ & \searrow^{p} & \downarrow \pi \\ & & U_\theta \subseteq B \end{array}$$

commutes, that is,

$$p = \pi \circ \varphi_{U_\theta}\ , \text{ over } p^{-1}(U_\theta)$$

where π *is the canonical projection of the product space* $U_\theta \times F$*, endowed with the product topology, onto the so–called* trivializing neighborhood U_θ*.*
We denote $F_\theta = p^{-1}(\theta)$, $\forall\theta \in B$ *which is called the* fiber over the point θ *of* B *and it is required that each* F_θ *be homeomorphic to* F*.*
We say that a fiber bundle (E, B, p, F) *is differentiable, respectively holomorphic, if* E*,* B*, and* F *are differentiable manifolds, respectively complex manifolds, and the trivializing maps are diffeomorphisms, respectively biholomorphisms.*

◇

Remark I.2

The product $B \times F$ *defines a trivial fiber bundle,* $(B \times F, B, \pi, F)$*, where* π *is the canonical projection over the base.*

Remark I.3

In the above definition we say that the homeomorphism φ_{U_θ} *is* fiber preserving *because it transforms fibers over* (E, B, p, F) *into the corresponding fibers over* $(B \times F, B, \pi, F)$*, that is* $\varphi_{U_\theta}(F_\theta) = \pi^{-1}(\theta)$*, where* $F_\theta = p^{-1}(\theta)$ *is the fiber of the point* $\theta \in B$ *with respect to* (E, B, p, F) *and* $\pi^{-1}(\theta) = \{\theta\} \times F$ *is the fiber of the same point with respect to the trivial fiber bundle* $(B \times F, B, \pi, F)$*.*

Proof.

($\subseteq$) Let $(\bar{\theta}, \bar{y}) \in \varphi_{U_\theta}(F_\theta) \subseteq U_\theta \times F$. Then $\exists \xi \in F_\theta$ such that $\varphi_{U_\theta}(\xi) = (\bar{\theta}, \bar{y})$. Since $\xi \in F_\theta$ then $p(\xi) = \theta$ and we have $\bar{\theta} = \pi(\bar{\theta}, \bar{y}) = \pi(\varphi_{U_\theta}(\xi)) = (\pi \circ \varphi_{U_\theta})(\xi) = p(\xi) = \theta$. Thus, $(\bar{\theta}, \bar{y}) \in \pi^{-1}(\theta)$.

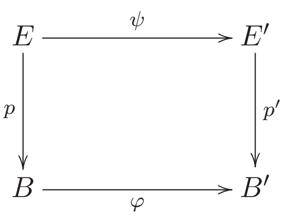
($\supseteq$) Let $(\bar{\theta}, \bar{y}) \in \pi^{-1}(\theta) \subseteq U_\theta \times F$. Then $\pi(\bar{\theta}, \bar{y}) = \theta \Rightarrow \bar{\theta} = \theta$. Since φ_{U_θ} is a homeomorphism there is a unique $\xi \in p^{-1}(U_\theta)$ such that $\varphi_{U_\theta}(\xi) = (\bar{\theta}, \bar{y})$. Let $\widetilde{\theta} = p(\xi) \in U_\theta$. Since $p = \pi \circ \varphi_{U_\theta}$ we have: $\widetilde{\theta} = p(\xi) = (\pi \circ \varphi_{U_\theta})(\xi) = \pi(\varphi_{U_\theta}(\xi)) = \pi(\bar{\theta}, \bar{y}) = \bar{\theta} = \theta$. Thence $\varphi_{U_\theta}(\xi) = (\bar{\theta}, \bar{y})$ with $p(\xi) = \theta$, i.e. $\xi \in p^{-1}(\theta) = F_\theta$. And this means that $(\bar{\theta}, \bar{y}) \in \varphi_{U_\theta}(F_\theta)$.

□

I.2 Bundle maps

Definition I.4 Bundle maps and isomorphisms of fiber bundles

A bundle map is a morphism in the category of fiber bundles.
More precisely, if (E, B, p, F) *and* (E', B', p', F') *are fiber bundles, then a bundle map between them is a couple of continuous functions* (φ, ψ) *with* $\varphi : B \longrightarrow B'$ *and* $\psi : E \longrightarrow E'$ *such that the following diagram commutes,*



i.e. $\varphi \circ p = p' \circ \psi$*.*
We define the composition of bundle maps as follows: if (φ, ψ) *is a bundle map between the fiber bundles* (E, B, p, F) *and* (E', B', p', F') *and* $(\widetilde{\varphi}, \widetilde{\psi})$ *is a bundle map between the fiber bundles* (E', B', p', F') *and* (E'', B'', p'', F'') *then the composition is the bundle map*

$$(\widetilde{\varphi}, \widetilde{\psi}) \circ (\varphi, \psi) = (\widetilde{\varphi} \circ \varphi, \widetilde{\psi} \circ \psi) ,$$

which is a bundle map between the the fiber bundles (E, B, p, F) *and* (E'', B'', p'', F'') *as it is shown* *in the following diagram.*

$$\begin{array}{ccccc} E & \xrightarrow{\psi} & E' & \xrightarrow{\widetilde{\psi}} & E'' \\ \Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle p'} & & \Big\downarrow{\scriptstyle p''} \\ B & \xrightarrow[\varphi]{} & B' & \xrightarrow[\widetilde{\varphi}]{} & B'' \end{array}$$

Since $\varphi \circ p = p' \circ \psi$ *and* $\widetilde{\varphi} \circ p' = p'' \circ \widetilde{\psi}$ *then* $(\widetilde{\varphi} \circ \varphi) \circ p = \widetilde{\varphi} \circ (\varphi \circ p) = \widetilde{\varphi} \circ (p' \circ \psi) = (\widetilde{\varphi} \circ p') \circ \psi = (p'' \circ \widetilde{\psi}) \circ \psi = p'' \circ (\widetilde{\psi} \circ \psi)$. *And this proves the assertion.*
On the other hand, a bundle map (φ, ψ) *between the fiber bundles* (E, B, p, F) *and* (E', B', p', F') *is an isomorphism of fiber bundles if there is another bundle map* $(\widetilde{\varphi}, \widetilde{\psi})$ *between* (E', B', p', F') *and* (E, B, p, F) *such that*

$$(\widetilde{\varphi}, \widetilde{\psi}) \circ (\varphi, \psi) = (\mathrm{I}_B, \mathrm{I}_E) \ ,$$

i.e. $\widetilde{\varphi} \circ \varphi = \mathrm{I}_B$ *and* $\widetilde{\psi} \circ \psi = \mathrm{I}_E$.

◇

I.3 Cross sections over fiber bundles

Definition I.5 Cross section

A cross section of a fiber bundle (E, B, p, F) *is a continuous map* $\sigma : B \longrightarrow E$ *such that* $p \circ \sigma = \mathrm{I}_B$.

◇

Proposition I.6 (Cross sections)

Every cross section of a trivial fiber bundle $(B \times F, B, \pi, F)$ *has the form:*

$$\begin{array}{rccl} \sigma : & B & \longrightarrow & E = B \times F \\ & \theta & \longmapsto & \sigma(\theta) = (\theta, \kappa(\theta)) \end{array} \quad ,$$

where $\kappa : B \longrightarrow F$ *is a continuous map uniquely determined by* σ.

Proof. Every continuous map $\sigma : B \longrightarrow E = B \times F$ has the form $\sigma(\theta) = (\iota(\theta), \kappa(\theta))$, $\theta \in B$, where $\iota : B \longrightarrow B$ and $\kappa : B \longrightarrow F$ are continuous maps uniquely determined by σ. Hence, σ is a cross section of $(E = B \times F, B, \pi, F)$ if, and only if, $\pi \circ \sigma = \mathrm{I}_B$, that is, $\forall \theta \in B, (\pi \circ \sigma)(\theta) = \pi(\iota(\theta), \kappa(\theta)) = \iota(\theta) = \theta$. Thus, $\sigma(\theta) = (\theta, \kappa(\theta))$, $\forall \theta \in B$.
Conversely, if $\kappa : B \longrightarrow F$ is a continuous map and

$$\begin{array}{rccl} \sigma : & B & \longrightarrow & E = B \times F \\ & \theta & \longmapsto & \sigma(\theta) = (\theta, \kappa(\theta)) \end{array} \quad ,$$

then σ is continuous. Furthermore, $\forall \theta \in B$, $(\pi \circ \sigma)(\theta) = \pi(\theta, \kappa(\theta)) = \theta = \mathrm{I}_B(\theta)$. Therefore, $\pi \circ \sigma = \mathrm{I}_B$ and σ is a cross section of the trivial fiber bundle $(B \times F, B, \pi, F)$. □

Remark I.7

It is usual for short to denote the fiber bundle by means of the bundle space, E*, when there is no ambiguity. Then, the set of cross sections is denoted by*

$$\Gamma(E) = \{\sigma : B \longrightarrow E \mid \sigma \text{ cross section over } (E, B, p, F)\} \ .$$

The latter proposition tells that when it comes to a trivial fiber bundle there is a bijection between the set of cross sections $\Gamma(B \times F)$ *and the set of continuous functions* $\mathcal{C}^0(B, F)$.

I.4 Vector bundles, vector bundle maps and cross sections over vector bundles

Definition I.8 Vector bundle

Let (E, B, p, F) be a fiber bundle. We say that it is a vector bundle *if:*

(i) F is a topological vector space over a field $\mathbb{K}$ (usually $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$);

(ii) For each $\theta \in B$, the fiber $F_\theta = p^{-1}(\theta) \subseteq E$ is a $\mathbb{K}-$vector space isomorphic to F;

(iii) The trivializing homeomorphisms restricted to the corresponding fibers are vector spaces isomorphisms, that is

$$\varphi_{U_\theta}|_{F_\theta} : \quad F_\theta \quad \longrightarrow \quad \{\theta\} \times F$$

is $\mathbb{K}-$linear and bijective for every fixed $\theta \in B$.

The dimension of F is called the rank *of the vector bundle.*

◇

A vector bundle may be viewed as a parameterized family of vector spaces.

Remark I.9

The prototypical example of vector bundle is the tangent bundle of a smooth manifold. More precisely, if $\mathbb{M}$ is a smooth $n-$dimensional manifold, then $(T\mathbb{M}, \mathbb{M}, \pi, \mathbb{R}^n)$ is a vector bundle of rank n, where $\pi : T\mathbb{M} \longrightarrow \mathbb{M}$ is the canonical projection of the tangent bundle onto the manifold. In this case, the cross sections are the vector fields over the manifold.

Remark I.10

A trivial fiber bundle $(B \times F, B, \pi, F)$ where F is a $\mathbb{K}-$vector space is a trivial vector bundle. In this case, all the fibers are isomorphic to F.
Furthermore, if F is a normed space and the base B is compact, then there is an induced normed structure over the set of cross sections of the trivial vector bundle $E = B \times F$ by means of the norm:

$$\begin{array}{rccl} \| \, \|_{\Gamma(E)} : & \Gamma(E) & \longrightarrow & \mathbb{R} \\ & \sigma = (\mathrm{I}_B, \kappa) & \longmapsto & \|\sigma\|_{\Gamma(E)} = \sup_{\theta \in B} \|\kappa(\theta)\|_F \end{array} \quad . \tag{I.1}$$

Proposition I.11 Module structure

Let (E, B, p, F) be an $n-$dimensional $\mathbb{K}-$vector bundle. Then the set of cross sections $\Gamma(E)$ is a module over the ring $\mathcal{C}^0(B, \mathbb{K})$ of continuous $\mathbb{K}-$valued functions on B.

Proof. In general, the existence of cross sections in fiber bundles can not be guaranteed. But in the particular case of a vector bundle, there are always cross sections. Indeed, the so–called *zero section* is defined as follows: Given $\theta \in B$, the fiber $F_\theta = p^{-1}(\theta)$ is a $\mathbb{K}-$vector space. Therefore the map

$$\begin{array}{rccl} e : & B & \longrightarrow & E \\ & \theta & \longmapsto & e(\theta) = 0_{F_\theta} \end{array}$$

is well defined. Furthermore, $(p \circ e)(\theta) = p(e(\theta)) = p(0_{F_\theta}) = \theta$. Thus, $p \circ e = \mathrm{I}_B$. Additionally, the zero map is continuous as we will see later. Consequently $e \in \Gamma(E)$ is a cross section.

Now we define the module operations.
Notice first that for every $\sigma \in \Gamma(E)$ and every $\theta \in B$, $\sigma(\theta) \in F_\theta = p^{-1}(\theta)$ which is a $\mathbb{K}-$vector space and then we can define the sum of cross sections and the product of continuous functions times cross sections in the following natural way.

$$\begin{array}{rccc} + : & \Gamma(E) \times \Gamma(E) & \longrightarrow & \Gamma(E) \\ & (\sigma, \widetilde{\sigma}) & \longmapsto & \sigma + \widetilde{\sigma} \end{array}$$

with $(\sigma + \widetilde{\sigma})(\theta) = \sigma(\theta) + \widetilde{\sigma}(\theta), \ \forall \theta \in B$;

$$\begin{array}{rccc} \cdot : & \mathcal{C}^0(B, \mathbb{K}) \times \Gamma(E) & \longrightarrow & \Gamma(E) \\ & (\lambda, \sigma) & \longmapsto & \lambda\sigma \end{array}$$

with $(\lambda\sigma)(\theta) = \lambda(\theta)\sigma(\theta), \ \forall \theta \in B$.

□

Definition I.12 Vector bundle map

A vector bundle map between two vector bundles (E, B, p, F) and (E', B', p', F') (over the same field $\mathbb{K}$) is a bundle map (φ, ψ) with the additional requirement that, when restricted to each fiber, ψ is $\mathbb{K}-$linear, that is, for every $\theta \in B$:

$$\psi|_{F_\theta \to F'_{\varphi(\theta)}} : \quad F_\theta = p^{-1}(\theta) \subseteq E \quad \longrightarrow \quad F'_{\varphi(\theta)} = (p')^{-1}(\varphi(\theta)) \subseteq E'$$

is well defined and it is a homomorphism of $\mathbb{K}-$vector spaces.

◇

I.5 Skew–product dynamical systems and invariant sections

Definition I.13 Skew–product dynamical system

Let $(B \times F, B, \pi, F)$ be a trivial fiber bundle with $E = B \times F$ the bundle space and let $\varphi : B \longrightarrow B$ be a homeomorphism. A skew–product dynamical system in F over φ is a bundle map (φ, ψ) of the fiber bundle (E, B, π, F) onto itself,

$$\begin{array}{ccc} E = B \times F & \xrightarrow{\ \psi\ } & E = B \times F \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ B & \xrightarrow[\ \varphi\ ]{} & B \end{array}$$

that is, $\varphi \circ \pi = \pi \circ \psi$.

◇

Remark I.14

From this definition we can identify the skew–product with a discrete dynamical system given by the map ψ which is of the form:

$$\begin{array}{rccc} \psi : & E = B \times F & \longrightarrow & E = B \times F \\ & (\theta, x) & \longmapsto & \psi(\theta, x) = (\varphi(\theta), f(\theta, x)) \end{array} ,$$

where

$$\begin{array}{rccl} f: & E = B \times F & \longrightarrow & F \\ & (\theta, x) & \longmapsto & f(\theta, x) = \widetilde{\pi}(\psi(\theta, x)) \end{array}$$

is continuous[1] *and* $\widetilde{\pi} : B \times F \longrightarrow F$ *the natural projection onto the second component.*
Indeed, $\pi(\psi(\theta, x)) = (\pi \circ \psi)(\theta, x)) = (\varphi \circ \pi)(\theta, x) = \varphi(\pi(\theta, x)) = \varphi(\theta),\ \forall(\theta, x) \in B \times F$.
Due to this fact, a skew–product (φ, ψ) *can be referred simply by the map* ψ *which is written as* $\psi = \varphi \times f$*, with* φ *a homeomorphism of the base* B *and* $f : E = B \times F \longrightarrow F$ *continuous.*

Definition I.15 Invariant section

Let (φ, ψ) *be a skew–product dynamical system over* $(E = B \times F, B, \pi, F)$ *and* $\sigma \in \Gamma(E)$ *a cross section. We say that* σ *is an invariant section if* $\sigma \circ \varphi = \psi \circ \sigma$*.*

◇

Proposition I.16 Invariant sections

Let $(B \times F, B, \pi, F)$ *be a trivial fiber bundle with* $E = B \times F$ *the bundle space,* (φ, ψ) *a skew–product, and* $\sigma \in \Gamma(E)$ *a cross section. Then,*

$$\sigma \circ \varphi = \psi \circ \sigma \Longleftrightarrow f(\theta, \kappa(\theta)) = \kappa(\varphi(\theta)),\ \forall \theta \in B\ , \tag{I.2}$$

where $f : E \longrightarrow F$ *and* $\kappa : B \longrightarrow F$ *are continuous maps such that* $\sigma = \mathrm{I}_B \times \kappa$ *and* $\psi = \varphi \times f$*.*

Proof. Lets consider the following commutative diagram:

$$\begin{array}{ccc} E = B \times F & \xrightarrow{\quad\psi\quad} & E = B \times F \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ B & \xrightarrow[\quad\varphi\quad]{} & B \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma} \\ E = B \times F & \xrightarrow{\quad\psi\quad} & E = B \times F \end{array}$$

By **Proposition I.6**, every cross section $\sigma \in \Gamma(E)$ of a trivial fiber bundle $(B \times F, B, \pi, F)$ has the form:

$$\begin{array}{rccl} \sigma : & B & \longrightarrow & E = B \times F \\ & \theta & \longmapsto & \sigma(\theta) = (\theta, \kappa(\theta)) \end{array} \quad ,$$

where $\kappa : B \longrightarrow F$ is a continuous map uniquely determined by σ. Hence, for every $\theta \in B$ we have on one side,

$$(\sigma \circ \varphi)(\theta) = \sigma(\varphi(\theta)) = (\varphi(\theta), \kappa(\varphi(\theta)))$$

and, on the other side, taking in account that (φ, ψ) is a skew–product, that is

$$\begin{array}{rccl} \psi : & E = B \times F & \longrightarrow & E = B \times F \\ & (\theta, x) & \longmapsto & \psi(\theta, x) = (\varphi(\theta), f(\theta, x)) \end{array} \quad ,$$

[1] We will assume, from the moment being, that f is $\mathcal{C}^1$ with respect to the second variable.

with $f : E \longrightarrow F$ continuous, we have:

$$(\psi \circ \sigma)(\theta) = \psi(\sigma(\theta)) = \psi(\theta, \kappa(\theta)) = (\varphi(\theta), f(\theta, \kappa(\theta))).$$

It follows that:

$$\sigma \circ \varphi = \psi \circ \sigma \Longleftrightarrow f(\theta, \kappa(\theta))) = \kappa(\varphi(\theta)),\ \forall \theta \in B.$$

□

Corollary

Let $(B \times F, B, \pi, F)$ be a trivial fiber bundle with $E = B \times F$ the bundle space, (φ, ψ) a skew–product, with $\psi = \varphi \times f$, and $\sigma = \mathrm{I}_B \times \kappa \in \Gamma(E)$ a cross section. Then,
σ is an invariant section under $(\varphi, \psi) \Longleftrightarrow \kappa$ is a fixed point of the graph functional

$$\begin{array}{rccl} \mathrm{G}: & \mathcal{C}^0(B, F) & \longrightarrow & \mathcal{C}^0(B, F) \\ & \kappa & \longmapsto & \mathrm{G}(\kappa) = \widetilde{\kappa} \end{array}$$

where

$$\begin{array}{rccl} \widetilde{\kappa}: & B & \longrightarrow & F \\ & \widetilde{\theta} & \longmapsto & \widetilde{\kappa}(\widetilde{\theta}) := f(\varphi^{-1}(\widetilde{\theta}), \kappa(\varphi^{-1}(\widetilde{\theta}))) \end{array} \quad ,$$

that is, $\mathrm{G}(\kappa) = \kappa$.

Proof.
Since $\varphi : B \longrightarrow B$ is a homeomorphism, for every $\widetilde{\theta} \in B$, $\exists! \theta \in B$ such that $\widetilde{\theta} = \varphi^{-1}(\theta)$. Hence, by **Proposition I.16**:

$$\begin{aligned} G(\kappa) = \kappa \quad & \Longleftrightarrow \quad \forall \widetilde{\theta} \in B,\ f(\varphi^{-1}(\widetilde{\theta}), \kappa(\varphi^{-1}(\widetilde{\theta}))) = \kappa(\widetilde{\theta}) \\ & \Longleftrightarrow \quad \forall \theta \in B,\ f(\varphi^{-1}(\varphi(\theta)), \kappa(\varphi^{-1}(\varphi(\theta)))) = \kappa(\varphi(\theta)) \\ & \Longleftrightarrow \quad \forall \theta \in B,\ f(\theta, \kappa(\theta)) = \kappa(\varphi(\theta)) \\ & \Longleftrightarrow \quad \sigma = \mathrm{I}_B \times \kappa \text{ is an invariant section over } (\varphi, \psi)\ . \end{aligned}$$

□

I.6 Invertibility of bundle maps and skew–products

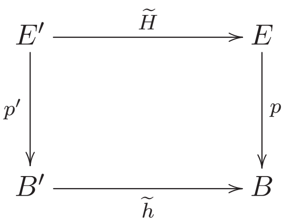
In this section, we describe the meaning of the invertibility of a bundle map and how it can be characterized for the particular case of a skew–product. Later we will come up with the application of these concepts to quasi–periodic skew–products and linear quasi–periodic skew–products, in order to enable the management of notions like topological conjugacy, linear conjugacy, and reducibility.

Definition I.18 Invertibility of a bundle map

Let (E, B, p, F) and (E', B', p', F') be two fiber bundles, and (h, H) be a bundle map between them. Thus, the diagram

$$\begin{array}{ccc} E & \xrightarrow{\ H\ } & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow[\ h\ ]{} & B' \end{array}$$

commutes, i.e. $h \circ p = p' \circ H$.
We say that (h, H) *is invertible if there exist another bundle map* $(\widetilde{h}, \widetilde{H})$ *from* (E', B', p', F') *onto* (E, B, p, F)*, of the form:*

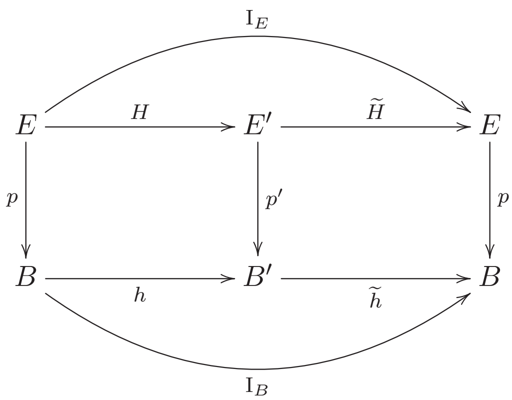


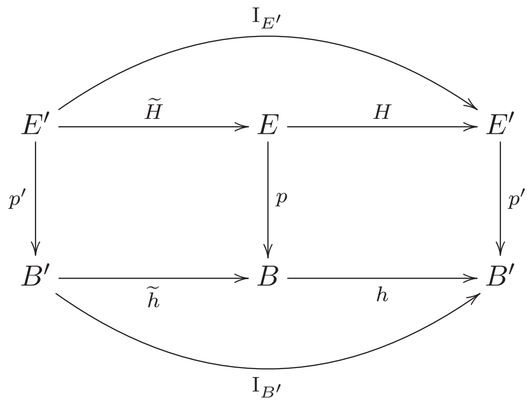
with $\widetilde{h} \circ p' = p \circ \widetilde{H}$*, and such that*

$$\begin{array}{rcl} (\widetilde{h}, \widetilde{H}) \circ (h, H) & = & (\mathrm{I}_B, \mathrm{I}_E) \\ (h, H) \circ (\widetilde{h}, \widetilde{H}) & = & (\mathrm{I}_{B'}, \mathrm{I}_{E'}) \end{array} \quad ,$$

or, equivalently:

$$\begin{array}{rclcrcl} \widetilde{h} \circ h & = & \mathrm{I}_B & , & \widetilde{H} \circ H & = & \mathrm{I}_E \\ h \circ \widetilde{h} & = & \mathrm{I}_{B'} & , & H \circ \widetilde{H} & = & \mathrm{I}_{E'} \end{array} \quad .$$





◇

Proposition I.19 Invertibility of skew–products

Let (h, H) *be a skew–product*[2] *defined over the trivial fiber bundle* $(E = B \times F, B, \pi, F)$*. Assume that*

$$H(\theta, x) = (h(\theta), g(\theta, x)), \ \forall (\theta, x) \in E \ ,$$

where $h : B \longrightarrow B$ *and* $g : E = B \times F \longrightarrow F$ *are continuous.*
Then, (h, H) *is invertible if and only if the following two conditions hold:*

(i) *h is invertible*[3].

---

[2]See **Definition I.13**.

[3]Notice that in our definition of skew–product we have assumed that the first component map h is, in any case, a homeomorphism, so it is always invertible and moreover, h^{-1} is also a homeomorphism. With these assumptions, the inverse of a skew–product remains being a skew–product.

(*ii*) *For every $\theta \in B$, the map*

$$\begin{array}{rccl} g_\theta : & F & \longrightarrow & F \\ & x & \longmapsto & g_\theta(x) := g(\theta, x) \end{array}$$

is invertible.

Moreover, in such a case, $(h,H)^{-1} = (h^{-1}, H^{-1})$, with

$$H^{-1}(\theta, x) = (h^{-1}(\theta), g^{-1}_{h^{-1}(\theta)}(x), \ \forall (\theta, x) \in E \ .$$

Proof.

($\Rightarrow$) Suppose that (h,H) is invertible. Then, there exist a skew–product $(\widetilde{h}, \widetilde{H})$, with

$$\widetilde{H}(\theta, x) = (\widetilde{h}(\theta), \widetilde{g}(\theta, x)), \ \forall (\theta, x) \in E \ ,$$

such that

$$\widetilde{h} \circ h = \mathrm{I}_B \tag{I.3}$$
$$h \circ \widetilde{h} = \mathrm{I}_B \tag{I.4}$$
$$\widetilde{H} \circ H = \mathrm{I}_E \tag{I.5}$$
$$H \circ \widetilde{H} = \mathrm{I}_E \tag{I.6}$$

From (I.3) and (I.4) we have that h is invertible and $h^{-1} = \widetilde{h}$.

From (I.5) and (I.6) we have that H is invertible and $H^{-1} = \widetilde{H}$, namely $\forall (\eta, y) \in E$, there exist a unique $(\theta, x) \in E$ such that $H(\theta, x) = (\eta, y)$.

Thus, given $\theta \in B$ and $y \in F$, we take $\eta = h(\theta) \in B$ and hence $\exists! x \in F$, such that $g(\theta, x) = g_\theta(x) = y$. Therefore, $\forall \theta \in B$, $\exists! x \in F$ such that $g_\theta(x) = y$ and g_θ is invertible for every $\theta \in B$.

($\Leftarrow$) Let us assume now that (*i*) and (*ii*) hold.

Define $\widetilde{h} = h^{-1}$ and

$$\begin{array}{rccl} \widetilde{H} : & E = B \times F & \longrightarrow & F \\ & (\theta, x) & \longmapsto & \widetilde{H}(\theta, x) := (h^{-1}(\theta), g^{-1}_{h^{-1}(\theta)}(x)) \end{array} \quad ,$$

i.e. $\widetilde{h}(\theta) = h^{-1}(\theta)$ and $\widetilde{g}(\theta, x) = g^{-1}_{h^{-1}(\theta)}(x))$.

Now we show that $\widetilde{H} \circ H = H \circ \widetilde{H} = \mathrm{I}_E$. Hence (h,H) is invertible as we stated, and $(h,H)^{-1} = (\widetilde{h}, \widetilde{H})$.

Indeed,

$(\widetilde{H} \circ H)(\theta, x) = \widetilde{H}(H(\theta, x)) = \widetilde{H}(h(\theta), g(\theta, x))$

$= (h^{-1}(h(\theta)), g^{-1}_{h^{-1}(h(\theta))}(g(\theta, x))) = (\theta, g_\theta^{-1}(g_\theta(x)) = (\theta, x), \ \forall (\theta, x) \in E.$

In the same manner,

$(H \circ \widetilde{H})(\theta, x) = H(\widetilde{H}(\theta, x)) = H(h^{-1}(h(\theta)), g^{-1}_{h^{-1}(\theta)}(x))$

$= (h(h^{-1}(\theta)), g(h^{-1}(\theta), g^{-1}_{h^{-1}(\theta)}(x))) = (\theta, g_{h^{-1}(\theta)}(g^{-1}_{h^{-1}(\theta)}(x))) = (\theta, x), \ \forall (\theta, x) \in E.$

□

I.7 Topological conjugacy and linear conjugacy of skew–products

Definition I.20 Topologically conjugate skew–products

Let (φ, ψ) *and* $(\widetilde{\varphi}, \widetilde{\psi})$ *be skew–product dynamical systems over the trivial fiber bundle* $(E = B \times F, B, \pi, F)$*, with*

$$\begin{array}{rcl} \psi(\theta, t) & = & (\varphi(\theta), f(\theta, t)) \\ \widetilde{\psi}(\theta, x) & = & (\widetilde{\varphi}(\theta), \widetilde{f}(\theta, x)) \end{array} \quad .$$

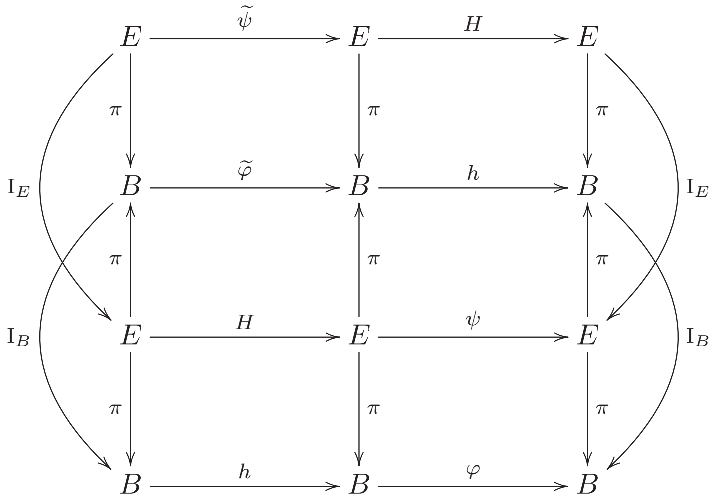
We say that (φ, ψ) *and* $(\widetilde{\varphi}, \widetilde{\psi})$ *are* topologically conjugate skew–products *if there is another skew–product* (h, H)*, with* $H(\theta, x) = (h(\theta), g(\theta, x))$*, which is invertible*[4] *and such that*

$$(\widetilde{\varphi}, \widetilde{\psi}) = (h, H)^{-1} \circ (\varphi, \psi) \circ (h, H) \quad .$$

◇

Remark I.21

According to the definition of bundle maps and their composition[5]*, the following diagram commutes:*



and, equivalently, we have,

$$\begin{array}{rcl} \widetilde{\varphi} & = & h^{-1} \circ \varphi \circ h \\ \widetilde{\psi} & = & H^{-1} \circ \psi \circ H \end{array} \quad .$$

whenever (φ, ψ) *and* $(\widetilde{\varphi}, \widetilde{\psi})$ *are topologically conjugate skew–products.*

Proposition I.22 Topologically conjugate skew–products

Let (φ, ψ) *and* $(\widetilde{\varphi}, \widetilde{\psi})$ *be skew–product dynamical systems over the trivial fiber bundle*

---

[4]See **Proposition I.19** for the characterization of the invertibility of a skew–product.

[5]See **Definition I.4**.

$(E = B \times F, B, \pi, F)$, *with*

$$\begin{array}{rcl} \psi(\theta, x) & = & (\varphi(\theta), f(\theta, x)) \\ \widetilde{\psi}(\theta, x) & = & (\widetilde{\varphi}(\theta), \widetilde{f}(\theta, x)) \end{array} .$$

Then, (φ, ψ) *and* $(\widetilde{\varphi}, \widetilde{\psi})$ *are topologically conjugate if and only if the following two conditions hold:*

(i) *There exist a homeomorphism* $h : B \longrightarrow B$ *such that* $\widetilde{\varphi} = h^{-1} \circ \varphi \circ h$, *i.e.* φ *and* $\widetilde{\varphi}$ *are topologically conjugate in* B.

(ii) *There exist a map,* $g : E = B \times F \longrightarrow F$, *such that* $\forall \theta \in B$, *the map*

$$\begin{array}{rccl} g_\theta : & F & \longrightarrow & F \\ & x & \longmapsto & g_\theta(x) = g(\theta, x) \end{array} ,$$

is a homeomorphism and

$$g(\varphi(h(\theta)), \widetilde{f}(\theta, x)) = f(h(\theta), g(\theta, x)), \ \forall (\theta, x) \in E.$$

In such a case, if we call $H(\theta, x) = (h(\theta), g(\theta, x))$, $(\theta, x) \in E$, *then:*

$$(\widetilde{\varphi}, \widetilde{\psi}) = (h, H)^{-1} \circ (\varphi, \psi) \circ (h, H) \ ,$$

Proof. □

Definition I.23 Linear skew–products

Let $(E = B \times F, B, \pi, F)$ *be a trivial vector bundle*[6] *and let* (φ, ψ) *be a skew–product where*

$$\psi(\theta, x) = (\varphi(\theta), f(\theta, x)), \ (\theta, x) \in E \ .$$

We say that (φ, ψ) *is a* linear skew–product *if* f *is linear w.r.t. the second component, that is, for every* $\theta \in B$, *the map,*

$$\begin{array}{rccl} f_\theta : & F & \longrightarrow & F \\ & x & \longmapsto & f_\theta(x) = f(\theta, x) \end{array} ,$$

is a homomorphism (of the $\mathbb{K}-$*vector space* F *onto itself).*

◇

Remark I.24

Whenever F *is finite dimensional, with* $n = \dim_{\mathbb{K}}(F)$, *and* f *is linear w.r.t the second component, it can be expressed as* $f(\theta, x) = m(\theta)x$, $\forall (\theta, x) \in E$, *where*

$$m : B \longrightarrow \mathcal{M}_n(\mathbb{K})$$

is continuous, i.e. $m(\theta)$ *is an* $n-$*dimensional square matrix, which is called* transfer matrix, *of continuous functions. Thus, a linear skew–product may be identified with a map of the form:*

$$\begin{array}{rccl} \psi : & E = B \times F & \longrightarrow & E = B \times F \\ & (\theta, x) & \longmapsto & \psi(\theta, x) = (\varphi(\theta), m(\theta)x) \end{array}$$

where φ *is a homeomorphism over the base* B *and* $m : B \longrightarrow \mathcal{M}_n(\mathbb{K})$ *is continuous.*

---

[6] Recall that in this case, by **Definition I.8**, we assume that F is a $\mathbb{K}-$vector space, where usually $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

Definition I.25 Linearly conjugate skew–products

Let (φ,ψ) *and* $(\widetilde{\varphi},\widetilde{\psi})$ *be linear skew–products over the vector bundle* $(E = B\times F, B, \pi, F)$*, with*

$$\begin{array}{rcl}\psi(\theta,x) & = & (\varphi(\theta), m(\theta)x)\\ \widetilde{\psi}(\theta,x) & = & (\widetilde{\varphi}(\theta), \widetilde{m}(\theta)x)\end{array} \quad .$$

We say that (φ,ψ) *and* $(\widetilde{\varphi},\widetilde{\psi})$ *are* linear conjugate skew–products *if there is another linear skew–product* (h,H)*, with* $H(\theta,x)=(h(\theta),c(\theta)x)$*, which is invertible and such that*

$$(\widetilde{\varphi},\widetilde{\psi}) = (h,H)^{-1}\circ(\varphi,\psi)\circ(h,H) \quad .$$

◇

Remark I.26

Observe that, according to ***Proposition I.19****, a linear skew–product* $H(\theta,x)=(h(\theta),c(\theta)x)$ *is invertible if and only if* $\det(c(\theta))\neq 0$*,* $\forall\theta\in B$*.*

Proposition I.27 Linearly conjugate skew–products

Let (φ,ψ) *and* $(\widetilde{\varphi},\widetilde{\psi})$ *be linear skew–products over the vector bundle* $(E = B\times F, B, \pi, F)$*, with*

$$\begin{array}{rcl}\psi(\theta,x) & = & (\varphi(\theta), m(\theta)x)\\ \widetilde{\psi}(\theta,x) & = & (\widetilde{\varphi}(\theta), \widetilde{m}(\theta)x)\ .\end{array}$$

Then, (φ,ψ) *and* $(\widetilde{\varphi},\widetilde{\psi})$ *are linearly conjugate if and only if there exist another linear skew–product,* (h,H)*, with*

$$H(\theta,x)=(h(\theta),c(\theta)x)\ \forall(\theta,x)\in E\ ,$$

such that the following properties hold:

(*i*) $\det(c(\theta))\neq 0,\ \forall\theta\in B$*;*

(*ii*) $\widetilde{\varphi}=h^{-1}\circ\varphi\circ h$*, i.e.* φ *and* $\widetilde{\varphi}$ *are topologically conjugate in* B*;*

(*iii*) $m(h(\theta))c(\theta)=c(\widetilde{\varphi}(\theta))\widetilde{m}(\theta),\ \forall\theta\in B$*.*

Proof.

($\Rightarrow$) Let (h,H), with $H(\theta,x)=(h(\theta),c(\theta)x)$, $(\theta,x)\in E$, be an invertible linear skew–product such that

$$(\widetilde{\varphi},\widetilde{\psi}) = (h,H)^{-1}\circ(\varphi,\psi)\circ(h,H) \quad .$$

Since (h,H) is invertible then, by **Proposition I.19**, h is invertible and for every $\theta\in B$ the map

$$\begin{array}{rccl} g_\theta: & F & \longrightarrow & F\\ & x & \longmapsto & g_\theta(x):=c(\theta)x\end{array} \quad ,$$

is invertible. Since g_θ is a homomorphism, then $c(\theta)\in\mathcal{M}_n(\mathbb{K})$ is invertible and hence $\det(c(\theta))\neq 0$. Furthermore, $H^{-1}(\theta,x)=(h^{-1}(\theta),c(h^{-1}(\theta))^{-1}x)$, $\forall(\theta,x)\in E$. Moreover, $\widetilde{\psi}(\theta,x)=(H^{-1}\circ\psi\circ H)(\theta,x),\forall(\theta,x)\in E$. Therefore,

$$\begin{array}{rcl}(\widetilde{\varphi}(\theta),\widetilde{m}(\theta)x) & = & H^{-1}(\psi(H(\theta,x))) = H^{-1}(\varphi(h(\theta),c(\theta)x))\\ & = & H^{-1}(\varphi(h(\theta)),m(h(\theta))c(\theta)x)\\ & = & (h^{-1}(\varphi(h(\theta))),c(h^{-1}(\varphi(h(\theta))))^{-1}m(h(\theta))c(\theta)x),\ \forall(\theta,x)\in E \quad .\end{array}$$

Identifying components, we have, on one side,

$\widetilde{\varphi}(\theta) = h^{-1}(\varphi(h(\theta))) = (h^{-1} \circ \varphi \circ h)(\theta),\ \forall \theta \in B$, that is, $\widetilde{\varphi} = h^{-1} \circ \varphi \circ h$.

On the other side,

$\widetilde{m}(\theta)x = c(h^{-1}(\varphi(h(\theta))))^{-1} m(h(\theta))c(\theta)x = c(\widetilde{\varphi}(\theta))^{-1} m(h(\theta))c(\theta)x,\ \forall (\theta, x) \in E.$

Consequently:

$\widetilde{m}(\theta) = c(\widetilde{\varphi}(\theta))^{-1} m(h(\theta))c(\theta),\ \forall \theta \in B$, which is equivalent to:

$m(h(\theta))c(\theta) = c(\widetilde{\varphi}(\theta))\widetilde{m}(\theta),\ \forall \theta \in B.$

Thus, (i), (ii), and (iii) are satisfied.

$(\Leftarrow)$ Assume that there exists a skew–product (h, H), with

$$H(\theta, x) = (h(\theta), c(\theta)x),\ (\theta, x) \in E$$

holding (i), (ii), and (iii).

Notice that h is invertible, since it is a homeomorphism. On the other hand, H is invertible since $\det(c(\theta)) \neq 0$. Moreover, $H^{-1}(\theta, x) = (h^{-1}(\theta), c(h^{-1}(\theta)))^{-1}t,\ \forall (\theta, x) \in E.$

Then, $(h, H)^{-1} \circ (\varphi, \psi) \circ (h, H) = (h^{-1} \circ \varphi \circ h, H^{-1} \circ \psi \circ H) = (\widetilde{\varphi}, H^{-1} \circ \psi \circ H)$.

It only remains to show that $H^{-1} \circ \psi \circ H = \widetilde{\psi}$.

Indeed, by means of (ii), and (iii) we have:

$$\begin{aligned}
(H^{-1} \circ \psi \circ H)(\theta, x) &= H^{-1}(\psi(H(\theta, x))) = H^{-1}(\psi(h(\theta), c(\theta)x)) \\
&= H^{-1}(\varphi(h(\theta)), m(h(\theta))c(\theta)x) \\
&= (h^{-1}(\varphi(h(\theta))), c(h^{-1}(\varphi(h(\theta))))^{-1} m(h(\theta))c(\theta)x) \\
&= (\widetilde{\varphi}(\theta), c(\widetilde{\varphi}(\theta))^{-1} m(h(\theta))c(\theta)x)) \\
&= (\widetilde{\varphi}(\theta), \widetilde{m}(\theta)x)) \\
&= \psi(\theta, x),\ \forall (\theta, x) \in E.
\end{aligned}$$

Thus, $H^{-1} \circ \psi \circ H = \widetilde{\psi}$, as we wanted to prove.

□

I.8 The framework under study: quasi–periodic skew–products.

Let $(E = B \times F, B, \pi, F)$ be a trivial fiber bundle. Let (φ, ψ) be a skew–product in F with $f : E = B \times F \longrightarrow F$ continuous, and $\sigma = \mathrm{I}_B \times \kappa \in \Gamma(E)$ a cross section where $\kappa : B \longrightarrow F$ is assumed to be continuous.

Henceforth, we shall mainly deal with a particular case. From now on we will consider under study only the frame in which the base space is the torus, $B = \mathbb{T}^d$, the fiber is the real line, $F = \mathbb{R}$, i.e. $E = \mathbb{T}^d \times \mathbb{R}$, $f : \mathbb{T}^d \times \mathbb{R} \longrightarrow \mathbb{R}$, and the homeomorphism φ is an *ergodic rigid rotation* , $\varphi = \mathcal{R}_\omega$, that is:

$$\begin{array}{rccl} \mathcal{R}_\omega : & \mathbb{T}^d & \longrightarrow & \mathbb{T}^d \\ & \theta & \longmapsto & \mathcal{R}_\omega(\theta) = \theta + \omega \end{array} \tag{I.7}$$

where the frequency $\omega \in \mathbb{T}^d$ is *rationally independent*, namely $a \cdot \omega \notin \mathbb{Z},\ \forall a \in \mathbb{Z} \setminus \{0\}$. In such a case we say that the frequency ω is ergodic or non–resonant.

In this setting, the system is being undergone to an external quasi–periodic force and we will refer to it as a *quasi–periodic skew–product*.

Thus $\psi = \mathcal{R}_\omega \times f$ is of the form:

$$\begin{array}{rccl} \psi: & \mathbb{T}^d \times \mathbb{R} & \longrightarrow & \mathbb{T}^d \times \mathbb{R} \\ & (\theta, x) & \longmapsto & \psi(\theta, x) = (\theta + \omega, f(\theta, x)) \end{array}$$

More particularly, we will concentrate our efforts on the case $d = 1$, and postpone the study of higher dimensions for later works. Furthermore, from now on we assume that the frequency ω is Diophantine (see **Definition 1.15**).

Summarizing, henceforth our target is the study of one-dimensional quasi–periodic skew–products of the form:

$$\begin{array}{rccl} \psi: & \mathbb{T} \times \mathbb{R} & \longrightarrow & \mathbb{T} \times \mathbb{R} \\ & (\theta, x) & \longmapsto & \psi(\theta, x) = (\theta + \omega, f(\theta, x)) \quad , \end{array} \tag{I.8}$$

where $\omega \in \mathbb{T}$ is a fixed Diophantine frequency and $f \in \mathcal{C}^r(\mathbb{T} \times \mathbb{R})$, with $r \geq 1$.

Our next goal is to analyze the existence of invariant curves for the skew–product (I.8), that is, according to **Proposition I.16**, continuous maps $\kappa : \mathbb{T} \longrightarrow \mathbb{R}$ such that:

$$f(\theta, \kappa(\theta)) = \kappa(\theta + \omega), \ \forall \theta \in \mathbb{T} \ . \tag{I.9}$$

Appendix II. Matlab® programming of numerical procedures

II.1 Orbits and the error function

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Author      :            Juan Pello García
% Degree      :            Universitat de Barcelona
%                          Facultat de Matemàtiques i Informàtica
%                          PhD Thesis 'Degenerate invariant tori in KAM theory'
% NUB         :            52485
% Date        :            December 2022
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% File        :             Orbit.m
% Input       :             a,b         Parameters of the skew product
%             :             omega       Diophantine frequency
%             :             theta_0,x_0 Initial point of the orbit
%             :             N           Number of iterates
%             :             N0          Number of discarded iterations
% Output      :             theta       Partition of the torus with N-N0
%                                       terminal points with
%                                       theta(n+1)=theta(n)+omega (mod 1)
%             :             x           vector with the values of the skew
%                                       product at the points of the partition
%                                       x(n+1)=f(theta(n),x(n))
% Synopsis    :             Computation of the forward orbit for the point
%                           (theta_0,x_0) by the given skew product f(theta,x)
% Syntax      :             [theta,x]=Orbit(a,b,omega,theta_0,x_0,N,N0)
% Example     :  [1]        a=6.8; b=1.62;
%                           omega=(1+sqrt(5))/2;
%                           theta_0=0; x_0=0;
%                           N=10^7;N0=2^7;
%                           [theta,x]=Orbit(a,b,omega,theta_0,x_0,N,N0);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function [theta,x]=Orbit(a,b,omega,theta_0,x_0,N,N0)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
theta=theta_0+omega*linspace(0,N-1,N);
theta=theta-floor(theta);
x=zeros(1,N);
x(1)=x_0;
for n=1:N-1
    x(n+1)=f(theta(n),x(n),a,b);
end
theta=theta(N0+1:end);
x=x(N0+1:end);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
[~, s] = sort(theta(1,:));
theta=theta(:,s);
x=x(:,s);
length(theta)
length(x)
plot(theta,x,'.','MarkerSize',1,'Color',[0.48 0.06 0.89])
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function F=f(theta,z,a,b)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
t=theta-floor(theta);
F=atan(a*z)+b*sin(2*pi*t);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Author      :            Juan Pello García
% Degree      :            Universitat de Barcelona
%                          Facultat de Matemàtiques i Informàtica
%                          PhD Thesis 'Degenerate invariant tori in KAM theory'
% NUB         :            52485
% Date        :            December 2022
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% File        :             BackwardOrbit.m
% Input       :             a,b         Parameters of the skew product
%             :             omega       Diophantine frequency
%             :             theta_0,x_0 Initial point of the orbit
%             :             N           Number of iterates
%             :             N0          Number of discarded iterations
% Output      :             theta       Partition of the torus with N-N0
%                                       terminal points with
%                                       theta(n+1)=theta(n)-omega (mod 1)
%             :             x           vector with the values of the inverse skew
%                                       product at the points of the partition
%                                       x(n+1)=f^(-1)(theta(n),x(n))
% Synopsis    :             Computation of the backward orbit for the point
%                           (theta_0,x_0) by the given skew product f(theta,x)
% Syntax      :             [theta,x]=BackwardOrbit(a,b,omega,theta_0,x_0,N,N0)
% Example     :  [1]        a=6.8; b=1.62;
%                           omega=(1+sqrt(5))/2;
%                           theta_0=0; x_0=0;
%                           N=10^7;N0=2^7;
%                           [theta,x]=BackwardOrbit(a,b,omega,theta_0,x_0,N,N0);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function [theta,x]=BackwardOrbit(a,b,omega,theta_0,x_0,N,N0)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
theta=theta_0-omega*linspace(0,N-1,N);
theta=theta-floor(theta);
x=zeros(1,N);
x(1)=x_0;
for n=1:N-1
    x(n+1)=h(theta(n),x(n),a,b);
end
theta=theta(N0:end);
x=x(N0:end);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
[~, s] = sort(theta(1,:));
theta=theta(:,s);
x=x(:,s);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function H=h(theta,z,a,b)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
omega=(1+sqrt(5))/2;
t=theta-floor(theta);
% F=atan(a*z)+b*sin(2*pi*t);
H=1/a*tan(z-b*sin(2*pi*(t-omega)));
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

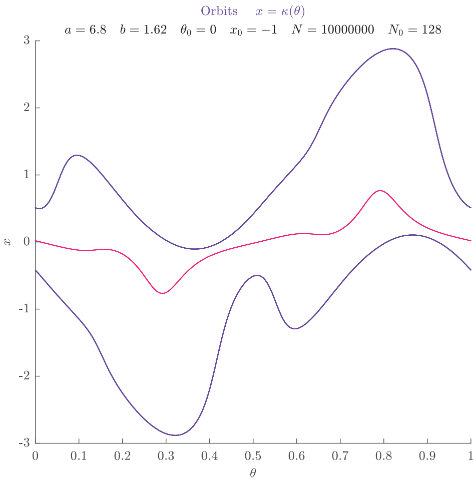


FIGURE II.1: Orbits of the skew–product with $a = 6.8,\ b = 1.62,$
$N = 10^7,\ N_0 = 2^7$
Forward orbit of the point $\theta_0 = 0\,,\ x_0 = 1$(top);
Backward orbit of the point $\theta_0 = 0\,,\ x_0 = 0$(middle);
Forward orbit of the point $\theta_0 = 0\,,\ x_0 = -1$(bottom).

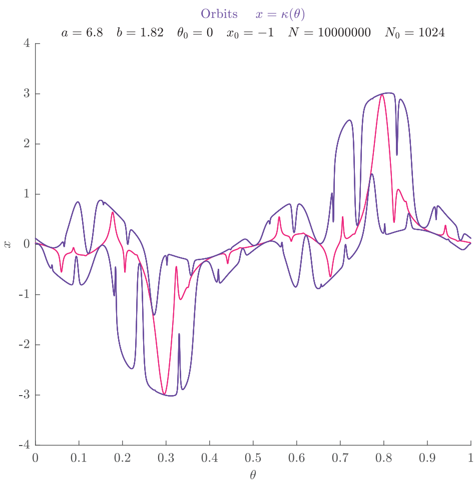


FIGURE II.2: Orbits of the skew–product with $a = 6.8,\ b = 1.82,$
$N = 10^7,\ N_0 = 2^10$
Forward orbit of the point $\theta_0 = 0\,,\ x_0 = 1$(top);
Backward orbit of the point $\theta_0 = 0\,,\ x_0 = 0$(middle);
Forward orbit of the point $\theta_0 = 0\,,\ x_0 = -1$(bottom).

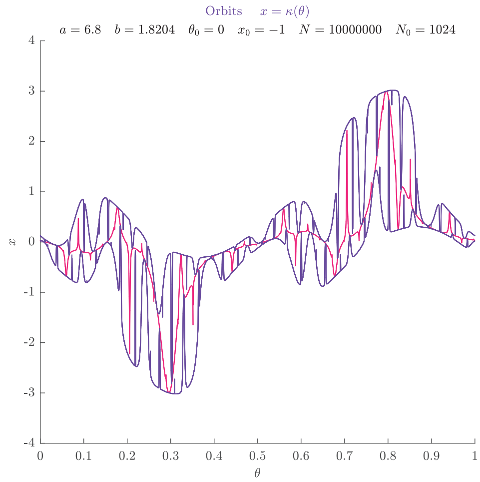


FIGURE II.3: Orbits of the skew–product with $a = 6.8,\ b = 1.8204,$
$N = 10^7,\ N_0 = 2^10$
Forward orbit of the point $\theta_0 = 0\ ,\ x_0 = 1$(top);
Backward orbit of the point $\theta_0 = 0\ ,\ x_0 = 0$(middle);
Forward orbit of the point $\theta_0 = 0\ ,\ x_0 = -1$(bottom).

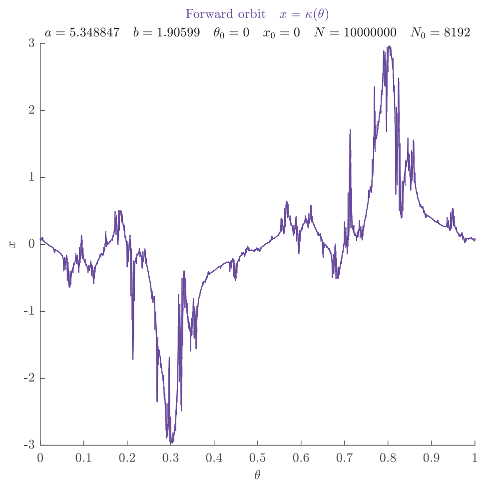


FIGURE II.4: Forward orbit of the skew–product with
$a = 5.348847,\ b = 1.905990, N = 10^7,\ N_0 = 2^{13},$
$\theta_0 = 0\ ,\ x_0 = 0.$

```
function [Error,NormError]=ErrorFunction(theta,kappa,omega,a,b)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% clear;a=6.8; b=1.62; omega=(1+sqrt(5))/2;
% theta_0=0; x_0=0;
% N=1e07;
% [theta,kappa]=Ordered_Curve(theta_0,x_0,omega,a,b,N);
% [Error,NormError]=ErrorFunction(theta,kappa,omega,a,b);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
[t,T_kappa]=Translation_omega(theta,kappa,omega);
y=f(t,kappa,a,b);
Error=y-T_kappa;
NormError=max(abs(Error));
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% PLOTS
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
figure
set(gca,'FontSize',8);
set(gcf,'Color',[1,1,1]);
axis square
hold on
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
plot(t,T_kappa,'--','MarkerSize',1,'Color',[0.7 0.7 0.7])
hold on
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
plot(t,T_kappa,'.','MarkerSize',2,'Color',[0.7 0.7 0.7])
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
plot(t,Error,'.','MarkerSize',1,'Color',[0 0 0])
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
plot(t,y,'.','MarkerSize',1,'Color',[1 0 0])
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% TITLE SUBTITLE AND LABELS
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
TITLE = '$E(\theta)=f(\theta,\kappa(\theta))-\kappa(\theta+\omega)$';
formatSpec = '%.8g';
SUBTITLE=['$\|E\|= \max(|E(\theta)|)=',num2str(NormError,formatSpec),'$'];
[TITLE,SUBTITLE]=title(TITLE,SUBTITLE,'interpreter','latex','FontSize',18);
TITLE.Color = '[0 0 0]';
SUBTITLE.Color = '[0 0 0]';
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
xlabel('$\theta$','interpreter','latex','FontSize',18);
ylabel('$x$','interpreter','latex','FontSize',18);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

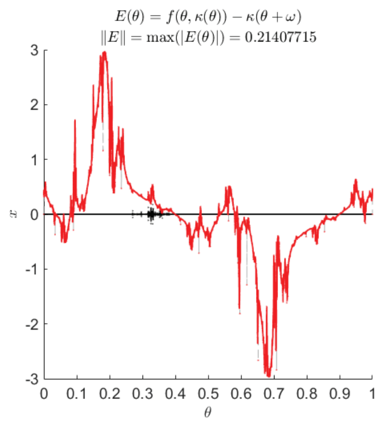


FIGURE II.5: Error of the orbit with $\theta_0 = 0$, $x_0 = 0$ with $N = 10^7$.

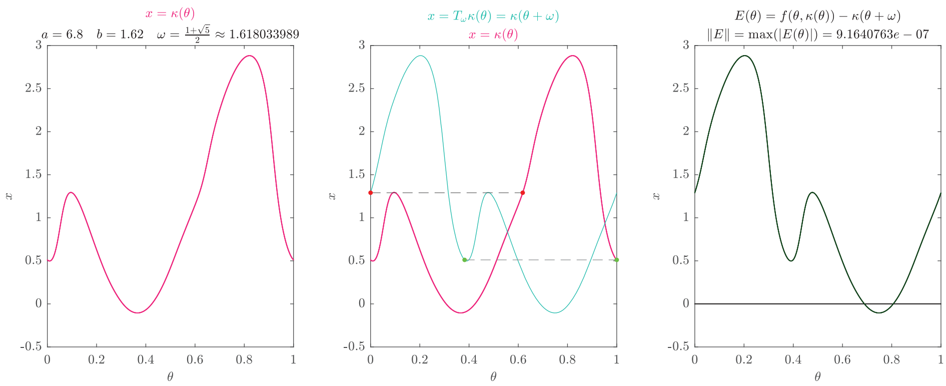


FIGURE II.6: Forward orbit of the point $\theta_0 = 0, x_0 = 1$
$x = \kappa(\theta)$(magenta) (left);
Translated curve $x = \kappa(\theta + \omega)$ (light green) (center);
$x = f(\theta, \kappa(\theta))$ (dark green) and the error function $E(\theta)$ (right).

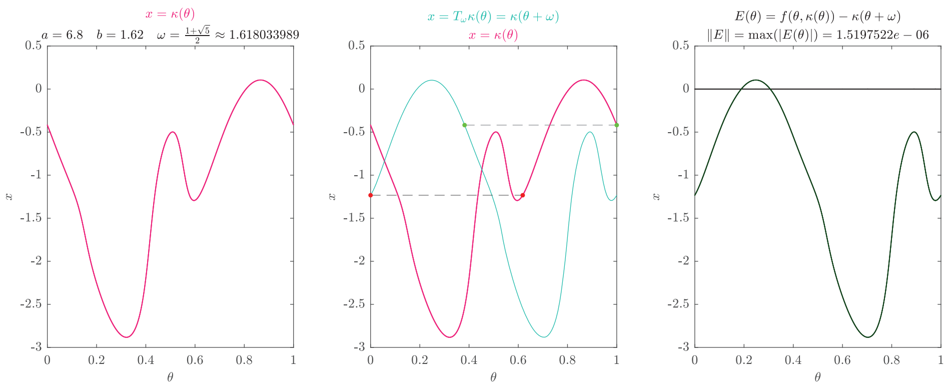


FIGURE II.7: Forward orbit of the point $\theta_0 = 0, x_0 = -1$
$x = \kappa(\theta)$(magenta) (left);
Translated curve $x = \kappa(\theta + \omega)$ (light green) (center);
$x = f(\theta, \kappa(\theta))$ (dark green) and the error function $E(\theta)$ (right).

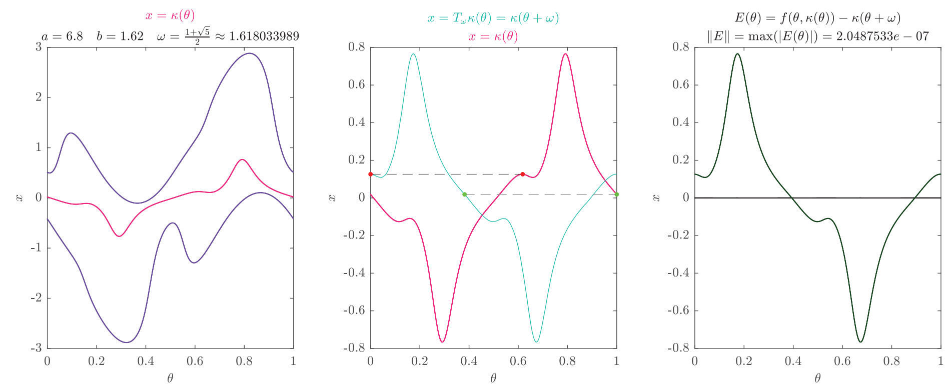


FIGURE II.8: Forward orbits (purple) and backward orbit
$x = \kappa(\theta)$(magenta) (left);
Translated curve $x = \kappa(\theta + \omega)$ (light green) (center);
$x = f(\theta, \kappa(\theta))$ (dark green) and the error function $E(\theta)$ (right).

II.2 Complex Fourier series estimates by means of the discrete Fourier transform (DFT)

The performance of the following algorithm is based on the properties of the DFT developed in **Section 5.5** and the computational error produced attends to **Corollary 5.11**.

Given W containing the Fourier coefficients of a function $(\widehat{u}_0, \widehat{u}_1, \ldots, \widehat{u}_M)$, the output is a vector w containing the values of the function $u(\theta) = \widehat{u}_0 + 2\,\mathrm{Re}\left(\sum_{k=1}^{M} \widehat{u}_k e^{2\pi k\theta i}\right)$, evaluated in an equidistributed partition of the torus of length $N = 2M$.

```
function w=IDFT_APPROX(W)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
tic
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
M=length(W)-1;
N=2*M;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% WM=exp(-2*pi*1i/M);
% WN=exp(-2*pi*1i/N);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
X=W(2:M+1);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
XN=[X,zeros(1,M)];
ZN=circshift(XN,1);
zN=ifft(ZN);
zN=circshift(zN,-1);
YN=fft(zN);
YN=circshift(YN,-1);
Y=YN(1:M);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
x=ifft(circshift(X,1));
y=ifft(circshift(Y,1));
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
w=[x;y];
w=w(:)';
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
w=W(1)+N*real(w);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
toc
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

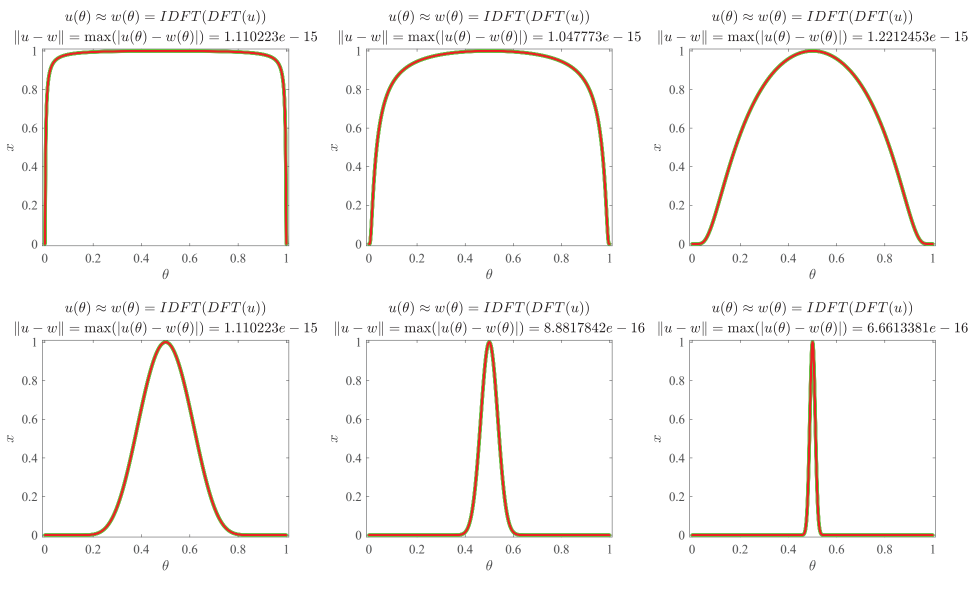


FIGURE II.9: Bump functions

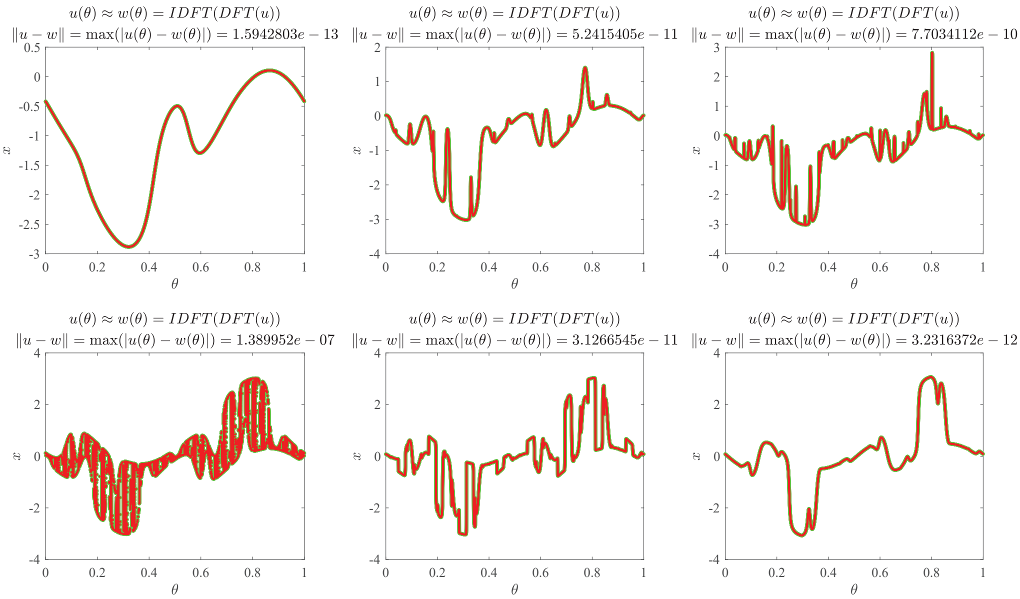


FIGURE II.10: Forward orbits: $u(\theta) = \widehat{u}_0 + 2\,\mathrm{Re}\left(\sum_{k=1}^{M} \widehat{u}_k e^{2\pi k\theta i}\right)$

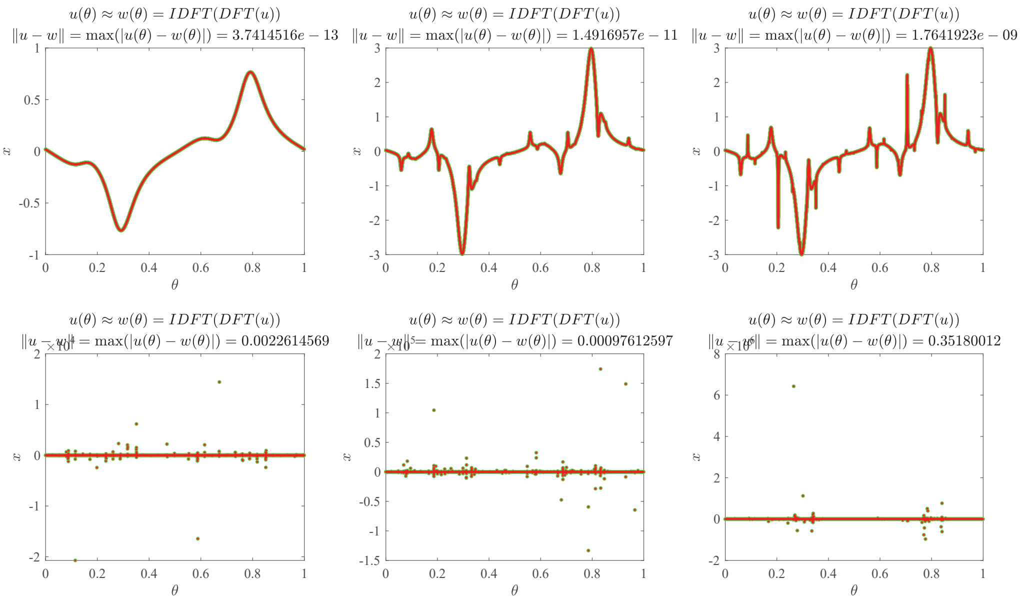


FIGURE II.11: Backward orbits: $u(\theta) = \widehat{u}_0 + 2\,\mathrm{Re}\left(\sum_{k=1}^{M} \widehat{u}_k e^{2\pi k\theta i}\right)$

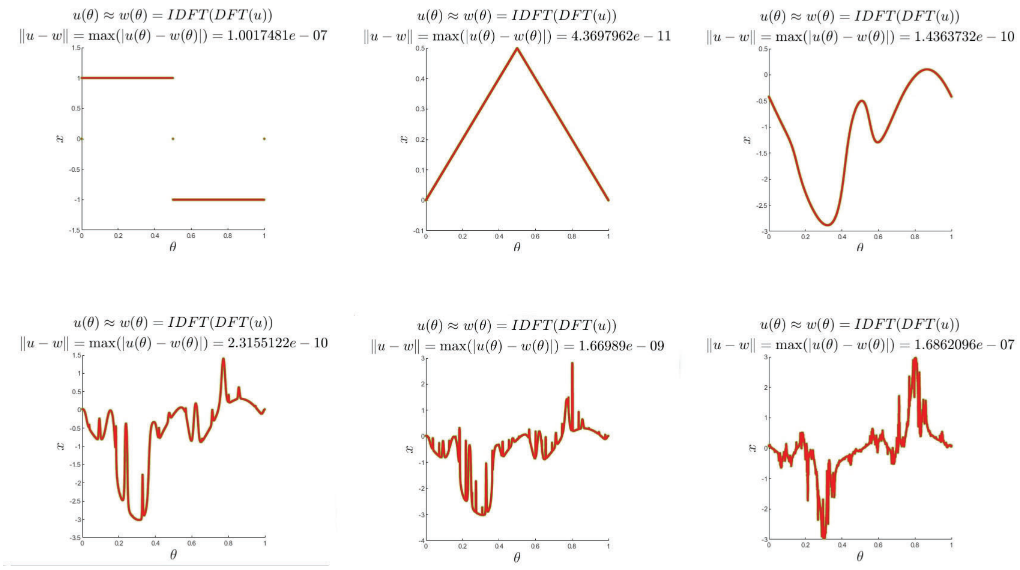


FIGURE II.12: $u(\theta) = \widehat{u}_0 + 2\,\mathrm{Re}\left(\sum_{k=1}^{M} \widehat{u}_k e^{2\pi k\theta i}\right)$

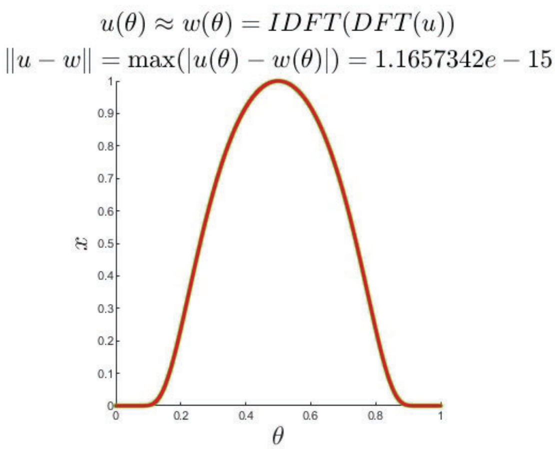


FIGURE II.13: Bump function $u(\theta) = e^{-\frac{(2\theta-1)^2}{1-(2\theta-1)^2}}, \ \theta \in \mathbb{T}$

II.3 Cohomological operator and the Floquet transformation

```
function [Rv,W]=R(lambda,omega,t,v)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% This function computes the solution u to the cohomological equation
% u(theta+omega)-lambda u(theta)= v~(theta), theta\in T=R/Z,
% by approximation of its Fourier series, where v=v^_0+v~, i.e. v~ is the
% (zero-average) oscillatory part of the given function v.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
N=length(t); % N must be even
M=N/2;
V=fft(v);
W=V(1:M+1)/N; % W=[v^_0,v^_1,...,v^_M] are fue Fourier coefficients of v
% Average_v=mean(v)
W(1)=0; % W=[0,v^_1,...,v^_M] Fourier coefficients of v~
for k=1:M
    W(k+1)=W(k+1)/(exp(2*pi*k*omega*1i)-lambda);
end % W=[0,u^_1,...,u^_M] Fourier coefficients of the solution u=R_lambda v
Rv=IDFT_APPROX(W);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

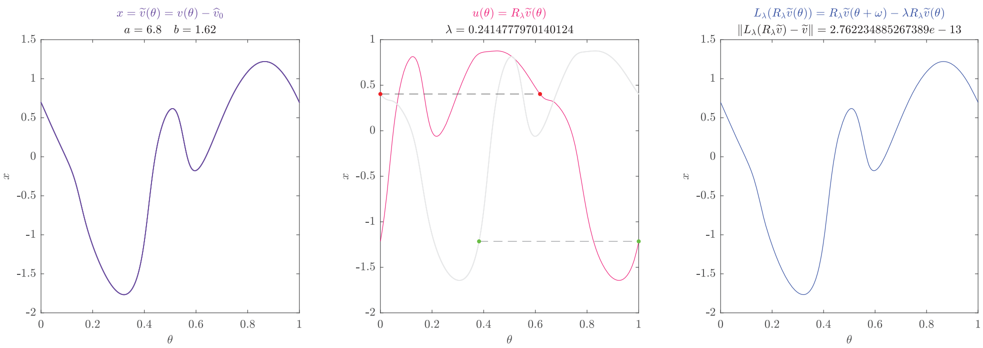


FIGURE II.14: Cohomological operator $\widetilde{v}(\theta) = v(\theta) - \widehat{v}_0$ where $v(\theta)$ is the orbit for $a = 6.8$ and $b = 1.62$ with $N = 10^7 + 1$, $N_0 = 2^8$ (left) ; $\mathfrak{R}_\lambda \widetilde{v}(\theta)$ in red and $\mathfrak{R}_\lambda \widetilde{v}(\theta+\omega)$ in gray (center); $\mathfrak{L}_\lambda \mathfrak{R}_\lambda \widetilde{v} \approx \widetilde{v}$ (right).

```
function [log_c,W,v]=FLOQUET(theta,kappa,omega,a)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
m=df(kappa,a);
v=log(m);
Lambda=mean(v);
v_TILDE=v-Lambda;
[log_c,W]=R(1,omega,theta,v_TILDE);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

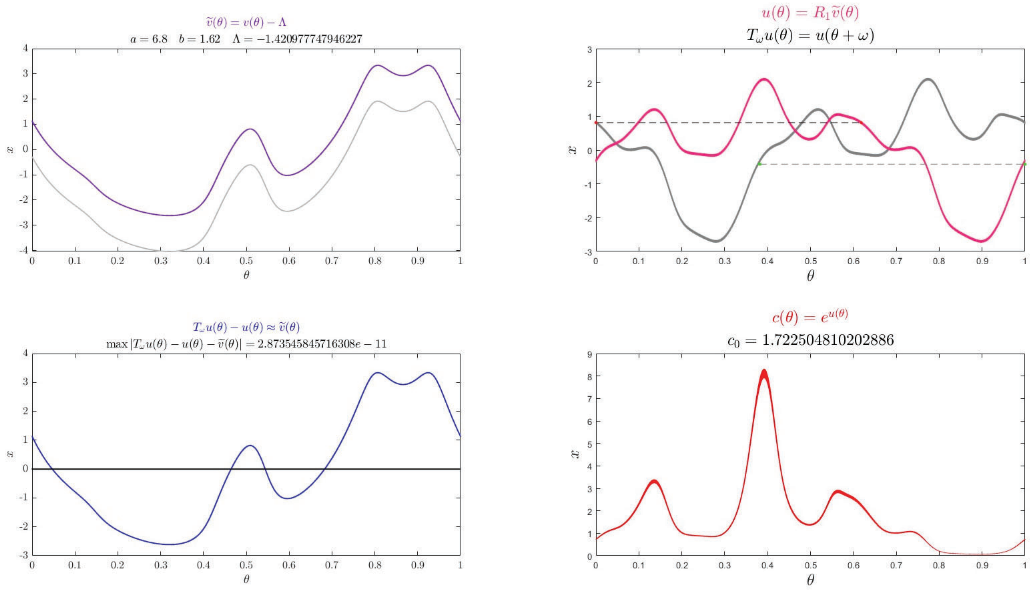


FIGURE II.15: Floquet transformation
$\widetilde{v}(\theta) = v(\theta) - \Lambda$ where $v(\theta) = \log(\frac{\partial f}{\partial x}(\theta, \kappa(\theta)))$ (top left);
$\mathfrak{R}_1\widetilde{v}(\theta)$ in magenta and $\mathfrak{R}_1\widetilde{v}(\theta+\omega)$ in gray (top right);
$\mathfrak{R}_1\widetilde{v}(\theta+\omega) - \mathfrak{R}_1\widetilde{v}(\theta) \approx \widetilde{v}(\theta)$ in blue (bottom left);
$c(\theta) = e^{\mathfrak{R}_1\widetilde{v}(\theta)}$ in red (bottom right).

II.4 KAM step

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function [Kappa,Tau,DET]=KAM_STEP(theta,kappa,tau,a,b,omega,p)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% [theta,kappa]=Orbit(theta_0,x_0,omega,a,b,N,N0); % 1.
% %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
m=df(kappa,a);                                     % 2.
v=log(m);
v=double(v);
Lambda=mean(v);                                    % 3.a
lambda=exp(Lambda);                                % 3.b
v_TILDE=v-Lambda;                                  % 4.
[u,~]=R(1,omega,theta,v_TILDE);                    % 5.
c=exp(u);                                          % 6.
Tc=Translation(c,omega);                           % 7.
eta=1./Tc;                                         % 8.
[E,~]=ERROR_FUNCTION(theta,kappa,omega,a,b,tau);
xi=E.*eta;                                         % 9.
eta_0=mean(eta);                                   % 10.a
xi_0=mean(xi);                                     % 10.b
c_0=mean(c);                                       % 11.a
c_TILDE=c-c_0;                                     % 11.b
eta_TILDE=eta-eta_0;                               % 12.a
[Reta,~]=R(lambda,omega,theta,eta_TILDE);          % 12.b
xi_TILDE=xi-xi_0;                                  % 13.a
[Rxi,~]=R(lambda,omega,theta,xi_TILDE);            % 13.b
ETA=c_TILDE.*Reta;                                 % 14.a
ETA_0=mean(ETA);                                   % 14.b
XI=c_TILDE.*Rxi;                                   % 15.a
XI_0=mean(XI);                                     % 15.b
e=mean(kappa)-p;                                   % 17.
A=[1-lambda,-eta_0 ; c_0,ETA_0 ];                  % 18.
B=[xi_0; e-XI_0];                                  % 19.
DET=det(A);                                        % 20.
S=B'/A';                                           % 21.
sigma_0=S(1);                                      % 22.a
Delta_tau=S(2);                                    % 22.b
sigma_TILDE=R(lambda,omega,theta,xi_TILDE+eta_TILDE*Delta_tau); % 23.
sigma=sigma_TILDE+sigma_0;                                      % 24.
Kappa=kappa+c.*sigma;                                           % 25.
Tau=tau+Delta_tau;                                              % 26.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function [Rv,W]=R(lambda,omega,t,v)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% This function computes the solution u to the cohomological equation
% u(theta+omega)-lambda u(theta)= v~(theta), theta\in T=R/Z,
% by approximation of its Fourier series, where v=v^_0+v~, i.e. v~ is the
% (zero-average) oscillatory part of the given function v.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
N=length(t); % N must be even
M=N/2;
V=fft(v);
W=V(1:M+1)/N; % W=[v^_0,v^_1,...,v^_M] are fue Fourier coefficients of v
W(1)=0; % W=[0,v^_1,...,v^_M] Fourier coefficients of v~
for k=1:M
    W(k+1)=W(k+1)/(exp(2*pi*k*omega*1i)-lambda);
end % W=[0,u^_1,...,u^_M] Fourier coefficients of the solution u=R_lambda v
Rv=IDFT_APPROX(W);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

Unstable invariant curve

n	$< \kappa_n >$	Λ_n	λ_n	$\|E_n\| = \sup_{\theta \in \mathbb{T}} \|E_n(\theta)\|$	$\|e_n(p)\| = \|< \kappa_n > -p\|$
0	$4.76471550321e - 10$	1.7811320816	5.93657330125	$2.31443690966e - 08$	0
1	$4.76471591067e - 10$	1.78113208167	5.93657330164	$6.44256392132e - 15$	$4.07464057767e - 17$

II.5 Bifurcations

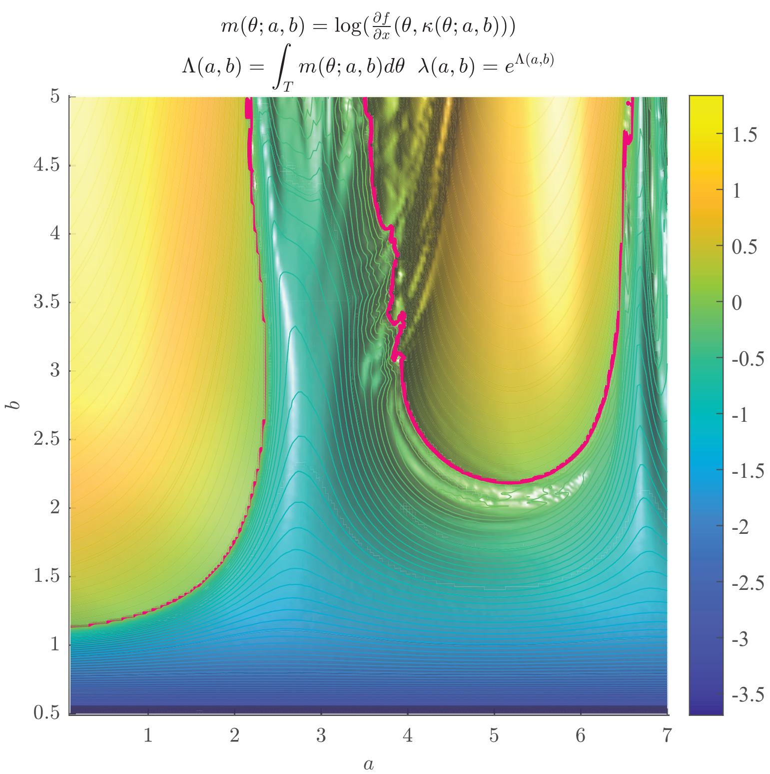


FIGURE II.16: Lyapunov exponents on the parameter space, based on the backward orbits of the origin.
In red those points with zero Lyapunov exponent.

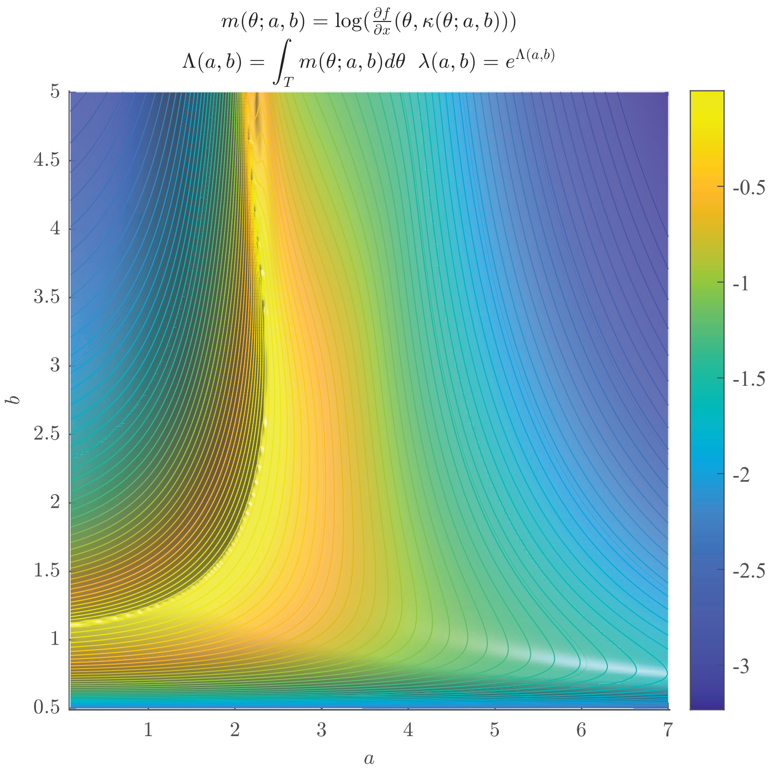


FIGURE II.17: Lyapunov exponents on the parameter space, based on the forward orbits of the origin.

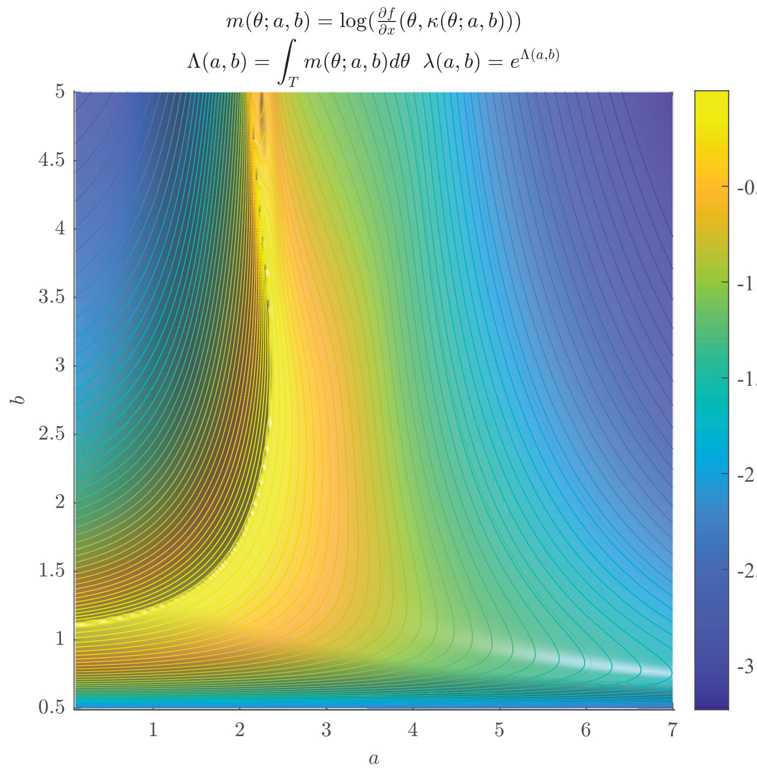


FIGURE II.18: Lyapunov exponents on the parameter space, based on the forward orbits of the point $(0, 1)$.

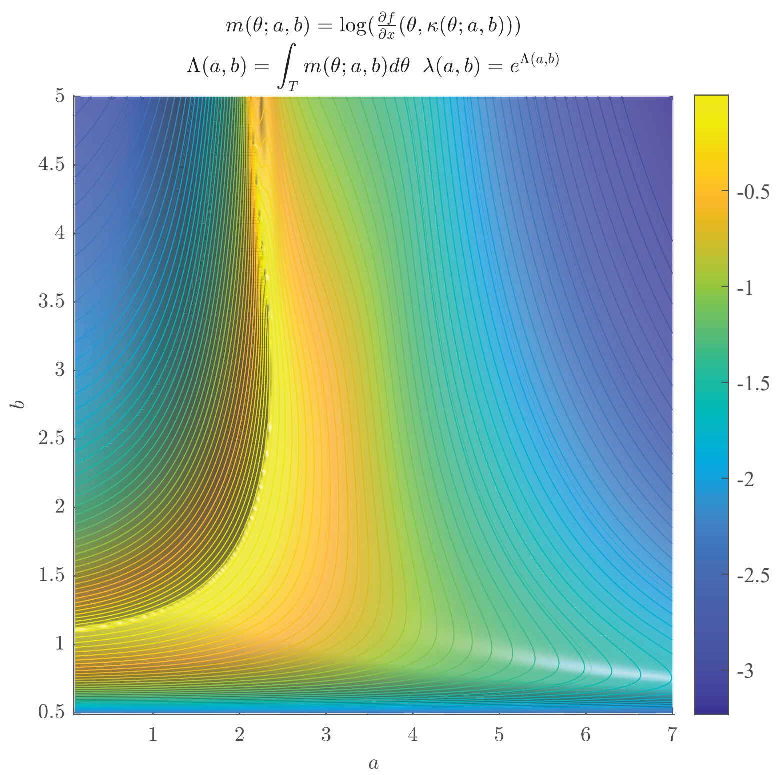


FIGURE II.19: Lyapunov exponents on the parameter space, based on the forward orbits of the point $(0, -1)$.

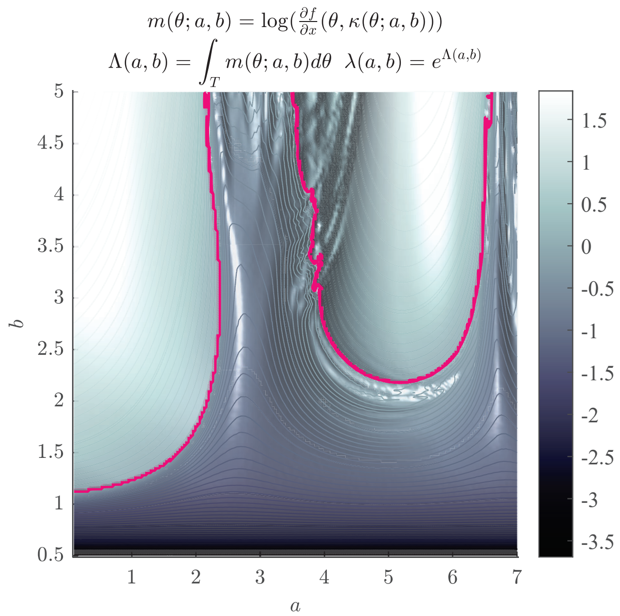


FIGURE II.20: Lyapunov exponents on the parameter space, based on the backward orbits of the origin.
In red those points with zero Lyapunov exponent.

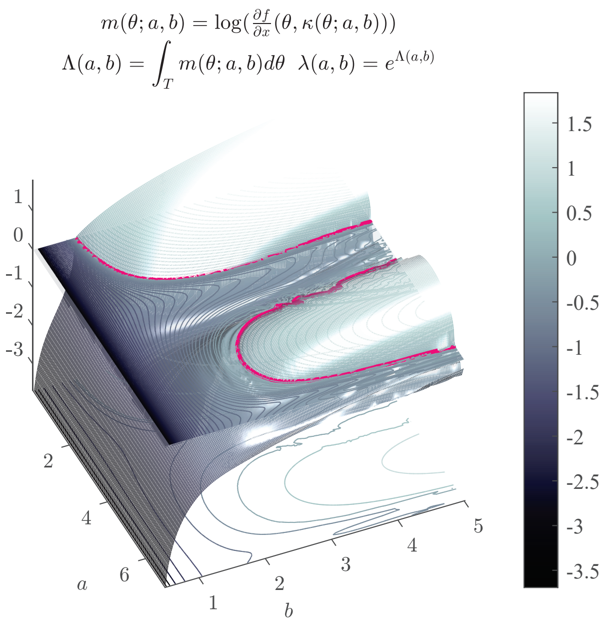


FIGURE II.21: Lyapunov exponents on the parameter space, based on the backward orbits of the origin.
In red those points with zero Lyapunov exponent.

Appendix III. Matrix condition numbers and estimates

III.1 Matrix condition numbers and estimates

The following lemma is often used to obtain several estimates needed in the KAM procedure.

Lemma III.1 Matrix condition numbers and estimates

Let $M \in \mathrm{GL}_n$*, i.e. a square invertible matrix* $(\det(M) \neq 0)$.
Assume that $\|M\| < \sigma_M$, $\|M^{-1}\| < \sigma_{M^{-1}}$ *and let* $M_1 = M + \Delta M$*, with* $\Delta M \in \mathcal{M}_n$.
If

$$\frac{\sigma_{M^{-1}}^2 \|\Delta M\|}{\sigma_{M^{-1} - \|M^{-1}\|}} < 1 \ , \tag{H}$$

then the following properties hold:

(a) $M_1 \in \mathrm{GL}_n$;

(b) $\|M_1^{-1}\| < \sigma_{M^{-1}}$.

(c) $\|M_1^{-1} - M^{-1}\| \leq \sigma_{M^{-1}}^2 \|\Delta M\|$.

Proof.

First, observe that the approximating matrix can be written as

$$M_1 = M + \Delta M = M(I + M^{-1}\Delta M) = M(I - A), \text{ where } A = -M^{-1}\Delta M \ ,$$

and consider the so–called Neumann series $S = \sum_{k=0}^{\infty} A^k$.
If $\|A\| < 1$, then S is normally convergent
Denoting $S_k = \sum_{l=0}^{k} A^l$, the sequence of partial sums of S, then $S_k(I - A) = I - A^{k+1}$, $k \in \mathbb{N}$.
Taking limits as $k \to \infty$, we get $S(I - A) = I$.
In such a case, $S \in \mathrm{GL}_n$, $I - A \in \mathrm{GL}_n$ and $M_1 = M(I - A) \in \mathrm{GL}_n$.
Furthermore, $(I - A)^{-1} = S$ and $M_1^{-1} = (I - A)^{-1}M^{-1} = SM^{-1}$.

(a) It is enough to see that $\|A\| < 1$.

From the hypothesis (H) we obtain

$$\sigma_{M^{-1}}^2 \ \|\Delta M\| < \sigma_{M^{-1}} - \|M^{-1}\| \Rightarrow \sigma_{M^{-1}}\|\Delta M\| < \frac{1}{\sigma_{M^{-1}}}(\sigma_{M^{-1}} - \|M^{-1}\|) = 1 - \frac{\|M^{-1}\|}{\sigma_{M^{-1}}} < 1,$$

since $\|M^{-1}\| < \sigma_{M^{-1}}$. Therefore, $\|A\| = \|M^{-1}\Delta M\| \leq \|M^{-1}\| \ \|\Delta M\| \leq \sigma_{M^{-1}}\|\Delta M\| < 1$.

(b) Again, from the hypothesis (H), we also have $\|M^{-1}\|\dfrac{\sigma_{M^{-1}}\|\Delta M\|}{\sigma_{M^{-1}-\|M^{-1}\|}} < 1.$

From this, we obtain, $\|M^{-1}\| < \sigma_{M^{-1}}(1 - \|M^{-1}\|\,\|\Delta M\|)$ and finally $\|M_1^{-1}\| = \|SM^{-1}\| \le \|S\|\|M^{-1}\| \le \frac{1}{1-\|A\|}\|M^{-1}\| \le \frac{\|M^{-1}\|}{1-\|M^{-1}\|\|\Delta M\|} < \sigma_{M^{-1}}.$

(c) The difference of the inverses can be expressed as

$$\begin{aligned} M_1^{-1} - M^{-1} &= (I - M^{-1}M_1)M_1^{-1} = (M^{-1}M - M^{-1}M_1)M_1^{-1} \\ &= M^{-1}(M - M_1)M_1^{-1} = -M^{-1}\Delta M M_1^{-1}. \end{aligned}$$

Then, $\|M_1^{-1} - M^{-1}\| \le \|M^{-1}\|\|\Delta M\|\|M_1^{-1}\| \le \sigma^2_{M^{-1}}\|\Delta M\|$.

□

Remark III.2

In fact,

$$\frac{\sigma^2_{M^{-1}}\|\Delta M\|}{\sigma_{M^{-1}-\|M^{-1}\|}} < 1 \;\Rightarrow \|A\| \le \frac{1}{4}\,,$$

and consequently,

$$\frac{\|\Delta M\|}{\|M\|} \le \frac{1}{4}.$$

Proof.

$$\frac{\sigma^2_{M^{-1}}\|\Delta M\|}{\sigma_{M^{-1}-\|M^{-1}\|}} < 1 \;\Leftrightarrow \|\Delta M\| < \frac{1}{\sigma_{M^{-1}}}\left(1 - \frac{\|M^{-1}\|}{\sigma_{M^{-1}}}\right)\,.$$

On the one hand,

$$\frac{\|\Delta M\|}{\|M\|} = \frac{\|MM^{-1}\Delta M\|}{\|M\|} \le \|M^{-1}\Delta M\| = \|A\|.$$

On the other hand,

$$\|A\| \le \|M^{-1}\|\,\|\Delta M\| \le \frac{\|M^{-1}\|}{\sigma_{M^{-1}}}\left(1 - \frac{\|M^{-1}\|}{\sigma_{M^{-1}}}\right) = t_M(1 - t_M)\,.$$

where $t_M = \frac{\|M^{-1}\|}{\sigma_{M^{-1}}} \in (0,1)$.
Calling $h(t) = t(1-t)$ with $t \in (0,1)$, we can see that h has an absolute maximum value at $t = \frac{1}{2}$.
Therefore, $\|A\| \le h(t_M) \le h(1/2) = 1/4$. □

Index

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