UNIVERSITAT DE BARCELONA

The Planar Limit of N=2 Superconformal Field Theories

Alan Rios Fukelman

- Tesi doctoral -

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a mis abuelos

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"In this age
which believes that there is a short-cut to everything,
the greatest lesson to be learned is that the most difficult way, in the long run,
is the easiest."
Henry Miller

Preface

This thesis gathers the main results during my PhD studies, from 2018 to 2022. These studies have been performed at the Gravity and Strings group of the Universitat de Barcelona, under the supervision of Prof. Bartomeu Fiol. We have developed new techniques to fully characterize the planar limit of a large family of Matrix models that appear in Lagrangian $\mathcal{N} = 2$ 4−dimensional superconformal field theories (SCFT) upon applying supersymmetric localization on S^4.

The thesis is organised as follows:

- Chapter 1 serves as an introduction to the framework utilized to study such problems. We start with a brief introduction to the main technical tools utilized such as supersymmetric gauge theories, supersymmetric localization and supersymmetric Wilson loop. When possible we present some basic example of the tools that we further explore down the road.

- In Chapter 2 we develop new matrix model techniques that allow us to fully characterize the planar limit of Lagrangian $\mathcal{N} = 2$ 4-dimensional SCFT and to characterize the planar Free Energy and the Wilson loop.

- In Chapter 3, utilizing the techniques developed in Chapter 2 we further solve a large family of matrix models with an infinite amount of both single and double trace deformations. This allows us to study and characterize the behaviour of 2− and 3−point functions of Chiral Primary operatos.

- In Chapter 4 we study the planar free energy of the Hermitian one matrix model and we characterize its convergence properties. We extend the techniques to discuss some examples of non-conformal $\mathcal{N} = 2$ super Yang-Mills theories.

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CHAPTER 1

Introduction

During the last decades Theoretical Physics has had a large impact in our understanding of nature. Ranging through large scale phenomena, to the smallest scales probed by experiments, we have seen a thriving interplay between old and new ideas that have deepened our grasp of the inner workings of the universe.

One of the most outstanding findings is that most of such phenomenema allows a description in terms of Quantum Fields. This is not only surprising given the large range of applicability of the theoretical framework but also, because more often than not, most of the a priori inconsistencies found within the theory lead us, once understood, to new and deep ideas about the Universe.

Over the last decades much progress has been made in our understanding of Quantum Field Theories (QFTs). We have been able to reach an unprecedented degree of precision and descriptive power trough the use of perturbative techniques, but of course we know that there is more to this than what meets the eye. Not only we have long been aware [1,2] that, in general, perturbative series of physical observables are asymptotic at best but also, that they totally fail to account for a large class of phenomena that we now call *nonperturbative effects*. This problem is not only theoretical, we know many physical systems realized in nature for which we require a description in terms of non-perturbative physics such as Quantum Chromodynamics (QCD). The more we study and comprehend QFTs the more we realize how constrained they are. It was long ago noted by Coleman and Mandula [3] that, under certain reasonable assumptions, we cannot build a QFT in which Poincarè invariance and internal symmetries are combined in any non-trivial way. While a priori stringent in its limitations, this no-go theorem opens up the possibility of studying theories with extended symmetries that arise as a consequence of by-passing one of the assumptions of the theorem and, most often than not, when characterizing these theories we find ourselves with the right tools at our disposal to obtain results that were previously out of reach both in the perturbative and non-perturbative regime of the theories.

Since all the fields are massless and there is no notion of a $S-$matrix, conformal field theories (CFT) avoid the Coleman-Mandula theorem and serve as an example of QFTs with extended symmetry. Much attention has been given to these theories in the last years: two dimensional CFTs have been heavily studied and partially classified motivated by applications in statistical physics and string theory, at present the most promising candidate for a theory of quantum gravity. However, most of the techniques developed for $2-$d CFTs do not extend to higher dimensions since in such cases the conformal group is finite dimensional and we do not have the powerful machinery that the infinite dimensional Virasoro algebra enables. In the modern context, CFTs can be seen to arise at the fixed points of the Renormalization Group (RG) flow of QFTs. At these points, the $\beta-$functions of the theory vanish, and if this leads to a zero trace of the energy momentum tensor of the theory the fixed point defines then a CFT. Many recent developments follow from the bootstrap equations, which are a consequence of just crossing symmetry and unitarity of theories with conformal symmetry. These equations are amenable to numerical computations from which significant constraints on the spectrum of operators, their dimensions and spins has been obtained.

Despite being originally introduced in early 1970, by considering additional spinorial generators and thus by-passing the Coleman-Mandula theorem, we still don't know how or if supersymmetry and supersymmetric gauge theories play a role in nature. Though in the early days the expectation was that new supersymmetric particles would be found in the next generation of experiments we have now realised that supersymmetry might be harder to detect than previously expected. Despite the apparent failure of the theory to meet the previous expectations, we have now understood that, due to the enourmous theoretical advances that supersymmetric field theories have enabled, these theories serve as a very valuable tool to deepen our understanding of general QFTs. Ranging from the absence of renormalization of many superpotentials, to the Seiberg duality, much of the original breakthroughs obtained in these theories were restricted to holomorphic observables or holomorphicity arguments. It wasn't until recently that we were able to compute other interesting observables in supersymmetric field theories by exploiting the results of the holographic duality and supersymmetric localization. In this case, the path integral of the theory placed on a compact manifold localizes to a certain $0-$dimensional theory described by a matrix model in which, despite the apparent simplifications, computations are still far from trivial.

This thesis aims to provide a first step in overcoming this difficulty. We study theories with superconformal symmetry in which supersymmetric localization can be applied. By developing novel matrix model techniques and a unified framework to understand the family of Lagrangian 4d $\mathcal{N}=2$ superconformal theories, we were able to fully solve the matrix integrals in the $0-$instanton sector. The techniques developed in this work allow us to obtain the full perturbative expansion of relevant observables, in the planar regime, in a purely combinatorial way. In addition, these techniques allowed us to exactly characterize certain general Hermitian one-matrix models that do not necesarily stem from supersymmetric theories. We managed to determine or bound the radius of convergence of the planar perturbative series as well as characterizing the planar regime of two non-conformal theories.

In the following sections of this Introduction, we present the basic ingredients to understand the technical tools and scope of the thesis. I have tried to make the presentation as self contained as possible, with examples when possible and references when needed in the hope of producing a fast review for the expert and a crashcourse for the newcomer.

§1.1 Quantum Field Theories with extended Symmetries

Quantum Field Theory is, in the most general scenario, the study of Renormalization Group (RG) flows, *i.e.* how the theory evolves from the UV to the IR regimes. Upon following the RG flow we could try to ask the following question: which is the behaviour of the theory in the deep IR region?. A priori we could expect to find three different phases

- a theory with a mass gap,
- a theory with massless particles,
- a scale invariant theory.

Theories of the first two kinds are traditionally more familiar than the later one, non-abelian Yang-Mills theory in $d = 4$ is one of the well known cases for which the low energy spectrum includes a *glueball*, a scalar particle of mass m that is dynamically generated upon flowing to the IR. On the other hand, Quantum Electrodynamics (QED) at energies $E \ll m_e$, the mass of the electron, serves as an example of a theory of the second kind since we are left with a theory of massless photons. Both cases are described as relativistic field theories in flat space-time, and as such we know that the symmetry group is given by transformations that leave the flat space metric $\eta_{\mu\nu} = \mathrm{diag}(-,+,+,+)$ invariant, the Poincaré group. Such transformations are of the form

$$x^\mu \to \Lambda^\mu_\nu x^\nu + a^\nu \,, \tag{1.1}$$

and are a combination of Lorentz transformations and translations parametrized by Λ and a respectively.

On the other hand, theories of the third kind are ones such that the scale transformation

$$x^\mu \to \lambda\, x^\mu \,, \tag{1.2}$$

is a symmetry. It is straightforward to note that these transformations are clearly *not* in the Poincaré group,

$$\eta_{\mu\nu} \to \lambda^{-2} \eta_{\mu\nu} \,, \tag{1.3}$$

from which we see that while lengths are rescaled, angles are preserved. Upon inspecting the transformation (1.2) we see that it is a special case of the most general conformal transformation

$$x^\mu \to \phi^\mu(x) \tag{1.4}$$

such that the metric is invariant up to a local scale factor

$$\eta_{\mu\nu}\frac{\partial\phi^\mu(x)}{\partial x^\alpha}\frac{\partial\phi^\nu(x)}{\partial x^\beta} = \lambda(x)^2\eta_{\alpha\beta}\,, \tag{1.5}$$

where now $\lambda(x)$ is the scale factor associated to the conformal transformation (1.4). Theories invariant under conformal transformations given by (1.4) are called Conformal Field Theories (CFT). In the case in which $\lambda(x) = 1$ we recover the invariance under the Poincarè group, thus the Conformal group is an extension of the former one.

From an RG flow point of view, theories with scale invariance arise at the fixed points in the IR of the β functions of the theory. While we do not necessarily have a UV fix point that defines also another CFT, it is nevertheles a useful framework to consider such scenario. We see thus that, studying CFTs let us map out the possible endpoints of RG flows, and thus understand the space of QFTs.

While more general than scale invariance, in $d = 2$ it can be proven that scale invariance is enhanced to the full conformal invariance [4, 5]. In $d = 4$, despite not having a formal proof of the enhancement, for unitary, Poincaré invariant theories with a discrete spectrum in scaling dimension there is perturbative evidence [6, 7] of such enhancement. We will therefore utilize the terms interchangeably and refer to the reviews [8, 9] for further details on the subject.

§1.1.1 Conformal Field theories

We can now consider a spacetime metric $\eta_{\mu\nu}$ on $\mathbf{R}^{m,n}$ with $d = m + n$, we have already discussed that the Poincaré group is a sub-group of the Conformal one, thus for $d \geq 3$ we have the following transformations

- Lorentz transformations:

  $$\phi^\mu(x) = M^\mu_\nu x^\nu\,, \tag{1.6}$$

  where M^μ_ν preserves $\eta_{\mu\nu}$ and is an element of SO(m,n). As previously noted, Lorentz transformations are isometries and hence the scale factor $\lambda(x) = 1$ is constant.

- Translations:

  $$\phi^\mu(x) = x^\mu + P^\mu\,, \tag{1.7}$$

  where once again they are isometries and have the same trivial scale factor.

- Dilatations:
$$\phi^\mu(x) = Dx^\mu \,, \tag{1.8}$$
this is the first new ingredient of the conformal group whose scale factor is $\lambda(x) = D$.

- Inversion, a discrete transformation:
$$\phi^\mu(x) = \frac{x^\mu}{|x|^2} \,, \tag{1.9}$$
whose scale factor is $\lambda(x) = |x|^{-2}$.

- Special conformal transformations:
$$\phi^\mu(x) = \frac{x^\mu + |x|^2 K^\mu}{1 + 2x^\nu K_\nu + |x|^2|K|^2} \,, \tag{1.10}$$
where now the scale factor is $\lambda(x) = \left(1 + 2x^\mu K_\mu + |x|^2|K|^2\right)^{-1}$, and it can be understood as a composition of a inversion plus a traslation plus another inversion.

The conformal group in $d \geq 3$ is isomorphic to SO(m+1,n+1), and we denote the corresponding generators of the Lie algebra by $M_{\mu\nu}$, P_μ, D and K_μ. They obey the following commutation relations

$$[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\nu\rho} M_{\mu\sigma} + \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\nu\sigma} M_{\mu\rho} \,, \tag{1.11a}$$
$$[M_{\mu\nu}, P_\rho] = \eta_{\nu\rho} P_\mu - \eta_{\mu\rho} P_\nu \,, \tag{1.11b}$$
$$[M_{\mu\nu}, K_\rho] = \eta_{\nu\rho} K_\mu - \eta_{\mu\rho} K_\nu \,, \tag{1.11c}$$
$$[D, P_\mu] = P_\mu \,, \tag{1.11d}$$
$$[D, K_\mu] = -K_\mu \,, \tag{1.11e}$$
$$[K_\mu, P_\nu] = 2(\eta_{\mu\nu} D - M_{\mu\nu}) \,, \tag{1.11f}$$

while all the other commutators vanish. From the commutation relations we can see that $M_{\mu\nu}$ defines the Lorentz Lie algebra so(m,n) and that both P_μ and K_μ transform in the vector representation of the algebra. The ones involving D tell us that we can think of P_μ, K_μ as raising and lowering operators for D, and hence every other generator has some weight, called *scaling dimension*, under it.

The extra symmetries that invariance under the conformal group (1.11) give us serve as powerful tools to organize and characterize the theory. Ranging from the study of its representations, unitarity conditions, to the bootstrap equations, the study of CFTs is a rich and active field of research whose full scope is out of the reach of this work and we thus refer the reader to the excellent review [10].

§1.1.2 Lagrangian $\mathcal{N} = 2$ Superconformal field theories

It has long been understood that supersymmetric quantum field theories enjoy many special properties that make them particularly useful testing grounds for more general ideas about quantum field theory. This is largely a consequence of the fact that many observables in such theories are *protected*, in the sense of being determined by a semiclassical calculation with a finite number of corrections taken into account, or alternatively by some related finite-dimensional problem that admits the type of closed form solution that is uncharacteristic of interacting quantum field theories.

In particular, the study of $\mathcal{N} = 2$ supersymmetric quantum field theories in four-dimensions has been a fertile ground for theoretical physicists for quite some time. These theories always have non-chiral matter representations, and therefore can never be directly relevant for describing the real world. That said, the existence of two sets of supersymmetries allows us to study their properties in much greater detail than both non-supersymmetric theories and $\mathcal{N} = 1$ supersymmetric theories. Starting from the seminal works of Seiberg and Witten [11, 12], where the infrared dynamics of $SU(2)$ gauge theory was exactly determined, to the work on instantons of Nekrasov [13], to the realization of Gaiotto [14] of the importance of strongly-coupled superconformal field theories arising from compactifications of the six-dimensional $\mathcal{N} = (2, 0)$ theories, we have seen the study of four dimensional $\mathcal{N} = 2$ theories as a fertile playground for mathematical physics. Still, supersymmetric UV-complete gauge theories composed of gauge groups and hypermultiples form a traditional subclass of all possible four-dimensional supersymmetric systems and are the main subject of these thesis.

In order to construct a supersymmetric QFT we want to enlarge the Poincaré algebra by generators that transform as spinors under the Lorentz group in such a way that commute with the translations P_μ. Considering the operators Q^I_α, $\bar{Q}^I_{\dot\alpha}$ transforming as $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ spinors respectively and considering the extra index $I = 1, \cdots, \mathcal{N}$ that labels the different spinorial generators, the supersymmetry algebra is given by

$$[M_{\mu\nu}, Q^I_\alpha] = i\,(\sigma_{\mu\nu})_\alpha^{\ \beta}\, Q^I_\beta\,, \tag{1.12a}$$

$$[M_{\mu\nu}, \bar{Q}^{I\dot\alpha}] = i\,(\bar\sigma_{\mu\nu})^{\dot\alpha}_{\ \dot\beta}\, Q^{\dot\beta I}\,, \tag{1.12b}$$

$$\{Q^I_\alpha, \bar{Q}^J_{\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta} P_\mu \delta^{IJ}\,, \tag{1.12c}$$

$$\{Q^I_\alpha, Q^J_\beta\} = \epsilon_{\alpha\beta} Z^{IJ}\,, \quad \{\bar{Q}^I_{\dot\alpha}, \bar{Q}^J_{\dot\beta}\} = \epsilon_{\dot\alpha\dot\beta} (Z^{IJ})^*\,. \tag{1.12d}$$

where the $Z^{IJ} = -Z^{JI}$ are central charges which means they commute with all generators of the full algebra. The simplest supersymmetry algebra has $\mathcal{N} = 1$ and no possibility of central charges. We can consider also theories with extended supersymmetry in which case $\mathcal{N} \geq 2$, in this work we will consider theories with $\mathcal{N} = 2$ supersymmetries.

In order to construct a supersymmetric QFT it is necessary to find representations of the susy algebra on fields. A standard way to do so is to introduce superspace and superfields in which we enlarge space-time by considering the usual coordinates x^μ with the addition of two anticommuting Grassmannian coordinates θ_α and $\bar{\theta}_{\dot{\alpha}}$. This method is particularly simple for theories with $\mathcal{N}=1$ supersymmetry, and can be extended to harmonic superspace in order to deal with theories with $\mathcal{N}=2$.

Instead of following that route we note that an $\mathcal{N}=2$ supersymmetric theory is in particular a $\mathcal{N}=1$ theory, therefore we can construct the Lagrangian of the theory utilizing $\mathcal{N}=1$ superfield language and imposing conservation of $SU(2)_R$ R-charge symmetry between the fields [15]. The $\mathcal{N}=1$ vector multiplet consists of a Weyl fermion λ_α and a vector field A_μ both in the adjoint of the gauge group G, while a chiral multiplet consists of a complex scalar Q and a Weyl fermion ψ_a both in the same representation R. With this field content we build the superfields

$$\begin{aligned} \mathcal{N}=1\,\text{vector multiplet}: W_\alpha &= \lambda_\alpha + F_{(\alpha\beta)}\theta^\beta + D\theta_\alpha + \cdots \\ \mathcal{N}=1\,\text{chiral multiplet}: Q &= Q\Big|_{\theta=0} + 2\psi_a\theta^\alpha + F\theta_\alpha\theta^\alpha \end{aligned} \tag{1.13}$$

where D,F are auxiliary fields and $F_{\alpha\beta} = \frac{i}{2}\sigma^{\mu\beta}_{\dot{\gamma}}\bar{\sigma}^{\nu\dot{\gamma}}_{\alpha}F_{\mu\nu}$ is the anti-self-dual part of the field strength $F_{\mu\nu}$. This two supersymmetric multiplets serve as the building block for the $\mathcal{N}=2$ vector multiplet

$$\mathcal{N}=2\ \text{vector multiplet:}\quad \begin{array}{ccc} & A_\mu & \\ \lambda^{I=1}_\alpha & & \lambda^{I=2}_\alpha \\ & \phi & \end{array}$$

where we see that the $\mathcal{N}=2$ vectormultiplet consists of an $\mathcal{N}=1$ vector multiplet and an $\mathcal{N}=1$ chiral multiplet, both sitting in the adjoint of the gauge group. For the hypermultiplet

$$\mathcal{N}=2\ \text{hypermultiplet:}\quad \begin{array}{ccc} & \psi_\alpha & \\ Q^{I=1}=q & & Q^{I=2}=\bar{\tilde{q}} \\ & \bar{\tilde{\psi}}_{\dot{\alpha}} & \end{array}$$

where now we can clearly see two $\mathcal{N}=1$ chiral multiplets, in the representation R and R^* respectively.

These are the fundamental blocks that allow us to build the theory with $\mathcal{N}=2$ supersymmetry. The Lagrangian of the theory will consist then on a pure $\mathcal{N}=2$ Super Yang-Mills sector and a matter contribution that we can build by choosing different representations R, R^* on which the hypermultiplets sit [16]. Thus we see that the theory we are considering is characterized by the matter content that we include in the theory. The theory has one complex coupling constant per gauge group

$$\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi} \tag{1.14}$$

where g is the Yang-Mills coupling constant and θ the instanton factor. Upon fixing the matter content of the theory we can study the beta function of the theory, it was shown by Novikov-Shifman-Vainshtein-Zakharov [17–19] that the beta function is $1-$loop exact and it vanishes given that

$$I_2(\mathrm{adj}) = \sum_{\mathcal{M}} n_{\mathcal{M}} I_2(\mathcal{M}) \,, \tag{1.15}$$

where $I_2(\mathcal{M})$ is the index of the representation $\mathcal{M}$ in which we have $n_{\mathcal{M}}$ hypermultiplets and $I_2(\mathrm{adj})$ is the index of the adjoint representation. We thus see that a suitable choice of matter content can produce the vanishing of the $\beta-$function in which case we have a theory with enhanced $\mathcal{N}=2$ superconformal symmetry. This allowed, for theories with semi-simple gauge groups G, to obtain a classification of $4d$ Lagrangian $\mathcal{N}=2$ superconformal symmetry [20,21].

§1.2 Supersymmetric Observables

The landscape of four-dimensional supersymmetric theories is vast and very rich, but due to the large amount of symmetry it is also highly constrained. Theories with $\mathcal{N}=1$ SUSY are very rich in their dynamics and physics seem, in the near future, to avoid a possible classification. Considering extra symmetries help to alleviate this problem, restricting our attention to Lagrangian $\mathcal{N}=2$ SCFTs we have already seen that a classification is possible just by characterizing their matter content, and it is also known that in the maximally symmetric case, *i.e* $\mathcal{N}=4$ we have a complete classification given by $\mathcal{N}=4$ Super-Yang-Mills.

Being able to constraint the space of possible theories, one would then hope to solve them by characterizing their operator product expansion (OPE) coefficients, scaling dimensions, correlations functions and so on. While daunting at first sight, much progress has been made in this direction by finding a protected subsector that is amenable to computations. In the case of $\mathcal{N}=4$ SYM there are many different sectors that allow to utilise and combine, in different regimes of validity, powerful techniques such as integrability, supersymmetric localization, the AdS/CFT correspondence, and the bootstrap equations for example. On the other hand, for theories with $\mathcal{N}=2$ things tend to be more complicated and in many cases the techniques developed for the maximally supersymmetric case do not extend inmidiately. Yet there has been advancements in this direction for integrability [22] or the bootstrap [23] of these theories. In this work we will follow another route and focus our attention to a set of operators belonging to the subsector that is captured by supersymmetric localization.

§1.2.1 Extremal correlation functions

For lagrangian superconformal field theories of the kind analyzed in the previous sector, we can study their supersymmetric vaccuum structure. To obtain the classical supersymmetric vacua one has to minimize the scalar potentials of the theory, which result in

$$\begin{aligned} [\phi^\dagger, \phi] = 0\,, &\quad (q_i q^{\dagger,i} - \tilde{q}_i^\dagger \tilde{q}^i)\Big|_{\text{traceless}} \\ \phi^\dagger q_i = 0 \;\;, &\quad \tilde{q}^i \phi^\dagger = 0\,, \end{aligned} \tag{1.16}$$

where we have added a flavour index i. There are two simple ways to satisfy these conditions

- Coulomb Branch: $q = 0$ and ϕ a normal matrix. This generally breaks the gauge group G to $U(1)^{\text{rank}(G)}$
- Higgs Branch: $q \neq 0$ and $\phi = 0$, subject that the q's verify the second condition.

Operators that parametrize the Coulomb branch deformations are known as Coulomb branch operators, for $G = SU(N)$ they are given by

$$\text{Tr}\left(\phi^k\right)\,, \quad k = 2, \cdots, N \tag{1.17}$$

studying representation theory for these family of theories it is easy to see that these operators preserve supersymmetry and thus sit in short representations of the superconformal algebra, in particular their scaling dimension is given in terms or their $r-$charge by $\Delta = \frac{r}{2}$ and we call them *chiral primaries*, in a similar way we can have *anti-chiral primaries* which $\Delta = -\frac{r}{2}$.

Coulomb branch operators have a ring structure, the so-called *chiral ring*. Their OPE has the form

$$\Phi_{r_1}(x_1)\Phi_{r_2}(x_2) = |x_1 - x_2|^{\Delta - \Delta_1 - \Delta_2}\Phi_{r_1+r_2}(x_2) + \cdots\,, \tag{1.18}$$

where, Φ_{r_i} denote the chiral primaries and the dots represent more regular terms. By conservation of $U(1)_r$ symmetry, $\Phi_{r_1+r_2}$ has charge $r_1 + r_2$ and scaling dimension $\Delta \geq \frac{1}{2}(r_1 + r_2) = \Delta_1 + \Delta_2$. We define then the ring structure by

$$\left(\Phi_{r_1} \cdot \Phi_{r_2}\right)(x_2) = \lim_{x_1 \to x_2} \Phi_{r_1}(x_1)\Phi_{r_2}(x_2) \tag{1.19}$$

One important family of observables that we can build with these operators are the extremal correlators, in which we consider n chiral primaries and one anti-chiral primary

$$\langle \Phi_{r_1}(x_1) \cdots \Phi_{r_n}(x_n)\bar{\Phi}_r(x)\rangle \tag{1.20}$$

where $\sum_i r_i + r = 0$. It can be shown that, due to the large amount of symmetry, this correlator is given by

$$\langle \Phi_{r_1}(x_1) \cdots \Phi_{r_n}(x_n)\bar{\Phi}_r(x)\rangle = \frac{\langle \Phi_{r_1} \ldots \Phi_{r_n}\bar{\Phi}_r(\tau, \bar{\tau})\rangle}{|x_1 - y|^{2\Delta_1} \ldots |x_{n-1} - y|^{2\Delta_{n-1}}} \tag{1.21}$$

where the position-independent coefficients $\langle \Phi_{r_1} \dots \Phi_{r_n} \bar{\Phi}_r(\tau, \bar{\tau}) \rangle$ are non-holomorphic functions of the complexified coupling (1.14).

§1.2.2 Supersymmetric Wilson Loops

Wilson loop operators are among the most interesting operators in any gauge theory. They are non-local gauge invariant operators which are essentially phase factors associated with the trajectory of a charged particle along a closed path. They can be mathematically understood as the holonomy of the gauge group and thus they are the parallel transporters for charged particles moving in a gauge field background. Physically we can understand them as codifying the response of the gauge field to the insertion of an external probe.

Originally proposed as order parameters in the lattice formulation of Quantum Chromodynamics [38], we now understand how the study of the expectation value of this observable is fundamental in our modern understanding of Quantum field theories. Ranging from the study of knots invariants [39], to the characterization of the cusp anomalous dimension [40, 41] and the development of precision tests of the holographic dualities [42]. In the scope of this thesis we will mostly be dealing with the Supersymmetric Wilson loop operator and we refer the reader to the traditional textbook material [43] for the treatment of the non-supersymmetric operator.

Let us be more explicit, in the case of $\mathcal{N} = 4$ Super Yang-mills theory, we define the operator as [44]

$$W_{\mathcal{R}}[\mathcal{C}] = \frac{1}{\dim \mathcal{R}} \mathrm{Tr}_{\mathcal{R}} \mathcal{P} \exp \left(i \oint_{\mathcal{C}} (A_\mu \dot{x}^\mu + |\dot{x}| \Phi_i \theta^i) \mathrm{d}s \right) , \tag{1.22}$$

where x^μ is the space-time trajectory described by the probe particle while following the curve $\mathcal{C}$. The particle sits in a representation $\mathcal{R}$ of the gauge group, and thus the operator (1.22) depends on the space-time trajectory $\mathcal{C}$, the representation $\mathcal{R}$ and the coupling of the theory. Note that it does not only couple to the gauge field A_μ of the theory as usual, but it also includes a coupling θ^i to the scalar fields Φ_i that sits in the same vectormultiplet of the gauge field. This, a priori subtle, modification is what allows the operator (1.22) to be preserve some amount supersymmetry [45].

Let us sketch how this can be seen, consider the supersymmetric variation δ_Q of the operator W and impose the supersymmetry preserving condition $\delta_Q W = 0$

$$\bar{\Psi}(i\Gamma^\mu \dot{x}^\mu + \Gamma^i \theta^i |\dot{x}|)\epsilon = 0 , \tag{1.23}$$

the term in the parenthesis squares to zero and thus this equation has 8 independent solutions, which will in general depend on the trajectory making the solution only *locally supersymmetric*. If instead we wish to consider *globally supersymmetric* configurations we have to impose that the variation parameter $\epsilon(x) = \epsilon$. The only two maximally (globally) supersymmetric trajectories are the straight line and the circle.

As with any observable in a quantum field theory, we generally only know how to compute it utilizing perturbation theory. For the case of some observables, such as the circular Wilson Loop[1], it is possible to develop techniques that allow us to surpass this technical limitation and obtain a result that is exact in the gauge coupling. Much has been said about the supersymmetric Wilson loops so we will refer the reader to the traditional works [46, 47] for the details and we will only briefly comment on the connection between the Wilson loop and matrix model computations.

§1.2.3 1/2-BPS Circular Wilson loop and matrix models

In order to compute the expectation value of the 1/2-BPS circular Wilson loop in the fundamental representation of $SU(N)$, $\mathcal{R} = \Box$ we would perform a perturbative expansion of the operator (1.22)

$$\langle W_{\Box}(\mathcal{C})\rangle = 1 + \frac{g^2 N}{4\pi^2}\oint \mathrm{d}s_1 \oint \mathrm{d}s_2 \frac{|\dot{x}_1||\dot{x}_2| - \dot{x}_1\cdot\dot{x}_2}{(x_1 - x_2)^2} + \mathcal{O}(\lambda^2) \tag{1.24}$$

where the first order contribution corresponds to a propagator connecting two points located at $x^\mu(s_i)$, $i = 1, 2$ within the loop and we still have to compute the space-time integral.

Although this integral could be very involved in the general case, for the circular trajectory we can utilize the usual parametrization of the unit circle $x^\mu = (\cos s, \sin s, 0, 0)$ and note the exact combination appearing in (1.24) is in fact constant

$$\langle \left(iA_\mu \dot{x}^\mu + \Phi_I \Theta^I |\dot{x}|\right)^2 \rangle_0 = \frac{g^2 \delta^{ab}}{8\pi^2}\,. \tag{1.25}$$

given that this expression is independent on the coordinates it is possible to consider all the contributions arising from the family of Feynman diagrams without internal vertices, known as *ladder* diagrams. We see then, that for the circular trajectory the problem reduces to a counting one: given an order in perturbation theory how many planar ladder diagrams we can include.

This counting problem can be attacked by introducing a recurrence relation for the number of diagrams at a given order and then solving it. This was in fact the original approach used in the seminal paper by Erickson-Semenoff-Zarembo (ESZ) [46] where they also found that the answer could be casted in terms of a Gaussian matrix model

$$\langle W\rangle_{\text{ladders}} = \frac{\int \mathrm{d}M e^{-\frac{2N}{\lambda}\mathrm{Tr}M^2} N^{-1}\mathrm{Tr}e^{M}}{\int \mathrm{d}M e^{-\frac{2N}{\lambda}\mathrm{Tr}M^2}}\,, \tag{1.26}$$

since the model is Gaussian, it is possible to compute (1.26) including $1/N$ corrections

$$\langle W\rangle_{\text{ladders}} = \frac{1}{N} L^1_{N-1}\left(-\frac{\lambda}{4N}\right) e^{-\frac{\lambda}{8N}} \tag{1.27}$$

[1] It can be shown that this observables preserves half of the supercharges, a combination of both the supersymmetry and conformal generators, and thus is a 1/2 BPS operator.

There is a small caveat, we have been neglecting the contribution of Feynman diagrams with internal vertices so it is far from clear that (1.26) gives the full answer to the expectation value of (1.22). In their original proposal (ESZ) verified that, up to $\mathcal{O}(\lambda^2)$, the interacting diagrams exactly cancel and thus, they conjectured (1.26) was the *exact* result for the circular Wilson loop. It was not until the work of Pestun [26], to be reviewed in the next section, in which this conjecture was later proven.

Secondly, the expression (1.26) is only valid in the case in which the Wilson loop sits in the fundamental representation of the gauge group. There are techniques that allow us to further compute the operator for other representations [48–51].

§1.3 Technical developments

§1.3.1 Supersymmetric localization

Exact results in quantum field theory tend to be very rare and particular. In addition, most often than not, we have to utilize a large amount of symmetry to constrain the system making the result trivial or uninteresting. Supersymmetric localization is a very powerful technique that allows us, in certain cases, to reduce 4 dimensional problems to a zeroth dimensional one and thus enables us to *exactly* compute the partition function and vacuum expectation values of certain operators in supersymmetric theories.

Four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories are known to be mathematicaly highly constrained, and yet they can accomodate a variety of interesting physical phenomena. For a long time, the most notable examples of exact results obtained in such theories were the ones obtained by performing a topological twist, in this case the path integral localizes to the $0-$dimensional moduli space of instantons and can be used to compute Donaldson-Witten invariants of four manifolds [24, 25], the Seiberg-Witten low energy action [11, 12] and Nekrasov's instanton partition function [13].

A major breakthrough was made by Pestun who constructed $\mathcal{N} = 2$ supersymmetric field theories on S^4 and derived a closed formulae for the partition function as well as expectation values of certain operators [26]. Supersymmetric localization *ala* Pestun can be thought of as an infinite-dimensional version of the Duistermaat-Heckman and Atiyah-Bott-Berline-Vergne localization formulae in equivariant cohomology.

Supersymmetric localization has evolved into a field of its own with many interesting applications which are outside the scope of this thesis. We refer the reader to [27] for an extensive review in the subject. Instead, in this section we will briefly introduce the basic notions of the techniques and the connection with Matrix models.

SUSY Localization overview

Consider a fermionic (Grassmann-odd) symmetry $\mathcal{Q}$ of a theory described by the action $S[\Phi]$, where Φ is a set of fields,

$$\delta S = \mathcal{Q} S[\Phi] = 0\,. \tag{1.28}$$

Since $\mathcal{Q}$ is fermionic, it squares to zero or a bosonic symmetry δ_B of the action[2]. Instead of considering the usual Euclidean path-integral we can perform a deformation of $S[\Phi]$ by a $\mathcal{Q}-$exact tern given by

$$Z(t) = \int \mathcal{D}\Phi\, e^{-S[\Phi]-t\mathcal{Q}V[\Phi]} \quad \text{with} \quad \delta_B V[\Phi] = 0\,, \tag{1.29}$$

where $V[\Phi]$ is some functional on the fields and the deformed action now depends on a free real parameter t.

We can now verify the dependence on t of this new partition function $Z(t)$

$$\frac{\mathrm{d}Z}{\mathrm{d}t} = -\int \mathcal{D}\Phi\, \mathcal{Q}V[\Phi] e^{-S[\Phi]-t\mathcal{Q}V[\Phi]} = -\int \mathcal{D}\Phi\, \mathcal{Q}\left(V[\Phi] e^{-S[\Phi]-t\mathcal{Q}V[\Phi]}\right)\,, \tag{1.30}$$

where we have used that the deformed action is $\mathcal{Q}-$invariant to integrate by parts. If now the measure is also invariant under $\mathcal{Q}$, *i.e.* the fermionic symmetry is non-anomalous we obtain

$$\frac{\mathrm{d}Z}{\mathrm{d}t} = 0\,. \tag{1.31}$$

Note that the last step may not hold if the boundary terms in field space do not decay sufficiently fast. We can verify that the same derivation still holds if we insert operators $\mathcal{O}$ that are in the $\mathcal{Q}-$cohomology class,

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\mathcal{O}\rangle_t = \frac{\mathrm{d}}{\mathrm{d}t}\int \mathcal{D}\Phi\, \mathcal{O}\, e^{-S[\Phi]-t\mathcal{Q}V[\Phi]} = -\mathcal{Q}\left(\int \mathcal{D}\Phi\, \mathcal{O}\, V[\Phi]\, e^{-S[\Phi]-t\mathcal{Q}V[\Phi]}\right) = 0\,. \tag{1.32}$$

Since both the modified partition function and the vev of the operator $\mathcal{O}$ do not depend on t we can compute them for any given value of t and they will all coincide with the original $t = 0$ case. If we can find a suitable $V[\Phi]$ such that the bosonic part $(\mathcal{Q}\,V[\Phi])_B > 0$, we can then take the $t \to \infty$ limit making all such field configurations infinitely suppressed. This of course does not mean that the partition function is trivial, we could have a zero measure number of contributions arising from the bosonic zeros of the deformation term.

We therefore say that the path-integral *localizes* to the bosonic zeros Φ_c that satisfy

$$(\mathcal{Q}\,V[\Phi])_B = 0\,. \tag{1.33}$$

As a matter of fact, in most cases the localized set of field configurations Φ_c are independent of the space-time coordinates thus leading to a $0-$dimensional

[2] A composition of translations with Lorentz, R-symmetry and flavor symmetry rotations

matrix model integral. After performing the localization procedure we end up with the following expression for the partition function of the theory

$$\mathcal{Z} = \int \mathcal{D}\,\Phi_c\,\mathcal{Z}_{1-\text{loop}}\,\mathcal{Z}_{\text{inst}}\,e^{-S[\Phi_c]}\,, \tag{1.34}$$

where $\mathcal{Z}_{1-\text{loop}}$ is the one-loop determinant of all field fluctuations arising from the saddle-point computation in the $t \to \infty$ limit, and $\mathcal{Z}_{\text{inst}}$ is Nekrasov's instanton partition function.

For $4-$dimensional Lagrangian $\mathcal{N}=2$ placed on S^4, Pestun showed [26] that upon localization the zero modes (1.33) correspond to the value of the scalar field of the vector multiplet at the poles of the sphere. Therefore, the partition function (1.34) reduces to a finite-dimensional matrix integral over the space of saddle points Φ_c parametrized by a Lie algebra-valued constant field a

$$\mathcal{Z}_{S^4} = \int \mathrm{d}a\,e^{-\frac{8\pi^2}{g^2_{\text{YM}}}\text{Tr}(a^2)}\mathcal{Z}_{1-\text{loop}}\,\mathcal{Z}_{\text{inst}}\,. \tag{1.35}$$

We will not need the general form of $\mathcal{Z}_{1-\text{loop}}$, though it can be found in the literature [26], since its quite involved and in general it needs a regularization. We will, however, present its expression for theories that are conformal at the quantum level, in such case we have

$$\mathcal{Z}_{1-\text{loop}}(a) = \frac{\prod_{\alpha\in\Lambda_{\mathcal{R}}(G)} H(i\alpha\cdot\hat{a})}{\prod_{\mathcal{R}}\prod_{\omega_{\mathcal{R}}\in\Lambda_\omega(G)} H(i\omega_{\mathcal{R}}\cdot\hat{a})^{n_{\mathcal{R}}(\omega)}} \tag{1.36}$$

where $\Lambda_{\mathcal{R}}(G)$ and $\Lambda_\omega(G)$ designate respectively the set of roots and the weight lattice of the algebra G. The label $\mathcal{R}$ follows from the representation of the matter hypermultiplets of the $\mathcal{N}=2$ theory under consideration, while $n_{\mathcal{R}}(\omega)$ accounts for the multiplicity of the weight ω in this representation.

We see that upon localization of the $\mathcal{N}=2$ theory the partition function (1.34) with the $1-$loop factor (1.36) reduces to an interacting matrix model, in the next section we develop the necessary techniques to attack such family of theories.

§1.3.2 Matrix Models techniques

The study of matrix models, such as the one characterizing the localized partition function (1.34), has a rich history both in physics and mathematics [28–34]. Being $0-$dimensional theories they are the simplest example of quantum gauge theories, this means that the fields have the group structure of a gauge connection *i.e.* they are matrices in the adjoint representation of a gauge group. We will, for simplicity, consider the gauge group to be $U(N)$[3], in which case the basic fields are $N\times N$ Hermitian matrices.

In order to define a theory for M we should consider an action composed of a potential $V(M)$ invariant under the action of the algebra on itself, the

[3]In the large N limit the difference between choosing $U(N)$ and another semi-simple classic gauge group is subleading so, for the scope of this work, there is no loss of generality.

transformations induced from it are then gauge transformations. The simplest action we can consider is of the form

$$S(M) = \frac{1}{g_s} \mathrm{Tr}\, V(M) \tag{1.37}$$

were g_s is a coupling constant and we take $V(M)$ to be a polynomial on the matrix M with, possible infinite, single and double trace deformations.

The partition function of the matrix model is then defined by

$$Z = \frac{1}{\mathrm{vol}\,(U(N))} \int \mathrm{d}M\, e^{-\frac{1}{g_s}\mathrm{Tr}\, V(M)}\,, \tag{1.38}$$

where $\mathrm{vol}\,(U(N))$ is the volume of the gauge group and the measure in the integral is given by

$$\mathrm{d}M = 2^{\frac{N(N-1)}{2}} \prod_{i=1}^{N} \mathrm{d}M_{ii} \prod_{1\leq i<j\leq N} \mathrm{d}\ \mathrm{Re}M_{ij} \mathrm{d}\ \mathrm{Im}M_{ij}\,. \tag{1.39}$$

Since the action (1.37) is invariant under

$$M \to UMU^{\dagger}\,, \tag{1.40}$$

the natural observables in the model are functions of the matrix satisfying the condition $f(UMU^{\dagger}) = f(M)$, it is then straightforward to compute expectation values

$$\langle f(M)\rangle = \frac{\int \mathrm{d}M f(M) e^{-\frac{1}{g_s}\mathrm{Tr}\, V(M)}}{\int \mathrm{d}M e^{-\frac{1}{g_s}\mathrm{Tr} V(M)}}\,. \tag{1.41}$$

There are many different techniques to tackle this family of models both at weak/strong coupling and finite/large N and we refer the interested reader to the references [35]. In the upcoming sections we outline the main aspects of the techniques that we have mostly used in this thesis.

Orthogonal Polynomials

Since both the potential (1.37) and the measure (1.39) are invariant under the gauge transformation (1.40), we can take advantage of this freedom to choose a gauge in which we diagonalize the matrix $M \to U\,M\,U^{\dagger} = D$, with $D = \mathrm{diag}(\lambda_1, \cdots, \lambda_N)$ and use the standard Faddeev-Popov techniques in order to compute the gauge-fixed integral. A standard computation allow us to obtain the Faddeev-Popov determinant

$$\Delta^2(\lambda) = \prod_{i<j} (\lambda_i - \lambda_j)^2\,, \tag{1.42}$$

where we see that it is the standard Vandermonde determinat. Now the computation of (1.38) reduces to

$$Z = \Omega_N \int \prod_{i=1}^{N} \mathrm{d}\lambda_i \Delta^2(\lambda) e^{-\frac{1}{g_s}\sum_i V(\lambda_i)}\,, \tag{1.43}$$

where for simplicity we introduced the Gauge group volume as Ω_N. For an arbitrary potential $V(\lambda)$, we can regard

$$\mathrm{d}\mu = e^{-\frac{1}{g_s}W(\lambda)}\mathrm{d}\lambda \tag{1.44}$$

as a measure in $\mathbb{R}$ and introduce the *orthogonal polynomials* $p_n(\lambda)$ defined by

$$\int \mathrm{d}\mu\, p_n(\lambda) p_m(\lambda) = h_n \delta_{nm}\,, \quad n \geq 0\,, \tag{1.45}$$

where we normalize the polynomials by requiring the behavior $p_n(\lambda) = \lambda^n + \cdots$. Utilizing the Leibniz formula for the determinant, we can recast (1.43) into

$$Z = \prod_{i=0}^{N-1} h_i = h_0^N \prod_{i=1}^{N} r_i^{N-i}\,, \quad \text{with} \quad r_k = \frac{h_k}{h_{k-1}}\,, \quad k \geq 1\,. \tag{1.46}$$

One of the most important propierties these polynomials satisfy is the three-term recursion relation given by

$$(\lambda + s_n) p_n(\lambda) = p_{n+1}(\lambda) + r_n p_{n-1}(\lambda)\,, \tag{1.47}$$

where the coeffients s_n depend on the exact form of the potential $V(\lambda)$. Knowing s_n and r_n is equivalent to find the exact form of the orthogonal polynomials and thus amounts to being able to exactly solve the theory.

One of the fundamental observables of interest for this thesis is the Free energy of the theory

$$\mathcal{F} = \log Z - \log Z_{\mathrm{G}}\,, \tag{1.48}$$

where we introduce the Free energy of the Gaussian theory. We can consider the perturbative series expansion $\mathcal{F} = \sum_{g\geq 0} \mathcal{F}_g g_s^{2g-2}$ and arrive at

$$g_s^2 \mathcal{F} = \frac{t^2}{N} \log \frac{h_0}{h_0^N} + \frac{t^2}{N} \sum_{k=1}^{N} \left(1 - \frac{k}{N}\right) \log \frac{r_k(N)}{k g_s} \tag{1.49}$$

where we have introduced the ’t Hooft coupling $t = g_s N$. This expression is valid at finite N and at strong coupling given our knowledge of the orthogonal polynomials of the theory.

It is known that in the large N limit only planar diagrams contribute to the computation of the partition function and that the right expansion parameter in this regime is the ’t Hooft coupling t. We can thus make further simpificacions to study the regime $N \to \infty$ by defining $\xi = \frac{k}{N}$ and considering the following asymptotic expansion of the coefficients $r_k(N)$

$$r_k(N) = \sum_{s=0}^{\infty} N^{-2s} R_{2s}(\xi) \tag{1.50}$$

from were we find the Planar free energy of the theory

$$\mathcal{F}_0(t) = -\frac{1}{2}t^2 \log t + t^2 \int_0^1 \mathrm{d}\xi (1-\xi) \log \frac{R_0(\xi)}{\xi} \,. \tag{1.51}$$

This gives us a closed expresion in terms of the large N limit of the recursion coefficients r_k. We see that in the planar limit, the basic quantity of interest is the coefficient $R_0(\xi)$ that we can compute recursively, as an example for a potential of the form

$$V(\lambda) = \sum_{p\geq 0} \frac{g_{2p+2}}{2p+2} \lambda^{2p+2} \tag{1.52}$$

one finds

$$\xi t = \sum_{p\geq 0} g_{2p+2} \binom{2p+1}{p} R_0^{p+1}(\xi) \tag{1.53}$$

which can in turn be formally solved by utilizing the Lagrange inversion formula.

The full Lie Algebra approach

Instead of reducing the Matrix integral (1.38) to a Cartan sub-algebra and dealing with the Vandermonde determinant (1.42) we can instead consider a flat integration metric and perform the integrals over the full Lie algebra. The main ingredient in the this approach to Matrix integrals is to note that the matrix M is a Lie algebra valued function and it can then be expanded as

$$M = m^a T^a \,, \tag{1.54}$$

with T_a the generators of the group G, and m^a are the coefficients of the matrix M. In the Gaussian model their two-point function is just

$$\langle m^a m^b \rangle = g_s \delta^{ab} \,, \quad a, b = 1, \cdots, d_A \tag{1.55}$$

where d_A is the dimension of the adjoint representation of the group G. Now the evaluation of any observable (1.41) reduces to the application of Wick's theorem and the computation of symmetrized traces, for which it is useful to introduce the fully symmetrized traces

$$d_R^{a_1 \cdots a_n} = \frac{1}{n!} \mathrm{Tr} \sum_{\sigma \in S_n} T_R^{a_{\sigma(1)}} \cdots T_R^{a_{\sigma(n)}} \tag{1.56}$$

which are generally known for the classical gauge groups [36]. Let us show an example, consider a Gaussian potential

$$V(M) = \frac{1}{g_s}\mathrm{Tr}M^2\,, \tag{1.57}$$

and an operator

$$f(M,R) = \frac{1}{d_R}\mathrm{Tr}_R e^M\,, \tag{1.58}$$

where now the operator sits at an arbitrary representation R and d_R is the dimension of such representation. Utilizing the two-point function (1.55) and the symmetrized traces (1.56) we can perform a perturbative computation to obtain

$$\langle f(M,R)\rangle = 1 + c_R\frac{g_s}{2} + \left(c_R^2 - \frac{1}{6}c_R c_A\right)\frac{g_s^2}{8} + \cdots\,, \tag{1.59}$$

where we have introduced c_R, c_A as the Casimir of the representation R and the adjoint respectively. With this technique we can actually obtain the exact expression valid even at finite N

$$\langle f(M,R)\rangle = \frac{1}{d_R}\sum_{k=0}^{\infty}\frac{1}{(2k)!}\langle m^{a_1}\dots m^{a_{2k}}\rangle \mathrm{Tr}T_R^{a_1}\dots T_R^{a_{2k}} = \frac{1}{d_R}\sum_{k=0}^{\infty} d_R^{a_1 a_1\dots a_k a_k}\frac{g^k}{k!}\,. \tag{1.60}$$

The appeal of this approach is not only that it allows us to carry out computations at once for any gauge group G and representation $\mathcal{R}$, but also that since the integrals are now Gaussian we can compute the full perturbative series in the planar limit. Let us be more explicit and consider once again the expectation value of the operator (1.58) in the fundamental representation of the group $G = SU(N)$, in this case we obtain

$$\langle f(M)\rangle = \frac{1}{N}\sum_{k=0}^{\infty}\frac{1}{(2k)!}\langle \mathrm{Tr}M^{2k}\rangle\,, \tag{1.61}$$

where we see that the problem effectively reduces to computations in the Gaussian matrix model. Luckily for us much is known for these models, in particular the classical result [37]

$$\langle \mathrm{Tr}M^{2k}\rangle = \frac{(2k)!}{(k+1)!k!}t^k N + \mathcal{O}(N^{-1})\,, \tag{1.62}$$

which in turn allows to fully characterize the leading contribution to the expectation value of $f(M)$

$$\langle f(M)\rangle = \sum_{k=0}^{\infty}\frac{t^k}{(2k)!}\frac{(2k)!}{(k+1)!k!} = \frac{1}{\sqrt{t}}I_1\left(2\sqrt{t}\right)\,. \tag{1.63}$$

Although this example is well known it serves the purpose of showing how by utilizing the full Lie algebra approach we can tackle the large N limit in a really efficient way. Of course while dealing with more general Matrix model potentials (1.38) this procedure is much more involved and it requires a proper characterization of the Gaussian correlators, such as the generalizations of (1.62), that one needs to consider. In the main body of the thesis we discuss how many observables arising in $\mathcal{N}=2$ 4−dimensional Lagrangian superconformal field theories, can be fully characterized by this technique in the planar limit.

§1.4 Summary of main results

In this section we briefly review the main results of the thesis. Throughout this work we mainly focus on a family of matrix models given by

$$\mathcal{Z} = \int \mathrm{d}a \, e^{-\frac{1}{2g}\mathrm{Tr}(a^2)} e^{-S_{int}} \tag{1.64}$$

where a is a Hermitian $\mathrm{N}\times\mathrm{N}$ matrix, $\mathrm{d}a$ is the flat measure and g is the matrix model coupling. The interacting part of the action consists of (possibly infinitely many) single and double trace terms,

$$S_{int} = \mathrm{N}\sum_{p\geq 3} c_p \mathrm{Tr}\; a^p + \sum_{mn} c_{mn}\mathrm{Tr}\; a^m\,\mathrm{Tr}\; a^n \,. \tag{1.65}$$

with the coefficients c_p, c_{mn} N-independent and otherwise arbitrary. Particular examples of this family of models have appeared in the study of two-dimensional quantum gravity [65–68, 70], and as reviewed in [69], they have also appeared in many other contexts, from two-dimensional statistical mechanics, to three-dimensional gauge theories, or M-theory. Without the N factor in front of the single-trace terms, they are relevant [71,72] in the application of supersymmetric localization to four dimensional undeformed $\mathcal{N}=2$ super Yang-Mills theories.

Instead of following the usual route, reducing the matrix model integrals to a Cartan subalgebra and working with the eigenvalue density, we tackle them in the original full Lie algebra formulation. Utilizing this approach we find a combinatorial expression for the planar free energy (1.48) of the models as a sum over a particular type of graphs, known as tree graphs

$$\begin{aligned}\mathcal{F} = \sum_{m=1}^{\infty}\frac{(-1)^m}{m!}\sum_{k=0}^{m}\binom{m}{k}\sum_{p_1,\dots,p_{m-k}} c_{p_1}\dots c_{p_{m-k}} \sum_{\substack{i_1,\dots,i_k\\ j_1,\dots,j_k}} c_{i_1 j_1}\dots c_{i_k j_k} \\ \sum_{\substack{\text{directed trees with}\\ \text{k labeled edges}}}\;\sum_{\substack{\text{single trace}\\ \text{insertions}}}\prod_{i=1}^{k+1} V_i\end{aligned} \tag{1.66}$$

where V_i is the plannar connected correlator on the $i-$th vertex on the tree, that contains the following operators: tr a^{i_s} if the directed edge labelled s

leaves that vertex; tr a^{j_s} if the directed edge labelled s arrives at that vertex; any single trace operators inserted on that vertex. For an arbitrary number of even operators, the explicit form of such plannar connected correlators is known [73, 74]

$$\langle \mathrm{Tr}a^{2k_1} \cdots \mathrm{Tr}a^{2k_n} \rangle_c = \tilde{\lambda}^d \frac{(d-1)!}{(d-n+2)!} \prod_{i=1}^{n} \frac{(2k_i)!}{k_i!(k_i-1)!} \mathrm{N}^{2-n} \tag{1.67}$$

where $d = \sum k_i$ and $\tilde{\lambda}$ is the 't Hooft coupling of the model. Let us introduce some notation for the numerical coefficients

$$V(k_1, \cdots, k_n) = \frac{(d-1)!}{(d-n+2)!} \prod_{i=1}^{n} \frac{(2k_i)!}{k_i!(k_i-1)!} \,. \tag{1.68}$$

In Chapter 4 we restrict our attention to potentials (1.65) where $c_{m,n} = 0$, and obtain new exact results for the planar free energy for the Hermitian one-matrix model for various choices of the potential. For potentials with an arbitrary number of single-trace terms $V = N(c_4 \mathrm{Tr}a^4 + \cdots + c_{2k}\mathrm{Tr}a^{2k})$ we obtain

$$\mathcal{F}_0(t) = \sum_{\substack{j_2,\ldots,j_k \\ j_2+\cdots+j_k>0}} \frac{1}{j_2! \ldots j_k!} \frac{(2j_2 + \cdots + kj_k - 1)!}{(j_2 + \cdots + (k-1)j_k + 2)!} (-x_2)^{j_2} \ldots (-x_k)^{j_k} \,. \tag{1.69}$$

For these models we manage to bound the radius of convergence of the perturbative expansion and in the large k limit we exactly obtain it.

For the models (1.65) that arise upon performing supersymmetric localization of 4−dimensional Lagrangian $\mathcal{N} = 2$ theories placed on S^4, we study the 1−loop partition function (1.36) for a classical gauge group G with conformal matter obeying (1.15) and we find that the asymptotic expansion of (1.36) can be re-written as

$$\begin{aligned} S^G_{int} = \sum_{n=2}^{\infty} \frac{\zeta(2n-1)(-1)^n}{n} \Bigg[&(4-4^n)\alpha_G \mathrm{Tr}a^{2n} + \beta_G \sum_{k=1}^{n-1} \binom{2n}{2k} \mathrm{Tr}a^{2(n-k)} \mathrm{Tr}a^{2k} \\ &+ \gamma_G \sum_{k=1}^{n-2} \binom{2n}{2k+1} \mathrm{Tr}a^{2(n-k)-1} \mathrm{Tr}a^{2k+1} \Bigg] \,, \end{aligned} \tag{1.70}$$

where G is the Gauge group and $\alpha_G, \beta_G, \gamma_G$ are order 1 parameters that depend on the particular choice of matter content. In the planar limit, for theories with a fraction of matter in the fundamental representation of G, we show that the leading contribution stems from the β_G term in (1.70). Thus we see that, in the language of (1.65), we have $c_p = 0$ and $c_{m,n}$ are obtained directly from (1.70). For these theories, we study the planar Free Energy (1.48) for which the full perturbative answer is given diagramatically by

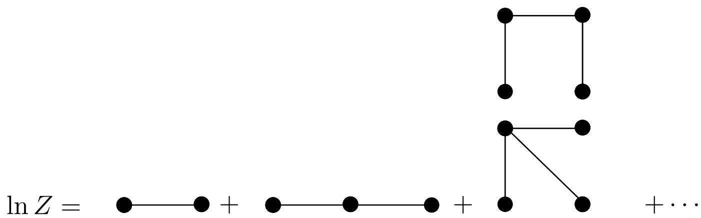

Figure 1.1: First orders of the perturbative expansion of the planar Free energy of 4−dimensional Lagrangian $\mathcal{N}=2$ superconformal field theories. Each graph is a tree graph in which each vertex contains the contribution to the planar regime of certain matrix correlators.

furthermore, it can be explicitly written as a perturbative expansion to all orders in the t' Hooft coupling λ

$$F_0(\lambda)-F_0(\lambda)^{\mathcal{N}=4}=\sum_{n=2}^{\infty}\left(-\frac{\lambda}{16\pi^2}\right)^n\sum_{\substack{\text{compositions of n}\\ \text{not containing 1}}}(-2\beta_G)^m\frac{\zeta(2n_1-1)\ldots\zeta(2n_m-1)}{n_1\ldots n_m}$$
$$\sum_{k_1=1}^{n_1-1}\binom{2n_1}{2k_1}\cdots\sum_{k_m=1}^{n_m-1}\binom{2n_m}{2k_m}\sum_{\substack{\text{unlabeled trees}\\ \text{with m edges}}}\frac{1}{|\mathrm{Aut(T)}|}\mathcal{V}_1\ldots\mathcal{V}_{m+1} \quad (1.71)$$

where m is the number of elements of a given composition of n[4], and $\mathcal{V}_i$ are combinatorial factors, to be defined in Chapter 2, attached to each of the $m+1$ vertices of the tree. In a similar way, the Wilson loop is given by

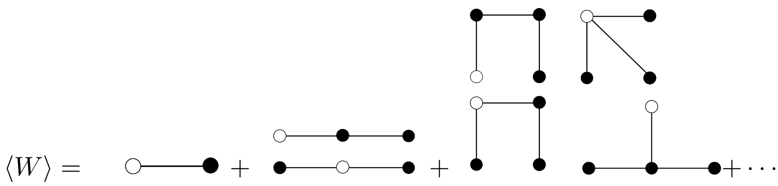

Figure 1.2: First orders of the perturbartive expansion of the Wilson loop in 4−dimensional Lagrangian $\mathcal{N}=2$ superconformal field theories. Each term is now a rooted tree, where the root showcases the insertion of the Wilson loop operator. Once again each vertex contains the contribution to the planar regime of certain matrix correlators.

and as a perturbative expansion to all orders in the t' Hooft coupling λ

[4] A composition is a partition where the order of the elements matters; *e.g.* $2+3$ and $3+2$ are different compositions of 5.

$$\langle W\rangle_{\mathcal{N}=2}-\langle W\rangle_{\mathcal{N}=4} = \sum_{l=1}^{\infty}\frac{b^{2l}}{(2l)!}\left(\frac{\lambda}{4}\right)^{l}\sum_{m=1}^{\infty}(-2\beta_G)^m\sum_{n_1,\dots,n_m=2}^{\infty}\left(\frac{-\lambda}{16\pi^2}\right)^{n_1+\dots}\prod_{i=1}^{m}\frac{\zeta(2n_i-1)}{n_i}$$
$$\sum_{k_1=1}^{n_1-1}\binom{2n_1}{2k_1}\cdots\sum_{k_m=1}^{n_m-1}\binom{2n_m}{2k_m}\sum_{\substack{\text{unlabeled rooted}\\ \text{trees with m edges}}}\frac{1}{|\mathrm{Aut(T)}|}\prod_{i=1}^{m+1}\mathcal{V}_i. \tag{1.72}$$

In Chapter 3 we focus our attention to the case of $G = SU(N)$ with $n_f = 2N$, namely Super-QCD. We study the planar limit of the extremal correlation functions of Chiral primary operators (1.21). To this end we interpret (1.66) as the generating functional of connected correlation functions of matrix operators. To obtain the correlation functions of Chiral primary operators we perform a standard unmixing procedure to relate the matrix observables, defined on S^4, to the gauge theory ones, defined on $\mathbb{R}^4$. We finally obtain for the 2− and 3−point functions of Chiral primary operators the following exact expressions

$$\langle O_k\bar{O}_k\rangle = k\left(\frac{\lambda}{16\pi^2}\right)^k\left(1-2k\sum_{n=2}^{\infty}\frac{\zeta_{2n-1}}{n}\left(\frac{-\lambda}{16\pi^2}\right)^n\binom{2n}{n}\left[(-1)^k\binom{2n}{n+k}+\binom{2n}{n+1}-n\right]+\dots\right. \tag{1.73}$$

$$\frac{\langle O_{k_1}O_{k_2}\bar{O}_{k1+k2}\rangle_n}{\sqrt{k_1\cdot k_2\cdot(k_1+k_2)}} \stackrel{?}{=} \mathrm{N}^{-1}\left[1-\sum_{n=2}^{\infty}\left(\frac{-\lambda}{16\pi^2}\right)^n\zeta_{2n-1}\binom{2n}{n}\right.$$
$$\left.\left(\binom{2n}{n+k_1}+\binom{2n}{n+k_2}+\binom{2n}{n+k_1+k_2}+(n-1)(\mathcal{C}_n-2)\right)\right]+\dots \tag{1.74}$$

Finally, in Chapter 4 we also consider models where now $c_{m,n} \neq 0$ and an infinite number of terms arising from non-conformal $\mathcal{N} = 2$ theories and compute their planar free energy, for SQCD with $n_f < 2N$ we obtain

$$\begin{aligned}\mathcal{F} = &-\left(\frac{\mathrm{N}_f}{\mathrm{N}}-2\right)(1+\gamma)\left(\frac{\lambda}{16\pi^2}\right)+\sum_{p=2}^{\infty}\frac{\zeta_{2p-1}}{p}\frac{(2p)!}{(p+1)!p!}\left(-\frac{\lambda}{16\pi^2}\right)^p\\ &-\sum_{i,j=1}^{\infty}\frac{\zeta_{2i+2j-1}}{(i+j)}\left(\frac{-\lambda}{16\pi^2}\right)^{i+j}\binom{2i+2j}{2i}\frac{(2i)!(2j)!}{(i+1)!i!(j+1)!j!}+\cdots,\end{aligned} \tag{1.75}$$

while for the massive deformation $\mathcal{N} = 2^*$ we are able to capture the whole leading M contribution

$$\mathcal{F}_0 = -\frac{4\pi^2}{\lambda}\int_0^{\infty}\mathrm{d}w\frac{\sinh^2(wM)}{w^3\sinh^2 w}\left(J_1\left(\frac{w\sqrt{\lambda}}{\pi}\right)^2-\frac{w^2\lambda}{4\pi^2}\right). \tag{1.76}$$

with J_1 a Bessel function.

CHAPTER 2

The planar limit of $\mathcal{N} = 2$ superconformal field theories

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The planar limit of $\mathcal{N} = 2$ superconformal field theories

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Abstract: We obtain the perturbative expansion of the free energy on S^4 for four dimensional Lagrangian $\mathcal{N} = 2$ superconformal field theories, to all orders in the 't Hooft coupling, in the planar limit. We do so by using supersymmetric localization, after rewriting the 1-loop factor as an effective action involving an infinite number of single and double trace terms. The answer we obtain is purely combinatorial, and involves a sum over tree graphs. We also apply these methods to the perturbative expansion of the free energy at finite N, and to the computation of the vacuum expectation value of the 1/2 BPS circular Wilson loop, which in the planar limit involves a sum over rooted tree graphs.

Keywords: Matrix Models, Supersymmetric Gauge Theory, Wilson, 't Hooft and Polyakov loops

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Contents

1 Introduction

Part of the theoretical appeal of supersymmetric gauge theories is that, for certain questions, they allow more analytical control than their non-supersymmetric counterparts. An outstanding example is supersymmetric localization, which allows to reduce the evaluation of certain quantities of 4d $\mathcal{N}=2$ super Yang-Mills (SYM) theories to matrix integrals [1]. For instance, the partition function on S^4 is reduced to

$$Z_{S^4} = \int da\, e^{-\frac{8\pi^2}{g^2_{\text{YM}}}\text{Tr}(a^2)} \mathcal{Z}_{\text{1-loop}}\, |\mathcal{Z}_{\text{inst}}|^2 \tag{1.1}$$

where $\mathcal{Z}_{\text{1-loop}}$ is a factor that arises from a 1-loop computation, while $\mathcal{Z}_{\text{inst}}$ is the instanton contribution. Similarly, the expectation value of a 1/2 BPS circular Wilson loop $\langle W_R\rangle$ is also reduced to a matrix integral [1].

The fact that four dimensional questions admit zero dimensional answers constitutes a dramatic simplification, but still leaves the formidable task of evaluating these matrix integrals. A first approach consists of restricting the integrals to a Cartan subalgebra of the Lie algebra. In a second approach [2–4], the integrals are over the full Lie algebra, and the 1-loop factor in (1.1) is rewritten as an effective action.

For $\mathcal{N}=4$ super Yang Mills theories, both $\mathcal{Z}_{\text{1-loop}}=1$ and $|\mathcal{Z}_{\text{inst}}|^2=1$ in (1.1) [1]. The free energy can be easily computed [5], but $\langle W_R\rangle$ is less trivial. Using the first approach mentioned above, the vev of the 1/2 BPS circular Wilson loop can be computed for different gauge groups G and representations R, on a case by case basis [6–8]. Recently, using the second approach, we derived a general expression for $\langle W_R \rangle$ valid for all gauge groups G and representations R, thus unifying and extending all previous exact results [9].

For generic $\mathcal{N}=2$ super Yang-Mills theories, the evaluation of (1.1) or $\langle W_R \rangle$ is considerably much more complicated. Within the first approach, the free energy and $\langle W_R \rangle$ for various representations have been evaluated with a saddle point approximation [5, 10–15]. In this note we will apply the second approach to the study of $\mathcal{N}=2$ Lagrangian superconformal field theories (SCFTs) for arbitrary gauge groups at finite N, and for classical gauge groups in the planar limit. Ideally, we would like to write the quantities of interest in terms of color invariants of the gauge group and matter representations. This is vastly more complicated than in the $\mathcal{N}=4$ case considered in our previous work [9], because the matrix model is interacting, and the identification of its perturbative expansion with the usual one in field theory — in terms of Feynman diagrams — is not immediate.

Let's outline our strategy and our results in some detail. Following ideas presented in [2–4], in section 2 we rewrite the 1-loop factor $\mathcal{Z}_{\text{1-loop}}$ in (1.1) as an effective action with an infinite number of single trace and double trace terms, where the traces are in the fundamental representation of the gauge group

$$S_{\text{int}}^G = -\ln \mathcal{Z}_{\text{1-loop}} = \sum_{n=2}^{\infty} \frac{\zeta(2n-1)(-1)^n}{n} \left[(4-4^n)\alpha_G \operatorname{Tr} a^{2n} + \beta_G \sum_{k=1}^{n-1} \binom{2n}{2k} \operatorname{Tr} a^{2(n-k)} \operatorname{Tr} a^{2k} \right.$$
$$\left. + \gamma_G \sum_{k=1}^{n-2} \binom{2n}{2k+1} \operatorname{Tr} a^{2(n-k)-1} \operatorname{Tr} a^{2k+1} \right], \tag{1.2}$$

where α_G, β_G and γ_G are constants that depend on the gauge group and the matter content of the conformal field theory. For the gauge group $\mathrm{SU}(N)$, this effective action has been independently derived in [4], and can find applications beyond the ones presented in this work.

Together with the kinetic term in (1.1), the interaction terms in (1.2) constitute a matrix model that is at the center of this work. Matrix models with single and double trace terms in the potential were discussed in the past [16–20], in the context of two dimensional quantum gravity. In the planar limit, these models present different phases, depending on the relative strengths of couplings of the single and double trace terms. For small coupling of the double trace term, the emerging geometry is that of a family of spheres connected by wormholes, created by the double trace terms [16]. More specifically, the planar limit imposes that the full surface has genus zero, so the spheres connected by wormholes must form a tree graph, in the sense that no wormhole connects a sphere with itself, no two spheres are connected by more than a wormhole, and there is no closed loop of spheres [18, 19]. As the coupling of the double trace increases in the matrix model, the system develops new phases, including a branched polymer phase [16].

The matrix model we encounter, with interaction terms (1.2), bears some differences with the ones studied in the past [16] and described above. First, the number of single and double trace terms in the effective action is now infinite. Moreover, the single trace terms in (1.2) do not have the right scaling to contribute to the planar limit. Additionally, the coefficients of the single trace terms $\mathrm{Tr}\, a^{2n}$ grow exponentially with n. On the other hand, the work [21] does consider — in their appendix B — a matrix model with an infinite number of double trace terms, and studies it in the planar limit with the technique of orthogonal polynomials.[1]

Armed with this effective action (1.2), we set out to evaluate various quantities of interest. The first one is the free energy of these SCFTs on S^4. As the integrals are Gaussian, they can be easily carried out. At finite N what is left is the evaluation of color invariants in the fundamental representation. There are well-known techniques to help with the evaluation of these traces [9, 22], but the expressions become more and more cumbersome as we go to higher orders in the perturbative expansion. Furthermore, the resulting expressions involve color invariants of the fundamental and adjoint representations, not of the matter representations of the various field theories. We then turn to the planar limit, and argue that only for theories with a finite fraction of matter in the fundamental representation — theories with $\beta_G \neq 0$ in (1.2) — the planar free energy differs from the $\mathcal{N}=4$ result. For these theories, we manage to write the full perturbative expansion to all orders in the 't Hooft coupling λ,

$$
\begin{aligned}
F_0(\lambda) - F_0(\lambda)^{\mathcal{N}=4} = \sum_{n=2}^{\infty}\left(-\frac{\lambda}{16\pi^2}\right)^n \sum_{\substack{\text{compositions of n}\\ \text{not containing 1}}} (-2\beta_G)^m \frac{\zeta(2n_1-1)\dots\zeta(2n_m-1)}{n_1\dots n_m} \\
\sum_{k_1=1}^{n_1-1}\binom{2n_1}{2k_1}\cdots\sum_{k_m=1}^{n_m-1}\binom{2n_m}{2k_m}\sum_{\substack{\text{unlabeled trees}\\ \text{with m edges}}}\frac{1}{|\mathrm{Aut(T)}|}\mathcal{V}_1\dots\mathcal{V}_{m+1} \qquad (1.3)
\end{aligned}
$$

where m is the number of elements of a given composition of n,[2] and $\mathcal{V}_i$ are combinatorial factors, to be defined below, attached to each of the $m+1$ vertices of the tree. As it turns out, (1.3) involves a sum over tree graphs, thus making contact with the picture encountered in the context of two dimensional gravity.

As a matter of fact, the particular values of the coefficients of the double trace terms in (1.2), including the binomial coefficients, don't play any role in our argument, so we have effectively shown that the planar free energy of any matrix model with just double trace terms in the potential will involve the same sum over trees as (1.3).

A basic question about this planar perturbative series (1.3), is whether it has a non-zero radius of convergence λ_c, as expected on general grounds, and what is its precise value. Recall that in full-fledged quantum field theories, perturbative series are usually asymptotic, due to the combinatorial explosion of the number of Feynman diagrams. In the case at hand, the perturbative series are presumably divergent, but they are Borel summable [23, 24]. On the other hand, there are generic arguments that in the planar limit, the drastic reduction of the number of diagrams implies that their number only grows powerlike with the number of loops, so the perturbative series has a finite radius of

[1] We would like to thank Marcos Mariño for pointing out this reference to us.

[2] A composition is a partition where the order of the elements matters; e.g. $2+3$ and $3+2$ are different compositions of 5.

convergence [25]. Finding the radius of convergence of (1.3), and more generally, unveiling the phase structure of these theories in the planar limit, as the 't Hooft coupling is varied, are important open questions.

In section 3 we tackle the evaluation of the expectation value of the 1/2 BPS circular Wilson loop. Again, we start by computing the first terms in the perturbative expansion at finite N. Then we turn to the planar limit, and restrict ourselves to Wilson loops in the fundamental representation. We argue that $\langle W_F \rangle$ differs from the $\mathcal{N}=4$ one only for theories with a finite fraction of matter in the fundamental representation. We again manage to derive the perturbative expansion to all orders in λ; it now involves a sum over rooted trees.

In this work we have restricted ourselves to superconformal theories for concreteness. Looking towards the future, the techniques we have used can be also applied to non-conformal theories, massive or not. It will be interesting to determine whether any of the phase transition encountered for these theories [5, 11, 12] can be detected with our methods.

2 The partition function of $\mathcal{N}=2$ superconformal Yang-Mills theories

In this section we discuss the partition function of four dimensional Lagrangian $\mathcal{N}=2$ superconformal field theories on S^4. The seminal work [1] showed that for Lagrangian $\mathcal{N}=2$ super Yang-Mills theories — not necessarily conformal — Z_{S^4} can be reduced, thanks to supersymmetric localization, to a matrix integral. In this work we will consider the perturbative expansion in the zero-instanton sector. We will follow the approach of [2–4] and consider the integrals over the full Lie algebra. Furthermore, following also [2–4] we rewrite the 1-loop factor of the integrand as an effective action. Our first result is a general expression for the complete effective action (see also [4] for the SU(N) case). Armed with this result, we apply it first to obtain in a unified way the first terms of the partition function for classical Lie groups at finite N. We then switch to the planar limit, and obtain the planar free energy to all orders in the 't Hooft coupling.

We start by identifying the theories we will be studying. Lagrangian $\mathcal{N}=2$ super Yang-Mills theories with semi-simple gauge group G and arbitrary matter hypermultiplets have been classified in [26, 27]. The necessary and sufficient condition for conformality of such theories is the vanishing β function at 1-loop order, which translates into the following condition for the matter hypermultiplets

$$I_2(\text{adj}) = \sum_{\mathcal{M}} n_{\mathcal{M}} I_2(\mathcal{M}) \tag{2.1}$$

where $n_{\mathcal{M}}$ is the number of matter multiplets and the index of the representation $I_2(\mathcal{M})$ is defined in (2.14). In this article we will be mainly interested in the classical groups for which (2.1) reads

$$\begin{aligned}
\text{SU}(N): &\qquad 2N = 2N n_{\text{adj}} + n_F + (N+2) n_{\text{sym}} + (N-2) n_{\text{asym}} \\
\text{SO}(2N): &\qquad 2N-2 = (2N-2) n_{\text{adj}} + n_{\text{v}} \\
\text{SO}(2N+1): &\qquad 2N-1 = (2N-1) n_{\text{adj}} + n_{\text{v}} \\
\text{Sp(2N)}: &\qquad 2N+2 = (2N+2) n_{\text{adj}} + n_{\text{v}} + (2N-2) n_{\text{asym}} \,.
\end{aligned} \tag{2.2}$$

2.1 The 1-loop factor as an effective action

As shown in [1], supersymmetric localization reduces the partition function of $\mathcal{N}=2$ SYM theories on S^4, to a matrix integral of the form

$$Z_{S^4} = \int da\, e^{-\frac{8\pi^2}{g_{\rm YM}^2}\mathrm{Tr}(a^2)} \mathcal{Z}_{\text{1-loop}} \left|\mathcal{Z}_{\rm inst}\right|^2 . \tag{2.3}$$

In (2.3), da denotes a flat integration measure, over all the matrix entries, $\mathcal{Z}_{\text{1-loop}}$ is a factor that arises from a 1-loop computation, while $|\mathcal{Z}_{\rm inst}|^2$ is the instanton contribution. From now on we will restrict ourselves to the zero-instanton sector, for which $|\mathcal{Z}_{\rm inst}|^2 = 1$.

The $\mathcal{Z}_{\text{1-loop}}$ factor contains all the information of the choice of G and matter, and it's given by products over the weights of the adjoint and of the matter representations

$$\mathcal{Z}_{\text{1-loop}} = \prod_{\alpha} H(i\alpha\cdot\hat{a}) \prod_{R}\prod_{\omega_R} H(i\omega_R\cdot\hat{a})^{-n_R} \tag{2.4}$$

where $H(x)$ is the Barnes function whose expansion is given by

$$\ln H(x) = -(1+\gamma)x^2 - \sum_{n=2}^{\infty} \zeta(2n-1)\frac{x^{2n}}{n} . \tag{2.5}$$

Following [2–4], our strategy will be to rewrite the 1-loop partition function $\mathcal{Z}_{\text{1-loop}}$ in (2.3) as an effective action

$$S_{\rm int}^G \equiv -\ln \mathcal{Z}_{\text{1-loop}} = S_2(a) + S_3(a) + \cdots \tag{2.6}$$

where each S_i term corresponds to a value of the series expansion (2.5), so effectively any computation is reduced to evaluation of correlators in the Gaussian theory. Due to the vanishing of the 1-loop β function (2.1) the sum of quadratic terms in (2.5) cancel among themselves, so the effective action starts at order $g_{\rm YM}^4$. The key step is that in (2.4), after carrying out the multiplications by the weights of the different representations, we get sums of products of eigenvalues of the matrix a. These products can be rewritten as products of traces of a in the fundamental representation of G. Since the weights involved in (2.4) have one or two non-zero entries, these products translate into single trace and double trace operators respectively. For instance,

$$\sum_{u,v=1}^{n} (a_u + a_v)^{2n} = \sum_{u,v=1}^{n}\sum_{k=0}^{2n} \binom{2n}{k} a_u^{2n-k} a_v^{k} = \sum_{k=0}^{2n}\binom{2n}{k} \mathrm{Tr}\, a^{2n-k}\mathrm{Tr}\, a^{k} . \tag{2.7}$$

Going through this procedure for all possible matter representations (2.2) is a straightforward but tedious exercise, so we leave the explicit form for appendix A. Here we present the general result, written in a unified manner for an arbitrary group G as,

$$\begin{aligned} S_{\rm int}^G = \sum_{n=2}^{\infty} \frac{\zeta(2n-1)(-1)^n}{n} \Bigg[&(4-4^n)\alpha_G \mathrm{Tr}\, a^{2n} + \beta_G \sum_{k=1}^{n-1}\binom{2n}{2k} \mathrm{Tr}\, a^{2(n-k)}\mathrm{Tr}\, a^{2k} \\ &+ \gamma_G \sum_{k=1}^{n-2} \binom{2n}{2k+1} \mathrm{Tr}\, a^{2(n-k)-1} \mathrm{Tr}\, a^{2k+1} \Bigg] , \end{aligned} \tag{2.8}$$

	α_G	β_G	γ_G
$\mathrm{SU}(N)$	$\frac{n_{\mathrm{sym}}-n_{\mathrm{asym}}}{2}$	$1-n_{\mathrm{adj}}-\frac{n_{\mathrm{sym}}+n_{\mathrm{asym}}}{2}$	$n_{\mathrm{adj}}-1-\frac{n_{\mathrm{sym}}+n_{\mathrm{asym}}}{2}$
$\mathrm{SO}(2N)$	$1-n_{\mathrm{adj}}$	$2(1-n_{\mathrm{adj}})$	0
$\mathrm{SO}(2N+1)$	$1-n_{\mathrm{adj}}$	$2(1-n_{\mathrm{adj}})$	0
$\mathrm{Sp}(2N)$	$-1+n_{\mathrm{adj}}-n_{\mathrm{asym}}$	$2(1-n_{\mathrm{adj}}-n_{\mathrm{asym}})$	0

Table 1. Value of the coefficients in (2.8) for the different Lie groups.

with coefficients α_G, β_G, γ_G that depend on the gauge group and matter content, see table 1. Eventually, we will be interested in taking the large N limit of various quantities. Since for $\mathcal{N}=2$ SCFTs, n_F scales with N, we have used the condition of the vanishing of the 1-loop β function, eq. (2.1), to eliminate n_F in the previous formulas; in this way, all these coefficients are of order one, independent of N. Furthermore, they vanish for $\mathcal{N}=4$, as they should. The coefficient β_G is essentially what was called ν in [15], the fraction of matter in the fundamental representation.

We conclude that for any Lagrangian $\mathcal{N}=2$ SCFT, the 1-loop factor in (2.3) can be expressed as the exponential of an action that includes infinitely many single and double trace terms. For $\mathrm{SU}(N)$, this all-order effective action was also recently derived in [4]. As mentioned in the introduction, matrix models with single and double trace terms in the action already appeared in the study of two-dimensional quantum gravity [16–18, 20], where these double traces were interpreted as wormholes connecting spheres. In the planar limit, for small enough coupling of the double trace term, the relevant surfaces were trees of spheres connected by these wormholes; we will see the reappearance of tree graphs in the planar limits of the free energy — section 2.3 — and of the expectation value of the Wilson loop, section 3.2.

It is important to appreciate a difference between the matrix model that results from the effective interacting action (2.8) and the matrix models just mentioned. Terms in the action of the matrix model that contribute of the large N limit are — after perhaps a rescaling of the matrix — of the form

$$S = N^2 W(\mathcal{O}) \tag{2.9}$$

where $W(\mathcal{O})$ is a function that has no explicit N dependence and $\mathcal{O}$ are normalized trace operators, e.g. $\mathcal{O}=\frac{1}{N}\mathrm{Tr}\, a^k$. Schematically, for a theory with kinetic term, single and double trace term interactions,

$$S = N^2\left(\frac{1}{N}\mathrm{Tr}\, a^2 + \frac{1}{N}\mathrm{Tr}\, a^k + \frac{1}{N^2}\mathrm{Tr}\, a^m\, \mathrm{Tr}\, a^n\right). \tag{2.10}$$

The kinetic term in (2.3) is already of this form, since $\lambda = g_{\mathrm{YM}}^2 N$. However, the single trace terms in (2.8) do not have the proper scaling (2.10) to contribute to the planar limit. On the other hand, the double trace terms in (2.8) do have the right scaling, and can contribute to the planar limit.

2.2 Partition function and color invariants at finite N

In this section we will compute the first terms of the zero-instanton sector of the normalized version of the partition function (2.3), using the explicit form of S^G_{int} (2.8), i.e.

$$Z_{S^4} = \frac{\left\langle e^{-\sum_{i=2}^{\infty} S_i(a)} \right\rangle_0}{\langle \mathbb{I} \rangle_0} \tag{2.11}$$

where the subscript 0 corresponds to the Gaussian matrix model over the full Lie algebra. The denominator is the partition function for the case with $\mathcal{Z}_{\text{1-loop}} = 1$, namely, the $\mathcal{N} = 4$ SYM theory. By not restricting the integrals to the Cartan subalgebra [2, 3], the Vandermonde determinant is not generated and the matrix integrals reduce to Gaussian ones that can be carried out by applying Wick's theorem. As discussed in [9], this approach has the advantage that it allows to deal with different gauge groups and matter content in a unified fashion.

We will actually compute the first terms of the free energy $F(\lambda, N)$, or to be more precise, due to the denominator in (2.11), $F(\lambda, N) - F(\lambda, N)^{\mathcal{N}=4}$. From a field theory perspective, it receives contributions only from connected Feynman diagrams. At a given order in g_{YM} the relevant interaction terms can be read off directly from (2.8). As mentioned above, for superconformal field theories, the effective action starts at order g^4_{YM}, so the first cancellation when considering the logarithm of the partition function takes place at order g^8_{YM}. Up to this order,

$$F(\lambda, N) - F(\lambda, N)^{\mathcal{N}=4} = -\langle S_2(a)\rangle - \langle S_3(a)\rangle - \langle S_4(a)\rangle + \frac{1}{2}\left(\langle S_2(a)^2\rangle - \langle S_2(a)\rangle^2\right) + \mathcal{O}(g^{10}_{\text{YM}}) \,. \tag{2.12}$$

The computation factorizes into a trivial evaluation of Gaussian correlators, and the evaluation of color traces. The first part just amounts to applying Wick's theorem with the following two-point function,

$$\langle a_a a_b \rangle_0 = \frac{g^2_{\text{YM}}}{8\pi^2} \delta_{ab} \,. \tag{2.13}$$

The second one consists of evaluating traces of the Lie algebra generators, and it can be carried out using the techniques described in [22]. Our conventions are

$$[T^a_R, T^b_R] = i f^{abc} T^c_R \qquad \text{Tr}(T^a_R T^b_R) = I_2(R)\delta^{ab} \qquad (T^a_R T^a_R)_{ij} = C_2(R)\delta_{ij} \tag{2.14}$$

with $a, b = 1, \dots, N_A$, N_R is the dimension of the representation R, and A denotes the adjoint representation. We further define fully symmetrized traces

$$d^{a_1 \dots a_n}_R = \frac{1}{n!} \text{Tr} \sum_{\sigma \in S_n} T^{a_{\sigma(1)}}_R \dots T^{a_{\sigma(n)}}_R \,. \tag{2.15}$$

To the order considered here, the relevant correlators (recall that all traces are in the fundamental representation) are

$$\left\langle \text{Tr}\, a^{2n} \right\rangle = (2n-1)!! d^{b_1 b_1 \dots b_n b_n}_F \tag{2.16}$$

$$\left\langle \text{Tr}\, a^2 \text{Tr}\, a^{2n} \right\rangle = I_2(F)(d_A + 2n)\left\langle \text{Tr}\, a^{2n} \right\rangle \tag{2.17}$$

$$\left\langle \text{Tr}\, a^3 \text{Tr}\, a^3 \right\rangle = 6 d^{abc}_F d^{abc}_F \tag{2.18}$$

$$\left\langle \text{Tr}\, a^3 \text{Tr}\, a^5 \right\rangle = (60 C_F - 15 C_A) d^{abc}_F d^{abc}_F, \tag{2.19}$$

and plugging them in (2.12) we obtain

$$
\begin{aligned}
F-F^{\mathcal{N}=4} = & -3\zeta(3)\frac{{g_{\rm YM}}^4}{(8\pi^2)^2}\left[\alpha_G(C_A-6C_F)+\beta_G I_2(F)(2+N_A)\right]I_2(F)N_A \\
& +\zeta(5)\frac{{g_{\rm YM}}^6}{(8\pi^2)^3}\left[-300\alpha_G d_F^{aabbcc}+30\beta_G I_2(F)(N_A+4)d_F^{aabb}+40\alpha_G d_F^{abc}d_F^{abc}\right] \\
& -7\zeta(7)\frac{g_{\rm YM}^8}{(8\pi^2)^4}\Bigg[-945\alpha_G d_F^{aabbccdd}+30\beta_G I_2(F)(N_A+6)d_F^{aabbcc}+60\gamma_G(4C_F-C_A)d_F^{abc}d_F^{abc} \\
& +5\beta_G\left(\frac{9}{2}d_F^{aabb}d_F^{ccdd}+I_2(F)^2N_A(6C_F-C_A)^2+12d_F^{abcd}d_F^{abcd}\right)\Bigg] \\
& +18\zeta(3)^2\frac{g_{\rm YM}^8}{(8\pi^2)^4}\Big[\alpha_G^2\left(2I_2(F)N_A(6C_F-C_A)^2+24d_F^{abcd}d_F^{abcd}\right) \\
& -4\alpha_G\beta_G I_2(F)^3N_A(N_A+3)(6C_F-C_A)+2\beta_G^2 I_2(F)^4N_A(N_A+2)(N_A+3)\Big]+\mathcal{O}(g_{\rm YM}^{10})\,.
\end{aligned}
\tag{2.20}
$$

If needed, the color invariants that appear in the expression above can be rewritten in terms of lower order color invariants, as discussed in [9, 22]. At order $g_{\rm YM}^{2n}$ the possible products of values of the ζ function that can appear are $\zeta(2n_1-1)\ldots\zeta(2n_m-1)$, where $\{n_1,\ldots,n_m\}$ is a partition of n not containing 1. The number of such partitions is $p(n)-p(n-1)$, where $p(n)$ is the number of partitions of n.[3]

The drawback of the approach we have pursued to carry out the integrals is that (2.20) involves color invariants in the fundamental and adjoint representations, and not color invariants of the original matter representations of the SCFT. Therefore, it is not straightforward to match the different terms we encounter with the perturbative series in field theory. At low orders in the pertubative expansion, we can undo this, by rewriting the coefficient in terms of the original invariants. For instance, for the coefficient at order $g_{\rm YM}^4$,

$$
\alpha(C_A-6C_F)+\beta I_2(F)(N_A+2)=C_A^2-\sum_R n_R C_R I_2(R)\,. \tag{2.21}
$$

If we now wish to study the large N expansion of (2.20), it is straightforward to evaluate this expression by fixing G, the matter content, computing the corresponding group factors and finally evaluating the large N limit. For $\mathcal{N}=2$ SQCD, i.e. taking $G=\mathrm{SU}(N)$ with $n_F=2N$, the corresponding group invariants are given by

$$
C_F=\frac{N^2-1}{2N}\,,\qquad N_A=N^2-1\,,\qquad I_2(F)=\frac{1}{2}, \tag{2.22}
$$

in addition, from table 1 we see that this case corresponds to $\alpha_G=0$, $\beta_G=1$ and $\gamma_G=-1$. All in all, taking the large N limit and neglecting subleading terms

$$
F_0(\lambda)-F_0(\lambda)^{\mathcal{N}=4}=-\frac{3\zeta(3)}{256\pi^4}\lambda^2+\frac{5\zeta(5)}{1024\pi^6}\lambda^3+\frac{9\zeta(3)^2-35\zeta(7)}{16384\pi^8}\lambda^4+\mathcal{O}(\lambda^5) \tag{2.23}
$$

where $F_0(\lambda)$ is the coefficient of N^2 in the $1/N$ expansion of the free energy.

[3] The generating function of the number of partitions of n not containing 1 is $\prod_{k=2}^\infty\frac{1}{1-x^k}=(1-x)\prod_{k=1}^\infty\frac{1}{1-x^k}=(1-x)\sum_n p(n)x^n=\sum_n\left(p(n)-p(n-1)\right)x^n$.

2.3 Free energy at large N

We turn now our attention to the large N limit of the free energy on S^4, $F(\lambda, N) = \ln Z_{S^4}$. The free energy admits a large N expansion, $F(\lambda, N) = F_0(\lambda)N^2 + \ldots$, and our goal is to determine $F_0(\lambda)$. We will argue that $F_0(\lambda)$ differs from the $\mathcal{N} = 4$ result only for theories with a finite fraction of matter in the fundamental representation, i.e. theories with $\beta_G \neq 0$ in (2.8). In general,

$$F(\lambda, N) = \ln Z_{S^4} = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \left(\sum_{k=1}^{\infty} \frac{1}{k!} \left\langle (-S_{\text{int}}^G)^k \right\rangle \right)^m . \tag{2.24}$$

In the previous expansion, $\left\langle (S_{\text{int}}^G)^k \right\rangle$ involves disconnected $2k$-point functions whose $1/N$ expansion has a leading N^{2k} term. On the other hand, the leading term in $F(\lambda, N)$ scales like N^2, so there are massive cancellations in (2.24). For actions with just single trace interactions, only planar connected diagrams contribute to $F_0(\lambda)$. The action (2.8) has however double trace terms, and we need to fully identify the N^2 terms that survive the cancellations in (2.24).

These contributions can be written as products of connected correlators, and as it turns out, the characterization of which products of connected correlators contribute to $F_0(\lambda)$ has a natural answer in terms of graph theory: for any product of connected correlators we introduce an associated graph, and we will argue that a product of connected correlators contributes to $F_0(\lambda)$ if and only if its associated graph is a tree. The perturbative expansion we find for $F_0(\lambda)$ of these theories is thus given by a sum over all tree graphs.

Once we accomplish the task of characterizing the contribution of any correlator to the planar free energy, we take advantage of the results of [28, 29] for the planar limit of connected correlators, and write the full perturbative expansion of the planar free energy of these theories.

As a starting point, notice that the different terms in S_{int}^G, eq. (2.8), have vevs with different large N scaling. The single trace terms have vevs that scale like N. The vev of double trace operators with even powers factorizes in the large N and it scales like N^2. On the other hand, the vev of a double trace of operators with odd powers does not have a disconnected contribution, and its leading term scales like N^0. This already suggests that the large N behavior of the free energy depends qualitatively of having $\beta_G \neq 0$ or not; this qualitative difference was already encountered with the saddle point approximation.

For $\mathcal{N} = 4$ SYM, all coefficients in (2.8) vanish, since the 1-loop factor is exactly one. The unnormalized partition function is then trivially given by Gaussian integrals, and the planar free energy takes the following form [5]

$$F_0(\lambda)^{\mathcal{N}=4} = \frac{1}{2} \ln \lambda \, . \tag{2.25}$$

Let us discuss now some genuinely $\mathcal{N} = 2$ SCFTs. When $\beta_G = 0$, there are no terms in S_{int}^G scaling like N^2, so $F_0(\lambda) = F_0(\lambda)^{\mathcal{N}=4}$. The last and most interesting case is that of theories with $\beta_G \neq 0$ in (2.8), that is, with a finite fraction of matter in the fundamental representation. Theories with $\beta_G \neq 0$ can have α_G and $\gamma_G \neq 0$. We argue below that the α_G, γ_G parts of S_{int}^G in eq. (2.8) do not contribute to the free energy in the planar limit.

Any disconnected correlator can be written as a sum of products of connected correlators

$$\left\langle \mathrm{Tr}\, a^{k_1} \ldots \mathrm{Tr}\, a^{k_n} \right\rangle = \sum \left\langle \mathrm{Tr}\, a^{k_1} \ldots \mathrm{Tr}\, a^{k_{r_1}} \right\rangle_c \ldots \left\langle \mathrm{Tr}\, a^{k_{r_s}} \ldots \mathrm{Tr}\, a^{k_n} \right\rangle_c . \tag{2.26}$$

Terms in the previous sum that grow faster than N^2 are too disconnected and cancel out when taking the logarithm. Terms that scale slower than N^2 do not contribute to the planar limit of the free energy. To characterize the terms in (2.26) that scale precisely like N^2, let's recall that at large N, planar diagrams can be drawn on a sphere, so they scale like N^2. They are associated to connected correlators, and in the conventions of (2.10), the Feynman rule for a trace operator inserts an additional factor of N, so

$$\left\langle N\mathrm{Tr}\, a^{k_1} \ldots N\mathrm{Tr}\, a^{k_n} \right\rangle_c \sim N^2 \tag{2.27}$$

or equivalently, the connected n-point function of trace operators scales as

$$\left\langle \mathrm{Tr}\, a^{k_1} \ldots \mathrm{Tr}\, a^{k_n} \right\rangle_c \sim N^{2-n} \tag{2.28}$$

as long as there is an even number of odd k_i; if the number of odd k_i is odd, the correlator vanishes. According to (2.28), a term in the expansion (2.26) that involves the product of s connected correlators of $r_1, r_2, \ldots r_s$ sizes, scales as

$$\left\langle \mathrm{Tr}\, a^{k_1} \ldots \mathrm{Tr}\, a^{k_{r_1}} \right\rangle_c \left\langle \mathrm{Tr}\, a^{k_1} \ldots \mathrm{Tr}\, a^{k_{r_2}} \right\rangle_c \ldots \left\langle \mathrm{Tr}\, a^{k_1} \ldots \mathrm{Tr}\, a^{k_{r_s}} \right\rangle_c \sim N^{2s-(r_1+\cdots+r_s)} . \tag{2.29}$$

For this term to have the right scaling as N^2, the total number of operators in the disconnected correlator, $r_1 + \cdots + r_s$ must be even, call it $2m$, and furthermore $2s - 2m = 2$, so $s = m + 1$. Therefore, for a disconnected $2m$-point function, the terms with the right N^2 scaling are products of precisely $m + 1$ connected correlators. A slightly different version of the argument is the following: if we rewrite the double trace as

$$\mathrm{Tr}\, a^{2n-2k} \mathrm{Tr}\, a^{2k} = \frac{1}{N^2} N\mathrm{Tr}\, a^{2n-2k}\, N\mathrm{Tr}\, a^{2k} \tag{2.30}$$

we observe that each double trace insertion comes with a $\frac{1}{N^2}$ factor. Then, the N^2 scaling comes from s connected blobs, each scaling like N^2, joined by m wormholes, each weighted by $\frac{1}{N^2}$

$$(N^2)^s \left(\frac{1}{N^2}\right)^m = N^2 \Rightarrow s = m + 1 . \tag{2.31}$$

Since we are partitioning $2m$ operators into $m+1$ correlators, the number of such products is given by the number of partitions of $2m$ into precisely $m+1$ parts, $p_{m+1}(2m)$. This can be shown to be the same as the number of partitions of $m - 1$, $p(m - 1)$.[4]

We have just argued that for a disconnected $2m$-point function, the terms that have the right large N scaling to contribute to $F_0(\lambda)$ are products of $m + 1$ connected correlators. But not all such terms do actually contribute to $F_0(\lambda)$, since they may not survive the

[4] $p_{m+1}(2m) = [x^{2m}] x^{m+1} \prod_{i=1}^{m+1} \frac{1}{(1-x^i)} = [x^{m-1}] \prod_{i=1}^{m+1} \frac{1}{(1-x^i)} = [x^{m-1}] \prod_{i=1}^{\infty} \frac{1}{(1-x^i)} = p(m-1)$.

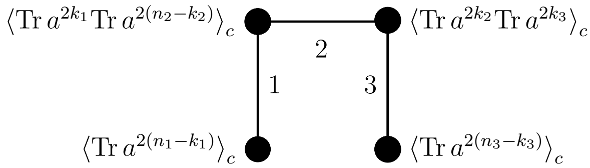

Figure 1. How to map a product of connected correlators to a tree with labeled edges: for each connected correlator, introduce a vertex. If two vertices contain operators in the same double trace, join them by an edge. The edges are labeled by the double trace that the respective vertices have in common.

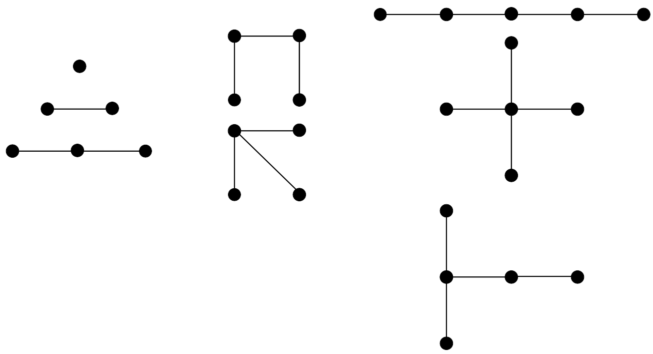

Figure 2. The list of trees up to 5 vertices.

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cancellations that take place in the sum (2.24). If a term in $\left\langle (S^G_{\text{int}})^n \right\rangle$ factorizes into pieces that appear in a product of $\left\langle (S^G_{\text{int}})^m \right\rangle$ of lower orders, it will be cancelled. So the terms in $\left\langle (S^G_{\text{int}})^n \right\rangle$ that contribute to $F_0(\lambda)$ are products of connected correlators, such that none of these correlators appears at lower orders. A succint way to describe this condition uses the language of graph theory. For this reason, we are going to associate a graph to any product of connected correlators.

Consider a particular product of $m+1$ connected correlators of lengths $r_1, \dots, r_{m+1}$ such that $r_1 + \cdots + r_{m+1} = 2m$. For each of them draw a vertex, so this is a graph with $m+1$ vertices. Then join two vertices by an edge if the correlators involve operators from the same double trace; there are then at most m edges. See figure 1 for an example of this procedure.

The condition on the correlators described above translates into the requirement that the graph is connected and has no loops; a connected graph with $m+1$ vertices and m edges is a tree [30]. See figure 2 for the list of trees with up to five vertices.

We can be more specific about the relevant types of trees. First, the edges are labeled by the double trace they represent. Furthermore, one has to distinguish two graphs coming from just swapping two operators in the same double trace. This can be taken into account by adding a direction (an arrow) to the edges, see figure 3. All in all, we have argued that the terms that contribute to $F_0(\lambda)$ from a disconnected $2m$-point function are in one-to-one correspondence with directed trees with $m+1$ vertices and labeled edges.

Let's collect some basic results about the enumeration of trees. There is no known formula for the number of unlabeled trees with n vertices. The sequence for the number of unlabeled trees with n vertices has the following first few terms [31]

$$1, 1, 1, 2, 3, 6, 11, 23, \ldots \tag{2.32}$$

A classical result by Cayley is that there are $(m+1)^{m-1}$ trees with labeled $m+1$ vertices [32]. Using this result, it is immediate to prove [33] that for $m \geq 2$, there are $(m+1)^{m-2}$ trees with labeled edges. Finally, if every edge is oriented (with an arrow), there is an additional factor of 2 for each edge, so the number of oriented trees with $m+1$ vertices and labeled edges is $2^m(m+1)^{m-2}$ [34]. See figure 3 for examples of these types of trees.

Let's go back to the expansion (2.24). Recall that the terms in the action (2.8) don't have any power of N in front of them. As discussed after eq. (2.10), that implies that double trace terms can contribute to the planar limit, but the single trace terms in (2.8) can't. Let's further argue that only the double traces of even powers — the β_G term in the action (2.8) — contribute to the planar limit. First, any non-zero correlator has an even number of odd powers, call it $2k$. In particular, no connected correlator can have just one odd power operator: it either has none, or at least two. Therefore there are at most k connected correlators with odd powers. The subgraph of connected correlators with odd powers has at most k vertices and precisely k edges, so it must contain loops. This implies that the full graph can't be a tree, and thus this product of connected correlators doesn't contribute to the large N limit.

After arguing that only double traces of even powers contribute to the planar limit of the free energy, we restrict our attention to just those terms,

$$\begin{aligned} Z_{S^4} = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m \beta_G^m}{m!} \sum_{n_1,\ldots,n_m=2}^{\infty} (-1)^{n_1+\cdots+n_m} \frac{\zeta(2n_1-1)\ldots\zeta(2n_m-1)}{n_1\ldots n_m} \\ \sum_{k_1=1}^{n_1-1}\binom{2n_1}{2k_1}\cdots\sum_{k_m=1}^{n_m-1}\binom{2n_m}{2k_m}\left\langle \operatorname{Tr} a^{2(n_1-k_1)}\operatorname{Tr} a^{2k_1}\ldots\operatorname{Tr} a^{2(n_m-k_m)}\operatorname{Tr} a^{2k_m}\right\rangle . \end{aligned} \tag{2.33}$$

To proceed, we need the coefficients of connected correlators in the planar limit. These coefficients give the number of connected planar fatgraphs one can draw with the corresponding operators, and are thus integer numbers. For one-point functions [35, 36]

$$\left\langle \frac{1}{N}\operatorname{Tr} a^{2k}\right\rangle \to C_k\left(\frac{\lambda}{16\pi^2}\right)^k \tag{2.34}$$

where C_k are the Catalan numbers. For connected n-point functions, the leading term at large N is [28] (see also [29] for an earlier, purely combinatorial derivation)

$$\left\langle \operatorname{Tr} a^{2k_1}\operatorname{Tr} a^{2k_2}\ldots\operatorname{Tr} a^{2k_n}\right\rangle_c = \frac{(d-1)!}{(d-n+2)!}\prod_{i=1}^{n}\frac{(2k_i)!}{(k_i-1)!k_i!}\left(\frac{\lambda}{16\pi^2}\right)^d N^{2-n}, \tag{2.35}$$

where $d = \sum k_i$. Notice that (2.35) reduces to (2.34) when $n = 1$. The results above were derived for the Hermitian matrix model, so in principle they apply to U(N)/SU(N) gauge

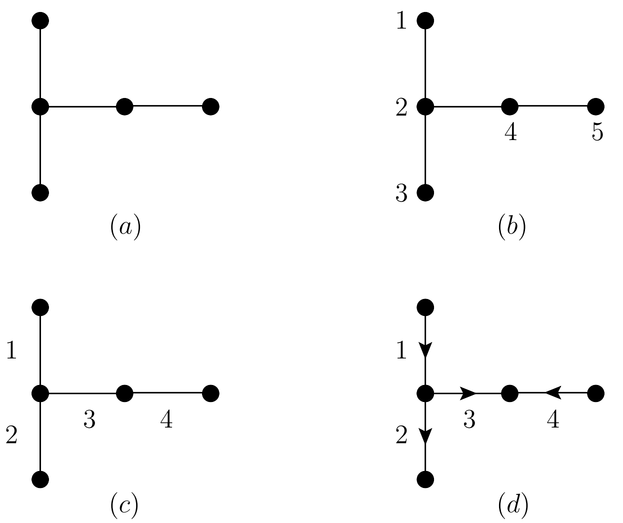

Figure 3. a) An unlabeled tree. b) A tree with labeled vertices. c) A tree with labeled edges. d) A directed tree with labeled edges.

theories. Nevertheless, since we are only concerned with planar diagrams, they apply also to SO(N), Sp(N) theories. For future use, let's give a name to the numerical coefficient in (2.35),

$$\mathcal{V}(k_1,\dots,k_n)=\frac{(d-1)!}{(d-n+2)!}\prod_{i=1}^{n}\frac{(2k_i)!}{(k_i-1)!k_i!}\,. \tag{2.36}$$

The contributions to $F_0(\lambda)$ at fixed order β_G^m are then obtained as follows. At this order, there are m pairs of traces, coming from m double trace terms, $\operatorname{Tr} a^{2n_1-2k_1}$, $\operatorname{Tr} a^{2k_1},\dots,$ $\operatorname{Tr} a^{2n_m-2k_m}$, $\operatorname{Tr} a^{2k_m}$. Draw all directed edge-labeled trees with m edges. Assign $\operatorname{Tr} a^{2n_i-2k_i}$ to the vertex at the start (i.e. origin of the arrow) of the i-th edge. Assign $\operatorname{Tr} a^{2k_i}$ to the vertex at the end i.e. end of the arrow of the i-th edge. This procedure assigns to each of the $m+1$ vertices a number of traces equal to the degree of the vertex, i.e. the number of edges connected to that vertex. For each vertex, consider now the connected correlator of all its trace operators and assign it its numerical factor $\mathcal{V}_i$, eq. (2.36). Then,

$$F_0(\lambda)-F_0(\lambda)^{\mathcal{N}=4}=\sum_{m=1}^{\infty}\frac{(-1)^m\beta_G^m}{m!}\sum_{n_1,\dots,n_m=2}^{\infty}\left(\frac{-\lambda}{16\pi^2}\right)^{n_1+\dots+n_m}\frac{\zeta(2n_1-1)\dots\zeta(2n_m-1)}{n_1\dots n_m}$$
$$\sum_{k_1=1}^{n_1-1}\binom{2n_1}{2k_1}\cdots\sum_{k_m=1}^{n_m-1}\binom{2n_m}{2k_m}\sum_{\substack{\text{directed trees}\\ \text{with labeled edges}}}\prod_{i=1}^{m+1}\mathcal{V}_i\,. \tag{2.37}$$

Let's illustrate this result by working out the lowest orders of (2.37). Notice that β_G counts the number of double trace terms, and it can be thought of as a wormhole counting parameter. When $m=1$, we have two-point functions, and we have to consider partitions of 2 into precisely 2 parts; the only possibility is 2=1+1, so the contribution comes from the product of two one-point functions. Applying (2.26) and (2.34) for the full series of (2.11), we obtain

$$\begin{aligned} F_0(\lambda)|_{\beta_G} &= -\beta_G \sum_{n_1=2}^{\infty} \frac{\zeta(2n_1-1)}{n_1} \left(\frac{-\lambda}{16\pi^2}\right)^{n_1} \sum_{k_1=1}^{n_1-1} \binom{2n_1}{2k_1} C_{n_1-k_1} C_{k_1} \\ &= -\beta_G \sum_{n=2}^{\infty} \frac{\zeta(2n-1)}{n} \left(\frac{-\lambda}{16\pi^2}\right)^{n} C_n (C_{n+1}-2) \end{aligned} \tag{2.38}$$

as expected from the finite N discussion we see that the leading β_G term is an infinite series that captures all the terms with only one ζ thus generalizing the result (2.23). At order β_G^2, the terms that contribute to $F_0(\lambda)$ come from distributing the 4 operators into precisely 3 correlators. The only possible partition is $4=1+1+2$, which corresponds to the only tree with 3 vertices in figure 2. There are 4 directed trees with labeled edges for this unlabeled tree, so the contributions are

$$\left\langle \mathrm{Tr}\, a^{2(n_1-k_1)} \right\rangle_c \left\langle \mathrm{Tr}\, a^{2(n_2-k_2)} \right\rangle_c \left\langle \mathrm{Tr}\, a^{2k_1} \mathrm{Tr}\, a^{2k_2} \right\rangle_c \tag{2.39}$$

and the corresponding permutations coming from exchanging $k_i \leftrightarrow n_i - k_i$. While the product of correlators is not invariant under this exchange, after summing over $k_{1,2}$ in (2.37), the answer is, so we can just take one of them and multiply by 4. The contribution to the planar free-energy is given by

$$\begin{aligned} F_0(\lambda)|_{\beta_G^2} = \frac{\beta_G^2}{2!} \sum_{n_1,n_2=2}^{\infty} \frac{\zeta(2n_1-1)\zeta(2n_2-1)}{n_1 n_2} \left(\frac{-\lambda}{16\pi^2}\right)^{n_1+n_2} \\ \sum_{k_1=1}^{n_1-1} \binom{2n_1}{2k_1} \sum_{k_2=1}^{n_2-1} \binom{2n_2}{2k_2} \frac{4}{k_1+k_2} \frac{(2k_1)!}{(k_1-1)!k_1!} \frac{(2k_2)!}{(k_2-1)!k_2!} C_{n_1-k_1} C_{n_2-k_2} \,. \end{aligned} \tag{2.40}$$

This expression recovers the term in (2.23) with a product of two values of ζ, and provides all the subsequent terms of this form.

After these examples, let's simplify the sums in (2.37). First, as we have seen in the example at order β_G^2, while the product $\mathcal{V}_1 \dots \mathcal{V}_{m+1}$, is not invariant under $k_i \leftrightarrow n_i - k_i$, after summing over all k_i in (2.37), the answer is the same for any of the choices of arrows of the tree, so one can just take any of the 2^m possible assignments, and replace the last sum by

$$2^m \sum_{\substack{\text{undirected trees} \\ \text{with labeled edges}}} \mathcal{V}_1 \dots \mathcal{V}_{m+1} \tag{2.41}$$

where now the sum is over undirected trees (no arrows) with m labeled edges. To further simplify this sum, note that for a given unlabeled tree T with $m \geq 2$ edges and with automorphism group Aut(T), there are $\frac{m!}{|\mathrm{Aut(T)}|}$ ways to label its edges [33]. They correspond to different rearrangements of the indices $1, 2, \dots, m$ in the traces placed in the $m+1$

correlators, so again, in general the values of the correlators are different. However, (2.37) contains a sum over all $n_1,\ldots,n_m$ so after this sum all such terms end up giving the same. The case $m=1$ has to be considered separately; the only tree with one edge, see figure 2, has $|\mathrm{Aut(T)}|=2$ and there is just one way to label its edge. On the other hand, for the directed version, changing the direction of the arrow does not change the tree, so these two factors cancel each other. These considerations allow to further simplify the sum over trees to

$$2^m \sum_{\text{unlabeled trees}} \frac{m!}{|\mathrm{Aut(T)}|}\mathcal{V}_1\ldots\mathcal{V}_{m+1} \tag{2.42}$$

finally arriving at,

$$\begin{aligned} F_0(\lambda)-F_0(\lambda)^{\mathcal{N}=4} = \sum_{m=1}^{\infty}(-2\beta_G)^m \sum_{n_1,\ldots,n_m=2}^{\infty}\left(\frac{-\lambda}{16\pi^2}\right)^{n_1+\cdots+n_m}\frac{\zeta(2n_1-1)\ldots\zeta(2n_m-1)}{n_1\ldots n_m} \\ \sum_{k_1=1}^{n_1-1}\binom{2n_1}{2k_1}\cdots\sum_{k_m=1}^{n_m-1}\binom{2n_m}{2k_m}\sum_{\substack{\text{unlabeled trees}\\ \text{with m edges}}}\frac{1}{|\mathrm{Aut(T)}|}\mathcal{V}_1\ldots\mathcal{V}_{m+1}\,. \end{aligned} \tag{2.43}$$

A physically more relevant expression comes from grouping all terms with the same power of λ. To write it down, first recall that a composition of n is a partition where order matters, so $3+2$ and $2+3$ are different compositions of 5. We will denote by m the number of non-zero elements of a given composition. Then

$$\begin{aligned} F_0(\lambda)-F_0(\lambda)^{\mathcal{N}=4} = \sum_{n=2}^{\infty}\left(-\frac{\lambda}{16\pi^2}\right)^{n}\sum_{\substack{\text{compositions of n}\\ \text{not containing 1}}}(-2\beta_G)^m\frac{\zeta(2n_1-1)\ldots\zeta(2n_m-1)}{n_1\ldots n_m} \\ \sum_{k_1=1}^{n_1-1}\binom{2n_1}{2k_1}\cdots\sum_{k_m=1}^{n_m-1}\binom{2n_m}{2k_m}\sum_{\substack{\text{unlabeled trees}\\ \text{with m edges}}}\frac{1}{|\mathrm{Aut(T)}|}\mathcal{V}_1\ldots\mathcal{V}_{m+1} \end{aligned} \tag{2.44}$$

where the second sum is over compositions $\{n_1,\ldots,n_m\}$ of n that don't contain 1. The number of such compositions of n is given by the Fibonacci number F_{n-1} [37]. Equation (2.44) is our result for the planar limit of the free energy of theories with $\beta_G\neq 0$, and the main result of this section. In appendix B we write explicitly its first terms, up to 13th order.

Let's comment now on the convergence of the perturbative expansion (2.44). Typically, perturbative series in quantum field theory are asymptotic, due to the combinatorial explosion of Feynman diagrams. The perturbative series of the full free energy of these theories is presumably divergent, but it is Borel summable [23, 24]. On the other hand, for generic quantum field theories, when we restrict to the planar limit, there is a drastic reduction in the number of Feynman diagrams that contribute in this limit, which now grows only powerlike. As a consequence, the planar perturbative series has a finite radius of convergence [25, 36]. A pertinent question is then what is the radius of convergence of (2.44).

We haven't been able to determine the radius of convergence of (2.44). Nevertheless, let us offer some comments on the convergence of the series that appear at every order in β_G in (2.43). At every fixed order in β_G, the coefficient is a series in λ. At order β_G, it follows immediately from the quotient criterion that the series (2.38) has radius of convergence $\lambda_c = \pi^2$. This is precisely the same value as the one found in [12] for the divergence of planar perturbation theory for $\mathcal{N} = 4$ SYM generic observables (in this sense, the 1/2 BPS $\mathcal{N} = 4$ Wilson loop turns out not to be a generic observable). We can sketch an argument proving that the series in λ at generic, but fixed, order in β_G in (2.43) have all the same radius of convergence. First, define $\tilde{\mathcal{V}}_i$ as the prefactor of $\mathcal{V}_i$ that does not factorize,

$$\tilde{\mathcal{V}}_i(k_1,\ldots,k_n) = \frac{(d-1)!}{(d-n+2)!} \tag{2.45}$$

with $d = \sum_i k_i$. Then, the last line in (2.43) can be rewritten as

$$\frac{(2n_1)!}{(n_1-1)!^2}\cdots\frac{(2n_m)!}{(n_m-1)!^2}\sum_{k_1=1}^{n_1-1}\binom{n_1-1}{k_1}\binom{n_1-1}{k_1-1}\cdots\sum_{k_m=1}^{n_m-1}\binom{n_m-1}{k_m}\binom{n_m-1}{k_m-1}$$
$$\sum_{\substack{\text{unlabeled trees}\\ \text{with m edges}}}\frac{\tilde{\mathcal{V}}_1\ldots\tilde{\mathcal{V}}_{m+1}}{|\mathrm{Aut(T)}|}\,. \tag{2.46}$$

The sum over trees in the equation above yields a rational function of the variables n_i, k_i of degree $-m-3$, let's call it $Q_m(n_i,k_i)$ For large n_i the m sums over k_i can be thought as a measure peaked around $k_i = n_i/2$, so we conjecture that when all ns are large, the effect of the sums is evaluating Q_m with all k_i taking the value $n_i/2$

$$\sum_{k_1}\sum_{k_m} Q_m(n_1,\ldots,n_m,k_1,\ldots,k_m) \to Q_m(n_1,\ldots,n_m,n_1/2,\ldots n_m/2)\sum_{k_1}\sum_{k_m}\,. \tag{2.47}$$

If this is true, assuming all the n_i are large enough and applying the Stirling approximation, it follows that for every m the series at order β_G^m in (2.43) has finite radius of convergence $\lambda_c = \pi^2$. Even if this argument can be made precise, proving that at every fixed order in β, the corresponding series in (2.43) has radius of convergence $\lambda_c = \pi^2$ doesn't prove that this is the radius of convergence of (2.44). Study of the convergence of (2.44) is under investigation.

Finally, let's point out that the number of trace insertions at every vertex in the tree graph is fixed. It is the degree of the vertex, i.e. the number of edges arriving at the vertex. This is due to the fact that single trace terms don't contribute to the planar limit.

3 The 1/2 BPS circular Wilson loop

Another milestone of supersymmetric localization is the possibility to compute the expectation value of certain class of protected operators. In the work [1] it was proven that the expectation value of the 1/2 BPS circular Wilson loop

$$W_R = \frac{1}{N_R}\mathrm{Tr}_R\mathcal{P}\exp\oint_C(A_\mu dx^\mu + i\Phi ds) \tag{3.1}$$

also reduces to a matrix model computation

$$\langle W \rangle = \frac{1}{Z_{S^4}} \int da \mathrm{Tr} e^{-2\pi b a} e^{-\frac{8\pi^2}{g_{\mathrm{YM}}^2}\mathrm{Tr}(a^2)} \mathcal{Z}_{\text{1-loop}} |\mathcal{Z}_{\text{inst}}|^2 \,. \tag{3.2}$$

It has been understood more recently that the correlator of the stress-energy tensor and a 1/2 BPS circular Wilson loop can also be determined by a matrix model computation. First, the two-point function of the stress-energy tensor and a straight 1/2 BPS Wilson line is determined by conformal invariance, up to a coefficient h_W [38]

$$\frac{\langle T^{00}(x) W \rangle}{\langle W \rangle} = \frac{h_W}{|\vec{x}|^4} \,. \tag{3.3}$$

This coefficient appears also in the two-point function of the stress-energy tensor and a circular Wilson loop [39]. It was conjectured in [40] that for $\mathcal{N}=2$ SCFTs

$$h_W = \frac{1}{12\pi^2} \partial_b \ln \langle W_b \rangle |_{b=1} \tag{3.4}$$

where the vev of the Wilson loop is computed in a squashed S^4 sphere of parameter b,

$$\langle W_b \rangle = \frac{1}{Z_{S^4}} \int da \mathrm{Tr} e^{-2\pi b a} e^{-\frac{8\pi^2}{g_{\mathrm{YM}}^2}\mathrm{Tr}(a^2)} \mathcal{Z}_{\text{1-loop}} |\mathcal{Z}_{\text{inst}}|^2 \,. \tag{3.5}$$

In principle, both $\mathcal{Z}_{\text{1-loop}}$ and $|\mathcal{Z}_{\text{ins}}|^2$ depend on the squashing parameter b, but this dependence starts at quadratic order in $(b-1)^2$ [40]. In practice, since the relation (3.4) is only sensitive to the linear dependence in $b-1$, in evaluating (3.4) it is valid to use the expressions for $\mathcal{Z}_{\text{1-loop}}$ and $|\mathcal{Z}_{\text{ins}}|^2$ of the ordinary S^4. Additional evidence for eq. (3.4) was provided in [41] and it was finally proven in [42]. It is also worth keeping in mind that for $\mathcal{N}=2$ theories it was conjectured in [40, 43] and proven in [44] that $B = 3h_W$, where B is the Bremsstrahlung function [45, 46].

The perturbative computation of the vev of this Wilson loop operator in $\mathcal{N}=2$ was studied before [47] by usual QFT techniques, in [41] by using the heavy quark effective theory and in [13] by matrix model techniques. All of this perturbative computations were done for the case of $G = \mathrm{SU}(N)$ with $n_F = 2N$, in addition going to higher orders in perturbation theory seems a daunting task in these formalisms.

As in the case of the partition function, we will attack this problem with localization techniques and once again we will not restrict the integration to the Cartan subalgebra of G. This will allow us to obtain both the Wilson loop operator and the Bremsstrahlung function in an unified manner for any choice of G obeying (2.1) in terms of color invariants. In the large N limit, we will be able to obtain an all order expression in λ, similar to the one found for the free energy in the previous section.

We will consider the generalized Wilson loop W_b, eq. (3.5), on an ordinary S^4 so we can apply (3.4) to obtain the one-point function of the stress-energy tensor in the presence of the Wilson loop. To obtain the expectation value of the Wilson loop on S^4, it is enough to set $b=1$.

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3.1 Wilson loops at finite N

In this section we will proceed as in the case of the free energy. We will perform a perturbative calculation of the lowest orders of the vev of the 1/2 BPS circular Wilson loop operator and we will be able to cast the result for arbitrary gauge group G at finite N. Setting from now on $|\mathcal{Z}_{\text{inst}}|^2 = 1$, from (3.5) we have

$$\langle W_b \rangle = \frac{1}{Z_{S^4}} \sum_{l=0}^{\infty} \frac{(-2\pi b)^l}{l! N_R} \langle \operatorname{Tr}_R a^l e^{-S(a)} \rangle_0, \tag{3.6}$$

since we are interested in corrections coming from the matter content it is convenient to subtract the expectation value of the Wilson loop operator of the $\mathcal{N}=4$ theory. Up to order g_{YM}^8 we find

$$\begin{aligned}\langle W\rangle_{\mathcal{N}=2}-\langle W\rangle_{\mathcal{N}=4} = &-\frac{1}{N_R}\frac{(2\pi b)^2}{2!}\Big[\langle \operatorname{Tr}_R a^2 S_2(a)\rangle - \langle \operatorname{Tr}_R a^2\rangle\langle S_2(a)\rangle + \langle \operatorname{Tr}_R a^2 S_3(a)\rangle - \langle \operatorname{Tr}_R a^2\rangle\langle S_3(a)\rangle\Big] \\ &-\frac{1}{N_R}\frac{(2\pi b)^4}{4!}\Big[\langle \operatorname{Tr}_R a^4 S_2(a)\rangle - \langle \operatorname{Tr}_R a^4\rangle\langle S_2(a)\rangle\Big] + \mathcal{O}(g_{\text{YM}}^{10}) \,. \end{aligned}\tag{3.7}$$

From (2.8) it's a straightforward calculation to obtain at finite N

$$\begin{aligned}\langle W\rangle_{\mathcal{N}=2}-\langle W\rangle_{\mathcal{N}=4} = &\frac{3\zeta(3) b^2 g_{\text{YM}}^6}{(8\pi^2)^2}\frac{I_2(R)}{N_R}\Big[6\alpha_G d_F^{aabb} - \beta_G I_2(F)^2 N_A(N_A+2)\Big] \\ &+\frac{\zeta(5) b^2 g_{\text{YM}}^8}{(8\pi^2)^3}\frac{I_2(R)}{N_R}\Big[-450\alpha_G d_F^{aabbcc} + 45\beta_G I_2(F)(N_A+4) d_F^{aabb} + 60\gamma_G d_F^{abc} d_F^{abc}\Big] \\ &+\frac{\zeta(3) b^4 g_{\text{YM}}^8}{4(8\pi^2)^4 N_R}\Big[-6\alpha_G\big(3\frac{d_R^{aabb} d_F^{ccdd}}{N_A} + d_R^{abcd} d_F^{abcd}\big) + 3\beta_G(N_A+3) d_R^{aabb}\Big] + \mathcal{O}(g_{\text{YM}}^{10}) \,. \end{aligned}\tag{3.8}$$

As a check, we can compare the order g_{YM}^6 general result with the computations carried out in [41, 47] for the special case of SQCD, this setup is the same as the one considered in (2.22), with this is straightforward to evaluate the order g_{YM}^6 term in (3.8) for this choice

$$\langle W\rangle_{\mathcal{N}=2}-\langle W\rangle_{\mathcal{N}=4} = -b^2 \frac{g_{\text{YM}}^6}{512\pi^4}(3\zeta(3))\frac{(N^2-1)(N^2+1)}{N}, \tag{3.9}$$

that precisely matches the result presented in [47] and further generalizes it to any gauge group G while preserving the finite N contributions.

As in the case of the free energy, a drawback of the result (3.8) is that it's written in terms of gauge invariants in the fundamental representation, and not of the matter representations of the theory. To fix this, we can look at the relevant diagrams in the quantum field theory computation, eq. (18) in [47], to find that at order g_{YM}^6 the color factor is

$$\frac{C_A^2}{2} - \sum_h n_h \left(C_h - \frac{C_A}{2}\right) I_2(h) = C_A^2 - \sum_h n_h C_h I_2(h) \tag{3.10}$$

where the sum is over the matter hypermultiplets. It can be easily checked that the color invariant and the coefficients in (3.8) do reproduce this color factor.

As a second example, consider the SCFT whose gauge group is SU(N) and has one rank-2 symmetric and one rank-2 antisymmetric hypermultiplet, with this matter content the degrees of freedom scale as $\frac{1}{2}N(N+1)+\frac{1}{2}N(N-1)\simeq N^2$ this is the same number as $\mathcal{N}=4$ SU(N) $\times$ U(1) gauge theory. This coincidence leads to a massive cancelation of Feynman diagrams from which it's expected [46, 48] that corrections from the expectation value of the $\mathcal{N}=4$ result scale as $1/N$, this is easy to check noting that this theory corresponds (1) to $\alpha_G=0$, $\beta_G=0$ and $\gamma_G=-2$, so (3.8) becomes

$$\langle W\rangle_{\mathcal{N}=2}-\langle W\rangle_{\mathcal{N}=4}=-\frac{120b^2g_{\mathrm{YM}}^8\zeta(5)}{N_R(8\pi^2)^3}d_f^{abc}d_f^{abc}I_2(R)\simeq-\frac{\lambda^4}{4N^2}\frac{15b^2\zeta(5)}{(8\pi^2)^3} \tag{3.11}$$

where we see that corrections scale as $1/N^2$.

Finally let us briefly discuss the large N limit and present the result corresponding to SQCD, from (3.7) and (3.8) we obtain

$$\langle W\rangle_{\mathcal{N}=2}-\langle W\rangle_{\mathcal{N}=4}=-\frac{3b^2\lambda^3}{512\pi^4}\zeta(3)-\lambda^4\left(\frac{2\pi^2b^4\zeta(3)-15b^2\zeta(5)}{4096\pi^6}\right)+\mathcal{O}(\lambda^5). \tag{3.12}$$

3.2 Wilson loops at large N

We want to determine now the planar limit of the expectation value of the Wilson loop operator. As found in the previous section for the planar free energy, the answer depends markedly on whether the SCFT has a finite fraction of matter in the fundamental representation, $\beta_G\neq 0$ in (2.8), or not.

Since we want to take advantage of the result for the planar connected correlators (2.35), we will restrict ourselves to the case $R=F$, so the Wilson loop is taken in the fundamental representation. Recall that the effective action (2.8) involves single trace terms, and double traces of even and of odd power operators. Note that when $\beta_G\neq 0$, the α_G and γ_G terms in the action give subleading contributions, and can be neglected in the planar limit.

First of all, let's argue that $\langle W_b\rangle$ scales like N^0 in the planar limit. Expanding the exponential of the Wilson loop insertion

$$\langle W_b\rangle=\sum_{l=0}^{\infty}\frac{(4\pi^2b^2)^l}{(2l)!}\frac{\left\langle\frac{1}{N}\mathrm{Tr}\,a^{2l}e^{-S}\right\rangle}{\langle e^{-S}\rangle}\,. \tag{3.13}$$

Now, the expression inside the sum is the one-point function of $\frac{1}{N}\mathrm{Tr}\,a^{2l}$ in an interacting matrix model, that scales like N^0. Following the same logic as in the previous section for the free energy, we aim to write this as a product of connected correlators with the right large N scaling. Moreover, as we did for the free energy, it proves convenient to subtract the result from the Gaussian matrix model, which in this context corresponds to the $\mathcal{N}=4$ theory.

After expanding the effective action, we want to extract the piece of $\left\langle\mathrm{Tr}\,a^{2l}S^m\right\rangle$ that scales like N. This correlator contains $2m+1$ traces, so by the same argument as in the previous section, a product of s connected correlators scales like $2s-2m-1$, which implies the relevant piece are again products of $m+1$ connected correlators. On the other hand,

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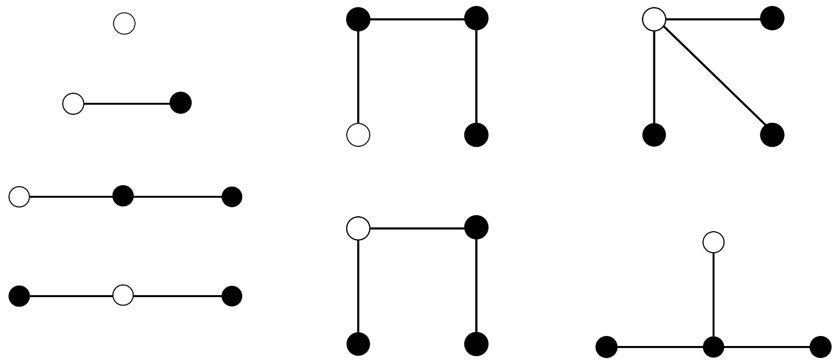

Figure 4. The list of rooted trees up to 4 vertices. In each tree, the distinguished vertex is represented by the white dot.

$\mathrm{Tr}\, a^{2l}$ can not be by itself in one of these correlators, because it will be cancelled by the $\mathcal{N}=4$ subtraction. As a consequence, after fixing the correlator that contains $\mathrm{Tr}\, a^{2l}$, we are again distributing $2m$ operators into $m+1$ connected correlators, so again we find a tree structure! There is however, an important difference: now, one of the connected correlators contains $\mathrm{Tr}\, a^{2l}$, so it is distinguished from the rest. When we translate the product of connected correlators to a tree graph, we have to distinguish one of the vertices, the one that correspond to the correlator containing $\mathrm{Tr}\, a^{2l}$. In the mathematical literature, trees with a distinguished vertex are call rooted trees [30]. See figure 4 for the list of rooted trees with up to four vertices. The number of rooted trees with n vertices is [49]

$$1, 1, 2, 4, 9, 20, 48, \dots \tag{3.14}$$

All in all, we find

$$\langle W\rangle_{\mathcal{N}=2} - \langle W\rangle_{\mathcal{N}=4} = \sum_{l=1}^{\infty} \frac{b^{2l}}{(2l)!}\left(\frac{\lambda}{4}\right)^{l} \sum_{m=1}^{\infty} \frac{(-\beta_G)^m}{m!} \sum_{n_1,\dots,n_m=2}^{\infty} \left(\frac{-\lambda}{16\pi^2}\right)^{n_1+\dots+n_m} \frac{\zeta(2n_1-1)\dots\zeta(2n_m-1)}{n_1\dots n_m}$$
$$\sum_{k_1=1}^{n_1-1}\binom{2n_1}{2k_1}\dots\sum_{k_m=1}^{n_m-1}\binom{2n_m}{2k_m} \sum_{\substack{\text{directed rooted trees}\\ \text{with m labeled edges}}} \prod_{i=1}^{m+1} \mathcal{V}_i\,. \tag{3.15}$$

By the same arguments that we used in the discussion of the free energy, this expression can be simplified to a sum over unlabeled rooted trees

$$\langle W\rangle_{\mathcal{N}=2} - \langle W\rangle_{\mathcal{N}=4} = \sum_{l=1}^{\infty} \frac{b^{2l}}{(2l)!}\left(\frac{\lambda}{4}\right)^{l} \sum_{m=1}^{\infty} (-2\beta_G)^m \sum_{n_1,\dots,n_m=2}^{\infty} \left(\frac{-\lambda}{16\pi^2}\right)^{n_1+\dots+n_m} \frac{\zeta(2n_1-1)\dots\zeta(2n_m-1)}{n_1\dots n_m}$$
$$\sum_{k_1=1}^{n_1-1}\binom{2n_1}{2k_1}\dots\sum_{k_m=1}^{n_m-1}\binom{2n_m}{2k_m} \sum_{\substack{\text{unlabeled rooted}\\ \text{trees with m edges}}} \frac{1}{|\mathrm{Aut(T)}|}\prod_{i=1}^{m+1} \mathcal{V}_i\,. \tag{3.16}$$

To illustrate (3.16), let's work out the terms up to β_G^3,

$$\langle W\rangle_{\mathcal{N}=2}-\langle W\rangle_{\mathcal{N}=4}=\sum_{l=1}^{\infty}\frac{b^{2l}}{l!(l-1)!}\left(\frac{\lambda}{4}\right)^l\Bigg[-\beta_G\sum_{n=2}^{\infty}\left(\frac{-\lambda}{16\pi^2}\right)^n\frac{\zeta(2n-1)}{n}\binom{2n}{n}\sum_{k=1}^{n-1}\binom{n}{k}\binom{n}{k-1}\frac{2}{l+k}$$
$$+\beta_G^2\sum_{n_1,n_2=2}^{\infty}\left(\frac{-\lambda}{16\pi^2}\right)^{n_1+n_2}\frac{\zeta(2n_1-1)\zeta(2n_2-1)}{n_1n_2}\binom{2n_1}{n_1}\binom{2n_2}{n_2}\sum_{k=1}^{n_1-1}\binom{n_1}{k_1}\binom{n_1}{k_1-1}\sum_{k=2}^{n_2-1}\binom{n_2}{k_2}\binom{n_2}{k_2-1}$$
$$\left(4\frac{(n_1-k_1+1)(n_1-k_1)}{(l+n_1-k_1)(k_1+k_2)}+2\right)-\frac{\beta_G^3}{6}\sum_{n_1,n_2,n_3=2}^{\infty}\left(\frac{-\lambda}{16\pi^2}\right)^{n_1+n_2+n_3}\frac{\zeta(2n_1-1)\zeta(2n_2-1)\zeta(2n_3-1)}{n_1n_2n_3}$$
$$\binom{2n_1}{n_1}\binom{2n_2}{n_2}\binom{2n_3}{n_3}\sum_{k=1}^{n_1-1}\binom{n_1}{k_1}\binom{n_1}{k_1-1}\sum_{k=2}^{n_2-1}\binom{n_2}{k_2}\binom{n_2}{k_2-1}\sum_{k=3}^{n_3-1}\binom{n_3}{k_3}\binom{n_3}{k_3-1}$$
$$\Bigg(48\frac{(n_1-k_1)(n_1-k_1+1)(n_3-k_3)(n_3-k_3+1)}{(l+k_1)(k_2+k_3)(n_1-k_1+n_3-k_3)}+48\frac{(n_1-k_1)(n_1-k_1+1)}{k_3+n_1-k_1}$$
$$+24\frac{(n_1-k_1)(n_1-k_1+1)}{l+n_1-k_1}+8(l+k_1+k_2+k_3-1)\Bigg)+\cdots\Bigg] \tag{3.17}$$

where the dots stand for terms with more than three values of the ζ function. We have checked that this expression correctly reproduces the explicit results of appendix A of [13], where this expectation value was computed up to λ^7. This match provides a very non-trivial check of our computation.

Finally, let's consider an example of a theory with $\beta_G=0$, namely the SU(N) gauge theory with a 2-symmetric and a 2-anti-symmetric hypermultiplet. This theory has $\alpha_G=\beta_G=0$ and $\gamma_G=-2$, so in the planar limit $\langle W\rangle_{\mathcal{N}=2}=\langle W\rangle_{\mathcal{N}=4}$, and we can compute some subleading $1/N^2$ terms in $\langle W\rangle_{\mathcal{N}=2}$. In particular, we will now derive the term linear in γ_G, so it contains all the terms with a single value of the ζ function. To do so note that in this case the effective action (2.8) has only odd powers so we need the equivalent result of (2.35) with two odd powers and a single even power. This is [28]

$$\left\langle \operatorname{Tr} a^{2k_1+1}\operatorname{Tr} a^{2k_2+1}\operatorname{Tr} a^{2k}\right\rangle_c=\frac{(2k_1+1)!}{k_1!^2}\frac{(2k_2+1)!}{k_2!^2}\frac{(2k_3)!}{(k_3-1)!k_3!}\,. \tag{3.18}$$

With this in mind we obtain

$$\langle W\rangle_{\mathcal{N}=2}-\langle W\rangle_{\mathcal{N}=4}=-\frac{\gamma_G}{N^2}\sum_{l=1}^{\infty}\frac{(4\pi^2b^2)^l}{l!(l-1)!}\sum_{n=3}^{\infty}\zeta(2n-1)(-1)^n\binom{2n}{n-1}(n+1)\left(\frac{\lambda}{16\pi^2}\right)^{n+l}\sum_{k=1}^{n-2}\binom{n-1}{k}^2, \tag{3.19}$$

where in fact the sums over k and l can be performed, which gives us

$$\langle W\rangle_{\mathcal{N}=2}-\langle W\rangle_{\mathcal{N}=4}=-\frac{\gamma_G}{N^2}\frac{b\sqrt{\lambda}}{2}I_1(b\sqrt{\lambda})\sum_{n=3}^{\infty}\zeta(2n-1)(-1)^n\binom{2n}{n-1}(n+1)\left[\binom{2(n-1)}{n-1}-2\right]\left(\frac{\lambda}{16\pi^2}\right)^n. \tag{3.20}$$

Where $I_1(x)$ is the modified Bessel function of the first kind and this result is computed for SU(N). In the large N limit, n-point functions of traces of odd powers don't coincide for U(N) and SU(N), so this result can't be compared with the large N limit of (3.8).

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A $\mathcal{Z}_{\text{1-loop}}$ and S^G_{int} for the classical groups

In this appendix we present the full expression of the $\mathcal{Z}_{\text{1-loop}}$ for (2.4) and the corresponding expression of the effective action

$$Z^{\mathrm{SU}(N)}_{\text{1-loop}} = \frac{\prod_{u<v=1}^N H(ia_u-ia_v)^2}{\prod_{u<v=1}^N H(ia_u-ia_v)^{2n_{\text{adj}}}\prod_{u=1}^N H(ia_u)^{n_f}\prod_{u\leq v=1}^N H(ia_u+ia_v)^{n_{\text{sym}}}\prod_{u<v=1}^N H(ia_u+ia_v)^{n_{\text{asym}}}} \tag{A.1}$$

$$Z^{\mathrm{SO}(2N)}_{\text{1-loop}} = \frac{\prod_{u<v}^N H^2(ia_u+ia_v)H^2(ia_u-ia_v)}{\prod_{u<v}^N H(ia_u+ia_v)^{2n_{\text{adj}}}H(ia_u-ia_v)^{2n_{\text{adj}}}\prod_{u=1}^N H(ia_u)^{2n_{\text{v}}}} \tag{A.2}$$

$$Z^{\mathrm{SO}(2N+1)}_{\text{1-loop}} = \frac{\prod_{u<v}^N H^2(ia_u+ia_v)H^2(ia_u-ia_v)\prod_{u=1}^N H(ia_u)^2}{\prod_{u<v}^N H(ia_u+ia_v)^{2n_{\text{adj}}}H(ia_u-ia_v)^{2n_{\text{adj}}}\prod_{u=1}^N H(ia_u)^{2n_{\text{adj}}+2n_{\text{v}}}} \tag{A.3}$$

$$Z^{\mathrm{Sp}(N)}_{\text{1-loop}} = \frac{\prod_{u<v}^N H^2(ia_u+ia_v)H^2(ia_u-ia_v)\prod_{u=1}^N H(2ia_u)^2}{\prod_{u<v}^N H(ia_u+ia_v)^{2n_{\text{adj}}+2n_a}H(ia_u-ia_v)^{2n_{\text{adj}}+2n_a}\prod_{u=1}^N H(2ia_u)^{2n_{\text{adj}}}\prod_{u=1}^N H(ia_u)^{2n_{\text{v}}}}, \tag{A.4}$$

$$\begin{aligned} S^{\mathrm{SU}(N)} = \sum_{n=2}^{\infty}\frac{\zeta(2n-1)(-1)^n}{n}\Bigg\{&\frac{4-4^n}{2}(n_{\text{sym}}-n_{\text{asym}})\operatorname{Tr}a^{2n} \\ &+\sum_{k=2}^{2n-2}\binom{2n}{k}\left((-1)^k(1-n_{\text{adj}})-\frac{n_{\text{sym}}+n_{\text{asym}}}{2}\right)\operatorname{Tr}a^{2n-k}\operatorname{Tr}a^k\Bigg\}, \end{aligned} \tag{A.5}$$

$$\begin{aligned} S^{\mathrm{SO}(2N)} &= \sum_{n=2}^{\infty}\frac{\zeta(2n-1)(-1)^n}{n}(1-n_{\text{adj}})\left\{\sum_{k=2}^{2n-2}\binom{2n}{k}\operatorname{Tr}a^{2n-k}\operatorname{Tr}a^k\left(1+(-1)^k\right)+(4-4^n)\operatorname{Tr}a^{2n}\right\} \\ S^{\mathrm{SO}(2N+1)} &= \sum_{n=2}^{\infty}\frac{\zeta(2n-1)(-1)^n}{n}(1-n_{\text{adj}})\left\{\sum_{k=2}^{2n-2}\binom{2n}{k}\operatorname{Tr}a^{2n-k}\operatorname{Tr}a^k\left(1+(-1)^k\right)+(4-4^n)\operatorname{Tr}a^{2n}\right\} \end{aligned} \tag{A.6}$$

$$\begin{aligned} S^{\mathrm{Sp}(2N)} = \sum_{n=2}^{\infty}\frac{\zeta(2n-1)(-1)^n}{n}\Bigg\{&(1-n_{\text{adj}}-n_{\text{asym}})\sum_{k=2}^{2n-2}\binom{2n}{k}\operatorname{Tr}a^{2n-k}\operatorname{Tr}a^k\left(1+(-1)^k\right) \\ &+(4-4^n)(n_{\text{adj}}-n_{\text{asym}}-1)\operatorname{Tr}a^{2n}\Bigg\}. \end{aligned} \tag{A.7}$$

B Explicit planar free energy up to 13th order

In this appendix we present the result of evaluating (2.44) up to λ^{13}. To do so, we use the shorthand $\tilde{\lambda} = -\frac{\lambda}{16\pi^2}$. Furthermore, we are not writing powers of β_G; to recover them, write one β_G for each ζ. The planar free energy is then

$$
\begin{aligned}
&F_0(\lambda) - F_0(\lambda)^{\mathcal{N}=4} = -3\zeta_3\tilde{\lambda}^2 - 20\zeta_5\tilde{\lambda}^3 + (36\zeta_3^2 - 140\zeta_7)\tilde{\lambda}^4 + (720\zeta_3\zeta_5 - 1092\zeta_9)\tilde{\lambda}^5 \\
&+(-720\zeta_3^3 + 3800\zeta_5^2 + 6720\zeta_3\zeta_7 - 9394\zeta_{11})\tilde{\lambda}^6 + (-25920\zeta_3^2\zeta_5 + 73360\zeta_5\zeta_7 + 65520\zeta_3\zeta_9 - 87516\zeta_{13})\tilde{\lambda}^7 \\
&+(18144\zeta_3^4 - 316800\zeta_3\zeta_5^2 - 282240\zeta_3^2\zeta_7 + 361620\zeta_7^2 + 732480\zeta_5\zeta_9 + 676368\zeta_3\zeta_{11} - 868725\zeta_{15})\tilde{\lambda}^8 \\
&+(967680\zeta_3^3\zeta_5 - (3920000\zeta_5^3)/3 - 6968640\zeta_3\zeta_5\zeta_7 - 3144960\zeta_3^2\zeta_9 + 7331520\zeta_7\zeta_9 + 7700880\zeta_5\zeta_{11} + \\
&7351344\zeta_3\zeta_{13} - 9072492\zeta_{17})\tilde{\lambda}^9 + (-(2612736/5)\zeta_3^5 + 19440000\zeta_3^2\zeta_5^2 + 11612160\zeta_3^3\zeta_7 - 43394400\zeta_5^2\zeta_7 \\
&-38478720\zeta_3\zeta_7^2 - 78180480\zeta_3\zeta_5\zeta_9 + 37570176\zeta_9^2 - 36523872\zeta_3^2\zeta_{11} + 77994840\zeta_7\zeta_{11} \\
&+84942000\zeta_5\zeta_{13} + 83397600\zeta_3\zeta_{15} - (493668032\zeta_{19})/5)\tilde{\lambda}^{10} \\
&+(-37324800\zeta_3^4\zeta_5 + 173952000\zeta_3\zeta_5^3 + 466502400\zeta_3^2\zeta_5\zeta_7 - 481376000\zeta_5\zeta_7^2 + 141523200\zeta_3^3\zeta_9 - 489014400\zeta_5^2\zeta_9 \\
&-865186560\zeta_3\zeta_7\zeta_9 - 912859200\zeta_3\zeta_5\zeta_{11} + 806319360\zeta_9\zeta_{11} - 441080640\zeta_3^2\zeta_{13} + 868659792\zeta_7\zeta_{13} \\
&+975477360\zeta_5\zeta_{15} + 979829136\zeta_3\zeta_{17} - 1111643260\zeta_{21})\tilde{\lambda}^{11} \\
&+(16422912\zeta_3^6 - 1064448000\zeta_3^3\zeta_5^2 + 584160000\zeta_5^4 - 479001600\zeta_3^4\zeta_7 + 6253228800\zeta_3\zeta_5^2\zeta_7 + 2787966720\zeta_3^2\zeta_7^2 \\
&-(5345751040\zeta_7^3)/3 + 5678830080\zeta_3^2\zeta_5\zeta_9 - 10857759360\zeta_5\zeta_7\zeta_9 - 4866160320\zeta_3\zeta_9^2 + 1785611520\zeta_3^3\zeta_{11} \\
&-5731228800\zeta_5^2\zeta_{11} - 10116912960\zeta_3\zeta_7\zeta_{11} + 4356229416\zeta_{11}^2 - 11075201280\zeta_3\zeta_5\zeta_{13} + 9045036000\zeta_9\zeta_{13} \\
&-5504241600\zeta_3^2\zeta_{15} + 10057407360\zeta_7\zeta_{15} + 11579728160\zeta_5\zeta_{17} + 11848032768\zeta_3\zeta_{19} - (38632924694\zeta_{23})/3)\tilde{\lambda}^{12} \\
&+(1478062080\zeta_3^5\zeta_5 - 15137280000\zeta_3^2\zeta_5^3 - 27186001920\zeta_3^3\zeta_5\zeta_7 + 27942656000\zeta_5^3\zeta_7 + 74609203200\zeta_3\zeta_5\zeta_7^2 \\
&-6227020800\zeta_3^4\zeta_9 + 75975782400\zeta_3\zeta_5^2\zeta_9 + 67576965120\zeta_3^2\zeta_7\zeta_9 - 60299164800\zeta_7^2\zeta_9 - 61206969600\zeta_5\zeta_9^2 \\
&+71569681920\zeta_3^2\zeta_5\zeta_{11} - 127317072960\zeta_5\zeta_7\zeta_{11} - 113820094080\zeta_3\zeta_9\zeta_{11} + 23289057792\zeta_3^3\zeta_{13} \\
&-69769814400\zeta_5^2\zeta_{13} - 122896597824\zeta_3\zeta_7\zeta_{13} + 98300538336\zeta_{11}\zeta_{13} - 138770723520\zeta_3\zeta_5\zeta_{15} + \\
&105367232880\zeta_9\zeta_{15} - 70547697792\zeta_3^2\zeta_{17} + 120227642080\zeta_7\zeta_{17} + 141264773520\zeta_5\zeta_{19} + 146736910320\zeta_3\zeta_{21} \\
&-152833845400\zeta_{25})\tilde{\lambda}^{13} + \mathcal{O}(\tilde{\lambda}^{14})\,.
\end{aligned}
$$

Using the method of orthogonal polynomials explained in appendix B of [21], we have checked this result up to $\tilde{\lambda}^7$.

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CHAPTER 3

The planar limit of $\mathcal{N} = 2$ chiral correlators

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The planar limit of $\mathcal{N} = 2$ chiral correlators

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Abstract: We derive the planar limit of 2- and 3-point functions of single-trace chiral primary operators of $\mathcal{N} = 2$ SQCD on S^4, to all orders in the 't Hooft coupling. In order to do so, we first obtain a combinatorial expression for the planar free energy of a hermitian matrix model with an infinite number of arbitrary single and double trace terms in the potential; this solution might have applications in many other contexts. We then use these results to evaluate the analogous planar correlation functions on $\mathbb{R}^4$. Specifically, we compute all the terms with a single value of the ζ function for a few planar 2- and 3-point functions, and conjecture general formulas for these terms for all 2- and 3-point functions on $\mathbb{R}^4$.

Keywords: 1/N Expansion, Supersymmetric Gauge Theory

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Contents

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1 Introduction

Correlation functions of local operators are among the most basic quantities of interest in any quantum field theory, yet in most cases their evaluation is prohibitively difficult. This state of affairs can improve in theories with additional symmetries, like conformal invariance and/or supersymmetry. In this work we are going to study a particular instance of such tractable correlation functions, the so-called extremal n-point functions of chiral primary operators (CPOs) of four dimensional Lagrangian $\mathcal{N} = 2$ superconformal field theories (SCFTs) [1]. For these correlation functions, the coupling and spacetime dependences factorize, and the spacetime dependence is completely fixed, thus reducing the problem to the — still very difficult — determination of the dependence on the marginal coupling.

In recent years, the study of these correlation functions has been approached from different angles, often in combination. A first approach [2–5] uses a 4d analog of the tt^* equations [1]. More recently, it has been shown [6, 7] that the evaluation of closely related n-point functions on S^4 can be reduced through supersymmetric localization to matrix model computations; in turn, a Gram-Schmidt orthogonalization procedure applied to these S^4 correlators yields the correlators on $\mathbb{R}^4$ [8–13]. Alternatively, the large R-charge limit [14] of these correlation functions has been studied in [15–20].

In the current work, we will focus on the planar limit of some of these extremal correlators. For concreteness, we will present explicit results for single-trace operators of $\mathcal{N} = 2$ SU(N) SYM with $N_F = 2N$ massless hypermultiplets in the fundamental representation, sometimes referred to as $\mathcal{N} = 2$ SQCD. The techniques we will use, however, can be easily extended to any other Lagrangian $\mathcal{N} = 2$ SCFT that admits a planar limit, and to correlation functions of multi-trace chiral operators. It was argued in [3] that extremal n-point functions are determined in terms of 2- and 3-point functions, so we will restrict our attention to these. We obtain what we believe are the first known all-order analytic expressions for coefficients in the perturbative expansion of the planar limit of these 2- and 3- point functions. *En route* to deriving these results, we deduce a combinatorial expression for the planar free energy of the relevant matrix model, and combinatorial expressions for the planar 2- and 3- point functions on S^4. In the rest of the introduction we briefly sketch a summary of these results and the methods used to derive them, and point out some possible extensions of the present work.

Four dimensional $\mathcal{N}=2$ SCFTs theories have various subsets of distinguished operators (see [21] for a thorough discussion on $\mathcal{N}=2$ SCFTs short multiplets). In particular, chiral primary operators (CPOs) are defined as being annihilated by all right chiral supercharges; similarly, anti-chiral operators are annihilated by all left chiral supercharges. CPOs have conformal dimension Δ fixed by their $\mathrm{U}(1)_R$ R-charge, $\Delta = R/2$ and are $\mathrm{SU}(2)_R$ singlets. Anti-chiral primary operators have $\Delta = -R/2$. We will consider CPOs that are Lorentz scalars, so they are characterized by their conformal dimension Δ. On $\mathbb{R}^4$, correlation functions of CPOs and anti-CPOs can be non-zero only if the sum of their R-charges is 0. In particular, this implies that n-point functions of chiral primary operators (with no anti-chirals) are zero. The simplest non-trivial case are the extremal correlation functions, involving $n-1$ CPOs O_i and a single anti-chiral operator $\bar{O}$

$$\left\langle O_{\Delta_1}(x_1)\dots O_{\Delta_{n-1}}(x_{n-1})\bar{O}_{\bar{\Delta}}(y)\right\rangle = \frac{\left\langle O_{\Delta_1}\dots O_{\Delta_{n-1}}\bar{O}_{\bar{\Delta}}\right\rangle(\tau,\bar{\tau})}{|x_1-y|^{2\Delta_1}\dots|x_{n-1}-y|^{2\Delta_{n-1}}} \tag{1.1}$$

with $\Delta_1+\dots\Delta_{n-1}=\bar{\Delta}$. The position-independent coefficients $\left\langle O_{\Delta_1}\dots O_{\Delta_{n-1}}\bar{O}_{\bar{\Delta}}\right\rangle(\tau,\bar{\tau})$ are non-holomorphic functions of the complexified coupling $\tau = \frac{2\theta}{\pi}+i\frac{4\pi}{g^2_{\mathrm{YM}}}$ and their determination is the driving question for this work.

In this paper we restrict to Lagrangian SCFTs. The CPOs we will consider are single-trace operators involving the complex scalar ϕ of the $\mathcal{N}=2$ vector multiplet, $O_m \propto \mathrm{Tr}\,\phi^m$. O_m has dimension $\Delta = m$. In the planar limit of theories with a single gauge coupling, extremal 2- and normalized 3- point functions on $\mathbb{R}^4$ are of the form

$$\langle O_k\overline{O_k}\rangle = k\left(\frac{\lambda}{16\pi^2}\right)^k\left[1+\sum_{m=1}^{\infty}\sum_{n_1,\dots,n_m=2}^{\infty}a_k(n_1,\dots,n_m)\zeta_{2n_1-1}\dots\zeta_{2n_m-1}\left(\frac{\lambda}{16\pi^2}\right)^{n_1+\dots+n_m}\right] \tag{1.2}$$

$$\frac{\left\langle O_{k_1}O_{k_2}\bar{O}_{k_1+k_2}\right\rangle_n}{\sqrt{k_1\cdot k_2\cdot(k_1+k_2)}} = \frac{1}{\mathrm{N}}\left[1+\sum_{m=1}^{\infty}\sum_{n_1,\dots,n_m=2}^{\infty}b_{k_1,k_2}(n_1,\dots,n_m)\zeta_{2n_1-1}\dots\zeta_{2n_m-1}\left(\frac{\lambda}{16\pi^2}\right)^{n_1+\dots+n_m}\right] \tag{1.3}$$

with ζ_i values of the ζ function, $\lambda = g^2_{\mathrm{YM}}\mathrm{N}$ the 't Hooft coupling and $a_k(n_i)$ and $b_{k_1,k_2}(n_1,\dots,n_m)$ rational numbers. For $\mathcal{N}=2$ SQCD, we have computed $a_k(n)$ explicitly for $k=2,4,6$, and the expressions we find suggest the following conjecture

$$\langle O_k\bar{O}_k\rangle \overset{?}{=} k\left(\frac{\lambda}{16\pi^2}\right)^k\left(1-2k\sum_{n=2}^{\infty}\frac{\zeta_{2n-1}}{n}\left(\frac{-\lambda}{16\pi^2}\right)^n\binom{2n}{n}\left[(-1)^k\binom{2n}{n+k}+\binom{2n}{n+1}-n\right]+\dots\right) \tag{1.4}$$

where the dots stand for terms with products of two or more values of the ζ function. Similarly, we have computed $\left\langle O_2O_2\bar{O}_4\right\rangle_n$ and $\left\langle O_2O_4\bar{O}_6\right\rangle_n$ and the results obtained suggest the following conjecture for even k_1, k_2

$$\frac{\langle O_{k_1}O_{k_2}\bar{O}_{k1+k2}\rangle_n}{\sqrt{k_1\cdot k_2\cdot(k_1+k_2)}} \stackrel{?}{=} \frac{1}{\mathrm{N}}\Bigg[1-\sum_{n=2}^{\infty}\left(\frac{-\lambda}{16\pi^2}\right)^n \zeta_{2n-1}\binom{2n}{n} \\ \left(\binom{2n}{n+k_1}+\binom{2n}{n+k_2}+\binom{2n}{n+k_1+k_2}+(n-1)(\mathcal{C}_n-2)\right)+\dots\Bigg] \tag{1.5}$$

where again the dots stand for terms with products of two or more values of the ζ function and $\mathcal{C}_n$ are Catalan numbers. If we assign transcendality n to ζ_n and in general $n_1+\cdots+n_m$ to $\zeta_{n_1}\dots\zeta_{n_m}$, then at every order in the planar perturbative series, our conjectures refer to the term with maximal transcendality. The two analytic expressions we propose are strikingly simple, and certainly simpler than the intermediate results used to arrive at them. This suggests that there may be a more direct way to obtain them than the one pursued in this work. We will come back to this point at the end of the introduction.

The technical tool that we have used to derive (1.4) and (1.5) is supersymmetric localization [22]. Supersymmetric localization has produced a plethora of exact results for supersymmetric quantum field theories in various dimensions (see [23] for a review) by reducing the evaluation of selected observables to matrix model computations. It is thus natural to try to apply it to the computation of chiral correlation functions. For CPOs with $\Delta = 2$, it was argued in [6] that this 2-point function can be obtained directly from the partition function of the CFT on S^4. For more general CPOs, the situation is more complicated: it was argued in [7] that correlation functions of CPOs on S^4 can be extracted from the S^4 partition function of a deformed theory. Furthermore, correlation functions on S^4 differ from those on $\mathbb{R}^4$; to obtain the latter from the former, one needs to apply the Gram-Schmidt orthogonalization procedure [7].

The path described above has been followed already in a number of papers [3–5, 8–13]. The novel ingredient that we introduce in this work is an alternative way of evaluating the free energy and correlators of the relevant matrix models, which allows us to obtain all-order analytic expressions in the planar limit. Usually, Hermitian matrix model integrals are solved by reducing them to a Cartan subalgebra, which reduces the number of integrals, at the price of introducing a non-trivial Jacobian, the Vandermonde determinant. Instead, it is possible to tackle them in the original full Lie algebra formulation, an approach that in the context of supersymmetric localization has been pioneered in [10, 24–26]. In this approach, the relevant matrix models for genuinely $\mathcal{N} = 2$ SCFTs can be rewritten in terms of an action with infinitely many single and double trace terms [26–28]. Furthermore, in the planar limit, it has been argued [27, 28] that the full perturbative series in λ for various observables can be written in terms of a sum over tree graphs. In this work, when applying this strategy to the relevant matrix model, the main novelty compared to [27] is that now the single-trace terms in the matrix model action also contribute to the planar limit, complicating the analysis. Nevertheless, the resulting expressions for the planar free energies and correlation functions still involve sums over tree graphs.

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This work leaves open a number of questions. First, it would be completely straightforward but rather tedious to extend the computations presented here to the terms involving a product of two values of ζ or higher in (1.2) and (1.3). In this work, we have focused on the terms with maximal transcendality; it might be possible to find analytic formulas for the coefficients of ofher terms with simple patterns, like those with just powers of ζ_3, as in [29]. It should also be possible to extend the analysis presented here for $\mathcal{N}=2$ SCQD to extremal correlators of other Lagrangian $\mathcal{N}=2$ SCFTs [11–13, 30]. A very interesting problem would be to prove our conjectures (1.4) and (1.5). Conceivably, a proof might just extend our computations for arbitrary values of the conformal dimensions; after all, the relevant ingredients are the coefficients of the correlators on S^4, and the Gram-Schmidt relation to correlation functions on $\mathbb{R}^4$, and both of these are known. A potentially more illuminating proof might bypass the relation to S^4 correlators, and work directly on $\mathbb{R}^4$. Indeed, the factor $\left(\frac{-\lambda}{16\pi^2}\right)^n\binom{2n}{n}\frac{\zeta_{2n-1}}{n}$ that appears in (1.4) and (1.5) coincides with the values of a certain family of Feynman diagrams considered in closely related work [11, 12] (see also [31]), so the form of (1.4) and (1.5) suggest that they can be proven by a combinatorial argument, counting the ways in which those particular Feynman diagrams enter the evaluation of $\left\langle O_k\bar{O}_k\right\rangle$ and $\left\langle O_{k_1}O_{k_2}\bar{O}_{k_1+k_2}\right\rangle$.

The structure of the paper is the following. In section 2 we consider a Hermitian matrix model with an action containing infinitely many single and double trace terms with arbitrary coefficients; we extend the analysis of [27], and manage to write the planar free energy and the planar 2- and 3-point functions as sums over tree graphs. In section 3 we consider the evaluation of correlation functions of $\mathcal{N}=2$ SCFTs on S^4 through supersymmetric localization. We argue that the relevant matrix model is a particular case of the one considered in section 2, thus obtaining expressions for the planar 2- and 3-point functions on S^4. Finally, in section 4 we apply the Gram-Schmidt procedure to the S^4 correlation functions found in the previous section, to obtain correlation functions on $\mathbb{R}^4$. The manipulations become quite involved, thus preventing us from obtaining closed expressions for the full planar 2- and 3-point functions. Nevertheless, by focusing on the terms with a single value of ζ, we compute them for operators of small conformal dimensions, and conjecture the formulas (1.4) and (1.5) for planar correlation functions of arbitrary single trace CPOs.

2 Matrix model with single and double traces

One of the main technical tools that we will use in the following sections to compute extremal correlation functions of CPOs is supersymmetric localization. As we will argue, the resulting matrix models can be written in terms of an action involving infinitely many single and double trace term deformations, the latter having very specific coefficients. In this section we study this type of matrix model with arbitrary coefficients, to highlight the generality of our arguments.

Let's consider a Hermitian matrix model

$$\mathcal{Z}=\int \mathrm{d}a\, e^{-\frac{1}{2g}\mathrm{Tr}(a^2)}e^{-S_{\mathrm{int}}} \tag{2.1}$$

where a is a Hermitian $\mathrm{N}\times\mathrm{N}$ matrix, $\mathrm{d}a$ is the flat measure and g is the matrix model coupling. The interacting part of the action consists of (possibly infinitely many) single and double trace terms,

$$S_{\text{int}} = \mathrm{N}\sum_{p\geqslant 3} c_p \mathrm{Tr}\, a^p + \sum_{mn} c_{mn} \mathrm{Tr}\, a^m\, \mathrm{Tr}\, a^n\,. \tag{2.2}$$

with the coefficients c_p, c_{mn} N-independent and otherwise arbitrary. Particular examples of this family of models have appeared in the study of two-dimensional quantum gravity [32–36], and as reviewed in [37], they have also appeared in many other contexts, from two-dimensional statistical mechanics, to three-dimensional gauge theories, or M-theory. Without the N factor in front of the single-trace terms, they are relevant [26, 27] in the application of supersymmetric localization to four dimensional undeformed $\mathcal{N}=2$ super Yang-Mills theories.

Our goals in this section are twofold: first, we will deduce the planar free energy for this family of models, as a function of the 't Hooft coupling and the coefficients c_p, c_{ij}. Then, in preparation for the next section, we will consider the coefficients c_p as external sources; this will allow us to obtain the planar 2- and 3-point functions of single trace operators of the matrix model (2.2) with just double-trace terms, by taking derivatives against the c_p and then turning them off.

As shown in [37], the planar free energy of these models can be deduced recursively, using the method of orthogonal polynomials. We will present an alternative expression for the planar free energy as a sum over tree graphs, generalizing [27]. More specifically, the matrix model considered in [27] was similar to (2.2), but without the power of N in front of the single trace terms, rendering them irrelevant in the planar limit. On the other hand, the matrix model we will encounter in the next section is precisely of the form (2.2). Nevertheless, we will show that the resulting planar free energy can still be written as a sum over tree graphs, albeit a more complicated one.

To study the planar limit of (2.2), start by defining the matrix model 't Hooft coupling by $\tilde{\lambda} = g\mathrm{N}$.[1] In the large N limit, the free energy of the matrix model admits an expansion of the form

$$\mathcal{F}(\tilde{\lambda}, N) = -\log \mathcal{Z} = -\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \left(\sum_{k=1}^{\infty} \frac{1}{k!} \Big\langle (-S_{\text{int}})^k \Big\rangle \right)^m = F_0(\tilde{\lambda}) N^2 + \cdots\,, \tag{2.3}$$

where the N^2 contribution is given by the planar free energy $F_0(\tilde{\lambda})$. At a given order we will have all the possible factorizations of a general correlator of the form $\langle S^m_{\text{int}}\rangle$, thus the terms contributing to the planar free energy will be those that scale as N^2 and survive all the cancellations arising from the logarithm. We wish then to characterize this set of terms. In particular, let's consider a term with $m-k$ single traces and k double trace; its contribution is of the form

$$\mathrm{N}^{m-k} \langle \mathrm{Tr}\, a^{p_1} \ldots \mathrm{Tr}\, a^{p_{m-k}} \mathrm{Tr}\, a^{m_1} \mathrm{Tr}\, a^{n_1} \ldots \mathrm{Tr}\, a^{m_k} \mathrm{Tr}\, a^{n_k} \rangle, \tag{2.4}$$

[1] This matrix model 't Hooft coupling $\tilde{\lambda}$ differs by a constant from the Yang-Mills 't Hooft coupling $\lambda = 16\pi^2\tilde{\lambda}$, to be introduced in the next section.

Since the planar free energy scales like N^2, we want to extract the part of the correlator that scales like N^{2-m+k}. Consider a contribution given by the product of s connected correlators of sizes $r_1, \ldots, r_s$. This product scales like $\mathrm{N}^{2s-(r_1+\cdots+r_s)}$, and since $r_1 + \cdots + r_s = m + k$, we find that $s = k + 1$.

We have learned that when there are k double traces in $\langle S^m \rangle$, the terms that scale as N^2 are products of $k+1$ connected correlators. As in [27], we can associate a graph to this product of connected correlators, with one vertex per correlator and one edge per double trace. Not all these terms contribute to the free energy, they must survive the logarithm. The argument from [27] still goes through, and the terms that survive are those whose graph is a tree, see [27] for the details of the argument.

However, there are various differences with respect to the case of a potential with just double traces. Now at order $\langle S^m \rangle$ we must consider trees with $0 \leqslant k \leqslant m$ edges. The case $k = 0$ corresponds to the connected correlator of just single traces; the term $k = m$ corresponds to the case with just double traces. So, for fixed k, we must sum over all the ways to distribute the $m - k$ single traces in the $k + 1$ correlators. More explicitly, the planar Free Energy for the family of theories such as (2.2) is given by[2]

$$\mathcal{F} = \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \sum_{k=0}^{m} \binom{m}{k} \sum_{p_1,\ldots,p_{m-k}} c_{p_1} \cdots c_{p_{m-k}} \sum_{\substack{i_1,\ldots,i_k \\ j_1,\ldots,j_k}} c_{i_1 j_1} \cdots c_{i_k j_k} \sum_{\substack{\text{directed trees with} \\ \text{k labeled edges}}} \sum_{\substack{\text{single trace} \\ \text{insertions}}} \prod_{i=1}^{k+1} V_i \tag{2.5}$$

where V_i is the planar connected correlator on the i-th vertex on the tree, that contains the following operators: $\mathrm{tr}\, a^{i_s}$ if the directed edge labelled s leaves that vertex; $\mathrm{tr}\, a^{j_s}$ if the directed edge labelled s arrives at that vertex; any single trace operators inserted on that vertex.

It is worth comparing this result with the one obtained in [27], valid for potentials with only double trace contributions in the large N limit. First, as already mentioned, now the sum at order m involves trees with $k \leqslant m$ edges. Second, in the case of just double-trace terms in the action, it is easy to argue [27] that double traces of odd powers don't contribute to the planar limit. The argument was based on the fact that a planar connected correlator must involve an even number of odd powers. However, the argument doesn't apply now, because the single traces in (2.2) can also have odd powers. So (2.5) includes also contributions coming from double traces of odd powers.

We have succeeded in writing the planar free energy as a sum over products of planar connected correlators of the free Gaussian model. To proceed, we need the explicit form of these planar connected correlators. They are known in some cases, but not all. For an arbitrary n-point function of even-power operators [38, 39]

$$\langle \mathrm{Tr} a^{2k_1} \cdots \mathrm{Tr} a^{2k_n} \rangle_c = \tilde{\lambda}^d \frac{(d-1)!}{(d-n+2)!} \prod_{i=1}^{n} \frac{(2k_i)!}{k_i!(k_i-1)!} \mathrm{N}^{2-n} \tag{2.6}$$

[2]Note that we are not including the free energy of the Gaussian model in this expression.

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where $d = \sum k_i$. Let us introduce some notation for the numerical coefficients

$$\mathcal{V}(k_1, \cdots, k_n) = \frac{(d-1)!}{(d-n+2)!} \prod_{i=1}^{n} \frac{(2k_i)!}{k_i!(k_i-1)!} \,. \tag{2.7}$$

The planar 2-point function of odd operators is

$$\langle \mathrm{Tr} a^{2k_1+1} \mathrm{Tr} a^{2k_2+1} \rangle_c = \frac{\tilde{\lambda}^{k_1+k_2+1}}{k_1+k_2+1} \frac{(2k_1+1)!}{(k_1!)^2} \frac{(2k_2+1)!}{(k_2!)^2} \tag{2.8}$$

A general formula is also known for the case of correlators involving and arbitrary number of even operators with two odd insertions [38, 39]. Finding the generalization to more than two odd insertions is an interesting open question.

As a check of (2.5), consider the expansion up to $m = 2$ in the case of just even traces

$$\begin{aligned}\mathcal{F} = &-\sum_p c_{2p} \frac{(2p)!}{(p+1)!p!} \tilde{\lambda}^p - \sum_{ij} c_{2i\,2j} \frac{(2i)!(2j)!}{(i+1)!i!(j+1)!j!} \tilde{\lambda}^{i+j} + \frac{1}{2} \sum_{p,q} \frac{c_{2p}\, c_{2q}}{p+q} \frac{(2p)!(2q)!}{(p-1)!p!(q-1)!q!} \tilde{\lambda}^{p+q} \\ &+2 \sum_p \sum_{ij} \frac{c_{2p}\, c_{2i\,2j}}{p+i} \frac{(2p)!}{(p-1)!p!} \frac{(2i)!}{(i-1)!i!} \frac{(2j)!}{(j+1)!j!} \tilde{\lambda}^{p+i+j} \\ &+2 \sum_{i,j,k,l} \frac{c_{2i,2j}\, c_{2k,2l}}{j+k} \frac{(2i)!}{(i+1)!i} \frac{(2j)}{(j-1)!j!} \frac{(2k)!}{(k-1)!k!} \frac{(2l)!}{(l+1)!l!} \tilde{\lambda}^{i+j+k+l} + \dots\end{aligned} \tag{2.9}$$

this model is now the one in appendix B of [37] and the expression above reproduces all the relevant terms in [37].

2.1 2- and 3-point functions

In preparation for the next section, we now compute the planar 2- and 3- point functions of the matrix model with interaction terms (2.2). Note that the expression (2.5) contains a sum over directed trees with k labeled edges and by taking a derivative with respect to any coupling we are selecting from the sum the trees that contain, in one of the vertices, an insertion of the operator associated to the aforementioned coupling. This distinguishes one of the vertices from the rest, turning the tree into a rooted tree, where the root vertex indicates the correlator that contains the selected operator. In the case of higher point functions we will have as many roots as operators we wish to consider, while bearing in mind that we can have multiple roots in the same vertex of the tree.[3]

Let us be more explicit for the correlation functions that we are interested in. For 2-point functions (2.5) reduces to

$$\langle \mathrm{Tr} a^p \mathrm{Tr} a^q \rangle = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{\substack{i_1,\dots,i_m \\ j_1,\dots,j_m}} c_{i_1 j_1} \cdots c_{i_m j_m} \sum_{\substack{\text{double rooted} \\ \text{directed trees} \\ \text{with m labeled edges}}} \prod_{i=1}^{m+1} V_i \,, \tag{2.10}$$

with the understanding that the tree with $m = 0$ edges is just a single vertex, corresponding to the connected Gaussian two-point function, $\langle \mathrm{Tr} a^p \mathrm{Tr} a^q \rangle_c$. The two distinguished vertices

[3]This is similar to a coloring of a given tree, but in that case it is not possible to paint the same vertex with multiple different colors.

— roots — of the tree correspond to the insertions of Tr a^p and Tr a^q (they can be inserted in the same vertex). The derivation of this formula guarantees its validity for $p > 2, q > 2$, but it is possible to check that it is also valid for $p = 2$ and/or $q = 2$ by using this relation

$$\langle \mathrm{Tr} a^2 \mathrm{Tr} a^{2k_2} \dots \mathrm{Tr} a^{2k_n} \rangle_c = \frac{2}{\mathrm{N}} \tilde{\lambda}^2 \partial_{\tilde{\lambda}} \langle \mathrm{Tr} a^{2k_2} \dots \mathrm{Tr} a^{2k_n} \rangle_c \tag{2.11}$$

which follows from (2.6). To illustrate (2.10), let's compute the first terms. We assume that $c_{pq} = c_{qp}$. In the even-even case, $\langle \mathrm{Tr} a^{2m} \mathrm{Tr} a^{2n} \rangle$ the first non-trivial contribution comes from two types of products of planar connected Gaussian correlators: $\langle \mathrm{Tr} a^{2m} \mathrm{Tr} a^{2n} \mathrm{Tr} a^{2p} \rangle_c \langle \mathrm{Tr} a^{2q} \rangle$, and $\langle \mathrm{Tr} a^{2m} \mathrm{Tr} a^{2p} \rangle_c \langle \mathrm{Tr} a^{2q} \mathrm{Tr} a^{2n} \rangle_c$. Both types of products correspond to a tree with a single edge and two vertices; in the first case, the two single trace operators are both inserted in the same vertex, and in the second case, each single trace operator in inserted in a different vertex. All in all,

$$\begin{aligned} \left\langle \mathrm{Tr} a^{2m} \mathrm{Tr} a^{2n} \right\rangle &= \frac{1}{m+n} \frac{(2m)!}{(m-1)!m!} \frac{(2n)!}{(n-1)!n!} \tilde{\lambda}^{m+n} \\ &- 2 \sum_{p,q} c_{2p,2q} \frac{\tilde{\lambda}^{p+q+n+m} (2p)!(2q)!(2m)!(2n)!}{p!(p-1)!q!(q-1)!m!(m-1)!n!(n-1)!} \left(\frac{1}{(p+1)p} + \frac{1}{(p+n)(q+m)} \right) + \cdots \end{aligned} \tag{2.12}$$

The odd-odd two-point function works similarly, with the difference that now when both single trace insertions are in different correlators, the double trace must be odd-odd, and if they are in the same correlator, the double-trace must be even-even,

$$\begin{aligned} \left\langle \mathrm{Tr} a^{2m+1} \mathrm{Tr} a^{2n+1} \right\rangle = {} & \frac{\tilde{\lambda}^{m+n+1}}{m+n+1} \frac{(2m+1)!(2n+1)!}{(m!)^2 (n!)^2} \\ & -2\tilde{\lambda}^{m+n+1} \frac{(2m+1)!(2n+1)!}{n!^2 m!^2} \sum_{ij} \Bigg(c_{2i+1,2j+1} \frac{\tilde{\lambda}^{i+j+1} (2i+1)!(2j+1)!}{(m+i+1)(i!)^2 (n+j+1)(j!)^2} \\ & + c_{2i,2j} \frac{\tilde{\lambda}^{i+j} (2i)!(2j)!}{(i!)(i-1)!(j+1)!j!} \Bigg) + \cdots \end{aligned} \tag{2.13}$$

In the case of 3-point functions we have

$$\langle \mathrm{Tr} a^p \mathrm{Tr} a^q \mathrm{Tr} a^l \rangle = \mathrm{N}^{-1} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{\substack{i_1,\dots,i_m \\ j_1,\dots,j_m}} c_{i_1 j_1} \cdots c_{i_m j_m} \sum_{\substack{\text{triple rooted} \\ \text{directed trees} \\ \text{with m labeled edges}}} \prod_{i=1}^{m+1} V_i \tag{2.14}$$

This formula applies to the two non-trivial cases: three even powers, and two odd powers and an even one. Let's illustrate it with the first case,

$$\begin{aligned} \langle \mathrm{Tr} a^{2p} \mathrm{Tr} a^{2q} \mathrm{Tr} a^{2l} \rangle &= \mathrm{N}^{-1} \frac{(2p)!(2q)!(2l)!}{p!(p-1)!q!(q-1)!l!(l-1)!} \tilde{\lambda}^{p+q+l} \\ & \left[1 - 2 \sum_{ij} c_{2i,2j} \tilde{\lambda}^{i+j} \frac{(2i)!(2j)!}{i!(i-1)!j!(j-1)!} \left(\frac{p+q+l+i-1}{(j+1)j} + \frac{1}{p+j} + \frac{1}{q+j} + \frac{1}{l+j} \right) + \dots \right] \end{aligned} \tag{2.15}$$

In the next section, we will evaluate these generic expressions for the specific matrix model of $\mathcal{N} = 2$ SQCD. As we will see, they reproduce and generalize known results [12], thus providing a non-trivial check of their validity.

3 Chiral correlators on S^4

In this section we derive planar 2- and 3- point functions of single-trace chiral primary operators of $\mathcal{N} = 2$ SQCD on S^4, using the results derived in the previous section.

Let us first quickly recall some basic facts about 4d $\mathcal{N} = 2$ SCFT theories and their chiral primary operators [21]. The generators of the superconformal algebra are given by the bosonic generators $P_\mu, K_\mu, M_{\mu\nu}, D$, the supercharges $Q^a_\alpha, \bar{Q}^a_{\dot\alpha}$, its superconformal partners $S^a_\alpha, \bar{S}^a_{\dot\alpha}$ and the generators of the $\mathrm{SU}(2) \times \mathrm{U}(1)$ R-symmetry. Highest weight representations are labelled by the quantum numbers $(\Delta; j_l, j_r; s; R)$ of the highest weight state under dilatations, the Lorentz group and the $SU2) \times \mathrm{U}(1)$ R-symmetry group. These highest weight states are created by superconfornal primary operators, annihilated by all $S^a_\alpha, \bar{S}^a_{\dot\alpha}$.

Among all of the superconformal primaries, there exists an interesting class given by the ones that are chiral, defined as $[\bar{Q}^a_{\dot\alpha}, O] = 0$. CPOs have $j_r = s = 0$ and $\Delta = R/2$. For Lagrangian SCFTs, one can further argue that $j_l = 0$, so they are Lorentz scalars [40]. Anti-chiral primary operators $\bar{O}$ are similarly defined, and satisfy $\Delta = -R/2$. We will denote chiral operators on S^4 by Ω, reserving O for chiral operators on $\mathbb{R}^4$.

3.1 $\mathcal{N} = 2$ SCFTs on S^4

It is possible to place any $\mathcal{N} = 2$ SCFT on S^4. Supersymmetric regularization of the resulting divergences implies that the theory preserves a subalgebra $osp(2|4)$ of the flat space supersymmetry algebra [41]. In particular, the flat space $\mathrm{U}(1)_R$ symmetry is broken on S^4 [41]. As a consequence, there is no $\mathrm{U}(1)_R$ selection rule for correlation functions on S^4: one-point functions are not vanishing, and similarly, two-point functions of operators of different dimension can also be non-zero. In [6] it was shown that the partition function on S^4 can be identified with the Kähler potential for the Zamolodchikov metric of the conformal manifold of the theory. Thus, the two-point function of CPOs with $\Delta = 2$ can be obtained by taking derivatives of this partition function on S^4.

Supersymmetric localization allows to evaluate efficiently this partition function, and consequently this very particular two-point function of CPOs, by reducing it to a matrix model integral [22],

$$Z_{S^4}(\tau_{\mathrm{YM}}) = \int da\, e^{-\frac{8\pi^2}{g^2_{\mathrm{YM}}}\mathrm{Tr}(a^2)} \mathcal{Z}_{1-loop}(a)\, |\mathcal{Z}_{inst}(a,\tau)|^2 \tag{3.1}$$

where $\mathcal{Z}_{1-loop}$ a factor arising from a 1-loop computation and $\mathcal{Z}_{inst}$ is the instanton contribution, that it is usually assumed to be negligible in the large N limit.

These results were extended in [7], where a method to exactly compute correlation functions of chiral primary operators on S^4 was developed. The starting point is to consider a deformation of the theory on S^4 that involves new couplings, one per generator of the chiral ring of the theory. This deformed SCFT still preserves $osp(2|4)$. It was argued in [7] that extremal correlators on S^4 of the undeformed theory can be obtained by taking derivatives of the partition function of the deformed theory.

Again, supersymmetric localization allows to efficiently evaluated this new partition function, and thus arbitrary extremal correlators. Indeed, it was proven in [7] that the deformed partition function can be obtained from a matrix model integral of the form

$$Z_{S^4} = \int da\, |e^{i\sum_{n=1}^{m}\pi^{n/2}\tau_n \mathrm{Tr}(a^n)}|^2 \mathcal{Z}_{1-loop}(a)\, |\mathcal{Z}_{inst}(a,\tau,\tau_n)|^2 \tag{3.2}$$

with τ_n a holomorphic coupling and $\tau_2 = \tau_{\mathrm{YM}} = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$. Note that the 1-loop partition function does not depend on the new couplings τ_n, but the instanton partition function does. The key point is that correlation functions of chiral operators on S^4 are given by correlation functions of this matrix model,

$$\langle \Omega_p \Omega_q \rangle_{S^4} = \langle \mathrm{Tr}a^p\, \mathrm{Tr}a^q \rangle_{\mathrm{MM}} \tag{3.3}$$

and similarly for higher n-point functions. This fixes the normalization of the CPOs.

In this work, we focus on the large N limit of these correlators, and that implies a number of simplifications: first, we can restrict the terms we add to the action to single-trace CPOs; second, we will neglect the instanton contribution, setting $\mathcal{Z}_{inst}(a,\tau,\tau_n) = 1$. We thus rewrite the deformation as

$$S = -i\sum_{n=2}^{m} \pi^{\frac{n}{2}}(\tau_n - \bar{\tau}_n)\mathrm{Tr}a^n = \frac{8\pi^2}{g^2}\mathrm{Tr}a^2 - i\sum_{n=3}^{m}\pi^{\frac{n}{2}}(\tau_n - \bar{\tau}_n)\mathrm{Tr}a^n\,, \tag{3.4}$$

where now we can identify $g = \frac{g_{\mathrm{YM}}^2}{16\pi^2}$ and we recognize the single trace deformation to be the one in (2.2). Following [10, 25–27] it is possible to rewrite $\mathcal{Z}_{1-loop} = e^{-S_{\mathrm{int}}}$, with S_{int} given by a sum of single and double trace terms. For $\mathcal{N} = 2$ SQCD

$$S_{\mathrm{int}} = \sum_{n=2}^{\infty} \frac{\zeta(2n-1)(-1)^n}{n}\left[\sum_{k=1}^{n-1}\binom{2n}{2k}\mathrm{Tr}a^{2(n-k)}\mathrm{Tr}a^{2k} - \sum_{k=1}^{n-2}\binom{2n}{2k+1}\mathrm{Tr}a^{2(n-k)-1}\mathrm{Tr}a^{2k+1}\right], \tag{3.5}$$

To sum up, the deformation of the $\mathcal{N} = 2$ SCFT, together with the rewriting of the 1-loop determinant as an effective action, show that the relevant matrix model is of the type (2.2) analyzed in the previous section.

3.2 Chiral correlators in $\mathcal{N} = 4$

As a warm-up, let's first recover the planar chiral 2- and 3- point functions of $\mathcal{N} = 4$ SU(N) SYM on S^4 with our techniques. In this case, supersymmetric localization reduces to the Gaussian matrix model, since the one-loop and the instanton contributions are trivial, $\mathcal{Z}_{1-loop} = 1$, $\mathcal{Z}_{inst} = 1$. Thus, the planar 2- and 3-point functions on S^4 are just particular cases of (2.6) and (2.8). Recalling the relation $16\pi^2\tilde{\lambda} = \lambda$ between the matrix model and the Yang-Mills 't Hooft couplings, we have

$$\begin{aligned}\langle \Omega_{2n}\Omega_{2m}\rangle &= \left(\frac{\lambda}{16\pi^2}\right)^{n+m}\frac{1}{n+m}\frac{(2m)!}{m!(m-1)!}\frac{(2n)!}{n!(n-1)!}\\ \langle \Omega_{2n+1}\Omega_{2m+1}\rangle &= \left(\frac{\lambda}{16\pi^2}\right)^{m+n+1}\frac{1}{m+n+1}\frac{(2m+1)!}{(m!)^2}\frac{(2n+1)!}{(n!)^2}\end{aligned} \tag{3.6}$$

$$\langle \Omega_{2m}\Omega_{2n}\Omega_{2p}\rangle = \left(\frac{\lambda}{16\pi^2}\right)^{m+n+p}\frac{(2m)!}{m!(m-1)!}\frac{(2n)!}{n!(n-1)!}\frac{(2p)!}{p!(p-1)!}\mathrm{N}^{-1} \tag{3.7}$$

which agrees with the results obtained in [8].

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3.3 Chiral operators in truly $\mathcal{N} = 2$ theories

Turning now our attention to truly $\mathcal{N} = 2$ theories, we can identify the coefficients c_{ij} in (2.2) with the ones appearing in the effective action (3.5) as

$$c_{pq} = \binom{2p+2q}{2p} \frac{\zeta(2p+2q-1)(-1)^{p+q}}{p+q} \tag{3.8}$$

For Lagrangian $\mathcal{N} = 2$ SCFTs theories, the planar free energy (2.5) was explicitly computed in [27], and for $\mathcal{N} = 2$ SQCD it was found to be given by

$$\begin{aligned} F_0(\lambda) = \frac{1}{2}\log\lambda + \sum_{n=2}^{\infty}\left(-\frac{\lambda}{16\pi^2}\right)^n \sum_{\substack{\text{compositions of n} \\ \text{not containing 1}}} (-2)^m \frac{\zeta(2n_1-1)\ldots\zeta(2n_m-1)}{n_1\ldots n_m} \\ \sum_{k_1=1}^{n_1-1}\binom{2n_1}{2k_1}\cdots\sum_{k_m=1}^{n_m-1}\binom{2n_m}{2k_m} \sum_{\substack{\text{unlabeled trees} \\ \text{with m edges}}} \frac{1}{|\mathrm{Aut(T)}|}\mathcal{V}_1\ldots\mathcal{V}_{m+1} \end{aligned} \tag{3.9}$$

Let us first note that we can obtain $\langle\Omega_2\Omega_2\rangle$ by taking two derivatives of the free energy with respect to the exactly marginal coupling g_{YM} of the theory. We obtain

$$\begin{aligned} \langle\Omega_2\Omega_2\rangle = \frac{2\lambda^2}{(4\pi)^4} + \frac{4\lambda^2}{(4\pi)^4}\sum_{n=2}^{\infty} n(n+1)\left(-\frac{\lambda}{16\pi^2}\right)^n \sum_{\substack{\text{compositions of n} \\ \text{not containing 1}}} (-2)^m \frac{\zeta(2n_1-1)\ldots\zeta(2n_m-1)}{n_1\ldots n_m} \\ \sum_{k_1=1}^{n_1-1}\binom{2n_1}{2k_1}\cdots\sum_{k_m=1}^{n_m-1}\binom{2n_m}{2k_m} \sum_{\substack{\text{unlabeled trees} \\ \text{with m edges}}} \frac{1}{|\mathrm{Aut(T)}|}\mathcal{V}_1\ldots\mathcal{V}_{m+1} \end{aligned} \tag{3.10}$$

where by expanding we see that the first terms match with eq. (4.13) of [12].

For the general planar 2- and 3-point functions on S^4 we can now use the results derived last section from the matrix model, eqs. (2.10), (2.14)

$$\langle\Omega_p\Omega_q\rangle = \sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\sum_{\substack{i_1,\ldots,i_m \\ j_1,\ldots,j_m}} c_{i_1j_1}\cdots c_{i_mj_m} \sum_{\substack{\text{double rooted} \\ \text{directed trees} \\ \text{with m labeled edges}}} \prod_{i=1}^{m+1} V_i\,, \tag{3.11}$$

$$\langle\Omega_p\Omega_q\Omega_l\rangle = \mathrm{N}^{-1}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\sum_{\substack{i_1,\ldots,i_m \\ j_1,\ldots,j_m}} c_{i_1j_1}\cdots c_{i_mj_m} \sum_{\substack{\text{triple rooted} \\ \text{directed trees} \\ \text{with m labeled edges}}} \prod_{i=1}^{m+1} V_i \tag{3.12}$$

While in most of this work we explicitly display terms with a single value of the ζ function, our formulas capture also all terms with products of two or more values of ζ. To illustrate this point (see the appendix for further examples), let's compute the ζ_3^2 term for the 2-point function of two even single trace CPOs,

$$\begin{aligned}\langle \Omega_{2m}\Omega_{2n}\rangle =&\frac{1}{m+n}\frac{(2m)!}{(m-1)!m!}\frac{(2n)!}{(n-1)!n!}\left(\frac{\lambda}{16\pi^2}\right)^{m+n}\\&-12\zeta_3\left(\frac{\lambda}{16\pi^2}\right)^{m+n+2}\frac{(mn+m+n+3)(2m)!(2n)!}{(m+1)!(m-1)!(n+1)!(n-1)!}\\&+72\zeta_3^2\left(\frac{\lambda}{16\pi^2}\right)^{m+n+4}\frac{(3+m+n)(5+m+n+mn)(2m)!(2n)!}{(m+1)!(m-1)!(n+1)!(n-1)!}\end{aligned} \tag{3.13}$$

which agrees with (4.33), (4.34), (4.35) of [8].

4 Chiral correlators on $\mathbb{R}^4$

In the previous section, we have provided combinatorial expressions for the full planar perturbative series of 2- and 3- point functions of $\mathcal{N}=2$ superconformal theories on S^4. As discussed above, it is not straightforward to read off the chiral correlators on $\mathbb{R}^4$ directly from the previous results. In order to do so we need to disentangle the mixing induced by the conformal anomaly through a Gram-Schmidt orthogonalization procedure [7]. In general, a given operator of dimension Δ_n will mix with all the operators with Δ_m such that $m<n$ differs from n by an even integer. To find the relation between $\mathbb{R}^4$ and S^4 operators, first introduce the matrix of two point functions on S^4 defined by

$$C_{n,m}=\langle \Omega_n\Omega_m\rangle, \tag{4.1}$$

then, the $\mathbb{R}^4$ operator O_n is given by

$$O_n(a)=\Omega_n(a)-\sum_{p,q}C_{n,p}\left(C^{-1}_{(n)}\right)^{p,q}\Omega_q(a)\,, \tag{4.2}$$

While in the previous section we provided a combinatorial expression for planar chiral correlators on S^4, for the analogous correlators on $\mathbb{R}^4$ obtained through the Gram-Schmidt procedure, a combinatorial description is no longer apparent. As discussed before, correlators of $\Delta=2$ operators can be extracted directly from the partition function of $\mathcal{N}=2$ theories [6] which in turn admits an exact combinatorial expression [27] so let's start discussing 2-point functions on $\mathbb{R}^4$.

On $\mathbb{R}^4$ the $\mathrm{U}(1)_R$ selection rule implies that the only non-zero two-point functions are $\left\langle O_k\bar{O}_k\right\rangle$. Since $\left\langle O_2\bar{O}_2\right\rangle$ is the same as on S^4, in this case we do have a full planar perturbative expression

$$\begin{aligned}\left\langle O_2\bar{O}_2\right\rangle =&\frac{4\lambda^2}{(4\pi)^4}\sum_{n=2}^{\infty}n(n+1)\left(-\frac{\lambda}{16\pi^2}\right)^n\sum_{\substack{\text{compositions of n}\\ \text{not containing 1}}}(-2)^m\frac{\zeta(2n_1-1)\dots\zeta(2n_m-1)}{n_1\dots n_m}\\&\sum_{k_1=1}^{n_1-1}\binom{2n_1}{2k_1}\dots\sum_{k_m=1}^{n_m-1}\binom{2n_m}{2k_m}\sum_{\substack{\text{unlabeled trees}\\ \text{with m edges}}}\frac{1}{|\mathrm{Aut(T)}|}\mathcal{V}_1\dots\mathcal{V}_{m+1}\end{aligned} \tag{4.3}$$

In the appendix we present the first few orders, that match with the results eq. (4.13) of [12]. For future reference, we note that all the terms in $\langle O_2\bar{O}_2\rangle$ with a single value of the ζ function can be rewritten as follows

$$\langle O_2\bar{O}_2\rangle = 2\left(\frac{\lambda}{16\pi^2}\right)^2\left(1-4\sum_{n=2}^{\infty}\frac{\zeta_{2n-1}}{n}\left(\frac{-\lambda}{16\pi^2}\right)^n\binom{2n}{n}\left[\binom{2n}{n+2}+\binom{2n}{n+1}-n\right]+\dots\right) \quad (4.4)$$

To evaluate $\langle O_k\bar{O}_k\rangle$ on $\mathbb{R}^4$ for $k>2$ we need to run the Gram-Schmidt procedure (4.2). Let us consider the first non-trivial case.

$$O_4 = \Omega_4 - \frac{C_{4,2}}{C_{2,2}}\Omega_2 \quad (4.5)$$

thus in order to compute $\langle O_4\bar{O}_4\rangle$ we require

$$\langle O_4\bar{O}_4\rangle = \langle\Omega_4\Omega_4\rangle - \frac{C_{4,2}^2}{C_{2,2}} \quad (4.6)$$

The fact that $\langle\Omega_2\Omega_2\rangle$ appears in the denominator complicates the task of finding a closed expression for $\langle O_4\bar{O}_4\rangle$. For concreteness, we will limit ourselves to present all the terms with a single value of the ζ function. Collecting all such terms we deduce

$$\langle O_4\bar{O}_4\rangle = 4\left(\frac{\lambda}{16\pi^2}\right)^4\left(1-8\sum_{n=2}^{\infty}\frac{\zeta_{2n-1}}{n}\left(\frac{-\lambda}{16\pi^2}\right)^n\binom{2n}{n}\left[\binom{2n}{n+4}+\binom{2n}{n+1}-n\right]+\dots\right) \quad (4.7)$$

We can repeat the same procedure for $\langle O_6\bar{O}_6\rangle$. A longer computation yields

$$\langle O_6\bar{O}_6\rangle = 6\left(\frac{\lambda}{16\pi^2}\right)^6\left(1-12\sum_{n=2}^{\infty}\frac{\zeta_{2n-1}}{n}\left(\frac{-\lambda}{16\pi^2}\right)^n\binom{2n}{n}\left[\binom{2n}{n+6}+\binom{2n}{n+1}-n\right]+\dots\right) \quad (4.8)$$

As a first test, these expressions reproduce the terms with a single value of ζ in [12]. While we are not writing them down, one can also check that the first terms with a product of two ζ also agree with the result of [12]. Now, looking at the explicit expressions (4.4), (4.7), (4.8) a pattern appears to emerge, so we are led to put forward the following conjecture for generic even k,

$$\langle O_k\bar{O}_k\rangle \stackrel{?}{=} k\left(\frac{\lambda}{16\pi^2}\right)^k\left(1-2k\sum_{n=2}^{\infty}\frac{\zeta_{2n-1}}{n}\left(\frac{-\lambda}{16\pi^2}\right)^n\binom{2n}{n}\left[\binom{2n}{n+k}+\binom{2n}{n+1}-n\right]+\dots\right) \quad (4.9)$$

where the dots stand for terms with two or more values of ζ. Using the Mathematica notebook available in [12], we have checked that this conjecture reproduces the first terms of $\langle O_8\bar{O}_8\rangle$. As for $\left\langle O_k\bar{O}_k\right\rangle$ for odd k, a bit of trial and error with the results available in [12] leads to the following generalized conjecture

$$\langle O_k\bar{O}_k\rangle \stackrel{?}{=} k\left(\frac{\lambda}{16\pi^2}\right)^k\left(1-2k\sum_{n=2}^{\infty}\frac{\zeta_{2n-1}}{n}\left(\frac{-\lambda}{16\pi^2}\right)^n\binom{2n}{n}\left[(-1)^k\binom{2n}{n+k}+\binom{2n}{n+1}-n\right]+\dots\right) \quad (4.10)$$

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The conjecture (4.10) is appealingly simple. The terms in square brackets has a k-dependent contribution, that is non-vanishing only at orders $n \geqslant k$, plus a universal, k-independent, contribution. Furthermore, the factor

$$\frac{\zeta_{2n-1}}{n}\left(\frac{-\lambda}{16\pi^2}\right)^n\binom{2n}{n}$$

coincides with the planar value of certain Feynman diagrams identified in section 7.3 of [11] (see also [12]). These ingredients hint at a diagrammatic derivation of (4.10). Finally, let us mention that [11, 13] have developed very efficient techniques to obtain analytic results for certain $\mathcal{N}=2$ SCFTs, which currently don't include $\mathcal{N}=2$ SCQD. In these works, a crucial role is played by an infinite matrix that can be written as an integral over Bessel functions. It is straightforward to check that our conjecture (4.10) can be written similarly,

$$\begin{aligned}&\sum_{n=2}^{\infty}\frac{\zeta_{2n-1}}{n}\left(\frac{-\lambda}{16\pi^2}\right)^n\binom{2n}{n}\left[(-1)^k\binom{2n}{n+k}+\binom{2n}{n+1}-n\right]=\\&\int_0^\infty dw\frac{J_k(\frac{w\sqrt{\lambda}}{\pi})^2-J_1(\frac{w\sqrt{\lambda}}{\pi})^2+\frac{w\sqrt{\lambda}}{2\pi}J_1(\frac{w\sqrt{\lambda}}{\pi})}{2w\sinh^2 w}\end{aligned} \tag{4.11}$$

It would be interesting to extend the techniques of [11, 13] to arbitrary $\mathcal{N}=2$ Lagrangian SCFTs; this would allow to prove (4.10) and extend it to terms with two or more values of ζ.

Let's now switch to the determination of planar 3-point functions on $\mathbb{R}^4$, repeating the same procedure. The first non trivial extremal 3-point function is given by

$$\langle O_2O_2\bar{O}_4\rangle=\langle\Omega_2\Omega_2\Omega_4\rangle-\frac{C_{4,2}}{C_{2,2}}\langle\Omega_2\Omega_2\Omega_2\rangle\,, \tag{4.12}$$

Upon collecting all the terms with a single ζ we obtain

$$\langle O_2O_2\bar{O}_4\rangle=\mathrm{N}^{-1}\frac{16\lambda^4}{(4\pi)^8}\left(1-2\sum_{n=2}^{\infty}\zeta_{2n-1}\left(\frac{-\lambda}{16\pi^2}\right)^n\binom{2n}{n}\left(\frac{(2n+1)!(n^2+3n+12)}{(n+3)!n!}-(n+3)\right)\right)+\dots \tag{4.13}$$

To get rid of the ambiguity associated to the normalization of the CPOs, it is convenient to define the normalized 3-point functions,

$$\langle O_{k_1}O_{k_2}\bar{O}_{k1+k2}\rangle_n=\frac{\langle O_{k_1}O_{k_2}\bar{O}_{k1+k2}\rangle}{\mathrm{N}\sqrt{\langle O_{k_1}\bar{O}_{k_1}\rangle\langle O_{k_2}\bar{O}_{k_2}\rangle\langle O_{k_1+k_2}\bar{O}_{k1+k2}\rangle}} \tag{4.14}$$

After doing so, we find

$$\begin{aligned}&\frac{\langle O_2O_2\bar{O}_4\rangle_n}{\sqrt{2\cdot2\cdot4}}\\&=\mathrm{N}^{-1}\left[1-\sum_{n=2}^{\infty}\zeta_{2n-1}\left(\frac{-\lambda}{16\pi^2}\right)^n\binom{2n}{n}\left[\binom{2n}{n+2}+\binom{2n}{n+2}+\binom{2n}{n+4}+(n-1)(C_n-2)\right]+\dots\right]\end{aligned} \tag{4.15}$$

Repeating all these steps for $\langle O_2 O_4 \bar{O}_6\rangle_n$ we find

$$\frac{\langle O_2 O_4 \bar{O}_6\rangle_n}{\sqrt{2\cdot 4\cdot 6}} \tag{4.16}$$
$$= \mathrm{N}^{-1}\left[1-\sum_{n=2}^{\infty}\zeta_{2n-1}\left(\frac{-\lambda}{16\pi^2}\right)^n\binom{2n}{n}\left[\binom{2n}{n+2}+\binom{2n}{n+4}+\binom{2n}{n+6}+(n-1)(C_n-2)\right]+\dots\right]$$

The first terms of these expressions reproduce the results presented in [5]. These two computations suggest the following general conjecture for planar 3-point functions of even-dimensional operators

$$\frac{\langle O_{k_1} O_{k_2} \bar{O}_{k1+k2}\rangle_n}{\sqrt{k_1\cdot k_2\cdot (k_1+k_2)}} \stackrel{?}{=} \mathrm{N}^{-1}\left[1-\sum_{n=2}^{\infty}\left(\frac{-\lambda}{16\pi^2}\right)^n\zeta_{2n-1}\binom{2n}{n}\right. \tag{4.17}$$
$$\left.\left(\binom{2n}{n+k_1}+\binom{2n}{n+k_2}+\binom{2n}{n+k_1+k_2}+(n-1)(\mathcal{C}_n-2)\right)\right]+\dots$$

As a first check, this conjecture correctly reproduces for arbitrary even k_1, k_2 the ζ_3 term found in [5]. We have checked that it also correctly reproduces the first terms of $\langle O_4 O_4 \bar{O}_8\rangle_n$ and $\langle O_4 O_6 \bar{O}_{10}\rangle_n$.[4] Again, we find the form of this conjecture remarkably simple, and suspect that it hints at the existence of a direct derivation of these results, that bypasses going through the route of computing first correlators on S^4. Finally, it is also possible to have a non-vanishing 3-point function involving two odd and one even operators. Motivated by (4.10) a possible guess is that in this case the $\binom{2n}{n+k}$ factors in (4.17) pick up a minus sign for odd k, but we haven't checked explicitly.

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A Explicit expansions

In the main text, when writing explicit results, we have mostly restricted to displaying only terms with a single value of the ζ function. Our techniques can equally well produce

[4] We have computed the frist terms of these 3-point functions explicitly, using the Gram-Schmidt procedure. The results obtained agree with the conjecture (4.17). However, we do not agree with the coefficient of ζ_7 for $\langle O_4 O_6 \bar{O}_{10}\rangle_n$ presented in [5].

terms with products of ζs. In this appendix we present two examples, the planar limit of $\langle O_2\overline{O_2}\rangle$ and $\langle O_2O_2\bar{O}_4\rangle$ on $\mathbb{R}^4$.

$$
\begin{aligned}
\langle O_2\overline{O_2}\rangle = \frac{2\lambda^2}{(4\pi)^4}\Bigg[& 1-\frac{9}{4}\zeta_3\frac{\lambda^2}{(2\pi)^4}+\frac{15}{2}\zeta_5\frac{\lambda^3}{(2\pi)^6} \\
& +\frac{5}{8}\left(9\zeta_3^2-35\zeta_7\right)\frac{\lambda^4}{(2\pi)^8}-\frac{45}{64}\left(60\zeta_3\zeta_5-91\zeta_9\right)\frac{\lambda^5}{(2\pi)^{10}} \\
& +\frac{21}{512}\left(-360\zeta_3^3+3360\zeta_7\zeta_3+1900\zeta_5^2-4697\zeta_{11}\right)\frac{\lambda^6}{(2\pi)^{12}} \\
& +\frac{7}{256}\left(20\left(\zeta_5\left(324\zeta_3^2-917\zeta_7\right)-819\zeta_3\zeta_9\right)+21879\zeta_{13}\right)\frac{\lambda^7}{(2\pi)^{14}} \\
& +\frac{27}{4096}\Bigg(6048\zeta_3^4-94080\zeta_7\zeta_3^2+528\left(427\zeta_{11}-200\zeta_5^2\right)\zeta_3 \\
& +140\left(861\zeta_7^2+1744\zeta_5\zeta_9\right)-289575\zeta_{15}\Bigg)\frac{\lambda^8}{(2\pi)^{16}} \\
& -\frac{15}{16384}\Bigg(560\left(\zeta_5\left(1296\zeta_3^3-9333\zeta_7\zeta_3-1750\zeta_5^2\right)+9\left(1091\zeta_7-468\zeta_3^2\right)\zeta_9\right) \\
& +5775660\zeta_5\zeta_{11}+5513508\zeta_3\zeta_{13}-6804369\zeta_{17}\Bigg)\frac{\lambda^9}{(2\pi)^{18}} \\
& -\frac{11}{32768}\Bigg(326592\zeta_3^5-7257600\zeta_7\zeta_3^3+4860\left(4697\zeta_{11}-2500\zeta_5^2\right)\zeta_3^2 \\
& +300\left(80164\zeta_7^2+162876\zeta_5\zeta_9-173745\zeta_{15}\right)\zeta_3-23481360\zeta_9^2 \\
& +525\zeta_7\left(51660\zeta_5^2-92851\zeta_{11}\right)-53088750\zeta_5\zeta_{13}+61708504\zeta_{19}\Bigg)\frac{\lambda^{10}}{(2\pi)^{20}}\Bigg]
\end{aligned}
\tag{A.1}
$$

$$
\begin{aligned}
\langle O_2O_2\bar{O}_4\rangle = 4\mathrm{N}^{-1}\Bigg(& 1-\frac{3\zeta_3\lambda^2}{64\pi^4}+\frac{45\zeta_5\lambda^3}{512\pi^6}+\frac{3(72\zeta_3^2-1085\zeta_7)}{32768\pi^8}\lambda^4+\frac{45(287\zeta_9-64\zeta_3\zeta_5)}{131072\pi^{10}}\lambda^5 \\
& +\frac{3\left(16164\zeta_3^3+19075\zeta_7\zeta_3+10500\zeta_5^2-65681\zeta_{11}\right)\lambda^6}{2097152\pi^{12}} \\
& -\frac{15\left(\zeta_5\left(80550\zeta_3^2+36911\zeta_7\right)+34272\zeta_3\zeta_9-99099\zeta_{13}\right)\lambda^7}{16777216\pi^{14}} \\
& +\Big(3822336\zeta_3^4+47231184\zeta_7\zeta_3^2+32\left(1633325\zeta_5^2+738969\zeta_{11}\right)\zeta_3 \\
& +245\left(48493\zeta_7^2+100032\zeta_5\zeta_9-243672\zeta_{15}\right)\Big)\frac{3\lambda^8}{2147483648\pi^{16}}+\cdots\Bigg)
\end{aligned}
\tag{A.2}
$$

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CHAPTER 4

On the planar Free Energy of Matrix models

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On the planar free energy of matrix models

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Abstract: In this work we obtain the planar free energy for the Hermitian one-matrix model with various choices of the potential. We accomplish this by applying an approach that bypasses the usual diagonalization of the matrices and the introduction of the eigenvalue density, to directly zero in the evaluation of the planar free energy. In the first part of the paper, we focus on potentials with finitely many terms. For various choices of potentials, we manage to find closed expressions for the planar free energy, and in some cases determine or bound their radius of convergence as a series in the 't Hooft coupling. In the second part of the paper we consider specific examples of potentials with infinitely many terms, that arise in the study of $\mathcal{N}=2$ super Yang-Mills theories on S^4, via supersymmetric localization. In particular, we manage to write the planar free energy of two non-conformal examples: SU(N) with $N_f < 2N$, and $\mathcal{N}=2^*$.

Keywords: $1/N$ Expansion, Matrix Models, Supersymmetric Gauge Theory

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Contents

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1 Introduction

In this work we study the planar limit of Hermitian matrix models with a single Hermitian N×N matrix ϕ,

$$\bar{Z}(g_s, c_i, c_{ij}) = \frac{Z(g_s, c_i, c_{ij})}{Z(g_s, 0, 0)} = \frac{\int d\phi\, e^{-\frac{1}{2g_s}\mathrm{tr}\,\phi^2 + V(\phi)}}{\int d\phi\, e^{-\frac{1}{2g_s}\mathrm{tr}\,\phi^2}}\,, \tag{1.1}$$

with a potential given by an arbitrary number of single- and double-trace terms of even powers of ϕ,

$$V(\phi) = \mathrm{N}\sum_i c_{2i}\mathrm{tr}\,\phi^{2i} + \sum_{i,j} c_{2i\,2j}\mathrm{tr}\,\phi^{2i}\mathrm{tr}\,\phi^{2j}\,, \tag{1.2}$$

where the sums can contain either a finite or an infinite number of terms and the coefficients $c_{2i}, c_{2i\,2j}$ are N-independent and arbitrary. The particular case where the potential has just a finite number of single-trace terms and no double-trace terms has been extensively studied for its relevance to various problems, from graph enumeration [1] to two-dimensional quantum gravity [2, 3]. Models with a finite number of double-trace terms in the potential have also been considered in the context of two-dimensional quantum gravity [4]. On the other hand, potentials of the form (1.2) with an infinite number of terms appear in statistical physics, in Chern-Simons theories with matter, or in duals to M-theory on certain backgrounds as reviewed in [5]. More recently, it has been realized [6–9] that supersymmetric localization [10] reduces the evaluation of certain observables of four dimensional $\mathcal{N}=2$ super Yang-Mills (SYM) theories on S^4 to matrix models that can be recast to take the form of (1.2).

One of the basic objects of interest in the study of these matrix models is the free energy

$$\mathcal{F} = F - F_{\text{gaussian}} = \log \bar{Z} = \log Z - \log Z_{\text{gaussian}} \tag{1.3}$$

and since the groundbreaking work [11] it has been understood that one can organize the perturbative evaluations of the free energy in a double series expansion

$$\mathcal{F} = \sum_{g\geqslant 0} \mathrm{N}^{2-2g}\mathcal{F}_g(t) = \sum_{g\geqslant 0} \mathrm{N}^{2-2g} \sum_{h\geqslant 0} F_{g,h} t^h \tag{1.4}$$

where $t = g_s N$ is the 't Hooft coupling of the matrix model. Over the years, many powerful methods have been developed to evaluate these expansions for various potentials (see [12–15] for reviews). It is however fair to note that there are preciously few potentials for which closed forms for the free energy are known. A first question is then whether one can obtain a closed form expression for the free energy — or its terms in the $1/\mathrm{N}$ expansion — for different choices of the potential. A second question that (1.4) immediately raises is the convergence of the perturbative series that appear. In Quantum Field Theory, the exponential growth of the number of Feynman diagrams at every order in the perturbative expansion implies that the default expectation is that, barring unusual cancellations, perturbative series are not convergent, but asymptotic. On the other hand, in the $1/\mathrm{N}$ expansion (1.4), the number of diagrams at a fixed order in $1/\mathrm{N}$ is drastically reduced, so it grows only as a power law [16]. This opens up the possibility that these series have a finite radius of convergence [16–18].

The purpose of this work is to present some progress on these two questions for various choices of potentials in (1.2). In the first part of this work, we focus on potentials with a finite number of terms, and for various examples we find a closed expression for the planar free energy $\mathcal{F}_0(t)$ as an all-order perturbative series in the 't Hooft coupling. In some of these cases, we either determine or bound the radius of convergence of the resulting series. In the second part of the paper we consider very particular instances of potentials with infinitely many single- and double-trace terms. The potentials we consider are relevant for the study of four dimensional $\mathcal{N}=2$ SYM theories via supersymmetric localization [10]. In particular, we provide the first examples of evaluation of $\mathcal{F}_0(t)$ for four dimensional non-conformal $\mathcal{N}=2$ SYM theories. In this second part we don't present any new result on the convergence of the resulting series; nevertheless, we suggest that the results in the first part of the paper may play an important role in deriving analytically the radius of convergence found numerically in the literature.

To arrive at the results announced above, we have followed an approach that bypasses the more usual route of diagonalizing the matrix ϕ and introducing a density of eigenvalues, to directly zero in the planar free energy. This approach relies on two key ingredients. The first one is the explicit form of the connected planar n-point function [19, 20],

$$\langle \mathrm{tr}\, \phi^{2k_1} \ldots \mathrm{tr}\, \phi^{2k_n} \rangle_c = \frac{(d-1)!}{(d-n+2)!} \prod_{i=1}^{n} \frac{(2k_i)!}{(k_i-1)!k_i!} t^d \mathrm{N}^{2-n}\,, \tag{1.5}$$

where $d = \sum_i k_i$. In the particular case when all the ks are the same, (1.5) reduces to

$$\langle \left(\text{tr}\, \phi^{2k} \right)^n \rangle_c = \frac{(nk-1)!}{(n(k-1)+2)!} \left(\frac{(2k)!}{k!(k-1)!} t^k \right)^n . \tag{1.6}$$

The second ingredient, which builds upon the first, is a recently found combinatorial expression for the planar free energy of matrix models with potentials of the form (1.2) [7, 9] as a sum over a particular type of graphs, known as tree graphs

$$\begin{aligned} \mathcal{F} = \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \sum_{k=0}^{m} \binom{m}{k} \sum_{p_1,\dots,p_{m-k}} c_{p_1} \dots c_{p_{m-k}} \sum_{\substack{i_1,\dots,i_k \\ j_1,\dots,j_k}} c_{i_1 j_1} \dots c_{i_k j_k} \\ \sum_{\substack{\text{directed trees with} \\ \text{k labeled edges}}} \sum_{\substack{\text{single trace} \\ \text{insertions}}} \prod_{i=1}^{k+1} V_i \end{aligned} \tag{1.7}$$

Let's explain the content of this expression (see [7, 9] for the derivation and further details). At a given order m, (1.7) instructs us to consider all tree diagrams (i.e. connected graphs without loops) with $k \leqslant m$ edges. Given one such tree, we have to consider all the ways to label the k edges with the integers $1, \dots, k$ and also equip each edge with an arrow that leaves one vertex and arrives at another, so the tree becomes a directed tree. Then, for each such tree, we assign to each vertex a connected correlator, according to the following rules: first, the i-th vertex on the tree is assigned tr ϕ^{i_s} if the directed edge labelled s leaves that vertex and tr ϕ^{j_s} if the directed edge labelled s arrives at that vertex. Then, we consider all the possible ways to distribute the remaining $m-k$ single trace operators among the $m+1$ vertices. Once we have distributed all operators among the vertices of the tree according to these rules, the contribution of each vertex is given by the planar connected correlator of its operators, given by (1.5). While in this work we will focus on the planar free energy, we have shown in previous work how to apply this approach to the planar limit of other observables, like the Wilson loop [7, 8] or extremal correlators [9].

The structure and main results of the paper are as follows. In section 2 we consider cases where the potential (1.2) contains only a finite number of terms. For instance, for a potential with a finite number of single-trace terms, $V = \mathrm{N} \left(c_4 \text{tr}\, \phi^4 + \dots + c_{2k} \text{tr}\, \phi^{2k} \right)$ we obtain

$$\mathcal{F}_0(t) = \sum_{\substack{j_2,\dots,j_k \\ j_2+\dots+j_k>0}} \frac{1}{j_2! \dots j_k!} \frac{(2j_2 + \dots + kj_k - 1)!}{(j_2 + \dots + (k-1)j_k + 2)!} (-x_2)^{j_2} \dots (-x_k)^{j_k} , \tag{1.8}$$

where $x_i = \frac{(2i)!}{(i-1)!i!} c_{2i} t^i$. This result was recently derived in [21] by different methods.

As a second example, for the potential $V = c_{2k2k} \text{tr}\, \phi^{2k} \text{tr}\, \phi^{2k}$ we obtain

$$\mathcal{F}_0(y_k) = \sum_{m=1}^{\infty} (-y_k)^m \frac{(m-1)!}{(2m)!} B_{2m,m+1}(1z_1, 2z_2, 3z_3, \dots, mz_m) , \tag{1.9}$$

where z_j is defined as[1]

$$z_n = \frac{(nk-1)!}{(n(k-1)+2)!}\,, \tag{1.10}$$

$y_k = 2\frac{(2k)!^2}{(k-1)!^2 k!^2} c_{2k2k} t^{2k}$ and $B_{n,k}$ are partial Bell polynomials [22]. In this last case, we haven't been able to determine analytically the radius of convergence, but we have found an analytic bound, and have proved that at large k, the radius of convergence tends to $t_c \to 1/4$.

In section 3, we turn our attention to various examples of potentials (1.2) with infinitely many terms, appearing in four dimensional $\mathcal{N} = 2$ supersymmetric gauge theories on S^4 upon localization. We first review the case of $\mathcal{N} = 2$ SU(N) with 2N multiplets in the fundamental representation of the gauge group. Being a superconformal theory, the matrix model contains only double-trace terms in the planar limit. Then, we let go of conformality to consider two examples of non-conformal $\mathcal{N} = 2$ theories. We show that their matrix models contain both single and double traces contributing to the planar limit. They thus fall in the category of theories for which (1.7) applies. We compute their planar free energy and comment on the convergence of the resulting perturbative series.

The appendix contains an alternative derivation of the results of section 2, using the more traditional approach. We derive the corresponding eigenvalue densities (or more precisely, their relevant moments). Besides taking more effort to derive, even after one obtains the eigenvalue densities, evaluating the planar free energies involves non-trivial mathematical identities, so it looks unlikely to us that one could have originally derived these planar free energies following this route, without knowing already the results.

This work can have a number of applications, and suggests a number of possible extensions. A promising application involves revisiting the derivation of exact glueball superpotentials in four-dimensional $\mathcal{N} = 1$ gauge theories, from the planar limit of an auxiliary matrix model [23]. As for extensions, [19, 20] obtained the planar connected correlator for an arbitrary number of even operators, and up to two odd operators. For this reason, in this paper we have restricted ourselves to even potentials; the results in [24] might allow to extend our work to include also odd potentials. A more interesting open question, on which we are currently working on, is to extend our method to subleading 1/N terms in (1.4). Finally, the techniques used here for Hermitian one-matrix models have been applied so far to very specific multi-matrix models [8]; it would be interesting to extend them to generic multi-matrix models.

2 Potentials with a finite number of terms

In this section, we apply the approach described in the introduction to various examples of potentials with a finite number of terms. In all of these examples, we find the all-order perturbative series for the planar free energy in closed form.

[1] Note that z_n depends on k, but since we use it only for potentials with a single value of k, we don't make explicit this dependence in the notation.

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2.1 Potential with one single-trace term

As a warm-up, consider first the case of a potential with just one single-trace term, $V = \mathrm{N}c_{2k}\mathrm{tr}\,\phi^{2k}$, besides the Gaussian quadratic term. The planar free energy is then the generating function of planar connected correlators, and in this case all $k_i = k$ so using (1.6) and recalling the definition of z_n (1.10), we arrive at

$$\mathcal{F}_0(t) = \sum_{n=1}^{\infty} \frac{(-c_{2k})^n}{n!} \left\langle \left(\mathrm{tr}\ \phi^{2k}\right)^n \right\rangle_c = \sum_{n=1}^{\infty} \frac{(nk-1)!}{n!(nk-n+2)!} (-x_k)^n = \sum_{n=1}^{\infty} \frac{z_n}{n!} (-x_k)^n \ , \quad (2.1)$$

where we have introduced the natural expansion parameter

$$x_k = \frac{(2k)!}{(k-1)!k!} c_{2k} t^k \,. \quad (2.2)$$

For the case $k = 2$, this result appears at the end of [11] while the general case appears in [21]. The power series in (2.1) are actually hypergeometric functions

$$\mathcal{F}_0(t) = -\frac{x_k}{k(k+1)}\ {}_{k+1}\mathrm{F}_k \left[\begin{matrix} 1\ 1\ \frac{k+1}{k} \ \ldots \ \frac{2k-1}{k} \\ 2\ \frac{k+2}{k-1} \ \ldots \ \frac{2k}{k-1} \end{matrix} ; -\frac{k^k x_k}{(k-1)^{k-1}} \right] . \quad (2.3)$$

It follows from the hypergeometric representation that the radius of convergence is $x_c = \frac{(k-1)^{k-1}}{k^k}$. In view of (2.2), the radius of convergence of the original 't Hooft coupling t tends to $t_c \to 1/4$ when $k \to \infty$.

2.2 Potential with a finite number of single-trace terms

Consider now the potential $V = \mathrm{N}(c_4\mathrm{tr}\,\phi^4 + \cdots + c_{2k}\mathrm{tr}\,\phi^{2k})$. Matrix models with these potentials were studied in [1] by different methods. In the context of 2d quantum gravity, these models gained further relevance after the work of [25]: in the double scaling limit [26–28] they reproduce the $(2, 2k-1)$ minimal models coupled to quantum gravity. A different context where the planar limit of these matrix models plays a crucial role is in the computation of the exact glueball superpotential of $\mathcal{N} = 1$ 4d gauge theories [23]. Very recently these models have been revisited in [21], where they also deduce the planar free energy using different methods.

Again, since there are no double-trace terms, the planar free energy is the generating function of connected correlators. It is convenient to write the connected correlators in terms of the multiplicities j_k of the operators $\mathrm{tr}\,\phi^{2k}$,

$$\begin{aligned} \mathcal{F}_0(t) &= \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \sum_{p_1,\ldots,p_m} c_{2p_1} \ldots c_{2p_m} \langle \mathrm{tr}\,\phi^{2p_1} \ldots \mathrm{tr}\,\phi^{2p_m} \rangle_c \\ &= \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \sum_{\substack{j_2,\ldots,j_k \\ j_2+\cdots+j_k=m}} \frac{m!}{j_2!\ldots j_k!} c_4^{j_2} \ldots c_{2k}^{j_k} \langle (\mathrm{tr}\,\phi^4)^{j_2} \ldots (\mathrm{tr}\,\phi^{2k})^{j_k} \rangle_c \,. \end{aligned} \quad (2.4)$$

Then, recalling the definition of the couplings x_k (2.2) and using (1.5) we arrive at a rather compact expression for the planar free energy of these models

$$\mathcal{F}_0(t) = \sum_{\substack{j_2,\ldots,j_k \\ j_2+\cdots+j_k>0}} \frac{1}{j_2!\ldots j_k!} \frac{(2j_2+\cdots+kj_k-1)!}{(j_2+\cdots+(k-1)j_k+2)!} (-x_2)^{j_2} \ldots (-x_k)^{j_k} \,. \quad (2.5)$$

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As a simple check, when only one of the terms in the potential is different from zero, (2.5) reduces to (2.1). As a first non-trivial example, when $V = \mathrm{N}(c_4 \mathrm{tr}\,\phi^4 + c_6 \mathrm{tr}\,\phi^6)$, eq. (2.5) reduces to

$$\mathcal{F}_0(t) = \sum_{m=1}^{\infty} \sum_{j=0}^{m} \frac{(3m-j-1)!}{j!(m-j)!(2m-j+2)!} (-x_2)^j (-x_3)^{m-j} \,, \tag{2.6}$$

which reproduces the result in [21]. It can be further rewritten as

$$\mathcal{F}_0(t) = \sum_{m=1}^{\infty} (-x_3)^m \frac{(3m-1)!}{m!(2m+2)!} F\left(-2-2m, -m; 1-3m; -\frac{x_2}{x_3}\right) \,, \tag{2.7}$$

which is simpler than the similar expression that appears in [21]. As a second example, for $k=4$ eq. (2.5) reduces to

$$\mathcal{F}_0(t) = \sum_{\substack{j_2, j_3, j_4 \\ j_2+j_3+j_4>0}} \frac{1}{j_2! j_3! j_4!} \frac{(2j_2+3j_3+4j_4-1)!}{(j_2+2j_3+3j_4+2)!} (-x_2)^{j_2} (-x_3)^{j_3} (-x_4)^{j_4} \,, \tag{2.8}$$

which upon expansion reproduces eq. (4.27) in [21].

2.3 Potential with one double-trace term

We now switch to examples of potentials with double-trace terms. As far as we are aware, the first appearance of such potentials was in [4], in the context of 2d quantum gravity, where they were introduced to take into account higher order curvature effects, see also [29–32]. Matrix models with double trace terms have been considered in the computation of glueball superpotential of 4d $\mathcal{N}=1$ gauge theories in [33].

The first example that we will consider is the potential with just one such term $V = c_{2k\,2k} \mathrm{tr}\,\phi^{2k} \mathrm{tr}\,\phi^{2k}$. As explained in [7], for potentials with just double traces, at order m in the perturbative expansion of the planar free energy, we have to sum over all the ways to distribute $2m$ operators into $m+1$ connected correlators, such that the same connected correlators don't appear as products of lower order terms. It was further argued in [7] that this sum can be represented as a sum over tree graphs with m edges. As we will show now, in the particular case at hand, since there is just one type of operator, $\mathrm{tr}\,\phi^{2k}$, the sum over trees simplifies drastically, and the contribution to the planar free energy is given at every order by a partial Bell polynomial.

To proceed, let's recall the basics of tree graphs (see [34, 35] for more details, or [7] for the bare minimum required in this work). A tree graph is a connected graph without loops. A tree with m edges has $m+1$ vertices. To each vertex we associate its *degree* d_i: the number of vertices it is connected to. A simple result is that a set of $m+1$ numbers $(d_1, \ldots, d_{m+1})$ is the degree sequence of a tree with $m+1$ vertices if and only if $\sum_i d_i = 2m$.

In general, for a potential with just double traces, (1.7) simplifies to [7]

$$\mathcal{F}_0(t) = \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \sum_{\substack{i_1,\ldots,i_m \\ j_1,\ldots,j_m}} c_{i_1 j_1} \ldots c_{i_m j_m} \sum_{\substack{\text{directed trees with} \\ \text{m labeled edges}}} \prod_{i=1}^{m+1} V_i \,, \tag{2.9}$$

where the product runs over the $m+1$ vertices of the tree. In the particular case when the potential contains just one double trace,

$$\mathcal{F}_0(t) = \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} c_{2k2k}^m \sum_{\substack{\text{directed trees with} \\ \text{m labeled edges}}} \prod_{i=1}^{m+1} z_{d_i} \left(\frac{(2k)!}{(k-1)!k!} \right)^{d_i} t^{d_i k} . \tag{2.10}$$

Recalling that for a tree with $m+1$ vertices, $\sum_i d_i = 2m$ [35], and taking into account that for $m > 1$ the directions of the arrows in the directed tree don't affect its contribution[2]

$$\mathcal{F}_0(t) = \sum_{m=1}^{\infty} \frac{1}{m!} \left(-2 \frac{(2k)!^2}{(k-1)!^2 k!^2} c_{2k2k} t^{2k} \right)^m \sum_{\substack{\text{trees with} \\ \text{m labeled edges}}} \prod_{i=1}^{m+1} z_{d_i} . \tag{2.11}$$

We define

$$y_k \equiv 2 \frac{(2k)!^2}{(k-1)!^2 k!^2} c_{2k2k} t^{2k} , \tag{2.12}$$

as the natural expansion parameter for these models. Next, given a tree, we denote by α_j the number of vertices with degree j. Then we make use of the fact that for $m > 1$ the number of trees with m labeled edges and a given degree sequence $(d_1, \ldots, d_{m+1})$ is [35][3]

$$\frac{1}{m+1} \binom{\sum_j \alpha_j}{\alpha_1 \ldots \alpha_m} \binom{\sum_i (d_i - 1)}{d_1 - 1 \ldots d_{m+1} - 1} , \tag{2.13}$$

to arrive at the following expression of the planar free energy as a sum over tree graphs

$$\mathcal{F}_0(t) = \sum_{m=1}^{\infty} \frac{(-y_k)^m}{m(m+1)} \sum_{\substack{\text{degree sequences} \\ \text{for trees with m edges}}} \binom{\sum_i \alpha_i}{\alpha_1 \ldots \alpha_m} \prod_{i=1}^{m+1} \frac{z_{d_i}}{(d_i - 1)!} . \tag{2.14}$$

The degree sequences in (2.14) are partitions of $2m$ elements (the total amount of operators) into exactly $m+1$ parts (the number of connected correlators), so the multiplicities satisfy $\alpha_1 + \cdots + \alpha_m = m+1$ and $1\alpha_1 + 2\alpha_2 + \cdots + m\alpha_m = 2m$. The planar free energy then can be rewritten as

$$\mathcal{F}_0(t) = \sum_{m=1}^{\infty} \frac{(-y_k)^m}{m(m+1)} \sum_{\substack{\alpha_1 + \cdots + \alpha_m = m+1 \\ 1\alpha_1 + \cdots + m\alpha_m = 2m}} \frac{(m+1)!}{\alpha_1! \ldots \alpha_m!} \left(\frac{1 z_1}{1!} \right)^{\alpha_1} \left(\frac{2 z_2}{2!} \right)^{\alpha_2} \cdots \left(\frac{m z_m}{m!} \right)^{\alpha_m} . \tag{2.15}$$

If we now recall the definition of the partial Bell polynomial [22]

$$\begin{aligned} &B_{n,k}(x_1, \ldots, x_{n-k+1}) \\ &= \sum_{\substack{\alpha_1 + \cdots + \alpha_{n-k+1} = k \\ 1\alpha_1 + \ldots (n-k+1)\alpha_{n-k+1} = n}} \frac{n!}{\alpha_1! \ldots \alpha_{n-k+1}!} \left(\frac{x_1}{1!} \right)^{\alpha_1} \left(\frac{x_2}{2!} \right)^{\alpha_2} \cdots \left(\frac{x_{n-k+1}}{(n-k+1)!} \right)^{\alpha_{n-k+1}} \end{aligned} \tag{2.16}$$

[2]As discussed in [7], the case $m = 1$ requires to be treated separately: in this case reversing the arrow in the edge does not change the tree, so we should not multiply by two. However, we are about to use the formula (2.13) that counts the number of labeled trees for a given degree sequence. That formula, valid for $m > 1$, is off by a factor $1/2$ when extended to $m = 1$. Thus, these two factors cancel each other, and the final results we present are also valid for $m = 1$.

[3]See the previous footnote for the case $m = 1$.

we realize that the planar free energy can be written in terms of partial Bell polynomials,

$$\mathcal{F}_0(y_k) = \sum_{m=1}^{\infty} (-y_k)^m \frac{(m-1)!}{(2m)!} B_{2m,m+1}(1z_1, 2z_2, 3z_3, \dots, mz_m)\,. \tag{2.17}$$

This is a pleasantly compact expression. In hindsight, the appearance of partial Bell polynomials is not surprising. At order m, the planar free energy receives contributions from the different ways to group a set of $2m$ operators into $m+1$ connected correlators, subject to the constraints mentioned above. The partial Bell polynomial $B_{n,k}$ enumerates all the ways to group a set of n elements into k groups, explaining the appearance of $B_{2m,m+1}$ at order m.

As a first check, in the particular case of $k=1$ the Bell polynomials in (2.17) can be evaluated

$$B_{2m,m+1}\left(\frac{1!}{2}, \frac{2!}{2}, \dots \frac{m!}{2}\right) = \frac{1}{2^{m+1}} \frac{(2m-1)!(2m)!}{(m+1)!m!(m-1)!}\,, \tag{2.18}$$

which gives the planar free energy

$$\mathcal{F}_0(y_1) = \frac{1}{2}\sum_{m=1}^{\infty}\left(-\frac{y_1}{2}\right)^m \frac{(2m-1)!}{m!(m+1)!} = \frac{1}{2}\log\left(\frac{\sqrt{1+16c_{22}t^2}-1}{8c_{22}t^2}\right) + \frac{(\sqrt{1+16c_{22}t^2}-1)^2}{64c_{22}t^2}\,, \tag{2.19}$$

where we have used $y_1 = 8c_{22}t^2$. This reproduces the result obtained in [4] for the $\mathrm{tr}\,\phi^2\mathrm{tr}\,\phi^2$ potential (see also the appendix). For arbitrary k, we can evaluate the first terms of (2.14),

$$\begin{aligned}\mathcal{F}_0(y_k) =& -\frac{y_k}{2k^2(k+1)^2} + \frac{y_k^2}{4k^3(k+1)^2} - \frac{y_k^3}{12}\left[\frac{6}{4k^4(k+1)^2} + \frac{2}{k^3(k+1)^3}\right] \\ &+ \frac{y_k^4}{20}\left[\frac{10}{8k^5(k+1)^2} + \frac{5}{k^4(k+1)^3} + \frac{5(4k-1)}{6k^4(k+1)^4}\right] + \dots\end{aligned} \tag{2.20}$$

or directly the first terms of (2.17),

$$\begin{aligned}\mathcal{F}_0(y_k) =& -\frac{y_k}{2k^2(k+1)^2} + \frac{y_k^2}{4k^3(k+1)^2} - \frac{(7k+3)y_k^3}{24k^4(1+k)^3} \\ &+ \frac{(23k^2+16k+3)y_k^4}{48k^5(1+k)^4} - \frac{(455k^3+405k^2+133k+15)y_k^5}{480k^6(1+k)^5} + \dots\end{aligned} \tag{2.21}$$

We now want to point out a relation between the planar free energies of two of the models discussed so far, the potential with one single-trace term, and the potential with one double-trace term. The fact that the arguments of the Bell polynomials in the planar free energy of the double-trace model, eq. (2.17), are the coefficients of the expression for the planar free energy of the single-trace model, eq. (2.1), implies that these two planar free energies $\mathcal{F}_0^{dt}$ and $\mathcal{F}_0^{st}$ are actually related. Indeed, it follows from (2.1), (2.17) and the Faà di Bruno's formula that

$$\mathcal{F}_0^{dt}(y_k) = \sum_{m=1}^{\infty}(-y_k)^m \frac{1}{(m+1)!}\frac{d^{m-1}}{dz^{m-1}}\left(\frac{d\mathcal{F}_0^{st}(-z)}{dz}\right)^{m+1}\Bigg|_{z=0}\,. \tag{2.22}$$

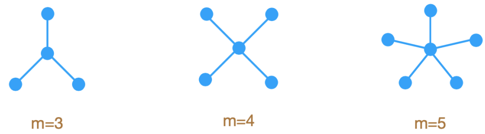

Figure 1. Examples of star graphs with $m = 3, 4, 5$ edges.

Using the Lagrange inversion theorem, this relation can be rewritten as

$$\frac{d\mathcal{F}_0^{dt}(-y_k)}{dy_k} = \frac{1}{2}\left(\frac{d\mathcal{F}_0^{st}(-z)}{dz}\right)^2 , \tag{2.23}$$

where $z(y_k)$ is obtained from inverting the equation

$$y_k = \frac{z}{\frac{d\mathcal{F}_0^{st}(-z)}{dz}} . \tag{2.24}$$

Let us conclude by mentioning that the free energies of the double scaling limits of these two models are also related [32].

2.3.1 Radius of convergence

We now want to discuss the radius of convergence of the planar free energy of this model, eq. (2.17). For $k = 1$, the radius of convergence of (2.19) is evidently $t_c^2 = \frac{1}{16c_{22}}$.

For any $k > 1$, we can derive an upper bound on the radius of convergence by considering at every order just the contribution from a single tree, the star graph. The star graph with $m+1$ vertices is the single tree with m vertices of degree 1 and one vertex of degree m, joined by edges to all the rest, see figure 1. Its degree sequence is $\vec{d} = (1, \ldots, 1, m)$ and the multiplicities of the degrees are $\vec{\alpha} = (m, 0, \ldots, 0, 1)$. In terms of planar connected correlators, this truncation amounts to, at order m, just consider the contribution from

$$\langle (\operatorname{tr}\phi^{2k})^m \rangle_c \, \langle \operatorname{tr}\phi^{2k} \rangle^m , \tag{2.25}$$

to the planar free energy.

At a given order m, the contribution to the sum over trees by the star graph is

$$\mathcal{F}_0^{\text{star}}(y_k) = \sum_{m=1}^{\infty} \frac{(mk-1)!}{m!(m(k-1)+2)!}\left(-\frac{y_k}{k(k+1)}\right)^m = \sum_{m=1}^{\infty} \frac{z_m}{m!}\left(-\frac{y_k}{k(k+1)}\right)^m . \tag{2.26}$$

This is exactly the same perturbative series as for the planar free energy of single trace models, eq. (2.1), which is a consequence of the kind of contributions we are keeping, eq. (2.25). If we truncate the sum over trees to just this contribution, it follows from the quotient criterion that the radius of convergence for this truncated series is

$$c_{2k2k} t_c^{2k}(\text{star}) = \frac{(k+1)(k-1)^{k-1}(k-1)!^2 k!^2}{2k^{k-1}(2k)!^2} . \tag{2.27}$$

Notice that at large k, $t_c^{2k}(star) \to 1/4$. At a given order in m, all the other trees besides the star graph contribute, so the full coefficient is larger than the contribution coming from the star graph. The true radius of convergence is thus smaller or equal than the one obtained by truncating the sum to just the contribution from the star graph

$$t_c^{2k} \leqslant t_c^{2k}(star)\,. \tag{2.28}$$

Finally, let's comment on the radius of convergene of (2.17) as $k \to \infty$. The large k limit of (2.17) simplifies to

$$\mathcal{F}_0(y_k) = \frac{1}{k^3}\sum_{m=1}^{\infty}\left(\frac{-y_k}{k}\right)^m \frac{(m-1)!}{(2m)!}B_{2m,m+1}(1^{1-2},2^{2-2},\ldots,m^{m-2})\,. \tag{2.29}$$

We learn that in the large k limit, the dependence on k factors out, something that one can see in the first terms of the expansion, eq. (2.21). It is possible to argue that the radius of convergence of (2.29) must be non-zero [36].[4] Recall from (2.23) that the planar free energy for this model is obtained by series reversion from the planar free energy of the single-trace model. By Lagrange's inversion theorem, since the later series has a non-zero radius of convergence, so does the first. We can do better, and bound the radius of convergence of (2.29) noting that for m large enough

$$B_{2m,m+1}(1,2,\ldots,m) < B_{2m,m+1}(1^{1-2},2^{2-2},\ldots,m^{m-2}) < B_{2m,m+1}(1^{1-1},2^{2-1},\ldots,m^{m-1})\,. \tag{2.30}$$

The Bell polynomials at the ends can be evaluated [37, 38], resulting in

$$\binom{2m}{m-1}m^{m-1} < B_{2m,m+1}(1^{1-2},2^{2-2},\ldots,m^{m-2}) < \binom{2m-1}{m}(2m)^{m-1}\,, \tag{2.31}$$

for m large enough. From these bounds we conclude that the radius of convergence of (2.29), for large k behaves as

$$y_{k,c} = Ck \qquad \frac{1}{2e} \leqslant C \leqslant \frac{1}{e}\,. \tag{2.32}$$

The $C \leqslant 1/e$ bound coincides with the large k limit of the bound obtained from the truncation to stargraph trees. It follows from (2.12) that the radius of convergence for t in the large k limit tends to $\frac{1}{4}$. Interestingly, this is the same limit found for the model with just a single trace.

2.4 Potential with one single and one double trace terms

As a last example, we will consider a potential with both single- and double-trace terms. Specifically, take $V = \mathrm{N}c_{2k}\,\mathrm{tr}\,\phi^{2k} + c_{2k2k}\,\mathrm{tr}\,\phi^{2k}\mathrm{tr}\,\phi^{2k}$. The advantage of considering the same power for the single-trace and the double-trace terms is that the sum over trees can again be simplified to yield Bell polynomials.

At order m, say we have n double traces and $m-n$ single traces: there is a total of $2n+(m-n) = m+n$ operators, to be distributed into $n+1$ connected correlators, with the constraint that no operators coming from the same double trace sit in the same connected correlator, and no pair of correlators share operators from more than one double-trace.

[4] We are very thankful to Max Alekseyev for providing this argument.

This translates into the combinatorial question of enumerating all trees with n labelled edges, to which we add $m-n$ extra labelled vertices of degree 1 (called leaves in the graph theory literature). This enumeration is again given by a partial Bell polynomial, $B_{m+n,n+1}(1z1,\ldots,mz_m)$, up to the overall normalization, so

$$\mathcal{F}(x_k,y_k)=\sum_{m=1}^{\infty}(-1)^m(m-1)!\sum_{n=0}^{m}\frac{x_k^{m-n}\,y_k^n}{(m-n)!(m+n)!}B_{m+n,n+1}(1z_1,\ldots,mz_m)\,. \tag{2.33}$$

When $c_{2k2k}=0$, only the terms with $n=0$ survive, and (2.33) reduces to (2.1), while when $c_{2k}=0$, only the $n=m$ terms survive and (2.33) reduces to (2.17).

3 Potentials with infinitely many terms

Matrix models whose potential (1.2) contains an infinite number of single an double trace deformations are relevant for two dimensional statistical physics models, Chern-Simons theories coupled to matter and models dual to M-theory backgrounds as discussed in [5]. More recently, it has been realized that supersymmetric localization [10] reduces the evaluation of certain observables of Lagrangian $\mathcal{N}=2$ super Yang-Mills theories on S^4 to matrix models that can be recast in this form [6, 7]. Most of these works deal with Lagrangian $\mathcal{N}=2$ superconformal theories. In what follows we comment on possible implications of the analysis of the previous section for these family of models, and apply these techniques to derive the planar free energies on S^4 of a couple of non-conformal gauge theories, namely $\mathcal{N}=2^*$ SYM and SU(N) with N$_f<2$N multiplets in the fundamental representation.

Recall that supersymmetric localization allows to write the partition function of a $\mathcal{N}=2$ SYM theory on S^4 as the following matrix model [10]

$$Z_{S^4}(\tau_{\rm YM})=\int da\, e^{-\frac{8\pi^2}{g^2_{\rm YM}}\mathrm{Tr}(a^2)}\mathcal{Z}_{\text{1-loop}}(a)\,|\mathcal{Z}_{\rm inst}(a,\tau)|^2\,, \tag{3.1}$$

where $\mathcal{Z}_{\rm inst}$ is the instanton contribution, that it is usually assumed to be negligible in the large N limit, and will be set to 1 one in what follows. $\mathcal{Z}_{\text{1-loop}}$ is a factor arising from a 1-loop computation in an auxiliary parameter, and itdepends on the choice of gauge group G and representations R of the matter multiplet. It is given by products over the weights of the adjoint and matter representations α, w_R respectively

$$\mathcal{Z}_{\text{1-loop}}=\prod_{\alpha}H(i\alpha\cdot\hat{a})\prod_{R}\prod_{w_R}H(iw_R\cdot\hat{a})^{-n_R}\,, \tag{3.2}$$

where H is related to the Barnes G-function by $H(x)=G(1+x)G(1-x)$. For any $\mathcal{N}=2$ SYM theory with matter in representations with up to two indices, this 1-loop factor can be recast as an effective action involving single and double trace terms [6, 7].

Let's now comment on the convergence of the resulting perturbative series for the planar free energies. For conformal theories that admit a large N expansion, we expect that in the planar limit the perturbative series for observable quantities have a finite radius of convergence in the 't Hooft coupling [16, 17]. In the case of $\mathcal{N}=2$ Lagrangian SCFTs, for observables that can be computed using supersymmetric localization, there is indeed strong numerical evidence that the radius of convergence of the perturbative planar series is given by $\lambda_c = \pi^2$ [39, 40]. As we will show, this value for the radius of convergence coincides with the large k limit of the radius of convergence for both the model with a single-trace term and the model with a double-trace term discussed in the previous section.

On the other hand, for asymptotically-free non-conformal theories, we generically expect that the theory has renormalons [41–43]. These renormalons can make every term in the 1/N expansion a divergent series [44]. One could then envision exploring the appearance of renormalons for non-conformal $\mathcal{N} = 2$ SYM using supersymmetric localization. However, it can be proven that for $\mathcal{N} = 2$ SU(N) SYM on S^4, observables that can be computed via supersymmetric localization are Borel summable at finite N [45–47]. Presumably, this implies that the perturbative planar series in λ is also Borel summable. One can even speculate that they may be convergent. The explicit perturbative series that we derive later in this section pave the way for a numerical analysis of this possibility.

3.1 Potential with infinitely many double-trace terms: $\mathcal{N} = 2$ SCFTs

It was shown in [6, 7] that for Lagrangian $\mathcal{N} = 2$ superconformal field theories, the matrix model that one obtains from supersymmetric localization contains both single- and double-trace terms,

$$S_{\text{int}} = \sum_{n=2}^{\infty} \frac{\zeta_{2n-1}(-1)^n}{n} \Bigg[(4-4^n)\alpha_G \operatorname{tr} a^{2n} + \beta_G \sum_{k=1}^{n-1} \binom{2n}{2k} \operatorname{tr} a^{2n-2k} \operatorname{tr} a^{2k} \\ + \gamma_G \sum_{k=1}^{n-2} \binom{2n}{2k+1} \operatorname{tr} a^{2n-2k-1} \operatorname{tr} a^{2k+1} \Bigg] , \tag{3.3}$$

where $\alpha_G, \beta_G, \gamma_G$ depend on the matter content of the theory. In theories where the matter in the fundamental representation scales with the rank of the gauge group, $\beta_G \neq 0$ in the planar limit. For those theories, only double-trace terms with even operators contribute to the planar limit, so for the purpose of studying the planar limit one can take [7]

$$S_{\text{int}} = \beta_G \sum_{i,j=1}^{\infty} \frac{\zeta_{2i+2j-1}}{i+j} (-1)^{i+j} \binom{2i+2j}{2i} \operatorname{tr} a^{2i} \operatorname{tr} a^{2j} . \tag{3.4}$$

In the previous section, we argued that for double-trace potentials with coefficients c_{2k2k} that don't scale with k, the radius of convergence tends to 1/4. In the case at hand, if we focus on terms with the same trace, they are essentially of the form

$$V = \binom{4k}{2k} \operatorname{tr} \phi^{2k} \operatorname{tr} \phi^{2k} , \tag{3.5}$$

so the coefficient c_{2k2k} scales like 4^{2k} at large k. Recalling the relation between y_k and t for these potentials, eq. (2.12), this simply implies that now $4t_c = 1/4$ for large k, or equivalently $t_c = 1/16$ in this limit. Comparing the kinetic terms in the original matrix model and in (3.1), we learn that the 't Hooft coupling of the matrix model t and the Yang-Mills 't Hooft coupling $\lambda = g_{\text{YM}}^2 \text{N}$ are related by $t = \frac{\lambda}{16\pi^2}$. So for the potential (3.5) with just one such term, at large k the radius of convergence tends to $\lambda_c = \pi^2$. This is actually the radius of convergence found for various planar series (or truncations thereof) of conformal $\mathcal{N}=2$ theories: it is the radius of convergence of the dispersion relation for the $\mathcal{N}=4$ magnon [48]. It is also the radius of convergence analytically found in [7] for the (uncontrolled) truncation of the planar free energy of $\mathcal{N}=2$ SYM theories on S^4 with $\beta_G \neq 0$ to terms with just one value of ζ. Finally, it is also the value found numerically in [39, 40] as the radius of convergence for the planar free energy of (3.3) when $\beta_G=0$, which corresponds to theories where the number of matter multiplets in the fundamental representation does not scale with N. We conjecture that for all observables of $\mathcal{N}=2$ SYM conformal theories captured by supersymmetric localization, the radius of convergence of the planar limit is $\lambda_c=\pi^2$. There is also evidence for this being the radius of convergence of some perturbative series for the non-planar terms in the 1/N expansion [49] and perhaps even non-conformal theories, as discussed below. The observation presented above, that this value coincides with the large k limit of the radius of convergence of (3.5) certainly does not constitute a proof of this conjecture, but we expect this observation to play an important role in a (yet to be developed) full analytic proof.

3.2 Potential with infinitely many single- and double-trace terms: $\mathcal{N}=2$ nonconformal theories

We consider now two specific examples of non-conformal $\mathcal{N}=2$ super Yang-Mills theories. We will see that in both cases, the partition function obtained from supersymmetric localization can be rewritten as a matrix model with a potential with infinitely many single and double trace terms. Opposite to what happens in the conformal case, eq. (3.3), now the single-trace terms have the right large N scaling to contribute to the planar limit. These two examples thus fall in the category of matrix models solved in the planar limit by (1.7).

3.2.1 $\mathcal{N}=2$ SU(N) with $\mathrm{N}_f<2\mathrm{N}$

As a first example of non-conformal $\mathcal{N}=2$ theory, we will consider $\mathcal{N}=2$ SU(N) SYM with $\mathrm{N}_f<2\mathrm{N}$ multiplets in the fundamental representation. The corresponding 1-loop determinant (3.2) is

$$Z_{\text{1-loop}}=\frac{\prod_{1=u<v}^{N} H(ia_u-ia_v)^2}{\prod_{u=1}^{N} H(ia_u)^{N_f}}\,. \tag{3.6}$$

Taking into account the perturbative expansion of the logarithm of the H function

$$\log H(x)=-(1+\gamma)x^2-\sum_{n=2}^{\infty}\zeta_{2n-1}\frac{x^{2n}}{n}\,, \tag{3.7}$$

the effective action can be rewritten as

$$\begin{aligned} S_{\text{int}}=\mathrm{N}&\left[\left(\frac{N_f}{N}-2\right)(1+\gamma)\mathrm{tr}a^2+\sum_{n=2}^{\infty}\frac{\zeta_{2n-1}(-1)^n}{n}\mathrm{tr}a^{2i}\right]\\ &+\sum_{n=2}^{\infty}\frac{\zeta_{2n-1}(-1)^n}{n}\left[\sum_{k=1}^{n-1}\binom{2n}{2k}\mathrm{tr}\,a^{2n-2k}\mathrm{tr}\,a^{2k}+\sum_{k=1}^{n-2}\binom{2n}{2k+1}\mathrm{tr}\,a^{2n-2k-1}\mathrm{tr}\,a^{2k+1}\right]. \end{aligned} \tag{3.8}$$

While this potential has single-trace terms, they are all even. Following the arguments in [9], it follows that the double-trace terms with odd powers don't contribute to the planar free energy. Thus, for the purpose of computing the planar free energy we can restrict to

$$
\begin{aligned}
S_{\text{int}} = \mathrm{N}\left[\left(\frac{\mathrm{N}_f}{\mathrm{N}}-2\right)(1+\gamma)\mathrm{tr}a^2+\sum_{n=2}^{\infty}\frac{\zeta_{2n-1}(-1)^n}{n}\mathrm{tr}a^{2i}\right] \\
+\sum_{i,j}\frac{\zeta_{2i+2j-1}(-1)^{i+j}}{i+j}\binom{2i+2j}{2i}\mathrm{tr}a^{2i}\mathrm{tr}a^{2j}\,,
\end{aligned}
\tag{3.9}
$$

so (3.1) can be rewritten as a Hermitian matrix model with a potential with an infinite number of single and double trace terms. The single-trace terms appearing in the potential are all proportional to the beta function of the theory, so they vanish in the particular case $\mathrm{N}_f = 2\mathrm{N}$, the conformal case. On the other hand, the double-trace terms are those of the conformal case. Note also that as long as $2\mathrm{N}-\mathrm{N}_f$ scales like N in the large N limit, the single-trace terms contribute to the planar limit. As an illustration, we apply (1.7) truncating the expansion of the planar free energy to the terms that contain only one zeta function

$$
\begin{aligned}
\mathcal{F} = -\left(\frac{\mathrm{N}_f}{\mathrm{N}}-2\right)(1+\gamma)\left(\frac{\lambda}{16\pi^2}\right)+\sum_{p=2}^{\infty}\frac{\zeta_{2p-1}}{p}\frac{(2p)!}{(p+1)!p!}\left(-\frac{\lambda}{16\pi^2}\right)^p \\
-\sum_{i,j=1}^{\infty}\frac{\zeta_{2i+2j-1}}{(i+j)}\left(\frac{-\lambda}{16\pi^2}\right)^{i+j}\binom{2i+2j}{2i}\frac{(2i)!(2j)!}{(i+1)!i!(j+1)!j!}+\cdots\,,
\end{aligned}
\tag{3.10}
$$

3.2.2 $\mathcal{N} = 2^*$

As our second example of non-conformal gauge theory, let's consider $\mathcal{N} = 2^*$ SU(N). This theory is the result of adding a mass term to the hypermultiplet of $\mathcal{N} = 4$ SU(N) SYM. $\mathcal{N} = 2^*$ SU(N) has already been studied using supersymmetric localization [50]. The 1-loop determinant (3.2) is now

$$
Z_{\text{1-loop}} = \frac{\prod_{u<v}H(ia_u-ia_v)^2}{\prod_{u<v}H(ia_u-ia_v-M)H(ia_u-ia_v+M)}\,. \tag{3.11}
$$

Taking the logarithm of the previous expression and recalling (3.7), this can be rewritten — up to a constant term — as

$$
\begin{aligned}
S_{\text{int}} = {} & 2\mathrm{N}\sum_{j=1}^{\infty}\sum_{n=j+1}^{\infty}\frac{\zeta_{2n-1}}{n}\binom{2n}{2j}(-1)^j M^{2n-2j}\mathrm{tr}\ a^{2j} \\
& +\sum_{i,j=1}^{\infty}\left[\binom{2i+2j}{2i}(-1)^{i+j}\sum_{n=i+j+1}\frac{\zeta_{2n-1}}{n}\binom{2n}{2i+2j}M^{2n-2i-2j}\right]\mathrm{tr}\ a^{2i}\mathrm{tr}\ a^{2j} \\
& +\sum_{i,j=1}^{\infty}\left[\binom{2i+2j+2}{2i+1}(-1)^{i+j}\sum_{n=i+j+2}\frac{\zeta_{2n-1}}{n}\binom{2n}{2i+2j+2}M^{2n-2i-2j-2}\right]\mathrm{tr}\ a^{2i+1}\mathrm{tr}\ a^{2j+1}\,,
\end{aligned}
\tag{3.12}
$$

so again the matrix model coming from localization can be recasted as a matrix model with a potential with single and double trace terms, and the single-trace terms come with a power of N, so they contribute to the planar limit. It is thus possible to write the planar free energy for this theory on S^4 using (1.7). In particular, the terms with a single value of ζ are

$$\mathcal{F}_0 = -\sum_{p=1}^{\infty} \frac{2(2p)!(2p+1)!}{p!p!(p+1)!(p+2)!} \left(\frac{-\lambda}{16\pi^2}\right)^p \sum_{m=1}^{\infty} \frac{\zeta_{2m+2p-1}}{m+p} \binom{2m+2p}{2p} M^{2m} + \dots \tag{3.13}$$

where the dots stand for terms with two or more values of ζ. It is possible to rewrite this result in integral form, which allows to explore the large λ regime. Upon performing the sums we obtain

$$\mathcal{F}_0 = -\frac{4\pi^2}{\lambda} \int_0^{\infty} \mathrm{d}w \frac{\sinh^2(wM)}{w^3 \sinh^2 w} \left(J_1 \left(\frac{w\sqrt{\lambda}}{\pi} \right)^2 - \frac{w^2 \lambda}{4\pi^2} \right), \tag{3.14}$$

with J_1 a Bessel function. As a check, if we keep only the M^2 term in the expression above, we reproduce the result of [50]. For this truncation at order M^2, it was proven in [50] that the radius of convergence is again $\lambda_c = \pi^2$. Because the theory is no longer conformal, the coupling runs, and this coupling should be understood as evaluated at the scale given by the radius of S^4. The result of [50] implies that at first order in conformal perturbation theory, the radius of convergence remains the same as the one found in the conformal cases reviewed above. It will be interesting to determine whether this is still the case for the full planar series.

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A Saddle point analysis

In this appendix we reproduce to some extent the results we found in the first section, for potentials with finitely many terms, using the methods introduced in [1]. [1] considered only potentials with single-trace terms, and the extension to potentials with double-trace terms was worked out in [5, 51]. We follow [5] closely. The matrix model considered is

$$V(M) = \frac{1}{2g} \mathrm{tr}\, M^2 + \mathrm{N} \sum_k c_{2k} \mathrm{tr}\, M^{2k} + \sum_{jk} c_{2j\,2k} \mathrm{tr}\, M^{2j} \mathrm{tr}\, M^{2k} . \tag{A.1}$$

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After diagonalization of the matrix M, introduce the density of eigenvalues $\rho(\lambda)$, and its moments

$$\rho_k = \int d\lambda \rho(\lambda)\lambda^k \,. \tag{A.2}$$

One of the basic quantities is $R_0(\xi, t)$, defined as the positive solution of

$$\xi = \frac{1}{t} R_0 + \sum_{k\geqslant 2} b_k R_0^k \,, \tag{A.3}$$

where ξ can be thought of as an auxiliary variable and

$$b_k = \frac{(2k)!}{k!(k-1)!} \left(c_{2k} + 2\sum_j c_{2j,2k}\rho_{2j} \right) \,. \tag{A.4}$$

Then, the planar free energy (after subtracting the Gaussian term) is given by

$$\mathcal{F}_0(t) = \int_0^1 d\xi (1-\xi) \log \frac{R_0(\xi,t)}{t\xi} + \sum_{j,k} c_{2j,2k}\rho_{2j}\rho_{2k} \,. \tag{A.5}$$

The starting point of our approach is to solve (A.3) by means of the Lagrange inversion theorem. In the case of potentials with just single-trace terms, this already yields an explicit expression for $R_0(\xi, t)$ and we can proceed to evaluate the planar free energy (A.5). The case of potentials with double-trace terms is *a priori* more complicated, since the coefficients b_k now depend on the moments of the eigenvalue density — see (A.4) — which at this stage is not known explicitly. In this case, one can further relate the eigenvalue density moments to R_0 through

$$\rho_{2l} = \frac{(2l)!}{l!^2} \int_0^1 d\xi R_0^l \,, \tag{A.6}$$

and this is enough to determine ρ_{2k} and R_0.

To proceed, define

$$g(x) = \frac{1}{t} + \sum_{k\geqslant 2} b_k x^{k-1} \,, \tag{A.7}$$

so according to Lagrange's inversion formula

$$R_0(\xi, t) = \sum_{n=0}^{\infty} \frac{\xi^{n+1}}{(n+1)!} \frac{d^n}{dx^n} \frac{1}{g(x)^{n+1}} \Big|_{x=0} \,. \tag{A.8}$$

The first terms of the perturbative expansion of $R_0(t, \xi)$ are

$$\frac{R_0}{\xi t} = 1 - b_2 t^2 \xi + (2b_2^2 t^4 - b_3 t^3)\xi^2 + (-b_4 t^4 + 5b_2 b_3 t^5 - 5b_2^3 t^6)\xi^3 + \ldots \tag{A.9}$$

and the coefficients that appear in this expansion constitute the integer sequence A111785 in [52]. Let's consider as a first application the case of the potential with finitely many single-trace terms. In this case, the functions b_k reduce to $b_k t^k = x_k$, with x_k defined in (2.2), so the first terms of the perturbative expansion of $R_0(t, \xi)$ are

$$\frac{R_0}{\xi t} = 1 - x_2\xi + (2x_2^2 - x_3)\xi^2 + (-x_4 + 5x_2 x_3 - 5x_2^3)\xi^3 + (14x_2^4 - 21x_2^2 x_3 + 3x_3^2 + 6x_2 x_4 - x_5)\xi^4 \ldots \tag{A.10}$$

Carrying out the integral for the planar free energy (A.5) we find

$$\mathcal{F}_0(x_i) = -\frac{x_2}{6} + \frac{x_2^2}{8} - \frac{x_2^3}{6} + \frac{7x_2^4}{24} - \frac{x_3}{12} + \frac{x_2 x_3}{5} - \frac{x_2^2 x_3}{2} + \frac{x_3^2}{12} - \frac{x_2 x_3^2}{2} - \frac{x_4}{20} + \dots \tag{A.11}$$

in agreement with the first terms in the expansion of the expression we found in the main text, eq. (2.5). The check we have just performed has an important consequence: it gives an expression for the integral in (A.5) in the general case. To understand why, notice that the perturbative series for $R_0(\xi, t)$ in the general case, (A.9), and in the particular case of just single-trace terms, (A.10), are the same, just with the substitution $b_k t^k \to x_k$. Therefore, the outcome of the integral in (A.5) for the general case is the same as for the particular case, with the substitution $b_k t^k \to x_k$. In the particular case, the planar free energy (A.5) is just given by the first term since $c_{2j,2k} = 0$, so it must coincide with the result found in the main text (2.5). In summary, we learn that

$$\int_0^1 d\xi(1-\xi)\log\frac{R_0(\xi,t)}{t\xi} = \sum_{\substack{j_2,\dots,j_k \\ j_2+\dots+j_k>0}} \frac{1}{j_2!\dots j_k!}\frac{(2j_2+\dots+kj_k-1)!}{(j_2+\dots+(k-1)j_k+2)!}(-b_2t^2)^{j_2}\dots(-b_kt^k)^{j_k} \tag{A.12}$$

Let's now move to the case of potentials with just one double-trace term. The $\mathrm{tr}\,\phi^2\mathrm{tr}\,\phi^2$ case can solved completely

$$R_0(\xi, t) = 2\xi t\frac{\sqrt{1+16c_{22}t^2}-1}{16c_{22}t^2}, \tag{A.13}$$

$$\rho_2(t) = \frac{\sqrt{1+16c_{22}t^2}-1}{8c_{22}t}, \tag{A.14}$$

$$\mathcal{F}_0(t) = \frac{1}{2}\log\left(\frac{\sqrt{1+16c_{22}t^2}-1}{8c_{22}t^2}\right) + c_{22}\rho_2^2\,. \tag{A.15}$$

This reproduces eqs. (32)-(34) of [4]. For generic $\mathrm{tr}\,\phi^{2k}\mathrm{tr}\,\phi^{2k}$, the equation (A.3) for R_0 simplifies to

$$\xi = \frac{1}{t}R_0 + b_k R_0^k\,, \tag{A.16}$$

which leads to

$$R_0(\xi, t) = \xi t\sum_{m=0}^{\infty}\frac{(km)!}{m!(m(k-1)+1)!}\left(-\xi^{k-1}b_k t^k\right)^m\,. \tag{A.17}$$

We can carry out the integral in (A.6), and we arrive at an implicit equation for the density moment $\rho_{2k}(t)$

$$\rho_{2k}(t) = -\frac{(2k)!}{k!(k-1)!}t^k\sum_{m=1}^{\infty}\frac{z_m}{(m-1)!}(-b_k t^k)^{m-1}\,. \tag{A.18}$$

If we define

$$\bar{\rho} = \frac{k!(k-1)!}{(2k)!t^k}\rho_{2k}\,, \tag{A.19}$$

this equation can be rewritten

$$\bar{\rho} = \sum_{m=1}^{\infty}\frac{z_m}{(m-1)!}(-y_k\bar{\rho})^{m-1} \tag{A.20}$$

Applying the Lagrange inversion formula to (A.20), we obtain an explicit expression for $\rho_{2k}(y_k)$ in terms of partial Bell polynomials

$$\rho_{2k}(y_k) = \frac{(2k)!}{k!(k-1)!} t^k \sum_{n=0}^{\infty} (-y_k)^n \frac{n!}{(2n+1)!} B_{2n+1,n+1}(1z_1, \ldots, (n+1)z_{n+1}) . \tag{A.21}$$

As a check, for $k = 1$ this reduces to the result for ρ_2 quoted above. From (A.12), we deduce that in this case

$$\mathcal{F}_0(y_k) = \sum_{m=1}^{\infty} \frac{z_m}{m!} (-y_k \bar{\rho})^m + \frac{1}{2} y_k \bar{\rho}^2 . \tag{A.22}$$

Taking the derivative against y_k we deduce that

$$-\frac{d\mathcal{F}_0(y_k)}{dy_k} = \frac{1}{2} \bar{\rho}^2 , \tag{A.23}$$

so the equivalence of both methods amounts to the mathematical identity

$$\sum_{m=0}^{\infty} u^m \frac{(m+1)!}{(2m+2)!} B_{2m+2,m+2}(x_1, \ldots, x_{m+1}) = \frac{1}{2} \left(\sum_{n=0}^{\infty} u^n \frac{n!}{(2n+1)!} B_{2n+1,n+1}(x_1, \ldots, x_{n+1}) \right)^2 \tag{A.24}$$

that can be proven by manipulating the generating function of Bell polynomials [36].[5]

The last example that we will consider in this appendix is $V = \mathrm{N} c_{2k} \operatorname{tr} \phi^{2k} + c_{2k,2k} \operatorname{tr} \phi^{2k} \operatorname{tr} \phi^{2k}$. The equation for ρ_{2k} is still (A.18), which in terms of $\bar{\rho}$ is now

$$\bar{\rho}(x_k, y_k) = \sum_{m=1}^{\infty} \frac{z_m}{(m-1)!} (-x_k - y_k \bar{\rho})^{m-1} , \tag{A.25}$$

whose solution is

$$\bar{\rho} = \sum_{m=1}^{\infty} (-1)^{m+1} \sum_{n=1}^{m} \frac{x_k^{m-n} y_k^{n-1}}{(m-n)!(n+m-1)!} B_{n+m-1,n}(1z_1, \ldots, mz_m) . \tag{A.26}$$

Using (A.12), the planar free energy can be written as

$$\mathcal{F}_0(x_k, y_k) = \sum_{m=1}^{\infty} \frac{z_m}{m!} (-x_k - y_k \bar{\rho})^m + \frac{1}{2} y_k \bar{\rho}^2 . \tag{A.27}$$

Taking partial derivatives with respect to x_k and y_k we find

$$\frac{\partial \mathcal{F}_0}{\partial x_k} = -\bar{\rho} , \qquad \frac{\partial \mathcal{F}_0}{\partial y_k} = -\frac{1}{2} \bar{\rho}^2 . \tag{A.28}$$

Taking the derivative of the result we found in the main text, eq. (2.33) against x_k, we do indeed recover eq. (A.26), proving the equivalence of both methods.

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CHAPTER 5

Conclusions

This thesis was mostly devoted to the development of new techniques to compute supersymmetric observables in 4−dimensional Lagrangian $\mathcal{N}=2$ superconformal field theories. Given the large amount of symmetry these family of theories posses, when we restrict our attention to a protected subsector we are able to obtain exact results that would be out of reach by traditional methods.

One of the most fascinating technical developments of the last years is supersymmetric localization. By localizing the 4−dimensional Lagrangian $\mathcal{N}=2$ superconformal theory on S^4, the path integral reduces to an interacting matrix model allowing thus to obtain 0-dimensional answers to 4−dimensional questions. The methods developed in this thesis are not only applicable to the specific matrix models describing $\mathcal{N}=2$ superconformal field theories, they are also well suited to study any model containing single and double trace deformations and, from a mathematical perspective, they pave the way to extend the framework beyond the Hermitian ensamble of random matrices.

When restricted to models that arise from localization of 4−dimensional $\mathcal{N}=2$ theories on S^4, the main achievement of this work is the characterization, in purely combinatorial terms, of the planar regime of BPS observables. We have been able to fully characterize the coupling dependence of the $\frac{1}{2}$−BPS Wilson loop and the planar Free Energy of generic Lagrangian $\mathcal{N}=2$ superconformal field theories. This result also allows us, utilizing the well known relation between the expectation value of the Wilson loop and the Brehmstrahlung function, to characterize the radiation emmited by a probe particle in such theories. In addition, when applied at finite N, the techniques developed in this work allow the characterization of these observables in terms of group theoretical factors such as the color invariants of the gauge group G.

One of the fundamental observable in any gauge theory is the correlation function of local operators. Chiral primary operators in 4−dimensional Lagrangian $\mathcal{N}=2$ superconformal field theories is an example of such operators for which, due to the large amount of symmetries, computing the n−point function reduces to a tractable problem. In addition we know that by computing the

$2-$point and $3-$point functions we have all the ingredients to characterize higher $n-$point correlation functions due to the bootstrap equations. Chiral primary operators are defined in $\mathbf{R}^4$ rather than on S^4, thus their computation through matrix model techniques is not straightforward. In order to compute them with our matrix model techniques we extended our combinatorial approach to models that now contain single trace contributions to the planar limit. With this consideration, a major milestone of this work is that we were able to obtain, for the first time, the exact expression in the planar limit for the $2-$ and $3-$point functions of CPOs of arbitrary scaling dimension k.

Furthermore, the techniques developed in this work allowed us to characterize the planar free energy for the Hermitian one-matrix model with various choices of the potential. Restricting our attention to potentials with finitely many terms we obtained novel closed expressions for the free energy as well as obtaining the exact radius of convergence as a series in the 't Hooft coupling. The study of this one-matrix models allowed us also to start the study of non-conformal supersymmetric theories such as $\mathcal{N}=2^*$ theory, or SQCD with $n_f \neq 2N$ ypermultiplets.

All through this work we have been mostly interested in the leading term of the large N expansion. Extending this framework to include $1/N$ corrections as well as odd matrix insertion is an open problem worth studying. On the other hand, we know that in $\mathcal{N}=1$ theories it is possible to obtain the exact glueball superpotential of the theory by stuyding the planar regime an auxiliary matrix model thus, revisiting this derivation with our combinatorial approach is a promising research venue.

Although we have been considering mostly matrix models that arise from a supersymmetric theory, the same models appear in the study of certain $2-$dimensional theories of gravity coupled to conformal matter. This opens up the possibility of extending the techniques that we have developed to address problems in such lower formulations of gravity. Even more, the new combinatorial interpretation of the planar regime of matrix models might be well suited to go beyond the Hermitian ensamble this will in turn allow us to study in a novel way theories with rich phase structure such as $\mathcal{N}=1$ or Fermionic models that serve as toy models for the static patch of de Sitter space.

Turning now our attention to correlation functions, the results obtained for the $2-$point and $3-$point functions of CPOs are intriguing. The result is not only strikingly simple, but it seems to be a solution to a counting problem namely, in how many ways we can add a given family of Feynman diagrams to the computation of the expectation value. Proving this would give a more solid ground to the possibility of summing all the contributions and extend the validity of the computation to the strong coupling regime. Being able to do so would be a breaktrhough in the study of correlation functions of Chiral primary operators.

Resumen

En esta tesis hemos desarrollado nuevas técnicas de cálculo en teorías de campo Lagrangianas en 4−dimensiones con $\mathcal{N} = 2$ simetría superconforme. Generalmente, los cálculos que se pueden llevar a cabo en una teoría de campos con menos simetría no permiten acceder al régimen de acoplamiento fuerte e inclusive, restringiendonos a acoplamiento débil, la dificultad de los cálculos crece exponencialmente. Al trabajar con teorías con una mayor cantidad de simetrías hemos podido obtener resultados extactos para una gran cantidad de observables físicos. En la introducción de esta tesis (Capítulo 1), hemos hecho un breve repaso a los ingredientes esenciales de teorías de campos supersimétricas, a la vez que introducimos algunos resultados previos que son el punto de partida de nuestras investigaciones.

En el segundo Capítulo de este trabajo recogemos los resultados obtenidos al caracterizar el régimen planar de la teoría. Al utilizar la técnica de localización supersimétrica, el cálculo de observables que preservan supersimetría se reduce a un modelo de matrices aleatorias. Hemos desarrollado nuevas técnicas para resolver este tipo de modelos característicos a teorías $\mathcal{N} = 2$ que han sido localizadas en S^4. El interes de estas técnicas, no solo radica en el hecho de que con ellas hemos podido caracterizar la Energía Libre, el operador de Wilson o la radiación emitida por una partícula de prueba, sino también en el hecho de que son aplicables a modelos de matrices más generales.

Un ejemplo en el cuál las técnicas desarrolladas en este trabajo son fundamentales es el estudio de funciones de correlación de operadores quirales primarios. En el Capítulo 3 recolectamos los resultados correspondientes al estudio de funciones de correlación de 2 y 3-puntos. Los resultados obtenidos son válidos para cualquier operador quiral de dimensión de escala arbitraria k, siendo este resultado el primero de su tipo en la literatura. Además de ser un resultado sorprendentemente simple, la expresión que hemos obtenido parece indicar la posible derivación del mismo mediante otras técnicas. Esto permitiría a su vez obtener por primera vez una predicción para dicha familia de operadores a acoplamiento fuerte.

Finalmente, en el Capítulo 4 utilizamos las técnicas desarrolladas para estudiar modelos de matrices que contienen un número finito de términos así como también los modelos que describen ciertas teorías de campo sin simetría conforme. En este caso hemos logrado obtener nuevos resultados exactos en el cálculo de la energía libre planar de estas teorías lo que nos ha permitido también caracterizar exactamente el radio de convergencia de las mismas. Para el caso de los modelos que se desprenden de las teorías no conformes y de forma más general, modelos con infinitos términos en el potencial, hemos obtenido cotas precisas para el radio de convergencia de la serie perturbativa en la constante de acoplamiento de 't Hooft.

A lo largo de este trabajo hemos enfocado nuestros esfuerzos en el cálculo del término dominante de la expansión sistemática en N. Una incipiente pregunta es generalizar las técnicas para poder incluir también correcciones en $1/N$, lo que permitiría a su vez realizar verificaciones no triviales de la dualidad holográfica.

De igual manera, hemos estado interesados en desarrollar técnicas combinatorias para modelos de matrices en donde la matriz es hermítica. Sin embargo, el mismo conjunto de ideas puede utilizarse para otros ensambles de matrices como el unitario. Este tipo de ensambles describen la dinámica de teorías físicas con un rico diagrama de fases como el modelo de Gross-Witten-Wadia o inclusive el índice supersimétrico de teorías con $\mathcal{N} = 1$ supersimetrías, por lo que extender el conjunto de ideas a este tipo de modelos es de gran interés.

Finalmente, los estudios realizados para modelos de matrices en los cuales el potencial es finito han sido recientemente estudiados en el contexto de teorías de gravedad en 2−dimensiones. Esto plantea la posibilidad de extender nuestros resultados y técnicas, derivados de teorías supersimétricas, a modelos de gravedad en donde la supersimetría es a priori no aparente.

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