UNIVERSITAT DE BARCELONA

Philosophy of Vagueness: A Topological Perspective

Nasim Mahoozi

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Supervisors

Dr. Thomas Mormann

Dr. José Martínez Fernández

Tutor

Dr. Manuel García Carpintero

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Rumi

While the mind sees only boundaries, love knows the secret way there

To my parents,

the typical cases of generosity, kindness and tolerance whose love and support have accompanied me in each moment of this long journey

Abstract

This dissertation is a contribution to the application of topology to the philosophical problem of vagueness. We pursue two main goals: The first goal is to give an account of the main features of vague concepts. In our proposed account, vague concepts that are structured around typical cases (called poles), are boundaryless and have borderline cases. We show that this account of vague concepts can be used to deal with the Sorites paradox and higher-order vagueness. The second goal is to provide a topological model for vague concepts in a conceptual space based on previous works by Ian Rumfitt and Thomas Mormann. We use this topological model to show how one can keep the two truth-valued semantics of classical logic while still reject the principle of Bivalence. While our main concern is to give an account of vagueness, Rumfitt cares about classical logic that has been threatened by vagueness, because it shakens the firm wall between the extensions of concepts. We share the idea with him that the principle of Bivalence does not hold, yet disagree with him in accepting the third truth value.

After the introduction, in Part II, through a literature review of some existing theories of vagueness, we settle what is expected from a theory of vagueness. In Part III, we review the fundamental notions of topology to show how they can be fruitfully applied to better understand the structure of vague concepts. Part IV consists of three sections. Sections 5 and 6 are dedicated to a critical analysis of two other topological proposals, namely the Kantian model by Boniolo and Valentini and the topological approach of Weber and Colyvan which presents a continuous version of the Sorites paradox. Section 7 is a critical review of a prominent geometrical framework in cognitive science, namely Gärdenfors' conceptual spaces, in which concepts are represented at a conceptual level. Conceptual spaces will be the base of our account. We discuss its pros and cons and following the recent works by Mormann on polar spaces, we show that conceptual spaces need a topological structure to be optimized. None of the previous views can answer or even aimed at answering all the questions relating to vagueness and finding a solution to the mentioned problems. Part V introduces a model for vagueness based on weakly scattered T_0 Alexandroff spaces. Alexandroff spaces have a tight relation to modal logic and applications in computer science and image processing, among other fields. The model is a refinement and expansion of Rumfitt's topological model and Mormann's generalization of it. In order to make it apt to deal with the phenomenon of vagueness, we improve the model to a 3-layer-model by taking a closer look at the previous topological models to reveal their hidden properties and deficiencies. This new model reveals three layers in a concept: the first layer contains the typical cases of the concept, the second layer contains the almost typical cases of the concept and the third layer the borderline cases of the concept. The extension of a concept contains the typical and almost typical cases, i.e., it consists of the first two layers. These layers were hidden in the previous models. Then, we define the notions of borderline case and similarity relations in this model and we use them to explain in detail Rumfitt's solution to the Sorites and sharp boundary paradoxes. The solution is based on the rejection of the tolerance principle in its strict sense. We propose a weak version of tolerance that holds in our model. We accept truth-value gaps but, pace Rumfitt, we do not accept a third alethic truth value. After that, we deal with the problem of higher-order vagueness and compare the proposed model to some of the dominant theories of vagueness. We end up with some suggestions to improve the model to overcome its limitations and to be able to answer further questions on vagueness.

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> I do not enjoy writing at all. If I can turn my back on an idea, out there in the dark, if I can avoid opening the door to it, I won't even reach for a pencil. But once in a while there's a great dynamite-burst of flying glass and brick and splinters through the front wall and somebody stalks over the rubble, seizes me by the throat and gently says, "I will not let you go until you set me, in words, on paper." That's how I met Illusions.
>
> Richard Bach, Illusions

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Part I

Introduction

> Any theory[of vagueness] needs to provide an account of borderline cases and apparently fuzzy boundaries; and this is closely tied up with the task of identifying the logic and semantics for a vague language. A theory must also tackle the sorites paradox. In addition we may expect it to deliver a criterion of vagueness.
>
> (Keefe and Smith 1997, p.6)

This dissertation is a contribution to the application of topology to some philosophical problems of vagueness. We pursue two main goals: The first goal is to give an account of the main features of vague concepts. The second goal is to provide a topological model for vague concepts in a conceptual space based on previous works by Ian Rumfitt and Thomas Mormann. In fact, the model is a refinement of the previous models to make it apt to model vagueness and deal with the phenomenon of vagueness. We share the idea with Rumfitt that the principle of Bivalence does not hold, yet disagree with him in accepting the third truth value.

Despite the fact that the literature on vagueness is extremely rich, hardly we can find a geometrical or topological approach to deal with the phenomena of vagueness. The connection between logic, algebra and topology is intricate and well-known. Therefore, the main idea is to investigate how topology can shed light into relevant questions concerning vague concepts and to see whether the topological semantics opens up a place for vagueness within the realm of classical logic. The thesis is a navigation in search of a suitable topological space in which vagueness can be defined and to answer the questions that any theory of vagueness should answer: How do we define vagueness? How do we define borderline cases? What is a blurred boundary? What are the main features of vagueness and which one(s) is salient? What is the source of vagueness? Why is the principle of tolerance, according to which vague concepts are tolerant to small changes, so appealing to us? Why is it problematic to accept this principle? How do we formulate the Sorites paradox, is it a valid argument or not and how do we deal with it? Is there higher-order vagueness? If so, how do we explain it? Our proposed account can be considered as a continuation of the idea of conceptual spaces proposed by Peter Gärdenfors in 1980s. We pursue the idea that to optimize a conceptual space we should endow it with a topological structure. To that end, we propose a layered model of conceptual space and define vagueness, tolerance principle and similarity relation.

The 3-layer-model is heavily indebted to Rumfitt's topological account of vagueness, presented in chapter 8 of his book "The boundary stones of thought" and its generalization that is recently proposed by Mormann(2020, 2021). In our proposed account, vague concepts that are structured around typical cases (called poles), are both boundaryless and have borderline cases. We shall show that this account of vague concepts can be used to deal with the Sorites paradox and higher-order vagueness. We use this topological model to show how one can keep the two truth-valued semantics of classical logic while still reject the principle of Bivalence. While our main concern is to give an account of vagueness, Rumfitt cares about classical logic that has been threatened by vagueness.

To pursue our goal, we shall understand the problem, analyse it from different angles and see in what ways other approaches to vagueness deal with the problem and what difficulties they are faced with. This is done in Part II, Vagueness.

0.1 Vagueness

Let us take a look at the phenomenon of vagueness through an example. Once upon a time there was a girl who measured 1.68-meter, Magi, who weighed 90 kilos and everybody told her that she was fat. She desperately went on a diet and lost 10 kilos. Good effort! But she got sick because of the hard diet. However, she was still considered as a fat girl. She decided to lose more weight but in a healthy way. Let us suppose that she went on a diet based on which every month she could lose half a kilo which means that every day she should have lost 16 grams. Also, suppose that for her record, she took a picture of herself every day in the same place and position. After 5 years she became a well-known model and started writing a guide book on how to lose weight without any bad effects. For 5 years, constantly, she had been losing 16 grams every day. She decided to hold an exhibition of her photos. She faded out her face. Below each picture there were three buttons, non-fat, fat and a white one for no position. The visitors were asked to press at least one of the buttons of each picture. Losing 16 grams seems nothing. If you weigh 80 kilos and you are fat, with 16 grams less you are still fat. In such series of pictures one can hardly say that Magi had lost weight from one day to the next .

But after a year the difference becomes clear and after 5 years she is obviously a non-fat thin girl. When did this change happen? When did she cease to be a fat girl? Is there a last day in which she was fat, such that in the next day she was no longer fat? Everything being equal, did everybody press the fat or non-fat buttons or some people did press the white button? If there are some borderline cases that have received the white button, then can we say that there is no sharp boundary between fat Magi and non-fat Magi? If we were omniscient, could we know which day was the last day in which Magi was fat?
In philosophy, there is a huge literature on how to answer the above questions. 'Fat' is a vague predicate. It seems that there are some borderline cases. Indeed, we are surrounded by many vague concepts. Predicates such as 'bald', 'hairy', 'red', 'tall', 'short', 'thin' are just some examples of vague predicates. According to the classical account of concepts, the extension of a predicate 'fat' is a set of all people who instantiate fatness. Furthermore, vagueness is not limited to predicates; nouns, singular terms, verbs, adjectives and adverbs also can be vague. Considering that there are borderline cases, one might think that although we cannot divide the pictures into two groups, the ones in which Magi is fat and the ones in which Magi is thin, we may divide the data into three; i.e., we add the pictures that received the white button into the third group of borderline cases. Even if it is possible, it does not solve the problem. It just adds one group more. The concept of boundary can also be vague. Again one may ask whether there is a sharp boundary between the pictures in which Magi is clearly fat and the borderline cases and whether there is a boundary between the pictures in which Magi is clearly non-fat and the ones in which she is a borderline case. This story can be continued, saying that 'borderline borderline', 'borderline borderline borderline', ... are also vague. This phenomenon is called higher-order vagueness. As Varzi (2001) says:

> We could distinguish between borderline cases and borderline borderline cases, or borderline borderline borderline cases, but things would only get worse... And this is a serious problem because it gives rise to a genuine logical puzzle. (Varzi 2001, p.136)

Here is the puzzle:

1. Magi in the first picture is fat

2. If Magi is fat in the first picture, she is still fat in the next picture (losing 16 grams).
From 1 and 2 and applying modus ponens after five years Magi is still fat. That is a result that many of us intuitively want to deny. This is called the Sorites paradox. Later, we will explain it in detail. It is widely accepted that one of the features of vagueness is being susceptible to the Sorites paradox.
Lack of a sharp boundary and having borderline cases are considered as two other features of vagueness. In the series of Magi's pictures there are some in which Magi is neither fat nor not fat but there is no single picture at which we pinpoint to say that from this one on she is not fat anymore. Most philosophers believe that vague concepts have blurred boundary or even do not have any boundary at all. Actually, as Williamson (1994) mentioned, the word 'vagueness' was "appreciated as a term of art for the phenomenon of blurred boundaries, which results in Sorites susceptibility" (Williamson 1994, p.36).

Another feature that is widely accepted, even by the ones for whom vague concepts have a sharp boundary, is that vague concepts have borderline cases. The ones that neither definitely belong to the extension of a concept nor definitely do not belong to it. As we will explain later a vague predicate might have just some of the features.
Higher-order vagueness is usually understood in terms of borderline cases. There is no consensus on the existence of vagueness of higher orders. Koons (1994) claims that there is just first-order vagueness. Wright (2010) also puts doubt on the existence of higher-order vagueness. For him, higher-order vagueness is an illusion. We shall explain briefly different views on higher-order vagueness.

Most commentators discuss the sources of vagueness. Does it stem from our inadequate representation of the world? Is it rooted in our ignorance and lack of knowledge? Is the world itself vague?
Epistemicists believe that there is a sharp borderline between tall people and not tall people. There is an exact day in which Magi is fat such that the next day she is non-fat. But we do not know where this borderline is. In their view, borderline cases arise because we do not know where exactly the borderline is (if there is one at all). We cannot definitely divide pictures in which Magi is fat and the rest because we cannot definitely know when Magi turns into a

non-fat girl (Williamson 1994).

On the contrary, the ones who find the sources of vagueness in the world or our language and our representation of the world share the idea that there is no such a sharp borderline. This lack of determinacy is either due to our 'loose talk'; for example, we have not defined exactly when somebody is fat, or it stems from the world itself that is vague (Van Inwagen 1988, Tye 1990, Barnes 2010).

Among these three views, the latter is less popular partly because it is hard to explain in what way the world itself is vague (cf. Keefe 2000b, Hawley 2001, Sainsbury 1996).

The most accepted view is the semantic one. Semanticists claim that vagueness is due to semantic indecision. A word does not have a unique semantic value, its referent is not unique (Hawley 2001, p.103). For example, 'Magi is fat' is vague because 'is fat' does not have a unique semantic value.

Another ancient problem is that vagueness is a big problem for classical logic. Frege, for example, wanted to eliminate vagueness to get rid of the heavy shadow of vagueness on classical logic. For him, concepts should have been precise with a sharp boundary. The possibility of eliminating vagueness completely from logic is doubtful, but even if it were possible, we would have a very limited language since natural language is replete with vague concepts. As Michael Dummett puts it:

> The great difficulty in discussing vagueness is that vagueness resembles dust, soot or sand: It gets into everything (Dummett 1995, p.207).

Why should vagueness be eliminated from classical logic? Why is it such a serious threat? Traditionally, since Aristotle, three laws of logic have been considered as laws of thought: The law of identity according to which everything is the same as itself. The law of non-contradiction which states that a proposition P and its negation cannot have the same truth value and finally, the law of excluded middle according to which a proposition or its negation is true. In classical logic, the proposition 'Magi is fat' is either true or false. The problem is that if Magi neither belongs to the set of fat people nor to the set of non-fat people, then this proposition cannot be either true or false and therefore that law is violated. Another problem is related to the law of excluded middle. 'Magi is fat' or 'Magi is not fat' is always true in classical logic. Again vagueness is problematic because it violates this principle since it opens up places in the middle of the extreme cases of fat and not fat. In a borderline case, ‘Magi is fat’ might be neither true nor false.

The incompatibility of vagueness with classical logic has convinced or motivated many philosophers to appeal to non-classical logics such as 3-valued logic, fuzzy logic, intuitionistic logic and probability logic, to name a few. Nevertheless, some philosophers like Sorensen (1988); Williamson (1994); Raffman (1994) and Rumfitt (2015) proposed an account of vagueness, compatible with classical logic.

Maybe philosophers are right in that vagueness threatens 2-valued logic, but does appealing to finite or infinite valued logic solve the problem? Many attempts have failed up to now. This brings to mind that maybe we should look at the problem from another perspective. There is an alternative to this maneuver. Considering concepts in a structural space may be an appropriate approach. This is actually not new at all. The inadequacy of set theory in distinguishing the natural concepts is quite well-known. Goodman(1954), for example, in his book “fact, fiction, and forecast ” showed that all predicates are not equally projectible, however set theory cannot make a difference between natural concepts like blue and non-natural ones such as grue(being blue until a certain time t and then after that green) and bleen (being blue until a certain time t and then after that green). As “grue” and “bleen” can be explained in terms of “green” and “blue” , “green” and “blue” can be explained equally well by “grue” and “bleen” and a temporal term. So, quantitativeness cannot distinguish between projective and non-projective predicates. In other words, it is not possible to logically distinguish those predicates and hence we cannot know which one is suitable to be used in induction. One of the solutions is to appeal to natural kinds, which is entangled with the notion of similarity. The notion of similarity plays an important role in the discussions on vagueness and particularly, the Sorites paradox. We shall explain how similarity is defined in conceptual spaces as well as Rumfitt’s topological account and shall propose our definition of similarity in the 3-layer topological model of conceptual spaces.

Topology has barely used in philosophical discussions. But we firmly believe that it can be a very useful tool to be used in philosophical debates. Hopefully, the current investigation plays a small role in bringing it to the core and show that it is not as strange and difficult as it might seem. In Part III, Topology, we review the fundamental notions of topology to show how they can be fruitfully applied to better understand the structure of vague concepts.

0.2 Topology

Since long time ago, philosophers have used geometry. The well-know sentence engraved at the door of Plato's academy: "Let no one ignorant of geometry enter" [1] is a prime example of the importance of geometry to philosophers. Topology is usually considered as a generalization of geometry. It is often called rubber geometry; a geometry in which shapes remain invariant under certain deformations.

In analysis, mathematicians like Poincaré, Cantor, Peano, Jordan, Fréchet, Hausdorff, Kuratowski, Sierpiński and Alexandroff introduced topological notions.
Hilbert wrote in the preface of the book of Alexandroff, "Elementary concepts in topology":

> The following book is to be greeted as a welcome complement to my *Anschauliche Geometri[e]* on the side of topological systematization; may it win new friends for the science of geometry Alexandroff (1932).

The hope is that topology also finds new friends in philosophy. As Sierpiński mentions, one of the advantages of studying abstract spaces is that we have options to use suitable axioms in order to apply the theorems to other fields of mathematics:

> Theorems obtained for a given abstract space are true for each set of elements, which satisfies the axioms of that space; however, the set may also satisfy other axioms. Herein lies the practical advantage of the study of abstract spaces. For, with a suitable choice of axioms for such a space, the theorems obtained in that space may be applied to different branches of mathematics (Sierpinski 1934, p.iii)

Even more broadly, the theorems can be applied to other fields. Topology has been applied to psychology, physics, biology and also philosophy. This work is a continuation of emphasis on the important role of topology in philosophical debates, particularly, vagueness. We propose that a

[1] Μηδείς αγεωμέτρητος εισίτω μου την στέγην

suitable topological space is useful to be applied to vagueness in particular. Topological spaces may provide an appropriate ground to define closeness and borderline cases as we intuitively understand via topological notions such as open, closed, and neighborhood. These essential notions in topology may make topology a good candidate to be considered as a philosophical tool to deal with vagueness. We propose that the right topological structure can provide an account of vagueness that does not raise the Sorites paradox.
There have been some attempts in philosophy, especially in epistemology to use topology. There are plenty of papers on belief and knowledge where the authors define belief using topological notions(Özgün 2013, Baltag et al. 2013, 2015). This seems quite natural, considering that epistemic logic is entangled with modal logic, interpreting box operator(it is necessary that) as "we know that". It is quite well-known since the work of McKinsey and Tarski (1944) that if in modal logic we interpret box as interior operator and its dual, diamond, as the closure operator, then the class of all topological spaces is defined by modal logic S4.
The hurdle, of course, will be to show the relation between the mathematical concepts and philosophical concepts that are related to vagueness that we will deal with it. Furthermore, one should find a suitable space for vagueness.

In psychology, Piaget sees the close relation between the first stages of children's cognitive intellectual developments and Bourbaki's mother structures, namely algebraic, order and topological structures. About the latter in particular, he points out that:

> There are operations that yield classes not in terms of resemblances and differences but in terms of "neighborhood ", "continuity "and "boundaries ", It is remarkable that, psychogenetically, topological structures antedate metric and projective structures, that psychogenesis inverts the historical development of geometry ...! These facts seem to suggest that the mother structures of the Bourbaki correspond to coordinations that are necessary to all intellectual activity. (Piaget 1970, pp.26-27)

Kurt Lewin in 1936 noticed that the notion of space goes far beyond the physical spaces and metric spaces such as Euclidean ones. One of the very first applications of topology goes back to his work. In "Principles of topological psychology" he applied topology to cognitive psychology in the hope that the topology- that was quite a young discipline back then- "make psychology a real science" (Lewin 1936, Preface). In his book he applied topology to psychology. He made a difference between things and regions and defined the notion of boundary between two regions (ibid, pp.116-121).

The geometric and topological approach to vagueness has been neglected to the large extent by philosophers in comparison to other approaches.

Although, in the main sources of vagueness in philosophy one cannot find a detailed application of topology on vagueness, there have been some attempts to use topological notions to deal with the problems of vagueness since 1980s.

As far as we know, the first trace of explicit topological approach to vagueness goes back to Walther Kindt's paper "Two approaches to vagueness: theory of interaction and topology", published in 1983. In this paper in the absence of convincing semantics to handle the phenomena of vagueness , he proposes a topological semantics to model the properties of vagueness via topological tools:

> Developing a theory of vagueness is one of the central aims of present semantics ... using a topological framework will be fruitful for handling some of the problems of vagueness unsolved up to now (Kindt 1983, pp. 361-362)

He proposes a topology based on prototypes as the " centre(maximum) " of a predicate. According to him, in order to decide whether something is red or someone is fat one should know the prototypes of 'red' or 'fat'. The prototypes of a predicate can be defined differently in coarser or finer scale. In a color scale where just ground colors exist, the difference between red and orange fades a way (Kindt 1983, p. 381).

Kindt introduces a topology that is compatible with fuzzy logic.

We will skip his view and will discuss in detail three more recent topological approaches to vagueness. Part IV, contains three sections. Sections 5 and 6 are dedicated to a critical analysis of two other topological proposals, namely the Kantian model by Boniolo and Valentini and the topological approach of Weber and Colyvan to present a continuous version of the Sorites paradox. Section 7 is a critical review of a prominent geometrical framework in cognitive science, namely Gärdenfors' conceptual spaces in which concepts are represented at a conceptual level. Conceptual spaces will be the base of our account. For that reason, we devote a separate subsection here.

0.3 Two topological approaches to vagueness

The first paper that is critically reviewed is proposed by Boniolo and Valentini (2008) under the name of "Vagueness, Kant and topology: A study of formal epistemology". They will use a kind of formal concept analysis, coined by Rudolf Wille in 1980, which was built on the mathematical theory of lattices and ordered sets. We shall give a mathematical reformulation of their account and critically analyze it.

The second one is proposed by Bueno and Colyvan under the name of : "Topological sorites". According to the authors, the main feature of vagueness is being Sorites susceptible. We shall see whether they give us enough information about the topological space in which vagueness is defined and how they formalize the continuous Sorites paradox. We shall give a critical analysis of this view.

0.4 Conceptual spaces: a geometrical/topological approach

The conceptual space approach is a geometrical account of concepts. It has a huge empirical support and application in different fields Zenker and Gärdenfors (2015). For example, Gärdenfors (1990) proposes a new solution to Goodman's "riddle of induction". His solution is based on what he calls "conceptual spaces" as a non-linguistic cognitive entity that has some quality dimensions like color, height, weight and time. These quality dimensions are endowed with a kind of topological or metrical structure(ibid, p.84). So, the main suggestion has been to turn the attention from the extensional logic to conceptual spaces that are structured. It is in these geometrical structured spaces that natural predicates become distinct. Gärdenfors claims that the problem of induction is a problem of knowledge representation and to analyze inductive inferences, logical tools in themselves are not sufficient.

> [I]n order to separate projectible predicates from non-projectible ones, we need a way of representing knowledge that goes beyond logic and language. (ibid, p.79)

In Gärdenfors (2000) he developed the idea of conceptual spaces. Section 7 is devoted to vagueness in geometrical conceptual spaces. In this approach elements of logic, geometry and cognitive science and philosophy are engaged. For Gärdenfors thought is not simply a mirror of logic; it can be explained geometrically. A natural concept is represented by a convex region in a geometrical topological conceptual space. Without going into detail, convexity here means that if two objects in a conceptual space such as the Euclidean space have the same property, all the objects between them also have that property. He takes for granted the notion of betweenness. The notion of similarity can be defined based on the distance function in the geometric metric space (see 7.2.3). The conceptual space approach has been used in artificial intelligence(AI) and robotics (Gärdenfors 2014). In artificial intelligence there is a lot of literature on the problem of knowledge representation. According to Gärdenfors, knowledge can be represented in terms of conceptual spaces in which projectible properties can be distinguished. This approach lies between two well-known approaches in AI, namely symbolic AI and connectionism.
The conceptual space approach was not originally proposed to be used to deal with vague concepts. It was first proposed by Gärdenfors as an alternative semantic theory. Gärdenfors endorsed that "the *meaning* of an expression is primarily determined by its relation to a conceptual space" (Gärdenfors 1988, p.26). In Gärdenfors (2000) he very briefly deals with the notion of vagueness. Later, the proponents of this view, such as Douven and Decock, in couple of papers expanded it to apply it to the philosophical issues such as dealing with the phenomenon of vagueness(See, for example, Douven et al. (2013), Douven (2016) and Decock and Douven (2015)).

In Decock and Douven (2015) the authors show that Gärdenfors' conceptual spaces that is mostly used in the fields of cognitive science and psychology, can be a useful tool for philosophers as well. Gärdenfors' use of topology is quite limited. Though he explicitly contends that the space can be non-Euclidean, he seems to stick to the Euclidean space and betweenness is conceived as betweenness in the Euclidean metric space. For example, by one metric, the points in the straight line that connects two points are between those points. Different points may be between those two points in another metric such as Manhattan metric(see figure 1). So, convexity is tightly related to the metric of the space.

Mormann (2021) argues that if he considers a topological structure, he does not need to focus that much on a metric. In other words, space can go beyond the Euclidean space. The basic elements of Euclidean spaces are collection of points that satisfy certain relations. Euclidean spaces are metric spaces. Topological spaces go beyond metric spaces. They are closely related to convex spaces and are more general and may encompass various subjects.

The three papers have one thing in common: the authors simply focus on Euclidean spaces. We follow Rumfitt and Mormann in going beyond the Euclidean space. In part V, we shall find a suitable topological conceptual space for vagueness.

0.5 Polar topological approach and its generalization

Our main focus will be on the recent topological approaches that has turned it into a hot topic in the area of vagueness.

Recently, Rumfitt (2015) has given a theory of vagueness for the predicates that are associated with a pole, the prototypical cases of that. A pole can be considered as typical exemplar of a predicate or a prototype. For example, sparrow is a typical example of 'bird' but penguin is not. This was considered by Gärdenfors as well in defining conceptual spaces. In vagueness we deal with the concept of approximation in the sense that if x has the property F and y approximates x, then y should have the property F or a property very similar to it.

This can be formulated topologically. In the polar space, when y approximates the pole of a concept p, it either has p as its unique pole or it has p and possible other poles.

Rumfitt defends classical logic without endorsing the principle of Bivalence by defining the extension of a predicate via some topological notions. This approach also formalizes what Sainsbury called boundarylessness according to which, roughly speaking, vague predicates do not have boundaries. For Sainsbury, the boundary exclusively is the sharp boundary and there is no such boundary for vague predicates. Rumfitt proposes a new(topological) semantics for classical logic that permits vague concepts. Unlike some philosophers such as Raffman and Fara whose theory of vagueness keeps classical logic, the topological theory of vagueness will provide a context endowed with a topological space in which the principle of bivalence does not hold.

As mentioned before, this thesis endeavors to discuss an alternative perspective towards vagueness which is topologic. We try to show that vague concepts can be defined in topological spaces. As Grosholz says:

> A topological space might be considered as an interpretation, a lattice of truth values, for logic (Grosholz 2007, p.269).

In topology, sets are endowed with certain structure. Finding the suitable topology on a set is not easy at all but worth trying. Rumfitt proposes a suitable topology, named polar topology. We shall first reformulate it. Then, we shall show in detail that topology is a very special kind of topology, namely T_0 Alexandroff and show what properties it has. Our reformulation is based on Mormann's(2020, 2021). We also follow him to give technical details of polar T_0 Alexandroff spaces and its generalization to T_0 weakly scattered Alexandroff spaces. In the next step, we shall critically analyze these topological models of vagueness when it comes to answering the questions related to vagueness and problems that should be tackled. We shall discuss the philosophical advantages of this account of vagueness in comparison with the other accounts of vagueness existing in the literature. Also, we shall discuss its limitations and drawbacks. Then we shall improve the model to make it more apt to deal with the phenomenon of vagueness. We shall define the relative notion of boundary. Then, we shall discuss how to formulate the similarity relation as the indistinguishability relation, how to use that to define a weaker version of the tolerance principle and how to deal with the hierarchical higher-order vagueness. Also, we shall compare the optimized model to some other theories of vagueness, discuss their similarities and differences and what makes this model a considerable alternative to other accounts of vagueness.

Part II

Vagueness in Philosophy

1 The nature of vagueness

1.1 Phenomena of vagueness

> [E]very proposition that can be framed in practice has a certain degree of vagueness.
>
> (Russell 1923)

In natural languages the phenomenon of vagueness is pervasive. We are surrounded by vague predicates such as ‘tall’, ‘bald’, ‘red’, ‘big’ and ‘tadpole’.
Russell mentions that not only predicates but also words in pure logic such as ‘or’, ‘if’, proper names, quantifiers such as ‘almost’, intensifiers such as ‘very’, ... are vague. So, a theory of vagueness should be general enough to encapsulate all of them. Russell claims that it is a representation (language, pictures, maps) that is vague or precise. According to him, there is no vagueness and precision beyond the representation (Russell 1923).
In contrast, some philosophers believe that vagueness is non-representational, it is language and mind-independent.

In this thesis we concentrate our attention on vague concepts and propositions that contain vague predicates. So, we are not claiming that we propose a theory of vagueness in its general sense to deal with all bearers of vagueness. Rather, we will present some topological approaches to vagueness. Then we discuss their pros and cons and following that, we will provide a suitable topological space in which vague concepts can be defined and we will lastly explain the phenomena of vagueness within that topological framework. At least we hope that the topological approach to vagueness will be the adequate one for vague expressions in language and thought.

To start off, let us review the main features of vagueness. There is a huge amount of literature on the features of vagueness. Generally, four main characteristics are mentioned for vague predicates:

1. Lacking sharp boundary

2. Having borderline cases

3. Being tolerant

4. Being susceptible to Sorites paradox.

In the following subsections we briefly explain them. [2]

1.1.1 Unsharp boundaries

One of the features of vague predicates is that apparently there is no sharp boundary between their positive and negative extensions. In other words, vague predicates, apparently, do not have well-defined extensions. So, it is said that vague predicates have fuzzy boundaries (Keefe 2000b).

For example, the predicate 'red' is vague in the sense that there is no sharp boundary between the things that are red and those that are not red. In the continuum of color spectrum from red to orange, it seems that there is no last point that is red such that its successors are not red. In the topological approach we consider such blurred boundaries. In a color spectrum a person can differentiate clear cases of red and clear cases of not red. Somewhere in the spectrum the change has occurred but they cannot pinpoint where exactly the change has happened. We will discuss it later in more detail what we mean by blurred boundaries. But what is wrong about the existence of concepts with blurred boundaries? According to Frege, concepts should have a sharp boundary. For Frege the extension of a concept is a set. It seems that if one accepts classical logic and the set theoretical account of concepts, then there will be no room for vague concepts. However, this classical view was questioned by philosophers and psychologists. For them concepts can be vague.

In classical logic, there is no room for fuzzy or blurred boundaries. Any proposition is either true or false. A vague proposition p (proposition that contains a vague predicate) apparently is neither true nor false because there are objects that neither belong to the set of objects that satisfy the predicate nor to the set of things that do not satisfy it.

The crux of the problem is that on the one hand, classical logic is, so to say, sacred. This simple, yet powerful and influential logic that guides our reasoning and plays a key role in

[2] For more detail see the classical books on vagueness: Keefe (2000b), Williamson (1994), Keefe and Smith (1997).

science and mathematics, traditionally was called "the one right logic" (Shapiro and Kouri Kissel (2021)). On the other hand, vagueness threatens classical logic and is so widespread that it cannot be easily put aside.

There are some philosophers who try to give a theory of vagueness based on the classical view of concepts in which the logic of vagueness is classical logic with its classical semantics. For them, every vague concept has an unknown sharp boundary (See subsubsection 1.2.2).

In defense of classical logic, there are some theories of vagueness in which vague concepts lack sharp boundaries, nevertheless this boundarylessness is reconcilable with classical logic and semantics. (Raffman (1996)).

Other proponents of classical logic, in dealing with such a hurdle, reject the classical view of concepts, contending that concepts can have no sharp boundaries. They try to provide non-classical semantics for the classical logic, though they contend that vague concepts lack a sharp boundary(See subsubsection 1.2.1).

In chapter V we will discuss in detail how Rumfitt(2015) introduces a topological non-classical semantics for the classical logic. As we will see, the extension of a concept can be a set endowed with a certain topological structure. Rumfitt intends to show that the topological structure provides some room for vague concepts within classical logic.

1.1.2 Borderline cases

If it is not clear whether a predicate applies to a certain case or not, that case is a borderline case. The more discussed definition of borderline cases is based on the modal connective 'definitely' that has different interpretations:
x is a borderline case of a predicate 'F' iff it is not definitely 'F' and it is not definitely not 'F' (Fine 1975, p.287).
For example, borderline cases of the predicate 'red' are the ones that are neither definitely red nor definitely not red. It is usually defined formally as:

Definition 1.1. *Let BD be the set of borderline cases of the predicate 'F' and 'Δ' an operator* *for 'it is definitely'. Then*

$$x \in BD(F) := \neg\Delta Fx \,\&\, \neg\Delta(\neg Fx). \qquad \text{(Standard boundary)}$$

The boundary of the vague predicate contains all borderline cases; those cases that are neither definitely F nor definitely not-F. Despite all disagreements on what vagueness is and what vagueness consists in, there is almost an agreement on this definition of borderline cases. (See, for example, Williamson (1994), Fine (1975), Bobzien (2010) and Bobzien (2010)) .
We will call BD, *the standard boundary.* In the next parts we will see other definitions of boundary and the comparison between them.

In general, there are different approaches towards what borderline cases are depending on an interpretation of 'definitely'.
According to the standard view, borderline cases of a predicate 'F' are the ones to which neither F nor not F are clearly applicable. For example, borderline cases of the predicate 'tall' are the ones that neither 'tall' nor 'not tall' are clearly applicable to them. A borderline case tall might be tall or might be not tall.
Michael Tye for example says:

> [The] concept of a borderline case is the concept of a case that is neither definitely in nor definitely out (Tye 1994a, p.18).

It is however good to mention that not all philosophers accept the definition of borderline cases via the modal operator 'clearly' or 'definitely'. The operators 'clearly' or 'determinately' or 'definitely' have different interpretations. For example, a is definitely red since there is a fact of the matter that it is red or because it definitely satisfies redness or because it is known that it is red or because it is red in all ways of making the vague predicate 'red' precise.[3] According to some philosophers, borderline cases of a predicate 'F' are the ones that are neither F nor not F. For example, a borderline case tall is neither tall nor not tall (Keefe 2000b).
This is different from the first definition in the sense that in the former, a borderline case may be 'F' or not 'F' but it is neither a clear case of 'F' nor a clear case of not 'F'. Epistemicists, for

[3] For a detailed discussion of the interpretation of the definitely operator see, for example, Greenough (2003), Keefe (2000b), Williamson (1994) and Zardini (2008).

instance, claim that something can be F or not F but we do not know whether it is F or not F. Considering these different interpretations, it is not clear what exactly borderline cases are. We will focus mostly on the standard definition and will introduce a topological interpretation of the operator 'determinately' in topological terms.

In spite of disagreements on the source of vagueness, borderline cases and boundary have been under consideration in defining vagueness. To witness, we mention some quotes from the main texts on vagueness.

Sorensen in the *Stanford Encyclopedia of Philosophy* mentions that the widely accepted fact that a vague concept has borderline cases turns the notion of a borderline case "crucial in accounts of vagueness" (Sorensen 2018).

Kit Fine defined vagueness as the following:

> More generally, a set is vague if it is not the case of every object that it either belongs or does not belong to the set (Fine 1975, p.285).

Even if a concept has a sharp boundary, it still can have borderline cases. For example, those who vindicate classical logic and contend that we cannot know where the sharp boundary is, endorse borderline cases, those to which we do not know whether a predicate applies or not. (Williamson 1994)

Varzi (2006) contends that whenever a concept has borderline cases it is vague. Dorr (2009), in his discussion about borderline cases says:

> The concept [of a borderline case] has its most basic application when we are faced with a question of the form 'is x F', but are unwilling to answer 'yes' or 'no' for a certain distinctive kind of reason. Wanting to be co-operative, we need to say something; by saying 'it's a borderline case' we excuse our failure to give a straightforward answer while conveying some information likely to be of interest to the questioner. Different views about this naturally lead to different answers to the question what it means to be a borderline case (Dorr 2009, p.550).

Sainsbury also claims that no matter what the borderline cases are, vague concepts have borderline cases:

> [T]heorists of all persuasions, if they believe that vagueness exists at all, should accept that there are borderline cases, objects which, when a vague expression is applied to them, are implicated in the phenomena of vagueness (whatever one thinks they are) (Sainsbury 1995, p.63).

He accepts that if a predicate is vague it has borderline cases but not vice versa. That is to say, non-vague predicates also can have borderline cases. He proposes that *vagueness is constituted in boundarylessness*; i.e, in not having a sharp boundary:

> [A] vague concept is boundaryless in that no boundary marks the things which fall under it from the things which do not, and no boundary marks the things which definitely fall under it from those which do not definitely do so; and so on. Manifestations are the unwillingness of knowing subjects to draw any such boundaries, the cognitive impossibility of identifying such boundaries, and the needlessness and even disutility of such boundaries (Sainsbury 1996, p.257).

This may seem to be in contrast with the traditional view of concepts according to which concepts necessarily have sharp boundaries; yet philosophers such as Daly (2011) try to accommodate boundarylessnes within the classical logic. Rumfitt (2015) also tries to achieve this aim.

If we follow Sainsbury in accepting boundarylessness, then the main task is to deal with what Wright (2016) calls "the characterization problem" saying that we must answer what the characterization of the borderline cases is. The task of finding out what is for something to be a borderline case is important since as Wright (2016) mentions, there are not many works on what borderline cases are and that is a gap to be filled. Rather, some philosophers attempted to propose new ways of confronting the paradoxes without giving philosophical discussions of what vagueness consists in. Wright mentions that:

> . . . these various kinds of view have not devoted the same degree of attention to elucidating and defending their (implicit) commitments concerning the nature of vagueness and borderline cases as they have devoted to the development of formal semantical theories, and to criticizing opposing views and attempting to address

> the paradoxes. Yet one would naturally suppose that the characterization problem should be a locus of developed discussion rather than one of presupposition. For until we have a properly argued account of what vagueness is, how can one possibly expect to know what kind of semantic theory for vague expressions might be best motivated, let alone how the most appropriate kind of semantic theory might assist with the disarming of the Sorites paradox and other problems? (Wright 2016, p.191)

He follows Schiffer in giving centrality to the characterization of borderline cases. Given such centrality role, we will try to give a new characterization of boundary and borderline cases from a topological point of view.

Recently, in defense of classical logic Rumfitt proposes a new approach in which he defines borderline cases in a topological space. We will explain this point in detail when we present Rumfitt's topological view of vagueness in which he provides topological semantics for the boundaryless account of vagueness that is surprisingly compatible with classical logic.

In his account classical logic holds but he gives a non-classical topological semantics for vagueness that is constituted in boundarylessness.

1.1.3 Tolerance

The principle of tolerance was coined by Crispin Wright:

> In these examples [infant, child, adult; red, orange; heap] we encounter the feature of a certain *tolerance* in the concepts respectively involved, a notion of a degree of change too small to make any difference, as it were (Wright 1975, p.333).

According to this principle, vague predicates are tolerant to very small changes. For example, very small changes do not have an impact on whether an amount of grains of sand is a heap or not, or whether someone is tall or not, or whether a person is fat or not. In Pagin's words:

> That a predicate is tolerant means roughly that it is insensitive to small differences.This lack of sensitivity can be expressed by so-called tolerance principles (Pagin 2017, p.3728).

Tolerance principle can be expressed in three classically equivalent ways:

a. If two things are similar in respect of F, then if F applies to one, it applies to the other one as well. Let us call it "Strict-TOLERANCE".

$$\forall x, y \in D\ (x \sim_F y \longrightarrow (Fx \longrightarrow Fy)) \qquad \text{(Strict-TOLERANCE)}$$

b. For all x and y in the domain of discourse,D, if x belongs to the extension of a predicate F and y is indistinguishable from x with respect to F, then y is also in the extension of F. Formally:

$$\forall x, y \in D\ (Fx \wedge x \sim_F y \longrightarrow Fy) \qquad \text{(}\forall\text{-TOLERANCE)}$$

c. It is not the case that there is an x, y in the domain of discourse such that x is F and y is not F. In other words, there is no sharp boundary. Formally:

$$\neg\exists x, y \in D\ (Fx \wedge x \sim_F y \wedge \neg Fy) \qquad \text{(}\exists\text{-TOLERANCE)}$$

As said before, if TOLERANCE[4] holds, then small changes will not change the applicability of a predicate. For example, if Peter is 200 cm and Robin is 199 cm, then Peter and Robin are similar with respect to the predicate 'tall'. So, it is quite intuitive that if Peter is tall, then Robin will be tall as well. However, the appealing and intuitive tolerance principle, surprisingly, has been rejected by many philosophers. The main reason is that in classical logic it raises a series of the so called Sorites paradoxes. In the next subsections we will explain what the Sorites paradox is and how philosophers deal with this. As we will see, TOLERANCE is at the heart of different formulations of the Sorites paradox. In most of the proposed solutions a weak version of TOLERANCE is proposed.

The rejection of TOLERANCE is also problematic because it says that vague concepts have a sharp boundary. The common belief is that vague concepts lack a sharp boundary. Facing

[4]Since the three presentations are equivalent in classical logic we use TOLERANCE whenever the difference among them is not important.

these problems, most philosophers often have intended to weaken the tolerance principle or limit it and revise the classical logic or(and) semantics.
In the following, after introducing different formulations of the Sorites paradox we will briefly explain some of the existing ways of dealing with Sorites paradoxes. In part V, we will discuss in detail the topological solution and will compare it to other views.

1.1.4 Sorites susceptibility

The name of the paradox comes from an ancient greek word σωρίτης , 'heaped up', from σωρός 'heap' or 'pile'. It is said that the Sorites paradox was proposed by Eubulides.[5] The classical example is that 'heap' gives rise to a Sorites paradox:

1.	10^4 grains of sand make a heap.	(Clear case)
2.	If n grains of sand make a heap, then $n-1$ grain of sands make a heap.	($\forall$-TOLERANCE)
3.	One grain of sand does not make heap.	(Clear non-case)
4.	One grain of sand makes a heap	

Intuitively 10^4 grains of sand do make a heap. But then as an instantiation of the $\forall$-TOLERANCE, 10^4-1 grains of sand also make a heap. By modus ponens, 10^4-1 grains of sand makes a heap. Again by 2 and modus ponens 10^4-1-1 makes a heap. Applying premise 2, and repeating the argument 10^4-2 times yields that one grain of sand makes a heap, which seems to be false.

There are different formulations of the Sorites paradox (Hyde, postnote; Dietz and Moruzzi, 2009).

Let X be a set. 'F' be a vague predicate(such as 'tall', 'red', 'heap'), $n \in \mathbb{N}$ and $A =< a_1, \ldots, a_n >$ be a series in which each element and its successor are similar with respect to F. This series is called Sorites series. $\forall\, 1 \leq i < n \; a_i \sim_F a_{i+1}$.

Mathematical Induction Sorites

[5]See Williamson (1994) for the history of the Sorites paradox.

1. Fa_1 (Clear case)
2. $\forall i(Fa_i \longrightarrow Fa_{i+1})$ ($\forall$-TOLERANCE)
3. $\neg Fa_n$ (Clear non-case)

---

4. $\forall n(Fa_n)$

In this form of the Sorites paradox we need universal instantiation and modus ponens to get the conclusion.

In another form of the Sorites the $\forall$-TOLERANCE is replaced by the $\exists$-TOLERANCE according to which there is no sharp boundary between the elements of the Sorites series that are F and the ones that are not F.

$\exists$-no sharp boundary

1.	$\neg\exists n(F\,a_n \wedge \neg F\,a_{n+1})$	(tolerance principle)
2.	Fa_1	(Clear case)
3.	$\neg Fa_i$	(Clear non-case)
4.	$F\,a_{i-1}$.	(Supposition).
5.	$F\,a_{i-1} \wedge \neg F\,a_i$.	($\wedge$ -introduction).
6.	$\exists n(F\,a_n \wedge \neg F\,a_{n+1})$.	($\exists$ -introduction) .
7.	$\neg Fa_{i-1}$.	$(1,4,6)$.
.		
.		
.		
8.	$\neg F\,a_1$.	(after $i-2$ times repeating the argument).
9.	$F\,a_1 \wedge \neg F\,a_1$.	$(2,8,\wedge$ - introduction) .
10.	$\neg\neg\exists n(F\,a_n \wedge \neg F\,a_{n+1})$.	$(1,2,9)$.
11.	$\exists n(F\,a_n \wedge \neg F\,a_{n+1})$	(10, double negation elimination).

In part V we will mostly focus on this form of the Sorites paradox to elaborate how Rumfitt (2015) dissolved this form of the paradox.(See subsection 8.4.)

Another formulation is the one in which in order to drive the conclusion from the premises just the rule of modus ponens is needed. It is called **Conditional Sorites**.

Consider the predicate 'F' and the sorites series mentioned above.

Conditional Sorites

1.	Fa_1	(Clear case)
2.	if Fa_1, then Fa_2	(tolerance principle)
3.	Fa_2	(1.,2., Modus Ponens)
4.	if Fa_2, then Fa_3	(tolerance principle)
5.	Fa_3	(3., 4., Modus Ponens)
.	.	.
.	.	.
.	.	.
2n-3.	Fa_{n-1}	(2n-5, 2n-4, Modus Ponens)
2n-2	if Fa_{n-1}, then Fa_n	(tolerance principle)
2n-1	$\neg Fa_n$	
2n.	Fa_n	(2n-3, 2n-2, Modus ponens)

Horgan proposed another version of the Sorites paradox, namely the forced march Sorites, that shows how deep the problem is. According to him this formulation is nearer to the original one. It shows that even if different interpretations of logical operators and quantifiers blocks the conditional, universal or existential Sorites paradox, the forced-march paradox still will be a hurdle (Horgan (1994)).

Forced march Sorites

Consider the Sorites series from a_1 to a_n and the predicate 'F'. A person is forced to march down and decide whether a_i is F or not. For example, consider a series of pictures of adult men whose heights range from 2 meters to 1.50 and the heights of each two adjacent members differ 1cm. Now the interrogator starts asking you questions and you should answer to all questions. Is person number 1 tall? Is person number 2 tall? ... Is person number n tall? We may start confidently with one answer, say yes. But as we march down somewhere we lose our confidence. This point may differ depending on many factors such as seeing the elements of the series all at once or one by one.[6] The point is that at some point we change the answer. As soon as we do that, we are forced to commit to the existence of a cut-off point between the tall persons and not-tall persons and in general, between the Fs and not Fs. However, it seems that there is no such a cut-off point in the Sorites series. If we do not change our answer, on the other hand, then someone with 150cm is considered tall! So, we are confronting a dilemma that cannot be solved simply by revising the classical logic and semantics or appealing to non-classical logic. In fact, the forced-march Sorites paradox demonstrates that the problem lies somewhere out of the realm of logic. More profoundly, the problem is rooted in explaining why it is so intuitive that there is no cut-off point in a Sorites series (Priest (2003);Horgan (1994)).

In the literature, we can find very different approaches to dissolve the Sorites paradox. Before introducing them let us see what vagueness stems from.

1.2 Sources of vagueness

In the literature, three main sources for vagueness are mentioned. Vagueness is rooted in the imperfection of our language or in our lack of knowledge or in the world.

1.2.1 Semantic vagueness

The proponents of semantic vagueness endorse that vagueness is due to our language. Supervaluationism is one of the most accepted semantic approaches. According to supervaluationists, there are different admissible ways of making a vague predicate precise but none of them has the priority over the others. Even if we were an omniscient being, and the world were precise, we could not semantically decide between these precisifications. Vagueness is due to this semantic indecision. Keefe, one of the most prominent defenders of supervaluationism says:

> No definite extension is settled to be the extension of a vague predicate such as ‘rich’ or ‘tall’ . . . , rather there is a range of possible extensions, and it is semantically unsettled

[6]For some thought experiments see Kamp and Monnich (1981); Egré et al. (2013)

> which is the extension. The various possible extensions correspond to different ways of making the predicate precise and the supervaluationist idea is that truth conditions involve quantifying over all those ways of making language precise (Keefe 2008, p.315).

Lewis also defended the semantic view as:

> If Fred is a borderline case of baldness, the sentence "Fred is bald" may have no determinate truth value. Whether it is true depends on where you draw the line. Relative to some perfectly reasonable ways of drawing a **precise boundary** between bald and not-bald, the sentence is true. Relative to other delineations, no less reasonable, it is false. Nothing in our use of language makes one of these delineations right and all the others wrong. We cannot pick a delineation once and for all (not if we are interested in natural language), but must consider the entire range of reasonable delineations (Lewis 1979, pp.351-352).

As Lewis contends, there is not a precise boundary but we have precise boundaries relative to ways of making a predicate precise and the truth value of a proposition can change relative to each precisification. If a proposition is true in all precisifications it is **super-true**, if it is false in all precisifications it is **super-false**, otherwise, it is neither true nor false. Russell has a looser condition for truth. According to him, a proposition is (more or less)true if "it is true over a large enough part of the range of delineations of its vagueness."(Lewis 1979, p.352) .

1.2.2 Epistemic vagueness

Epistemicists claim that even if the semantics of our language were precise, still there is vagueness since vagueness is rooted in our lack of knowledge. There is a sharp boundary but we do not know where it is ((Sorensen 1988);(Williamson 1994);(Schiffer 1998)). In this view, the rules of classical logic hold. For each proposition either it or its negation is true. Furthermore, a proposition such as " Magi is fat" is either true or false. So, bivalence holds. Yet it does not preclude the phenomenon of vagueness because in borderline cases we do not know whether it is true or false, though in fact it is either true or false. Williamson (1994) claims that there is a fact of the matter about where the boundaries are, we are just ignorant of it. More precisely, he sticks to the classical logic and contends that vague predicates have well-defined extensions of which we are ignorant. This ignorance is due to the fact that our knowledge is inexact (Williamson 1994, p.216). Inexact knowledge is rooted in the fact that indiscriminability is non-transitive. When there is inexact knowledge, it seems reasonable to assume some margin for error principle.

Margin for error principle:

MEP : A is true in all cases similar to cases in which '*it is known that A*' is true (Williamson 1994, p.227).

When it comes to vagueness, the operator *I know that* is to be replaced by *It is clear that* and likewise, *I do not know* by *it is not clear that.*

1.2.3 Ontic vagueness

According to the proponents of ontic vagueness, vagueness is in the world. Even if we were omniscient beings and our language were precise, still we would have vagueness. To put it in another way, vagueness is not a feature of thought and language. Rather it is a feature of the world. Objects as well as properties can be vague. Kilimanjaro is vague since it has fuzzy boundaries. It is not clear where it starts and where it ends (Tye (1990)).

The very well-known argument against ontic vagueness was proposed by Evans in his one-page article, published in 1978 where he provided an argument that ontic vagueness is inconsistent and that pace their idea, there is no vague object. Nevertheless, that influential argument has not knocked down ontic vagueness. Today there are still proponents of ontic vagueness (Barnes 2010, Tye 1990, Akiba 2004 .

We will not delve deeply into this view of vagueness. In the topological approach our concern is not worldly vague objects and properties. For the defense of ontic vagueness see for example: Tye (1990), Barnes (2010) and Keefe and Smith (1997).

1.3 Higher-Order vagueness

> The fact is that all words are attributable without doubt over a certain area, but become questionable within a penumbra, outside which they are again certainly not attributable. Someone might seek to obtain precision in the use of words by saying that no word is to be applied in the penumbra, but unfortunately the penumbra is itself not accurately definable, and all the vagueness which apply to the primary use of words apply also when we try to fix a limit to their indubitable applicability.
>
> (Russell 1923, pp.63-64)

In the literature there are two main paradigms of higher-order vagueness, hierarchical higher-order vagueness and columnar higher-order vagueness. The former has a long history in the discussions on vagueness while the latter, recently, has been proposed and discussed in a series of papers by Bobzien (See Bobzien (2010, 2011, 2012, 2013, 2015)).

Hierarchical higher-order vagueness has been characterized differently based on the focus on some mentioned features of vagueness. The most discussed characterization of hierarchical approach is what Raffman (2009) calls "higher-order borderline cases".

In this view, in the Sorites series when we march from definitely F to definitely not F, there is no sharp line between things that are definitely F and the ones that are borderline F. As Russell(1923) mentioned, even though there is a penumbral(borderline) area, its border with definitely F area and definitely not F area is also blurred.

In contrast to the hierarchical view in which the area of borderline cases decreases at each order, in the columnar account of higher order vagueness, that area remains the same in all orders. Something that is borderline F will always remain borderline and something that is definitely F will always remain definitely F, no matter in which order we are.

Any theory of vagueness should explain higher-order vagueness and should consider higher-order vagueness when it deals with the Sorites paradox.

In the following subsections we will explain these two paradigms of higher-order vagueness in turn. Then, we will explain some proposed solutions to the Sorites paradox and we will see their difficulties in explaining higher-order vagueness. In part V, we will discuss higher-order vagueness from a topological approach and will discuss whether it can be apt to model columnar and higher-order vagueness and whether the topological solution to the Sorites paradox can better deal with higher-order vagueness.

1.3.1 Hierarchical higher-order vagueness

The received view is that there is a hierarchy of higher-order vagueness. In this view, borderline cases themselves have borderline cases. In other words, 'boundary' is vague. There are clear cases of the borderline cases, clear cases of not borderline cases and borderline cases of borderline cases.

In dealing with a vague predicate 'fat', we may be able to divide people into obese(definitely fat) ones, skinny(definitely not fat) ones and borderline fat ones;the ones who are neither definitely fat nor definitely non-fat. But again 'borderline fat' may be vague. So, we can continue the story mentioned in the introduction. In the process of losing weight, Magi is clearly fat at first, clearly borderline fat in another stage and borderline borderline fat in another;i.e., there might be some stages in which Magi is neither definitely borderline fat nor definitely not borderline fat. This classification can be continued. The first one is called 1st-order vagueness, the second one is called 2nd-order vagueness and in general, the nth iteration is called nth-order vagueness. This is the common idea of higher-order vagueness, called hierarchical view, that, as it can be seen, is mostly based on the view that vague predicates have borderline cases. In this account the iteration of boundaries form a pyramid.The extension of the boundary decreases as we move upwards in the pyramid.

The idea that 'borderline case', 'borderline borderline case',... are vague can be traced back to Russell(1923) who argued that borderline area (penumbra) do not have a sharp boundary. The hierarchical view has received more attention in the late 20th century by works of Sainsbury (1991) and Wright (1992).

In the literature, philosophers have presented different higher-order vagueness paradoxes. They show that higher-order vagueness is incoherent and contradictory (Sainsbury (1991)).

The Sorites paradox raises at higher orders as well. For example, if Magi is borderline fat and loses one gram, she won't become a not-fat girl. But if we march down the Sorites series, a 30-kg-girl will not be not-fat. However, the proposed solutions of the Sorites paradox usually refer to the first order and cannot be equally generalized to solve the Sorites at higher orders. In the next subsection we will explain it in detail.

In the literature on vagueness, higher-order vagueness is tightly related to operators 'definitely', 'determinately' or 'clearly'. Just as there is no sharp boundary between the cases that are determinately F and the cases that are determinately not F, there is no sharp boundary between the cases that are definitely F and the ones that are borderline F, or between the cases that are definitely not F and the borderline cases. For example, 'F' is first order vague; i.e, where δ denotes the operator 'definitely', 'determinately' or 'clearly', $BD(F) = \neg \vartriangle F \wedge \neg \vartriangle \neg F$. It may be second order vague;i.e, $BD^2(F) = \neg \vartriangle BD(F) \wedge \neg \vartriangle \neg BD(F)$. The hierarchical view of higher-order vagueness is widely accepted. However, it has been problematic for many accounts of vagueness, especially those that endorse non-classical logic. The reason is that even if they consider more possible borderlines for a vague predicate, they should again account for a new Sorites paradox at a higher order. As Williamson (1994) claims, they have not been successful in giving an account for higher-order vagueness.

Higher-order vagueness also is problematic for the theories of vagueness in which TOLERANCE is the main feature of vagueness (see, for example, Wright (1975); Kamp and Monnich (1981); Greenough (2003) and Pagin (2017)). The higher-order vagueness again is problematic because the tolerance principle holds for borderline cases as well. There is no sharp boundary between determinate cases of F and borderline cases of F, and between borderline cases of F and determinate cases of not F. If there were a sharp boundary, then TOLERANCE would not hold at least for two adjacent members in the series that are located in two sides of the sharp boundary.

In dealing with hierarchical higher-order vagueness, -regardless of whether tolerance is a constitutive feature of vagueness or the existence of borderline cases- philosophers have come up with different strategies; denying its existence, limiting the orders to a finite number, putting certain restrictions or considering it as a serious problem to be dealt with.

One theory may deny the genuineness of higher-order vagueness showing that higher-order vagueness is just an illusion (Wright 2010). Raffman (2013) denies the existence of higher-order vagueness. According to her, ordinary speakers do not use vagueness of higher orders.

> I have not heard an ordinary speaker call something a borderline case of a borderline case, much less a borderline borderline borderline etc. case. In fact I doubt that ordinary speakers would make much sense of the idea (Raffman 2013, p.61).

Another theory of vagueness might endorse that the orders will be finished at some finite order. For example, Burgess claims that there is no order of vagueness higher than two or "higher-order vagueness terminates at a fairly low order". (Burgess 1990, p.427)

Others endorse that if there are borderline cases in the first order, then there will be borderline cases at all infinite number of orders. This is called radical higher-order vagueness. Keefe (2000b) defends the radical higher-order vagueness. In Williamson (1999) he demonstrates that if there is second-order vagueness, there will be higher-order vagueness as well. According to him, any theory of vagueness should explain higher-order vagueness, but apart from the epistemic account, the theories of vagueness have not been successful in fulfilling this aim. (See also Williamson (1994))
Sorensen (2010) argues that 'vague' is vague, in the sense that borderline cases have borderline borderline cases, borderline borderline cases,.... Following Sorensen, Hyde (1994, p.39) also claims that there is radical higher-order vagueness and that it is a real problem, yet unproblematic, having the right conception of borderline cases.
In Tye (1994b) the author criticises Hyde's argument and gives an argument for the denial of higher-order vagueness (Tye 1994b, p.44) .

Sainsbury (1991) shows that the hierarchical higher-order vagueness leads to an incoherence within the classical conception of concepts according to which concepts must have sharp boundaries.

> This notion[higher-order vagueness] is essentially a manifestation of the grip which classical conception has upon us, a grip which I am trying to loosen. (Sainsbury 1991, p.168)

Not to cling to the classical view of concepts, he proposes to drop off the classificatory role of concepts that, in its classical sense, leads to the "problem of higher-order vagueness". Instead, he suggests that concepts are boundaryless. Concepts classify but they do not mark any sharp boundary. According to him, for boundaryless concepts there is "no real higher-order vagueness"(Sainsbury 1991, p.179).
As mentioned before, Rumfitt (2015) proposes topological semantics for boundarylessness in Sainsbury's sense. In part V we discuss whether the topological semantics provides a right account for hierarchical higher-order vagueness.

1.3.2 Columnar higher-order vagueness

Bobzien proposed another approach to higher-order vagueness, called "Columnar higher-order vagueness", according to which whenever something is a borderline case of a vague predicate 'F', it is borderline borderline F, etc. She contends that this view is immune to the higher-order paradox. For example, if Magi is borderline fat, then she will be borderline borderline fat, borderline borderline borderline fat, etc. If she is clearly fat, then she will be clearly clearly fat, etc. So, the iteration does not affect the positive, negative and borderline extensions of a predicate.

Like Sainsbury, she defends the absence of a sharp boundary between the cases that are clear cases of a predicate and those that are not. According to her, there is higher-order vagueness. In fact, "vagueness is higher-order vagueness"(Bobzien 2015, p.61). However, she does not accept the hierarchical view. Instead, according to her, the extension of borderline cases, clear cases of F and clear cases of not F at all orders completely overlap and so, there will be columns of clear cases, borderline cases of F and clear cases of not F.
Bobzien (2013) distinguishes between two kinds of borderlinness, classificatory and epistemic.

According to the former, if Magi is identified as neither clearly fat nor clearly not fat, she can be classified as a borderline case. Based on the epistemic borderlinness, Magi is borderline fat only if it is not possible to know that she is fat and it is not possible to know that she is not fat. Epistemically, the possibility operator functions differently. This differentiate it from the classificatory account. For example, epistemically, if Magi is a borderline case, she can be both borderline fat and fat. Classificatorily, if Magi is a borderline case of fat, she is not fat and she is not not-fat. So, the epistemic account which considers the epistemic limits of human knowledge, unlike the classificatory account, is compatible with the classical logic. Bobzien appeals to this incapability of knowing where the sharp borderline is located, to dissolve the Sorites paradox (Bobzien 2013, pp.7-10).

Bobzien (2015) argues that ignoring such a distinction makes us argue for the existence of clear borderline cases. She introduces a new logical structure of borderlineness. It is this logic that does not permit us to identify an object in a clear category. (See section V for discussion on her proposed logic and its comparison with the topological account.)

As mentioned before,in the columnar account of higher-order vagueness, if something is a borderline case it is a borderline borderline ,... and if it is a clear case it is radically clear, i.e., it is clearly clearly the case. So, in this view, there is neither clear borderline cases nor borderline clear cases (Bobzien 2015, p.63). Something that is borderline F always remains borderline F and something that is clearly F always remains so.[7] According to this view, the borderline cases of a vague predicate such as 'red' form a column since extensionally, there is just one kind of borderline cases such that "each borderline case is radically higher-order or radically borderline" (Bobzien 2015, p.63). In other words, vagueness of a first-order coincides with the higher-order vagueness. Bobzien contends that, unlike most of the hierarchical views of vagueness, her view does not lead to the Sorites paradoxes.

[7]See Keefe (2015) for criticisms of this view. She claims that for any theory of vagueness to be coherent, the presence of clear borderline cases and borderline clear cases is necessary.

Later, in part V, following Mormann (2020), we will show that Rumfitt's polar topological approach to vagueness models Bobzien's columnar view and we will discuss whether we can reconcile the generalized topological approach with the hierarchical view.

2 Different ways of dealing with the Sorites paradox

To dissolve the Sorites paradox, philosophers have proposed a large number of theories. Some accept the consequence as true contradictions or reject the reasoning by denying the rule of modus ponens (Priest (2002, 2017)). However, the majority reject one of the premises, esp. the tolerance principle(see page 24) for different reasons.
The denial of the tolerance principle has created difficulties for the philosophers who are inclined to keep classical logic yet deny that vague predicates have a sharp boundary.
Some other approaches keep the principle of tolerance. van Rooij, for example, proposes that "the tolerance principle is valid with respect to a natural notion of truth and consequence. What we should give up is that this notion of consequence is transitive "(Van Rooij et al. 2010, p.206) .
Rumfitt's topological view, however, keeps classical logic, accepts the tolerance principle in the sense of the boundarylessness of vague predicates but rejects Strict-TOLERANCE and gives a solution to the Sorites paradox without denying the transitivity condition (See part V). In the rest of this part, we will introduce some solutions to the Sorites paradox, grouping them based on whether they keep classical logic or not. The ones that keep classical logic are divided into two groups: the ones that also keep classical semantics and the ones that propose a new semantics for the classical logic. Since the topological account keeps classical logic we will focus on such theories and just briefly explain the non-classical logic approaches.

2.1 Classical solutions to the Sorites paradox

To set the stage, let us review the main principles of classical logic and semantics.

In classical logic, among other principles, there are two fundamental principles: the law of excluded middle and the principle of bivalence .

Definition 2.1. ***The law of excluded middle:*** *(LEM)* $\models F \vee \neg F$.
In other words, the disjunction of F *and its negation is logically true.*

Definition 2.2. ***Principle of bivalence:*** *Each proposition 'F' is either true or false.*

Classical logic is a two-valued logic. A proposition that contains borderline cases, seems to lack a truth value. In classical logic a proposition such as ‘Peter is tall’ is either true or false. But as we saw before, Peter might be a borderline case in the sense that he neither does belong to the clear cases of tall nor to the clear cases of not tall. It seems that vagueness puts classical logic in danger. According to Frege, predicates that do not have sharp boundaries must remain out of the logical discourse. However, on the one hand, vagueness is a pervasive phenomenon in natural language and its denial would affect a huge part of natural language. On the other hand, abandoning classical logic is not appreciated by some philosophers. According to them, there is no need to deviate from classical logic which is simple and well-established. Not to detach from a big part of natural language by denying vagueness, they propose theories in which vagueness can be explained within the classical logic discourse. For example, Williamson (1994) argues that none of the non-classical approaches were successful in explaining the features of vagueness. The epistemic view of vagueness keeps the classical logic and provides an account of vagueness in which there is a sharp line between the positive and negative extensions of a predicate (cf. (Keefe 2000b, Williamson 1994)). Fara (2000) argues to the same line. Raffman (1994, 1996) claims that her contextualist approach reconciles classical logic and semantics with vagueness. Rumfitt, however, keeps classical logic but rejects the existence of a sharp line, proposing a non-classical semantics. We will discuss in some more details the suggested solutions of the Sorites paradox that stick to the classical logic in the sense that they accept the law of excluded middle. In this sense, supervaluationism also keeps classical logic. We will call them “Classical solutions”.

We start off by epistemicism that keeps both classical logic and semantics. In the following subsection we focus on mathematical induction Sorites.

2.1.1 Epistemicism

Epistemicists reject the tolerance principle (major premise). Considering that the argument is valid in classical logic and that the two other premises, clear case and clear non-case, are quite undoubtable, the tolerance principle is more vulnerable to be rejected. So far, so good. However, in classical logic the negation of the tolerance principle means that there is a sharp cut-off between the cases that are F and the ones that are not F. Epistemicists accept that there is a sharp boundary but we do not know where it is. We are ignorant of the truth-value of vague statements, though they are either true or false. So, there is no need to revise classical logic and semantics. This ignorance exists in higher orders as well. So, at each order there are sharp boundaries between clear cases and borderline cases of F but we are ignorant of them.

Williamson (1994) weakens the tolerance principle in the following way:

if we *know that* x is F and x is similar to y (denoted by $x \sim y$), then y is F. In this way, there is no guarantee that if y is F and $y \sim z$, then z is F.

Advantages: The main advantage of this view is that it keeps classical logic and semantics and it can homogeneously explain higher-order vagueness. In other words, just like we do not know where the boundary between tall people and not tall people is, we do not and cannot know where the boundary between clear cases of tall and borderline cases of tall lies. The same argument applies to higher orders.

Disadvantages: It is quite unintuitive that vague predicates have sharp boundaries yet we do not and cannot know where they are. But even if it were intuitive, epistemicists need to explain what ignorance is. Furthermore, they need to explain what is the difference between vague and non-vague predicates. According to them, both of them have sharp boundaries. The question is why we are ignorant about one and not about the other. What is the difference between the precise predicates such as 'prime', 'even', 'higher than 178' and vague predicates such as 'tall', 'fat', ...?

Williamson (1994) responds to these criticisms.[8] According to him, it is exactly our ignorance that explains why it seems unintuitive to us that there is a sharp boundary but we cannot know where it is. For him, we have already seen a lot of examples of mathematical truths that are metaphysically necessary, yet ,"for all we know",can be unknowable. For example, Goldbach's conjecture:

> Consider mathematical truths. They are all metaphysically necessary; there is no presumption that they are all knowable. A standard example is Goldbach's Conjecture, which says that every even number greater than 2 is the sum of two prime numbers. The Conjecture has been neither proved nor refuted; for all we know, no humanly intelligible method of argument can decide it one way or the other... For all we know, Goldbach's Conjecture is a humanly unknowable, metaphysically necessary truth. Vague truths can be in that position too. It is integral to the epistemic view that metaphysically necessary claims like 'Everyone with physical measurements m is thin' can be as unknowable as physically contingent ones like 'TW is thin' (Williamson 1994, p.204).

We think that this analogy is not adequate because in the case of Goldbach's conjecture, it may be

[8] In the current work we are not going to defend or reject a specific theory of vagueness. Here, we just briefly go through different solutions to the Sorites paradox. Then, in chapter V we compare the topological solution to these solutions.

unknowable or may be knowable. In some years, it may happen that someone proves it. However, a vague predicate, according to epistemicists, is unknowable. The unknowability is the main feature of vagueness. 'tall' is vague because it has sharp boundaries that are unknown to us. No intelligible argument may help us to know where the boundary is. In contrast, we have seen some seemingly unknowable conjectures that were proved by an intelligible argument. For example, Fermat's conjecture according to which no three positive integers a, b, and c satisfy the equation $a^n + b^n = c^n$, for any integer n greater than 2. This, of course, does not mean that epistemicism is wrong. It is just to say that to defend this view, better analogies are needed.
The idea that vagueness is constituted by our ignorance has been targeted to criticism. Sorensen (2001), as a prominent epistemicist, believes that such an ignorance explains why we are ignorant about the truth-value of vague predicates. In the appendix of the book "vagueness", Williamson gives a logic for ignorance, as the real essence of the phenomenon of vagueness. Mormann (2020) shows that in this logic vague concepts are columnar (See part V).

An objection is that we are ignorant about many things that are not vague. So, Epistemicists need to explain what is the difference between these kinds of ignorance. Shapiro (2006) says:

> Of course, it will not do to characterize vagueness as ignorance, even for the epistemicist, since there are plenty of things we are ignorant of— and plenty of things that we are necessarily ignorant of— other than borderline cases of vague predicates. Moreover, there is no consensus that ignorance is necessary for vagueness (Shapiro 2006, p.3).

2.1.2 Supervaluationism

Like epistemicists, supervaluationists reject the second premise of the Sorites paradox, namely the tolerance principle. However, unlike epistemicists, they reject the principle of bivalence. So, for them LEM does not imply that there is a sharp boundary.
The intuition behind supervaluationism, the dominant view of vagueness, is that vague propositions can be precisified in different ways. In other words, there are many equally good admissible precisifications (or specifications in Fine's terms) of a vague predicate and we are not able to decide between them. For Fine the admissibility is a primitive notion (Fine 1975, p.272). Not all precisifactions are admissible. They need to satisfy certain requirements. For example, the clear cases of a vague predicate, F, should be in its extension. For example, someone whose height is more than 2 meters in the

(normal condition) should be considered tall. The classical logic holds in all precisifications. Every proposition in a certain precisification is either true or false. So, someone who is tall cannot be not-tall as well.

Supervaluationists, however, reject the principle of bivalence. They accept truth-value gaps. A proposition is super-true if it is true on all precisifications. Likewise, a proposition is super-false if it is false on all precisifications. A vague proposition ‘P’ is neither super-true nor super-false. It is not super-true(super-false) since it is not true(false) in all ways of making the vague predicate precise. Suppose that Peter is 1.8 meters. ‘Peter is tall’ is true in a precisification in which people with more than 1.78 are tall. But it is false in the one in which only men with more than 1.82 m are tall. Supervaluationists, in its standard sense that was proposed by Fine, identify truth with super-truth. Supervaluational logic does not radically deviate from classical logic.[9] In particular, LEM holds because it is true on all ways of making a predicate precise. Even if P and ¬ P are vague propositions, P ∨ ¬ P holds because it is true in all admissible precisifications. So, the disjunction may hold without any of disjuncts being true. Therefore, supervaluational logic is not truth-functional.

As mentioned before, the standard boundary (See definition 1.1). is defined formally via the ‘definitely’ operator and it facilitates formalizing the higher-order vagueness. Kit Fine expands his proposed supervaluational logic, adding the definitely(determinately) operator,‘Δ’, to the language (Fine 1975, 287).

The interpretation of Δ is as follows:

A proposition is definitely true iff it is true on all admissible precisifications(or in Fine’s word, specifications).

This means that a proposition is definitely true iff it is super-true. Likewise, a proposition is definitely false iff it is super-false. A vague proposition, then, is true under some precisifications and false under other precisifications. So,

1. ΔP iff ‘P’ is super-true.

2. $\Delta\neg P$ iff ‘P’ is super-false.

3. ∇P iff $\neg\Delta P \wedge\neg\Delta\neg P$

[9]There are several variaties of supervaluationism. To put it in Varzi’s words, “Supervaluationism is a mixed bag”(Varzi 2007, p.623). They differ in their proposed semantics. They define different notions of truth and validity. Some supervaluationists reject LEM as well. Some accept the degree of truth. For the sake of space we will not consider all of them in the current work. We will focus on what is usually called standard supervaluationism (Fine 1975).

With the aid of the definitely operator, 'Δ', supervaluationists define the super-truth, super-falsity and truth value gap in the object language.

The notion of validity is defined classically as the preservation of truth. Validity for supervaluationists, however, may be defined in different ways. We consider two mostly discussed ones, namely global and local validity. (cf. Keefe 2000b, p.174; Williamson 1994, p.148.)[10]

> An argument is valid iff, necessarily if every premise is super-true, then its conclusion is super-true.

In other words, an argument is valid if it preserves super-truth. This is called the global validity in contrast to the local validity:

> An argument is valid iff, necessarily, on all precisifications, if every premise is true in a precisification, then its conclusion is true in it.

Supervaluationists like Fine endorse the notion of global validity.

All valid consequences in classical logic are valid in supervaluational logic when the language is not extended (cf. Williamson 1994, pp.148-149). Some classical rules fail when the language is enriched to contain ΔF. For example, in supervaluational logic if p is super-true, Δp is super-true, but unlike in classical logic, in supervaluational logic from that we cannot infer the conditional $p \rightarrow \Delta p$. Neither $\neg\Delta p \models \neg p$.

It is important to note that this logic is different from three-valued logic because it does not endorse a third value. As Van Fraassen(1966) claimed, the fact that a statement such as "The king of France is wise"-in which the singular term lacks a reference- is 'neither true nor false' is not on a par with it 'has a third value which is neither true nor false'.

Keefe mentions that in the non-truth-functional logic that she proposes the sentences that are neither true nor false should fall into a truth-value gap (Keefe 2000b, p.152).

According to supervaluationists, the Sorites paradox is valid but unsound. It is valid because truth is defined via precisifications and within precisifications the classical logic holds. But it is unsound since the second premise is not true (Fine 1975, Varzi 2001). The universal tolerance is false without having any false instances. It is false because it is false under all ways of precisifying the vague predicate. Likewise, the existential quantifer can be true without having any true instance. They do not commit to the sharp cut-off because they reject the principle of bivalence. So, it is super-true that there is an

[10] For more detailed discussion on the notion of validity for supervaluationists see Keefe (2000a) and Varzi (2007).

n such that Fa_n and $\neg Fa_{n+1}$. However, there is not a specific n of which $Fa_n \wedge \neg Fa_{n+1}$. is super-true. In this way, they do not trap into the sharp boundary paradox.
Advantages:

The idea that vague propositions are neither true nor false seems quite intuitive.

Supervaluationism is a popular view that slightly deviates from classical logic. In each admissible precisification LEM holds; i.e. there is a sharp cut-off. Then, in comparison to other views of vagueness that propose non-classical semantics, the minimum deviation from classical logic usually is considered as a point in favor of supervaluationism.

Disadvantages:

The expansion of the language, adding the definitely operator with the mentioned interpretation, has raised some problems for supervaluationists. In fact, supervaluational logic coincides with classical logic when the determinately operator is put aside. The proponents of other non-classical theories of vagueness claim that the deviation of supervaluational logic from classical logic is actually remarkable and therefore, contend that supervaluationism is no better than their views in this sense.

Supervaluationists need to explain higher-order vagueness. Even if they had a solution for the Sorites paradox in the vague language, the Sorites paradox would have arised again at the higher levels for super-truth. Some supervaluationists claim that the range of precisifications of a vague predicate is itself vague and this will help them to argue that the boundary in the first order won't be precise and therefore, one can explain higher-order vagueness. (Keefe 2000b, p.161) The question is then why the determinately F cases are vague? They are true on all precisifications, no matter what their range is. (Varzi 2007, Fara 2002, Williamson 1994).

Supervaluational logic denies bivalence. Then, given the identification of super-truth with truth, certain objections are raised. For example, Williamson (1994) argues that the denial of bivalence and the acceptance of Tarskian truth condition: "P" is true iff P, will lead to a contradiction. Later we will discuss that Rumfitt tries to show that those objections will not affect his theory of vagueness.

Supervaluationists need explain why the tolerance principle is so intuitive to us. (Fara 2010, p.374) claims that this question is to be answered.

2.2 Non-classical Solutions

Some philosophers do not find two-valued classical logic sufficient to give an account of vagueness. According to them a proposition P can either accept other values than 0 and 1, or have both true and false values.
For example, subvaluationists claim that P can be both true and false. And some like Tye (1990) and Smith (2008) accept three-valued and fuzzy logics respectively.

2.2.1 Paraconsistent logic

Some philosophers contend that the suitable logic for vagueness is paraconsistent logic; the logic that admits true contradictions. They find the Sorites argument unsound, rejecting the rule of modus ponens.

Paraconsistent logic is a logic that admits true contradictions. In this logic it is not the case that anything comes out from the contradiction. So, the main aim is to control the contradiction (Priest 2007, 2008, Hyde 1998).
Some philosophers contend that a paraconsistent logic can model the phenomenon of vagueness. For example, proposition 'Peter is tall', when Peter is a borderline case, is both true and false. For any proposition either it or its negation is true. The proposition 'Peter is tall' is not true because Peter is a borderline case of tall and therefore its negation is true. But it is neither false so its negation is true and hence both the proposition 'Peter is tall' and its negation will be true and false.
In the case of a vague predicate, there is an overlap between the extension of a predicate and its anti-extension. This overlap occurs in the borderline area. In this logic the classical logic rule modus ponens fails. It is then clear what their solution to the Sorites paradox would be. Priest as a dialethist for example, contends that all the premises are true but the modus ponens is invalid so the conclusion is false.

Some of the paraconsistent approaches to vagueness are the following:

Subvaluationism

According to subvaluationism, a vague proposition is both true and false; vagueness is nothing but the overdetermination of meaning. This view is usually considered as a dual of supervaluationism in which instead of truth-value gap, there is truth-value glut. Hyde and Colyvan (2008), for example, contend that subvaluationism is the paraconsistent dual of supervaluationism and admits truth value

glut. They argue that gappy and glutty approaches are two sides of a coin (Hyde and Colyvan 2008, p.93). However, their differences are considerable when we categorize theories of vagueness based on to what extent they keep classical logic.
Subvaluationists define truth and falsity as:

A proposition is sub-true (sub-false) iff it is true (false) on some precisification (Keefe 2000b, p.197).

Subvaluationists dissolve the Sorites paradox because it does not preserve validity, where validity is defined as the preservation of sub-truth. In subvaluational logic modus ponens fails because the premises might be true in different precisifications but this does not entail that they are both true in a precisification (Hyde 2010, 4.2). For example, suppose x is borderline F and y is not F. Then, F_x and $F_x \to F_y$ are true in some precisifications because F_x is both true and false. However, F_y is false. So, modus ponens fails. More acurately, modus ponens for material implication, i.e. disjunctive syllogism, is not valid in Subvaluationism.
To solve the Sorites paradox, then, subvaluationists contend that the argument is not valid. They reject the existential tolerance principle, though they find each instance of the conjunction sub-true. (Oms and Zardini 2019, p.10).
2. *Dialetheism*

Dialetheism is the view according to which both a proposition and its negation can be true. So, the law of non contradiction, that holds in classical logic, fails. In classical logic anything follows from a contradiction; This is usually called the explosion that fails in dialetheism. In this view, Sorites argument is not valid because the modus ponens fails. Priest in several papers and books defends this theory (Priest 2002, 2003, Priest et al. 2018). He introduces the logic LP. An inference is valid if it preserves truth in all interpretations.In the Sorites argument all the premises are true but the modus ponens is not valid. So, somewhere in the Sorites series Fa_n is both true and false but it is not entailed by the premises or a specific n (Priest 2017, pp.228-230).
Subvaluational logic and LP are similar for atomic sentences but the evaluation complex sentences differs.(Hyde and Colyvan 2008).

Advantages:

Since subvaluationism is the dual of supervaluationism, usually it has the merits and the drawbacks of supervaluationism.[11] Moreover, denying the law of non contradiction is not intuitive to many

[11]For recent discussion on the similarities and differences of supervaluationism, subvaluationism in dealing with the Sorites paradox see Oms and Zardini 2019, pp.38-62.

philosophers. Maybe that is why the supervaluationim is much more popular. It seems quite unintuitive that a proposition be both true and false. However, in recent years there have been more interest towards subvaluationism. For example, Cobreros (2011) argues that subvaluationism better explains higher-order vagueness.

Priest claims that dialetheism provides the same solution for all logical paradoxes such as the Liar paradox and to him that is a big advantage. [12].

Disadvantages For the proponents of classical logic the invalidity of modus ponens is a big price to pay. Also, many philosophers find the acceptance of true contradiction absurd:

> Many philosophers would soon discount the paraconsistent option (almost) regardless of how well it treats vagueness on the grounds of. . . the absurdity of $p \wedge \neg p$ both being true for many instances of p. (Keefe 2000b, p.197).

Keefe also criticies the proponents of paraconsistent logic on the ground that they do not have a uniform solution for seemingly equivalent Sorites arguments (Keefe 2000b, p.200).

2.2.2 Intuitionism

Intuitionistic logic is a non-classical logic in which the double negation elimination does not hold; i.e., it is not the case that $\neg\neg P \models P$. In this logic the law of excluded middle does not hold. That is to say, from the fact that a proposition is not true one cannot infer that that proposition is false. They solve the Sorites paradox by denying the second premise, the tolerance principle. In “Vagueness and alternative logic”, Putnam proposes that in dealing with the phenomenon of vagueness, using intuitionistic logic provides a solution to the Sorites paradox because in this logic $\neg\forall n\, \phi_n$ does not entail that $\exists n \phi_n$:

> My proposal *keeps* mathematical induction, even for “vague” predicates, but distinguishes between saying “it is false that *all n* are such that ———” and saying “There exist an n such that not ——-” (Putnam 1983, p.313).[13]

Crispin Wright also indicates that intuitionistic logic is a good candidate for the logic of vagueness. According to him, any logic that denies the double negation elimination can be a logic of vagueness

[12]For more detail on this view see Priest 1979, Priest et al. 2006, Oms and Zardini 2019, Oms Sardans 2016

[13]For more discussion on Putnam's proposal and its criticism see Williamson (1996).

because by that denial one can solve the Sorites paradox. He agrees with Williamson that the source of vagueness is cognitive rather than semantic but he contends that rejecting the tolerance principle does not commit us to accept the "superstitious" idea of the existence of a sharp boundary:

> ...the intuitionist credits the epistemicist with a crucial insight: that vagueness is indeed a cognitive rather than a semantic phenomenon; that our inability to apply the concepts on either side of a vague distinction with consistent mutual precision is not a *consequence* of some kind of indeterminacy or incompleteness in the semantics of vague expressions but is constitutive of the phenomenon. (Wright 2018, pp.357-358)

In Wright (2007) the author proposes a solution to the Sorites paradox according to which the tolerance principle is rejected. Its negation, however, does not entail the existence of a sharp cut-off. The reason, roughly, is that the negation of the tolerance principle does not entail that there is an n such that $Fa_n \wedge \neg Fa_{n+1}$ because the double negation elimination is not valid in intuitionistic logic.
Rumfitt (2015) argues that one does not need to appeal to intuitionistic logic to deal with the sharp boundary problem. In part V, we will discuss in detail how intuitionism deals with the Sorites paradox and why, at least for concepts that have poles, the topological approach is a better approach. Recently, Rumfitt (2018b) contends that for other concepts the intuitionistic logic is the adequate logic for vagueness.
Advantages: Intuitionistic logic is a very interesting approach to vagueness that is being more and more popular. The proponents of this view are trying to advance the theory. The intuitionistic approach drastically deviates from classical logic. Nevertheless, it seems an appealing approach to vagueness for some philosophers, like Rumfitt, who are inclined to keep classical logic. At face value, it seems that it can explain the sharp boundary paradox and the Sorites paradox.
Disadvantages:
The main drawback of this view, concerning Wright's proposal, is that he does not give a semantics for intuitionistic logic. In Wright (2021) in support of his view and its possible problem Wright quotes from Dummett's paper, published in 2007, in which he replies to Wright :

> I am left, then, with admiration for the beautiful solution of the Sorites paradox advocated by Crispin Wright, clouded by a persistent doubt whether it is correct . . . I do not say that Wright's proposed solution of the Sorites is wrong; I say only that we need a more far-going explanation than Wright has given us of why intuitionistic logic is the right logic for statements containing vague expressions before we can acknowledge it as correct. It is not

> enough to show that the Sorites paradox can be evaded by the use of intuitionistic logic: what is needed is a theory of meaning, or at least a semantics, for sentences containing vague expressions that shows why intuitionistic logic is appropriate for them rather than any other logic . . . If Crispin Wright is to persuade us that he has the true solution to the Sorites paradox, he must give a more convincing justification of the use of intuitionistic logic for statements containing vague expressions: a justification namely, that does not appeal only to the ability of that logic to resist the Sorites slide into contradiction. We need a justification that would satisfy someone who was puzzled about vagueness but had never heard of the Sorites: a justification that would sketch a convincing semantics for sentences involving vague expressions (Dummett 2007, pp. 453–4).

Rumfitt (2018b) provides such semantics for sentences containing vague expressions for the intuitionistic logic. In his recent paper, "Intuitionism and the Sorites paradox", Wright considers Dummett's and also Rumfitt's criticisms concerning the lack of semantics (Oms and Zardini 2019, pp.95-117) .

2.2.3 Many-valued, fuzzy and probabilistic logic

The proponents of many-valued logic, like epistemicists and supervaluationists, find the Sorites paradox valid but they deny the tolerance principle and therefore claim that the Sorites argument is unsound. In each interpretation of the language, one promise is not true simpliciter.

If the two-valued logic cannot be the logic of vagueness, then it seems natural to think of a logic that allows more truth values. Almost all theories of vagueness deny the principle of bivalence. Naturally, then, many philosophers have appealed to many valued logics. They consider three, four or more truth values. [14]

Tye (1990), in defense of ontic vagueness, accepts three-valued logic.[15] The truth values are true, false and indefinite. In this view an argument is valid if it preserves truth. The Sorites argument is valid because it preserves truth but the tolerance principle is indefinite and its negation also is indefinite. So, the Sorites argument is unsound. The view is quite similar to supervaluationism but there is a third value as well.

In another account of vagueness, the value of a proposition ranges over the interval $[0, 1]$ in the real line. So, there are infinite number of truth values. In the fuzzy approach there are degrees of truth

[14] For the historical review of many-valued logics see (Williamson 1994, ch.4).

[15] Tye is one of the main defenders of three valued logic. For previous defenders of this view such as Halldén and Körner see Williamson 1994, pp.103-110.

from 0 to 1. It is based on Zadeh's work on fuzzy logic. Machina (1976) is one of the main defenders of this view.

In this view, an argument is valid if and only if the conclusion is at least as true as the least true premise. In the Sorites argument the premises are almost true but the conclusion is false. So, the Sorites argument is not valid.

Smith (2008) finds fuzzy logic the adequate logic of vagueness. He proposes a theory of vagueness based on a notion weaker than the tolerance principle, namely closeness:

Closeness: If a and b are very similar in F-relevant respects, then Fa and Fb are very similar in respect of truth.

A proposition is vague if it satisfies Closeness condition. So, to deal with the Sorites, he denies the tolerance principle in its strict sense according to which if a and b are very similar in F-relevant respects, then Fa and Fb are *identical* in respect of truth. He rejects the principle of bivalence, otherwise tolerance principle and closeness will coincide.

In part V, we argue that Smith's theory of vagueness is very similar to Rumfitt's account. However, pace Smith, one does not need to appeal to fuzzy logic in the topological approach.

In general, many-valued logic approach usually refers to the truth-functional logics with more than two truth-values. So, the truth values of the components of a compound sentence determine its truth value. It is good to mention that not all many-valued logics are truth functional. For example, Edgington's probabilistic approach also considers degrees of closeness to truth but the logical connectives are not truth-functional just like in the probability theory that the probability of $A \wedge B$ is different from the probability of A and the probability of B (Edgington 1997).

Advantages:

Many approaches of vagueness, despite their differences in accepting the source of vagueness, either endorse three-valued strong Kleene logic or fuzzy logic (See, for example, Shapiro (2006), Rumfitt (2018b), Smith (2008), Edgington (1997)). If the truth value of a vague proposition is neither true nor false and in the absence of LEM, it seems quite natural to add more values. They explain why the tolerance principle seems so appealing to us. For example, for the proponents of Fuzzy logic the reason is that all instances of the tolerance principle are of the degree very close to 1.

Disadvantages:

In general, in these approaches the law of excluded middle fails. So, for the vague predicate F there is no sharp line between F-things and not-F things. For them, LEM just is valid for precise predicates.

But the language is replete of vague predicates. Many critics believe that the proponents of many-valued logic approaches do nothing but adding more sharp lines between the things that are F and the things that are not F:

> ... you do not improve a bad idea by iterating it. In more detail, suppose we have a finished account of a predicate, associating it with some possibly infinite number of boundaries, and some possibly infinite number of sets. Given the aims of the description, we must be able to organize the sets in the following threefold way: one of them is the set supposedly corresponding to the things of which the predicate is absolutely definitely and unimpugnably true, the things to which the predicate's application is untainted by the shadow of vagueness; one of them is the set supposedly corresponding to the things of which the predicate is absolutely definitely and unimpugnably false, the things to which the predicate's non-application is untainted by the shadow of vagueness; the union of the remaining sets would supposedly correspond to one or another kind of borderline case. So the old problem re-emerges: no sharp cut-off to the shadow of vagueness is marked in our linguistic practice, so to attribute it to the predicate is to misdescribe it (Sainsbury 1996, p.255).

The point is that the Sorites paradox can be generalized for the more fine-grained truth values. That is to say, the generalised Sorites paradox would show the inadequacy of many-valued logic approaches. This time by showing that in the absence of LEM again the argument leads to a paradox considering the sharp lines between F-cases and neither F nor not-F cases,...

Another problem is that these views cannot explain higher-order vagueness. (See, for example, Keefe (2000b), Williamson (1994)).

As serious challenge for degree theorists is to explain why they assign a unique, precise degree of truth to each proposition. For example, why 'Peter is tall' is true to 0,75 degree. The unique assignment of degrees of truth to the propositions is problematic. Smith endorses that this objection applies to fuzzy logic and proposes a theory of vagueness based on a more complicated fuzzy logic which escapes that objection (Smith 2008, 2011).

Part III

Topology

3 On the general concept of space

Since antiquity, the nature and structure of space has been studied in various fields. Hausdorff (1903) in his inaugural lecture in the university of Leipzig starts off his talk considering at least five sciences "involved and interested" in solving the problem of space, namely physics, mathematics, psychology, physiology and epistemology. To that list we may add computer science and artificial intelligence(AI) as well. Different types of "logical spaces", "conceptual spaces" and "possible worlds spaces" are some instances of such studies. So, there are well-established views about space and spatial objects. There are different kinds of spaces such as Euclidean, Riemannian, Hilbert and Minkowski spaces among others that have been used in physics; for example, Minkowski space-time structures in Einstein's special relativity, Riemannian space in general relativity and Hilbert spaces in quantum mechanics. A well-known and broadly used space is the Euclidean space. It was used in the Newtonian mechanics and was dominant for more than two millennia.In the Euclidean geometry the distance between two points can be measured by the Euclidean metric. This metric was considered as a constitutive element in science and so, a priori.
As is well-known, for Kant space as well as (Euclidean)geometry(considered as a theory of space) were synthetic a priori. That is to say, they were built into our cognitive capacities and the scientific theories could not be stated without them. Yet, they could not be tested in a direct way and they were independent of experience (cf. Stump (2015)).

However, the advent of the non-Euclidean geometries put an end to the role of Euclidean geometry as the unique universal structure of the physical space. This, of course, put into doubt the notion of synthetic a priori. Friedman (2001) refers to Reichenbach (1920) who stated that Kant's notion of synthetic a priori consists of two components: 1- Synthetic a priori knowledge is firmly fixed and unrevisable. 2- It is constitutive of any scientific knowledge. Philosophers such as Poincaré, Cassirer and Carnap rejected the first claim but for them geometry still is synthetic a priori in the second sense. Euclidean geometry was not anymore necessary. There were non-Euclidean geometries and each one of them had its own application. Then, one of the main concerns of the philosophers and mathematicians was how to choose between them. Poincaré denied the necessity and unrevisibility of geometry and proposed conventionalism according to which the choice of one geometry among others is a matter of convention. According to Poincaré, we are free to choose one geometry based on criteria such as simplicity. This was against Helmholtz's idea that the choice between Euclidean and non-Euclidean

geometry is based on experience (cf. Friedman 2001, p.62).
For the sake of space, we cannot go into the details of the attempts to solve the problem of finding criteria to choose between different geometries. Instead, we will focus on another problem of "underdetermination".

In Euclidean space one can define the distance between two points in different ways. Using the Euclidean metric[16] is one way to define such a distance. Minkowski defined a series of metrics on Euclidean space one of which was the Euclidean metric. Again a question is in what way one can choose between different metrics on a single space. This, as we will see raises some difficulties for Gärdenfors' view of concepts defined in a geometrically(topologically) structured conceptual space.
Another important question regarding space is whether we can talk about space and analyse it if we do not have a metric. In other words, are metrics constitutive of the notion of space? The answer is no. Space need not have a metric.

In fact, topology, as a general mathematical framework for spaces, gives us a much wider and flexible notion of space. Many defined metrics on a set induce the same topology. However, there are topologies that are not induced by metrics. So, topological spaces include metric spaces as well as non-metric spaces; we can talk about space even in the absence of metrics.
In topology, instead of quantitative distance we talk about neighborhoods. Topology defines a notion of neighborhood[17] on a set and turns it into a space-like object. In this way, instead of points that lack extension and are too small to be located neighborhoods are to be observed. This means that the relation between points are characterized by neighborhoods or open sets. The neighborhoods of points may overlap or may be completely disjoint. So, it is not the case that the existence of space depends firmly on points.[18]
So, topology is not limited to what is called point topology. We may even go further and put aside the Euclidean idea that points are the constitutive part of space. Whitehead claimed that regions are the basic fundamental part of space. So, for Whitehead space is not a collection of points endowed with certain relations, rather it is a collection of regions with certain properties and points can be constructed (cf.Mormann (1998)).
In a nutshell, topology, like geometry, may be considered as a theory of space that is a necessary presupposition and constitutive of experience and science.

[16]For the definition of metric see definition 3.1 on page 53.
[17]For the definition of neighborhood see page 60.
[18]This is against the ancient belief that "Practically, it is the point that gives space its excuse for being." (de Laguna 1922, p.448)

Philosophers have analyzed space from different perspectives; for example, spatial logic and geometry. Whereas, topology is almost a terra incognita for philosophers. If the notion of space is important in philosophy, then topology, just like spatial logic, can play a crucial role in philosophy as well. We hope that topology finds its place among philosophers.

In this work we are going to present one application of topology in philosophy. In particular, we argue that the phenomenon of vagueness has a topological character. To set the stage, we will go through some basic notions of metric spaces and topological spaces.

To do so, we will first define what metric is and will give some examples of metric spaces and then we will focus on different topological spaces and their properties. We will then explain the tight relation between logic, lattice theory and topology. This can reveal that topology is not so strange as one might think it is and it is worth to become a new continent for philosophers to discover and enjoy its benefits.

Metric spaces

The notion of metric space was first introduced by Fréchet (1906) as a generalization of the Euclidean space. In the Euclidean space the distance between two points is defined as the length of the straight line segment that connects those points. However, we will see that in a metric space the distance between points can be defined in different ways.

As usual, $\mathbb{R}$ will denote the set of real numbers.

Definition 3.1. *A* ***metric*** *on a set* X *is a function* $d : X \times X \longrightarrow \mathbb{R}$ *that satisfies the following conditions: for all* $x, y, z \in X$

1. $d(x, y) \geqslant 0$,

2. $d(x, y) = 0 \quad \textit{iff} \quad x = y$,

3. $d(x, y) = d(y, x)$,

4. $d(x, y) + d(y, z) \geqslant d(x, z)$.

(X, d) *is called a* ***metric space*** *and* $d(x, y)$ *is called the* ***distance*** *between* x *and* y.

The following examples show different metrics defined on a given set. We start by the typical example of a metric space, namely the Euclidean one.

Example 3.2. *Let $\mathbb{R}^n$ be the n-th Cartesian power of $\mathbb{R}$ and $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n)$ be two points of $\mathbb{R}^n$. Then, the Euclidean metric in $\mathbb{R}^n$ is defined as the following:*

$$d_E(x, y) := \|x - y\| = \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2}$$

The Euclidean metric is a special case of Minkowski's metric.

Example 3.3. *Minkowski defined, in a more general way, the distance function between two points $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n)$ and $k \geqslant 1$ as :*

$$d_k(x, y) := \sqrt[k]{\sum_{i=1}^{n} |x_i - y_i|^k}$$

In addition to the Euclidean metric, there are also non-Euclidean metrics. One of the most well-known examples of such metrics is the taxicab metric.

Example 3.4. *Consider the Cartesian plane, $\mathbb{R}^2$. Then a distance between two points $x = (x_1, x_2)$ and $y = (y_1, y_2)$ can be defined as:*

$$d_t(x, y) := |x_1 - y_1| + |x_2 - y_2|$$

This metric also is another case of Minkowski's metric where $k = 1$. It is called Taxicab Metric or Manhattan Metric for the following reason: suppose that you are in Manhattan, the heart of NewYork city, where the streets run north-south and east-west in a regular grid. A taxi driver can choose different ways to get from one point to the destination point. There is no way to pass through houses to get to the destination by choosing the straight line. Unlike

Euclidean metric, the taxicab metric measures the distance of these points based on the way the taxi driver chooses to get to the destination and therefore, makes it possible for the driver to choose different ways with the same distance to get to the destination. For instance, in a Euclidean space $\mathbb{R}^2$, let $x(1,1)$ and $y(4,5)$ be the starting point and the destination point respectively. By definition of taxicab metric $d_t(x,y) = |1-4| + |1-5| = 7$

So, to go from x to y the driver may choose the route x-z-y or the stairs one. As it is shown in figure 1 the distance is the same. In contrast, if we consider the Euclidean metric, the distance between these two points is the straight line from x to y.

$d_E = \sqrt{\sum_{i=1}^{2}(x_1 - y_1)^2 + (x_2 - y_2)^2} = \sqrt{(1-4)^2 + (1-5)^2} = 5.$

Figure 1 shows the absolute value of the sum of the differences of movement on both axis versus Euclidean metric.

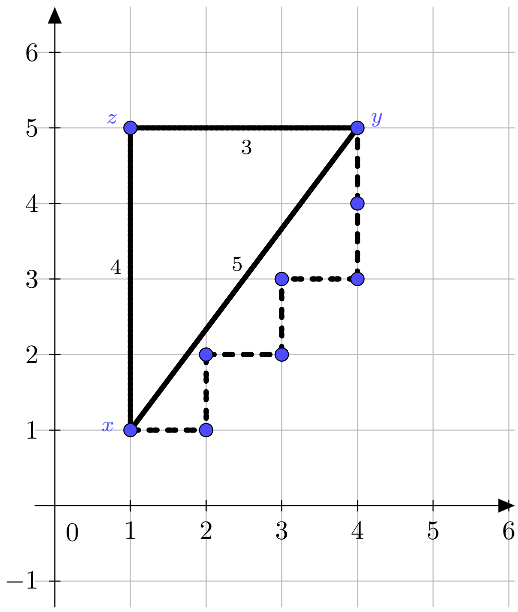


Figure 1: Manhattan metric

There are many other metrics. One of them is the discrete metric.

Example 3.5. *Every set X has a (trivial) discrete metric:*

$$d_0(x,y) := \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

It is easy to see that d_0 is a metric.

The distance between two points can be of interest to a philosopher who wants to define the similarity of two objects based on how far they are from each other in a certain space. For example, the less distance they have the more similar they are. In section IV, we will explain an alternative geometrical theory of vagueness based on such a similarity relation in a conceptual space.

In a metric space one can define continuous functions. But before that we need some other definitions.

Definition 3.6. *Let (X,d) be a metric space. For $\epsilon > 0, x \in X, B_d(x,\epsilon) = \{y \in X | d(x,y) < \epsilon\}$ is called open ball.*

An open ball is the set of all points y such that their distance from x is less than ϵ.

Definition 3.7. *Let (X,d) be a metric space. A sequence $\{x_n\}_{n\in\mathbb{N}} = \{x_1, x_2, \ldots, x_n, \ldots\}$ converges to x if for all $\epsilon > 0$, the elements of the sequence, x_i, eventually end up inside an open ball $B(x,\epsilon)$ around x. Equivalently:*

$$\forall \epsilon > 0 \; \exists k \in \mathbb{N} \; \forall n \geqslant k \quad d(x_n - x) \leqslant \epsilon.$$

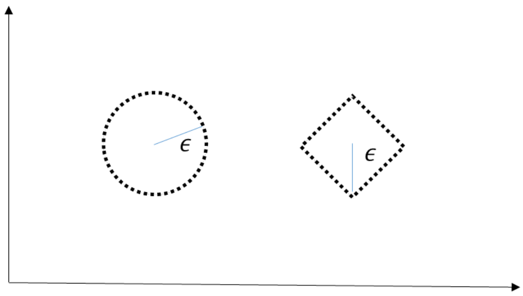


Figure 2: Open balls in $\mathbb{R}^2$ for d_E and d_t.

If a sequence converges to a point x, then x is the limit point of the sequence. Now we define a fundamental notion namely, continuity.

Sometimes we do not have a lot of information about a function. It is like a black box that takes some inputs and gives back some outputs. Even so, we may gain more information about how it works. For example, when a function is continuous, if we slightly move from a point x to x', we can be sure that $f(x')$ will not be far from $f(x)$. This intuitive idea is formalized in the following definition:

Definition 3.8. *Let (X, d_1) and (Y, d_2) be two metric spaces. A function $f : X \longrightarrow Y$ is **continuous at a point** x in X iff it satisfies the following condition:*

$$\forall \epsilon > 0 \ \exists \delta > 0 : \forall y \in X \ \textit{if} \ d_1(x, y) < \delta \ \textit{then} \ d_2(f(x), f(y)) < \epsilon.$$

*The function f is called **continuous** if it is continuous at all points x in X.*

Informally, points close to x in the metric d_1 are mapped close to $f(x)$ considering the metric d_2.

Continuity is a central notion in topology. In the next section we will introduce topology and topological spaces in detail. A topological space abstracts away from metrics. Topological structures are generalizations of metric spaces in that any metric space gives rise to (induces) a topology but not all topological spaces are metrizable. In topology some basic relations and properties of space are important, not the metric structure of space. There is the following relation between the classes of these spaces with respect to inclusion:

$$\text{Euclidean spaces} \subset \text{Metric spaces} \subset \text{Topological spaces.}$$

A metric is helpful to verify the distance between two points in a space. In the case of vagueness, two similar points with respect to a predicate share the same properties. As said before, similarity relation between two objects is usually defined based on their distance. However, it might happen that there is no such a metric or we cannot find it out. Then the question is: How a similarity relation can be defined in such spaces? Topology provides us with an appropriate tool to define the similarity relation based on the so called neighborhood or closeness and therefore, at face value, it is a more general tool to formulate similarity or the tolerance principle. In the next section we introduce some fundamental notions of topology that will be used to give a theory of vagueness.

4 Topological spaces

4.1 Basic topological notions

The aim of this section is to introduce more general spaces, namely topological spaces. The generality of these spaces stems from the fact that not all topological spaces are metric spaces yet any metric space determines a topological space. The concept of continuity is defined in topology without appealing to a metric. However, as we will show later, it is compatible with the notion of continuity defined in metric spaces. The outline of this section is the following:

First, we will give the definitions of basic notions of topology such as open sets, closed sets, topological spaces and will give different equivalent definitions of topology. We will start with a definition that is usually used in the textbooks on topology. This definition is based on open sets. Intuitively, an open set is a set such that if from any point in the set we move a little bit in any direction, we are still in the set. Another definition uses the notion of neighborhood of a point, which is the set of points very close to that point. This makes it a good candidate for formulating the tolerance principle. In the next step, we will give examples of topologies that are based on metrics and some that are not. Furthermore, we will briefly introduce the main properties of topologies that will later be applied to the phenomena of vagueness. We will then introduce another definition of topology via operators. This new perspective was first proposed by Kuratowski and will be the basic definition in the rest of the sections. We start by giving a definition of topology based on open sets.

Definition 4.1. *Let X be a set and $\mathcal{O}X$ a family of subsets of X, called **open sets**. Then, $\mathcal{O}X$ is a topology on X if it satisfies the following conditions:*

i. $\emptyset, X \in \mathcal{O}X$.

ii. $\forall A_i \in \mathcal{O}X, i \in I, \cup A_i \in \mathcal{O}X$.

iii. if $A_1, A_2 \in \mathcal{O}X$ *then* $A_1 \cap A_2 \in \mathcal{O}X$.

The set X is a space, $\mathcal{O}X$ is called a topology *on X and the pair $(X, \mathcal{O}X)$ is called a topological space.*

According to this definition, the empty set and the whole set are open sets. The second condition implies that the union of any family of open sets is an open set. According to the third condition, the finite intersection of open sets is open.

A topological space is a lattice of open sets (See 10.26).

By definition, a set $A \subseteq X$ is closed if the set $X - A$ ($\complement A$) is open; i.e., if its set-theoretical complement belongs to $\mathcal{O}X$. The set of all closed sets $\mathcal{C}X$ is defined as:

$$\mathcal{C}X := \{A | X - A \in \mathcal{O}X\}.$$

The open sets are set-theoretical complements of closed sets, i.e.

$$\mathcal{O}X = \{X - B | B \in \mathcal{C}X\}$$

In a topological space we can talk about the neighborhoods of a point. In a topological space $(X, \mathcal{O}X)$ a neighborhood of a point $x \in X$ is a set $U \subseteq X$ that includes an open set V containing x. A neighborhood $U \subseteq X$ is a neighborhood of $A \subseteq X$ iff U is a neighborhood of each point of A.

Definition 4.2. *Let $(X, \mathcal{O}X)$ be a topological space and $A \subseteq X$. Then, the largest open set contained in A is called the interior of that set and the smallest closed set containing A is called the closure of A and the boundary of A contains the points that are shared between its closure and the closure of its complement. Formally:*

$$int(A) := \bigcup\{B \subseteq A, B \in \mathcal{O}X\}.$$
$$cl(A) := \bigcap\{B | A \subseteq B, B \in \mathcal{C}X\}.$$
$$bd(A) := cl(A) \cap cl(\complement A).$$

It is good to mention that by the definition of topology (definition 4.1), the intersection of a *finite* family of open sets is open in any topological space. But in these spaces it may happen that the intersection of an infinite family of open sets not be open.

Definition 4.3. *Let $(X, \mathcal{O}X)$ be a topological space. Then, $\mathcal{O}X$ is Alexandroff iff $\mathcal{O}X$ is closed under arbitrary intersections. That is to say:*

$$\textit{for all } A_i \in \mathcal{O}X, i \in I, \bigcap A_i \in \mathcal{O}X.$$

In Alexandroff topologies open sets are closed under arbitrary intersections. It is good to mention that not all topologies are Alexandroff.

Example 4.4. *Let $(\mathbb{R}, \mathcal{O}\mathbb{R})$ be the topological space with the standard topology on $\mathbb{R}$ (see example 4.13). The open sets are not closed under infinite intersection because:*

$$\textstyle\bigcap_{n\in\mathbb{N}}(\frac{-1}{n}, \frac{1}{n}) = \{0\}$$

$\{0\}$ *is closed in that topology.*

Proposition 4.5. *Let $(X, \mathcal{O}X)$ be a topological. Then X is an Alexandroff space iff each point in X has a minimal open neighborhood.*

Proof. From left to right:
Suppose that X is Alexandroff, $x \in X$. Let $U_i(x), i \in I$, be the open neighborhoods of $x \in X$ and $O(x) = \{U_i(x), i \in I\}$. Define $M(x) = \bigcap\limits_{i\in I} U_i(x), x \in X, U_i(x) \in O(x)$. We show that $M(x)$ is the minimal open neighborhood of x. Since X is Alexandroff, $M(x)$, by definition, is an open neighborhood of x. Now we need to prove that it is a minimal neighborhood of x. By the definition of $M(x)$, it is clear that it is minimal because it is a subset of any open set containing x.
From right to left:
Suppose that each point in X has a minimal open neighborhood, $M(x)$. Let A

be an arbitrary set of open sets of X containing $x \in X$. We show that for any arbitrary $U_i \in A, i \in I$, $V = \bigcap_{i \in I} U_i$ is open in X.

If $V = \emptyset$ it is open and we are done. Now suppose that $V \neq \emptyset$. Let $x \in V$. Then by definition, for all $i \in I \quad x \in U_i$. Since $M(x)$ is a minimal neighborhood of x, $M(x) \subseteq U_i$ for all $i \in I$. Therefore, V is open since each member of it contains an open set around each of its points. $\square$

Proposition 4.6. *Let $\mathfrak{B}$ be a collection of subsets of X such that for each $x \in X$ there is a minimal set $S(x) \in \mathfrak{B}$ containing x, then $\mathfrak{B}$ is a unique basis for a topology on X and X is an Alexandroff space with this topology. In addition, $S(x) = M(x)$.*

Proof. By definition 4.12, it is easy to see that $\mathfrak{B}$ covers X. We prove that for any two members of $\mathfrak{B}, B_1, B_2$ that contain x, there is $B_3 \subset B_1 \cap B_2$ such that $x \in B_3$. Since B_1, B_2 contain x, $x \in B_1 \cap B_2$. $S(x)$ is a minimal neighborhood of x. Therefore, $S(x) \subset B_1 \cap B_2$. We now prove that $\mathfrak{B}$ is unique. Let $\mathfrak{C}$ be a minimal basis as well.

$\square$

In this thesis, we will argue that Alexandroff spaces have a particular importance in the topological account of vagueness. Here, we just introduce the definition of Alexandroff space and postpone the detailed discussions to the section V.

A topology $\mathcal{O}X$ on a set X is usually defined based on open sets. However, it is sometimes more appropriate to define a topology based on its closed sets. Since closed sets are set theoretic complements of open sets it is easily proved that the following definition is equivalent to the previous definition.

Definition 4.7. *Let X be a non-empty set, $\mathcal{C}X$, be a family of subsets of X, called* ***closed sets****, such that it satisfies the following conditions:*

i. $\emptyset, X \in \mathcal{C}X$.

ii. $\forall B_i \in \mathcal{C}X, i \in I, \bigcap B_i \in \mathcal{C}X$.

iii. If $B_1, B_2 \in \mathcal{C}X$, *then* $B_1 \cup B_2 \in \mathcal{C}X$.

Define $\mathcal{O}X$ as the complement of $\mathcal{C}X$. Then, $\mathcal{O}X$ is a topology on X by De Morgan's laws. $(X, \mathcal{C}X)$ is a topological space.

The topological space $(X, \mathcal{C}X)$ is Alexandroff iff $\mathcal{C}X$ is closed under *arbitrary* union.

Up to now we have introduced two ways of defining a topology on a given set, namely a topological space defined as a system of open (closed) sets. A topology can also be defined based on neighborhood (Willard 1970, p: 31).

The following examples introduce some of the topologies on X as a finite, countably infinite and uncountable infinite set.

Example 4.8. *Let X be a set. Then, $\{\emptyset, X\}$ is a topology on X, called* ***(trivial) indiscrete topology*** *and is denoted by $\mathcal{O}_0X$.*
The power set $\mathcal{P}X$ is a topology on X, called the ***(trivial)discrete topology*** *and is denoted by $\mathcal{O}_1X$. In these spaces every open set is also closed. In the indiscrete space the interior of every proper subset of X is empty and therefore, its closure is the whole set. If X is discrete, then the interior and closure of any subset of X is the set itself.*

The next example will introduce topologies on a finite set.

Example 4.9. *There are some topologies on* $X = \{0, 1, 2\}$*. For example,* $\mathcal{O}_{\hat{0}}X = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, X\}$ *is a topology on* X *because it is closed under arbitrary union and finite intersections of its members and it contains both the empty set and the whole set.*

Remark 4.10. *In ordinary language, if something is closed it cannot be open and vice versa. For example, a window is either closed and therefore, not open, or it is open and therefore not closed. It cannot be both closed and open or neither closed nor open. However, in topology, it is not the case that if a set is not open (closed) it is closed (open). For example, in the real line, with the topology induced by the metric* d_E *(see definition 4.15), an interval* (a, b) *is open and an interval* $[a, b]$ *is closed but the half open intervals,* $[a, b)$ *and* $(a, b]$ *are neither open nor closed since their complements are not closed and open respectively. Moreover, a set can be both open and closed. For instance, in a topological space* $(X, \mathcal{O}X)$*,* $\emptyset$ *and* X *are both open and closed. So, the definition of open and closed sets does not preclude that a set be both open and closed. These open and closed sets are called clopen.*
These sets have interesting properties. We will come back to this point when we deal with the concept of vagueness.

Example 4.11. *The* ***cofinite topology*** *on a set* X *is defined as:*

$$\mathcal{O}_{cf}X = \{\emptyset\} \cup \{A \subseteq X \mid X - A \text{ is finite } \}.$$

For instance, the cofinite topology on the set of natural numbers $\mathbb{N}$ *is defined as:*

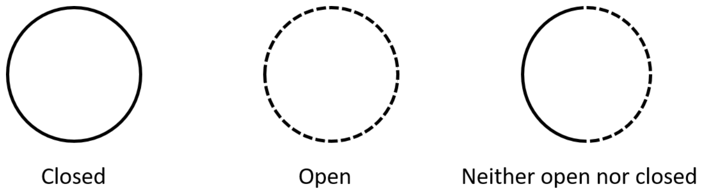


Figure 3: Closed, open and neither open nor closed sets

$$\mathcal{O}_{cf}\mathbb{N} = \{\emptyset\} \cup \{A \subseteq \mathbb{N} \mid \mathbb{N} - A \text{ is finite }\}$$

Informally, $\mathcal{O}_{cf}\mathbb{N}$ is the set of all subsets of $\mathbb{N}$ whose complements are finite. So, the only closed sets are $\emptyset, X$ and finite sets.

In $(\mathbb{N}, \mathcal{O}_{cf}\mathbb{N})$, there are subsets such as $2\mathbb{N}$ and $2\mathbb{N}+1$ that are neither open nor closed. If X is a finite set, the cofinite topology is just the discrete topology because all the subsets of X are such that their complements are finite.
Most of the time it is difficult to find out all the members of $\mathcal{O}X$ or $\mathcal{C}X$. Usually, a certain collection of subsets of X is considered and topology is determined by defining open subsets of X based on the elements of that collection (Munkres 2000, p.78).

Definition 4.12. *Let X be a set. A collection $\mathfrak{B}$ of subsets of X is a **basis** for a topology on X if it satisfies the following two conditions:*
1. For each $x \in X$, there exists at least one element of the basis, B_1, containing x.
2. For $B_1, B_2 \in \mathfrak{B}$ such that $x \in B_1$ and $x \in B_2$, there is a basis element $B_3 \subset B_1 \cap B_2$ such that $x \in B_3$.

Define a topology on $(X, \mathcal{O}X)$, by a basis $\mathfrak{B}$ in the following way:

A subset $U \subseteq X$ is open in X if for all $x \in U, \exists B \in \mathfrak{B}$ such that $x \in B, B \subset U$. $\mathcal{O}X$ satisfies the conditions mentioned in def 4.1. [19]

$\mathcal{O}X$ is called a topology ***generated or induced*** *by $\mathfrak{B}$. The generated topology can be written as the collection of all sets which can be written as an arbitrary union of elements of $\mathfrak{B}$ (Steen and Seebach 1995, p.4).*

Example 4.13. *The collection of all open intervals $(a, b) = \{x \mid a < x < b\}$ of the real line is a basis that generates the* ***standard topology*** *on $\mathbb{R}$.*

Definition 4.14. *Let (X, d) be a metric space. Then, the topology induced by the collection of all open balls as basis is called the* ***metric topology*** *induced by d.*

A prime example of metric topology is the Euclidean topology.

Definition 4.15. *The* ***Euclidean topology*** *is the topology induced by the Euclidean metric d_E.*

Every metric space induces a topology based on open balls. Different metrics may define the same topology. That becomes important when we discuss conceptual spaces. In examples 3.5, 3.2 and 3.4, d_0 induces the discrete topology (Munkres 2000, p.120). Moreover, d_E and d_t induce the same topology, that is the standard topology on $\mathbb{R}$ (Munkres 2000, p.126).

If X is an ordered set and $<$ the ordered relation, there is a topology on X, using $<$. Before defining the topology, we need to define intervals.

Definition 4.16. *Let X be an ordered set, $<$ its order relation , $a, b \in X$ and $a < b$. The following four subsets of X are called* intervals *determined by a and b:*

[19] For the proof see Munkres (2000, p.79).

$$(a, b) = \{x|a < x < b\}$$
$$(a, b] = \{x|a < x \leq b\}$$
$$[a, b) = \{x|a \leq x < b\}$$
$$[a, b] = \{x|a \leq x \leq b\}$$

The first one is called an open interval, *the last one is called a* closed interval *and the other two are called* half-open *intervals (Munkres 2000, p.84).*

Definition 4.17. *Let $(X, <)$ be a non-empty ordered set. Let $\mathfrak{B}$ be the collection of all sets of the following types:*

1- All open intervals (a, b) in X.

2- All intervals $[a_0, b)$ where a_0 is the smallest element(if any) of X.

3- All intervals $(a, b_0]$ where b_0 is the largest element(if any) of X.

The collection $\mathfrak{B}$ is a basis for a topology on X, called ***order topology*** *(cf. (Munkres 2000, p.84)).*

[20] The order topology on $\mathbb{R}$ is also equivalent to the standard topology on $\mathbb{R}$.

Definition 4.18. *Let $(X, \mathcal{O}X)$ be a topological space. Then $\mathcal{O}X$ is called metrizable if $\mathcal{O}X$ is a metric topology for some metric d.*

Not all topologies are metric topologies. In the subsection 4.5 at the page 77 we will give some examples of not metrizable topologies.

The next section will be devoted to the comparison between different topologies on a set.

[20]For the proof see (Munkres 2000, 84).

4.2 Comparison of topologies and Continuity

As mentioned before, there can be many topologies defined on a given set. Two topologies can be compared:

Definition 4.19. *For any two topologies T_1 and T_2 on a given set X, if $T_1 \subseteq T_2$, then T_2 is* ***finer*** *than T_1 and T_1 is* ***coarser*** *than T_2. T_1 and T_2 are* ***comparable*** *if either $T_1 \subseteq T_2$ or $T_2 \subseteq T_1$.*

Obviously, the coarsest topology is the indiscrete one and the finest is the discrete one.

All topologies can be partially ordered by inclusion. They are all included in the discrete topology and include the indiscrete one.

The topologies on a set X form a complete lattice where the least element is the indiscrete topology and the greatest one is the discrete topology. Furthermore, any set of topologies on X have the least upper bound and the greatest lower bound (Larson and Andima 1975, p.177).
In general, the number of topologies on a set sharply increases with the increase of the number of members of the set, but there is no general rule to find out all the topologies on a set. For example, let's start with finite sets. A set with one member, $X = \{0\}$ has one topology. It is both indiscrete and discrete since the only open sets are the empty set and the singleton itself.

The Polish mathematician, Sierpiński, introduced this set as the smallest set that contains a topology other than the extreme ones, in the sense that either $\{0\}$ is open and $\{1\}$ is closed or vice versa. It does not matter which one. So, in fact, there are three topologies that are not equivalent. It is considered as the smallest non-trivial topology. On a set with 3 elements, $X = \{0, 1, 2\}$ there will be 8 subsets and 29 distinct topologies. This number sharply increases for a set with 4 elements on which there are 355 distinct topologies. Larson and Andima (1975) show that although there is not a fixed formula to calculate the number of topologies on a finite set, it can be shown that this number is greater than 2^n and less than $2^{n(n-1)}$ for $n = |X|$, where $|X|$ denotes the cardinal number of X, the number of members of X. However, for an infinite set X, there are infinite cardinality of topologies. For example, each singleton defines a topology on $\mathbb{R}$ such that for any $x \in X, \{\emptyset, \{x\}, X\}$ is a topology. In fact, the cardinality of the lattice of topologies on an infinite set X is $2^{2^{|X|}}$ (Larson and Andima 1975, p.179). Finding out the number of topologies on a set is quite difficult. However, the important and very complicated task is finding or constructing one of these topologies that is convenient for the main purpose that one is seeking for. Some of these topologies are comparable with each other in the sense that one is finer or larger than the other one. It is good to mention that there are some topologies on a given set that are not comparable. For example, Sierpiński´s topologies 2. and 3., defined on X are not comparable.

One of the most useful characteristics of topology is that it studies structure preserving maps between topological spaces. Very specifically, topology studies continuous maps between topological spaces. One advantage of topology over set theory is that in the former it is possible to define continuity. Continuity is the fundamental concept of topology. In set theory, there is no suitable structure to be able to talk about continuity of functions. Topology provides such structure so that a function that maps a topological space to another one can be continuous.

4.3 Continuity

In the previous section, continuity was defined in metric spaces using the distance function. The continuous function preserves the distance between objects. However, in topology continuity is also phrased in terms of open sets. In topology, functions need not preserve the distances; they need to be continuous in the sense that if x and y are near each other, $f(x)$ and $f(y)$ are near each other as well. Nevertheless, structures are preserved. Since each metric space gives rise to a topology -which as mentioned before is called metric topology- functions that are continuous with respect to metric spaces are also continuous with respect to the corresponding metric topology. But the definition of continuity that is given in this section is more general and includes topologies that are not metrizable as well.

Definition 4.20. *Let $(X, \mathcal{O}X)$ and (Y, OY) be two topological spaces, $f : X \longrightarrow Y$ be a map and $f^{-1} : \mathcal{P}Y \longrightarrow \mathcal{P}X$ be the inverse image of f.*
A function f is ***continuous*** *if it satisfies the following condition:*

$$\text{If } A \in \mathcal{O}Y, \text{ then } f^{-1}(A) \in \mathcal{O}X.$$

This is a general definition regardless of whether $\mathcal{O}X$ and $\mathcal{O}Y$ are metric topologies or non-metric. It is good to notice that continuity of a function $f : X \longrightarrow Y$ depends on the topologies the set X and Y are endowed with. It is trivial to say that f is continuous without mentioning the topologies on X and Y because we can always make a topology continuous either by endowing X with the discrete topology or Y with the indiscrete topology (see example 4.22, a).

Definition 4.21. *A function f is **continuous at the point** x if for any open set $U \subseteq Y$ such that $f(x) \in U$, the preimage $f^{-1}(U)$ is an open set of x.*[21]

Example 4.22. *a. Let $f : (X, \mathcal{O}_1 X) \longrightarrow (Y, \mathcal{O}Y)$. Then, f is continuous regardless of the topology that Y is endowed with. Another extreme case is when Y is endowed with $\mathcal{O}_0 Y$. In this case, f, regardless of the topology on X is continuous.*

b. Let $X = \mathbb{R}$. Let $\mathcal{O}\mathbb{R}$ be the standard topology. Define f as:

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$
$$f(x) = -x$$

Then, f is continuous. But f under another topology on $\mathbb{R}$ may not be continuous. For example, define $\mathcal{O}\mathbb{R} = \{\emptyset, \mathbb{R}, [0, a)\}$. Then, f is not continuous.

c. Let f be the identity function, then f is continuous.

d. Let $f : X \to Y$ and $g : Y \to Z$ be continuous. Then, $g \circ f : X \to Z$ is continuous.

Continuous functions play an important role in studying properties of spaces and constructing new spaces from old ones.

We will now define the notion of homeomorphmism in topology which corresponds to the notion of isomorphism in algebras. Homeomorphism may be considered as the main equivalence relation between topological spaces.

Definition 4.23. *Let $(X, \mathcal{O}X)$ and $(Y, \mathcal{O}Y)$ be two topological spaces. Then, a continuous function $f : X \longrightarrow Y$ is a **homeomorphism** if:*

1. f is bijective

[21] Compare this definition with the definition 3.8.

2. f^{-1} *is continuous.*

f *is a* ***local homeomorphism*** *iff for each* $x \in X$*, there is an open neighborhood U(x) of* x *in* X *that is mapped homeomorphically by* f *onto* $f(U(x)) = \{f(y) : y \in U(x)\}$ *and* $f(U(x)) \in \mathcal{O}Y$*.*

If $f : X \longrightarrow Y$ is a homeomorphism(local homeomorphism), X and Y are called homeomorphic(locally homeomorphic).
If a property of a topological space remains the same for any two homeomorphic spaces, it is called a **topological invariant**. If two topological spaces are homeomorphic, one can get from one of these spaces to the other through continuous deformations. It is usually said that two homeomorphic topological spaces are topologically indistinguishable or identical.

Topological spaces can be used to construct new topological spaces such that the latter has different and useful properties. In the next subsection we give some prominent examples of such new spaces.

4.4 New Spaces from old spaces

Definition 4.24. *Let* $(X, \mathcal{O}X)$ *be a topological space and* $Y \subseteq X$*. Consider the set* $\mathcal{O}Y$ *of the intersection of* Y *with all open sets of* X*. This can be formulated as:*

$$\mathcal{O}Y = \{Y \cap B \mid B \in \mathcal{O}X\}$$

The pair $(Y, \mathcal{O}Y)$ *is called a* ***subspace*** *of* $(X, \mathcal{O}X)$*, and* $\mathcal{O}Y$ *is called the* ***subspace topology*** *for* Y*.*

Example 4.25. *Let $\mathbb{Q}$ be the set of rational numbers. Then we can define the induced topology on $\mathbb{Q}$ by the standard topology on $\mathbb{R}$ in the following way: $\mathcal{O}\mathbb{Q} = \{\mathbb{Q} \cap B \mid B \in \mathcal{O}\mathbb{R}\}$. The subspace topology for $\mathbb{Q}$ is $\mathcal{O}\mathbb{Q}$*

The inclusion map $i : Y \longrightarrow X$ is continuous with respect to the subspace topology $\mathcal{O}Y$.

Definition 4.26. *Let $(X, \mathcal{O}X)$ be a topological space, Y a set and $g : X \longrightarrow Y$ be an onto function. Then, the collection $\mathcal{O}_g X$ of subsets of Y defined by*

$$\mathcal{O}_g Y = \{G \subset Y | g^{-1}(G) \text{ is open in } X\}$$

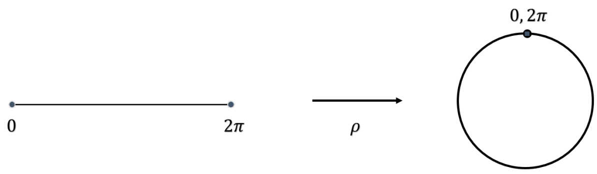
is a topology on Y, called the quotient topology induced on Y by g (Willard 1970, pp.59-60).

Definition 4.27. *Let $(X, \mathcal{O}X)$ be a topological space and $\simeq$ an equivalence relation on X, $g : X \longrightarrow X/_{\simeq}$ that maps each $x \in X$ to its equivalence class $[x]$ in $X/_{\simeq}$.*

A quotient topology on $X/_{\simeq}$ such that g is continuous is defined as $\mathcal{O}(X/_{\simeq}) := \{U \in X/_{\simeq} : g^{-1}(U) \in \mathcal{O}X\}$.

Intuitively, a quotient topology $\mathcal{O}(X/_{\simeq})$ is used for gluing together points of a set. For example, we can glue two end points of a segment.

Definition 4.29. *Let X and Y be two topological spaces. A topology on the Cartesian product of $X \times Y$ can be defined based on some basis of X and Y. If basis of X is U and some basis of Y is V, then the basis of $X \times Y$ is defined as the following collection: $B = \{U_1 \times V_1 \mid U_1 \in U \text{ and } V_1 \in V\}$.*

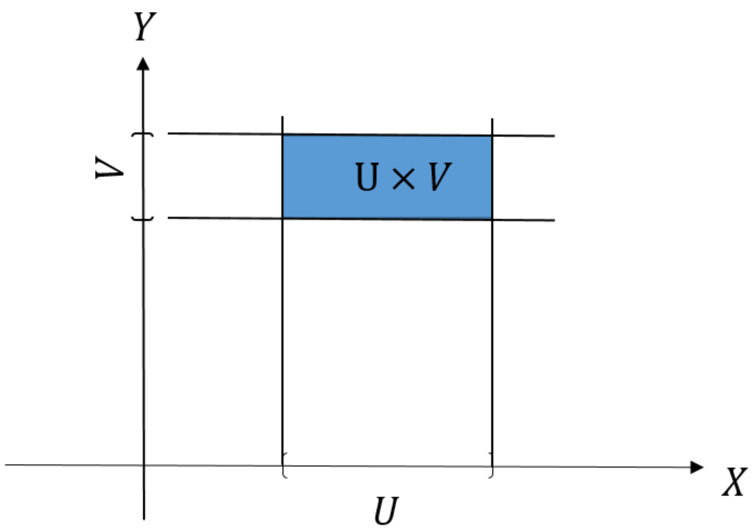


Example 4.28.

Figure 4: The quotient space with equivalence relation $[0, 2\pi]/(0 \simeq 2\pi)$ identifies the two end points of the interval.

This topology is called ***product topology****.*

The two canonical projections $\rho_1 : X \times Y \longrightarrow X$ and $\rho_2 : X \times Y \longrightarrow Y$, embed $\mathcal{O}X$ and $\mathcal{O}Y$ into $\mathcal{O}(X \times Y)$ respectively.



Example 4.30. *Let $\mathbb{R}$ be the set of real numbers, $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. The basis for the topology on $\mathbb{R}^2$ can be defined based on two basis of $\mathbb{R}$. The collection of all products of all open intervals of $\mathbb{R}$, $(a, b) \times (c, d)$, can be a basis for a topology on $\mathbb{R}^2$. Likewise, the basis for $\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n}$ can be the collection of the product of all open intervals of (a_i, b_i), $1 \leqslant i \leqslant n$.*

We will see that the product topology plays an important role for topological account of vagueness in dealing with the sorites paradox.

4.5 Separation axioms

We have seen different equivalent ways of defining a topology on a set and we have seen that there might be different topologies on a given set. Now one might want to consider a specific topology to be used for a certain aim but the definition of topology is too general to distinguish different types of topological spaces. So, it is expedient to ask for some restrictions on topologies. Separation axioms put some restrictions on topologies. They focus on distinct points and sets. According to Aull and Thron (1962), Urysohn for the first time studied the properties of such axioms. The restriction is on whether these points and sets are topologically distinguishable or not. For example, an indiscrete topology lacks such condition in the sense that points are not topologically distinguishable but the discrete topology does have such condition. Yet it is interesting to see how far the topologies between these two extreme topologies satisfy separation axioms. It is easy to see that in metric topologies points are distinguishable using the available metric but sometimes one looks for weaker or stronger topologies. Furthermore, as mentioned before, there are non-metric interesting topologies. The following axioms are the most well-known separation axioms that provide us some information about all topological spaces with regard to their distinguishability power:

Definition 4.31. *A topological space* $(X, \mathcal{O}X)$ *is called a* $T_0 -$ *space or a* ***Kolmogorov*** *space if for every two distinct points* x*,* $y \in X$*, there exists an open set* $U \in \mathcal{O}X$ *such that either* $x \in U$ *and* $y \notin U$ *or* $y \in U$ *and* $x \notin U$*.*

For example, let $X = \{a, b\}$ be endowed with a topology $\mathcal{O}X = \{\emptyset, \{a\}, X\}$ (Sierpiński topology).Then, X is T_0.

Definition 4.32. *A topological space $(X, \mathcal{O}X)$ is called a T_1- space if for every distinct $x, y \in X$ there exists open sets $U, V \in \mathcal{O}X$ containing x and y respectively such that $y \notin U$ and $x \notin V$.*

Lemma 4.33. *A topological space $(X, \mathcal{O}X)$ is T_1 iff each one-point set in X is closed.*

Proof. $(X, \mathcal{O}X)$ is $T_1 \iff \forall x \in X \quad \forall y \in X - \{x\}$, $\exists U$: $x \notin U, y \in U \iff \forall x \in X, X - \{x\}$ is open $\iff \forall x \in X, \{x\}$ is closed. $\square$

Every T_1-space is T_0. But not vice versa. For example, the Sierpiński topology is T_0, but not T_1.

Definition 4.34. *A topological space $(X, \mathcal{O}X)$ is called a T_2-**space** or a **Hausdorff space** if for every distinct $x, y \in X$ there exists open sets $U, V \in \mathcal{O}X$ containing x and y respectively such that $U \cap V = \emptyset$.*

In the Hausdorff space also every finite point set is closed. Every T_2-space is T_1 and T_0, but not vice versa. For example, a set X endowed with the cofinite topology is T_1 because all the points are closed, but it is not Hausdorff because no two nonempty open sets are disjoint.
However, it can easily be shown that all metric spaces are Hausdorff because we can separate each two distinct point of a metric space by two distinct open sets, having the distance between them. Since each Hausdorf space is T_1, the points in

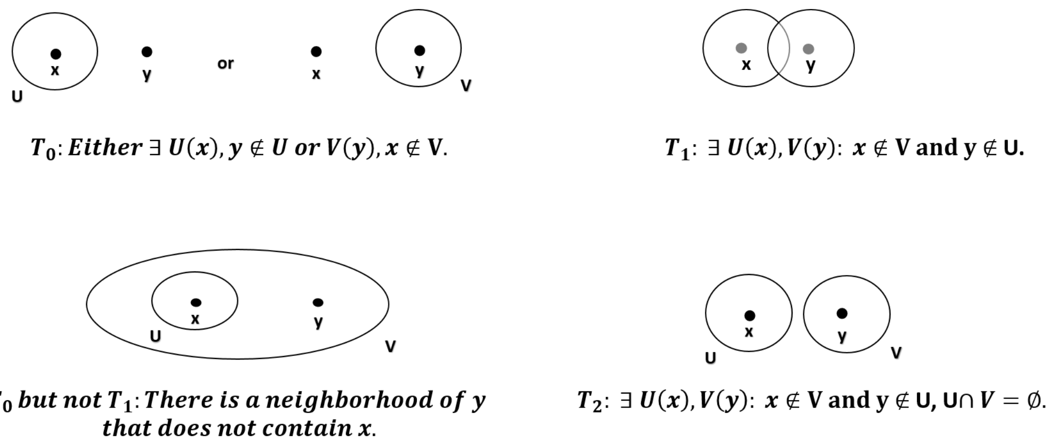


Figure 5: Topological separation axioms

a metric space are closed.

There are more separation axioms above T_2 but since they are not related to our discussions on vagueness we won't mention them. In this thesis we are interested in some non-Hausdorff topological spaces that are between T_0 and T_1. [22] In particular, in part V it is to be argued that polar topology is Alexandroff. For Alexandroff spaces the most interesting separations axioms are the ones between T_0 and T_1. We will see some examples of such axioms in part V.

$$T_2 \Rightarrow T_1 \Rightarrow T_{\frac{1}{2}} \Rightarrow T_0.$$

We now give some examples of topological spaces that are not metrizable.

Example 4.35. *a. The Sierpiński topology $\mathcal{O}X$ on a finite set X with two members, 0 and 1, such that $\mathcal{O}X$ contains the empty set, X and just the singleton of one of the members of X, $\{0\}$ or $\{1\}$, is not metrizable. Suppose that $\mathcal{O}X := \{\emptyset, \{1\}, X\}$. Since the only open set containing $\{0\}$ is X and $\{1\}$ is open, $\{0\}$ and*

[22] Axioms $T_0 < T_i < T_1$ can be found in modern works on topology such as Aull and Thron (1962) and Picado and Pultr (2011). Also the applications of non-Hausdorff spaces can be seen in Johnstone (1982).

$\{1\}$ are not separable by two disjoint open sets. So, this space is not Hausdorff and therefore, is not metrizable.

b. Consider the set of natural numbers $\mathbb{N}$ endowed with the topology $\mathcal{O}\mathbb{N}$ such that $\mathcal{O}\mathbb{N}= \{\emptyset\} \cup \{A \subseteq \mathbb{N} : 1 \in A\}$. It is easy to see that $\mathcal{O}\mathbb{N}$ is a topology on $\mathbb{N}$. It is not metrizable.

In general, one can get the series of such non metrizable topologies in the following way:

Let X be a set. Let $\mathcal{O}_p X := \{\emptyset\} \cup \{A \in X : p \in A\}$. Then, the topology is not metrizable since the singleton $\{p\}$ is open but all non-empty open sets contain it and so the points are not separable from p by disjoint open sets.[23]

4.6 Connectedness

Connectedness is a very important concept in topology. Intuitively, a topological space is connected if it cannot be decomposed into two (or more) disconnected parts. Before giving the formal definition of connectedness, we define clopen sets that we have mentioned before; the sets that are both open and closed. The reason is that the notion of connectedness in entangled with the notion of clopen sets.

As said before some sets may be both open and closed with respect to the given topology.

Definition 4.36. *Let $(X, \mathcal{O}X)$ be a topological space. A set in $\mathcal{O}X$ is called* ***clopen*** *iff it is both open and closed.*

In any topological space $(X, \mathcal{O}X)$, $\emptyset$ and X are clopen. But there are also

[23] It can be shown that in all these examples, the topologies are Alexandroff.

many other examples of clopen sets. For instance, $A = \{x \in \mathbb{Q} \mid \sqrt{2} < x < \sqrt{3}\}$ is a clopen set in $\mathbb{Q}$ with the topology induced by the standard topology in $\mathbb{R}$. It is closed since the irrational numbers, $\sqrt{2}$ and $\sqrt{3}$ do not belong to the set of rational numbers,$\mathbb{Q}$, so the complement of A in $\mathbb{Q}$ is open, because it is $[(-\infty, \sqrt{2}) \cup (\sqrt{3}, +\infty)] \cap \mathbb{Q}$. On the other hand, A is open because it is $(\sqrt{2}, \sqrt{3}) \cap \mathbb{Q}$. Therefore, $A \subseteq \mathbb{Q}$ is clopen. Even though the only clopen sets in $\mathbb{R}$ are the empty set and $\mathbb{R}$ itself, $\mathbb{Q}$ endowed with this topology contains a lot of clopen sets. For example, all the sets of the form (a, b); i.e., $\{x|a < x < b; a, b \in \mathbb{R} - \mathbb{Q}\} \subseteq \mathbb{Q}$ are clopen.

Definition 4.37. *Let $(X, \mathcal{O}X)$ be a topological space. Then, X is **connected** if and only if the whole set and the empty set are the only clopen subsets of X (Munkres 2000, p.148).*

Definition 4.38. *a. A topological space $(X, \mathcal{O}X)$ is **disconnected** if there are two disjoint non empty open(closed) sets, U and V, such that $X = U \cup V$.*

*b. A topological space $(X, \mathcal{O}X)$ is **totally disconnected** if for each distinct $x, y \in X$ there exists two disjoint open sets U and V containing x and y, respectively such that $U \cup V = X$ [p.191](Willard (1970)).*

Example 4.39. *Let $\mathcal{O}\mathbb{R}$ be the standard topology on the set of real numbers $\mathbb{R}$. Then*

a. $\mathbb{R}$ is connected.

b. $A = \mathbb{R} - \{0\}$ with the subspace topology is not connected.

c. The discrete space is totally disconnected while the indiscrete space is connected.

4.7 Topology via operators

There are different equivalent definitions of topology. We have mentioned three of them based on open sets, closed sets and neighborhood. However, topology can also equivalently be defined differently based on closure, interior and boundary operators (cf. Zarycki (1927)).
Kuratowski and Mostowski (1968) first proposed a definition of topology based on a closure operator.
There are different kinds of closure operators. We will start by general conditions that each closure operator should satisfy and then we will define other closure operators.

Definition 4.40. *A* ***closure operator*** *on X is a map $CL : \mathscr{P}X \longrightarrow \mathscr{P}X$ such that for any subsets of X, A and B, satisfies the following axioms:*

1. $A \subseteq B \Longrightarrow CL(A) \subseteq CL(B)$	(isotone or monotone)
2. $A \subseteq CL(A)$	(extensive)
3. $CLCL(A) = CL(A)$.	(idempotent)

Table 1: The general conditions for the closure operator

(X, CL) with the above conditions is called a **closure space**.
Similarly, there are different kinds of interior operators. In the following definition we can find the general conditions that each interior and closure operator should satisfy.

Definition 4.41. *A* ***interior operator*** *on X is a map $INT : \mathscr{P}X \longrightarrow \mathscr{P}X$ such that for any subsets of X, A and B, satisfies the following axioms:*

1. $A \subseteq B \Longrightarrow INT(A) \subseteq INT(B)$	(isotone or monotone)
2. $INT(A) \subseteq A$	(Decreasing)
3. $INTINT(A) = INT(A)$.	(idempotent)

Table 2: The general conditions for the interior operator

A set $A \subseteq X$ is closed iff $CL(A) = A$. It is open if its set-theoretical complement is closed with respect to CL. Similarly, a set $A \subseteq X$ is open iff $INT(A) = A$. It is closed if its set-theoretical complement is open with respect to INT.

We will later show that the extension of a predicate is a closed set with respect to a special closure operator.

Each special kind of closure operator adds some axioms to the three axioms mentioned above. In the course of the thesis we will introduce some closure operators, will compare them and choose one in particular to define vague concepts.

One of the most well-known closure operators is the one introduced by Kuratowski. It is stronger than CL; in the sense that Kuratowski's closure operator, cl, is closed under the union. There are other closure operators that have been used in different fields such as cognitive science, logic and mathematics among others. The concept of vagueness that deals with the concept of boundary is highly dependent on how we define the closure operator since the boundary of a set is usually defined as the intersection of closure of the set and closure of its complement. So, it is important to find out the right closure operator to deal with vagueness.

Definition 4.42. *(Kuratowski's topological closure operator)*

A topological closure operator on X is a map $cl : \mathscr{P}(X) \longrightarrow \mathscr{P}(X)$ such that for

$A, B \subseteq X$:

$K1)$ $cl(A \cup B) = cl(A) \cup cl(B)$ *(Union preservation),*

$K2)$ $clcl(A) = cl(A)$ *(idempotent),*

$K3)$ $A \subseteq cl(A)$, *(extensive)*

$K4)$ $cl(\emptyset) = \emptyset$ *(The empty set preservation).*

(X, cl) is called a topological space.

It is easy to see that monotonicity follows from axioms K1 to K4.

Topology can be equivalently defined via the topological interior operator :

Definition 4.43. *A topological interior operator on X is a map $int : \mathscr{P}X \longrightarrow \mathscr{P}X$ such that for $A \subseteq X$:*

1- $int(A \cap B) = int(A) \cap int(B)$ (Distributive over conjunction)

2- $intint(A) = int(A)$ (Idempotent)

3- $int(A) \subseteq A$ (Decreasing)

4- $int(X) = X$ (Total)

Also, monotonicity holds for int. If $A \subseteq B$ then $int(A) \subseteq int(B)$ (Monotone).

The topological spaces (X, cl) and (X, int) are equivalent (For the proof see Zarycki (1927)). Interior(Kernel) and closure(hull) are dual operators; i.e., $int = \mathtt{C}cl\mathtt{C}$ and $cl = \mathtt{C}int\mathtt{C}$. So, naturally, closed sets are the ones that remain invariant with respect to the closure operator and open sets are the ones that remain invariant with respect to the interior operator. Formally:

$$\mathcal{C}_k X = \{B \subseteq X; cl(B) = B\}$$
$$\mathcal{O}_k X = \{A \subseteq X; int(A) = A\}$$

Having a topological space $(X, \mathcal{O}X)$, one can define the closure and interior operators that satisfy Kuratowski's axioms, using the definition of closure and interior given in definition 4.2. And the other way around, $\mathcal{O}_k X$ defined in (X, cl) satisfies the three conditions, mentioned in definition 4.1, and therefore, defines the topology $(X, \mathcal{O}_k X)$.

In a nutshell, a topological space can be defined as a collection of points along with the set of open(closed) sets satisfying certain axioms or equivalently can be defined as a collection of points along with an operator(closure, interior, boundary) that satisfies certain conditions. In part V, we will show how Rumfitt uses the topological interior operator to introduce the polar topology for vagueness.

Proposition 4.44. *The composition of closure and interior of a set A is a closure operator* $\mathrm{int}\,\mathrm{cl} : \mathcal{O}X \longrightarrow \mathcal{O}X$.

Proof. To prove that $\mathrm{int}\,\mathrm{cl}$ is a closure operator, we should prove that it satisfies the three conditions mentioned in 4.40.
We show that $A \subseteq B$ implies $\mathrm{int}\,\mathrm{cl}(A) \subseteq \mathrm{int}\,\mathrm{cl}(B)$. Let us suppose that $A \subseteq B$. Then, by monotonicity of cl, $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$. By monotonicity of int, $\mathrm{int}\,\mathrm{cl}(A) \subseteq \mathrm{int}\,\mathrm{cl}(B)$.

Next we prove that $A \subseteq \mathrm{int}\,\mathrm{cl}(A)$.

Since cl is extensive, $A \subseteq \mathrm{cl}(A)$. By monotonicity of int, $\mathrm{int}(A) \subseteq \mathrm{int}\,\mathrm{cl}(A)$. Since A is open, $A = \mathrm{int}(A)$. Therefore, $A \subseteq \mathrm{int}\,\mathrm{cl}(A)$.

Finally, we show that *intcl* is idempotent; i.e., $\operatorname{int}\operatorname{cl}\operatorname{int}\operatorname{cl}(A) = \operatorname{int}\operatorname{cl}(A)$. Since cl is extensive, $\operatorname{int}\operatorname{cl}(A) \subseteq \operatorname{cl}\operatorname{int}\operatorname{cl}(A)$. By monotonicity and idempotency of int, $\operatorname{int}\operatorname{cl}(A) \subseteq \operatorname{int}\operatorname{cl}\operatorname{int}\operatorname{cl}(A)$.

Now it is enough to show that $\operatorname{int}\operatorname{cl}\operatorname{int}\operatorname{cl}(A) \subseteq \operatorname{int}\operatorname{cl}(A)$.
Since int is decreasing, $\operatorname{int}\operatorname{cl}(A) \subseteq \operatorname{cl}(A)$. By monotonicity and idempotency, $\operatorname{cl}\operatorname{int}\operatorname{cl}(A) \subseteq \operatorname{cl}\operatorname{cl}(A) = \operatorname{cl}(A)$. By monotonicity, $\operatorname{int}\operatorname{cl}\operatorname{int}\operatorname{cl}(A) \subseteq \operatorname{int}\operatorname{cl}(A)$. □

Proposition 4.45. *Let $(X, \mathcal{O}X)$ be a topological space, $A, B \in \mathcal{O}X$. The interior closure operator* $\operatorname{int}\operatorname{cl} : \mathcal{O}X \longrightarrow \mathcal{O}X$ *satisfies the following condition:*

$\operatorname{int}\operatorname{cl}(A) \cap \operatorname{int}\operatorname{cl}(B) \subseteq \operatorname{int}\operatorname{cl}(A \cap B)$ *for all A, B in $\mathcal{O}X$.*

Proof. From left to right:
A, B are open sets. So, $\operatorname{int}(A) = A$, $\operatorname{int}(B) = B$. It is easy to see that if A is open, $A \subseteq \operatorname{int}\operatorname{cl}(A)$. B is open, so, $B \subseteq \operatorname{int}\operatorname{cl}(B)$. $(A \cap B) \subseteq A \subseteq \operatorname{int}\operatorname{cl}(A)$ and $(A \cap B) \subseteq B \subseteq \operatorname{int}\operatorname{cl}(B)$. Therefore, by idempotency of int cl, $\operatorname{int}\operatorname{cl}(A \cap B) \subseteq \operatorname{int}\operatorname{cl}(A) \cap \operatorname{int}\operatorname{cl}(B)$.

□

The int cl operator differs from the Kuratowski closure operator cl in the sense that it is not closed under union. For example, in the standard topology on $\mathbb{R}$, let $A = (1, 2), B = (2, 4)$. Then, $\operatorname{int}\operatorname{cl}((1, 2) \cup (2, 4)) = (1, 4) \neq \operatorname{int}\operatorname{cl}(A) \cup \operatorname{int}\operatorname{cl}(B) = (1, 2) \cup (2, 4)$.

Certain properties of a set are defined based on these compositions. We are interested in this closure operator since, following Rumfitt, the extension of a predicate is a closed set with respect to int cl. This means that it is regular open.

Definition 4.46. *Let X be a topological space and $A \in \mathcal{O}X$. A is **regular open** if* $\operatorname{int}\operatorname{cl}(A) = A$. *It is called **regular closed** if* $\operatorname{cl}\operatorname{int}(A) = A$.

Regular open sets lack cracks:

Definition 4.47. *Let $(X, \mathcal{O}X)$ be a topological space, $A \in \mathcal{O}X$. Then, x is a **crack** of A iff $x \notin A \wedge x \in \operatorname{int}\operatorname{cl}(A)$.*

For example, consider the real line and the usual topology on it based on the open intervals. Then $A = (a, b) \cup (b, c)$ is not regular open because $\operatorname{int}\operatorname{cl}(A) = \operatorname{int}[a, c] = (a, c) \neq A$. In this example b is a crack. However, A is regular open if $X = \mathbb{R} - \{b\}$ endowed with the subspace topology. In this case A does not have any crack. An open circle is regular open in $\mathbb{R}^2$. Intuitively, all the regular open subsets of $\mathbb{R}^2$ are the ones that do not have any cracks or pinholes.

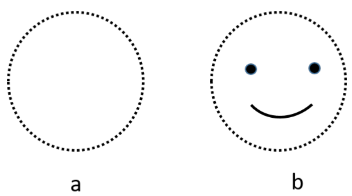


Figure 6: a. Regular open set , b. Non-regular open set with some cracks as point or line

$B = [a, c]$ is regular closed in $\mathbb{R}$ because $\operatorname{cl}\operatorname{int}(B) = cl((a, c)) = [a, c]$ whereas $C = \{b\}$ is not regular closed because $\operatorname{cl}\operatorname{int}(C) = cl(\emptyset) = \emptyset$.

Proposition 4.48. *The set of all regular open sets, denoted by $\mathcal{O}_{reg}X$ and the set of all regular closed sets, denoted by $\mathcal{C}_{reg}X$ are complete Boolean algebras.*

The family of all regular open subsets of a topological space $(X, \mathcal{O}X)$ forms a Boolean algebra with top and bottom as X and $\emptyset$ respectively and operations as: $x \vee y = \operatorname{int}\operatorname{cl}(x \cup y)$; $x \wedge y = x \cap y$; $\neg x = int(\complement X)$. Analogously, you can define the operators of regular closed sets.[24]

According to this proposition, the behaviour of the members of the family of regular open sets conform to the rules of the Boolean algebra. By the definition of negation, it is easy to see that the topological interpretation of double negation on open sets of a topological space is int cl of it: For $U \in \mathcal{O}X, \neg\neg U = \operatorname{int}\operatorname{cl}(U)$. So, for regular open sets $\neg\neg U = U$.

Definition 4.49. *Let X be a topological space and $A \subseteq X$. A point p is a **limit point** of A if every open set containing p contains at least one point of A distinct from p. In other words, p is a limit point of A if every neighborhood of p contains a different point q in A. If we drop the requirement of the distinction of the point of p and q, p is called an **adherent** point.*

A closure operator adds all the limit points to a given set.

Definition 4.50. *Let $(X, \mathcal{O}X)$ be a topological space and $A \subseteq X$. Boundary is defined as:*

$$bd(A) := \operatorname{cl}(A) \cap \operatorname{cl}(\complement A).$$

Proposition 4.51. *The boundary of any set is closed.*

Proof. Since boundary is the intersection of two closed sets it is closed. □

Remark 4.52. *In general, an infinite union of closed sets may not be closed.*

[24]For the proof see chapter 10 of Givant and Halmos (2008).

For example, consider the closed interval $[\frac{1}{n}, 1]$. Then, $\bigcup_{n=1}^{\infty}[\frac{1}{n}, 1] = (0, 1]$ which is not closed. Applying the boundary operator to some sets sometimes amplifies the set. For example, consider $\mathbb{R}$ and the standard topology on it. For $\mathbb{Q} \subset \mathbb{R}$, $\mathrm{cl}(\mathbb{Q}) = \mathbb{R}$. Also, $\mathrm{cl}(\mathbb{R} - \mathbb{Q}) = \mathbb{R}$. So, the boundary of $\mathbb{Q}$ is $\mathbb{R}$ which shows that the boundary of $\mathbb{Q}$ is much bigger than $\mathbb{Q}$ itself. And it is the same for the complement of $\mathbb{Q}$, the set of irrational numbers.

Definition 4.53. *Two sets A and B are separated if*

$$\mathrm{cl}(A) \cap B = A \cap \mathrm{cl}(B) = \emptyset.$$

Proposition 4.54. *The family of all clopen subsets of a topological space is a Boolean algebra.*[25].

Boolean algebras give semantics to the classical propositional calculus. As explained before, other complete lattice is a Heyting algebra. Heyting algebras provide a semantics for the intuitionistic propositional calculus. All Boolean algebras are Heyting but not vice versa.

Proposition 4.55. *$bd(A) = \emptyset$ iff A is clopen.*

Proof. $\Longleftarrow$

If A is clopen then it is closed and so its complement is open. But A is also open and so its complement is closed so $\mathrm{cl}(\mathtt{C}(A)) = \mathtt{C}(A)$.

therefore, $bd(A) = \mathrm{cl}(A) \cap \mathrm{cl}(\mathtt{C}A) = A \cap \mathtt{C}(A) = \emptyset$.

$\Longrightarrow$

Suppose $bd(A) = \emptyset$. We should prove that A is clopen. By definition, a set is

[25] For the proof see Givant and Halmos (2008)

clopen if it is both open and closed. By reductio ad absurdum, suppose A is not clopen. Suppose A is not closed. Then, $\exists a \in \text{cl}(A) : a \notin A$. Hence, $a \in \text{cl}(\complement(A))$. Therefore, $a \in \text{cl}(A) \cap \text{cl}(\complement(A))$. This contradicts the assumption that $bd(A) = \emptyset$. If A is not open, there exists $a \in A$, such that $a \notin \text{int}(A)$. Since $a \in A$, by the extension, $a \in \text{cl}(A)$. Since $a \notin \text{int}(A), a \in \text{cl}(\complement A)$. Therefore, $a \in \text{cl}(A) \cap \text{cl}(\complement A)$ and therefore, $a \in bd(A)$.

□

Definition 4.56. *Let $(X, \mathcal{O}X)$ be a topological space and $A \subseteq X$ $x \in X$.*

1. *A is **dense** in X if $\text{cl}(A) = X$.*
2. *X is **dense-in-itself** if for all $x \in X, x \in \text{cl}(X - \{x\})$.*
3. *A is **nowhere dense** if $\text{int}\,\text{cl}(A) = \emptyset$.*
4. *x is an **isolated** point of A if $\{x\}$ is open in the subspace topology on A. This means that there is an open set that contains x and does not contain any other member of A. The set of isolated points is denoted by $ISO(A)$.*
5. *X is **weakly scattered** if $ISO(X)$ is dense in X.*
6. *X is scattered if every non-empty subspace of X contains an isolated point.*

It can easily be seen that a dense-in-itself set does not contain any isolated points (Steen and Seebach 1995, p.6).

Example 4.57. *a. Let $\mathcal{O}\mathbb{R}$ be the standard topology on $\mathbb{R}$. Then, $\mathbb{Z} \subseteq \mathbb{R}$ is nowhere dense. The set A is nowhere dense if none of the nonempty open sets of X are contained in $\text{cl}(A)$. Since the $\text{cl}(\mathbb{Z}) = \mathbb{Z}, \text{int}\,\text{cl}(\mathbb{Z}) = int(\mathbb{Z}) = \emptyset$.*
$\mathbb{Q} \subseteq \mathbb{R}$ and $\mathbb{R} - \mathbb{Q} \subseteq \mathbb{R}$, the set of rational and irrational numbers, are dense subsets of $\mathbb{R}$ because $\text{cl}(\mathbb{Q}) = \text{cl}(\mathbb{R} - \mathbb{Q}) = \mathbb{R}$

b. Let $X = \{1, 2, 3, 4, 5\}$ *and* $\mathcal{O}X = \{\emptyset, \{1\}, \{3, 4\}, \{1, 3, 4\}, \{2, 3, 4, 5\}, X\}$. *It can easily be checked that* $(X, \mathcal{O}X)$ *is a topological space.* $\{1, 3\}$ *is dense in* X *because* $\mathrm{cl}(\{1, 3\}) = X$.

Part IV

A critical review on some topological approaches to vagueness

5 Vagueness, topology and formal epistemology

In "Vagueness, Kant and topology: A study of formal epistemology " that was published in 2008 Boniolo and Valentini give a topological definition of vagueness. They said:

> We show that its [vagueness] natural home rests into a topological formalism (Boniolo and Valentini 2008, p.142).

They propose a new perspective to vagueness based on the Kantian idea that the first and most important factor in the cognitive process is the knowing subject that has a conceptual apparatus with which he can cognitively grasp the world (ibid, p.142). Their main idea is that:

> knowing subject *constitutes* the world in a cognitively significant way... Concepts *constitute* objects by giving them *cognitive significance* and thus inserting them into a class (ibid: 143).

Then they formalize topologically the constitution as ″the process by which the knowing subject imposes his conceptual apparatus on the world to render it cognitively significant.″ibid.

So, in their transcendental approach to vagueness it is the knowing subject with the conceptual apparatus that is significant. In their view each concept has a finite number of characteristics. For example, ‘basketball' is an object that is ‘spherical', ‘elastic', has certain weight....

In the next part of the paper they give the topological explication of vagueness. We will first give their main definitions by which they define the boundary of a concept and vague concepts.Then we will show that their definition of boundary is a dual of Zarycki's notion of border. Furthermore, we will show that, contrary to what they claim, the higher order vagueness still will be problematic for their account of vagueness.

5.1 Topological definition of vagueness

As we have said, in Boniolo and Valentini's approach the cognitive role of the knowing subject is very significant. If one wants to render the world cognitively significant, they need to impose their concept apparatus.
Now let's see how they formalize their transcendental approach and how they define vagueness topologically. Following the authors, we start off by some preliminaries. They consider a set of basic concepts, called characteristics. Concepts, then, are defined as finite sets of characteristics. Formally,

Definition 5.1. *Let C be a countable set of basic concepts, called characteristics. Then a concept ν over C is a non-empty finite subset of C , that is, for $c_i \in C$, $1 \leq i \leq n$*

$$\nu = \{c_1, ..., c_n\}$$

So, each concept over C is "a *name* given to a finite set of characteristics" (Boniolo and Valentini 2008, p.145).
They define 'Δ' as a set of concepts that belong to the conceptual apparatus of the knowing subject:

$$\triangle := \{\{c_1, ..., c_n\} | c_1, ..., c_n \in C\}$$

Definition 5.2. *The conjunction of concepts* $\nu_1 = \{c_1, ..., c_n\}$ *and* $\nu_2 = \{d_1, ..., d_n\}$ *is defined as:*

$$\nu_1 \wedge \nu_2 := \{c_1, ..., c_n, d_1, ..., d_n\}$$

So, the binary relation of conjunction between two concepts is defined as the union of characteristics of those concepts.
Before defining a topology on a set of objects, they define a crucial relation, namely, constitution relation:

Definition 5.3. *Let* Δ *be the set of concepts, and* X *be the set of objects. Given any object* $o \in X$ *and any* $\nu \in \Delta$*, define a binary relation* $\Vdash$*, called* constitution relation *as:*

$$o \Vdash \nu$$

It means that the concept ν is one of the concepts cognitively constituting o. Naturally, o is constituted by a concept if and only if it is constituted by all finite characteristics of the concept. This is formalized as the constitution rules:

Constitution rules

$$\frac{o \Vdash \nu \quad c \in \nu}{o \Vdash \{c\}}$$

$$\frac{\forall c \in \nu \quad o \Vdash \{c\}}{o \Vdash \nu}$$

All objects are constituted at least by a concept:

$$\forall o \in X \quad \exists \nu \in \Delta \quad o \Vdash \nu \qquad \text{(Concept condition)}$$

This condition reveals the core idea of the authors according to which among the objects in the world "the only objects which are cognitively *visible* to the knowing subject are those that he constitutes by means of some concepts of his."(ibid. p. 147)
From the concept condition and the fact that a characteristic c can be defined as a concept $\{c\}$, they introduce the characteristic condition:

$$\forall o \in X \quad \exists c \in C \quad o \Vdash \{c\} \qquad \text{(Characteristic condition)}$$

Then they define Extension of a characteristic and a concept:

Definition 5.4. *Let X be a set of objects, $c \in C$ be a characteristic. Then, the extension of c is the set of all the objects constituted by $\{c\}$*

$$Ext : C \longrightarrow 2^X$$
$$Ext(c) := \{o \in X | \quad o \Vdash \{c\}\}$$

So, for each given object o, there is always a characteristic that constitutes it (o is in the extension of the characteristic).
The function Ext can be expanded. The extension of a concept, ν, is defined as the subset of the objects in X constituted by ν:

Definition 5.5. *Let X be a set of objects, $\nu \in \Delta$.*

$$Ext : \Delta \longrightarrow 2^X$$
$$Ext(\nu) := \{o \in X | \quad o \Vdash \nu\}$$

The authors prove that

$$Ext(\nu) = \bigcap_{c \in \nu} ext(c)$$

In this way they formulate the extension condition as :

$$\forall o \in X \quad \exists \nu \in \Delta \quad o \in Ext(\nu) \qquad \text{(Extension condition)}$$

The extension $Ext(\nu)$ [26] will give us exactly the objects that satisfy all characteristics of a concept and $Ext(c)$ contains all objects that have the characteristic 'c'. For example, basketball, volleyball and tennis ball are all round but they are different because they do not share the characteristic of having certain size.
Using the definitions and associativity of intersection the following important lemma can be proved[27]:

Lemma 5.6. *Let $\nu_1, \nu_2 \in \Delta$. Then*

$$Ext(\nu_1 \wedge \nu_2) = Ext(\nu_1) \cap Ext(\nu_2)$$

It follows from the definition that the extension of the empty set contains all objects in X.

They use the function Ext to provide a base for a topology on X.
Now let U be a set of concepts. Then again the function Ext can be expanded in the following way:

Definition 5.7. *Let $U \subseteq \triangle$, and X be a set of objects. Then, the extension of U is defined by:*

[26] 'Ext' is what is called extent in the formal concept analysis. To see the details and to compare it with Boniolo and Valentini's approach see Davey and Priestley 2002, ch.3.

[27] See (Boniolo and Valentini 2008, p.148)

$$Ext : 2^{\Delta} \longrightarrow 2^{X}$$
$$Ext(U) := \bigcup_{\nu \in U} Ext(\nu)$$

Define $U \vee V := U \cup V$ and $U \wedge V := \{\nu_1 \wedge \nu_2; \nu_1 \in U, \nu_2 \in V\}$.

So, the extension of a set of concepts is the objects that are constituted at least by one of those concepts. For example, if $U = \{\nu_1, \nu_2\}$, $Ext(U) = Ext(\nu_1) \cup Ext(\nu_2)$.

This is, of course, different from the extension of two sets of concepts and the union of two sets of concepts. We do not look for objects that are constituted by both concepts. Rather, we are interested in finding the objects that are constituted at least by one of those concepts.

The authors show that: for $U_i \in \Delta$

$$Ext(U_1 \wedge U_2) = Ext(U_1) \cap Ext(U_2)$$
$$Ext(\bigcup_{i \in I} U_i) = \bigcup_{i \in I} Ext(U_i)$$

This shows that $Ext(U)$ is closed under the union and finite intersection.

We will call the triple (X, C, Ext) t-distribution.[28] A t-distribution classifies the members of X based on the characteristics that constitute them. Using a t-distribution the authors define a topology on X:

Theorem 5.8. *Let (X, C, Ext) be a t-distribution. If the map Ext satisfies the extension condition, then the family of $\{Ext(U) | U \subseteq \Delta\}$ is a topology on X [29] (Boniolo and Valentini 2008, p.153).*

[28] t-distribution stands for the distribution in the transcendental account of vagueness.

[29] For the proof, see Boniolo and Valentini 2008, pp. 149-50

The philosophical transcendental idea is formalized by that extension map that they call the "constitutive morphism":

> We start from a set of concepts structured in a certain manner and we arrive at a set of objects which, in this way, has been rendered both cognitively significant and cognitively structured (ibid, p.150).

In this topology, the operations closure and interior are characterised as:

Proposition 5.9. *Let (X, C, Ext) be a t-distribution, $A \subseteq X$. The closure* $\mathrm{cl} : 2^X \longrightarrow 2^X$ *is characterized as:*

$\mathrm{cl}(A) := \{o \in X | \forall \nu \in \Delta(o \in Ext(\nu) \to Ext(\nu) \cap A \neq \emptyset)\}$ *(Boniolo and Valentini 2008, p.152)*

Also, one can characterise the interior, $\mathrm{int} : 2^X \longrightarrow 2^X$ *as:*

$\mathrm{int}(A) = \{o \in X | o \in A \wedge \exists \nu \in \Delta\, (o \in Ext(\nu) \wedge Ext(\nu) \subseteq A.)\}$

To give an account of vagueness they define the concept of border:

Definition 5.10. *Let $Bd(A)$ denotes the border of A. Then,*

$$Bd(A) := \mathrm{cl}(A) \cap \complement A = \mathrm{cl}(A) - A.$$

It is good to mention that many years ago, Zarycki (1927) made a difference between the standard boundary and the border. Using, as usual, bd for the standard boundary and $b_z(A)$ for Zarycki's border, we have:

$$b_z(A) := A \cap bd(A) = A \cap \mathrm{cl}(\complement A).$$

So, $Bd(A)$ is Zarycki's $b_z(\complement A)$. That is to say, $Bd(A) = b_z(\complement A) = bd(A) \cap \complement A$. For Zarycki, the notion of boundary is broader than the notion of border whereas the intuitive notion of boundary for Boniolo and Valentini should be formalized as border. Nevertheless, following Zarycki they could show that Bd, satisfying certain axioms is equivalent to Kuratowski's system. In other words, Bd can be considered as a primitive notion and other operators can be defined based on Bd.

They define vagueness as:

Definition 5.11. *Let ν be any concept and U be any set of concepts. Then ν is a vague concept if $Bd(Ext(\nu))$ is not empty, and U is a vague set of concepts if $Bd(Ext(U))$ is not empty*

5.2 Criticism

The transcendental approach to vagueness is quite interesting. Yet, it seems to us that it is not as successful in solving problems of vagueness as the authors claim.

One might argue that $BdBd(A)$ for $A \subseteq X$, is not always empty. More precisely, it is empty if A is open or A is closed . $BdBd(A) = \mathrm{cl}(Bd(A)) \cap \complement Bd(A)$.

Now if A is closed then $cl(A) = A$. So,

$BdBd(A) = \mathrm{cl}(A \cap \complement A) \cap \complement(A \cap \complement A) = \emptyset$.

if A is open, then $\complement A$ is closed and $Bd(A)$ is also closed. $BdBd(A) = \mathrm{cl}(Bd(A)) \cap \complement Bd(A) = \emptyset$.

However, the following example shows that $BdBd(A)$ is not always empty:

Let $(\mathbb{R}, \mathcal{O}\mathbb{R})$ be the standard topological space, $\mathbb{Q}$ the set of rational numbers. Then,

$Bd(\mathbb{Q}) = \mathrm{cl}(\mathbb{Q}) \cap \complement(\mathbb{Q}) = \complement(\mathbb{Q}) = \mathbb{R} - \mathbb{Q} \neq \emptyset$.

$BdBd(\mathbb{Q}) = Bd(\mathbb{R} - \mathbb{Q}) = \mathrm{cl}(\mathbb{R} - \mathbb{Q}) \cap \complement(\mathbb{R} - \mathbb{Q}) = \mathbb{R} \cap \mathbb{Q} = \mathbb{Q} \neq \emptyset$.

Calculating further iterations of Bd shows that $Bd^n(\mathbb{Q}) = \mathbb{R} - \mathbb{Q}$ if n is odd. If n is even, then $Bd^n(\mathbb{Q}) = \mathbb{Q}$.

This counterexample, though, may not be accepted by the authors because for them, the extension of a concept is an open set while $\mathbb{Q}$ is not open in the standard topology on $\mathbb{R}$. So, as we showed, if A is open, $BdBd(A) = \emptyset$.[30]

This shows that in their view there is no higher-order vagueness at all to be explained. [31] This leads us to the second possible criticism.

Boniolo and Valentini contend that if one gives a precise definition of vagueness, then there will not be any problem of higher-order vagueness. According to them, their definition of boundary and vagueness is precise and therefore there will not be any problem of higher-order vagueness:

> Note that our explication of the notion of 'border of subset of objects' is precise and rigorous, and not vague at all. Therefore, there is no vagueness in stating which objects are internal, external, and in the border of the extension of a given concept.... Thus, by using our formalization, the problem of higher order vagueness seems to be totally vanished (Boniolo and Valentini 2008, p.154).

We think that they are too quick in their inference from having a precise notion of border and vagueness to solving the problem of higher-order vagueness. We can mention two main reasons. For one thing, we have the intuition that there is higher-order vagueness. The problem is that even if there is no vagueness of

[30] This might seem a limitation for their account but we think it makes sense that the extension of rational numbers has a sharp boundary. A number is either rational or irrational. There is no borderline case. The same goes for irrational numbers. It is good to know whether we can limit the extension of concepts to regular open sets, as Rumfitt does. See section V for more discussions.

[31] Note that this is different from the idea that there are higher-order vagueness but they collapse to the first order. We will discuss in the next part to compare this view with Rumfitt's topological approach.

higher order, many theories of vagueness cannot explain why it seems so intuitive to us. Therefore, claiming that the higher- order vagueness is totally vanished does not solve the problem of higher-order vagueness.
They may explain that since the notion of border highly depends on the conceptual apparatus of the knowing subject, one may consider an object as a borderline case of a concept while another person consider the same object as a clear case of it.[32] So, it may seem to the knowing subject that the notion of border is vague. This answer, however, is not convincing. Even if there is just one knowing subject with his conceptual apparatus, it is quite intuitive that there is no sharp line between the clear cases of a concept and its boundary. They refer to the questions that any theory of vagueness should answer. One of them is to explain why we are ignorant about boundaries. The authors believe that "there is no vague boundaries in nature but only in the way we cognitively constitute nature by the conceptual apparatus" (Boniolo and Valentini 2008, p.166).
As we saw, they explained the nature of boundary by giving the precise definition of border. Border is defined within the conceptual apparatus of a human being. So, still they need to explain, why it seems to me, as a human being (not in communication with others) that, for example, in the spectrum of color there is no sharp, precise boundary between red and orange.

As for the second reason, the precise definition of border divides the objects into three sharp groups, those that are in the extension of the concept, those that are completely out of its extension and the borderline cases. It seems that, like

[32]The authors consider this point as a possible problem for their approach because it might seem that the communication won't be possible when the knowing subjects have different conceptual apparatus (Boniolo and Valentini 2008, p.154). We do not consider it as a serious problem. See how Gärdenfors (2014) is faced with the same problem and in what way he deals with it in favor of his conceptual space approach.

the proponents of three valued logic, they just add more sharp lines.

In the next part, we propose another topological account of vagueness.

6 Topological sorites

Another topological definition of vagueness is given by Weber and Colyvan (2010). Using the topological apparatus, they formulate the Sorites paradox without limiting themselves to a numerical ordering. Their general approach, also, is supposed to be neutral in respect of theories of vagueness such as supervaluationism, epistemicism, etc. When the definition of vagueness is based on having borderline cases a gappy approach already puts aside glutty approach according to which there is an inconsistent boundary(in which borderline cases of a predicate both belong to its extension and anti-extension) or if one defines vagueness in terms of having unknown sharp boundary, then already supervaluationism will stay out of the game. So, they give a definition of vagueness based on sorites susceptibility. According to them:

> A predicate is vague just in case it can be employed to generate a sorites argument (Bueno and Colyvan 2012, p.29).

It is good to mention that, unlike Boniolo and Valentini, the borderline cases for Weber and Colyvan are included in the extension of a concept. In search of a unified characterization of the Sorites paradox Weber and Colyvan propose a topological approach in which the extension of a concept is a closed set.

In this part we will focus on the topological definition of vagueness that they propose, we will discuss its pros and cons and later in part V we will compare it with Rumfitt's topological account of vagueness.

To start off, let's consider two kinds of the Sorites paradox:

1. Numerical Sorites
   a. Discrete
   b. Continuous
2. non-numerical Sorites

The discrete numerical Sorites is quite well-known. For example, the vague predicate 'bald'. They consider 'tall' as an example for continuous numerical Sorites. Though, as we have seen in part II, it is usually considered as the discrete one. One of the examples of non-numerical Sorites is converting gradually from one religion to another, the very gradual transition from Buddhism to Zoroastrianism is a continuous non-numerical example.

Weber and Colyvan (2010) give a topological definition of vagueness, taking into account not only the discrete version of the Sorites paradox, but also the continuous one. Furthermore, they consider family resemblance examples of non-numerical cases of the Sorites paradox. Their idea is that:

> ... a narrow focus on the discrete, numerical versions such as heap of sand obscures what really drives the paradox (Weber and Colyvan 2010, p.312).

As we saw in part II, classically, vague predicates give rise to the Sorites paradox.

In tallness, baldness and heap, the number of height, hair and grain are in order. The authors present a very general characteristic of the Sorites paradox, namely a topological version of the Sorites paradox in which the order in the sorites series is not necessarily important. (Weber and Colyvan 2010, p.313). Considering the Euclidean idea that the science of space is related to the qualitative notions like closeness rather than quantitative ones, they find topological spaces suitable spaces in which vagueness can be defined. In Weber and Colyvan (2010) the authors center their attention on point-set topology to construct a general Sorites paradox which can be continuous. Instead of suggesting a treatment for a disease, they try to deeply assess the disease, its causes and symptoms. According to them, what the Sorites paradox, generally, shows is that our intuition that the space is connected and our intuition that in the sorites series somewhere a change happens apparently lead to a contradiction. The aim of Weber and Colyvan is not to resolve the Sorites paradox. Rather, they try to explain and formulate a general form of the Sorites paradox.[33] In that regard, they discuss the problem set-theoretically and topologically.

In part V, following Rumfitt, we will show that there is no need to limit ourselves to point-set topology and we do not need to deviate drastically from classical logic to resolve the Sorites paradox.

6.1 Continuous Sorites

Following James Chase, Weber and Colyvan propose the continuous Sorites paradox, focusing on the properties of the real line.[34]

Let us see how Weber and Colyvan (2010) use Dedekind's cuts[35], to define the

[33]Recently, Weber (2021) has proposed the paraconsistent topological approach to vagueness to resolve the paradox. In the current work, we just focus on the formulation of the Sorites that Weber and Colyvan suggested in their paper.

[34]They have used a manuscript of Chase. See also Chase (2016) in defence of paraconsistent account of vagueness.

[35]In 1872 Dedekind proposed a method to construct the real numbers from the rational numbers. In this method the elements of $\mathbb{R}$ are some subsets of $\mathbb{Q}$ called cuts. Each real number can be identified with a specific

continuous version of the Sorites paradox.

In particular, they use the following properties of $\mathbb{R}$:

i. **The least upper bound principle:** Any non-empty set of real numbers bounded from above has a least upper bound(Sup).

ii. **greatest lower bound principle:** Any non-empty set of real numbers bounded from below has a greatest lower bound(Inf).

iii. The real numbers are dense in the sense that if $\forall x, y \in \mathbb{R}$ if $x < y$, then $\exists z \in \mathbb{R}\ x < z < y$ (See definition 4.56).

For the authors the tolerance principle for the continuous Sorites can be defined as:

Definition 6.1. ***Continuous tolerance principle:***

For a vague predicate F, if a is F and b is vanishingly close to F, then b is also F.

Now we are ready to see how they formulate the continuous Sorites paradox. The rough idea is that the least upper bound of Fs is both F and not F, that is a contradiction.

Let 'F' be a vague predicate, whose extension is included in $[0, 1] \subseteq \mathbb{R}$. Define two non-empty subsets of $[0, 1]$ as

$$A = \{x \in [0, 1] : F(x)\}$$
$$B = \{x \in [0, 1] : \neg F(x)\}$$

such that the following properties hold:

1. $A \cap B = \emptyset$

cut.

2. $F(0)$
3. $\neg F(1)$
4. $\forall a \in A, b \in B \qquad a < b.$
5. $\forall a \in A, b \in B \qquad \neg F(a) \wedge b > a \rightarrow \neg F(b).$

So, $[0, 1]$ is divided into two disjoint non-empty sets.[36] Everything on the left is F and everything on the right is ¬ F. If something is F, everything before that is F and if something is not F, everything after it is not F. By 4 and due to the properties of $\mathbb{R}$,

A has the least upper bound; call it $SupA$. Similarly, B has the greatest lower bound; call it $InfB$.[37] By definition 6.1, since the members of A that are very close to $SupA$ are F, $F(SupA)$. Similarly, since the members of B that are very (vanishingly) close to $InfB$ are not F, $\neg F(InfB)$. This version of the Sorites paradox is highly based on the linearly ordered connected dense real line. Since $\mathbb{R}$ is linear, if $InfB \neq SupA$, either $InfB < SupA$ or $SupA < InfB$. Since $\mathbb{R}$ is dense, if $InfB < SupA \ , \exists z_1$ such that $InfB < z_1 < SupA$. If $SupA < InfB, \quad \exists\, z_2\ SupA < z_2 < InfB$. It can easily be seen the problem because z_1 and z_2 are both F and ¬ F. If $SupA = InfB$, then $F(SupA)$ and $\neg F(SupA)$ which is a contradiction.

Up to now, the authors have given a continuous version of the Sorites based on the properties of the real line which has the property of being connected in

[36]The subsets A and B constitute a cut in Dedekind's sense. Remind that Dedekind used cuts to construct $\mathbb{R}$ from $\mathbb{Q}$. A cut is a pair $(A, B), A, B \subseteq \mathbb{Q}$ that satisfies the following properties:
$A, B \neq \emptyset$; If $a \in A$ and $c < a$, then $c \in A$; If $b \in B$ and $c > b$, then $c \in B$; If $b \notin B$ and $a < b$, then $a \in A$; If $a \notin A$ and $a < b$, then $b \in B$; $\forall a \in A, \exists b > a$ so that $b \in A$. $\mathbb{R}$ is defined to be the set of all Dedekind cuts.

[37]See definition 10.16.

the sense that if $\mathbb{R}$ is the union of two non-empty disjoint closed sets A, B, there is at least a number that is adherent [38] to both sets (Weber 2010, p.316).

Weber and Colyvan do not stop at this point. They suggest a more general formulation of the Sorites paradox in which there might be no order and no metric. As we saw in part III, there are topologies that are not induced by metrics. An important requirement in the topological formalization of the Sorites is that the space needs to be connected. In the next subsection we explain the mathematical details of their topological approach.

6.2 Topological Sorites

Weber and Colyvan define local constancy as the generalization of the tolerance principle. For them the characteristic function of a vague predicate is locally constant but fails to be globally constant. Roughly, this, as we will explain in detail in this subsection, means that a vague predicate 'F' is constant in a specific neighborhood.

Definition 6.2. *Let $(X, \mathcal{O}X)$ be a topological space. A function $f : X \to Y$ is **locally constant** iff for each $x \in X$ there is a neighborhood U_x such that the restriction of f to U_x is constant. A **globally constant** function always takes the same value, without restriction (ibid, p.318).*

In a discrete version of the Sorites paradox, the tolerance principle holds for very small increments. The smaller the increments, the more plausible seems the tolerance principle. In the color space, for example, the finer the continuous space is divided into the strips, the more plausible is to say that in moving from red to

[38] See definition ??

orange if one strip is red, the next one is also red. In the continuous version of the Sorites, there is no need to break up the color space into strips. It will be enough to consider a neighborhood of each point. Being locally constant is identified with being tolerant.

In the paper, the topological Sorites, like other varieties of the Sorites paradox, is formalized within the realm of classical logic.

In classical logic, if X is not connected, it is separated. The authors again consider connectedness as usual(the space X is connected if it cannot be partitioned into two non-empty, disjoint open sets or, equivalently, the space X is connected iff the only clopen sets are X and $\emptyset$ (Also see definition 4.37).

To generalize the formulation of the continuous sorites to the topological Sorites paradox, the authors prove a key lemma:

Lemma 6.3. *Let X be a connected space, Y a set, and f a function from X to Y. Suppose that f is locally constant. Then f is globally constant. A fortiori, if y is in the range of f, then $X = \{x : f(x) = y\}$ (ibid, p. 319).*

In lemma 6.3, let $Y = \{0, 1\}$. Define the characteristic function σ as:

$$\sigma_A(x) = \begin{cases} 1 \quad if \qquad x \in A \\ 0 \quad if \qquad x \notin A \end{cases}$$

When it comes to the extension of a predicate, the authors endorse the classical view according to which the extension of a predicate is a set. If A is the extension of the predicate F:

$$\sigma_F(x) = \begin{cases} 1 & if \quad F(x) \\ 0 & if \quad \neg F(x) \end{cases}$$

Using local constancy and the characteristic function, Weber and Colyvan (2010) introduce their new definition of vagueness:

Definition 6.4. *A predicate is vague iff its characteristic function is locally constant but not globally constant (ibid).*

As mentioned before, local constancy is a generalization of the tolerance principle. This means that a predicate is vague iff the tolerance principle in its general sense holds. If x is red, then there is a neighborhood of x all members of which are red as well. Nevertheless, it does not hold for all neighborhoods of x. In the words of the authors:

> The definition says that a vague predicate is tolerant of small changes but does run out somewhere (ibid, p.318).

According to the authors, this definition of vagueness has two main advantages: First, it does not beg the question in presupposing one special point of view and excluding others (like presupposing gaps rather than gluts). Second, it is general enough to include continuous cases as well. It is quite similar to the ones who define vagueness based on the tolerance principle. A predicate is vague if it is tolerant in its continuous form.

Lemma 6.3 shows that in a connected space everything will be either F or ¬ F(there will be no vague concept). So, the authors formalize different versions of the Sorites paradox in topological terms:

Inductive topological Sorites:

Let F be a vague predicate in a connected space X. Then, if a member of X is F, all members of it will be F.

This is the direct result of the key lemma 6.3 and the definition 6.4.

Topological line-drawing sorites:

Let X be a topological space and F be a vague predicate. If for some distinct points x, y in X, x is F and y is not F, then X is disconnected.
This is the contraposition of the inductive topological Sorites (Weber 2021, p.58).

The idea is that if we can divide the space into two non-empty open subsets , then the space is disconnected. It is so because if the space is connected, then the only clopen sets will be the whole set and the empty set but in this case the space is divided into two clopen subsets (the extension of F and its complement) and the boundary set is empty (See subsection 4.6). The tolerance principle indicates that in a topological space X the extension of F and the anti-extension of F are both non-empty and closed. This, however, is in contrast with a premise that X is a connected topological space. Up to now they wanted to show in what way, given all classical assumptions, vagueness can be represented topologically. Where the extension of a predicate is a set,

> classical topology predicts that the host-space of a vague predicate is not connected, because otherwise vague predicates would apply to everything (Weber and Colyvan 2010, p.321).

6.3 Criticism

Weber and Colyvan (2010) suggest a general structure for the Sorites paradox.

Their attention to the continuous version of the Sorites paradox is remarkable. The underlying space neither needs not be discrete, nor needs to be linearly ordered. Scientists deal with the continuous spaces as well. The space of endangered species is an example of such spaces(ibid. p.324). So, according to them, it is enough to have a set of objects X, a subset A as the extension of a predicate 'F' and the characteristic function whose domain is a connected topological space X. The tolerance principle is identified with locally constancy of the characteristic function of F which in turn is equivalent to saying that A is a clopen set (Rizza 2013, p.362). We will argue that their topological approach either trivializes the notion of vagueness or vague concepts won't be sorites susceptible. Furthemore, there is not enough information on what the extension of a concept is and the topology it is endowed with.

1. What the authors suggest is that the Sorites paradox shows that the space in which vague concepts are defined cannot be connected because if it is connected the only clopen sets will be the whole set and the empty set. But we want the extension of a vague concept to be non-trivial. So, either the space is connected and all predicates are precise because their boundary is empty or the space in which vagueness is defined should be disconnected and the generality of the definition of vagueness that the authors where looking for fails.

   In a connected space with a classical topology any local property that has a clopen extension collapses to the global as well. As Rizza points out, in this case:

> the transition from the extension of a vague property to the extension of its negation resembles a discrete step. In a connected space, this step is vacuous, because one of the relevant extensions is empty. Since a Sorites-type argument should arise from a series of steps or transitions, it looks as if, in this case, no Sorites can arise because no transition can take place. This observation suggests that one may take Weber's and Colyvan's topological argument to prove that certain generalisations of the Sorites paradox are not possible.

They were supposed to give a general definition of vagueness and a general formulation of the Sorites paradox. In the first case, the definition of vagueness in a connected space does not differentiate vague concepts from the precise ones. Vague concepts are locally constant but not globally constant and this is the case iff the extension of the concept is a clopen set which means that the boundary of it is empty. So, the space in which there are vague concepts will contain distinct points and therefore, the aim of the authors in defining vagueness in a cohesive, connected space fails. This is problematic for the authors because connectedness was crucial in their formalization of the continuous Sorites. For example in the formalization of the inductive Sorites they say:

> The 'induction step' is that X is connected, because connected spaces support the local-global property of Lemma 1 (6.3, now built into the definition of vagueness (Weber and Colyvan 2010, p.320).

In a nutshell, if the space is connected and the tolerance principle holds, concepts are sorites susceptible, but they won't be vague because they are globally constant as well. If the space is not connected, then although according to their definition there will be vague concepts, they won't be sorites susceptible.

2. Unlike Boniolo and Valentini, Weber and Colyvan (2010) simply suppose that there will be a topology on a given set of objects. They just consider the Euclidean spaces and their inherited topology (Weber and Colyvan 2010, p.323). We are not saying that they claim that vagueness should exclusively be defined in the Euclidean spaces. The point is just that they could have said in detail how to define a topology in which the extension of a concept is a closed set. Rizza rightly claims that the point-set topology is not the suitable topology:

   > Weber and Colyvan are interested in working within a cohesive environment, i.e., one that cannot be decomposed into detachable parts. Their reliance on point-set topology, however, is not in line with this project, insofar as it leaves them with spaces that are aggregates of distinct points (Rizza 2013, p.363).

   They might answer that their aim in the paper was just to show how one can formulate the continuous Sorites paradox, not to resolve the paradox. For that aim they just needed to suppose that there was a non-trivial connected topological space in which vagueness could be defined and the paradox could be formulated. They find it plausible to suppose that the extension of a concept can be endowed with a non-trivial topology. This might be a good answer to Smith's objection according to which there is no such topology (see Smith 2008, p.152; Weber and Colyvan 2010, p.323). However, our point here is that neither we do know anything about the extension of a vague predicate nor about the suitable topology on the domain of the characteristic function of a vague concept. Boniolo and Valentini, for example, give us information about the extension of a predicate and how to create a topology on it. In Weber and Colyvan's view, on the other hand, we just suppose that there is a set on which hopefully a topology can be defined and the sets which are the extensions of the concepts are closed in that topology. One worry about lack of enough information about the extension of a concept except that it is closed is the following:

If they accept the classical view of a concept (in contrast to the prototype theory, for example), then the extension of a concept is a set such that none of its members is more typical than the other. Since the set is closed, it contains typical cases and borderline cases of the concept. However, there is no criterion to discriminate borderline cases from typical cases. So, a borderline red thing would be as red as a typical red thing or a 2-meter man would be as tall as a 1,80-meter(a borderline case tall) man. This is unacceptable because we usually discriminate the typical cases of a concept from its non-typical cases. Imagine that you are sitting in a room and some men randomly will enter the room and you have to tell whether the man who enters to the room is tall or not. When a 2-meter man enters a room, you immediately will call him tall but when a 180-meter man enters the room you might call him tall with hesitance. So, it is more intuitive to think that the extension of a concept not be homogeneous in the sense that some elements of the extension are more typical than the rest.

In the next section, we will introduce another approach toward vagueness considering geometrical conceptual spaces. The geometrical tool has been used to define borderline cases of a vague concept. In particular, in several papers since the publication of the paper “Vagueness: A Conceptual Spaces Approach”, published in 2003, there has been big attempts to model vagueness in the conceptual spaces approach (Decock et al. 2013). Given the high experimental support of this approach and its popularity in cognitive science and recently in philosophy, we will explain in detail what a conceptual space is and how vagueness is defined in these conceptual spaces. Then, following Mormann (2021) we will show the advantages of the topological approach over the geometrical approach and will explain in what way the topological approach provides an optimization of conceptual spaces.

7 Vagueness in conceptual spaces

7.1 Introduction

The conceptual spaces approach is a geometrical account of concepts. According to this view, concepts are represented in geometrically constructed spaces. This quite well-known approach in cognitive science, artificial intelligence and psychology recently has found its place among philosophers as well(Decock et al. (2013); Zenker and Gärdenfors (2015); Mormann (2021)). With a huge experimental support, it has been used as a powerful tool to solve some philosophical problems such as vagueness (Zenker and Gärdenfors (2015)).

Peter Gärdenfors in a couple of papers in the 1980s and mainly in his book "Conceptual spaces: The Geometry of Thought", published in 2000, has developed the idea of conceptual spaces.
As far as we know, Gärdenfors for the first time presented the idea of conceptual spaces in the paper "Semantics, Conceptual Spaces and the Dimensions of Music" as an alternative to existing prominent theories that treat semantics either as a relation between language and the external world (Fregean-Tarskian view) or as a relation between language and a set of possible worlds (Kripkean-Montagovian view). In his theory, semantics is treated as a "relation between language and a conceptual structure which can be conceived as a kind of mental model" (Gärdenfors 1988, p.9).

In this view, semantics is a relation between language and a cognitive structure. In another words, language is related to conceptual spaces and then to the external world. A property can be defined by the structure of a conceptual space without assuming the truth in actual or possible worlds:

> The purpose of the mapping from the language to a cognitive structure is to provide acceptance criteria for the sentences of the language. These acceptance criteria and their components determine the meaning of an expression in the language. ... The truth of a sentence can only be determined afterwards via some form of connection between the cognitive structure and the external world (Gärdenfors 1988, p.12).

According to Gärdenfors, the truth of a sentence is independent of its meaning.

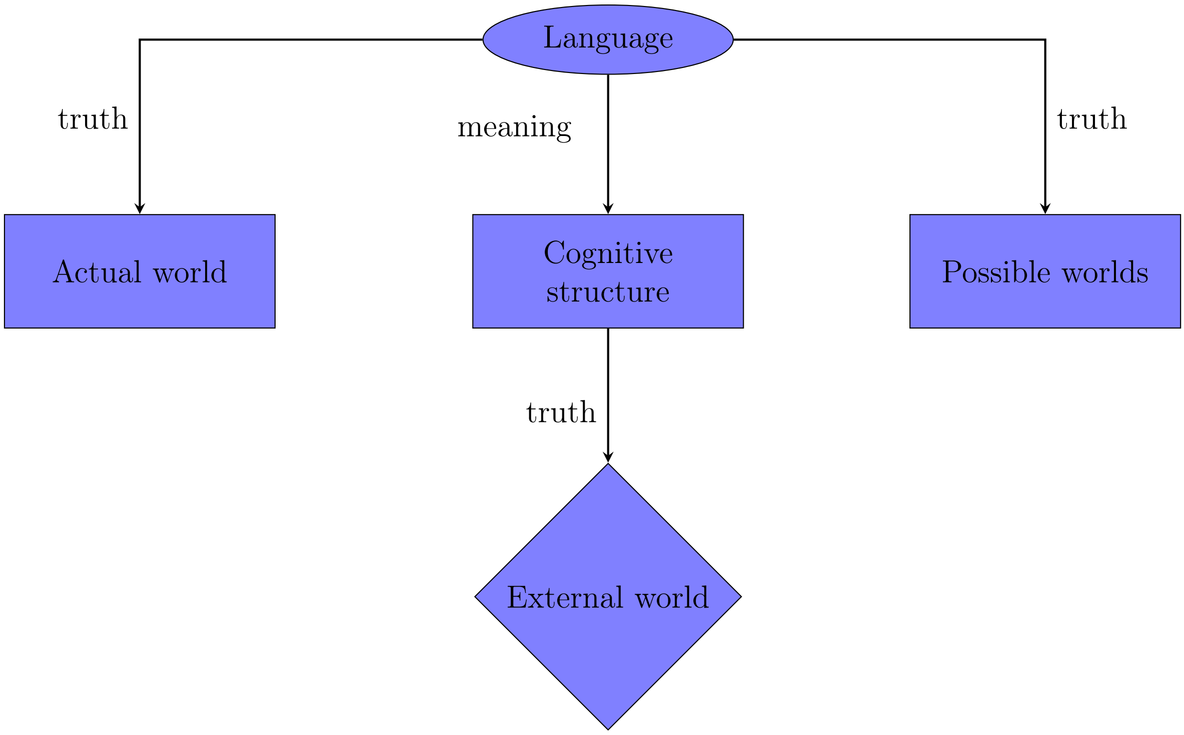


Figure 7: The relation between different kinds of semantics

Figure 7 shows the difference between the conceptual space approach and the well-established Fregean and Kripkean approaches. Semantics in this cognitive approach is a relation between language and a conceptual space and therefore, is a cognitive notion.

In this approach, concepts can be represented in so-called similarity spaces which may be one or multi-dimensional. In the following subsection we explain where this approach stands among the well-known representation approaches in cognitive science and artificial intelligence(AI).

7.1.1 Representation

Information representation and learning have special importance in cognitive science and AI. One aim in these fields is to find out how machines, using language, can solve problems that human beings are able to solve.

In cognitive science and, particularly, in artificial intelligence, there are two approaches towards representation that are often considered as rivals: the symbolic approach and the associationist approach (whose main case is connectionism). Gärdenfors does consider them as two different approaches that model different aspects of cognitive phenomena at two different levels, symbolic and sub-conceptual. He contends that although these approaches complete each other, there are some phenomena that cannot be modelled by neither of them. Concept learning is one of those cognitive phenomena. Connectionists focused on learning. Nevertheless, Gärdenfors claims that they have not been able to give a model for fast concept learning. Furthermore, they cannot model similarity that is closely related to learning.

Gärdenfors proposes an approach at the conceptual level that stands in the middle of the two mentioned ones and fill in the gaps:

> Again, the conceptual representations should not be seen as competing with symbolic or connectionist (associationist) representations. There is no unique correct way of describing cognition. Rather, the three kinds mentioned here can be seen as three levels of representations of cognition with different scales of resolution. Which level provides the best explanation or ground for technical constructions depends on the cognitive problem area that is being modeled (Gärdenfors 2000, p.2).

This approach is conceptual because as we will see concepts are explicitly defined in a conceptual space that is geometrically or topologically structured.

We will now briefly explain the three mentioned levels of representation and the relation between them. Then we concentrate on the conceptual level which has been used to define vague concepts.

- **Symbolic level.** In the symbolic approach thinking is nothing but symbol manipulation, governed by certain rules. Thinking is a form of computation. According to this view, the mind is like a Turing machine. The main idea of the symbolic approach can be found in the physical symbol system(PSS) hypothesis proposed by Newell and Simon:

> A physical symbol system consists of a set of entities, called symbols, which are physical patterns that can occur as components of another type of entity called an expression(symbol structure). Thus a symbol structure is composed of a number of instances(or tokens) of symbols related in some physical way(such as one token being next to the other) (Newell and Simon 1976, p.116).

A physical symbol system consists of a set of symbols that have meaning and represent concepts, things,... and some processes and rules that apply to the symbols and produce new structures. Symbol manipulation is independent of the environment. When the machine receives the input, it operates on symbols based on some given rules without considering the possible effects of the environment on the agent. This view is based on the representational theory of mind according to which the human mind is a tool that does computation on the representations of mind. Thought in its lowest level occurs in a common code, called the language of thought(LOT). According to Fodor- one of the most prominent proponents of this view- computation requires language of thought, and learning and perception are based on computational processes. Fodor (1975) contends that thought is information processing within the LOT (often called Mentalese): “there is no computation without representation” (Fodor 1981, p,122).

The main characteristics of LOT are compositionality, systematicity and productivity. According to compositionality, the determination of a content of a thought depends on the content of its constituent concepts and the structure of the thought. Systematicity is the property of a human cognition to process Mentalese sentences.

Systematicity of mental representations means that if we are able to entertain a certain thought, then we are also capable of entertaining the thought that have semantically related contents. If we entertain the thought Mary loves John, by systematicity we are able to entertain John loves Mary as well. Productivity is based on Chomsky’s idea that human beings, in principle, are able to produce many sentences. A symbol system can encode infinitely many propositions (Fodor and Pylyshyn (1988)).

Fodor (2008) claims that compositionality guarantees the other features and therefore is more fundamental:

> Most of what we know about concepts follows from the compositionality of thoughts (Fodor 2008, p.20).

According to Fodor, Mentalese is innate. He grants that "there is something fishy about learning" (ibid). The mental language is not learnt:

> ... nothing has prejudiced the claim that learning, including first

> language learning, essentially involves the use of an unlearned internal representational system.(Fodor 1975, p.87)

In the symbolic approach the extension of a concept is a set of elements that satisfy certain necessary and sufficient conditions. A well-known example is bachelor. Someone who satisfies the conditions of being man, adult and unmarried belongs to the extension of the concept bachelor. This classical view of concept is not always accepted. In psychology and philosophy it has been criticised (see Margolis and Laurence (2021)). In prototype theory, for example, it is argued that some objects are more typical cases of a concept than others. For instance, pigeon is more typical than penguin in respect of being a bird.

Another difficulty of this view is that despite the very good explanatory power of this rule-based approach, the learning process cannot be explained. The question is how one can learn the meaning of a concept? Why computation over meaningless symbols provides a meaningful concept? Harnad (1990) criticises the symbolic approach under the title "The symbol grounding problem". Following Searle [39], Harnad contends that cognition goes beyond simple symbol manipulation (Harnad 1990, p.339).

Another concern that Gärdenfors has is that even if concepts have meaning

[39] Searle believed that syntax in itself is not enough to attribute meaning to symbols in a physical symbol system:

> If my thoughts are to be about anything, then the strings must have a meaning which makes the thoughts about those things. In a word, the mind has more than a syntax, it has a semantics. The reason that no computer program can ever be a mind is simply that a computer program is only syntactical, and minds are more than syntactical. Minds are semantical, in the sense that they have more than a formal structure, they have a content (Searle 1984, p.29).

His well-known Chinese room thought experiment is to show this point.

how do we explain the dynamics of concepts. Furthermore, even if we can model the change of the meaning of a concept, this model does not explain why we learn so fast.

Another approach that, roughly speaking, does not have the symbol grounding problem and to some extent can answer the above questions related to learning is connectionism.[40]

Sub-conceptual level. Another dominant approach is associationism according to which the building blocks of representation are associations between the information elements. Connectionism is a special case of associationism which models associations by artificial neuron networks. In this approach the connection between neurons and their weights is important. The information, in this approach is distributed over the system and neurons process the distributed information in a parallel way. A neuron or a bunch of neurons can represent a concept. The proponents of this view try to model brain as a dynamic non-linear system. It is based on the architecture of the brain where there are several neurons connected in a parallel fashion. The artificial neural model consists of 3 main parts (see figure 8):

Input layer: There are input units (or perceptrons). The data is received from the nature or from other neurons. Each connection has a certain weight, the real numbers that describe the strength of the connection. At first they are randomly chosen but they can be changed during the learning process.

Hidden layer(s): The units of each layer are not connected among themselves

[40]We do not go into the details of these two approaches. Neither we discuss whether the proponents of the symbolic approach and later connectionism can defend their theories. Definitely, these criticisms have not knocked down these approaches. They have had their undoubtable success.

but they are connected to the next layer. The information is distributed over these units and is processed in a parallel way and at last, the final information is sent to the output layer.

Output layer(s): It may have one or more units. For example, you receive a picture to process and say whether it is a face of a woman or man. Given the result, the network can be trained just like how a child learns. The weights may be chosen randomly at first but during the learning process we finally get the right output. This learning process may take a long time.

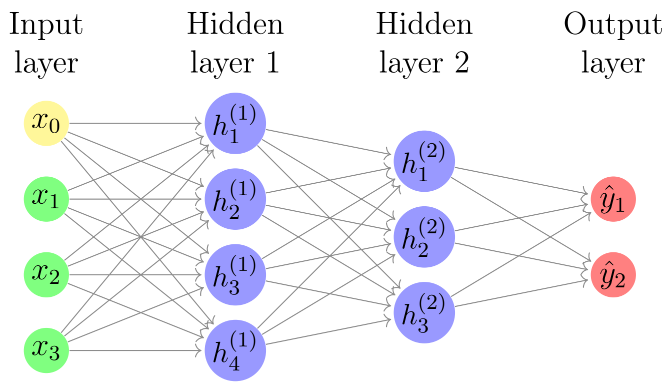


Figure 8: Artificial neural network

One of the advantages of connectionism is that it is a more flexible model. If a neuron dies or some part of the neural network damages, the system still can work properly to give the right output because the output emerges as a result of the parallel distributed processing of many neurons just like how the neural system of the human being works. Furthermore, it can explain the learning process, though in this model, the learning process is very slow. Last but not least, the symbol grounding problem and Chinese room argument cannot be applied to this view because it is not limited to the symbolic level.

41

However, this view has some drawbacks:

- The low explanatory power of what occurs in the hidden layer. The input layer is connected to the hidden layer and the hidden layer is connected to the output layer. We have access to the information in the input and output layer but the hidden layer acts as a black box.

- Similarity plays a crucial role in cognitive psychology and philosophy among other fields. According to some theories of categorization and concept, categorization is grounded by similarity. Paul Churchland, as a prominent defender of connectionism, sees one of the advantages of connectionism (in comparison to the symbolic account) in that it can give a natural account of similarity.[42] Nevertheless, the similarity relation in artificial neural network is learnt slowly. On the contrary, we learn very fast. This cannot be explained by connectionism. The learning process may take a long time showing the system thousands of pictures of a person to the system to be able to correctly recognize it.[43]

- Conceptual level.

As we saw, both symbolic and associationistic approaches have their advantages and disadvantages. They are often presented as competing paradigms.

[41] Associationism is a very successful approach and the model is much more complicated that what we described here. In this work we will focus more on the conceptual space approach. For more information about the history of artificial intelligence and in particular, connectionism see (Rumelhart et al. 1988, Bringsjord and Govindarajulu 2020, Buckner and Garson 2019).

[42] In philosophy, similarity plays a significant role in Carnap's logical construction of the world. On the contrary, the concept of similarity was harshly criticized by well-known philosophers such as Quine and Goodman. The former found similarity "repugnant" and the latter finds similarity "an impostor" and a "quack" Goodman (1972, pp.437 – 447).

[43] For more criticisms of connectionism see Fodor and Pylyshyn (1988).

Sometimes one became the dominant approach due to a huge success, sometimes one failed due to lack of explanation power but then came back strongly. The latter happened to connectionism, for example. Due to very successful programs such as PROLOG and criticisms towards the simple version of connectionism[44], the symbolic approach became the prominent approach. Nevertheless, it was not the end of story. Rumelhart et al. (1988) wrote "Parallel Distributed Processing" that turned the page in favor of connectionism. However, Gärdenfors rightly believes that they are not rivals. Rather, they deal with cognitive problems on different levels, symbolic and sub-conceptual.

This is in line with what Papert, one of the main critics of connectionism says about why he thinks Minsky and he himself didn't have the intention to kill the rival.

> In fact, more than half of our book is devoted to "properceptron" findings about some very surprising and hitherto unknown things that perceptrons can do. But in a culture set up for global judgment of mechanisms, being understood can be a fate as bad as death. A real understanding of what a mechanism can do carries too much implication about what it cannot do (Papert 1988, p.8).

While they were trying to understand how the artificial neural network can model the mind, they found some difficulties and they came up with a constructive criticism that finally led to the progress of that approach.

The conceptual spaces approach lies between these two levels. Our brain has

[44] In 1968 Marvin Minsky and Seymour Papert published a book "Perceptrons: An Introduction to Computational Geometry" in which they argued against the simple artificial neural network based on one perceptron(node). They claimed that the neural network approach cannot model the mind while computer programs are able to do that. The triumph of computer programs at that time was in favor of the symbolic approach.

certain geometrical structure in which similarity is defined based on the distance of objects in the conceptual space. The meaning of many words can be described in such similarity structures in conceptual spaces. Gärdenfors adapts prototype theory of concepts and suggests a model that can explain why we learn so fast. Of course, this semantic approach is immune to criticisms such as Chinese room or grounding problem.

In the following sections we first explain in detail what conceptual spaces are. Then, we will explain convexity, as a crucial restriction on spaces. Later, we overview the expansion of conceptual spaces proposed by Douven et al. that relates conceptual spaces with the problem of vagueness and finally, we criticize this view on the basis of the fact that to give a theory of vagueness, the topological structure is more appropriate.

7.2 Conceptual spaces

Conceptual spaces is an alternative framework for knowledge representation. As mentioned before, Gärdenfors proposes a third level of representation that is conceptual. In this approach the building block of representation is a quality dimension rather than symbols or neurons. The quality dimensions are abstract representations that have certain geometric structure and represent different qualities of objects. Some dimensions build up domains. Concepts then, are defined as geometric objects in a conceptual space. In the conceptual spaces approach representations are based on geometrical(topological) structures. These structures let us measure the distance between two objects or talk about their closeness in a space. This distance is easily calculated in a metric space and Gärdenfors often considers such spaces.

There is a tight relation between the similarity relation and the distance function. The smaller the distance between two objects in a conceptual space, the more similar they are. On the other hand, concept learning is closely related to the notion of similarity in the sense that representations are representations of similarities. For Gärdenfors, a concept consists of a group of things that are similar in a conceptual space.

The structure of the dimensions also plays an important role in the discretization of space into convex regions. Roughly speaking, a region is convex if for any two points in the region, all points between those points belong to that region. So, in the identification of a concept, the structure of the dimensions, similarity and betweenness are crucial. Also with regard to the theories of concepts the prototype theory of concepts is adopted. Roughly speaking, in the extension of a concept some members are more salient. Also, a mathematical technique, Voronoi-Tessellation, is used to carve up the conceptual space into convex regions.

Gärdenfors argues that the conceptual spaces approach gives a better model for how children quickly learn new concepts. It is enough to show them a typical example of dog such as a husky to have the concept of dog. The more dog breeds the child sees, the more exact he(she) will grasp the concept of dog. Concepts can be learned. Gärdenfors claims that a conceptual space is adjustable. When you show a Husky to a child and tell him(her) that it is a dog, it is possible that the child sees a wolf and says dog because it is very similar to the husky. But little by little his or her conceptual space changes by seeing more dog breeds and becomes more exact.

In this section the aim is to clarify what conceptual spaces are, and to mention some difficulties that the proponents of the conceptual space approach have to deal with. The topological approach, defended in the current work, is quite similar to this approach yet, as we will discuss in the next part, it does not have the difficulties of this view.

7.2.1 Quality dimensions

The quality dimensions are prelinguistic and conceptual spaces are made of these prelinguistic quality dimensions.

> The dimensions are taken to be independent of language and symbolic representations in the sense that we and other animals can represent the qualities of objects, for example when planning an action, without presuming an internal language in which these qualities are expressed. The quality dimensions should be seen as abstract representations used as a modeling factor in describing mental activities of organisms. They are thus not assumed to have any immediate physical realisation. However, they will hopefully be useful constructs when developing artificial systems (Gärdenfors 2009, p.5).

Gärdenfors introduces quality dimensions via some examples. Weight, temperature and time has one dimension. Color has three dimensions: brightness, hue and saturation. Pitch, loudness and timbre are dimensions of sound. The three dimensions of space are length, width and height. (Gärdenfors 1988, p.12).

In the framework of quality dimensions each object has its attributed properties and the relation between the objects is specified. In other words, object's qualities are identified through such quality dimensions. Each dimension has a certain geometric or topological structure. For example, there is an isomorphism between the dimension of "weight" and positive real numbers. Time is one dimensional but has different topological structure. These structures may depend on cultures but Gärdenfors argues that it won't cause difficulties for people coming from different cultures to communicate and understand each other.

The origin of the quality dimensions is not clear. Some are innate, some are learned, some depend on culture, some may be introduced by scientists ... (Gardenfors 2004, p.15).

Gärdenfors(1988) makes a difference between two interpretations of quality dimensions: **psychological** and **scientific**. The latter is a kind of idealization of the space. A visual space is not exactly a 3 dimensional Euclidean space. It refers to the dimensions that are introduced in scientific theories. The former, which is mostly used in AI "generally concerns how humans structure their perceptions."(Gärdenfors and Zenker 2015, p.4; Gärdenfors 2000, p.89). The dimensions of color, hue, saturation and brightness are commonly used examples of the psychological interpretation. The number of dimensions may change. The quality space of "taste" was thought to be a four dimensional space where the dimensions were sour, saline, sweet and bitter and now one more dimension, namely umami, is added to the other four dimensions. [45]

Some dimensions are interrelated in the sense that they are inseparable ; i.e, a

[45] The dimensions are chosen by psychologists through some empirical research. For example, in 1916, a German psychologist Hans Henning proposed four primary tastes. After 80 years Lindemann proposed that the fifth taste be called umami.

change in one affects the other dimension. Gärdenfors calls them **"integral dimensions"**. The non-integral dimensions are called separable. For example, the dimensions of space, width and length are separable. It may happen that an integral dimension is considered separable after some time and vice versa.

The integral quality dimensions are categorized into different domains. For example, hue, saturation, and brightness are dimensions of the color domain. A conceptual space models these domains.

7.2.2 Domains

Domains are defined based on integral dimensions:

> a *Domain* can be defined as a set of integral dimensions that are separable from all other dimensions. (Gärdenfors 2014, p.22)

Properties are represented as a region in one of the dimensions of the domain. A concept (e.g. apple) can be represented as a collection of regions from possibly different spaces. These domains are informed by empirical psychological research (e.g. hue, saturation and brightness for the color domain).

7.2.3 Similarity

The similarity relation is crucial in the representation of concepts and concept learning. A conceptual space in fact is a kind of similarity space with some restrictions on the space to get the best results out of it. The geometrical(topological) structure allows us to talk about similarity. For example, in a color space green is more similar to blue than red because it is closer to blue. A conceptual space is defined geometrically using a metric space. In a metric space similarity is defined

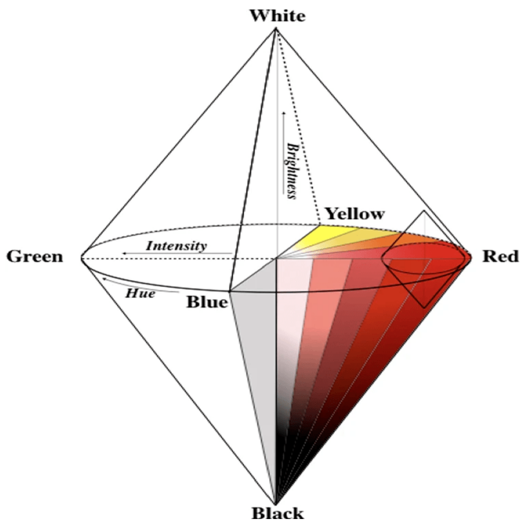


Figure 9: The color spindle with three dimensions

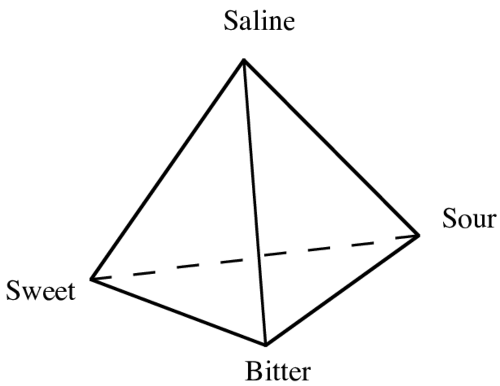


Figure 10: Hennings' taste tetrahedron

through the distance function. Two things are similar if they are within a certain distance. An object x is more similar to y than to z if it has less distance from y than from z.

The similarity relation may be defined differently in two geometric structures with different metrics. For example, two things may be similar in a Euclidean space with the Euclidean metric and not similar in Euclidean space with the taxi-cab metric.

Similarity can also be used to define another crucial notion, namely betweenness that is used to define convexity. As recognized by Gärdenfors himself, betweenness can be defined differently depending on the distance function. The points between a and b in the Euclidean metric space lie in the straight line between a and b, whereas the set of points between a and b in a taxicab(Manhattan) metric will form a rectangle (Gärdenfors 2000, p.18).

This, in turn, shows that a point may be between two points in a metric space but not in another one.[46] Gärdenfors and Williams (2001) point out that the betweenness relation may be considered as primitive but also may be defined in terms of the similarity relation. In a similarity space S, $B(a, b, c)$ means that b is between a and c. In a specific case that the space is metric, the betweenness relation can be defined as:

Definition 7.1. *Let S be a conceptual space and the distance measure is a metric d, the betweenness relation $B(a, b, c)$ is defined as:*

$$B(a, b, c) := d(a, c) = d(a, b) + d(b, c) \text{ (Gärdenfors and Williams 2001, p.386).}$$

Another notion that also depends on the metric space is the notion of being equidistant. The distance between a and b can be equal to the one between c and d in a space with the Euclidean metric but not in that space with the taxicab metric.

In several works of Gärdenfors it is crucial that concepts are convex regions in a conceptual space. For example, if two things are red all the things between them are red as well. Euclidean space endowed with standard metric or the taxicab(Manhattan) metric is one of these spaces.

[46]For example, in figure 1 z is between x and y considering the Manhattan metric but is not between those points when we consider the Euclidean metric.

As we will see, the point that similarity and convexity highly depend on the metric will be problematic for the conceptual space approach (this is usually referred to as the uniqueness problem.[47]

Before discussing the critical points, in the following subsection we will explain convexity and its possible weaker versions.

7.2.4 Convexity

Convexity is a very well-known notion in mathematics. The convexity criterion plays a crucial role in the conceptual space approach. Gärdenfors proposed that concepts are not any regions in a conceptual space. Rather, they need to be convex. At first it was an empirical hypothesis that later was confirmed through many experiments. In fact, many common examples of concepts such as taste, color, time ... turns out to be convex regions. Convexity is important for two main reasons. The first reason is related to learnability process. Convexity speeds up the learning process. It is enough to show a child a few examples of dog to have the concept of dog. The second reason is that it plays a pivotal role in acquiring a concept and the effectiveness of communication.

> As we shall see, the learnability and effectiveness of communication clearly interact in complementary ways in the acquisition of concepts. Strong support for this idea has been provided by Jäger (2008, p.552). He argues that "languages where meanings are convex regions (of a special kind) are . . . optimally adapted to communication. The preference for convex meanings can thus be seen as the result of some

[47] For more detail see Douven and Gärdenfors (2018).

> process of (cultural) evolution”[48] (Gärdenfors 2014, p.26).

The standard formal definition of convexity is:

Definition 7.2. *Let* (X, d, B) *be a metric space where* $B(x, y, z)$ *is a betweenness relation, and* $A \subseteq X$*. Then* A *is a convex set iff for all* $x, y \in A$*, for all* $z \in X$*, if* $B(x, z, y)$*, then* $z \in A$*.*

In the figures 7.2.4 and 7.2.4 you can see an example of convex and non-convex sets. The definitions show that convexity is tightly related to betweenness relation.

Figure 11: Convex vs non-convex sets

This is considered as a weak assumption about the structure of the dimensions in a conceptual space (Gärdenfors 1988, p.14).

There are two less restricting conditions that Gärdenfors and Williams (2001) mentions: connectedness and star-shapedness. The former is a topological notion (see definition 4.38. The latter is a weaker constraint according to which there is at least one point, x, (called Kernel point) in the region such that any point between x and another point of the region belongs to the region. Formally:

Definition 7.3. *Let* R *be a region in a conceptual space.Then,* R *is star-shaped iff*

$$\exists x(x \in R \wedge \forall y \forall z((z \in R \wedge B(x, y, z)) \rightarrow y \in R)).$$

[48]For more details about empirical supports see (Gärdenfors 2014, p.26) and Douven (2016).

All convex sets are star-shaped but not all star-shaped sets are convex. Star-shapedness also relies on betweenness. Gärdenfors finds this condition "desirable" for the categorization. However, despite the criticisms that target the convexity condition, he still prefers the convexity relation as a criterion that is better to be used as the necessary condition for a natural property and a natural concept. So, he defines concepts and properties the conceptual space by the following two criteria:

Criterion P: A natural property is a convex region in some domain.

Criterion C: A concept is represented as a set of convex regions in a number of domains together with information about how the regions in different domains are correlated.

Convexity is a necessary condition for a concept to be natural or a good concept. Therefore, the convexity restriction helps us to explain why Grue, that does not correspond to a convex region, is not a natural concept.[49]

However, convexity is not a sufficient condition because there are some convex regions that do not represent a natural property. For example, red and non-red both are convex regions in a conceptual space but unlike the former, the latter is too wide and intuitively is not considered as a concept. The reason is that there are several ways to decompose a conceptual space into convex regions and not all of them correspond to natural properties or natural concepts (Douven and Gärdenfors 2018, p.9).

Adopting the prototype theory of concepts helps to find the right regions corresponding to natural properties. So, let's see the role of prototypes in the con-

[49]See page 13 of the current work.

ceptual space approach.

7.2.5 Conceptual spaces and prototype theory

In psychology, Rosch(1975) developed "prototype theory". Unlike the classical view in which the elements of the extension of a concept 'F' have the same status, in her view some elements are better examples than others. For example, robins represent the concept bird better than chickens, though both belong to the extension of bird. Prototypes of a concept are its typical examples or representatives. But there is not a sharp line between the typical examples and the rest of the members. As Lakoff puts it:

> The existence of focal colors shows that color categories are not uniform. Some members of the category RED are better examples of the category than others. Focal red is the best example. Color categories thus have central members. There is no general principle, however, for predicting the boundaries from the central members. They seem to vary, somewhat arbitrarily, from language to language (Lakoff 1987, p.26).

A focal red is a shade of the RED category that represents the prime example of the RED category.

Rosch says:

> Another way to achieve separateness and clarity of actually continuous categories is by conceiving of each category in terms of its clear cases rather than its boundaries (Rosch 1975, p.306).

Gärdenfors endorses that there is a close relation between the framework of conceptual spaces and prototype theory:

> describing properties as convex regions of conceptual spaces fits very well with the so called *prototype theory* of categorization developed by Rosch and her collaborators (Gardenfors 2004, p.20).

The geometrical structure makes some predictions possible that are not possible in the prototype theory (Gärdenfors 2000, Decock and Douven 2015).

In a conceptual space the regions are convex and the prototype of each concept is in the center of the region:

> When natural properties are defined as convex regions of a conceptual space, prototype effects are indeed to be expected. In a convex region one can describe positions as being more or less central. In particular, if the space has a metric, one can calculate the “center of gravity” of a region (Gärdenfors and Zenker 2015, p.7).

The "centre of gravity" is considered as a prototype point or a prototype region.

In order to decompose the space into convex regions, Gärdenfors appeals to a mathematical technique, called Voronoi-tessellation.

7.2.6 Voronoi-tessellation

In the conceptual space approach, concepts are nothing but convex regions in a conceptual space. Roughly speaking, each concept has a prototype or some typical exemplars. Conceptual spaces contain some points that represent the prototypes associated with each concept. Then as is shown in figure 12, Voronoi-tessellation application divides the space into cells the center of which are those points. Concepts are defined as cells in such a space. The Voronoi-tessellation of X generated by the set of poles $P = \{p_1, ..., p_n\}$ is the set of cells $\{C_1, ..., C_n\}$

such that the points in each cell, C_i are the only points that are at least as close to p_i as they are to p_j for all $p_i \neq p_j$ where closeness is measured by the metric associated with X (see Figure 12). Formally:

Definition 7.4. *Let (X, d) be a metric space and $P \subseteq X$ be a set of poles. Then, the Voronoi tessellation $V(P)$ is a family of sets:*

$$V(P) := \{C_i | p_i \in P\}$$
$$C_i := \{x \in X | d(p_i, x) \leq d(p_j, x) \quad \forall j \in \{1, ..., n\}\}.$$

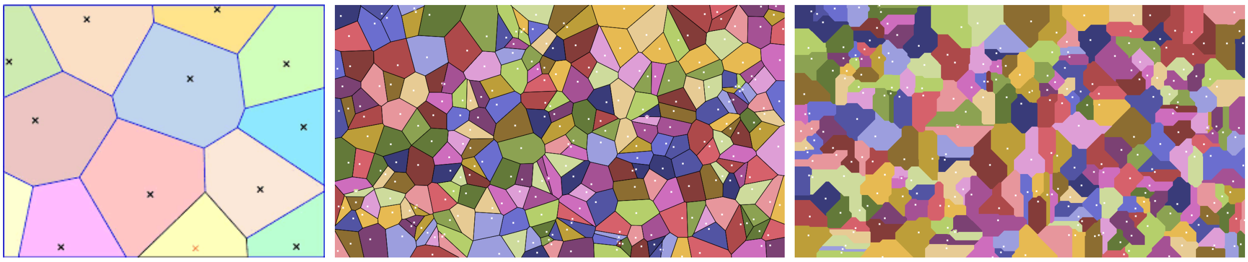


Figure 12: The two first pictures from left to right are Voronoi tessellation of a space with Euclidean metric and the third one is the same space as the second picture but with Manhattan metric.

The dividing lines, called edges, are considered as thresholds for the similarity space in the sense that two objects that are on the two sides of an edge are not similar. Each point on an edge is equidistant from exactly two prototypes. Those points represent borderline cases. The cells form polygons. A region Reg(p) is a cell that contains p at its center. The regions are not empty because at least p belongs to the region. If there are n poles, the space is divided into n regions. The regions obtained in this way are closed sets (Okabe et al. (1992)). So, if concepts are defined as convex regions in a conceptual space, their extension must be a closed set. In a nutshell, given a set of poles the Voronoi tessellation technique divides the space into convex regions. A point belongs to Reg(p) if it is closer to p in comparison to other poles and it is a borderline if it is as close as p as to other pole, q.

It is good to notice that not in all Voronoi tessellations the generator points are located in the center of the cells. In conceptual spaces approach it is supposed that the generator points are far enough.

Now we are ready to see how the conceptual space approach is used to deal with the phenomenon of vagueness.

7.3 Definition of vagueness in conceptual spaces

For Gärdenfors, the conceptual space approach can explain why language is replete of vague terms. The Voronoi-tessellation technique helps us to know the mechanism of vagueness. Vagueness might be due to the change of the location of prototypes. As we see more examples of a concept, the location of the prototype changes. Another reason is that the integrity of dimensions may change for different reasons. For example, dimensions in different contexts may have different weights. So, vagueness is due to some cognitive limitations. For example, if we are comparing apple with pear, the sweetness of the taste domain and being roundish of the shape domain receive more weight than hue, saturation and brightness of the color domain.

To deal with the phenomenon of vagueness within the conceptual space approach, Douven and his colleagues generalize Gärdenfors's definition of a conceptual space so that concepts can have wider boundary. To find out what borderline cases are in a conceptual space, they look for a shared property of all borderline cases of a concept. The authors agree with Gärdenfors that concepts can be represented in a conceptual space, which is a metric space and that natural concepts are convex regions in the conceptual space. However, they argue that Gärdenfors has not correctly addressed vague concepts (Douven et al. (2013)).

The criticism is based on the fact that the definition of borderline cases is too thin:

> ... the borderline cases that result from the conceptual space-cum-prototypes-cum-Voronoi-diagrams model are too 'thin' ((Decock and Douven 2015, p.214).

The borderlines are thin because no point in the neighborhood of a borderline case is a borderline case. However, intuitively, in the color spectrum 'red' has more than one borderline case, a case that is not definitely red and not definitely orange. Each borderline red case is surrounded by other borderline red cases. In other words, the boundary is "thick"; there is no abrupt switch from borderline red things to non borderline cases of red and orange for example.

To expand the borderline, Decock et al. (2013), relying on some psychological experiments done by Berlin and Key(1969), claim that prototypical points should be replaced by prototypical regions because according to the experiment, people choose more than one color as typical examples of a certain color. They modify the Voronoi technique so that it can be applied to prototypical regions. They call it " Collated Voronoi diagram".[50] Informally, each concept has a prototypical region. Consider each point of the region. It will generate a Voronoi tessellation on the space. To construct the Collated Voronoi tessellation, generated by the

[50]For the experiments with results in favor of the collated Voronoi tessellation see Douven (2016), Douven (2018), Douven and Decock (2017).

prototype region, it is enough to consider all the Voronoi tessellations generated by the points of the region at the same time.[51]

This method, as is claimed, divides the space into convex regions. Concepts in this new conceptual space can have a 'thick' boundary in which there can be borderline cases in the neighborhood of any borderline case.

Figure 13 shows the difference between standard Voronoi tessellation and the collated Voronoi tessellation. In this way, borderlines are replaced by border

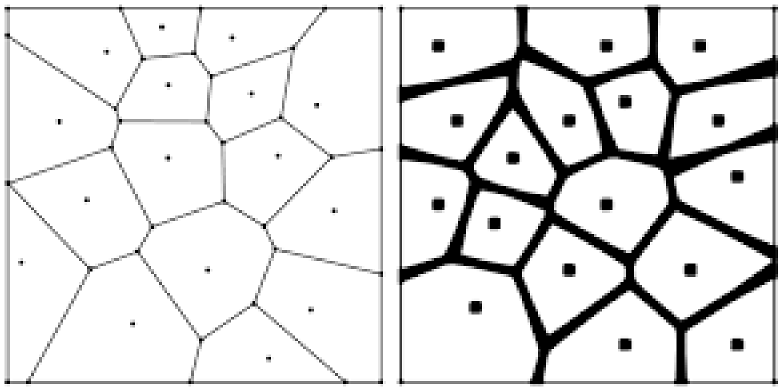


Figure 13: Multiple prototypes generate a thick boundary in collated Voronoi tessellation

regions and borderline cases are surrounded by other borderline cases. The thick boundary approach also has an advantage of showing that some borderline cases are nearer to one prototype than to the other. A borderline red case can be more reddish than orangish.

To deal with the Sorites paradox Douven and Deckock, following Edgington (1997), endorse that a point in a conceptual space belongs to certain region to some degree. The degree of membership of clear cases of a concept is 1 and is 0 for the clear non-cases. For borderline cases the degree gradually changes as one distances from the prototype region. However, they argue that their view lacks

[51]For the formal definition of collated Voronoi tessellation see Douven et al. (2013).

the problems of degree theory approach to Vagueness (Decock and Douven 2014; Douven and Decock 2017).

Also, according to the authors, this methodology helps them to explain higher-order vagueness:

> there are no abrupt transitions from clear cases to borderline cases, given that the borderline cases neighboring the clear cases fall under the given concept to a degree still very close to 1 (and thus very close to the degree of any of the clear cases (Douven and Gärdenfors 2018, p.15).

This is an appropriate movement. They draw their attention towards borderline cases and the boundary of the regions. This point was missed by Gärdenfors (2000).

In a nutshell, vagueness arises when we apply some rationality principles to the cognitive domain :

> ...the occurrence of vagueness is not just excusable, or understandable, or inevitable; the "structure" of vagueness—by which I mean, which predicates are vague and where their borderlines are found— can be seen as following from principles of rational design applied in the cognitive realm (Douven 2018, p.2).

The idea is that in an optimised conceptual space, vagueness arises when a concept is vague in the sense that there will be thick boundaries between regions. The optimised conceptual space has some criteria and constraints posed upon it in order to enable any creature with limited memory to identify natural concepts from non-natural ones, to categorize objects properly, to communicate about concepts and to reason. Here are the criteria suggested by Douven and Gärdenfors (2018):

1. **Parsimony:** The conceptual structure should not overload the system's memory.

2. **Informativeness:** The concepts should be informative, meaning that they should jointly offer good and roughly equal coverage of the domain of classification cases.

3. **Representation:** The conceptual structure should be such that it allows the system to choose for each concept a prototype that is a good representative of all items falling under the concept.

4. **Contrast:** The conceptual structure should be such that prototypes of different concepts can be so chosen that they are easy to tell apart.

5. **Learnability:** The conceptual structure should be learnable, ideally from a small number of instances (Douven and Gärdenfors 2018, pp.6-7).

By these criteria, the conceptual space should provide enough information, not too much and not too little for a person to be able to act and communicate in a proper way. The sufficiency of the information depends on the context. A painter needs to differentiate more colors than a mathematician. For a mushroom farmer it is indispensable distinguishing edible mushrooms from the non-edible poisonous mushrooms. The first criterion is to guarantee the right categorisation of objects and the second one considers the limitation of memory. The representation and contrast criteria make sure that we do not mix up the very clear cases of two concepts and therefore, do not make mistakes in object categorization. This also helps the learning process. We can memorize the sufficient information about the clear cases of a concept and differentiate it from the typical cases of other concepts. One of the main aims of proposing the conceptual space approach was to explain the fast learning process in children. By the last criterion, it is enough to show a child a few instances of a concept to grasp that concept.

Apart from the designed principles, Douven and Gärdenfors (2018) contend that for the optimal categorization of similar objects as well as dissimilar objects, the conceptual space should be well-formed:

Well-formedness: The concepts should be "well-formed" in that the items falling under any one of them are maximally similar to each other and maximally dissimilar to the items falling under the other concepts represented in the same space (Douven and Gärdenfors 2018, p.11).

Natural concepts, then, are defined as convex regions in an *optimized* conceptual space and vague concepts are the ones that have borderline cases (that belong to the thick boundary) in such an optimized conceptual space.

In the next subsection we will see that even this designed architecture is not without difficulties.

7.4 Criticisms

The conceptual space approach provides a framework for concept representation. Optimising such spaces, Douven and his colleagues since 2013 have attempted to deal with the phenomenon of vagueness. A conceptual space contains some points each of which corresponds to an observation. There are some quality dimensions. Some of them are interrelated, called integral dimension. These integral dimensions form a domain. Concepts are defined as convex regions in some of these domains. The distance between two points in a domain is usually calculated by the Euclidean metric and the distance between two points of different domains by the Manhattan metric. The well-known examples such as color usually are one-domain concepts. Considering the multitude of one-domain concepts, Gärdenfors proposed that the meaning of a word (except nouns) depends on a single domain (Gärdenfors 2014, p.239).

Despite a huge success of this view and its application in many fields such as cognitive science, cognitive linguistics, artificial intelligence and robotics, recently it has been targeted to some criticisms that has led to the improved optimised version of the conceptual space approach. However, this better designed version is not without problems because, as we will see, its main assumption that the regions are convex has recently been put into doubt.

1- **Uniqueness problem**

According to Gärdenfors, given a metric on the subspace that is subject to categorization and a set of prototypes, one can use Voronoi-tessellation technique to generate a unique partitioning of the subspace into convex regions. So, defining concepts as a convex region in the conceptual space is tightly related to the structure of the conceptual space and the distribution of prototypes in the space.

This problem can be divided into different parts:

1. There is no unique way of choosing a dimension to represent a particular quality. The weight of a dimension and the integrity of dimensions are not unique.

2. There is no unique way of choosing a metric: Betweenness and distance both ask for a metric. Even on a metric space, different metrics can be defined. The categorization of a concept is tightly related to the metric defined on a conceptual space. As we stated, the crucial condition of convexity highly depends on the metric defined on a conceptual space.

Confronting the uniqueness problem one may ask the following questions:

Which space is more salient? Are people with different conceptual spaces able to communicate? A convex region in the Euclidean space with the standard metric might be not convex in the Euclidean space with the taxicab metric. So, the question is: how can we uniquely define concepts? This concern is noted by the proponents of the conceptual space approach. Gärdenfors (2014) shows that there won't be any problem of communication and the recent optimised design is supposed to lead us to a unique way of conceptualization.

> The concern is that such constraints may not be enough to guarantee uniqueness, in that there may be many partitions of a given similarity space that satisfy the constraints to the same maximal extent (Douven and Gärdenfors 2018, p.12).

There may be more than one optimal conceptual space or, as the authors mention, there might be an almost optimized conceptual space that is worth using it and not go for a better one. Up to now color spaces have been deeply studied and the results have been in favor of the conceptual space approach. Nevertheless, more works should be done to see whether it works for other concepts.

2- **The convexity constraint is too strong.**

According to Gärdenfors (2000) convexity is a "principle of cognitive economy".

Furthermore, in comparison with arbitrarily chosen shapes, the convex ones help us to learn fast (Gärdenfors 2000, p.70). Hernández-Conde (2017) criticizes the conceptual spaces approach on the ground that accepting prototype theory and a context-dependent conceptual space is enough for the concept categorization, there is no need to the convexity constraint because if having a prototype is considered as a constraint on regions, then it is enough for regions to be star-shaped and star-shapedness is enough for categorization. Hernández-Conde targets his criticisms towards the convexity constraint. He argues that there is no guarantee to have a convex region in a conceptual space with a non-Euclidean metric. In addition, not all properties are convex and the combination of convex properties of the same domain may be non-convex.

The problem of convexity of multi-domain concepts is also considered by (Bechberger and Kühnberger 2017, 2019). Concepts are represented in a conceptual space. If we consider multi-domain concepts, then the correlation between the domains should be represented as well. However, according to the authors, the representation of correlated concepts is not possible, if the concepts are convex regions in a conceptual space. They consider height and age domains to define the concepts child and adult. As mentioned before, for multi-domain concepts the metric is Manhattan. They show that the representation of the correlation of those two domains is in conflict with the convexity restriction in a Manhattan metric space. The reason is that it does not show that up to a certain age the height increases and then becomes stable. They suggest to weaken the convexity restriction to star-shapedness of the regions in a conceptual space. The authors argue that if concepts are defined as convex regions, then their representations

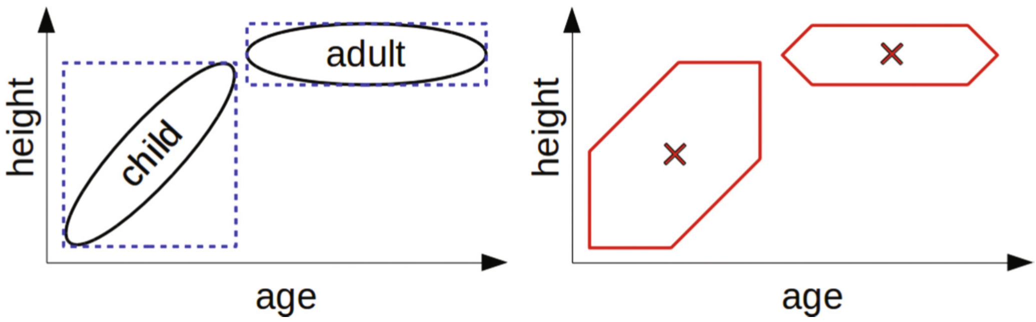


Figure 14: Left: The solid lines represent the regions for the concepts of 'adult' and 'child'. The dashed lines represent those concepts as *convex* regions when the metric is Manhattan. Right: The red lines represent those concepts as *star-shaped* regions in a Manhattan metric conceptual space. The crosses represent the central points.(The figure is taken from Bechberger and Kühnberger 2017, p.61).

in a Manhattan metric space would be parallel-axis cuboids and therefore, the intuitive intuition of the correlation between age and height of child will be dismissed. If the metric is Euclidean, there won't be any problem but then replacing Manhattan metric by the Euclidean metric is too much of a change and is not necessary. They suggest to replace convexity by star-shapedness. According to them that would be the best choice for two main reasons. On the one hand, there will be small deviation from the convexity constraint and on the other hand, it solves the mentioned problem to a large extent. Figure 14 shows why the representation of correlated concepts as star-shaped regions in a conceptual space with Manhattan metric is more apt. In the left part, considering the square, we cannot see the increase of height when the age goes up. On the right side, we see the star-shaped region in a Manhattan metric space that, roughly speaking, gives us the right information related to the age and height correlation.

In his reply to Hernández-Conde, Gärdenfors (2019) defends convexity as an empirical testable hypothesis that has been confirmed by many experiments done in cognitive science, psychology, etc, and yet can be falsified.

> It is important to note, however, that the convexity criterion is not proposed as something that necessarily holds of an application, but as an empirical law that generates testable predictions (Gärdenfors 2000, Gärdenfors 2014). The convexity criterion is what furnishes the theory of conceptual spaces with most of its empirical content. The fact that the consequences of the convexity criterion could be false simply means that the criterion is testable (Gärdenfors 2019, pp.77-78).

In general, Gärdenfors thinks that Hernández-Conde has attributed a stronger thesis to him. Unlike what Hernández-Conde argues against, for Gärdenfors the convexity constraint has been proposed as an empirical hypothesis. There is no claim about mutual relationship between prototype theory, Voronoi-tessellation and convexity. He does not deny that it may happen that the space is partitioned into non-convex regions by Voronoi-tessellation technique and it may happen that the space is divided into convex regions by other methods. Some dimensions, such as length and weight do not have a prototype. For Gärdenfors, convexity has advantage over star-shapedness because it has had more empirical support and supposing convexity in the Euclidean space guarantees that Voronoi generators locate at the center of the Voronoi cells Gärdenfors (2019).

One of the reasons to appeal to prototypes and applying Voronoi technique was to help to choose between a huge number of ways of partitioning the space into convex regions to be able to identify natural concepts. Suppose that the mentioned criticisms of convexity just show the hypothesis is empirically testable. We think that at least for concepts that have prototypes, in an optimized conceptual space where prototypes of concepts are far from each other, star-shapedness is a more suitable constraint on multiple-domain concepts and convexity for one-domain concepts. This can be considered as an empirical hypothesis. Maybe further experiments on other domains (with specific focus to see whether the regions are star-shaped and not convex or are convex[52]) such as sound or taste confirm it.

As far as we know, Gärdenfors has not replied to Bechberger and Kühnberger (2019).[53] We think that one possible answer, regarding their example, would be denying that the dimensions that play the role in defining the concept of child are age and height. So, their criticism would be an argument even in favor of the convexity constraint. It is for psychologists or cognitive psychologists to say when one ceases to be a child. Someone whose height stops at early age may still be a child. The definition of adult in the Oxford dictionary is: 'a fully grown person who is legally responsible for their actions'. A child is a young person who is not an adult. So, a child is a young person who is not legally responsible for their actions. In this definition the relation between age and height does not play a crucial role.[54]Consider a different example, a tadpole and a frog. A tadpole becomes bigger and bigger up to a point that it becomes a frog but the length of the tadpole and its age does not categorize it as a tadpole. There are many tadpoles that do not convert into a frog. They lack a gene that produces the hormone thyroxine that is responsible for their growth and therefore, they remain

[52]Another possibility would be that the regions be non-star-shaped. But we discard it due to the high empirical support for convexity and that if star-shapedness of regions find empirical support, then natural concepts can be identified by being star-shaped regions in a conceptual space.

[53]Recently, Strößner (2022) has proposed a new way to deal with the representation of multi-domain concepts in a conceptual space, considering the mentioned criticisms regarding the convexity constraint. They suggest to represent the concept child in a product space of the two domains with, age and height where the metric is Manhattan.

[54]It might be said that some young children be responsible for their acts or a grown up person not be responsible for her acts. In both cases, if someone is a child if she(he) is not an adult, then he(she) will be a child. We are not here looking for the right definition of child or adult. The point is that the height and age does not play any role in the mentioned official definition of child and adult.

tadpoles for ever. So, the concept cannot be defined by their growth of their tails when they are at first stages of their life.

This, of course, does not answer the more general worry behind the author's example, that the conceptual space approach needs to deal with the multi-domain concepts even if most concepts are one domain. We will go back to this point when we compare the conceptual space approach with the topological approach. In the next part we discuss that in the topological approach natural concepts can be identified as specific regions in a topological space. Mormann (2021) shows the tight relation between these regions and convex regions. We will also show how to deal with multi-domain concepts.

3- **The priority of concepts to prototypes**

Recently, Douven and Gärdenfors (2018) and more specifically Douven (2018) criticize and expand the notion of conceptual space. As we saw, conceptual spaces are based on prototypes and concepts are defined as convex regions in a conceptual space that is endowed with a geometrical topological structure. However, Douven and Gärdenfors question which is made primitive: prototypes or concepts?
The idea is that in order to have some typical examples one needs to have already a concept. So, according to the authors it seems that concepts have the priority over the prototypes. Prototypes are typical exemplars of a concept. That can be problematic for the conceptual space approach in identifying natural concepts. Because if convexity is not sufficient to identify natural concepts and these convex regions need to be a result of applying Voronoi technique to a set of prototypes in a metric space, then prototypes have priority over concepts.

After all, prototypes are not supposed to be prior to concepts. Rather, a

> prototype is said to be the best representative, the most typical instance, of a concept ... Thus, the concept must be there before the prototype can come into existence (Douven 2018, p.12).

This is a problem that should be considered by any theory of concepts that is based on prototype theory. Douven and Gärdenfors (2018) do not define a natural concept as a convex region in a conceptual space per se; the conceptual space that is partitioned into regions that have prototypes at their center to which the Voronoi-tessellation technique is applied to uniquely define a natural concept. Rather, according to them, natural concepts are convex regions in an optimized conceptual space. In this way, they do not commit themselves to the priority of concepts to prototypes. Though, in practice, the hypothesis that natural concepts are convex regions in a prototype-based- cum- Voronoi-tessellation technique conceptual space has been confirmed.

4- **The problem of graded membership**

The optimized advanced conceptual space approach may be able to deal with the uniqueness problem, the priority of concepts to prototypes and thickness problem but still cannot explain higher-order vagueness, endorsing that the objects belong to the extension of a concept to some degree.

As mentioned before, the advocates of conceptual space and gradedness of membership, unlike Gärdenfors's original conceptual spaces, claim that their view can give an account for at least first-order vagueness. In the original account the boundary is thin and the convex regions are precise. So, even talking about borderline cases is problematic. In the generalized approach, however, there are thick boundary and sharp regions.

The question, then, is :

What is the difference between boundary regions (if the thick boundary is considered as a region) and other regions that correspond to clear cases of a concept? By gradedness, borderline cases of red maybe very near red. So, their membership is almost 1. But the gradedness of membership will be problematic for the following reason:

Either the very small difference is important or not. If it is not important, then why don't we consider them as red? In this case, the extension of a concept will be a closed set. This is compatible with the application of Voronoi tessellation technique. Then, the objects that belong to the red region will be more similar to each other than the objects of other regions.

If the borderline cases do not belong to the extension of the concept, then the region is open. The problem is that as we move from prototype region to the edge of the region, the membership gradedness decreases. So, for example, in the red region, it might happen that a non-typical red object be more similar to a borderline red-orange object than to a red object. This violates the well-formedness criterion.

5. **A plea for a better design to encompass the perplexity of vagueness**

Conceptual spaces are not sufficient to encompass the perplexity of vagueness. The new design of conceptual spaces might be able to provide spaces with thick boundaries but still the conceptual space approach need to explain higher-order vagueness and to deal with the Sorites paradox. The latter, is not discussed by the proponents of this approach and we claim that their account of higher-order vagueness is problematic. If we want to talk about higher-order vagueness, we already suppose that "boundary" is a concept. In the conceptual space approach, a concept needs to be a convex region in one or more domains. One can suppose that there is a prototype region of typical borderline cases. As soon as we consider that point or region, the geometrical structure of the conceptual space in question changes. In the same way, one can explain higher-order vagueness. Nevertheless, it is not possible to explain higher-order vagueness in the object language. Suppose that you show some typical red things to a child and also some typical orange things. So, by Voronoi-tessellation techniques the conceptual space of a child is sharply divided into red region and orange region or into two sharp regions with a thick boundary. Now, suppose that you show the child some borderline cases. It takes time to conceive the concept of borderline. When he grasps such concept, teaching him some borderline cases, the topology of his conceptual space changes. It is not clear which dimensions are involved in the conceptualization of "boundary" and therefore, is not clear what the domain is. So, even if the location of borderline cases in the conceptual space is calculated by the collated Voronoi tessellation and they are defined in a conceptual space, it is not clear how to define boundary of boundary and further iterations.

The conceptual space approach is a very promising approach and we will adopt this approach in this thesis. However, pace Gärdenfors, who focuses mostly on Euclidean metric spaces, we define concepts in a conceptual space endowed with a topology. As argued by Mormann (2021) the topological approach provides us an optimized conceptual space in which concepts can be defined uniquely no matter what metric is chosen. Furthermore, the approach will be extended to non-metric spaces. So, even though the above criticisms regarding the convexity criterion and uniqueness just strengthen the criterion, showing its testability, conceptual spaces are not sufficient to encompass the perplexity of vagueness. In part V, we will discuss in detail whether the topological optimization of the conceptual spaces helps us to explain higher-order vagueness and to dissolve the Sorites paradox. A more general topological view can give us a better design for the conceptual spaces. The importance of the topological approach has already been mentioned in the literature on psychology. Lewin (1936) notes this point in the application of topological spaces to psychology.

> By this term [topological space] is meant that we are dealing with mathematical relationships which can be characterized without measurement. No distances are defined in topological space. A drop of water and the earth are, from a topological point of view, fully equivalent" (Lewin 1936, p.53).

Kurt Lewin's book ″Principles of topological psychology″, published in 1936, is a good example of applying topology to a field other than mathematics, namely psychology. The idea occurred to him when he became interested in the concept of space. Lewin uses topological spaces that might not be metric. According to Lewin, the concept of space has gone beyond physical spaces and Euclidean metric spaces (Lewin 1936, p.52).

He relates this concept of space to psychological facts:

> As far as mathematics is concerned there is therefore no fundamental

> objection to applying the mathematical concept of space to psychological facts. The crucial point is whether the relationships that characterize space in mathematics can be applied adequately to psychological facts, and whether one can coordinate psychological processes uniquely to mathematical operations (ibid,p.53).

In the next part we will introduce the topological approach, developed by Ian Rumfitt and recently, expanded by Thomas Mormann and will discuss the philosophical advantages of this approach to deal with the phenomenon of vagueness and the hurdles on the way of the proponents of this view to overcome.

Part V

An appropriate topological semantics for vagueness

8 Rumfitt's topological semantics for vague concepts

8.1 Introduction

In this part, we demonstrate in technical detail how vagueness binds up with classical logic via topological semantics. We propose that the appropriate topology may be a special kind of T_0-Alexandroff topology, namely the weakly scattered T_0- Alexandroff. We start off by recalling the apparent inconsistency of vagueness with classical logic. Then, we explain Rumfitt's recent topological view of vagueness. After some philosophical discussions, we show that Rumfitt's topology is a special case of weakly scattered T_0- Alexandroff and can be generalized to weakly scattered T_0- Alexandroff to deal with the phenomenon of vagueness.

Natural language is full of vague concepts. Any theory of vagueness should provide logic and semantics for a vague language. As we saw in part two, vagueness has cast doubt on the universal applicability of classical logic to the extent that many philosophers have deviated from classical logic, proposing a new logic for vagueness. Nevertheless, there have been some efforts to keep classical logic and semantics for vague predicates(cf. Fara (2000);Raffman (1994); Williamson (1994)).

Recently, Rumfitt proposed an approach according to which one can vindicate classical logic without accepting bivalence:

> I think it is a strategic mistake to rest the case for classical logic on the Principle of Bivalence: the soundness of the classical logical rules is far more compelling than the truth of Bivalence (Rumfitt 2015, p.13).

He rejects bivalence and gives a non-classical semantics for vagueness. Nevertheless, he keeps classical logic accepting the law of the excluded middle. This account of vagueness according to him dissolves the Sorites paradox.
Let us recall the Sorites paradox. Consider the well-known color example. Let X be a spectrum of color with a series of 100 patches such that the first one is clearly red and the last one is clearly orange and any two adjacent patches are indiscriminable in color. That is to say, when viewing any pair in the series in isolation (without considering other patches) no normal, unaided, sharp-eyed person can detect a difference in color between them. (See the tolerance principle in part II). So, to that person, if one is red the other is red as well. If this is so, it seems plausible to say that in this series there is no pair such that one is red and the other is not red. So, if the first one is red, then the last one will be red as well. This is a contradiction because the last one was assumed to be clearly not red.

As mentioned before, one way to dissolve the paradox is to deny the tolerance principle. For example, epistemicists claim that there is a sharp boundary but we do not know where it is. In this view, any patch in the above series is either red or not-red and somewhere in the series there is a pair such that one is red and the other is not red but we do not know where it is. One may say that the boundary is like a slippery fish that can never be caught. Why should we accept that there is such an inaccessible boundary?
Williamson(1994) contends that deviating from classical logic and semantics is not worth the price in the presence of the epistemic view, since the non-classical accounts so far proposed are all doomed to failure, either because the rejection

of bivalence leads to an absurdity, or because they cannot argue for higher-order vagueness:

> It begins to look as though abandoning the assumption that vague utterances are bivalent makes vagueness no easier to understand. If one abandons bivalence for vague utterances, one pays a high price. One can no longer apply classical truth-conditional semantics to them, and probably not even classical logic. Yet classical semantics and logic are vastly superior to the alternatives in simplicity, power, past success, and integration with theories in other domains. It would not be wholly unreasonable to insist on these grounds alone that bivalence must somehow apply to vague utterances, attributing any contrary appearances to our lack of insight. Not every anomaly falsifies a theory. That attitude might eventually cease to be tenable, if some non-classical treatment of vagueness was genuinely illuminating. No such treatment has been found (Williamson 1994, p.186).

Rumfitt apparently has found such treatment. He claims that his theory of vagueness escapes the criticisms of Williamson against the rejection of bivalence.[55] This section is devoted to this new topological account in which the truth value of a vague proposition can be indeterminate, yet does not generate a non-classical logic.

Rumfitt's theory of vagueness is based on Sainsbury's account according to which

[55]Williamson finds the topological accounts unnecessarily complicated.
Some philosophers such as Williamson do not find the logic S4 a suitable logic for vagueness. Also, according to him, when classical logic and semantics can be kept by his proposed logic the topological approach is not necessary. Bobzien does not find S4 problematic. Since the topological semantics presented by Rumfitt and ours generalizes the usual Kripke semantics for S4, we shall argue why it is not problematic.

vague predicates lack any (sharp) boundaries.(Sainsbury (1996)). According to Sainsbury, nobody even looks for a boundary for a vague predicate. If we accept Sainsbury's view that it is boundarylessness that constitutes vagueness, can we still keep classical logic?

At first glance, it seems quite clear that the answer is "no". The reason goes back to the dominant classical view of concepts. Frege in "Grundgesetze der Arithmetik" explicitly contends that the law of excluded middle entails that concepts have sharp boundaries:

> A definition of a concept (of a possible predicate) must be complete; it must unambiguously determine, as regards any object, whether or not it falls under the concept (whether or not the predicate is truly assertible of it). Thus there must not be any object as regards which the definition leaves in doubt whether it falls under the concept... We may express this metaphorically as follows: the concept must have a sharp boundary. If we represent concepts in extension by areas in a plane, this is admittedly a picture that may be used only with caution, but here it can do us good service. To a concept without a sharp boundary there would correspond an area that didn't have a sharp boundary-line all around, but in places just vaguely faded away into the background. This would not really be an area at all, and likewise, a concept that is not sharply defined is wrongly termed a concept. Such quasi-conceptual constructions cannot be recognized as concepts by logic; it is impossible to lay down precise laws for them. The law of excluded middle is really just another form

> of the requirement that the concept should have a sharp boundary [56] (Geach and Black 1952, p.159).

It is important to remark two points that Frege mentions with regards to concepts:

1- Concepts should have a sharp boundary.

2- The law of excluded middle entails the existence of a sharp boundary.

If one considers boundarylessness as one of the constitutive features of vagueness, then the first point wipes out from the language the concepts that are not sharply defined. It implies that concepts cannot be vague. If we want to accept vague concepts in our language, the second point deprives us of retaining classical logic. So, according to Frege, boundaryless concepts do not have any place in the language based on classical logic.
In set-theoretical terms, the positive and negative extensions of a predicate are sets. The former is a set of things that fall under the concept and the latter is the set-theoretical complement of the former: the set that contains the rest of the objects of the domain. This set-theoretic account of concepts, of course, cannot tolerate any vague concept. However, natural language is full of vague concepts. A topological account of vagueness is supposed to make some room for vague concepts. The existence of a sharp boundary on which Frege insisted, topologically means that the boundary is empty. However, we showed that there are topologies with non-empty boundaries. Vague concepts can be defined in these topological spaces.

In this part, we discuss Rumfitt's topological view on vagueness that maintains

[56] Gottlob Frege, The Fundamental Laws of Arithmetic, Vol. II, §56.

distance from the Fregean view, rejecting both points mentioned above.
Rumfitt gives a new definition of membership or falling under a concept, such that the law of excluded middle does not mean that a concept should have a sharp boundary. He provides a new formulation for the classical logical laws, such that the denial of bivalence does not lead to the denial of the law of excluded middle.

> From a logical point of view, analyzing statements in terms of function and argument is a great improvement on the traditional theory, but we cannot exclude the possibility of achieving a yet more powerful formulation of logical laws using a quite different set of fundamental notions (Rumfitt 2015, p.18).

To put it in a nutshell, Rumfitt has proposed an account of vagueness in which classical logic holds, the essence of vagueness is boundarylessness and the principle of bivalence does not hold. His account is based on prototypes or paradigms, the most representative instances of the concept.
Rumfitt takes a psychological view of concepts, namely prototype theory, as a departure point. He considers the concepts that have prototypes, or paradigms or typical exemplars that he, following Sainsbury, calls “poles”. Sainsbury uses the analogy of "magnetic poles" for how concepts become classified:

> As like magnetic poles exerting various degrees of influence: some objects cluster firmly to one pole, some to another, and some, though sensitive to the forces, join no cluster (Sainsbury 1996, p.258).

Rumfitt, however, thinks that it is enough to consider objects that are attracted to a pole. Therefore, he uses a simpler analogy, namely gravitational poles (Rumfitt

2015, p.236).

More precisely, each concept has one pole, p. It is called the pole associated with the concept. Of course, this does not mean that Rumfitt contends that all concepts have just one pole. However, he simply gives a polar topology for such concepts. Following Rumfitt, for simplification we will only consider concepts with one pole. But in general, we think that concepts may have more than one pole. In our generalization of polar topology, we will later not only consider the concepts with a pole, but also concepts with more than one pole and concepts without poles.

Just like a gravitational pole that attracts any mass sufficiently close to it, the pole associated with a concept attracts all the objects sufficiently close to it. Roughly speaking, there is a set of poles. The closer the object is to a pole, the gravity will be stronger so that it cannot be attracted by other poles. To find out whether an object belongs to the extension of a concept, the object is compared with the pole of that concept. An object is *maximally close* to a pole if it is closer to it than to other poles. An object may have more than one pole maximally close to it. The objects which are maximally close to just the pole of one concept belong to the extension of that concept. For example, suppose that the concept 'red' has a pole, r. All objects to which r is *the only* pole maximally close are considered as red. The borderline cases red-orange to which r and o, the pole of orange, are maximally close neither belong to the extension of red nor to that of orange.

According to Rumfitt, sharp boundaries do not play an important role in the categorization of a concept. Rumfitt takes this idea to use those representatives of best examples of a concept as poles. An object belongs to the extension of a concept with respect to the pole of that concept.

The elements of the extension of a predicate are related to the prototype or pole of that predicate. That is to say, they are exclusively maximally close to the prototype. In other words, the objects belong to the extension of a color not because they satisfy the necessary and sufficient conditions of being a member of the positive extension of the color. Rather, they are considered as that color (say red) because they are attracted *exclusively* by the pole associated with that color; i.e., r.

Rumfitt supports Sainsbury's idea of boundarylessness of vague concepts. Sainsbury contends that the notion of boundarylessness is entangled with the semantic for vagueness. Nevertheless, he asks for an appropriate logic and semantics for vague concepts:

> We do not know what our actual logic, which would be reapplied homophonically, is. We do not know, for example, whether every instance of P or not-P is counted true in our language and thought, and one pertinent reason for this doubt stems from vagueness. Secondly, even if we knew what our actual logic is, we could not uncritically reuse it in a semantic project, for the existence of Sorites reasoning casts doubt upon whether we are right to subscribe to the logic to which we actually subscribe. The logic of vagueness, characterized as boundarylessness, thus remains to be described (Sainsbury 1996, p.16).

Rumfitt's polar topology can be considered as an attempt to achieve that goal. For Rumfitt, in the spectrum of colors, main vague predicates such as 'red', 'blue' and 'orange' are mutually exclusive. Thus, saying that 'it is red' implies that 'it is not orange'. So, even assuming that for the borderline cases one is warranted both to say 'it is red' and 'it is orange', it is never acceptable to say 'it is red and orange' because these two colors are mutually exclusive.[57] The law of excluded middle remains since 'it is red or not red' holds even for borderline cases(ibid). Rumfitt argues technically for this view.

He uses the entangled relation between logic, order theory, algebra, and topology. It is well known that Boolean algebras are the basic algebraic structures of classical logic. Furthermore, regular open subsets of a topological space have the structure of a Boolean algebra. Rumfitt defines the extension of a concept as a regular open set. In that way he can provide a semantics for vague concepts that justifies the retention of the classical logic.
He concentrates on a very concrete usual example of vague concepts, namely color. In the spectrum of colors,'red' has a pole, r. Each predicate has a pole in the sense that it has cases that are typical examples of 'red'. Consider a limited language consisting of seven colors; Red, Orange, Indigo, Green, Blue, Yellow and Violet and their poles provided by the color spectrum; r, o, i, g, b, y, v, respectively.
Consider all possible colored objects such that each possible colored object is at least close enough to one pole. An object is red if it is maximally close only to the red pole r, it is not red if it is not maximally close to r A borderline case of 'red' is the one which is as maximally close to r as to another pole.

As Rumfitt puts it:

[57] Fine also has the intuition that it is false to say that a red-orange object be both red and orange. But this intuition is not completely shared. For example, for Machina (1976) and (Ostertag 2016, 461) it is quite intuitive. Also, see Priest (2017).

> The meaning of these color predicates is to be understood in relation to the poles or paradigms provided by a spectrum (ibid, 238).

His proposed semantics explains how the meaning of a color predicate is related to its pole.

Rumfitt defines a topology on the set of possible colored objects X, putting a restriction on the extension of a vague predicate. Objects that belong to the extension of a vague predicate, associated with the pole p are those to which p is the only pole maximally close. That is to say, they are closer to p than to any other pole in the color spectrum (ibid, p.239).

We will show that the polar topology is a special case of Alexandroff topology in which elements of the set of the colored objects are either a pole or sufficiently close to a pole. The poles are mutually exclusive. For example, r and o as typical examples of red and orange are far from each other.

Closeness is a topological concept. When there is a space with a certain metric, closeness can be defined with respect to that metric. Topology, however, is more general. It can be defined on a metric space or a non-metric one. There is no reason to limit us to the metric spaces (See part III). Furthermore, there is a tight relation between open sets in topology and the principle of tolerance that makes it plausible to formulate the latter in terms of the former.

As seen before, a topology can be defined based on open sets. By definition, any member of an open set has a neighborhood, all of whose members belong to that set. The following quote from Bourbaki may show the possible relation between neighborhoods in a topological space and being maximally close:

> The everyday sense of the word "neighborhood" is such that many of the

> properties which involve the mathematical idea of neighborhood appear as the mathematical expression of intuitive properties; the choice of this term thus has the advantage of making the language more expressive. For this purpose, it is also permissible to use the expressions "sufficiently near" and "as near as we please" in some statements. For example, Proposition I [A set is a neighborhood of each of its points if and only if it is open.] can be stated in the following form: a set A is open if and only if, for each $x \in A$, all the points sufficiently near x belong to A. More generally, we shall say that a property holds for all points sufficiently near a point x, if it holds for all points of some neighborhood of x (Bourbaki 1966, p.112).

Back to the principle of tolerance, if x belongs to the extension of 'red', so do all the objects indistinguishable from x with respect to its color. If the extension of 'red' is open and x is maximally close only to r, then there is a neighborhood of x such that all objects in that neighborhood are also maximally close to r. On the other hand, there is an affinity between closeness and continuity. Smith (2008) argues that such a relation between closeness and continuity trivializes the concept of vagueness in the sense that every predicate is vague. Later on, we will show that this is not the case.

Rumfitt considers the problem related to vague predicates that is commonly formulated as the Sorites paradox in which certain true premises using classical logic lead to a conclusion that seems to be false.

To dissolve the paradox, many philosophers appeal to non-classical logic. One of these attempts was proposed by Crispin Wright in his paper "Wang's paradox". According to Rumfitt, Wright argues that intuitionistic logic is *the* logic of vagueness.[58] Rumfitt claims that even if we assume that intuitionistic logic was the unique logic of vagueness, it would need a semantics that is not given by Wright. Without going into the detail of his semantics for the intuitionistic logic, at the end of part V, we briefly compare Rumfitt's semantics for intuitionistic and classical logic. Here we just roughly explain why intuitionistic logic does not seem the best choice.

Rumfitt proposes a semantics for intuitionistic logic based on the following conjecture[59]:

> Objects which satisfy a vague predicate may be expected to form an open set in a suitable topology.

However, he contends that openness is not restrictive enough for a concept. The reason is that open sets may have some cracks.

It is not appropriate for a concept to contain cracks or holes. For example, in the color spectrum, we do not expect to find a non-red object in the middle of red things. From part III, we know that regular open sets lack such cracks. So, Rumfitt's conjecture is that:

Objects which satisfy a vague predicate may be expected to form a regular open set in a suitable topology.

The relation between the regular open sets, Boolean algebras and classical logic, discussed in part III, gives us a clue on why appealing to regular open sets

[58]We think that the idea of Wright is weaker. In particular, he does not contend that it is *the* logic of vagueness, rather he finds any logic in which the double negation elimination is not valid, a suitable logic for vagueness.

[59]For more detail on why by considering the extension of vague predicates as an open set we get to the intuitionistic logic see appendix B.

seems a good choice for Rumfitt, who aims at keeping classical logic.

Rumfitt proposes the polar topology as the suitable topology for vague concepts.

In the Sorites series, objects are ordered. For example, the people are ordered from tall to not-tall, from thin to not-thin, or there is an order in the color spectrum such that the color of objects can vary from violet to green. Also, there is a relation, namely, indistinguishability between two adjacent members of the series. We will explain how polar topology deals with the Sorites paradox.

Rumfitt is aware that his topological account is restricted to the concepts with a unique pole, not for possible concepts that lack poles or have more than one pole and so he can only prove that such vague concepts have a place in classical logic.

> I argued there [in Rumfitt(2015)] that the extension of a vague predicate will be regular open whenever the predicate is 'polar' in Mark Sainsbury's sense—that is, when its meaning is given by reference to a system of contrary paradigms or poles (see Sainsbury (1991)). I stand by that argument. However, I never pretended to have an argument for the conclusion that every vague predicate is polar. So I have no argument for the thesis that every vague predicate has a regular open extension (Rumfitt 2018b, p.24).

For that reason he thinks that probably the intuitionistic logic is the appropriate logic for vague concepts in general:

> Perhaps having an open set as its semantic value is the strongest general requirement that any vague predicate must satisfy. If the topological space in which the set is open is Euclidean, then this requirement will

> sustain intuitionistic logic. That logic, then, may be the strongest logic that we are entitled to use in reasoning with any vague predicate (ibid).

It is expedient to ask whether we can improve his theory to include all concepts, considering an appropriate non-Euclidean topological space. Indeed, we will show that Alexandroff spaces provide a suitable bedrock for those who are going to define vague concepts.
We discuss the disadvantages of polar topology. In the last section, we propose a generalization of polar topology. Following Mormann(2020) we argue that weakly scattered Alexandroff topology is a good option for vagueness. However, we discuss its disadvantages.

We start off by elaborating the technical details of Rumfitt's polar topology.

8.2 Rumfitt's polar topology and its generalization

In this subsection, we explore in detail Rumfitt's topological account of vagueness. It reveals how topology is relevant to deal with the phenomenon of vagueness. More precisely, following Rumfitt we claim that vague concepts have a topological structure. Through detailed mathematical explanations, we demonstrate that polar topology is $T_{\frac{1}{2}}$ and therefore T_0- Alexandroff. Then, we go beyond that and propose one of the main claims of the thesis: that a suitable topology for vagueness is weakly scattered Alexandroff.

8.2.1 Rumfitt's polar topology as an Alexandroff topology

The main idea is that the sets are accompanied by a certain structure. What is the suitable structure of a set to confront the problems of vagueness, discussed in the second part? To give an answer, we need some preliminaries.

To introduce a suitable topology for vague predicates take a space X containing objects to be classified and a subset of X as the fixed set of poles. To define the topology of polar spaces, Rumfitt focuses on the typical examples of color spaces. Let X be a space that contains all possible colored objects. Let $P \subseteq X$ be the set of poles, provided by the spectrum of color. In particular, for colors red, orange, indigo, green, blue, yellow and violet, the set of poles is : $P = \{r, o, i, g, b, y, v\}$. Each object x is maximally close to at least one pole $p \in P$. Maximal closeness means that there is no other pole closer to x than p. This is intuitive because each object has a color in the color spectrum. So, it will at least be close to one of the poles. It is important to notice that it may happen that more than one pole be maximally close to x. For example, if in the color spectrum there exists a borderline red-orange object, then both r and o are maximally close to it. However, if the spectrum is divided sharply into certain colors, each object will have just one pole. In this case, the problem of vagueness will not occur. Furthermore, the poles are distinguishable so that if p is a pole, then no other pole is maximally close to it.

Below, we introduce our reformulation of the topology Rumfitt(2015, 8.4) proposed. The main aim of this reformulation is to mathematically elaborate Rumfitt's view to see some interesting properties of this topology.Rumfitt assumes that the maximal closeness relation can be defined in a metric space. We will show that this is an unnecessary limitation.

Definition 8.1. *Let X be a non-empty set of objects. $P \subseteq X$ the non-empty set of poles. Define $M : X \longrightarrow 2^P$, such that it satisfies the following two properties:*

1- $\forall x \in X \quad M(x) \neq \emptyset$

2- $\forall p \in P \quad M(p) = \{p\}$.

*The function M is called a **polar function** and (X, P, M) is called a **polar distribution**.*

The first condition in 8.1 captures the idea that each object is maximally close to at least one pole. The second condition formalizes the idea that the only pole maximally close to the pole p is p itself.

The polar function attributes to each element of a set X the set of poles maximally close to it. So, the distribution (X, P, M) classifies the members of X based on their set of poles maximally close to them.

In the next step, we will show that (X, P, M) defines a topology on X. Rumfitt suggests that (X, P, M) defines a topology via the interior operator (Rumfitt

2015, p.243).

Definition 8.2. *Let (X, P, M) be a polar distribution, $A \subseteq X$. Then, define the interior operation* Int *on X as:*

$$\text{Int} : 2^X \longrightarrow 2^X$$
$$\text{Int}(A) := \{x \in A | M(x) \subseteq A\}$$

The members of Int(A) are the ones that are in A and their set of poles maximally close to them also belongs to A. In other words, if x is in the interior of A, it does not have any pole that is not in A. For instance, if x is in the interior of 'red' associated with r, it does not have any pole maximally close to it other than r. So, x is far from the poles of other colors such as orange; rather, it is close to the typical examples of red.

Proposition 8.3. *The operator* Int $: 2^X \longrightarrow 2^X$ *is a Kuratowski's topological interior operator.*

Proof. Let X be a set and $A, B \subseteq X$. We show that Int satisfies the conditions mentioned in the definition 4.43.

(1) Decreasing:

By definition of Int,

$$\forall y \in \text{Int}(A), y \in A.$$

So, Int is decreasing.

(2) Total: Since Int is decreasing, Int$(X) \subseteq X$. On the other hand, all members of X are also members of Int(X) because for all $x \in X$, $M(x) \subseteq X$. So, Int is total.

(3) Idempotent (Rumfitt 2015, p.243):

Since Int is decreasing, $\operatorname{Int}\operatorname{Int}(A) \subseteq \operatorname{Int}(A)$. To prove the other direction, suppose that $y \in \operatorname{Int}(A)$. By definition of Int, $\operatorname{Int}\operatorname{Int}(A) = \{x \in \operatorname{Int}(A) | M(x) \subseteq \operatorname{Int}(A)\}$. We should prove that $y \in \operatorname{Int}\operatorname{Int}(A)$, so if we prove that $M(y) \subseteq \operatorname{Int}(A)$ we are done. By hypothesis, $y \in \operatorname{Int}(A)$. So $M(y) \subseteq A$. The elements of the set $M(y)$ are the poles maximally close to y. We show that all its members belong to $\operatorname{Int}(A)$. Suppose $p \in M(y)$. Since $M(y) \subseteq A$, $p \in A$. By definition 8.1 $M(p) = \{p\} \subseteq A$. So, $p \in \operatorname{Int}(A)$ and therefore, $\operatorname{Int}(A) \subseteq \operatorname{Int}\operatorname{Int}(A)$.

(4) Monotonicity : Assume by hypothesis $A \subseteq B$. Suppose $y \in \operatorname{Int}(A)$. By definition of Int, $y \in A$ and $M(y) \subseteq A$ By hypothesis $A \subseteq B$, therefore, $y \in B$ and $M(y) \subseteq B$. So, by definition of Int, $y \in \operatorname{Int}(B)$.

Finally, we prove that Int is distributive over conjunction.

(5) Distributive (See ibid, p.244):

$A \cap B \subseteq A$, and $A \cap B \subseteq B$. By monotonicity of Int , $\operatorname{Int}(A \cap B) \subseteq \operatorname{Int}(A)$ and $\operatorname{Int}(A \cap B) \subseteq \operatorname{Int}(A)$. Therefore, $\operatorname{Int}(A \cap B) \subseteq \operatorname{Int}(A) \cap \operatorname{Int}(B)$.

Conversely, suppose $z \in \operatorname{Int}(A) \cap \operatorname{Int}(B)$. Then, $z \in \operatorname{Int}(A)$ and, since Int is decreasing, $z \in A$. Also $M(z) \subseteq A$. Similarly, $z \in \operatorname{Int}(B)$ so $z \in B$ and $M(z) \subseteq B$. So, $z \in A \cap B$ and $M(z) \subseteq A \cap B$. So, $z \in \operatorname{Int}(A \cap B)$ and therefore, $\operatorname{Int}(A) \cap \operatorname{Int}(B) \subseteq \operatorname{Int}(A \cap B)$.

$\square$

The following theorem indicates that polar topology is closed under arbitrary intersection and, therefore, is Alexandroff. This provides much more information about the structure of polar topology, but it has not been mentioned by Rumfitt.

Theorem 8.4. *Let $(X, \mathcal{O}X)$ be the polar topology induced by a polar distribution (X, P, M). Consider an arbitrary family of open sets of X, $A_i \in \mathcal{O}X$. Then, the topology $\mathcal{O}X$ is Alexandroff :*

$$\mathrm{Int}(\bigcap A_i) = \bigcap A_i$$

Proof. Since Int is decreasing, $\mathrm{Int}(\bigcap A_i) \subseteq \bigcap A_i$.

Let us now prove the other direction: $\bigcap A_i \subseteq \mathrm{Int}(\bigcap A_i)$.

Suppose $y \in \bigcap A_i$. We show that $y \in \mathrm{Int}(\bigcap A_i)$. Since $\forall i \in I A_i$ is open, by the definition of Int, it is easy to see that $M(y) \subseteq \bigcap A_i$ and therefore, $y \in \mathrm{Int}(\bigcap A_i)$. □

Given any set A one can, as usual, define the closure operator via the interior operator since the closure of A, denoted by $\mathrm{Cl}(A)$, is the dual of $\mathrm{Int}(A)$:

$$\mathrm{Cl}(A) := A \cup \{x | M(x) \cap A \neq \emptyset\}.$$

Closure of A contains all members of A plus the ones that have at least one pole in A to which it is maximally close.

The polar topology is a special kind of Alexandroff topology. It has the following properties:

Proposition 8.5. *Let (X, P, M) be a polar distribution and $p \in P$. Then,*

1. $\forall x \in X - P$ $\mathrm{Int}(\{x\}) = \emptyset$, *equivalently,* $\mathrm{Cl}(\{x\}) = \{x\}$.
2. $\mathrm{Int}(\{p\}) = \{p\}$.
3. $\mathrm{Cl}(\{p\}) = \{x | p \in M(x)\}$.

4. $\operatorname{Int}\operatorname{Cl}(\{p\}) = \{x | \{p\} = M(x)\}$.

Proof. Considering the definitions of Int and Cl, it is easy to prove 1-3. We just give a proof for 4.
By the definition of Int, $\operatorname{Int}\operatorname{Cl}(\{p\}) = \{x \in \operatorname{Cl}(\{p\}) | M(x) \subseteq \operatorname{Cl}(\{p\})\}$. By the definition of Cl, $\operatorname{Cl}(\{p\}) = \{p\} \cup \{x | M(x) \cap \{p\} \neq \emptyset\}$. It is easy to see that no pole other than p belongs to the $\operatorname{Cl}(\{p\})$ because for any $q \in P, M(\{q\}) = \{q\}$. So, for $x \in \operatorname{Int}\operatorname{Cl}(\{p\})$, $M(x)$ as a subset of $\operatorname{Cl}(\{p\})$ should be $\{p\}$. Therefore, $\operatorname{Int}\operatorname{Cl}(\{p\}) = \{x | \{p\} = M(x)\}$.

□

The closure of $\{p\}$ contains the objects that at least have p as a pole maximally close to them but it may happen that they have another pole(s) maximally close to them. For example, consider the color red with its pole r. The closure of $\{r\}$ contains all the objects that are exclusively maximally close to r or the ones that are maximally close to another pole, say orange, as well. We may consider the latter objects as reddish or orangish. The interior closure of $\{p\}$ contains all the objects that just have p as the pole maximally close to them. The interior closure of 'red' is the set of objects that have no pole other than r maximally close to them. So, they are clearly red.

The open sets are the ones that are equal to their interiors. In polar topology the singletons are either open or closed. Next proposition immediately follows from proposition 8.5 :

Proposition 8.6. *Let (X, P, M) be a polar distribution. Then*
the poles and the set-theoretical complements of the non-pole singletons are open

sets. Formally:

$$\text{If } p \in P, \text{ then } \{p\} \in \mathcal{O}X \text{ and if } x \notin P, \text{ then } X - \{x\} \in \mathcal{O}X.$$

Recall that a topological space X is Alexandroff iff each point in X has a minimal open neighborhood. Since polar topology is Alexandroff, every singleton in a polar distribution has the minimal neighborhood.

Definition 8.7. *Let (X, P, M) be a polar distribution. Define an open set $U(x)$ of x as:*

$$\forall x \in X \; U(x) := \{x\} \cup M(x).$$

Proposition 8.8. *Let (X, P, M) be a polar distribution. Then, the open set $U(x)$ defined in the definition 8.7 is the smallest open set of the polar topology $\mathcal{O}X$ containing x; i.e.,it is the intersection of all open sets of x.*

Proof. By the definition of Int and $U(x)$, it is easy to see that $\text{Int}(U(x)) = \{y \in U(x) | M(y) \subseteq U(x)\} = U(x)$. So, $U(x)$ is open.
Let $U_i(x)$ be the collection of all open sets containing x. Since polar topology is Alexandroff, $\bigcap U_i(x)$ is a unique minimal open set containing x. We show that $\bigcap U_i(x) = U(x)$. From left to right is trivial, given that $U(x)$ is open. We prove the other direction. Suppose $y \in U(x)$. It is clear that if $y = x$, then $y \in \bigcap U_i(x)$. If $y \neq x$, then by the definition of $U(x)$, it is a pole of x. Let $y = p \in M(x)$. Since $x \in \bigcap U_i(x)$, and $\bigcap U_i(x)$ is open, $x \in \text{Int}(\bigcap U_i(x)) = \{z \in \bigcap U_i(x) | \, M(z) \subseteq \bigcap U_i(x)\}$. So, $M(x) \subseteq \bigcap U_i(x)$. Hence, $p \in \bigcap U_i(x)$.

□

Now we are ready to give the main connection between vague language and a polar distribution and the induced polar topology:

Principle 8.9. *Given a polar distribution* (X, P, M)*, the extension of a concept,*$\mathscr{C}$*, is the interior closure of its pole.*

The theorem can be expanded to the concepts with more than one pole. But for now we stick to the concepts with one pole.

Rumfitt claims that the interior closure of the pole of a concept is a better option than the interior of that pole to define the extension of a concept. As we said in the introduction, one reason is that $\mathrm{Int}\,\mathrm{Cl}$ just considers the objects that are very similar to the typical cases of the concept as the elements of that concept.

We do want that the objects that are not exactly red but very close to red be counted as red. If a person has 5 or 6 hairs we are still inclined to call him or her bald (of course it depends on the pole).
Another option may be the closed sets. Baskent (2013) defines the extension of a proposition as a closed set and introduces a paraconsistent semantics. However, if the extension of a concept is the closure of its pole, then it contains all the borderline cases. However, it is not palatable to have the borderline cases within the extension of a concept. We do not want to have a borderline red-orange or a red-violet object within the extension of "red".[60]
One more option is to define the extension of a concept as the closure interior of its pole. In this case, the extension of the concept will be regular closed. In fact, this is an option that has been vastly discussed in topological approaches to

[60] There is no consensus on putting aside the borderline cases from the extension of a concept. Priest (2017) finds it natural to accept them in the extension of a concept.

belief. In a series of papers Parikh and his colleagues relate topological reasoning with the logics of knowledge and belief (see Parikh et al. (2007), Moss and Parikh (1992)).

> We provide a topological semantics for belief, in particular, for Stalnaker's notion of belief defined as 'epistemic possibility of knowledge', in terms of the closure of the interior operator on extremely disconnected spaces (Baltag et al. (2018)).

According to Stalnaker, if someone believes that ϕ she believes also that she knows ϕ.

Definition 8.10. *A topological space* $(X, \mathcal{O}X)$ *is* ***extremely disconnected*** *if the closure of each open subset of* X *is open.*

The topological space $(X, \mathcal{O}X)$ is not, in general, extremely disconnected.

Proposition 8.11. *Let* (X, P, M) *be a polar distribution. Then,* $(X, \mathcal{O}X)$ *is extremely disconnected iff all elements of* X *have exclusively one pole.*

Proof. Let X be extremely disconnected. Then, by definition 8.10, $\forall A \in \mathcal{O}X$ $\operatorname{Int}\operatorname{Cl}(A) = \operatorname{Cl}(A)$. We prove that each element of X has a unique pole. By reductio, suppose that there is an x that has more than one pole maximally close to it. Suppose that $p \in P$ is one of the poles maximally close to x. Then, by proposition 8.5, $\operatorname{Int}\operatorname{Cl}(\{p\}) \neq \operatorname{Cl}(\{p\})$. This contradicts the hypothesis because $\{p\}$ is an open set. Therefore, all elements of X have exclusively one pole.

Suppose that all elements of X have just one pole, we prove that the space is extremely disconnected, namely $\forall A \in \mathcal{O}X$ $\operatorname{Int}\operatorname{Cl}(A) = \operatorname{Cl}(A)$. Clearly, $\operatorname{Int}\operatorname{Cl}(A) \subseteq$

$\mathrm{Cl}(A)$. So, it is enough to prove $\mathrm{Cl}(A) \subseteq \mathrm{Int}\,\mathrm{Cl}(A)$. Suppose $x \in \mathrm{Cl}(A) = A \cup \{y | M(y) \cap A \neq \emptyset\}$. By hypothesis, $M(x) = \{p\}, p \in P$. We consider two cases: either x is in A or it is not.

i) $x \in A$. By hypothesis, $A \in \mathcal{O}X$. Therefore, $A = \mathrm{Int}(A) \Rightarrow M(x) = \{p\} \subseteq A \subseteq \mathrm{Cl}(A) \Rightarrow M(x) \subseteq \mathrm{Cl}(A)$. By the definition, $\mathrm{Int}\,\mathrm{Cl}(A) = \{z \in \mathrm{Cl}(A) | M(z) \subseteq \mathrm{Cl}(A)\}$. Therefore, $x \in \mathrm{Int}\,\mathrm{Cl}(A)$.

ii) $x \notin A$. Then, $x \in \{y | M(y) \cap A \neq \emptyset\} \Rightarrow M(x) \cap A \neq \emptyset \Rightarrow p \in A \subseteq \mathrm{Cl}(A) \Rightarrow M(x) = \{p\} \subseteq \mathrm{Cl}(A) \Rightarrow x \in \mathrm{Int}\,\mathrm{Cl}(A) = \{z \in \mathrm{Cl}(A) | M(z) \subseteq \mathrm{Cl}(A)\}$. □

In extremely disconnected spaces there is no room for vagueness. On the other hand, for the defenders of classical logic like Rumfitt it is not a good choice because the set of regular closed sets forms a co-Heyting algebra (see Appendix B: Topological vs classical and intuitionistic semantics).

Baltag et al. (2013) following Parikh et al. (2007) interpret the belief modality as the interior of the closure of the interior operator on extremely disconnected spaces. They consider it as a generalization of Parikh's topological space.

In polar topology, however, $\mathrm{Int}\,\mathrm{Cl}\,\mathrm{Int}(\{p\}) = \mathrm{Int}\,\mathrm{Cl}(\{p\})$ and, as we mentioned before, the space is not, in general, extremely disconnected.

So, it seems more appropriate that we appeal to regular open sets to define the extensions of vague concepts.

Defining the extension of a predicate as a regular open set has some advantages for the ones who want to defend classical logic as well.

Proposition 8.12. *Let (X, P, M) be a polar distribution. The Boolean algebra $\mathcal{O}_{reg}X$ of regular open sets is isomorphic to 2^P by:*

$$\phi : 2^P \longrightarrow \mathcal{O}_{reg}X$$
$$\phi(A) := \operatorname{Int}\operatorname{Cl}(\bigcup_{p\in A}\operatorname{Int}\operatorname{Cl}(\{p\}))$$

Proof. First, we prove that ϕ is 1-1. Suppose $\phi(A) = \phi(B)$. Take $q \in A$. By hypothesis, $\phi(A) = \phi(B) \Rightarrow$
$\operatorname{Int}\operatorname{Cl}(\bigcup_{p\in A}\operatorname{Int}\operatorname{Cl}(\{p\})) = \operatorname{Int}\operatorname{Cl}(\bigcup_{p\in B}\operatorname{Int}\operatorname{Cl}(\{p\}))$.
By supposition, $q \in A$. Therefore, $\operatorname{Int}\operatorname{Cl}(\{q\}) \subseteq \bigcup_{p\in A}\operatorname{Int}\operatorname{Cl}(\{p\})$. By monotonicity of $\operatorname{Int}\operatorname{Cl}$, $\operatorname{Int}\operatorname{Cl}\operatorname{Int}\operatorname{Cl}(\{q\}) \subseteq \operatorname{Int}\operatorname{Cl}(\bigcup_{p\in A}\operatorname{Int}\operatorname{Cl}(\{p\})) = \operatorname{Int}\operatorname{Cl}(\bigcup_{p\in B}\operatorname{Int}\operatorname{Cl}(\{p\}))$.
So, by idempotency of $\operatorname{Int}\operatorname{Cl}$, $\operatorname{Int}\operatorname{Cl}(\{q\}) \subseteq \operatorname{Int}\operatorname{Cl}(\bigcup_{p\in B}\operatorname{Int}\operatorname{Cl}(\{p\}))$. Therefore, since $q \in \operatorname{Int}\operatorname{Cl}(\{q\}), q \in \operatorname{Int}\operatorname{Cl}(\bigcup_{p\in B}\operatorname{Int}\operatorname{Cl}(\{p\}))$.
By definition of Int, $M(q) = \{q\} \subseteq \operatorname{Cl}(\bigcup_{p\in B}\operatorname{Int}\operatorname{Cl}(\{p\}))$, i.e., $q \in \operatorname{Cl}(\bigcup_{p\in B}\operatorname{Int}\operatorname{Cl}(\{p\}))$.
By definition of Cl, $q \in \bigcup_{p\in B}\operatorname{Int}\operatorname{Cl}(\{p\}) \cup \{y : M(y) \cap (\bigcup_{p\in B}\operatorname{Int}\operatorname{Cl}(\{p\}) \neq \emptyset\}$.
It follows that $q \in \bigcup_{p\in B}\operatorname{Int}\operatorname{Cl}(\{p\})$. Take $p \in B$ such that $q \in \operatorname{Int}\operatorname{Cl}(\{p\})$.
By definition of Int, $M(\{q\}) = \{q\} \subseteq \operatorname{Cl}(\{p\})$, so $q \in \operatorname{Cl}(\{p\}) = \{p\} \cup \{y : M(y) \cap \{p\} \neq \emptyset\}$. It follows that $q = p \in B$. So $q \in B$. This shows that $A \subseteq B$ and, analogously, $B \subseteq A$. So, $A = B$.

Second, we prove that ϕ is onto.

By definition 10.6, we shall show that for each regular open set, A, there is a set B in the powerset of P, such that $\phi(B) = A$. Let $A \in \mathcal{O}_{reg}X$. Then, $\operatorname{Int}\operatorname{Cl}(A) = A$. Define $B \in 2^P$ such that $B := \bigcup_{p\in A\cap P}\{p\}$. We show that $\phi(B) = \operatorname{Int}\operatorname{Cl}(\bigcup_{p\in B}\operatorname{Int}\operatorname{Cl}(\{p\})) = A$. Suppose that $x \in A = \operatorname{Int}\operatorname{Cl}(A)$. By the definition of $\operatorname{Int}\operatorname{Cl}$, $x \in \operatorname{Cl}(A)$ and $M(x) \subseteq \operatorname{Cl}(A) = A \cup \{y \in A | M(y) \cap A \neq \emptyset\}$. Let $q \in M(x)$. By supposition, $x \in A$, since A is open, by definition of Int, $M(x) \subseteq A$ and therefore, $q \in A$. Therefore, $q \in \bigcup_{p\in B}\operatorname{Int}\operatorname{Cl}(\{p\})$. So, $M(x) \subseteq \bigcup_{p\in B}\operatorname{Int}\operatorname{Cl}(\{p\})) \subseteq \operatorname{Cl}(\bigcup_{p\in B}\operatorname{Int}\operatorname{Cl}(\{p\}))$. So, $M(x) \subseteq \operatorname{Cl}(\bigcup_{p\in B}\operatorname{Int}\operatorname{Cl}(\{p\}))$.

Since $q \in M(x)$, $x \in \mathrm{Cl}(\{q\})$.

Therefore, $x \in \operatorname{Int}\mathrm{Cl}(\bigcup_{p\in B}\operatorname{Int}\mathrm{Cl}(\{p\}))$. So, $A \subseteq \operatorname{Int}\mathrm{Cl}(\bigcup_{p\in B}\operatorname{Int}\mathrm{Cl}(\{p\}))$.

Now we show that $\operatorname{Int}\mathrm{Cl}(\bigcup_{p\in B}\operatorname{Int}\mathrm{Cl}(\{p\})) \subseteq A$.

$\forall p \in A$ by the monotonicity of $\operatorname{Int}\mathrm{Cl}$, $\operatorname{Int}\mathrm{Cl}(\{p\}) \subseteq \operatorname{Int}\mathrm{Cl}(A) = A \Rightarrow \bigcup_{p\in B}\operatorname{Int}\mathrm{Cl}(\{p\}) \subseteq \operatorname{Int}\mathrm{Cl}(A) = A$. By monotonicity and idempotency of $\operatorname{Int}\mathrm{Cl}$, $\operatorname{Int}\mathrm{Cl}(\bigcup_{p\in B}\operatorname{Int}\mathrm{Cl}(\{p\})) \subseteq \operatorname{Int}\mathrm{Cl}(A) = A$. □

As mentioned in the proposition 4.48, the set of regular open sets $\mathcal{O}_{reg}X$ forms a complete Boolean algebra. If X is finite, the only finite Boolean algebra is the power set of X, $\mathscr{P}X$ (Givant and Halmos 2008, p.127).

All finite topological spaces are Alexandroff. However, the infinite sets endowed with an Alexandroff topology have certain merits. In Alexandroff spaces we can study infinite sets as if they were finite. This does not happen if X is a Euclidean space (Gierz et al. 2003, p.14).
The Boolean algebra of regular open subsets of an infinite topological space $(X, \mathcal{O}X)$, as an Alexandroff space, behaves quite differently from the Boolean algebras in the Euclidean space. For example, it is quite well known that the Boolean algebra of regular open sets in the Euclidean space is atomless whereas, by proposition 8.12, in the Alexandroff space they are atomic; i.e., every non-zero element contains a minimal non-zero element (Givant and Halmos 2008, p.125).
In a polar space $\mathcal{O}_{reg}X$ is a complete Boolean algebra generated by $\operatorname{Int}\mathrm{Cl}(\{p\}) \quad p \in P$.

Proposition 8.13. *Let (X, P, M) be a polar distribution. Then, the induced polar topology $\mathcal{O}X$ is T_0.*

Proof. By definition 4.31, $(X, \mathcal{O}X)$ is T_0 if for every two distinct points x, $y \in X$, there exists an open set $U \in \mathcal{O}X$ such that either $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$.

a. If x is a pole, then $\{x\}$ is open and does not contain y. Analogously, when y is a pole, $\{y\}$ is open and does not contain x.

b. If neither of x and y are poles, then since $(X, \mathcal{O}X)$ is Alexandroff, x, y have minimal neighborhoods. For a polar space we showed that those neighborhoods are the open sets $U(x) = \{x\} \cup M(x)$ and $U(y) = \{y\} \cup M(y)$. Clearly, $y \notin U(x)$ and $x \notin U(y)$.

□

Proposition 8.14. *Let $(X, \mathcal{O}X)$ be an Alexandroff topology. If the topology is also T_1, then it is discrete.*

Proof. Since $(X, \mathcal{O}X)$ is T_1, for $x, y \in X, x \neq y$, there is an open set U such that $x \in U$ and $y \notin U$. Since $(X, \mathcal{O}X)$ is Alexandroff, the intersection of all open sets containing x,call it $U(x)$, is also open and $U(x) \subseteq U$. So, $y \notin U(x)$. Since y is arbitrary, $U(x) = \{x\}$. So, any singleton set is open and therefore, all subsets of X are open and the topology on X is discrete. □

By proposition 8.14, the only T_0 Alexandroff topology that is T_1, is the discrete topology (Arenas 1995, p.2). So, we just consider T_0 Alexandroff topologies lower than T_1.

In particular, we are interested in one of these separation axioms, namely $T_{1/2}$.

Definition 8.15. *Let $(X, \mathcal{O}X)$ be a topological space. Then X is $T_{1/2}$ space if every singleton set is either open or closed.*

Proposition 8.16. *Let (X, P, M) be the polar distribution. Then the polar topology is $T_{1/2}$.*

Proof. By proposition 8.5 any singleton is either a singleton of a pole and therefore open or it is not and it is closed. So, by 8.15 it is $T_{1/2}$. □

Up to now, following Mormann (2020) and Mormann (2021) we have reformulated a topology that Rumfitt (2015) considers as a suitable one for a theory of vagueness. However, Rumfitt does not explain in detail what kind of topology it is. We have shown in detail that the polar topology is $T_{1/2}$-Alexandroff. In the rest of this subsection, we will go further and will introduce some properties of polar topology considering that it is $T_{1/2}$-Alexandroff.
For general topological spaces, not all nowhere dense sets are closed. However, considering the definition of Int Cl, one can prove that in polar spaces, any nowhere dense set is closed. Recall that a set is nowhere dense if its closure has empty interior and a set is closed if it is identified with its closure.

Definition 8.17. *Let $(X, \mathcal{O}X)$ be a topological space. X is* ***nodec*** *iff for all $A \subseteq X$*

$$\textit{if } \operatorname{Int}\operatorname{Cl}(A) = \emptyset \textit{ then } \operatorname{Cl}(A) = A \textit{ (Van Douwen 1993, p.129).}$$

In words, X is nodec iff every nowhere dense subset of X is closed.

The following proposition shows that X endowed with the polar topology is nodec.

Proposition 8.18. *Let $(X, \mathcal{O}X)$ be a polar topology. Then, X is nodec; i.e. every nowhere dense set is closed.*

Proof. Suppose $A \subseteq X$ is nowhere dense. So $\operatorname{Int}\operatorname{Cl}(A) = \{x \in \operatorname{Cl}(A) \,|\, M(x) \subseteq \operatorname{Cl}(A)\} = \emptyset$. To show that A is closed we should show that $\operatorname{Cl}(A) = A$. From right to left is clear. So, we just show that $\operatorname{Cl}(A) \subseteq A$.

Take $x \in \operatorname{Cl}(A)$. By the definition of Cl, $x \in A \cup \{y |\, M(y) \cap A \neq \emptyset\}$. By reductio ad absurdum, suppose $x \notin A$. So, $x \in \{y |\, M(y) \cap A \neq \emptyset\}$. Let $p \in M(x) \cap A$. By the definition of $\operatorname{Int}\operatorname{Cl}, \operatorname{Int}\operatorname{Cl}(A) = \{z \in \operatorname{Cl}(A) | M(z) \subseteq \operatorname{Cl}(A)\}$. $p \in A \subseteq \operatorname{Cl}(A)$. In polar topology, where p is a pole $M(p) = \{p\}$. So, $p \in \operatorname{Int}\operatorname{Cl}(A)$.

This contradicts the supposition that A is nowhere dense. Therefore, $x \in A$.

□

One can observe that when a polar topology is connected, given that a polar topology is a nodec space, the only nowhere dense open set is the empty set.

Definition 8.19. *A topological space $(X, \mathcal{O}X)$ is* ***sub-maximal*** *if each dense subset is open.*

Proposition 8.20. *Let (X, P, M) be a polar distribution. Then, the polar space is sub-maximal.*

Proof. Suppose that $A \subseteq X$ is dense. So, $\operatorname{Cl}(A) = X$. To show that polar topology is sub-maximal, we should prove that A is open; i.e, $\operatorname{Int}(A) = A$.
Since $\operatorname{Int}(A) \subseteq A$, we just need to prove that $A \subseteq \operatorname{Int}(A)$. By reductio, suppose that $A \nsubseteq \operatorname{Int}(A)$. Then $\exists x \in A : x \notin \operatorname{Int}(A) = \{x \in A | M(x) \subseteq A\}$. Therefore, $M(x) \nsubseteq A$. So, $\exists q \in M(x) : q \notin A$. Since by hypothesis, $\operatorname{Cl}(A) = X$, $q \in \operatorname{Cl}(A) = A \cup \{y : M(y) \cap A \neq \emptyset\}$. So, $M(q) \cap A \neq \emptyset$. Since q is a pole $M(q) = \{q\}$. So, $q \in A$ which is contradiction. Therefore, $A \subseteq \operatorname{Int}(A)$. □

Since polar topology is sub-maximal, the only closed dense set in X is X itself.

For further analysis of the polar topology and to give the topological semantics that Rumfitt proposed, and to generalize the polar topology, we need some preliminaries. First, we recall the relation between modal logic and Alexandroff spaces and some necessary modal systems. Then, we will explain the tight relation between pre-ordered sets and Alexandroff topology and we will see to what extent polar topology, order theory and modal logic are integrated.

8.2.2 Modal logic and Alexandroff topology

Modal logic has found its place in philosophy. Various interpretations of modal operations have been used in different areas of philosophy. The philosophical papers with the special topological point of view usually consider the epistemic reading of the modal operators. Kishida (2011), for example, considers a possible world interpretation of propositional epistemic logic and shows that his proposed topological semantics for propositional modal logic, unlike Kripke's semantics, solves some epistemic problems. (Also see Parikh et al. (2007)).

Rumfitt also tries to give a topological semantics for modal logic, different from Kripke's possible world semantics in which bivalence holds.

> I have also included—mainly in footnotes—proofs of some of the basic facts about the semantic models that I use. Those models draw on lattice-theoretic and topological results that are not as well known among philosophers as they ought to be. Many of the proofs are short and simple, and I hope thereby to encourage philosophers to explore alternatives to the familiar, but often too restrictive, possible-worlds

semantics (Rumfitt 2015, 17).

In this section, we will show the close relation between Kripke frames $S4$ and Alexandroff topology. This may suggest that topology can be a suitable toolkit for philosophers in tackling philosophical hurdles, just like modal logic is.

Modal logic is the logic of possibility and impossibility, necessity and contingency, the topics that are discussed in philosophy to a large extent. There are different modal systems. C.I. Lewis proposed different axiomatic modal systems $S1$ to $S5$, using a modal operator box($\Box$). McKinsey and Tarski (1944) found a tight relation between the axioms of the modal system $S4$ and Kuratowski's axioms for the topological interior operator if $\Box$ is interpreted as the topological interior operator. Since then, it is quite well-known that the modal logic $S4$ is the logic of all topological spaces. Then, in the late 1950's Kripke proposed the possible world semantics for modal logic. Every topological semantics generalizes Kripke's semantics for $S4$. Proposing a topological as well as algebraic framework for intuitionistic logic, McKinsey and Tarski showed that topological spaces provide a suitable semantics for intuitionistic logic and modal systems.

On the other hand, it is well-known that there is a one-one correspondence between a Kripke frame that is reflexive and transitive($S4$) and Alexandroff spaces (Alexandroff (1937)). Alexandroff showed that there is a 1-1 correspondence between pre-orders and Alexandroff spaces. In the next subsection we will introduce more definitions to relate pre-order lattices to Alexandroff topological spaces as a suitable space in which vague concepts can be defined. In particular, we will explain in detail in what way from a Kripke frame (X, R) where R is a reflexive and transitive relation, one can construct an Alexandroff space and vice versa.

Let us remind what a Kripke frame is.

Definition 8.21. *A* ***Kripke frame*** *is an ordered pair, $\mathcal{F} = (X, R)$, where X is a set of points and R is a binary relation on X. The elements of X are called possible worlds and R is called accessibility relation. In a language of modal logic, $\mathcal{L}$, $\mathcal{M} = (X, R, \nu)$ is called a* ***Kripke model*** *where (X, R) is a frame, and ν is a valuation function that takes a world x and an atomic formula P and gives as value 0 or 1, to determine which atomic formulas are true at what particular worlds.*

Modal system	Frame properties	Axioms
K	Non	$K : \Box(\phi \to \psi) \longrightarrow (\Box\phi \to \Box\psi)$
T	reflexive	$T : \Box\phi \longrightarrow \phi, K$
B	reflexive and symmetric	$B : \phi \longrightarrow \Box\Diamond\phi, K, T$
S4	reflexive and transitive	$4 : \Box\phi \longrightarrow \Box\Box\phi, K, T$
S5	equivalence	$5 : \Diamond\phi \longrightarrow \Box\Diamond\phi, K, T$

Table 3: Some modal systems

There are different systems of modal logics, differentiated by the properties of the accessibility relation. In the table 3 we summarize some of the modal systems that we will use in our discussions on vagueness:

8.2.3 Order theory, Alexandroff topology and Polar topology

In this subsection we will show in what way T_0 Alexandroff topologies can be characterized by partial orders and how a pre-order on a set induces an Alexandroff topology on that set. This is due to the work of Alexandroff (1937).

Trivially, all finite topological spaces are Alexandroff. Some Alexandroff spaces like discrete and indiscrete spaces are metric but most of them are not metric. The Alexandroff topology puts a restriction on a set X so that even if it is infinite it has some properties of a finite set. Alexandroff (1937) refers to these spaces as "quasi-discrete" spaces. These spaces have special properties that connect them to the familiar fields of modal logic and order theory, In particular, to a S4 Kripke frame with a special pre-order relation. Let us start by showing that given any pre-order $\leq$ on a set X, one can define different topologies on X. We just consider one of them in particular that turns out to be Alexandroff and is appropriate for our discussions on vagueness.

Proposition 8.22. *Let $(X, \leq)$ be a pre-order. Define $\mathcal{O}X := \{\uparrow A; A \subseteq X\}$. Remind that $\uparrow A := \{x \in X| \quad (\exists a \in A) \quad a \leq x\}$, (see definition 10.27).Then, $(X, \mathcal{O}X)$ is a topology on X, called "upper topology".*

From the proposition 8.22, one can see that $\forall U \in \mathcal{O}X$ if $x \in U$ and $x \leq y$, then $y \in U$.

Proposition 8.23. *The upper topology $(X, \mathcal{O}X)$ induced by the pre-order $(X, \leq)$ is Alexandroff.*

Proof. To show that it is Alexandroff it is enough to show that $\mathcal{O}X$ is closed under infinite intersection,i.e., if $A_i \in \mathcal{O}X$ for all $A_i, i \in I$, then $\bigcap_{i\in I} A_i \in \mathcal{O}X$, i.e., $\uparrow \bigcap_{i\in I} A_i = \bigcap_{i\in I} A_i$.
$\supseteq$ is trivial. We prove $\subseteq$,i.e., $\uparrow \bigcap_{i\in I} A_i \subseteq \bigcap_{i\in I} A_i$.
Take $x \in \uparrow \bigcap_{i\in I} A_i$. Then, there exists $y \in \bigcap_{i\in I} A_i$ such that $y \leq x$. It follows that $y \in A_i$ for all $i \in I$ and $x \in \uparrow y$. Since $\uparrow y \subseteq \uparrow A_i$, it follows that $x \in \uparrow A_i$

for all $i \in I$. By hypothesis, $A_i \in \mathcal{O}X$ for $i \in I$, so $A_i = \uparrow A_i$ and, therefore, $x \in \bigcap_{i \in I} A_i$.

□

Definition 8.24. *Let $(X, \leq)$ be a poset and $(X, \mathcal{O}X)$ the corresponding Alexandroff topology. Then, the minimal neighborhood of x, and interior and closure of a subset of X are defined as:*

$$U(x) := \uparrow x = \{y \in X : x \leq y\}.$$
$$\mathrm{int}(A) := \{x | \uparrow x \subseteq A\} = \{x \in X : \forall y \in X \ (x \leq y \Longrightarrow y \in A)\}.$$
$$\mathrm{cl}(A) := \bigcup\{\downarrow x : x \in A\} = \{x \in X : \exists y \in A, x \leq y \ \}.$$

Comparing these definitions with the definition of the interior and closure operators that Rumfitt gives makes it clear in what way polar topology is naturally defined, given a certain relation and why it is Alexandroff.

Proposition 8.25. *Let $(X, \leq)$ be a pre-order. Then, $\leq$ is a poset iff $(X, \mathcal{O}X)$ is T_0.*

Proof. Let $(x, \leq)$ be a poset. Then, it is reflexive, anti-symmetric and transitive. By the definition 4.31, we should prove that for any $x, y \in X, x \neq y \ \exists U \in \mathcal{O}X, x \in U, y \notin U$ or $y \in U, x \notin U$. Let $U_x = \uparrow x = \{z \in X : x \leq z\}$.

Since $x \neq y$, by anti-symmetry, $x \nleq y \vee y \nleq x$. If $x \nleq y$, then $x \in U_x$ and $y \notin U_x$. If $y \nleq x$, then $y \in U_y$ and $x \notin U_y$. Therefore, the space is T_0.

Suppose that $(X, \mathcal{O}X)$ is T_0. Then, it is enough to show that $\leqslant$ is an anti-symmetric relation. By hypothesis, for all x and y in X, $x \neq y$, $\exists U \in \mathcal{O}X, x \in U, y \notin U$ or $y \in U, x \notin U$. Suppose that there exists $U \in \mathcal{O}X$ such that $x \in U, y \notin U$. By reductio, suppose $x \leq y$. Since by hypothesis $U = \uparrow U$ and $x \in U$, it follows that $y \in U$. This contradicts the supposition that $y \notin U$. Therefore, $x \nleq y$. Analogously, if $y \in U, x \notin U$, then $y \nleq x$. So, $x \neq y \to x \nleq y \vee y \nleq x$. By contraposition, $x \leq y \wedge y \leq x \to x = y$. So, $(X, \leq)$ is a poset. $\square$

Interestingly, Alexandroff also showed that we can start from an Alexandroff space and generate a poset. Let us explain it in more detail.

If we start with an Alexandroff topology, we can generate a $S4$ frame by defining the pre-order as the following:

Definition 8.26. *Let $(X, \mathcal{O}X)$ be a topological space. Then, $\mathcal{O}X$ defines the* ***specialization pre-order*** *$\leqslant$ on X by:*

$$x \leqslant y \textit{ iff } x \in \mathrm{cl}(\{y\}) \ .$$

It is easy to see that the specialization relation is a pre-order.

Given any topology, by the specialization order we obtain a pre-order. On the other hand, given a pre-order $(X, \leq)$, there are many topologies on X that induce that order as their specialization order. We saw one of them, namely upper topology. Similarly, one can define the open sets as the set of down sets. This topology is called "lower topology". Lower topology is also Alexandroff. The following crucial proposition says that there is an isomorphism between T_0 Alexandroff spaces and posets.

Proposition 8.27. *Let $(X, \leq)$ be a poset. Then, the corresponding T_0 Alexandroff topology $\mathcal{O}_{\leq}X$ can be defined as the set of all upsets of $\leq$. ($\mathcal{C}_{\leq}X$ will be the set of downsets). On the other hand, let $(X, \mathcal{O}X)$ be a T_0 Alexandroff topology. Define the specialization order $\leq_{\mathcal{O}X}$ on X by $x \leq_{\mathcal{O}X} y$ iff $x \in \mathrm{cl}(\{y\})$. Then,* *for all partial orders $\leq$ on X, $\leq_{\mathcal{O}X}=\leq$ and for all T_0 Alexandroff topologies $\mathcal{O}X, \mathcal{O}_{\leq}X = \mathcal{O}X$.*

(see (Erné et al. 2007, Bezhanishvili et al. 2004, Mormann 2021)).

For example, one can show that if $f : (X, \leq) \to (X, \leq')$ is monotone, then the corresponding function $\phi : (X, \mathcal{O}_{\leq}X) \to (X, \mathcal{O}_{\leq'}X)$ is continuous.

Now let's go back to the polar space. We have shown that polar topology is Alexandroff. Now define an order: (see Mormann (2021))

Definition 8.28. *Let (X, P, M) be a polar distribution, $x \in X$ and $p \in P$. Then define the* ***polar order*** *as: $x \prec y$ iff $x \in \mathrm{Cl}(\{y\})$.*

We call polar order the specialization order of the polar topology (that as we know is $T_{1/2}$ Alexandroff). So, it is clear that $(X, \prec)$ is a poset.

One can easily see that by the definition of Cl, for $y \neq x$, $x \prec y$ iff $y \in M(x)$ because $M(x) \cap \{y\} \neq \emptyset$. The set of isolated points (ISO) of a T_0 Alexandroff topological space $(X, \mathcal{O}X)$ corresponds to the maximal elements with respect to the specialization order. We denote the set of maximal elements by Max. In particular, in the polar space an element $x \in X$ is maximal with respect to the specialization order iff $\{x\} \in \mathcal{O}X$;i.e, if x is a pole. This shows that the specialization order in a polar space is of depth 2. The elements are either poles or non-poles. So, it is not possible to order the closed sets neither the poles by their prototypicality. As figure 8.2.3 shows, in polar space there are some objects that are poles and some that are non-poles. Poles are not comparable, neither are the non-poles.

Proposition 8.29. *Let (X, P, M) be a polar space. Then the space is scattered.*

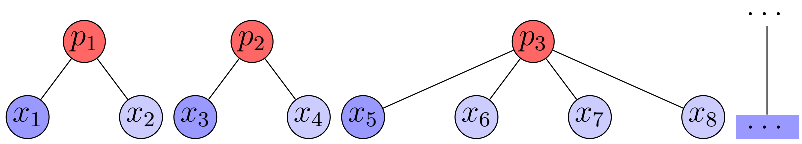


Figure 15: The relation between poles and non-poles in a polar space that is $T_{1/2}$ Alexandroff.

Proof. We show that any subset of X, endowed with the induced topology, has an isolated point. Let S be a subset of X. Either S contains $p \in P$ and therefore, has an isolated point or S does not contain any pole and therefore, is nowhere dense. Then, the induced topology will be the discrete topology. It follows that any point of S, endowed with the induced topology is isolated. Therefore, any subset of X contains an isolated point and so, is scattered. $\square$

Not all weakly scattered spaces are scattered. Mormann (2021, 2020) generalizes the polar space to weakly scattered spaces that may not be scattered. The spaces in which the specialization pre-order might be of order more than one.

In a nutshell, what Rumfitt did and we reformulated in definition 8.7 is to define $U(x)$ as a set of all up-sets with respect to a pre-order $\prec$ as a basis for polar topology. This topology, as we showed, is $T_{1/2}$ Alexandroff. This makes the polar spaces very limited in the sense that the maximal elements are open and other singletons closed. In the Euclidean space the specialization order is defined as $x \leq y$ iff $x = y$. So, $Max(X) = X$. Therefore, it is not apt if we want to distinguish some elements in the extension of a set. In weakly scattered spaces not only there are closed and open sets but the ones that are neither open nor closed. Mormann (2021, 2020) proposed that one should go beyond polar spaces. He consider Weakly scattered T_0 Alexandroff spaces for several reasons that will be mentioned shortly.

8.2.4 Rumfitt's topological semantics for classical logic

Rumfitt shows that topological spaces provide semantics for classical logic to be flexible enough to accept vague predicates. Following Tarski, he connects two-valued classical logic with topology for polar predicates such that:
Any sentence ϕ is valid in classical logic iff it holds in every topological space (Tarski 1956, p.421).

Let us define the syntax before giving the topological semantics for classical logics, proposed by Rumfitt.

Syntax

Let $\mathcal{L}$ be a language with countable proposition letters, $p, q, \dots,$ Boolean connectives, $\wedge, \vee, \rightarrow, \neg$ and the propositional constant for falsity $\bot = p \wedge \neg p$.

Denote a topological model by $\mathcal{M} = (X, \mathcal{O}X, \nu_{ro})$ where $(X, \mathcal{O}X)$ is a topology and $\nu_{ro} : \mathcal{L} \longrightarrow \mathcal{O}_{reg}X$ is an interpretation that takes any propositional letter of $\mathcal{L}$ to a regular open set of X.

Definition 8.30. *Let $\mathcal{M} = (X, \mathcal{O}X, \nu_{ro})$ be a topological model. Define the interior closure semantics for the language $\mathcal{L}$ as:*

$$\nu_{ro}(\phi \wedge \psi) = \nu_{ro}(\phi) \cap \nu_{ro}(\psi)$$
$$\nu_{ro}(\phi \vee \psi) = \operatorname{Int}\operatorname{Cl}(\nu_{ro}(\phi) \cup \nu_{ro}(\psi))$$
$$\nu_{ro}(\phi \rightarrow \psi) = \operatorname{Int}\operatorname{Cl}(\nu_{ro}(\psi) \cup \operatorname{Int}(X - \nu_{ro}(\phi)))$$
$$\nu_{ro}(\neg\phi) = \operatorname{Int}(X - \nu_{ro}(\phi))$$

This interpretation is suitable , according to Rumfitt, because the model $(X, \mathcal{O}_{reg}X, \nu_{ro})$ satisfies the axioms of Boolean algebra.

Definition 8.31. *Denote the satisfaction of a formula ϕ at a point x in a model $\mathcal{M}$ by $\mathcal{M}, x \models \phi$. Define the extension of a formula ϕ in the model $\mathcal{M}$, denoted by $[\phi]^{\mathcal{M}}$, as the points in $\mathcal{M}$ at which ϕ is satisfied.*

ϕ is true in a topological model $\mathcal{M} = (X, \mathcal{O}X, \nu_{ro})$ if $[\phi]^{\mathcal{M}} = X$.

ϕ is valid in $(X, \mathcal{O}X)$ if $[\phi]^{\mathcal{M}} = X$ for all topological models $\mathcal{M}$. And ϕ is valid in a class of topological spaces if ϕ is valid in every member of the class.

Before going to the reformulation of the Sorites paradox, we prove some of the classical logic rules hold in polar topology.

Lemma 8.32. *Let (X, P, M) be a polar distribution, $\nu_{ro} : \mathcal{L} \longrightarrow \mathcal{O}_{reg}X$, ϕ, ψ two sentential variables.* [61]

a. $\nu_{ro}(\phi \longrightarrow \psi) = \nu_{ro}(\neg\phi \vee \psi)$

b. The law of excluded middle holds: $\models \phi \vee \neg\phi$

Proof. a. By interpretation of the '$\rightarrow$':
$\nu_{ro}(\phi \longrightarrow \psi) = \operatorname{Int}\operatorname{Cl}(\nu_{ro}(\psi) \cup \operatorname{Int}(\complement\nu_{ro}(\phi))) = \nu_{ro}(\neg\phi \vee \psi)$.

b. Take any space $(X, \mathcal{O}X)$ and any interpretation ν_{ro}. cf $\nu_{ro}(\phi \vee \neg\phi) = \operatorname{Int}\operatorname{Cl}(\nu_{ro}(\phi) \cup \operatorname{Int}\complement\,\nu_{ro}(\phi)) = \operatorname{Int}(\operatorname{Cl}\nu_{ro}(\phi) \cup \operatorname{Cl}\complement\operatorname{Cl}\nu_{ro}(\phi)) = \operatorname{Int}(X) = X$. □

In the next subsection we will introduce the generalization of the polar spaces to weakly scattered spaces proposed by Mormann (2020), its characteristics that

[61] Whenever it is clear we just use ν instead of ν_{ro}

according to Mormann makes these spaces more apt to deal with the phenomenon of vagueness and in particular, its application to vagueness in a conceptual space and to deal with the higher-order vagueness.

8.2.5 The generalization of polar spaces to weakly scattered spaces

In the previous section, we defined an extension of a vague concept associated with a pole p as an interior closure of the pole in the polar topology. We showed that polar topology is $T_{1/2}$ Alexandroff. In any Alexandroff topology, each element has a minimal neighborhood. Considering a set of poles, provided for example by the color spectrum, we defined the minimal neighborhood of an element x, as $U(x) = \{x\} \cup M(x)$. The space is nodec and submaximal, any nowhere dense set is closed in X and the specialization order is of depth 2.

In weakly scattered spaces, however, one may make a difference between the elements of a set. There is a degree of prototypicality with respect to a certain property. For example, in the set of red objects some are the most typical cases of red, some are less prototypical than the previous ones but still are more prototypical than some other objects.

Let us remind the definition of a weakly scattered topological space:

$(X, \mathcal{O}X)$ is weakly scattered iff $ISO(X)$, the set of isolated points in X, is dense in x, i.e., $\mathrm{Cl}(ISO(X)) = X$ (see definition 4.56). It is quite well-known that the logic of the class of weakly scattered spaces is S4.1, the extended logic of S4 by adding McKinsey axiom:

$$\Box\Diamond\phi \longrightarrow \Diamond\Box A \qquad (1.)$$

$$\operatorname{int}\operatorname{cl}(A) \subseteq \operatorname{cl}\operatorname{int}(A) \tag{1'.}$$

(1'.) is the usual topological interpretation of (1.) (where the interpretation of statements are not necessarily regular open sets).

Before delving into weakly scattered spaces, let us recall some properties of any topological space.

Lemma 8.33. *Let $(X, \mathcal{O}X)$ be a topological space, $A \subseteq X$.*

$$bdbd(A) = bd(A) \textit{ iff } \operatorname{int}(bd(A)) = \emptyset.$$

Proof. $\leftarrow$: Suppose $\operatorname{int}(bd(A)) = \emptyset$.

$bdbd(A) = \operatorname{cl}(bd(A)) \cap \operatorname{cl}(\complement bd(A))$

$= bd(A) \cap \complement \operatorname{int}(bd(A))$ ($bd(A)$ is closed)

$= bd(A)$ (by hypothesis, $\operatorname{int}(bd(A)) = \emptyset$.)

$\rightarrow$: Suppose $bdbd(A) = bd(A)$. Then, $bd(A) \subseteq bdbd(A)$.

$\operatorname{int}(bdbd(A)) = \operatorname{int}(\operatorname{cl}(bd(A)) \cap cl(\complement bd(A)))$

$= \operatorname{int}(bd(A) \cap \complement \operatorname{int}(bd(A)))$ ($bd(A)$ is closed).

$= \operatorname{int}(bd(A)) \cap \operatorname{int} \complement \operatorname{int}(bd(A))$ (By distributivity of int over conjunction).

$\subseteq \operatorname{int}(bd(A)) \cap \complement \operatorname{int}(bd(A))$ (int is Decreasing (see definition 4.41)).

Therefore, $\operatorname{int}(bdbd(A)) \subseteq \emptyset$. It follows that $\operatorname{int}(bdbd(A)) = \emptyset$.

By monotonicity of the interior operator, from the hypothesis it follows that $\operatorname{int}(bd(A)) \subseteq \operatorname{int}(bdbd(A))$. Since $\operatorname{int}(bdbd(A)) = \emptyset, \operatorname{int}(bd(A)) = \emptyset$. □

Definition 8.34. *Let $(X, \mathcal{O}X)$ be a topological space, $A \subseteq X$. A has a thin boundary bd if* $\operatorname{int}(bd(A)) = \emptyset$. *Otherwise, A has a thick boundary.*

In the Appendix B of his paper Mormann (2020) gives some examples of weakly scattered spaces that are not Alexandroff and some Alexandroff spaces that are not weakly scattered spaces. In that paper he uses one special characteristics of weakly scattered spaces with respect to the boundary operator:

Proposition 8.35. *Let $(X, \mathcal{O}X)$ be a weakly scattered Alexandroff space. Then for all $A \subseteq X$, $\operatorname{int}\operatorname{cl}(A) \subseteq \operatorname{cl}\operatorname{int}(A)$ iff $bdbd(A) = bd(A)$.*

Proof. By 8.33, it is enough to show that the Mckinsey axiom holds for A iff it has a thin boundary.

$\operatorname{int}\operatorname{cl}(A) \subseteq \operatorname{cl}\operatorname{int}(A)$

$$\begin{aligned}
&\Longleftrightarrow \operatorname{int}\operatorname{cl}(A) \cap \complement \operatorname{cl}\operatorname{int}(A) = \emptyset \\
&\Longleftrightarrow \operatorname{int}\operatorname{cl}(A) \cap \operatorname{int} \complement \operatorname{int}(A) = \emptyset \\
&\Longleftrightarrow \operatorname{int}\operatorname{cl}(A) \cap \operatorname{int}\operatorname{cl}(\complement A) = \emptyset \\
&\Longleftrightarrow \operatorname{int}(\operatorname{cl}(A) \cap \operatorname{cl}(\complement A)) = \emptyset \,\text{(By distributivity of int)} \\
&\Longleftrightarrow \operatorname{int}(bd(A)) = \emptyset \;\; \text{(by the definition of } bd\text{).}
\end{aligned}$$ □

By the definition 4.56, it is easy to see that polar topology is a specific example of weakly scattered spaces since the poles are the only isolated points and the closure of the set of poles is equal to the whole set. We copy this proposition from Mormann (2020):

Proposition 8.36. *Let (X, P, M) be a polar distribution. Then the polar topology defined on X is weakly scattered.*

Proof. We know that $ISO(X) = P$. So, $\mathrm{Cl}(ISO(X)) = \mathrm{Cl}(P) = P \cup \{x|M(x) \cap P \neq \emptyset\}$. Since each x has at least one pole, $\mathrm{Cl}(P) = X$. Therefore, the space is weakly scattered. □

The weakly scattered T_0 Alexandroff space with the specialization order $(X, \leq)$, defined as $x \leq y$ iff $x \in \mathrm{cl}(y)$ orders the space X, in a different way in comparison to the polar topology. In polar topology, each chain is of depth 2, namely $x \leq p, p \in P$. However, in weakly scattered T_0 Alexandroff spaces, the depth of a chain can be more than 2. In fact, the space need not be $T_{1/2}$. This means that we may have orders between poles and finitely many sub poles: $x \leq y \leq z ... \leq q$. Figure 16 shows the simplest poset of depth more than two: $x \leq y \leq z$.

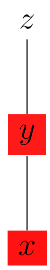


Figure 16: A weakly scattered space of depth 3.

For example, in the polar color space one can differentiate red things from not-red things without considering its shades. In the weakly scattered color space, however, she can differentiate a shade of red like crimson as a sub-pole.

All shades belong to the closure of $\{r\}$ and the extension of the concept 'crimson' is included in the extension of 'red'. In figure 16, y can be considered as the sub-pole crimson and z as the pole of red, r.

In general, in weakly scattered T_0 Alexandroff spaces the specialization order has maximal elements, i.e., the space is Noetherian (see definition 10.23). These maximal elements coincide with the set of poles in the polar space (Mormann 2021, proposition 4.1).

In a nutshell, a conceptual space endowed with a weakly scattered T_0 Alexandroff topology, as an atomic Boolean lattice, leads us to the optimised design of the conceptual spaces. We get the tessellation of the space into atomic regular open sets in which the extension of a concept is $\operatorname{int}\operatorname{cl}(\{p\}) := \{x| \uparrow x \subseteq \downarrow p, p \in P\}$ where P is dense in X. This discretization of the space coincides with the one suggested by Gärdenfors (2000), applying Voronoi-tessellation technique, as proved in Mormann (2021).

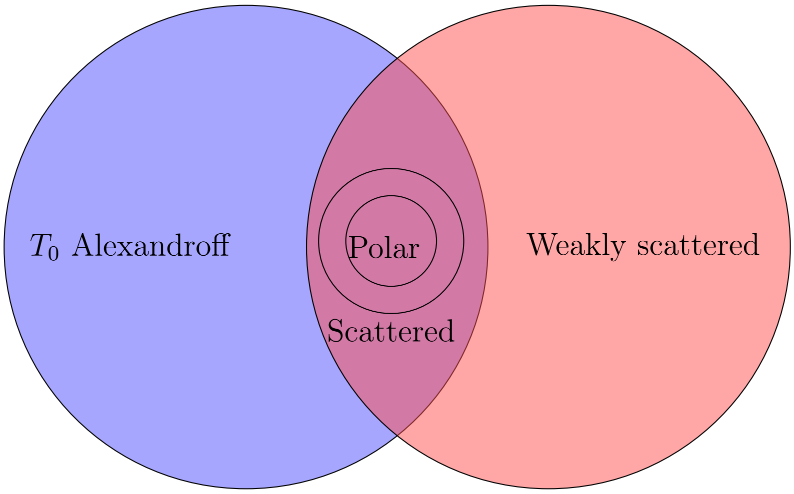


Figure 17: Polar space and its possible generalizations: Polar spaces are small part of weakly scattered Alexandroff spaces, namely the scattered spaces. We generalize it to the weakly scattered Alexandroff spaces. One can generalize it to T_0 Alexandroff spaces.

Motivations to go beyond polar spaces

Rumfitt considers a set of poles. The relation between the objects is defined by their closeness to the poles. The poles are mutually disjoint. The set of poles defines the polar topology that, as we showed, is Alexandroff. Up to here, everything goes well. The problem is that it is not general enough to cope with our conceptual framework. We may, for example, say that the polar topology for the color space is a topology for an idealized cognitive experience of color or in Rumfitt's terms gives a "lower-resolution banding". Let us explain it in more detail. Our perception of color is limited but not so limited as to not differentiate the shades of a color. This means that the order can be defined more specifically in such a way that the objects in the extension of 'red' are also ordered with respect to some other (finite) poles (call them sub-poles) such as scarlet, crimson, carmine, etc. In polar topology the classification of objects in a color space is based on the pole of the concept without considering its shades. Being less red or redder is not important. The weakly scattered Alexandroff space provides a good model for the "higher-resolution" of the extension of a concept in which the observer differentiates shades of the color. In fact, this is what Rumfitt had in mind but the polar space is not able to model it. Though, apparently what Rumfitt is looking for is to consider these shades when the observer focuses on each band:

> That is to say, he[the viewer] sees in S[the spectrum] bands of colour. This perceived banding has a complex structure. At the crudest level, he discerns seven bands within S, which he labels r, o, y, g, b, i, v from left to right across S. But if he focuses on the leftmost band, r, he will discern bands within it, bands that can serve as paradigms of different varieties of red, such as scarlet or crimson. We further suppose, however, that when our viewer looks again at the whole of S he continues to discern the seven main bands he had already labelled. So the higher-resolution banding does not undermine or conflict with the lower-resolution banding (Rumfitt 2015, p.238).

The polar space and its generalization, weakly scattered Alexandroff space model lower and higher resolution banding, respectively.

On the other hand, starting with prototypes is quite restrictive. We can move the other way around. If we suppose that the structure of a space is weakly scattered T_0 Alexandroff, then by definition, the maximal sets are all open. These elements may be considered the poles and they define the specialization order which generates that topology.

Advantages:

We adopt weakly scattered T_0 Alexandroff spaces as the spaces that provide a better theory of vagueness. If we simply accept T_0 Alexandroff spaces, then there is no guarantee for the existence of maximal elements. Maximal elements correspond to the set of poles in the polar space. So, we need to restrict the T_0 Alexandroff space to weakly scattered T_0 Alexandroff spaces to get the poles.

For any object in the space, there is at least one pole that is maximally close to it. Polar $T_{1/2}$ Alexandroff space is just a specific example of the weakly scattered Alexandroff spaces since the latter permits the existence of chains of poles of different but finite depths. In fact, weakly Alexandroff spaces better model the finer resolution of colors.

Also, as showed by Mormann (2021), it better designs the conceptual spaces. The extension of a concept in the tessellation of the topological space into atomic regular open sets is the same as extension of a concept when the Voronoi tessellation is applied in the geometrical conceptual space. Nevertheless, it does not have the uniqueness problem of the conceptual spaces approach. Conceptual spaces endowed with the weakly scattered T_0 Alexandroff topology provide us with optimised conceptual spaces to which many criticisms mentioned for the geometrical conceptual space does not apply. For example, the criticisms related to the convexity and variety of metrics. Since while the concepts in both frameworks have the same extensions, the topological framework provides a unique space for different metrics defined on a set. Moreover, weakly scattered T_0 Alexandroff spaces are well-behaved spaces. They have maximal elements that permit us to consider them as poles, the intersection of arbitrary open sets is still open, the elements of the extension of a concept can be compared with respect to their prototypicality.

Furthermore, it provides a logic that can be the logic of columnar higher-order vagueness. Mormann (2020) shows in detail that the logic of the generalized polar topology is the logic of Bobzien's columnar vagueness, S4.1.

Given the huge application of Alexandroff spaces and weakly scattered ones in computer science and digital topology and image processing[62], it is expedient to work on these spaces more profoundly.

Disadvantages

Despite all its advantages, still we need to deal with the "thickness problem". This topological approach explains that the boundary can be thick with regard to the cardinality of the boundary. That is to say, if thin boundary is interpreted as the boundary that cannot have many members, then the topological approach can dissolve the problem. However, if the topological interpretation of the problem is that there is no neighborhood all elements of which belong to the boundary ($\mathrm{int}(bd(A)) = \emptyset$), then obviously, weakly scattered spaces will not be adequate. So, we may differentiate three kinds of "thickness problem":

[62]See, for example, Khalimsky (1987) and Mormann (2021).

1. **Euclidean-thickness:** If we move a little bit from a Borderline case, we still find other borderline cases.

   This is the one that proposed by Douven to criticize the original version of conceptual spaces. As we discussed before, conceptual spaces are usually limited to the Euclidean spaces.

2. **Numerical-thickness:** Mormann's interpretation is that if we consider the Euclidean space, then the boundary is thin in the sense that there are a few borderline cases. So, to solve D-thickness with his interpretation, he showed that in the topological approach there might be many borderline cases. So, his solution in fact, works for the Numerical-thickness problem. However, the topological approach cannot solve the Euclidean-thickness problem. This leads us to a topological thickness problem:

3. **Topological-thickness:** In the weakly scattered spaces the boundary is thin because the interior of the boundary is empty. So, there is no neighborhood around a borderline case such that all of his members belong to the boundary. This can be considered as a topological version of Euclidean thickness, proposed by Douven. The problem is not the cardinality of the set of borderline cases, rather the problem is that intuitively, in the smooth transition from the typical cases of a concept to the typical cases of its negation we usually find more borderline cases in the neighborhood of a borderline case. If we can come over this problem, then we can argue that we have saved the conceptual spaces from the thickness problem by imposing a topology on the space.

Also, if we would like to explain hierarchical higher-order vagueness we need to sacrifice this well-behaved space for more general one, namely T_0 Alexandroff spaces. We still keep the relation between Boolean algebra, order theory and topology but we lose atomicity of the spaces and well-behavedness of the space. Yet we can define vagueness and precision for all concepts not just the ones that have poles. In the current work we will just consider weakly scattered spaces as the generalization of polar spaces (see figure 17). We will now define borderline cases. As discussed in part II, this is an important task in dealing with the phenomenon of vagueness.

8.3 Borderline cases in polar topology

The extension of a concept was defined as the set of elements of X that just have one pole maximally close to them, say p. Formally, those elements belong to $\operatorname{Int}\operatorname{Cl}(\{p\})$. So, they are the objects in the domain that are very close to the pole of the concept so that they are attracted by it but they are not necessarily typical examples of it; in the case of the concept 'red', they are the objects that we classify them as red with respect to the pole of red.

Borderline cases are usually defined as the ones that neither definitely belong to the extension of a concept nor to the extension of its complement. The boundary of a set A is defined in the formal way as $\operatorname{Cl}(A) \cap \operatorname{Cl}(\complement A)$. We will show that the boundary of a concept, associated with the pole p, in polar topology is not empty if the space is connected. The color space is connected. But suppose for example that there were three poles red, orange and blue such that the blue is so far from the other poles that there are no objects to which both blue and another pole

are maximally close. Then, 'blue' would be precise and the color space would be disconnected. This does not happen to two adjacent colors such as red and orange or blue and green in the normal color space because the color space is connected. As we mentioned before, in the philosophical literature on vagueness usually the boundary of a concept is defined with respect to the boundary of its complement. For example, the boundary between the red things and not red things, the boundary between tall and not tall people,.... However, in many cases we also look for the boundary of a concept with respect to another concept that is not necessarily its complement. For instance, we would like to define the boundary between red and orange, or between red and violet. We would like to differentiate the boundary between red and orange from the boundary between red and violet. For that reason, we think the boundary of a concept, in general, should be defined with respect to another concept such that the extreme case be the absolute boundary between the concept and its complement. So, we define borderline cases of a predicate associated with a pole p with respect to another concept associated with a pole q as the ones whose set of poles contains both p and q.
For example, x is a borderline case of 'red' with respect to 'orange' if it has r and o maximally close to it.
Now we present our formalization of these notions.

Definition 8.37. *Let (X, P, M) be a polar distribution, $A, B \subseteq X$ such that $A \cap P = \{p\}, B \cap P = \{q\}$. Define the boundary of A with respect to B as:*

$$Bd(A|B) := \mathrm{Cl}(\{p\}) \cap \mathrm{Cl}(\{q\}).$$

Proposition 8.38. *Let (X, P, M) be a polar distribution, $p, q \in P$ be two distinct poles. Then,*

$$Bd(\{p\}|\{q\}) = \{x|p, q \in M(x)\}.$$

Proof. $Bd(\{p\}|\{q\}) = \mathrm{Cl}(\{p\}) \cap \mathrm{Cl}(\{q\}) = \{x|p \in M(x)\} \cap \{y|q \in M(y)\} = \{z|p, q \in M(z)\}$.

□

Definition 8.39. *Let (X, P, M) be a polar distribution, $A, B \subseteq X$ such that $A \cap P = \{p\}, B \cap P = \{q\}$. Define the boundary of A with respect to B as:*

$$Bd(A|B) := \mathrm{Cl}(A) \cap \mathrm{Cl}(B).$$

Proposition 8.40. *Let (X, P, M) be a polar distribution. Then, the boundary defined in definition 8.37 coincides with the boundary defined in definition 8.39.*

Proof. We need to prove that $\mathrm{Cl}(\mathrm{Int}\,\mathrm{Cl}(\{p\})) \cap \mathrm{Cl}(\mathrm{Int}\,\mathrm{Cl}(\{q\})) = \mathrm{Cl}(\{p\}) \cap \mathrm{Cl}(\{q\})$. It is enough to prove that $\mathrm{Cl}(\mathrm{Int}\,\mathrm{Cl}(\{p\})) = \mathrm{Cl}(\{p\})$.

First we prove that $\mathrm{Cl}(\mathrm{Int}\,\mathrm{Cl}(\{p\})) \subseteq \mathrm{Cl}(\{p\})$. By Decreasing, $\mathrm{Int}\,\mathrm{Cl}(\{p\}) \subseteq \mathrm{Cl}(\{p\})$. By monotonicity of Cl, $\mathrm{Cl}\,\mathrm{Int}\,\mathrm{Cl}(\{p\}) \subseteq \mathrm{Cl}\,\mathrm{Cl}(\{p\})$. By idempotency of Cl, it follows that $\mathrm{Cl}\,\mathrm{Int}\,\mathrm{Cl}(\{p\}) \subseteq \mathrm{Cl}(\{p\})$.

Second, we show that $\mathrm{Cl}(\{p\}) \subseteq \mathrm{Cl}(\mathrm{Int}\,\mathrm{Cl}(\{p\}))$.

By the definition of Int Cl, $p \in \mathrm{Int}\,\mathrm{Cl}(\{p\})$. Therefore, $\{p\} \subseteq \mathrm{Int}\,\mathrm{Cl}(\{p\})$. By monotonicity of Cl, $\mathrm{Cl}(\{p\}) \subseteq \mathrm{Cl}(\mathrm{Int}\,\mathrm{Cl}(\{p\}))$. □

This is, actually, what we were looking for. The borderline cases of a concept with a certain pole are the ones to which that pole and at least one other pole is maximally close. In the case of color space, the closure of $\{p\}$ is the set of colored objects to which p is maximally close but they might have other poles as well. The boundary of ‘red’ with respect to ‘orange’ contains the objects to which at least r and o are maximally close to. Obviously, this boundary is different from the boundary between 'red' and 'violet'.

The whole boundary of A, associated with the pole $\{p\}$, is defined as:

$$Bd(A) = \bigcup_{1 \leq i \leq |P|-1} Bd(A|B_i),\ B_i = \operatorname{Int}\operatorname{Cl}(\{q\}),\ q \in P - \{p\}.$$

This means that the whole boundary of a concept is the union of the boundaries of that concept with respect to all other concepts. For example, the boundary of 'red' is the union of its boundary with respect to orange and its boundary with respect to violet, etc. Two concepts with poles p, q are adjacent iff $Bd(p|q) = B(q|p) \neq \emptyset$. The meaning of 'red' is determined with respect to its pole and other poles in the space. So, it is natural that its boundary is defined with respect to the closeness to its pole with respect to other poles. It is obvious that the boundary of a concept is precise if there is no object in the closure of p that is marginally close to another pole (Rumfitt 2015, p.239).[63]

By definition 4.37 the space is connected if the only clopen sets are the empty set and the whole set. On the other hand, by proposition 4.36, a set is clopen iff its boundary is empty. So, if the space is connected, no concept has an empty boundary.

The following proposition is based on theorem 2.7 in (Arenas 1995, pp.19-20) where the author shows that any T_0 Alexandroff is connected iff $\forall a, b \in X$, $\exists a_0, \ldots a_n \in X$ such that $a_0 = a$ and $a_n = b$ and for $|i - j| \leqslant 1$:

$$V(a_i) \cap V(a_j) \neq \emptyset. \quad (*)$$

[63]Rumfitt relates the meaning of a color concept with its perceived pole but , as he mentions in the footnote 10 of his book, "exactly how to bring the perceiver into the story is a delicate question". Given the tight relation between the conceptual spaces and the topological approach discussed in this thesis, one may investigate when a child or a learner perceives the pole of a concept. We leave it for further research.

where for any $x \in X$, $V(x)$ is the minimal neighborhood of x.

Proposition 8.41. *Let (X, P, M) be a polar distribution, such that $|P| \geqslant 2$. The space is connected iff $\forall a, b \in X$, $\exists a_0, \ldots a_n \in X$ such that $a_0 = a$ and $a_n = b$ and for $|i - j| \leqslant 1$*

$$M(a_i) \cap M(a_j) \neq \emptyset.$$

Proof. By proposition 8.8, in a polar space, which is T_o Alexandroff, for each $x \in X$ there is a minimal neighborhood, $V(x)$ of the form $\{x\} \cup M(x)$. Since in a polar space for any $a_i \in X, M(a_i) \neq \emptyset$, it is obvious that if $a_i = a_j$, then $V(a_i) = \{a_i\} \cup M(a_i) \neq \emptyset$ and $M(a_i) \cap M(a_j) \neq \emptyset$. Considering $(*)$, it is enough to show that in a polar space $V(a_i) \cap V(a_j) = M(a_i) \cap M(a_j)$ for $a_i \neq a_j$.
Now let us suppose that $a_i \neq a_j$.

$V(a_i) \cap V(a_j) = (\{a_i\} \cap \{a_j\}) \cup (\{a_i\} \cap M(a_j)) \cup (\{a_j\} \cap M(a_i)) \cup (M(a_i) \cap M(a_j))$.

We consider the following possible cases:

a. $a_i, a_j \notin P$: In this case obviously $\{a_i\} \cap M(a_j) = \{a_j\} \cap M(a_i) = \{a_i\} \cap \{a_j\} = \emptyset$. Therefore, $V(a_i) \cap V(a_j) = M(a_i) \cap M(a_j)$.

b. $a_i = p \in P$, $a_j \notin P$:

In this case, $\{a_i\} \cap \{a_j\} = \emptyset$, $V(a_i) = \{a_i\} \cup M(a_i) = \{a_i\}, \{a_i\} = M(a_i)$ So, $V(a_i) \cap V(a_j) = \{a_i\} \cap (\{a_j\} \cup M(a_j)) = (\{a_i\} \cap \{a_j\}) \cup (\{a_i\} \cap M(a_j)) = M(a_i) \cap M(a_j)$.

The same result holds for $a_j = p \in P$, $a_i \notin P$.

c. $a_i, a_j \in P$:

In this case, $\{a_i\} = M(a_i), \{a_j\} = M(a_j), \{a_i\} \cap \{a_j\} = \emptyset$. So, $\{a_i\} \cap M(a_j) = \{a_j\} \cap M(a_i) = M(a_i) \cap M(a_j)$. Therefore, $V(a_i) \cap V(a_j) = M(a_i) \cap M(a_j)$.

□

Let us now define the precise and vague concepts in a polar space.

Definition 8.42. *Let* (X, P, M) *be a polar distribution and* $\mathscr{C} = \operatorname{Int}\operatorname{Cl}(\{p\})$ *a concept. Then,*

1. *The concept* $\mathscr{C}$ *is precise with respect to another concept with the pole* q *iff*

   Either $\operatorname{Cl}(\{p\}) \cap \operatorname{Cl}(\{q\}) = \emptyset$, *or* $\operatorname{Int}\operatorname{Cl}(\{p\}) - \{p\} = \emptyset$.

2. *The concept* $\mathscr{C}$ *is precise iff it is precise with respect to each distinct pole* $q \in P$.

3. *The concept* $\mathscr{C}$ *is vague with respect to another concept if it is not precise with respect to the other concept.*

4. *The concept* $\mathscr{C}$ *is vague if it is not precise.*

In fact, the main idea is that there is no abrupt jump in the space (the space is boundaryless) and that the concept has an adjacent concept with which it shares the boundary (there are borderline cases). In a totally disconnected space, every concept is precise because it is clopen (see definition 4.38). According to this definition of vagueness, a concept is vague if it is not extremely disconnected (see definition 8.10), and also if there is no gap between p and its interior closure. This means that in the extension of a concept there should be objects that are not typical cases of the concept. In the color space, for example, the concept “red” needs to have some shades of red as well. That seems quite natural because if we just have clear cases of red and its borderline cases, we can easily differentiate them.

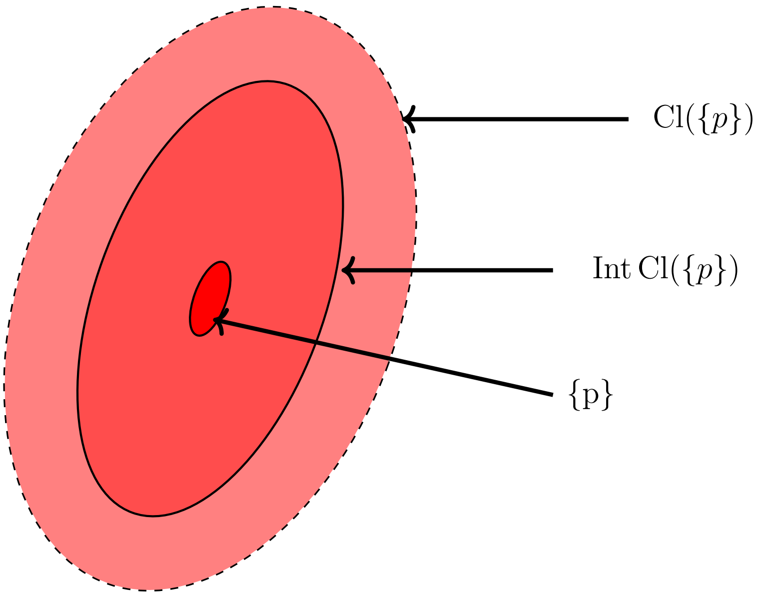


Figure 18: Three-layer model: $L_1 := \{p\}, L_2 := \mathrm{Int}\,\mathrm{Cl}(\{p\}) - \{p\}, L_3 := \mathrm{Cl}(\{p\}) - \mathrm{Int}\,\mathrm{Cl}(\{p\})$. In this model, L_3 represents the boundary of p with respect to other poles and L_2 represents the border of red with respect to the subpoles within the realm of the concept with pole p.

The definition of precision is compatible with the idea that vagueness of a concept depends on other relevant concepts in the space. If there is just one color pole, everything will be of that color because all elements must have a pole maximally close to it. So, trivially there will be no borderline cases. Now suppose that there are two boxes, red and blue and you are supposed to put red objects into the red box and the blue ones into the other box. This can be easily done because objects that are maximally close to the pole b are not attracted by the other pole. But if boxes were red and orange, there might have remained some objects that can be equally well put in the red box or the orange box.

One worry is that the definition of vagueness implies that everything in the space is vague. In other words, it may exclude all precise concepts. The example below shows that there might be a precise concept in a connected space. Furthermore, space is not always connected. This does not necessarily occur in a space with more than 2 poles. If the space is extremely disconnected, it is totally disconnected but the converse does not hold.[64]

The space may be totally disconnected but then all the concepts will be precise. So, obviously to be able to open a place for vague concepts we consider spaces that are not disconnected. Furthermore, a concept may be precise with respect to one concept and vague with respect to another one.

The following example [65] shows that a concept can be precise yet have borderline cases.

Example 8.43. *Let* $X = \{N, 1, 2, \dots, 100, S\}$ *and* (X, P, M) *be the polar distribution defined by* $M(N) = \{N\}, M(S) = \{S\}$, *and* $M(i) = \{S, N\}$ *for* $1 \leq i \leq 100$. *It can be easily calculated that:* $\operatorname{Int}\operatorname{Cl}(\{N\}) = \{N\}$ *and* $\operatorname{Int}\operatorname{Cl}(\{S\}) = \{S\}$. *So, for both concepts* $\operatorname{Int}\operatorname{Cl}(\{N\}), \operatorname{Int}\operatorname{Cl}(\{S\})$ *with the poles* N, S, *respectively* $L_2 = \emptyset$. *However,* $\operatorname{Cl}(\{N\}) - \operatorname{Int}\operatorname{Cl}(\{N\}) = \operatorname{Cl}(\{S\}) - \operatorname{Int}\operatorname{Cl}(\{S\}) = \{1, \dots, 100\}$.

In this example, the second layer collapses to the first one, i.e., $\operatorname{Int}\operatorname{Cl}(\{N\}) = \{N\}, \quad \operatorname{Int}\operatorname{Cl}(\{S\}) = \{S\}$.

In fact, X is not extremely disconnected because there are borderline cases that have two poles maximally close to them. It is neither totally disconnected because no two borderline cases in $Bd(X)$ can be separated by two disjoint open

[64] For instance, $\mathbb{Q} \subseteq \mathbb{R}$ be a set of rational numbers endowed with the subspace topology (see example 4.25). $\mathbb{Q}$ is totally disconnected but not extremely disconnected.See also (Steen and Seebach 1995, p.32).

[65] The example is given by Mormann (2020)(example 5.3) in favor of the rejection of bivalence in polar topology. Also it was mentioned by (Mormann 2021, example 3.7), to show that in a polar space the cardinality of boundary can be enough large to say that the boundary is thick.

sets in X. The space is connected because there is no clopen sets other than X and $\emptyset$. The example shows that even if the boundary is not empty, the concept may be precise. That is an interesting result that we will discuss in detail shortly. It shows that having borderline cases is not the necessary and sufficient condition for a concept to be vague. In our view of vagueness, there are two main criteria for a concept to be vague. One is that to go from a point to another point there should be a smooth path. In the example, there is no such a path from clear cases to borderline cases. The two criteria $\operatorname{Int}\operatorname{Cl}(\{p\}) - \{p\} \neq \emptyset$. and $\operatorname{Cl}(\{p\}) - \operatorname{Int}\operatorname{Cl}(\{p\}) \neq \emptyset$ are to smoothing the space. It also guarantees that there are borderline cases. Furthermore, not all concepts in a connected space are vague. So, in our view, neither connectedness nor having borderline cases alone can be a necessary and sufficient condition for vagueness. It shows that the existence of the middle layer, L_2, is crucial. That is in line with what Rumfitt was looking for:

> However, while the rough set approach may be the best we can do to characterize some vague predicates, I hope in the next two sections to say rather more about the set of red objects than that it includes the set of things that are clearly red and is included in the set of things that are arguably red (Rumfitt 2015, p.235).

The way we defined vagueness is also in favor of Sainsbury's view that having borderline cases does not guarantee the vagueness of a concept. Nevertheless, we argue that his example of precise concepts with borderline cases can be resisted by our definition of vagueness. In the following subsection we rule out his example. In fact, we show that what he calls a borderline case is not really a borderline case. Rather, it is a pseudo borderline case. Furthermore, if there is no borderline cases, there won't be vagueness.

8.3.1 Topology in defence of the importance of boundary and borderline cases

Vague concepts are usually considered as the ones having borderline cases. Sainsbury claimed that the other direction is not always true because there are some precise concepts that have borderline cases. So, according to him, having borderline cases is not a necessary and sufficient condition for a concept to be vague. Let us abbreviate Sainsbury's child argument by SCHA.

> (SCHA):
>
> [S]uppose that *child** is true just of persons who have not reached their sixteenth birthday, false of persons who have reached their eighteenth birthday, and neither true nor false of all other persons. Intuitively, this is not a vague predicate, despite the existence of borderline cases (Sainsbury 1991, p.173).

He contends that a predicate is vague if and only if it is boundaryless. Vague concepts are boundaryless if there is no set of which they are true.

Rumfitt finds Sainsbury's account appealing. Intuitively, we do not see any boundaries in the Sorites series. For example, in the spectrum of color it seems that there is no sharp boundary between colors. The predicate *child** is not vague because it has sharp boundaries.

We think that Sainsbury's counterexample for the essentiality of borderline cases in defining vagueness does not work. We will show that the above definition of borderline cases exclude *child** borderline cases.

The concept *child** has some borderline cases that are maximally close to the poles of *child** and *adult** but they are neither child nor adult. Then how can we say that it is precise? Is it a real definition of child/adult? In this example, it is precise who is a child and who is an adult. The borderline cases are somehow pseudo because for x to be a borderline case of A, not only should it belong neither to the extension of A nor to the extension of not-A, but also $M(x)$ should have an intersection both with A and not-A. In the case of *child**, someone between 16 and 17 cannot be a borderline case because neither of the poles are maximally close to it.

When there is a sharp boundary like less than 16, and more than 18 the poles of the concept and of the complement of the concept become repelling. They do not let the borderline cases enter into the extension of the concept. So, they are not real borderline cases. If they were, they would have been allowed to pertain to one of the extensions. In Sainsbury's term, these borderline cases are not "loose" objects because they cannot even in principle be attached loosely to the poles of the concept.

We think Sainsbury's example against the essentiality of borderline cases for vagueness does not work if we restrict the usual definition of borderline case to the one that also has the property of being maximally close to the poles of *child** and *adult**. A 16-17-year-old person does not have this property because they are not maximally close to neither of the poles. They are in a transition phase but it is clear when they will not be.

Let (X, P, M) be the polar distribution. $P = \{c, a\}$ where c, a are the poles of *child** and *adult** respectively. $\mathrm{Cl}(\{c\}) = \{x | c \in M(x)\}$. In this case there are two poles. Obviously, if x is maximally close to the pole of *child**, it is not maximally close to the pole of *adult** since there is a gap between them. So, $Bd(\{c\}|\{a\})$ is empty. It follows that $\mathrm{Cl}(\{c\})$ collapses to $\mathrm{Int}\,\mathrm{Cl}(\{c\})$. By definition 8.42, the concept *child** is precise and does not have borderline cases. In fact, this example has some elements that are not maximally close to no pole in the space. So, the space is not polar at all.

In the case of the ordinary concept "child", the situation is different because there is no gap. The space is connected and the closure of the pole of the concept is not open.

Comparing those borderline cases with the ones of the vague concept 'child' makes it clear why the former is the pseudo-borderline case. Someone who has had his sixteenth birthday just three months ago is loosely attached to the extension of child, so that if we define a new pole they easily detach from it, but the pseudo borderline cases do not need to detach because they are already dangling between childhood and adulthood.

Patrick Greenough makes a difference between a kind of indeterminacy that can be seen in examples such as SCHA and vagueness. According to him, a proposition such as 'x is a child' is "semantically incomplete", rather than vague:

> This species of indeterminacy per se is not vagueness since the term 'oldster'[66] draws a perfectly sharp and clearly identifiable three-fold division across its associated dimension of comparison (Greenough 2003, p.245).

[66] The open sentence 'x is an oldster' is determinately true of every person sixty-eight years of age and over, determinately false of those persons sixty-five years of age and under, and neither determinately true nor determinately false of the remainder.

We think that in this indefinable division there is no borderline case in the sense of the ones we consider in vagueness, probably one can call them intermediate cases.

Since the space is connected, every two adjacent concepts have a shared neighborhood. In the case of child* this fails. There is a pair such that the neighborhood of one is disjointed from the neighborhood of the other. We can say that the poles of "child*" and "adult*" are not maximally close to a 17-year- old person. So, that person is not a borderline case of child*/adult*. Either we should add another concept such as middle age, so that we have 3 vague concepts or we have a case in a disconnected space that is not of interest to us, because then we are faced with an uninteresting trivial case in which each object belongs to the extension of one and only one pole, the topology will be discrete and the Sorites paradox does not arise in the first place.

One can argue that by our definition, every concept will be vague in a connected space. We will not argue that the space is always connected but if it is extremely disconnected, of course there is no room for vagueness. Even in a connected color space, we may talk about the boundary between two concepts red and blue and by the definition the boundary is empty. The point is that in the case of precise concepts such as *odd* number, the concept is precise because numbers can be divided into odd and even numbers. Do we have typical cases of oddness or evenness? It seems that in these cases all numbers are either typical case of oddness or evenness. So, such concepts do not have a pole. But in the red and blue example, we can put the reddish things in the red box and bluish things in the blue box. In this case, red things and non-red things just like even and not-even numbers, can be distinguished.

8.4 Polar topology and the Sorites paradox

Vague predicates are susceptible to the Sorites paradox. Any theory of vagueness should explain why in the sorites series it seems to us that there is no sharp boundary. In other words, it seems that the tolerance principle holds for vague predicates, i.e, no element that is F in the sorites sequence is followed by an immediate successor that is not F. If this is so, then in the sequence that starts with an object that has the property F, and ends with the one that does not by applying rules and laws of classical logic a contradiction arises. However, if we reject the tolerance principle, then it seems that we should accept that vague predicates have sharp boundaries. This is against our intuition that in the sorites sequence there is no sharp line between the objects that are in the extension of the vague predicate F and those that are not.

In this section, we explain how Rumfitt's topological approach can deal with the Sorites paradox.

We start by one of the classical formulations of Sorites paradox, namely $\exists$-no sharp boundary.

We reformulate Rumfitt's polar topology to provide the detailed version of his view in dealing with the sorites argument in a language in which $\wedge, \vee$ and $\neg$ are sentential connectives and $\forall$ and $\exists$ are quantifiers. Then through some lemmas, we show that the laws of classical logic, used in the sorites argument are valid in Rumfitt's topological reasoning except bivalence. Following that, we reformulate the version of Sorites paradox whose first premise is that there is no sharp boundary. Then, we go beyond that and will look for a suitable topology that embraces the polar topology as a specific kind of Alexandroff topology.[67]

Sorites argument:

Let us copy the $\exists$- no-sharp boundary argument presented in 1.1.4:

1.	$\neg\exists n(F\,a_n \wedge \neg F\,a_{n+1})$	(tolerance principle)
2.	Fa_1	(Clear case)
3.	$\neg Fa_i$	(Clear non-case)
4.	$F\,a_{i-1}$.	(Supposition).
5.	$F\,a_{i-1} \wedge \neg F\,a_i$.	($\wedge$ -introduction).
6.	$\exists n(F\,a_n \wedge \neg F\,a_{n+1})$.	($\exists$ -introduction) .
7.	$\neg Fa_{i-1}$.	$(1, 4, 6)$.
.		
.		
.		
8.	$\neg F\,a_1$.	(after $i-2$ times repeating the argument).
9.	$F\,a_1 \wedge \neg F\,a_1$.	$(2, 8, \wedge$ - introduction) .
10.	$\neg\neg\exists n(F\,a_n \wedge \neg F\,a_{n+1})$.	$(1, 2, 9)$.
11.	$\exists n(F\,a_n \wedge \neg F\,a_{n+1})$	(10, double negation elimination).

The argument shows that accepting the principle of tolerance leads to a contradiction (9.) and rejecting it leads to what Wright calls “the paradox of sharp boundary”(11.). The latter seems problematic because denying the first premise will entail that a vague predicate has a sharp boundary which seems counterintuitive to many.

[67] In our investigation the ultimate aim is to look for a topological space in which not only concepts with one pole, but also any kind of concepts can be defined. But given the complexity of the topologies we would need and for the sake of space, in the current work we concentrate on the concepts that have poles.

Wright claims that by rejecting double negation elimination one can deny the inference from 10. to 11. In this way, the denial of the tolerance principle does not lead to the existence of a sharp boundary for a vague predicate. In intuitionistic logic, the double negation elimination is not valid. So Wright suggests that:

> ... something like an intuitionistic logic — at the least, a logic in which double negation elimination does not hold unrestricted— will be required in any fully satisfactory treatment (Wright 2007, p.22) .

Rumfitt (2015), however, contends that appealing to intuitionistic logic to solve the Sorites paradox is a big price to pay and his polar topology is more suitable. In particular, for certain concepts that have poles, one can keep classical logic and change the classical semantics. So, 10. correctly infers 11. but polar topology explains why 11. does not entail that there is a sharp boundary.
The semantics he proposes for classical logic is limited to the vague *polar* concepts. 68

As we saw earlier, he shows that the extension of a vague predicate is a regular open set in a suitable topology. In the previous section, we showed that the polar topology as the suitable topology for singular polar predicates is $T_{1/2}$ Alexandroff. To deal with the Sorites paradox, it is important to have a suitable notion of validity to argue whether the Sorites reasoning is valid or not. Classically, an argument is valid if it preserves the truth. An argument '$A_1, \ldots, A_n$; so B' is valid if and only if it is impossible in virtue of logical form that A_i s are true and B is false.

[68] It seems that there is no argument to show that all concepts are polar. In fact, one can say that concepts such as 'even'are not polar.The discussion on the nature of concepts goes beyond the aim of this thesis. We just consider concepts with poles.

To dissolve the Sorites paradox, not only one should show which premise is false or which law is invalid but also they should argue why that premise or law looks plausible to us (Fara (2000)). For example, the tolerance principle seems quite intuitive. Although rejecting that premise may solve the paradox, it needs a persuasive argument.

As said before, the tolerance principle along with undoubtable premises, 2. and 3., applying the rules of classical logic leads to the paradox. One way to tackle the paradox is to deny the tolerance principle. Rumfitt claims that the denial of the first premise is not problematic because 11. is equivalent to:

$$\textbf{11'.} \qquad (Fa_1 \wedge \neg Fa_2) \vee (Fa_2 \wedge \neg Fa_3) \vee \cdots \vee (Fa_{n-1} \wedge \neg Fa_n).$$

In polar topology, a disjunction of two atomic predicates can be true without either of its disjuncts being true. For example, something can be red or orange while being neither red nor orange.

The Sorites paradox occurs when we consider a series of objects, starting from the one that satisfies F to the one that does not satisfy F. Also, the adjacent objects are indiscriminable with respect to F. It is true that somewhere a pair $< a_i, a_{i+1} >$ is such that $(F\,a_i \wedge \neg F\,a_{i+1})$ otherwise, every object in the series would have satisfied ‘F’. But it is not true for one particular pair. For example, in the color spectrum from red to orange, at some point, an object ceases to be red while its predecessor is red but this does not entail that there is a specific pair in the sequence the first element of which is red while its second element is not red. Rumfitt truly notices that the connectives in the Sorites paradox are sentential while he was considering just predicate operators. In case of vague predicates(particularly, color) Rumfitt claims that to dissolve the Sorites paradox we should "move from predicates that classify individual objects by color to predicates that effect classification of N-tuples of objects"(Rumfitt 2015, p.256). Each step in the Sorites argument should be considered in comparison to the whole series rather than isolated. So, instead of the topology of pairs, we need a product topology. As Zach (2018) mentions,

> Rumfitt does not provide a detailed account of this topology, how it is generated from the original topology, and what the defining clauses for disjunction, conjunction and negation amount to in the resulting topology (Zach 2018, p.2088).

In the following subsection, we explain in more detail in what way the proposed product topology helps to show the validity of the Sorites argument and how to dissolve it.

8.4.1 Topology for an n-place predicate

We proved that the polar topology is Alexandroff. The following theorem shows that if two topologies are Alexandroff their product topology also is Alexandroff.

Theorem 8.44. *Let $(X, \mathcal{O}X), (Y, \mathcal{O}Y)$ be two Alexandroff spaces. Then, $X \times Y$ is an Alexandroff space, with $U(x, y) = U(x) \times U(y)$*

Proof. As defined in part III, $X \times Y$ has as basis $\beta = \{U \times V : U \in \mathcal{O}X$ and $V \in \mathcal{O}Y\}$. Let $(x, y) \in X \times Y$, then $U(x) \times U(y)$ is in β. We show that $U(x) \times U(y)$ is a minimal open set in β containing (x, y). If $(x, y) \in U \times V \in \beta$, then $x \in U$ and $y \in V$ so $U(x) \subseteq U$ and $V(y) \subseteq V$. Therefore, $U(x) \times V(y) \subseteq U \times V$. So, since each element in $X \times Y$ has a minimal open set, $X \times Y$ is an Alexandroff space and $U(x, y) = U(x) \times U(y)$. $\square$

This can easily be generalized to n spaces:

Corollary 8.45. *Let $X_1, ..., X_n$ be Alexandroff spaces. Then the product topology $X_1 \times ... \times X_n$ also is Alexandroff. Furthermore, $U(x_1, ..., x_n) = U(x_1) \times ... \times U(x_n)$.*

So, if for all i, X_i is an Alexandroff space, then the minimal open set of $\prod X_i$ is the set of products $\prod U_i$, where $U_i s$ are the minimal open sets of X_is.

We showed that for polar monadic predicates the minimal open neighborhood is $U(x) := \{x\} \cup M(x)$.
The definition of M can be expanded to the one for n-place predicates, $\prod M_i$.

Proposition 8.46. *Let (X_i, P_i, M_i) be finitely many polar distributions, i=1,...,n. Then $(\prod_{i=1}^{n} X_i, \prod_{i=1}^{n} P_i, \prod_{i=1}^{n} M_i)$ is a polar distribution with respect to the product topology (called product polar distribution), where*

$$\prod_{i=1}^{n} M_i : \prod_{i=1}^{n} X_i \longrightarrow \prod_{i=1}^{n} 2^{P_i}$$
$$(x_1, \ldots, x_n) \longrightarrow M(x_1) \times M(x_2) \cdots \times M(x_n) \ .$$

It is easy to see that $\prod_{i=1}^{n} M_i \neq \emptyset$ and if $(x_1, \ldots, x_n)$ belong to the extension of a concept with pole p, then $\prod_{i=1}^{n} M_i(x_1, \ldots, x_n) = (\{p\}, \ldots, \{p\})$. This means that the only pole maximally close to the pole of an n-place predicate is that pole.

In case of polar topology, X_is , P_is and M_is, $1 \leq i \leq n$ are the same. So, the product polar distribution will be (X^n, P^n, M^n). For example, in the color space, the set of poles is always P with its fixed number of poles. However, in general, it may happen that we want to consider two different spaces. For example, a polar topology on the color space and a polar topology on the set of geometrically shaped objects. A red round ball can be defined in the product topology. We first will talk about the simple "homogeneous" case.

By 8.45, if X_is are endowed with polar topologies, then the topology generated by the product of the minimal neighborhoods of X_is is a polar topology and it is $T_{1/2}$ Alexandroff. Let us call this topology $\mathcal{O}_\pi X$.[69].

Let us consider the simplest case of ordered pairs in the color space. Furthermore, suppose that in the spectrum of colors there are just two colors, red and orange. Then, the number of poles of objects in X^2 will be four, namely,$< r, o >$ $, < r, r >, < o, r >, < o, o >$. Each pair is maximally close to at least one of these poles. In this case, the product polar distribution is (X^2, P^2, M^2). The extensions of atomic predicates of ordered pairs, 'red-red', 'red-orange', 'orange-red', 'orange-orange' are regular open sets in $\mathcal{O}_\pi X$. For example, consider a 2-place predicate 'F', 'first orange, second orange' ('orange, orange', for short). An ordered pair $< a, b >$ satisfies 'F' if the only pole maximally close to it is $< o, o >$.

Just as the set of borderline cases of a vague predicate is not empty, the set of poles of an n-place predicate also is not empty.

Principle 8.47. *Given a product polar distribution (X^n, P^n, M^n), the extension of concepts are regular open in the product polar topology.*

It is supposed that the set of poles of a space is always fixed. So, consider the color space. Denote the cardinal number of P by $|P|$. Then,

Let $X = \prod X_i$, $1 \leqslant i \leqslant n$, be a space , $P \subseteq X$ the set of poles of X. If

[69]Interior, closure and interior closure for finite product spaces are defined in this topology in a usual way as the following:

$\mathrm{Int}(\prod_{i=1}^{n} A_i) = \prod_{i=1}^{n} \mathrm{Int}(A_i)$. In particular, $\mathrm{Int}(\prod_{i=1}^{n} p_i) = \prod_{i=1}^{n} \mathrm{Int}(\{p_i\})$

$\mathrm{Cl}(\prod_{i=1}^{n} A_i) = \prod_{i=1}^{n} \mathrm{Cl}(A_i)$. In particular, $\mathrm{Cl}(\prod_{i=1}^{n} p_i) = \prod_{i=1}^{n} \mathrm{Cl}(\{p_i\})$

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X is uni-color; i.e., $|P| = 1$, then $\mathrm{Cl}(\{(p, \ldots, p)\}) = X$. In this case, since each n-tuple has a pole maximally close to it, $(p, ..., p)$ is maximally close to all points in the product space and therefore, the classification of objects does not have any sense. Remember that the objects are classified by reference to the pole of a concept in comparison to other poles of the space. When there is no other pole, the classification won't be possible and even does not make any sense.
If X has two colors; i.e., $|P| = 2$, then the number of possible poles of each member of X is 2^n. In general, the possible number of poles for each member of X is $|P|^n$. According to Rumfitt, the sorites argument is valid. If we accept the premises 1., 2., and 3, then the conclusion comes out immediately by classical reasoning. Since the regular open sets of a topological space form a complete Boolean algebra, the classical logic holds. In particular, modus ponens holds. If F_{a_i} and $F_{a_i} \to F_{a_{i+1}}$ it is inferred that $F_{a_{i+1}}$.

Any two objects that belong to the interior of the closure of a pole are indiscriminable. In the color space, all objects in the extension of 'red' are indiscriminable with respect to its pole r if they both belong to the regular open set generated by $\{r\}$.

Roughly speaking, what happens in the Sorites is that each premise adds a restriction on space. For example, in the color space ranging from red to orange, the premises 2. and 3. restrict n-place predicates to the ones whose first element is red and the last one is orange. So, the number of possible poles of n-place predicate reduces to 2^{n-2}. The tolerance principle also restricts the number of poles because for instance, consider a predicate whose first element is red. The poles of that predicate do not contain the ones whose first element is r and the second one is o. In the above example, the extensions of atomic predicates of ordered pairs, 'red, red' , 'red, orange' , 'orange, red' and 'orange, orange' are regular open sets in the product polar topology. For $< a_1, \ldots a_n >$ to satisfy the n-place predicate red, is to be maximally close to $< red, ..., red >$. These predicates are also tolerant to small changes. If 'a_i is red', then the possible poles of the predicate are the ones that their i-th position is r. So, in the Sorites paradox, 'a_1 is red' implies that $< a_1, a_2 >$ is either 'red-red' or 'red-orange'. Then, by the induction step, $< a_1, a_2 >$ is not 'red-orange'. therefore, $< a_1, a_2 >$ is 'red-red' and therefore, a_2 is red.

What is the suitable interpretation of the connectives of such polar predicates? The semantics Rumfitt gives for the connectives of monadic predicates, namely 'and', 'or' and 'not' is expanded in a natural way to the semantics of sentential connectives '$\wedge$', '$\vee$' and '$\neg$'. As in the monadic case, when the operators are applied to regular open sets, the result will be a regular open set as well.[70]

So, the Sorites paradox can be reformulated.

The idea is to cover all the possible boundaries. The disjunction of the sentences is true without any one of them being true.

Just like the logic for the monadic polar predicate, the logic for n-place polar predicates is classical if the set of poles is fixed.

[70]In a product space, $(\mathrm{A}\times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

8.4.2 Reformulation of the Sorites paradox

1. $F\,a_1$
2. $\neg F\,a_n$
3. $\forall n(F\,a_n \longrightarrow F\,a_{n+1})$
4. $[(F\,a_1 \longrightarrow F\,a_2) \wedge (F\,a_2 \longrightarrow F\,a_3) \wedge \cdots \wedge (F\,a_{n-1} \longrightarrow F\,a_n)]$
5. $F\,a_1 \longrightarrow F\,a_2$ $\quad$ ($\wedge-$ elimination)
6. $F\,a_2$ (1, 5, modus ponens)
7. $F\,a_2 \longrightarrow F\,a_3$
8. $F\,a_3$

   ...

9. $F\,a_n$ (Contradiction with 2.) Since 1 and 2 seem correct and modus ponens holds,
10. $\neg\forall n(F\,a_n \longrightarrow F\,a_{n+1})$
11. $\neg((F\,a_1 \longrightarrow F\,a_2) \wedge (F\,a_2 \longrightarrow F\,a_3) \wedge \cdots \wedge (F\,a_{n-1} \longrightarrow F\,a_n))$
12. $\neg(F\,a_1 \longrightarrow F\,a_2) \vee \neg(F\,a_2 \longrightarrow F\,a_3) \vee \cdots \vee \neg(F\,a_{n-1} \longrightarrow F\,a_n)$ $\quad$ (11., De Morgan)
13. $(F\,a_1 \wedge \neg F\,a_2) \vee (F\,a_2 \wedge \neg F\,a_3) \vee \cdots \vee (F\,a_{n-1} \wedge \neg F\,a_n)$

It can easily be seen that if $(X, \mathcal{O}X)$ is a polar space, the interior closure operator, is a closure operator. However, it is different from Kuratowski's closure operator in which for all $A, B \subseteq X, \operatorname{int}\operatorname{cl}(A \cup B) = \operatorname{int}\operatorname{cl}(A) \cup \operatorname{int}\operatorname{cl}(B)$.

Lemma 8.48. *Let for all* $1 \leqslant i \leqslant n$, $X = \prod X_i, (X, \mathcal{O}_\pi X)$ *be the product polar topology,*$A, B \subseteq X$. *Then, in general:*

$$\operatorname{Int}\operatorname{Cl}(A \cup B) \neq \operatorname{Int}\operatorname{Cl}(A) \cup \operatorname{Int}\operatorname{Cl}(B).$$

$\operatorname{Int}\operatorname{Cl}(A)$ is the set of things whose unique pole is that of A and $\operatorname{Int}\operatorname{Cl}(B)$ is the set of things whose unique pole is that of B. So, their union does not contain borderline cases. Whereas, $\operatorname{Int}\operatorname{Cl}(A \cup B)$ includes the objects whose set of poles can contain both poles of A and B.

The point is that 13. in the reformulation of the Sorites on page 228 does not entail that one of the disjuncts is true:

In Rumfitt's words,

> ... that way, (10)[11'] is entirely innocuous. All it says is that when classifying entire sequences of coloured objects, whose members are arranged in order of gradually decreasing redness from something clearly red to something clearly not red, either the second or the third or . . . or the 100th object will be the first object not classified as red. That claim is obviously true. It does not, however, entail the existence of a sharp boundary to the concept red, which is a mode of classifying a single object in respect of its colour. Indeed, it does not say anything directly about that latter mode of classification at all (Rumfitt 2015, p.253).

According to Rumfitt truth does not distribute over disjunction. As we will discuss, truth for him is vague as well and its extension is a regular open set in a suitable topology.

One can reach the same results for the product topology of a finite number of polar spaces generated by different sets of poles. For example, topologies generated by poles of color, chair, tall, small,...
The "key technical notion" Rumfitt uses to find the semantic values of complex sentences such as 'x is red and y is small' or 'x is either child or adult' is again using the product topology. This time, for the clear reason he uses the product topology of different topological spaces. For example, the product topology of two topologies generated by the poles of tall and the poles of young.
Before criticising polar topology, we need to mention some remarks on polar topology. One is related to the indistinguishability of a_is in the Sorites paradox and the tolerance principle. The other is to clarify how to assign truth values to vague propositions.

Similarity relation and indistinguishability relation in weakly scattered polar spaces

[71] Rumfitt, plausibly, endorses that the Sorites paradox is valid but unsound. So, in this regard, he is in the same camp as supervaluationists, epistemicists among others. He accepts that vague predicates are tolerant in the sense that they are boundaryless. The extension of a vague concept in his theory is a regular open set. So, if x is in the extension of a concept, then there is a neighborhood of it such that all of its members belong to that extension as well. His semantics for boundarylessness of vague concepts reveals that none of the instances of the tolerance principle and the sharp boundary principle is true and that the tolerance

[71] Thanks to Javier Belastegui for his invaluable comments.

principle in its strongest sense is untrue. In his view, vague concepts are tolerant because they are boundaryless. Rumfitt does not explicitly say that boundarylessness alone is not a necessary and sufficient condition for a concept to be vague. Nevertheless, we think that his topological account of vagueness shows that the existence of borderline cases is crucial to define vagueness. From the very beginning we suppose that for vague concepts there are objects that have at least to pole maximally close to them. Vague concepts do not draw any boundaries. It is actually quite interesting that we appeal to topology in which the concept of boundary is crucial to defend boundarylessness of vague concepts. But the clever use of regular open sets and the importance of borderline cases shows that topology can be an appropriate tool to be used by philosophers in defining vagueness. For Sainsbury whenever we talk about boundary, we talk about a sharp boundary. Nevertheless, it is quite intuitive to think of a vague boundary, in the sense that we can move around a borderline case in the boundary and still we find more borderline cases. Rumfitt models Sainsbury's boundarylessness endorsing thin boundary. Nevertheless, we need to go beyond that to deal with hierarchical higher-order vagueness. The 3-layer model is in line with Rumfitt's project and reveals how smoothly we move from typical cases of a vague concept to the typical cases of another vague concept.

In the polar space, as we saw, we always compare the object to the pole. The specialization order was of order 1.

Rumfitt defines "comparative similarity relation" as a 3-place relation that is defined based on maximally closeness relation: x is closer to p than to any other pole (Rumfitt 2015, p.239). Similar objects, then, belong to the extension of the concept. So, the only objects that are similar to the pole are the ones that are in the interior of the closure of the pole. This relation is crucial to the classification of objects but is not appropriate as the indistinguishability relation that is used in the Sorites sequence. Let us make this point clear by focusing on the similarity(indistinguishability) relation, Maxsim, that we think Rumfitt might suggest:

Maxsim : Let (X, P, M) be a polar distribution. Then

a. $x \sim p$ iff $M(x) = \{p\}$.

b. If $p \sim x$, then $x = p$.

According to Maxsim a. the objects in the interior closure of p are similar or indistinguishable from p. An object, x, is closer to p than to any other pole if it is in the interior closure of the pole. It is important that for the classification of the objects we compare them with the pole of the concept in the polar space in which the distribution of the poles is settled. So, a pole is just compared with itself. Nothing is closer to p than itself. This point is represented by Maxsim b.

This similarity relation is not symmetric and does not let us compare non-pole members of the extension. It even does not let us compare a non-pole object with itself. For the categorization that makes sense because, for example, to see whether an object satisfies a predicate we compare it with the pole of that predicate not with the non-typical object itself. But it does not make sense when we consider the similarity relation as indistiguishability relation. Furthermore, in the Sorites series we need to compare non-pole objects. So, Rumfitt may define the similarity relation, Sim, based on Maxsim:

$$For\, x, y \in X - P,\, x \sim_p y \quad iff \quad x \sim p \wedge y \sim p. \qquad (\textbf{Sim})$$

Sim tells us how to compare two non-pole objects. Two objects that are not poles are indistinguishable if, and only if they belong to the interior closure of p. We prefer this formulation to the following one:

$$For x, y \in X - P, x \sim_p y \quad iff \quad x, y \in \mathrm{Int\,Cl}(\{p\}). \qquad (\textbf{Sim}_1)$$

The reason is that Sim more explicitly shows that x and y should be compared with p. Nevertheless, by considering Sim_1, Sim and Maxsim can be perfectly merged into:

$$a'. \quad For \quad x \in X - P, y \in X \quad x \sim_p y \quad iff \quad x, y \in \mathrm{Int\,Cl}(\{p\}). \quad (\textbf{Sim}_2)$$

b'. If $p \sim_p x$, then $x = p$.

The condition b' is just b. in Maxsim. A pole is closer to itself than to any other pole. a'. says that if x is not a pole, then y is similar to x just in case both of them are closer to p than to any other pole. For example, a shade of red is similar to the pole of 'red'.

Rumfitt does not explain in detail how to define indiscriminability. If the relation is what we defined by Sim, Sim_1 or Sim_2 based on Maxsim, then the shades of the color red are indistinguishable from typical cases of red. We are supposed to fix the poles. When we do that, the poles are the ones who cannot be compared with any other object but themselves.

This definition, no matter which formulation we choose, is quite intuitive. Yet it is problematic because in a color space, for example, a borderline case is not indiscriminable from a red object and so, there is a sharp line between the objects that are red and the ones that are borderline cases. [72]

Weakly scattered spaces, though, are more flexible. In these spaces the boundary still remains thin but we may compare two objects that are not typical cases. When we consider a certain concept, naturally, for the objects to be similar with respect to that concept, its pole needs to be maximally close to them. One natural move to solve the problem is to define the similarity relation in a weakly scattered polar space in a loose way as:

$$x \sim_p y \quad iff \quad p \in M(x) \cap M(y). \tag{\textbf{Sim*}}$$

In this way, borderline cases of 'red' are similar to the shades of red and even to its typical cases. By this definition, similarity is symmetric and transitive. Usually, the similarity relation is considered as a reflexive, symmetric but not necessarily transitive relation. The definition has two main problems: One is that if we consider similarity relation as indistinguishability relation, then the pole would be indistinguishable from the borderline cases because the relation is transitive. This is something that obviously we want to avoid. We can differentiate a red object from a reddish-orangish object. In the forced march Sorites, for example, at first there is no doubt that someone is tall. The doubt starts after we get far from the typical cases.

[72]Maybe in his model of boundarylessness, the boundary is very thin and is neglected.

> The second reason is that if we consider circular color space, an object that is borderline red-violet and the object that is red-orange are similar according to this definition but are discriminable. We need to find a way to exclude such cases to be able to consider the Sorites sequence.

To deal with these problems we may define the similarity relation as:

$$x \sim_p y \quad iff \quad M(x) \subseteq M(y), p \in M(x). \tag{\textbf{Sim**}}$$

In this case, we always start from a concept to one of its adjacent concepts. So, the journey starts from a pole, p, and ends up at another pole, say q, in a smooth path. The objects in the closure of p are not similar with respect to p to the objects in the interior of the closure of q.

In the Sorites sequence we move from prototypical cases to the borderline cases that are similar to the previous cases because they share the same pole and the degree of prototypicality with respect to that pole decreases while this degree increases with respect to another pole in the destination. So, borderline cases are quite indistinguishable from the points near the boundary. We start from typical red things to crimson,... to light red,... till we get to the borderline cases and shades of orange and finally we get to the typical cases of orange. The borderline cases of red are not similar with respect to p to the shades of orange, the ones that belong to the extension of orange. This is, on the one hand, good because we can distinguish the red things from non-red things. By this definition, the second problem is solved. Yet, still the poles remain indistinguishable from borderline cases. Furthermore, similarity relation is reflexive and transitive but not symmetric, neither anti-symmetric. The symmetry holds just in case both

x and y belong to the extension of the concept. In general, that is a one way road from one concept to its adjacent concept. It may work in the Sorites series when we move from a clear case of a concept towards a clear case of its negation. Nevertheless, if similarity means indistinguishability, then it is quite intuitive to look for a symmetric relation. So, we keep this similarity relation with the interpretation that if two objects are similar with respect to a certain pole, they are in the closure of the pole. So, borderline cases are somehow similar to a concept because still are under the gravity power of the pole. Now we make a difference between similarity relation and indistinguishability relation.

Definition 8.49. *Let (X, P, M) be a polar distribution 3-layer model in a weakly scattered space. Then, define the indistinguishability relation as:*

$$x \approx_p y \quad iff \quad x, y \in \mathrm{Cl}(\{p\}) \wedge |L_j - L_i| \leq 1, \quad for \quad x \in L_i,\, y \in L_j. \quad (\textbf{Simi})$$

x, y are distinguishable, if they are not indistinguishable.

It can easily be seen that $\approx_p$ is reflexive and symmetric but not transitive. Now we have a similarity relation as indistinguishability that is reflexive, symmetric and not necessarily transitive, the one that we were looking for. In this sense, Simi is different from the family of similarity relations that we defined before. Two things may be similar when we move from the pole of red towards the pole of orange but not similar when we move in the opposite direction. That is actually a merit for a similarity relation that is not considered as an indistinguishbility relation. Nevertheless, when we consider the Sories paradox, the similarity relation as indistinguishability relation is usually considered as a reflexive and symmetric relation. Furthermore, a borderline case of 'red' is somehow similar to a typical case of 'red' in having r in their set of poles but they are distinguishable from the pole. In our 3-layered model by **Simi**, poles are distinguishable from the objects in $X - \operatorname{Int}\operatorname{Cl}(\{p\})$ while the borderline cases are indistinguishable from the objects in the second layer, the ones that belong to $\operatorname{Int}\operatorname{Cl}(\{p\}) - \{p\}$.

It might be objected that this definition is not appropriate for the categorization of a concept since one cannot differentiate borderline cases from almost clear cases.

We can reply in the following way: the indistinguishability relation is not transitive. So, even if the objects in the second and third layer are indistinguishable, the third layer is distinguishable from the pole. Since for the categorization we always compare the object to the pole of the concept, only the ones that are indistinguishable from the pole will belong to the extension of the concept and this is what we were looking for: the extension of a concept is a regular open set in a weakly scattered Alexandroff space. It can easily be seen that if two objects, x, y are indistinguishable with respect to p, they are similar(**Sim***) or **Sim****) hold and if the objects are in the interior closure of p, then **Sim** holds as well). Yet the converse does not hold. Two similar things (like a borderline case and a typical case) may be distinguishable.[73]

Having **Simi** at hand, let's see in what way we can weaken the tolerance principle not to get to a contradiction within the realm of classical logic.

[73]Recently, Belastegui (2022) points out that considering the extension of a concept as an open set is in contrast to well-formedness criterion. We leave the discussion for another time, but just as a quick answer, we might argue that in this model, similarity between two adjacent layers is strong while there is a loose similarity between the first and the third layers. So, well-formedness holds.

Some remarks on the tolerance principle

Rumfitt shows that the rejection of tolerance principle does not lead to the sharp boundary problem. Usually, philosophers offer an alternative for the tolerance principle. This is missing in discussions on polar topology. The only thing that we know is that for any object, x, in the extension of a predicate there is a neighborhood, U, such that all of its members belong to the extension of the concept. The same holds for the negation of the concept. If so, then Fy holds for all y in U. and therefore, the Sorites paradox arises. A concept and its negation are vague and therefore, boundaryless. The symmetry between the extension of a concept and its negation in polar topological semantics is quite remarkable. It is one of the reasons to prefer it to the intuitionistic approach in which the extension of a predicate is an open set in a suitable topology. However, we need to know more about the tolerance principle. In particular, we need to know how to revise the tolerance principle when we reject it to dissolve the Sorites paradox.

To the best of our knowledge, this is the first time that the weaker version of the tolerance principle in a polar model is being proposed. In the following, we compare this weak version of tolerance principle to some other versions.

Weak tolerance principle

Let us remind the original tolerance principle: ∀-TOLERANCE in our 3-layer model can be formulated as:

$$Tol : \forall x, y \in X\, (Fx \wedge x \approx_p y) \to Fy.$$

Williamson rejects Tol and weakens the tolerance principle by saying that if x is *clearly* F and y is indistinguishable from x, then y is in the extension of F[74]:

$$Tol_W : \forall x, y \in X\, (x = p \wedge x \approx y) \rightarrow Fy.$$

If this interpretation is right, then Williamson's weak version of tolerance says that if x is in the first layer (is the pole) and y is indistinguishable from x, then y at most will be in the second layer (the extension of F). This interpretation matches with our model. Nevertheless, we need to say something more. In particular we want to answer the following question:

What happens when y is indistinguishable from x when x is not a pole but is F?

The weak version of tolerance that we propose in the current work answers this question:

$$Tol_M\colon\ \forall x, y \in X, (Fx \wedge x \approx_p y) \rightarrow y \in \mathrm{Cl}(\{p\}).$$

In words, if x is F and y is indistinguishable from x, then y is in the closure of the pole. For example, it might happen that x be red and the next one loosely attaches to the extension of red. The point is that in the safe area, when we are still not far from the pole, p, Tol holds. But as we go towards the realm of other pole, q, the gravitational power of p loses its effect. In the Sorites series if something is close enough to the pole, the next one cannot be maximally close to any other pole. There is no jump. So, the viewer when starts off from p, judging the property of the next object without looking back to the pole, after a while suspends the judgement. This is the time of thinking, having a_{n+1} at hand:

[74]The original formulation of Williamson's tolerance principle is: $(\Box Fx \wedge x \sim y) \rightarrow Fy$. If it is known that(it is clearly the case) that x is F and y is indistinguishable from x, then y is F.(See Mormann (2020), Van Rooij et al. (2010).)

if a_n is F, so is a_{n+1}. But let me check the pole. hmmm a_{n+1} is not that red. Let me check the pole of orange. hmmmm neither is clearly orange. Now that I am looking at r, maybe I shouldn't have judged a_n as red. Never mind, there is no way to change it. I wish I had looked at the pole before! Let's be coherent at least.

He(she) continues judging other objects in the Sorites series but this time having in mind that now there are two poles in the play, he has seen the gleam of the "lighthouse" in the distance. He(she) continues judging knowing that somewhere should break his silence and firmly judge but the only things that he(she) is shown are the 2 adjacent objects. Eventually, he(she) sees a typical case of orange and asks: when did I get to the realm of o? And the answer is simply: somewhere!

There is no abrupt jump from red to non-red.[75] It is true that the viewer does not realize when he(she) enters to the realm of another incompatible concept but this transition is very smooth.

Our weak version of tolerance is very similar to the one that Shapiro proposes:

$$Tol_S : \forall x, y \in X(Fx \wedge x \approx y) \rightarrow \neg\overline{F}y.^{76}$$

$\overline{F}$ is incompatible with F meaning that $\overline{F} \cap F = \emptyset, \overline{F} \cup F \neq X$.

In polar topological semantics what Shapiro suggests is that if x is F and y

[75] Maybe it can be compared with what Raffman (1994) calls "a kind of gestalt switch" (p.53).

[76] He defines the tolerance relation as:

In general, a sorites series arises when we have a (prima facie) tolerant predicate P, and a series of objects running from a clear (or determinate) instance of P to *a* clear non-instance of P, with each differing marginally from its neighbors. I propose, instead, this principle of tolerance:
Suppose that two objects a, a_0 in the field of P differ only marginally in the relevant respect (on which P is tolerant). Then if one competently judges a to have P, then she cannot competently judge a_0 in any other manner. This, I submit, is the key to avoiding contradiction (Shapiro 2006, p.23)

While Shapiro defends 3-valued logic, having the same idea about the weakening of the tolerance principle, following Rumfitt, we can keep 2-valued classical logic.

is similar to x, then y cannot be in the interior closure of any pole incompatible with F. y is not in the interior closure of any other incompatible concept. This definition implies that if 'x is F' is true, even if y is indistinguishable from x, 'y is F' does not need to be true and neither can be false because y does not belong to the extension of any other pole.

In order to assign truth values to statements in the proposed weak tolerance that we suggest, it is expedient to know more about the truth value of a statement in Rumfitt's polar topological account.

Truth values in polar topology: two or more truth values?

According to Rumfitt 'true' and 'false' are two vague predicates. In Rumfitt's view "statement" and its truth value plays an important role. In his own terms:

a statement is an utterance or inscription that expresses a complete thought(Rumfitt 2015, p.303).[77]

The extensions of truth values of statements are regular open sets in the space of truth values. According to Rumfitt, we need to know the number of poles provided by the space to be able to assign truth values to statements. If there are just two poles, $p_{\mathcal{T}}$ and $p_{\mathcal{F}}$, then the truth value of the vague statement is as maximally close to $p_{\mathcal{T}}$ as to $p_{\mathcal{F}}$. But if the set of poles contain indeterminate, $p_{\mathcal{I}}$, as well, then the truth value of a vague statement is maximally close to that pole. In other words, the truth value of a statement depends on the fixed set of poles of the truth values provided by the space that generates a topology on that space.

[77]In the first part of the book Rumfitt explains in detail what he means by a statement. In the following discussions on Rumfitt's polar topology, for the sake of argument, we follow Rumfitt in considering statements as the truth bearers.

For example, let the space of truth values X' contains two poles: $P' = \{p_{\mathcal{T}}, p_{\mathcal{F}}\}$. Let (X', P', M') be the polar distribution in a weakly scattered space. In the color space consider the statement 'x is red'. The truth value of this statement is true if it is in the extension of $p_{\mathcal{T}}$, false, if it is in the extension of $p_{\mathcal{F}}$ and neither true nor false otherwise. Note that in this case there is no third value. In the Sorites paradox, as we go along from typical cases of red to typical cases of orange the truth value of 'x is red' moves slightly from typical cases of true statements towards the typical cases of false statements.

Surprisingly, even if Rumfitt defends classical logic, his topological semantics permits a statement to have a third value. In fact, the space of truth values may have three poles, $p_{\mathcal{T}}, p_{\mathcal{F}}$ and $p_{\mathcal{I}}$. In the Sorites series from $a_1, \ldots, a_{100}$ of color tubes such that the first tube is typically red and the last one is typically orange, let's say that a_{50} is typically a borderline case. Then, "a_{50} is F" can be considered as a typical case of indeterminate truth value.

Given that the interior of the boundary is thin in polar topology and its generalization to weakly scattered spaces, it is not clear what the typical case of the borderline cases would be.(See the third criticism below on page 255 where we elaborate on this point.)

At this point we would like to consider Smith's closeness condition for two main reasons: On the one hand, we see a close relation between Tol_M and Smith's closeness condition as a weaker version of tolerance principle. On the other hand, we would like to show that his pessimistic view about considering topological structure on the space of objects and on the space of truth values is not well-supported and moreover, by endowing such structures on such sets, we can endorse degrees of truth within the realm of classical logic. So, pace Frege, [78], in a suitably topologically structured space, truth do tolerate "more or less" even if there are only two truth values, true and false.

Let's first see what Smith's closeness condition is and how he defines vagueness.

In defining vagueness Smith (2005, 2008) uses a ternary relation of closeness: x is at least as close to z as y, in F-relevant respects. Formally: $x \leqslant_z^F y$. To understand closeness in respect of truth, he defines closeness as the following:

Closeness:

If a and b are very close in F-relevant respects, then 'Fa' and 'Fb' are close in respect of truth (Smith 2008, p.165).

Formally, Let D be the domain of discourse, $\mathcal{T}$ the truth values, $\approx^F$ and $\approx^{\mathcal{T}}$ be the relation on D of being very close in F-relevant respects and the relation on $\mathcal{T}$ of being very close in $\mathcal{T}$-relevant respects respectively. Then, closeness condition will be:

$$x \approx^F y \Longrightarrow [Fx] \approx^{\mathcal{T}} [Fy]$$

If two objects a and b are very close in F-relevant respects, the truth value of their characteristic function $[Fa]$ and $[Fb]$ will be very close in $\mathcal{T}$-relevant respects.

Then he defines vagueness as:

A predicate is vague just in case its characteristic function[79] is continuous.

[78] "... what is only half true is untrue. Truth cannot tolerate a more or less" (Frege 1956, p.291).

[79] Let D be the domain of discourse, $\mathcal{T}$ the truth values and $[Fx]$ denote the value of the characteristic function for F at the object x. Define the characteristic function of F as the following:

$$\chi : D \longrightarrow \mathcal{T}$$
$$\chi(x) = [Fx].$$

Then he claims that although the structure of this relative closeness is not limited to the metric structures, topological structures are not suitable because the only topological space in which the characteristic function is continuous is the discrete topology. So, all predicates become vague.

> So we need a different proposal concerning the origin of the topologies on the domain—and I do not know what such a proposal would look like. In the absence of such a proposal, we are left with the bare stipulation that for each predicate F, there is an associated topology on the domain of discourse.(Smith 2008, p.153)

On the contrary, we show that if we consider polar topology, we can define a continuous characteristic function. Polar topology is a non-discrete topology on the domain of discourse.

Let X be a set endowed with polar topology, $S = \{0, 1\}$ and $\mathcal{O}S = \{\emptyset, \{1\}, S\}$ be the Sierpiński topology on S. Let p be the pole of the predicate F. By definition, the extension of F is $\operatorname{Int}\operatorname{Cl}(\{p\})$. One can define the characteristic function $\chi : X \longrightarrow S$ in the following way:

$$\chi_{\operatorname{int}\operatorname{cl}(p)}(x) = \begin{cases} 1 & x \in \operatorname{Int}\operatorname{Cl}(\{p\}) \\ 0 & x \notin \operatorname{Int}\operatorname{Cl}(\{p\}) \end{cases}$$

The characteristic function $\chi_{\operatorname{Int}\operatorname{Cl}(\{p\})} : X \longrightarrow S$ is continuous because $\chi^{-1}_{\operatorname{Int}\operatorname{Cl}(\{p\})}(1) = \operatorname{Int}\operatorname{Cl}(\{p\})$ which is open in X.

In the above definition, a non pole object x whose only pole is p, the pole of the predicate F, falls under F and $[Fx] = 1$.

Sierpiński topology is a very well-known non-trivial topology. It is T_0- Alexandroff. The characteristic function is not always continuous. It is continuous if P is open and not closed. Therefore, it is not the case that all concepts are vague. The predicate 'F', associated with the pole p is exact if $\{p\}$ is clopen. This does not happen always. As we saw before, $\{p\}$ is open but not always closed. Rarely happens that p is clopen. For example, if there are just two colors and all the objects are either the clear example of red or that of green. Then P will be clopen.

Tol_M is very similar to closeness condition. From Tol_M one can infer that if x is in the extension of a concept 'F' and y is indistinguishable from x, then the truth value of 'x is F' and 'y is F' **are not necessarily the same** but they cannot be very far away in the space of truth values endowed with a polar topology. In our 3-layer model, we can formulate it as:

Let (X, P, M) be a weakly scattered polar space, $F \cap P = \{p\}$ and $[Fx]$ denotes the truth value of the characteristic function for F at the object x.

$$\text{If } a \approx_p b \to [Fa] \approx_{p_{\mathcal{T}}} [Fb]$$

If a is very similar to b in respect of p, then Fa, Fb will be very similar in the topology generated by the poles that are provided by the truth-value space.

In fact, in a polar space the way we defined the characteristic function, there are two truth values but just one of them is open. So, there is just one pole.[80] By χ a vague proposition is false. This may be a good formulation for Raffman's account(Raffman (1994)). But Rumfitt considers true, false,... as poles. So, all

[80] The other one can perfectly be considered as a pole.

of them need to be open. For example, if we consider two poles, $P = \{p_{\mathcal{T}}, p_{\mathcal{F}}\}$, then the characteristic function can be defined as:

$$\chi(x) = \begin{cases} \text{T} & x \in \operatorname{Int}\operatorname{Cl}(\{p_{\mathcal{T}}\}) \\ \text{F} & x \in \operatorname{Int}\operatorname{Cl}(\{p_{\mathcal{F}}\}) \\ Neither\ true\ nor\ false & otherwise \end{cases}$$

In a polar distribution this function is continuous because both $\operatorname{Int}\operatorname{Cl}(\{p_{\mathcal{T}}\})$ and $\operatorname{Int}\operatorname{Cl}(\{p_{\mathcal{F}}\})$ are open. So, vague propositions are neither true nor false. In our model it says more than that. The layer-model together with **Simi** represent how the statements move from definitely true statements to definitely false statements in a very smooth way. For vague concepts there is no jump from true statements to false statements. The good thing is that this "somehow true", "somehow false", "almost true",... can be modelled without endorsing degrees of truth. So, although Tol_M is quite similar to Smith's closeness condition, we are not forced to accept fuzzy logic. This can be considered as an advantage of this view over all other gappy and degree theories of vagueness. Unlike supervaluationists, we do not consider precisifications to assign a truth value to statements. No delineation is needed. Vague concepts are boundaryless. In this view, one can escape one of the main criticisms to degree theory, namely how to assign an exact degree of truth to a statement. The reason is that one does not need to assign a truth value quantitatively. This makes sense because if Magi says "this curtain is red", she might be 100 percent right or 100 percent wrong or somehow right or wrong. In the borderline cases what matters is that she is not right, she is not wrong though, to some extent she is right and to some extent right. The exact degree does not matter.

8.5 Criticisms to Rumfitt's polar topology

Despite all the advantages of defining vague concepts in a polar space, it has serious drawbacks. We will mention some of them and will discuss whether our suggested ways of clarifying, elaborating and optimizing Rumfitt's account can confront these criticisms. Furthermore, we will come up with new challenges to the optimised version of the polar topology, presented in this thesis.

1. Zach (2018) finds Rumfitt's semantics quite appreciable. However, he puts doubt on the extension of it to multi-place predicates to solve the Sorites paradox :

   > The intuitive appeal of the polar semantics in the case of monadic predicates is quickly lost when considering the product topology for multi-place relation(Zach 2018, p.2087).

   He has two main reasons:

   a- Rumfitt does not talk in detail about the disjunction, conjunction, and negation in the product topology. Furthermore, defining them may fail classical logic.

   b- In Rumfitt's polar view " in order to evaluate an argument, we have to know how many poles there are" (Zach 2018, p.2088).

   He argues that the number of poles increases exponentially just by adding one shade of color. For example, consider a predicate '$(a_1, ..., a_{100})$' such that a_1 is red and a_{100} is orange, then there will be 2^{100} poles. By adding one object, a_k, between two objects such as a_2 and a_3 the number of poles will become twice more. The situation of course becomes worse by the increase of the cardinality of set of poles.

In his reply to Zach, Rumfitt agrees with Zach in that, though polar topology works well for monadic predicates, its extension to sentential logic through the product topology is inconvenient and artificial. We think that polar topological approach can be defended, at least in its limited version to solve the Sorites paradox. If someone accepts the polar topology for monadic predicates, naturally the product topology arises and conjunction, disjunction and negation in product topology can be defined in a usual way that is found in texts on polar topology. This of course does not deny that there should be further work on the semantics of product topology for infinite products. Nevertheless, given that in the Sorites paradox in question two poles are considered and the number of cases in the sorites sequence is finite, polar topology works. For the same reason polar topology can resist the criticism that for infinite products there is no guarantee that the product topology be polar.[81]

With respect to the increase of the number of poles in the product topology, which is an interesting observation, one may respond in two ways: the first one is based on the fact that the number of related poles, the ones that are maximally close to the objects in $\prod X_i$, will be restricted. As soon as we realize the color of an object, the number of related poles decreases. We can also apply modus ponens to reduce that number. It may be an answer for the case $|P| = 2$. Though, admittedly, mathematically speaking, it does not

[81] In digital topology, Khalimsky space is an example that shows that the product of the polar spaces may not be polar (see Mormann (2021)).

answer the criticism, esp. for considerable number of poles. This criticism also applies to weakly scattered spaces since the set of maximal elements coincides with the set of poles in a polar space. In defense, we may say that in our life experience sometimes adding even one object changes the whole space. Imagine that scientists discover a new planet, or we go to Africa and find a new species of animal, then the topology defined on the space will change. This observation leads us to another way to resist the criticism.

We may say that this is of course a mathematical result but in practice many of these poles will be dismissed. Then, that should be considered as a cognitive or epistemic problem. In practice, we need to know why we can dismiss the unrelated options rapidly. Apparently, when we add a new item in the sorites series our cognitive power leads us directly to the more related poles. This is then for the cognitive scientists or cognitive psychologists to see how we do that. We can consider it as a future task to advance the topological conceptual space approach to explain why we are able to ignore several unrelated poles rapidly.

2. Rumfitt considers poles as bands.[82] This makes two problems. Firstly, it does not fit with his proposed semantics for vagueness where he supposes that poles are points. Rumfitt may answer that in his polar topological account he consider poles as an idealization of the situation when the viewer simply differentiates seven poles in the color space in a low resolution. The generalization of Rumfitt's polar space to Weakly scattered polar space, though,

[82]For example, he says: "a normal viewer looking at the spectrum continues to perceive the original seven colour bands r, o, y, g, b, i, and v (Rumfitt 2015, p.259)."

can model what Rumfitt was looking for in a higher resolution. There are poles and sub-poles that could have been poles.

Secondly, a more serious problem is that if we accept that poles are bands, the notion of maximal closeness is not well defined. An object x might be maximally close to the beginning point of the band orange but might be maximally close to the center of the band red and not maximally close to the center of band orange. So, depending on the definition of maximal closeness, the extension of the predicate changes and therefore, the meaning of a concept changes.

Rumfitt is aware of this point and actually, uses that to explain the indeterminacy of the extension of a predicate. In other words, in defending the idea that his semantic theory does not provide a necessary and sufficient condition under which an object belongs to the extension of a concept he argues that the poles are bands and that the closeness relation will be indeterminate. As a result, for him the extension of a concept is vague.

> ... the satisfaction of a vague predicate will itself be vague. But so is the proposed condition for x to satisfy 'red': even if the relation of perceived closeness in colour were perfectly precise (which I doubt), the property of being closer in perceived colour to r than to any other pole will be indeterminate because the paradigm r is a colour band which lacks precise boundaries. So vagueness provides no objection to our proposed principle about the satisfaction of colour

terms (ibid. p.240).

The question now is whether really we need the indeterminacy of the closeness relation to justify that the criterion of being in the interior of the closure of the pole is vague as well. Let's review Rumfitt's idea using the color space example.

In the color space, according to Rumfitt, the meaning of 'red' highly depends on choosing a point in the band red as the pole of red. Since the colors in the band are indistinguishable with respect to the color red, we may choose any point in the band as the pole of red. The closeness relation, though precise, is related to the selected point and its relation to other poles. If one chooses a point at the end of the spectrum, rather than the one in the center, the set of objects that belong to the extension of the concept will be different.

Suppose that the perceiver chooses a point at the end of the spectrum of red band as r and the first point in the orange band as o. It is true that choosing the first pole affects the space because the other poles should be far enough from each other. So, it seems that the choice of the points in the band is not that arbitrary. In case of conceptual spaces, by the slight change of the distribution of the poles, the conceptual space changes. Applying the Voronoi-tessllation to a slightly different set of poles, the discretization of the space changes. But there, the closeness relation is defined as having less distance to one pole than to other poles. We have mentioned the difficulties of defining the closeness relation in a metric space. Rumfitt's topological approach goes beyond the Euclidean space and escapes the criticisms related to the convexity and metric spaces. Nevertheless, as Gärdenfors, in order to justify his claim that the apparent boundaries in the space are not sharp, Rumfitt appeals to the indeterminacy of the choice of poles and the closeness relation. The criteria for falling under a concept is vague because there are different appropriate ways of choosing poles. Our point is that the fact that poles are bands and any point in the band may be considered as the pole of the concept does not make the concept vague. It may show that the categorization of concepts depends on the set of poles in the space and the maximally closeness relation. In other words, it may relativise the satisfaction criteria to the choice of the pole and its gravitational power but it doesn't make it vague. Let us explain this point in more detail. Of certain concern to Rumfitt as a philosopher is to clarify under what conditions an object satisfies the predicate, say 'red', a criterion that relates the meaning of 'red' to the perceived color pole, $\{r\}$.

> ... when we are classifying objects in relation to colour poles, an object will satisfy 'red' if and only if it is perceived to be closer in colour to the pole r than to any other colour pole (Rumfitt 2015, p.239).

Furthermore, the pole is vague as well.

> The paradigm r is a color band which lacks precise boundaries (Rumfitt 2015, p.240).

It is quite natural, then, to justify the indeterminacy of such criterion by the indeterminacy of the closeness or similarity relation. The point is that we may have a better justification. We will explain what it might be.

It is true that in the polar space the meaning of a predicate is understood in virtue of the poles in the corresponding space. In the example of colors, the color space provides 7 poles. A normal viewer sees a band of colors in the color space and differentiates these 7 colors. These 7 poles, though, have some sub-poles that can be considered as poles of scarlet, crimson,... So, in Rumfitt's view, the viewer when zooms out can differentiate different poles. Of course, there is a difference between the typical case red and typical orange. But is there really a borderline between them? No, because when the viewer zooms in, he(she) observes bunch of colors in the red territory that gradually distantiate from the pole (Rumfitt 2015, p.258).

Our point is that it is the existence of these bunch of colors and the borderline cases in a connected space that makes the predicate vague. Metaphorically speaking, for vague concepts there are some objects that are in a state of limbo, not belonging to the extension of any concept. It is this condition, in a boundaryless space that makes the predicate vague. The upshot is that even if we fix the context by fixing the set of poles and clarifying what the closeness relation is, still there will be vagueness. The semantics, then, paves the way to pass through the limbo to get to the extension of another concept.

In a nutshell, the weakly scattered Alexandroff topological approach shows that the choice of the poles are not that arbitrary. The maximal points with respect to the specialization order are the poles. Furthermore, within the fixed context we can justify the indeterminacy of the satisfaction criterion.

In the presented topological account, to argue for the **indeterminacy** of the satisfaction of a vague predicate, one can appeal to the fact that the pole, in fact, is a band that contains objects very similar to the pole. But **vagueness** is not due to the choice of a pole or that the maximally closeness depends on the chosen pole.

3. According to Rumfitt, the truth value of a vague statement depends on how many poles there are in the truth-value space. On the other hand, the poles of the truth-value space depends on the polar space in which the vague concept in question is defined(call it S). The truth-value space may have two poles or three poles, etc. Our point is that while it perfectly make sense to have two poles, it is not clear to us in what way we can have the third truth value. In particular, Rumfitt does not explain what would be the polar space for the truth values. The question is: "why is indeterminate, $\mathcal{I}$ a vague concept in the truth-value space?"

   In the color space, for example, there cannot be a pole for intermediate, if there is no borderline case or if there is just one borderline case. We need to have sufficiently borderline cases to have "shades" of indeterminate pole. Otherwise, the space would be precise. If for a predicate 'F' there is just one borderline case,x, then $[Fx]$ will be the pole of the indeterminate but there will be no y such that $[Fy]$ is maximally close to the indeterminate pole because there is just one borderline case of F. Furthermore, even if there are more borderline cases, since the boundary of F in S is thin and closed, boundary is not a concept at all and there is no open neighborhood around x, in which there are more borderline cases. So, it is not clear what would be the statement that is maximally close to the pole of indeterminate. In another words, as soon as we have the pole of indeterminate truth value, it is distinguishable from other poles, $P_{\mathcal{T}}$ and $P_{\mathcal{F}}$. Then,$P_{\mathcal{I}}$ will be the only pole maximally close to $[Fx]$ of borderline cases of F. Indeterminate needs to have borderline cases and it is not clear whether there are such cases to which the poles of indeterminate and true or indeterminate and false is maximally close.

This, in fact, is better for Rumfitt's aim to keep classical logic. By his topological semantic, vagueness can be defined in two-valued classical logic. Truth-value space is a connected boundaryless space with two poles and bunch of statements whose truth values range over $\mathcal{T}$ and $\mathcal{F}$. Probably, one can say that in the metalanguage in higher orders one can add another pole like $\mathcal{I}$, considering a pole for the boundary.

9 Philosophical discussion

9.1 Polar theory and the law of excluded middle

Rumfitt contends that the polar theory of vagueness will keep the law of the excluded middle. However, as mentioned before, he does not accept the very traditional account of the law of excluded middle, proposed by Frege according to which accepting LEM is nothing but accepting that there is a sharp boundary. Furthermore, the set of truth values can contain more than two members. Rumfitt's proposed semantics for classical logic keep classical logic because the family of regular open sets in any non-empty topological space forms a Boolean algebra.

The question is: "what is good about keeping classical logic that can have a third truth value?" His proposed semantics makes it clear why LEM holds but does not entail sharp boundary. However, it seems quite unclear to us why unlike Fregean classical logic, we can have more than two truth values. We can say a statement is somehow false or somehow true, even if we keep the two-valued classical logic. In the absence of the principle of bivalence, the gappy approach is quite justifiable and appreciated. But why can we keep classical logic and accept a third value? The pole of the third value will be located in the linearly ordered truth-value space between the poles of true and false. So, a vague statement cannot be either true or false. In the presence of the third value, it can be indeterminate or either true or indeterminate. So, why does LEM hold in this case, given that the pole of indeterminate is far from the pole of false. If a statement is in the extension of indeterminate, it cannot be false. So, in the end what is left from classical logic and what does LEM mean?

Rumfitt compares his account with Raffman's account who is also proposed to defend the boundarylessness of vague concept. Nevertheless, she keeps bivalence as well. The main reason is that "Raffman's treatment of negation conflicts with the principle that both a predicate, and its negation, have open extensions." For him, the negation of a predicate is also a predicate and if the extension of a predicate is open, the extension of its negation also needs to be open. He finds this point more important than keeping the bivalence principle.

This makes sense. A statement may be neither true nor false and this can be explained in the absence of a third value.

Rumfitt also considers Edgington's approach who also is a kind of polar approach and sees the problem of Sorites in the accumulation of true instances of the tolerance principle. Nevertheless, in Rumfitt's account one does not need look for the numerical degree of closeness to truth(verities). He does not consider Smith's account. But as we discussed, his argument also works against Smith's account.

Finally, we compare polar approach with another gappy approach, namely supervaluationism who accepts two valued classical logic. If, as Rumfitt does, the poles are points, then a concept may be vague in the sense that the set of poles can be fixed in different admissible ways and all of them may be the poles of the space. For example, the spectrum of color may provide a fixed set of poles in slightly different ways. If we accept this interpretation, we should show in what way this view differs from supervaluationism. Suppose that X is a space. The subset of X, the set of poles defines a topology on X. Different sets of poles define different topologies on X. Supervaluationsts find vagueness in the deficiency of meaning. Rumfitt defends boundarylessness of vague concepts. Therefore, like Sainsbury, he contends that in the polar space we do not look for the right precisification rather, we compare an object with the pole of the concept. Nevertheless, according to Rumfitt, the meaning of a concept "lies in its association with the pole or paradigm"(Rumfitt 2015, 255).

In a polar space concepts are associated with a pole. If the pole can be changed slightly, it seems that we still face the deficiency of meaning because there is no way of choosing the right pole for the predicate.

One way to deal with this problem is to say that in a certain context, the space provides us the set of poles. This can be in favor of weakly scattered spaces that provide poles as their maximal elements. For example, the spectrum of color space

S, provides us with 7 poles. We fix these poles and define vagueness in this context. If vagueness is due to the choice of the poles, then a kind of supervaluationistic argument can be given: vagueness is indecision between different ways of choosing the set of poles and the gravitational power of the poles. A statement is true if it is true in all admissible ways of choosing poles. However, we do not need to appeal to bands to explain vagueness. Even within a fixed "precisification", a concept can be vague in the sense that the boundary is not empty and it is it is boundaryless. So, the meaning of 'red', for example, highly depends on a particular spectrum. If that is the reason that the extension of a concept is vague, vagueness apparently would be again due to the deficiency of meaning, this time not because we can make the concept precise in different admissible ways, but because we are not able to find the right pole of the concept in the band. In saying that 'Fred is bald' we do not fail in finding the precise delineation but we fail in finding the right pole to compare Fred with it. A borderline x in one topology may not be a borderline case anymore in the other one. So, what will be the truth value of a statement containing x? How do we choose between those set of poles? Our point is that we do not need to answer these questions if we fix the context and justify why the extension of a predicate is vague, saying that the smooth movement through three layers shows that there is no boundary that sharply identifies the things which fall under a concept from the things which do not.

Our account of vagueness, like supervaluationists is gappy but we do not make the concept precise drawing a sharp line between the closure of the pole of the concept and its interior closure in different suitable ways. Nevertheless, the space provides us certain poles and we fix them. So, we do not need to endorse super truth or super false. As Rumfitt mentions:

> Unlike supervaluationism, we can allow that a vague statement expresses just one thought: the statement A_{50} (e.g.) says that the tube a_{50} is red-no more, no less (Rumfitt 2018a, p.430).

9.2 Polar theory of vagueness and the principle of bivalence

On the contrary to the traditional view, Rumfitt claims that the principle of bivalence is not the building block of the classical logic. One can keep classical logic and maintain a non-classical semantics that is suitable enough to account for vagueness. With this idea, he departs from the traditional view of the classical logic discussed by Aristotle and Frege.

It seems that he is defending a very light account of classical logic in which the law of excluded middle has lost its power and the principle of bivalence has lost its validity. So, we are left with a weakened version of classical logic. This, in itself, is not problematic. But he considers his account better than supervaluationism or other non-classical views because it keeps classical logic. In particular, one may ask "What is the advantage of Rumfitt's polar account over supervaluationism? " Supervaluationism was attacked by Williamson, a defender of the classical logic. His two main arguments are as follows:

(1) Denial of bivalence is absurd. (DBA)

(2) Higher-order vagueness is not explained.

The law of excluded middle and T-schema together yield bivalence. So, the ones who accept LEM and T-schema cannot deny bivalence. Those who deny T-schema then are challenged to give a truth condition. Furthermore, higher-order vagueness has been the Achilles' heel of many accounts of vagueness that deviate from classical logic Williamson (1994).

Williamson's argument for the denial of bivalence(DBA)

In Williamson and Simons (1992) the authors argue that the denial of the principle of bivalence leads to a contradiction. Also, Williamson (1994) uses this argument in favor of an epistemic account of vagueness.

> By abandoning the assumption that vague utterances are bivalent, it is suggested, we free ourselves to understand the phenomena of vagueness (Williamson 1994, p.185).

The main idea in defense of vagueness as ignorance against most theories of vagueness is the following:

the claim that it is not the case that a value of a vague statement is either true or false leads to a contradiction:

> ... suppose that 'TW is thin' is neither true nor false. If I were thin,'TW is thin' would be true; since it isn't, I'm not. But if I'm not thin, 'TW is not thin' is true, and so 'TW is thin' false. The supposition seems to contradict itself. Yet on the majority view, it is true. (Williamson and Simons 1992, p.145)

In this section we would like to discuss how Rumfitt gets a way with Williamson's argument. Furthermore, we will discuss what the truth condition of predicates will be in Rumfitt's account. We will start with an example.

Imagine that the sky is borderline blue. Let A be the statement 'The sky is blue'. The principle of bivalence is defined as:

(B) 'A' has one truth value which is either true or false.

Those who reject the principle of bivalence deny that the truth value of 'A' is exactly either true or false which means that the truth value of 'A' can be:

(a) Both true and false

(b) Neither true nor false

(c) It has a third value (there are more than two truth values for a statement) (Pelletier and Stainton (2003)).

The subvaluationists choose the first option, the supervaluationists the second one and the proponents of 3-valued logic, degree theorists, many valued and fuzzy logic among others endorse the third one.

In his argument, Williamson uses Tarski's disquotational schema for truth(T-schema) according to which considering any sentence S,

(T-schema) The sentence 'S' is true if, and only if, S.

For example, the sentence 'the sky is blue' is true if, and only if, the sky is blue . Instances of T-schema will be used in DBA. One of the ways to reject the argument is to deny T-schema. This calls for an explication of what the truth condition for a statement is.

In the following, we first give two formalizations of Williamson's argument[83], the ways to tackle the argument and then we will explain how Rumfitt's account

[83] For more formalization see Williamson (1994), Pelletier and Stainton (2003) and Keefe(2000).

escapes the tramp of the argument and what the truth conditions of a predicate are.

On arguments against DBA

The first version of the argument, $(DBA)_1$ takes statements as the truth-bearers. Let F be a statement and $\mathcal{T}$(F) denotes that 'F' is true. Williamson's argument goes as follows: $(DBA)_1$

(1) $\neg(\mathcal{T}(F) \vee \mathcal{T}(\neg F))$ (Denial of B)

(2.a) $\mathcal{T}$(F)$\Longleftrightarrow F$ (T-schema)

(2.b) $\mathcal{T}(\neg F) \Longleftrightarrow \neg F$ (T-schema)

(3) $\neg(F \vee \neg F)$ (1, 2a, 2b)

(4) $\neg F \wedge \neg\neg F$ (3, De Morgan)

Williamson claims that (4) is a contradiction no matter whether we accept double negation elimination rule and how we interpret the negation. Also, it is a contradiction regardless of accepting classical logic or intuitionistic logic. Pelletier and Stainton (2003) argue that the rejection of bivalence is not incoherent because one may deny De Morgan laws and therefore blocks the inference from (3) to (4). This cannot be a suitable argument for Rumfitt since De Morgan laws hold in Rumfitt's theory. Then, they take a look at more serious objections to DBA. The first objection relies on the fact that (1) is not what the proponents of non-bivalence argue for. In other words, it is the denial of a version of the law of excluded middle.

Rumfitt does deny (1). Bivalence is a stronger claim about the statement itself, that it is either true or its negation is true.

Even if (1) is accepted, there are other ways to reject $(DBA)_1$.

Another argument against DBA targets the premises (2a) and (2b), namely the T-schemas. Pelletier and Stainton contend that this definition of T-schema already presupposes bivalence and hence cannot be used in DBA. The ones who deny bivalence also deny the biconditional. For example, if one accepts three-valued logic with values True, False and Indeterminate, with the usual negation, then just the left to the right side of the biconditional holds.

Later, in 1994 Williamson chooses utterances instead of statements as the bearers of truth:

> A philosopher might endorse bivalence for propositions, while treating vagueness as to the failure of an utterance to express a unique proposition. On this view, a vague utterance in a borderline case expresses some true propositions and some false ones (a form of supervaluationism might result). There is no commitment to a bivalent classification of utterances, or to the ignorance on our part that such a classification implies. *The problem of vagueness is a problem with the classification of utterances. To debate a form of bivalence in which the truth-bearers are statements is to miss the point of the controversy.* In a relevant form of bivalence, the truth-bearers are (perhaps with a little artificiality) the utterances themselves. (Williamson 1994, p.187)(My emphasis)

Even if one considers utterances as the truth-bearers, Rumfitt claims that DBA begs the question. In other words, he claims that Williamson presupposes bivalence in his argument. The failure of the argument is independent of what the truth-bearers are.

Before seeing in what way the polar account denies the redundancy of the non-bivalent account, let us see the second argument proposed by Williamson based on utterances:

Consider that u is an utterance and F a vague statement. Denote 'u says F' by S(u,F). For example, u says that it is the case that 'the sky is blue'.

The principle of bivalence will be:

(B') If u says that F, then either u is true or u is false.

(T-schema)

($\mathcal{T}$) If u says that F, then u is true if and only if F.

($\mathcal{F}$) If u says that F, then u is false if and only if not F.

If u says that 'the sky is blue' then u is true if the sky is blue and vice versa. And if the sky is not blue then u is false.

$(DBA)_2$

(0) S(u,F)

(1) $\neg(\mathcal{T}(u) \vee \mathcal{F}(u))$ (Denial of B')

(2.a) S(u,F)$\Longleftrightarrow$ ($\mathcal{T}(u)$ iff F) (T-schema)

(2.b) S(u,F)$\Longleftrightarrow$ ($\mathcal{F}(u)$ iff $\neg$ F) (T-schema)

(3) $\mathcal{T}$(u) iff F (0, 2a, Modus ponens)

(4) $\mathcal{F}(u)$ iff $\neg F$ (0, 2b,Modus ponens)

(5) $\neg(F \vee \neg F)$ (1,3,4, substitution)

(6) $\neg F \wedge \neg\neg F$ (5, De Morgan)

Again, by DBA_2, supposing that the principle of bivalence is not true leads to a contradiction. In the footnote of his book, Williamson gives a formal argument of DBA. We will quote his argument and in the next part, we will show how polar theory reveals why Williamson presupposes that a statement either is true or false in his arguments.

> A formal version of the argument is as follows:
> Each formula 'P' is assigned a semantic value [P]. The semantics values form a lattice under a partial ordering $\leqslant$, i.e. each pair of values has a greatest lower bound (glb) and least upper bound (lub). $[P \wedge Q] = glb\{[P], [Q]\}; [P \vee Q] = lub\{[P], [Q]\}$; if $[P] \leqslant [Q]$ then $[\sim Q] \leqslant [\sim P]$. These assumptions are met by standard classical, supervaluational, intuitionist and many-valued treatments, and others. It is then easy to show that $[\mathcal{T}(u)] = [P]$ and $[\mathcal{F}(u)] = [\sim P]$ imply $[\sim |\mathcal{T}(u) \vee \mathcal{F}(u)|] \leqslant [\sim P \wedge \sim\sim P]$ (Williamson 1994, PP. 300-301).

Rumfitt and his answers to DBA arguments

The truth predicate, like other predicates, is defined in a topological space generated by a fixed set of poles. Rumfitt finds the problem in the truth principles:

> ... there is a classically valid argument for the conclusion that a true disjunction contains at least one true disjunct which seems to rely on only innocuous principles about truth (Rumfitt 2015, 257).

Williamson says that "(2a) is to be read as implying that 'u is true' and 'F' are true on exactly the same admissible interpretations, and similarly for (2b). "

Even if one accepts that the truth value of 'u' and 'F' are the same, the argument fails unless we accept that 'F' has only two values, true or false, which is what was supposed to be proven. In other words, in both arguments DBA_1 and DBA_2, 2a and 2b are questionable. In particular, Rumfitt finds B' too strong. Let us explain it in more detail. (2a) as well as (2b) have two directions:

$(2a)^* : S(u, F) \wedge \mathcal{T}(u) \longrightarrow F$

$(2a)^{**} : S(u, F) \wedge F \longrightarrow \mathcal{T}(u)$

$(2b)^* : S(u, F) \wedge \mathcal{F}(u) \longrightarrow \neg F$

$(2b)^{**} : S(u, F) \wedge \neg F \longrightarrow \mathcal{F}(u)$

Rumfitt like many other philosophers who reject bivalence does not admit $(2a)^{**}$ and $(2b)^{**}$ because they both presuppose bivalence. He claims that his account reveals why it is so in a different and more appropriate way. He accepts truth-value gap but instead of appealing to super-truth like supervaluationists, he contends that 'true' as well as 'false' are vague concepts, their extensions depend on the set of poles. If the only poles are true and false, (2a) and (2b) are true premises and the argument is valid. Nevertheless, it will be question-begging because the only truth values for an utterance are supposed to be true and false. However, in the polar account, one can legitimately suppose that there are three poles, say true, false and indeterminate. For example, if my dress is borderline red(red-orange), then it is either red or orange. Now consider u says that 'Nasim's dress is orange'. The utterance is simply false because my dress is not orange. So, the conditional, $S(u, F) \wedge F \longrightarrow \mathcal{T}(u)$ has an indeterminate antecedent and false consequent. Since the poles of the color of my dress are red and orange, the pole indeterminate is maximally close to u.

When we consider the formal version of Williamson's argument it becomes clear that the semantics of disjunction is different from the one Williamson assumes as the one that all accounts of vagueness share.

Rumfitt endorses truth-value gap. There is a gap not because there is a red-orange pole but because 'true' and 'false' are vague predicates and their extensions do not exhaust the whole space. This is enough to reject 2b.

As discussed before, considering a third pole is problematic. Nevertheless, he can resist against Williamson's argument by rejecting the T-schema(2b). The following quote from Rumfitt reveals why he endorses a third truth value while there is no pole of borderline cases.

> If the vague terms 'true' and 'false' are related to a system of poles that includes the Indeterminate as well as the True and the False, each of these conditionals has an indeterminate antecedent and a false consequent and is consequently unacceptable. Admittedly, if the only poles in the relevant system are the True and the False, then Bivalence holds. Both these poles will be maximally close to a vague statement, so every statement will belong to the interior of the closure of the union of the extension of 'true' with the extension of 'false', and hence (given the recommended semantics) will satisfy 'is either true or false'. However,

> to assume that the True and the False are the only poles in the relevant system is to assume what was to be proved. So the present argument for the bivalence of vague statements begs the question.(Rumfitt 2015, p.309).

Bivalence says that the space of truth values is precise and sharply bounded. What Rumfitt can deny, then, is that 'true' and 'false' are precise, that the union of the interior closure of poles of the set of truth values (in this case true and false) exhaust the space. He can deny that in the topology on the set of truth values, there is a sharp boundary between the extension of 'true' and the extension of 'false'. He can do that without endorsing the third value. So, the disjunction is true while none of the disjuncts are true. It may happen that the truth value of a statement neither be true nor false, not because it is maximally close to the pole of the third value, Indeterminate, but because it is a borderline case of true with respect to false.

In his later paper he explains this point:

> It[bivalence] implies... that there is a cut-off point in the sequence at which the statements switch from being true to being false. But that in turn implies that there is a cut-off point at which tubes switch from being red to being not red- a grossly implausible conclusion (Rumfitt 2018a, p.419).

9.3 Dynamic vagueness, a possible solution to higher-order vagueness

In this section, we would like to deal with the problem of higher-order vagueness from a new perspective. Any theory of vagueness has to deal with the problem of higher-order vagueness according to which the concepts boundary, boundary of boundary ... are also vague. Therefore, neither there is a sharp line between clear cases of a concept, A, and its boundary nor there is a sharp line between the clear cases of $\mathtt{C}A$ and the borderline cases.

For Sainsbury concepts are boundaryless. There is no boundary between clear cases and the borderline cases, there is no boundary between borderline cases and clear non-cases. Nevertheless higher-order vagueness is not an illusion, but there is no hierarchy:

> Nothing gives rise to substantive issues about the level of vagueness appropriate to our familiar examples(Sainsbury 1991, p.180).

Mormann (2020) shows that these spaces are related to Bobzien's columnar vagueness, in that as soon as something is a borderline case, it is a borderline case in other orders as well. So, vagueness at higher orders collapse to the first level in weakly scattered spaces. In weakly scattered spaces, $bdbd(A) = bd(A)$. It is a remarkable result.

We do not want to argue in favor of columnar vagueness or hierarchical higher-order vagueness. However, we think the latter seems plausible and should not be neglected. So, the question is whether we can topologically model hierarchical higher-order vagueness. At first glance, the answer is no because in all topological spaces- even if they are not weakly scattered- $bdbdbd(A) = bdbd(A)$. This means that $int(bdbd(A))$ is empty in all topological spaces. So, it appears that it is not possible to model vagueness of order higher than 2 in the hierarchical higher-order vagueness.

We propose a new perspective to overcome this limitation. It may seem an ad-hoc movement but when we focus on the boundary the movement looks quite plausible.

The main idea is that when we think about higher-order vagueness in a polar space, we realize that at each level the set of poles changes, and therefore, the topological space corresponding to the new set of poles changes as well. But the orders are not independent of each other. In the following, we will show how to model higher-order vagueness. First, let's see how the cardinality number of poles grows. We will define a sequence of polar distributions (X, P_i, M_i). Denote the cardinal number of a set of poles P by $|P|$. Let $|P_1| = n$.

$|P_i| = 2|P_{i-1}| - 1, i \geqslant 2$.

For example, for $n = 2$:

$|P_2| = (2 \times 2) - 1 = 3$. So, in the second level, there are three poles.

$|P_3| = (2 \times 3) - 1 = 5$. So, in the third level, there are five poles.

If the space is circular, then:

$|P_i| = 2|P_{(i-1)}|$ for $i \geqslant 2$.

For example, Let $|P_1| = 2$. Then $|P_2| = 2 \times 2 = 4$ $|P_3| = 2 \times 4 = 8$

If in the hierarchy somewhere two orders coincide, then it stabilizes:

If $\exists i \exists j, P_i = P_j$, then $\forall j \geqslant i\ P_i = P_j$.

Just like the polar topology in the first level, M_i in i-th level, takes each element of X to the set of poles at that level maximally close to them. In the polar topological space and its generalization, the set of poles are fixed, poles are open and for a concept with the pole p, the objects that are almost clearly A belong to $\operatorname{int}\operatorname{cl}(A)$. The interior of A in higher orders coincide. This means that when we focus on the boundary in one level, the changes occur around the boundary and almost clear cases. But the pole(s) of a concept remains the same. In other words, even if in other levels there appear more poles, they won't affect the typical cases of a concept. For example, the typical cases of the concept of the boundary between red and orange, the ones that are typically, neither definitely red nor definitely not red, do not make us modify prototypes of red things and prototypes of orange things. In fact, borderline cases are far enough from the poles and as a result, any change in it does not affect the poles. Though, it affects their closure and interior closure of the poles. We know that $bd(A)$ is not a concept in a polar conceptual space because it is not regular open. Therefore, to go to the second-order we have to consider $\operatorname{int}\operatorname{cl}(bd(A))$. Since $bd(A)$ is closed, $\operatorname{int}\operatorname{cl}(bd(A)) = \operatorname{int}(bd(A))$. In polar topology we consider concepts with a pole. It is quite intuitive that if the interior of boundary is not empty, we have some clear cases of borderline cases. So, we consider typical cases of the boundary as its paradigm. Therefore, even though we do not have a concept of boundary at the first level, in other levels this concept appears. In the first level, the focus is on 2 concepts and the boundary contains the elements that do not definitely belong to the extension of neither of them. In other levels, however, we focus on the boundary. If it doesn't have topological thin boundary, that is to say, if its interior is not empty, if there are other borderline cases in the neighborhood of a borderline case, then it can be considered as a new concept. We can consider that borderline case as a typical borderline case (as the pole of the boundary) and interior closure of the boundary in the new topological space would be the extension of the new vague concept, boundary.

One might object that if we consider weakly scattered spaces, the boundary would be thin and therefore, all orders collapses to the first order. So, how can we go to the second order? In this case, we may say that when we zoom in to take a close look at the boundary, we will find some borderline cases. We may choose one of them as the typical borderline. For example, in the color space we may choose the one that is among the objects that if we were forced to call it red or orange we would deny it with more hesitation, say a_{50} in the sorites series.

We may say that if we stay in one level and take a look at the levels above us, borderline cases seem the same, just like the columnar vagueness but if we take a look at levels below us, we see the hierarchy of boundaries. When we zoom in a pole, say r, we will see the reddish objects around it. But it seems that when we zoom in the boundary, we may find a bunch of objects some of which are nearer to the reddish objects, some nearer to the orangish objects and some in the middle. We may choose a pole among the latter ones.[84] Suppose that when we zoom in the boundary we can find a typical case. It is good to mention again that in this view, the set of poles at any level will exist in all higher orders. So, for example, if something is really clearly red; i.e., it belongs to the $\mathrm{int}(\{r\})$, it will be really clearly red in other levels as well. But the elements of the $\mathrm{cl}(\{r\})$ or of the $intcl(\{r\})$ may attach to a new concept $\mathrm{int}\,\mathrm{cl}(bd(\{r\}))$ in the next level. When we focus on the new concept $\mathrm{int}\,\mathrm{cl}(bd(\{r\}))$ we realize that some members

[84]Admittedly, this idea needs to be developed. We leave it for the future work.

that used to belong to the $\operatorname{int}\operatorname{cl}(bd(\{r\}))$ in the first level, in fact, do not belong to the $\operatorname{int}\operatorname{cl}(bd(\{r\}))$ in the second level because they belong to the boundary of the interior closure of the pole of the boundary. It is just like how we learn a new concept. What is ignored in the discussions on higher-order vagueness is that in each order we acquire at least a new concept that affects our conceptual space. In other words, in each level, the context changes but it is related to the previous context because there are some stable poles in all levels. In the following, we will formally show how the conceptual spaces changes. Let $P_1 = \{p, q\}$. We will use parenthesis in order to create a pole between two poles. For example, (p, q) is the pole of borderline cases to which p and q are maximally close.

$P_2 = P_1 \cup \{(p, q)\} = \{p, (p, q), q\}$

$P_3 = P_1 \cup \{(p, (p, q)), ((p, q), q)\} = \{p, (p, (p, q)), (p, q), ((p, q), q), q\}$

$P_4 = P_2 \cup \{(p, (p, (p, q))), ((p, (p, q)), (p, q)), ((p, q), ((p, q), q)), (((p, q), q), q)\} =$
$\{p, (p, (p, (p, q))), (p, (p, q)), ((p, (p, q)), (p, q)), (p, q), ((p, q), ((p, q), q)), ((p, q), q),$
$(((p, q), q), q), q\}$

The following relation holds between the set of poles. $P_i \subseteq P_j$ $i \leq j$. The function $M_i : X \to 2^{P_i}$ is defined just like M_1. (X, P_i, M_i) in each level is a different polar distribution. Note that the extension of a concept $\operatorname{int}\operatorname{cl}(\{p\})$ may decrease in higher orders by the increase of the number of poles.

It is good to notice that this dynamic version of vagueness differs from what Gärdenfors considers as a distinction between scientific and psychological representation of a concept. A painter may differentiate more colors than others. According to Gärdenfors, a scientific representation of color would require a different representation, however, one that captures important scientific features of the electromagnetic spectrum such that the wave properties of wavelength and amplitude constitute integral dimensions.

This difference can be seen at the same level and explains the different conceptual spaces of human beings. There are a few people like Jean des Esseintes- the protagonist of the well-known novel “Against nature”[85]- who is able to distinguish and name many different shades of green. Mormann(2021) shows how the specialization order in a weakly scattered spaces gives us different shades of a color. So, Jean des Esseintes within the realm of green things may make subtle differences between shades of green. The point is that each person has a distinct conceptual space. For example, if I cannot differentiate dark blue from black, my conceptual space will be different from my sister’s who is able to do that. At the first level, we have two conceptual spaces with two different sets of poles with different cardinalities and elements. As Gärdenfors argues, we understand each other after communicating. Imagine that my sister has two bags, white and another one. If my sister asks me to bring her dark blue bag and I just see the black one, I would take her the black one and if that is the one she had asked for, I will learn that she differentiates more colors than me. Next time that she asks me to bring her a dark blue thing I will go for a black one, perhaps thinking that I am somehow a color-blind person. But in the case of higher-order vagueness, we consider the same person who has reflected on the boundary and borderline cases and has acquired new concepts related to the boundary. For example, what happens to the conceptual space of Jean des Esseintes who has a fine-grained color space, if he focuses on the borderline cases between two concepts? He might consider the

[85]See Huysmans (2011).

typical borderline cases as the pole of the boundary between two concepts. He might also consider different shades of the new concept.

In our view, the extension of a concept changes in each level with different conceptual space but some elements, namely the typical cases always belong to the extension of that concept. As we showed formally, some new concepts are added in a hierarchical space. So, in any order, space has more elements than in the previous orders. The new space along with the new set of poles and the function M that maps each member of the space to its set of poles to which they are maximally close defines a new topology at each order different from the other orders. However, they are all polar topological spaces and therefore T_0 Alexandroff.

Part VI

Conclusions

The aim of the thesis was to give a topological account of vagueness. A pervasive phenomenon that apparently, like soot, swirling up around us, casts a shadow on the omnipotence of classical logic in which concepts have sharp boundaries. Shall we keep classical logic and semantics and sweep them up or shall we make classical logic more flexible to tolerate them? Answering these questions is important and we have proposed an answer. Nevertheless, our first concern was the phenomenon of vagueness itself, and then clarifying its logic. We needed to know what is expected from an account of vagueness. In part II we reviewed some theories of vagueness. The main questions to be answered by any theory of vagueness were: How do we define vagueness? How do we define borderline cases? What is a blurred boundary? What are the main features of vagueness and which one(s) is salient? What is the source of vagueness? Why is the principle of tolerance, according to which vague concepts are tolerant to small changes, so appealing to us? Why is it problematic to accept this principle? How do we formulate the Sorites paradox, is it a valid argument or not and how do we deal with it? Is there higher-order vaguenes? If so, how do we explain it? Epistemicism, Supervaluationism, many-valued logic, degree theories and fuzzy logic were just some of the approaches to vagueness that we briefly reviewed.

Geometry and topology had no role in the theories of vagueness that were discussed in that part. They mostly took a look at the phenomenon of vagueness from a logical point of view and accepted the classical notion of concept rather than prototype theory. There was a need to change the perspective to shed a new light on the philosophical discussions of vagueness. Topology seemed a very useful tool to be employed to model vagueness. After all, one of the main notions of topology is that of boundary and the tolerance principle is another way of saying that for each object that satisfies a certain property, there are some other objects in its neighborhood that have the same property. Furthermore, some topological spaces are tightly connected to modal logic. So, we aimed at showing the friendly part of topology not to be thought of as an isolated, abstract and unknown frightening field. Also, to explore a topological approach to the phenomenon of vagueness, we had to get familiar with the topological notions to be able to know how to use this tool. Part III was to briefly overview some topological notions such as boundary, connected spaces, neighborhood and closeness, open and closed sets, metric spaces, etc. We also introduced closure operators and some topological spaces and their characteristics. The main reason for finding topology an appropriate tool to deal with the phenomenon of vagueness was motivated by the idea that all the characteristics of vague concepts could be formulated in topological terms. Tolerance has a tight relation to the notions of approximation and closeness. The importance of *neighborhood* of objects rather than that of objects, makes it a good choice to be used to formulate the tolerance relation between the objects. Borderline cases are the ones that neither definitely belong to the extension of a concept nor definitely to its set-theoretic complement. Since on any given set several topologies can be defined, the hurdle was finding a suitable topology for the task at hand.

Part IV was the result of our navigation into the literature to find out some topological approaches to vagueness. Topology had been applied to other areas in philosophy such as epistemology and mereotopology. nevertheless, at the beginning, there were very few papers on this topic whose concern was not to cover all the questions we had set to answer. The first two sections are devoted to two such papers by Boniolo and Valentini and Weber and Colyvan. The third section was devoted to the Gärdenfors' conceptual spaces approach, and Douven and his colleagues' contribution to improve it and apply it to vagueness. Getting familiar with the conceptual spaces approach to concepts, as a dominant geometrical approach in cognitive science and its application to vagueness paved the way to reach our goal. We explained in detail what a conceptual space is, what are its goals and its achievements and in what way it had applied to vagueness. Recent valuable works on the application of the optimized conceptual spaces approach to vagueness, convinced us that vague concepts should be defined in a conceptual space. Concepts were defined as convex regions in a conceptual space. Many criticisms towards this approach stemmed from the fact that convexity depends on a similarity relation defined on the space and similarity depends on the metric of the space. Since in these views the space is usually considered to be Euclidean, and different metrics can be defined in a Euclidean space, a region can be convex in one metric space and not convex in another. Topology could save the conceptual space approach from these kinds of criticisms. It was enough to endow the space with a topology to embrace all those metric spaces. In part V, we accomplished this task. Rumfitt's topological semantics facilitated our way. It could be the bedrock of our work.

Following Rumfitt, we defended boundarylessness, yet we also put emphasis on the importance of having borderline cases as the main features of vagueness. In our three-layer topological model vagueness was defined in a topological space in which one can pass through concepts smoothly in a "silky road", from one concept to another. It has some things in common with dominant theories of vagueness, yet different from them. The source of vagueness is cognitive but there is no sharp boundary to be ignored. Like supervaluationists we endorse truth value gaps; some propositions are neither true nor false yet there is no need to precisify vague concepts. These are all in line with Sainsbury's boundarylessness account of vagueness. We also agreed with him that borderline cases alone is not a sufficient condition for a concept to be vague. Nevertheless, we argued that his example does not show this fact. We replaced it by our example that was not partially defined like Sainsbury's and thus was immune to justified criticisms of philosophers such as Greenough.

This work described up to now originates an expansion of Rumfitt's topological account of vagueness in defense of the classical logic. Pace Rumfitt, the main goal of the thesis is not retaining classical logic. Nevertheless, in our 3-layered model in which the extension of vague concept is also defined as the interior closure of its pole (the set of objects that are typical cases of concepts or very similar cases), it turned out that the classical logic can be kept but Bivalence fails.

The failure of the principle of Bivalence has been target to criticisms, especially by Williamson. Rumfitt claimed that his arguments beg the question. We explained it in detail and elaborated on that. The topological approach, presented in the current work is " just a drop of water in an endless sea" of discussions on vagueness. A colored drop that attempts to open upon a new horizon in approaches towards vagueness turning the attention to the importance and usefulness of the geometrical and topological structures. We give an account of vagueness in which the boundarylessness and having borderline cases both together are the prominent features of vagueness. The tolerance principle holds in the sense that concepts lack a sharp boundary. That is why it seems so intuitive to us. But there is more into the strict tolerance principle that leads to the Sorites paradox: if two objects x, y are similar with respect to the pole of a certain concept,F, they both belong to the extension of that concept and therefore, Fx and Fy have the same truth value. This is too strong. As we discuss, we smoothly move from the pole of F towards another pole passing through similar objects that little by little lose their attachment to the pole. So, the truth value of propositions Fx and Fy of adjacent objects in this "silky" road are similar but maybe not the same. This was implicit in Rumfitt's account. We made it explicit by 3-layer- model and the definition of similarity and explaining what would be a topology defined on the set of truth values. Contrary to what Smith believes, we showed that it is possible to define a topology if the set of truth values contain true and false. Nevertheless, we argued that Rumfitt is not right in assuming a third value because the boundary in his model is too thin. Actually, he did not even need that and that makes his theory even more acceptable. It seemed to us quite weird to defend a classical logic with three or more truth values. We need to emphasis that the 3-layer-model was not proposed as a new model. It was simply an improvement of the generalization of the polar topology to Weakly scattered T_0 Alexandroff spaces to apply it to vagueness. We did that by focusing on the properties of the space, revealing the hidden parts of it and refining the model to be able to answer further questions related to the phenomenon of vagueness. It was proposed as a small contribution and attempt to give a topological account of vagueness that could be stand among other theories of vagueness in the huge literature on vagueness.

Main results

The main results of this thesis are:

Part II: Clearing up the task of a theory of vagueness and reviewing the pros and cons of some existing theories of vagueness.

Part III: Introducing basic notions of topology as a useful tool to deal with the problem of vagueness.

Part IV: Exploring three topological/geometrical approaches to vagueness and arguing why they are not suitable approaches.

Part IV: Choosing Gardenfors' conceptual spaces as a suitable space to land in to explore, critically analyse, defend and optimize.

Part V: Using Rumfitt's topological semantics for boundarylessness of vague concepts, analyzing his view from a mathematical and philosophical point of view. Expanding his account to be able to answer some of the criticisms while coming up with new constructive criticisms to be considered in improving his model.

Part V: Making a difference between Mormann's numerical thickness problem and qualitative thickness problem proposed by Douven et al. (2013) for geometrical conceptual spaces and proposing the qualitative topological thickness problem.(See 204)

Part V: A close look at the Sorites paradox. Slightly improve and refine the generalized polar topology to the 3-layer model in which vagueness and similarity relation are defined and a weaker version of tolerance principle proposed.

Part V: It turned out that the logic of vagueness that was defined in a topologically structured conceptual space is classical. Yet the rejection of bivalence leaves some room for gappy propositions, the ones that are neither true nor false. The value of a proposition also changes smoothly in the space of truth values. Pace Rumfitt, we denied that there can be a third truth value.

Part V: We compared the 3-layer model with some prominent theories of vagueness. Our view was different from Shapiro's and Rumfitt's in not accepting the third alethic truth value. Also, we explained in what way the refined generalization of Rumfitt's topological account of vagueness was different from supervaluationism, a gappy approach that also rejects bivalence. The reason was that in the boundaryless account of vagueness, we do not endorse super-truth and super-falsity. The view also escapes from criticisms to degree theory because it does not assign a specific degree of truth to a proposition. It was also different from Smith's closeness account of vagueness. We showed that our weak version of tolerance is quite similar to Smith's colseness condition. Nevertheless, we argued that there is no need to endorse fuzzy logic.

Part V: We explained and critically analysed Mormann's recent generalization of Rumfitt's polar spaces to weakly scattered spaces T_0 Alexandroff ones that can model Bobzien's columnar higher-order vagueness. As we discussed, in these spaces, boundary of boundary is just the boundary(bdbd(A)=bd(A)).

Part V: We proposed a a dynamic account of vagueness as a way to go beyond one of the limitations of topological spaces. In general, in any topological space boundary of boundary of boundary is equal to boundary of boundary (bdbdbd(A)=bdbd(A)). This limitation of topology may discard it as a useful tool to be used to model vagueness. At least, it can model the columnar higher-order vagueness. Set this aside, we found hierarchical higher-order vagueness quite intuitive and admitting the limitation of topology. Therefore, we proposed a way to cope with this limitation. In fact, it is a demonstration of what happens when we zoom in a boundary and we consider the boundary as a concept and go to the next level.

For Rumfitt the gist of his work is "classical logic is good, classical semantics is bad", we may say that the gist of the dissertation is:

"Vagueness is good, sharp boundary is bad. Topological semantics is good, classical semantics is bad".

Future work

We argued that our proposed topological account of vagueness, as a refinement of the generalization of Rumfitt's topological account, can deal with some constantly recurring problems which any theory of vagueness is faced with. Topological account is quite new and vagueness is a complicated issue. Future work may encompass some fields such as computer science, cognitive science, logic, and philosophy.

Alexandroff topology has had huge application in cognitive science, computer science, artificial intelligence and digital images. In cognitive science and psychology, the conceptual space account of vagueness has had huge experimental support. The topological account of vagueness lacks such experiments and may be of interest for the researchers in those fields. It is quite interesting to see how our proposed optimization of conceptual spaces works in practice.

In this dissertation we modeled vague concepts that had prototypical cases(poles). It is expedient to consider other concepts as well. The space won´t be as well-behaved as T_0 weakly scattered Alexandroff spaces and needs further research.

Vague existence is one of the controversial issues in philosophy. Rumfitt briefly mentions this problem. Like him, we left the issue for future. Since we defend the idea that the extension of a concept can be vague, we need to deal with the criticisms towards vague identity. A well-known work of Evans against vague existence seemed to be the knockdown argument Evans (1978). But probably it is not. We have defined Noetherian T_0- Alexandroff spaces as a suitable topology for vagueness. Now, we may ask for a suitable topology for vague existence. Our guess is that we need to free ourselves from the points and that point-free topology may help.

Part VII

Appendices

10 Appendix A: Set theory

The material of this section is introduced to recall some of the terminologies in set theory.

A set is a collection of things. These things are called elements or points or members of that set. Membership relation is denoted by the Greek letter $\in$(epsilon). The expression "$a \in A$" means that a is an element of the set A or it belongs to A. "$a \notin A$" means that a is not an element of A or it does not belong to A. Set theory rests on the membership relation;i.e., in set theory the properties of a collection are defined by their membership relation. A set can be specified by a property that is shared by all members of the set. A set of all elements that have the property ϕ is denoted by $\{x : \phi x\}$ or $\{x| \, \phi x\}$. For two sets A and B, A is a subset of B if every element of A belongs to B. It is denoted by $A \subseteq B$. The collection of all subsets of A is called **the power set** of A and is denoted by $\mathscr{P} A$. If A is a subset of B and B a subset of A, then they are equal.

The **union of** A and B, denoted by $A \cup B$ is a collection of all members of A and B. In other words, each member of $A \cup B$ either belongs to A or to B:

$$A \cup B = \{a : a \in A \text{ or } a \in B\}.$$

The intersection of A and B, denoted by A ∩ B , is the collection of elements that belong both to A and B.

$$A \cap B = \{a : a \in A \text{ and } a \in B\}.$$

Given the set X, **the complement of** A in X, denoted by $\complement_X A$ is the set whose members do not belong to A. When X is a clear set in the context we will write $\complement A$:

$$\complement A = \{a \in X : a \notin A\}$$

The empty set, the set that does not have any member, is denoted by $\emptyset$:

$$\emptyset = \{a : a \neq a\}.$$

Two sets are called disjoint if $A \cap B = \emptyset$ (cf. Goldblatt 2014; Willard 1970).

As we said before, a set is a collection of things. By definition, there is no order between the elements of a set. In other words, for any two distinct elements of a set, $\{x, y\} = \{y, x\}$. This stems from the axiom of extensionality according to which two sets are equal if they have the same elements. But sometimes we need an order so that no longer that equality holds. The notion of the ordered pair (x, y) puts an order into the elements of the set. An ordered pair (x, y) has x as first element and y as second. Two ordered pairs (x, y) and (z, t) are equal if and only if $x = z$ and $y = t$.

Definition 10.1. *The* ***Cartesian product*** $X \times Y$ *of* X *and* Y *is defined as the set of all ordered pairs* (x, y) *such that* $x \in X$ *and* $y \in Y$*:*

$$X \times Y := \{(x, y) | x \in X, y \in Y\}$$

The ***Cartesian power*** *of a set* X*, denoted by* X^n*, is defined as:*

$$X^n = \underbrace{X \times ...X}_{n} = \{(x_1, ..., x_n) | x_i \in X \textit{ for all } i = 1, ..., n\}.$$

Functions can be defined based on ordered pairs and their Cartesian product. Informally, it is said that a function is a kind of "black box". It receives an object as an input and assigns to it a *unique* object as an output.

Formally, a function is defined as:

Definition 10.2. *Let X and Y be two sets, then a **function(map)** from X to Y, denoted by $f : X \longrightarrow Y$, is a rule that assigns a unique element $f(x) = y$ of Y to every element x of X. (Munkres 2000, p.15)*

In other words, f is a subset of the Cartesian product $X \times Y$ such that:

$$\forall x \in X, \exists y \in Y : (x, y) \in f.$$
$$\text{if } (x, y) \in f \text{ and } (x, z) \in f, \text{ then } y = z.$$

X is called the **domain** of f and Y the **co-domain or range** of f. It is good to mention that the domain and the co-domain can be the same set.

Definition 10.3. *Let $f : X \longrightarrow Y$ be a function from X to Y and $A \subseteq X$. Then $f(A) = \{y \in Y : \exists a \in A \quad f(a) = y\}$ is called the **direct image** of A. It is sometimes denoted by $f^{\rightarrow}(A)$ Similarly, the **inverse image(preimage)** of $B \subseteq Y$, with respect to f, denoted by $f^{-1}(B)$(or $f^{\leftarrow}(B)$), is the set of all elements of X that are mapped into B:*

$$f^{-1} : \mathscr{P}Y \longrightarrow \mathscr{P}X$$
$$f^{-1}(B) := \{x \in X : f(x) \in B\}.$$

*The inverse image of a singleton is called a **Fiber**.* [86]

Inverse images commute with respect to union and intersection. In formulas:

$$\forall A_i, B_j \subseteq Y,\ f^{-1}(\bigcup A_i) = \bigcup f^{-1}(A_i).$$
$$f^{-1}(\bigcap B_j) = \bigcap f^{-1}(B_j).$$

[86] Inverse image is different from inverse function.

So, the inverse image behaves well with respect to the union and intersection. Also, it behaves well with respect to the complement, i.e., For $B \subseteq Y$, $f^{-1}(Y - B) = X - f^{-1}(B)$.

This is an important point that will be used in defining continuity.

Definition 10.4. *For any set X, there is a map $id : X \longrightarrow X$ such that $id(x) = x$.*

This map is called the identity map. If $Y \subset X$, the inclusion $i : Y \longrightarrow X$ is a function such that for each $y \in Y, i(y) = y$. To denote the inclusion map, usually a hooked arrow is used:

$$i : Y \hookrightarrow X.$$

Definition 10.5. *Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$. Then* ***the composition*** *of f and g, denoted by $g \circ f$[87], is defined as:*

$$g \circ f : A \longrightarrow C$$

$$g \circ f(a) = g(f(a)), \text{ for } a \in A.$$

'$g \circ f$' is read 'g following f' or 'g of f'

For any functions $f : A \longrightarrow B$, $g : B \longrightarrow C$, $h : C \longrightarrow D$ the associative law holds:

$$h \circ (g \circ f) = (h \circ g) \circ f.$$[88]

Another important point about composition of functions is the identity law for composition according to which for any two functions $f : A \longrightarrow B$, $g : B \longrightarrow C$ and for the identity function $id_B : B \longrightarrow B$:

[87] Sometimes it is just written as gf.

[88] See (Goldblatt 2014, p.20-21)

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "h \circ (g \circ f)"'] \arrow[dr, "g \circ f"] & B \arrow[d, "g"] \arrow[dl, "h \circ g"'] \\
D & C \arrow[l, "h"]
\end{tikzcd}
$$

$$id_B \circ f = f, \text{ and } g \circ id_B = g$$

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "f"'] & B \arrow[d, "id_B"] \arrow[dr, "g"] & \\
& B \arrow[r, "g"'] & C
\end{tikzcd}
$$

Definition 10.6. *a. A function $f : X \longrightarrow Y$ is called **surjective(onto or epi)** if for every element y of Y, there is an element x of X such that $f(x) = y$. Equivalently, f is surjective if $f(X) = Y$.*

*b. A function $f : X \longrightarrow Y$ is **injective(1-1 or mono)** if $f(x) = f(y) \Longrightarrow x = y$ or equivalently $x \neq y \Longrightarrow f(x) \neq f(y)$.*

*c. A function is **bijective** if it is injective and surjective.*

Informally, a function $f : X \longrightarrow Y$ is injective if there are no two elements of the domain that are mapped to the same element of the co-domain. This function takes distinct elements of X to different elements of Y.

In set theory every function can be a factorization of a mono and an epi function;i.e., for any $f : X \longrightarrow Y$ there are two functions $e : X \longrightarrow f(X)$ where $f(X) = \{f(x); x \in X\}$ and $m : f(X) \hookrightarrow Y$ such that $f = m \circ e$.

Definition 10.7. ***(Binary)relations***_on a set X are defined as subsets of a Cartesian product $X \times X$; i.e., a (binary)relation R is:_

$$R \subseteq X \times X.$$

If X is a set and R is a relation on X, $(x,y) \in R$ is denoted by xRy. For example, the relation "less than", denoted by $<$, is a relation on the set of real numbers;i.e., it is a subset of $\mathbb{R} \times \mathbb{R}$ defined by:

$$x < y := \{(x,y) \in \mathbb{R}^2 |\, x \text{ is less than } y\}$$

Definition 10.8. *A relation R on X is **reflexive** iff for each $x \in X, xRx$; **symmetric** iff for all $x, y \in X, xRy$ implies yRx ; **anti-symmetric** iff for all $x, y \in X$, xRy and yRx implies $x = y$ and **transitive** iff for all $x, y, z \in X$, xRy and yRz implies xRz. For example, the relation smaller than or equal on the set of real numbers, denoted by $\leqslant$ is reflexive, anti-symmetric and transitive.*

Definition 10.9. *A relation R on a set X is an **equivalence relation** if it is reflexive, symmetric and transitive.*

Definition 10.10. *Let $x \in X$ and $\simeq$ be an equivalence relation. Then, the equivalence class of x determined by $\simeq$ is denoted by $[x]$ and is defined as:*

$$[x] := \{y \in X; x \simeq y\}$$

Equivalence classes are either identical or disjoint, i.e. $[x] = [y]$ or $[x] \neq [y]$ and $[x] \cap [y] = \emptyset$.
The sets $[x]$, for $x \in X$, form a partition of X, i.e., they are disjoint sets whose union is X.

For example, Let $\mathbb{Z}$ be the set of integers. Let $x \simeq y := \exists k \in \mathbb{Z} : y = 3k + x$. Then the equivalence class of 2 is: $[2] = \{3k + 2 : k \in \mathbb{Z}\}$.

Relations, just like functions, can be composed and form a new relation. The composition of relations is defined as the following:

Definition 10.11. *Let $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ be binary relations. Then, the* ***composition*** *of R and S is a binary relation $S \circ R \subseteq X \times Z$, defined by:*

$$x(S \circ R)z := \{(x,z) \in X \times Z : \exists y \in Y \ (x,y) \in R \text{ and } (y,z) \in S.\}$$

10.1 Ordered structures, Lattices and Algebraic structures

In section V, we discuss the relation between order theory and topology. In particular, we show that the tolerance principle can be formulated as an order relation on a given set of objects in which each two adjacent objects share the same properties.

Definition 10.12. *a. A relation R on a set X is a* ***quasi-order*** *or* ***pre-order*** *if R is reflexive and transitive.*
b. A quasi-order relation R on a set X is a ***partial order*** *if it is anti-symmetric.*
c. A strict order on a set X is an irreflexive transitive relation.

For example, $\leq$ is a partial relation on $\mathbb{R}$. Usually, any partial order relation on any set is denoted by $\leq$, a partially ordered set(poset) is denoted by $(X, \leq)$ and a strict order by $x < y$.

Definition 10.13. *A set X is called* ***linearly ordered*** *by a partial order $\leq$ if for any $x, y \in X$ exactly one of $x < y$, $y < x$ or $x = y$ holds (Willard 1970, p.5; James 1999, p.12).*

Definition 10.14. *Let $(X, \leq)$ be a poset.*

a. If there exists $a \in X$ such that for all $b \in X, a \leq b$, then a is called the largest(greatest) element and is denoted by 1 or $\top$.

b. If there exists $a \in X$ such that for all $b \in X, b \leq a$, then a is called the least element and is denoted by 0 or $\perp$.

Definition 10.15. *Let $(X, \leq)$ be a poset.*

a. An element a_0 is a ***minimal element*** *if $\forall a \in X$, if $a \leq a_0$, then $a = a_0$.*

b. An element b_1 is a ***maximal element*** *if $\forall b \in X$, if $b_1 \leq b$, then $b = b_1$.*

There may be more than one minimal(maximal) elements but if the smallest(largest)element exists, then it is the unique minimal(maximal)element.

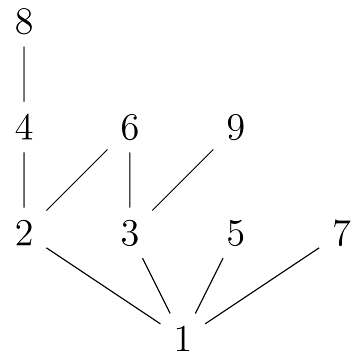
Definition 10.16. *(Willard 1970, p.6) The* ***least upper bound(lub) or supremum(sup)*** *of a subset Y of a partially ordered set X is the smallest element of the set of upper bounds $\{x \in X | \forall y \in Y,\ y \leq x\}$.*

Similarly, the ***greatest lower bound(glb) or infimum(inf)*** *of a subset Y of a partially ordered set X is the largest element of the set of all lower bounds $\{x \in X | \forall y \in Y,\ x \leq y\}$.*

The supremum of a set is denoted by $\bigvee$ and the infimum is denoted by $\bigwedge$. If $A = \{x, y\}$, then inf and sup are denoted by $x \wedge y$ and $x \vee y$ respectively.

The least upper bound and greatest lower bound do not always exist. If they do, they are unique but they might not belong to Y. Y is called *bounded above*(bounded below) if it has an upper bound(lower bound). (Davey and Priestley 2002, p.33)

Example 10.17. *1. Let X= 1,2, ...,9 and the partial order be the integer division . The Hasse diagram for this poset is given by:*



i. $A = \{5, 7\}$ *does not have an upper bound but it has a lower bound , 1.*
$B = \{1, 2, 3\}$ *has an inf, 1 and an upper bound 6. Note that 4 is not an upper bound of B because 4 does not follow 3.*
$C = \{1, 2, 8\}$ *has both sup and inf since 1 is the only element less than all members of C and 8 is the unique largest element of C.*

ii. Let $\mathbb{R}^-$ *be the set of negative real numbers. Then the least upper bound of it is 0.Yet 0 does not belong to the set. But it does not have the greatest lower bound.*

Another important notion is a directed set.

Definition 10.18. *A directed set of a partial order* $(X, \leq)$ *is a non-empty subset* $Y \subseteq X$ *such that* $\forall x, y \in Y\ \exists z \in Y\ x \leq z\ \&\ y \leq z$.

In domain theory, it is famous that directed sets are "going somewhere". It means that given two pieces of information one can grasp a further piece of information that contains the other ones. The set of all subsets of X ,$\mathscr{P}X$, is directed. Each pair of elements of S has an upper bound in S.

If a directed set S has a supremum then it is denoted by $\bigvee^{\uparrow} S$ (see (Winskel 2009, p. 4160)).

Definition 10.19. *Let $(X, \leq)$ be an ordered set, $x, y \in X$ and $A \subseteq X$, $a, b \in A$. Then,*

a. *A subset A of X is a chain(linearly ordered set or totally ordered set) if for any two elements of A, either $a \leq b$ or $b \leq a$. In this case we say that a and b are comparable.*

b. *The length of a chain is the number of elements of the chain.*

c. *The depth of $(X, \leq)$ is n if the largest chain in X is of length n.*

Example 10.20. *The typical example of directed set is given by the set of finite subsets of an arbitrary set. Abramsky and Jung (1994)*
Chains also are directed sets. The set of natural numbers $\mathbb{N}$ endowed with the partial relation less than $\leq$ is a chain.

Figure 19: The chain of natural numbers $\mathbb{N}$ endowed with $\leq$

Definition 10.21. *Goubault-Larrecq (2007)*
Let $(X, \leq)$ be a poset. We say that X has(satisfies) the ***Ascending Chain***

Condition(*ACC*) *iff every infinite ascending chain* $a_1 \leq a_2 \leq \cdots \leq a_k \leq \ldots$ *stabilizes, i.e., there is an integer* N *such that* $a_k = a_N$ *for all* $k \geqslant N$.

The dual of ACC is called **Descending Chain Condition**(DCC):

Definition 10.22. *Let* $(X, \leq)$ *be a poset. We say that* X *has(satisfies) the descending chain condition(DCC) iff every infinite descending chain* $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_k \geqslant \ldots$ *stabilizes, i.e., there is an integer* N *such that* $a_k = a_N$ *for all* $k \geqslant N$.

If X satisfies both ACC and DCC then it is of finite chain condition (FCC).

Definition 10.23. *(Picado and Pultr 2011, p.22) A poset* $(X, \leq)$ *is* ***Noetherian*** *iff the order satisfies the ACC condition. That is, there is no strictly increasing sequence*

$$a_1 < a_2 < \cdots < a_k < \ldots.$$

According to the definition, a Noetherian poset satisfies ACC. In a Noetherian poset each directed subset has a greatest element.(Erné et al. (2007))
The dual of a Noetherian space is called Artinian. So, Artinian spaces satisfy DCC.

Definition 10.24. *(Abramsky and Jung (1994)) A poset* D *in which every directed subset has a supremum is called a directed-complete partial order, or dcpo for short.*

In figure 19 the set $\{1, 2, 3, \ldots\}$ is directed but $(\mathbb{N}, \leq)$ lacks supremum and therefore is not a dcpo.

Continuous functions between dcpos are defined as the following:

Definition 10.25. *Let L, L' be two dcpos, and $f : L \longrightarrow L'$. The function f is continuous iff for all directed subsets $X \subseteq L$,*

1. *There exists $\bigvee f(X)$.*
2. *$f(\bigvee X) = \bigvee(f(X))$.*

Intuitively, continuity of f means that f does not take the elements of subsets of a dcpo with a supremum to a surprising place because the range of the function also has a supremum that is exactly $f(\bigvee X)$. So, the supremum is preserved.

Definition 10.26. *A partially ordered set $(L, \leq)$ is called a **lattice** if any two elements in the set have a greatest lower bound and a least upper bound.*

Definition 10.27. *Let $(X, \leq)$ be a quasi-order, $A \subseteq X$ and $x \in X$. Then, define **up-set**(upper set, increasing set) $\uparrow A$ as:*

$$\uparrow A := \{x \in X | \quad (\exists a \in A) \quad a \leq x\}.$$

The set A is up-set if $A = \uparrow A$.

If A is an up-set, then for any member of A, a, if $a \leq b$, then b is in A. The dual of up-sets are called down-sets:

Definition 10.28. *Let $(X, \leq)$ be a quasi-order, $A \subseteq X$ and $x \in X$. Then define **down-set**(lower set, decreasing set) $\downarrow A$ as:*

$$\downarrow A := \{x \in X | \quad (\exists a \in A) \quad x \leq a\}.$$

The set A is down-set if $A = \downarrow A$.

If A is a down-set, then for any member of A, a, if $x \leq a$, then x is in A.

Example 10.29. *Let $(\mathbb{Z}, \leq)$ be a poset and $\mathbb{Z}^+$ and $\mathbb{Z}^-$ be sets of positive and negative integers respectively.*

$A = \mathbb{Z}^+$ is an up-set and $A = \mathbb{Z}^-$ is a down-set.

$A = \{2k|k > 0\}$ is not an up-set because $3 \in \uparrow A$ but it is not in A. Likewise, $B = \{2k|\, k \leq -1\}$ is not a down-set.

A very special example of down-sets and up-sets are filters and ideals. In particular, directed down-sets(up-sets)are ideals and filters respectively.

Definition 10.30. ***a.*** *Let L be a lattice and $I \subseteq L$. I is called an **ideal** if:*

(i) If $a, b \in I$, then $a \vee b \in I$.

(ii) I is a lower set[down-set]; i.e. If $a \in I$ and $b \leq a$, then $b \in I$.

b. *Let L be a lattice and a non-empty subset $F \subseteq L$. F is called a **filter** if:*

(i) If $a, b \in F$, then $a \wedge b \in F$.

(ii) F is a upset; i.e. If $a \in L, b \in F$ and $b \leq a$, then $a \in F$.

$\downarrow\{x\} := \{a \in X|a \leq x\}$ is an ideal, called a principal ideal. As usual, it is denoted by $\downarrow$x.

$\uparrow\{x\} := \{a \in X| \quad x \leq a\}$ is a filter, called a principal filter. As usual, it is denoted by $\uparrow$x.

Definition 10.31. *A lattice L is **complete** if any of its subsets have a greatest lower bound and a least upper bound in the set. A lattice L is **distributive** iff for all $a, b, c \in L$,*

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

Definition 10.32. *A lattice L is **bounded** if it has both a least element (0) and a greatest element(1). A bounded lattice usually is denoted by $(L, \vee, \wedge, 0, 1)$.*

Definition 10.33. *A distributive bounded lattice is* ***Boolean*** *iff for any a there is $a^{'}$ such that:*

$1 - a \wedge a^{'} = 0$

$2 - a \vee a^{'} = 1$

The prime example of a Boolean lattice is $(\mathscr{P}X, \subseteq)$.

The structure of a lattice can be characterized in two mathematically equivalent ways: as an ordered set $(L, \leq)$ or as an algebraic structure $(L, \wedge, \vee)$ where $\wedge : L \times L \longrightarrow L$ and $\vee : L \times L \longrightarrow L$ satisfy the following laws:(Davey and Priestley 2002, p.39)

1. $(a \vee b) \vee c = a \vee (b \vee c)$ (associativity)

2. $a \vee b = b \vee a$ (commutativity)

3.$a \vee a = a$ (idempotency)

4. $a \vee (a \wedge b) = a$ (absorption)

One can see that the same will hold for the logical operator $\wedge$.

1. $(a \wedge b) \wedge c = a \wedge (b \wedge c)$

2. $a \wedge b = b \wedge a$

3.$a \wedge a = a$

4. $a \wedge (a \vee b) = a$

The following lemma relates the relation $\leq$ with logical operators $\vee$ and $\wedge$.

Lemma 10.34. *Let L be a lattice and $a, b \in L$. Then the following items are equivalent:*

(i) $a \leq b$

(ii) $a \vee b = b$

(iii) $a \wedge b = a$

Boolean algebra is structurally isomorphic to the standard semantics of classical logic, where $\neg$, $\wedge$ and $\vee$ correspond to set theoretical complement, intersection and union respectively.
Heyting algebra that is structurally isomorphic to intuitionistic logic is another example of a lattice. Co-Heyting algebra is another lattice that corresponds to the logic LP that Graham Priest proposed as an appropriate logic for vagueness.

Definition 10.35. *A bounded distributive lattice H is said to be a* ***Heyting algebra*** *if, for each pair of elements (y, z), there exists an element* $(y \longrightarrow z)$ *such that* $x \leq (y \longrightarrow z)$ *iff* $x \wedge y \leq z$.

The dual of it is called Co-Heyting.(Johnstone 1982, p.8)

Definition 10.36. *A* ***co-Heyting*** *algebra is a bounded distributive lattice with a "subtraction"* $\backslash : L \times L \longrightarrow L$ *satisfying the following property:*
$\forall x, y, z \in L \quad x \backslash y \leq z$ *iff* $x \leq y \vee z$

Definition 10.37. *Reyes and Zolfaghari (1996) A bounded distributive lattice that is both Heyting and Co-Heyting is called bi-Heyting*

Negations in Heyting and co-Heyting algebras are defined by the above operators as:

Definition 10.38. *Let 0 and 1 be bottom and top of Heyting and co-Heyting lattices L and L' respectively.*

$$\neg x = x \longrightarrow 0$$

$$\sim x = x \backslash 0$$

The first negation $\neg$ is the intuitionistic negation or pseudo-complement. The second one, called supplementation, is used in dialethic logics such as Priest's LP logic. We focus mostly on Heyting algebras, since the intuitionistic negation will be used in the topological view of vagueness. We will show in detail that not only negation is different from the one defined classically, but also the disjunction is defined differently. As is well-known, in intuitionistic logic double negation elimination fails because generally $\neg\neg x \neq x$.

By the following proposition we lay out some properties of the Heyting algebra as far as we need Reyes and Zolfaghari (1996).

Proposition 10.39. *Let $(H, \leq)$ be a complete Heyting algebra. Then H has the following properties:*

(i) $x \leq (y \longrightarrow z)$ *iff* $x \wedge y \leq z$

(ii) $x \leq \neg\neg x$

(iii) $\neg\neg\neg x = \neg x$

(iv) $\neg 0 = 1$

(v) $\neg 1 = 0$

(vi) $\neg(x \vee y) = \neg x \wedge \neg y$

(vii) $\neg\neg(x \wedge \neg x) = 1$

$\neg x$ in Heyting algebra is the largest element disjoint from x.

All Boolean algebras are Heyting but not vice versa.

Lemma 10.40. *A Heyting algebra $(H, \leq)$ is a Boolean algebra iff:*

i. $x \vee \neg x = 1$

ii. $\neg\neg x = x$

For the proof see (James 1999, p.9).

In Part V we relate these lattices with the topological notions.

11 Appendix B: Topological vs classical and intuitionistic semantics

Let X be a set and $(X, \mathcal{OX})$ be a topological space, L a propositional language along with sentential connectives '$\wedge$','$\vee$','$\wedge$', '$\neg$' and '$\rightarrow$'.

The following table shows the interpretation functions ν_c, ν_i and ν_{ro} for the standard classical, intutionistic logic and the logic Rumfitt proposes based on the set of regular open sets respectively.

The first column from the left, is the way the interpretation $\nu : L \longrightarrow \mathscr{P}X$ maps each well-formed formula of the language L to the power set of X.

In the second and third columns, the interpretations map each well-formed formula of the language L to a topological space X. In the second column the right side of the equations are all open sets whereas the ones in the last column are regular open. We have seen that if two sets are regular open then their union is open but it might be the case that their union is not regular open. So, the interpretation function maps the disjunction of two sets to the interior closure of their union.

In topology $(\mathcal{O}X, \vee, \wedge)$ forms a Heyting algebra, $(\mathcal{C}X, \vee, \wedge)$ forms a Co-Heyting algebra and $(\mathcal{O}_{reg}X, \vee, \wedge)$ forms a Boolean algebra. Not all Heyting or Co-Heyting algebras are Boolean.

We showed that polar topology is Alexandroff. If (X, OX) is Alexandroff topological space, then $(\mathcal{O}X, \cap, \cup)$ is bi-Heyting.

Semantics		
Standard Classical logic $\nu_c : L \longrightarrow \mathscr{P}X$	**Intuitionistic logic** $\nu_i : L \longrightarrow \mathcal{O}X$	**Regular open logic** $\nu_{ro} : L \longrightarrow \mathcal{O}_{reg}X$
$\nu_c(A \wedge B) = \nu_c(A) \cap \nu_c(B)$	$\nu_i(A \wedge B) = \nu_i(A) \cap \nu_i(B)$	$\nu_{ro}(A \wedge B) = \nu_{ro}(A) \cap \nu_{ro}(B)$
$\nu_c(A \vee B) = \nu_c(A) \cup \nu_c(B)$	$\nu_i(A \vee B) = \nu_i(A) \cup \nu_i(B)$	$\nu_{ro}(A \vee B) = IntCl(\nu_{ro}(A) \cup \nu_{ro}(B))$
$\nu_c(A \to B) = \neg(\nu_c(A)) \cup \nu_c(B)$	$\nu_i(A \to B) = Int(\nu_i(B) \cup \complement\nu_i(A)$	$\nu_{ro}(A \to B) = IntCl(\nu_{ro}(B) \cup Int(\complement(\nu_{ro}(A))$
$\nu_c(\neg A) = \complement(\nu_c(A))$	$\nu_i(\neg A) = Int(\complement(\nu_i(A)))$	$\nu_{ro}(\neg A) = Int(\complement(\nu_{ro}(A)))$

Part VIII

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