UNIVERSITAT DE BARCELONA

On symplectic linearization of singular Lagrangian foliations

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Juny 2003

Memòria presentada per aspirar al grau de Doctor en Matemàtiques per la Universitat de Barcelona.

Certifico que la present memòria ha estat realitzada per Eva Miranda Galcerán i dirigida per mi.

Barcelona, 10 de Juny de 2003

Carlos Currás Bosch

A mi madre...

*"Pirata de Mar y cielo
si no lo fui ya lo seré,
si no robé la aurora de los mares,
si no la robé, ya la robaré".*
Rafael Alberti

Preface

In this thesis we study the problem of classification of symplectic structures in a neighbourhood of a singular compact orbit of a completely integrable system on a symplectic manifold (M^{2n}, Ω) for which the foliation determined by the moment map is generically Lagrangian. The foliation is determined by the orbits of the distribution generated by the symplectic gradients of the components of a proper moment map $F : M^{2n} \longrightarrow \mathbb{R}^n$. We also assume that the singularity is non-degenerate in the Morse-Bott sense. Under these assumptions, we prove that any two symplectic structures for which this foliation is generically Lagrangian are equivalent in the following sense: there exists a diffeomorphism defined in a neighbourhood of a compact orbit preserving the foliation, fixing the singular orbit and sending one symplectic form to the other. In the case there exists a symplectic action of a compact Lie group preserving the moment map we prove that the diffeomorphism can be chosen to be G-equivariant.

We also give an application of this result to contact geometry. We consider a contact manifold (M^{2n+1}, α) for which the Reeb vector field admits n first integrals generically independent and commuting with respect to the Jacobi bracket. The horizontal parts of the contact vector fields associated to these n functions determine a foliation $\mathcal{F}$. We consider the enlarged foliation $\mathcal{F}'$ generated by this foliation and the Reeb vector field. The Reeb vector field is assumed to be the infinitesimal generator of an S^1-action. We study the problem of classification of contact forms α in a neighbourhood of a singular orbit having the same Reeb vector field and for which $\mathcal{F}$ is Legendrian. Then under the assumption that the singular orbit is compact and non-degenerate in the Morse-Bott sense we prove that any two contact forms are equivalent. In other words, we show that there exists a diffeomorphism preserving the foliation $\mathcal{F}$, fixing the singular orbit and taking one contact form to the other. In the case there exists a contact action of a compact Lie group preserving the functions and preserving the Reeb vector field this diffeomorphism can be chosen to be G-equivariant.

Acknowledgements

First of all, I would like to take this opportunity to thank my advisor Carlos Currás Bosch for his help, endless patience and advice. In fact, the problem of studying singular Lagrangian foliations is a joint endeavour with him. I owe him lots of hours devoted to make me understand my mistakes and also a lot of "lunches" discussing some of the problems tackled in this thesis. I learned a lot from him. In the end, I hope the result fulfills his expectations on the topic.

I would also like to thank Nguyen Tien Zung for his help and for being so inspiring. In fact, part of the results obtained in this thesis have been obtained jointly with him. It has been a great pleasure to collaborate with him.

I also wish to thank Jaume Amorós and Viktor Ginzburg for stimulating discussions about some contents in this thesis and for giving me another perspective on the topic.

This is the time to thank my parents and Jordi. A very special thank must go to my mother who didn't loose the faith on me even when I had already lost it. In particular, my mother has been all her life cheering me on my projects and I wish to thank her for that.

I suppose I wouldn't be here if I didn't have such inspiring maths teachers as I have had from the cradle. I am specially grateful to Ángel Pérez, my maths teacher at the High School for making me love mathematics so much and all the great teachers in the University.

Although my graduate studies have been held at the Departament d'Àlgebra i

Geometria, I have also been working in the following maths Departments: Departament de Matemàtiques de la Universitat de Lleida, Departament de Matemàtica Aplicada i Telemàtica de la Universitat Politècnica de Catalunya, Departament de Matemàtica Aplicada I de la Universitat Politècnica de Catalunya. In those places, I met not only some wonderful professionals but also some wonderful people to learn from.

I also wish to thank my friends Marc Fransoy, Marco Castrillón, Ferran Espuny and Carlos Calvo for all their support. I remember having shared a lot of coffee breaks with Josep Maria Ribo, Susana Clara Lopez, Juan Carlos Naranjo and Frederic Gabern. Thanks for your advice. Thanks also to José Gil for helping me with the typesetting problems associated to the thesis (thanks also for the fancyhdr package although I didn't use it in the end).

Contents

Introduction

"Around the new position, a circle, somewhat larger than in the former instance was now described, and we again set to work with the spades. I was dreadfully, weary, but scarcely understanding what had occasioned the change in my thoughts..... I dug eagerly, and now and then caught myself actually looking with something that very much resembled expectation for the fancied treasure."

"The Gold Bug" by Edgar Allan Poe

This thesis is mainly concerned with the geometry underlying a completely integrable Hamiltonian system. A Hamiltonian system on a symplectic manifold (M^{2n}, ω) is the system defined by the symplectic gradient of a function H which is called the Hamiltonian function of the system. The study of the integrability of such systems is relevant in many areas of mathematics and has its own story.

In June 29*th* of 1853 Joseph Liouville presented a communication entitled "Sur l'intégration des equations différentielles de la Dynamique" at the "Bureau des longitudes". In the resulting note [40] he relates the notion of integrability of the system to the existence of n integrals in involution with respect to the Poisson bracket attached to the symplectic form. These systems come to the scene with the classical denomination of "completely integrable systems". In another language, a particular choice of n-first integrals in involution determines the n components of a moment map $\mathbf{F} : M^{2n} \longrightarrow \mathbb{R}^n$. A lot of work has been done in the subject after

Liouville. Let us outline some of the remarkable achievements from a geometrical and topological point of view.

Consider a completely integrable Hamiltonian system. The symplectic gradients of the components of the moment map define an involutive distribution. Assume that the moment map is proper. Let L be a regular orbit of this distribution then this orbit is a Lagrangian submanifold. Moreover, it is a torus and the neighbouring orbits are also tori. Those tori are called Liouville tori. This is the topological contribution of a theorem which has been known in the literature as Arnold-Liouville theorem. The geometrical contribution of the above-mentioned theorem ensures the existence of symplectic normal forms in the neighbourhood of a compact regular orbit. To the author's knowledge, the works of Henri Mineur [44, 45, 46] already gave the a complete description of the Hamiltonian system in a neighbourhood of a compact regular orbit. That is why we will refer to the classical Arnold-Liouville theorem as Liouville-Mineur-Arnold theorem. Let us state the theorem below,

Theorem 0.0.1 (Liouville-Mineur-Arnold Theorem)

Let (M^{2n}, ω) be a symplectic manifold and let $F : M^{2n} \longrightarrow \mathbb{R}^n$ be a proper moment map. Assume that the components f_i of F are pairwise in involution with respect to the Poisson bracket associated to ω and that $df_1 \wedge \cdots \wedge df_n \neq 0$ on a dense set. Let $N = F^{-1}(c)$, $c \in \mathbb{R}^n$ be a connected levelset. Then there exists a neighbourhood $U(N)$ of N and a diffeomorphism $\phi : U(N) \longrightarrow D^n \times \mathbb{T}^n$ such that,

1. *$\phi(N) = \{0\} \times \mathbb{T}^n$.*
2. *A set of coordinates μ_i in D^n and a set of coordinates β_i in $\mathbb{T}^n$ for which, $\phi^*(\sum_{i=1}^n d\mu_i \wedge d\beta_i) = \omega$.*
3. *F depends only on $\phi^*(\mu_i) = p_i$ and it does not depend on $\phi^*(\beta_i) = \theta_i$.*

The new coordinates p_i obtained are called action coordinates. The coordinates θ_i are called angle coordinates. Mineur also showed that the action functions p_i

can be defined via the period integrals. Let x be a point in a small neighbourhood of N, the period integrals are defined by the following formula:

$$p_i(x) = \int_{\Gamma_i(x)} \alpha \tag{0.0.1}$$

where α fulfills the condition $d\alpha = \omega$, and $\Gamma_i(x)$ is a closed curve which depends smoothly on x and which lies on the Liouville torus containing x. The homology classes of $\Gamma_1(x), ..., \Gamma_n(x)$ form a basis of the first homology group of the Liouville torus.

The existence of action-angle coordinates in a neighbourhood of a compact orbit provides a symplectic model for the Lagrangian foliation $\mathcal{F}$ determined by the symplectic gradients of the n component functions f_i of the moment map F. In fact, Liouville-Mineur-Arnold theorem entails a "uniqueness" result for the symplectic structures making $\mathcal{F}$ into a Lagrangian foliation. In other words, if ω_1 and ω_2 are two symplectic structures defined in a neighbourhood of N for which $\mathcal{F}$ is Lagrangian then there exists a symplectomorphism preserving the foliation, fixing N and carrying ω_1 to ω_2. This is due to the following observation: Let X_{f_i} be the symplectic gradients of the functions f_i for any $1 \leq i \leq n$, then the Lagrangian condition implies that in fact $\mathcal{F} =< X_{f_1}, \dots, X_{f_n} >$, further $\{f_j, f_k\}_i = 0$ where $\{.,.\}_i$ stands for the Poisson bracket attached to ω_i , $i = 1, 2$. Then by virtue of Liouville-Mineur-Arnold theorem there exists a foliation-preserving symplectomorphism ϕ_i taking ω_i to $\omega_0 = \sum_{i=1}^n dp_i \wedge d\theta_i$. In all, the diffeomorphism $\phi_2^{-1} \circ \phi_1$ does the job. It takes ω_1 to ω_2, it fixes N and it is foliation preserving.

So if the orbit is regular the existence of action-angle coordinates enables to classify the symplectic germs, up to foliation-preserving symplectomorphism, for which $\mathcal{F}$ is Lagrangian in a neighbourhood of a compact orbit. There is just one class of symplectic germs for which the foliation is Lagrangian.

One could look at the problem from a global perspective. There are topological obstructions to the existence of global action-angle coordinates as it was shown by Duistermaat in [22].

The problem of classification of symplectic germs for regular Lagrangian foliations can be taken further to consider the case of foliations not necessarily determined by a completely integrable system. Curras-Bosch and Molino have considered the following concomitant problem: They consider the problem of classification for germs of Lagrangian foliation defined in a neighbourhood of an torus equipped with an affine structure. The motivation for considering an affine structure on the torus is the Bott-Weinstein connection attached to the regular leaves of a Lagrangian foliation [59]. In the case the germ of Lagrangian foliation is determined by a completely integrable system this affine structure is trivial. In the above mentioned papers it is proved that there is no uniqueness result for the symplectic germ if the affine structure on the torus is non-trivial.

After this review for regular Lagrangian foliations, the following question arises:

What can be said about the corresponding classification problem for symplectic germs if the completely integrable systems has singularities?

This question is quite natural because singularities are present in many well-known examples of integrable systems. In fact, if the completely integrable system is defined on a compact manifold then the singularities cannot be avoided.

One of the main goals of this thesis is to prove that the uniqueness result for symplectic germs for which the foliation determined by a completely integrable system is generically Lagrangian holds when L is a singular orbit.

In the singular case, the problem can be posed at three different levels:

1. At the orbit level: In the neighbourhood of a compact singular orbit.

2. At a semi-local level: In the neighbourhood of a compact singular leaf.

3. At a global level.

Throughout this thesis we will only deal with the first situation. We will always assume that the singularity is non-degenerate. In any case, let us say a few words about the semi-local and global problem first.

The problem of topological classification of integrable Hamiltonian systems began with Fomenko [25] in some particular cases. Nguyen Tien Zung [61] studied the general case for the semi-local problem for non-degenerate singularities. It turns out that from a topological point of view we have a product-like description of the singularities in terms of the Williamson type. Nguyen Tien Zung also proved in [61] the existence of partial action-angle coordinates. The symplectic classification in the semi-local case for non-elliptic singularities has been studied in the hyperbolic case by Dufour, Molino and Toulet in [20]. The focus-focus case has been studied recently by San Vu Ngoc in [58]. In the hyperbolic and focus-focus case there are more invariants attached to the singularity. The symplectic germ in the hyperbolic case is determined by the jet of a function depending on a variable and in the focus-focus case is determined by the jet of a function in two variables. The singular global case has been studied by Nguyen Tien Zung in the paper [63] where the notion of Duistermaat-Chern class and monodromy (introduced by Duistermaat for regular foliations) is extended in order to include the singularities into the picture.

The condition of non-degeneracy is always present in the works cited above. There are also some contributions for degenerate singularities in the world of integrable systems. A recent contribution in that direction is contained in the paper [7] by Colin de Verdière. In that paper, among other things, the problem of classification of germs of singular Lagrangian manifolds is posed for more general singularities with a special emphasis on quasi-homogeneous singularities. For instance in this paper an explicit classification is obtained in the case of the cusp. The singular achievements formerly specified often have a semiclassical version. Their semiclassical counterpart has been obtained by Colin de Verdière and San Vu Ngoc in [8, 56, 57, 7].

After this digression we will focus on the orbit-like case. The main goal of this thesis is to study problems of classification in the neighbourhood of an orbit.

The singularity of the orbit can be described in terms of the singularity of the functions f_i.

Let us start with the case L is reduced to a point.

Observe that the Poisson bracket induces a Lie algebra structure in the set of functions. Since the functions f_i are in involution with respect to the Poisson bracket, the quadratic parts of the functions f_i commute defining in this way an abelian subalgebra of $Q(2n, \mathbb{R})$ (the set of quadratic forms on $2n$-variables). In the case the singularity of the functions f_i is of Morse type this subalgebra is indeed a Cartan subalgebra. We call these singularities of non-degenerate type.

The problem of classification of singularities for the quadratic parts of the functions f_i can be therefore converted into the problem of classification of Cartan subalgebras of $Q(2n, \mathbb{R})$. The singularities for the quadratic parts are well-understood thanks to a result of Williamson [60] where Cartan subalgebras of $Q(2n, \mathbb{R})$ are classified. Let us recall its precise statement,

Theorem 0.0.2 (Williamson)

For any Cartan subalgebra $\mathcal{C}$ of $Q(2n, \mathbb{R})$ there is a symplectic system of coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ in $\mathbb{R}^{2n}$ and a basis $f_1, \ldots, f_n$ of $\mathcal{C}$ such that each f_i is one of the following:

$$
\begin{aligned}
&f_i = x_i^2 + y_i^2 \qquad \text{for } \ 1 \leq i \leq k_e \ , && \text{(elliptic)}\\
&f_i = x_i y_i \qquad \text{for } \ k_e + 1 \leq i \leq k_e + k_h \ , && \text{(hyperbolic)}\\
&\begin{cases} f_i = x_i y_{i+1} - x_{i+1} y_i, & \text{(focus-focus pair)} \\ f_{i+1} = x_i y_i + x_{i+1} y_{i+1} & \text{for } \quad i = k_e + k_h + 2j - 1, \ 1 \leq j \leq k_f \end{cases}
\end{aligned}
\tag{0.0.2}
$$

The linear system given by the quadratic parts of the f_i is called the linear model for a singularity. Williamson's Theorem can be seen as a normal form theorem for the linear model.

We may attach a triple of natural numbers (k_e, k_h, k_f) to a non-degenerate singularity p of F, where k_e stand for the number of elliptic components in the linear model, k_h and k_f the number of hyperbolic and focus-focus components in the linear model respectively.

By virtue of Williamson theorem this triple is an invariant of the linear system. That is why this triple is often called the Williamson type of the singularity.

Now that the classification in the linear model has been carried out a natural question arises:

Can we linearize the completely integrable system symplectically in a neighbourhood of a point p?

We can reformulate the question as follows,

Problem 1

Consider a foliation $\mathcal{F}$ defined by a completely integrable system defined in a neighbourhood of a non-degenerate singular 0-dimensional orbit of $\mathcal{F}$. Assume that we are given two symplectic forms ω_1 and ω_2 for which the foliation $\mathcal{F}$ is Lagrangian. Does there exist a local diffeomorphism fixing p and taking ω_1 to ω_2?

This problem of symplectic linearization is closely related to another problem in the spirit of Morse lemma which was solved succesfully by Vey for analytic systems and by Vey and Colin de Verdière for smooth systems.

Problem 2

Given a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ with a non-degenerate singularity at the origin and let ω be a volume form on $\mathbb{R}^n$ and let Q be its quadratic part at the origin. Does there exist a diffeomorphism $\phi : (R^n, 0) \longrightarrow (R^n, 0)$ such that $\phi^*(f) = Q$ and such that ω is taken to the volume form $\omega_0 = dx_1 \wedge \cdots \wedge dx_n$?

In [6] Colin de Verdière and Vey prove that there exists a smooth function χ such that $\phi^*(\omega) = \chi(Q) \cdot \omega_0$.

In that paper it is also proved that the function χ is characteristic of the pair (f, ω) if Q is definite, otherwise only the jet is characteristic for the pair.

As a corollary of this result we obtain normal forms for foliations defined by the levelsets of f because we can find a foliation-preserving diffeomorphism sending the volume form $\chi(Q)\cdot\omega_0$ to the volume form ω_0 as was observed in the paper cited above. Notice as well that this result provides an affirmative answer to Problem 1 in the case $n = 2$ because a volume form on a 2-dimensional manifold is a symplectic form and the Lagrangian condition for a curve is automatic in that dimension.

The affirmative answer to Problem 1 in any dimension was provided by Eliasson in [23] and [24]. As a matter of fact the proof provided by Eliasson seems complete just in the case the singularity is completely elliptic (of Williamson type $(k_e, 0, 0)$).

In this thesis we will give another proof of Eliasson theorem with all the details for singularities whose Williamson type is $(k_e, k_h, 0)$. We will also sketch a proof for the focus-focus components.

Observe that Eliasson's theorem can be seen as a symplectic linearization result which ensures that the initial completely integrable system can be taken to the linear system and that the symplectic form can be taken to the standard one. As a byproduct we obtain a multiple differentiable linearization result for n commuting vector fields with singularities of non-degenerate type.

The symplectic linearization in a neighbourhood of an orbit L with $\dim L > 0$ is due to Ito in the analytic case [32]. In this thesis we present the result in the smooth case. Partial results in that direction (with $\dim L = 1$ in a manifold of dimension 4) where obtained by Currás-Bosch and the author of this thesis in [13] and independently by Colin de Verdière and San Vu Ngoc in [8]. The final result in any dimension was obtained by Nguyen Tien Zung and the author of this thesis in [48]. In [48] it is also included a G-equivariant version of the symplectic linearization.

Symmetries are present in many physical problems and therefore they show up in integrable systems theory as well. Those symmetries are encoded in actions of Lie groups.

A special emphasis has been given to Hamiltonian actions of tori in symplectic geometry. Along the way many results of symplectic uniqueness are obtained. A good example of this is Delzant's theorem [19] which enables to recover information of a compact $2n$-dimensional manifold by looking at the image of the moment map of a Hamiltonian torus action which is, surprisingly, a convex polytope in $\mathbb{R}^n$. A lot of contributions in the area of Hamiltonian actions of Lie groups have been done ever since. Let us mention some of the references of the large list of results in that direction: the works of Lerman and Tolman to extend those result to symplectic orbifolds ([37]) and the works of Karshon and Tolman for complexity one Hamiltonian group actions ([33], [34]).

In this thesis actions of compact Lie group are also considered. We assume also that the group acts symplectically and preserves the moment map which is underlying in the foliation.

We end up proving the equivariant version of the symplectic uniqueness result in a neighbourhood of a singular compact orbit. A nice consequence is the abelianity of the group of symplectomorphisms preserving the moment map. In particular, in the case the action of the group is effective then this group is Abelian, in all, since it is also compact it is a product of a torus with a finite group. In the end, in the case the group is connected we recover actions by tori in the spirit of the theorem of Delzant.

Loosely speaking, the odd-dimensional counterpart of the theorems obtained would be considered in the contact case. That is we can consider foliations on a contact manifold as close as possible to the ones described by completely integrable systems on symplectic manifolds. The regular case started with Lutz ([41]) who studies the problem of classification for contact structures in a compact contact manifold under the constraint that they are invariant under the action of a torus. This problem is naturally linked with the analogous problem for symplectic manifolds exposed above. Recent contributions to that problem in the setting of contact orbifolds are due to Lerman [36] where a convexity result is also established. This problem has been considered by Molino and Banyaga in [3] and [4] also for singular foliations. The common property of the foliations considered by Lutz, Lerman, Molino and Banyaga is that their orbits are given by a torus action. In this thesis we prove a similar result in the neighbourhood of a compact orbit but for foliations whose orbits are not necessarily given by a torus action but fulfill hypothesis of non-degeneracy. The foliation is determined as the enlarged foliation of a Legendrian foliation described by the horizontal parts of contact vector fields together with a Reeb vector field. We also assume that the Reeb vector field is the infinitesimal generator of an S^1-action. We study the problem of classification for Legendrian foliations under the assumption that the contact form has the same Reeb vector field. This assumption is a bit constraining. The natural generalization of this result would be to study the problem of classification under the less-constraining assumption that the Reeb vector field belongs to the enlarged foliation instead. This result has been left in the pipeline and it is not included in this thesis. It uses an adaptation of Gray's path method in contact geometry adapted to foliations.

Organization of this thesis:

In **Chapter 1** we make a review of the differentiable linearization result (theorem 1.3.1) for the foliations considered. We provide our own proof for the corank 1 case.

This differentiable linearization allows to work in a linear model in the covering.

In **Chapter 2** the analytic tools necessary to face the problem are developed. We also present our own proof for the symplectic linearization in dimension 2.

In **Chapter 3** we study the corank 1 case in dimension 4. We present two proofs for the symplectic uniqueness. One of the proofs is based on the construction of a symplectic orthogonal decomposition to reduce the problem to a 2 dimensional case. The tecniques of decomposition of functions introduced in chapter 2 are used to construct the symplectic orthogonal decomposition.

In **Chapter 4** we study the rank 0 case in dimension 4. We prove the symplectic uniqueness again using the geometrical tecniques of symplectic orthogonal decomposition. In the construction of the symplectically orthogonal distributions we use Moser tecniques and geometrical tricks relying on the Bott-Weinstein connection.

In **Chapter 5** we use induction, Liouville-Mineur-Arnold Theorem and the results obtained in the previous chapters to prove the general case in any rank and in any dimension.

In **Chapter 6** we present the equivariant version of the symplectic uniqueness attained in Chapter 5. This equivariant version allows to conclude the symplectic linearization in a neighbourhood of the initial compact orbit considered. We also present a slice statement of the equivariant symplectic linearization result in the neighbourhood of an orbit.

Finally, in **Chapter 7** we consider the contact case and prove the contact linearization result in the covering. We also present the G-equivariant contact version of the theorem which yields in particular the contact linearization in the initial neighbourhood considered.

Part of the results contained in this thesis are contained in the publications and preprints that we cite below,

- Publications:

    1. C. Currás-Bosch and E. Miranda, *Symplectic linearization of singular Lagrangian foliations in M^4*, Differential Geom. Appl. **18** (2003), no. 2, 195-205.

2. E. Miranda, *On the symplectic classification of singular Lagrangian foliations.* Proceedings of the IX Fall Workshop on Geometry and Physics (Vilanova i la Geltrú, 2000), 239–244, Publl. R. Soc. Mat. Esp., **3**, R. Soc. Mat. Esp., Madrid, 2001.

- Preprints:

1. E. Miranda and Nguyen Tien Zung, *Equivariant normal forms for non-degenerate singular orbits of integrable Hamiltonian systems*, preprint 2003, http://xxx.arxiv.org/abs/math.SG/0302287.

2. C. Currás-Bosch and E. Miranda, *Symplectic germs of singular Lagrangian Foliations,* preprint 265 de la Facultat de Matemàtiques. Universitat de Barcelona, 1999.

Resum en català

0.1 Introducció

Objectius de la tesi

L'objectiu d'aquesta tesi és estudiar dos problemes de classificació de foliacions. El primer problema es planteja per a foliacions definides per sistemes completament integrables a varietats simplèctiques. El segon problema es planteja en l'àmbit de les varietats de contacte per a foliacions també de tipus completament integrable la definició de les quals està fortament inspirada en el cas simplèctic.

Tot seguit anem a establir els objectius amb precisió que seran tractats en aquesta tesi.

Primer estudiarem el problema de classificació d'estructures simplèctiques definides a un entorn d'una òrbita compacta d'un sistema completament integrable per les quals la foliació definida per l'aplicació moment és genèricament Lagrangiana. Quan diem que una foliació amb singularitats és genèricament Lagrangiana volem dir que les fulles regulars són subvarietats Lagrangianes. Per continuïtat, les fulles singulars (de dimensió inferior a la meitat de la dimensió de la varietat) són subvarietats isòtropes. Al llarg de tota la tesi treballarem a nivell de germes. És a dir tots els objectes es consideren definits a un entorn tubular de l'òrbita compacta. Suposem que l'entorn considerat és una varietat simplèctica. Sigui Ω una forma simplèctica fixada inicialment a l'entorn. La foliació està definida de la següent manera: és la foliació determinada per les òrbites de la distribució generada pels gradients simplèctics respecte de la forma simplèctica Ω de les components de l'aplicació moment. La forma simplèctica inicial Ω només és necessària per a definir la foliació. El tipus de singularitats que considerarem són no degenerades en el sentit de Morse-Bott. Sota aquestes hipòtesis demostrem que qualssevol dues estructures simplèctiques per a les quals la foliació és genèricament Lagrangiana són equivalents. La noció d'equivalència per al problema de classificació plantejat és el següent: Dues formes simplèctiques definides a un entorn de la fulla són equivalents si existeix un simplectomorfisme preservant la foliació que envia una forma simplèctica a l'altra i que fixa l'òrbita singular. En el cas que existeixi una acció d'un grup de Lie compacte preservant l'aplicació moment provem que aquesta equivalència és G-equivariant.

El segon problema que ens plantegem és un problema de classificació per a formes de contacte. Considerem una varietat de contacte (M^{2n+1}, α) que compleix les següents hipòtesis:

1. El camp de Reeb és el generador infinitesimal d'una acció del grup de Lie S^1.

2. El camp de Reeb admet n integrals primeres f_i funcionalment independents en un conjunt dens.

3. Les integrals primeres commuten respecte del parèntesi de Jacobi.

En aquesta varietat de contacte hi considerem dues foliacions: la foliació $\mathcal{F}$ definida per les parts horitzontals dels camps de contacte associats a les n integrals primeres f_i considerades i la foliació $\mathcal{F}'$ definida com la foliació generada per $\mathcal{F}$ conjuntament amb el camp de Reeb.

Un cop definida la foliació anem a plantejar el problema de classificació: Volem classificar les formes de contacte α definides a un entorn d'una òrbita compacta singular de la foliació $\mathcal{F}'$ que tenen el mateix camp de Reeb i per a les quals la foliació $\mathcal{F}$ és Legendriana. Sota la hipòtesis de que la singularitat sigui no degenerada en el sentit de Morse-Bott provem que qualssevol dues formes de contacte verificant les condicions anteriorment esmentades són equivalents. És a dir, provem que existeix un difeomorfisme definit a un entorn de l'òrbita que envia una forma de contacte a l'altra, preservant la foliació $\mathcal{F}$ i fixant l'òrbita singular . En el cas en què existeixi una acció d'un grup de Lie compacte preservant la forma de contacte α provem que es pot trobar un difeomorfisme G-equivariant, així doncs, la equivalència és G-equivariant.

Ubicació del problema

Aquesta tesi es centra en l'estudi de la geometria que està encoberta als sistemes Hamiltonians totalment integrables. Un sistema Hamiltonià en una varietat simplèctica (M^{2n}, ω) és el sistema definit pel gradient simplèctic d'una funció H anomenada funció Hamiltoniana del sistema. L'estudi de la integrabilitat d'aquests sistemes és rellevant en moltes àrees de les matemàtiques i té la seva pròpia història.

El 29 de Juny de 1853 Joseph Liouville va presentar una comunicació titulada "Sur l'intégration des equations différentielles de la Dynamique"al "Bureau des longitudes". A la nota resultant [40] es relaciona la noció d'integrabilitat del sistema amb l'existència de n integrals primeres en involució respecte el parèntesi de Poisson associat a la forma simplèctica. Aquests sistemes apareixen amb la denominació clàssica de "sistemes completament integrables". En un altre llenguatge l'elecció de n integrals determina les components de l'aplicació moment $\mathbf{F} : M^{2n} \longrightarrow \mathbb{R}^n$. Els treballs de Joseph Liouville constitueixen el punt de partida de tot un seguit de treballs posteriors. Anem a destacar algunes de les fites aconseguides en aquest terreny des d'un punt de vista geomètric i topològic.

Considerem, d'entrada, un sistema Hamiltonià completament integrable en una varietat simplèctica. Els gradients simplèctics de les components de l'aplicació moment defineixen una distribució involutiva. Suposem que aquesta aplicació moment és pròpia. Sigui L una òrbita regular d'aquesta distribució, la condició de completa integrabilitat implica que aquesta òrbita és una subvarietat Lagrangiana. A més a més es tracta d'un torus i les òrbites a un entorn d'aquesta són també torus. Aquests torus es diuen torus de Liouville. Aquesta és la contribució topològica d'un teorema conegut com teorema d'Arnold-Liouville. L'aportació geomètrica d'aquest teorema és l'existència de formes normals simplèctiques a un entorn d'una òrbita regular compacta. De fet sembla ser que els treballs de Henri Mineur [44, 45, 46] donaven una descripció del sistema Hamiltonià en un entorn d'una òrbita compacta regular. Per aquest motiu ens referirem al clàssic teorema d'Arnold-Lioville com a teorema de Liouville-Mineur-Arnold. Recordem-ne l'enunciat,

Teorema de Liouville-Mineur-Arnold

Sigui (M^{2n}, ω) una varietat simplèctica i sigui $F : M^{2n} \longrightarrow \mathbb{R}^n$ una aplicació moment. Suposem que les components f_i de F estan en involució dos a dos respecte el parèntesi de Poisson associat a ω i que $df_1 \wedge \cdots \wedge df_n \neq 0$ en un conjunt dens. Sigui $N = F^{-1}(c)$, $c \in \mathbb{R}^n$ un nivell connex de l'aplicació moment. Llavors existeix un entorn $U(N)$ de N i un difeomorfisme $\phi : U(N) \longrightarrow D^n \times \mathbb{T}^n$ tal que,

1. $\phi(N) = \{0\} \times \mathbb{T}^n$.
2. *Existeixen coordenades μ_i a un disc D^n i coordenades β_i definides en un torus $\mathbb{T}^n$ tals que,* $\phi^*(\sum_{i=1}^n d\mu_i \wedge d\beta_i) = \omega$.
3. *F només depèn de $\phi^*(\mu_i) = p_i$ i no depèn de $\phi^*(\beta_i) = \theta_i$.*

Les noves coordenades p_i s'anomenen coordenades acció. Les coordenades θ_i s'anomenen coordenades angle. Mineur va donar la fòrmula de les integrals de període que s'utilitzen per a definir les coordenades d'acció. Sigui x un punt a un entorn de N, definim les integrals de període mitjançant la fòrmula:

$$p_i(x) = \int_{\Gamma_i(x)} \alpha \tag{0.1.1}$$

on α és una forma de Liouville per a la forma simplèctica, és a dir, ve donada per la condició $d\alpha = \omega$ i $\Gamma_i(x)$ és una corba tancada que depèn diferenciablement de x i està continguda en un torus de Liouville. Les classes d'homologia $\Gamma_1(x), ..., \Gamma_n(x)$ formen una base del primer grup d'homologia del torus de Liouville.

L'existència de coordenades acció-angle a un entorn d'una òrbita compacta donen un model simplèctic per a la foliació Lagrangiana determinada per les òrbites del gradient simplèctic de les n funcions components de l'aplicació moment F. De fet, el teorema de Liouville-Mineur-Arnold porta implícit un resultat d'unicitat simplèctica, llevat de simplectomorfisme preservant la foliació, de formes simplèctiques que fan que $\mathcal{F}$ sigui Lagrangiana.

Dit d'una altra manera, si ω_1 i ω_2 són dues formes simplèctiques definides a un entorn de N per a les quals la foliació $\mathcal{F}$ és Lagrangiana, llavors existeix un simplectomorfisme preservant la foliació, fixant N i enviant ω_1 a ω_2. Això es degut a la següent observació: Siguin $X_{f_i}^{\omega_k}$ els gradients simplèctics de les funcions f_i respecte ω_k per a qualsevol $1 \leq i \leq n$ i $k = 1, 2$, llavors la condició de Lagrangianitat implica que de fet es té la següent igualtat $\mathcal{F} =< X{f_1}^{\omega_k}, \ldots, X_{f_n}^{\omega_k} >$, $k = 1, 2$, a més a més $\{f_i, f_j\}_k = 0$ on $\{., .\}_k$ és el parèntesi de Poisson associat a ω_k , $k = 1, 2$. Llavors, pel teorema de Liouville-Mineur-Arnold existeix un simplectomorfisme ϕ_k, preservant la foliació i enviant la forma simplèctica ω_k a $\omega_0 = \sum_{i=1}^{n} dp_i \wedge d\theta_i$. Finalment el difeomorfisme $\phi_2^{-1} \circ \phi_1$ envia ω_1 a ω_2, fixa N i preserva la foliació.

Per tant si l'òrbita és regular l'existència de coordenades acció angle permet classificar els germes simplèctics per als quals la foliació $\mathcal{F}$ és Lagrangiana llevat de simplectomorfisme preservant la foliació. Com acabem de comprovar només existeix una classe de germes simplèctics pels quals la foliació és Lagrangiana.

Podem mirar aquest problema des d'un punt de vista global. Existeixen obstruccions topològiques a l'existència de coordenades acció-angle global tal i com va provar Duistermaat a [22].

El problema de classificació de germes simplèctics de foliacions Lagrangianes es pot portar més enllà i considerar foliacions Lagrangianes regulars que no provenen, necessàriament, d'un sistema completament integrable. Currás-Bosch i Molino ([9, 10, 14, 15, 16]) han considerat el problema de classificació per a germes de foliacions Lagrangianes definides a un entorn d'un torus amb una estructura afí fixada. La motivació per a considerar una estructura afí al torus és la connexió de Bott-Weinstein associada a les fulles regulars d'una foliació Lagrangiana [59].

En el cas que el germe de foliació Lagrangiana quedi determinat per un sistema completament integrable aquesta estructura afí és trivial. Als treballs esmentats anteriorment es prova que no hi ha resultat d'unicitat per als germes simplèctics en el cas que la estructura afí no sigui trivial.

Després d'aquest repàs de resultats per a foliacions Lagrangianes regulars ens plantegem la següent pregunta.

Què podem dir sobre el problema de classificació de germes simplèctics si el sistema completament integrable té singularitats?

Aquesta pregunta és bastant natural perquè les singularitats són presents en molts sistemes integrables coneguts. De fet si el sistema completament integrable està definit en una varietat compacta les singularitats són inevitables.

Un dels principals objectius d'aquesta tesi és provar que es té també un resultat d'unicitat per a germes simplèctics per als quals la foliació determinada pel sistema completament integrable és Lagrangiana a un entorn d'una òrbita singular compacta.

En el cas singular, ens podem plantejar el problema a tres nivells diferents:

1. A nivell d'òrbita. En un entorn d'una òrbita singular compacta.

2. A nivell semilocal. En un entorn d'una fulla singular compacta.

3. A nivell global.

En aquesta tesi només ens preocuparem de la primera situació.

És a dir, farem un estudi a un entorn d'una òrbita compacta singular que suposarem singular no-degenerada en el sentit de Morse-Bott. Abans d'endinsar-nos en l'estudi a un entorn de l'òrbita, anem a fer esment ràpidament d'alguns resultats destacables a nivell semilocal i global.

El problema de classificació topològica dels sistemes Hamiltonians completament integrables va començar amb Fomenko [25] en alguns casos particulars. L'estudi de la topologia d'aquests sistemes a l'entorn d'una fulla no-degenerada en el cas general es deu a Nguyen Tien Zung [61]. La descripció de les singularitats des d'un punt de vista topològic és de tipus producte de singularitats el·líptiques, hiperbòliques i focus-focus. El tipus de Williamson és per tant l'unic invariant topològic semi-local. En aquest treball, Nguyen Tien Zung també prova l'existència de coordenades acció-angle parcials. La classificació simplèctica en el cas semi-local per a singularitats de tipus no el·líptic ha estat estudiat per Dufour, Molino i Toulet [20], [53] en el cas hiperbòlic i per San Vu Ngoc en el cas focus-focus [58]. La conclusió d'aquests treballs és que hi ha més invariants associats a la singularitat caracteritzats per jets de funcions d'una variable (en el cas hiperbòlic) i pels jets de funcitons de dues variables (en el cas focus-focus). El cas global singular va ésser estudiat per Nguyen Tien Zung a [63]. En aquest treball s'estén el concepte de classe de Duistermaat-Chern i el concepte de monodromia en el cas singular. La condició de no-degeneració està sempre present als treballs anteriorment citats. Però també cal destacar algunes contribucions en el camp de sistemes integrables amb singularitats de tipus degenerat. En aquesta direcció apunta el treball de Colin de Verdière [7]. En aquest article, entre altres moltes coses es planteja el problema de classificació de germes de varietats Lagrangianes singulars per a singularitats de tipus més general que les no degenerades, destacant especialment les singularitats quasi-homogènies. Per exemple, s'estudia el problema de classificació associat al cas de la cúspide.

La majoria de resultats esmentats tenen la seva versió semiclàssica. Els resultats en l'àmbit semiclàssic han estat desenvolupats per Colin de Verdière i San Vu Ngoc a [8, 56, 57, 7].

0.2 Resultats

L'objectiu principal d'aquesta tesi és estudiar problemes de classificació en un entorn de l'òrbita.

Les singularitats queden descrites en termes de les singularitats de les components de l'aplicació moment f_i.

Comencem pel cas que L sigui reduïda a un punt. Observem que el parèntesi de Poisson indueix una estructura de àlgebra de Lie en el conjunt de funcions diferenciables. Com que les funcions f_i estan en involució respecte el parèntesi de Poisson, les parts quadràtiques de les funcions f_i commuten definint, d'aquesta manera, una estructura d'àlgebra abeliana al conjunt de formes quadràtiques en $2n$ variables que denotarem per $Q(2n, \mathbb{R})$. En el cas que la singularitat de les funcions sigui de tipus Morse aquesta subàlgebra és una subàlgebra de Cartan. Aquestes singularitats es diuen singularitats de tipus no degenerat.

Per tant el problema de classificació de singularitats per les parts quadràtiques de les components de l'aplicació moment s'ha convertit en un problema merament algebraic: la classificació de les subàlgebres de Cartan de $Q(2n, \mathbb{R})$. La classificació d'aquestes singularitats es deu al següent resultat de Williamson ([60])

Teorema (Williamson)

Donada una subàlgebra de Cartan $\mathcal{C}$ *de* $Q(2n, \mathbb{R})$ *existeix un sistema simplèctic de coordenades* $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ *a* $\mathbb{R}^{2n}$ *i una base* $f_1, \ldots, f_n$ *de* $\mathcal{C}$ *tal que cada* f_i *és del següent tipus:*

$$
\begin{aligned}
&f_i = x_i^2 + y_i^2 \qquad \text{si} \quad 1 \leq i \leq k_e \ , && \text{(el·líptic)} \\
&f_i = x_i y_i \qquad \text{si} \quad k_e + 1 \leq i \leq k_e + k_h \ , && \text{(hiperbòlic)} \\
&\begin{cases} f_i = x_i y_{i+1} - x_{i+1} y_i, & \text{(parell focus-focus)} \\ f_{i+1} = x_i y_i + x_{i+1} y_{i+1} & \text{si} \quad i = k_e + k_h + 2j - 1, \ 1 \leq j \leq k_f \end{cases}
\end{aligned}
\tag{0.2.1}
$$

El sistema lineal donat per les parts quadràtiques de les f_i es diu model lineal per a les singularitats. El teorema de Williamson es pot veure com un teorema de formes normals per al model lineal. Podem adjuntar un triplet de nombres naturals (k_e, k_h, k_f) a una singularitat no degenerada de F on k_e és el nombre de components el.líptiques al model lineal i k_h i k_f són el nombre de components hiperbòliques i focus-focus respectivament.

Com a conseqüència del teorema de Williamson aquest triplet és un invariant del sistema lineal. Per aquest motiu s'anomena tipus de Williamson de la singularitat.

Ara que ja tenim la classificació al model lineal la pregunta natural és:

Podem linealitzar simplècticament un sistema completament integrable en un entorn d'un punt singular p?

Podem reformular la pregunta de la manera següent,

Problema 1

Considerem una foliació $\mathcal{F}$ definida per un sistema completament integrable definit en un entorn d'una òrbita singular no degenerada de dimensió 0 de $\mathcal{F}$. Suposem que tenim donades dues formes simplèctiques ω_1 i ω_2 per a les quals la foliació $\mathcal{F}$ és Lagrangiana. Existeix un difeomorfisme local fixant p i portant ω_1 a ω_2?

El problema de linealització simplèctica està íntimament relacionat amb un altre problema en l'ordre d'idees del lema de Morse. L'altre problema és el següent:

Problema 2

Donada una funció $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ amb una singularitat no degenerada a l'origen i sigui ω una forma de volum a $\mathbb{R}^n$. Notem per Q la seva part quadràtica a l'origen. Existeix un difeomorfisme $\phi : (R^n, 0) \longrightarrow (R^n, 0)$ tal que $\phi^*(f) = Q$ i enviant la forma de volum ω a la forma $\omega_0 = dx_1 \wedge \cdots \wedge dx_n$?

A l'article [6] Colin de Verdière i Vey demostren que existeix una funció diferenciable χ tal que $\phi^*(\omega) = \chi(Q) \cdot \omega_0$.

A l'article esmentat anteriorment es prova que la funció χ caracteritza el parell (f, ω) en el cas que la forma quadràtica Q sigui definida, en cas contrari només el jet de la funció caracteritza el parell.

Com a corol·lari d'aquest resultat obtenim formes normals per a foliacions definides pels nivells de f perquè podem trobar un difeomorfisme preservant la foliació i enviant la forma de volum $\chi(Q) \cdot \omega_0$ a la forma de volum ω_0 com s'observa a la publicació anteriorment citada.

Observem que aquest resultat dóna una resposta afirmativa al Problema 1 en el cas $n = 2$ perquè una forma de volum en una varietat de dimensió 2 és una forma simplèctica i la condició de Lagrangianitat en el cas d'una corba és automàtica en aquesta dimensió.

La resposta afirmativa al Problema 1 en qualsevol dimensió és conseqüència del teorema d'Eliasson [23] i [24]. De fet, cal remarcar que la demostració donada per Eliasson és completa només en cas què la singularitat sigui completament el·líptica (tipus de Williamson $(k_e, 0, 0)$).

En aquesta tesi donem una altra demostració amb tots els detalls per a singularitats que tenen tipus de Williamson $(k_e, k_h, 0)$. També donem un esboç de la demostració en el cas d'existir components focus-focus.

Observem que el Teorema d'Eliasson es pot mirar com un resultat de linealització simplèctica que assegura que el sistema completament integrable inicial es pot portar a un sistema lineal amb la forma simplèctica estàndard. Com a resultat obtenim un resultat de linealització múltiple per a n camps vectorials que commuten i que tenen singularitats de tipus no degenerat. La linealització simplèctica a un entorn de l'òrbita L en el cas $\dim L > 0$ es degut a Ito en el cas analític [32]. En aquesta tesi presentem el resultat en el cas diferenciable. Resultats parcials en aquesta direcció (en el cas en que la dimensió de l'òrbita singular sigui 1 en una varietat de dimensió 4) vàren ser obtinguts per Currás-Bosch conjuntament amb l'autora d'aquesta tesi conjuntament ([13]) i independentment per Colin de Verdière i San Vu Ngoc [8]. El resultat final en qualsevol dimensió ha estat obtingut per Nguyen Tien Zung conjuntament amb l'autora d'aquesta tesi a [48]. En aquest paper també està continguda la versió G-equivariant de la linealització simplèctica.

L'estudi de simetries té rellevància en molts problemes físics i, en conseqüència, també apareix a la teoria de sistemes completament integrables. Aquestes simetries queden codificades en forma d'accions diferenciables de grups de Lie. El cas d'accions Hamiltonianes de torus mereix especial atenció. En aquest teoria apareixen molts resultats d'unicitat simplèctica. El Teorema de Delzant n'és un exemple clar. El Teorema de Delzant permet recuperar informació en una varietat compacta de dimensió $2n$ a partir de la imatge de l'aplicació moment, que curiosament, és un politop convex. S'han produit moltes contribucions en aquest camp darrerament. En destaquem dues: Els treballs de Lerman i Tolman per extendre aques resultat al cas d'orbifolds simplèctiques [37] i els treballs de Karshon i Tolman per a generalitzar aquests resultats en el cas d'accions Hamiltonianes de complexitat 1.

En aquesta tesi també considerem accions de grups de Lie compactes. Suposarem que el grup actúa simplècticament i preserva l'aplicació moment que està oculta a la foliació. En aquesta tesi demostrem la versió equivariant dels resultats d'unicitat simplèctica en un entorn de l'òrbita compacta singular. Una consequència curiosa és que el grup de simplectomorfismes preservant l'aplicació moment és un grup abelià. En particular, si l'acció del grup G és efectiva d'aquest resultat en podem extreure l'abelianitat de G. Donat que el grup G és compacte és un producte d'un torus per un grup finit. En el cas que el grup G sigui connex recuperem l'acció d'un torus en la línia del Teorema de Delzant.

La contrapartida dels anteriors resultats en dimensió imparella venen donats pel cas de contacte. Aquest és el segon problema de classificació plantejat a la tesi. Considerem foliacions en una varietat de contacte semblants a les donades per sistemes completament integrables en varietats simplèctiques.

La motivació en el cas regular va ser donada per Lutz ([41]) que va estudiar el problema de classificació per a estructures de contacte en una varietat compacta sota la hipòtesi que siguin invariants per l'acció d'un torus.

Aquest problema està lligat amb el problema anàleg per a varietats simplèctiques que hem exposat abans. Cal destacar les següents contribucions recents en aquest camp: en el cas d'orbifolds de contacte destaquem els treballs de Lerman [36] on es dóna un resultat de convexitat. Aquest problema ha estat considerat per Molino i Banyaga a [3] i [4] en el cas de foliacions singulars. El denominador comú de les foliacions considerades per Lutz, Lerman, Molino i Banyaga és que les òrbites vénen donades per l'acció d'un torus. En aquesta tesi demostrem un resultat similar en l'entorn d'una òrbita compacta en el cas que la foliació no estigui necessàriament donada per l'acció d'un torus. Les singularitats les suposem no degenerades i la foliació queda deteminada com la foliació donada per les parts horitzontals del camps de contacte de n integrals primeres del camp de Reeb conjuntament amb el camp de Reeb. En aquesta tesi suposem que el camp de Reeb ve donat com a generador infinitesimal d'una acció del grup S^1. Estudiem el problema de classificació de la foliació Legendriana descrita per les parts horitzontals dels camps de contacte sota la hipòtesi que la forma de contacte tingui el mateix camp de Reeb. Aquesta condició és una mica restrictiva. Una generalització natural seria estudiar el problema més general en què el camp de Reeb pertanyi a la foliació. Aquest resultat que requereix les tècniques de Gray per deformació de formes de contacte no està inclòs a la tesi.

Organització de la tesi

Al **capítol 1** estudiem el problema de linealització diferenciable en un recobriment de l'entorn inicialment considerat. Donem també una demostració diferent a la donada per l'Eliasson en el cas de foliacions amb corang 1. Aquest resultat de linealització diferenciable permet treballar en un model lineal al recobridor. En aquest capítol donem una demostració del següent teorema en el cas de singularitats de corang 1.

Teorema 1.3.1

A $\widetilde{U(L)}$ la foliació Lagrangiana definida pels gradients simplèctics de l'aplicació moment és difeomorfa a la foliació linealitzada.

On notem per $\widetilde{U(L)}$ un recobriment finit d'un entorn de l'òrbita singular compacta.

L'objectiu del **capítol 2** és doble; per una banda s'introdueixen les eines analítiques necessàries per resoldre el problema també es dóna una demostració de la linealització simplèctica en dimensió 2.

El primer objectiu d'aquest capítol és provar que donada una funció diferenciable g es pot trobar una descomposició del tipus,

$$g = g_1 + X(g_2) \quad , \quad X(g_1) = 0 \tag{0.2.2}$$

per a determinats tipus de camps singulars, els que corresponen als camps lineals de la foliació (components el.lítiques i hiperbòliques). Aquest tipus de descomposicions de funcions seran útils més endavant per a trobar deformacions de l'estructura simplèctica “à la Moser”preservant la foliació.

El cas el.liptic ja havia estat considerat per Eliasson. El cas hiperbòlic en dimensió 2 també. Donem demostracions pels casos el.liptics i hiperbòlics en qualsevol dimensió. Els resultats principals provats són els següents:

En el cas que X sigui un camp corresponent a una singularitat el.líptica,

Proposició 2.2.1

Sigui M una varietat diferenciable i sigui g un germe de funció diferenciable en un entorn del punt p. Donat un camp X que en coordenades locals s'expressa $X = x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}$ llavors existeixen funcions diferenciables g_1 i g_2 tals que:

$$g = g_1(x_1^2 + x_2^2, x_3 \dots, x_n) + X(g_2)$$

En el cas que X sigui un camp corresponent a una singularitat hiperbòlica,

Proposició 2.2.2

Sigui M una varietat diferenciable i sigui g un germe de funció diferenciable en un entorn del punt p. Considerem un camp vectorial X que en coordenades locals s'expressa $Y = -x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}$ llavors existeixen funcions diferenciables g_1 i g_2 tals que,

$$g = g_1(x_1 x_2, x_3 \dots, x_n) + Y(g_2)$$

En el cas hiperbòlic aquest resultat estén el resultat en el cas en dimensió 2 provat per Colin de Verdière i Vey a [6]. El cas hiperbòlic és més complicat, cal reduïrse primer al cas de funcions planes al llarg d'un subespai i després usar tècniques d'integració semblants a les usades al Teorema de linealització de Sternberg.

Usant aquests resultats analítics en el cas 2 dimensional i tècniques de deformació d'estructures simplèctiques usant camins "à la Moser"demostrem, per últim, el resultat de linealització simplèctica en dimensió 2,

Teorema 2.3.1

Sigui (M^2, ω_1) *una varietat simplèctica* 2*-dimensional amb coordenades* (x, y) *i sigui* $\mathcal{F}$ *una foliació Lagrangiana amb singularitats de tipus el.líptic o hiperbòlic a l'origen* $(0, 0)$*, llavors existeix un difeomorfisme local* ϕ *preservant la foliació* $\mathcal{F}$ *tal que* $\phi^*(dx \wedge dy) = \omega_1$*.*

Aquest resultat de linealització simplèctica en dimensió 2 es desprén de [6] pero en donem la nostra pròpia demostració.

Al **capítol 3** estudiem el cas de corang 1 en dimensió 4. Donem dues demostracions de la unicitat simplèctica.

El punt clau per a provar la unicitat simplèctica rau en recuperar una acció Hamiltoniana de S^1 tangent a la foliació. Per tal d'aconseguir aquesta acció usem el Lema de Poincaré i provem un lema tipus Moser de deformació d'estructures simplèctiques preservant la foliació que citem a continuació:

Lema 3.2.3

Sigui α *una* 1*-forma, que s'anul.la a* L*, i que es* $\mathcal{F}_0$*-bàsica i sigui* ω_1 *una forma simplèctica a* M_0^4 *tal que* $\mathcal{F}_0$ *és Lagrangiana. Llavors:*

1. *La* 2*-forma* $\omega_o = \omega_1 - d\alpha$ *es una estructura simplèctica en un entorn de* L *per la qual la foliació és Lagrangiana.*

2. *Existeix un difeomorfisme* η *entre dos entorns de* L *a* M_0^4 *preservant* $\mathcal{F}_0$ *i tal*

que $\eta^*(\omega_1) = \omega_0$.

El resultat principal de la segona secció d'aquest capítol és el següent,

Proposició:

Existeix una acció Hamiltoniana de S^1 *tangent a la foliació. De fet, existeixen coordenades* (θ, p, x, y) *en un entorn de* L *tal que* $\omega = d(pd\theta + C(p, x, y)dx + D(p, x, y)dy)$ *i l'acció Hamiltoniana es produeix per translacions en la coordenada* θ.

Un cop provada l'existència d'aquesta acció donem dues demostracions del següent teorema,

Teorema 3.1.1

Sigui $M_0^4 = S^1 \times D^3$, *amb coordenades* (θ, p, x, y). *Sigui* $\mathcal{F}_0$ *la foliació donada per:*

$$Y_1 = \frac{\partial}{\partial \theta}$$

$$Y_2 = y\frac{\partial}{\partial x} - \epsilon x\frac{\partial}{\partial y}$$

$\epsilon \in \{-1, 1\}$ *(*$\epsilon = 1$ *cas el.líptic,* $\epsilon = -1$ *cas hiperbòlic).*

Sigui $L = S^1 \times (0, 0, 0)$. *Llavors qualssevol dues formes simplèctiques* ω_1 *i* ω_2 *a* M^4 *per a les quals* $\mathcal{F}_0$ *és Lagrangiana són equivalents.*

La primera demostració implica un treball de deformació de la forma simplèctica pel mètode del camí. La segona demostració usa l'existència d'una acció Hamiltoniana per a contruir una descomposició ortogonal simplèctica (lema 3.3.2) utilitzada per a reduir el problema a dos problemes de classificació 2-dimensional.

Al **Capítol 4** estudiem el cas de rang 0 en dimensió 4. Donem un resultat d'unicitat simplèctica usant les tècniques geomètriques de descomposició ortogonal.

Els resultats més importants d'aquest capítol són els següents,

Teorema 4.2.1 (Descomposició simplèctica ortogonal)

Sigui ω una forma simplèctica per la qual $\mathcal{F}$ és genèricament Lagrangiana. Llavors existeix un germe simplèctic $\overline{\omega}$ equivalent a ω i existeixen dues distribucions simplèctiques D_1 i D_2 tals que,

1. *D_1 i D_2 són involutives i simplècticament ortogonals respecte $\overline{\omega}$.*
2. *$X_{1,\epsilon_1} \in D_1$ i $X_{2,\epsilon_2} \in D_2$.*

Teorema 4.2.2 (Unicitat simplèctica)

Sigui ω una forma simplèctica en un entorn de p per a la qual $\mathcal{F}$ és genèricament lagrangiana llavors ω és equivalent a $\omega_0 = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$.

Per a demostrar l'existència de la descomposició ortogonal recuperem accions Hamiltonianes tangents a les fulles usant el mètode del camí. Un dels resultats bàsics en la prova d'aquest teorema es la següent Proposició:

Proposició 4.4.1

Existeix un germe simplèctic $\overline{\omega}_1$ equivalent a ω tal que,

$$i_{X_{1,\epsilon_1}}\overline{\omega}_1 = H_1 df_1 + H_2 df_2 \ .$$

per a funcions $\mathcal{F}$-bàsiques H_1 i H_2.

Un cop demostrada aquesta proposició utilitzem tècniques de normalització per a trobar camps Hamiltonians convenients, tangents a la foliació. La demostració és diferent en el cas que la foliació contingui components el.líptiques o en el cas que la foliació sigui completament hiperbòlica. Usem al llarg de la demostració els resultats de el capítol 2 i el mètode del camí per a deformar formes simplèctiques. El següent pas per a demostrar l'existència de la descomposició ortogonal és provar que un dels camps lineal és paral.lel respecte de la connexió de Bott-Weinstein definida a les fulles Lagrangianes regulars properes a la fulla singular. El fet que els camps siguin paral.lels ens permet donar una demostració geomètrica del teorema de descomposició ortogonal simplèctic. De fet en el cas que la foliació tingui alguna component el.líptica donem dues demostracions d'aquest fet, una basada en raonaments geomètrics usant la connexió de Bott-Weinstein i un altra usant la forma explícita de l'estructura simplèctica en un entorn de la fulla i el Lema de Poincaré. En el cas completament hiperbòlic cal aplicar el mètode del camí diverses vegades per a trobar una estructura simplèctica tal que el Hamiltonià corresponent a f_1 sigui el camp lineal X_1. Aquest procés constitueix el contingut en les proposicions 4.6.1, 4.5.1 i permet demostrar el teorema 4.2.1. Per a demostrar 4.2.1 contruïm dues distribucions ortogonals simplèctiques que contenen cadascun dels camps lineals de la distribució. A partir d'això per a demostrar el teorema 4.2.2 usem els resultats d'unicitat simplèctica en dimensió 2 demostrats al capítol 2.

Al **Capítol 5** usem inducció, el teorema de Liouville-Mineur-Arnold, el mètode de la descomposició simplèctica ortogonal i els resultats de capítols anteriors per a demostrar el cas general en qualsevol rang i qualsevol dimensió.

En aquest capítol demostrem els següents teoremes,

Per al cas de foliacions de rang 0,

Teorema 5.1.1

Sigui ω un forma simplèctica definida en un entorn de l'origen i tal que la foliació lineal $\mathcal{F}$ és Lagrangiana, llavors existeix un difeomorfisme local $\phi : (U,p) \longrightarrow (\phi(U),p)$ preservant la foliació i tal que $\phi^*(\sum_i dx_i \wedge dy_i) = \omega$, essent x_i, y_i coorde-

nades locals a $(\phi(U), p)$.

Per a foliacions de rang diferent de zero provem el teorema d'unicitat simplèctica següent,

Teorema 5.2.1

Siguin ω i ω_0 dues formes simplèctiques que en un entorn d'una òrbita singular compacta per a les quals la foliació lineal és Lagrangiana llavors ω i ω_0 són equivalents.

Al **Capítol 6** donem la versió equivariant de la unicitat simplèctica obtinguda al Capítol 5. Aquesta versió equivariant permet concloure la linealització simplèctica en un entorn de l'òrbita inicial compacta. També donem un enunciat tipus "slice"de la linealització simplèctica equivariant.

Al llarg d'aquest capítol suposem que G és un grup de Lie compacte que actúa simplècticament en la varietat i deixa invariant l'aplicació moment. Els resultats principals obtinguts en aquest capítol són els següents.

En la primera secció es s'estudia el problema de linealització de l'acció a l'entorn d'un punt fix.

Per a fer això primer estudiem els simplectomorfismes locals que deixen invariant l'aplicació moment. Provem el següent teorema,

Teorema 6.3.2

Sigui $\psi : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0)$ un simplectomorfisme local de $\mathbb{R}^{2n}$ que preserva l'aplicació moment en el model $\mathbf{h}$. Llavors la part lineal $\psi^{(1)}$ és un simplectomorfisme que preserva l'aplicació moment i existeix una única funció diferenciable definida en un entorn de l'origen $\Psi : (\mathbb{R}^{2n}, 0) \to \mathbb{R}$ que s'anul.la a l'origen, que es una integral primera del sistema lineal definit per $\mathbf{h}$ i tal que $\psi^{(1)} \circ \psi^{-1}$ es el flux a *temps 1 del camp Hamiltonià X_Ψ de Ψ. Si ψ és real analítica llavors Ψ és també real analítica. Si ψ depèn diferenciablement del paràmetres (resp analíticament) Ψ també.*

Com a corol.lari obtenim la versió amb paràmetres,

Corol.lari 6.3.3

Sigui D_p un disc centrat a l'origen 0 en els paràmetres $p_1, \ldots, p_k$. Notem per $\mathbf{p} = (p_1, \ldots, p_k)$. Suposem que $\psi_{\mathbf{p}} : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0)$ és un simplectomorfisme local de $\mathbb{R}^{2n}$ que preserva l'aplicació quadràtica $\mathbf{h}$ i que depèn diferenciablement dels paràmetres $\mathbf{p}$. Llavors existeix una única funció local diferenciable $\Psi\mathbf{p} : (\mathbb{R}^{2n}, 0) \to \mathbb{R}$ que s'anul.la a 0 i que depèn diferenciablement en els paràmentres $\mathbf{p}$ i que és una integral primera del sistema lineal definit per $\mathbf{h}$ i tal que $\psi_0^{(1)} \circ \psi_{\mathbf{p}}^{-1}$ es el flux a temps 1 del camp Hamiltonià $X_{\Psi_{\mathbf{p}}}$ de $\Psi_{\mathbf{p}}$. En cas que $\psi_{\mathbf{p}}$ sigui real analítica i depengui analíticament en els paràmetres, la funció $\Psi_{\mathbf{p}}$ també.

Aquests resultats permetem provar el següent teorema,

Teorema 6.3.4

Existeix un canvi de coordenades a $\mathbb{R}^{2n}$ que preserva el sistema $(\mathbb{R}^{2n}, \sum_{i=1}^n dx_i \wedge dy_i, \mathbf{h})$ i que linealitza l'acció de G.

També demostrem com a corol.lari la versió amb paràmetres que queda recollida al corol.lari 6.3.5.

Corol.lari 6.3.5

Si l'acció $\rho_{\mathbf{p}}$ depèn de paràmetres diferenciablement (resp. analíticament) existeix una transformació local simplèctica a $\mathbb{R}^{2n}$, $\Phi_{\mathbf{p}}$ que preserva el sistema i que verifica,

$$\Phi_{\mathbf{p}} \circ \rho_{\mathbf{p}}(h) = \rho_0(h)^{(1)} \circ \Phi_{\mathbf{p}}$$

Si notem $\mathcal{G}$ com el grup de simplectomorfismes preservant l'aplicació moment finalment provem el teorema,

Teorema 6.3.6

El grup $\mathcal{G}$ és abelià.

Com a corol.lari en deduïm que si l'acció es efectiva el grup G és abelià.

Un cop obtingut el resultat local de linealització l'òrbita provem el teorema de linealització en un entorn de l'òrbita.

Teorema 6.4.1

Sigui G un grup compacte preservant el sistema $(D^k \times \mathbb{T}^k \times D^{2(n-k)}, \sum_{i=1}^k dp_i \wedge d\theta_i + \sum_{i=1}^{n-k} dx_i \wedge dy_i, \mathbf{F})$ llavors existeix Φ_G un difeomorfisme en un entorn de l'òrbita $L = \mathbb{T}^k$ que preserva el sistema $(D^k \times \mathbb{T}^k \times D^{2(n-k)}, \sum_{i=1}^k dp_i \wedge d\theta_i + \sum_{i=1}^{n-k} dx_i \wedge dy_i, \mathbf{F})$ i que linealitza l'acció G.

També es dona com a corol.lari un resultat tipus "slice"en un entorn de l'òrbita (Corolari 6.4.2).

Si prenem com a grup G el grup de transformacions recobridores del recobriment $\widetilde{U(L)}$ considerat al capítol 1 es prova el teorema de linealització en un entorn de l'òrbita inicialment considerada que enunciem de manera abreujada com,

Teorema 6.5.1

La foliació $\mathcal{F}$ és simplècticament linealitzable en un entorn d'un òrbita com-

pacta singular.

Finalment al **capítol 7** considerem el cas de contacte i provem el resultat de linealització de contacte al recobridor. També donem la versió G-equivariant de contacte del teorema que ens dóna en particular el cas de linealització de contacte a l'entorn inicialment considerat.

Els resultats més importants d'aquest capítol són els següents:

Primer provem un resultat per a linealització diferenciable de foliacions legendrianes verificant les condicions especificades a la secció 7.3.1.

Teorema 7.3.1

Existeixen coordenades $(\theta_0, \ldots, \theta_k, p_1, \ldots, p_k, x_1, y_1, \ldots, x_{n-k}, y_{n-k})$ *en un recobriment finit d'un entorn tubular de* $\mathcal{O}$ *tal que*

- *El camp de Reeb és* $Z = \frac{\partial}{\partial \theta_0}$.
- *Existeix un triplet de nombres naturals* (k_e, k_h, k_f) *amb* $k_e + k_h + 2k_f = n - k$ *i tal que les integrals primeres* f_i *són* $f_i = p_i, \quad 1 \leq i \leq k$ *i*

$$f_{i+k} = x_i^2 + y_i^2 \ \text{ si } \ 1 \leq i \leq k_e \ ,$$
$$f_{i+k} = x_i y_i \ \text{ si } \ k_e + 1 \leq i \leq k_e + k_h \ ,$$
$$f_{i+k} = x_i y_{i+1} - x_{i+1} y_i \ \text{ i}$$
$$f_{i+k+1} = x_i y_i + x_{i+1} y_{i+1} \ \text{ si } \ i = k_e + k_h + 2j - 1, \ 1 \leq j \leq k_f$$

- *La foliació* $\mathcal{F}$ *ve descrita per les òrbites de la distribució* $\mathcal{D} =< Y_1, \ldots Y_n >$ *on* $Y_i = X_i - f_i Z$ *essent* X_i *el camp de contacte* f_i *respecte la forma de contacte estàndard* $\alpha = d\theta_0 + \sum_{i=1}^{n-k} \frac{1}{2}(x_i dy_i - y_i dx_i) + \sum_{i=1}^{k} p_i d\theta_i$.

Un cop demostrada la linealització diferenciable procedim a provar la linealització simplèctica

Teorema 7.4.1

Sigui α una forma de contacte a la varietat model M_0^{2n+1} per la qual la foliació $\mathcal{F}$ és legendriana i tal que el camp de Reeb és $\frac{\partial}{\partial \theta_0}$. Llavors existeix un difeomorfisme ϕ definit a un entorn de l'òrbita singular $\mathcal{O} = (\theta_0, \ldots, \theta_k, 0, \ldots, 0)$ preservant $\mathcal{F}'$ i enviant α a α_0.

En el cas que existeixi una acció d'un grup compacte preservant la foliació i el camp de Reeb tenim una versió G-equivariant del teorema anterior que enunciem resumidament,

Teorema 7.5.1

Existeix un contactomorfisme que linealitza la foliació $\mathcal{F}$ i l'acció del grup.

Si apliquem aquest resultat al cas que el grup sigui el grup de transformacions recobridores obtenim el Teorema 7.5.2 que assegura que la linelització de contacte és pot dur a terme a l'entorn inicial de l'òrbita. Enunciem aquest teorema a continuació,

Teorema 7.5.2

Sigui $\mathcal{F}$ una foliació verificant totes les condicions especificades a la secció 7.3.1, sigui $\mathcal{F}'$ la foliació ampliada amb el camp de Reeb Z i sigui α una forma de contacte per la qual $\mathcal{F}$ és Legendriana i tal que Z és el seu camp de Reeb llavors existeix un difeomorfisme definit en un entorn de $\mathcal{O}$ que porta $\mathcal{F}'$ a la foliació lineal, l'òrbita $\mathcal{O}$ al torus $\{x_i = 0, y_i = 0, p_i = 0\}$ i la forma de contacte a la forma de contacte de Darboux α_0.

Alguns dels resultats continguts en aquesta tesi estan continguts a les publicacions i prepublicacions que citem a continuació,

- Publicacions:

  1. C. Currás-Bosch i E. Miranda, *Symplectic linearization of singular Lagrangian foliations in M^4*, Differential Geom. Appl. **18** (2003), no. 2 , 195-205.
  2. E. Miranda, *On the symplectic classification of singular Lagrangian foliations.* Proceedings of the IX Fall Workshop on Geometry and Physics (Vilanova i la Geltrú, 2000), 239–244, Publ. R. Soc. Mat. Esp., **3**, R. Soc. Mat. Esp., Madrid, 2001.

- Prepublicacions:

  1. E. Miranda i Nguyen Tien Zung, *Equivariant normal forms for nondegenerate singular orbits of integrable Hamiltonian systems*, preprint 2003, http://xxx.arxiv.org/abs/math.SG/0302287.
  2. C. Currás-Bosch i E. Miranda, *Symplectic germs of singular Lagrangian Foliations,* preprint 265 de la Facultat de Matemàtiques. Universitat de Barcelona, 1999.

0.3 Conclusions

En aquesta tesi estudiem dos problemes de classificació per a foliacions definides a varietats simplèctiques i de contacte.

En quant al primer problema de classificació: Com s'ha detallat a la secció de resultats provem que una foliació Lagrangiana definida pels gradients simplèctics d'una aplicació moment pròpia és equivalent a la foliació linealitzada amb la forma simplèctica de Darboux en un entorn d'una òrbita compacta singular no degenerada.

En relació a aquest problema també provem un teorema de linealització simplèctica per accions simplèctiques de grups compactes que preserven l'aplicació moment.

En quant al segon problema de classificació: Provem que una foliació Legendriana completament integrable amb camp de Reeb definit per una acció de una S^1 és equivalent a la foliació linealitzada amb la forma de Darboux en un entorn d'una òrbita compacta no degenerada de la foliació ampliada amb el camp de Reeb. També provem un teorema de linealització per accions de grups compactes per contactomofismes a varietats de contacte.

Chapter 1

Differentiable linearization

1.1 Introduction

The aim of this chapter is to recall results of differentiable linearization for foliations given by a certain class of singular integrable Hamiltonian systems.

Throughout this thesis and otherwise stated all the objects considered will be $\mathcal{C}^\infty$.

In this chapter and throughout the thesis, we will consider germ-like foliations in a symplectic manifold (M^{2n}, ω) defined by n first integrals in a neighbourhood of a compact submanifold L. Since we are considering germs-like objects, the foliation is defined in a neighbourhood $U(L)$ of L. We denote by $f_1, \dots f_n$ the n-first integrals. The leaves of the foliation are $L_{(c_1,\dots,c_n)} = \{p \in U(L), \quad f_1(p) = c_1, \dots f_n(p) = c_n\}$. We denote by F the function $F = (f_1, \dots, f_n)$. We will require the following condition on the functions f_i. We will assume that the functions f_i are in involution with respect to the Poisson bracket associated to ω in the neighbourhood considered. That is to say, $\{f_i, f_j\} = 0$ for any pair i, j. When this condition is fulfilled we say that F defines a completely integrable Hamiltonian system on $U(L)$ and the mapping F is called the moment map. Namely,

Definition 1.1.1 *A completely integrable Hamiltonian system on a symplectic manifold* (M^{2n}, ω) *is a* C^∞ *Poisson* $\mathbb{R}^n$*-action, generated by a moment map* $F : M^{2n} \longrightarrow \mathbb{R}^n$.

The foliation defined by the level sets of F can also be considered as a pair $(\chi, \mathcal{A})$, where χ is an n-dimensional commutative Lie algebra of vector fields and $\mathcal{A}$ is a vector space of first integrals of the vector fields of χ. This presentation of the foliation is specially interesting when the foliation has singularities. Then the foliation obtained from χ is called the <u>singular Lagrangian foliation</u>.

The compact submanifold L that we will consider is an orbit of χ through a singular point. Let us introduce the definition of singular point,

Definition 1.1.2 *A point* $x_0 \in M^{2n}$ *is a singular point of the integrable Hamiltonian systems if the rank of* $d_{x_0}F = (d_{x_0}f_1, \ldots, d_{x_0}f_n)$ *is less than n.*

Remark:

Since L is an orbit of χ, all the points in L are singular points for F and L is contained in a singular leaf of the foliation. On the other hand, observe that singular orbits do not necessary coincide with singular leaves of the foliation as the following example shows. Consider $M = \mathbb{R}^2$ endowed with coordinates (x, y). Let $\mathcal{F} =< x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x} >$. A first integral for $\mathcal{F}$ is $f = x^2 - y^2$. The only singular point is $(0, 0)$. The orbit through this point is just the point, but the leaf containing the singular point is $L = \{(x, y), \quad x^2 - y^2 = 0\}$ which consists of a pair of lines through the origin.

When we talk about <u>differentiable linearization</u> in a neighbourhood of L, we mean that there exists a diffeomorphism in a neighbourhood of L, fixing L and taking the given foliation to a simpler foliation defined by a linear model. Linearization is not always possible. We will need additional assumptions on the functions

f_i. Namely, the completely integrable Hamiltonian system considered fulfills the following three hypotheses:

1. The global moment map $F : \mathbb{M}^{2n} \longrightarrow \mathbb{R}^n$ is a proper map.
2. The singular orbits L of minimal rank are tori.
3. The singularities considered are of "non-degenerate" type.

Remarks

- As a consequence of Liouville-Mineur-Arnold theorem, if $\mathcal{F}$ is a regular foliation given by a completely integrable system and L is an n-dimensional compact leaf. Then this leaf is a torus and the foliation in a neighbourhood of L is a foliation by n-dimensional tori. Those tori are Lagrangian for the symplectic form considered.

  Now assume that the foliation is allowed to have singularities. The first example that comes to our mind is to create a singular foliation by collapsing some of the cycles of the regular tori L and leaving the rest of the foliation invariant. In this way, the resulting foliation will be a foliation by regular tori except for the singular one $\overline{L}$ which will be a torus with dimension $r < n$. When all the cycles of the initial torus L are collapsed we obtain an isolated singular leaf whose dimension has been decreased to 0, that is, a point.

  From a symplectic point of view, this torus is an isotropic submanifold, that is to say it preserves all the properties of Lagrangianity except for the maximal dimension.

  We will take this example as a starting point. The foliation that we will consider has a torus as an isolated singular leaf but the neighbouring orbits are not always tori. The example described above corresponds to the picture of a "completely elliptic" singularity of corank $n - r$. As we will see this example is one of the "differentiable models" for our foliation. In fact we will see that our foliation is differentiably equivalent in a finite covering of a neighbourhood of the singular leaf to a direct product type foliation of a regular Lagrangian foliation by tori with a singular foliation which is also a direct type foliation of k_e components of elliptic type, k_h components of hyperbolic type and k_f components of focus-focus type.

- The third condition (non-degeneracy) is a condition on the quadratic parts of the components of the moment map. Its role in the linearization process is similar to that of non-degeneracy for Morse-like theorem for single functions. In fact the differentiable linearization that we prove is a kind of "multiple Morse" theorem. That is, we can find a diffeomorphism, in a finite covering of the initial neighbourhood considered, such that the foliation determined by the moment map can be taken to the foliation determined by the quadratic parts of the components of the moment map. This is the main difference with the result of Morse for non-degenerate singularities of differentiable functions. The involution of the components of the moment map make this simultaneous linearization possible.

In this chapter, the symplectic properties of the foliation will be temporarily left aside and our attention will be focused on the differentiable side of the story. In any case we will need some facts from symplectic geometry which we introduce in the section called "Preliminaries" of this chapter.

The linearization will be carried out in a neighbourhood of the singular orbit.

1.2 Preliminaries.

Let us recall some notations and definitions:

1.2.1 Hamiltonian vector fields and the Poisson bracket

Let us start with the definition of symplectic manifold.

Definition 1.2.1 *A symplectic manifold is a pair* (M, ω) *where* M *is a differentiable manifold and* ω *is a closed non-degenerate* 2*-form.*

Remarks:

- As a consequence of the definition all symplectic manifolds are even dimensional.
- In contrast to Riemannian manifolds, symplectic manifolds have no local invariants. This is due to the Theorem of Darboux which establishes the uniqueness of a local model.

Theorem 1.2.1 *Let* (M, ω) *be a* $2n$*-dimensional symplectic manifold and let* p *be a point in* M *then there exists local coordinates* $(x_1, y_1, \dots, x_n, y_n)$ *in a neighbourhood* U *of* p *such that,*

$$\omega_{|U} = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$$

In the spirit of this theorem the main goal of this thesis is to establish models for symplectic manifolds with Lagrangian foliations in a neighbourhood of a singular orbit of the foliation.

Let (M, ω) be a symplectic manifold. Consider the set of differentiable functions on M, $\mathcal{C} = \mathcal{C}^\infty(M)$.

Let us introduce the notion of Hamiltonian vector field associated to a function $f \in \mathcal{C}$ and the notion of Poisson bracket associated to a pair of functions f and g contained in $\mathcal{C}$.

Definition 1.2.2 *Let $f \in \mathcal{C}$, we define the Hamiltonian vector field associated to f as the unique vector field X_f satisfying,*

$$i_{X_f}\omega = -df.$$

Definition 1.2.3 *Let $f, g \in \mathcal{C}$ we define the Poisson bracket of f and g as*

$$\{f, g\} = \omega(X_f, X_g).$$

Remarks:

- The following formula can be derived from the definition of Poisson bracket,

$$X_{\{f,g\}} = [X_f, X_g]$$

- Take $M = \mathbb{R}^{2n}$ endowed with coordinates $(x_1, y_1, \ldots, x_n, y_n)$ and let ω be the Darboux symplectic form $\omega = \sum_i dx_i \wedge dy_i$. The standard Poisson bracket is the one associated to ω. Given two functions $f, g \in \mathcal{C}^\infty(\mathbb{R}^{2n})$, the standard Poisson bracket $\{f, g\}$ equals

$$\sum_{i=1}^{n}(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i}\frac{\partial g}{\partial x_i}).$$

  The pair $(\mathcal{C}^\infty(\mathbb{R}^{2n}), \{.,.\})$ is a Lie algebra.

  Now consider $Q(2n, \mathbb{R})$ the set of quadratic forms in the variables $x_1, y_1, \ldots, x_n, y_n$ then the standard Poisson bracket of two quadratic forms is again a quadratic form. Therefore the pair $(Q(2n, \mathbb{R}), \{.,.\})$ is a Lie subalgebra of $(\mathcal{C}^\infty(\mathbb{R}^{2n}), \{.,.\})$.

1.2.2 Completely integrable systems and regular Lagrangian foliations

Recall that a system is completely integrable if it is defined by n first integrals in involution with respect to the Poisson bracket. The following proposition relates completely integrable Hamiltonian systems to Lagrangian foliations,

Proposition 1.2.2 *Let $f_1, \dots, f_n$ be n functions such that $\{f_i, f_j\} = 0, \forall i, j$. Assume that $p \in M$ is a point for which $d_p f_1 \wedge \dots \wedge d_p f_n \neq 0$. Then the distribution generated by the Hamiltonian vector fields $\mathcal{D} =< X_{f_1}, \dots, X_{f_n} >$ is involutive and the leaf through p is a Lagrangian submanifold.*

Proof:

Since $[X_{f_i}, X_{f_j}] = X_{\{f_i, f_j\}}$, the condition $\{f_i, f_j\} = 0$ implies $[X_{f_i}, X_{f_j}] = 0$, $\forall i, j$ and the distribution is involutive. On the other hand, from the definition of Poisson bracket $\{f_i, f_j\} = \omega(X_{f_i}, X_{f_j})$. So the foliation defined by $\mathcal{D}$ is isotropic. The condition $d_p f_1 \wedge \dots \wedge d_p f_n \neq 0$ implies that the Hamiltonian vector fields X_{f_i} span an n-dimensional vector space at the point p. Therefore the leaf through p is Lagrangian.

□

Remark

From the definition of Hamiltonian vector fields $i_{X_{f_i}} \omega = -df_i$ and since $\omega(X_{f_i}, X_{f_j}) = 0$ for any pair of vector fields tangent to the Lagrangian foliation then $X_{f_i}(f_j) = 0, \forall i, j$. Those conditions imply that the functions f_i are first integrals for the foliation defined by the distribution $\mathcal{D}$.

1.2.3 Orbit versus leaf

In this thesis we will deal with problems of equivalence for symplectic structures in the neighbourhood of an orbit of a foliation $\mathcal{F}$.

Observe that for foliations given by a completely integrable systems there are two ways of describing the foliation: the set of orbits and the set of levelsets of the moment map F.

An orbit of the foliation is the orbit of the distribution X_{F_i}, where F_i is the ith component of the moment map.

A leaf of the foliation is a levelset of the moment map F.

1.2.4 Transversal linearization at a singular point

Let x_0 be as singular point of the foliation defined by F, we start by defining the rank and corank of a singular point.

Definition 1.2.4 *Let $x_0 \in M^{2n}$ be a singular point of the integrable Hamiltonian system we say that the rank of x_0 is k if the rank of the moment map at x_0 is k, that is to say if* $\text{rank } d_{x_0}F = \text{rank } (d_{x_0}f_1, \ldots, d_{x_0}f_n) = k$.

We say that a singular point of rank k has corank $n-k$.

Recall that the foliation can be thought as a pair $(\chi, \mathcal{A})$, where χ is an n-dimensional commutative Lie algebra of vector fields and $\mathcal{A}$ is a vector space of first integrals of the vector fields of χ. The foliation obtained from χ is called the singular Lagrangian foliation.

We follow Nguyen Tien Zung [61] for the definitions concerning the notion of transversal linearization at a singular point.

Let x_0 be a singular point and let χ_{x_0} be the subspace of $T_{x_0}M$ generated by $X_{x_0}, \forall X \in \chi$. Let $K_{x_0} = \cap_{f\in\mathcal{A}} Ker d_{x_0} f$, and let $\mathcal{B}_{x_0}$ the set of $f \in \mathcal{A}$ such that $d_{x_0}f = 0$. Then $\forall f \in \mathcal{B}_{x_0}$ the 2-order jet of $f - f(x_0)$ gives a quadratic form on K_{x_0}, such that its kernel contains χ_{x_0}, so it gives a quadratic form $f_{x_0}^{\prime T}$ on K_{x_0}/χ_{x_0}, the set of quadratic forms obtained in this way which we denote by $\mathcal{A}_{x_0}^{\prime T}$, is a commutative subalgebra under the Poisson bracket, which is often called the transversal linearization of F.

Notice that K_{x_0}/χ_{x_0} carries a natural symplectic structure $\overline{\omega}_{x_0}$, and it is symplectomorphic to a subspace $R_{x_0} \subset T_{x_0}M^{2n}$.

We are going to introduce the notion of nondegenerate point but first we need to recall the definition of Cartan subalgebra.

Definition 1.2.5 *A Cartan subalgebra is a maximal self-centralizing abelian subalgebra.*

Definition 1.2.6 *A singular point of corank k is called non-degenerate if $A'^{T}_{x_0}$ is a Cartan subalgebra of the algebra of quadratic forms on R_{x_0}.*

1.2.5 The linear model

Let us recall the following classical result of Williamson [60], which will be the starting point for the linearization.

Theorem 1.2.3 (Williamson)

For any Cartan subalgebra $\mathcal{C}$ of $Q(2n,\mathbb{R})$ there is a symplectic system of coordinates $(x_1,\dots,x_n,y_1,\dots,y_n)$ in $\mathbb{R}^{2n}$ and a basis $f_1,\dots,f_n$ of $\mathcal{C}$ such that each f_i is one of the following:

$$
\begin{aligned}
&f_i = x_i^2 + y_i^2 \qquad \text{for} \quad 1 \le i \le k_e\ , && \text{(elliptic)}\\
&f_i = x_i y_i \qquad \text{for} \quad k_e + 1 \le i \le k_e + k_h\ , && \text{(hyperbolic)}\\
&\begin{cases} f_i = x_i y_{i+1} - x_{i+1} y_i, & \text{(focus-focus pair)}\\ f_{i+1} = x_i y_i + x_{i+1} y_{i+1} & \text{for} \quad i = k_e + k_h + 2j - 1,\ 1 \le j \le k_f \end{cases}
\end{aligned}
\tag{1.2.1}
$$

This result ensures the existence of a transversal linear model on K_{x_0}/χ_{x_0}. This basis is often called Williamson basis.

The Williamson type of an orbit

In order to prove the existence of a linear model in a whole neighbourhood one must consider in general a finite normal covering of the initial neighbourhood. First let us introduce the notion of Williamson type of an orbit of the integrable system.

Observe that because of theorem 1.2.3 the triple (k_e, k_h, k_f) at p is an invariant of the point.

This triple is called the Williamson type of the singular point p. As it has been shown by Nguyen Tien Zung in [61], this triple is also an invariant of the orbit. That is why it is also called the Williamson type of the orbit L.

A Hamiltonian free action of $\mathbb{T}^k$ in a covering

In order to introduce the linear model we need to recall a result of Nguyen Tien Zung [61] which ensures the existence of a locally free action of $\mathbb{T}^k$ (k is the dimension of the orbit) in a neighbourhood of the orbit which preserves the foliation. There exists a normal finite covering of a neighbourhood of the orbit such that this action can be lifted to a free action in the covering.

In ([61]) Nguyen Tien Zung proves the following results concerning the existence of Hamiltonian actions of tori in a neighbourhood of a singular leaf of a Hamiltonian system.

Let N be a singular leaf (not a singular orbit of the foliation). In [61] the pair $(U(N), \mathcal{F})$ stands for a foliated neighbourhood of a singular leaf ($\mathcal{F}$ is the singular Lagrangian foliation).

Theorem 1.2.4 (Nguyen Tien Zung)

Let $(U(N), \mathcal{F})$ be a nondegenerate singularity of Williamson type (k_e, k_h, k_f) and of corank $n - k = k_e + k_h + 2k_f$ of an integrable system with n degrees of freedom. Then there exists an effective Hamiltonian action of a torus $\mathbb{T}^{k+k_e+k_f}$ in $U(N)$ which preserves the moment map. This action is unique up to automorphisms of the torus.

As observed in [61] in order for this action to be free one must consider a finite covering of $(U(N), \mathcal{L})$ and choose a subtorus $\mathbb{T}^k$ of $\mathbb{T}^{k+k_e+k_f}$. Then the following theorem is proved in [61],

Theorem 1.2.5 (Nguyen Tien Zung)

Let $(U(N), \mathcal{F})$ be a strongly nondegenerate singularity of corank $n-k = k_e + k_h + 2k_f$ of an integrable system with n degrees of freedom. Then there exists a normal finite covering $(\widetilde{U(N)}, \widetilde{\mathcal{F}})$ of $(U(N), \mathcal{F})$ and a free Hamiltonian action of the torus $\mathbb{T}^k$ in the covering $(\widetilde{U(N)}, \widetilde{\mathcal{F}})$ which preserves the moment map.

Remark:

In [61] a nondegenerate singularity $(U(N), \mathcal{F})$ of an integrable system is called strongly nondegenerate singularity if the set of singular values of the moment map when restricted to $U(N)$ coincides with the set of singular values of a singular point of maximal corank in N.

In the case N coincides with an orbit this condition is automatically satisfied.

Therefore we may apply this result to a neighbourhood of a nondegenerate orbit of an integrable Hamiltonian system. Namely, since the dimension of the orbit equals k, the isotropy group of the action is a finite abelian group so there exists a finite covering $\widetilde{U(L)}$ of the neighbourhood of the orbit such that the foliation, the symplectic form and the action of $\mathbb{T}^k$ can be lifted to $\widetilde{U(L)}$.

And if L is an orbit of an integrable Hamiltonian system we may restate the theorem above as,

Theorem 1.2.6 *Let $U(L)$ be a neighbourhood of a nondegenerate singular orbit of an integrable system with n degrees of freedom. Assume the corank of the orbit is $n-k = k_e + k_h + 2k_f$. Let $\mathcal{F}$ be the singular Lagrangian foliation defined by the integrable system. Then there exists a normal finite covering $\widetilde{U(L)}$ of $U(L)$ such that the foliation can be lifted to $\widetilde{\mathcal{F}}$ and a free Hamiltonian action of the torus $\mathbb{T}^k$ in the covering $\widetilde{U(L)}$ which preserves the moment map.*

Now we can introduce the linear model associated to the orbit L. Later, we will see that the invariants associated to the linear model are the Williamson type of the orbit and a twisting group Γ attached to it.

First we introduce the linear model in the covering,

The linear model in the covering

Denote by $(p_1,...,p_k)$ a linear coordinate system of a small ball D^k of dimension k, $(\theta_1,\ldots,\theta_k)$ is a standard periodic coordinate system of the torus $\mathbb{T}^k$, and $(x_1,y_1,...,x_{n-k},y_{n-k})$ a linear coordinate system of a small ball $D^{2(n-k)}$ of dimension $2(n-k)$. Now we consider the manifold

$$V = D^k \times \mathbb{T}^k \times D^{2(n-k)} \tag{1.2.2}$$

with the standard symplectic form $\sum dp_i \wedge d\theta_i + \sum dx_j \wedge dy_j$, and the following moment map:

$$F = (p_1,...,p_k,f_1,...,f_{n-k}) : V \to \mathbb{R}^n \tag{1.2.3}$$

where

$$\begin{aligned} &f_i = x_i^2 + y_i^2 \ \text{ for } \ 1 \le i \le k_e \ , \\ &f_i = x_i y_i \ \text{ for } \ k_e + 1 \le i \le k_e + k_h \ , \\ &f_i = x_i y_{i+1} - x_{i+1} y_i \ \text{ and} \\ &f_{i+1} = x_i y_i + x_{i+1} y_{i+1} \ \text{ for } \ i = k_e + k_h + 2j - 1, \ 1 \le j \le k_f \end{aligned} \tag{1.2.4}$$

The linearized foliation in the covering is the foliation determined by the above moment map. This presentation of the foliation would be the one of $\mathcal{A}$, that is, the above components of the moment map are the first integrals of the system. We can also look for generators of χ to define the linearized foliation in the covering. After performing a linear change of coordinates in such a way that the hyperbolic functions can be written as $f_i = x_i^2 - y_i^2$, the following vector fields form a basis of χ,

$$\begin{aligned} &Y_i = \tfrac{\partial}{\partial \theta_i} \text{ for } \ 1 \le i \le k \ , \\ &X_i = -y_i \tfrac{\partial}{\partial x_i} + x_i \tfrac{\partial}{\partial y_i} \ \text{ for } \ 1 \le i \le k_e \ , \\ &X_i = y_i \tfrac{\partial}{\partial x_i} + x_i \tfrac{\partial}{\partial y_i} \ \text{ for } \ k_e + 1 \le i \le k_e + k_h \ , \\ &X_i = x_i \tfrac{\partial}{\partial x_{i+1}} - y_{i+1} \tfrac{\partial}{\partial y_i} - x_{i+1} \tfrac{\partial}{\partial x_i} + y_i \tfrac{\partial}{\partial y_{i+1}} \ \text{ and} \\ &X_{i+1} = -x_i \tfrac{\partial}{\partial x_i} + y_i \tfrac{\partial}{\partial y_i} - x_{i+1} \tfrac{\partial}{\partial x_{i+1}} + y_{i+1} \tfrac{\partial}{\partial y_{i+1}} \ \text{ for } \ i = k_e + k_h + 2j - 1, \ 1 \le j \le k_f \end{aligned}$$

As a matter of notation, when we talk about rank r foliations or corank $n-r$ foliations, we mean that k=r and that the foliation is defined in a neighbourhood of an r-dimensional torus.

When we refer to completely elliptic foliations we mean that the Williamson type of the orbit L is $(n-k,0,0)$. We denote by completely hyperbolic foliations in any corank those foliations for which the Williamson type of the orbit L is $(0,n-k,0)$. The linear model in the neighbourhood will be determined by the following two data: The linear model in the covering and a twisting group attached to the isotropy group of the Hamiltonian $\mathbb{T}^k$-action along the singular orbit L. In any case the role of the twisting group in the linearization process will be clarified when we prove the equivariant version of the symplectic linearization. The linearized foliation in the initial neighbourhood considered $U(L)$ is the linearized foliation in the covering quotiented by the action of the twisting group.

1.2.6 The parametrized Morse lemma

In this section we are going to recall the parametrized Morse lemma. The Morse lemma without parameters establishes the existence of a diffeomorphism in a neighbourhood of a nondegenerate singular point of a smooth function which takes the given function to its quadratic part. If the function depends on parameters then the above diffeomorphism also exists and depends smoothly on the parameters. We include here the proof of the Morse lemma without parameters provided by Richard S. Palais in [51], the Morse lemma with parameters will be a consequence of it. So let us recall the content and proof of the Morse lemma. But before let us outline the following: Palais also proved the theorem for Banach spaces; we will stick to the differentiable case.

Theorem 1.2.7 *Let f be a smooth function defined in convex neighbourhood W of the origin in a finite dimensional vector space V. Let* 0 *stand for the origin of the vector space. Suppose that $f(0)=0$, $d_0 f=0$ and that $\frac{1}{2}d_0^2 f$ is a nonsingular* *quadratic form Q. Then there exists a neighbourhood U of the origin and a diffeomorphism $\phi : U \longrightarrow W$ with $\phi(0) = 0$ and $d_0\phi = Id$ such that for $x \in U$, $f(\phi(x)) = Q(x,x)$.*

Proof: The proof uses the path method. Put $f = f_1$ and define $f_0(x) = Q(x,x)$. We define the path $f_t = f_0 + t(f_1 - f_0)$ for $t \in [0,1]$. Notice that $\dot{f}_t = \frac{df_t}{dt} = f_1 - f_0$. We look for a one parameter family of diffeomorphisms ϕ_t such that $f_t \circ \phi_t = f_0$ and satisfying the condition $\phi_0 = Id$. Once the one parameter family is found, the diffeomorphism ϕ that we are looking for will be ϕ_1. Now we introduce the t-parametric vector field associated to the family ϕ_t,

$$X_{t_0}(\phi_{t_0}(x)) = \frac{d}{dt}(\phi_t(x))_{|t=t_0}, \tag{1.2.5}$$

From this expression the following relation is obtained (lemma 1 in [51]),

$$\frac{d}{dt}(f_t \circ \phi_t(x)) = (\dot{f}_t + X_t(f_t)) \circ \phi_t(x)$$

From this equation in particular if $X_t(f_t) = -\dot{f}_t$, $\forall t \in [0,1]$ then $f_t \circ \phi_t = f_0$. So going back to the problem posed at the beginning, it is enough to find a dependent vector field X_t such that

$$X_t(f_t) = -(f_1 - f_0). \tag{1.2.6}$$

Given a smooth mapping g we will denote by $d_x g$ the differential of g at the point x. In order to see which is the convenient vector field X_t observe that since $d_0 f_t = 0$, we can write

$$d_x f_t(v) = \int_0^1 \frac{d}{ds} d_{sx} f_t(v) ds = \int_0^1 d^2_{sx} f_t(x,v) ds$$

And this last term is $B^t_x(x,v)$ where,

$$B^t_x(u,v) = \int_0^1 d^2_{sx} f_t(u,v) ds$$

Observe that $B^t_x = B^0_x + t(B^1_x - B^0_x)$. On the other hand, since $B^t_0 = 2Q$ and Q is a nonsingular quadratic form for all t then B^t_x is nonsingular $\forall x$ in a neighbourhood of 0 and $\forall t \in [0,1]$. Now equation 1.2.6 can be rewritten as,

$$B_x^t(x, X_t) = f_0 - f_1,$$

so if we could identify $g = f_0 - f_1$ with a quadratic form then the equation 1.2.6 has a well-defined solution. We make similar computations to the ones we did with $d_x f_t$,

$$f_0 - f_1 = \int_0^1 \frac{d}{ds} g(sx) ds = \int_0^1 d_{sx} g(x) ds = \int_0^1 \int_0^1 d^2_{rsx} g(sx, x) dr ds$$

This last expression equals the quadratic form $C_x(x,x)$ being C_x the bilinear form,

$$C_x(u, v) = \int_0^1 \int_0^1 d^2_{rsx} g(su, v) dr ds$$

Finally, the X_t satisfying the equation is the unique solution of the equation,

$$B_x^t(u, X_t) = C_x(u, x), \quad \forall u \in V.$$

Now the diffeomorphism ϕ_t defined by the equation 1.2.5 is such that ϕ_1 is a solution. Observe also that because ϕ_0 is the identity mapping then $d_0\phi_0$ is also the identity mapping and since $d_0 X_t = 0$ then $d_0\phi_t$ is the identity mapping $\forall t$ in particular for $t = 1$, as desired. This ends the proof of the theorem. □

Remark: As observed by Guillemin and Sternberg in [30], the proof provided by Palais allows to claim that if f depends smoothly on parameters then the diffeomorphism obtained ϕ will depend smoothly on the same parameters because the vector field X_t depends differentiably on them (all the operations performed to find X_t are differentiable). This is the content of the so-called parametrized Morse Lemma.

1.2.7 Our notion of equivalent symplectic germs

We say that a foliation is generically Lagrangian if its regular leaves are Lagrangian submanifolds and its singular leaves are isotropic. Let L be an orbit of the foliation.

Let us introduce the notion of equivalence for the singular Lagrangian foliation in a neighbourhood of the orbit that we will use in the sequel.

Definition 1.2.7 *Let (U_1, ω_1) and (U_2, ω_2) be two symplectic germs such that $\mathcal{F}$ is generically Lagrangian for both ω_1 and ω_2. We say that ω_1 and ω_2 are equivalent if there exists a diffeomorphism $\phi : U_1 \longrightarrow U_2$ such that:*

1. *$\phi^*(\omega_2) = \omega_1$.*
2. *ϕ preserves $\mathcal{F}$.*
3. *ϕ fixes L.*

Notation 1.2.8 *We write $\omega_1 \sim_{\mathcal{F}} \omega_2$ to denote equivalent symplectic germs.*

1.3 Differentiable equivalence in a finite normal covering

In this section we recall the result of linearization in a finite normal covering. That is, we will see that there exists a finite covering in which the foliation can be linearized in a neighbourhood of a singular orbit for the foliation. This result was proved by Eliasson. We give our own proof in the corank 1 case which uses Morse methods to linearize in the covering. The objective of this section is to provide the following main result,

Theorem 1.3.1 *In $\widetilde{U(L)}$ the singular Lagrangian foliation is diffeomorphic to the linearized one.*

Remarks:

- The linearization result for analytical systems was proved by Vey.
- The differentiable linearization for maximal corank singularities, that is when L is reduced to a point, was proved by Eliasson in [23].

- The differentiable linearization for any corank was proved by Eliasson in [24] and [23]. Another proof in the completely elliptic case for any corank was provided by Dufour and Molino in [21].

Before proceeding to the proof of the theorem for rank $n-1$ foliations, we recall the result for rank 0 foliations which was proved by Eliasson. We would like to remark that Eliasson's theorem of linearization is also valid for a set of functions $h_1, \ldots, h_k$ in involution with respect to the Poisson bracket attached to the corresponding symplectic form with $k \leq n$. In any case, we state here the version for n commuting functions. In the statement of the theorem, the set $q_1, \ldots, q_n$ stands for a Williamson basis of the Cartan subalgebra attached to the singularity as guaranteed by theorem 1.2.3 and $\{,\}_0$ stands for the standard Poisson bracket.

Theorem 1.3.2 (Eliasson) *Let (M, ω) be a symplectic manifold and let $h_1, \ldots, h_n$ be a set of functions in involution with respect to the Poisson bracket $\{,\}$ attached to ω. Assume p is a non-degenerate critical point of rank 0. Then there exists a local chart $\phi : T_0M \longrightarrow M$ such that $d\phi(0) = Id$ and such that $\{h_j \circ \phi, q_i\}_0 = 0$ for all i, j. If there are no hyperbolic elements among the q_i then there exists germs of smooth functions $\psi_1, \ldots, \psi_n$ such that, $h_j \circ \phi = \psi_j(q_1, \ldots, q_n)$.*

Remark:

Although the exception made for the hyperbolic components in the Cartan subalgebra in the statement of the theorem above, the condition $\{h_j \circ \phi, q_i\}_0 = 0$ is enough to guarantee that the foliation is linearizable. This is due to the fact that the condition $\{h_j \circ \phi, q_i\}_0 = 0$ is equivalent to the condition $X_i(h_j \circ \phi) = 0$, $\forall i, j$ and this, in turn, implies that the foliation is generated, in the new coordinates provided by ϕ, by the vector fields of the linearized foliation.

Finally we proceed to prove the linearization theorem for rank $n-1$ foliations in a $2n$-dimensional manifold,

Proof of 1.3.1 in the corank 1 case:

In $\widetilde{U}(L)$, the algebra $\mathcal{A}$ is generated by n functions $g_1, \dots, g_{n-1}, f$. Let H_{g_i}, $1 \le i \le n-1$ be the infinitesimal generators of the Hamiltonian free $\mathbb{T}^{n-1}$ -action. Recall that L is diffeomorphic to a torus $\mathbb{T}^{n-1}$. Then one can take coordinates

$$(\theta_1, \dots, \theta_{n-1}, p_1, \dots, p_{n-1}, x, y)$$

in $\widetilde{U}(L)$ such that,

$H_{g_i} = \frac{\partial}{\partial \theta_i}$ and $g_i = p_i$ for $1 \le i \le n-1$.

Let f be a singular first integral. Since $\{f, g_i\} = 0$, in particular we obtain $\frac{\partial f}{\partial \theta_i} = 0$ and the function f does not depend on θ_i for any i. Let p be a point in L, since p is nondegenerate we may apply the result of Williamson and there exists coordinates on the vector space K_{x_0}/χ_{x_0} such that $\overline{f}$ is one of the following:

- If the Williamson type of the orbit is $(1,0,0)$ then $\overline{f} = x^2 + y^2$.
- If the Williamson type of the orbit is $(0,1,0)$ then $\overline{f} = x^2 - y^2$.

Let us set the following simplifying notation which we will use throughout the proof, $\theta = (\theta_1, \dots, \theta_{n-1})$ and $p = (p_1, \dots, p_{n-1})$. The notation $D_\epsilon(p)$ stands for a disk of radius ϵ in the coordinates $p_1, \dots, p_{n-1}$ centered at the origin.

In order to prove the theorem we need the following lemma:

Lemma 1.3.3 *Let $N \subset \widetilde{U}(L)$ be defined as*

$$N = \{(\theta, p, x, y) \in \widetilde{U}(L) \quad | \quad \frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0\}.$$

Then, under the non-degeneracy assumptions, there exist functions $h_1 : \mathbb{T}^{n-1} \times D_\epsilon(p) \longrightarrow \mathbb{R}$ and $h_2 : \mathbb{T}^{n-1} \times D_\epsilon(p) \longrightarrow \mathbb{R}$ and a tubular neighbourhood $W(L)$ of L, $W(L) \subset \widetilde{U}(L)$ such that

$$N \cap W(L) = \{(\theta, p, x, y) \in W(L) \quad | \quad x = h_1(\theta, p), y = h_2(\theta, p)\}.$$

Proof:

Let $\hat{N}$ be the set $\{(\theta, p, x, y) \in \widetilde{U}(L) \quad | \quad x = y = 0\}$.

There exists a neighbourhood $V(L)$ of L such that the differential of the mapping

$$\begin{array}{rccc} H: & V(L) & \longrightarrow & \hat{N} \times \mathbb{R}^2 \\ & (\theta, p, x, y) & \longrightarrow & (\theta, p, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) \end{array}$$

is non-singular along L. So there is an open neighbourhood $W(L)$ of L in $V(L)$, such that $H_{|W(L)}$ is a diffeomorphism.

We use this diffeomorphism to define h_1 and h_2.

Notice that $H(N) = \hat{N} \times (0,0)$.

Finally defining $h_1(\theta, p) = \pi_3 \circ H^{-1}(\theta, p, 0, 0)$ and $h_2(\theta, p) = \pi_4 \circ H^{-1}(\theta, p, 0, 0)$ (where π_3 and π_4 stand for the projections on the x-axis and y-axis respectively) we have

$$N \cap W(L) = \{(\theta, p, x, y) \in W(L) \quad | \quad x = h_1(\theta, p), y = h_2(\theta, p)\}.$$

And this concludes the proof of the lemma.

□

The proof of the theorem continues as follows:

The following diffeomorphism

$$\begin{array}{rccc} G: & W(L) & \longrightarrow & G(W(L)) \\ & (\theta, p, x, y) & \longrightarrow & (\theta, p, x - h_1, y - h_2) \end{array}$$

takes $N \cap W(L)$ to $\{(\theta, p, 0, 0) \in G(W(L))\}$. Let (θ, p, x_1, y_1) stand for coordinates on $G(W(L))$.

After applying the parametrized Morse lemma we get coordinates $(\theta, p, \bar{x}, \bar{y})$ in a neighbourhood $T(L)$ of L, $T(L) \subset W(L)$, such that

$$f = \overline{x}^2 + \epsilon \overline{y}^2.$$

where $\epsilon = 1$ if f is elliptic and $\epsilon = -1$ in the hyperbolic case. So, the singular Lagrangian foliation becomes differentiably equivalent to the one described by the orbits of the distribution generated by Y_i and X being $Y_i = \frac{\partial}{\partial \theta_i}$, $1 \leq i \leq n-1$ and $X = -\epsilon \overline{y} \frac{\partial}{\partial \overline{x}} + \overline{x} \frac{\partial}{\partial \overline{y}}$, where $\epsilon = 1$ if f is elliptic and $\epsilon = -1$ in the hyperbolic case. This ends the proof of the theorem in the corank 1 case.

□

Chapter 2

Analytic tools and symplectic linearization in dimension 2

2.1 Introduction

In this chapter we will prove some results which will play an important role in the symplectic linearization process in any dimension concluding also the symplectic linearization in dimension 2.

The chapter is organized as follows: In the first section we prove some results concerning two special decomposition for functions. The kind of tools are those of analysis. Some of this results have already been proved by Eliasson [23] and Colin de Verdière and Vey in dimension 2 [6]. In any case, we extend those results to any dimension. This generalization will be needed in the chapters that follow.

In the last section we prove a *symplectic linearization* result for singular Lagrangian foliations fulfilling the hypotheses posed in the first chapter in dimension 2.

When we talk about symplectic linearization we mean the following: We consider a foliation given by a completely integrable system with singularities of non-degenerate type. In the first chapter we saw that those foliations are differentiably linearizable in a finite covering of a neighbourhood of an orbit of the distribution generated by the Hamiltonian vector fields. We consider the linearized foliation in the covering and we pose the following problem. Given a symplectic form for which the foliation is Lagrangian, does there exist a diffeomorphism in the covering taking the given symplectic form to the Darboux symplectic form and preserving the foliation?

When the answer to the question is affirmative we say that the foliation is symplectically linearizable in the covering. To attain the symplectic linearization in the initial manifold we will need to talk about an equivariant linearization result. This will be done in a further chapter.

The aim of the second section is to give an affirmative answer to that matter in dimension 2.

2.2 Two special decompositions for functions

Let g be a smooth function if X is a smooth vector field on a manifold M and $p \in M$ such that $X(p) \neq 0$, then it is a well-known result that g admits a local smooth decomposition of the following type:

$$g = g_1 + X(g_2) \quad , \quad X(g_1) = 0 \tag{2.2.1}$$

In order to do that just take local coordinates $(x_1, \ldots, x_n)$ centered at a point p such that $X = \frac{\partial}{\partial x_1}$ and apply the classical integration trick. That is, if we consider the smooth function $g_1(x_1, \ldots, x_n) = g(0, x_2, \ldots, x_n)$ and the smooth function

$$g_2 = \int_0^1 g(tx_1, \ldots, x_n)dt$$

we obtain the desired decomposition 2.2.1.

Now the question arises: Can we obtain similar local decomposition for singular vector fields?

In this section we are going to prove that similar decompositions can be obtained for the following vector fields $X = x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}$ or $Y = -x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}$.

This special decomposition for functions are going to become a key point in the proof of the local uniqueness theorem for the elliptic-elliptic, elliptic-hyperbolic and hyperbolic-hyperbolic cases.

Let us state and proof the following propositions. The first proposition is proved by Eliasson in [24] and [23] in any dimension whereas a proof for the second proposition is proved by Eliasson when $n = 2$ in [23]. Let us point out that when the manifold is $M = \mathbb{R}^2$ a proof of this decomposition had been formerly given by Guillemin and Schaeffer [28] and by Colin de Verdière and Vey [6]. This generalization to any dimension seems to be new in the non-elliptic case. In any case the techniques used here are fairly inspired in those of the paper of Colin de Verdière and Vey.

Proposition 2.2.1 *Let M be a differentiable manifold and let g be a germ of smooth function in a neighbourhood of a point p. Consider X a vector field which in local coordinates can be written as $X = x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}$ then there exist differentiable functions g_1 and g_2 such that:*

$$g = g_1(x_1^2 + x_2^2, x_3 \ldots, x_n) + X(g_2)$$

Proof:

We follow Eliasson's recipe [23] for this proof:

Let ϕ_t be the flow of the vector field X. Since the orbits of X are circles, after shrinking the neighourhood U of the point p if necessary we can assume that $\phi_t(U) \subset U$. On the other hand, the orbits of X are periodic of period 2π. Thus we can consider the following well-defined function,

$$g_1(x_1, \ldots, x_n) = \frac{1}{2\pi}\int_0^{2\pi} g(\phi_t(x_1, \ldots, x_n))dt$$

and

$$g_2(x_1,\ldots,x_n) = \frac{1}{2\pi}\int_0^{2\pi}(tg(\phi_t(x_1,\ldots,x_n)) - g_1(x_1,\ldots,x_n))dt.$$

Clearly, these functions are differentiable. Let us check that these g_1 and g_2 give the decomposition sought.

First we check $X(g_1) = 0$ and since the orbits of the vector field are connected this implies that $g_1 = g_1(x_1^2 + x_2^2, x_3, \ldots, x_n)$.

Since

$$X(g_1) = \lim_{s\to 0}\frac{g_1(\phi_s(x_1,\ldots,x_n)) - g_1(x_1,\ldots,x_n)}{s},$$

we compute this derivative:

$$g_1(\phi_s(x_1,\ldots,x_n)) = \frac{1}{2\pi}\int_0^{2\pi} g(\phi_t(\phi_s(x_1,\ldots,x_n)))dt$$

Since ϕ_s is a one-parameter subgroup we get:

$$g_1(\phi_s(x_1,\ldots,x_n)) = \frac{1}{2\pi}\int_0^{2\pi} g(\phi_{t+s}(x_1,\ldots,x_n))dt$$

Now we perform the change of variable $\bar{t} = t + s$ and the right hand side becomes:

$$\frac{1}{2\pi}\int_s^{2\pi+s} g(\phi_{\bar{t}}(x_1,\ldots,x_n))d\bar{t}$$

Now we differentiate this expression with respect to s to get:

$$\frac{1}{2\pi}(g(\phi_{2\pi+s}(x_1,\ldots,x_n)) - g(\phi_{2\pi}(x_1,\ldots,x_n)))$$

Since ϕ_t is 2π-periodic this expression equals 0 for all s, in particular, for $s = 0$ and this proves $X(g_1) = 0$.

Now we perform the same kind of calculations for g_2. We have to check that

$$X(g_2) = g - g_1.$$

We have,

$$g_2(\phi_s(x_1,\ldots,x_n)) = \frac{1}{2\pi}\int_0^{2\pi}((t{+}s)g(\phi_t(\phi_s(x_1,\ldots,x_n))){-}(t{+}s)g_1(\phi_s(x_1,\ldots,x_n)))dt$$

We split this into two integrals.

The first integral $\frac{1}{2\pi}\int_0^{2\pi}(t+s)g(\phi_t(\phi_s(x_1,\ldots,x_n)))dt$ becomes

$$\frac{1}{2\pi}\int_s^{2\pi+s}(\bar{t}g(\phi_{\bar{t}}(x_1,\ldots,x_n)))d\bar{t}$$

under the change of variable $\bar{t} = t+s$. Now differentiating in s we obtain

$$\frac{1}{2\pi}((2\pi+s)g(\phi_{2\pi+s}(x_1,\ldots,x_n)) - sg(\phi_s(x_1,\ldots,x_n)))$$

Again, since ϕ_t is 2π-periodic this expression equals

$$\frac{1}{2\pi}(2\pi g(\phi_s(x_1,\ldots,x_n)))$$

Finally, put $s=0$; since $\phi_0 = Id$ we get

$$g(x_1,\ldots,x_n).$$

As for the second integral,

$$\frac{1}{2\pi}\int_0^{2\pi}(t+s)g_1(\phi_s(x_1,\ldots,x_n))dt = g_1(\phi_s(x_1,\ldots,x_n))\int_0^{2\pi}(t+s)dt$$

$$= g(\phi_s(x_1,\ldots,x_n))(\frac{(2\pi)^2}{2}+2\pi s)$$

Finally differentiating in s and setting $s=0$ this expression equals,

$$\frac{1}{2\pi}(\frac{(2\pi)^2}{2}X(g_1(x_1,\ldots,x_n)) + 2\pi g_1(x_1,\ldots,x_n))$$

But since $X(g_1)=0$ this integral is $g_1(x_1,\ldots,x_n)$. This proves $X(g_2) = g - g_1$ and we are done.

□

Now let us prove a similar result but for a vector field of type $Y = -x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}$.

We will prove,

Proposition 2.2.2 *Let M be a differentiable manifold and let g be a germ of smooth function in a neighbourhood of a point p. Consider X a vector field which in local coordinates can be written as $Y = -x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}$ then there exist differentiable functions g_1 and g_2 such that*

$$g = g_1(x_1x_2, x_3 \ldots, x_n) + Y(g_2)$$

Before we will need some lemmas concerning the smooth resolution of the equation $Y(f) = g$ for a given smooth g.

Lemma 2.2.1 *Let g be a smooth function, the equation $Y(f) = g$ admits a formal solution along the subspace $S = \{(0, 0, x_3, \ldots, x_n)\}$ if and only if*

$$\frac{\partial^{2k} g}{\partial x_1^k \partial x_2^k}(0, 0, x_3, \ldots, x_n) = 0.$$

Proof:

Let us construct a solution considering the (x_1, x_2)-jets. That is, assume the (x_1, x_2)-jet of f along $S = \{(0, 0, x_3, \ldots, x_n)\}$ is $\sum_{ij} f_{ij}x_1^i x_2^j$, the coefficients f_{ij} being functions in the variables $(x_3, \ldots, x_n)$. Denote by $\sum_{ij} g_{ij}x_1^i x_2^j$ the (x_1, x_2)-jet of g along $S = \{(0, 0, x_3, \ldots, x_n)\}$.

Then the condition $X(f) = g$ implies the following conditions for the coefficient functions

$$(-i + j)f_{ij} = g_{ij} \quad , \quad \forall i, j$$

Particularizing $i = j$ in this equation we obtain $g_{ii} = 0$; so in order to have a solution by jets of the equation $Y(f) = g$, the terms $\frac{\partial^{2k} g}{\partial x_1^k \partial x_2^k}(0, 0, x_3, \ldots, x_n)$ have to vanish necessarily.

On the other hand if $i \neq j$ from the above relation, the following relation is met $f_{ij} = \frac{g_{ij}}{-i+j}$. Therefore, if the condition $\frac{\partial^{2k} g}{\partial x_1^k \partial x_2^k}(0,0,x_3,\ldots,x_n) = 0$ is fulfilled this gives a solution by jets to the equation $Y(f) = g$. □

According to Borel's theorem there exists a smooth function $\widehat{f}$ with the (x_1, x_2)-jets previously found. It remains to solve this equation for functions for which

$$\frac{\partial^{i+j} g}{\partial x_1^i \partial x_2^j}(0,0,x_3,\ldots,x_n) = 0.$$

We will refer to this functions as (x_1, x_2)-flat functions along the subspace $S = \{(0,0,x_3,\ldots,x_n)\}$.

Lemma 2.2.2 *Let g be a (x_1, x_2)-flat function along the subspace $S = \{(0,0,x_3,\ldots,x_n)\}$ then there exists a smooth function f for which $Y(f) = g$.*

Proof:

Consider the function,

$$T(x_1,\ldots,x_n) = \begin{cases} \frac{1}{2}\ln\frac{x_1}{x_2} & x_1x_2 > 0 \\ \frac{1}{2}\ln\frac{-x_1}{x_2} & x_1x_2 < 0 \end{cases}$$

Denote by $\phi_t(x_1,\ldots,x_n)$ the flow of the vector field Y, being $Y = -x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}$. Observe that $\phi_t(x_1,\ldots,x_n) = (e^{-t}x_1, e^t y_1, \ldots, x_n)$.

Now we define

$$f(x_1,\ldots,x_n) = -\int_0^{T(x_1,\ldots,x_n)} g(\phi_t(x_1,\ldots,x_n))dt. \qquad (2.2.2)$$

This function is defined outside the set $\Omega = \Omega_1 \cup \Omega_2$ being $\Omega_1 = \{(x_1,\ldots,x_n), x_1 = 0\}$ and $\Omega_2 = \{(x_1,\ldots,x_n), x_2 = 0\}$. Let us prove that f admits a smooth continuation in the whole neighbourhood considered and that it is a solution to our problem.

In order to check that it admits a smooth extension. We compute the derivatives.

Formally differentiating under the integral sign, the computation of the first derivatives reads,

- If $i = 1, 2$

$$\frac{\partial}{\partial x_i} f = -g(\phi_{T(x_1,x_2,\ldots,x_n)}) \frac{\partial}{\partial x_i} T - \int_0^{T(x_1,\ldots,x_n)} \frac{\partial}{\partial x_i} g(\phi_t(x_1,\ldots,x_n)) dt \quad (2.2.3)$$

- When $i \neq 1$ and $i \neq 2$,

$$\frac{\partial}{\partial x_i} f = - \int_0^{T(x_1,\ldots,x_n)} \frac{\partial}{\partial x_i} g(\phi_t(x_1,\ldots,x_n)) dt \quad (2.2.4)$$

Observe that the set S equals $S = \Omega_1 \cap \Omega_2$. Observe that f is smooth outside the set $\Omega = \Omega_1 \cup \Omega_2$.

The first term in 2.2.3 is smooth outside the set $\Omega = \Omega_1 \cup \Omega_2$. And observe that if p lies in Ω then from the definition of T, the point $\phi_T(p)$ lies in S. On the other hand, the function g is flat along the subspace S. Thus the first term in 2.2.3 $-g(\phi_{T(x_1,x_2,\ldots,x_n)}) \frac{\partial}{\partial x_i} T$ is smooth in the whole neighbourhood of the origin considered.

As for the second term, we could reproduce word by word the proof supplied by Eliasson in [23] in the two dimensional case. The proof can be adapted because the function g is flat along S. In fact, it is just the parametric version of Eliasson's result. In the same way, Eliasson's proof yields that the integral 2.2.4 is a smooth function.

The same arguments applied to the successive derivatives prove that f admits a $\mathcal{C}^\infty$ continuation.

In fact in [23] it is proved that the integral defining f is absolutely integrable, thus we can differentiate with respect to s. This lets us prove that f is, in fact, a solution to the equation $Y(f) = g$.

Now let us check that this is a solution to the equation.

First,

$$f(\phi_s(x_1,\ldots,x_n)) = -\int_0^{T(\phi_s(x_1,\ldots,x_n))} g(\phi_s(\phi_t(x_1,\ldots,x_n)))dt \qquad (2.2.5)$$

The relations $\ln\frac{e^{-s}x_1}{e^s x_2} = \ln\frac{x_1}{x_2} - 2s$ when $x_1x_2 \geq 0$ and $\ln\frac{-e^{-s}x_1}{e^s x_2} = \ln\frac{-x_1}{x_2} - 2s$ when $x_1x_2 \leq 0$ imply $T(\phi_s(x_1,\ldots,x_n)) = T(x_1,\ldots,x_n) - s$. On the other hand, since ϕ_s is a one-parameter subgroup. Equation 2.2.5 can be written as,

$$f(\phi_s(x_1,\ldots,x_n)) = -\int_0^{T(x_1,\ldots,x_n)-s} g(\phi_{t+s}(x_1,\ldots,x_n))dt$$

Now we perform the change of variable $\bar{t} = t + s$ and this equation reads,

$$f(\phi_s(x_1,\ldots,x_n)) = -\int_s^{T(x_1,\ldots,x_n)} g(\phi_{\bar{t}}(x_1,\ldots,x_n))d\bar{t}$$

Now after differentiating in s this equation yields,

$$\frac{df(\phi_s(x_1,\ldots,x_n))}{ds} = g(\phi_s(x_1,\ldots,x_n))$$

Finally, put $s = 0$ to obtain $Y(f) = g$ as we wanted.

This ends the proof of the lemma. □

Let us go back to the proof of proposition 2.2.2. Given a differentiable function g, we want to find smooth functions g_1 and g_2 such that

$$g = g_1(x_1x_2, x_3\ldots,x_n) + Y(g_2).$$

The strategy for finding this decomposition will be to find a solution by (x_1,x_2)-jets and then apply the second lemma to gather all the remaining (x_1,x_2)-flat terms as $Y(f)$ for a certain smooth f.

So let $\sum_{ij} g_{ij}x_1^i x_2^j$ be the (x_1,x_2)-Taylor expand for g at a point $(0,0,x_3,\ldots,x_n)$ lying in the subspace $S = \{(0,0,x_3,\ldots,x_n)\}$.

Now we split this Taylor expand in two. The first one, $\sum_{ii} g_{ij} x_1^i x_2^i$, and the second one $\sum_{i \neq j} g_{ij} x_1^i x_2^j$. Denote by $\widehat{r}_1$ and $\widehat{r}_2$ two smooth functions with the previous jets. Then we can assert that

$$\widehat{r}_1 = g_1(x_1 x_2, x_3, \ldots, x_n) + \phi(x_1, \ldots, x_n),$$

being $\phi(x_1, \ldots, x_n)$ a (x_1, x_2)-flat function along $S = \{(0, 0, x_3, \ldots, x_n)\}$. Further, using the two above lemmas (2.2.1,2.2.2), the function $\widehat{r}_2$ can be written as $\widehat{r}_2 = Y(R_2)$. Now since ϕ is (x_1, y_1)-flat, according to lemma 2.2.2 we can write $\phi(x_1, \ldots, x_n) = Y(R)$. Finally define $g_2 = R_2 + R$ and g_1 and g_2 satisfy the decomposition sought $g = g_1(x_1 x_2, x_3 \ldots, x_n) + Y(g_2)$. And this completes the proof of proposition 2.2.2.

Observation 2.2.1 *Observe that the function defined by formula 2.2.2 is not smooth if g is not flat along the subspace S.*

If g is only flat at the origin then we can find examples which show that f does not admit a smooth continuation.

For instance consider $n = 4$, the function $g = e^{-(\frac{1}{x_3})^2}$ is flat at the origin but it is not flat along the subspace $S = \{(0, 0, x_3, x_4)\}$. Observe that the integral does not extend to a smooth function at points of the form $(0, x_2^0, x_3^0, x_4^0)$ with $x_2^0 \neq 0$ and $x_3^0 \neq 0$.

This integral has been used by some authors without the condition of flatness along the subspace and just the condition of flatness at the origin (see proposition 2.13 in [57]). Thus, the functions defined by those integrals in [57] do not always admit a smooth continuation unless the function g is flat along S.

2.3 Symplectic linearization in dimension 2

In this section we consider a foliation given a completely integrable system with singularities of non-degenerate type in a 2-dimensional manifold.

As we observed in the first chapter, the foliation defined as above can be differentiably linearized because of Eliasson's theorem for rank 0 singularities.

Then we know that the foliation is differentiably equivalent to the foliation defined by the orbits of the vector field,

$$X_\epsilon = x\frac{\partial}{\partial y} - \epsilon y\frac{\partial}{\partial x}.$$

Where $\epsilon = 1$ in the elliptic case and $\epsilon = -1$ in the hyperbolic case. That is, the foliation that we will consider will be $\mathcal{F}_\epsilon =< x\frac{\partial}{\partial y} - \epsilon y\frac{\partial}{\partial x} >$.

The problem that we want to solve in this section is the following:

Problem

Given two symplectic structures ω_1 and ω_2, we want to find a local foliation preserving diffeomorphism defined in a neighbourhood of the origin such that $\phi^*(\omega_1) = \omega_2$

This problem is not new.

The affirmative answer was given by Vey [55] in the analytical case and by Colin de Verdière and Vey [6] in the smooth case. A proof for the smooth elliptic case was given by Eliasson in [23].

In any case, we provide our own proof here.

Observe that in dimension 2 this problem is equivalent to the problem of symplectic linearization of singular Lagrangian foliations. This second problem is more constraining in dimensions greater than 2.

Let us recall what is the problem of symplectic linearization of singular Lagrangian foliations about,

Problem

Let $\mathcal{F}$ be a foliation given by a completely integrable system with singularities of non-degenerate type.

Given two symplectic structures ω_1 and ω_2 for which $\mathcal{F}$ is Lagrangian, we want to find a local foliation preserving diffeomorphism defined in a neighbourhood of the origin such that $\phi^*(\omega_1) = \omega_2$.

This result has an utter importance in the forthcoming chapters. It can be considered as the first step of an inductive process valid for integrable systems without focus-focus components which will allow us to conclude the symplectic linearization.

Now we can state and prove the following,

Theorem 2.3.1 *Let (M^2, ω_1) be a 2-dimensional symplectic manifold endowed with coordinates (x, y) and let $\mathcal{F}$ be a singular Lagrangian foliation with an elliptic or hyperbolic singularity at the origin $(0,0)$, then there exists a local diffeomorphism ϕ preserving $\mathcal{F}$ such that $\phi^*(dx \wedge dy) = \omega_1$.*

Proof:

Let $X_\epsilon = x\frac{\partial}{\partial y} - \epsilon y \frac{\partial}{\partial x}$. We denote by f_ϵ be the function $f_\epsilon = x^2 + \epsilon y^2$. Now assume $\omega_1 = A(x,y) dx \wedge dy$ Then $i_{X_\epsilon}\omega = -A df_\epsilon$.

In the elliptic case ($\epsilon = -1$) lemma 2.2.1 shows that we can write $A = A_1 + X_1(A_2)$ with A_1 basic for convenient functions A_1 and A_2.

In the hyperbolic case ($\epsilon = -1$) we use lemma 2.2.2 to find a similar decomposition. But we have to perform a change of coordinates first, consider $\overline{x} = x + y, \overline{y} = x - y$ and now apply lemma 2.2.2 which guarantees the existence of functions A_1 and A_2 such that $A = A_1 + X_{-1}(A_2)$ with A_1 basic.

Now in both cases, we define $\alpha = A_2 df_\epsilon$. Observe that α is a basic 1-form.

The next lemma shows that we can deform ω_1 to an equivalent $\overline{\omega}_1$ with the coefficient function A basic for $\mathcal{F}$. More exactly, we prove the following lemma which is a foliation-preserving version of the Moser path method.

Lemma 2.3.2 *Let α be an $\mathcal{F}$-basic 1-form and let ω_1 be a symplectic germ on a 2-dimensional manifold for which $\mathcal{F}$ is Lagrangian. Then:*

1. *The 2-form* $\omega_0 = \omega_1 - d\alpha$ *is a symplectic structure in a neighbourhood of* p.

2. *There is a diffeomorphism* η *between two neighbourhoods of* p *preserving* $\mathcal{F}$ *and such that* $\eta^*(\omega_1) = \omega_0$.

Proof:

First, let us check that ω_0 is a symplectic form in a neighbourhood of p. Clearly, ω_0 is a closed 2-form. Let us see that it is non-degenerate; Since α is basic for the foliation, $\alpha = g \cdot (df_\epsilon)$. In particular α vanishes at $p = (0,0)$ and $\omega_{0_{|p}}$ equals $\omega_{1_{|p}}$. Therefore, since ω_1 is non-degenerate at p, the 2-form ω_0 is non-degenerate in a neighbourhood of p. This ends the proof of the first assertion. In order to prove the second assertion we consider the following family of 2-forms:

$$\omega_t = \omega_0 + t(\omega_1 - \omega_0), \quad t \in [0,1]$$

Let us see that these 2-forms are symplectic germs. Clearly, the 2-forms ω_t are closed. And since $\omega_{t_{|p}} = \omega_{0_{|p}}$, we can repeat the argument above to see that the 2-forms are non-degenerate. Therefore, they are symplectic in a neighbourhood of the point p.

Now we are going to use Moser's path method to conclude. First, we consider the well-defined vector field X_t by the following equality:

$$i_{X_t}\omega_t = -\alpha.$$

Recall that α vanishes at p, this guarantees [59] that the time-dependent vector field X_t is integrable. Let ϕ_t stand for the "flow" of the time dependent vector field X_t defined by the conditions: $\phi_0 = Id$, $X_t = \frac{d\phi_s}{ds}_{|s=t}$ We check that this field is tangent to the foliation, in this way the flow of the time-dependent vector field will preserve the leaves of the foliation.

The singular set for the foliation is reduced to the origin p. We will see that the vector field is tangent to the foliation in two steps:

- At points $q \neq (0,0)$: Since α is basic for the foliation X_t verifies $\omega_t(X_t, X_\epsilon) = -\alpha(X_\epsilon) = 0$. Therefore, X_t belongs to its symplectic orthogonal and thus it has to be tangent to the foliation along the regular leaves.

- At the point $p = (0,0)$: The vector field X_t is tangent to the foliation because α vanishes at $p = (0,0)$ and therefore X_t vanishes at p.

So, we conclude that its flow preserves the leaves of the foliation. Further, remember that we are looking for a symplectomorphism; this symplectomorphism will be given by the flow of the vector field X_t at time $t = 1$. Remember that the flow ϕ_t gives us a family of diffeomorphisms verifying:

1. $\phi_t(p) = p$.

2. $\phi_t^*\omega_t = \omega_0$; that is to say, as a particular case, we have: $\phi_1^*(\omega_1) = \omega_0$.

3. ϕ_t preserves the leaves of the foliation.

So ϕ_1 is the symplectomorphism we are looking for and the two symplectic forms ω_0 and ω_1 define equivalent symplectic structures. This proves the second assertion of the lemma.

$\square$

Now we continue with the proof of the theorem. We apply the lemma taking $\alpha = A_2 df_\epsilon$ and the symplectic form $\overline{\omega}_1 = \omega_1 - d\alpha$ is equivalent to the initial ω_1 and so far

$$i_{X_1}\omega = -A_1 df_1$$

with A_1 basic.

The theorem will be proved once we prove the following lemma,

Lemma 2.3.3 *For any 2-form on $\mathbb{R}^2$ of the form $\omega = \chi(f_\epsilon)dx \wedge dy$ verifying $\chi(0) \neq 0$, there is a germ of diffeomorphism ν preserving the origin and also preserving the foliation given by $df_\epsilon = 0$, such that*

$$\nu^*(\omega) = dx \wedge dy$$

Proof of the lemma:

To start with, observe that if $\psi(f_\epsilon)$ is any differentiable function of f_ϵ such that $\psi(0) \neq 0$, the mapping

$$\begin{array}{rccc} G: & (\mathbb{R}^2, 0) & \longrightarrow & (\mathbb{R}^2, 0) \\ & (x, y) & \longrightarrow & (x \cdot \psi(f_\epsilon), y \cdot \psi(f_\epsilon)) \end{array}$$

defines a germ of diffeomorphism preserving the foliation $df_\epsilon = 0$. Moreover, observe that

$$G^*(dx \wedge dy) = (\psi^2 + 2\psi\psi' f_\epsilon)dx \wedge dy.$$

Consider the equation

$$\frac{d}{du}(\psi^2(u) \cdot u) = \chi(u),$$

where $u = f_\epsilon$. Observe that after integrating in u we obtain,

$$\psi^2(u) = \frac{\int_0^u (\chi(u))}{u}$$

which is a smooth function.

On the other hand since

$$\psi^2(0) = \lim_{u \to 0} \frac{\int_0^u (\chi(u))}{u} = \chi(0)$$

and $\chi(0) \neq 0$ (it is the coefficient of a symplectic 2-form) we can assert that ψ is a smooth function in a neighbourhood of the origin and that $\psi(0) \neq 0$. So taking as ψ the solution of the equation

$$\frac{d}{du}(\psi^2(u) \cdot u) = \chi(u)$$

where $u = f_\epsilon$, we have the desired diffeomorphism ν and that finishes the proof of the lemma and therefore the proof of the theorem.

$\square$

Chapter 3

Rank 1 singularities in dimension 4

3.1 Introduction

In the first chapter we attained the differentiable equivalence between the singular Lagrangian foliation and the linearized one. In this chapter we shall see that this equivalence becomes symplectic in a covering of a neighbourhood of a non-degenerate singular periodic orbit L. That is to say, we consider a foliation $\mathcal{F}$ given by a completely integrable system on a four dimensional manifold fulfilling the hypotheses of non-degeneracy established in the first chapter. Then we consider a symplectic germ in a neighbourhood of the non-degenerate singular periodic orbit for which the foliation is Lagrangian.

We prove that there exists a diffeomorphism defined in a neighbourhood of L then the foliation can be symplectically linearized in a neighbourhood of a singular periodic orbit.

Namely, we will prove the following theorem,

Theorem 3.1.1 *Let* $M_0^4 = S^1 \times D^3$*, endowed with the coordinates* (θ, p, x, y)*. Let*

$\mathcal{F}_0$ be the foliation given by:

$$Y_1 = \frac{\partial}{\partial \theta}$$

$$Y_2 = y\frac{\partial}{\partial x} - \epsilon x \frac{\partial}{\partial y}$$

$\epsilon \in \{-1, 1\}$ ($\epsilon = 1$ elliptic case, $\epsilon = -1$ hyperbolic case).

Let $L = S^1 \times (0,0,0)$. Then any two symplectic two forms ω_1 and ω_2 in M^4 such that $\mathcal{F}_0$ becomes Lagrangian are equivalent, i.e, there exists a diffeomorphism between two neighbourhood of S^1 such that ϕ preserves $\mathcal{F}_0$ and $\phi^*(\omega_2) = \omega_1$.

Observation 3.1.1 *Once this theorem has been proved, as we proved in the first chapter that the singular Lagrangian foliation is differentiable equivalent to the linearized one then it is symplectically equivalent to a neighbourhood of L in M_0^4, with a certain symplectic structure on it, but as this symplectic form is equivalent to the standard one, we finally get the desired symplectic equivalence which we formulate as:*

Theorem 3.1.2 *Given an integrable Hamiltonian system on a symplectic manifold (M^4, ω) with a non-degenerate singular periodic orbit L. Let $\mathcal{F}$ be the singular Lagrangian foliation associated to it. Let ω_0 be the canonical symplectic structure on M_0^4 given by: $\omega_0 = dp \wedge d\theta + dx \wedge dy$.*

- *If the singularity on L is elliptic there are neighbourhoods of L in M^4 and M_0^4 and a diffeomorphism between them ϕ such that $\phi^*(\omega_0) = \omega$, sending $\mathcal{F}$ to $\mathcal{F}_0$.*

- *If the singularity on L is hyperbolic there are in general a double covering of a neighbourhood of L in M^4, a neighbourhood of L in M_0^4 and a diffeomorphism ϕ between them such that $\phi^*(\omega_0) = \omega$, sending $\mathcal{F}$ to $\mathcal{F}_0$.*

The results contained in this section have been obtained jointly with Carlos Currás-Bosch and are contained in the paper [13].

In any case we include two proofs of this fact here one of them is different from the one contained in the publication.

Let us outline how the chapter is organized.

In the first section, we prove the existence of a Hamiltonian S^1-action tangent to the foliation which will be a key point in the proof of linearization. Namely, according to the first chapter, we may assume that the foliation is generated by $Y_1 = \frac{\partial}{\partial\theta}$ and $Y_2 = y\frac{\partial}{\partial x} - \epsilon x\frac{\partial}{\partial y}$ $\epsilon \in \{-1, 1\}$ being $\epsilon = 1$ in the elliptic case and $\epsilon = -1$ in the hyperbolic case. Let ω be a symplectic structure such that $\mathcal{F}$ is Lagrangian. In order to achieve the symplectic linearization we prove first the existence of a Hamiltonian S^1-action by translations which is tangent to the foliation.

In the second section we give two proofs of the theorem: The first one uses some of the analytical tools contained in the first chapter and defines some diffeomorphisms ad hoc. The second proof is based on the idea of finding a sort of splitting which separates clearly the singular part from the regular part of the foliation. Namely, we define two symplectic orthogonal distributions D_1 and D_2 such that $Y_1 \in D_1$ and $Y_2 \in D_2$. Then we show that these two distributions are integrable. In this way we obtain new coordinates in a neighbourhood of the singular circle. Finally the symplectic form ω may be written as $\omega = \omega_1 + \omega_2$ being ω_1 and ω_2 be two symplectic forms in the 2-dimensional submanifolds integrating the distributions D_1 and D_2 respectively. Since each distribution contains a vector field of the foliation, once reached this point, the symplectic linearization results in dimension 2 obtained in chapter 2 let us conclude the symplectic linearization process. This second proof sets a precedent for induction which will be used later.

3.2 Recovering a Hamiltonian S^1-action

In order to prove theorem 3.1.1 we need first to find a Hamiltonian S^1-action by translations preserving the foliation.

Let ω be a symplectic form in M_0^4 such that $\mathcal{F}_0$ is Lagrangian. Let f be a function defined on $S^1 \times D^3$, we denote by H_f^ω the Hamiltonian vector field associated to f.

Lemma 3.2.1 *There exist coordinates $(\overline{\theta}, \overline{p}, x, y)$ in a neighbourhood of $L \cong S^1 \times (0,0,0)$, such that $H^\omega_{\overline{p}}{}_{|N} = \frac{\partial}{\partial \overline{\theta}}$ on $N = \{(\overline{\theta}, \overline{p}, 0, 0)\}$.*

Proof: Let us consider $N \subset M_0^4$, $N = \{(\theta, p, 0, 0)\}$ and let $i : N \longrightarrow M_0^4$ be the inclusion of N in M_0^4. One can easily check that $i^*\omega$ endows N with a symplectic structure and that $dp = 0$ defines a regular Lagrangian foliation on N by circles; by a simple continuity argument: $i_{\frac{\partial}{\partial\theta}}\omega_{|N} = \lambda dp$, $\lambda \neq 0$, so one can take coordinates $(\overline{\theta}, \overline{p})$ in N such that $i_{\frac{\partial}{\partial\overline{\theta}}}(i^*\omega) = d\overline{p}$. Considering $(\overline{\theta}, \overline{p}, x, y)$ as coordinates in M_0^4 (after shrinking M_0^4 if necessary), we have

$$i_{\frac{\partial}{\partial\overline{\theta}}}\omega_{|N} = d\overline{p}.$$

$\square$

To avoid unnecessary changes of notation, we will consider from now on M_0^4 endowed with coordinates (θ, p, x, y) verifying $H_p^\omega = \frac{\partial}{\partial\theta}$ on N.

By using the generalized Poincaré Lemma one can write

$$\omega = d(Ad\theta + Bdp + Cdx + Ddy)$$

where A, B, C, D are differentiable functions vanishing on L.

Now we need the following

Lemma 3.2.2 *There exist coordinates (θ, p, x, y) in a neighbourhood of L such that*

$$\omega = d(pd\theta + Bdp + Cdx + Ddy)$$

where B, C, D vanish along L.

Proof:

Considering the decomposition

$$A(\theta, p, x, y) = A_0(p, x, y) + \frac{\partial \overline{A}}{\partial \theta}.$$

We can write

$$\omega = d(A_0(p, x, y)d\theta + d(\overline{A}) + (B - \frac{\partial \overline{A}}{\partial p})dp + (C - \frac{\partial \overline{A}}{\partial x})dx + (D - \frac{\partial \overline{A}}{\partial y})dy).$$

Now $\omega = d(A_0(p, x, y)d\theta + \overline{B}dp + \overline{C}dx + \overline{D}dy)$.

Let us see that A_0 is basic for the foliation $\mathcal{F}_0$. As $\mathcal{F}_0$ is Lagrangian for ω, we have

$$(-y\frac{\partial}{\partial x} + \epsilon x\frac{\partial}{\partial y})A_0 + \frac{\partial}{\partial \theta}(y\overline{C} - \epsilon x\overline{D}) = 0.$$

So this yields the following two equalities

$$(y\frac{\partial}{\partial x} - \epsilon x\frac{\partial}{\partial y})A_0 = 0$$
$$-y\overline{C} + \epsilon x\overline{D} = f(p, x, y).$$

The first condition together with $\frac{\partial A_0}{\partial \theta} = 0$ implies that A_0 is basic for the foliation.

On the other hand, as $H_p^\omega = \frac{\partial}{\partial \theta}$ on N, in particular we obtain $\frac{\partial A_0}{\partial p} = 1$ on N.

So the following mapping

$$\begin{array}{cccc} \varphi: & M_0^4 & \longrightarrow & M_0^4 \\ & (\theta, p, x, y) & \longrightarrow & (\theta, A_0(p, x, y), x, y) \end{array}$$

is a foliation preserving diffeomorphism.

Finally,

$$\varphi^*(d(pd\theta + B_2dp + C_2dx + D_2dy)) = \omega.$$

Notice that as on N, $H_p^\omega = \frac{\partial}{\partial\theta}$, the following functions $\frac{\partial B_2}{\partial\theta}$, $\frac{\partial C_2}{\partial\theta}$ and $\frac{\partial D_2}{\partial\theta}$ vanish on N. So, in particular B_2, C_2, D_2 are constant on L.

As

$$\omega = d(pd\theta + (B_2 - B_2(\theta,0,0,0))dp + (C_2 - C_2(\theta,0,0,0))dx + (D_2 - D_2(\theta,0,0,0))dy),$$

we can assume that the coefficients B, C, D are zero along L. □

We will need the following lemma which is an application of Moser's path method:

Lemma 3.2.3 *Let α be a 1-form, vanishing on L, and $\mathcal{F}_0$-basic and let ω_1 be a symplectic structure on M_0^4 such that $\mathcal{F}_0$ is Lagrangian. Then:*

1. *The 2-form $\omega_o = \omega_1 - d\alpha$ is a symplectic structure in a neighbourhood of L and makes the foliation Lagrangian.*

2. *There is a diffeomorphism η between two neighbourhoods of L in M_0^4 such that it preserves $\mathcal{F}_0$ and $\eta^*(\omega_1) = \omega_0$.*

Proof: Let $\omega_1 = d(pd\theta + Bdp + Cdx + Ddy)$. As α is basic for the foliation,

$$\alpha = F(xdx + \epsilon ydy) + Gdp.$$

Consider the following family of 2-forms $\omega_t = \omega_0 + t(\omega_1 - \omega_0)$, $t \in [0,1]$. So,

$$\omega_t = d(pd\theta + Bdp + Cdx + Ddy + (t-1)F(xdx + \epsilon ydy) + (t-1)Gdp),$$

where B, C, D and G vanish along L. And therefore $\omega_t|_L$ is the 2-form

$$\omega_t|_L = dp \wedge d\theta + (\frac{\partial B}{\partial x} - \frac{\partial C}{\partial p} + (t-1)\frac{\partial G}{\partial x})dx \wedge dp +$$

$$+(\frac{\partial B}{\partial y} - \frac{\partial D}{\partial p} + (t-1)\frac{\partial G}{\partial y})dy \wedge dp + (\frac{\partial D}{\partial x} - \frac{\partial C}{\partial y})dx \wedge dy,$$

$\omega_1|_L$ non-degenerate implies that $(\frac{\partial D}{\partial x} - \frac{\partial C}{\partial y})|_L \neq 0$ and one can check that this implies that ω_t is non-degenerate along L, for all $t \in [0,1]$. Therefore, we may assume that $\forall t \in [0,1]$ ω_t is symplectic in a tubular neighbourhood of L. Moreover, as

$$i_{\frac{\partial}{\partial\theta}}\omega_t = (\frac{\partial B}{\partial \theta} - 1 + (t-1)\frac{\partial G}{\partial \theta})dp + \frac{\partial C}{\partial \theta}dx + \frac{\partial D}{\partial \theta}dy + (t-1)\frac{\partial F}{\partial \theta}(xdx + \epsilon ydy)$$

we conclude that the foliation given by Y_1, Y_2 is Lagrangian for all ω_t.

In particular, taking $t = 0$ we have proved the first assertion claimed in the lemma.

Now, using non-degeneracy, we have a well-defined vector field X_t by the following equality

$$i_{X_t}\omega_t = -\alpha. \quad (I)$$

Notice that as we have assumed that $\alpha_{|L} = 0$, this guarantees ([59]) that the time-dependent vector field X_t is integrable (as it is integrable on L). Let ϕ_t stand for the "flow" of the time dependent vector field X_t defined by the conditions

$$\phi_0 = Id, \quad X_t = \frac{d\phi_s}{ds}_{|s=t}.$$

We check that this field is tangent to the foliation, in this way the flow of the time-dependent vector field will preserve the leaves of the Lagrangian foliation. Let $\mathcal{B} = \{(\theta, p, 0, 0)\}$ be the singular set for the foliation. We will see that the vector field is tangent to the foliation in two steps:

- Outside $\mathcal{B}$:

  Notice that since the regular leaves of the foliation are Lagrangian submanifolds for all ω_t, any vector field belonging to its symplectic orthogonal has to be tangent to the foliation; but as α is basic for the foliation X_t verifies $\omega_t(X_t, Y_1) = -\alpha(Y_1) = 0$ and $\omega_t(X_t, Y_2) = -\alpha(Y_2) = 0$. So X_t is tangent to the foliation along the regular leaves.

- In the singular set $\mathcal{B}$:

  Let $c = (\theta_0, c_0, 0, 0) \in \mathcal{B}$. A singular leaf for the foliation through c is the circle $L_c = \{(\theta, c_0, 0, 0)\}$. So a time-dependent vector field tangent to L_c at c has the form $\gamma_t(\theta, p, x, y)\frac{\partial}{\partial\theta}$. Let us check that X_t has this form: On $\mathcal{B}$, X_t has to be tangent to $\mathcal{B}$ otherwise its flow would reach a regular Lagrangian leaf of the foliation. So $X_{t|\mathcal{B}} = \alpha_t\frac{\partial}{\partial p} + \beta_t\frac{\partial}{\partial\theta}$.

  Finally, as X_t has to verify (I), α_t has to be zero, and therefore X_t is tangent to the foliation along $\mathcal{B}$.

So, we conclude that its flow preserves the leaves of the foliation.

Further, remember that we are looking for a symplectomorphism; this symplectomorphism will be given by the flow of the vector field X_t at time $t = 1$.

The flow ϕ_t gives us a family of diffeomorphisms verifying:

1. It is equal to the identity on L.

2. $\phi_t^*\omega_t = \omega_0$; that is to say, as a particular case, we have $\phi_1^*(\omega_1) = \omega_0$.

3. ϕ_t preserves the leaves of the foliation.

So $\eta = \phi_1$ is the symplectomorphism we are looking for and the two symplectic forms ω_0 and ω_1 define equivalent symplectic structures. This proves the second assertion of the lemma. □

Now we are going to apply this lemma to prove that there is a Hamiltonian S^1-action tangent to the foliation:

Proposition 3.2.1 *There is a Hamiltonian S^1-action tangent to the foliation. In fact, there exist coordinates (θ, p, x, y) in a neighbourhood of L such that $\omega = d(pd\theta + C(p,x,y)dx + D(p,x,y)dy)$ and the Hamiltonian S^1-action is performed by translations with respect to θ.*

Proof:

We will apply lema 3.2.3 in two stages:

First stage:

Consider $\omega_1 = d(pd\theta + Bdp + Cdx + Ddy)$ and $\omega_0 = \omega_1 - d(Bdp)$.

As B vanishes along L, applying lemma 3.2.3, ω_0 defines a symplectic structure in a neighbourhood of L such that makes $\mathcal{F}_0$ into a Lagrangian foliation and it is equivalent to ω_1 .

And so far, we can assume

$$\omega = \omega_0 = d(pd\theta + C(\theta, p, x, y)dx + D(\theta, p, x, y)dy).$$

Second stage:

As

$$i_{\frac{\partial}{\partial\theta}}\omega = -dp + \frac{\partial C}{\partial\theta}dx + \frac{\partial D}{\partial\theta}dy,$$

then

$$\frac{\partial C}{\partial\theta}dx + \frac{\partial D}{\partial\theta}dy = \lambda(xdx + \epsilon ydy)$$

for a certain function λ.

Therefore,

$$\frac{\partial C}{\partial\theta} = x\lambda, \quad \frac{\partial D}{\partial\theta} = \epsilon y\lambda.$$

This leads us to the following decomposition

$$C = C_0(p, x, y) + x\frac{\partial \overline{H}}{\partial \theta}$$

$$D = D_0(p, x, y) + \epsilon y\frac{\partial \overline{H}}{\partial \theta}.$$

Now

$$\omega = d(pd\theta + C_0(p, x, y)dx + D_0(p, x, y)dy + \frac{\partial \overline{H}}{\partial \theta}(xdx + \epsilon ydy)).$$

Let $\omega_0 = d(pd\theta + C_0(p, x, y)dx + D_0(p, x, y)dy)$. So, we can apply lemma 3.2.3 again and we can assume

$$\omega = d(pd\theta + C(p, x, y)dx + D(p, x, y)dy)$$

Notice that $i_{\frac{\partial}{\partial\theta}}\omega = -dp$.

So, S^1 acts on this neighbourhood in a Hamiltonian fashion and p is the moment map.

This concludes the proof of the proposition.

□

3.3 Two proofs for theorem 3.1.1

In this section we present two different proofs of the symplectic linearization for rank 1 singularities in dimension 4 which use the Hamiltonian S^1-action obtained in the previous section.

The first one is the one which appears in our joint paper with Carlos Currás-Bosch [13] but we also include here a proof for the elliptic rank 1 case which was already proved in a different way by Eliasson and Molino and Dufour.

The second proof is based on a geometrical idea of finding a decomposition which singles out the singular part and the regular part. This technique has turned out to be very useful to give a general proof for the symplectic linearization in any dimension. We define two symplectic orthogonal integrable distributions which allow to reduce the proof to the 2-dimensional case.

3.3.1 First proof

Let us go on with the proof of 3.1.1.

Without loss of generality, we may assume that the symplectic form on $M_0^4 = S^1 \times D^3$ can be expressed as:

$\omega = dp \wedge d\theta + A(p,x,y)dp \wedge dx + B(p,x,y)dp \wedge dy + C(p,x,y)dx \wedge dy$

Recall that $\mathcal{F}_0$ is, $\mathcal{F}_0 =< \frac{\partial}{\partial\theta}, \epsilon y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} >$.

Observation 3.3.1 *As ω is symplectic $C(0,0,0) \neq 0$ so after shrinking the neighbourhood, if necessary, we may assume $C(p,0,0) \neq 0$.*

Let us introduce the following notation:

For any function $f(p,x,y)$, $d_T f$ will stand for the 1-form:

$d_T f = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$. We use the notation f_ϵ to refer to the function $f_\epsilon = x^2 + \epsilon y^2$, where $\epsilon = 1$ in the elliptic case and $\epsilon = -1$ in the hyperbolic case.

Lemma 3.3.1 *Given $C(p,x,y)$ there exist $\mathcal{C}^\infty$-functions χ and f such that:*

$$C(p,x,y)dx \wedge dy = \chi(p, f_\epsilon)dx \wedge dy + d_T f \wedge d(f_\epsilon)$$

Proof of the lemma:

According to lemma 2.2.1 in the elliptic case we can write the following decomposition,

$$C(p,x,y) = C_1(p, x^2 + y^2) + X_1(C_2)$$

In the hyperbolic case, after performing the change of coordinates $\overline{x} = x + y, \quad \overline{y} = x - y$, we can apply lemma 2.2.2 to write a similar decomposition,

$$C(p,x,y) = C_1(p, x^2+y^2) + X_{-1}(C_2)$$

Now take $\chi = C_1$ and $f = \frac{-H}{2}$ and the function decompositions above yield the desired decomposition,

$$C(p,x,y)dx \wedge dy = \chi(p, f_\epsilon)dx \wedge dy + d_T f \wedge d(f_\epsilon)$$

And this ends the proof of the lemma.

□

Once reached this point, we can assert that ω can be written as:

$$\omega = dp \wedge d\theta + A(p,x,y)dp \wedge dx + B(p,x,y)dp \wedge dy + \chi(p, f_\epsilon)dx \wedge dy + d(f df_\epsilon)$$

Now applying lemma 3.2.3 we can eliminate $d(f df_\epsilon)$ by using a diffeomorphism which preserves the foliation.

The next step will be to "normalize" the coefficient of $dx \wedge dy$, we achieve this applying lemma 2.3.3 with parameters. Namely,

Observe that if $\psi(p, f_\epsilon)$ is a smooth function such that $\psi(0,0) \neq 0$. Then the mapping:

$$\begin{array}{cccc} G: & (M_0^4, 0) & \longrightarrow & (G(M_0^4), 0) \\ & (\theta, p, x, y) & \longrightarrow & (\theta, p, x \cdot \psi(p, f_\epsilon), y \cdot \psi(p, f_\epsilon)) \end{array}$$

defines a germ of diffeomorphism preserving the foliation $df_\epsilon = 0$.

As we saw on the proof of lemma 2.3.3 we can take as ψ the solution of the equation:

$$\frac{d}{du}(\psi^2(p,u) \cdot u) = \chi(p,u)$$

where $u = f_\epsilon$, we have the desired ϕ, and that finishes the proof of the lemma.

Using the previous lemma we can find a diffeomorphism preserving $\mathcal{F}_0$, such that the pull-back of ω is:

$$\omega = dp \wedge d\theta + \overline{A}(p,x,y)dp \wedge dx + \overline{B}(p,x,y)dp \wedge dy + dx \wedge dy$$

Finally, as ω is closed, there exists $g(p,x,y)$ such that,

$$\overline{A} = \frac{\partial g}{\partial x} \quad \overline{B} = \frac{\partial g}{\partial y}$$

And using the diffeomorphism:

$$\begin{array}{cccc} \phi : & M_0^4 & \longrightarrow & M_0^4 \\ & (\theta, p, x, y) & \longrightarrow & (\theta + g(p,x,y), p, x, y) \end{array}$$

which preserves $\mathcal{F}_0$, we can write $\omega = dp \wedge d\theta + dx \wedge dy$. And this ends the proof of the theorem.

3.3.2 Second proof

In this section we construct two symplectic orthogonal distributions. Those distributions will be 2-dimensional regular distributions and will allow us to reduce the proof to the 2-dimensional case.

Recall that we may assume that the symplectic form on $M_0^4 = S^1 \times D^3$ can be expressed as:

$$\omega = dp \wedge d\theta + A(p,x,y)dp \wedge dx + B(p,x,y)dp \wedge dy + C(p,x,y)dx \wedge dy$$

Lemma 3.3.2 *The distribution $\mathcal{D}_1 =< X, Y >$ defined by the relations:*

$$i_X\omega = dp$$

$$i_Y\omega = d\theta$$

is $\mathcal{C}^\infty$, symplectic in a neighbourhood of p and involutive everywhere.

Proof: First of all, since ω is symplectic and the forms dp and $d\theta$ are differentiable and independent, the distribution D_1 is clearly $\mathcal{C}^\infty$ and regular. Now let us prove that this distribution is symplectic. Observe that this distribution is symplectically orthogonal to the distribution $D_2 =< \frac{\partial}{\partial x}, \frac{\partial}{\partial y} >$. As a consecuence of observation 3.3.1, D_2 is symplectic in a neighbourhood of the origin then its symplectic orthogonal is also symplectic.

Now let us see that this distribution is involutive. We have to check that $[X, Y] \in D_1, \forall X, Y \in D_1$. In fact, it is enough to prove that $[X, Y] \in D_1$ for vector fields which are independent on a dense set in the neighbourhood considered. So we can take $X = Y_1 = \frac{\partial}{\partial \theta}$. By Leibnitz's rule:

$$L_{Y_1}(\omega(Y, \frac{\partial}{\partial x})) = L_{Y_1}(\omega)(Y, \frac{\partial}{\partial x}) + \omega(L_{Y_1}Y, \frac{\partial}{\partial x}) + \omega(Y, L_{Y_1}(\frac{\partial}{\partial x}))$$

Now if we take any $Y \in D_1$ then the left hand side of the equality above equals zero. As for the right hand side: The first term is zero because Y_1 is Hamiltonian and, in particular, it is symplectic; the third term vanishes because $L_{Y_1}(\frac{\partial}{\partial x}) = 0$. So we are led to $\omega(L_{Y_1}Y, \frac{\partial}{\partial x}) = 0$. In the same way, we prove that $\omega(L_{Y_1}Y, \frac{\partial}{\partial y}) = 0$ and therefore the distribution is involutive.

□

Now consider two distributions D_1 and D_2 those distributions are symplectically orthogonal distributions and contain Y_1 and Y_2, respectively. Since these regular distributions are involutive, there are regular foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ integrating D_1 and D_2 respectively. Now take a point p in the singular orbit, Frobenius Theorem provides new coordinates $(\overline{p}, \overline{\theta}, \overline{x}, \overline{y})$ in a neighbourhood of p such that the leaves of $\mathcal{F}_1$ are $L_{1b} = \{(\overline{p}, \overline{\theta}, b_1, b_2), b_1, b_2 \in \mathbb{R}\}$ and the leaves of $\mathcal{F}_2$ are $L_{2a} = \{(a_1, a_2, \overline{x}, \overline{y}), a_1 \in \mathbb{R}\, a_2 \in [0, 2\pi)\}$. Since the vector field $\frac{\partial}{\partial \theta}$ is Hamiltonian, in particular its flow is a symplectomorphism and preserves the symplectic orthogonal decomposition. Thus, sliding along the singular circle, we can extend these coordinates to a whole neighbourhood of the singular orbit. Now let us see what the expression of ω is in these coordinates; Since D_1 and D_2 are symplectically ortogonal and since $d\omega = 0$, in these new coordinates the symplectic form can be written as:

$$\overline{\omega}_2 = A(\overline{p}, \overline{\theta}) d\overline{p} \wedge d\overline{\theta} + B(\overline{x}, \overline{y}) d\overline{x} \wedge d\overline{y}.$$

Observe on the other hand, that Y_1 defines a foliation by circles on the submanifolds L_{1b} and Y_2 defines a foliation on the submanifolds L_{2a} with a singularity of elliptic type when $\epsilon = 1$ and hyperbolic type when $\epsilon = -1$. It remains to apply the known results of symplectic uniqueness in dimension 2. Namely, Y_1 defines a regular Lagrangian foliation by circles on L_{1b}. Thus after the theorem of Liouville-Mineur-Arnold, there exists a local diffeomorphism defined in a neighbourhood of the singular circle such that $\phi_1^*(dp \wedge d\theta) = A(\overline{p}, \overline{\theta}) d\overline{p} \wedge d\overline{\theta}$, being p, θ the coordinates defined by ϕ_1. The vector field Y_2 defines a nondegenerate singularity in each 2-dimensional submanifold L_{2a}. So we may apply the symplectic linearization result in dimension 2 (Theorem 2.3.1) to find a local diffeomorphism in a neighbourhood of the origin ϕ_2 such that $\phi_2^*(dx \wedge dy) = B(\overline{x}, \overline{\theta}) d\overline{x} \wedge d\overline{y}$, being x, y the coordinates provided by the diffeomorphism ϕ_2. Finally, we define a diffeomorphism in a neighbourhood of L, $\phi(\overline{p}, \overline{\theta}, \overline{x}, \overline{y}) = (\phi_1(\overline{p}, \overline{\theta}), \phi_2(\overline{x}, \overline{y}))$. This diffeomorphism preserves the foliation $\mathcal{F}$ and satisfies that $(\phi^*)(dp \wedge d\theta + dx \wedge dy) = \overline{\omega}_2$.

This ends the second proof of the theorem.

Chapter 4

Rank 0 singularities in dimension 4

4.1 Introduction

In this section we consider the symplectic linearization problem for foliations defined by a completely integrable system with rank 0 singularities in dimension 4. According to the first chapter we will assume that the foliations are already linear. That is, $\mathcal{F} = < X_1, X_2 >$. Recall that there are 4 possible cases from a differentiable point of view:

- **Elliptic-elliptic case:** In this case there are coordinates (x_1, y_1, x_2, y_2) centered at the point p such that two first integrals of the Hamiltonian system are $f_1 = \frac{x_1^2+y_1^2}{2}$ and $f_2 = \frac{x_2^2+y_2^2}{2}$.

- **Elliptic-hyperbolic case:** There are coordinates (x_1, y_1, x_2, y_2) centered at the point p such that two first integrals of the Hamiltonian system are $f_1 = \frac{x_1^2+y_1^2}{2}$ and $f_2 = \frac{x_2^2-y_2^2}{2}$.

- **Hyperbolic-hyperbolic case:** There are coordinates (x_1, y_1, x_2, y_2) centered at the point p such that two first integrals of the Hamiltonian system are

$f_1 = \frac{x_1^2 - y_1^2}{2}$ and $f_2 = \frac{x_2^2 - y_2^2}{2}$.

- **The focus-focus case:** There are coordinates (x_1, y_1, x_2, y_2) centered at the point p such that two first integrals of the Hamiltonian system are $f_1 = x_1 y_1 + x_2 y_2$ and $f_2 = x_1 y_2 - x_2 y_1$.

For the sake of brevity, we will refer to the first three types as decomposable at p and we will say that $\mathcal{F}$ is non-decomposable at p if its Williamson type is of focus-focus type.

Following the spirit of the preceding chapters, we look for a local symplectic classification of a foliation defined by the "linearized" model under the only constraint that the regular leaves of the foliation are Lagrangian. We give a complete proof for the symplectic uniqueness result in all the decomposable cases. Recall that this symplectic uniqueness for the "linearized" model is what we call *symplectic linearization*.

We do not provide a proof for the focus-focus case. In fact, so far, two proofs for the symplectic linearization in the focus-focus case are known to the author: The one provided by Eliasson in his thesis [23] and another proof provided by Nguyen Tien Zung, recently [65].

We would like to point out that this symplectic uniqueness is a consequence of Eliasson's Theorem. One of the main motivations for providing another proof for this fact is the following: the key point of Eliasson's proof is Proposition 4 in his thesis [23] but this proposition is stated without proof. The analogous proposition when the system is of analytical nature was proved by Vey [54] (lemma 1). The transition from the analytical case to the smooth case entails a non-trivial work with flat functions which in our opinion cannot be neglected.

A proof for this Proposition when the system is completely elliptic is contained in Eliasson's paper [24]. In the case that the dimension of the manifold is equal to 2 or the dimension of this manifold is 4 and the foliation corresponds to a foliation of focus-focus type, a proof of this fact is contained in Eliasson's thesis. In fact when the dimension of the manifold is equal to 2 and the foliation is of hyperbolic type this had been formerly proved by Colin de Verdière and Vey [6]. As far as the remaining cases are concerned, to our knowledge the only attempt to provide a proof in the completely hyperbolic case is due to San Vu Ngoc [57]. Unfortunately, the proof provided by San Vu Ngoc in that paper is based on a construction of a solution to a differential equation which is only smooth under some additional conditions concerning the flat terms which are not necessarily fulfilled under the hypotheses considered as we showed in Observation 2.2.1. Thus, from out point of view the proof provided in [57] does not fill the gap. Therefore, in our opinion, the problem remains unsolved in the decomposable cases which are not completely elliptic.

In this chapter we are going to provide a proof for the local symplectic uniqueness in dimension 4 for all the cases which does not rely on Proposition 4 of Eliasson's theorem. We do not attempt to give a proof of this Proposition but to prove the local uniqueness result.

In order to do that, we look for a symplectic orthogonal decomposition in the decomposable cases to reduce the problem of local uniqueness in dimension 4 to a problem in dimension 2 as we did in chapter 3. Although the proof is inspired by ideas coming from geometry, we had to plunge into analytical problems in order to find an orthogonal symplectic decomposition. Some of those analytical questions have already been presented in chapter 2.

The chapter is organized as follows: In the second section we pose the problem for the decomposable cases and we outline the strategy of the proof. In the third section we prove three basic lemmas that will be used in the uniqueness proof. One of them is a foliation preserving version of Moser path's method for corank 2 singularities. In the fourth section we prove a common proposition concerning basic 1-forms attached to the foliation and the symplectic form. In the fifth section, we look for a special Hamitonian vector field in the elliptic-elliptic case and elliptic-hyperbolic cases that will lead us to the symplectic orthogonal decomposition. We give two proofs of this fact: one is based on computation with forms and the second one is a geometrical proof which uses Bott-Weinstein connection and results for rank 1 singularities in dimension 4 obtained in chapter 3. In section 6 we find a special Hamiltonian vector field for hyperbolic-hyperbolic singularities. Finally, in section 7 we prove the existence of a symplectic orthogonal decomposition which yields the symplectic uniqueness for the decomposable cases (elliptic-elliptic, elliptic-hyperbolic and hyperbolic-hyperbolic cases).

As a side remark, let us point out that this technique of reduction to lower dimensional cases can be exported to dimension higher than 4 using induction to give a complete proof for the local uniqueness theorem in any dimension as we will see in the next chapter.

4.2 Strategy of the proof

Let M be a 4-dimensional manifold endowed with coordinates (x_1, y_1, x_2, y_2) and let p be the point $p = (0, 0, 0, 0)$. In the sequel we will deal with germ-like objects in a neighbourhood of the point p. Consider the vector fields $X_{1,\epsilon_1} = x_1 \frac{\partial}{\partial y_1} - \epsilon_1 y_1 \frac{\partial}{\partial x_1}$ and $X_{2,\epsilon_2} = x_2 \frac{\partial}{\partial y_2} - \epsilon_2 y_2 \frac{\partial}{\partial x_2}$ where ϵ_1 and ϵ_2 can be either $+1$ or -1. Throughout this section, $\mathcal{F}$ will stand for the germ of foliation given by the vector fields X_{1,ϵ_1} and X_{2,ϵ_2} and f_1 and f_2 will stand for the first integrals $f_1 = x_1^2 + \epsilon_1 y_1^2$ and $f_2 = x_2^2 + \epsilon_2 y_2^2$. The pair (ϵ_1, ϵ_2) labels the foliation. When $(\epsilon_1, \epsilon_2) = (1, 1)$ we say that $\mathcal{F}$ is of elliptic-elliptic type. If (ϵ_1, ϵ_2) is $(1, -1)$ or $(-1, 1)$, we talk about a foliation of elliptic-hyperbolic type. And finally when $(\epsilon_1, \epsilon_2) = (-1, -1)$ the foliation is referred to as a foliation of hyperbolic-hyperbolic type. Our goal is to study the germs of symplectic structures at p for which the decomposable $\mathcal{F}$ is generically Lagrangian. We say that a foliation is generically Lagrangian if its regular leaves are Lagrangian submanifolds and its singular leaves are isotropic. In this section we prove that if ω is such a symplectic germ then there exists a local diffeomorphism ϕ preserving $\mathcal{F}$, fixing p and such that $\phi^*(\omega) = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$. We organize the proof in two stages: In the first stage we find a symplectically orthogonal decomposition adapted to $\mathcal{F}$. The proof of the construction of this symplectic orthogonal decomposition differs slightly in each of the cases (elliptic-elliptic, elliptic-hyperbolic, hyperbolic-hyperbolic). In the second stage we use this decomposition and the known results in dimension 2 to prove that any two symplectic germs are equivalent. Let us state the main concluding results contained in this chapter:

Theorem 4.2.1 (Symplectically orthogonal decomposition) *Let ω be a symplectic germ for which $\mathcal{F}$ is generically Lagrangian. Then there exists a symplectic germ $\overline{\omega}$ equivalent to ω and there exist two symplectic distributions D_1 and D_2 such that:*

1. *D_1 and D_2 are involutive and symplectically orthogonal with respect to $\overline{\omega}$.*
2. *$X_{1,\epsilon_1} \in D_1$ and $X_{2,\epsilon_2} \in D_2$.*

Theorem 4.2.2 (Symplectic Uniqueness) *Let ω be a symplectic germ at p for which $\mathcal{F}$ is generically Lagrangian then ω is equivalent to $\omega_0 = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$.*

In order to prove Theorem 4.2.1, we will find an equivalent $\overline{\omega}$ for which the vector field X_1 will be Hamiltonian with Hamiltonian function f_1. But first we need a few propositions and lemmas.

4.3 Three common lemmas

In this section we are going to prove several lemmas which will be used later to prove the local uniqueness of the symplectic germs in the decomposable cases.

When we write X_1, X_2 we mean a basis of the Lagrangian foliation. Let us set the particular basis we are going to use throughout the section:

- In the elliptic-elliptic case, $X_1 = x_1\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_1}$ and $X_2 = x_2\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial x_2}$.
- In the elliptic-hyperbolic case, $X_1 = x_1\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_1}$ and $X_2 = x_2\frac{\partial}{\partial y_2} + y_2\frac{\partial}{\partial x_2}$.
- In the hyperbolic-hyperbolic case, $X_1 = x_1\frac{\partial}{\partial y_1} + y_1\frac{\partial}{\partial x_1}$ and $X_2 = x_2\frac{\partial}{\partial y_2} + y_2\frac{\partial}{\partial x_2}$.
- In the focus-focus case, $X_1 = -x_1\frac{\partial}{\partial x_1} + y_1\frac{\partial}{\partial y_1} - x_2\frac{\partial}{\partial x_2} + y_2\frac{\partial}{\partial y_2}$ and $X_{2,\epsilon_2} = x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1} + y_1\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial y_1}$.

The first lemma is a foliation-preserving version of the Moser path method.

Lemma 4.3.1 *Let α be an $\mathcal{F}$-basic 1-form and let ω_1 be a symplectic germ for which $\mathcal{F}$ is Lagrangian. Then:*

1. *The 2-form $\omega_0 = \omega_1 - d\alpha$ is a symplectic structure in a neighbourhood of p and makes the foliation Lagrangian.*
2. *There is a diffeomorphism η between two neighbourhoods of p preserving $\mathcal{F}$ and such that $\eta^*(\omega_1) = \omega_0$.*

Proof:

First, let us check that ω_0 is a symplectic form in a neighbourhood of p. Clearly, ω_0 is a closed 2-form. Let us see that it is non-degenerate; Since α is basic for the foliation, $\alpha = F(df_1) + G(df_2)$. In particular α vanishes at $p = (0,0,0,0)$ and $\omega_{0_{|p}}$ equals $\omega_{1_{|p}}$. Therefore, since ω_1 is non-degenerate at p, the 2-form ω_0 is non-degenerate in a neighbourhood of p. In order to prove that the foliation is Lagrangian for ω_0 we have to check that $\omega_0(X_1, X_2) = 0$. Since $\omega_0(X_1, X_2) = \omega_1(X_1, X_2) - d\alpha(X_1, X_2)$ and $\mathcal{F}$ is Lagrangian for ω_1, we have to see that $d\alpha(X_1, X_2)$ vanishes. Since α is $\mathcal{F}$-basic and $[X_1, X_2] = 0$, the formula $d\alpha(X_1, X_2) = X_1\alpha(X_2) - X_2\alpha(X_1) - \alpha([X_1, X_2])$ implies that $d\alpha(X_1, X_2) = 0$.

This ends the proof of the first assertion. In order to prove the second assertion we consider the following family of 2-forms:

$$\omega_t = \omega_0 + t(\omega_1 - \omega_0), \quad t \in [0,1]$$

Let us see that these 2-forms are symplectic germs for which the foliation $\mathcal{F}$ is Lagrangian. Clearly, the 2-forms ω_t are closed. And since $\omega_{t_{|p}} = \omega_{0_{|p}}$, we can repeat the argument above to see that the 2-forms are non-degenerate. Therefore, they are symplectic in a neighbourhood of the point p. Further, since $\omega_0(X_1, X_2) = 0$ and $\omega_1(X_1, X_2) = 0$, the foliation $\mathcal{F}$ is Lagrangian for ω_t, $\forall t \in [0,1]$.

Now we are going to use Moser's path method to conclude. First, we consider the well-defined vector field X_t by the following equality:

$$i_{X_t}\omega_t = -\alpha.$$

Recall that α vanishes at p, this guarantees [59] that the time-dependent vector field X_t is integrable. Let ϕ_t stand for the "flow" of the time dependent vector field X_t defined by the conditions: $\phi_0 = Id$, $X_t = \frac{d\phi_s}{ds}_{|s=t}$ We check that this field is tangent to the foliation, in this way the flow of the time-dependent vector field will preserve the leaves of the Lagrangian foliation. Consider the sets: $\mathcal{B}_1 = \{(x_1, y_1, 0, 0)\} \setminus \{(0,0,0,0)\}$ and $\mathcal{B}_2 = \{(0,0,x_2,y_2)\} \setminus \{(0,0,0,0)\}$. Then the singular set for the foliation is $\mathcal{B} = \{(0,0,0,0)\} \cup \mathcal{B}_1 \cup \mathcal{B}_2$ if the foliation is decomposable. In the focus-focus case the singular set for the foliation reduces to $(0,0,0,0)$.

We will see that the vector field is tangent to the foliation in two steps:

- Outside $\mathcal{B}$: Notice that since the regular leaves of the foliation are Lagrangian submanifolds for all ω_t, any vector field belonging to its symplectic orthogonal has to be tangent to the foliation; but as α is basic for the foliation X_t verifies $\omega_t(X_t, X_1) = -\alpha(X_1) = 0$ and $\omega_t(X_t, X_2) = -\alpha(X_2) = 0$. So X_t is tangent to the foliation along the regular leaves.

- Along $\mathcal{B}$, the singular set: We check that the vector field X_t is tangent to the foliation in the following cases:
  - The focus-focus case: In this case the only singular point is the origin. The vector field X_t is tangent to the foliation because α vanishes at $p = (0,0,0,0)$ and therefore X_t vanishes at p.
  - The decomposable cases. In this case the singular set is $\mathcal{B} = \{(0,0,0,0)\} \cup \mathcal{B}_1 \cup \mathcal{B}_2$. Let us check that X_t is tangent to the foliation:
    1. At the point $p = (0,0,0,0)$: The vector field X_t is tangent to the foliation because α vanishes at $p = (0,0,0,0)$ and therefore X_t vanishes at p.
    2. At a point $q_1 = (a_1, b_1, 0, 0) \in \mathcal{B}_1$: Since $\alpha_{|p} = F(q_1)(a_1 dx_1 + b_1 dy_1)$, the symplectic orthogonal to X_t at the point q_1 is generated by the vector fields $X_1, \frac{\partial}{\partial x_2}$ and $\frac{\partial}{\partial y_2}$ and since $X_{t_{|q_1}}$ we can write $X_{t_{|q_1}} = f_t(a_1, b_1, 0, 0)X_1 + g_t(a_1, b_1, 0, 0)\frac{\partial}{\partial x_2} + h_t(a_1, b_1, 0, 0)\frac{\partial}{\partial y_2}$. If $X_{t_{|q_1}}$ is not a multiple of X_1, the flow of X_t starting at q_1 would reach a regular Lagrangian leaf of the foliation. So $X_{t_{|q_1}} = f_t(a_1, b_1, 0, 0)X_1$ and, in particular, it is tangent to the foliation.
    3. At a point $q_2 = (0, 0, a_2, b_2) \in \mathcal{B}_2$: We proceed as in the previous case to see that $X_{t_{|q_2}} = r_t(0, 0, a_2, b_2)X_2$, and again X_t is tangent to the foliation at the point q_2.

So, we conclude that its flow preserves the leaves of the foliation. Further, remember that we are looking for a symplectomorphism; this symplectomorphism will be given by the flow of the vector field X_t at time $t = 1$. Remember that the flow ϕ_t gives us a family of diffeomorphisms verifying:

1. $\phi_t(p) = p$.
2. $\phi_t^* \omega_t = \omega_0$; that is to say, as a particular case, we have: $\phi_1^*(\omega_1) = \omega_0$.

3. ϕ_t preserves the leaves of the foliation.

So ϕ_1 is the symplectomorphism we are looking for and the two symplectic forms ω_0 and ω_1 define equivalent symplectic structures. This proves the second assertion of the lemma.

□

The following lemma will be used in the proof of the symplectic uniqueness for the decomposable cases. It also holds in the focus-focus case but we have included the proof only for the decomposable cases. The lemma proves that the forms $i_{X_1}\omega$ and $i_{X_2}\omega$ are basic for the foliation. That is, they are a combination of df_i for f_i defining the foliation.

Under these assumptions,

Lemma 4.3.2 *There exists $\mathcal{C}^\infty$-functions h_1, h_2, g_1 and g_2 such that:*

$$\begin{array}{rcl} i_{X_1}\omega & = & h_1 df_1 + h_2 df_2 \\ i_{X_2}\omega & = & g_1 df_1 + g_2 df_2 \end{array}.$$

Proof:

Let us check that $i_{X_1}\omega = H_1 df_1 + H_2 df_2$ and that $i_{X_2}\omega = G_1 df_1 + G_2 df_2$ for certain differentiable functions. Let the symplectic form ω be

$$\omega = A dx_1 \wedge dy_1 + B dx_1 \wedge dx_2 + C dx_1 \wedge dy_2 + D dy_1 \wedge dx_2 + E dy_1 \wedge dy_2 + F dx_2 \wedge dy_2.$$

In the decomposable cases, the foliation is generated by $X_{1,\epsilon_1} = x_1 \frac{\partial}{\partial y_1} - \epsilon_1 y_1 \frac{\partial}{\partial x_1}$ and $X_{2,\epsilon_2} = x_2 \frac{\partial}{\partial y_2} - \epsilon_2 y_2 \frac{\partial}{\partial x_2}$ where ϵ_1 and ϵ_2 can be either $+1$ or -1. If ϵ_1 and ϵ_2 have different sign, we say that the foliation is of elliptic-hyperbolic type. If the pair $(\epsilon_1, \epsilon_2) = (1,1)$, we say that the foliation is elliptic-elliptic. Finally, if the pair $(\epsilon_1, \epsilon_2) = (-1,-1)$, we say that the foliation is hyperbolic-hyperbolic.

Now let us look at the contractions $i_{X_{1,\epsilon_1}}\omega$ and $i_{X_{2,\epsilon_2}}\omega$:

$$i_{X_{1,\epsilon_1}}\omega = -A(x_1 dx_1 + \epsilon_1 y_1 dy_1) + (D x_1 - \epsilon_1 B y_1) dx_2 + (E x_1 - \epsilon C y_1) dy_2$$

$$i_{X_{2,\epsilon_2}}\omega = (\epsilon_2 By_2 - Cx_2)dx_1 + (\epsilon_2 Dy_2 - Ex_2)dy_1 - F(x_2dx_2 + \epsilon_2 y_2 dy_2)$$

But since $\mathcal{F}$ is Lagrangian, we have $i_{X_1}\omega(X_2) = 0$; and so we are led to the equality:

$$y_2(Dx_1 - \epsilon_1 By_1) = (\epsilon_1 Cy_1 - Ex_1)x_2 \quad (I).$$

From here it is clear that if we take $H_2 = \frac{Dx_1 - \epsilon_1 By_1}{x_2} = \frac{\epsilon_1 Cy_1 - Ex_1}{y_2}$, the following equalities hold: $Dx_1 - \epsilon_1 By_1 = H_2x_2$ and $\epsilon_1 Cy_1 - Ex_1 = H_2y_2$. To check that this H_2 is a $\mathcal{C}^\infty$-function, we apply a classical integration trick: Consider $\phi(x_1, y_1, x_2, y_2) = y_2(Dx_1 - \epsilon_1 By_1)$. Then we can write the following decomposition:

$$\phi(x_1, y_1, x_2, y_2) = \phi(x_1, y_1, 0, y_2) + x_2 \int_0^1 \frac{\partial\phi}{\partial x_2}(x_1, y_1, tx_2, y_2)dt.$$

Due to (I) the function ϕ vanishes on $(x_1, y_1, 0, y_2)$, this implies that H_2 equals the function $\int_0^1 \frac{\partial\phi}{\partial x_2}(x_1, y_1, tx_2, y_2)dt$ which is $\mathcal{C}^\infty$. So taking $H_1 = -A$ and $H_2 = \frac{Dx_1 - \epsilon_1 By_1}{x_2}$ we have proven that $i_{X_{1,\epsilon_1}}\omega = H_1 df_1 + H_2 df_2$. The condition of Lagrangianity can also be written as: $i_{X_{2,\epsilon_2}}\omega(X_{1,\epsilon_1}) = 0$; and this leads us now to the equality: $y_1(\epsilon_2 By_2 - Cx_2) = (\epsilon_2 Dy_2 - Ex_2)x_1$. As before, we can prove that $G_1 = \frac{\epsilon_2 By_2 - Cx_2}{x_1}$ is $\mathcal{C}^\infty$. And taking $G_2 = -F$ we have that: $i_{X_{2,\epsilon_2}}\omega = G_1 df_1 + G_2 df_2$. □

Finally, let us state and proof a very simple lemma which is a consequence of Cartan's formula and the Lagrangianity condition.

Lemma 4.3.3 *The following equality holds*

$$L_{X_2} i_{X_1}\omega = L_{X_1} i_{X_2}\omega.$$

Proof:

First, since $i_{[X_1,X_2]}\omega = L_{X_1} i_{X_2}\omega - i_{X_2} L_{X_1}\omega$ and $[X_1, X_2] = 0$, then $L_{X_1} i_{X_2}\omega = i_{X_2} L_{X_1}\omega$.

Now we compute:

$$i_{X_2} L_{X_1}\omega \underset{d\overline{\omega}=0}{=} i_{X_2} d i_{X_1}\overline{\omega} \underset{L_X = di_X + i_X d}{=} L_{X_2} i_{X_1}\overline{\omega} - d i_{X_2} i_{X_1}\overline{\omega} \underset{i_{X_2} i_{X_1}\overline{\omega}=0}{=} L_{X_2} i_{X_1}\overline{\omega}$$

And this proves the formula.

□

Observe that this lemma holds for all the cases, decomposable or non-decomposable and also in any dimension.

4.4 A common proposition

We will assume throughout the section that the foliation is decomposable since the proofs are supplied just in the decomposable cases. As a matter of fact, as we saw on the preceeding sections the proposition holds also for the focus-focus case. But we do not give a proof for this fact.

In subsequent sections we will try to identify the Hamiltonian vector field associated to f_1. The main goal will be to find new coordinates in such a way that X_1 can be identified with the Hamiltonian vector field of the function f_1 in convenient coordinates. The first step is given by the following proposition,

Proposition 4.4.1 *There exists a symplectic germ $\overline{\omega}_1$ equivalent to ω such that,*

$$i_{X_{1,\epsilon_1}}\overline{\omega}_1 = H_1 df_1 + H_2 df_2 \ .$$

for $\mathcal{F}$-basic functions H_1 and H_2.

Proof:

First by lemma 4.3.2 we can write

$$i_{X_{1,\epsilon_1}}\omega = H_1 df_1 + H_2 df_2 \ .$$

We distinguish the following cases:

4.4.1 Proof of proposition 4.4.1 in the non-completely hyperbolic cases

We prove 4.4.1 in the elliptic-hyperbolic case and the elliptic-elliptic case:

In this case, we can assume $\epsilon_1 = 1$. For the sake of simplicity, we write X_1 instead of $X_{1,1}$. Applying proposition 2.2.1 we can write:

$$\begin{array}{rcl} H_1(x_1,y_1,x_2,y_2) & = & h_1(x_1^2+y_1^2,x_2,y_2)+X_1(\overline{h}_1) \\ H_2(x_1,y_1,x_2,y_2) & = & h_2(x_1^2+y_1^2,x_2,y_2)+X_1(\overline{h}_2) \end{array}.$$

Now consider the 1-form $\alpha = \overline{h}_1 df_1 + \overline{h}_2 df_2$. Since α is $\mathcal{F}$-basic, we can apply lemma 4.3.1 taking $\omega_1 = \omega$. As a consequence, $\overline{\omega}_1 = \omega - d\alpha$ will be a symplectic germ equivalent to the initial ω. Let us check that for this $\overline{\omega}_1$ the conditions stated in the proposition are fulfilled. First, we calculate $i_{X_1}\overline{\omega}_1$. We have $i_{X_1}\overline{\omega}_1 = i_{X_1}\omega - i_{X_1}d\alpha$. Due to Cartan's formula, we have $i_{X_1}d\alpha = di_{X_1}\alpha + L_{X_1}\alpha$. But since α is $\mathcal{F}$-basic, in particular $i_{X_1}\alpha = 0$. So in the end, $i_{X_1}d\alpha = X_1(\overline{h}_1)df_1 + X_1(\overline{h}_2)df_2$. Finally, we have that $i_{X_1}\overline{\omega}_1 = h_1(x_1^2+y_1^2,x_2,y_2)df_1 + h_2(x_1^2+y_1^2,x_2,y_2)df_2$.

Now

$$\begin{array}{rcl} i_{X_1}\overline{\omega}_1 & = & h_1 df_1 + h_2 df_2 \\ i_{X_{2,\epsilon_2}}\overline{\omega}_1 & = & g_1 df_1 + g_2 df_2 \end{array}$$

for certain differentiable functions g_1 and g_2.

For the sake of simplicity we write X_2 instead of X_{2,ϵ_2}.

According to lemma 4.3.3 the following formula (5.1) takes place

$$L_{X_2} i_{X_1}\overline{\omega}_1 = L_{X_1} i_{X_2}\overline{\omega}_1.$$

From this formula we obtain the following relations,

$$\begin{array}{rcll} X_2(h_1) & = & X_1(g_1), & (IIa) \\ X_2(h_2) & = & X_1(g_2), & (IIb) \end{array}$$

Now, apply L_{X_1} to these equalities and use that $[X_1, X_2] = 0$ and the fact that $X_1(h_1) = 0$ and $X_1(h_2) = 0$ to get

$$\begin{aligned} X_1(X_1(g_1)) &= 0, \quad (A) \\ X_1(X_1(g_2)) &= 0, \quad (B) \end{aligned}$$

Now we want to prove that this implies $X_1(g_1) = 0$ and $X_1(g_2) = 0$. In order to do this, we consider the change to polar coordinates given by the equalities $x_1 = r\cos\theta, y_1 = r\sin\theta, x_2 = x_2, y_2 = y_2$. This change of coordinates is valid outside the meagre set $(0, 0, x_2, y_2)$. In these new coordinates, equations (A) and (B) are written as $\frac{\partial^2}{\partial\theta^2}(g_1) = 0$ and $\frac{\partial^2}{\partial\theta^2}(g_2) = 0$, respectively. So from these equations, the functions g_1 and g_2 are affine functions in the θ-coordinate. Since they are 2π-periodic in the coordinate θ, they have to be constant in the coordinate θ. And as a consequence the conditions $X_1(g_1) = 0$ and $X_1(g_2) = 0$ are satisfied in the whole neighbourhood of p considered. Finally, turning back to (II a) and (II b), we are led to the equalities $X_2(h_1) = 0$ and $X_2(h_2) = 0$. This completes the proof of the proposition in the elliptic-hyperbolic and the elliptic-elliptic cases.

4.4.2 Proof of proposition 4.4.1 in the completely hyperbolic cases

We prove proposition 4.4.1 in the hyperbolic-hyperbolic case.

For the sake of simplicity, we write X_1 instead of $X_{1,-1}$.

We consider the change of coordinates, $x_1 = \frac{u_1+v_1}{2}, y = \frac{u_1-v_1}{2}, x_2 = x_2, y_2 = y_2$. And X_1 in the new coordinates can be written as, $X_1 = -u_1\frac{\partial}{\partial u_1} + v_1\frac{\partial}{\partial v_1}$.

Applying proposition 2.2.2 we can write:

$$\begin{aligned} H_1(u_1, v_1, x_2, y_2) &= h_1(u_1.v_1, x_2, y_2) + X_1(\overline{h}_1) \\ H_2(u_1, v_1, x_2, y_2) &= h_2(u_1.v_1, x_2, y_2) + X_1(\overline{h}_2) \end{aligned}.$$

Now consider the 1-form $\alpha = \overline{h}_1 df_1 + \overline{h}_2 df_2$. Since α is $\mathcal{F}$-basic, we can apply lemma 4.3.1 taking $\omega_1 = \omega$. As a consequence, $\overline{\omega}_1 = \omega - d\alpha$ will be a symplectic germ equivalent to the initial ω. Let us check that for this $\overline{\omega}_1$ satisfies the conditions stated in the proposition. First, we compute $i_{X_1}\overline{\omega}_1$. We have $i_{X_1}\overline{\omega}_1 = i_{X_1}\omega - i_{X_1}d\alpha$. Due to Cartan's formula, we have $i_{X_1}d\alpha = di_{X_1}\alpha + L_{X_1}\alpha$. But since α is $\mathcal{F}$-basic, in particular $i_{X_1}\alpha = 0$. So, $i_{X_1}d\alpha = X_1(\overline{h}_1)df_1 + X_1(\overline{h}_2)df_2$. Finally, we have that

$$i_{X_1}\overline{\omega}_1 = h_1(u_1.v_1, x_2, y_2)df_1 + h_2(u_1.v_1, x_2, y_2)df_2.$$

Now

$$\begin{aligned} i_{X_1}\overline{\omega}_1 &= h_1 df_1 + h_2 df_2 \\ i_{X_{2,\epsilon_2}}\overline{\omega}_1 &= g_1 df_1 + g_2 df_2 \end{aligned}$$

for certain differentiable functions g_1 and g_2.

For the sake of simplicity we write X_2 instead of X_{2,ϵ_2}.

According to lemma 4.3.3 the following formula (5.1) takes place

$$L_{X_2} i_{X_1}\overline{\omega}_1 = L_{X_1} i_{X_2}\overline{\omega}_1.$$

From this formula we obtain the following relations,

$$\begin{array}{rcll} X_2(h_1) &=& X_1(g_1), & (IIa) \\ X_2(h_2) &=& X_1(g_2), & (IIb) \end{array}.$$

We are going to use this relations to draw conclusions about the (u_1, v_1)-jets of h_1 and g_1 along $\{(0, 0, x_2, y_2)\}$ using lemma 2.2.1.

By construction and as it was seen in the proof of lemma 2.2.2 the (u_1, v_1)-jet of h_1 along $\{(0, 0, x_2, y_2)\}$ is of the form

$$\sum_i h_{ii} u_1^i v_1^i.$$

On the other hand as it was seen in the proof of lemma 2.2.1, the (u_1, v_1)-jet of $X_1(g_1)$ along $\{(0, 0, x_2, y_2)\}$ has the form

$$\sum_{i \neq j} r_{ij} u_1^i v_1^j$$

for certain differentiable functions $r_{ij}(x_2, y_2)$. With all this information at hand, we can look at the equation (IIA), $X_2(h_1) = X_1(g_1)$ at the level of (u_1, v_1)-jets along $\{(0, 0, x_2, y_2)\}$. We obtain $\sum_i X_2(h_{ii})u_1^i v_1^i = \sum_{i \neq j} r_{ij} u_1^i v_1^j$. In particular, $X_2(h_{ii}) = 0, \forall i$. And from this relations $h_1 = S_1 + \phi_1$ where S_1 satisfies $X_2(S_1) = 0$ (and $X_1(S_1) = 0$) and ϕ_1 is an (u_1, v_1)-flat function along $\{(0, 0, x_2, y_2)\}$. Finally, apply lemma 2.2.2 to ensure that we can write $\phi_1 = X_1(R_1)$ for a smooth R_1. Therefore, so far we have $h_1 = S_1 + X_1(R_1), \quad X_1(S_1) = 0, \quad X_2(S_1) = 0$.

We may proceed in the same way for h_2 to write the following decomposition $h_2 = S_2 + X_1(R_2), \quad X_1(S_2) = 0, \quad X_2(S_2) = 0$.

Now,

$$i_{X_1}\overline{\omega}_1 = (S_1 + X_1(R_1))df_1 + (S_2 + X_1(R_2))df_2$$

for basic S_1 and S_2.

Finally, we apply Moser again 4.3.1 with $\alpha = R_1 df_1 + R_2 df_2$ to obtain a new symplectic form $\overline{\omega}_2$ equivalent to ω such that $i_{X_1}\overline{\omega}_1 = S_1 df_1 + S_2 df_2$ for basic functions S_1 and S_2. This ends the proof of the proposition in the hyperbolic-hyperbolic case and therefore the proof of the proposition.

□

4.4.3 A normalization result

Observe that $h_1 = -A$, that is to say $-h_1$ coincides with the coefficient function of $dx_1 \wedge dy_1$. If we could "normalize" our symplectic form in the (x_1, y_1)-direction (that is to say, find a foliation preserving symplectomorphism such that $A = 1$) we would be closer to our result. The following lemmas will ensure that we can "normalize" our symplectic form in a foliation preserving way.

Lemma 4.4.1 *Let ω be a symplectic form such that the foliation $\mathcal{F}$ is Lagrangian. Let $\overline{D}_1$ and D_2 stand for the distributions $\overline{D}_1 =< \frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1} >$ and $D_2 =< \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_2} >$.*

Let $\omega_{|\overline{D}_1}$ and $\omega_{|D_2}$ be the restriction of ω to the planes integrating the distributions $\overline{D}_1$ and D_2, respectively. Then $\omega_{|\overline{D}_1}$ and $\omega_{|D_2}$ are symplectic forms in a neighbourhood of the point p.

Proof:

The condition $\omega(X_{1,\epsilon_1}, X_{2,\epsilon_2}) = 0$ for all q in the neighbourhood considered, implies in particular the following relations:

$$w(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_2})_{|q_{(y_1,x_2)}} = 0, \quad q_{(y_1,x_2)=(0,y_1,x_2,0)} \quad (I)$$

$$w(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})_{|q_{(y_1,y_2)}} = 0, \quad q_{(y_1,y_2)=(0,y_1,0,y_2)} \quad (II)$$

$$w(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2})_{|q_{(x_1,x_2)}} = 0, \quad q_{(x_1,x_2)=(x_1,0,x_2,0)} \quad (III)$$

$$w(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial x_2})_{|q_{(y_1,x_2)}} = 0, \quad q_{(y_1,x_2)=(x_1,0,0,y_2)} \quad (IV)$$

In particular all these relations are fulfilled at the point $p = (0,0,0,0)$. If $\omega_{|\overline{D}_1}$ was not symplectic at p, then the $w(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1})_{|p} = 0$. But this condition together with conditions (I) and (III) would imply that $\omega_{|p}$ would vanish on a 3-dimensional vector space, which is not possible because the initial ω is symplectic. In the same way, we can prove that $\omega_{|D_2}$ is symplectic at p. Since we have proved that $\omega_{|\overline{D}_1}$ and $\omega_{|D_2}$ are symplectic at p and the condition of being symplectic is an open condition, they are also symplectic in a neighbourhood of p.

□

Lemma 4.4.2 *There exists a symplectic germ $\overline{\omega}_2$ equivalent to $\overline{\omega}_1$ such that:*

$$i_{X_1}\overline{\omega}_2 = -df_1 + \overline{h}_2 df_2.$$

Proof: Due to 4.4.1, the plane $\Pi = (x_1, y_1, 0, 0)$ is symplectic and this implies that $A \neq 0$ in a neighbourhood of p. We apply the same trick as in the proof of lemma 2.3.3 in chapter 2 but with two variables u and v corresponding to the first integrals of the foliation f_1 and f_2. That is, observe that if $\psi(f_1, f_2)$ is any differentiable function of f_1 and f_2 such that $\psi(0,0) \neq 0$ and U stands for a neighbourhood of the origin where everything is defined, the mapping

$$\begin{array}{cccc} G: & (U,0) & \longrightarrow & (G(U),0) \\ & (x_1,y_1,x_2,y_2) & \longrightarrow & (x_1 \cdot \psi(f_1,f_2), y_1 \cdot \psi(f_1,f_2), x_2, y_2) \end{array}$$

defines a germ of diffeomorphism preserving the foliation defined by f_1 and f_2. Consider the equation

$$\frac{d}{du}(\psi^2(u,v) \cdot u) = A(u,v),$$

where $u = f_1$ and $v = f_2$. As we saw in the proof of 2.3.3 ψ is smooth and normalizes the coefficient function of $dx_1 \wedge dy_1$. Furthermore, since the function A is basic, this diffeomorphism is foliation preserving. Now if we define $\overline{\omega}_2 = (\phi^{-1})^* \overline{\omega}_1$. Then $\overline{\omega}_2$ is equivalent to $\overline{\omega}_1$ and satisfies $i_{X_1}\overline{\omega}_2 = -df_1 + \overline{h}_2 df_2$ for a certain differentiable function $\overline{h}_2$.

□

4.5 A special Hamiltonian for the non-completely hyperbolic cases

The aim of this section is to prove the existence of a diffeomorphism taking the initial symplectic form to a symplectic form for which the vector field X_1 is the Hamiltonian vector field associated to f_1. The proof uses the preceeding lemmas but is shorter if the singularity is non-completely hyperbolic. The hyperbolic-hyperbolic case will be treated separately in the next section.

The result is summed up in the following proposition,

Proposition 4.5.1 *Let $\mathcal{F}$ be a foliation of elliptic-elliptic type or elliptic-hyperbolic type. Let X_1 be the vector field $X_1 = x_1\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_1}$ belonging to $\mathcal{F}$. Under the assumptions of lemma 4.4.2, we have:*

$$i_{X_1}\overline{\omega}_2 = -df_1.$$

We present two proofs of this proposition. The first one uses decompositions of symplectic 2-forms while the second one is based on a geometrical argument.

4.5.1 First proof of proposition 4.5.1

Proof:

Let us recap information on the symplectic form.

Since $\overline{\omega}_2$ is locally exact, we can write:

$$\overline{\omega}_2 = dx_1 \wedge dy_1 + d(A_1 dx_1 + B_1 dy_1 + A_2 dx_2 + B_2 dy_2).$$

Given a smooth function f, we denote by $d_{(1)}(h) = \frac{\partial h}{\partial x_1}dx_1 + \frac{\partial h}{\partial y_1}dy_1$ and $d_{(2)}(h) = \frac{\partial h}{\partial x_2}dx_2 + \frac{\partial h}{\partial y_2}dy_2$. By lemma 4.4.2,

$$i_{X_1}\overline{\omega}_2 = -df_1 + \overline{h}_2 df_2.$$

This yields $d_{(1)}(A_1 dx_1 + B_1 dy_1) = 0$ and therefore, $A_1 dx_1 + B_1 dy_1 = d_{(1)}(g_1)$ for a certain differentiable function. On the other hand, taking into account that $d(g_1) = d_{(1)}(g_1) + d_{(2)}(g_1)$, the above symplectic form becomes,

$$\overline{\omega}_2 = dx_1 \wedge dy_1 + d(d(g_1) - d_{(2)}(g_1) + A_2 dx_2 + B_2 dy_2).$$

So after gathering coefficients of the terms dx_2 and dy_2 we obtain,

$$\overline{\omega}_2 = dx_1 \wedge dy_1 + d(C_2 dx_2 + D_2 dy_2),$$

for certain smooth functions C_2 and D_2.

Now, we compute the contraction $i_{X_1}\overline{\omega}_2$ again with this expression to get:

$$i_{X_1}\overline{\omega}_2 = -df_1 + X_1(C_2)dx_2 + X_1(D_2)dy_2.$$

The Lagrangian condition yields in the elliptic-elliptic case,

$$\begin{aligned} X_1(C_2) &= \overline{h}_2(x_2) \\ X_1(D_2) &= \overline{h}_2(y_2) \end{aligned}$$

And

$$\begin{aligned} X_1(C_2) &= \overline{h}_2(x_2) \\ X_1(D_2) &= -\overline{h}_2(y_2) \end{aligned}$$

in the elliptic-hyperbolic case.

In both cases, since $X_1(\overline{h}_2) = 0$, we can apply L_{X_1} in these relations to obtain,

$$X_1(X_1(C_2)) = 0 \quad , \quad X_1(X_1(D_2)) = 0.$$

Using these equations, we want to deduce that $X_1(C_2) = 0$ and $X_1(D_2) = 0$.

In order to do this, we consider the change to polar coordinates given by the equalities $x_1 = r\cos\theta, y_1 = r\sin\theta, x_2 = x_2, y_2 = y_2$. This change of coordinates is valid outside the meagre set $(0, 0, x_2, y_2)$. In these new coordinates the equations above are written as $\frac{\partial^2}{\partial\theta^2}(C_2) = 0$ and $\frac{\partial^2}{\partial\theta^2}(D_2) = 0$, respectively. So from these equations,C_2 and D_2 are affine functions in the α-coordinate. Since they are 2π-periodic in the coordinate α, they have to be constant in the coordinate α. And as a consequence the conditions $X_1(C_2) = 0$ and $X_1(D_2) = 0$ are satisfied in the whole neighbourhood of p considered.

Finally, from this equations we obtain $\overline{h}_2 = 0$, this proves that X_1 is a Hamiltonian vector field with Hamiltonian function f_1 and this ends the proof of the proposition.

□

4.5.2 The Bott-Weinstein connection and a geometrical proof of proposition 4.5.1

In this short section we propose a digression. We will give another proof of proposition 4.5.1 based on geometrical arguments concerning the Bott-Weinstein connection. Observe that, a posteriori, the vector field X_1 is Hamiltonian. Hamiltonian vector fields are a special class of parallel vector fields with respect to the Bott-Weinstein connection defined in the neighbouring regular leaves of the Lagrangian foliation. Let us introduce the notion of Bott-Weinstein connection for a regular Lagrangian foliation.

The Bott-Weinstein connection associated to a Lagrangian foliation. Let $\mathcal{F}$ be a regular n-dimensional Lagrangian foliation. We denote by ∇ the Bott-Weinstein [59] connection associated to $\mathcal{F}$. We recall that the Hamiltonian vector fields of the functions which locally define $\mathcal{F}$ are parallel with respect to ∇. Now the question arises: Is the converse true? That is to say, Can we assert that a parallel vector field is locally Hamiltonian? The following innocuous example can help us to see that this is false in general. For example take $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$, consider the regular Lagrangian foliation generated by the vector fields $X = \frac{\partial}{\partial x_1}$ and $Y = \frac{\partial}{\partial x_2}$. Those vector fields define in turn a basis of parallel vector fields with respect to the Bott-Weinstein connection. Consider now the vector field $Z = e^{y_2} \frac{\partial}{\partial x_1}$. This vector field is parallel but since $i_Z \omega = e^{y_2} dy_1$, it is not locally Hamiltonian. As this example shows the affirmative answer is far from being true; But in the case the foliation is given by the vector fields X_1 and X_2 of elliptic-elliptic type or hyperbolic-elliptic type, we use the existence of a Hamiltonian S^1-action tangent to the regular leaves of the foliation to prove that the parallel vector field X_1 is indeed Hamiltonian. Namely, recall that in the second section of chapter 3 we recovered a Hamiltonian S^1-action for the elliptic-hyperbolic case and the elliptic-elliptic case. We will use the existence of this action in the elliptic-hyperbolic case to prove the result. In the completely elliptic case we will use the Liouville-Mineur-Arnold theorem applied to the compact regular leaves. This, in particular, leads to another proof of proposition 4.5.1.

Second proof of 4.5.1

Let $\mathcal{S}$ be the singular set for the foliation $\mathcal{S} = \{(0,0,x_2,y_2)\} \cup \{(x_1,y_1,0,0)\}$. We denote by $\mathcal{B}$ the dense set $\mathcal{B} = M \setminus \mathcal{S}$. Then $\mathcal{F}' = \mathcal{F} \cap \mathcal{B}$ is a regular Lagrangian foliation. Let ∇ be the Bott-Weinstein connection associated to $\mathcal{F}'$. We are going to prove that X_1 is Hamiltonian in $\mathcal{B}$.

We are going to distinguish cases:

- The Elliptic-elliptic case.

  The foliation $\mathcal{F}'$ is a regular foliation by tori on $\mathcal{B}$ and the functions f_1 and f_2 are regular functions. According to Liouville-Mineur-Arnold, there exist a basis of Hamiltonian vector fields Z_1 and Z_2 which are periodic of constant period 2π and which are tangent to the foliation by tori. Those vector fields form a basis of parallel vector fields, so we may write the vector field $X_1 = g_1 Z_1 + g_2 Z_2$ for basic functions g_1 and g_2. Now the vector fields X_1, Z_1 and Z_2 are periodic vector fields of constant period, hence $\frac{g_1}{g_2}$ takes values in $\mathbb{Q}$ therefore using continuity the quotient $\frac{g_1}{g_2}$ is a rational number $\frac{p}{q}$ with $(p,q)=1$. Summing up, we can write $X_1 = g_2(\frac{p}{q} Z_1 + Z_2)$. If we proof that g_2 is constant then as a consequence X_1 will be a Hamiltonian vector field.

  In order to do this we need the following well-known sublemma which will be useful later.

  Sublemma 4.5.1 *Let X_{G_1} and X_{G_2} be two Hamiltonian vector fields tangent to $\mathcal{F}$. Denote by $\phi^s_{X_{G_1}}$ and $\phi^s_{X_{G_2}}$ the time-s-map of X_{G_1} and X_{G_2} respectively.*

Then

$$\phi^s_{X_{G_1}+X_{G_2}} = \phi^s_{X_{G_1}} \circ \phi^s_{X_{G_2}}.$$

Proof Since $\{G_1, G_2\} = \omega(X_{G_1}, X_{G_2})$ and X_{G_1} and X_{G_2} are tangent to the Lagrangian fibration $\mathcal{F}$ then $\{G_1, G_2\}_L = 0$ for any regular fiber L of $\mathcal{F}$. On the other hand, since the set of regular fibers is dense and X_{G_1} and X_{G_2} are also tangent along the singular fibers, the bracket $\{G_1, G_2\}$ vanishes everywhere.

This implies in turn that $[X_{G_1}, X_{G_2}] = 0$ and therefore

$$\phi^s_{X_{G_1}} \circ \phi^t_{X_{G_2}} = \phi^t_{X_{G_2}} \circ \phi^s_{X_{G_1}}, \quad \forall s, t. \tag{4.5.1}$$

Now consider $\alpha_s = \phi^s_{X_{G_1}} \circ \phi^s_{X_{G_2}}$. Due to 4.5.1, α_s is a one-parameter subgroup. It remains to compute the infinitesimal generator.

Since

$$\frac{d(\phi^s_{X_{G_1}} \circ \phi^s_{X_{G_2}})}{ds} = \frac{d(\phi^t_{X_{G_1}} \circ \phi^s_{X_{G_2}})}{dt}_{|t=s} + \frac{d(\phi^s_{X_{G_1}} \circ \phi^r_{X_{G_2}})}{dr}_{|r=s},$$

setting $s = 0$ this expression implies that the infinitesimal generator of α_s is $X_{G_1} + X_{G_2}$.

In particular this proves that

$$\phi^s_{X_{G_1}+X_{G_2}} = \phi^s_{X_{G_1}} \circ \phi^s_{X_{G_2}}, \quad \forall s.$$

□

Now going back to the vector field X_1 applying this sublemma the period of X_1 is $\frac{2\pi q}{g_2}$. But X_1 is a periodic vector field with constant period. Therefore g_2 is constant and the vector field is Hamiltonian on $\mathcal{B}$. Observe that since X_1 is Hamiltonian on $\mathcal{B}$ we obtain $d(i_{X_1}\overline{\omega}_2) = 0$ on $\mathcal{B}$. This implies that

$\frac{\partial}{\partial x_1}h_2$ and $\frac{\partial}{\partial y_1}h_2$ vanish on B and hence, using the density of the set B, they vanish everywhere. So h_2 does not depend on the variables x_1 and y_1. But since $i_{X_1}\overline{\omega}_{2|(0,0,x_2,y_2)} = 0$, the function h_2 vanishes.

Therefore, $i_{X_1}\overline{\omega}_2 = -df_1$ as we wanted.

- The elliptic-hyperbolic case

  In this case we cannot go straight to the regular foliation $\mathcal{F}'$ (Liouville-Mineur-Arnold only works when we consider regular foliations with compact leaves and in the elliptic-hyperbolic case those leaves are cylinders). So let us consider the auxiliary foliation $\mathcal{F}'' = \mathcal{F} \cap (M \setminus \{(0,0,x_2,y_2)\}$, By virtue of proposition in chapter 3 , we know that there exists a unique Hamiltonian S^1-action which is tangent to the leaves of the $\mathcal{B}''$. Consider now Y_1, a vector field generated by this Hamiltonian S^1-action with constant period $2\pi k$. Complete Y_1 to a basis Y_1, Y_2 of Hamiltonian vector fields tangent to the leaves of $\mathcal{F}'$. Observe that Y_2 cannot be periodic because the leaves of $\mathcal{F} \cap \mathcal{B}$ are cilinders. Using lemma 4.4.2, the vector field X_1 can be expressed as $X_1 = X_{f_1} - h_2 X_{f_2}$ for a basic function h_2. As a consequence, the vector field X_1 is parallel with respect to ∇ and therefore we can write $X_1 = \alpha_1 Y_1 + \alpha_2 Y_2$ for basic functions α_1 and α_2. Since X_1 and Y_1 have periodic orbits but Y_2 does not, α_2 has to vanish. Now as $X_1 = \alpha_1 Y_1$, the period of X_1 has to be $\frac{2\pi k}{\alpha_1}$. But the period of X_1 is 2π, so $\alpha_1 = k$ and therefore, X_1 is Hamiltonian on $\mathcal{B}$. Finally, X_1 is Hamiltonian on $\mathcal{B}$ yields $d(i_{X_1}\overline{\omega}_2) = 0$ on $\mathcal{B}$. This implies that $\frac{\partial}{\partial x_1}h_2$ and $\frac{\partial}{\partial y_1}h_2$ vanish on B and hence, using the density of the set B, they vanish everywhere. So h_2 does not depend on the variables x_1 and y_1. But since $i_{X_1}\overline{\omega}_{2|(0,0,x_2,y_2)} = 0$, the function h_2 vanishes. And in the end, the condition $i_{X_1}\overline{\omega}_2 = -df_1$ is met.

This ends the second proof of proposition 4.5.1.

□

4.6 A special Hamiltonian for the completely hyperbolic cases

It remains to prove an equivalent proposition for the hyperbolic-hyperbolic case. In the hyperbolic-hyperbolic case we do not have a privileged S^1-action. This means that to reach a similar result we need to apply our Moser type lemma again.

Proposition 4.6.1 *Let $\mathcal{F}$ be a foliation of hyperbolic-hyperbolic type. Let X_1 be the vector field $X_1 = x_1\frac{\partial}{\partial y_1} + y_1\frac{\partial}{\partial x_1}$ belonging to $\mathcal{F}$. Under the assumptions of lemma 4.4.2, there exists an equivalent symplectic form $\overline{\omega}_3$ for which we have:*

$$i_{X_1}\overline{\omega}_3 = -df_1.$$

Proof:

Since $\overline{\omega}_2$ is locally exact, we can write:

$$\overline{\omega}_2 = dx_1 \wedge dy_1 + d(A_1dx_1 + B_1dy_1 + A_2dx_2 + B_2dy_2).$$

Using lemma 4.4.2,

$$i_{X_1}\overline{\omega}_2 = -df_1 + \overline{h}_2 df_2.$$

This yields $d_{(1)}(A_1dx_1 + B_1dy_1) = 0$ and therefore, $A_1dx_1 + B_1dy_1 = d_{(1)}(g_1)$ for a certain differentiable function. And collecting the coefficients of the terms dx_2 and dy_2 as it was seen in the first proof of 4.5.1, we can write

$$\overline{\omega}_2 = dx_1 \wedge dy_1 + d(C_2dx_2 + D_2dy_2)$$

for certain smooth functions C_2 and D_2.

Now, we compute the contraction $i_{X_1}\overline{\omega}_2$ again with this expression to get:

$$i_{X_1}\overline{\omega}_2 = -df_1 + X_1(C_2)dx_2 + X_1(D_2)dy_2.$$

After making a change of coordinates in such a way that $f_1 = x_1y_1$ and $f_2 = x_2y_2$, the Lagrangian condition implies,

$$X_1(C_2) = \overline{h}_2(y_2)$$
$$X_1(D_2) = \overline{h}_2(x_2)$$

Let us use the first equation, for instance, to draw conclusions about the (x_1, y_1)-jets of the function h_2 along $\{(0,0,x_2,y_2)\}$.

As observed in the proof of 2.2.2, the (x_1, y_1)-Taylor expand of the function $X_1(C_2)$ along $\{(0,0,x_2,y_2)\}$ has the form $\sum_{i\neq j} c_{ij}x_1^i y_1^j$ where c_{ij} are smooth functions in the variables (x_2, y_2). Therefore the funtion h_2 has a (x_1, y_1)-Taylor expand of the form $\sum_{i\neq j} h_{ij}x_1^i y_1^j$ for certain differentiable h_{ij}.

But, since the function h_2 is basic, $X_1(h_2) = 0$. This implies that h_2 is (x_1, y_1)-flat along $\{(0,0,x_2,y_2)\}$. Now we apply 2.2.2 to ensure that there exists a smooth h such that $h_2 = X_1(h)$.

So far,

$$i_{X_1}\overline{\omega}_2 = -df_1 + (X_1(h))df_2.$$

Finally, we can apply lemma 4.3.1 with the basic 1-form $\alpha = hdf_2$ to obtain a new symplectic form $\overline{\omega}_3$, equivalent to $\overline{\omega}_2$ for which,

$$i_{X_1}\overline{\omega}_3 = -df_1.$$

This ends the proof of the proposition.

□

4.7 Two symplectic orthogonal distributions and symplectic linearization

For the sake of simplicity, let us unify the notation in all the cases and denote $\overline{\omega}$ the symplectic form equivalent to the initial ω for which X_1 is Hamiltonian with

Hamiltonian function f_1. Once reached this point, we are close to the symplectic orthogonal decomposition. Let us proof the following lemma before:

Lemma 4.7.1 *The distribution $D_1 =< X, Y >$ defined by the relations:*

$$i_X\overline{\omega} = dx_1$$

$$i_Y\overline{\omega} = dy_1$$

is $\mathcal{C}^\infty$, symplectic in a neighbourhood of p and involutive everywhere.

Proof: First of all, since $\overline{\omega}$ is symplectic and the forms dx_1 and dy_1 are differentiable and independent, the distribution D_1 is clearly $\mathcal{C}^\infty$ and regular. Now let us prove that this distribution is symplectic. Observe that this distribution is symplectically orthogonal to the distribution $D_2 =< \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_2} >$ defined in lemma 4.4.1. Since D_2 is symplectic in a neighbourhood of p (lemma 4.4.1), the distribution D_1 is also symplectic in a neighbourhood of p. Now let us see that this distribution is involutive. We have to check that $[X, Y] \in D_1, \forall X, Y \in D_1$. In fact, it is enough to prove that $[X, Y] \in D_1$ for vector fields which are independent on a dense set in the neighbourhood considered. So we can take $X = X_1$. By Leibnitz's rule:

$$L_{X_1}(\overline{\omega}(Y, \frac{\partial}{\partial x_2})) = L_{X_1}(\overline{\omega})(Y, \frac{\partial}{\partial x_2}) + \overline{\omega}(L_{X_1}Y, \frac{\partial}{\partial x_2}) + \overline{\omega}(Y, L_{X_1}(\frac{\partial}{\partial x_2}))$$

Now if we take any $Y \in D_1$ then the left hand side of the equality above equals zero. As for the right hand side: The first term is zero because X_1 is Hamiltonian and, in particular, it is symplectic; the third term vanishes because $L_{X_1}(\frac{\partial}{\partial x_2}) = 0$. So we are led to $\overline{\omega}(L_{X_1}Y, \frac{\partial}{\partial x_2}) = 0$. In the same way, we prove that $\overline{\omega}(L_{X_1}Y, \frac{\partial}{\partial y_2}) = 0$ and therefore the distribution is involutive.

□

Now let us use the distribution above to prove Theorem 4.2.1. We recall the precise statement of Theorem 4.2.1.

Theorem 4.2.1(Symplectically orthogonal decomposition)

Let ω be a symplectic germ for which $\mathcal{F}$ is generically Lagrangian. Then there exists a symplectic germ $\overline{\omega}$ equivalent to ω and there exist two symplectic distributions D_1 and D_2 such that:

1. *D_1 and D_2 are involutive and symplectically orthogonal with respect to $\overline{\omega}$.*

2. *$X_{1,\epsilon_1} \in D_1$ and $X_{2,\epsilon_2} \in D_2$.*

Proof of Theorem 4.2.1: Consider D_1 the distribution defined in the above lemma. Observe that propositions 4.5.1 and 4.6.1 prove that this distribution contains the vector field X_1 in the elliptic-elliptic, elliptic-hyperbolic and hyperbolic-hyperbolic cases. On the other hand, we consider the distribution $D_2 =< \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_2} >$, this distribution contains the vector field X_2. The distribution D_2 is symplectic due to lemma 4.4.1 and trivially involutive. The distribution D_1 is symplectically orthogonal to D_2 by construction. This ends up proving Theorem 4.2.1 in all the cases.

□

Next step, we use the symplectic orthogonal decomposition to prove that there is just one symplectic germ making the foliation $\mathcal{F}$ into a Lagrangian foliation.

For the sake of clarity, we recall the precise statement of Theorem 4.2.2:

Theorem 4.2.2

Let ω be a symplectic germ at p for which $\mathcal{F}$ is generically Lagrangian then ω is equivalent to $\omega_0 = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$.

Proof:

Firstly, by virtue of Theorem 4.2.1 there exist symplectically orthogonal distributions D_1 and D_2 containing X_1 and X_2, respectively. Since these regular distributions are involutive, there are regular foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ integrating D_1 and D_2 respectively. Furthermore, Frobenius Theorem provides new coordinates $(\overline{x}_1, \overline{y}_1, \overline{x}_2, \overline{y}_2)$ in a neighbourhood of p such that the leaves of $\mathcal{F}_1$ are $L_{1b} = \{(\overline{x}_1, \overline{y}_1, b_1, b_2), b_1, b_2 \in \mathbb{R}\}$ and the leaves of $\mathcal{F}_2$ are $L_{2a} = \{(a_1, a_2, \overline{x}_2, \overline{y}_2), a_1, a_2 \in \mathbb{R}\}$.

Since D_1 and D_2 are symplectically ortogonal and since $d\overline{\omega}_2 = 0$, in these new coordinates the symplectic form can be written as:

$$\overline{\omega}_2 = A(\overline{x}_1, \overline{y}_1) d\overline{x}_1 \wedge d\overline{y}_1 + B(\overline{x}_2, \overline{y}_2) d\overline{x}_2 \wedge d\overline{y}_2.$$

Since X_1 belongs to D_1 and X_2 belongs to D_2 it remains to apply the known results of symplectic uniqueness in dimension 2 (theorem 2.3.1 in section 2) in the $(\overline{x}_1, \overline{y}_1)$-coordinates and in the $(\overline{x}_2, \overline{y}_2)$-coordinates separately.

More exactly, let us recall this results in dimension 2 and then we perform a composition of symplectomorphisms.

Theorem 2.3.1

Let (M^2, ω_1) be a 2-dimensional symplectic manifold endowed with coordinates (x, y) and let $\mathcal{F}$ be a singular Lagrangian foliation with an elliptic or hyperbolic singularity at the origin $(0, 0)$, then there exists a local diffeomorphism ϕ preserving $\mathcal{F}$ such that $\phi^*(dx \wedge dy) = \omega_1$.

Let ϕ_1 and ϕ_2 be the diffeomorphisms provided by the above theorem, attached to D_1 and D_2 respectively.

We define a local diffeomorphism

$$\phi(\overline{x}_1, \overline{y}_1, \overline{x}_2, \overline{y}_2) = (\phi_1(\overline{x}_1, \overline{y}_1), \phi_2(\overline{x}_2, \overline{y}_2)).$$

This diffeomorphism preserves the foliation $\mathcal{F}$ and satisfies that

$$\phi^*(dx_1 \wedge dy_1 + dx_2 \wedge dy_2) = \overline{\omega}_2.$$

□

Chapter 5

Higher dimensions

The aim of this chapter is to give a general result for symplectic linearization in arbitrary dimension and for foliations defined by completely integrable systems in a neighbourhood of a rank k-orbit of the system. In the preceeding chapter we set a precedent for induction. The idea of the proof relies on an inductive process and the symplectic orthogonal decomposition. Those tecniques were thoroughly studied in the last chapter. Most of the lemmas contained in this chapter will be claimed without proof. In fact, they are the higher dimensional counterparts to the lemmas contained in chapter 4. That is why we omit their proofs understanding that the dimension makes no difference in that matter.

The chapter is organized as follows: In the first section we study the rank 0-foliations and we prove the symplectic uniqueness for those foliations in the case there are no focus-focus components. In the second section we pose the problem for rank k-foliations and we prove that the linear foliation in the covering is symplectically linearizable. We do it by defining a splitting of the regular and singular parts. The splitting is again a symplectic orthogonal decomposition. For the regular part, we apply the classical Liouville-Mineur-Arnold theorem and we apply the result of symplectic uniqueness established in the first section for the singular part.

5.1 Rank 0 foliations in any dimension

In this section we deal with the rank 0 case. That is assume that $\mathcal{F}$ is the linear foliation defined by $\mathcal{F} =< X_1, \dots, X_n >$ where the vector fields are the linear vector fields introduced in the first chapter.

This foliation is a linear foliation on M^{2n} with a rank 0 singularity at the origin p. Assume that the Williamson type of the singularity is (k_e, k_h, k_f). Recall that the foliation is then generated by the following vector fields,

$$\begin{array}{l} X_i = -y_i\frac{\partial}{\partial x_i} + x_i\frac{\partial}{\partial y_i} \ \text{ for } \ 1 \leq i \leq k_e \ , \\ X_i = y_i\frac{\partial}{\partial x_i} + x_i\frac{\partial}{\partial y_i} \ \text{ for } \ k_e+1 \leq i \leq k_e+k_h \ , \\ X_i = x_i\frac{\partial}{\partial x_{i+1}} - y_{i+1}\frac{\partial}{\partial y_i} - x_{i+1}\frac{\partial}{\partial x_i} + y_i\frac{\partial}{\partial y_{i+1}} \ \text{ and} \\ X_{i+1} = -x_i\frac{\partial}{\partial x_i} + y_i\frac{\partial}{\partial y_i} - x_{i+1}\frac{\partial}{\partial x_{i+1}} + y_{i+1}\frac{\partial}{\partial y_{i+1}} \ \text{ for } \ i = k_e+k_h+2j-1, \ 1 \leq j \leq k_f \end{array}$$

We want to prove the following theorem,

Theorem 5.1.1 *Let ω be a symplectic form defined in a neighbourhood of the origin for which $\mathcal{F}$ is Lagrangian, then there exists a local diffeomorphism ϕ : $(U,p) \longrightarrow (\phi(U),p)$ such that ϕ preserves the foliation and $\phi^*(\sum_i dx_i \wedge dy_i) = \omega$, being x_i, y_i local coordinates on $(\phi(U),p)$.*

Remarks

Observe that this theorem has already been proved in dimension 2 in chapter 2 and in dimension four in chapter 4. So we propose here to prove it by induction. Before starting the induction process we claim the following two lemmas (we omit the proof) which are the higher dimensional analogues of lemmas 4.3.2 and 4.3.1, respectively. We include the proof here just in the case that the Williamson type is $(k_e, k_h, 0)$.

Lemma 5.1.2 *There exists $\mathcal{C}^\infty$-functions h_i^j, $\forall i, j \in \{1, n\}$ such that*

$$i_{X_i}\omega = \sum_{j=1}^{n} h_j^i df_j$$

The following lemma is a foliation-preserving version of the Moser path method in the general case.

Lemma 5.1.3 *Let α be an $\mathcal{F}$-basic 1-form and let ω_1 be a symplectic germ for which $\mathcal{F}$ is Lagrangian. Then:*

1. *The 2-form $\omega_0 = \omega_1 - d\alpha$ is a symplectic structure in a neighbourhood of p and makes the foliation Lagrangian.*

2. *There is a diffeomorphism η between two neighbourhoods of p preserving $\mathcal{F}$ and such that $\eta^*(\omega_1) = \omega_0$.*

We will also need the following lemma. We already proved this lemma in any dimension in the previous chapter.

Lemma 4.3.3
The following equality holds

$$L_{X_i} i_{X_j}\omega = L_{X_j} i_{X_i}\omega, \quad \forall i, j \in \{1, n\}.$$

Now we can start the proof of the theorem.

Proof of theorem 5.1.1

We prove it by induction on n. Recall that $\dim M = 2n$.

- Case $n = 1$. This is nothing but the symplectic linearization result (theorem 2.3.1) in dimension 2 which is contained in chapter 2.

- Now let us assume that the theorem holds for $r \leq n-1$ let us prove that the result is true for $r = n$. Observe that when we pass from $n-1$ to n we attach a component of the foliation which can be elliptic or hyperbolic. In the case we attach an elliptic or hyperbolic component we are adding a vector field (generically independent from the others) to the distribution.

  So we need to make out the following cases

  1. **Attaching an elliptic component.**

     According to lemma 5.1.2 we can write,

     $$i_{X_i}\omega = \sum_{k=1}^{n} h_k^i df_k, \quad \forall i \leq n$$

     On the other hand for each function h_k^n we can apply the decomposition of proposition 2.2.1 in page 23 (chapter 2) to write,

     $$h_k^n = \overline{h}_k^n + X_n(H_k^n), \quad X_n(\overline{h}_k^n) = 0,$$

     for convenient differentiable functions. Now consider the 1-form $\alpha = \sum_{k=1}^{n} H_k^n df_k$. We can apply lemma 5.1.3 with this 1-form and ω will be equivalent to $\overline{\omega} = \omega - d\alpha$.

     Now we compute $i_{X_n}\overline{\omega} = i_{X_n}\omega - i_{X_n}d\alpha$. We can compute the second term in the right hand side of the equality using Cartan's formula to obtain,

     $$i_{X_n}d\alpha = L_{X_n}\alpha - di_{X_n}\alpha = \sum_{k=1}^{n} X_n(H_k^n)df_k$$

     where in the last equality we have used the fact that the 1-form is $\mathcal{F}$-basic.

     In the end this yields,

$$i_{X_n}\overline{\omega} = \sum_{k=1}^{n} \overline{h}_k^n df_k, \quad X_n(\overline{h}_k^n) = 0.$$

For the sake of simplicity we will keep the notation h_k^n instead of $\overline{h}_k^n$ bearing in mind that we can assume that $X_n(h_k^n) = 0$. Observe that formula 5.1 $L_{X_i} i_{X_j}\omega = L_{X_j} i_{X_i}\omega$ of lemma 4.3.3 yields the relations,

$$X_j(h_k^i) = X_i(h_k^j), \quad \forall i, j, k \tag{5.1.1}$$

After taking $i = n$ in the relation above and applying L_{X_n} both sides we obtain,

$$X_n^2(h_k^j) = 0$$

now since X_n is a periodic vector field we can reproduce the arguments exposed in the first proof of proposition 4.5.1 in page 69 to obtain $X_n(h_k^j) = 0$, now going back to equation 5.1.1 we get,

$$X_j(h_k^n) = 0, \quad \forall j.$$

Next step, we normalize in the (x_n, y_n)-direction, that is to say as we did in the proof of 2.3.3 in page 34 first we consider the smooth solution of the differential equation

$$\frac{d}{du}(\psi^2(u, \hat{x}_n, \hat{y}_n) \cdot u) = h_n^n(u, \hat{x}_n, \hat{y}_n),$$

where $u = x_n^2 + y_n^2$, $\hat{x}_n = (x_1, \ldots, x_{n-1})$ and $\hat{y}_n = (y_1, \ldots, y_{n-1})$ and then we define the foliation preserving diffeomorphism

$$\phi(\hat{x}_n, \hat{y}_n, x_n, y_n) = (\hat{x}_n, \hat{y}_n, \psi \cdot x_n, \psi \cdot y_n).$$

This diffeomorphism takes the symplectic form to a new symplectic form $\overline{\omega}_1$ such that,

$$i_{X_n}\overline{\omega}_1 = \sum_{k=1}^{n-1} \overline{h}_k^n df_k - df_n, \quad X_n(\overline{h}_k^n) = 0. \tag{5.1.2}$$

Taking into account the expression above and since $\overline{\omega}_1$ is locally exact, we can write:

$$\overline{\omega}_1 = dx_n \wedge dy_n + d(\sum_{k<n}(A_k dx_k + B_k dy_k)).$$

We use the notation by $d_{(1)}(h) = \frac{\partial h}{\partial x_n}dx_n + \frac{\partial h}{\partial y_n}dy_n$ and $d_{(2)}(h) = \sum_{k<n} \frac{\partial h}{\partial x_k}dx_k + \frac{\partial h}{\partial y_k}dy_k$ with $d_{(1)}(\sum_{k<n}(A_k dx_k + B_k dy_k)) = 0$ and therefore, $\sum_{k<n}(A_k dx_k + B_k dy_k) = d_{(1)}(g_n)$ for a certain differentiable function. After gathering coefficients conveniently this yields,

$$\overline{\omega}_1 = dx_n \wedge dy_n + d(\sum_{k<n}(\overline{A_k} dx_k + \overline{B}_k dy_k)),$$

for certain smooth functions $\overline{A}_k$ and $\overline{B}_k$.

Now, we compute the contraction $i_{X_n}\overline{\omega}_1$ again with this expression to get:

$$i_{X_n}\overline{\omega}_1 = -df_n + \sum_{k<n}(X_n(\overline{A}_k)dx_k + X_n(\overline{B}_k)dy_k).$$

Comparing this expression with equation 5.1.3 we obtain,

$$\begin{aligned} X_n(\overline{A}_k) &= h_k^n(x_k) \\ X_n(\overline{B}_k) &= \epsilon_k h_k^n(y_k) \end{aligned}$$

for any k, where $\epsilon_k = 1$ if the function f_k is elliptic and $\epsilon_k = -1$ for an hyperbolic function f_k.

Since $X_n(h_k^n) = 0$, we can apply L_{X_n} to these relations to obtain,

$$X_n^2(\overline{A}_k) = 0, \quad X_n^2(\overline{B}_k) = 0,$$

again since X_n is periodic these relations yield,

$$X_n(\overline{A}_k) = 0, \quad X_n(\overline{B}_k) = 0,$$

proving so far

$$i_{X_n}\overline{\omega}_1 = -df_n.$$

In order to finish the proof it remains to define a symplectically orthogonal decomposition and apply the induction hypothesis.

As we did in chapter 4 we use the lemma,

Lemma 5.1.4 *The distribution $D_1 =< X, Y >$ defined by the relations:*

$$i_X\overline{\omega}_1 = dx_n$$

$$i_Y\overline{\omega}_1 = dy_n$$

is $\mathcal{C}^\infty$, symplectic in a neighbourhood of the origin and involutive everywhere.

The proof can be treated along the same lines that is why we omit it.

This lemma allows us to talk about symplectically orthogonal decomposition. Notice that from the definition the 2-dimensional distribution D_1 is symplectically orthogonal to the $2(n-2)$ dimensional distribution D_2 generated by the vector fields $\frac{\partial}{\partial x_k}$ and $\frac{\partial}{\partial y_k}$, for $k \neq n$. Both distributions are involutive. The integral submanifolds M_c integrating the first one are symplectic whereas the integral submanifolds N_c integrating the second distribution are symplectic because they are symplectically orthogonal to the former.

These two orthogonal distributions provide coordinates $\overline{x}_i, \overline{y}_i$. Such that the symplectic form may be expressed as $\overline{\omega}_1 = \omega_1 + \omega_2$, where ω_1 defines a symplectic 2-form on $T_p(M_c)$ for each $p \in M_c$ and, in the same way, ω_2 defines a symplectic 2-form on $T_p(N_c)$ for each $p \in N_c$. The condition $d\overline{\omega}_1 = 0$ implies that ω_1 depends only on the $\overline{x}_n, \overline{y}_n$ variables and ω_2 depends only on the $\overline{x}_k, \overline{y}_k$ variables (for $k \neq n$).

Observe that (M_c, ω_1) is a 2-dimensional manifold endowed with the foliation $\mathcal{F}_1$ defined by X_n. In the same way, (N_c, ω_2) is a $2(n-1)$-dimensional manifold endowed with the foliation $\mathcal{F}_2$ generated by X_k for $k \neq n$.

The lagrangian condition imposed on the initial foliation $\mathcal{F}$ implies the Lagrangian condition for $\mathcal{F}_1$ and $\mathcal{F}_2$. So we may apply the hypothesis for induction to the symplectic submanifolds N_c and M_c to define foliation preserving diffeomorphisms ϕ_1 and ϕ_2 for which $\phi^*(\omega_1) = dx_n \wedge dy_n$ and $\phi^*(\omega_2) = \sum_{k<n} dx_k \wedge dy_k$.

Finally define $\phi = (\phi_1, \phi_2)$ to get the desired foliation preserving diffeomorphism.

2. **Attaching an hyperbolic component.**

   The proof follows the guidelines of the elliptic case with some differences. Let us point them out.

   First, using lemma 2.2.2 in page 26 and lemma 5.1.3 we can assume that,

$$i_{X_n}\overline{\omega} = \sum_{k=1}^{n} \overline{h}_k^n df_k, \quad X_n(\overline{h}_k^n) = 0.$$

   As we observed in the proof for completely hyperbolic case in chapter 4, this implies

$$X_n^2(h_k^i) = 0, \quad \forall i, k$$

Since X_n is a vector field corresponding to an hyperbolic singularity, this does not imply necessarily that $X_n(h_k^i) = 0$. It remains to apply Moser again exactly as we did on chapter 4 to obtain $X_n(h_k^i) = 0, \quad \forall i$. As a consequence of the commutation relations this yields $X_j(h_k^n) = 0$, $\forall i, k$ and then we can proceed to normalization. That is to say we consider the foliation preserving diffeomorphism

$$\phi(\hat{x}_n, \hat{y}_n, x_n, y_n) = (\hat{x}_n, \hat{y}_n, \psi \cdot x_n, \psi \cdot y_n).$$

where ψ is the smooth solution of the differential equation

$$\frac{d}{du}(\psi^2(u, \hat{x}_n, \hat{y}_n) \cdot u) = h_n^n(u, \hat{x}_n, \hat{y}_n),$$

and $u = x_n^2 - y_n^2$, $\hat{x}_n = (x_1, \ldots, x_{n-1})$ and $\hat{y}_n = (y_1, \ldots, y_{n-1})$ This diffeomorphims takes the symplectic form to a new symplectic form $\overline{\omega}_1$ such that,

$$i_{X_n}\overline{\omega}_1 = \sum_{k=1}^{n-1} \overline{h}_k^n df_k - df_n, \quad X_n(\overline{h}_k^n) = 0. \tag{5.1.3}$$

Now in view of this expression and following the same arguments as in the elliptic case, the symplectic form can be written,

$$\overline{\omega}_1 = dx_n \wedge dy_n + d(\sum_{k<n}(\overline{A_k} dx_k + \overline{B}_k dy_k)),$$

for certain smooth functions $\overline{A}_k$ and $\overline{B}_k$.

The contraction $i_{X_n}\overline{\omega}_1$ reads,

$$i_{X_n}\overline{\omega}_1 = -df_n + \sum_{k<n}(X_n(\overline{A}_k)dx_k + X_n(\overline{B}_k)dy_k).$$

From here and the Lagrangian condition we can write

$$\begin{array}{rcl} X_n(\overline{A}_k) & = & h_k^n(x_k) \\ X_n(\overline{B}_k) & = & \epsilon_k h_k^n(y_k) \end{array},$$

for any k, where $\epsilon_k = 1$ if the function f_k is elliptic and $\epsilon_k = -1$ for an hyperbolic function f_k.

Since $X_n(h_k^n) = 0$, we can apply L_{X_n} to these relations to obtain,

$$X_n^2(\overline{A}_k) = 0, \quad X_n^2(\overline{B}_k) = 0,$$

from this expressions we can draw conclusions about the (x_n, y_n)-jet of the function along the subspace $x_n = 0, y_n = 0$ and then apply Moser again as we did in the proof for the completely hyperbolic case in chapter 4. Then we will be taken to a new symplectic form and for new coefficients $\overline{A}_k$ and $\overline{B}_k$ for which

$$X_n(\overline{A}_k) = 0, \quad X_n(\overline{B}_k) = 0,$$

proving so far

$$i_{X_n}\overline{\omega}_1 = -df_n.$$

From this moment on, we can define the symplectically orthogonal decomposition as we did in the proof of the elliptic case and follow the same proof which takes the initial symplectic form to the standard one.

Remark

This theorem was proved with all the details by Eliasson just in the completely elliptic case. Here we have included a different proof for foliations of Williamson type $(k_e, k_h, 0)$. We have not include here the more general case $k_f \neq 0$. But let us add a few words about this case, when $k_f \neq 0$ we can prove the theorem using induction as well. Observe that when we attach a focus-focus component we are adding two vector fields to the distribution. One of this vector fields X_{n-1} is periodic. For this vector field, we can obtain a decomposition as the one obtained in chapter 2 proposition 2.2.1, page 23. Then using Moser and tecniques similar to the elliptic case we can prove that X_{n-1} is a Hamiltonian vector field with hamiltonian function f_{n-1}. This enables to define a symplectic distribution which obviously contains X_{n-1}. Since this distribution is symplectically orthogonal to the distribution generated by $\frac{\partial}{\partial x_k}, \frac{\partial}{\partial y_k}$ $k \neq n$, $k \neq n-1$. This distribution also contains X_n. Once reached this point, we have an orthogonal symplectic decomposition and we can apply the induction hypothesis exactly as we did in the other cases.

5.2 Rank k foliations in any dimension

The goal of this section is to give a symplectic uniqueness result in the neighbourhood of an orbit for the linearized foliation. This result is contained in the joint paper [48] with Nguyen Tien Zung. We provide a different proof here. Recall that the foliation is given by the vector fields,

$Y_i = \frac{\partial}{\partial \theta_i}, \quad 1 \leq i \leq k$
$X_i = -y_i \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial y_i}$
for $1 \leq i \leq k_e$,
$X_i = y_i \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial y_i}$ for $k_e + 1 \leq i \leq k_e + k_h$,
$X_i = x_i \frac{\partial}{\partial x_{i+1}} - y_{i+1} \frac{\partial}{\partial y_i} - x_{i+1} \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_{i+1}}$ and
$X_{i+1} = -x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i} - x_{i+1} \frac{\partial}{\partial x_{i+1}} + y_{i+1} \frac{\partial}{\partial y_{i+1}}$ for $i = k_e + k_h + 2j - 1, \ 1 \leq j \leq k_f$

We denote this foliation by $\mathcal{F}$. We want to prove the following theorem,

Theorem 5.2.1 *Let ω and ω_0 two symplectic forms in a neighbourhood of a singular orbit for which the foliation $\mathcal{F}$ is Lagrangian then ω and ω_0 are equivalent.*

Remarks:

- Taking into account the notion of equivalence, we could also state this theorem in the following way,

  Theorem 5.2.2 *Let ω be a symplectic form defined in a neighbourhood of a singular orbit L, then there exists coordinates $(\theta_1, \dots, \theta_k, p_1, \dots, p_k, x_1, y_1, \dots, x_{n-k}, y_{n-k})$, a diffeomorphism ϕ in a neighbourhood of L preserving the foliation $\mathcal{F}$ such that*

$$\phi^*(\sum_{i=1}^{k} dp_i \wedge d\theta_i + \sum_{i=1}^{n-k} dx_i \wedge dy_i) = \omega.$$

- The idea of the proof is to define an splitting at each point of the symplectic manifold into two symplectic manifolds M_p and N_p which are symplectically orthogonal and such that the regular distribution (generated by the vector fields Y_i) is tangent to M_p and the singular distribution (generated by the vector fields X_i) is tangent to N_p.

Proof of theorem 5.2.1:

Let ω be a symplectic form for which $\mathcal{F}$ is Lagrangian. In order to define the symplectic orthogonal decomposition we need to prove the existence of a Hamiltonian $\mathbb{T}^k$ action tangent to the leaves of the foliation. In section 3, proposition 4.5.2 in page 75 we proved the existence of a Hamiltonian S^1-action tangent to the leaves of the foliation using Moser path method and the Lagrangian condition.

Iterating this procedure and using 5.1.3 we have the following proposition, we omit the proof because it is a straightforward generalization of proposition 4.5.2.

Let us introduce the following simplifying notation,

$\theta = (\theta_1, \dots, \theta_k)$, $p = (p_1, \dots, p_k)$ $x = (x_1, \dots, x_{n-k})$ and $y = (y_1, \dots, y_{n-k})$

Proposition 5.2.1 *There is a Hamiltonian $\mathbb{T}^k$-action tangent to the foliation. In fact, there exist coordinates (θ, p, x, y) in a neighbourhood of L such that $\omega = d(\sum_{i=1}^k p_i d\theta_i + \sum C(p,x,y)dx + D(p,x,y)dy)$ and the Hamiltonian $\mathbb{T}^k$-action is performed by translations with respect to θ_i, for $1 \leq i \leq k$.*

Under these conditions we can prove the following lemma,

Lemma 5.2.3 *The distribution $\mathcal{D} =< Z_1, T_1, \dots, Z_k, T_k >$ defined by the relations:*

$$i_{Z_i}\omega = dp_i$$

$$i_{T_i}\omega = d\theta_i$$

is $\mathcal{C}^\infty$, symplectic in a neighbourhood of L and involutive everywhere.

Proof:

Clearly this distribution is smooth and regular. First we prove that the distribution is symplectic. From the definition, this distribution is the symplectic orthogonal to the distribution $\mathcal{D}'$ generated by $\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}$ for $1 \leq i \leq n-k$. If we prove that $\mathcal{D}'$ is symplectic in a neighbourhood of the orbit L then we will be done.

In order to check that $\mathcal{D}'$ is symplectic we use the same strategy as in the proof of lemma 4.4.1 in page 67. From the Lagrangian conditions at any point in p in a neighbourhood of L the following relations are fulfilled, $\omega(Y_i, X_j) = 0$, $\omega(Y_i, Y_j) = 0$ and $\omega(X_i, X_j) = 0$.

Now along L the coordinates x_i and y_i vanish so in the absence of focus-focus components, the relations above read,

$$w(\frac{\partial}{\partial \theta_i}, \frac{\partial}{\partial \theta_j})_{|q} = 0, \quad q \in L$$

$$w(\frac{\partial}{\partial \theta_i}, \frac{\partial}{\partial x_j})_{|q} = 0, \quad q \in L$$

$$w(\frac{\partial}{\partial \theta_i}, \frac{\partial}{\partial y_j})_{|q} = 0, \quad q \in L$$

$$w(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})_{|q} = 0, \quad q \in L$$

$$w(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j})_{|q} = 0, \quad q \in L$$

$$w(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_j})_{|q} = 0, \quad i \neq j \quad q \in L$$

Following the same arguments as in proposition 4.4.1 these relations imply that $\mathcal{D}'$ is symplectic and therefore its symplectic orthogonal distribution $\mathcal{D}$ is also symplectic. In the case there are focus-focus components the last relation changes for $j = i+1$ in a focus-focus pair f_i, f_{i+1} and following similar arguments we conclude that $\mathcal{D}'$ is also symplectic.

In order to see that the distribution is involutive observe first that the vector fields Z_i coincide with the vector fields $Y_i = \frac{\partial}{\partial \theta_i}$. Therefore, $[Z_i, Z_j] = 0$. On the other hand according to proposition 5.2.1 the coefficients of the symplectic form do not depend on the angular coordinates θ_i. From here, $[Z_i, T_j] = 0$ because of the expression of the symplectic form obtained in Proposition 5.2.1.

It remains to check that $[T_i, T_j] = 0$.

We use the formula,

$$i_{[T_i,T_j]}\omega = L_{T_i} i_{T_j}\omega - i_{T_j} L_{T_i}\omega. \quad *$$

The second term vanishes because from the definition of T_i the vector field T_i is locally Hamiltonian. As for the first term, applying the definition of T_i, we obtain the following chain of equalities,

$$L_{T_i} i_{T_j}\omega = L_{T_i} d\theta_j = d(T_i(\theta_j)) = 0,$$

where in the last equality we have used again the explicit expression of ω in proposition 5.2.1.

Now going back to $*$ we obtain,

$$i_{[T_i,T_j]}\omega = 0$$

Finally this yields $[T_i, T_j] = 0$ and the distribution is involutive. This ends the proof of the lemma. □

Once this key lemma has been proved we are already done. Because we have two symplectic orthogonal distributions $\mathcal{D}$ and $\mathcal{D}'$.

Now through each point p in L there are two symplectic submanifolds, symplectically orthogonal to each other, M_p and N_p integrating $\mathcal{D}$ and $\mathcal{D}'$ respectively. Observe that the distribution generated by the singular vector fields X_i is tangent to M_p thus defining a foliation tangent to M_p. We call this foliation $\mathcal{F}_1$. In the same way, the distribution generated by the regular vector fields Y_i is tangent to

N_p thus defining a foliation tangent to N_p. We call this foliation $\mathcal{F}_2$. Note, as well, that the Lagrangian condition imposed on $\mathcal{F}$ implies that the foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ are Lagrangian with respect to the symplectic forms ω_1 and ω_2 induced by restriction of ω to M_p and N_p, respectively. Further, since M_p and N_p are symplectically orthogonal to each other we may write,

$$\omega = \omega_1 + \omega_2.$$

Let us have a look at the expression of ω in local coordinates. Frobenius theorem provides coordinates $(\overline{\theta}_1, \ldots, \overline{\theta}_k, \overline{p}_1, \ldots, \overline{p}_k, \overline{x}_1, \overline{y}_1 \ldots, \overline{x}_{n-k}, \overline{y}_{n-k})$ in a neighbourhood of p such that $(\overline{\theta}_1, \ldots, \overline{\theta}_k, \overline{p}_1, \ldots, \overline{p}_k)$ are coordinates in N_p and

$(\overline{x}_1, \overline{y}_1 \ldots, \overline{x}_{n-k}, \overline{y}_{n-k})$ are coordinates in M_p. The condition $d\omega = 0$ implies that the coefficient functions of ω_1 just depend on $\overline{x}_i$ and $\overline{y}_i$ and that the coefficient functions of ω_2 just depend on $\overline{p}_i$ and $\overline{\theta}_i$.

Once reached this point, we can apply theorem 5.1.1 to the pair (M_p, ω_1) and there exists a diffeomorphims ϕ_1 preserving the foliation $\mathcal{F}_1$ and coordinates (x_i, y_i) such that $\phi^*(\sum_i dx_i \wedge dy_i) = \omega_1$. We can apply Liouville-Mineur-Arnold theorem to the pair (N_p, ω_2) to obtain a diffeomorphims ϕ_1 preserving the foliation $\mathcal{F}_2$ and coordinates (p_i, θ_i) such that $\phi^*(\sum_i dp_i \wedge d\theta_i) = \omega_2$. Observe that these coordinates can be extended to a whole neighbourhood of the orbit using the flow of the vector fields Y_i which are symplectomorphisms.

Finally the desired preserving foliation diffeomorphism is $\phi = (\phi_1, \phi_2)$.

This ends the proof of theorem 5.2.1.

□

Chapter 6

Equivariant linearization and symplectic equivalence

6.1 Introduction

In the previous chapters we have attained the symplectic linearization of the foliation $\widetilde{\mathcal{F}}$ in a finite normal covering $\widetilde{U}(L)$ of the initial neighbourhood of the orbit L.

As we observed in the first chapter, the group Γ of deck transformation attached to the covering preserves the symplectic structure and the fibration given by the mapping $\mathbf{F} = (f_1, \dots, f_n)$. Therefore, in order to prove the symplectic equivalence in the initial neighbourhood of the orbit we have to check that the symplectomorphism which provides the linearization result in the covering can be chosen to be Γ-equivariant.

We can pose the problem in the following terms,

Denote by α the initial symplectic action of the group Γ in the covering $\widetilde{U(L)}$ and we denote by ω the symplectic form in the covering. This action preserves the fibration given by $\mathbf{F}$. Let $\phi : \widetilde{U(L)} \longrightarrow \phi(\widetilde{U(L)})$ be the symplectomorphism attained in the previous chapter. This symplectomorphism preserves the foliation

$\widetilde{\mathcal{F}}$ and satisfies $(\phi^*)^{-1}(\omega) = \omega_0$, being $\omega_0 = \sum_{i=1}^k dp_i \wedge d\theta_i + \sum_{i=1}^{n-k} dx_i \wedge dy_i$.

Then the initial action α of Γ becomes an action ρ on $\phi(\widetilde{U(L)})$ preserving ω_0 and the fibration given by $\mathbf{F} = (f_1, \dots, f_n)$.

Summing up, if we prove that the Γ-equivariant equivalence taking the initial action to a linear action can be attained in the linear model with the symplectic structure $\omega_0 = \sum_{i=1}^k dp_i \wedge d\theta_i + \sum_{i=1}^{n-k} dx_i \wedge dy_i$, then we will be done.

Observe that the twisting group Γ is a finite group. So it would be sufficient to prove that the Γ-equivariance holds for finite groups.

We will prove a more general theorem which will yield the desired result as a corollary.

The general result that we prove provides an equivariant version of the symplectic linearization. In other words, we will assume that there exists a symplectic action of a compact Lie group in a neighbourhood of L preserving $\mathbf{F} = (f_1, \dots, f_n)$ and we will show that the symplectic linearization can be carried out in an equivariant way.

As a consequence of this result we end up proving the symplectic linearization result in the original neighbourhood of the orbit.

Another byproduct of the proof of this theorem is that if the action is effective then the group G fulfilling all the above-mentioned hypotheses has to be abelian.

As a matter of fact, the equivariant symplectic linearization result turns out to have interest by itself. It follows the general philosophy of the large list of linearization results for compact group actions preserving additional structures. From our point of view, the first one in this long list is the one for fixed points due to Bochner. Let us state Bochner's linearization theorem.

Theorem 6.1.1 (Bochner)

Let α be a smooth action of a compact group G on a manifold M and let $x_0 \in M$ be a fixed point for the action, i.e $\alpha(g, x_0) = x_0$, $\forall g \in G$. Denote by α_g^1 the differential at x_0 of the diffeomorphism $\alpha_g : M \longrightarrow M$ induced by α. Then there *exists a G-invariant neighbourhood U of x_0 and a diffeomorphism ϕ from U onto an open neighbourhood V of the origin 0 in $T_{x_0}M$, such that,*

$$\phi(x_0) = 0, \quad d_{x_0}\phi = Id$$

and

$$\phi \circ \alpha_g = \alpha_g^{(1)} \circ \phi, \quad \forall g \in G, \quad x \in U$$

The orbit-like version of this theorem was given by Koszul [35]. This theorem has been known in the literature as the slice theorem. It guarantees that for any smooth action of a compact Lie group on a manifold M there exists a slice S through every point $x \in M$. Furthermore one can choose coordinates on S so that S is an open invariant disk in a vector space upon which the isotropy group acts linearly. Namely,

Theorem 6.1.2 (Koszul)

Let G be a compact connected Lie group acting on a manifold M and let $x \in M$ be a point. A neighbourhood of the orbit $G \cdot x$ through the point x, is G-equivariantly diffeomorphic to a neighbourhood of the zero section of the homogeneous vector bundle $G \times_{G_x} W$ where $W = T_x(M)/T_x(G \cdot x)$ and where the action of G_x on W is linear.

An extension of this theorem to proper actions of groups was provided by R. Palais [50], [49].

Another G-equivariant result concerning also symplectic forms is the G-equivariant Darboux theorem. We state it below,

Theorem 6.1.3 *Let G be a compact Lie group acting smoothly on a manifold M, let Y be a G-invariant compact submanifold of M, and let ω_1 and ω_2 be two G-invariant symplectic forms on M such that $\omega_{0p} = \omega_{1p}$ for all $p \in Y$. Then there* *exists a G-invariant neighbourhood U of Y and a G-equivariant diffeomorphism f of U onto another G-invariant neighbourhood of Y such that $f(y) = y$ for all $y \in Y$ and $f^*(\omega_1) = \omega_0$.*

The first two theorems can be considered as linearization theorems for actions of compact Lie groups whereas the last theorem is also concerned with an additional geometrical structure (the symplectic form).

In this spirit we will prove similar linearization theorems in the neighbourhood of a point and in the neighbourhood of an orbit under the more constraining condition that the diffeomorphism preserves the map $\mathbf{F}$ and the symplectic structure.

This chapter is organized as follows: in the first section we introduce the notion of the linear action on the linear model. In the second section we study the case of a fixed point and we prove that the action can be linearized. As a by-product, we prove that the group of symplectomorphisms preserving the system is abelian. In the third section we prove the G-linearization in the neighbourhood of an orbit. As a corollary we obtain the symplectic equivalence in a neighbourhood of an orbit, this result is included in the last section.

Throughout the chapter there will be two different concepts that show up. The concept of foliation preserving symplectomorphism and the concept of system preserving diffeomorphism. They are slightly different concepts. Let us point out the difference in advance.

When we say that a symplectomorphism is foliation preserving we mean that it preserves the foliation. That is, it sends leaves to to leaves. When we refer to a system preserving diffeomorphism we mean that the diffeomorphism preserves the symplectic form considered and $\mathbf{F}$. In particular, a system preserving diffeomorphism is foliation preserving.

The results contained in this section have been obtained jointly with Nguyen Tien Zung. The proofs provided here (with the only exception of the proof of the linearization theorem in the neighbourhood of an orbit and the parametric version of theorem 6.3.2 and theorem 6.3.4 (corollaries 6.3.3 and 6.3.5)) are contained in the joint paper [48].

6.2 The linear action on the linear model

We are going to introduce the notion of linear action on the linear model associated to the orbit L for a given symplectic action preserving the system. Later, we will see that the invariants associated to the linear model are the Williamson type of the orbit and a twisting group Γ attached to it.

We recall the notion of linear model. Denote by $(p_1, ..., p_k)$ a linear coordinate system of a small ball D^k of dimension k, $(\theta_1(mod 1), ..., \theta_k(mod 1))$ a standard periodic coordinate system of the torus $\mathbb{T}^k$, and $(x_1, y_1, ..., x_{n-k}, y_{n-k})$ a linear coordinate system of a small ball $D^{2(n-k)}$ of dimension $2(n-k)$. Consider the manifold

$$V = D^k \times \mathbb{T}^k \times D^{2(n-k)} \tag{6.2.1}$$

with the standard symplectic form $\omega_0 = \sum dp_i \wedge d\theta_i + \sum dx_j \wedge dy_j$, and the following moment map:

$$\mathbf{F} = (p_1, ..., p_k, f_{k+1}, ..., f_n) : V \to \mathbb{R}^n \tag{6.2.2}$$

where

$$\begin{aligned}
&f_{i+k} = x_i^2 + y_i^2 \ \text{ for } \ 1 \le i \le k_e \ , \\
&f_{i+k} = x_i y_i \ \text{ for } \ k_e + 1 \le i \le k_e + k_h \ , \\
&f_{i+k} = x_i y_{i+1} - x_{i+1} y_i \ \text{ and} \\
&f_{i+k+1} = x_i y_i + x_{i+1} y_{i+1} \ \text{ for } \ i = k_e + k_h + 2j - 1, \ 1 \le j \le k_f
\end{aligned} \tag{6.2.3}$$

For the sake of simplicity we will denote by $\mathbf{p}$ the mapping whose components are the k regular first integrals p_i and $\mathbf{h}$ will stand for the mapping whose components are the singular first integrals f_i, $i \ge k$; following this convention we will write $\mathbf{F} = (\mathbf{p}, \mathbf{h})$. Let Γ be a group with a symplectic action $\rho(\Gamma)$ on V, which preserves the moment map $\mathbf{F}$. We will say that the action of Γ on V is linear if it satisfies the following property:

Γ acts on the product $V = D^k \times \mathbb{T}^k \times D^{2(n-k)}$ componentwise; the action of Γ on D^k is trivial, its action on $\mathbb{T}^k$ is by translations (with respect to the coordinate system $(\theta_1, ..., \theta_k)$), and its action on $D^{2(n-k)}$ is linear with respect to the coordinate system $(x_1, y_1, ..., x_{n-k}, y_{n-k})$.

Suppose now that Γ is a finite group with a free symplectic action $\rho(\Gamma)$ on V, which preserves the moment map and which is linear. Then we can form the quotient symplectic manifold V/Γ, with an integrable system on it given by the induced moment map as above:

$$\mathbf{F} = (p_1, ..., p_k, f_{k+1}, ..., f_n) : V/\Gamma \to \mathbb{R}^n \tag{6.2.4}$$

The set $\{p_i = x_i = y_i = 0\} \subset V/\Gamma$ is a compact orbit of Williamson type (k_e, k_f, k_h) of the above system. We will call the above system on V/Γ, together with its associated singular Lagrangian foliation, the linear system (or linear model) of Williamson type (k_e, k_f, k_h) and twisting group Γ (or more precisely, twisting action $\rho(\Gamma)$). We will also say that it is a direct model if Γ is trivial, and a twisted model if Γ is nontrivial.

A symplectic action of a compact group G on V/Γ which preserves the moment map $(p_1, ..., p_k, f_{k+1}, ..., f_n)$ will be called linear if it comes from a linear symplectic action of G on V which commutes with the action of Γ. In our case, let $\mathcal{G}'$ denote the group of linear symplectic maps which preserve the moment map then this group is abelian and therefore this last condition is automatically satisfied. In fact $\mathcal{G}'$ is isomorphic to $\mathbb{T}^m \times G_1 \times G_2 \times G_3$ being G_1 the direct product of k_e special orthogonal groups $SO(2, \mathbb{R})$, G_2 the direct product of k_h components of type $SO(1, 1, \mathbb{R})$ and G_3 the direct product of k_f components of type $\mathbb{R} \times SO(2, \mathbb{R})$, respectively.

6.3 G-linearization for rank 0 foliations

In this section we consider the action of a compact Lie group on the linear model corresponding to a rank 0 point of Williamson type (k_e, k_h, k_f). We assume that this action preserves the symplectic form and the mapping $\mathbf{F} = (f_1, \ldots, f_n)$ where f_i are of elliptic, hyperbolic or focus-focus type as specified in the section above (formula 6.2.3). We prove that the action can be linearized in a foliation preserving way. We will also provide the analytic version of the theorem.

The proof of this linearization theorem resembles very much the proof of Bochner's linearization theorem. We will use the averaging method. Nevertheless, we have to make sure that the linearization can be carried out using symplectomorphisms and that all the symplectomorphisms preserve the Lagrangian foliation. In order to do that, we will often consider flows of Hamiltonian vector fields which are tangent to the foliation.

Let us fix some notation that we will use throughout the chapter. The vector field X_Ψ will stand for a Hamiltonian vector field with associated Hamiltonian function Ψ. We will denote by $\phi^s_{X_t}$ the time-s-map of the vector field X_t. Let ψ be a local diffeomorphism $\psi : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0)$. In the sequel, we will denote by $\psi^{(1)}$ the linear part of ψ at 0. That is to say, $\psi^{(1)}(x) = d_0\psi(x)$.

In this section we will linearize the action of G using the averaging method and a theorem about local automorphisms of the linear integrable system $(\mathbb{R}^{2n}, \sum_{i=1}^n dx_i \wedge dy_i, \mathbf{h})$.

First we recall this well-known sublemma (sublemma 4.5.1) of chapter 4.

Sublemma 6.3.1 *Let X_{G_1} and X_{G_2} be two Hamiltonian vector fields tangent to $\mathcal{F}$. Denote by $\phi^s_{X_{G_1}}$ and $\phi^s_{X_{G_2}}$ the time-s-map of X_{G_1} and X_{G_2} respectively. Then*

$$\phi^s_{X_{G_1}+X_{G_2}} = \phi^s_{X_{G_1}} \circ \phi^s_{X_{G_2}}.$$

Now we can state and prove the following theorem,

Theorem 6.3.2 *Suppose that $\psi : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0)$ is a local symplectic diffeomorphism of $\mathbb{R}^{2n}$ which preserves the quadratic moment map $\mathbf{h}$. Then the linear part $\psi^{(1)}$ is also a system-preserving symplectomorphism, and there is a unique local smooth function $\Psi : (\mathbb{R}^{2n}, 0) \to \mathbb{R}$ vanishing at 0 which is a first integral for the linear system given by $\mathbf{h}$ and such that $\psi^{(1)} \circ \psi^{-1}$ is the time-1 map of the Hamiltonian vector field X_Ψ of Ψ. If ψ is real analytic then Ψ is also real analytic. If ψ depends smoothly (resp analytically) on parameters so does Ψ.*

Proof

We are going to construct a path connecting ψ to $\psi^{(1)}$ contained in $\mathcal{G} = \{\phi : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0), \text{ such that } \quad \phi^*(\omega) = \omega, \quad \mathbf{h} \circ \phi = \mathbf{h}\}$. Given a function $\psi \in \mathcal{G}$, we consider

$$S_t^\psi(x) = \begin{cases} \dfrac{\psi \circ g_t}{t}(x) & t \in (0, 1] \\ \psi^{(1)}(x) & t = 0 \end{cases}$$

being g_t the homothecy $g_t(x_1, \ldots, x_n) = t(x_1, \ldots, x_n)$.

Observe that in case ψ is smooth, this mapping S_t^ψ is smooth and depends smoothly on t. In case ψ is real analytic, the corresponding S_t^ψ is also real analytic and depends analytically on t. If ψ depends smoothly or analytically on parameters so does S_t^ψ.

First let us check that $h \circ S_t^\psi = h$ when $t \neq 0$. We do it component-wise.

Let $x = (x_1, \ldots, x_n)$ and let f_j be one of the components of $\mathbf{h}$, then

$$f_j \circ (\frac{\psi \circ g_t}{t})(x) = \frac{(f_j \circ \psi \circ g_t)(x)}{t^2} = \frac{f_j \circ g_t(x)}{t^2} = f_j(x)$$

where in the first and the last equalities we have used the fact that each component f_j of the moment map h is a quadratic polynomial whereas the condition $h \circ \psi = h$ yields the second equality.

Now we check that $(S_t^\psi)^*(\omega_0) = \omega_0$ when $t \neq 0$. Since $\omega_0 = \sum dx_i \wedge dy_i$, then

$g_t^*(\omega_0) = t^2\omega_0$. But since ψ preserves ω_0 then

$$(S_t^\psi)^*(\omega) = (\frac{\psi \circ g_t}{t})^*\omega_0 = \omega_0$$

when $t \neq 0$.

So far we have checked the conditions $h \circ S_t^\psi = h$ and $(S_t^\psi)^*(\omega_0) = \omega_0$ when $t \neq 0$ but since S_t^ψ depends smoothly on t we also have that $h \circ {S_0}^\psi = h$ and $(S_0^\psi)^*(\omega_0) = \omega_0$. So, in particular, we obtain that $S_0^\psi = \psi^{(1)}$ preserves the moment map and the symplectic structure and therefore $\psi^{(1)}$ is also contained in $\mathcal{G}$.

Now consider

$$R_t = \psi^{(1)} \circ S_t^{(\psi^{-1})}$$

with $t \in [0,1]$, this path connects the identity to $\psi^{(1)} \circ \psi^{-1}$ and is contained in $\mathcal{G}$.

We are going to use this path to define a Hamiltonian vector field such that its time-1-map is $\psi^{(1)} \circ \psi^{-1}$. First, we consider the t-dependent vector field X_t satisfying

$$X_t(p) = \frac{d}{ds}(R_s(q))_{|s=t}, \quad q = R_t^{-1}(p) \tag{6.3.1}$$

with $t \in [0,1]$. Since R_s is a symplectomorphism for any s contained in $[0,1]$, the vector field X_t is locally Hamiltonian. Then the vector field

$$X = \int_0^1 X_t dt$$

is also locally Hamiltonian. Since the symplectic manifold considered is a neighbourhood U of the origin, the vector field X is indeed Hamiltonian in U. There is a unique local Hamiltonian function Ψ associated to X satisfying $\Psi(0) = 0$. This Hamiltonian function is a first integral for the system since $\{\Psi, h_i\} = 0$. If ψ is real analytic (respectively differentiable) X_t is also real analytic (respectively differentiable) and the same property holds for Ψ. Observe that in the case ψ depends smoothly (resp. analytically) on parameters then by construction the t-dependent vector field X_t depends smoothly (resp. analytically) on that parameters and therefore so does the vector field X and its time one map. It remains to check that the time-1-map of X is $\psi^{(1)} \circ \psi^{-1}$.

We are going to prove that the time-1-map of the vector field X coincides with R_1 by performing a partition of the interval $[0,1]$ into n pieces and then approximating the integral by a sum.

First observe that formula 6.3.1 shows that the vector $X_{1-\frac{k+1}{n}}(p)$ is tangent to the curve $R_s \circ R^{-1}_{1-\frac{k+1}{n}}(p)$ at the point p. On the other hand by the definition of flow the vector is also tangent to the curve $\phi^s_{X_{1-\frac{k+1}{n}}}(p)$ at the point p.

So in fact,

$$X_{1-\frac{k+1}{n}}(p) = \lim_{s\to 0} \frac{R_{1-\frac{k+1}{n}+s} \circ R_{1-\frac{k+1}{n}}{}^{-1}(p) - p}{s} \tag{6.3.2}$$

and also

$$X_{1-\frac{k+1}{n}}(p) = \lim_{s\to 0} \frac{\phi^s_{X_{1-\frac{k+1}{n}}}(p) - p}{s} \tag{6.3.3}$$

Therefore,

$$\lim_{s\to 0} \frac{R_{1-\frac{k+1}{n}+s} \circ R^{-1}_{1-\frac{k+1}{n}}(p) - \phi^s_{X_{1-\frac{k+1}{n}}}(p)}{s} = 0 \tag{6.3.4}$$

In other words, $R_{1-\frac{k+1}{n}+s} \circ R^{-1}_{1-\frac{k+1}{n}}(p) = \phi^s_{X_{1-\frac{k+1}{n}}}(p) + o(s^1)$. After refining the initial partition if necessary, we can particularize $s = \frac{1}{n}$ to obtain

$$R_{1-\frac{k}{n}} \circ R^{-1}_{1-\frac{k+1}{n}} = \phi^{\frac{1}{n}}_{X_{1-\frac{k+1}{n}}} + o(\frac{1}{n}). \tag{6.3.5}$$

Since R_1 can be written as,

$$R_1 = (R_1 \circ R^{-1}_{1-\frac{1}{n}}) \circ R_{1-\frac{1}{n}} \cdots \circ (R_{1-\frac{n-1}{n}} \circ R_0^{-1}). \tag{6.3.6}$$

we can perform the necessary substitutions of 6.3.5 in 6.3.6 and we are led to

$$R_1 = (\phi^{\frac{1}{n}}_{X_{1-\frac{1}{n}}} + o(\frac{1}{n})) \circ (\phi^{\frac{1}{n}}_{X_{1-\frac{2}{n}}} + o(\frac{1}{n})) \circ \cdots \circ (\phi^{\frac{1}{n}}_{X_0} + o(\frac{1}{n})) \tag{6.3.7}$$

Assuming that $\prod$ here stands for the composition of diffeomorphisms, we can write this expression in a reduced form as

$$R_1 = (\prod_{k=0}^{k=n-1} \phi^{\frac{1}{n}}_{X_{1-\frac{k}{n}}}) + o(1). \tag{6.3.8}$$

Observe that the vector field X_t is tangent to the fibration $\mathcal{F}$ for any t contained in $[0,1]$ because the diffeomorphisms R_s preserve the fibration $\mathcal{F}$, $\forall s$. On the other hand, each vector field X_t is Hamiltonian. Therefore we may apply sublemma 4.5.1 for any pair t, t' contained in $[0,1]$ and the following equation holds $\phi^s_{X_t+X_{t'}} = \phi^s_{X_t} \circ \phi^s_{X_{t'}}$. Now this expression reads,

$$R_1 = (\phi^{\frac{1}{n}}_{\sum_{k=0}^{k=n-1} X_{1-\frac{k}{n}}}) + o(1). \tag{6.3.9}$$

Since the time-1-map of $\frac{X}{n}$ and the time-$\frac{1}{n}$-map of X are related by the formula $\phi^{\frac{1}{n}}_X = \phi^1_{\frac{X}{n}}$, we obtain,

$$R_1 = \lim_{n\to\infty} (\phi^1_{\sum_{k=0}^{k=n-1} \frac{X_{1-\frac{k}{n}}}{n}}) \tag{6.3.10}$$

But,

$$\lim_{n\to\infty} \sum_{k=0}^{k=n-1} \frac{X_{1-\frac{k}{n}}}{n} = \int_0^1 X_t dt.$$

This identity shows that $R_1 = \phi^1_X$ and this ends the proof of the theorem.

□

As a corollary of the above theorem we obtain a local linearization result of symplectomorphism depending on parameters. The corollary below will be a key point in the proof of the linearization in a neighbourhood of the orbit.

Corollary 6.3.3 *Let D_p stand for a disk centered at 0 in the parameters $p_1, \dots, p_k$. We denote by $\mathbf{p} = (p_1, \dots, p_k)$. Assume that $\psi_{\mathbf{p}} : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0)$ is a local symplectic diffeomorphism of $\mathbb{R}^{2n}$ which preserves the quadratic moment map $\mathbf{h}$ and which depends smoothly on the parameters $\mathbf{p}$. Then there is a unique local* *smooth function $\Psi\mathbf{p} : (\mathbb{R}^{2n}, 0) \to \mathbb{R}$ vanishing at 0 depending smoothly on $\mathbf{p}$ which is a first integral for the linear system given by $\mathbf{h}$ and such that $\psi_0^{(1)} \circ \psi_{\mathbf{p}}^{-1}$ is the time-1 map of the Hamiltonian vector field $X_{\Psi_{\mathbf{p}}}$ of $\Psi_{\mathbf{p}}$. If $\psi_{\mathbf{p}}$ is real analytic and depends analytically on the parameters then $\Psi_{\mathbf{p}}$ is also real analytic and depends analytically on the parameters.*

Proof:

According to theorem 6.3.2 there exists a first integral $F_{1,\mathbf{p}}$ such that the time-one-map of the Hamiltonian vector field $X_{F_{1,\mathbf{p}}}$ is $\psi_{\mathbf{p}}^{(1)} \circ \psi_{\mathbf{p}}^{-1}$. We will apply the same trick that we applied to the path R_t in the proof of theorem 6.3.2 but applied to the path in $\mathcal{G}$, M_t defined as follows,

$$M_t = \psi_0^1 \circ (\psi_{g_t(\mathbf{p})}^{(1)})^{-1},$$

where $g_t(p) = (tp_1, \ldots, tp_k)$.

Observe that the path is contained in $\mathcal{G}$ and is well defined (since the disk is convex). This path is smooth (resp. analytic) if ψ is smooth (resp. analytic) and depends analytically on t. Now we can apply the same reasoning as in the proof of theorem 6.3.2.

Namely, we can consider the t-dependent vector field,

$$X_t(p) = \frac{d}{ds}(M_s(q))_{|s=t}, \quad q = M_t^{-1}(p) \tag{6.3.11}$$

And also the averaged vector field

$$X = \int_0^1 X_t dt.$$

As we pointed out in the proof of theorem 6.3.2, this vector field is Hamiltonian and it is tangent to the foliation. Denote by $F_{2,\mathbf{p}}$ the only Hamiltonian function attached to X such that $F_{2,\mathbf{p}}(0) = 0$. This function is a first integral of the system. And the time-1-map of the vector field $X_{F_{2,\mathbf{p}}}$ coincides with $\psi_0^1 \circ (\psi_{\mathbf{p}}^{(1)})^{-1}$.

Now we consider the composition of the two time-1-maps associated to $X_{F_{1,\mathbf{p}}}$ and $X_{F_{2,\mathbf{p}}}$ respectively.

The composition equals,

$$(\psi_0^{(1)} \circ (\psi_{\mathbf{p}}^{(1)})^{-1}) \circ (\psi_{\mathbf{p}}^{(1)} \circ \psi_{\mathbf{p}}^{-1})$$

In the end we obtain the desired diffeomorphism,

$$(\psi_0^{(1)} \circ \psi_{\mathbf{p}}^{-1}).$$

This diffeomorphism has been presented as a composition of two time-1-maps. The time-1-map associated to $X_{F_{1,\mathbf{p}}}$ and the time-1-map associated to $X_{F_{2,\mathbf{p}}}$. It remains to identify this composition as the time-1-map of a Hamiltonian vector field tangent to the foliation.

On the one hand, according to sublemma 4.5.1 with $s = 1$

$$\phi^1_{X_{F_{2,\mathbf{p}}}+X_{F_{1,\mathbf{p}}}} = \phi^1_{X_{F_{2,\mathbf{p}}}} \circ \phi^1_{X_{F_{1,\mathbf{p}}}}$$

On the other hand, $X_{F_{1,\mathbf{p}}} + X_{F_{2,\mathbf{p}}} = X_{F_{1,\mathbf{p}}+F_{2,\mathbf{p}}}$; in view of this decomposition the Hamiltonian vector field to consider is $G = F_{1,\mathbf{p}} + F_{2,\mathbf{p}}$. Since $F_{1,\mathbf{p}}$ and $F_{2,\mathbf{p}}$ are first integrals for the system so is G.

This ends the proof of the corollary.

□

By abuse of language, we will denote the local (a priori nonlinear) action of our compact group G on $(\mathbb{R}^{2n}, \sum_{i=1}^n dx_i \wedge dy_i, \mathbf{h})$ by ρ. For each element $g \in G$, denote by $X_{\Psi(g)}$ the Hamiltonian vector field whose time-1 map is $\rho(g)^{(1)} \circ \rho(g)^{-1}$, where $\rho(g)^{(1)}$ denotes the linear part of $\rho(g)$, as provided by the previous lemma.

Consider the averaging of the family of vector fields $X_{\Psi(g)}$ over G with respect to the Haar measure $d\mu$ on G. That is to say,

$$X_G(x) = \int_G X_{\Psi(g)}(x) d\mu, \quad x \in \mathbb{R}^{2n} \tag{6.3.12}$$

This vector field is Hamiltonian with Hamiltonian function $\int_G \Psi(g)d\mu$. It is also tangent to the foliation. Denote by Φ_G the time-1 map of this vector field X_G. Observe that this mapping preserves the system.

Finally we can prove the local linearization theorem,

Theorem 6.3.4 *Φ_G is a local symplectic variable transformation of $\mathbb{R}^{2n}$ which preserves the system $(\mathbb{R}^{2n}, \sum_{i=1}^n dx_i \wedge dy_i, \mathbf{h})$ and under which the action of G becomes linear.*

Proof.

Since Φ_G is the time-1 map of a Hamiltonian vector field then it is a diffeomorphism satisfying $\Phi_G^*(\omega) = \omega$. Therefore it defines a local symplectic variable transformation. Let us check that this transformation linearizes the action of G. We want to show that for any $h \in G$ the following relation is fulfilled $\Phi_G \circ \rho(h) = \rho(h)^{(1)} \circ \Phi_G$. From the definition of Φ_G and formula 6.3.12,

$$\Phi_G(x) = \phi^1_{X_G}(x) = \int_G \phi^1_{X_{\Psi(g)}}(x)d\mu$$

But since, $\phi^1_{X_{\Psi(g)}} = \rho(g)^{(1)} \circ \rho(g)^{-1}$ we have,

$$\Phi_G(x) = \int_G \rho(g)^{(1)} \circ \rho(g)^{-1}(x)d\mu$$

Now we write,

$$(\rho(h)^{(1)} \circ \Phi_G \circ \rho(h)^{-1})(x) = \rho(h)^{(1)} \circ \int_G (\rho(g)^{(1)} \circ \rho(g)^{-1})(\rho(h)^{-1}(x))d\mu.$$

Using the linearity of $\rho(h)^{(1)}$ and the fact that ρ stands for an action, the expression above can be written as,

$$\int_G (\rho(h) \circ \rho(g))^{(1)} \circ (\rho(h) \circ \rho(g))^{-1}(x)d\mu$$

Finally this expression equals Φ_G due to the left invariance property of averaging. So $\Phi_G \circ \rho(h) = \rho(h)^{(1)} \circ \Phi_G$ and this ends the proof of the lemma.

□

As a consequence of this theorem and corollary 6.3.3 we obtain the following parametric version of the theorem,

Corollary 6.3.5 *In the case the action $\rho_{\mathbf{p}}$ depends smoothly (resp. analytically) on parameters there exists a local symplectic variable transformation of $\mathbb{R}^{2n}$, $\Phi_{\mathbf{p}}$ which preserves the system and which satisfies,*

$$\Phi_{\mathbf{p}} \circ \rho_{\mathbf{p}}(h) = \rho_0(h)^{(1)} \circ \Phi_{\mathbf{p}}$$

Let

$$\mathcal{G} = \{\phi : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0), \text{ such that } \quad \phi^*(\omega) = \omega, \quad \mathbf{h} \circ \phi = \mathbf{h}\}$$

be the group of germs of smooth symplectomorphisms that preserve ω and $\mathbf{h}$, i.e the symmetry group for the system. We will denote by $\mathcal{G}'$ the subgroup of linear transformations contained in $\mathcal{G}$. As we have observed in the introduction, this group is abelian.

A direct consequence of theorem 6.3.2 is that any two diffeomorphisms contained in $\mathcal{G}$ whose linear part is the identity commute because they are the time-1-map of hamiltonian vector fields tangent to the foliation and those in turn commute by virtue of sublemma 4.5.1. In fact, this property extends to the whole $\mathcal{G}$. This is the content of the theorem below.

Before stating the theorem, let us point our some general observations which we will apply later.

Let f be a diffeomorphism and let X be a vector field, we denote by f_* and f^* the push-forward and pullback associated to f. Recall that the following formula holds

$$i_{(f_* X)}\omega = (f^{-1})^*(i_X(f^*(\omega))).$$

Now let f be a diffeomorphism contained in $\mathcal{G}'$. In particular $f^*(\omega) = \omega$. Therefore the above formula shows that the pushforward of a Hamiltonian vector field is also a Hamiltonian vector field. In fact, $f_*(X_\Psi) = X_{\Psi \circ f^{-1}}$.

Theorem 6.3.6 *The group $\mathcal{G}$ is abelian.*

Proof

We will prove that any two diffeomorphisms contained in $\mathcal{G}$ commute by stages.

First we prove that any diffeomorphism ψ contained in $\mathcal{G}$ commutes with any linear transformation A contained in $\mathcal{G}'$.

We will denote by $\mathcal{G}'_0$ the connected component of the identity of $\mathcal{G}'$. Observe that if the system does not contain any hyperbolic component $\mathcal{G}' = \mathcal{G}'_0$.

Let X_Ψ be the Hamiltonian vector field with associated Hamiltonian Ψ whose time-1-map is $\psi^{(1)} \circ \psi^{-1}$. The existence of this vector field is guaranteed by theorem 6.3.2. Now due to the above observation, the vector field $Y = (A)_*(X_\Psi)$ is a Hamiltonian vector field with Hamiltonian function $\Psi \circ A^{-1}$. Assume first that A is contained in $\mathcal{G}'_0$. We are going to show that for such a linear transformation, $\Psi \circ A^{-1} = \Psi$.

In order to do that we start by observing that since the vector field X_Ψ is tangent to the fibration $\mathcal{F}$ then $\{\Psi, h_i\} = 0$.

Let us assume first that there are no hyperbolic components among the h_i. Then according to Vey [54] for an analytical h and Eliasson [23] (Proposition 1) in the differentiable case, we can assert that $\Psi = \phi(h_1, \dots, h_n)$ being ϕ an analytical (respectively differentiable) function.

In the case there are hyperbolic components this assertion is no longer true for differentiable functions as was observed by Eliasson in [23]. But the property remains true if we restrict Ψ to each orthant. We label each orthant with an n-tuple of signs $(\epsilon_1, \dots, \epsilon_{2n})$, $\epsilon_i \in \{-1, +1\}$ following the convention $\epsilon_i = +1$ if $x_i \geq 0$ and $\epsilon_i = -1$ if $x_i \leq 0$. Then we can assert that Ψ restricted to each orthant is a function $\phi_{(\epsilon_1, \dots, \epsilon_{2n})}(h_1, \dots, h_n)$.

After making this distinction, observe that in both cases since A^{-1} belongs to the connected component of the identity $\mathcal{G}$, then it leaves the connected components of the fibration $\mathcal{F}$ invariant. Therefore, the transformation A^{-1} leaves the function Ψ invariant when restricted to each orthant. So in fact, $\Psi \circ A^{-1} = \Psi$ and the vector fields Y and X_Ψ coincide. As a consequence their flows coincide as well.

Recall that if X is a vector field whose flow is ϕ_X then for any diffeomorphism the flow of $(f_*)X$ is $f \circ \phi_X \circ f^{-1}$. The same is true replacing flows by time-1-maps.

Therefore, since $Y = X_\Psi$ we obtain the following relation

$$\psi^{(1)} \circ \psi^{-1} = A \circ \psi^{(1)} \circ \psi^{-1} \circ A^{-1}.$$

But since A and $\psi^{(1)}$ commute this expression reads

$$\psi^{(1)} \circ \psi^{-1} = \psi^{(1)} \circ A \circ \psi^{-1} \circ A^{-1}$$

which leads to the commutation of A with ψ.

If A does not belong to $\mathcal{G}'_0$ then we can write

$$A = I_{2k,2k+1} \circ \cdots \circ I_{2l,2l+1} \circ B \tag{6.3.13}$$

with B belonging to $\mathcal{G}'_0$ and for certain diagonal matrices $I_{2r,2r+1}$ (corresponding to hyperbolic involutions) whose entries a_{ij} satisfy the following relations $a_{2r,2r} = -1, a_{2r+1,2r+1} = -1$ and $a_{i,i} = 1 \quad i \neq 2r \quad , \quad i \neq 2r+1$. It can be checked that the linear transformations of type $I_{2k,2k+1}$ commute with any diffeomorphism contained in $\mathcal{G}$. Now since $\mathcal{G}'$ is abelian the expression 6.3.13 shows that the linear transformation A commutes with any ψ contained in $\mathcal{G}$.

In particular, taking $A = {\psi^{(1)}}^{-1}$ we obtain that ψ commutes with $\psi^{(1)}$. And as a consequence, given two diffeomorphisms ψ_1 and ψ_2 contained in $\mathcal{G}$ the diffeomorphisms $f = (\psi_1^{(1)})^{-1} \circ \psi_1$ and $g = \psi_2 \circ (\psi_2^{(1)})^{-1}$, commute because they are the time-1-map of Hamiltonian vector fields.

Finally we are going to show that any two diffeomorphisms ψ_1 and ψ_2 contained in $\mathcal{G}$ commute.

We can write,

$$\psi_1 \circ \psi_2 = \psi_1^{(1)} \circ ((\psi_1^{(1)})^{-1} \circ \psi_1) \circ (\psi_2 \circ (\psi_2^{(1)})^{-1}) \circ \psi_2^{(1)} \tag{6.3.14}$$

As we have explained before, the diffeomorphisms within brackets commute and this expression reads,

$$\psi_1^{(1)} \circ (\psi_2 \circ (\psi_2^{(1)})^{-1}) \circ ((\psi_1^{(1)})^{-1} \circ \psi_1) \circ \psi_2^{(1)}$$

Due to the commutation of any linear transformation contained in $\mathcal{G}'$ with any diffeomorphims contained in $\mathcal{G}$, we can write this expression as,

$$\psi_1^{(1)} \circ (\psi_2^{(1)})^{-1} \circ (\psi_1^{(1)})^{-1} \circ \psi_2^{(1)} \circ (\psi_2 \circ \psi_1)$$

But since $\mathcal{G}'$ is abelian the expression above equals $\psi_2 \circ \psi_1$ and therefore coming back to equation 6.3.14, ψ_1 and ψ_2 commute.

This completes the proof of the theorem.

□

A smooth action of a group G on a manifold M is called effective if the condition $\rho(h,p) = p, \quad \forall p \in M$ implies that $h = e$.

Now assume that we are given an effective action of a group ρ preserving the system. If the action of the group is effective the abelianity of $\mathcal{G}$ implies the abelianity of G. As a direct corollary we obtain,

Corollary 6.3.7 *Let G be a compact Lie group which acts effectively on the linear model of a rank 0 singularity preserving the system $(\mathbb{R}^{2n}, \sum_i dx_i \wedge dy_i, \mathbf{h})$ then G is abelian.*

Proof: Consider the group of diffeomorphisms $\rho(h), h \in G$. According to theorem

6.3.6 this group of diffeomorphisms is abelian. Therefore $\rho(h_1) \circ \rho(h_2) = \rho(h_2) \circ \rho(h_1)$, $\forall h_1, h_2 \in G$. Since ρ is an action, this relation yields,

$$\rho(h_1 h_2 h_1^{-1} h_2^{-1}) = \rho(e), \quad \forall h_1, h_2 \in G$$

But $\rho(e)$ is the identity mapping and since the action is effective this implies $h_1 h_2 h_1^{-1} h_2^{-1} = e, \quad \forall h_1, h_2 \in G$ and therefore the group G is abelian. □

6.4 Linearization in the neighbourhood of an orbit

In this section we prove a linearization theorem in the neighbourhood of an orbit.

Recall the meaning of linear action of a group G in the neighbourhood of an orbit,

G acts on the product $V = D^k \times \mathbb{T}^k \times D^{2(n-k)}$ componentwise; the action of Γ on D^k is trivial, its action on $\mathbb{T}^k$ is by translations (with respect to the coordinate system $(\theta_1, ..., \theta_k)$), and its action on $D^{2(n-k)}$ is linear with respect to the coordinate system $(x_1, y_1, ..., x_{n-k}, y_{n-k})$

Now we are ready to state and prove the linearization theorem in the neighbourhood of an orbit.

Theorem 6.4.1 *Let G be a compact Lie group preserving the system $(D^k \times \mathbb{T}^k \times D^{2(n-k)}, \sum_{i=1}^k dp_i \wedge d\theta_i + \sum_{i=1}^{n-k} dx_i \wedge dy_i, \mathbf{F})$ then there exists Φ_G a diffeomorphism defined in a tubular neighbourhood of the orbit $L = \mathbb{T}^k$ which preserves the system $(D^k \times \mathbb{T}^k \times D^{2(n-k)}, \sum_{i=1}^k dp_i \wedge d\theta_i + \sum_{i=1}^{n-k} dx_i \wedge dy_i, \mathbf{F})$ and under which the action of G becomes linear.*

Proof:

After shrinking the original neighbourhood if necessary, we may assume without loss of generality that we are considering a G-invariant neighbourhood of L. First of all, let us express in local coordinates how the action looks like. We denote by ρ the action of G. For convenience, we use the simplifying notation $p = (p_1, \ldots, p_k)$ and $(x, y) = (x_1, y_1, \ldots, x_{n-k}, y_{n-k})$. Since G preserves the system, in particular ρ preserves p and sends $\frac{\partial}{\partial \theta_i}$ to $\frac{\partial}{\partial \theta_i}$ because it preserves the symplectic form and it sends the Hamiltonian vector fields associated to p_i to the same vector fields. After all these considerations, for each $h \in G$ the diffeomorphism $\rho(h)$ can be written as,

$$\rho(h)(p, \theta_1, \ldots, \theta_k, x, y) = (p, \theta_1 + g_1^h(p, x, y), \ldots, \theta_k + g_k^h(p, x, y), \alpha^h(x, y, p))$$

where the functions g_i^h and α^h are subdued to more constraints given by the preservation of the system. Before considering these constraints, it will be most convenient to simplify the expression of α^h first. This will be done using the local linearization theorem with parameters (corollary 6.3.5).

In order to do that, we restrict our attention to the induced mapping,

$$\overline{\rho}(h)(p, x, y) = (p, \alpha^h(p, x, y))$$

and we consider the family of diffeomorphisms $\overline{\rho}(h)_p : D^{2(n-k)} \longrightarrow D^{2(n-k)}$ defined as follows,

$$\overline{\rho}(h)_p(x, y) = \alpha^h(p, x, y).$$

We may look at $p = (p_1, \ldots, p_k)$ as parameters. For each p the mapping $\overline{\rho}(h)_p(x, y)$ induces an action of G on the disk $D^{2(n-k)}$ which preserves the induced system $(D^{2(n-k)}, \sum_{i=1}^n dx_i \wedge dy_i, \mathbf{h})$. Observe that the preservation of the induced system implies, in particular, that the action fixes the origin.

According to corollary 6.3.5 we can linearize the action $\overline{\rho}(h)_p$ in such a way that it is taken to the parametric-free linear action $\overline{\rho}(h)_0^{(1)}$. We can extend trivially the diffeomorphism Φ in the disk provided by the corollary 6.3.5 to a diffeomorphism Ψ in the whole neighbourhood considered, simply by declaring,

$\Psi(p, \theta_1, \ldots, \theta_k, x, y) = (p, \theta_1, \ldots, \theta_k, \Phi(x, y))$. After this linearization in the (x, y)-direction the initial expression of $\rho(h)$ looks like,

$$\rho(h)(p, \theta_1, \ldots, \theta_k, x, y) = (p, \theta_1 + g_1^h(p, x, y), \ldots, \theta_k + g_k^h(p, x, y), \overline{\rho}(h)_0^{(1)}(x, y)),$$

Since the action preserves the symplectic form $\sum_{i=1} dp_i \wedge d\theta_i + \sum_{i=1}^n dx_i \wedge dy_i$ we conclude that the functions g_i^h do not depend on (x, y) and so far just depend on the parameters $(p_1, \ldots, p_k)$.

That is,

$$\rho(h)(p, \theta_1, \ldots, \theta_k, x, y) = (p, \theta_1 + g_1^h(p), \ldots, \theta_k + g_k^h(p), \overline{\rho}(h)_0^{(1)}(x, y)),$$

Observe that if we prove that these functions g_i^h do not depend on p then we will be done because then the induced action on $\mathbb{T}^k$ will be performed by translations. And, in all, the action will be linear.

Consider $\mathcal{H} = \{\rho(h), h \in G\}$, we are going to prove that this group is abelian.

We have to check that $\rho(h_1) \circ \rho(h_2) = \rho(h_2) \circ \rho(h_1)$

We compute

$$\rho(h_1) \circ \rho(h_2)(p, \theta_1, \ldots, \theta_k, x, y) =$$

$$(p, \theta_1 + g_1^{h_2}(p) + g_1^{h_1}(p), \ldots, \theta_k + g_k^{h_2}(p) + g_k^{h_1}(p), \overline{\rho}(h_1)_0^{(1)}(x, y) \circ \overline{\rho}(h_2)_0^{(1)}(x, y))$$

on the other hand,

$$\rho(h_2) \circ \rho(h_1)(p, \theta_1, \ldots, \theta_k, x, y) =$$

$$(p, \theta_1 + g_1^{h_1}(p) + g_1^{h_2}(p), \ldots, \theta_k + g_k^{h_1}(p) + g_k^{h_2}(p), \overline{\rho}(h_2)_0^{(1)}(x, y) \circ \overline{\rho}(h_1)_0^{(1)}(x, y))$$

Clearly, the first $2k$ components coincide. As for the $2(n-k)$ last components, we can use theorem 6.3.6 to conclude the commutation.

So far we know that the group $\mathcal{H}$ is abelian. It is also compact, therefore it is a direct product of a torus $\mathbb{T}^r$ with finite groups $\mathbb{Z}/m_r\mathbb{Z}$. We are going to check that for each $\rho(h) \in \mathcal{H}$ the functions g_i^h do not depend on p. It is enough to check it for $\rho(h)$ in one of the components $\mathbb{Z}/n\mathbb{Z}$ and $\mathbb{T}^r$. So we distinguish two cases,

- $\rho(h)$ belongs to $\mathbb{Z}/n\mathbb{Z}$.

  Then $\rho(h)^n = Id$ this condition yields, $ng_i^h(p) = 2\pi m_i$, for all $1 \leq i \leq k$ and $m_i \in \mathbb{Z}$. In particular, $g_i^h(p) = \frac{2\pi m_i}{n}$ and g_i^h does not depend on p.

- $\rho(h)$ belongs to $\mathbb{T}^r$. We can consider a sequence $\rho(h_n)$ lying on the torus which belong to a finite group $\mathbb{Z}/k_n\mathbb{Z}$ and which converge to $\rho(h)$. For each of these points $\rho(h_n)$ we can apply the same reasoning as before to obtain, $g_i^{h_n}(p) = \frac{2\pi m_i}{k_n}$.

  Now for each n, the diffeomorphism $\rho(h_n)$ does not depend on p, we may write this condition as,

$$\frac{\partial \rho(h_n)}{\partial p_i} = 0, \quad 1 \leq i \leq k$$

  Now since the action is smooth we can take limits in this expression to obtain that

$$\frac{\partial \rho(h)}{\partial p_i} = 0, \quad 1 \leq i \leq k$$

  and finally $g_i^h(p)$ does not depend on p.

And this ends the proof of the theorem.

□

We may look at this theorem as a slice theorem for integrable systems.

In order to announce this result as a slice theorem we need some notation.

6.4.1 The classical slice theorem

Let G be a compact Lie group and let H be a closed subgroup. Let φ be a representation of H on a vector space E. Then we have an induced action of H on the product $G \times E$. The action is given by $\rho(h,(g,u)) = (gh^{-1}, \varphi(h,u))$. We can consider the space of orbits by this action; it is the quotient $G \times E/H$. Observe that this quotient is a vector bundle over G/H with typical fiber E. Classically, this quotient in denoted by $G \times_H E$.

As a matter of notation a class in the quotient is denoted by $[g,u]$. Observe that the action of G on $G \times E$ defined as $\alpha(a,(g,u)) = (ag,u), \quad a \in G$ commutes with the action of H and hence descends to the quotient and the projection $\pi : G \times_H E \longrightarrow G/H$ is G-equivariant.

Example

Assume that β stands for an action of a compact Lie group on a manifold M. Let p be a point in M. We denote by $G \cdot p$ the orbit through the point p. We denote by G_p the isotropy group for the action at the point p, $G_p = \{g \in G, \quad \alpha(g,p) = p\}$. It is a well-known fact that the isotropy group is a compact subgroup of G. Now consider a G_p-invariant Riemannian metric in a G_p invariant neighbourhood of the orbit $G \cdot p$. Define E as the subspace of the tangent space at the point p which is orthogonal to the tangent space to the orbit. Since the metric chosen is G_p-invariant. The action of G_p induces an action of G_p on E. Denote by $\alpha_p^{(1)}$ the differential of the action of G_p at the point p. As before, $\alpha_p^{(1)}$ defines a representation when restricted to E. So if we take $H = G_p$ and $\varphi = \alpha_p^{(1)}$ the vector bundle defined above becomes, $G \times_{G_p} E$.

This example is more than an example. It is the standard model for the action of a compact Lie group on a manifold. The classical slice theorem [35] asserts that a neighbourhood of the orbit is diffeomorphic to $G \times_{G_p} E$. The linear representation of G_p on E induced by the action of G is called the slice representation.

In the case the action of the manifold preserves the fibration defined by F and the symplectic structure, we have a similar "slice theorem" in the neighbourhood of an orbit whenever the orbit L coincides with the orbit of the action of the group.

6.4.2 The slice statement of the linearization

Now, let us go back to our situation. Let p be a point lying on the orbit L. Observe that the preservation of the system yields that the orbit of the action through a point $p \in L$ is contained in L but it does not have to coincide with L.

From now on we will assume that L coincides with an orbit of the action. We take coordinates centered at p. We consider the isotropy group at p, G_p. Since G_p preserves the symplectic structure leaves the symplectic orthogonal to the orbit invariant. On the other hand, the isotropy group G_p fixes L thus it induces an action of G_p on $D^{2(n-k)}$, where $D^{2(n-k)}$ is endowed with the (x_i, y_i) coordinates. On $D^{2(n-k)}$ we have the induced system $(U(L), \sum_i dx_i \wedge dy_i, \mathbf{h})$ being $\mathbf{h} = (f_{k+1}, \ldots, f_n)$. By virtue of theorem 6.3.4 we can linearize the induced action by the isotropy group in a foliation preserving way. Observe that this linear action can be extended trivially to a linear action β_p on a vector space $E_1^{2(n-k)}$ containing the disk $D^{2(n-k)}$. In the same way the foliation defined by $\mathbf{h}$ can be extended to a foliation $\mathcal{F}'$ on $E_1^{2(n-k)}$. We denote this extension by $\widetilde{\beta}_p$. Now let E_2^k a vector space containing the disk D^k endowed with coordinates $(p_1, \ldots, p_k)$. Finally, we define the linear representation of G_p on $E_1^{2(n-k)} \times E_2^k$ as $\gamma_p(u, v) = (\widetilde{\beta}_p(u), v)$.

Now form the bundle $G \times_{G_p} (E_1^{2(n-k)} \times E_2^k)$ attached to this linear representation. On this bundle we can consider the foliation induced by the product foliation $G \cdot p \times \mathcal{F}'$ on $G \times (E_1^{2(n-k)} \times E_2^k)$.

With all this notations, now we are ready to give another presentation of theorem 6.4.1 "à la slice". We do it in form of corollary,

Corollary 6.4.2 *Let G be a compact group preserving the system $(D^k \times \mathbb{T}^k \times D^{2(n-k)}, \sum_{i=1}^k dp_i \wedge d\theta_i + \sum_{i=1}^{n-k} dx_i \wedge dy_i, \mathbf{F})$ and let p be a point in L. Assume that $L = G \cdot p$, then a neighbourhood of the orbit L is G-equivariantly diffeomorphic to* *a neighbourhood of the zero section of the bundle* $G \times_{G_p} (E_1^{2(n-k)} \times E_2^k)$*. Further, this diffeomorphism can be chosen to be foliation preserving.*

6.5 Equivariant symplectic equivalence

As a corollary of the G-linearization results in the linear model obtained in the last section and the symplectic linearization in the covering obtained in the last chapter we obtain the equivariant symplectic equivalence. This equivariant symplectic equivalence is valid also for analytical systems (since the results for G-linearization are valid for analytical systems and the results of symplectic equivalence in the covering are valid for analytical systems [54]). Now we can formulate the equivariant symplectic linearization theorem for nondegenerate singular orbits of integrable Hamiltonian systems, that we have been envisaging for chapters:

Theorem 6.5.1 *Consider* $\mathcal{F}$ *the foliation defined by a completely integrable system and consider* L*, a compact orbit of Williamson type* (k_e, k_h, k_f)*. Let* ω *be a symplectic for which the foliation* $\mathcal{F}$ *is Lagrangian. Then there exists a finite group* Γ *and a diffeomorphism taking the foliation to the linear foliation on* V/Γ *given by (6.2.1,6.2.2,6.2.3,6.2.4), and taking* ω *to* ω_0*, which sends* L *to the torus* $\{p_i = x_i = y_i = 0\}$*. The smooth symplectomorphism* ϕ *can be chosen so that via* ϕ*, the system-preserving action of the compact group* G *near* L *becomes a linear system-preserving action of* G *on* V/Γ*. If the moment map* $\mathbf{F}$ *is real analytic and the action of* G *near* L *is analytic, then the symplectomorphism* ϕ *can also be chosen to be real analytic. If the system depends smoothly (resp., analytically) on a local parameter (i.e. we have a local family of systems), then* ϕ *can also be chosen to depend smoothly (resp., analytically) on that parameter.*

Observation 6.5.1 *A proof for the twisted hyperbolic case when* $n = 2$ *and* $k = 1$ *was provided by Currás-Bosch in [11].*

Chapter 7

Contact linearization of singular Legendrian foliations

7.1 Introduction

The aim of this chapter is to prove an analogue to the linearization result for singular Lagrangian foliations which was studied in the previous chapters but in the case of singular Legendrian foliations in contact manifolds.

Consider a contact manifold M^{2n+1} together with a contact form. We assume that the Reeb vector field associated to α coincides with the infinitesimal generator of an S^1 action. We assume further than there exists n-first integrals of the Reeb vector field which commute with respect to the Jacobi bracket. Then there are two foliations naturally attached to the situation. On the one hand, we can consider the foliation associated to the distribution generated by the contact vector fields. We call this foliation $\mathcal{F}'$. On the other hand we can consider a foliation $\mathcal{F}$ given by the horizontal parts of the contact vector fields. The functions determining the contact vector fields may have singularities. We will always assume that those singularities are of non-degenerate type.

Observe that $\mathcal{F}'$ is nothing but the enlarged foliation determined by the foliation $\mathcal{F}$ and the Reeb vector field.

Let α' be another contact form in a neighbourhood of a compact orbit $\mathcal{O}$ of $\mathcal{F}'$ for which $\mathcal{F}$ is Legendrian and such that the Reeb vector field with respect to α' coincides with the Reeb vector field associated to α. In this chapter we prove that then there exists a diffeomorphism from a neighbourhood of $\mathcal{O}$ to a model manifold taking the foliation $\mathcal{F}'$ to a linear foliation in the model manifold with a finite group attached to it and taking the initial contact form to the Darboux contact form. As it was done in the last chapter for Lagrangian foliations determined by a completely integrable system, we also prove the G-equivariant version of this fact for Legendrian foliations. That is, we prove that in the case there exists a compact Lie group preserving the first integrals of the Legendrian foliation and preserving the contact form then the contactomorphism can be chosen to be G-equivariant.

The problem of determining normal forms for foliations related to Legendrian foliations has its own story. P. Libermann in [38] established a local equivalence theorem for α-regular foliations. Loosely speaking, those foliations are regular foliations containing the Reeb vector field and a Legendrian foliation. The problem of classifying contact forms is different from the problem of classification of contact structures. As a example of this, if M is a compact manifold then any two contact structures are equivalent as Gray's theorem asserts ([27]). Whereas one can find examples of two contact forms which are not equivalent (see for example [26]). The problem of classifying contact structures which are invariant under a Lie group was considered by Lutz in [41]. In particular he proves that two contact structures in a compact manifold M^{2n+1} which are invariant by a locally free action of $\mathbb{R}^{n+1}$ are equivalent in the sense that there exists an equivariant contactomorphism taking one to the other.

The foliations studied by Libermann and Lutz are regular. The singular counterpart to the result of Lutz was proved by Banyaga and Molino in [4] but for contact forms.

Namely, Banyaga and Molino study the problem of finding normal forms under the additional assumption of transversal ellipticity. The assumption of transversal ellipticity allows to relate the foliation $\mathcal{F}'$ of generic dimension $(n+1)$ with the foliation given by the orbits of a torus action.

This chapter pretends to extend these results for foliations which are related in the same sense to $(n+1)$-foliations but which are not necessarily identified with the orbits of a torus action. All our study of the problem is done in a neighbourhood of a compact orbit. Global results for contact manifolds admiting torus action have been obtained by Banyaga and Molino in [4] and recently by Lerman in [36]. Linearization results for contact vector fields in $\mathbb{R}^{2n}$ with an hyperbolic zero were considered by Guillemin and Schaeffer in [28].

The chapter is organized as follows: In the first section we make a review of the basic facts in contact geometry that we will need later. In section 2 we define two foliations, $\mathcal{F}$ and $\mathcal{F}'$ and we prove that we can find coordinates in a finite covering such that the foliations have a particularly simple form. In section 3 we prove that for any two contact forms for which $\mathcal{F}$ is Legendrian and having the same Reeb vector field, we can find a foliation preserving contactomorphism taking one to the other. It turns out that the Legendrian condition imposed on the foliation for the contact form α becomes a Lagrangian condition for the same foliation with the symplectic form $d\alpha$ defined in a convenient submanifold. The result appears then as an application of the symplectic equivalence results for Lagrangian foliations which we have been working out in the previous chapters.

In the last Section we establish the G-equivariant version of contact equivalence. Applying this G-equivariant version to the particular case of the finite group attached to the finite covering, we obtain as a consequence the contact equivalence of any two contact forms fulfilling the above mentioned conditions.

7.2 Basics in contact geometry

In this section we recall some basic definitions in contact geometry.

Definition 7.2.1 *Let M^{2n+1} be a $2n+1$-dimensional manifold. A 1-form on a manifold M^{2n+1} is a contact form if the set $E = \{(p,u) \in T(M), \quad \alpha_p(u) = 0\}$ is a smooth subbundle of $T(M)$ and $d\alpha_{|E}$ is a symplectic structure on the vector bundle $E \longrightarrow M$.*

When we talk about a contact pair we consider a pair (M, α) where α is a contact form on M.

Remark:

- The classical definition of contact manifold is the following. It is a pair (M, α) where α satisfies the condition $\alpha \wedge (d\alpha)^n_p \neq 0$, $\forall p \in M$. In turn, this condition implies the nonintegrability of the subbundle $E = \{(p,u) \in T(M), \quad \alpha_p(u) = 0\}$. That is it is not possible to find a symplectic submanifold S such that $T(S) = E$.

Suppose that α is a contact form on a manifold M. Then if f is a positive function the 1-form $f\alpha$ is also a contact form.

This motivates the definition of contact structure,

Definition 7.2.2 *A contact structure on a manifold M is a subbundle E of the tangent bundle of the form $E = \{(p,u) \in T(M), \quad \alpha_p(u) = 0\}$ for some contact form α.*

The problem of classification of contact structures is different from that of contact forms. There are a lot of results in the literature concerning the classification of contact structures from a local, global or semilocal point of view. Finding their counterparts for contact forms is not always possible.

Our problem of classification will always be focused on contact forms.

In contrast to symplectic manifolds (M, ω) where the condition $i_X(\omega) = 0$ implies $X = 0$, in a contact manifold we can find non-trivial solutions X to the equation $i_X(\omega) = 0$. A privileged solution of this equation has the particular name of Reeb vector field. It is a concept attached to the contact form rather than the contact structure.

Definition 7.2.3 *Given a contact pair (M, α), the Reeb vector field Z is the unique vector field satisfying the following two conditions,*

- $i_Z d\alpha = 0$.
- $\alpha(Z) = 1$.

The Reeb vector field is a particular case of what we call contact vector field.

Definition 7.2.4 *Let f be a smooth function on the contact pair (M, α) the contact vector field associated to f is the unique vector field X_f fulfilling the following two conditions*

- $i_{X_f} d\alpha_{|E} = -df_{|E}$.
- $\alpha(X_f) = f$.

Observe that the contact vector field associated to the function 1 is precisely the Reeb vector field.

As it is proved in [38], we can express any vector field X in $T(M)$ as a sum of two vector fields X_1 and X_2 where the vector field X_1 belongs to the subbundle E and its called the horizontal part of X and the vector field X_2 is the component in the direction of the Reeb vector field. The standard notation for the horizontal vector field associated to X is $\widehat{X}$.

We can now define the notion of Jacobi bracket of two functions, which is the contact counterpart to the Poisson bracket of two functions.

Definition 7.2.5 *Let f, g be two smooth functions on a contact pair (M, α), we define the Jacobi bracket as,*

$$[f, g] = \alpha([X_f, X_g]).$$

The following relations are proved in [38],

- $$X_{[f,g]} = [X_f, X_g] \tag{7.2.1}$$

- $$[f, g] = d\alpha(X_f, X_g) + f(Z(g)) - g(Z(f)) \tag{7.2.2}$$

Definition 7.2.6 *A submanifold $N \subset M^{2n+1}$ is Legendrian if $dimN = n$ and $\alpha(X) = 0$ for any $X \in T(N)$.*

7.3 The foliation and its differentiable linearization

In this section we define the foliations that we will work with throughout the chapter and we will also define the linear model.

7.3.1 Posing the problem

Let (M^{2n+1}, α) be a contact pair and let Z be its Reeb vector field. We make the following assumptions,

- We assume Z coincides with the infinitesimal generator of an S^1 action. Let S be one of its orbits.

- We assume that there are n first integrals $f_1, \dots, f_n$ of Z (that is $Z(f_i) = 0$) which fulfill the following additional hypotheses:

  1. The first integrals are independent in an open dense set. That is, $df_1 \wedge \cdots \wedge df_n \neq 0$ in an open dense set.
  2. The n-first integrals are in involution with respect to the Jacobi bracket associated to α. That is to say,
  $$[f_i, f_j] = 0 \quad , \forall i, j.$$
  3. The minimum rank of the differential $(df_1, \dots, df_n)$ is k. Let p be a point in M^{2n+1} such that the rank is exactly k. Let $\mathcal{O}$ be the orbit of the contact vector fields through p. We will assume the following,
     (a) $\mathcal{O}$ is diffeomorphic to a torus of dimension $k+1$.
     (b) The first integrals $f_1, \dots, f_k$ are non-singular along $\mathcal{O}$ and the first integrals $f_{k+1}, \dots, f_n$ have a non-degenerate singularity in the Morse-Bott sense along $\mathcal{O}$.

Since $[f_i, f_j] = 0$ then due to formula 7.2.1, $[X_{f_i}, X_{f_j}] = 0$ and this implies that the distribution $< Z, X_{f_1}, \dots, X_{f_n} >$ is involutive because the functions f_i are first integrals of the Reeb vector field. Thus, we can talk about the foliation generated by the contact vector fields of the functions $1, f_1, \dots, f_n$. This foliation will be denoted by $\mathcal{F}'$.

On the other hand, consider the horizontal parts of the contact vector fields. They have the form $\widehat{X}_f = X_f - fZ$. Thus the distribution $< \widehat{X}_{f_1}, \dots \widehat{X}_{f_n} >$ defines an involutive distribution. The foliation defined by this distribution will be denoted by $\mathcal{F}$. Observe that since $\alpha(X_f) = f$ and $\alpha(Z) = 0$ then the regular leaves of this foliation are Legendrian submanifolds with respect to α.

That is why this foliation will be called the singular Legendrian foliation.

In fact we will work with germ-like foliations. That is, we will assume that the foliation is defined in a neighbourhood of $\mathcal{O}$. Now let $p \in M$ be a singular point. We will say that the point has rank r if the dimension of the orbit through p is r.

Once the two foliations $\mathcal{F}$ and $\mathcal{F}'$ are defined we are ready to pose the following problem.

Problem

Study the contact forms α' defined in a neighbourhood of $\mathcal{O}$ for which $\mathcal{F}$ is Legendrian and such that the Reeb vector field with respect to α' coincides with the Reeb vector field with respect to α.

As far as this problem is concerned we will prove the following.

There exists a diffeomorphism ϕ defined in a neighbourhood of $\mathcal{O}$ such that $\phi^*(\alpha') = \alpha$ and ϕ preserves the foliations $\mathcal{F}$ and $\mathcal{F}'$.

In order to deal with this problem we will need to introduce coordinates in such a way that the foliations $\mathcal{F}$ and $\mathcal{F}'$ are really simple. This judicious choice of coordinates leads us to the linear model.

7.3.2 Differentiable linearization

In this section we want to prove that under the above assumptions there exist coordinates in a neighbourhood of $\mathcal{O}$ such that the foliation can be linearized.

We prove the following theorem,

Theorem 7.3.1 *There exist coordinates* $(\theta_0, \dots, \theta_k, p_1, \dots, p_k, x_1, y_1, \dots, x_{n-k}, y_{n-k})$ *in a finite covering of a tubular neighbourhood of* $\mathcal{O}$ *such that*

- *The Reeb vector field is* $Z = \frac{\partial}{\partial \theta_0}$.
- *There exists a triple of natural numbers* (k_e, k_h, k_f) *with* $k_e + k_h + 2k_f = n - k$ *and such that the first integrals* f_i *are of the following type,* $f_i = p_i, \quad 1 \leq i \leq k$ *and*

$$
\begin{aligned}
&f_{i+k} = x_i^2 + y_i^2 \quad \text{for} \quad 1 \leq i \leq k_e \ , \\
&f_{i+k} = x_i y_i \quad \text{for} \quad k_e + 1 \leq i \leq k_e + k_h \ , \\
&f_{i+k} = x_i y_{i+1} - x_{i+1} y_i \quad \text{and} \\
&f_{i+k+1} = x_i y_i + x_{i+1} y_{i+1} \quad \text{for} \quad i = k_e + k_h + 2j - 1, \ 1 \leq j \leq k_f
\end{aligned}
$$

- *The foliation* $\mathcal{F}$ *is given by the orbits of the distribution* $\mathcal{D} = < Y_1, \dots Y_n >$ *where* $Y_i = X_i - f_i Z$ *being* X_i *the contact vector field of* f_i *with respect to the contact form* $\alpha = d\theta_0 + \sum_{i=1}^{n-k} \frac{1}{2}(x_i dy_i - y_i dx_i) + \sum_{i=1}^{k} p_i d\theta_i$.

Proof:

First of all, since Z is the infinitesimal generator of an S^1-action, according to the Slice Theorem [50] a neighbourhood of $\mathcal{O}$ in M^{2n+1} is diffeomorphic to the bundle $S^1 \times_{S^1_x} W$ where S^1_x denotes the isotropy group at a point in the orbit. Thus we can choose coordinates

$$(\theta_0, \dots, \theta_k, p_1, \dots, p_k, x_1, y_1, \dots, x_{n-k}, y_{n-k})$$

in a finite covering of a neighbourhood of $\mathcal{O}$ such that the Reeb vector field has the form $Z = \frac{\partial}{\partial \theta_0}$. Now the 1-form α can be written as

$$\alpha = d\theta_0 + \overline{\alpha}.$$

Observe that since Z is the Reeb vector field in particular we obtain

$$i_Z d\overline{\alpha} = 0$$

Using Cartan's formula $L_Z(d\overline{\alpha}) = di_Z(\overline{\alpha}) + i_Z d\overline{\alpha}$ we deduce that $\overline{\alpha}$ does not depend on θ_0.

Further, the condition on the contact form $\alpha \wedge d\alpha^n \neq 0$ implies that $d\overline{\alpha}$ is a symplectic form in the submanifold $N_0 = \{p \in U(\mathcal{O}), \quad \theta_0 = 0\}$. Let f_i be the n first integrals. The equation

$$i_Y d\overline{\alpha} = -df_i$$

has a unique well-defined solution when restricted to the symplectic submanifold N_0. We denote by $X^s_{f_i}$ the n Hamiltonian vector fields of the functions f_i with respect to the symplectic structure $d\overline{\alpha}$ on N_0. We denote by $X^c_{f_i}$ the n contact vector fields of the functions f_i with respect to the contact structure α. With all these information at hand we can write

$$X^c_{f_i} = X^s_{f_i} + g_i Z \tag{7.3.1}$$

for certain smooth functions g_i.

We are going to focus our attention in the symplectic submanifold N_0 and in the Hamiltonian vector fields $X^s_{f_i}$ for a while.

First of all, we will check that $\{f_i, f_j\} = 0$ where $\{,\}$ stands for the Poisson bracket attached to $d\overline{\alpha}$. Thus, the vector fields $X^s_{f_i}$ define a completely integrable Hamiltonian system on N_0 and the foliation they define is a singular Lagrangian foliation.

We are going to check

$$\{f_i, f_j\} = [f_i, f_j]$$

Because of the definition of Poisson bracket,

$$\{f_i, f_j\} = d\overline{\alpha}(X^s_{f_i}, X^s_{f_j})$$

Since $d\alpha = d\overline{\alpha}$, we can write this last equality as, $d\alpha(X^s_{f_i}, X^s_{f_j})$

Taking into account this observation and due to 7.3.1 this equality can be written as,

$$\{f_i, f_j\} = d\alpha(X^c_{f_i} - g_i Z, X^c_{f_j} - g_i Z)$$

But Z is the Reeb vector field and the last expression reads

$$d\alpha(X^c_{f_i}, X^c_{f_j})$$

which is, by definition, the Jacobi bracket of the functions f_i and f_j. Thus $\{f_i, f_j\} = [f_i, f_j] = 0$

Denote by $\mathcal{O}_N$ a singular compact orbit of minimal rank of the singular Lagrangian foliation in N_0. According to the symplectic linearization theorem (theorem 6.5.1) for Lagrangian foliations whose proof was concluded in the last chapter. There exists a diffeomorphism in a neighbourhood of a singular compact orbit which takes the foliation to the linearized one and the symplectic structure $d\overline{\alpha}$ to the Darboux symplectic structure. Recall that the linearized foliation has a finite group attached to it. In particular, we can find a diffeomorphism in a covering of a tubular neighbourhood of $\mathcal{O}_N$, $\phi : (\widetilde{U(\mathcal{O}_N)}) \longrightarrow \phi((\widetilde{U(\mathcal{O}_N)}))$ such that in the new coordinates provided by the diffeomorphism the first integrals have the following simple form:

$f_{i+k} = x_i^2 + y_i^2$ for $1 \leq i \leq k_e$,
$f_{i+k} = x_i y_i$ for $k_e + 1 \leq i \leq k_e + k_h$,
$f_{i+k} = x_i y_{i+1} - x_{i+1} y_i$ and
$f_{i+k+1} = x_i y_i + x_{i+1} y_{i+1}$ for $i = k_e + k_h + 2j - 1,\ 1 \leq j \leq k_f$

Now define,

$$\begin{array}{rccc} \varphi: & S^1 \times (\widetilde{U(\mathcal{O}_N)}) & \longrightarrow & \varphi(S^1 \times (\widetilde{U(\mathcal{O}_N)})) \\ & (\theta_0, z) & \longrightarrow & (\theta_0, \phi(z)) \end{array}$$

Observe that since

$$\phi^*(\sum_{i=1}^{k} dp_i \wedge d\theta_i + \sum_{i=1}^{n-k} dx_i \wedge dy_i) = d\overline{\alpha}$$

Then $\phi^*(\sum \frac{1}{2}(x_i dy_i - y_i dx_i) + \sum p_i d\theta_i + dH) = \overline{\alpha}$
this yields,

$$\varphi^*(d\theta_0 + \sum \frac{1}{2}(x_i dy_i - y_i dx_i) + \sum p_i d\theta_i + dH) = d\theta_0 + \overline{\alpha}$$

Thus we may assume that in the new coordinates

$$\alpha = d\theta_0 + \sum_{i=1}^{n-k} \frac{1}{2}(x_i dy_i - y_i dx_i) + \sum_{i=1}^{k} p_i d\theta_i + dH$$

Now consider the path of contact forms

$$\alpha_t = d\theta_0 + \sum_{i=1}^{n-k} \frac{1}{2}(x_i dy_i - y_i dx_i) + \sum_{i=1}^{k} p_i d\theta_i + t dH$$

Observe that $\alpha_1 = \alpha$ and α_0 is the Darboux contact form.

Let ψ_t be the flow of the vector field $X = -HZ$. Note that as a matter of fact, $\phi_1(\theta_0, \theta_1, \ldots, \theta_k, p_1, \ldots, p_k, x_1, \ldots, y_{n-k}) = (\theta_0 - H, \theta_1, \ldots, \theta_k, p_1, \ldots, p_k, x_1, \ldots, y_{n-k})$. So $\psi_1(\alpha_1) = \alpha_0$. Thus, $\psi_1^*(\alpha_1) = \alpha_0$ and in the new coordinates provided by ψ_1 we can assume that α is the Darboux contact form. That is to say, we can assume that $\alpha = d\theta_0 + \sum_{i=1}^{n-k} \frac{1}{2}(x_i dy_i - y_i dx_i) + \sum_{i=1}^{k} p_i d\theta_i$.

Observe that $L_{-HZ}(f_i) = 0$ this implies that $\frac{d}{dt}\psi_t^*(f_i) = 0$ and therefore $\psi_t^*(f_i)$ does not depend on t thus $\psi_1^*(f_i) = \psi_0^*(f_i) = f_i$. So in the new coordinates f_i have the same form.

Finally the foliation we are considering is generated by the horizontal parts of X_{f_i} which in the new coordinates are $Y_i = X_i - f_i Z$ being X_i the contact vector field of f_i with respect to the contact form $\alpha = d\theta_0 + \sum_{i=1}^{n-k} \frac{1}{2}(x_i dy_i - y_i dx_i) + \sum_{i=1}^{k} p_i d\theta_i$. This ends the proof of the theorem.

□

This theorem establishes the existence of a linear foliation and a model manifold.

The model manifold is the manifold $M_0^{2n+1} = \mathbb{T}^{k+1} \times U^k \times V^{2(n-k)}$, where U^k and $V^{2(n-k)}$ are k-dimensional and $2(n-k)$ dimensional disks respectively. Now we introduce a contact form in this model manifold. We take coordinates $(\theta_0, \dots, \theta_k)$ on $\mathbb{T}^{k+1}$, $(p_1, \dots, p_k)$ on U^k and $(x_1, \dots, x_{n-k}, y_1, \dots y_{n-k})$ on $V^{2(n-k)}$ and we consider the following contact form

$$\alpha_0 = d\theta_0 + \sum_{i=1}^{k} p_i d\theta_i + \sum_{i=1}^{(n-k)} \frac{1}{2}(x_i dy_i - y_i dx_i).$$

The pair (M_0^{2n+1}, α_0) is called the contact model manifold. The Reeb vector field in the contact model manifold is the vector field $\frac{\partial}{\partial \theta_0}$.

Now consider functions of the following type, $f_i = p_i, \quad 1 \leq i \leq k$ and

$$f_{i+k} = x_i^2 + y_i^2 \text{ for } 1 \leq i \leq k_e \ ,$$
$$f_{i+k} = x_i y_i \text{ for } k_e + 1 \leq i \leq k_e + k_h \ ,$$
$$f_{i+k} = x_i y_{i+1} - x_{i+1} y_i \text{ and}$$
$$f_{i+k+1} = x_i y_i + x_{i+1} y_{i+1} \text{ for } i = k_e + k_h + 2j - 1, \ 1 \leq j \leq k_f$$

The linear foliation is the foliation given by the orbits of the distribution $\mathcal{D} = < Y_1, \dots Y_n >$ where $Y_i = X_i - f_i Z$ being X_i the contact vector field of f_i in the contact model manifold.

In all, we have proved that there exists a finite covering of a neighbourhood $U(\mathcal{O})$ of the compact orbit considered such that the lifted foliation in the covering is differentiably equivalent to the linear foliation in the contact model manifold.

The linear model for the foliation $\mathcal{F}'$ is the foliation expressed in the coordinates provided by the theorem together with a finite group attached to the finite covering.

The different smooth submodels corresponding to the model manifold are labeled by a finite group which acts in a contact fashion and preserves the foliation in the model manifold. This is the only differentiable invariant. Therefore, our problem of contact equivalence will be studied in this model manifold and the equivalence will be established via the equivariant version equivalence which will be considered in the last section.

7.4 Contact linearization in the model manifold

The aim of this section is to prove the following theorem,

Theorem 7.4.1 *Let α be a contact form on the model manifold M_0^{2n+1} for which $\mathcal{F}$ is a Legendrian foliation and such that the Reeb vector field is $\frac{\partial}{\partial\theta_0}$. Then there exists a diffeomorphism ϕ defined in a neighbourhood of the singular orbit $\mathcal{O} = (\theta_0, \dots, \theta_k, 0, \dots, 0)$ preserving $\mathcal{F}'$ and taking α to α_0.*

Proof:

We are going to solve the problem by adjusting the contact form to a point where we can apply our symplectic linearization result.

Let us start by considering the contact 1-form α,

$$\alpha = Ad\theta_0 + \sum B_i dp_i + \sum C_i d\theta_i + \sum D_i dx_i + \sum E_i dy_i$$

Observe that the fact that the Reeb vector field is $\frac{\partial}{\partial\theta_0}$ imposes the following two conditions on α,

- $\alpha(\frac{\partial}{\partial\theta_0}) = 1$, that is to say $A = 1$.

  So far we can write $\alpha = d\theta_0 + \alpha'$, being $\alpha' = \sum B_i dp_i + \sum C_i d\theta_i + \sum D_i dx_i + \sum E_i dy_i$.

- $i_{\frac{\partial}{\partial\theta_0}} d\alpha = 0$,

Since $d\alpha = d\alpha'$ the condition becomes,

$$i_{\frac{\partial}{\partial\theta_0}} d\alpha' = 0$$

Now Cartan's formula yields,

$$0 = i_{\frac{\partial}{\partial\theta_0}} d\alpha' = L_{\frac{\partial}{\partial\theta_0}} \alpha' - d i_{\frac{\partial}{\partial\theta_0}} \alpha'$$

Since the last term vanishes this chain of equalities give the condition,

$$L_{\frac{\partial}{\partial\theta_0}} \alpha' = 0$$

Therefore, the coefficient functions do not depend on θ_0. Let us see that the submanifold $\theta_0 = 0$ equipped with the form $d\alpha'$ is a symplectic submanifold of the model contact manifold. We denote this submanifold by N.

Since α is a contact form $d\alpha$ has to be symplectic in the vector bundle E defined by $E = \{(p, u) \in T(M), \quad \alpha_p(u) = 0\}$ and $d\alpha = d\alpha'$ then $d\alpha'$ defines a symplectic structure on N.

Observe that the vector fields $X_i = X_{f_i}$ are tangent to the submanifold N. Next step, we check that the vector fields X_i are Lagrangian for N, observe that $\alpha(X_i) = f_i$.

Now since, $d\alpha'(X_i, X_j) = X_i\alpha(X_j) - X_j\alpha(X_i) - \alpha([X_i, X_j])$

According to the computation above $X_i\alpha(X_j) = X_i(f_j)$ but f_i are first integrals for the foliation and therefore this term vanishes. Symmetrically, the second term vanishes. And since the Lie bracket of the vector fields are zero we obtain,

$$d\alpha'(X_i, X_j) = 0$$

Therefore, the foliation $\mathcal{F}$ is Lagrangian for $d\alpha'$ and we may apply the symplectic linearization result in a neighbourhood of $L = \mathbb{T}^k$ (theorem 5.2.1) to find a local diffeomorphism $\varphi : U(L) \longrightarrow \varphi(U(L))$ in a neighbourhood of the leaf L, preserving the foliation $\mathcal{F}$ and satisfying $\varphi^*(\omega_0) = d\alpha'$, where $\omega_0 = \sum_i dp_i \wedge d\theta_i + \sum dx_i \wedge dy_i$. After shrinking the initial neighbourhood if necessary, the neighbourhood of $\mathbb{T}^{k+1}$ in the initial manifold M can be decomposed as a product, $\mathbb{S}^1 \times U(L)$. The $\mathbb{S}^1$ corresponds to an orbit of the Reeb vector field. We denote by z a point in $U(L)$. Now we define a diffeomorphism in the following way,

$$\begin{array}{rccc} \phi : & \mathbb{S}^1 \times U(L) & \longrightarrow & \phi(\mathbb{S}^1 \times U(L)) \\ & (\theta_0, z) & \longrightarrow & (\theta_0, \varphi(z)) \end{array}$$

Since φ preserves $\mathcal{F}$ it is clear that this diffeomorphism is foliation-preserving.

Now consider $\phi(\mathbb{S}^1 \times U(L))$ endowed with the Darboux contact form. That is with the contact form $\alpha_0 = d\theta_0 + \sum_{i=1}^k p_i d\theta_i + \sum_{i=1}^{(n-k)} \frac{1}{2}(x_i dy_i - y_i dx_i)$. It remains to check that the diffeomorphism above is indeed a contactomorphism.

First observe that since

$$\varphi^*(\omega_0) = d\alpha'$$

and $\omega_0 = d(\beta)$, being $\beta = (\sum_{i=1}^k p_i d\theta_i + \sum_{i=1}^{(n-k)} \frac{1}{2}(x_i dy_i - y_i dx_i))$ we can assert that $\varphi^*(\beta) = \alpha' + df$ for a smooth function f. Observe that since φ preserves the foliation the function f is a basic function for the foliation. Now consider the path $\alpha_t = \alpha_0 + t df$ being α_0 the contact form $\alpha_0 = d\theta_0 + \alpha'$.

Now, consider the vector field $X = -f\frac{\partial}{\partial \theta_0}$. Denote by ψ_t its flow. Since $\psi_1(\theta_0, \theta_1, \ldots, \theta_k, p_1, \ldots, p_k, x_1, \ldots, y_{n-k}) = (\theta_0 - H, \theta_1, \ldots, \theta_k, p_1, \ldots, p_k, x_1, \ldots, y_{n-k})$, we obtain $\psi_1^*(\alpha_1) = \alpha_0$.

Therefore ϕ is a contactomorphism. And clearly it preserves the foliation because $[X, X_i] = 0$ and therefore the flow ψ_t preserves the foliation.

And this ends the proof of the theorem.

□

7.5 Equivariant contact linearization

In this section we consider a compact Lie group G acting on a contact model manifold in such a way that preserves the n first integrals of the Reeb vector field and preserves the contact form as well. We want to prove that there exists a diffeomorphism in a neighbourhood of $\mathcal{O}$ preserving the n first integrals , preserving the contact form and linearizing the action of the group. This result is a consequence of the equivariant symplectic linearization theorem of the last chapter.

The notion of linear action of a Lie group on the contact model manifold is analogous to the equivalent notion for the symplectic model manifold.

Let G be a group defining a smooth action $\rho : G \times M_0^{2n+1} \longrightarrow M_0^{2n+1}$ on M_0^{2n+1}. We assume that this action preserves the contact form α_0 of the contact model manifold. That is to say $\rho_g^*(\alpha_0) = \alpha_0$. Assume further that it preserves the n-first integrals $(f_1, \ldots, f_n)$, where $f_i = p_i, \quad 1 \leq i \leq k$. For the sake of simplicity we denote by F the collective mapping $F = (p_1, \ldots, p_k, f_{k+1}, \ldots, f_n)$. We will say that the action of G on M_0^{2n+1} is *linear* if it satisfies the following property:

G acts on the product $M_0^{2n+1} = D^k \times \mathbb{T}^{k+1} \times D^{2(n-k)}$ componentwise; the action of G on D^k is trivial, its action on $\mathbb{T}^{k+1}$ is by translations (with respect to the coordinate system $(\theta_0, \ldots, \theta_k)$), and its action on $D^{2(n-k)}$ is linear with respect to the coordinate system $(x_1, y_1, ..., x_{n-k}, y_{n-k})$.

Under the above notations and assumptions. Now we can state and prove the following theorem,

Theorem 7.5.1 *There exists a diffeomorphism ϕ defined in a tubular neighbourhood of $\mathcal{O}$ such that,*

- *it preserves the contact form α_0 i.e $\phi^*(\alpha_0) = \alpha_0$.*
- *it preserves F.*
- *it linearizes the action of G. That is to say $\phi \circ \rho_g = \rho_g^{(1)} \circ \phi$.*

Proof:

Recall that $\alpha_0 = d\theta_0 + \overline{\alpha}_0$ being $\overline{\alpha}_0$ the 1-form $(\sum_{i=1}^{k} p_i d\theta_i + \sum_{i=1}^{(n-k)} \frac{1}{2}(x_i dy_i - y_i dx_i))$. Consider the symplectic manifold $S = M_0^{2n+1} \times (-\epsilon, \epsilon)$ endowed with the symplectic form $\omega_0 = dt \wedge d\theta_0 + d\overline{\alpha}_0$, where t stands for a coordinate function on $(-\epsilon, \epsilon)$. An action of G on M_0^{2n+1} can be extended in a natural way to an action of G on S as follows,

$$\begin{array}{rccc} \widehat{\rho}: & G \times M_0^{2n+1} \times (-\epsilon, \epsilon)+ & \longrightarrow & M_0^{2n+1} \times (-\epsilon, \epsilon) \\ & (g, z, t) & \longrightarrow & (\rho_g(z), t) \end{array}$$

On S we consider the moment mapping $\widehat{F} = (F, t)$. We can apply the equivariant linearization theorem to obtain a symplectomorphism $\widehat{\varphi}$ preserving $\widehat{F}$ and linearizing the action $\widehat{\rho}$. From the definition of the action $\widehat{\rho}$ and the definition of $\widehat{F}$, this symplectomorphism clearly descends to a diffeomorphism φ on M_0^{2n+1} which linearizes the action ρ and which satisfies $\varphi^*(d\alpha_0) = d\alpha_0$.

Therefore,

$$\varphi^*(\alpha_0) = \alpha_0 + dh$$

Finally the diffeomorphism,

$$\phi(\theta_0, \ldots, \theta_k, p_1, \ldots, p_k, x_1, \ldots, y_{n-k}) = (\theta_0 - h, \ldots, \theta_k, p_1, \ldots, p_k, x_1, \ldots, y_{n-k})$$

takes the form $\alpha_0 + dh$ to α_0 and provides new coordinates for which the action is linear.

□

In the previous section we have attained the contact linearization in the covering. Now applying the theorem of equivariant linearization to the group of deck transformations we obtain as a corollary the following theorem,

Theorem 7.5.2 *Let $\mathcal{F}$ be a foliation fulfilling the hypotheses specified in section 7.3.1, let $\mathcal{F}'$ be the enlarged foliation with the Reeb vector field Z and let α be a* *contact form for which* $\mathcal{F}$ *is Legendrian and such that* Z *is the Reeb vector field then there exists a diffeomorphism defined in a neighbourhood of* $\mathcal{O}$ *taking* $\mathcal{F}'$ *to the linear foliation, the orbit* $\mathcal{O}$ *to the torus* $\{x_i = 0, y_i = 0, p_i = 0\}$ *and taking the contact form to the Darboux contact form* α_0.

"Where am I? Metaphysics says
No question can be asked unless
It has an answer, so I can
Assume this maze has got a plan."

"The Maze" W.H Auden

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