PHD THESIS

Resource Theories of Quantum Dynamics

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Abstract

As one of our most successful theories, quantum theory has greatly strengthened our understanding of nature and significantly advanced technologies. Specifically, quantum effects provide advantages in a broad range of information processing tasks. The exploration of the interplay of quantum phenomena and information theory is the interdisciplinary field dubbed quantum information theory. Since its inception, quantum information theory has revolutionised our understanding of quantum phenomena and shown that various quantum properties act as resources for performing useful tasks, such as computation, information transmission, energy extraction, cryptography, metrology, and information storage. These findings set the stage for the theoretical approaches termed quantum resource theories, which allow in a mathematically rigorous fashion to describe a wide range of quantum phenomena. Quantum information theory identifies several intriguing quantum properties, and quantum resource theories provide the means to construct the 'rulers' to measure these properties operationally. However, despite their great success in describing 'static' quantum phenomena, it was unknown whether resource theories would be as powerful in their descriptions of physical systems that 'evolve in time', namely, when we consider 'dynamical' quantum properties. Recent results have allowed us to extend quantum resource theories to the dynamical regime, which has already revealed novel links between quantum communication, quantum memory, and quantum thermodynamics. This thesis aims at substantially developing this newly-established, interdisciplinary research direction that is called *dynamical resource theories*.

The main contributions of the thesis are divided into three parts. The first part (Chapter 3) focuses on improving our understanding of dynamical resource theories' general structures. Adopting the resource-theoretical approaches, we formulate 'the ability of quantum dynamics to preserve a physical property' as a dynamical resource. The resulting framework is called *resource preservability theories*. We systematically study their theoretical structures and further explore their applications to communication and thermodynamics. More specifically, resource preservability theories enable us to understand the connections between (a) the ability of a quantum dynamics to keep systems out of thermal equilibrium, (b) the smallest heat bath size needed to thermalise all outputs of a quantum dynamics, and (c) the ability of a quantum dynamics to transmit classical information. This draws one of the first quantitative links between quantum communication and thermodynamics.

In the second part (Chapter 4), we upgrade our discussion from a single quantum

dynamics to *a collection of* local quantum dynamics. In this regime, an important question is whether the given local dynamics can be realised *simultaneously*; namely, as the *marginals* of a single, global dynamics. To systematically address this question, we introduce the *channel marginal problems* (CMPs), which are dynamical generalisations of the well-known state marginal problems. Using the resource-theoretical approach, we analyze CMPs' general solutions via semi-definite programming, which helps us derive a witness form and operational interpretations of CMPs. Our findings also show that fundamental principles such as entanglement monogamy and no-cloning theorem play an essential role in CMPs.

Finally, in the last part (Chapter 5), we consider a specific question that is behind the structures of dynamical resource theories and channel marginal problems: We ask whether globally distributed quantum entanglement can survive locally performed thermalisation when shared randomness is the only allowed resource to assist the process. Such a dynamics, whenever it exists, is called *entanglement preserving local thermalisation* (EPLT). We show that EPLTs exist for *every* nonzero local temperatures and non-degenerate finite-energy local Hamiltonians. Our findings illustrate the generality of EPLTs. Moreover, we discuss the physical mechanism behind EPLTs, and our calculation suggests that EPLTs' existence relies on a 'speed-up effect' of thermalisation that happens locally.

In summary, in this thesis we contribute to the field of dynamical resource theories by introducing general frameworks to describe quantum resource preservation and compatibility of local quantum dynamics. Our general results have implications across quantum physics, quantum communication, thermodynamics of quantum systems, and causal structures.

Resumen

Siendo una de nuestras teorías con más éxito, la teoría cuántica ha fortalecido en gran medida nuestra comprensión de la naturaleza y las tecnologías significativamente avanzadas. En concreto, los efectos cuánticos traen consigo ventajas en una amplia gama de tareas de procesamiento de información. La exploración de la interacción de los fenómenos cuánticos y la teoría de la información es el campo interdisciplinario denominado teoría cuántica de la información. Desde sus inicios, la teoría de la información cuántica ha revolucionado nuestra comprensión de los fenómenos cuánticos y ha demostrado que varias propiedades cuánticas actúan como recursos para realizar tareas útiles, como computación, transmisión de información, extracción de energía, criptografía, metrología y almacenamiento de información. Estos hallazgos preparan el escenario para los enfoques teóricos denominados teorías cuánticas de recursos, que permiten describir de manera matemáticamente rigurosa una amplia gama de fenómenos cuánticos. La teoría de la información cuántica identifica varias propiedades cuánticas intrigantes, y las teorías de recursos cuánticos proporcionan los medios para construir las ”reglas” para medir estas propiedades. Sin embargo, a pesar de su gran éxito en la descripción de fenómenos cuánticos ”estáticos”, se desconocía si las teorías de recursos serían tan poderosas en sus descripciones de sistemas físicos que ”evolucionan en el tiempo”, es decir, cuando consideramos las propiedades cuánticas ”dinámicas”. Los resultados recientes nos han permitido extender las teorías cuánticas de recursos al régimen dinámico, lo que ya ha revelado novedosos vínculos entre la comunicación cuántica, la memoria cuántica y la termodinámica cuántica. Esta tesis tiene como objetivo desarrollar sustancialmente esta dirección de investigación interdisciplinaria recientemente establecida que se llama em teorías dinámicas de recursos.

Las principales contribuciones de la tesis se dividen en tres partes. La primera parte (Capítulo 3) se enfoca en mejorar nuestra comprensión de las estructuras generales de las teorías dinámicas de recursos. Adoptando los enfoques teóricos de los recursos, formulamos ”la capacidad de la dinámica cuántica para preservar una propiedad física” como un recurso dinámico. El marco resultante se denomina *teorías de conservación de recursos*. Estudiamos sistemáticamente sus estructuras teóricas y exploramos más a fondo sus aplicaciones en la comunicación y la termodinámica. Más específicamente, las teorías de conservación de recursos nos permiten comprender las conexiones entre (a) la capacidad de una dinámica cuántica para mantener los sistemas fuera del equilibrio térmico, (b) el tamaño más pequeño del baño de calor que es necesario para termalizar todas las salidas de la dinámica cuántica y (c) la capacidad de la dinámica

cuántica para transmitir información clásica. Esto dibuja uno de los primeros vínculos cuantitativos entre la comunicación cuántica y la termodinámica.

En la segunda parte (Capítulo 4), generalizamos nuestra discusión de una única dinámica cuántica a *una colección de* dinámicas cuánticas locales. En este régimen, una pregunta importante es si la dinámica local dada puede realizarse *simultáneamente*; es decir, como los *marginales* de una única dinámica global. Para abordar sistemáticamente esta pregunta, presentamos los *problemas marginales de canal* (CMP), que son generalizaciones dinámicas de los conocidos problemas marginales de estado. Utilizando el enfoque teórico de recursos, analizamos las soluciones generales de los CMP a través de lo que se conoce como programación semidefinida, lo que nos ayuda a derivar una forma de comprobar los CMP y nos da interpretaciones operativas de los mismos. Nuestros hallazgos también muestran que los principios fundamentales como la monogamia del entrelazamiento y el teorema de no-clonación juegan un papel esencial en los CMP.

Finalmente, en la última parte (Capítulo 5), consideramos una pregunta específica que está detrás de las estructuras de las teorías de recursos dinámicos y los problemas marginales del canal: nos preguntamos si el entrelazamiento cuántico distribuido globalmente puede sobrevivir a la termalización realizada localmente cuando la aleatoriedad compartida es el único recurso permitido para ayudar en el proceso. Esa dinámica, siempre que exista, se denomina *entrelazamiento que preserva la termalización local* (EPLT). Demostramos que los EPLT existen para *todas las* temperaturas locales distintas de cero y hamiltonianos locales de energía finita no degenerados. Nuestros hallazgos ilustran la generalidad de los EPLT. Además, discutimos el mecanismo físico detrás de los EPLT, y nuestro cálculo sugiere que la existencia de los EPLT se basa en un ”efecto de aceleración” de la termalización que ocurre localmente.

En resumen, en esta tesis contribuimos al campo de las teorías de recursos dinámicos mediante la introducción de marcos generales para describir la preservación de recursos cuánticos y la compatibilidad de la dinámica cuántica local. Nuestros resultados generales tienen implicaciones en la física cuántica, la comunicación cuántica, la termodinámica de los sistemas cuánticos y las estructuras causales.

Resum

Essent una de les nostres teories amb més èxit, la teoria quàntica ha enfortit molt la nostra comprensió de la natura i les tecnologies significativament avançades. Concretament, els efectes quàntics proporcionen avantatges en una àmplia gamma de tasques de processament de la informació. L'exploració de la interacció dels fenòmens quàntics i la teoria de la informació és el camp interdisciplinari anomenat teoria de la informació quàntica. Des dels seus inicis, la teoria de la informació quàntica ha revolucionat la nostra comprensió dels fenòmens quàntics i ha demostrat que diverses propietats quàntiques actuen com a recursos per realitzar tasques útils, com ara la computació, la transmissió d'informació, l'extracció d'energia, la criptografia, la metrologia i l'emmagatzematge d'informació. Aquests descobriments han establert l'escenari per als enfocaments teòrics que s'anomenen teories de recursos quàntics, i que permeten descriure de manera matemàticament rigorosa una àmplia gamma de fenòmens quàntics. La teoria de la informació quàntica identifica diverses propietats quàntiques intrigants, i les teories dels recursos quàntics proporcionen els mitjans per construir les "normes" per mesurar aquestes propietats de manera operativa. Tanmateix, malgrat el seu gran èxit a l'hora de descriure fenòmens quàntics "estàtics", es desconeixia si les teories dels recursos serien tan poderoses en les seves descripcions de sistemes físics que "evolucionen en el temps", és a dir, quan considerem les propietats quàntiques "dinàmiques". Els resultats recents ens han permès estendre les teories dels recursos quàntics al règim dinàmic, que ja ha revelat vincles novedosos entre la comunicació quàntica, la memòria quàntica i la termodinàmica quàntica. Aquesta tesi té com a objectiu desenvolupar substancialment aquesta direcció de recerca interdisciplinària recentment establerta que s'anomena *teories de recursos dinàmics*.

Les principals contribucions d'aquesta tesi es divideixen en tres parts. La primera part (Capítol 3) se centra en millorar la nostra comprensió de les estructures generals de les teories de recursos dinàmics. Adoptant els enfocaments teòrics dels recursos, formulem "la capacitat de la dinàmica quàntica de preservar una propietat física" com a recurs dinàmic. El marc resultant s'anomena *teories de preservabilitat de recursos*. Estudiem sistemàticament les seves estructures teòriques i explorem més les seves aplicacions a la comunicació i la termodinàmica. Més concretament, les teories de la preservació dels recursos ens permeten entendre les connexions entre (a) la capacitat d'una dinàmica quàntica per mantenir els sistemes fora de l'equilibri tèrmic, (b) la mida més petita del bany de calor necessària per termalitzar totes les sortides d'una dinàmica quàntica i (c) la capacitat d'una dinàmica quàntica per transmetre informació clàssica.

Això dibuixa un dels primers vincles quantitatius entre la comunicació quàntica i la termodinàmica.

A la segona part (Capítol 4), anem més enllà de la nostra discussió d'una única dinàmica quàntica, considerant *una col·lecció de* dinàmiques quàntiques locals. En aquest règim, una qüestió important és si la dinàmica local donada es pot realitzar *simultàniament*; és a dir, com els *marginals* d'una única dinàmica global. Per abordar aquesta qüestió sistemàticament, introduïm els *problemes marginals de canal* (CMP), que són generalitzacions dinàmiques dels coneguts problemes marginals d'estat. Fent ús de l'enfocament teòric dels recursos, analitzem les solucions generals dels CMP mitjançant el que es coneix com a programació semidefinida, que ens ajuda a obtenir una forma de corroborar els CMP i les seves interpretacions operatives. Els nostres resultats també mostren que principis fonamentals com la monogàmia d'entrellaçament i el teorema de no-clonació tenen un paper essencial en els CMP.

Finalment, a l'última part (Capítol 5), considerem una qüestió específica que hi ha darrere de les estructures de les teories dels recursos dinàmics i dels problemes marginals del canal: ens preguntem si l'entrellaçament quàntic distribuït globalment pot sobreviure a la termalització realitzada localment quan l'aleatorietat compartida és l'únic recurs permès per ajudar el procés. Aquesta dinàmica, sempre i quan existeixi, s'anomena *entrellaçament que preserva la termalització local* (EPLT). Mostrem que els EPLT existeixen per a *qualsevol* temperatura local diferent de zero i hamiltonians locals d'energia finita no degenerada. Els nostres resultats il·lustren la generalitat dels EPLT. A més, discutim el mecanisme físic darrere dels EPLT, i el nostre càlcul suggereix que l'existència dels EPLT depèn d'un "efecte d'acceleració" de la termalització que es produeix localment.

En resum, en aquesta tesi contribuïm al camp de les teories de recursos dinàmics introduint marcs generals per descriure la preservació dels recursos quàntics i la compatibilitat de la dinàmica quàntica local. Els nostres resultats generals tenen implicacions en la física quàntica, la comunicació quàntica, la termodinàmica dels sistemes quàntics i les estructures causals.

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Do or die, you'll never make me
Because the world will never take my heart
Go and try, you'll never break me
We want it all, we wanna play this part
I won't explain or say I'm sorry
I'm unashamed, I'm gonna show my scars
Give a cheer for all the broken
Listen here, because it's who we are

— from *Welcome to the Black Parade*, My Chemical Romance

List of Publications

Publications forming part of the thesis

- *Entanglement preserving local thermalization*
  C.-Y. Hsieh, M. Lostaglio, and A. Acín
  Physical Review Research **2**, 013379 (2020) – Ref. [1]
- *Resource preservability*
  C.-Y. Hsieh
  Quantum **4**, 244 (2020) – Ref. [2]
- *Communication, dynamical resource theory, and thermodynamics*
  C.-Y. Hsieh
  PRX Quantum **2**, 020318 (2021) – Ref. [3]
- *Quantum channel marginal problem*
  C.-Y. Hsieh, M. Lostaglio, and A. Acín
  Physical Review Research **4**, 013249 (2022) – Ref. [4]

Publications/Preprints relevant to the thesis, but not forming part of it

- *Gaussian thermal operations and the limits of algorithmic cooling*
  A. Serafini, M. Lostaglio, S. Longden, U. Shackerley-Bennett,
  C.-Y. Hsieh, and G. Adesso
  Physical Review Letter **124**, 010602 (2020) – Ref. [5]
- *Complete classification of steerability under local filters and its relation with measurement incompatibility*
  H.-Y. Ku, C.-Y. Hsieh, S.-L. Chen, Y.-N. Chen, and C. Budroni
  Nature Communications **13**, 4973 (2022) – Ref. [6]
- *Entanglement transitivity problems*
  G. N. M. Tabia, K.-S. Chen, C.-Y. Hsieh, Y.-C. Yin, and Y.-C. Liang
  npj Quantum Information **8**, 98 (2022) – Ref. [7]
- *Resource marginal problems*
  C.-Y. Hsieh, G. N. M. Tabia, Y.-C. Yin, and Y.-C. Liang
  arXiv:2202.03523 – Ref. [8]
- *Thermodynamic criterion of transmitting classical information*
  C.-Y. Hsieh
  arXiv:2201.12110 – Ref. [9]

Contents

Chapter 1

Introduction

As one of our most experimentally well-tested theories, *quantum theory* has greatly deepened our understanding of nature and significantly advanced technology. It reveals intriguing phenomena that are not only fundamentally important, but they can also be understood as *resources* for performing useful tasks, such as computation, cryptography, and information processing. For instance, quantum entanglement enables one to teleport unknown quantum systems deterministically and faithfully. The connections between information science and quantum theory have revolutionised our understanding of the latter, opening the door to *quantum information theory*.

One central goal of quantum information theory is to identify different notions of 'resources' in various operational tasks. Since the development of quantum entanglement theory, it was recognised that many quantum phenomena are able to serve as different types of resources. Consequently, it is of great importance to search for a general, systematic umbrella covering them all under a single framework. These efforts led to the inception of theoretical approaches dubbed *quantum resource theories*, or simply *resource theories*, which have shown their ability in providing a quantitative understanding of properties of physical phenomena that do not fit the usual notion of quantum observable. In other words, quantum information theory and resource theories provide the 'rulers' enabling researchers to measure properties such as entanglement, which, differently from energy, angular momentum, etc., is not associated to any natural observable.

The resource-theoretic approach [10] has succeeded in quantitatively describing a broad range of *static* physical phenomena, such as, but not limited to, entanglement [11], coherence [12], athermality [13], nonlocality [14], and steering [15, 16]. Its generality and flexibility make it a commonly-used, powerful underpinning of the study of quantum information theory and related areas. Nevertheless, despite its success in characterising *static* quantum phenomena, it was unknown whether it would be as powerful when things start to *change in time*, namely, when we consider *dynamical* features of quantum systems, which play vital roles in the study of, e.g., quantum communication, quantum memory, and open quantum system. Let us make a concrete example. The resource-theoretic approach has provided us with an analytical way to quantify and compare entanglement contents of different quantum systems. Nevertheless, it was unclear how the same approach applies to understanding the ability of a quantum dynamics to maintain entanglement, a property which is indispensable in developing the theory of quantum memories [17]. Hence, in 2019, researchers started to extend the applicability of resource theories to the dynamical regime, initiating the active and thriving direction called *dynamical resource theories* [18, 19]. This newly-established direction opens the door for systematic, analytical understandings of dynamical quantum phenomena. For example, it offers a natural platform to study quantum communication [20, 21, 22], quantum memories [17, 23], and dynamical generalisations of various static physical properties. The recent progress suggests that significant novel insights can be obtained by building dynamical resource theories further, just like what have been gained by studying resource theories of static properties.

With the above motivations, this thesis aims at developing dynamical resource theories, focusing on two major objectives: To generalise the theoretical structure of dynamical resource theories, and to investigate their multidisciplinary applications.

1.1 Motivation & Contributions

1.1.1 Resource Preservability Theories and Their Applications

In 2019, Liu-Winter [18] and Liu-Yuan [19] gave the first systematic generalisation of static resource theories to the dynamical regime, initiating the direction dubbed ‘dynamical resource theory’ in the quantum information theory community. Despite the generality of their approaches, it was still unknown how to use a resource-theoretic approach to quantitatively understand the ability of a quantum dynamics to preserve static physical properties (e.g., entanglement, coherence, athermality). An appropriate answer can further provide insights for, e.g., communication and thermodynamics. This thus motivated us to initiate this research project.

Main Results [2, 3]

We provide the first general and quantitative resource theory of *channels’* ability to maintain, or say to *preserve*, a given static physical property. The framework is dubbed *resource preservability* theory. We systematically investigate its resource-theoretic structures and various quantifiers. Furthermore, we show that specific quantifiers have different interpretations in thermodynamics and classical communication.

As applications, we study the connection between resource preservability theory and classical communication theory. We study classical communication via channels unable to generate a given static resource. In this setting, a commonly used measure of classical communication, the so-called one-shot classical capacity, is shown to be upper bounded by resource preservability quantifiers. As an implication to thermodynamics, the smallest bath needed to thermalise all outputs of a Gibbs-preserving coherence-annihilating channel provides an upper bound on this channel’s one-shot classical capacity. In this sense, a connection between thermodynamics and classical information transmission is established.

1.1.2 Quantum Channel Marginal Problem

Apart from resources carried by a *single* state, there are certain resources that are possessed by a *collection* of *local* states of a larger *global* system. When the underlying objects are such collections, it is crucial to know whether those local states in a given collection can exist *simultaneously*; namely, being the *marginals* of a single, global state. If they can, then they are called *compatible*; otherwise, they are *incompatible*. Incompatibility turns out to be a fundamental property in quantum theory, and the study of its manifestation at the level of states goes under the name of *state marginal problem* (SMP) [24, 25]. It is thus both interesting and important to know how to formally extend the SMP to the dynamical regime.

Main Results [4]

We study the dynamical generalisation of the SMP, entitled *channel marginal problem* (CMP): It asks, whether a given set of local dynamics is compatible with a global dynamics. We provide a complete characterisation of the CMP as well as operational interpretation in a state discrimination task. Furthermore, our findings identify a gap between classical and quantum channel marginal problems and show that the CMP is irreducible to the SMPs.

1.1.3 Entanglement Preserving Local Thermalisation

Thermalisation is the physical process that forces a system to evolve toward thermal equilibrium with an environment. Being a many-to-one mapping, thermalisation completely erases the information originally carried by the physical system. On the other hand, entanglement, as an iconic quantum-informational resource, is known to be fragile and hard to be maintained. Consequently, from a foundational perspective, it is interesting to understand how thermalisation processes affect entanglement. This motivates us to ask: *Can entanglement survive after thermalisation?* An appropriate, quantitative answer to this question can tell us the fundamental relation between entanglement preservability and the ability to locally thermalise a global system, giving potential insights to the studies of dynamical resource theory as well as thermalisation in many-body systems.

Main Results [1]

We answer this question in the positive by showing that for every positive temperature and non-degenerate local Hamiltonians, there exists a local operations plus pre-shared randomness channel that can (1) locally thermalise arbitrary global input to the desired local thermal states, and (2) preserve entanglement for certain global entangled input states. We call such channels *entanglement preserving local thermalisations*. In other words, locally performed thermalisation processes do not necessarily imply that the resulting global state is not entangled.

1.2 Outline of The Thesis

We start with Chapter 2, including preliminary notions necessary to understand the content of this thesis. In Chapter 3, we detail the resource preservability theories [2] and their applications to communication and thermodynamics [3]. In Chapter 4 we introduce the channel marginal problems [4]. In Chapter 5, we detail the results related to entanglement preserving local thermalisation. Finally, we conclude the thesis in Chapter 6, which also includes outlook and discussions of open questions. Throughout this thesis, main results are marked in blue.

Chapter 2

Preliminary Notions

2.1 Quantum Theory

We quickly go through some basic notions of quantum theory. A complete introduction to quantum theory and its role in quantum information theory can be found in textbooks such as Refs. [26, 27].

Quantum States

The first postulate of quantum theory says that every physical system S has a corresponding Hilbert space $\mathcal{H}_S$. In this thesis, we always consider finite-dimensional Hilbert spaces $\mathcal{H}_S \simeq \mathbb{C}^d$ for some $d \in \mathbb{N}$. The *physical states* of S can be described by normalised vectors in this Hilbert space: $|\psi\rangle \in \mathcal{H}_S$ and $\langle\psi|\psi\rangle = 1$. Here, we have used the conventional Dirac bra-ket notation; namely, $\langle\psi| := |\psi\rangle^\dagger = |\psi\rangle^{*,t}$ and $\langle\psi|\psi\rangle := \mathrm{tr}(|\psi\rangle\langle\psi|)$ is the inner product (see, e.g., Refs. [26, 27] for details). $|\psi\rangle$ called a *pure state*, which contains the full knowledge of a system.

Practically, it is not always easy to have access to the whole system. One postulate asserts that a composite system, e.g., AB, has its Hilbert space as the *tensor product* of A and B's Hilbert space: $\mathcal{H}_A \otimes \mathcal{H}_B$. In such a composite system, it is also vital to know how to describe *part of it*. This requires the mathematical description of "ignoring part of the system," which is achieved by the *partial trace* operation. More precisely, in a composite system AB, the local behavior of a global pure state $|\psi_{AB}\rangle$ in system A is given by

$$\mathrm{tr}_B(|\psi_{AB}\rangle\langle\psi_{AB}|) := \sum_{n=1}^{d_B}(\mathbb{I}_A \otimes \langle b_n|)|\psi_{AB}\rangle\langle\psi_{AB}|(\mathbb{I}_A \otimes |b_n\rangle), \tag{2.1}$$

where $\{|b_n\rangle\}_{n=1}^{d_B}$ is any orthonormal basis of system B. An important physical message is that *the local behavior of a global pure state is not necessarily pure anymore.* In general, it is a positive semidefinite operators $\rho : \mathcal{H}_S \to \mathcal{H}_S$ with unit trace $\mathrm{tr}(\rho) = 1$.

These operators are called *mixed states*, or simply *states*. By the spectral decomposition theorem [26], one can diagonalise a state ρ as $\rho = \sum_{i=0}^{d-1} a_i|\phi_i\rangle\langle\phi_i|$. From here one can see that the positivity condition, $\rho \geq 0$, means that the eigenvalues a_i's cannot be negative; the unit-trace condition, $\mathrm{tr}(\rho) = 1$, means the normalisation condition $\sum_i a_i = 1$ and there is no missing information of the system. Physically, this a classical probabilistic mixture of pure states $|\phi_i\rangle$'s with probability weights a_i's, which justifies the name 'mixed state'.

Throughout this thesis, the notation STATE denotes the set of all states, and STATE_S with the subscript is the set of all states of the system S:

$$\mathrm{STATE} := \{\rho \mid \rho \geq 0,\ \mathrm{tr}(\rho) = 1\}\,; \tag{2.2}$$

$$\mathrm{STATE}_S := \{\rho \mid \rho : \mathcal{H}_S \to \mathcal{H}_S\,,\ \rho \in \mathrm{STATE}\}\,. \tag{2.3}$$

Also, we constantly use subscripts to denote the corresponding system. For instance, by writing ρ_S we mean a quantum state in STATE_S. The same convention also applies to other operators; for example, $\mathbb{I}_S$ means the identity operator in $\mathcal{H}_S$. Finally, we always use the notation d_S to denote the dimension of the Hilbert space $\mathcal{H}_S$.

Quantum Measurements

Knowing that the physical state of a system is described by a unit-trace positive semidefinite operator ρ, quantum theory further prescribes that every physical observable, such as energy and momentum, can be described by a Hermitian operator, e.g., $H = \sum_{i=1}^{N} H_i|i\rangle\langle i|$ with $\langle i|j\rangle = \delta_{ij}$. Given a pure state $|\psi\rangle$, the *average* measurement outcome of this observable is $\langle\psi|H|\psi\rangle = \sum_{i=1}^{N} |\langle\psi|i\rangle|^2 H_i$. The postulates of quantum theory tell us that the collection of projective operators $\{|i\rangle\langle i|\}_{i=1}^{N}$ appropriately characterise this measurement process: with probability $|\langle\psi|i\rangle|^2 = \mathrm{tr}(|\psi\rangle\langle\psi|i\rangle\langle i|)$ the measurement outcome reads H_i, and after the measurement the system collapses to the eigenstate $|i\rangle$. This is called a (rank-one) *projective measurement*.

Similar to the case of pure states, the most general situation is to look at 'part of a' projective measurement. More precisely, in an d dimensional system, projective measurements are forced to have at most d outcomes; however, one can obtain a measurement with arbitrarily many outcomes by interacting the system with a larger environment so that the properties of the former get encoded in the latter, and then do a projective measurement on the system plus environment. It can be proved that, in general, a quantum measurement can be characterised by a collection of positive semidefinite operators $\{E_i \geq 0\}_{i=1}^{N}$ satisfying $\sum_{i=1}^{N} E_i = \mathbb{I}_S$. This collection, called a *positive operator-valued measure* (POVM), describes a measurement on the state ρ that returns with probability $\mathrm{tr}(\rho E_i)$ the ith outcome[1]. From here one can see that positivity of E_i's guarantees the positivity of each $\mathrm{tr}(\rho E_i)$, and $\sum_{i=1}^{N} E_i = \mathbb{I}_S$ implies that when we sum over probabilities, we have $\sum_i \mathrm{tr}(\rho E_i) = 1$.

[1] Note that in the notion of POVM, we mainly focus on the statistics of measurement outcomes, rather than the states the system collapses to.

Quantum Channels

Finally, the dynamics of a system is given by linear mappings that bring states to states. Based on the postulates of quantum theory, a *closed* system in a pure state evolves as $|\psi\rangle \mapsto e^{-\frac{i}{\hbar}Ht}|\psi\rangle$, where $\hbar$ is the Planck's constant, H is the system Hamiltonian, and t is time. Since H is Hermitian, the operator $e^{-\frac{i}{\hbar}Ht}$ is always unitary. In other words, closed system dynamics are *unitary*; namely, $|\psi\rangle\langle\psi| \mapsto U|\psi\rangle\langle\psi|U^\dagger$ with an unitary operator U.

The dynamics of a system A in contact with an environment B, on the other hand, can be described as $(\cdot)_A \otimes |0_B\rangle\langle 0_B| \mapsto U_{AB}[(\cdot)_A \otimes |0\rangle\langle 0|_B]U_{AB}^\dagger$, where $|0_B\rangle$ is a fixed pure state in B. Then the reduced dynamics in A, i.e., $\mathrm{tr}_{B'}\left\{U_{AB}[(\cdot)_A \otimes |0\rangle\langle 0|_B]U_{AB}^\dagger\right\}$, is *not* unitary in general[2]. Note that B' is the environment that we want to ignore at the end of the dynamics, which is not necessarily the same with B. Just like the cases of states and measurements, we need a notion to describe the *local* evolution of the *global* system AB. Formally, such a dynamics with input system S' and output system S is given by a linear map $\mathcal{E}_{S|S'} : \mathcal{S}_{S'} \to \mathcal{S}_S$ called *quantum channel*, or simply *channel*, satisfying

- (Complete Positivity) $(\mathcal{E}_{S|S'} \otimes \mathcal{I}_{A|A})(\rho_{S'A}) \geq 0$ for every state $\rho_{S'A}$ and ancillary system A, where $\mathcal{I}_{A|A}$ is the identity channel acting on A.

- (Trace Preserving) $\mathrm{tr} \circ \mathcal{E}_{S|S'}(\rho_{S'}) = \mathrm{tr}(\rho_{S'})$ for every state $\rho_{S'}$.

Let us briefly discuss the physical meaning of complete positivity and trace preservation. First, in order to describe a physical dynamics, $\mathcal{E}_{S|S'}$ must map a quantum state to a quantum state. Furthermore, since it is a physical evolution, it can also only act on part of a larger system. In other words, when we look at the input and output as $S'A$ and SA, respectively, with some external system A, the locally performed dynamics, represented by $\mathcal{E}_{S|S'} \otimes \mathcal{I}_{A|A}$, still needs to make sure the *global output* will be a quantum state for *every global input state*. The "global positivity" is guaranteed by the complete positivity condition, and the unit-trace is ensured by the trace-preserving condition. Finally, we remark that linearity is imposed to respect probability rules; namely, when two states ρ, σ are prepared with probabilities $p, 1-p$ and sent via a channel $\mathcal{E}$, we expect the outcome to be the mixture $p\mathcal{E}(\rho) + (1-p)\mathcal{E}(\sigma)$.

In this thesis, we use the notation CPTP to denote the set of all linear completely positive and trace-preserving maps; namely, all channels. Also, $\mathrm{CPTP}_{S|S'}$ stands for all channels from S' to S; that is,

$$\mathrm{CPTP}_{S|S'} \coloneqq \{\mathcal{E} \mid \mathcal{E} : \mathcal{S}_{S'} \to \mathcal{S}_S, \mathcal{E} \in \mathrm{CPTP}\}. \tag{2.4}$$

The subscript $S|S'$ denotes an input-output pair, explicitly showing the input and output systems of a channel. Finally, note that when $S' = S$, we will simply write $\mathcal{E}_S$ for the channel.

[2]Note that one can always purify the state in B, absorb an additional unitary in U_{AB}, and include an additional partial-trace. Hence, it suffices to always adopt $|0\rangle_B$ as the initial state of B.

Channel-State Duality

Notably, there is a duality between states and channels, which is given by the *Choi-Jamiołkowski isomorphism* [28, 29]. Formally, it is a linear map $\mathcal{J} : \text{CPTP}_{S|S'} \to \text{STATE}_{SS'}$, where SS' denotes a bipartite system consisting of two subsystems S and S'. The *Choi state* of a channel $\mathcal{E}_{S|S'}$ is hence defined by

$$\mathcal{J}\left(\mathcal{E}_{S|S'}\right) \coloneqq \mathcal{E}^{\mathcal{J}}_{SS'} \coloneqq \left(\mathcal{E}_{S|S'} \otimes \mathcal{I}_{S'}\right)\left(|\Psi^+_{S'S'}\rangle\langle\Psi^+_{S'S'}|\right), \tag{2.5}$$

where

$$|\Psi^+_{S'S'}\rangle \coloneqq \frac{1}{\sqrt{d_{S'}}}\sum_{i=0}^{d_{S'}-1}|ii\rangle \tag{2.6}$$

is maximally entangled in S′S′, which is a bipartite system consisting of two copies of S′. The inverse map, which brings a bipartite state $\text{STATE}_{SS'}$ to a channel in $\text{CPTP}_{S|S'}$, is given by

$$\mathcal{J}^{-1}(\rho_{SS'}) \coloneqq d_{S'}\text{tr}_{S'}\left(\left[\mathbb{I}_S \otimes (\cdot)^t_{S'}\right]\rho_{SS'}\right), \tag{2.7}$$

where superscript t denotes the transpose operation with the basis used to defined the maximally entangled state given in Eq. (2.6). The channel-state duality can then be summarised by the following theorem, whose proof is given in Appendix A for the completeness of this thesis:

Theorem 2.1.1. (Choi-Jamiołkowski Isomorphism Theorem [28, 29])

- $\rho_{SS'} \geq 0$ *and* $\text{tr}_S(\rho_{SS'}) = \frac{\mathbb{I}_{S'}}{d_{S'}}$ *if and only if* $\mathcal{J}^{-1}(\rho_{SS'}) \in \text{CPTP}_{S|S'}$.
- $\mathcal{J}^{-1} \circ \mathcal{J}(\mathcal{E}_{S|S'}) = \mathcal{E}_{S|S'}$ *for every channel* $\mathcal{E}_{S|S'} \in \text{CPTP}_{S|S'}$.

Distance Measures

Since quantum states carry the information about physical systems, it is of great importance to analytically understand how different two quantum states are. One common solution to this problem in quantum information theory is to adopt the geometrical notion of *distance* and use it to measure *how far away two states are from each other*. Formally, we have the following definition:

Definition 2.1.2. (Generalised State Distance Measure) *A non-negative function* $D(\cdot,\cdot) : \text{STATE} \times \text{STATE} \to [0,\infty]$ *is said to be a* generalised distance measure *for states if it satisfies the following two conditions:*

- *For* $\rho,\sigma \in \text{STATE}_S$, $D(\rho,\sigma) \geq 0$ *and equality holds if and only if* $\rho = \sigma$.
- *(Data-Processing Inequality)* $D[\mathcal{E}(\rho),\mathcal{E}(\sigma)] \leq D(\rho,\sigma)$ *for every states* $\rho,\sigma \in \text{STATE}_{S'}$ *and channel* $\mathcal{E} \in \text{CPTP}_{S|S'}$.

Note that it is called a "generalised" distance because it is not required to satisfy the triangle inequality and it needs not be symmetric. Hence, in general, it is not a proper distance. Still, it captures two key properties that are relevant in quantum information theory. First, it is *faithful*, in the sense that the input states ρ, σ coincide when, and only when, the function D vanishes. Second, it respects the *data-processing inequality*, which intuitively means that after information processing (described by the channel $\mathcal{E}$), two states can only become less distinguishable.

One way to measure the distance between two states is to adopt relative entropies. First, the *von Neumann entropy* of a quantum state ρ is defined by $S(\rho) := -\mathrm{tr}(\rho \log_2 \rho)$. It characterises the degree of uncertainty, or say the amount of inaccessible information contained in ρ. This can be further used to compare the difference between two states. Formally, the *quantum relative entropy*, or simply *relative entropy*, of ρ conditioned on σ is given by $S(\rho \,||\, \sigma) := \mathrm{tr}[\rho(\log_2 \rho - \log_2 \sigma)]$. One can prove that this is indeed a generalised state distance measure. There are other choices of relative entropies, which will be introduced in later chapters when we need to use them.

Apart from entropic quantities, other commonly-used distance measures are those induced by operator norms. Here we mention three as examples. For a normal operator M, $||\rho - \sigma||_\infty$ is a distance measure induced by the *sup norm* $||M||_\infty := \max_{|\psi\rangle} |\langle\psi|M|\psi\rangle|$. The second example is the *trace distance* $\frac{1}{2}\,||\rho - \sigma||_1$ induced by the *trace norm* $||M||_1 := \mathrm{tr}|M| := \mathrm{tr}\sqrt{M^\dagger M}$. Finally, the third one, $||\rho - \sigma||_2$, is a distance measure induced by the *Hilbert-Schmidt norm* $||M||_2 := \sqrt{\mathrm{tr}(M^\dagger M)}$.

2.2 Quantum Resource Theories

Entanglement Theory: The First Resource Theory

In the development of quantum information theory, it is both important and useful to identify advantages in various operational tasks enabled by different quantum effects. For instance, one can achieve teleportation and superdense coding by consuming entanglement. In these tasks, entanglement acts as a *resource*. This motivates researchers to seek a quantitative understanding of it. The outcome, which is called *entanglement theory*, is briefly summarised as follows.

The very first step is to know the rigorous definition of entanglement. Formally, *a state is entangled if and only if it cannot be written as a convex mixture of product pure states, i.e.,* $\sum_i a_i \rho_i \otimes \eta_i$. This mathematical definition enables one to know, in principle, whether a given state is entangled or not. However, it turns out that knowing if there is entanglement is not enough in many relevant situations. In fact, to achieve perfect teleportation and superdense coding, one not only needs entanglement, but also *maximal entanglement*. This means that when we are given several systems, we would also like to know which one can provide the best performance in a given operational task. That is, one needs to know how to *compare* the entanglement content of two systems. This can be done by an operationally motivated set of channels that *cannot generate entanglement*, e.g., *local operations plus classical communication* (LOCC) channels. If one can map ρ to σ through one such channel, then we *define* σ as having the entanglement content no greater than ρ. In this sense, all LOCC channels jointly define a partial order on the set of states, helping us to compare the entanglement content of quantum states. Still, in many cases, it is convenient and useful if there is a quantitative way to *quantify* entanglement. More precisely, we are asking for a way to use *numbers* to quantitatively measure the entanglement content of different systems. This can be done by looking for a real-valued function D satisfying

- (Entanglement Detection) $D(\rho) \geq 0$ and $D(\rho) = 0$ if ρ is not entangled.
- (Entanglement Comparison) $D[\mathcal{L}(\rho)] \leq D(\rho)$ $\forall$ state ρ and LOCC $\mathcal{L}$.

Any function D equipped with the above two properties can not only detect and compare entanglement contents[3], but it also acts as a *ruler* that quantifies entanglement by real values [11]. Such functions allow researchers to analytically and numerically understand entanglement, and are called *entanglement monotones* or *entanglement quantifiers*.

From Entanglement to General Quantum Resources

Apart from entanglement, there are various quantum properties providing operational advantages in different tasks. The success of entanglement theory soon inspired researchers to apply a similar approach to quantitatively understand *different quantum phenomena*. By following the same steps, one is able to apply a systematic procedure to formulate a wide range of physical phenomena into quantitative, mathematical notions. This general approach is now called *quantum resource theory*, or simply *resource theory*, and it plays a central role in this thesis.

To introduce the resource-theoretic approach, suppose there is a given *physical property*. This physical property can be a physical feature shared by some quantum states, just like entanglement [11], coherence [12], nonlocality [14], etc. Or, alternatively, it can be a property possessed by some quantum dynamics, such as the ability to maintain entanglement. At a general and abstract level, we simply denote this physical property as R, and use $\mathbb{U}$ to denote the *universal set* under consideration, i.e., the set of *physical objects* that may or may not have R. For instance, when R = bipartite entanglement, $\mathbb{U}$ is the set of bipartite states in the given bipartition.

Similar to the structure of entanglement theory, the very first step to quantitatively understand R is to formally define it. This can be done by identifying the set of all physical objects that *do not possess* R. This set, denoted by $\mathcal{F}_R \subseteq \mathbb{U}$, is called the *free set*. This is the set of *free physical objects*, and *a physical object $q \in \mathbb{U}$ has the property R if and only if $q \notin \mathcal{F}_R$.*

When one treats R as a *resource* in certain tasks, usually one not only wants to know whether a physical object possesses R, but also whether *the amounts of R is high enough* to promise advantages in the given task. For instance, when one plans to extract certain amounts of energy from a given quantum state, the state not only needs to be out of thermal equilibrium (i.e., *athermal*), but also needs to be athermal enough to guarantee a high enough extractable work. Hence, after knowing how to appropriately

[3] Note that the partial order of LOCC channels is only partially captured by entanglement monotones.

describe R, the next crucial step is to understand how to compare the *resource content* of different physical objects. This can be done by considering *operations that cannot generate the resource* R, which are called *resource non-generating*, or simply *R-non-generating* operations. These operations help us to *compare* resource contents: If we can use one such operation to map a physical object q to another one r, then we learn that r cannot be more resourceful than q. We say that an operation 'cannot generate R' when it 'maps free objects to free objects'. This leads to the following formal definition:

Definition 2.2.1. (Resource Non-Generating Operations) *Given a resource R and a universal set $\mathbb{U}$, an operation $\mathcal{E}$ is called* resource non-generating *for R, or simply* R-non-generating, *if and only if*

$$\mathcal{E}(q) \in \mathcal{F}_R \quad \forall\, q \in \mathcal{F}_R. \tag{2.8}$$

We use $O_{R|\max}$ to denote the set of all R-non-generating operations.

Similar to entanglement theory, additional physical constraints may be considered for different purposes, resulting in a subset $O_R \subseteq O_{R|\max}$. Members of O_R are called *free operations* of the resource R, and $O_{R|\max}$ is the *largest possible set* of free operations. Specifying free operations allows us to compare resource content under the given physical constraints. More precisely, we can compare resource contents of two *resourceful* objects only *after* we have specified O_R — before specifying free operations, such an ordering *does not* exist. With the above ingredients, a resource theory can be defined as follows:

Definition 2.2.2. (Resource Theory) *A* resource theory of R *is a pair* $(\mathcal{F}_R, O_R)$ *consisting of the set of free objects $\mathcal{F}_R \subseteq \mathbb{U}$ and the set of free operations $O_R \subseteq O_{R|\max}$.*

Once $\mathcal{F}_R$ is specified, R becomes well-defined; namely, an object $q \in \mathbb{U}$ is said to be *R-resourceful* if and only if $q \notin \mathcal{F}_R$. To explicitly indicate what the resource is, we sometimes write a resource theory of R as a triplet $(R, \mathcal{F}_R, O_R)$. Here we give an important remark. Definition 2.2.2 illustrates *one way* to formulate a resource theory. It is also possible to go in the opposite direction; namely, introducing the set of free operations first, and then identifying 'all objects that can be prepared by free operations' as the set of free objects. For instance, one can first consider LOCC channels as operations that are free to implement, and then define free states as those that can be prepared by LOCC channels. The resulting resource, i.e., those that cannot be prepared by LOCC channels, are entangled states. However, in the recent resource-theoretic studies of channel resources, it is more intuitive to start with a given set of free objects, which is usually simple to know, and then discuss possible free operations. This motivates us to adopt Definition 2.2.2.

Similar to entanglement theory, for a resource theory $(R, \mathcal{F}_R, O_R)$, it is natural to ask: *How to quantify the resource R?* One can axiomatically construct measures, termed *resource monotones*, that quantify the resource content of R in the same way that entanglement monotones do for entanglement:

Definition 2.2.3. (Resource Monotone) *For a resource theory* $(R, \mathcal{F}_R, \mathcal{O}_R)$*, a* resource monotone *of* R*, or simply an* R-monotone*, is a non-negative function* $Q_R : \mathbb{U} \to [0, \infty]$ *satisfying*

- (Resource Detection) $Q_R(q) \geq 0$ *and the equality holds if* $q \in \mathcal{F}_R$.
- (Resource Comparison) $Q_R[\mathcal{E}(q)] \leq Q_R(q)$ *for every* $q \in \mathbb{U}$ *and* $\mathcal{E} \in \mathcal{O}_R$.

Q_R *is further called* faithful *when* $Q_R(q) = 0$ *only if* $q \in \mathcal{F}_R$.

An R-monotone is a ruler that can measure R, and different R-monotones give different operational meanings as well as different partial orders of resource content of R according to the performances in certain operational tasks. An important remark that should be made here is that *different* monotones may provide *different* orderings; that is, it is common to have two monotones $Q_{1|R}, Q_{2|R}$ of the same resource R such that $Q_{1|R}(q) > Q_{1|R}(r)$ and $Q_{2|R}(q) < Q_{2|R}(r)$ for some physical objects q, r. Physically, this is because $Q_{1|R}, Q_{2|R}$ measure R based on performance in *different* operational tasks, resulting in a different hierarchy. It also means that there is no operation in $\mathcal{O}_R$ mapping either q to r or r to q. The two objects are known as *incomparable*.

Finally, it is worth mentioning that *every* generalised state distance measure (Definition 2.1.2) induces a monotone for *every* state resource:

Theorem 2.2.4. (Distance-Induced Resource Monotone) *Given a state resource theory* $(R, \mathcal{F}_R, \mathcal{O}_R)$ *and a generalised distance measure* D *for states, the function*

$$Q_{D|R}(\rho) := \inf_{\eta \in \mathcal{F}_R} D(\rho, \eta) \tag{2.9}$$

is an R*-monotone. It is also faithful if* $\mathcal{F}_R$ *is closed in the topology induced by* D [4].

Proof. First, by definition we have $Q_{D|R}(\rho) \geq 0$ for every $\rho \in$ STATE and equality holds if $\rho \in \mathcal{F}_R$. Note that if $Q_{D|R}(\rho) = 0$, it means that, by the definition of infimum, there exists a sequence of free states $\{\eta_n\}_{n=1}^{\infty} \subseteq \mathcal{F}_R$ such that $D(\rho, \eta_n) \to 0$ when $n \to \infty$[5]. Hence, we can conclude that $\rho \in \mathcal{F}_R$ when $\mathcal{F}_R$ is closed in the topology induced by D. This shows the claim of the faithfulness. Finally, the second condition of R monotone can be seen by using data-processing inequality as follows:

$$Q_{D|R}[\mathcal{E}(\rho)] := \inf_{\eta \in \mathcal{F}_R} D[\mathcal{E}(\rho), \eta] \leq \inf_{\eta \in \mathcal{F}_R} D[\mathcal{E}(\rho), \mathcal{E}(\eta)] \leq \inf_{\eta \in \mathcal{F}_R} D(\rho, \eta) = Q_{D|R}(\rho). \tag{2.10}$$

Note that this computation holds for every state ρ and free operations $\mathcal{E} \in \mathcal{O}_R$. □

[4] The *topology induced by* D on STATE_S is referred to the set

$$\{\emptyset\} \cup \{V \mid \forall\, q \in V, \exists\, \delta > 0 \text{ s.t. } \mathcal{B}_D(\rho, \delta) \subset V\},$$

where $\mathcal{B}_D(\rho, \delta) := \{\sigma \in \text{STATE}_S \mid D(\rho, \sigma) < \delta\}$. See, e.g., pages 76-78 of Ref. [30].

[5] One can see that this is also true when $\mathcal{F}_R$ is a finite set. For example, when we consider the resource theory of athermality, the only free state (with a given system dimension) is the thermal state γ; namely, $\mathcal{F}_R = \{\gamma\}$. Then, in this case, $Q_{D|R}(\rho) = 0$ means that $\rho = \gamma$, and one can choose the sequence $\{\eta_n\}_{n=1}^{\infty}$ satisfying $\eta_n = \gamma$ for every n.

Chapter 3

Resource Preservability Theories and Their Applications

In the development of state resource theories, it is important to estimate the remaining resource *after* implementing free operations. An appropriate description can reveal the mechanism of how resources are maintained, or say preserved (see Fig. 3.1 for a schematic illustration), by allowed physical transformations, which is both foundationally and practically relevant. This motivates us to seek an analytical and quantitative answer of the following question for a given state resource theory $(R, \mathcal{F}_R, \mathcal{O}_R)$:

For a given free operation $\mathcal{E} \in \mathcal{O}_R$, what is its ability to preserve R?

This chapter aims to axiomatically formulate this ability as a *channel* resource theory induced by the given *state* resource theory $(R, \mathcal{F}_R, \mathcal{O}_R)$. The resulting theory, termed *resource preservability theory*, can be understood as a dynamical generalisation of the static resource R.

3.1 Formulation

To address the ability of channels to maintain a given state resource, we need to start with a given state resource theory, which is denoted by $(R, \mathcal{F} = \mathcal{F}_R, \mathcal{O} = \mathcal{O}_R)$ in this chapter, where we drop the R dependency since most of the time only one state resource is considered. Because we aim at understanding the ability of a channel to maintain resources, it makes sense to focus on channels that do not have the ability to generate them. Consequently, we mainly focus on channels in $\mathcal{O}$ unless otherwise stated. To achieve a analytical study, we have to consider state resource theories satisfying certain assumptions. Before detailing those assumptions, we need to introduce the following notion:

Definition 3.1.1. (Absolutely Free State) *For a given state resource theory* $(R, \mathcal{F}, \mathcal{O})$, *a free state* $\widetilde{\eta} \in \mathcal{F}$ *is said to be an* absolutely free state *if*

$$\widetilde{\eta} \otimes \eta \in \mathcal{F} \quad \forall\, \eta \in \mathcal{F}. \tag{3.1}$$

We denote the set of all absolutely free states by $\mathcal{F}_{\rm abs}$.

Absolutely free states are those without *hidden* resources [31, 32]. For example, in the resource theory of entanglement, all separable states are absolutely free. Nevertheless, there are state resources with free states that are not absolutely free[1]. We note that $\mathcal{F}_{\rm abs}$ is closed under tensor product, in the sense that $\widetilde{\eta}_1 \otimes \widetilde{\eta}_2 \in \mathcal{F}_{\rm abs}$ if $\widetilde{\eta}_1, \widetilde{\eta}_2 \in \mathcal{F}_{\rm abs}$.
With the above notion, we can list the basic assumptions imposed in this chapter.

Basic Assumptions 3.1.2. *In this chapter, we always consider state resource theories* $(R, \mathcal{F}, \mathcal{O})$ *satisfying*

(R1) With a given system S, *the set of free states* $\mathcal{F}$ *can be written as*

$$\mathcal{F} = \bigcup_{N\in\mathbb{N}} \mathcal{F}_{(N)}, \tag{3.2}$$

where $\mathcal{F}_{(N)} \subseteq {\rm STATE}_{S^{\otimes N}}$ *defines the free states in the* N*-copy system* $S^{\otimes N}$. *Similarly, the set of free operations* $\mathcal{O}$ *can be written as*

$$\mathcal{O} = \bigcup_{N,M\in\mathbb{N}} \mathcal{O}_{(M|N)}, \tag{3.3}$$

where $\mathcal{O}_{(M|N)} \subseteq {\rm CPTP}_{S^{\otimes M}|S^{\otimes N}}$ *defines the free operations from* $S^{\otimes N}$ *to* $S^{\otimes M}$. *We say* $\{S^{\otimes N}\}_{N=1}^{\infty}$ *are* allowed systems *of the given state resource theory.*

(R2) For every allowed system $S^{\otimes N}$, $\mathcal{F}_{\rm abs} \cap {\rm STATE}_{S^{\otimes N}} \neq \emptyset$ *and* $\mathcal{F}_{(N)}$ *is convex.*

(R3) Identity channel and partial trace are free operations.

(R4) Tensoring with absolutely free states, i.e., $(\cdot) \mapsto (\cdot) \otimes \widetilde{\eta}$ *with a given* $\widetilde{\eta} \in \mathcal{F}_{\rm abs}$, *are free operations.*

(R5) The set of free operations $\mathcal{O}$ *is closed under tensor products, convex sums, and compositions. Namely, for every* $\mathcal{E}_1, \mathcal{E}_2 \in \mathcal{O}$, $p \in [0,1]$, *we have*

$$\mathcal{E}_1 \otimes \mathcal{E}_2 \in \mathcal{O}, \quad p\mathcal{E}_1 + (1-p)\mathcal{E}_2 \in \mathcal{O}, \quad \mathcal{E}_1 \circ \mathcal{E}_2 \in \mathcal{O}. \tag{3.4}$$

Let us comment on the physical motivations behind Basic Assumptions 3.1.2. First, Assumption (R1) is imposed to address multi-copy cases. Intuitively, not *every* system is allowed in a given state resource theory. For instance, when R = bipartite entanglement and S = a two-qubit system, an allowed system must be a bipartite system with equal local dimension 2^N with some $N \in \mathbb{N}$. Also, if R = athermality with the

[1] This can be seen by the so-called *superactivation* of nonlocality [33] and steering [34, 35].

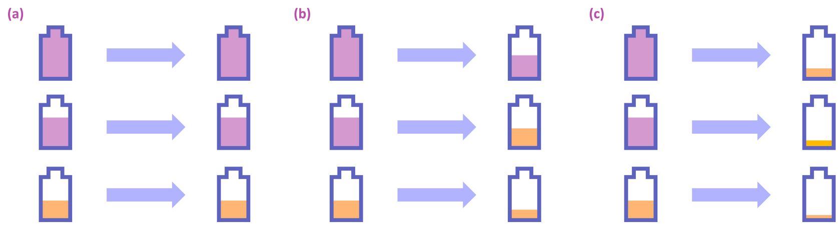

Figure 3.1: We say a channel preserves a state resource if it does not completely destroy the resource for every inputs. In the figure, each battery icon represents a state. The heights of colored columns qualitatively represents the resource content. Purple means that the state is resourceful, and yellow means the opposite. When the channel is a free operation, we have three possible cases: (a) The channel maintains the resource for every input. (b) The channel partially degrades the resource content but can still output some resourceful states. (c) The channel totally destroys the resource for every input. Here, (a) and (b) are channels with the ability to preserve the resource, but not (c).

thermal state γ_S, then all allowed systems are $S^{\otimes N}$ for some $N \in \mathbb{N}$. Regarding Assumption (R2), we expect the existence of states that are free even after combining with other free states. Furthermore, the convex sums of free states is expected to be free again. These are phrased as Assumption (R2) for every allowed system in the given state resource theory. Assumption (R3) originates from the expectation that 'doing nothing' and 'ignoring part of the system' are both free to implement and unable to generate the given state resource R. Assumption (R4) guarantees that, for a given initial resource content, adding an absolutely free state does not generate any extra resource. Finally, we expect that the simultaneous applications (i.e., tensor product), classical probabilistic mixture (i.e., convex sum), and sequential applications (i.e., composition) of free operations are again unable to generate the resource. These lead to Assumption (R5). As expected, Assumption (R5) is commonly shared by many state resource theories[2], including the ones of entanglement, nonlocality, and athermality.

3.1.1 Resource Preservability Theories

With a given state resource theory $(R, \mathcal{F}, \mathcal{O})$ satisfying basic assumptions listed in Definition 3.1.2, we can start to formulate the resource preservability theory induced by it. We abbreviate the outcome by *R-preservability theory* once the state resource theory is given and clear, and we follow Definition 2.2.2 to formulate R-preservability theory as a resource theory of *channels*. To this end, the first thing to address is the range of

[2] There are cases where Assumption (R5) is violated. To see this, consider the state resource theory of nonlocality with all nonlocality non-generating channels as the set of free operations. Suppose ρ_0 is local such that $\rho_0^{\otimes 2}$ is nonlocal [33], i.e., ρ_0's nonlocality can be superactivated. Then the channel $\Phi_{\rho_0} : (\cdot) \mapsto \rho_0$ is nonlocality non-generating, while two copies of it, $\Phi_{\rho_0} \otimes \Phi_{\rho_0}$, can always output nonlocal states.

objects that is considered. As mentioned at the beginning of this chapter, we mainly focus on channels in $\mathcal{O}$. Using the language of Sec. 2.2, this means that we set $\mathbb{U} = \mathcal{O}$ for the resource preservability theory induced by $(R, \mathcal{F}, \mathcal{O})$. To complete the formulation, we still need to specify two sets in this channel resource theory; namely, the one of free objects, and the one of free operations.

Resource-Annihilating Channels

Intuitively, free objects of R-preservability are those channel that cannot preserve the resource of *any* input. Formally, we introduce the following notion:

Definition 3.1.3. (Resource-Annihilating Channels) *With a given state resource theory* $(R, \mathcal{F}, \mathcal{O})$*,* resource-annihilating channels *of* R*, or simply* R-annihilating channels*, are those in the set*

$$\mathcal{A} := \{\mathcal{E} \in \mathcal{O} \mid \mathcal{E}(\rho) \in \mathcal{F} \ \forall \rho\}. \tag{3.5}$$

Every channel in $\mathcal{O} \setminus \mathcal{A}$ has certain ability to maintain the given resource R. Consequently, channels in $\mathcal{O} \setminus \mathcal{A}$ are said to have R-preservability. In this sense, members of $\mathcal{A}$ are the desired free objects of the R-preservability theory. Finally, note that Assumption (R5) implies $\mathcal{A}$ is convex.

It is worth mentioning that if the given state resource theory admits the so-called 'superactivation' [33, 34, 35], then the induced resource preservability theory could also inherit this property. This is formally addressed as follows (see Appendix B.1.3 for the proof when we consider the state resource theory of nonlocality):

Lemma 3.1.4. (Superactivation of Resource Preservability) *There exist state resource theories with a free operation* $\mathcal{L} \in \mathcal{A}$ *and* $N \in \mathbb{N}$ *such that* $\mathcal{L}^{\otimes N} \notin \mathcal{A}$*.*

This observation means that, if we aim to formulate resource preservability theories applicable to a wide range of state resources, we need to respect properties such as superactivation. This also suggests us to extend the notion of absolutely free states (Definition 3.1.1) to the dynamical regime:

Definition 3.1.5. (Absolutely Resource Annihilating Channel) *Given a state resource theory* $(R, \mathcal{F}, \mathcal{O})$*, we say* $\widetilde{\Lambda} \in \mathcal{A}$ *is an* absolutely resource annihilating channel *of* R*, or simply* absolutely R-annihilating channel*, if*

$$\widetilde{\Lambda} \otimes \Lambda \in \mathcal{A} \quad \forall \Lambda \in \mathcal{A}. \tag{3.6}$$

We denote the set of all such channels by $\mathcal{A}_{\rm abs}$.

In other words, absolutely R-annihilating channels are those whose ability to preserve the resource R *cannot be superactivated*. As an example, one can actually show that a channel that is entanglement-annihilating [36] as well as entanglement-breaking[3] [37] must be an absolutely entanglement-annihilating channel (we detail the

[3] A channel $\mathcal{E}_S$ is said to be *entanglement-breaking* if $\mathcal{E}_S \otimes \mathcal{E}_A$ is entanglement-annihilating for every ancillary system A. It is shown in Ref. [37] that a channel is entanglement-breaking if and only if it is a measure-and-prepare channel; i.e., a channel of the form $\sum_x \rho_x {\rm tr}[E_x(\cdot)]$ with a POVM $\{E_x\}$ and states ρ_x's.

argument in Appendix B.2). Finally, we observe that

$$\widetilde{\Lambda}\circ\mathcal{E}\in\mathcal{A}_{\rm abs}\quad\&\quad\mathcal{E}\circ\widetilde{\Lambda}\in\mathcal{A}_{\rm abs}\quad\forall\,\mathcal{E}\in\mathcal{O}\;\&\;\widetilde{\Lambda}\in\mathcal{A}_{\rm abs};$$
$$\widetilde{\Lambda}_S\otimes\widetilde{\Lambda}_{S'}\in\mathcal{A}_{\rm abs}\quad\forall\,\widetilde{\Lambda}_S,\widetilde{\Lambda}_{S'}\in\mathcal{A}_{\rm abs}. \tag{3.7}$$

The first line implies that a sequential application of free operations cannot maintain any resource, even with the assistance of ancillary resource-annihilating channels, if an absolutely resource-annihilating channel has been applied in the sequence. Also, since absolutely resource-annihilating channels forbid activation, the second line in Eq. (3.7) means that simultaneous applications of two such channels still forbid any activation.

Free Operations of Resource Preservability

Now it remains to identify free operations of R-preservability with a given state resource theory $(R,\mathcal{F},\mathcal{O})$. To this end, the first thing to know is *how to map channels to channels*. The general structure of such a mapping, called *super-channels*, are [38, 39]:

$$\mathcal{E}\mapsto\mathcal{M}\circ(\mathcal{E}\otimes\mathcal{I}_A)\circ\mathcal{N}, \tag{3.8}$$

where A is an ancillary system, and $\mathcal{M}$, $\mathcal{N}$ are some pre-processing and post-processing channels. This means that a general way to manipulate a quantum dynamics is to adopt operations before and after the given dynamics. With this notion in hand, one option of free operations is to consider all super-channels that map $\mathcal{A}$ into $\mathcal{A}$, which corresponds to the role of '$\mathcal{O}_{R|\rm max}$' in Definition 2.2.2. However, it is unclear whether they map $\mathcal{O}$ into $\mathcal{O}$, as $\mathcal{O}$ is the universal set of the R-preservability theory. This means these super-channels would not be valid free operations for R-preservability. For instance, when R = bipartite entanglement and $\mathcal{O}$ = local operations channels, the superchannel that always output a state preparation channel of a non-product separable state is invalid.

Due to the above observation, we try to impose physical conditions on Eq. (3.8) and obtain free operations of resource preservability theories. First, we expect that free operations of R-preservability cannot generate 'the ability to generate R'. This suggests that all steps in Eq. (3.8) should be *members of* $\mathcal{O}$; that is, $\mathcal{N},\mathcal{M}\in\mathcal{O}$. Furthermore, when both $\mathcal{N},\mathcal{M}$ are identity channels, the ancillary identity channel $\mathcal{I}_A$ in Eq. (3.8) may provide artificial R-preservability. Since free operations of R-preservability cannot generate R-preservability, we expect the ancillary system A should not maintain any resource R. This motivates us to replace $\mathcal{I}_A$ in Eq. (3.8) by an appropriate, alternative channel. Concerning the existence of superactivation properties discussed in the previous section (see also Appendix B.1), we impose the condition that *the process in the ancillary system A is an absolutely R-annihilating channel in $\mathcal{A}_{\rm abs}$*. The above discussions can be summarised as follows:

Definition 3.1.6. (Free Operations of Resource Preservability Theories) *Given a state resource theory $(R,\mathcal{F},\mathcal{O})$, free operations of the resource preservability theory induced by $(R,\mathcal{F},\mathcal{O})$ are mappings $\mathfrak{F}:\mathcal{O}\to\mathcal{O}$ of the form*

$$\mathfrak{F}(\mathcal{E}):=\Lambda_+\circ\left(\mathcal{E}\otimes\widetilde{\Lambda}_A\right)\circ\Lambda_-, \tag{3.9}$$

where $\Lambda_+,\Lambda_-\in\mathcal{O}$ and $\widetilde{\Lambda}_A\in\mathcal{A}_{\rm abs}$.

Note that asking Λ_+, Λ_- to be R-annihilating trivialise the setting, and one can check that $\mathfrak{F}(\Lambda) \in \mathcal{A}$ if $\Lambda \in \mathcal{A}$. We use the notation $\mathbb{F}$ to denote the set of superchannels satisfying Eq. (3.9) with the given state resource theory $(R, \mathcal{F}, O)$ (note that we again ignore the R dependency). Then the corresponding resource preservability theory can be written as, using the language of Definition 2.2.2,

$$(R\text{-preservability}, \mathcal{A}, \mathbb{F}).$$

3.2 Resource Preservability Monotones

As shown in Theorem 2.2.4, one natural way to measure the resource content of an object is to measure its distance from the free set. Such a geometrical intuition also applies to resource preservability theories, and we need to extend the notion of state distance to the dynamical regime. Suppose D is a general state distance measure defined in Definition 2.1.2, then we define the following function:

$$D^R(\mathcal{E}_S, \Lambda_S) \coloneqq \sup_{A;\rho_{SA},\widetilde{\Lambda}_A \in \mathcal{A}_{\text{abs}}} D\left[(\mathcal{E}_S \otimes \widetilde{\Lambda}_A)(\rho_{SA}), (\Lambda_S \otimes \widetilde{\Lambda}_A)(\rho_{SA})\right]; \tag{3.10}$$

Geometrically, this is a distance between $\mathcal{E}_S$ and Λ_S that is *adjusted by* absolutely resource-annihilating channels. Note that the maximisation is taken over all ancillary system A with *allowed system dimensions* [see Assumption (R1) in Basic Assumptions 3.1.2 and the discussion below]. Now we state the following main result. In what follows, we say a set of channels $\mathcal{C}$ is *closed under D* if for any sequence $\{\Lambda_k\}_{k=1}^\infty \subseteq \mathcal{C}$ satisfying $\lim_{k\to\infty} \sup_\rho D[\mathcal{E}(\rho), \Lambda_k(\rho)] = 0$, we have $\mathcal{E} \in \mathcal{C}$.

Main Theorem 3.2.1. (Distance-Induced Resource Preservability Monotones) *Consider a state resource theory $(R, \mathcal{F}, O)$ satisfying Basic Assumptions 3.1.2 and D a generalised state distance measure as in Definition 2.1.2. Define*

$$P_{D|R}(\mathcal{E}_S) \coloneqq \inf_{\Lambda_S \in \mathcal{A}} D^R(\mathcal{E}_S, \Lambda_S). \tag{3.11}$$

Then we have

1. *$P_{D|R}(\mathcal{E}) \geq 0$ and $P_{D|R}(\mathcal{E}) = 0$ if $\mathcal{E} \in \mathcal{A}$. When $\mathcal{A}$ is closed under D, we further have $P_{D|R}(\mathcal{E}) = 0$ if and only if $\mathcal{E} \in \mathcal{A}$.*

2. *$P_{D|R}[\mathfrak{F}(\mathcal{E})] \leq P_{D|R}(\mathcal{E})$ for every channel $\mathcal{E}$ and free super-channel $\mathfrak{F} \in \mathbb{F}$.*

3. *$P_{D|R}(\mathcal{E} \otimes \mathcal{E}') \geq P_{D|R}(\mathcal{E})$ for every $\mathcal{E}, \mathcal{E}' \in O$. The equality holds if $\mathcal{E}' \in \mathcal{A}_{\text{abs}}$.*

Theorem 3.2.1 implies that, for every generalised state distance measure D, the function $P_{D|R}$ is a monotone of the resource theory (R-preservability, $\mathcal{A}$, $\mathfrak{F}$) according to Definition 2.2.3. Furthermore, condition 3 is an additional property of resource preservability monotones induced by distance measures. Note that condition 2 actually works for *every* channel, including those outside the set O. This is useful when one needs to consider the smooth version of resource preservability monotones.

Proof. In the proof, we adopt the notation $\overline{\sup}_{\mathrm{A}} := \sup_{A;\rho_{SA},\widetilde{\Lambda}_A\in\mathcal{A}_{\mathrm{abs}}}$.

Proof of Condition 1.– The first part holds automatically due to the definition. It suffices to prove the faithfulness condition. Suppose we have $P_{D|R}(\mathcal{E}) = 0$ for a given channel $\mathcal{E}$. By definition, this means that, by considering the one-dimensional ancillary system, $\inf_{\Lambda_S\in\mathcal{A}}\sup_\rho D[\mathcal{E}(\rho),\Lambda_S(\rho)] = 0$. Consequently, by the definition of infimum, there exists a sequence $\{\Lambda_k\}_{k=1}^\infty \subseteq \mathcal{A}$ such that $\lim_{k\to\infty}\sup_\rho D[\mathcal{E}(\rho),\Lambda_k(\rho)] = 0$. If $\mathcal{A}$ is closed under D, this implies $\mathcal{E}\in\mathcal{A}$. This proves that $P_{D|R}$ is faithful; namely, $P_{D|R}(\mathcal{E}) = 0$ if and only if $\mathcal{E}\in\mathcal{A}$.

Proof of Condition 2.– For a given channel $\mathcal{E}_S$ and a free operation of resource preservability (Definition 3.1.6) $\mathfrak{F}:\mathcal{E}_S\mapsto\Lambda_+\circ(\mathcal{E}_S\otimes\widetilde{\Lambda}_B)\circ\Lambda_-$ with $\Lambda_+,\Lambda_-\in\mathcal{O}$ and $\widetilde{\Lambda}_B\in\mathcal{A}_{\mathrm{abs}}$, direct computation shows that:

$$\begin{aligned}
P_{D|R}[\mathfrak{F}(\mathcal{E}_S)] &= \inf_{\Lambda_{S'}\in\mathcal{A}}\overline{\sup_{\mathrm{A}}}\, D\left[(\mathfrak{F}(\mathcal{E}_S)\otimes\widetilde{\Lambda}_A)(\rho_{S'A}),(\Lambda_{S'}\otimes\widetilde{\Lambda}_A)(\rho_{S'A})\right]\\
&\le \inf_{\Lambda_{SB}\in\mathcal{A}}\overline{\sup_{\mathrm{A}}}\, D\left\{(\mathfrak{F}(\mathcal{E}_S)\otimes\widetilde{\Lambda}_A)(\rho_{S'A}),\left[(\Lambda_+\circ\Lambda_{SB}\circ\Lambda_-)\otimes\widetilde{\Lambda}_A\right](\rho_{S'A})\right\}\\
&\le \inf_{\Lambda_{SB}\in\mathcal{A}}\overline{\sup_{\mathrm{A}}}\, D\left\{\left[(\mathcal{E}_S\otimes\widetilde{\Lambda}_B\otimes\widetilde{\Lambda}_A)\circ(\Lambda_-\otimes\mathcal{I}_A)\right](\rho_{S'A}),\left[(\Lambda_{SB}\otimes\widetilde{\Lambda}_A)\circ(\Lambda_-\otimes\mathcal{I}_A)\right](\rho_{S'A})\right\}\\
&\le \inf_{\Lambda_{SB}\in\mathcal{A}}\overline{\sup_{\mathrm{A}}}\, D\left[(\mathcal{E}_S\otimes\widetilde{\Lambda}_B\otimes\widetilde{\Lambda}_A)(\rho_{SBA}),(\Lambda_{SB}\otimes\widetilde{\Lambda}_A)(\rho_{SBA})\right]\\
&\le \inf_{\Lambda_S\in\mathcal{A}}\overline{\sup_{\mathrm{A}}}\, D\left[(\mathcal{E}_S\otimes\widetilde{\Lambda}_B\otimes\widetilde{\Lambda}_A)(\rho_{SBA}),(\Lambda_S\otimes\widetilde{\Lambda}_B\otimes\widetilde{\Lambda}_A)(\rho_{SBA})\right]\\
&\le \inf_{\Lambda_S\in\mathcal{A}}\overline{\sup_{\mathrm{A}}}\, D\left[(\mathcal{E}_S\otimes\widetilde{\Lambda}_A)(\rho_{SA}),(\Lambda_S\otimes\widetilde{\Lambda}_A)(\rho_{SA})\right] = P_{D|R}(\mathcal{E}_S).
\end{aligned}\tag{3.12}$$

The second line is because $\Lambda_+\circ\Lambda_{SB}\circ\Lambda_-\in\mathcal{A}$ [Assumption (R5)]. The third line is due to data-processing inequality of D (Definition 2.1.2). The fifth line is due to $\Lambda_S\otimes\widetilde{\Lambda}_B\in\mathcal{A}$ (Definition 3.1.5). The sixth line is because of Eq. (3.7).

Proof of Condition 3.– Direct computation shows that

$$\begin{aligned}
P_{D|R}(\mathcal{E}_S\otimes\mathcal{E}_{S'}) &= \inf_{\Lambda_{SS'}\in\mathcal{A}}\overline{\sup_{\mathrm{A}}}\, D\left[(\mathcal{E}_S\otimes\mathcal{E}_{S'}\otimes\widetilde{\Lambda}_A)(\rho_{SS'A}),(\Lambda_{SS'}\otimes\widetilde{\Lambda}_A)(\rho_{SS'A})\right]\\
&\ge \inf_{\Lambda_{SS'}\in\mathcal{A}}\overline{\sup_{\mathrm{A}}}\, D\left[(\mathcal{E}_S\otimes\mathcal{E}_{S'}\otimes\widetilde{\Lambda}_A)(\rho_{SA}\otimes\widetilde{\eta}_{S'}),(\Lambda_{SS'}\otimes\widetilde{\Lambda}_A)(\rho_{SA}\otimes\widetilde{\eta}_{S'})\right]\\
&\ge \inf_{\Lambda_{SS'}\in\mathcal{A}}\overline{\sup_{\mathrm{A}}}\, D\left\{(\mathcal{E}_S\otimes\widetilde{\Lambda}_A)(\rho_{SA}),\mathrm{tr}_{S'}\left[(\Lambda_{SS'}\otimes\widetilde{\Lambda}_A)(\rho_{SA}\otimes\widetilde{\eta}_{S'})\right]\right\}\\
&\ge \inf_{\Lambda_S\in\mathcal{A}}\overline{\sup_{\mathrm{A}}}\, D\left[(\mathcal{E}_S\otimes\widetilde{\Lambda}_A)(\rho_{SA}),(\Lambda_S\otimes\widetilde{\Lambda}_A)(\rho_{SA})\right] = P_{D|R}(\mathcal{E}_S).
\end{aligned}\tag{3.13}$$

In the second line, we pick an absolutely free state $\widetilde{\eta}_{S'}\in\mathcal{F}_{\mathrm{abs}}$, which is possible due to Assumption (R2). The third line is because of the data-processing inequality of D. In the last line, we use the fact that $\mathrm{tr}_{S'}\circ\Lambda_{SS'}[(\cdot)\otimes\widetilde{\eta}_{S'}]$ is an R-annihilating channel [Assumptions (R3), (R4), and (R5)].

To show the equality, for a given $\widetilde{\Lambda}_{S'} \in \mathcal{A}_{\rm abs}$, one can compute that:

$$P_{D|R}(\mathcal{E}_S \otimes \widetilde{\Lambda}_{S'}) = \inf_{\Lambda_{SS'} \in \mathcal{A}} \sup_{\rm A} D\left[(\mathcal{E}_S \otimes \widetilde{\Lambda}_{S'} \otimes \widetilde{\Lambda}_A)(\rho_{SS'A}), (\Lambda_{SS'} \otimes \widetilde{\Lambda}_A)(\rho_{SS'A})\right]$$
$$\leq \inf_{\Lambda_S \in \mathcal{A}} \sup_{\rm A} D\left[(\mathcal{E}_S \otimes \widetilde{\Lambda}_{S'} \otimes \widetilde{\Lambda}_A)(\rho_{SS'A}), (\Lambda_S \otimes \widetilde{\Lambda}_{S'} \otimes \widetilde{\Lambda}_A)(\rho_{SS'A})\right]$$
$$\leq \inf_{\Lambda_S \in \mathcal{A}} \sup_{\rm A} D\left[(\mathcal{E}_S \otimes \widetilde{\Lambda}_A)(\rho_{SA}), (\Lambda_S \otimes \widetilde{\Lambda}_A)(\rho_{SA})\right] = P_{D|R}(\mathcal{E}_S), \quad (3.14)$$

which is because $\Lambda_S \otimes \widetilde{\Lambda}_{S'} \in \mathcal{A}_{\rm abs}$ with the fixed $\widetilde{\Lambda}_{S'}$ and $\widetilde{\Lambda}_{S'} \otimes \widetilde{\Lambda}_A \in \mathcal{A}_{\rm abs}$ [Eq. (3.7)]. □

3.2.1 Resource Preservability Robustness

Theorem 3.2.1 provides a general characterisation of resource preservability monotones induced by generalised state distance measure. When specific distance measures are considered, one is able to extract different physical implications. As an explicit example, we focus on a generalised state distance measure that is relevant in various operational tasks: Formally, the *max-relative entropy* of a state ρ conditioned on another state σ is defined by [40]:

$$D_{\max}(\rho\|\sigma) := \log_2 \inf\{\lambda \geq 1 \,|\, \rho \leq \lambda\sigma\}. \quad (3.15)$$

Note that we adopt the convention notion $\inf \emptyset := +\infty$. To interpret max-relative entropy physically, let us rewrite it as

$$D_{\max}(\rho\|\sigma) = -\log_2 \sup\{p \in [0,1] \,|\, p\rho + (1-p)\eta = \sigma,\ \eta \in {\rm STATE}\}. \quad (3.16)$$

From here, one can observe that $D_{\max}(\rho\|\sigma)$ represents the minimal amount of *noise*, in terms of the probability weighting $(1-p)$ in front of the noise term η, needed to mix with ρ in order to realise the state σ. Now, we note that [40]

- For every states ρ, σ, $D_{\max}(\rho\|\sigma) \geq 0$, and the equality holds if and only if $\rho = \sigma$.
- (Data-processing inequality) $D_{\max}[\mathcal{E}(\rho)\|\mathcal{E}(\sigma)] \leq D_{\max}(\rho\|\sigma)$ for every channels $\mathcal{E}$ and states ρ, σ.

Hence, by Definition 2.1.2, it is a generalised state distance measure. Applying Theorem 3.2.1, we learn that the function $P_{D_{\max}}$ is a resource preservability monotone with any state resource theory $(R, \mathcal{F}, \mathcal{O})$ satisfying Basic Assumptions 3.1.2. To understand its physical meaning, one can rewrite it as (see Appendix B.3)

$$P_{D_{\max}|R}(\mathcal{E}) = \inf_{\Lambda \in \mathcal{A}} -\log_2 \sup\left\{p \in [0,1] \,\middle|\, \frac{1}{1-p}(\Lambda - p\mathcal{E}) \otimes \widetilde{\Lambda}_A \in {\rm CPTP}\ \forall A\ \&\ \widetilde{\Lambda}_A \in \mathcal{A}_{\rm abs}\right\}. \quad (3.17)$$

Hence, $P_{D_{\max}|R}(\mathcal{E})$ is the minimal amount of noise needed to turn $\mathcal{E}$ into resource-annihilating *under extensions with every possible* $\widetilde{\Lambda}_A \in \mathcal{A}_{\rm abs}$. $P_{D_{\max}|R}$ is a robustness-like measure and, therefore, we call it *resource preservability robustness* in this thesis.

Note that resource-annihilating channels achievable by adding the smallest amount of noise to $\mathcal{E}$ can be interpreted as $\mathcal{E}$'s 'resource-annihilating counterparts'. Namely, they are some channels $\Lambda \in \mathcal{A}$ such that $D_{\max}^{R}(\mathcal{E}\|\Lambda) = P_{D_{\max}|R}(\mathcal{E})$. Geometrically, they are the resource-annihilating channels 'closest to $\mathcal{E}$' when we use $D_{\max}^{R}$ to measure the distance between channels. When a given error $\kappa > 0$ is allowed, we formally define them as the following set [recall the notation from Eq. (3.10)]:

$$\mathcal{A}(\kappa;\mathcal{E}) \coloneqq \left\{\Lambda \in \mathcal{A} \;\middle|\; \left|D_{\max}^{R}(\mathcal{E}\|\Lambda) - P_{D_{\max}|R}(\mathcal{E})\right| \le \kappa\right\}. \tag{3.18}$$

Members of $\mathcal{A}(\kappa;\mathcal{E})$ are resource-annihilating channels that are closest to $\mathcal{E}$, up to the given error κ. We call them *resourceless versions* of $\mathcal{E}$ up to the error κ, and use the notation $\Lambda^{\mathcal{E}} \in \mathcal{A}(\kappa;\mathcal{E})$ to emphasise their dependence on $\mathcal{E}$.

3.3 Implications and Applications

It turns out that the resource preservability robustness is able to provide applications and implications to classical communication, thermodynamics, and their connections, which we detail in the following sections. As the first application, the resource preservability robustness can be used to estimate the ability to transmit classical information. In order to state the formal result, we briefly recap the relevant ingredient from classical communication theory.

3.3.1 Applications to Classical Communication

Classical Communication Through Quantum Channels

Classical information can be described by a set of integers $\{m\}_{m=0}^{M-1}$, where M is the total number of possible classical messages. Intuitively, if one can use physical processes to reliably send messages to another agent, the sender is able to *communicate with* the receiver. The aim is to analytically describe how to do so via a quantum dynamics, or say a quantum channel, $\mathcal{E}$. To this end, the sender needs to first *encode* the classical information into a set of quantum states $\{\rho_m\}_{m=0}^{M-1}$ in the input space of the channel $\mathcal{E}$. Then $\mathcal{E}$ can act on those states and proceed with the transformation. In the output space of $\mathcal{E}$, the receiver has to *decode* the classical information. This can be done by applying a POVM $\{E_m\}_{m=0}^{M-1}$ (Sec. 2.1). If the receiver's measurement outcome coincides with the label of classical message initially encoded by the sender, the transmission is successful. Define an *M-code* as $\Theta_M = (\{\rho\}_{m=0}^{M-1}, \{E_m\}_{m=0}^{M-1})$. To *quantitatively* understand $\mathcal{E}$'s ability to transmit classical information in this scenario, a commonly used measure, the *one-shot classical capacity* of $\mathcal{E}$ with error ϵ, can be defined as (see, e.g., Ref. [41]):

$$C_{(1)}^{\epsilon}(\mathcal{E}) \coloneqq \max\{\log_2 M \mid \exists\Theta_M,\ p_s(\Theta_M,\mathcal{E}) \ge 1-\epsilon\}, \tag{3.19}$$

where

$$p_s(\Theta_M,\mathcal{E}) \coloneqq \frac{1}{M}\sum_{m=0}^{M-1} \mathrm{tr}\left[E_m\mathcal{E}(\rho_m)\right] \tag{3.20}$$

is the *average success probability* of the transmission via $\mathcal{E}$ with the M-code Θ_M.

Bounds On the Classical Capacity

To estimate the one-shot classical capacity subject to non-vanishing errors, we need to appropriately "smooth" the resource preservability robustness. Formally, with a given error $\delta \geq 0$, the *smoothed version* of $P_{D|R}(\mathcal{E})$ is defined by

$$P^{\delta}_{D|R}(\mathcal{E}) := \inf_{\|\mathcal{E}-\mathcal{E}'\|_{\diamond}\leq 2\delta} P_{D|R}(\mathcal{E}'). \tag{3.21}$$

It smooths the original optimisation over all channels $\mathcal{E}'$ close to $\mathcal{E}$, and

$$\|\mathcal{E}_S\|_{\diamond} := \sup_{A,\rho_{SA}} \|(\mathcal{E}_S \otimes \mathcal{I}_A)(\rho_{SA})\|_1 \tag{3.22}$$

is the *diamond norm*. Finally, define

$$\Gamma_{\kappa}(\mathcal{E}) := \log_2 \inf_{\Lambda^{\mathcal{E}}\in\mathcal{A}(\kappa;\mathcal{E})} \sup_{\Theta_M} \sum_{m=0}^{M-1} \mathrm{tr}\left[E_m \Lambda^{\mathcal{E}}(\rho_m)\right], \tag{3.23}$$

where $\sup_{\Theta_M}$ maximises over every natural number M and M-code Θ_M. $2^{\Gamma_{\kappa}(\mathcal{E})}$ is related to the highest number of discriminable states at the output space of every resourceless version of $\mathcal{E}$, up to the error κ [4]. Similar to Eq. (3.21), we define its smoothed version as $\Gamma^{\delta}_{\kappa}(\mathcal{E}) := \sup_{\|\mathcal{E}-\mathcal{E}'\|_{\diamond}\leq 2\delta} \Gamma_{\kappa}(\mathcal{E}')$. Then one can show the following result, serving as a connection between classical communication and resource preservability theory:

Main Theorem 3.3.1. (Resource Preservability and One-Shot Classical Capacity) *Given a state resource theory $(R,\mathcal{F},\mathcal{O})$ satisfying Basic Assumptions 3.1.2 and $\epsilon,\delta \geq 0$ & $0<\kappa<1$ satisfying $\epsilon+\delta<1$, then for every $\mathcal{E}\in\mathcal{O}$ we have*

$$C^{\epsilon}_{(1)}(\mathcal{E}) \leq P^{\delta}_{D_{\max}|R}(\mathcal{E}) + \Gamma^{\delta}_{\kappa}(\mathcal{E}) + \log_2 \frac{2^{\kappa}}{1-\epsilon-\delta}. \tag{3.24}$$

Proof. Consider a channel $\mathcal{E}'$ satisfying $\|\mathcal{E}-\mathcal{E}'\|_{\diamond} \leq 2\delta$ and a given error $\kappa > 0$. Then for every resourceless version $\Lambda^{\mathcal{E}'} \in \mathcal{A}(\kappa;\mathcal{E}')$ [recall from Eq. (3.18)], there exists a positive map $\mathcal{P}$ such that (see Lemma B.3.2 in Appendix B)

$$\mathcal{P}\otimes\widetilde{\Lambda}_A \text{ is a positive map } \forall A \,\&\, \widetilde{\Lambda}_A \in \mathcal{A}_{\mathrm{abs}} \quad \& \quad \mathcal{E}'+\mathcal{P} = 2^{D^R_{\max}\left(\mathcal{E}'\|\Lambda^{\mathcal{E}'}\right)}\Lambda^{\mathcal{E}'}. \tag{3.25}$$

Now, with a given M-code $\Theta_M = (\{\rho_m\}_{m=0}^{M-1},\{E_m\}_{m=0}^{M-1})$, we have [recall the definition and notation from Eqs. (3.10) and (3.20)]:

$$\begin{aligned} p_s\left(\Theta_M,\mathcal{E}'\right) &= \frac{2^{D^R_{\max}\left(\mathcal{E}'\|\Lambda^{\mathcal{E}'}\right)}}{M} \sum_{m=0}^{M-1} \mathrm{tr}\left[E_m\Lambda^{\mathcal{E}'}(\rho_m)\right] - \frac{1}{M}\sum_{m=0}^{M-1}\mathrm{tr}\left[E_m\mathcal{P}(\rho_m)\right] \\ &\leq \frac{2^{\left[P_{D_{\max}|R}(\mathcal{E}')+\kappa\right]}}{M} \sup_{\Theta_{M'}} \sum_{m=0}^{M'-1} \mathrm{tr}\left[E'_m\Lambda^{\mathcal{E}'}(\rho'_m)\right], \end{aligned} \tag{3.26}$$

[4] Note that Eq. (3.23) is *not* a minimisation over one-shot classical capacities of R-annihilating channels $\Lambda^{\mathcal{E}} \in \mathcal{A}(\kappa;\mathcal{E})$.

where the facts that $\Lambda^{\mathcal{E}'} \in \mathcal{A}(\kappa,\mathcal{E}')$ and $\mathrm{tr}\left[E_m\mathcal{P}(\rho_m)\right] \geq 0$ for all m imply the inequality, and $\sup_{\Theta_{M'}}$ maximises over every natural number M' and M'-code $\Theta_{M'} = (\{\rho'_m\}_{m=0}^{M'-1},\{E'_m\}_{m=0}^{M'-1})$. This is true for every $\Lambda^{\mathcal{N}'} \in O_R^N(\kappa,\mathcal{E}')$, and we conclude that:

$$p_s\left(\Theta_M,\mathcal{E}'\right) \leq \frac{1}{M} \times 2^{\left[P_{D_{\max}|R}(\mathcal{E}')+\Gamma_\kappa(\mathcal{E}')+\kappa\right]}. \tag{3.27}$$

Now, using the estimate $\sup_\rho \sup_{0\leq E\leq\mathbb{I}} 2\mathrm{tr}[E(\mathcal{N}'-\mathcal{N})(\rho)] \leq \|\mathcal{N}'-\mathcal{N}\|_\diamond$ [20] for arbitrary channels $\mathcal{N},\mathcal{N}'$, we note that

$$\left|p_s\left(\Theta_M,\mathcal{E}'\right)-p_s\left(\Theta_M,\mathcal{E}\right)\right| = \left|\frac{1}{M}\sum_{m=0}^{M-1}\mathrm{tr}\left[E_m(\mathcal{E}'-\mathcal{E})(\rho_m)\right]\right| \leq \frac{1}{2}\left\|\mathcal{E}'-\mathcal{E}\right\|_\diamond. \tag{3.28}$$

Hence, for every channel $\mathcal{E}'$ satisfying $\|\mathcal{E}-\mathcal{E}'\|_\diamond \leq 2\delta$ and M-code Θ_M achieving $p_s\left(\Theta_M,\mathcal{E}\right) \geq 1-\epsilon$, we have

$$1-\epsilon \leq p_s\left(\Theta_M,\mathcal{E}\right) \leq p_s\left(\Theta_M,\mathcal{E}'\right)+\delta \leq \frac{1}{M} \times 2^{\left[P_{D_{\max}|R}(\mathcal{E}')+\Gamma_\kappa(\mathcal{E}')+\kappa\right]}+\delta. \tag{3.29}$$

In other words, for every given $\epsilon,\delta \geq 0$ & $0<\kappa<1$ satisfying $\epsilon+\delta<1$ we have

$$\begin{aligned} C_{(1)}^\epsilon(\mathcal{E}) &\leq \log_2\frac{1}{1-\epsilon-\delta}+\kappa+\inf_{\|\mathcal{E}-\mathcal{E}'\|_\diamond\leq 2\delta}\left[P_{D_{\max}|R}(\mathcal{E}')+\Gamma_\kappa(\mathcal{E}')\right] \\ &\leq \log_2\frac{2^\kappa}{1-\epsilon-\delta}+\inf_{\|\mathcal{E}-\mathcal{E}'\|_\diamond\leq 2\delta}P_{D_{\max}|R}(\mathcal{E}')+\Gamma_\kappa^\delta(\mathcal{E}), \end{aligned} \tag{3.30}$$

and the result follows. □

The upper bound in Theorem 3.3.1 contains two terms: $P_{D_{\max}|R}^\delta(\mathcal{E})$ is the contribution from $\mathcal{E}$'s ability to maintain R, and $\Gamma_\kappa^\delta(\mathcal{E})$ is the highest amount of classical information *that can be carried* by *every* resourceless version of $\mathcal{E}$. To illustrate the connection between resource preservability theory and classical communication, let us rewrite the upper bound as $C_{(1)}^\epsilon(\mathcal{E})-\Gamma_\kappa^\delta(\mathcal{N}) \lesssim P_{D_{\max}|R}(\mathcal{E})$, up to an one-shot error term containing ϵ,δ,κ. Then the term '$C_{(1)}^\epsilon(\mathcal{E})-\Gamma_\kappa^\delta(\mathcal{E})$' is related to the amount of transmissible classical information via $\mathcal{N}$'s ability to preserve R. As expected, it is quantitatively controlled by the resource preservability of $\mathcal{E}$.

We remark that when the optimal amount of classical information can be encoded into free states, the optimal capacity should be attainable by channels without resource preservability. Consequently, resource preservability robustness cannot upper bound classical capacity *solely*, and Theorem 3.3.1 is consistent with this fact with the help of $\Gamma_\kappa^\delta(\mathcal{E})$. Concerning tightness, up to one-shot error terms, the inequality is tight, since it is saturated by all state preparation channels of free states; namely, $(\cdot)\mapsto\eta$ with $\eta\in\mathcal{F}$. As an alternative instance to saturate the bound, consider a d-dimensional system with R = coherence. Then the dephasing channel $(\cdot)\mapsto\sum_{i=1}^d |i\rangle\langle i|\cdot|i\rangle\langle i|$ saturates the upper bound with value $\log_2 d$. Finally, we note that $\Gamma_\kappa^\delta(\mathcal{E})$ largely depends on the given state resource theory. In certain cases, it can be explicitly estimated, and the upper bound can be simplified. As a simple example, when R = the athermality with the thermal state γ (see next section for further detail), γ is the only free state and we have $\Gamma_\kappa(\mathcal{E}) = 0 \;\forall\; \mathcal{E}, \kappa$.

Finally, we remark that although Theorem 3.3.1 is stated for resource-preservability theories, the technique adopted in the proof should be able to be extended to other resource theory of channels as long as appropriate assumptions are imposed (e.g., Basic Assumptions 3.1.2). We leave this to the future research.

3.3.2 Applications to Thermodynamics

In this section, we move to the regime of thermodynamics. As we demonstrate below, the ability to keep a system out of thermal equilibrium can be naturally linked to the bath needed for thermalisation. Before stating the main result, we first recap relevant ingredients, including the formulation of the state resource theory of thermodynamics, and the definition of thermalisation bath.

Resource Theory of Thermodynamics

We briefly recap the state resource theory of thermodynamics; namely, the one of athermality. More in-dept introductions can be found in, e.g., Refs. [13, 42, 43, 44], and here we simply mention ingredients relevant to this thesis. *Athermality* depicts the status of a system out of thermal equilibrium. For a given system S with dimension d, the unique free state in this state resource theory is the thermal equilibrium state, or say the *thermal state*. With a given system Hamiltonian H_S and temperature T, the thermal state is uniquely given by

$$\gamma = \frac{e^{-\beta H_S}}{\mathrm{tr}(e^{-\beta H_S})}, \tag{3.31}$$

where $\beta = \frac{1}{k_B T}$ is the inverse temperature and k_B is the Boltzmann constant. It is also possible to allow composite systems in this resource theory. In this case, all allowed systems are $S^{\otimes k}$ [recall Assumption (R1) in Definition 3.1.2], and all free states in this resource theory read $\gamma^{\otimes N}$ with some $N \in \mathbb{N}$. In this thesis, we adopt *Gibbs-preserving channels* as the free operations of this state resource theory. They are channels $\mathcal{E}$ map thermal states to thermal states:

$$\mathcal{E}(\gamma^{\otimes N}) = \gamma^{\otimes M}, \tag{3.32}$$

where d^N and d^M are the input and output dimensions, respectively. Physically, they are dynamics unable to drive thermal equilibrium away from equilibrium. In the language of Sec. 2.2, they form the largest possible set of free operations; namely, $\mathcal{O}_{\gamma|\max}$. Finally, as mentioned previously, '$R = \gamma$' denotes the athermality induced by a given thermal state γ.

Thermalisation Bath Size

In order to address baths for thermalisation, we adopt the thermalisation model proposed by Ref. [45]. A brief introduction is given in this section, and we refer readers to the original article for further details. Consider a system S with Hilbert space $\mathcal{H}_S$ and a bath system B with Hilbert space $\mathcal{H}_S^{\otimes(n-1)}$ ($n \in \mathbb{N}$ is a natural number). The bath system is assumed to possess a given, well-defined temperature T. Moreover, in this model, the bath system has the Hamiltonian $H_B = \sum_{i=1}^{n-1} \mathbb{I}_1 \otimes ... \otimes \mathbb{I}_{i-1} \otimes H_S \otimes \mathbb{I}_{i+1} \otimes ... \otimes \mathbb{I}_{n-1}$, where H_S is the Hamiltonian of the given system S. Let γ be the thermal state associated with T and H_S, as defined in Eq. (3.31). Then the bath is assumed to initially be in the state $\gamma^{\otimes(n-1)}$. Our aim now is to understand how large the bath needs to be in order to successfully thermalise the system S. To this end, a global channel $\mathcal{L}_{SB} : SB \to SB$ is said to *ϵ-thermalise* the system S in the state ρ_S if [45]

$$\left\| \mathcal{L}_{SB} \left[\rho_S \otimes \gamma^{\otimes(n-1)} \right] - \gamma^{\otimes n} \right\|_1 \leq \epsilon. \tag{3.33}$$

In this definition, *thermalisation* of a state in S means that there is a channel that can map the system plus a global bath, namely, SB, to a global thermal state.

In the present model, the system-bath interaction for thermalisation is modeled by the following master equation [45]:

$$\frac{\partial \rho_{SB}(t)}{\partial t} = \sum_k \lambda_k \left[U_{SB}^{(k)} \rho_{SB}(t) U_{SB}^{(k),\dagger} - \rho_{SB}(t) \right], \tag{3.34}$$

where $\rho_{SB}(t)$ is the state of the global system SB at time t, $U_{SB}^{(k)}$ is a unitary operator acting on the global system satisfying $[U_{SB}^{(k)}, H_S + H_B] = 0$ (that is, it is energy-preserving), and λ_k is the rate for $U_{SB}^{(k)}$ to happen[5]. Each unitary $U_{SB}^{(k)}$ models an elastic collision between certain subsystems in SB. Hence, not every channel is allowed in this thermalisation model. For instance, one cannot simply discard the original input state and prepare a global thermal state in SB. We refer the readers to Ref. [45] for the complete framework.

Now, let C_n be the set of all channels $SB \to SB$ that can be generated by the model Eq. (3.34) with a bath system $\mathcal{H}_S^{\otimes(n-1)}$ and a realisation time t. C_n then characterises all allowed physical transformations in the present thermalisation model with a given size of the bath n. From here, one can introduce the following quantity [45]:

$$n_\epsilon(\rho_S) := \inf\{n \in \mathbb{N} \,|\, \exists\, \mathcal{L}_{SB} \in C_n \text{ s.t. Eq. (3.33) holds}\}. \tag{3.35}$$

$n_\epsilon(\rho_S) - 1$ is the smallest number of copies the bath system needs to possess in order to ϵ-thermalise the given state ρ_S in the present model. In other words, $n_\epsilon(\rho_S)$ quantitatively describes the smallest size of the bath needed to thermalise ρ_S.

It turns out that this concept can be directly generalised to channels. For a channel $\mathcal{E} : S \to S$, define the *thermalisation bath size* of a channel $\mathcal{E}$ as

$$\mathcal{B}^\epsilon(\mathcal{E}) := \sup_\rho n_\epsilon[\mathcal{E}(\rho)] - 1, \tag{3.36}$$

which maximises over all the smallest bath sizes among all outputs of $\mathcal{E}$. $\mathcal{B}^\epsilon(\mathcal{E})$ can be understood as the smallest bath size needed to ϵ-thermalise *all* outputs of $\mathcal{E}$ in the

[5] One can see this by checking Eqs. (A2) and (A3) in Appendix A of Ref. [45], which imply that $U_{\mathrm{SB}}^{(k)}$ occurs according to a Poisson distribution with mean value $\lambda_k t$.

given thermalisation model. Finally, before detailing the main result, we mention a core assumption made in Ref. [45]:

Definition 3.3.2. (Energy Subspace Condition [45]) *A given Hamiltonian H with energy levels $\{E_i\}_{i=1}^d$ is said to satisfy the* energy subspace condition *if for every natural number $M \in \mathbb{N}$ and every pair of different vectors $\{\vec{m} \neq \vec{m}'\} \subset (\mathbb{N} \cup \{0\})^d$ satisfying $\sum_{i=1}^d m_i = \sum_{i=1}^d m_i' = M$, we have $\sum_{i=1}^d m_i E_i \neq \sum_{i=1}^d m_i' E_i$.*

The energy subspace condition ensures that the energy levels cannot be integer multiples of each other. Consequently, energy degeneracy is also forbidden.

Thermalisation Bath Size and Athermality Preservability

First, we need to recall the main result of Ref. [45]. To do so, we define the smoothed version of max-relative entropy as [recall Eq. (3.15)]

$$D_{\max}^{\epsilon}(\rho\|\sigma) := \inf_{\frac{1}{2}\|\rho'-\rho\|_1 \le \epsilon} D_{\max}(\rho'\|\sigma). \tag{3.37}$$

Suppose again that γ is the thermal state associated with the given temperature T and system Hamiltonian H_S. Then we have the following thermodynamic interpretation of max-relative entropy from Ref. [45]:

Theorem 3.3.3. (Thermodynamic Meaning of Max-Relative Entropy [45]) *For a given state ρ_S, we have*

$$n_\epsilon(\rho_S) \le \frac{1}{\epsilon^2} 2^{D_{\max}(\rho_S\|\gamma)} + 1. \tag{3.38}$$

Moreover, if the system Hamiltonian H_S satisfies the energy subspace condition given in Definition 3.3.2 and ρ_S is energy-incoherent, i.e., diagonal in the energy eigenbasis of H_S, then we also have

$$D_{\max}^{\sqrt{\epsilon}}(\rho_S\|\gamma) \le \log_2 n_\epsilon(\rho_S). \tag{3.39}$$

Before proving the main result of this section, we need one more lemma to address the continuity of the max-relative entropy. In a finite dimensional case, we say a state is *full-rank* if it has only positive eigenvalues; that is, its support is the whole Hilbert space. Then one can show that:

Lemma 3.3.4. (Bounding Max-Relative Entropy By Trace Norm) *Given three states ρ, ρ', σ, where σ is full-rank. Then we have*

$$\left|2^{D_{\max}(\rho'\|\sigma)} - 2^{D_{\max}(\rho\|\sigma)}\right| \le \frac{\|\rho - \rho'\|_1}{p_{\min}(\sigma)}, \tag{3.40}$$

where $p_{\min}(\sigma)$ is the smallest eigenvalue of σ.

Proof. Define the set $\mathcal{L}(\rho\|\sigma) := \{\lambda \geq 1 \mid \rho \leq \lambda\sigma\}$. Then the max-relative entropy [Eq. (3.15)] reads $D_{\max}(\rho\|\sigma) = \inf_{\lambda\in\mathcal{L}(\rho\|\sigma)} \log_2 \lambda$. Now we observe that $\lambda \in \mathcal{L}(\rho\|\sigma)$ if and only if $\lambda\sigma - \rho \geq 0$, which is true if and only if

$$\inf_{|\phi\rangle} \langle\phi|(\lambda\sigma - \rho)|\phi\rangle \geq 0. \tag{3.41}$$

This means that, for any $\lambda \in \mathcal{L}(\rho\|\sigma)$,

$$\begin{aligned} &\inf_{|\phi\rangle} \langle\phi|(\lambda\sigma - \rho)|\phi\rangle = \inf_{|\phi\rangle} \left[\langle\phi|(\lambda\sigma - \rho')|\phi\rangle + \langle\phi|(\rho' - \rho)|\phi\rangle\right] \\ &\geq \inf_{|\phi\rangle} \langle\phi|(\lambda\sigma - \rho')|\phi\rangle + \inf_{|\phi\rangle} \langle\phi|(\rho' - \rho)|\phi\rangle \geq \inf_{|\phi\rangle} \langle\phi|(\lambda\sigma - \rho')|\phi\rangle - \left\|\rho - \rho'\right\|_1, \end{aligned} \tag{3.42}$$

where we have used the relation (recall that $\|\cdot\|_\infty := \sup_{|\psi\rangle} |\langle\psi| \cdot |\psi\rangle|$ and $\|\cdot\|_\infty \leq \|\cdot\|_1$)

$$\inf_{|\phi\rangle} \langle\phi|(\rho' - \rho)|\phi\rangle = -\sup_{|\phi\rangle} \langle\phi|(\rho - \rho')|\phi\rangle \geq -\left\|\rho - \rho'\right\|_\infty \geq -\left\|\rho - \rho'\right\|_1. \tag{3.43}$$

Since the argument works when we exchange the roles of ρ and ρ', we conclude that

$$\left|\inf_{|\phi\rangle} \langle\phi|(\lambda\sigma - \rho')|\phi\rangle - \inf_{|\phi\rangle} \langle\phi|(\lambda\sigma - \rho)|\phi\rangle\right| \leq \left\|\rho - \rho'\right\|_1. \tag{3.44}$$

Consequently, for a given $\lambda \in \mathcal{L}(\rho\|\sigma)$,

$$\begin{aligned} 0 &\leq \inf_{|\phi\rangle} \langle\phi|(\lambda\sigma - \rho)|\phi\rangle \leq \inf_{|\phi\rangle} \langle\phi|(\lambda\sigma - \rho')|\phi\rangle + \left\|\rho - \rho'\right\|_1 \\ &= \inf_{|\phi\rangle} \langle\phi| \left[\left(\lambda + \frac{\|\rho - \rho'\|_1}{\langle\phi|\sigma|\phi\rangle}\right)\sigma - \rho'\right] |\phi\rangle \leq \inf_{|\phi\rangle} \langle\phi| \left[\left(\lambda + \frac{\|\rho - \rho'\|_1}{p_{\min}(\sigma)}\right)\sigma - \rho'\right] |\phi\rangle. \end{aligned} \tag{3.45}$$

Note that $\langle\phi|\sigma|\phi\rangle > 0$ since σ is full-rank and hence has only strictly positive eigenvalues. Also, $\langle\phi|\sigma|\phi\rangle \geq p_{\min}(\sigma) = \inf_{|\psi\rangle} \langle\psi|\sigma|\psi\rangle$ for all $|\phi\rangle$. This computation implies $\lambda + \frac{\|\rho-\rho'\|_1}{p_{\min}(\sigma)} \in \mathcal{L}(\rho'\|\sigma)$ whenever $\lambda \in \mathcal{L}(\rho\|\sigma)$ [recall Eq. (3.41)]. Hence, we have

$$2^{D_{\max}(\rho'\|\sigma)} = \inf_{\lambda\in\mathcal{L}(\rho'\|\sigma)} \lambda \leq \inf_{\lambda\in\mathcal{L}(\rho\|\sigma)} \left(\lambda + \frac{\|\rho - \rho'\|_1}{p_{\min}(\sigma)}\right) = 2^{D_{\max}(\rho\|\sigma)} + \frac{\|\rho - \rho'\|_1}{p_{\min}(\sigma)}. \tag{3.46}$$

Then the desired bound follows by exchanging the roles of ρ and ρ'. □

We remark that Lemma 3.3.4 implies that the function $2^{D_{\max}(\cdot\|\sigma)}$ is Lipschitz continuous when σ is full-rank. Now we present the main result of this section, which is a dynamical generalisation of Theorem 3.3.3. In what follows, a channel is said to be *coherence-annihilating* if it only outputs states diagonal in the given energy eigenbasis corresponding to a given Hamiltonian. Also, recall again that $p_{\min}(P)$ is the smallest eigenvalue of the operator P.

Main Theorem 3.3.5. (Athermality Preservability and Thermalisation Bath) *Given a Gibbs-preserving map $\mathcal{E}$ and $0 \le \epsilon < 1$. Then we have*

$$\mathcal{B}^{\epsilon}(\mathcal{E}) \le \frac{1}{\epsilon^2} 2^{P_{D_{\max}|\gamma}(\mathcal{E})}. \tag{3.47}$$

Moreover, if the thermal state γ is full-rank, $\mathcal{E}$ is coherence-annihilating, and the system Hamiltonian H_S satisfies the energy subspace condition given in Definition 3.3.2, then we have

$$2^{P_{D_{\max}|\gamma}(\mathcal{E})} \le \mathcal{B}^{\epsilon}(\mathcal{E}) + \frac{2\sqrt{\epsilon}}{p_{\min}(\gamma)} + 1. \tag{3.48}$$

Proof. First, in the state resource theory of athermality with the thermal state γ in S, we have $\mathcal{A} = \mathcal{A}_{\text{abs}}$ ($R = \gamma$), which only contains state preparation channels of thermal state of the form $\gamma^{\otimes N}$, where $N \in \mathbb{N}$ is some natural number. Then, with $R = \gamma$ and a channel $\mathcal{E}$ having S as its input system, direct computation shows

$$\begin{aligned} P_{D_{\max}|\gamma}(\mathcal{E}) &:= \inf_{\Lambda_S \in \mathcal{A}} \sup_{A;\rho_{SA},\widetilde{\Lambda}_A \in \mathcal{A}_{\text{abs}}} D_{\max}\left[(\mathcal{E}\otimes\widetilde{\Lambda}_A)(\rho_{SA})\middle\|(\Lambda_S\otimes\widetilde{\Lambda}_A)(\rho_{SA})\right] \\ &= \sup_{A;\rho_{SA}} D_{\max}[(\mathcal{E}\otimes\Phi_{\gamma_A})(\rho_{SA})\|\Phi_{\gamma\otimes\gamma_A}(\rho_{SA})] \\ &= \sup_{A;\rho_S} D_{\max}[\mathcal{E}(\rho_S)\otimes\gamma_A\|\gamma\otimes\gamma_A] = \sup_{\rho} D_{\max}[\mathcal{E}(\rho)\|\gamma]. \end{aligned} \tag{3.49}$$

Note that the ancillary systems A's must be of allowed system dimension [Assumption (R1)], meaning that A's can only be $S^{\otimes N}$ with $N \in \mathbb{N}$ [see also the comment below Eq. (3.10)]. To see the last equality, note that for any operator K and any positive operator E, $K \ge 0$ if and only if $K \otimes E \ge 0$. This implies $D_{\max}(\rho\otimes\eta\|\sigma\otimes\eta) = D_{\max}(\rho\|\sigma)$ for all states ρ,σ,η. Combining Eq. (3.49) with Theorem 3.3.3 and Eq. (3.36), we conclude the bound Eq. (3.47):

$$\mathcal{B}^{\epsilon}(\mathcal{E}) := \sup_{\rho} n_{\epsilon}[\mathcal{E}(\rho)] - 1 \le \sup_{\rho} \frac{1}{\epsilon^2} 2^{D_{\max}[\mathcal{E}(\rho)\|\gamma]} = \frac{1}{\epsilon^2} 2^{P_{D_{\max}|\gamma}(\mathcal{E})}. \tag{3.50}$$

To show Eq. (3.48), we apply Theorem 3.3.3 and Lemma 3.3.4 to conclude

$$\begin{aligned} 1 + \mathcal{B}^{\epsilon}(\mathcal{E}) &:= \sup_{\rho} n_{\epsilon}[\mathcal{E}(\rho)] \ge \sup_{\rho} 2^{D_{\max}^{\sqrt{\epsilon}}[\mathcal{E}(\rho)\|\gamma]} = \sup_{\rho} \inf_{\frac{1}{2}\|\rho'-\mathcal{E}(\rho)\|_1 \le \sqrt{\epsilon}} 2^{D_{\max}(\rho'\|\gamma)} \\ &\ge \sup_{\rho}\left(2^{D_{\max}[\mathcal{E}(\rho)\|\gamma]} - \frac{2\sqrt{\epsilon}}{p_{\min}(\gamma)}\right) = 2^{P_{D_{\max}|\gamma}(\mathcal{E})} - \frac{2\sqrt{\epsilon}}{p_{\min}(\gamma)}, \end{aligned} \tag{3.51}$$

and the proof is completed. □

As expected, the thermalisation bath size of a Gibbs-preserving channel $\mathcal{E}$, i.e., $\mathcal{B}^{\epsilon}(\mathcal{E})$, can measure how robust the channel is against thermalisation effect. Equation (3.47) tells us that the weaker the channel is at maintaining athermality, the smaller the bath needed to thermalise all its output. Theorem 3.3.5 provides a link to connect the ability to preserve athermality and the resource needed to thermalise all outputs of a given quantum dynamics. It also gives to the athermality preservability robustness a thermodynamic interpretation in the context of thermalisation.

3.3.3 Connecting Thermodynamics and Classical Communication

As the last application of resource preservability theories, we combine Theorem 3.3.1 and Theorem 3.3.5 to bridge classical communication and thermodynamics. To this end, we need the following lemma that connects classical communication, resource preservability theories, and thermodynamic properties:

Lemma 3.3.6. (Athermality Bound Lemma) *Consider a given state resource theory $(R,\mathcal{F},\mathcal{O})$ satisfying Basic Assumptions 3.1.2, $0\le\epsilon<1$ & $0<\kappa<1$, and a full-rank thermal state γ. For a Gibbs-preserving $\mathcal{E}\in\mathcal{O}$ and every $\Lambda^{\mathcal{E}}\in\mathcal{A}(\kappa;\mathcal{E})$, we have*

$$C_{(1)}^{\epsilon}(\mathcal{E})\le P_{D_{\max}|R}(\mathcal{E})+P_{D_{\max}|\gamma}\left(\Lambda^{\mathcal{E}}\right)+\log_2\frac{2^{\kappa}}{1-\epsilon}. \tag{3.52}$$

Proof. Equation (3.49) implies that $P_{D_{\max}|\gamma}(\mathcal{E})=\sup_\rho D_{\max}[\mathcal{E}(\rho)\|\gamma]$. This means $\mathcal{E}(\rho)\le 2^{P_{D_{\max}|\gamma}(\mathcal{E})}\gamma\ \forall\rho$, which further means that

$$\begin{aligned}\Gamma_\kappa(\mathcal{E})&:=\log_2\inf_{\Lambda^{\mathcal{E}}\in\mathcal{A}(\kappa;\mathcal{E})}\sup_{\Theta_M}\sum_{m=0}^{M-1}\mathrm{tr}\left[E_m\Lambda^{\mathcal{E}}(\rho_m)\right]\\&\le\log_2\inf_{\Lambda^{\mathcal{E}}\in\mathcal{A}(\kappa;\mathcal{E})}2^{P_{D_{\max}|\gamma}(\Lambda^{\mathcal{E}})}\sup_{\Theta_M}\sum_{m=0}^{M-1}\mathrm{tr}\left(E_m\gamma\right)=\inf_{\Lambda^{\mathcal{E}}\in\mathcal{A}(\kappa;\mathcal{E})}P_{D_{\max}|\gamma}\left(\Lambda^{\mathcal{E}}\right),\end{aligned} \tag{3.53}$$

and the result follows from Theorem 3.3.1. □

Lemma 3.3.6 tells us that the ability of a Gibbs-preserving $\mathcal{E}\in\mathcal{O}$ to transmit classical information is limited by its ability to maintain R, plus the ability of its resourceless version (to R) to maintain athermality. Using this lemma, we are in position to show the main result of this section. In what follows, '$R=\mathrm{Coh}$' denotes coherence with respect to the energy eigenbasis of the given system Hamiltonian that induces the thermal state, and $\mathcal{A}_{R=\mathrm{Coh}}(\kappa,\mathcal{E})$ denotes the set defined in Eq. (3.18) when we set $R=\mathrm{Coh}$.

Main Theorem 3.3.7. (Thermalisation and Classical Communication) *Consider $0\le\epsilon,\delta<1$ & $0<\kappa<1$ and a full-rank thermal state γ. Assume the system Hamiltonian satisfies the energy subspace condition given in Definition 3.3.2. Then for any Gibbs-preserving channel $\mathcal{E}$ of γ that is also coherence non-generating, that is, $\mathcal{E}\in\mathcal{O}_{\gamma|\max}\cap\mathcal{O}_{\mathrm{Coh}|\max}$, we have*

$$C_{(1)}^{\epsilon}(\mathcal{E})\le P_{D_{\max}|\mathrm{Coh}}(\mathcal{E})+\log_2\left(\mathcal{B}_{\gamma}^{\delta}\left(\Lambda^{\mathcal{E}}\right)+\frac{2\sqrt{\delta}}{p_{\min}(\gamma)}+1\right)+\log_2\frac{2^{\kappa}}{1-\epsilon} \tag{3.54}$$

for every $\Lambda^{\mathcal{E}}\in\mathcal{A}_{R=\mathrm{Coh}}(\kappa,\mathcal{E})$.

Proof. Setting $R=\mathrm{Coh}$ in Lemma 3.3.6 leads to

$$C_{(1)}^{\epsilon}(\mathcal{E})\le P_{D_{\max}|\mathrm{Coh}}(\mathcal{E})+P_{D_{\max}|\gamma}\left(\Lambda^{\mathcal{E}}\right)+\log_2\frac{2^{\kappa}}{1-\epsilon} \tag{3.55}$$

for every $\Lambda^{\mathcal{E}} \in \mathcal{A}_{R=\mathrm{Coh}}(\kappa, \mathcal{E})$. Since $\Lambda^{\mathcal{E}}$ is coherence-annihilating, a direct application of Theorem 3.3.5 implies that

$$C^{\epsilon}_{(1)}(\mathcal{E}) \leq P_{D_{\max}|\mathrm{Coh}}(\mathcal{E}) + \log_2\left(\mathcal{B}^{\delta}_{\gamma}(\Lambda^{\mathcal{N}}) + \frac{2\sqrt{\delta}}{p_{\min}(\gamma)} + 1\right) + \log_2 \frac{2^{\kappa}}{1-\epsilon}, \tag{3.56}$$

which completes the proof. □

Theorem 3.3.7 illustrates how dynamical resource theories connect a thermodynamic property and a measure in the classical communication theory. To illustrate the physical meaning of this result, let us first focus on the special case when $\mathcal{E}$ is coherence-annihilating; namely, $\mathcal{E} \in \mathcal{A}$ with $R = \mathrm{Coh}$ (recall Definition 3.1.3). Then one can choose $\mathcal{E} = \Lambda^{\mathcal{E}}$ and $\kappa = 0$ in Theorem 3.3.7. In this case, if $\mathcal{N}$ can transmit a high amount of classical information [i.e., the left-hand-side of Eq. (3.54) is high], it necessarily requires a large bath to thermalise all its outputs [i.e., the right-hand-side of Eq. (3.54) is forced to be high]. On the other hand, if $\mathcal{E}$ only need a small thermalisation bath, it unavoidably has a weak ability to transmit classical information. Theorem 3.3.7 provides a quantitative description of this qualitative intuition; furthermore, it suggests the following physical message in the present setting:

If a channel can transmit n bits of classical information, then thermalising its output requires a bath size at least $2^n - 1$*, up to one-shot error terms.*

In other words, thermalisation bath size is a *thermodynamic prerequisite* needed to *transmit* classical information. When $\mathcal{E}$ is able to preserve coherence, interestingly, the prerequisite to transmit classical information becomes a hybrid term, containing the thermalisation bath size of $\mathcal{N}$'s incoherent version, i.e., $\Lambda^{\mathcal{E}} \in \mathcal{A}_{R=\mathrm{Coh}}(\kappa, \mathcal{E})$, *plus* the ability of $\mathcal{E}$ to maintain coherence. Namely, it is a combination of a thermodynamic property of $\mathcal{E}$'s 'classical counterpart', and the quantum effect maintained by $\mathcal{E}$. In this sense, we interpret Theorem 3.3.7 as a connection between thermodynamics and classical communication.

Note that, as expected, state preparation channels of the given thermal state cannot transmit any amount of classical information, since there is no need to have any bath for thermalisation. This means that the inequality in Theorem 3.3.7 is tight. Still, the inequality cannot be saturated in general, and it is natural to ask whether one can improve this bound by using different entropy quantities and alternative thermodynamic properties. This is in fact achievable by adopting the so-called hypothesis testing relative entropy–using this entropy measure, one is able to connect one-shot classical capacity with an one-shot work extraction task. Since these results are beyond the scope of this thesis, we refer the readers to Ref. [9] for further details.

Chapter 4

Quantum Channel Marginal Problem

A fundamental question in the study of quantum theory is whether a given set of local states are compatible with a global one; namely, can the former be the *marginals* of a single, global state? This kind of questions are known as *quantum marginal problems*, or *state marginal problems* (SMPs) in this thesis. A well-known example of SMPs is the 2-*body N-representability problem* asking which 2-body reduced states can be marginals of a global state of N particles. This problem is motivated by finding ground states of 2-body Hamiltonians (see, e.g., [24, 25]). Due to its relevance, SMPs have been studied in many different contexts, such as entanglement detection [99, 47], nonlocality detection [48, 49], and efficient measurement construction strategies for estimating marginal states [50, 51, 52]. SMPs are only concerned with *static* properties carried by states, and we aim at understanding how compatibility between local and global physical descriptions extends to the *dynamical* regime. Namely, our major objective is

To seek a natural dynamical generalisation of SMPs.

4.1 Formulation

To extend SMPs to the dynamical regime, the very first step is to formally know what SMPs are (see also Fig. 4.1):

Definition 4.1.1. (State Marginal Problems) *Consider a global system S and a set of local states $\{\rho_X\}_{X\in\Lambda}$, where Λ is a collection of subsystems X of S and each ρ_X is a state in the system X. Then a* state marginal problem (SMP) *asks whether there exists a global state ρ_S in S* compatible *with all of them, that is,*

$$\exists\, \rho_S \in \mathrm{STATE}_S \quad \text{such that} \quad \mathrm{tr}_{S\setminus X}(\rho_S) = \rho_X \quad \forall\, X \in \Lambda. \tag{4.1}$$

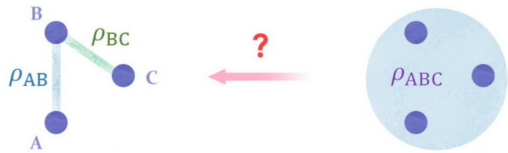

Figure 4.1: A *state marginal problem* (Definition 4.1.1) asks whether a given set of local states (e.g., ρ_{AB}, ρ_{BC}) are marginal states of a global state (ρ_{ABC}).

An immediate observation is that SMPs always have a trivial solution via tensor product if marginals do not overlap, i.e., $\rho_S = \bigotimes_{X\in\Lambda} \rho_X$. Hence, SMPs become much more interesting and non-trivial in the overlapping regions. For SMPs to be well-posed in the overlapping cases, ρ_{X}'s must be compatible in the overlapping regions; namely, their reduced states must be equal. Formally, these are called *local compatibility conditions*:

$$\mathrm{tr}_{X\setminus(X\cap Y)}(\rho_X) = \mathrm{tr}_{Y\setminus(X\cap Y)}(\rho_Y) \quad \forall\, X, Y \in \Lambda. \tag{4.2}$$

Note that local compatibility are necessary and easy to verify but, unfortunately, not sufficient to guarantee solutions to SMPs.

4.1.1 Channel Marginal Problems

To formulate the dynamical version of SMPs, one may be tempted to define it as follows: given a set of local channels $\{\mathcal{E}_X\}_{X\in\Lambda}$, where Λ is a collection of subsystems X of S and each $\mathcal{E}_X$ is a channel acting as $X \to X$, the dynamical version of SMP asks whether there exists a global channel $\mathcal{E}_S$ *compatible with all of them*. To formalise this compatibility condition, however, is not obvious because the concept of *marginals* for channels is not as clear as that for states. In fact, being an input-output process, the existence of well-defined marginals for a dynamics needs to satisfy certain no-signaling conditions. To see this, consider the classical case first. A dynamical map from the inputs $S' = \{s'_i\}$ to the outputs $S = \{s_j\}$ is given by a so-called stochastic matrix $P_{S|S'}$, which is simply a matrix whose element (i, j) is a conditional transition probability from state i to j. Given $X' \subset S'$ and $X \subset S$, $P_{S|S'}$ has a well-defined marginal from X' to X, or simply a reduced map $P_{X|X'}$, if and only if for every input probability distribution $p_{S'}$ we have

$$P_{X|X'} \sum_{s'_i\in S'\setminus X'} p_{S'} = \sum_{s_i\in S\setminus X} P_{S|S'} p_{S'}, \tag{4.3}$$

where $P_{B|A} p_A \coloneqq \sum_{a_1,\ldots,a_k} P_{b_1,\ldots,b_k|a_1,\ldots,a_k} p_{a_1,\ldots,a_k}$. In fact, this is equivalent to the condition that the map $P_{S|S'}$ is *no-signaling* from $S'\setminus X'$ to X; that is, $\sum_{s_i\in S\setminus X} P_{S|S'}$ is independent of the input in $S' \setminus X'$. Equation (4.3) has a clear generalisation in the quantum regime:

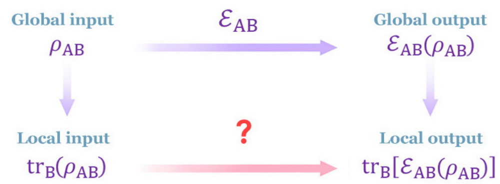

Figure 4.2: Commutation diagram used to define the *marginal channel* in a subsystem (Definition 4.1.2). For a local input coming from part of a global input, $\mathrm{tr}_B(\rho_{AB})$, the marginal channel $\mathrm{Tr}_{B|B}(\mathcal{E}_{AB})$ of a global channel $\mathcal{E}_{AB}$ needs to map it to the marginal state of the global output, $\mathrm{tr}_B[\mathcal{E}_{AB}(\rho_{AB})]$. Note that, unlike the case for states, the marginal of a channel does not always exist.

Definition 4.1.2. (Marginal Channels) *Given a global channel $\mathcal{E}_{S|S'}$ and subsystems $X' \subseteq S', X \subseteq S$, $\mathcal{E}_{S|S'}$ is said to have a well-defined* marginal channel *from X' to X, or reduced channel $\mathcal{E}_{X|X'}$, if for every state $\rho_{S'}$*

$$\mathcal{E}_{X|X'} \circ \mathrm{tr}_{S'\setminus X'}(\rho_{S'}) = \mathrm{tr}_{S\setminus X} \circ \mathcal{E}_{S|S'}(\rho_{S'}). \tag{4.4}$$

On the other hand, a local channel $\mathcal{E}_{X|X'}$ and a global channel $\mathcal{E}_{S|S'}$ are called compatible *if they can achieve Eq. (4.4).*

See Fig. 4.2 for a brief illustration. If Eq. (4.4) holds, we denote the marginal channel by the following notation

$$\mathrm{Tr}_{S\setminus X|S'\setminus X'}(\mathcal{E}_{S|S'}) := \mathcal{E}_{X|X'}. \tag{4.5}$$

Interestingly, we obtain Eq. (4.4) once we *quantise* inputs and outputs of no-signaling correlations in Eq. (4.3). Channels satisfying Eq. (4.4) are known in the literature as *semi-causal channels* in X with respect to $S' \setminus X'$ or *no-signaling* from $S' \setminus X'$ to X [53, 54, 55, 56]. In fact, these channels are also equivalent to quantum dynamics achievable by one-way communication from X' to $S \setminus X$ [54], also known as *semi-localisable* channels in X.

It has been shown [53, 54, 55, 56, 57, 58] that Eq. (4.4) can equivalently be expressed in terms of Choi states [Eq. (2.5)] as follows:

Lemma 4.1.3. (Compatibility Lemma) *$\mathcal{E}_{S|S'}$ is compatible with $\mathcal{E}_{X|X'}$ if and only if*

$$\mathrm{tr}_{S\setminus X}\left(\mathcal{E}^{\mathcal{J}}_{SS'}\right) = \mathcal{E}^{\mathcal{J}}_{XX'} \otimes \frac{\mathbb{I}_{S'\setminus X'}}{d_{S'\setminus X'}}. \tag{4.6}$$

Proof. For every input state $\rho_{X'}$ in X', one can use Eq. (2.7) to write $\mathcal{E}_{X|X'}(\rho_{X'}) = d_{X'}\mathrm{tr}_{X'}[(\mathbb{I}_X \otimes \rho^t_{X'})\mathcal{E}^{\mathcal{J}}_{XX'}]$, where $(\cdot)^t$ is the transpose map. This means

$$\frac{1}{d_{S'}}\mathrm{tr}_{S\setminus X} \circ \mathcal{E}_{S|S'}(\rho_{S'}) = \mathrm{tr}_{S\setminus X} \circ \mathrm{tr}_{S'}\left[\left(\mathbb{I}_S \otimes \rho^t_{S'}\right)\mathcal{E}^{\mathcal{J}}_{SS'}\right] = \mathrm{tr}_{S'}\left[\left(\mathbb{I}_X \otimes \rho^t_{S'}\right)\mathrm{tr}_{S\setminus X}\left(\mathcal{E}^{\mathcal{J}}_{SS'}\right)\right]. \tag{4.7}$$

On the other hand, we have

$$\frac{1}{d_{X'}}\mathcal{E}_{X|X'}\circ \mathrm{tr}_{S'\setminus X'}(\rho_{S'}) = \mathrm{tr}_{X'}\left\{\left[\mathbb{I}_X\otimes(\mathrm{tr}_{S'\setminus X'}(\rho_{S'}))^t\right]\mathcal{E}^{\mathcal{J}}_{XX'}\right\} = \mathrm{tr}_{S'}\left[\left(\mathbb{I}_X\otimes\rho^t_{S'}\right)\left(\mathcal{E}^{\mathcal{J}}_{XX'}\otimes\mathbb{I}_{S'\setminus X'}\right)\right], \tag{4.8}$$

where we use the identity $\mathrm{tr}_{\mathrm{A}}(\rho_{\mathrm{AB}})^t = \mathrm{tr}_{\mathrm{A}}(\rho^t_{\mathrm{AB}})$ [1]. Since $d_{X'}d_{S'\setminus X'} = d_{S'}$, we learn that Eq. (4.6) ensures that $\mathcal{E}_{S|S'}$ is compatible with $\mathcal{E}_{X|X'}$.

To show the necessity, note that $\mathcal{E}_{X|X'}\circ\mathrm{tr}_{S'\setminus X'}(\rho_{S'}) = \mathrm{tr}_{S\setminus X}\circ\mathcal{E}_{S|S'}(\rho_{S'})$ implies

$$\mathrm{tr}_{S\setminus X}\left(\mathcal{E}^{\mathcal{J}}_{SS'}\right) = \left[(\mathcal{E}_{X|X'}\circ\mathrm{tr}_{S'\setminus X'})\otimes\mathcal{I}_{S'}\right](\Psi^+_{S'S'}) = \mathcal{E}^{\mathcal{J}}_{XX'}\otimes\frac{\mathbb{I}_{S'\setminus X'}}{d_{S'\setminus X'}}. \tag{4.9}$$

This completes the proof. □

With the above notion of marginal channels, we are now in the position to formally state the dynamical generalisation of SMPs, termed *channel marginal problems*:

Definition 4.1.4. (Channel Marginal Problems) *Consider global systems S', S and a set of local channels $\{\mathcal{E}_{X|X'}\}_{X|X'\in\Lambda}$, where $\Lambda := \{X|X'\}$ is a collection of input-output pairs with $X'\subseteq S', X\subseteq S$. Then a* channel marginal problem (CMP) *asks whether there exists a global channel $\mathcal{E}_{S|S'}$ compatible with all of them, that is,*

$$\exists\ \mathcal{E}_{S|S'}\in \mathrm{CPTP}_{S|S'}\quad \text{such that}\quad \mathrm{Tr}_{S\setminus X|S'\setminus X'}(\mathcal{E}_{S|S'}) = \mathcal{E}_{X|X'}\quad \forall\ X|X'\in\Lambda. \tag{4.10}$$

We say the collection $\{\mathcal{E}_{X|X'}\}_{X|X'\in\Lambda}$ is compatible *if there exists at least one global channel $\mathcal{E}_{S|S'}$ achieving Eq. (4.10).*

See also Fig. 4.3. With this definition, the analogies between CMP and SMP are clear. Similar to the case of SMPs, when the systems X's and X''s are non-overlapping, CMPs have a trivial solution, which is the product channel $\bigotimes_{X|X'\in\Lambda}\mathcal{E}_{X|X'}$. When overlapping marginals are considered, we again need to first verify whether CMPs are well-posed. This includes checking whether the overlapping channels coincide in the common region. In analogy to Eq. (4.2), for every $X|X', Y|Y'\in\Lambda$ we need,

$$\mathrm{Tr}_{X\setminus Y|X'\setminus Y'}(\mathcal{E}_{X|X'}) = \mathrm{Tr}_{Y\setminus X|Y'\setminus X'}(\mathcal{E}_{Y|Y'}). \tag{4.11}$$

As for states, a set of channels $\{\mathcal{E}_{X|X'}\}_{X|X'\in\Lambda}$ satisfying this condition is said to be *locally compatible*. In fact, applying Lemma 4.1.3 to X, Y and $X\cap Y$ (and hence also X', Y' and $X'\cap Y'$) shows a characterisation for this condition:

Lemma 4.1.5. (Local Compatibility Lemma) *If $\mathcal{E}_{X|X'}$ and $\mathcal{E}_{Y|Y'}$ are compatible, then they are also locally compatible; namely,*

$$\mathrm{Tr}_{X\setminus Y|X'\setminus Y'}(\mathcal{E}_{X|X'}) = \mathrm{Tr}_{Y\setminus X|Y'\setminus X'}(\mathcal{E}_{Y|Y'}). \tag{4.12}$$

[1] To see this identity, we write $\rho_{AB} = \sum_{ijkl} f_{ij|kl}|i\rangle\langle j|_A\otimes|k\rangle\langle l|_B$, where $\{|i\rangle_A\}_i$ and $\{|k\rangle_B\}_j$ are the bases for the definition of the transpose map $(\cdot)^t$. Then direct computation shows $\mathrm{tr}_A(\rho_{AB})^t = \left(\sum_n\sum_{ijkl} f_{ij|kl}\langle n|i\rangle_A\langle j|n\rangle_A\times|k\rangle\langle l|_B\right)^t = \sum_{nkl} f_{nn|kl}|l\rangle\langle k|_B = \mathrm{tr}_A\left(\sum_{ijkl} f_{ij|kl}|j\rangle\langle i|_A\otimes|l\rangle\langle k|_B\right) = \mathrm{tr}_A(\rho^t_{AB})$.

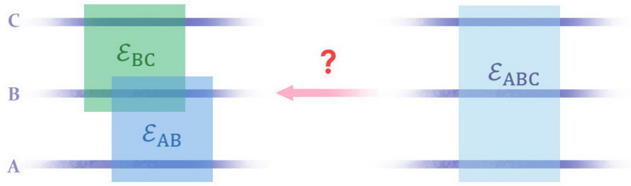

Figure 4.3: A *channel marginal problem* (Definition 4.1.4) asks whether a given set of local channels (e.g., $\mathcal{E}_{AB}, \mathcal{E}_{BC}$) are marginal channels of a global channel ($\mathcal{E}_{ABC}$).

Proof. To see this, we redefine the systems as $A = X \setminus Y$, $B = X \cap Y$, $C = Y \setminus X$, and $S = A \cup B \cup C$ (similar definitions apply to A', B', C', S'). Now, suppose that $\mathcal{E}_{X|X'} = \mathcal{E}_{AB|A'B'}$ and $\mathcal{E}_{Y|Y'} = \mathcal{E}_{BC|B'C'}$ are compatible, then there exists a global channel $\mathcal{E}_{S|S'}$ such that $\mathrm{Tr}_{C|C'}(\mathcal{E}_{S|S'}) = \mathcal{E}_{AB|A'B'}$ and $\mathrm{Tr}_{A|A'}(\mathcal{E}_{S|S'}) = \mathcal{E}_{BC|B'C'}$. Then Lemma 4.1.3 implies that $\mathrm{tr}_A\left(\mathcal{E}^{\mathcal{J}}_{XX'}\right) \otimes \frac{\mathbb{I}_{C'}}{d_{C'}} = \mathrm{tr}_{AC}\left(\mathcal{E}^{\mathcal{J}}_{SS'}\right) = \mathrm{tr}_C\left(\mathcal{E}^{\mathcal{J}}_{YY'}\right) \otimes \frac{\mathbb{I}_{A'}}{d_{A'}}$. Tracing out the system C' and using Lemma 4.1.3 again, we learn that $\mathcal{E}_{X|X'}$ is compatible with a channel in $B|B'$ [$\mathrm{Tr}_{X\setminus Y|X'\setminus Y'}(\mathcal{E}_{X|X'})$] with the Choi state $\mathrm{tr}_{CC'}\left(\mathcal{E}^{\mathcal{J}}_{YY'}\right) = \mathrm{tr}_{ACA'C'}\left(\mathcal{E}^{\mathcal{J}}_{SS'}\right)$. Similarly, by tracing out A', one can show that $\mathcal{E}_{Y|Y'}$ is compatible with a channel in $B|B'$ [$\mathrm{Tr}_{Y\setminus X|Y'\setminus X'}(\mathcal{E}_{Y|Y'})$] with the same Choi state $\mathrm{tr}_{AA'}\left(\mathcal{E}^{\mathcal{J}}_{XX'}\right) = \mathrm{tr}_{ACA'C'}\left(\mathcal{E}^{\mathcal{J}}_{SS'}\right)$. □

As a remark on Definition 4.1.4, by using Lemma 4.1.3 it is clear that CMPs can be rephrased via Choi states as SMPs with overlapping marginals, but the formulation involves an *additional tensor product* structure taking into account the quantum non-signaling constraints associated to the dynamical problem (see Lemma 4.6). Hence, the CMP is not equivalent to the SMP for the Choi states. However, in the special case of broadcasting compatibility that will be discussed later, where all X' coincide with the input global system S', the 'tensor identity' parts in Lemma 4.1.3 disappear. In this special case, CMPs reduces to SMPs for Choi states, recovering the result of Ref. [59].

Now we showcase how the CMPs include as special cases several problems considered before in the studies of quantum information theory and physics. Consider the case where all X' coincide with S'; that is, $\Lambda = \{X|S'\}$. In this case, the no-signaling condition in Definition 4.1.2 trivialises, and, consequently, the global channels $\mathcal{E}_{S|S'}$ automatically have well-defined marginals $\mathcal{E}_{X|S'}$. CMPs then reduces to the question of the existence of a global channel $\mathcal{E}_{S|S'}$ such that $\mathrm{tr}_{S\setminus X} \circ \mathcal{E}_{S|S'} = \mathcal{E}_{X|S'}$ $\forall$ X. This is known as *broadcasting (in)compatibility*, which is a notion that has been studied extensively as a natural generalisation of measurement (in)compatibility [59, 60, 61, 62, 63, 64, 65]. When the channels under consideration are identity channels, the non-existence of a global channel is the well-known no-broadcasting theorem [66]. Next, consider the case where $X'_i = X_i = AB_i$, where all B_i's are isomorphic. Given identical channels $\mathcal{E}_{AB_i}$, the CMP asks whether there exists an *extension* to a global channel $\mathcal{E}_{AB_1 \ldots B_n}$. This notion of *channel extendibility* was recently introduced to extend the state extendibility to the dynamical regime. Also, it has been used in the studies of quantum communication scenarios [67] and testing symmetries on a quantum computer [68].

Finally, before proceeding to quantitatively study channel incompatibility, we specify notations. From now on, we denote the vector of local channels defining a given CMP by $\boldsymbol{\mathcal{E}} := \{\mathcal{E}_{X|X'}\}_{X|X'\in\Lambda}$. We say that $\boldsymbol{\mathcal{E}}$ is *compatible* whenever the corresponding CMP has a solution; that is, there exists a global channel $\mathcal{E}_{S|S'}$ compatible with each $\mathcal{E}_{X|X'}$ in $\boldsymbol{\mathcal{E}}$. The set of compatible local channels is denoted by $\mathfrak{C}$.

4.2 Channel Incompatibility Robustness

To quantitatively understand dynamical incompatibility, a suitable measure is needed. Following an approach similar to the one adopted in Sec. 3.2.1, we introduce a robustness measure, dubbed *channel incompatibility robustness*, which gives an efficient solution to CMPs and provides a quantitative measure of incompatibility:

$$R(\boldsymbol{\mathcal{E}}) := \max\{\lambda \in [0,1] \,|\, \lambda\boldsymbol{\mathcal{E}} + (1-\lambda)\boldsymbol{\mathcal{N}} \in \mathfrak{C}\}, \tag{4.13}$$

where the maximisation runs over vectors of local channels $\boldsymbol{\mathcal{N}} = \{\mathcal{N}_{X|X'}\}_{X|X'\in\Lambda}$. Note that the linear combination of $\boldsymbol{\mathcal{E}},\boldsymbol{\mathcal{N}}$ is defined component-wise, i.e., $a\boldsymbol{\mathcal{E}} + b\boldsymbol{\mathcal{N}} := \{a\mathcal{E}_{X|X'} + b\mathcal{N}_{X|X'}\}_{X|X'\in\Lambda}$. Channel incompatibility robustness serves as a measure of incompatibility in the following sense: $R(\boldsymbol{\mathcal{E}}) = 1$ if and only if $\boldsymbol{\mathcal{E}} \in \mathfrak{C}$, meaning that the CMP for $\boldsymbol{\mathcal{E}}$ admits a solution. Furthermore, a value $R < 1$ can detect instances in which the local compatibility condition is *not* sufficient to ensure the existence of a global channel, and we refer the reader to Sec. 4.3.1 for examples.

Note that being locally *incompatible* does not imply $R = 0$. For example, in a 3-qubit setting, consider

$$\mathcal{E}^{\mathcal{J}}_{AA'BB'} = |00\rangle\langle 00|_{\mathrm{AB}} \otimes \frac{\mathbb{I}_{A'B'}}{4} \quad \& \quad \mathcal{E}^{\mathcal{J}}_{CC'BB'} = |11\rangle\langle 11|_{CB} \otimes \frac{\mathbb{I}_{C'B'}}{4}, \tag{4.14}$$

which are not locally compatible in B. However, by considering the noise channels

$$\mathcal{N}^{\mathcal{J}}_{AA'BB'} = |01\rangle\langle 01|_{AB} \otimes \frac{\mathbb{I}_{A'B'}}{4} \quad \& \quad \mathcal{N}^{\mathcal{J}}_{CC'BB'} = |10\rangle\langle 10|_{CB} \otimes \frac{\mathbb{I}_{C'B'}}{4}, \tag{4.15}$$

one can check that $\frac{1}{2}\mathcal{E}^{\mathcal{J}}_{XX'BB'} + \frac{1}{2}\mathcal{N}^{\mathcal{J}}_{XX'BB'}$ $(X = A, C)$ is the marginal of the global 3-qubit channel, whose Choi state is

$$|0\rangle\langle 0|_A \otimes \frac{\mathbb{I}_B}{2} \otimes |1\rangle\langle 1|_C \otimes \frac{\mathbb{I}_{A'B'C'}}{8}. \tag{4.16}$$

Hence, they have $R \geq 0.5$ even though they are not locally compatible.

Importantly, one can show that the computation of the channel incompatibility robustness is a *semi-definite program* (SDP) (see, e.g., Ref. [27]). This means that channel incompatibility robustness not only provides an analytical way to understand CMPs, but also admit a numerically feasible form. Before stating the main result, we briefly go through the basic formulation of SDP in the following section.

4.2.1 Semi-Definite Program: A Brief Introduction

SDP is a central tool, both analytically and numerically, in the study of quantum information theory. An SDP contains two formulations, named the *primal* and the *dual* problems, respectively. Usually, the study of the dual SDP can provide insights about operational interpretations of the problems under consideration. Formally, from Ref. [27], one can write the *primal problem* of an SDP as follows:

$$\begin{aligned} \max_{V} \quad & \langle V, A\rangle \\ \text{s.t.} \quad & \Phi(V) = B; \quad \Psi(V) \le C; \quad V \ge 0. \end{aligned} \tag{4.17}$$

Here, both Ψ and Φ are hermitian-preserving linear maps, A, B, C are fixed operators, and V is the variable (which is again an operator in general). The corresponding *dual problem* is (see, e.g., Sec. 1.2.3 in Ref. [27])

$$\begin{aligned} \min_{H,W} \quad & \langle H, B\rangle + \langle W, C\rangle \\ \text{s.t.} \quad & \Phi^{\dagger}(H) + \Psi^{\dagger}(W) \ge A; \quad H^{\dagger} = H; \quad W \ge 0, \end{aligned} \tag{4.18}$$

Again, H, W are variables, which are operators. Now, it is important and useful to know when the primal and the dual SDPs output the same optimum. This is called *strong duality*, which is guaranteed when (i) the primal problem Eq. (4.17) is finite and feasible (i.e., $\exists\, V \ge 0$ satisfying $\Phi(V) = B, \Psi(V) \le C$); (ii) the dual problem Eq. (4.17) is strictly feasible (i.e., $\exists\, W > 0, H^{\dagger} = H$ satisfying $\Phi^{\dagger}(H) + \Psi^{\dagger}(W) > A$). These are the so-called *Slater's conditions* (see Theorem 1.18 in Ref. [27]).

4.2.2 Channel Incompatibility Robustness As An SDP

Now we state the following result, which explicitly provides the SDP form of the channel incompatibility robustness defined in Eq. (4.13):

Main Theorem 4.2.1. (SDP Form of Channel Incompatibility Robustness) *For every $\mathcal{E}$, its channel incompatibility robustness $R(\mathcal{E})$ is the solution of the following SDP*

$$\begin{aligned} \max_{\rho_{SS'},\lambda} \quad & \lambda \\ \text{s.t.} \quad & \rho_{SS'} \ge 0; \quad \mathrm{tr}_S(\rho_{SS'}) = \frac{\mathbb{I}_{S'}}{d_{S'}}; \quad \lambda \in [0,1]; \\ & \mathrm{tr}_{S\setminus X}(\rho_{SS'}) \ge \lambda \mathcal{E}^{\mathcal{J}}_{XX'} \otimes \frac{\mathbb{I}_{S'\setminus X'}}{d_{S'\setminus X'}} \quad \forall\, X|X' \in \Lambda; \\ & \mathrm{tr}_{S\setminus X}(\rho_{SS'}) = \mathrm{tr}_{SS'\setminus XX'}(\rho_{SS'}) \otimes \frac{\mathbb{I}_{S'\setminus X'}}{d_{S'\setminus X'}} \quad \forall\, X|X' \in \Lambda. \end{aligned} \tag{4.19}$$

Proof. By definition Eq. (4.13), $R(\mathcal{E})$ is the solution of the following maximisation

$$\begin{aligned} \max_{\mathcal{N},\mathcal{L},\lambda} \quad & \lambda \\ \text{s.t.} \quad & \mathcal{L} \in \mathfrak{C}; \quad \lambda \in [0,1]; \quad \mathcal{N} : \text{vector of channels}; \quad \lambda\mathcal{E} + (1-\lambda)\mathcal{N} = \mathcal{L}, \end{aligned} \tag{4.20}$$

where the last condition holds if and only if $\lambda\mathcal{E}^{\mathcal{J}}_{XX'}+(1-\lambda)\mathcal{N}^{\mathcal{J}}_{XX'} = \mathcal{L}^{\mathcal{J}}_{XX'}\ \forall\ X|X' \in \Lambda$. By Lemma 4.1.3, there exists a global channel $\mathcal{L}_{S|S'}$ such that $\mathrm{tr}_{S\setminus X}\left(\mathcal{L}^{\mathcal{J}}_{SS'}\right) = \mathcal{L}^{\mathcal{J}}_{XX'} \otimes \frac{\mathbb{I}_{S'\setminus X'}}{d_{S'\setminus X'}}$ $\forall\ X|X' \in \Lambda$. From here we learn that

$$\mathrm{tr}_{S\setminus X}\left(\mathcal{L}^{\mathcal{J}}_{SS'}\right) = \left[\lambda\mathcal{E}^{\mathcal{J}}_{XX'} + (1-\lambda)\mathcal{N}^{\mathcal{J}}_{XX'}\right] \otimes \frac{\mathbb{I}_{S'\setminus X'}}{d_{S'\setminus X'}} \quad \forall\ X|X' \in \Lambda. \tag{4.21}$$

This means, for every $X|X' \in \Lambda$, we have

$$\begin{aligned} \mathrm{tr}_{S\setminus X}\left(\mathcal{L}^{\mathcal{J}}_{SS'}\right) &\geq \lambda\mathcal{E}^{\mathcal{J}}_{XX'} \otimes \frac{\mathbb{I}_{S'\setminus X'}}{d_{S'\setminus X'}}; \\ \mathrm{tr}_{S\setminus X}\left(\mathcal{L}^{\mathcal{J}}_{SS'}\right) &= \mathrm{tr}_{SS'\setminus XX'}\left(\mathcal{L}^{\mathcal{J}}_{SS'}\right) \otimes \frac{\mathbb{I}_{S'\setminus X'}}{d_{S'\setminus X'}}; \\ \mathrm{tr}_{S}\left(\mathcal{L}^{\mathcal{J}}_{SS'}\right) &= \mathrm{tr}_{X}\left(\mathcal{L}^{\mathcal{J}}_{XX'} \otimes \frac{\mathbb{I}_{S'\setminus X'}}{d_{S'\setminus X'}}\right) = \frac{\mathbb{I}_{S'}}{d_{S'}}. \end{aligned} \tag{4.22}$$

Hence, when $(\mathcal{N}, \mathcal{L}, \lambda)$ is feasible for Eq. (4.20), the pair $\left(\mathcal{L}^{\mathcal{J}}_{\mathrm{SS'}}, \lambda\right)$ is feasible for Eq. (4.19).

Conversely, if the pair $(\rho_{SS'}, \lambda)$ is feasible for Eq. (4.19), then the state $\rho_{SS'} = \mathcal{L}^{\mathcal{J}}_{SS'}$ is a Choi state of a global channel $\mathcal{L}_{S|S'}(\cdot) := d_{S'}\ \mathrm{tr}_{S'}\left\{[\mathbb{I}_S \otimes (\cdot)^T]\rho_{SS'}\right\}$ [see Eq. (2.7); also, note that the first condition in Eq. (4.19) implies $\mathrm{tr}_{\mathrm{S}}(\rho_{\mathrm{SS'}}) = \frac{\mathbb{I}_{S'}}{d_{S'}}$]. By Lemma 4.1.3 and the last condition in Eq. (4.19), the global channel $\mathcal{L}_{S|S'}$ has a well-defined marginal in each $X|X' \in \Lambda$, denoted by $\mathcal{L}_{X|X'}$, with Choi state $\mathcal{L}^{\mathcal{J}}_{XX'} = \mathrm{tr}_{SS'\setminus XX'}(\rho_{SS'})$. Tracing out $S' \setminus X'$ in the second condition in Eq. (4.19), we obtain

$$\mathcal{L}^{\mathcal{J}}_{XX'} - \lambda\mathcal{E}^{\mathcal{J}}_{XX'} \geq 0. \tag{4.23}$$

On the other hand, since $\mathcal{L}^{\mathcal{J}}_{XX'}$ and $\mathcal{E}^{\mathcal{J}}_{XX'}$ are both Choi states, we learn that, when $\lambda < 1$,

$$\frac{1}{1-\lambda}\mathrm{tr}_{X}\left(\mathcal{L}^{\mathcal{J}}_{XX'} - \lambda\mathcal{E}^{\mathcal{J}}_{XX'}\right) = \frac{\mathbb{I}_{X'}}{d_{X'}}. \tag{4.24}$$

Equations (4.23) and (4.24) imply that $\frac{1}{1-\lambda}\left(\mathcal{L}^{\mathcal{J}}_{\mathrm{XX'}} - \lambda\mathcal{E}^{\mathcal{J}}_{\mathrm{XX'}}\right)$ is a legal Choi state for *every* $\lambda < 1$. In other words, Hence, $\frac{1}{1-\lambda}\left(\mathcal{L}_{X|X'} - \lambda\mathcal{E}_{X|X'}\right)$ is a channel from X' to X when $\lambda < 1$. For each $X|X' \in \Lambda$, by defining the channel

$$\mathcal{N}_{X|X'} := \frac{1}{1-\lambda}\left(\mathcal{L}_{X|X'} - \lambda\mathcal{E}_{X|X'}\right) \quad \text{if } \lambda < 1 \quad \& \quad \mathcal{N}_{X|X'} : \text{arbitrary} \quad \text{if } \lambda = 1, \tag{4.25}$$

we have $\mathcal{L}_{X|X'} = \lambda\mathcal{E}_{X|X'} + (1-\lambda)\mathcal{N}_{X|X'}\ \forall\ X|X' \in \Lambda$. From here we conclude that the pair $(\{\mathcal{N}_{X|X'}\}_{X|X'\in\Lambda}, \{\mathcal{L}_{X|X'}\}_{X|X'\in\Lambda}, \lambda)$ is feasible for Eq. (4.20) for every $(\rho_{SS'}, \lambda)$ that is feasible for Eq. (4.19). Thus, the two optimisation problems Eqs. (4.20) and (4.19) have the same optimum. □

Theorem 4.2.1 provides a general and quantitative strategy to tackle CMPs. Being an SDP form, it is numerically feasible at least for small systems. The solution to the SDP also returns the global physical process that best approximates the marginal channels in $\boldsymbol{\mathcal{E}}$. As a remark, in the broadcasting scenario ($X' = S'$), the channel incompatibility robustness recovers as a special case the *consistent robustness* introduced in Ref. [59] and Theorem 1 in the same article.

4.2.3 Dual Problem of Channel Incompatibility Robustness

After proving that the channel incompatibility robustness can be cast into an SDP, we now study its dual form. In what follows, when an operator is written as $V^{(X)}_{Y|W}$, it means that it is acting on the system X with dependency on the input-output pair $Y|W$.

Lemma 4.2.2. (Duel Channel Incompatibility Robustness) *The dual of Eq. (4.19) is*

$$\begin{aligned}
\min_{w,H_{S'},\left\{H^{(XS')}_{X|X'}\right\},\left\{W^{(XS')}_{X|X'}\right\}} \quad & \frac{\operatorname{tr}(H_{S'})}{d_{S'}} + w \\
\text{s.t.} \quad & f\left(H_{S'},\left\{H^{(XS')}_{X|X'}\right\},\left\{W^{(XS')}_{X|X'}\right\}\right) \geq 0; \\
& w + \sum_{X|X'\in\Lambda} \operatorname{tr}\left[W^{(XS')}_{X|X'}\left(\mathcal{E}^{\mathcal{J}}_{XX'} \otimes \frac{\mathbb{I}_{S'\setminus X'}}{d_{S'\setminus X'}}\right)\right] \geq 1; \\
& H^{(XS'),\dagger}_{X|X'} = H^{(XS')}_{X|X'} \quad \forall\, X|X' \in \Lambda; \quad H^{\dagger}_{S'} = H_{S'}; \\
& W^{(XS')}_{X|X'} \geq 0 \quad \forall\, X|X' \in \Lambda; \quad w \geq 0,
\end{aligned} \tag{4.26}$$

where

$$\begin{aligned}
& f\left(H_{S'},\left\{H^{(XS')}_{X|X'}\right\},\left\{W^{(XS')}_{X|X'}\right\}\right) := \\
& \mathbb{I}_S \otimes H_{S'} + \sum_{X|X'\in\Lambda}\left(H^{(XS')}_{X|X'} - \operatorname{tr}_{S'\setminus X'}\left(H^{(XS')}_{X|X'}\right) \otimes \frac{\mathbb{I}_{S'\setminus X'}}{d_{S'\setminus X'}} - W^{(XS')}_{X|X'}\right) \otimes \mathbb{I}_{S\setminus X}.
\end{aligned} \tag{4.27}$$

Also, strong duality holds; namely, Eq. (4.19) and (4.26) have the same optimum.

Proof. The first step is to write Eq. (4.19) in the standard form of SDP. In what follows, we use the notation $\mathcal{L}(X)$ to denote the set of all linear maps on X. Then let $V := \rho_{SS'} \oplus \lambda$ and $A := 0 \oplus 1$, both in $\mathcal{L}(SS') \oplus \mathbb{R}$, such that $\langle V, A\rangle := \operatorname{tr}(V^{\dagger}A) = \lambda$. Note that the direct sum operation is defined as $x \oplus y := \begin{pmatrix} x & \cdot \\ \cdot & y \end{pmatrix}$, where the off-diagonal terms are irrelevant to the definition. $\langle V, A\rangle$ is the objective function for the standard form that will be detailed soon. The feasible set can be characterised by defining the following functions: $\Phi := \left(\bigoplus_{X|X'\in\Lambda} \Phi_{X|X'}\right) \oplus \Phi_0$ and $\Psi := \left(\bigoplus_{X|X'\in\Lambda} \Psi_{X|X'}\right) \oplus \Psi_0$. For each $X|X' \in \Lambda$ we have

$$\Phi_{X|X'}(V) := \operatorname{tr}_{S\setminus X}(\rho_{SS'}) - \operatorname{tr}_{SS'\setminus XX'}(\rho_{SS'}) \otimes \frac{\mathbb{I}_{S'\setminus X'}}{d_{S'\setminus X'}}; \tag{4.28}$$

$$\Phi_0(V) := \operatorname{tr}_S(\rho_{SS'}); \tag{4.29}$$

$$\Psi_{X|X'}(V) := \lambda\mathcal{E}^{\mathcal{J}}_{XX'} \otimes \frac{\mathbb{I}_{S'\setminus X'}}{d_{S'\setminus X'}} - \operatorname{tr}_{S\setminus X}(\rho_{SS'}); \tag{4.30}$$

$$\Psi_0(V) := \lambda, \tag{4.31}$$

where $\Phi_{X|X'} : \mathcal{L}(SS')\oplus\mathbb{R} \to \mathcal{L}(XS')$, $\Phi_0 : \mathcal{L}(SS')\oplus\mathbb{R} \to \mathcal{L}(S')$, $\Psi_{X|X'} : \mathcal{L}(SS')\oplus\mathbb{R} \to \mathcal{L}(XS')$, and $\Psi_0 : \mathcal{L}(SS') \oplus \mathbb{R} \to \mathbb{R}$. As one can check, all of them are hermitian-preserving linear maps. Now we further choose

$$B := \left(\bigoplus_{X|X'\in\Lambda} 0_{XS'}\right) \oplus \frac{\mathbb{I}_{S'}}{d_{S'}}; \qquad C := \left(\bigoplus_{X|X'\in\Lambda} 0_{XS'}\right) \oplus 1. \tag{4.32}$$

Gathering all the above ingredients, Eq. (4.19) can be rewritten as the standard form of SDP as in Eq. (4.17). From Eq. (4.18), its dual problem reads

$$\begin{aligned} \min_{H,W} \quad & \langle H, B\rangle + \langle W, C\rangle \\ \text{s.t.} \quad & \Phi^\dagger(H) + \Psi^\dagger(W) \geq A; \quad H^\dagger = H; \quad W \geq 0, \end{aligned} \tag{4.33}$$

where

$$H = \left(\bigoplus_{X|X'\in\Lambda} H_{X|X'}^{(XS')}\right) \oplus H_{S'} \in \left(\bigoplus_{X|X'\in\Lambda} \mathcal{L}(XS')\right) \oplus \mathcal{L}(S'); \tag{4.34}$$

$$W = \left(\bigoplus_{X|X'\in\Lambda} W_{X|X'}^{(XS')}\right) \oplus w \in \left(\bigoplus_{X|X'\in\Lambda} \mathcal{L}(XS')\right) \oplus \mathbb{R}. \tag{4.35}$$

Now it remains to find $\Phi^\dagger$ and $\Psi^\dagger$ to complete the proof. First, we note that

$$\begin{aligned} \langle \Phi^\dagger(H), V\rangle &= \left\langle \left(\bigoplus_{X|X'\in\Lambda} H_{X|X'}^{(XS')}\right) \oplus H_{S'}, \left(\bigoplus_{X|X'\in\Lambda} \Phi_{X|X'}(V)\right) \oplus \Phi_0(V) \right\rangle \\ &= \langle H_{S'}, \Phi_0(V)\rangle + \sum_{X|X'\in\Lambda} \left\langle H_{X|X'}^{(XS')}, \Phi_{X|X'}(V)\right\rangle = \left\langle \Phi_0^\dagger(H_{S'}) + \sum_{X|X'\in\Lambda} \Phi_{X|X'}^\dagger\left(H_{X|X'}^{(XS')}\right), V\right\rangle. \end{aligned} \tag{4.36}$$

This means $\Phi^\dagger(H) = \Phi_0^\dagger(H_{S'}) + \sum_{X|X'\in\Lambda} \Phi_{X|X'}^\dagger\left(H_{X|X'}^{(XS')}\right)$, and, similarly, $\Psi^\dagger(W) = \Psi_0^\dagger(w) + \sum_{X|X'\in\Lambda} \Psi_{X|X'}^\dagger\left(W_{X|X'}^{(XS')}\right)$. Consequently, it suffices to find the adjoint of each $\Phi_{X|X'}$, Φ_0, $\Psi_{X|X'}$, and Ψ_0 separately. Direct computation shows $\langle \Phi_0^\dagger(H_{S'}), V\rangle = \mathrm{tr}\left[H_{S'}\mathrm{tr}_S\left(\rho_{SS'}\right)\right] = \langle (\mathbb{I}_S \otimes H_{S'}) \oplus 0, V\rangle$, which implies that $\Phi_0^\dagger(H_{S'}) = (\mathbb{I}_S \otimes H_{S'}) \oplus 0$. For a given $X|X' \in \Lambda$, we have

$$\begin{aligned} \langle \Phi_{X|X'}^\dagger\left(H_{X|X'}^{(XS')}\right), V\rangle &= \left\langle H_{X|X'}^{(XS')}, \mathrm{tr}_{S\setminus X}(\rho_{SS'}) - \mathrm{tr}_{SS'\setminus XX'}\left(\rho_{SS'}\right) \otimes \frac{\mathbb{I}_{S'\setminus X'}}{d_{S'\setminus X'}} \right\rangle \\ &= \mathrm{tr}\left[(H_{X|X'}^{(XS')} \otimes \mathbb{I}_{S\setminus X})\rho_{SS'}\right] - \mathrm{tr}\left[\left(\frac{\mathrm{tr}_{S'\setminus X'}\left(H_{X|X'}^{(XS')}\right)}{d_{S'\setminus X'}} \otimes \mathbb{I}_{SS'\setminus XX'}\right)\rho_{SS'}\right] \\ &= \left\langle \left(H_{X|X'}^{(XS')} \otimes \mathbb{I}_{S\setminus X} - \frac{\mathrm{tr}_{S'\setminus X'}\left(H_{X|X'}^{(XS')}\right)}{d_{S'\setminus X'}} \otimes \mathbb{I}_{SS'\setminus XX'}\right) \oplus 0, V\right\rangle. \end{aligned} \tag{4.37}$$

From here we learn that $\Phi^{\dagger}_{X|X'}(H^{(XS')}_{X|X'}) = \left[\left(H^{(XS')}_{X|X'} - \mathrm{tr}_{S'\setminus X'}\left(H^{(XS')}_{X|X'}\right) \otimes \frac{\mathbb{I}_{S'\setminus X'}}{d_{S'\setminus X'}}\right) \otimes \mathbb{I}_{S\setminus X}\right] \oplus 0$. On the other hand, we have $\langle \Psi^{\dagger}_0(w), V\rangle = w\lambda$ and hence $\Psi^{\dagger}_0(w) = 0 \oplus w$. Also, for a given $X|X' \in \Lambda$, we have

$$\begin{aligned}\langle \Psi^{\dagger}_{X|X'}(W^{(XS')}_{X|X'}), V\rangle &= \left\langle W^{(XS')}_{X|X'}, \lambda\mathcal{E}^{\mathcal{J}}_{XX'} \otimes \frac{\mathbb{I}_{S'\setminus X'}}{d_{S'\setminus X'}} - \mathrm{tr}_{S\setminus X}(\rho_{SS'})\right\rangle \\ &= \lambda \mathrm{tr}\left[W^{(XS')}_{X|X'}\left(\mathcal{E}^{\mathcal{J}}_{XX'} \otimes \frac{\mathbb{I}_{S'\setminus X'}}{d_{S'\setminus X'}}\right)\right] - \mathrm{tr}\left[\left(W^{(XS')}_{X|X'} \otimes \mathbb{I}_{S\setminus X}\right)\rho_{SS'}\right] \\ &= \left\langle \left(-W^{(XS')}_{X|X'} \otimes \mathbb{I}_{S\setminus X}\right) \oplus \mathrm{tr}\left[W^{(XS')}_{X|X'}\left(\mathcal{E}^{\mathcal{J}}_{XX'} \otimes \frac{\mathbb{I}_{S'\setminus X'}}{d_{S'\setminus X'}}\right)\right], V\right\rangle, \end{aligned} \tag{4.38}$$

meaning that

$$\Psi^{\dagger}_{X|X'}(W^{(XS')}_{X|X'}) = \left(-W^{(XS')}_{X|X'} \otimes \mathbb{I}_{S\setminus X}\right) \oplus \mathrm{tr}\left[W^{(XS')}_{X|X'}\left(\mathcal{E}^{\mathcal{J}}_{XX'} \otimes \frac{\mathbb{I}_{S'\setminus X'}}{d_{S'\setminus X'}}\right)\right]. \tag{4.39}$$

Combining everything in Eq. (4.18), one is able to obtain the desired form, i.e., the one in Eq. (4.26). Finally, note that the primal problem Eq. (4.17) is finite and feasible [27] by taking $\rho_{SS'} = \frac{\mathbb{I}_{SS'}}{d_{SS'}}$ and $\lambda = 0$. Also, the dual is strictly feasible [27] by taking any $H^{(XS')}_{X|X'}$ Hermitian, $W^{(XS')}_{X|X'} > 0$, and $H_{S'} = h\mathbb{I}_{S'}$ for $h > 0, w > 0$ large enough. By Slater's conditions (Theorem 1.18 in Ref. [27]), strong duality holds. □

4.2.4 Channel Incompatibility Witness

The dual of channel incompatibility robustness (Lemma 4.2.2) leads to a simple operational interpretation of channel incompatibility robustness [Eq. (4.13)], helping us to single out the physical meaning behind incompatibility of quantum dynamics. In fact, one is able to use it to derive a necessary and sufficient characterisation of channel incomaptibility through an operational *witness*; that is, an operational method to certify the existence of the given quantum resource. For instance, an entanglement witness is an operator W such that $\mathrm{tr}(W\rho) \geq 0$ if ρ is separable, and $\mathrm{tr}(W\rho) < 0$ for some ρ. To proceed, we need to prove a lemma first. Let $\boldsymbol{\mathcal{E}}^{\mathcal{J}} := \{\mathcal{E}^{\mathcal{J}}_{XX'}\}_{X|X'\in\Lambda}$ be the vector of *Choi states* of the channels in $\boldsymbol{\mathcal{E}}$. Also, for a set of operators $\boldsymbol{A} := \{A_{X|X'}\}_{X|X'\in\Lambda}$, where $A_{X|X'}$ is in $X|X'$, we define $\left\langle \boldsymbol{A}, \boldsymbol{\mathcal{E}}^{\mathcal{J}}\right\rangle := \sum_{X|X'\in\Lambda} \mathrm{tr}\left(A_{X|X'}\mathcal{E}^{\mathcal{J}}_{XX'}\right)$.

Lemma 4.2.3. (Channel Incompatibility Witness via Choi States) *$\boldsymbol{\mathcal{E}}$ is incompatible if and only if there exists a set of positive operators $\boldsymbol{H} := \{H_{X|X'}\}_{X|X'\in\Lambda}$ such that*

$$\left\langle \boldsymbol{H}, \boldsymbol{\mathcal{E}}^{\mathcal{J}}\right\rangle > \max_{\boldsymbol{\mathcal{L}}\in\mathfrak{C}} \left\langle \boldsymbol{H}, \boldsymbol{\mathcal{L}}^{\mathcal{J}}\right\rangle. \tag{4.40}$$

Proof. it suffices to show that incompatibility implies Eq. (4.40). First, we note that when $\left\{W^{(XS')}_{X|X'}\right\}_{X|X'\in\Lambda}$ and w satisfy the second constraint in Eq. (4.26), we have $w \geq 1 - \sum_{X|X'\in\Lambda} \mathrm{tr}\left[W^{(XS')}_{X|X'}\left(\mathcal{E}^{\mathcal{J}}_{XX'} \otimes \frac{\mathbb{I}_{S'\setminus X'}}{d_{S'\setminus X'}}\right)\right]$. Then Lemma 4.2.2 and strong duality jointly imply that $R(\mathcal{E})$ is lower bounded by

$$\begin{aligned}
\min_{H_{S'},\{H^{(XS')}_{X|X'}\},\{W^{(XS')}_{X|X'}\}} \quad & \frac{\operatorname{tr}(H_{S'})}{d_{S'}} + 1 - \sum_{X|X'\in\Lambda} \operatorname{tr}\left[W^{(XS')}_{X|X'}\left(\mathcal{E}^{\mathcal{J}}_{XX'} \otimes \frac{\mathbb{I}_{S'\setminus X'}}{d_{S'\setminus X'}}\right)\right] \\
\text{s.t.} \quad & f\left(H_{S'},\left\{H^{(XS')}_{X|X'}\right\},\left\{W^{(XS')}_{X|X'}\right\}\right) \geq 0; \quad W^{(XS')}_{X|X'} \geq 0 \ \forall\ X|X' \in \Lambda; \\
& H^{(XS'),\dagger}_{X|X'} = H^{(XS')}_{X|X'} \ \forall\ X|X' \in \Lambda; \quad H^{\dagger}_{S'} = H_{S'}.
\end{aligned} \tag{4.41}$$

Note that since the objective function becomes independent of w, the second constraint in Eq. (4.26) always holds and hence can be removed. Now we note that

$$\begin{aligned}
\frac{\operatorname{tr}(H_{S'})}{d_{S'}} &= \max_{\mathcal{L}\in\mathfrak{C}} \operatorname{tr}\left[(\mathbb{I}_S \otimes H_{S'})\mathcal{L}^{\mathcal{J}}_{SS'}\right] \\
&\geq \max_{\mathcal{L}\in\mathfrak{C}} \sum_{X|X'\in\Lambda} \operatorname{tr}\left[\left(\left(-H^{(XS')}_{X|X'} + \operatorname{tr}_{S'\setminus X'}\left(H^{(XS')}_{X|X'}\right) \otimes \frac{\mathbb{I}_{S'\setminus X'}}{d_{S'\setminus X'}} + W^{(XS')}_{X|X'}\right) \otimes \mathbb{I}_{S\setminus X}\right)\mathcal{L}^{\mathcal{J}}_{SS'}\right] \\
&= \max_{\mathcal{L}\in\mathfrak{C}} \sum_{X|X'\in\Lambda} \operatorname{tr}\left[\left(-H^{(XS')}_{X|X'} + \operatorname{tr}_{S'\setminus X'}\left(H^{(XS')}_{X|X'}\right) \otimes \frac{\mathbb{I}_{S'\setminus X'}}{d_{S'\setminus X'}} + W^{(XS')}_{X|X'}\right)\left(\mathcal{L}^{\mathcal{J}}_{XX'} \otimes \frac{\mathbb{I}_{S'\setminus X'}}{d_{S'\setminus X'}}\right)\right] \\
&= \max_{\mathcal{L}\in\mathfrak{C}} \sum_{X|X'\in\Lambda} \operatorname{tr}\left[W^{(XS')}_{X|X'}\left(\mathcal{L}^{\mathcal{J}}_{XX'} \otimes \frac{\mathbb{I}_{S'\setminus X'}}{d_{S'\setminus X'}}\right)\right].
\end{aligned} \tag{4.42}$$

In the first line $\mathcal{L}^{\mathcal{J}}_{SS'}$ denotes the Choi state of a global channel $\mathcal{L}_{S|S'}$ compatible with $\mathcal{L} = \{\mathcal{L}_{X|X'}\}_{X|X'\in\Lambda}$, and the equality follows from the property of a Choi state [Eq. (2.5)]. The inequality in the second line is due to the first constraint in Eq. (4.26). The third line is due to Lemma 4.1.3, and the last line is because of the equality:

$$\operatorname{tr}\left[H^{(XS')}_{X|X'}\left(\mathcal{L}^{\mathcal{J}}_{XX'} \otimes \frac{\mathbb{I}_{S'\setminus X'}}{d_{S'\setminus X'}}\right)\right] = \operatorname{tr}\left[\operatorname{tr}_{S'\setminus X'}\left(H^{(XS')}_{X|X'}\right) \otimes \frac{\mathbb{I}_{S'\setminus X'}}{d_{S'\setminus X'}}\left(\mathcal{L}^{\mathcal{J}}_{XX'} \otimes \frac{\mathbb{I}_{S'\setminus X'}}{d_{S'\setminus X'}}\right)\right]. \tag{4.43}$$

From here we conclude that the channel incompatibility robustness $R(\mathcal{E})$ is lower bounded by

$$\begin{aligned}
\min_{\left\{W^{(XS')}_{X|X'}\right\}} \quad & 1 - \sum_{X|X'\in\Lambda} \operatorname{tr}\left[W^{(XS')}_{X|X'}\left(\mathcal{E}^{\mathcal{J}}_{XX'} \otimes \frac{\mathbb{I}_{S'\setminus X'}}{d_{S'\setminus X'}}\right)\right] \\
& \qquad + \max_{\mathcal{L}\in\mathfrak{C}} \sum_{X|X'\in\Lambda} \operatorname{tr}\left[W^{(XS')}_{X|X'}\left(\mathcal{L}^{\mathcal{J}}_{XX'} \otimes \frac{\mathbb{I}_{S'\setminus X'}}{d_{S'\setminus X'}}\right)\right] \\
\text{s.t.} \quad & W^{(XS')}_{X|X'} \geq 0 \quad \forall\ X|X' \in \Lambda.
\end{aligned} \tag{4.44}$$

Since the objective function is independent of $H_{S'},\left\{H^{(XS')}_{X|X'}\right\}_{X|X'\in\Lambda}$, the first constraint in Eq. (4.41) is dropped because it always holds with an appropriately selected $H_{S'}$. When $\mathcal{E}$ is incompatible, i.e., $R(\mathcal{E}) < 1$, there exist positive operators $W^{(XS')}_{X|X'}$ such that

$$\max_{\mathcal{L}\in\mathfrak{C}} \sum_{X|X'\in\Lambda} \operatorname{tr}\left[W^{(XS')}_{X|X'}\left(\mathcal{L}^{\mathcal{J}}_{XX'} \otimes \frac{\mathbb{I}_{S'\setminus X'}}{d_{S'\setminus X'}}\right)\right] < \sum_{X|X'\in\Lambda} \operatorname{tr}\left[W^{(XS')}_{X|X'}\left(\mathcal{E}^{\mathcal{J}}_{XX'} \otimes \frac{\mathbb{I}_{S'\setminus X'}}{d_{S'\setminus X'}}\right)\right]. \tag{4.45}$$

Finally, let $\widetilde{H}_{X|X'} := \frac{\mathrm{tr}_{S'\setminus X'}\left(W^{(XS')}_{X|X'}\right)}{d_{S'\setminus X'}}$, which is again positive. The above inequality implies

$$\max_{\mathcal{L}\in\mathfrak{C}} \sum_{X|X'\in\Lambda} \mathrm{tr}\left(\widetilde{H}_{X|X'}\mathcal{L}^{\mathcal{J}}_{XX'}\right) < \sum_{X|X'\in\Lambda} \mathrm{tr}\left(\widetilde{H}_{X|X'}\mathcal{E}^{\mathcal{J}}_{XX'}\right), \tag{4.46}$$

and the proof is completed. □

Lemma 4.2.3 gives a witness form for channel incompatibility in terms of local channels' Choi states. It can be understood as the dynamical version of the *state* incompatibility witness given by Ref. [69]. However, intuitively, one may expect the possibility to define witnesses in terms of *channels*, rather than their Choi states. This can be done by applying Proposition 7 in Ref. [17], which is formally stated as follows:

Theorem 4.2.4. [17] *W_{AB} is a hermitian operator acting on a bipartite system AB. Then there exist states $\xi^{(i)}$ in A, $\rho^{(j)}$ in B, and real numbers ω_{ij} such that $W_{AB} = \sum_{i,j}\omega_{ij}\xi^{(i),t}\otimes\rho^{(j),t}$, where $(\cdot)^t$ is the transpose operation. Also, the number of nonzero ω_{ij} is at most $d^2_{\min}+3$, where $d_{\min}$ is the smallest system dimension among A, B.*

Then we have the following result, serving as a witness of channel incompatibility:

Main Theorem 4.2.5. (Channel Incompatibility Witness) *$\mathcal{E}$ is incompatible if and only if for every $X|X'\in\Lambda$ there exist Hermitian operators $\{H^{(i)}_{X|X'}\}_{i=1}^{N_\Lambda}$ in X and states $\{\rho^{(i)}_{X|X'}\}_{i=1}^{N_\Lambda}$ in X' such that, with $N_\Lambda := (\max_{X|X'\in\Lambda}\{d_X, d_{X'}\})^2+3$,*

$$\sum_{X|X',i} \mathrm{tr}\left[H^{(i)}_{X|X'}\mathcal{E}_{X|X'}\left(\rho^{(i)}_{X|X'}\right)\right] > \max_{\mathcal{L}\in\mathfrak{C}} \sum_{X|X',i} \mathrm{tr}\left[H^{(i)}_{X|X'}\mathcal{L}_{X|X'}\left(\rho^{(i)}_{X|X'}\right)\right]. \tag{4.47}$$

Proof. For every $\mathcal{E}$ and $\{H_{X|X'}\}_{X|X'\in\Lambda}$ with $H^\dagger_{X|X'} = H_{X|X'}$ in the system XX', Theorem 4.2.4 implies the existences of states $\xi^{(i)}_{X|X'}$ in X, $\rho^{(j)}_{X|X'}$ in X', and real numbers $\{\omega^{(ij)}_{X|X'}\}_{i,j=1}^{N_{X|X'}}$ [we can choose $N_{X|X'} \le (\min\{d_X, d_{X'}\})^2+3$ for every $X|X'\in\Lambda$], such that

$$\sum_{X|X'\in\Lambda} \mathrm{tr}\left(H_{X|X'}\mathcal{E}^{\mathcal{J}}_{XX'}\right) = \sum_{X|X'\in\Lambda}\sum_{i,j=1}^{N_{X|X'}} \omega^{(ij)}_{X|X'}\mathrm{tr}\left[\left(\xi^{(i),t}_{X|X'}\otimes\rho^{(j),t}_{X|X'}\right)(\mathcal{E}_{X|X'}\otimes\mathcal{I}_{X'})(|\Psi^+_{X'X'}\rangle\langle\Psi^+_{X'X'}|)\right]$$

$$= \sum_{X|X'\in\Lambda}\sum_{i,j=1}^{N_{X|X'}} \frac{\omega^{(ij)}_{X|X'}}{d_{X'}}\mathrm{tr}\left[\xi^{(i),t}_{X|X'}\mathcal{E}_{X|X'}\left(\rho^{(j)}_{X|X'}\right)\right] = \sum_{X|X'\in\Lambda}\sum_{j=1}^{N_{X|X'}} \mathrm{tr}\left[E^{(j)}_{X|X'}\mathcal{E}_{X|X'}\left(\rho^{(j)}_{X|X'}\right)\right], \tag{4.48}$$

where $E^{(j)}_{X|X'} := \sum_{i=1}^{N_{X|X'}} \frac{\omega^{(ij)}_{X|X'}}{d_{X'}}\xi^{(i),t}_{X|X'}$ is again a Hermitian operator in X. The result follows by using Lemma 4.2.3, and noticing that for every $X|X'$ we have that $N_{X|X'} \le (\min\{d_X, d_{X'}\})^2+3 \le (\max_{X|X'\in\Lambda}\{d_X, d_{X'}\})^2+3$. □

4.3 Implications and Applications

4.3.1 Local Compatibility Does Not Imply Compatibility

As we mentioned previously, there exist locally compatible channels [Eq. (4.11)] that are actually incompatible. Here we provide examples in a tripartite setting ABC with

$A \simeq A' \simeq C \simeq C'$ and $B \simeq B'$. In what follows, $|\phi_{XBB'}\rangle$ is a pure state satisfying the condition $\mathrm{tr}_{XB}(|\phi_{XBB'}\rangle\langle\phi_{XBB'}|) = \frac{\mathbb{I}_{B'}}{d_{B'}}$[2]. Define a channel $\mathcal{M}_{XB}$ through its Choi state ($X = A, C$; recall that $\mathcal{E}_X = \mathcal{E}_{X|X'}$ when $X = X'$):

$$\mathcal{M}^{\mathcal{J}}_{XX'BB'} := |\phi_{XBB'}\rangle\langle\phi_{XBB'}| \otimes \frac{\mathbb{I}_{X'}}{d_{X'}}, \tag{4.49}$$

Then, by construction, $\mathcal{M}_{AB}$ and $\mathcal{M}_{CB}$ are locally compatible; i.e., $\mathrm{Tr}_{A|A'}(\mathcal{M}_{AB}) = \mathrm{Tr}_{C|C'}(\mathcal{M}_{CB})$. On the other hand, using Lemma 4.2.3, one can show that

Lemma 4.3.1. (Locally Compatible Incompatibility Lemma) *$\mathcal{M}_{AB}$ and $\mathcal{M}_{CB}$ are incompatible if and only if $|\phi_{XBB'}\rangle$ is non-product in the X vs. BB' bipartition.*

Proof. To show the sufficiency, suppose $|\phi_{XBB'}\rangle = |\phi\rangle_X \otimes |\xi\rangle_{BB'}$ is product in X *vs.* BB' bipartition. Then by Lemma 4.1.3 ($S = ABC$) $\mathcal{E}^{\mathcal{J}}_{SS'} = |\phi\rangle\langle\phi|_A \otimes |\phi\rangle\langle\phi|_C \otimes |\xi\rangle\langle\xi|_{BB'} \otimes \frac{\mathbb{I}_{A'C'}}{d_{A'C'}}$ is the Choi state of a global channel compatible with $\{\mathcal{M}_{AB}, \mathcal{M}_{CB}\}$.

To show the necessity, consider the Hermitian operators $H_{XB|X'B'} := d_{X'}\mathcal{M}^{\mathcal{J}}_{XX'BB'}$ ($X = A, C$). Then we have

$$\begin{aligned}&\max_{\mathcal{L}\in\mathfrak{C}} \left[\mathrm{tr}\left(H_{AB|A'B'}\mathcal{L}^{\mathcal{J}}_{AA'BB'}\right) + \mathrm{tr}\left(H_{CB|C'B'}\mathcal{L}^{\mathcal{J}}_{CC'BB'}\right)\right] \\ &\le \max_{\rho_{SS'}} \mathrm{tr}\left[(|\phi_{ABB'}\rangle\langle\phi_{ABB'}| \otimes \mathbb{I}_{A'CC'} + |\phi_{CBB'}\rangle\langle\phi_{CBB'}| \otimes \mathbb{I}_{AA'C'})\rho_{SS'}\right] \le 2,\end{aligned} \tag{4.50}$$

where the last inequality is saturated if and only if there exists a state $\rho_{ABB'C}$ such that $\mathrm{tr}\left[(|\phi_{ABB'}\rangle\langle\phi_{ABB'}| \otimes \mathbb{I}_C)\rho_{ABB'C}\right] = 1 = \mathrm{tr}\left[(|\phi_{CBB'}\rangle\langle\phi_{CBB'}| \otimes \mathbb{I}_A)\rho_{ABB'C}\right]$. This is true if and only if $\mathrm{tr}_C(\rho_{ABB'C}) = |\phi_{ABB'}\rangle\langle\phi_{ABB'}|$ and $\mathrm{tr}_A(\rho_{ABB'C}) = |\phi_{CBB'}\rangle\langle\phi_{CBB'}|$. Now, by entanglement monogamy [70] (see also Ref. [71]), it is *impossible* for such $\rho_{ABB'C}$ to exist when $|\phi_{XBB'}\rangle$ is *non-product* in X *vs.* BB' bipartition. Since, by assumption, $|\phi_{\mathrm{XBB'}}\rangle$ is non-product, we conclude that

$$\max_{\mathcal{L}\in\mathfrak{C}} \left[\mathrm{tr}\left(H_{AB|A'B'}\mathcal{L}^{\mathcal{J}}_{AA'BB'}\right) + \mathrm{tr}\left(H_{CB|C'B'}\mathcal{L}^{\mathcal{J}}_{CC'BB'}\right)\right] < 2. \tag{4.51}$$

On the other hand, a direct computation shows that

$$\mathrm{tr}\left(H_{AB|A'B'}\mathcal{M}^{\mathcal{J}}_{AA'BB'}\right) + \mathrm{tr}\left(H_{CB|C'B'}\mathcal{M}^{\mathcal{J}}_{CC'BB'}\right) = 2. \tag{4.52}$$

Using Lemma 4.2.3, the desired result follows. □

A potential physical explanation behind this result is entanglement monogamy [70], which suggests that it is *impossible* to *clone quantum correlations*; i.e., the mapping $|\psi_{AB}\rangle \mapsto |\psi_{ABC}\rangle$ such that $|\psi_{AB}\rangle = |\psi_{BC}\rangle$ is possible only when $|\psi_{AB}\rangle$ is product in the A *vs.* B bipartition.

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[2]Here, we use subscripts to indicate the *same* state distributed among *different* local systems.

Remarks on the Optimal Universal Cloning Machine

As a special case, consider $|\phi_{XBB'}\rangle = \frac{1}{\sqrt{2}}(|000\rangle_{XBB'} + |111\rangle_{XBB'}) = |\mathrm{GHZ}_{XBB'}\rangle$ [72] in Eq. (4.49). Then we have

$$\mathcal{M}_{XB}(\cdot) \coloneqq \mathrm{CNOT}_{XB}\left[|0\rangle\langle 0|_X \otimes \mathrm{tr}_X(\cdot)\right], \tag{4.53}$$

where $\mathrm{CNOT}_{XB} : |i\rangle_X \otimes |j\rangle_B \mapsto |(i+j)_{\mathrm{mod}\,2}\rangle_X \otimes |j\rangle_B$ is a qubit CNOT-gate. In this case, $\mathcal{M}_{AB}$ can be realised by B (i) implementing a CNOT gate between their incoming qubit and an extra ancillary qubit prepared in the state $|0\rangle$ (controlled on the former) and (ii) sending the ancillary qubit to A, who uses it as their output. However, this channel is incompatible with $\mathcal{M}_{CB}$, since the SDP in Theorem 4.2.1 returns $R(\{\mathcal{M}_{AB}, \mathcal{M}_{CB}\}) = 0.75$. From here, a natural question is to ask:

Is the global channel achieving this optimal value achieved by optimal quantum cloning at the input B and distributing it to A and C?

In fact, the SDP in Theorem 4.2.1 can also numerically return such a global channel, and it is indeed related to quantum cloning. Formally, an *optimal universal cloning machine* that clones arbitrary states in X into AC is a unitary operator acting as [73]

$$|0\rangle_X \otimes |00\rangle_{CM} \mapsto \sqrt{\frac{2}{3}}|001\rangle_{ACM} - \sqrt{\frac{1}{3}}|\psi_+\rangle_{AC} \otimes |0\rangle_M; \tag{4.54}$$

$$|1\rangle_X \otimes |00\rangle_{CM} \mapsto -\sqrt{\frac{2}{3}}|110\rangle_{ACM} + \sqrt{\frac{1}{3}}|\psi_+\rangle_{AC} \otimes |1\rangle_M. \tag{4.55}$$

where $|\psi_+\rangle \coloneqq \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$ and M is an ancillary system (denoting the "machine") that will be dropped in the end. Let $C_{AC|X} : X \to AC$ denote the map obtained by tracing out M after the above cloning unitary. Then, numerically, the following global channel is returned by the SDP in Theorem 4.2.1:

$$\widetilde{\mathcal{M}}_{ABC}(\cdot) \coloneqq (C_{AC|X} \otimes \mathcal{I}_B) \circ \mathrm{CNOT}_{XB}[|0\rangle\langle 0|_X \otimes \mathrm{tr}_{AC}(\cdot)]. \tag{4.56}$$

It can be understood as follows: B gets an input state, takes an ancillary system X initially prepared in $|0\rangle_X$, and performs a CNOT_{XB} controlled on B. Then B applies the optimal universal cloning machine ($C_{AC|X}$) to X and sends one copy to A and one to C. Finally, we remark that the local noise model can also be found explicitly.

4.3.2 CMP Is Irreducible to SMPs

One may conjecture that channel compatibility of $\mathcal{E} = \{\mathcal{E}_{X|X'}\}$ can be reduced to the *state* compatibility of the image states $\{\mathcal{E}_{X|X'}(\rho_{X'})\}$ for *every* set of compatible inputs $\{\rho_{X'}\}$. Namely, one may be tempted to ask:

Can the CMP be reduced to SMPs for the image states of local channels?

Perhaps unexpectedly, we can disprove this conjecture by constructing a family of counterexamples. Consider a bipartite state $\omega_{XB'}$ satisfying (1) $\omega_{XB'}$ is not *2-extendible* with respect to B' [67, 74]; (2) $\mathrm{tr}_X(\omega_{XB'}) = \frac{\mathbb{I}_{B'}}{d_{B'}}$. Also, define a channel $\mathcal{W}_{XB} : XB \to XB$ whose Choi state reads

$$\mathcal{W}^{\mathcal{J}}_{XX'BB'} := \sigma_B \otimes \omega_{XB'} \otimes \frac{\mathbb{I}_{X'}}{d_{X'}}, \tag{4.57}$$

where σ_B is a fixed (but arbitrarily chosen) state in B. Then we can prove that:

Lemma 4.3.2. (CMP Is Not Reducible To SMPs) *The channel $\mathcal{W}_{XB}$ satisfies*

1. *$\{\mathcal{W}_{AB}, \mathcal{W}_{CB}\}$ is locally compatible but incompatible.*
2. *$\{\mathcal{W}_{AB}(\eta_{AB}), \mathcal{W}_{CB}(\eta_{CB})\}$ is a compatible pair of states for every locally compatible pair of input states $\{\eta_{AB}, \eta_{CB}\}$.*

Proof. First, since locally in B both $\mathcal{W}_{AB}, \mathcal{W}_{CB}$ are the state preparation channel of σ_B, they are locally compatible. Now, assume by contradiction that the set $\{\mathcal{W}_{AB}, \mathcal{W}_{CB}\}$ is compatible, then there exists a global channel whose Choi state contains a 2-extension of the state $\omega_{XB'}$, resulting in a contradiction. The first claim is proved. To prove the second claim, we note that for any given set of locally compatible states $\{\eta_{AB}, \eta_{CB}\}$ with marginal $\mathrm{tr}_A(\eta_{AB}) = \mathrm{tr}_C(\eta_{CB}) = \kappa$, we have that, for $X = A, C$, $\mathcal{W}_{XB}(\eta_{XB}) = \Omega_{X|B}(\kappa) \otimes \sigma_B$, where $\Omega_{X|B} : B \to X$ is a channel with Choi state $\Omega^{\mathcal{J}}_{XB'} = \omega_{XB'}$. These two image states are *always compatible* with the tripartite state $\Omega_{A|B}(\kappa) \otimes \sigma_B \otimes \Omega_{C|B}(\kappa)$. □

Lemma 4.3.2 illustrates the fact that channel incompatibility cannot be always detected from incompatibility of local channels' image states, thereby disproving the previously-mentioned conjecture.

Remarks On No-Cloning Theorem

Here we consider a special case of Lemma 4.3.2, which connects the CMP and no-cloning theorem. Take the bipartite channel

$$\mathcal{K}_{XB}(\cdot) := \mathrm{SWAP}\left[|0\rangle\langle 0|_X \otimes \mathrm{tr}_X(\cdot)\right], \tag{4.58}$$

where SWAP : $|ij\rangle \mapsto |ji\rangle$ is the swap operation. This channel has the Choi state $\mathcal{K}^{\mathcal{J}}_{XX'BB'} = |\Psi^+_{XB'}\rangle\langle\Psi^+_{XB'}| \otimes |0\rangle\langle 0|_{\mathrm{B}} \otimes \frac{\mathbb{I}_{X'}}{d_{X'}}$ [recall from Eq. (2.6) the definition of a maximally entangled state]. Hence, Lemma 4.3.2 implies that $\{\mathcal{K}_{AB}, \mathcal{K}_{CB}\}$ is a counterexample to the above-mentioned conjecture. In fact, it is rewarding to see the following alternative proof for this special case. First, by contradiction, suppose that there was a tripartite channel $\mathcal{K}_{ABC}$ simultaneously realising $\mathcal{K}_{AB}$ and $\mathcal{K}_{CB}$. By considering all inputs of the form $|\psi\rangle\langle\psi|_B \otimes |00\rangle\langle 00|_{AC}$, this tripartite channel $\mathcal{K}_{ABC}$ realises $|\psi\rangle \mapsto |\psi\rangle \otimes |\psi\rangle$, in violation of the *no-cloning theorem* [73, 75]. Hence, $\mathcal{K}_{\mathrm{ABC}}$ cannot exist. Still, for every compatible input pair $\{\rho_{AB}, \rho_{CB}\}$ with marginal in B as $\sigma = \mathrm{tr}_A(\rho_{AB}) = \mathrm{tr}_C(\rho_{BC})$, the image states under $\{\mathcal{K}_{AB}, \mathcal{K}_{CB}\}$ read $\{\sigma_A \otimes |0\rangle\langle 0|_B, |0\rangle\langle 0|_B \otimes \sigma_C\}$, which are compatible with the tripartite state $\sigma_A \otimes |0\rangle\langle 0|_B \otimes \sigma_C$. Hence, we again conclude that incompatibility of local channels cannot be always detected from incompatibility of their image states, and a potential physical mechanism behind is the well-known no-cloning theorem. Note that this also means that channel incompatibility is *not equivalent to* violation of entanglement monogamy [70] for the image states of local channels.

4.3.3 Gap Between Classical and Quantum CMPs

Previous sections have demonstrated the existence of non-trivial examples of quantum channel incompatibility in the AB/BC scenario. Remarkably, such an incompatibility structure *never* occurs *classically*. In fact, given two classical channels (i.e., stochastic matrices) $P_{AB|XY}$, $P_{BC|YW}$ with a well-defined marginal $P_{B|Y}$ in $B|Y$, the CMP always has a solution, which is the following global classical channel

$$P_{ABC|XYW} = \frac{P_{AB|XY}P_{BC|YW}}{P_{B|Y}}, \tag{4.59}$$

as one can show by taking the corresponding marginals [3]. A natural question at this point is:

Is the gap between classical and quantum CMPs simply due to the fact that the former is always trivial?

that is, are classical CMPs simply decided by local compatibility? The following example shows that this is not the case. Take a *Popescu-Rohrlich box* [76, 77], also known as *PR-box*, on AB, which is given by: $P_{AB|XY} = \frac{1}{2}$ if $A \oplus B = XY$ and 0 otherwise, where all random variables are bits. Similarly, define PR-boxes in AC and BC. One can check that all marginals are well-defined and coincide on A, B and C. Now, one can show that there *doe not exist* any joint classical channel $P_{ABC|XYW}$ compatible with all of them. To show this, by contradiction, suppose it does. Then since all its 2-party marginals are well-defined, one can show that this joint classical channel $P_{ABC|XYW}$ must be a no-signaling distribution. However, it is known that the PR-box cannot be shared *without* violating no-signaling condition [78]. Consequently, $P_{ABC|XYW}$ cannot exist. This proves that classical CMPs are not trivial either, but they are structurally different from the quantum CMPs[4].

4.3.4 Operational Advantage in State Discrimination Tasks

Finally, we can also relate CMPs to discrimination tasks, demonstrating operational advantages of channel incompatibility–namely, in such a task, channel incompatibility can be treated as a resource. We introduce the following tasks, termed *ensemble state discrimination task*. Given Λ, consider a scenario where we have an agent for each systems X and X'. With probability $p_{X|X'}$, the input-output pair $X|X'$ is announced,

[3] Write $P_{AB|XY}(\cdot) = \sum_{a,b,x,y} P(ab|xy)|ab\rangle\langle ab|\langle xy| \cdot |xy\rangle$ with a probability distribution $\{P(ab|xy)\}$. Similar for $P_{BC|YW}$ with $\{P(bc|yw)\}$ and $P_{B|Y}$ with $\{P(b|y)\}$ satisfying $\sum_c P(bc|yw) = P(b|y) = \sum_a P(ab|xy)$. Then $P_{AB|XY}, P_{BC|YW}$ are compatible via a global classical channel $P_{ABC|XYW}$ with $\{P(ab|xy)P(bc|yw)/P(b|y)\}$.

[4] In fact, dynamical incompatibility in the classical regime is a consequence of loops [79], which are not required in the quantum domain.

and the agent in X' is given a set of states $\{\rho^{(i)}_{X|X'}\}_i$ sent with probabilities $\{q^{(i)}_{X|X'}\}_i$ (i.e., $q^{(i)}_{X|X'} \geq 0$ and $\sum_i q^{(i)}_{X|X'} = 1$). The agent in X' needs to send them to the agent in X via an arbitrary channel $\mathcal{E}_{X|X'}$. After the agent in X receives the states, they need to perform a discriminating measurement described by the POVM $M^{(i)}_{X|X'}$ (recall Sec. 2.1). The agent then guesses that the state originally sent was $\rho^{(i)}_{X|X'}$ in case of measurement outcome i.

Setting $D := \left(\{p_{X|X'}\}, \{q^{(i)}_{X|X'}, \rho^{(i)}_{X|X'}\}, \{M^{(i)}_{X|X'}\}\right)$, which denotes a specific task, the corresponding success probability in the task D reads

$$P(D, \boldsymbol{\mathcal{E}}) := \sum_{X|X'} p_{X|X'} \sum_i q^{(i)}_{X|X'} \mathrm{tr}\left[M^{(i)}_{X|X'} \mathcal{E}_{X|X'}\left(\rho^{(i)}_{X|X'}\right)\right]. \tag{4.60}$$

To avoid trivial scenarios, we further consider discrimination tasks D in which the weights are all strictly positive, i.e., $p_{X|X'} > 0, q^{(i)}_{X|X'} > 0, M^{(i)}_{X|X'} > 0 \; \forall\, i \; \forall\, X|X'$. Such a task D is called *strictly positive*. Then channel incompatibility is equivalent to an advantage in an ensemble state discrimination task (note that the subscripts of the following Hermitian operators and local states are showing the dependency on the input-output pair $X|X'$ rather than the system they belong to):

Main Theorem 4.3.3. (Advantage in Ensemble State Discrimination Tasks) *For every $\boldsymbol{\mathcal{E}}$, the following two statements are equivalent:*

1. *For every $X|X' \in \Lambda$ there exist Hermitian operators $\left\{H^{(i)}_{X|X'}\right\}_{i=1}^N$ in X and states $\left\{\rho^{(i)}_{X|X'}\right\}_{i=1}^N$ in X', where $N \in \mathbb{N}$ is independent of $X|X'$, such that*

$$\sum_{X|X'\in\Lambda} \sum_{i=1}^N \mathrm{tr}\left[H^{(i)}_{X|X'} \mathcal{E}_{X|X'}\left(\rho^{(i)}_{X|X'}\right)\right] > \max_{\boldsymbol{\mathcal{L}}\in\mathfrak{C}} \sum_{X|X'\in\Lambda} \sum_{i=1}^N \mathrm{tr}\left[H^{(i)}_{X|X'} \mathcal{L}_{X|X'}\left(\rho^{(i)}_{X|X'}\right)\right]. \tag{4.61}$$

2. *There exists a strictly positive D such that $P(D, \boldsymbol{\mathcal{E}}) > \max_{\boldsymbol{\mathcal{L}}\in\mathfrak{C}} P(D, \boldsymbol{\mathcal{L}})$.*

In other words, $\boldsymbol{\mathcal{E}}$ is incompatible if and only if there exists a strictly positive ensemble state discrimination task D such that

$$P(D, \boldsymbol{\mathcal{E}}) > \max_{\boldsymbol{\mathcal{L}}\in\mathfrak{C}} P(D, \boldsymbol{\mathcal{L}}). \tag{4.62}$$

Proof. Since Statement 2 implies Statement 1 with the Hermitian operator $H^{(i)}_{X|X'} := p_{X|X'} q^{(i)}_{X|X'} M^{(i)}_{X|X'} \geq 0$, it remains to show the opposite direction. As the first step, we note that Statement 1 holds if and only if

$$\sum_{X|X'\in\Lambda} \sum_{i=1}^N \mathrm{tr}\left[\kappa \times (H^{(i)}_{X|X'} + \Delta^{(i)}_{X|X'}\mathbb{I}_X)\mathcal{E}_{X|X'}\left(\rho^{(i)}_{X|X'}\right)\right]$$

$$> \max_{\boldsymbol{\mathcal{L}}\in\mathfrak{C}} \sum_{X|X'\in\Lambda} \sum_{i=1}^N \mathrm{tr}\left[\kappa \times (H^{(i)}_{X|X'} + \Delta^{(i)}_{X|X'}\mathbb{I}_X)\mathcal{L}_{X|X'}\left(\rho^{(i)}_{X|X'}\right)\right], \tag{4.63}$$

for every $\kappa > 0$ and real numbers $\{\Delta^{(i)}_{X|X'}\}$. This is because $\mathcal{E}_{X|X'}$'s are trace-preserving, and $\sum_{i,X|X'} \mathrm{tr}\left[\Delta^{(i)}_{X|X'}\mathbb{I}_X \times \mathcal{E}_{X|X'}\left(\rho^{(i)}_{X|X'}\right)\right] = \sum_{i,X|X'} \Delta^{(i)}_{X|X'}$ is a fixed real number. Now, let $W^{(i)}_{X|X'} := \kappa\left(H^{(i)}_{X|X'} + \Delta^{(i)}_{X|X'}\mathbb{I}_X\right)$, which is a Hermitian operator in X with dependency on $X|X'$. Then we can choose $\kappa, \Delta^{(i)}_{X|X'}$ such that $W^{(i)}_{X|X'} > 0$ and $\sum_{i=1}^N W^{(i)}_{X|X'} < \mathbb{I}_X$ for every $i, X|X'$. This means that, for each $X|X'$, $\{W^{(i)}_{X|X'}\}_{i=1}^N$ can be interpreted as part of a POVM. Then Statement 1 implies that there exists a set of states $\{\rho^{(i)}_{X|X'}\}$ such that

$$\sum_{X|X'\in\Lambda}\sum_{i=1}^N \mathrm{tr}\left[W^{(i)}_{X|X'}\mathcal{E}_{X|X'}\left(\rho^{(i)}_{X|X'}\right)\right] > \max_{\mathcal{L}\in\mathfrak{C}} \sum_{X|X'\in\Lambda}\sum_{i=1}^N \mathrm{tr}\left[W^{(i)}_{X|X'}\mathcal{L}_{X|X'}\left(\rho^{(i)}_{X|X'}\right)\right]. \tag{4.64}$$

Now, consider the task $D = \left(\{p_{X|X'}\}, \{q^{(i)}_{X|X'}, \sigma^{(i)}_{X|X'}\}, \{M^{(i)}_{X|X'}\}\right)$ given by (what follows holds for every $X|X'$; also, the subscript now denotes the dependency on $X|X'$ rather than the systems they live in):

$$p_{X|X'} = \frac{1}{|\Lambda|}; \tag{4.65}$$

$$q^{(i)}_{X|X'} = \frac{1-\epsilon}{N} \quad \text{if } i = 1, ..., N \quad \& \quad q^{(N+1)}_{X|X'} = \epsilon; \tag{4.66}$$

$$\sigma^{(i)}_{X|X'} = \rho^{(i)}_{X|X'} \quad \text{if } i = 1, ..., N \quad \& \quad \sigma^{(N+1)}_{X|X'} = \eta_{X|X'}; \tag{4.67}$$

$$M^{(i)}_{X|X'} = W^{(i)}_{X|X'} \quad \text{if } i = 1, ..., N \quad \& \quad M^{(N+1)}_{X|X'} = \mathbb{I}_X - \sum_{i=1}^N M^{(i)}_{X|X'}, \tag{4.68}$$

where $\epsilon \in [0,1]$ is a real number whose range will be set later, and $\eta_{X|X'}$'s are states in X' that can be chosen arbitrarily (but they still depend on $X|X'$). Then one can check that $\{M^{(i)}_{\mathrm{X|X'}}\}_{i=1}^{N+1}$ is a POVM in the output system X for every $X|X' \in \Lambda$, which means D is an ensemble state discrimination tasks. Also, D is strictly positive once $0 < \epsilon < 1$.

For any set of channels $\mathcal{N} = \{\mathcal{N}_{X|X'}\}_{X|X'\in\Lambda}$, we write

$$P(D, \mathcal{N}) = \widetilde{P}(\mathcal{N}) + \epsilon \times \Gamma(\mathcal{N}), \tag{4.69}$$

where

$$\widetilde{P}(\mathcal{N}) := \frac{1}{N|\Lambda|}\sum_{X|X'\in\Lambda}\sum_{i=1}^N \mathrm{tr}\left[W^{(i)}_{X|X'}\mathcal{N}_{X|X'}\left(\rho^{(i)}_{X|X'}\right)\right]; \tag{4.70}$$

$$\Gamma(\mathcal{N}) := \frac{1}{|\Lambda|}\sum_{X|X'\in\Lambda}\left(\mathrm{tr}\left[\left(\mathbb{I}_X - \sum_{i=1}^N W^{(i)}_{X|X'}\right)\mathcal{N}_{X|X'}(\eta_{X|X'})\right] - \frac{1}{N}\sum_{i=1}^N \mathrm{tr}\left[W^{(i)}_{X|X'}\mathcal{N}_{X|X'}(\rho^{(i)}_{X|X'})\right]\right). \tag{4.71}$$

Equation (4.64) implies that $\widetilde{P}(\mathcal{E}) = \Delta + \max_{\mathcal{L}\in\mathfrak{C}} \widetilde{P}(\mathcal{L})$ for some $\Delta > 0$, which further means

$$\begin{aligned}\max_{\mathcal{L}\in\mathfrak{C}} P(D,\mathcal{L}) &\le \max_{\mathcal{L}\in\mathfrak{C}} \widetilde{P}(\mathcal{L}) + \epsilon \times \max_{\mathcal{L}\in\mathfrak{C}} \Gamma(\mathcal{L}) \\ &= \widetilde{P}(\mathcal{E}) - \Delta + \epsilon \times \max_{\mathcal{L}\in\mathfrak{C}} \Gamma(\mathcal{L}) = P(D,\mathcal{E}) - \Delta + \epsilon \times \left[\max_{\mathcal{L}\in\mathfrak{C}} \Gamma(\mathcal{L}) - \Gamma(\mathcal{E})\right],\end{aligned} \tag{4.72}$$

Set $\Delta' \coloneqq \max_{\mathcal{L}\in\mathfrak{C}} \Gamma(\mathcal{L}) - \Gamma(\mathcal{E})$, which is finite since Γ is bounded. Then we conclude that $\max_{\mathcal{L}\in\mathfrak{C}} P(D,\mathcal{L}) \leq P(D,\mathcal{E}) - \Delta + \epsilon\Delta'$. If $\Delta' \leq 0$, then $\max_{\mathcal{L}\in\mathfrak{C}} P(D,\mathcal{L}) < P(D,\mathcal{E})$ $\forall\epsilon\in[0,1]$. If $\Delta' > 0$, take $\epsilon < \min\left\{\frac{\Delta}{\Delta'}, 1\right\}$ which gives $\max_{\mathcal{L}\in\mathfrak{C}} P(D,\mathcal{L}) < P(D,\mathcal{E})$. □

As a remark, such operational advantages in ensemble state discrimination tasks can be extended to a general dynamical resource theory setups recently investigated in, e.g., Ref. [18, 19]. Setting $X' = S'$ for every member in Λ, Theorem 4.3.3 implies that every set of broadcast incompatible channels gives an advantage over compatible ones in some ensemble state discrimination tasks. This recovers results from Refs. [60, 80]. Also, in the particular case of quantum-to-classical channels, Theorem 4.3.3 recovers results on the discrimination advantages of incompatible measurements [61, 81, 82].

Chapter 5

Entanglement Preserving Local Thermalisation

As one of the most representative resources in quantum information science, entanglement is at the basis of quantum advantages in various operational tasks such as metrology [83], cryptography [84], communication [85], computation [86], and quantum therodynamics [90, 91, 87, 88, 89]. While being a powerful resource, it often does not survive interactions with an external environment. Consequently, it is crucial to know whether entanglement can be maintained by certain classes of dynamics. From a thermodynamic perspective, one important class is thermalisation, which describes the evolution of states toward thermal equilibrium. Formally, a channel is called a *full thermalisation* if it maps *every* input state to a fixed output state – the thermal state. While entanglement is distributed at spatially separated locations, thermalisation often acts locally and is known to destroy quantum correlations. This motivates us to ask:

Can globally distributed entanglement survive locally performed full thermalisation?

Ultimately, it is a fundamental question concerning the structure of quantum theory, specifically about the interplay between subsystem thermalisation and quantum correlations.

5.1 Formulation

One way to formalise this question is as follows. Suppose two agents in a bipartite system AB are restricted to perform *local operations plus shared randomness* (LOSR) channels in AB, which takes the form

$$\mathcal{E}_{AB} = \int (\mathcal{E}_{\lambda,A} \otimes \mathcal{E}_{\lambda,B}) p_\lambda d\lambda, \tag{5.1}$$

with $p_\lambda \geq 0$ and $\int d\lambda p_\lambda = 1$ is a probability distribution and $\mathcal{E}_{\lambda,A}, \mathcal{E}_{\lambda,B}$ are local channels in A, B, respectively, with λ dependency. Physically, an LOSR channel can be realised as follows: At the beginning of the operation, two local agents share classical randomness, e.g., a third party samples λ and distributes the outcomes to them according to the probability p_λ. Each λ corresponds to a specific local channel, i.e., $\mathcal{E}_{\lambda,A}, \mathcal{E}_{\lambda,B}$. The local agents apply the corresponding local channels conditioned on the received λ.

Now, suppose that each local agent holds a system with local Hamiltonian H_X, where we write $X = A, B$, and the total Hamiltonian is $H_A \otimes \mathbb{I}_B + \mathbb{I}_A \otimes H_B$; namely, there is no interaction between A, B. By means of local processes and pre-shared classical resources (i.e., LOSR channels), they want to fully thermalise their local systems to some local environment temperatures T_A and T_B, respectively. Recall from Eq. (3.31) that $\gamma_X = \frac{e^{-\beta_X H_X}}{\mathrm{tr}(e^{-\beta_X H_X})}$ is the definition of a thermal state in X, where $\beta_X = \frac{1}{k_B T_X}$ is the inverse temperature and k_B is the Boltzmann constant. Then we can state the main definition as follows:

Definition 5.1.1. (Local Thermalisation) *A channel $\mathcal{E}_{AB}$ acting on AB is called a* local thermalisation *to a pair of thermal states (γ_A, γ_B) if*

1. *$\mathcal{E}_{AB}$ is an LOSR channel.*
2. *$\mathrm{tr}_A \mathcal{E}_{AB}(\rho_{AB}) = \gamma_B$ and $\mathrm{tr}_B \mathcal{E}_{AB}(\rho_{AB}) = \gamma_A$, for every ρ_{AB}.*

Namely, a local thermalisation is an LOSR channel in AB whose marginal channels in A, B are both exist full thermalisation channels. This also means that a local thermalisation is *local* in two senses: It is a channel involving local actions that locally thermalises every input. We further say a local thermalisation $\mathcal{E}_{AB}$ is an *entanglement preserving local thermalisation* (EPLT) if, furthermore, there exists some input ρ_{AB} such that the output $\mathcal{E}_{AB}(\rho_{AB})$ is entangled. In other words, an EPLT is a local thermalisation with non-vanishing entanglement preservability.

Note that dropping either of the two conditions trivialises the dynamical question. If we drop condition 1, any channel can be used, including state preparation channels of entangled locally thermal states (see, e.g., Ref. [91]). Also, dropping condition 2 means that the existence of entangled locally thermal states and the identity channel would trivially satisfy the requirements. On the other hand, one may ask whether we could strengthen condition 1 by asking $\mathcal{E}_{AB}$ to be a local operation *without* shared randomness. However, as expected, no correlation can be preserved in this case:

Lemma 5.1.2. (Product Local Thermalisation Is Trivial) *Any product local thermalisation to (γ_A, γ_B) coincides with the state preparation channel $(\cdot) \mapsto \gamma_A \otimes \gamma_B$.*

Proof. Suppose $\mathcal{E}_{AB} = \mathcal{E}_A \otimes \mathcal{E}_B$ is a product local thermalisation. Then $\mathcal{E}_X$ is identical to the state preparation channel $(\cdot) \mapsto \gamma_X$, which is entanglement-breaking [37]. This means (1) $(\mathcal{E}_A \otimes \mathcal{E}_B) \circ (\mathcal{E}_A \otimes \mathcal{E}_B) = \mathcal{E}_A \otimes \mathcal{E}_B$ and (2) $\mathcal{E}_{AB}(\rho_{AB}) = (\mathcal{E}_A \otimes \mathcal{E}_B)(\rho_{AB})$ is always separable for every ρ_{AB}. Hence, for an arbitrary ρ_{AB}, we can write $(\mathcal{E}_A \otimes \mathcal{E}_B)(\rho_{AB}) = \sum_i f_i \rho_A^i \otimes \rho_B^i$ for some $f_i \geq 0$, $\sum_i f_i = 1$, and direct computation shows that

$$\mathcal{E}_{AB}(\rho_{AB}) = (\mathcal{E}_A \otimes \mathcal{E}_B) \circ (\mathcal{E}_A \otimes \mathcal{E}_B)(\rho_{AB}) = (\mathcal{E}_A \otimes \mathcal{E}_B)(\sum_i f_i \rho_A^i \otimes \rho_B^i) = \gamma_A \otimes \gamma_B, \tag{5.2}$$

which completes the proof. □

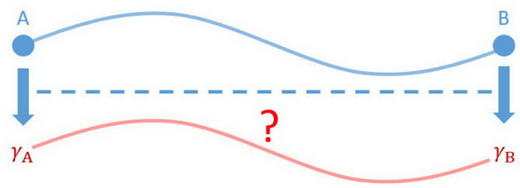

Figure 5.1: *Information-theoretic formulation.* In this formulation, we ask whether entanglement can survive after an LOSR channel that is locally indistinguishable from a full thermalisation process.

Hence, the simplest EPLT, if it exists, must exploit shared randomness to preserve entanglement during a local thermalisation. Our main question is then as follows:

Do EPLTs exist?

See also Fig. 5.1 for a schematic interpretation. Here, we ask if classical correlations/shared randomness alone allow for the preservation of entanglement in thermalisations.

5.1.1 Existence of EPLT

We now turn to the existence of EPLT at *every* positive local temperatures. From now on, we assume equal finite local dimension d, no degeneracies, and finite energies. First, we need to recall the $(U \otimes U^*)$-*twirling operation* [92, 93], which is defined by

$$\mathcal{T}_{AB}(\rho_{AB}) := \int (U \otimes U^*)\rho_{AB}(U \otimes U^*)^\dagger dU, \tag{5.3}$$

where the integration is taken over the group of unitary operators in dimension d with the Haar measure dU. Operationally, twirling results from the application of coordinated random local unitaries, turning local systems into maximally entropic states while still partially maintaining correlations between them. An important property of the twirling operation is that its output is always an isotropic state [93] [recall from Eq. (2.6) the definition of maximally entangled states]:

$$\rho_{\text{iso}}(p) := p|\Psi^+_{AB}\rangle\langle\Psi^+_{AB}| + (1-p)\frac{\mathbb{I}_{AB}}{d^2}, \tag{5.4}$$

where $p \in [-\frac{1}{d^2-1}, 1]$ due to the positivity of quantum states. $\mathcal{T}_{AB}$ can preserve entanglement since $\langle\Psi^+_{AB}|\mathcal{T}_{AB}(\cdot)|\Psi^+_{AB}\rangle = \langle\Psi^+_{AB}| \cdot |\Psi^+_{AB}\rangle$ [93] together with the fact that an isotropic state ρ_{iso} is entangled if and only if $\langle\Psi^+_{AB}|\rho_{\text{iso}}|\Psi^+_{AB}\rangle > \frac{1}{d}$ [93]. Hence, $\mathcal{T}_{AB}(\rho_{AB})$ is entangled if and only if $\langle\Psi^+_{AB}|\rho_{AB}|\Psi^+_{AB}\rangle > \frac{1}{d}$.

Lemma 5.1.3. (Local Thermalisation Lemma) *Define the following channel in AB:*

$$\mathcal{E}^{\epsilon}_{AB}(\cdot) := \mathcal{D}^{(1-\epsilon)}_{\eta^{\epsilon}_A \otimes \eta^{\epsilon}_B} \circ \mathcal{T}_{AB}(\cdot), \tag{5.5}$$

where

$$\mathcal{D}^{p}_{\sigma}(\cdot) := p\sigma + (1-p)\mathcal{I}_{AB}(\cdot) \tag{5.6}$$

and, for $X = A, B$,

$$\eta^{\epsilon}_X := \gamma_X + \frac{\epsilon}{1-\epsilon}\left(\gamma_X - \frac{\mathbb{I}_X}{d}\right). \tag{5.7}$$

Then $\mathcal{E}^{\epsilon}_{AB}$ is a local thermalisation to (γ_A, γ_B) for all $0 \le \epsilon \le \epsilon_* := d\overline{P}_{\min}$, where $\overline{P}_{\min}$ is the smallest eigenvalue among γ_A and γ_B.

Proof. First, the definition of η^{ϵ}_X implies that $\mathcal{E}^{\epsilon}_{AB}$ locally behaves as a full thermalisation. This is because $\mathrm{tr}_A \circ \mathcal{E}^{\epsilon}_{AB}(\cdot) = (1-\epsilon)\eta^{\epsilon}_B + \epsilon\frac{\mathbb{I}_B}{d} = \gamma_A$, and the same by exchanging A and B. It remains to show that $\mathcal{E}^{\epsilon}_{AB}$ is an LOSR channel, which means that it suffices to prove that η^{ϵ}_X is a state when ϵ falls into the prescribed region. Write $\gamma_X = \sum_{n=0}^{d-1} P^X_n |n\rangle\langle n|_X$ with $1 \ge P^X_0 \ge P^X_1 \ge ... \ge P^X_{d-1} \ge 0$. Then we have $\eta^{\epsilon}_X = \sum_{n=0}^{d-1} Q^X_n |n\rangle\langle n|$ with

$$Q^X_n = \frac{1}{1-\epsilon}P^X_n - \frac{\epsilon}{d(1-\epsilon)}. \tag{5.8}$$

Since $\epsilon \le 1$, we have the hierarchy $Q^X_0 \ge Q^X_1 \ge ... \ge Q^X_{d-1}$ and the normalisation condition $\sum_{n=0}^{d-1} Q^X_n = 1$. This means that it suffices to impose $Q^X_{d-1} \ge 0$ to make sure η^{ϵ}_X is a state, which leads to the desired range $0 \le \epsilon \le \epsilon_* := d\overline{P}_{\min}$. □

With the above lemma in hand, we are in the position to answer our central question: (note again that we only consider finite-energy Hamiltonians):

Main Theorem 5.1.4. (Existence of EPLTs) *$\mathcal{E}^{\epsilon_*}_{AB}$ is an EPLT to (γ_A, γ_B) for all local temperatures $T_A, T_B > 0$.*

Proof. It suffices to show that the output state is entangled when the input state is $|\Psi^+_{AB}\rangle$. To this end, we adopt the *positive partial transpose* (PPT) criterion [94, 95] to detect the entanglement of the output. Again, write $\eta^{\epsilon}_X = \sum_{n=0}^{d-1} Q^X_n |n\rangle\langle n|$; then we have

$$\begin{aligned}\mathcal{E}^{\epsilon_*}_{AB}(|\Psi^+_{AB}\rangle\langle\Psi^+_{AB}|) &= \epsilon_*|\Psi^+_{AB}\rangle\langle\Psi^+_{AB}| + (1-\epsilon_*)\sum_{n,m=0}^{d-1} Q^A_n Q^B_m |nm\rangle\langle nm| \\ &= \sum_{n,m=0}^{d-1}\left[\frac{\epsilon_*}{d}|nn\rangle\langle mm| + (1-\epsilon_*)Q^A_n Q^B_m |nm\rangle\langle nm|\right].\end{aligned} \tag{5.9}$$

Taking the partial transpose on B gives

$$\sum_{n,m=0}^{d-1}\left[\frac{\epsilon_*}{d}|nm\rangle\langle mn| + (1-\epsilon_*)Q_n^A Q_m^B |nm\rangle\langle nm|\right] = \left(\bigoplus_{n\neq m} M_{nm}\right)\oplus D, \tag{5.10}$$

where

$$M_{nm} := \begin{pmatrix} (1-\epsilon_*)Q_n^A Q_m^B & \frac{\epsilon_*}{d} \\ \frac{\epsilon_*}{d} & (1-\epsilon_*)Q_m^A Q_n^B \end{pmatrix}, \tag{5.11}$$

and $D := \bigoplus_{n=0}^{d-1}\left[\frac{\epsilon_*}{d} + (1-\epsilon_*)Q_n^A Q_n^B\right]$ is the contribution of the diagonal terms. To see that the output is entangled, it suffices to show that there exists a negative eigenvalue of at least one M_{nm}. Note that $Q_{d-1}^A = 0$ when we substitute $\epsilon = \epsilon_* = dP_{\min}$ in Eq. (5.8) (without loss of generality, we assume $P_{\min} = P_{d-1}^A$). This means that, for every $m < d-1$, we have

$$M_{d-1,m} = \begin{pmatrix} 0 & \frac{\epsilon_*}{d} \\ \frac{\epsilon_*}{d} & (1-\epsilon_*)Q_m^A Q_{d-1}^B \end{pmatrix}, \tag{5.12}$$

which has a negative eigenvalue if the off-diagonal terms are positive; namely, when $\epsilon_* > 0$. This completes the proof. □

Theorem 5.1.4 shows the existence of EPLT in the most general case for bipartite systems, apart from the special case of $T_A = 0$ or $T_B = 0$. Note that we have $\epsilon_* \to 1$ and $\mathcal{E}_{AB}^{\epsilon_*} \to \mathcal{T}$ when $T_A, T_B \to +\infty$. Hence. the twirling operation is an EPLT with infinite local temperatures or fully degenerate local Hamiltonians. In this sense, $\mathcal{E}_{AB}^{\epsilon}$ can be treated as a finite temperature extension of the twirling operation.

Theorem 5.1.4 leaves out only the case $T_A = T_B = 0$[1], which we treat separately here. We separately consider two cases, depending on whether or not there is any ground-state degeneracy on the systems. In the latter case, the corresponding local thermal state is given by the unique *pure* ground state of the local Hamiltonian. Then one can immediately conclude that no entanglement can be preserved, because a pure state cannot be correlated with any other system. Hence, no EPLT exists in such a case. On the other hand, if both local systems admit ground-state degeneracy, then EPLTs exist even when $T_{\mathrm{A}} = T_{\mathrm{B}} = 0$. The idea is to perform the twirling operation in the *ground energy subspace*. To illustrate the idea, consider the following protocol, where we assume two-fold ground state degeneracy on both local systems: In X, where $X = A, B$, consider the POVM $\{\Pi_0^X, \mathbb{I}_X - \Pi_0^X\}$, where $\Pi_0^{\mathrm{X}} := \sum_{g=0,1}|0,g\rangle\langle 0,g|_X$ is the projector onto the ground energy subspace, g is a degeneracy index, and $\{|0,g\rangle\}_{g=0,1}$ span the ground energy subspace of the local Hamiltonian H_X. The first step of the protocol is to measure $\{\Pi_0^A, \mathbb{I}_A - \Pi_0^A\}\otimes\{\Pi_0^B, \mathbb{I}_B - \Pi_0^B\}$. For each local agent, if the outcome reads Π_0^X, nothing is done; on the other had, if the outcome reads $\mathbb{I}_X - \Pi_0^X$, then the agent discards the original input and prepares $\frac{\Pi_0^X}{2}$. The second step of the protocol is to use shared randomness to achieve a twirling operation on the ground

[1] The case when, e.g., $T_A = 0$ and $T_B > 0$ can be addressed in a similar way.

energy subspace, denoted by $\mathcal{T}^0_{AB}$. Formally, the channel corresponding to the above protocol is $\mathcal{T}^0_{AB} \circ (\mathcal{L}_A \otimes \mathcal{L}_B)$, where $\mathcal{L}_X(\cdot) := \Pi_0^X(\cdot)\Pi_0^X + \Phi_{\frac{\Pi_0^X}{2}}[(\mathbb{I}_X - \Pi_0^X)(\cdot)(\mathbb{I}_X - \Pi_0^X)]$ and $\Phi_\rho : (\cdot) \mapsto \rho$ is the state preparation channel of ρ. This protocol indeed gives a local thermalisation because the output states have, independently of the input, marginal $\frac{\Pi_0^X}{2}$ in X, which is the desired thermal state in this case. Finally, the entanglement preservation can be seen by choosing the input state as $\frac{1}{\sqrt{2}}(|0,0\rangle_A \otimes |0,0\rangle_B + |0,1\rangle_A \otimes |0,1\rangle_B)$, which is invariant under the whole protocol. This confirms that EPLTs exist in all nontrivial scenarios.

5.1.2 Alternative Formulation

So far we have analysed the *information-theoretic* formulation (Fig. 5.1), and it turns out that our central question admits an *equivalent*, *thermodynamic* reformulation as follows. Suppose an unknown input state is distributed to two agents in a bipartite system AB. We assume that the agents neither share additional quantum resources, such as another entangled state, nor can they communicate with each other. Each of them has access to a local heat bath, and we allow for the two baths to be *classically correlated* across the bipartition. Each party thermalises their half of the (unknown) input state by coupling their local systems to their local baths. Then our central question reads

Can entanglement survive when local systems A and B are both fully thermalised?

Schematically, this formulation depicts two local systems interacting with their individual heat baths and the thermalising. The heat baths do not interact between them, but can share some initial classical correlations. The resulting dynamics can hence be characterised as follows:

Definition 5.1.5. *A channel C_{AB} acting on a bipartite system AB is called a* local bath thermalisation *to (γ_A, γ_B) if*

1. $C_{AB}(\rho_{AB}) = \mathrm{tr}_{A'B'}[\mathcal{V}_{AA'} \otimes \mathcal{V}_{BB'}(\rho_{AB} \otimes \gamma_{A'B'})]$, *where $\mathcal{V}_{XX'}(\cdot) := U_{XX'}(\cdot)U^\dagger_{XX'}$ are local unitary dynamics on XX' and $\gamma_{A'B'}$ is a separable thermal state.*

2. $\mathrm{tr}_A \circ C_{AB}(\rho_{AB}) = \gamma_B$ *and* $\mathrm{tr}_B C_{AB}(\rho_{AB}) = \gamma_A$ *for all ρ_{AB}.*

The above-defined channel C_{AB} is further called an *entanglement preserving local bath thermalisation* if there exists some state ρ_{AB} such that $C_{AB}(\rho_{AB})$ is entangled. The above notion illustrates the thermodynamic formulation, and the alternative form of our central question is:

Do entanglement preserving local bath thermalisations exist?

The following result allows us to rephrase the results in this alternative formulation:

Main Theorem 5.1.6. (Thermodynamic Formulation) *A bipartite channel is a local bath thermalisation if and only if it is a local thermalisation.*

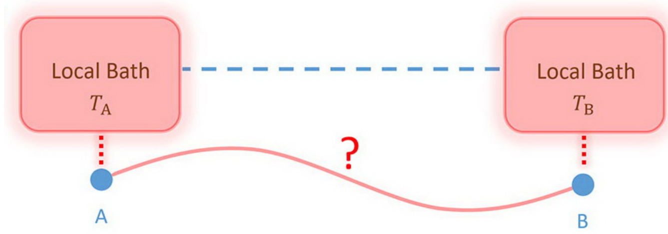

Figure 5.2: *Thermodynamic formulation.* In this formulation, we ask whether entanglement can survive after subsystem thermalisations are achieved by coupling to classically correlated heat baths.

Proof. Note that every local bath thermalisation is by definition a local thermalisation, so it remains to prove the inverse statement. To this end, first, we recall that all LOSR channels in AB are in the convex hull of the set of all product channels. Being embedded in a finite Euclidean space, Carathéodory theorem implies that for each LOSR channel $\mathcal{E}_{AB}$, there exists a finite set of product channels and a probability distribution, $\{\mathcal{E}^i_A \otimes \mathcal{E}^i_B, p_i > 0\}_{i=1}^D$, such that $\mathcal{E}_{AB} = \sum_{i=1}^D p_i(\mathcal{E}^i_A \otimes \mathcal{E}^i_B)$, where D only depends on the local dimensions. Then, for a given i and $X = A, B$, the Stinespring dilation theorem [96] guarantees the existence of an ancillary system X'_i with dimension d^2 and a unitary operator $U_{XX'_i}$ acting on XX'_i such that $\mathcal{E}^i_X(\cdot) = \mathrm{tr}_{X'_i}\left\{U_{XX'_i}\left[(\cdot)\otimes|0\rangle\langle 0|_{X'_i}\right]U^\dagger_{XX'_i}\right\}$. Since $X'_i \simeq \mathbb{C}^{d^2}$ for all i, we simply choose them to be the same Hilbert space, denoted by $X' \simeq \mathbb{C}^{d^2}$, and write the corresponding unitary operator as $U^i_{XX'}$. Then we have $\mathcal{E}_{AB}(\cdot) = \mathrm{tr}_{A'B'}\left\{\sum_{i=1}^D p_i\left(U^i_{AA'}\otimes U^i_{BB'}\right)[(\cdot)\otimes|00\rangle\langle 00|_{A'B'}]\left(U^i_{AA'}\otimes U^i_{BB'}\right)^\dagger\right\}$. Define a Hilbert space $\mathcal{H}_D := \mathrm{span}\,\{|i\rangle\}_{i=1}^D$ and introduce two additional ancillary systems $A'' \simeq \mathcal{H}_D$, $B'' \simeq \mathcal{H}_D$, we conclude that

$$\mathcal{E}_{AB}(\cdot) =$$

$$\mathrm{tr}_{A'B'A''B''}\left\{(V_{AA'A''}\otimes V_{BB'B''})\left[(\cdot)\otimes|00\rangle\langle 00|_{A'B'}\otimes\sum_{i=1}^D p_i|ii\rangle\langle ii|_{A''B''}\right](V_{AA'A''}\otimes V_{BB'B''})^\dagger\right\}, \tag{5.13}$$

where $V_{XX'X''} := \sum_i U^i_{XX'}\otimes|i\rangle\langle i|_{X''}$ is a unitary operator acting on $XX'X''$. The separable state $\sum_{i=1}^D p_i|ii\rangle\langle ii|_{A''B''}$ is full rank, hence it can be treated as a thermal state in $A''B''$ by an appropriate choice of the Hamiltonian. □

See Fig. 5.2 for a schematic interpretation. Theorem 5.1.6 shows that, as it is intuitive, the set of local bath thermalisations coincides with the set of local thermalisations. It implies that if two local agents perform local interactions with a thermal bath that fully thermalise their local states for every input, then even knowing that the bath has no entanglement across the bipartition, they still *cannot* conclude that their shared output is separable. In other words, classical correlations alone in the bath can allow for the entanglement preservation even after locally performed full thermalisation. We note that Lemma 5.1.2 and Theorem 5.1.6 imply that no entanglement preserving local bath thermalisation exists if we restrict $\gamma_{A'B'} = \gamma_{A'}\otimes\gamma_{B'}$ in the setting of Theorem 5.1.6; namely, if we remove the classical correlation in the bath.

5.2 Mechanism

Given the existence of EPLT, it is then natural and important to ask:

What is the mechanism behind EPLT?

To answer this question, we introduce another family of EPLTs and use it to study the underlying physical mechanism:

$$\Lambda_{AB}^{(\epsilon_A,\epsilon_B)} := \left[\mathcal{D}_{\eta_A^{\epsilon_A}}^{(1-\epsilon_A)} \otimes \mathcal{D}_{\eta_B^{\epsilon_B}}^{(1-\epsilon_B)}\right] \circ \mathcal{T}, \tag{5.14}$$

where $\mathcal{D}_\sigma^p$ is defined in Eq. (5.6). The same proof shows that $\Lambda^{(\epsilon_A,\epsilon_B)}$ is a local thermalisation to (γ_A, γ_B) for all $0 \le \epsilon_X \le dP_{\min}^X$, where $P_{\min}^X$ is the smallest eigenvalue of γ_X. Eq. (5.14) has a clear thermodynamic interpretation. First, local agents A and B perform the twirling operation. Then, they perform a sudden quench of the local system Hamiltonians $H_X \mapsto H_X^{\epsilon_X}$ (that is, energies are tuned, but not the eigenstates), with $\eta_X^{\epsilon_X} \propto e^{-H_X^{\epsilon_X}/k_B T_X}$. At this point, by thermal contact with their local environments, they let their local system undergo a *partial thermalisation*[2],

$$\mathcal{P}_\gamma^t(\cdot) := e^{-\frac{t}{\tau_\gamma}}(\cdot) + \left(1 - e^{-\frac{t}{\tau_\gamma}}\right)\gamma, \tag{5.15}$$

where $\tau_\gamma \in (0,\infty)$ is the thermalisation time scale corresponding to the thermal state γ. Note that $\mathcal{P}_\gamma^t = \mathcal{D}_\gamma^p$ with $p = 1 - e^{\frac{-t}{\tau_\gamma}}$, and hence $\mathcal{D}_\gamma^p$ can be realised by a partial thermalisation. Finally, they quench their Hamiltonians back to H_X, and the local states are always γ_X, i.e., A and B both have fully thermalised their local systems. However, entanglement can survive once local thermality is reached. This is in contrast to what happens if local agents each let their local systems fully thermalise to an independent bath according to Eq. (5.15): In this case, local thermality is reached only when the global state is $\gamma_A\otimes\gamma_B$. This observation suggests that, by taking the EPLT of Eq. (5.14) as a model, the existence of ELPTs potentially rely on a 'local speed-up' of the full thermalisation process. Roughly speaking, we want to show that such an EPLT can thermalise the local system $X = A, B$ to γ_X at a time instance t_1 that is *strictly smaller* than the time instance t_2 for a partial thermalisation model to thermalise X to γ_X.

To gather evidence for this intuition, first we note from Eq. (5.15) that a partial thermalisation takes *infinite* time to fully thermalise the subsystem to γ_X. On the other

[2]The partial thermalisation model can be derived from collision models [97] and can be seen as a particular realisation of the Davies dynamical semigroup [98].

hand, the local behavior of Eq. (5.14) on X is a random unitary followed by an *incomplete* partial thermalisation with the completion time $t = -\tau_{\eta_X^{\epsilon_X}} \ln \epsilon_X$, which is *always finite*. Hence, we conclude that the subsystem thermalisation is faster in the EPLT scheme when the time to implement random unitaries is finite.

One may, however, suspect that in practice the exact twirling operation requires again infinite time. Let us show that the same speed-up argument holds even when one accounts for the time needed to implement the twirling operation. In fact, with a finite number N of unitaries, one can realise an approximation $\mathcal{T}_{AB}^{(N)}$ of the twirling operation $\mathcal{T}_{AB}$ with an exponentially good precision in N [99]. The completion time of this approximation is Nt_U, with t_U the time necessary to perform a single unitary. Hence, as long as t_U is sufficiently small compared with the typical time scale of thermalisation τ_{γ_X}, $\delta > 0$ is small enough, and N is large enough, we expect a shorter time in the EPLT thermalisation scheme compared with the partial thermalisation described by Eq. (5.15). In fact, we are able to show an analytical result. Before proving it, we still need a lemma describing a finite-time one-shot approximate implementation of the twirling operation, which is detailed in the following section.

5.2.1 Approximating the Twirling Operation

To approximate the twirling operation $\mathcal{T}_{AB}$, we consider its implementation by means of a finite sequence of unitaries, as introduced in Ref. [99],

$$\mathcal{T}_{\boldsymbol{U}}^{(N)} := \prod_{k=1}^{N} \mathcal{T}_k, \tag{5.16}$$

with

$$\mathcal{T}_k(\cdot) := \frac{1}{2}\mathcal{I}_{AB}(\cdot) + \frac{1}{2}(U_k \otimes U_k^*)(\cdot)(U_k \otimes U_k^*)^\dagger, \tag{5.17}$$

where each U_k represents a random unitary and $\boldsymbol{U} = (U_1, ..., U_N)$ is a vector of random variables. Setting $\|\mathcal{E}\|_\infty := \sup_\rho \|\mathcal{E}(\rho)\|_\infty$ for a given channel $\mathcal{E}$, it was proven in Ref. [99] that

$$\left\langle \left\| \mathcal{T}_{AB} - \mathcal{T}_{\boldsymbol{U}}^{(N)} \right\|_\infty^2 \right\rangle < \frac{1}{2^N}, \tag{5.18}$$

where $\langle(\cdot)\rangle := \int (\cdot) dU_1 dU_2 ... dU_N$ is the average over the Haar measure. In order to establish the speed-up result, we give a more detailed version of the result in Ref. [99] by assessing the probability that a given realisation of $\mathcal{T}_{\boldsymbol{U}}^{(N)}$ is close to $\mathcal{T}_{AB}$:

Lemma 5.2.1. (Twirling Implementation Lemma) *For every $\lambda > 0$, we have*

$$\mathrm{Prob}\left(\left\| \mathcal{T}_{AB} - \mathcal{T}_{\boldsymbol{U}}^{(N)} \right\|_\infty^2 - \frac{1}{2^N} > \lambda \right) < \frac{1}{\lambda^2 2^N}. \tag{5.19}$$

Proof. This lemma can be seen by applying Chebyshev's inequality on the random variable $\left\| \mathcal{T}_{AB} - \mathcal{T}_{\boldsymbol{U}}^{(N)} \right\|_\infty^2$, whose variance can be shown to be upper bounded by $\frac{1}{2^N}$. To see this, we let $\Delta := \left\|\mathcal{T}_{AB} - \mathcal{T}_{\boldsymbol{U}}^{(N)}\right\|_\infty^2$. Then direct computation shows

$$\left\langle (\Delta - \langle \Delta \rangle)^2 \right\rangle = \left\langle \Delta^2 \right\rangle - \langle \Delta \rangle^2 \leq \left\langle \Delta^2 \right\rangle \leq \left\langle \left\|\mathcal{T}_{AB} - \mathcal{T}_{\boldsymbol{U}}^{(N)}\right\|_2^2 \right\rangle < \frac{1}{2^N}, \tag{5.20}$$

where $\|\mathcal{E}\|_2 := \sup_\rho \|\mathcal{E}(\rho)\|_2$ for the Hilbert-Schmidt norm $\|\cdot\|_2$, and the last inequality is due to the relation $\left\langle \left\|\left(\mathcal{T}_{AB} - \mathcal{T}_{\boldsymbol{U}}^{(N)}\right)(\rho)\right\|_2^2 \right\rangle = \frac{1}{2^N}\left(\|\rho\|_2^2 - \|\mathcal{T}_{AB}(\rho)\|_2^2\right) < \frac{1}{2^N}$ given by Eq. (22) in Ref. [99]. Hence, it remains to check the applicability of Chebyshev's inequality, which requires the given random variable to be integrable. It suffices to show the continuity of $\left\|\mathcal{T}_{AB} - \mathcal{T}_{\boldsymbol{U}}^{(N)}\right\|_\infty$ in the argument $\boldsymbol{U} = (U_1, ..., U_N)$ with respect to the metric d_N defined by $d_N(\boldsymbol{U}, \boldsymbol{V}) := \sum_{i=1}^N \|U_i - V_i\|_\infty$. To this end, consider a given pair of sequences of unitaries $\boldsymbol{U}$ and $\boldsymbol{V}$. Using the notations $\mathcal{U}_i(\cdot) := (U_i \otimes U_i^*)(\cdot)(U_i \otimes U_i^*)^\dagger$, $\mathcal{V}_i(\cdot) := (V_i \otimes V_i^*)(\cdot)(V_i \otimes V_i^*)^\dagger$, we learn that

$$\left|\left\|\mathcal{T}_{AB} - \mathcal{T}_{\boldsymbol{U}}^{(N)}\right\|_\infty - \left\|\mathcal{T}_{AB} - \mathcal{T}_{\boldsymbol{V}}^{(N)}\right\|_\infty\right| \leq \left\|\mathcal{T}_{\boldsymbol{U}}^{(N)} - \mathcal{T}_{\boldsymbol{V}}^{(N)}\right\|_\infty \leq \frac{1}{2^N}\sum_{\boldsymbol{s}} \left\|\prod_{i=1}^{j_{\boldsymbol{s}}} \mathcal{U}_{s_i} - \prod_{i=1}^{j_{\boldsymbol{s}}} \mathcal{V}_{s_i}\right\|_\infty, \tag{5.21}$$

where we repeatedly used the triangle inequality and the summation $\sum_{\boldsymbol{s}}$ is over all the possible strings of ordered indices $\boldsymbol{s} = \{s_1, s_2, ..., s_{j_{\boldsymbol{s}}}\} \subseteq \{1, 2, ..., N\}$ with $j_{\boldsymbol{s}} \leq N$. Since

$$\begin{aligned}
\left\|\prod_{i=1}^{j_{\boldsymbol{s}}} \mathcal{U}_{s_i} - \prod_{i=1}^{j_{\boldsymbol{s}}} \mathcal{V}_{s_i}\right\|_\infty &= \left\|\mathcal{U}_{s_{j_{\boldsymbol{s}}}} \circ \prod_{i=1}^{j_{\boldsymbol{s}}-1} \mathcal{U}_{s_i} - \mathcal{V}_{s_{j_{\boldsymbol{s}}}} \circ \prod_{i=1}^{j_{\boldsymbol{s}}-1} \mathcal{V}_{s_i}\right\|_\infty \\
&\leq \left\|\mathcal{U}_{s_{j_{\boldsymbol{s}}}} \circ \left(\prod_{i=1}^{j_{\boldsymbol{s}}-1} \mathcal{U}_{s_i} - \prod_{i=1}^{j_{\boldsymbol{s}}-1} \mathcal{V}_{s_i}\right)\right\|_\infty + \left\|\left(\mathcal{U}_{s_{j_{\boldsymbol{s}}}} - \mathcal{V}_{s_{j_{\boldsymbol{s}}}}\right) \circ \prod_{i=1}^{j_{\boldsymbol{s}}-1} \mathcal{V}_{s_i}\right\|_\infty \\
&= \left\|\prod_{i=1}^{j_{\boldsymbol{s}}-1} \mathcal{U}_{s_i} - \prod_{i=1}^{j_{\boldsymbol{s}}-1} \mathcal{V}_{s_i}\right\|_\infty + \left\|\mathcal{U}_{s_{j_{\boldsymbol{s}}}} - \mathcal{V}_{s_{j_{\boldsymbol{s}}}}\right\|_\infty,
\end{aligned} \tag{5.22}$$

we conclude that $\left\|\prod_{i=1}^{j_{\boldsymbol{s}}} \mathcal{U}_{s_i} - \prod_{i=1}^{j_{\boldsymbol{s}}} \mathcal{V}_{s_i}\right\|_\infty \leq \sum_{i=1}^{j_{\boldsymbol{s}}} \left\|\mathcal{U}_{s_i} - \mathcal{V}_{s_i}\right\|_\infty$. The continuity in the argument $\boldsymbol{U}$ in the metric d_N thus follows from the following computation

$$\begin{aligned}
&\|\mathcal{U}_i - \mathcal{V}_i\|_\infty \\
&\leq \left\|(U_i \otimes U_i^* - V_i \otimes V_i^*)\rho(U_i \otimes U_i^*)^\dagger\right\|_\infty + \left\|-(V_i \otimes V_i^*)\rho(V_i \otimes V_i^* - U_i \otimes U_i^*)^\dagger\right\|_\infty \\
&\leq 2\left\|U_i \otimes U_i^* - V_i \otimes V_i^*\right\|_\infty \\
&\leq 2(\|U_i \otimes \mathbb{I}\|_\infty \|\mathbb{I} \otimes (U_i^* - V_i^*)\|_\infty + \|(U_i - V_i) \otimes \mathbb{I}\|_\infty \|\mathbb{I} \otimes V_i^*\|_\infty) = 4\,\|U_i - V_i\|_\infty,
\end{aligned} \tag{5.23}$$

where in the first step we added and subtracted $(V_i \otimes V_i^*)\rho(U_i \otimes U_i^*)^\dagger$ and used the triangle inequality; in the second step, we used the fact that for any two linear operators C and D, $\|CD\|_\infty \leq \|C\|_\infty \|D\|_\infty$ (submultiplicativity); in the third step, we added and subtracted $U_i \otimes V_i^*$ and again used the triangle inequality and submultiplicativity; and in the last step we used $\|C \otimes \mathbb{I}\|_\infty = \|C\|_\infty$. □

The above lemma implies that, for an arbitrarily small λ, with probability $1 - O(\lambda^{-2}e^{-N})$, the realisation $\mathcal{T}_{\boldsymbol{U}}^{(N)}$ is $\lambda + O(e^{-N})$ close to $\mathcal{T}$.

5.2.2 Local Speed-Up Effect

Now we are in the position to prove the local thermalisation speed-up result. To formally state the result, We define the following notations: In the subsystem X, we consider the following implementation of EPLT:

$$\Lambda_X^{(N,t_{\mathrm{total}})} = \mathcal{P}^{t_X}_{\eta_X^{dP^X_{\min}}} \circ \mathcal{T}_{\boldsymbol{U},X}^{(N)}, \tag{5.24}$$

where $\mathcal{T}_{\boldsymbol{U},X}^{(N)} := \prod_{k=1}^{N} (\mathrm{tr}_{AB\backslash X} \circ \mathcal{T}_k)$, $\mathcal{P}_\gamma^t(\cdot) := e^{-\frac{t}{\tau_\gamma}}(\cdot) + \left(1 - e^{-\frac{t}{\tau_\gamma}}\right)\gamma$ is the partial thermalisation model defined in Eq. (5.15), and $\eta_X^\epsilon := \gamma_X + \frac{\epsilon}{1-\epsilon}\left(\gamma_X - \frac{\mathbb{I}_X}{d}\right)$ is defined in Eq. (5.7). The total time to implement this channel, i.e., t_{total}, reads

$$t_{\mathrm{total}} = t_X + N \times t_U, \tag{5.25}$$

where t_U is the time to implement each $\mathcal{T}_k$. In what follows, we say a channel Λ *fully δ-thermalises* a state ρ to a thermal state γ if $\|\Lambda(\rho) - \gamma\|_\infty < \delta$. Also, $\lceil x \rceil$ is the smallest integer larger than x.

Main Theorem 5.2.2. (Mechanism of EPLTs) *Let γ_X be the thermal state with $X = A, B$. If*

$$\tau_{\gamma_X} > t_U \times \frac{8}{\ln 2} \approx t_U \times 11.5416, \tag{5.26}$$

then for every $\rho_X \neq \gamma_X$ and $p_* \in (0,1)$, there exists $\delta' > 0$ such that for every $\delta \in (0, \delta')$:

1. *there exists an integer*

$$N_\delta := \left\lceil 8 \log_2 \frac{d^2 P^X_{\min} \sqrt{2}}{\delta} \right\rceil \tag{5.27}$$

and a time t_1 such that $\Lambda_X^{(N_\delta, t_1)}$ δ-thermalises ρ_X to γ_X with success probability

$$1 - \left[\frac{\delta}{d^2 P^X_{\min} \sqrt{2}}\right]^4, \tag{5.28}$$

which can be chosen to be larger than p_*.

2. *$\mathcal{P}^t_{\gamma_X}$ fully δ-thermalises ρ_X to γ_X only if $t \geq t_2$.*

3. *$t_1 < t_2$.*

To interpret the above theorem, note that t_1 is the total time for the EPLT model given in Eq. (5.24) to δ-thermalise a local system. On the other hand, t_2 is the minimal time for the partial thermalisation model Eq. (5.15) to δ-thermalise the same local system. The strict inequality $t_1 < t_2$ suggests that, with the imposed assumptions, EPLT scheme provides a thermalisation faster than the partial thermalisation model. In this sense, a potential physical mechanism behind the existence of EPLTs is the local speed-up of thermalisation processes.

Proof. We use the shortcut notation η_X for the state $\eta_X^{dP^X_{\min}}$. To get an explicit estimate on time, we take $\lambda = 2^{-\frac{N}{4}}$ and use Lemma 5.2.1. By noting that $t_X = -\tau_{\eta_X} \ln dP^X_{\min}$, we start with the computation of the local state,

$$\begin{aligned}
&\left\|\Lambda_X^{(N,t_{\text{total}})}(\rho_X) - \gamma_X\right\|_\infty = \left\|\mathcal{P}_{\eta_X}^{t_X} \circ \mathcal{T}_{\boldsymbol{U},X}^{(N)}(\rho_X) - \gamma_X\right\|_\infty \\
&= \left\|dP^X_{\min}\mathcal{T}_{\boldsymbol{U},X}^{(N)}(\rho_X) + \left(1 - dP^X_{\min}\right)\eta_X - \gamma_X\right\|_\infty \\
&= dP^X_{\min}\left\|\mathcal{T}_{\boldsymbol{U},X}^{(N)}(\rho_X) - \frac{\mathbb{I}_X}{d}\right\|_\infty \leq d^2 P^X_{\min}\left\|\mathcal{T}_{\boldsymbol{U}}^{(N)} - \mathcal{T}\right\|_\infty < d^2 P^X_{\min}\sqrt{2} \times 2^{-\frac{N}{8}},
\end{aligned} \tag{5.29}$$

which holds with probability $1 - 2^{-\frac{N}{2}}$. The first inequality follows from the relations $\|\text{tr}_Y(\cdot)\|_\infty \leq \|\text{tr}_Y(\cdot)\|_1 \leq \|\cdot\|_1 \leq d\,\|\cdot\|_\infty$ and $\|\mathcal{E}(\rho)\|_\infty \leq \|\mathcal{E}\|_\infty$. The second inequality is due to the estimate $\sqrt{\lambda + 2^{-N}} < \sqrt{2\lambda} = \sqrt{2} \times 2^{-\frac{N}{8}}$. The above computation means that for any given $\delta \in (0,1)$, there exists a sufficiently large $N = N_\delta$ to let the above upper bound be smaller than δ. In other words, this choice of N ensures full δ-thermalisation of *every* local input ρ_X with the success probability $1 - 2^{-\frac{N}{2}}$. It suffices to take $N = N_\delta$ as the one defined in Eq. (5.27).

Now, consider a given $\delta \in (0,1)$ and a given local input state ρ_X. Then, $t_X = -\tau_{\eta_X} \ln dP^X_{\min}$ and N_δ gives us the following total time to implement the channel $\Lambda_X^{(N_\delta,t_1)}$:

$$t_1 = t_X + N_\delta t_U = \tau_{\eta_X} \ln \frac{1}{dP^X_{\min}} + N_\delta t_U. \tag{5.30}$$

If local agents in A and B simply leave their local systems in contact with local independent baths, the partial thermalisation model fully δ-thermalises the local state ρ_X in a time

$$t_2 = \tau_{\gamma_X} \ln \frac{\|\rho_X - \gamma_X\|_\infty}{\delta}. \tag{5.31}$$

Combining Eqs. (5.27), (5.30), and (5.31), we learn that $t_1 < t_2$, with probability $1 - 2^{-\frac{N_\delta}{2}}$, if $0 < \tau_{\gamma_X} \ln \frac{\|\rho_X - \gamma_X\|_\infty}{\delta} + \tau_{\eta_X} \ln(dP^X_{\min}) - N_\delta t_U$. This is true if

$$\|\rho_X - \gamma_X\|_\infty > f \times \delta^{\left(1 - \frac{t_U}{\tau_{\gamma_X}}\frac{8}{\ln 2}\right)}, \tag{5.32}$$

where $f := (dP^X_{\min})^{-\frac{\tau_{\eta_X}}{\tau_{\gamma_X}}} e^{\frac{t_U}{\tau_{\gamma_X}}} \left(d^2 P^X_{\min}\sqrt{2}\right)^{\frac{t_U}{\tau_{\gamma_X}}\frac{8}{\ln 2}}$ is a constant in δ. Note that f is finite for all possible values of $\frac{\tau_{\eta_X}}{\tau_{\gamma_X}}$. Hence, when the exponent of δ in Eq. (5.32) is positive, it is always possible to find a small enough δ to achieve Eq. (5.32). Specifically, suppose Eq. (5.26) is satisfied. Then, for any given $\rho_X \neq \gamma_X$, a successful implementation of twirling operation demonstrates $t_1 < t_2$ (i.e., a speed-up effect) for every $\delta > 0$ small enough, where the success probability is precisely given by Eq. (5.28). □

In most cases we have $t_U \ll \tau_{\gamma_X}$. Hence, δ-thermalisation with small enough δ is faster in the EPLT scheme than in the partial thermalisation model, even taking into account the time to implement random unitaries. Theorem 5.2.2 means that for any $\rho_X \neq \gamma_X$ and $\delta > 0$ small enough, the EPLT scheme achieves a speed-up of δ-thermalisation with probability $1 - O(\delta^4)$ whenever Eq. (5.26) holds. In practice, the thermalisation time scale τ_{γ_X} is often much longer than the time scale t_U of applying a single unitary operator. Consequently, Eq. (5.26) holds in various physical settings.

We conclude the discussion by providing an example. Suppose $\tau_{\eta_X} = \tau_{\gamma_X} = 100t_U$, which means it is possible to establish the speed-up. If one sets $P_{\min} = \frac{2}{d^2}$ and $\delta = 10^{-3}$, then we have $N = 92$ [from Eq. (5.27)]. This means we have speed-up for all $\rho_X \neq \gamma_X$ satisfying $\|\rho_X - \gamma_X\|_\infty > d\times 0.00126$ with success probability of implementation higher than $1 - 10^{-13}$.

Chapter 6

Conclusion

This thesis aims at unifying and bridging various dynamical quantum phenomena by the resource-theoretic approach. This includes 'resource preservability theories', a dynamical resource theory framework to describe the ability to preserve a given static quantum property, such as entanglement, coherence, athermality, and nonlocality. It also includes 'quantum channel marginal problems', a framework to describe local behaviours of global quantum dynamics which includes, e.g., state marginal problem, channel extendibiltiy, and causal channels, as particular cases. As applications, resource preservability theories allow us to reveal a thermodynamic criterion underlying transmitting classical information, providing a quantitative link between communication and thermodynamics. Also, quantum channel marginal problems highlight the relations between marginal problems of quantum states, classical channels, and quantum channels, demonstrating the importance of fundamental quantum principles such as entanglement monogamy and no-cloning theorem but also showing the irreducibility of dynamical problems to static ones. Our findings suggest that significant novel insights can be obtained by developing and building dynamical resource theories further. In this chapter, we briefly mention several on-going/future directions, including results that are relevant to this thesis, while not forming part of it.

Thermodynamic Criterion of Transmitting Information

As proved in Chapter 3, one can build a quantitative connection between a thermodynamic property and a classical communication task. This motivates us to ask: Can we upgrade the connection into an *equivalence relation*? Namely, can we equate the performances of two operational tasks, one from thermodynamics, and one from communication theory? Such an equivalence relation can potentially help us to uncover a quantitative thermodynamic description of classical information transmission, and vice versa. It turns out that, as reported in our recent pre-print [9], we proved such an equivalence relation between a wide range of classical communication scenarios and work extraction tasks. This recent progress motivates us to explore the following two research topics:

- **The Roles of Thermodynamics in Quantum Communication.–** Findings of this thesis and Ref. [9] provide quantitative links between tasks in communication and thermodynamics. Treating them as a bridge, it is interesting to know whether one is able to use thermodynamic approaches to tackle questions in the study of quantum communication theory. In particular, it is foundationally relevant to know whether thermodynamic principles play a key role in quantum communication.

- **Communication Cost of Thermodynamic Tasks.–** Findings of this thesis and Ref. [9] provide a potential platform to study the 'classical communication cost' of different thermodynamic tasks. The findings can potentially uncover the role of information transmission in thermodynamics.

Channel Resource Theories + Channel Marginal Problems

From the structural perspective of quantum resource theories, it is important to know whether a single framework can cover 'resource theory of channels' and 'channel marginal problems' simultaneously. In our recent pre-print [8], we have introduced a unified framework, dubbed *resource marginal problems*, to include resource theory of states, state marginal problems, resource theory of channels, and channel marginal problems at the same time. We provide a systematic approach to obtain robustness measures, witness form, and operational advantages in discrimination tasks. Based on this work, it is then interesting to explore the following topic:

- **Transitivity of General Quantum Resources.–** In our recent pre-print [7], we study the *entanglement transitivity problem*, whose simplest form can be seen in a tripartite setting ABC: For two *compatible* entangled states ρ_{AB}, ρ_{BC}, is it true that *every* global state ρ_{ABC} compatible with them *must* have an entangled marginal state in AC? Questions of this kind can be cast into a resource marginal problem, and it is interesting to know how to generalise such questions to the dynamical regime, and what kinds of operational advantages can be guaranteed when entanglement is transitive. Also, since entanglement is just one of the many quantum resources, it is natural to ask whether such a transitivity problem can be generalised to *arbitrary quantum resources*. This gives rise to the future project aiming at understanding transitivity properties of both static and dynamical quantum resources.

Appendix A

Proofs for Chapter 2

A.1 Properties of Choi States

First, we need the following lemma:

Lemma A.1.1. $(\mathcal{E}_{S|S'}\otimes\mathcal{I}_{S'})\left(|\Psi^+_{S'S'}\rangle\langle\Psi^+_{S'S'}|\right)\geq 0$ *if and only if* $\mathcal{E}_{S|S'}$ *is complete positive.*

Proof. First, note that $\mathcal{E}_{S|S'}\otimes\mathcal{I}_A$ is a positive map if and only if we have that

$$\mathrm{tr}\left[\eta_{SA}(\mathcal{E}_{S|S'}\otimes\mathcal{I}_A)(\rho_{S'A})\right]\geq 0\quad\forall\,\rho_{S'A}\in\mathrm{STATE}_{S'A},\ \eta_{SA}\in\mathrm{STATE}_{SA}. \tag{A.1}$$

Now, direct computation shows:

$$\begin{aligned}
&\mathrm{tr}\left[\eta_{SA}(\mathcal{E}_{S|S'}\otimes\mathcal{I}_A)(\rho_{S'A})\right]\\
&=d_{S'}\mathrm{tr}\left[(\eta_{SA}\otimes\mathbb{I}_{S'})(\mathcal{E}_{S|S'}\otimes\mathcal{I}_{AS'})[(\rho_{S'A}\otimes\mathbb{I}_{S'})(\mathbb{I}_A\otimes|\Psi^+_{S'S'}\rangle\langle\Psi^+_{S'S'}|)]\right]\\
&=d_{S'}d_A\mathrm{tr}\left[(\eta_{SA}\otimes\mathbb{I}_{S'A})(\mathcal{E}_{S|S'}\otimes\mathcal{I}_{AS'A})[(\rho_{S'A}\otimes\mathbb{I}_{S'A})(|\Psi^+_{AA}\rangle\langle\Psi^+_{AA}|\otimes|\Psi^+_{S'S'}\rangle\langle\Psi^+_{S'S'}|)]\right]\\
&=d_{S'}d_A\mathrm{tr}\left[(\eta_{SA}\otimes\rho^t_{S'A})[|\Psi^+_{AA}\rangle\langle\Psi^+_{AA}|\otimes(\mathcal{E}_{S|S'}\otimes\mathcal{I}_{S'})(|\Psi^+_{S'S'}\rangle\langle\Psi^+_{S'S'}|)]\right],
\end{aligned}\tag{A.2}$$

where we have inserted maximally entangled states on $S'S'$ and AA in the first and the second equalities; in the last line, we have used the property $(\rho_{S'A}\otimes\mathcal{I}_{S'A})|\Psi^+_{AA}\rangle\otimes|\Psi_{S'S'}\rangle=(\mathcal{I}_{S'A}\otimes\rho^t_{S'A})|\Psi^+_{AA}\rangle\otimes|\Psi_{S'S'}\rangle$, where $(\cdot)^t$ is the transpose operation. From here we conclude that Eq. (A.1) holds if $(\mathcal{E}_{S|S'}\otimes\mathcal{I}_{S'})\left(|\Psi^+_{S'S'}\rangle\langle\Psi^+_{S'S'}|\right)\geq 0$. □

Proof of Theorem 2.1.1

Proof. For a state $\rho_{SS'}$ satisfying $\mathrm{tr}_S(\rho_{SS'})=\frac{\mathbb{I}_{S'}}{d_{S'}}$, let us write $\mathcal{L}_{S|S'}(\cdot)=\mathcal{J}^{-1}(\rho_{SS'})$. Then we need to show that $\mathcal{L}_{S|S'}$ is a channel. First, one can see that

$$\mathrm{tr}\circ\mathcal{L}_{S|S'}(\cdot)_{S'}=d_{S'}\mathrm{tr}\left(\left[\mathbb{I}_S\otimes(\cdot)^t_{S'}\right]\rho_{SS'}\right)=d_{S'}\mathrm{tr}\left[(\cdot)^t_{S'}\mathrm{tr}_S(\rho_{SS'})\right]=\mathrm{tr}(\cdot)_{S'}. \tag{A.3}$$

This means $\mathcal{L}_{S|S'}$ is trace-preserving. Also, the complete positivity is guaranteed by Lemma A.1.1. From here we learn that for a channel $\mathcal{E}_{S|S'}$, the positivity of $\mathcal{E}^{\mathcal{J}}_{S|S'}$ corresponds to the complete positivity of the channel, and the condition that its marginal in the system S' is identical to $\frac{\mathbb{I}_{S'}}{d_{S'}}$ corresponds to the trace-preserving condition. □

Appendix B

Proofs and Remarks for Chapter 3

B.1 Activation Properties of Resource Preservability

B.1.1 Resource Theory of Nonlocality

To demonstrate the activation property of resource preservability, we need to briefly recap the definition of nonlocality and its corresponding state resource theory [14], which is denoted by $R = \mathrm{NL}$ in this section.

Recap of Bell Nonlocality

To begin with, recall that a bipartite probability distribution $\boldsymbol{P} := \{P(ab|xy)\}_{a,b,x,y}$ s said to admit a *local hidden variable* (LHV) model, denoted by $\boldsymbol{P} \in \mathrm{LHV}$, if

$$P(ab|xy) = \int P(a|x,\lambda)P(b|y,\lambda)p_\lambda d\lambda \tag{B.1}$$

for some variable λ and probability distributions $\{p_\lambda\}, \{P(a|x,\lambda)\}, \{P(b|y,\lambda)\}$. With this notion, we say a bipartite state ρ_{AB} in AB is *local* if for every set of local POVMs $\{E^{a|x}\}_{a,x}$ in A (with $\sum_a E_{a|x} = \mathbb{I}_A\ \forall\ x$) and $\{E_{b|y}\}_{b,y}$ in B (with $\sum_b E_{b|y} = \mathbb{I}_B\ \forall\ y$), there exists a probability distribution $\boldsymbol{P}^{\mathrm{LHV}} \in \mathrm{LHV}$ such that $\mathrm{tr}\left[(E_{a|x} \otimes E_{b|y})\rho_{AB}\right] = P^{\mathrm{LHV}}_{ab|xy}$. In other words, a state is local if all statistics induced by it with local POVMs are indistinguishable from LHV models. States that are not local are said to be *nonlocal*–each of them can violate a so-called *Bell inequality*. More precisely, every collection of real numbers $\boldsymbol{B} := \{B_{ab|xy}\}$ induces a Bell inequality which reads

$$\sum_{a,b,x,y} B_{ab|xy}\mathrm{tr}\left[(E_{a|x} \otimes E_{b|y})\rho_{AB}\right] \le \max_{\boldsymbol{P}\in\mathrm{LHV}} \sum_{a,b,x,y} P(ab|xy)B_{ab|xy}. \tag{B.2}$$

Its violation can witness the existence of quantumness. From now on, we use the notation LOCAL_{AB} to denote all local states in the bipartite system AB.

Nonlocality As A State Resource

With the above notions, one is able to define nonlocality as a state resource theory. In this resource theory, we consider all bipartite states with equal local dimensions $d^{\otimes k}$, where $k \in \mathbb{N}$ is a positive integer, and $d \in \mathbb{N}$ is the minimal possible system dimension under consideration. Within this setting, free states are all local states; namely, let S be an d dimensional system, then

$$\mathcal{F}_{R=\mathrm{NL}} = \bigcup_{k\in\mathbb{N}} \mathrm{LOCAL}_{(SS)^{\otimes k}}. \tag{B.3}$$

To define free operations, we use *local operations plus shared randomness* (LOSR) channels (see, e.g., Ref. [17]). Formally, an LOSR channel acting on a bipartite system AB is defined to take the following form:

$$\mathcal{E}_{AB} = \int (\mathcal{E}_{\lambda,A} \otimes \mathcal{E}_{\lambda,B}) p_\lambda d\lambda, \tag{B.4}$$

where $\{p_\lambda\}$ is some probability distribution and $\mathcal{E}_{\lambda,A}, \mathcal{E}_{\lambda,B}$ are local channels in A, B, respectively, with λ dependency. Now we note the following observation:

Lemma B.1.1. *LOSR channels map local states to local states.*

Proof. For a given LOSR channel $\mathcal{E}_{AB}$, we have

$$\begin{aligned}\mathrm{tr}\left[(E_{a|x} \otimes E_{b|y})\mathcal{E}_{AB}(\rho_{AB})\right] &= \int \mathrm{tr}\left[(E_{a|x} \otimes E_{b|y})(\mathcal{E}_{\lambda,A} \otimes \mathcal{E}_{\lambda,B})(\rho_{AB})\right] p_\lambda d\lambda \\ &= \int \mathrm{tr}\left\{\left[\mathcal{E}^\dagger_{\lambda,A}(E_{a|x}) \otimes \mathcal{E}^\dagger_{\lambda,B}(E_{b|y})\right](\rho_{AB})\right\} p_\lambda d\lambda, \end{aligned}\tag{B.5}$$

Since $\mathcal{E}^\dagger$ is unital (i.e., maps identity operator to identity operator) if $\mathcal{E}$ is a channel, we learn that$\mathcal{E}^\dagger_{\lambda,A}(E_{a|x})$ and $\mathcal{E}^\dagger_{\lambda,B}(E_{b|y})$ again form local sets of POVMs. Since ρ_{AB} is local, we conclude that the probability distribution $\mathrm{tr}\left\{\left[\mathcal{E}^\dagger_{\lambda,A}(E_{a|x}) \otimes \mathcal{E}^\dagger_{\lambda,B}(E_{b|y})\right](\rho_{AB})\right\}$ must admit a LHV model, and the result follows. □

The above lemma shows that all LOSR channels form a suitable set of free operations for nonlocality, which is adopted in the rest of this section; namely, we consider

$$\mathcal{O}_{R=\mathrm{NL}} = \{\text{all LOSR channels}\}. \tag{B.6}$$

B.1.2 Superactivation of Nonlocality

To demonstrate the activation property of nonlocality preservability, it remains to recall a phenomenon called *superactivation*, which is proved for nonlocality [33] and generalised to quantum steering [34, 35]. Formally, in a bipartite system AB with equal local dimension d, a local state ρ_{AB} is said to admit *superactivation of nonlocality* if there exists a finite integer $k \in \mathbb{N}$ such that $\rho_{AB}^{\otimes k}$ is nonlocal in the $A^{\otimes k}$ vs. $B^{\otimes k}$ bipartition with equal local dimension d^k. Notably, it is shown that $\rho_{AB}^{\otimes k}$ is nonlocal for some $k \in \mathbb{N}$

if its *fully entangled fraction* (FEF) is higher than $\frac{1}{d}$ [100], where the FEF is defined by [101, 102] [recall from Eq. (2.6) the definition of a maximally entangled state]:

$$\mathrm{FEF}(\rho_{AB}) := \max_{U_B} \langle \Psi^+_{AB}| \left(\mathbb{I}_A \otimes U_B^\dagger\right) \rho_{AB} \left(\mathbb{I}_A \otimes U_B\right) |\Psi^+_{AB}\rangle, \tag{B.7}$$

where the maximisation is taken over all unitary operator U_B in B. Note that It is equivalent to maximising over all maximally entangled states in AB.

To see superactivation of nonlocality indeed exists, we still need to use the $(U\otimes U^*)$-*twirling operation* on AB [92, 93], which is defined in Eq. (5.3), and we restate it here:

$$\mathcal{T}_{AB}(\cdot) := \int (U \otimes U^*)(\cdot)_{AB}(U \otimes U^*)^\dagger dU, \tag{B.8}$$

where the integration is taken over the group of $d \times d$ unitary operators with the Haar measure dU. One can directly check that $\mathcal{T}_{AB}$ is by definition an LOSR channel, thereby being a free operation of nonlocality. Other direct observations are

- $\mathcal{T}_{AB}$ maintains the overlap with the maximally entangled state; namely, we have $\langle \Psi^+_{AB}|\mathcal{T}_{AB}(\cdot)|\Psi^+_d\rangle = \langle \Psi^+_{AB}| \cdot |\Psi^+_{AB}\rangle$.
- $\mathcal{T}_{AB}$ only outputs *isotropic states* [93] [which is defined in Eq. (5.4)]; namely, $\rho_{\mathrm{iso}}(p) := p|\Psi^+_{AB}\rangle\langle\Psi^+_{AB}| + (1-p)\frac{\mathbb{I}_{AB}}{d^2}$. $p \in \left[-\frac{1}{d^2-1}, 1\right]$ due to positivity of states.

Now, if we choose the parameter p such that (1) $\rho_{\mathrm{iso}}(p)$ cannot have its FEF larger than the threshold for nonlocality [14], meaning that $\rho_{\mathrm{iso}}(p)$ is thus local; and (2) $\rho_{\mathrm{iso}}(p)$ has its FEF larger than $\frac{1}{d}$, then any such isotropic state can demonstrate superactivation of nonlocality. More precisely, when we choose [14, 103]

$$\frac{1}{d+1} < p < \frac{(d-1)^{(d-1)}(3d-1)}{(d+1)d^d}, \tag{B.9}$$

then the above two conditions are guaranteed. This shows the existence of superactivation of nonlocality.

B.1.3 Superactivation of Nonlocality Preservability

Now we are in the position to construct an example to demonstrate that nonlocality preservability can be superactivated, which is also the proof of Lemma 3.1.4:

Proof of Lemma 3.1.4. In AB with equal local dimension d, consider the following channel:

$$\widetilde{\mathcal{T}}_{AB}(\cdot) := \widetilde{p}\mathcal{T}_{AB}(\cdot) + (1-\widetilde{p})\frac{\mathbb{I}_{AB}}{d^2}, \tag{B.10}$$

and we choose $\widetilde{p}$ from the interval Eq. (B.9). By construction, $\widetilde{\mathcal{T}}_{AB}$ is thus a LOSR channel that can only output local states in AB. This means $\widetilde{\mathcal{T}}_{AB} \in \mathcal{O}^N_{\mathrm{NL}}$; that is, it is resource-annihilating (Definition 3.1.3). When the input state is $|\Psi^+_{AB}\rangle$, $\widetilde{\mathcal{T}}_{AB}(|\Psi^+_{AB}\rangle\langle\Psi^+_{AB}|)$ is an entangled isotropic state $\rho_{\mathrm{iso}}(\widetilde{p})$ with the parameter $\widetilde{p}$ lying in the interval Eq. (B.9), thereby admitting superactivation of nonlocality. From here we conclude that there exists a natural number $k \in \mathbb{N}$ such that $\widetilde{\mathcal{T}}^{\otimes k}_{AB}\left(|\Psi^+_{AB}\rangle\langle\Psi^+_{AB}|^{\otimes k}\right) = \left[\widetilde{\mathcal{T}}_{AB}\left(|\Psi^+_{AB}\rangle\langle\Psi^+_{AB}|\right)\right]^{\otimes k}$ is nonlocal in $A^{\otimes k}$ vs. $B^{\otimes k}$ bipartition. In other words, we have that $\widetilde{\mathcal{T}}^{\otimes k} \notin \mathcal{O}^N_{\mathrm{NL}}$. □

B.2 Absolutely Resource-Annihilating Channels

Fact B.2.1. *If a bipartite channel is entanglement-annihilating and entanglement-breaking, then it is absolutely entanglement-annihilating.*

Proof. We rewrite this channel as $\mathcal{E}_{A_1B_1}$, which is in the bipartite system A_1B_1. Then it suffices to show that there is no entanglement in the A vs. B bipartition of the output of $\mathcal{E}_{A_1B_1} \otimes \Lambda_{A_2B_2}$ for any entanglement-annihilating channel $\Lambda_{A_2B_2}$ in the bipartite system A_2B_2. Since $\mathcal{E}_{A_1B_1}$ is entanglement-breaking, $\mathcal{E}_{A_1B_1} \otimes \mathcal{I}_{A_2B_2}$ is entanglement-annihilating in the 1 vs. 2 bipartition. In other words, no entanglement in the 1 vs. 2 bipartition after $\mathcal{E}_{A_1B_1} \otimes \Lambda_{A_2B_2}$, meaning that the global output is always of the form $\sum_i \alpha_i \rho_{i|A_1B_1} \otimes \eta_{i|A_2B_2}$ with some probability weights α_i and states $\rho_{i|A_1B_1}$ in A_1B_1 and $\eta_{i|A_2B_2}$ in A_2B_2. From here we observe that the remaining possibility for the preserved entanglement are in subsystems A_1B_1 and A_2B_2. Now we note that both $\mathcal{E}_{A_1B_1}$ and $\Lambda_{A_2B_2}$ are entanglement-annihilating in the A vs. B bipartition, implying that no entanglement can exist in A_1B_1 and A_2B_2. This shows that the output is separable. □

B.3 Remarks on Resource Preservability Robustness

We start with a lemma that helps us to characterise the resource preservability robustness. In what follows, $\widetilde{\Lambda}_A$ always denotes an absolutely R-annihilating channel (i.e., in $\mathcal{A}_{\rm abs}$).

Lemma B.3.1. *Given two channels $\mathcal{N}$ and $\mathcal{E}$, then we have*

$$\sup_{\widetilde{\Lambda}_A,\rho_{SA}} \inf\left\{\lambda \geq 1 \,|\, 0 \leq [(\lambda\mathcal{E} - \mathcal{N}) \otimes \widetilde{\Lambda}_A](\rho_{SA})\right\}$$
$$= \inf\left\{\lambda \geq 1 \,|\, 0 \leq [(\lambda\mathcal{E} - \mathcal{N}) \otimes \widetilde{\Lambda}_A](\rho_{SA})\ \forall A, \widetilde{\Lambda}_A, \rho_{SA}\right\}. \tag{B.11}$$

Proof. Let $\mathcal{L}_{\boldsymbol{A}} := \left\{\lambda \geq 1 \,|\, 0 \leq [(\lambda\mathcal{E} - \mathcal{N}) \otimes \widetilde{\Lambda}_A](\rho_{SA})\right\}$, where $\boldsymbol{A} := (A, \widetilde{\Lambda}_A, \rho_{SA})$. Then in Eq. (B.11) the left-hand-side can be rewritten as $\sup_{\boldsymbol{A}} \inf\{\lambda \geq 1 \,|\, \lambda \in \mathcal{L}_{\boldsymbol{A}}\}$, and the right-hand-side reads $\inf\{\lambda \geq 1 \,|\, \lambda \in \bigcap_{\boldsymbol{A}} \mathcal{L}_{\boldsymbol{A}}\}$. The inequality "≤" follows since $\bigcap_{\boldsymbol{A}} \mathcal{L}_{\boldsymbol{A}} \subseteq \mathcal{L}_{\boldsymbol{A}'}$ for all $\boldsymbol{A}'$. To show the opposite direction, consider an arbitrary natural number $k \in \mathbb{N}$. Then there exist $\boldsymbol{A}_k$ and $\lambda_k \in \mathcal{L}_{\boldsymbol{A}_k}$ such that

$$\inf\{\lambda \,|\, \lambda \in \mathcal{L}_{\boldsymbol{A}_k}\} \leq \sup_{\boldsymbol{A}} \inf\{\lambda \geq 1 \,|\, \lambda \in \mathcal{L}_{\boldsymbol{A}}\} < \inf\{\lambda \,|\, \lambda \in \mathcal{L}_{\boldsymbol{A}_k}\} + \frac{1}{k}; \tag{B.12}$$

$$\lambda_k - \frac{1}{k} < \inf\{\lambda \,|\, \lambda \in \mathcal{L}_{\boldsymbol{A}_k}\} \leq \lambda_k. \tag{B.13}$$

This implies that $\inf\{\lambda \,|\, \lambda \in \mathcal{L}_{\boldsymbol{A}}\} < \lambda_k + \frac{1}{k}$ for all $\boldsymbol{A}$, which means $\lambda_k + \frac{1}{k} \in \bigcap_{\boldsymbol{A}} \mathcal{L}_{\boldsymbol{A}}$. Hence, we conclude that

$$\inf\left\{\lambda \,|\, \lambda \in \bigcap_{\boldsymbol{A}} \mathcal{L}_{\boldsymbol{A}}\right\} \leq \lambda_k + \frac{1}{k} \leq \inf\{\lambda \,|\, \lambda \in \mathcal{L}_{\boldsymbol{A}_k}\} + \frac{2}{k} \leq \sup_{\boldsymbol{A}} \inf\{\lambda \,|\, \lambda \in \mathcal{L}_{\boldsymbol{A}}\} + \frac{2}{k}, \tag{B.14}$$

and the desired claim follows by considering every possible $k \in \mathbb{N}$. □

Combining Eq. (3.15) and Lemma B.3.1, we have the following characterisation:

$$D^R_{\max}(\mathcal{N}\|\mathcal{E}) := \sup_{A,\widetilde{\Lambda}_A,\rho_{SA}} D_{\max}\left[(\mathcal{N}\otimes\widetilde{\Lambda}_A)(\rho_{SA})\,\|\,(\mathcal{E}\otimes\widetilde{\Lambda}_{\mathrm{A}})(\rho_{\mathrm{SA}})\right]$$
$$:= \log_2 \sup_{A,\widetilde{\Lambda}_A,\rho_{SA}} \inf\left\{\lambda\geq 1\,|\,0\leq[(\lambda\mathcal{E}-\mathcal{N})\otimes\widetilde{\Lambda}_A](\rho_{SA})\right\}$$
$$= \log_2 \inf\left\{\lambda\geq 0\,|\,(\lambda\mathcal{E}-\mathcal{N})\otimes\widetilde{\Lambda}_A \text{ is a positive map } \forall A,\widetilde{\Lambda}_A\right\}. \tag{B.15}$$

A direct observation from Eq. (B.15) is the following lemma:

Lemma B.3.2. $\left(2^{D^R_{\max}(\mathcal{N}\|\mathcal{E})}\mathcal{E}-\mathcal{N}\right)\otimes\widetilde{\Lambda}_A$ *is a positive map* $\forall A,\widetilde{\Lambda}_A$.

Proof. Suppose the opposite was correct. Then there exists an ancillary system A_*, an absolutely resource-annihilating channel $\widetilde{\Lambda}_{A_*}$, and two states ρ_{SA_*}, $|\phi\rangle$ such that $\langle\phi|\left(2^{D^R_{\max}(\mathcal{N}\|\mathcal{E})}\mathcal{E}-\mathcal{N}\right)\otimes\widetilde{\Lambda}_{A_*}(\rho_{SA_*})|\phi\rangle<0$. Nevertheless, due to Eq. (B.15), we have

$$\langle\phi|\left(2^{\left[D^R_{\max}(\mathcal{N}\|\mathcal{E})+\frac{1}{k}\right]}\mathcal{E}-\mathcal{N}\right)\otimes\widetilde{\Lambda}_{A_*}(\rho_{SA_*})|\phi\rangle\geq 0\quad\forall\, k\in\mathbb{N}. \tag{B.16}$$

This leads to a contradiction when $k\to\infty$. □

Finally, we note the following alternative form of the resource preservability robustness, which directly implies Eq. (3.17):

$$P_{D_{\max}|R}(\mathcal{E}) = \log_2 \inf_{\Lambda\in\mathcal{A}} \inf\left\{\lambda\geq 1\,|\,(\lambda\Lambda-\mathcal{E})\otimes\widetilde{\Lambda}_A \text{ is a positive map } \forall A,\widetilde{\Lambda}_A\right\}. \tag{B.17}$$

Bibliography

[1] C.-Y. Hsieh, M. Lostaglio, and A. Acín, *Entanglement preserving local thermalization*, Phys. Rev. Research **2**, 013379 (2020).

[2] C.-Y. Hsieh, *Resource preservability*, Quantum **4**, 244 (2020).

[3] C.-Y. Hsieh, *Communication, dynamical resource theory, and thermodynamics*, PRX Quantum **2**, 020318 (2021).

[4] C.-Y. Hsieh, M. Lostaglio, and A. Acín, *Quantum channel marginal problem*, Phys. Rev. Research **4**, 013249 (2022).

[5] A. Serafini, M. Lostaglio, S. Longden, U. Shackerley-Bennett, C.-Y. Hsieh, and G. Adesso, *Gaussian thermal operations and the limits of algorithmic cooling*, Phys. Rev. Lett. **124**, 010602 (2020).

[6] H.-Y. Ku, C.-Y. Hsieh, S.-L. Chen, Y.-N. Chen, and C. Budroni, *Complete classification of steerability under local filters and its relation with measurement incompatibility*, Nat. Commun. **13**, 4973 (2022).

[7] G. N. M. Tabia, K.-S. Chen, C.-Y. Hsieh, Y.-C. Yin, and Y.-C. Liang, *Entanglement transitivity problems*, npj Quantum Inf. **8**, 98 (2022).

[8] C.-Y. Hsieh, G. N. M. Tabia, Y.-C. Yin, and Y.-C. Liang, *Resource marginal problems*, arXiv:2202.03523.

[9] C.-Y. Hsieh, *Thermodynamic criterion of transmitting classical information*, arXiv:2201.12110.

[10] E. Chitambar and G. Gour, *Quantum resource theories*, Rev. Mod. Phys. **91**, 025001 (2019).

[11] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, *Quantum entanglement*, Rev. Mod. Phys. **81**, 865 (2009).

[12] A. Streltsov, G. Adesso, and M. B. Plenio, *Colloquium: Quantum coherence as a resource*, Rev. Mod. Phys. **89**, 041003 (2017).

[13] M. Lostaglio, *An introductory review of the resource theory approach to thermodynamics*, Rep. Prog. Phys. **82**, 114001 (2019).

[14] N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani, and S. Wehner, *Bell nonlocality*, Rev. Mod. Phys. **86**, 419 (2014).

[15] R. Uola, A. C. S. Costa, H. C. Nguyen, and O. Gühne, *Quantum steering*, Rev. Mod. Phys. **92**, 015001 (2020).

[16] D. Cavalcanti and P. Skrzypczyk, *Quantum steering: a review with focus on semidefinite programming*, Rep. Prog. Phys. **80**. 024001 (2016).

[17] D. Rosset, F. Buscemi, and Y.-C. Liang, *A resource theory of quantum memories and their faithful verification with minimal assumptions*, Phys. Rev. X **8**, 021033 (2018).

[18] Z.-W. Liu and A. Winter, *Resource theories of quantum channels and the universal role of resource erasure*, arXiv:1904.04201.

[19] Y. Liu and X. Yuan, *Operational resource theory of quantum channels*, Phys. Rev. Research **2**, 012035(R) (2020).

[20] R. Takagi, K. Wang, and M. Hayashi, *Application of the Resource Theory of Channels to Communication Scenarios*, Phys. Rev. Lett. **124**, 120502 (2020).

[21] B. Regula and R. Takagi, *One-shot manipulation of dynamical quantum resources*, Phys. Rev. Lett. **127**, 060402 (2021).

[22] B. Regula and R. Takagi, *Fundamental limitations on distillation of quantum channel resources*, Nat. Commun. **12**, 4411 (2021).

[23] X. Yuan, Y. Liu, Q. Zhao, B. Regula, J. Thompson, and M. Gu, *Universal and operational benchmarking of quantum memories*, npj Quantum Inf. **7**, 108 (2021).

[24] C. Schilling, *Quantum marginal problem and its physical relevance*, Ph.D. thesis, ETH Zurich, 2014.

[25] R. M. Erdahl and B. Jin, in *Many-Electron Densities and Density Matrices*, edited by J. Cioslowski (Kluwer, Boston, 2000).

[26] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*, (Cambridge University Press, Cambridge, UK, 2000).

[27] J. Watrous, *The Theory of Quantum Information*, Cambridge University Press, 2018.

[28] M.-D. Choi, *Completely positive linear maps on complex matrices*, Linear Algebr. Appl. **10**, 285 (1975).

[29] A. Jamiołkowski, *Linear transformations which preserve trace and positive semidefiniteness of operators*, Rep. Math. Phys. **3**, 275 (1972).

[30] J. R. Munkres, *Topology*, 2nd edition, (Prentice Hall Press, 2000).

[31] Ll. Masanes, Y.-C. Liang, and A. C. Doherty, *All bipartite entangled states display some hidden nonlocality*, Phys. Rev. Lett. **100**, 090403 (2008).

[32] Y.-C. Liang, Ll. Masanes, and D. Rosset, *All entangled states display some hidden nonlocality*, Phys. Rev. A **86**, 052115 (2012).

[33] C. Palazuelos, *Superactivation of quantum nonlocality*, Phys. Rev. Lett. **109**, 190401 (2012).

[34] C.-Y. Hsieh, Y.-C. Liang, and R.-K. Lee, *Quantum steerability: Characterization, quantification, superactivation and unbounded amplification*, Phys. Rev. A **94**, 062120 (2016).

[35] M. T. Quintino, M. Huber, and N. Brunner, *Superactivation of quantum steering*, Phys. Rev. A **94**, 062123 (2016).

[36] L. Moravčíková and M. Ziman, *Entanglement-annihilating and entanglement-breaking channels*, J Phys. A: Math. Theor. **43**, 275306 (2010).

[37] M. Horodecki, P. W. Shor, and M. B. Ruskai, *Entanglement breaking channels*, Rev. Math. Phys. **15**, 629 (2003).

[38] G. Chiribella, G. M. D'Ariano, and P. Perinotti, *Transforming quantum operations: quantum supermaps*, EPL (Europhysics Letters) **83**, 30004 (2008).

[39] G. Chiribella, G. M. D'Ariano, and P. Perinotti, *Quantum circuit architecture*, Phys. Rev. Lett. **101**, 060401 (2008).

[40] N. Datta, *Min- and max-relative entropies and a new entanglement monotone*, IEEE Trans. Inf. Theory **55**, 2816 (2009).

[41] L. Wang and R. Renner, *One-shot classical-quantum capacity and hypothesis testing*, Phys. Rev. Lett. **108**, 200501 (2012).

[42] F. G. S. L. Brandão, M. Horodecki, J. Oppenheim, J. M. Renes, and R. W. Spekkens, *Resource theory of quantum states out of thermal equilibrium*, Phys. Rev. Lett. **111**, 250404 (2013).

[43] M. Horodecki and J. Oppenheim, *Fundamental limitations for quantum and nanoscale thermodynamics*, Nat. Commun. **4**, 2059 (2013).

[44] F. G.S.L. Brandão, M. Horodecki, N. H. Y. Ng, J. Oppenheim, and S. Wehner, *The second laws of quantum thermodynamics*, PNAS **112**, 3275 (2015).

[45] C. Sparaciari, M. Goihl, P. Boes, J. Eisert, and N. H. Y. Ng, *Bounding the resources for thermalising many-body localized systems*, Commun. Phys. **4**, 3 (2021).

[46] G.Tóth, C. Knapp, O. Gühne and H. J. Briegel, *Optimal spin squeezing inequalities detect bound entanglement in spin models*, Phys. Rev. Lett **99**, 250405 (2007).

[47] M. Navascués, F. Baccari and A. Acín, *Entanglement marginal problems*, Quantum 5, 589 (2021).

[48] J. Tura, R. Augusiak, A. B. Sainz, T. Vértesi, M. Lewenstein and A. Acín, *Detecting nonlocality in many-body quantum states*, Science **344**, 1256 (2014).

[49] Z. Wang, S. Singh and M. Navascués, *Entanglement and nonlocality in infinite 1D systems*, Phys. Rev. Lett. **118**, 230401 (2017).

[50] J. Cotler and F. Wilczek, *Quantum overlapping tomography*, Phys. Rev. Lett. **124**, 100401 (2020).

[51] X. Bonet-Monroig, R. Babbush and T. E. O'Brien, *Nearly optimal measurement scheduling for partial tomography of quantum states*, Phys. Rev. X **10**, 031064 (2020).

[52] G. García-Pérez, M. A. C. Rossi, B. Sokolov, E.-M. Borrelli and S. Maniscalco, *Pairwise tomography networks for many-body quantum systems*, Phys. Rev. Research **2**, 023393 (2020).

[53] D. Beckman, D. Gottesman, M. A. Nielsen, and J. Preskill, *Causal and localizable quantum operations*, Phys. Rev. A **64**, 052309 (2001).

[54] T. Eggeling, D. Schlingemann, and R. F. Werner, *Semicausal operations are semilocalizable*, Europhys. Lett. **57**, 782 (2002).

[55] M. Piani, M. Horodecki, P. Horodecki, and R. Horodecki, *On quantum non-signalling boxes*, Phys. Rev. A 74, 012305 (2006).

[56] R. Duan and A. Winter, *No-signalling-assisted zero-error capacity of quantum channels and an information theoretic interpretation of the Lovász number*, IEEE Trans. Inf. Theory **62**, 891 (2016).

[57] X. Wang, W. Xie, R. Duan, *Semidefinite programming strong converse bounds for classical capacity*, IEEE Trans. Inf. Theory **64**, 640 (2018).

[58] M. J. Hoban, A. B. Sainz, *A channel-based framework for steering, non-locality and beyond*, New J. Phys. **20**, 053048 (2018).

[59] E. Haapasalo, T. Kraft, N. Miklin, and R. Uola, *Quantum marginal problem and incompatibility*, Quantum **5**, 476 (2021).

[60] C. Carmeli, T. Heinosaari, and A. Toigo, *Quantum incompatibility witnesses*, Phys. Rev. Lett. **122**, 130402 (2019).

[61] C. Carmeli, T. Heinosaari, T. Miyadera, and A. Toigo, *Witnessing incompatibility of quantum channels*, J. Math. Phys. **60**, 122202 (2019).

[62] M. Girard, M. Plávala, and J. Sikora, *Jordan products of quantum channels and their compatibility*, Nature Commun. **12**, 2129 (2021).

[63] E. Haapasalo, T. Heinosaari, and J.-P. Pellonpää, *When do pieces determine the whole? Extremal marginals of a completely positive map*, Rev. Math. Phys. **26**, 1450002 (2014).

[64] E. Haapasalo, *Robustness of incompatibility for quantum devices*, J. Phys. A: Math. Theor. **48**, 255303 (2015).

[65] P. Johnson and L. Viola, *On state vs. channel quantum extension problems: exact results for $U \otimes U \otimes U$ symmetry*, J. Phys. A: Math. Theor. **48**, 035307 (2015).

[66] T. Heinosaari, T. Miyadera, and M. Ziman, *An invitation to quantum incompatibility*, J. Phys. A: Math. Theor. **49**, 123001 (2016).

[67] E. Kaur, S. Das, M. M. Wilde, and A. Winter, *Extendibility limits the performance of quantum processors*, Phys. Rev. Lett. **123**, 070502 (2019).

[68] M. L. LaBorde and M. M. Wilde, *Testing symmetry on quantum computers*, arXiv:2105.12758.

[69] W. Hall, *Compatibility of subsystem states and convex geometry*, Phys. Rev. A **75**, 032102 (2007).

[70] V. Coffman, J. Kundu, and W. K. Wootters, *Distributed entanglement*, Phys. Rev. A **61**, 052306 (2000).

[71] A. C. Doherty, *Entanglement and the shareability of quantum states*, J. Phys. A: Math. Theor. **47**, 424004 (2014).

[72] D. M. Greenberger, M. A. Horne, and A. Zeilinger, *Going Beyond Bell's Theorem in Bell's Theorem, Quantum Theory, and Conceptions of the Universe* (Kluwer Academic, Dordrecht, 1989).

[73] V. Scarani, S. Iblisdir, N. Gisin, and A. Acín, *Quantum cloning*, Rev. Mod. Phys. **77**, 1225 (2005).

[74] E. Kaur, S. Das, M. M. Wilde, and A. Winter, *Resource theory of unextendibility and nonasymptotic quantum capacity*, Phys. Rev. A **104**, 022401 (2021).

[75] W. Wootters and W. Zurek, *A single quantum cannot be cloned*, Nature **299**, 802 (1982).

[76] L. A. Khalfi and B. S. Tsirelson, *Quantum and quasi-classical analogs of Bell inequalities*, in *Symposium on the Foundations of Modern Physics* edited by P. Lahti and P. Mittelstaedt (World Scientific, Singapore, 1985), pp. 441-460.

[77] S. Popescu and D. Rohrlich, *Quantum nonlocality as an axiom*, Found. Phys. **24**, 379 (1994).

[78] J. Barrett, N. Linden, S. Massar, S. Pironio, S. Popescu, and D. Roberts, *Nonlocal correlations as an information-theoretic resource*, Phys. Rev. A **71**, 022101 (2005).

[79] M. M. Wolf, F. Verstraete, and J. I. Cirac, *Entanglement and frustration in ordered systems*, Int. J. Quantum Inf. **1**, 465 (2003).

[80] R. Uola, T. Kraft, J. Shang, X.-D. Yu, and O. Gühne, *Quantifying quantum resources with conic programming*, Phys. Rev. Lett. **122**, 130404 (2019).

[81] P. Skrzypczyk, I. Šupić, and D. Cavalcanti, *All sets of incompatible measurements give an advantage in quantum state discrimination*, Phys. Rev. Lett. 122, 130403 (2019).

[82] J. Mori, *Operational characterization of incompatibility of quantum channels with quantum state discrimination*, Phys. Rev. A **101**, 032331 (2020).

[83] R. Demkowicz-Dobrzański and L. Maccone, *Using Entanglement Against Noise in Quantum Metrology*, Phys. Rev. Lett. **113**, 250801 (2014).

[84] S. Pironio, A. Acín, S. Massar, A. Boyer de la Giroday, D. N. Matsukevich, P. Maunz, S. Olmschenk, D. Hayes, L. Luo, T. A. Manning, and C. Monroe, *Random numbers certified by Bell's theorem*, Nature **464**, 1021 (2010).

[85] R. Cleve and H. Buhrman, *Substituting quantum entanglement for communication*, Phys. Rev. A **56**, 1201 (1997).

[86] S. Bravyi, D. Gosset, and R. Köenig, Quantum advantage with shallow circuits, Science **362**, 308 (2018).

[87] J. Oppenheim, M. Horodecki, P. Horodecki, and R. Horodecki, *Thermodynamical Approach to Quantifying Quantum Correlations*, Phys. Rev. Lett. **89**, 180402 (2002).

[88] D. Jennings and T. Rudolph, *Entanglement and the thermodynamic arrow of time*, Phys. Rev. E **81**, 061130 (2010).

[89] L. del Rio, J. Åberg, R. Renner, O. Dahlsten, and V. Vedral, *The thermodynamic meaning of negative entropy*, Nature **474**, 61 (2011).

[90] K. Funo, Y. Watanabe, and M. Ueda, *Thermodynamic work gain from entanglement*, Phys. Rev. A **88**, 052319 (2013).

[91] M. Perarnau-Llobet, K. V. Hovhannisyan, M. Huber, P. Skrzypczyk, N. Brunner, and A. Acín, *Extractable Work from Correlations*, Phys. Rev. X **5**, 041011 (2015).

[92] C. H. Bennett, G. Brassard, S. Popescu, B. Schumacher, J. A. Smolin, and W. K. Wootters, *Purification of Noisy Entanglement and Faithful Teleportation via Noisy Channels*, Phys. Rev. Lett. **76**, 722 (1996).

[93] M. Horodecki and P. Horodecki, *Reduction criterion of separability and limits for a class of distillation protocols*, Phys. Rev. A **59**, 4206 (1999).

[94] A. Peres, *Separability Criterion for Density Matrices*, Phys. Rev. Lett. **77**, 1413 (1996).

[95] M. Horodecki, P. Horodecki, and R. Horodecki, *Separability of mixed states: necessary and sufficient conditions*, Phys. Lett. A **223**, 1 (1996).

[96] M. M. Wolf, *Quantum Channels & Operations Guided Tour* (2012), https://www-m5.ma.tum.de/foswiki/pub/M5/ Allgemeines/MichaelWolf/QChannelLecture.pdf.

[97] V. Scarani, M. Ziman, P. Štelmachovič, N. Gisin and V. Bužek, *Thermalising Quantum Machines: Dissipation and Entanglement*, Phys. Rev. Lett **88**, 097905 (2002).

[98] W. Roga, M. Fannes, and K. Życzkowski, *Davies maps for qubits and qutrits*, Rep. Math. Phys. **66**, 311 (2010).

[99] G. Tóth and J. J. García-Ripoll, *Efficient algorithm for multiqudit twirling for ensemble quantum computation*, Phys. Rev. A **75**, 042311 (2007).

[100] D. Cavalcanti, A. Acin, N. Brunner, and T. Vertesi, *All quantum states useful for teleportation are nonlocal resources*, Phys. Rev. A **87**, 042104 (2013).

[101] M. Horodecki, P. Horodecki, and R. Horodecki, *General teleportation channel, singlet fraction, and quasidistillation*, Phys. Rev. A **60**, 1888 (1999).

[102] S. Albeverio, S.-M. Fei, and W.-L. Yang, *Optimal teleportation based on bell measurements*, Phys. Rev. A **66**, 012301 (2002).

[103] M. L. Almeida, S. Pironio, J. Barrett, G. Tóth, and A. Acín, *Noise robustness of the nonlocality of entangled quantum states*, Phys. Rev. Lett. **99**, 040403 (2007).