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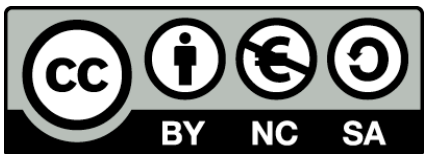
Truthlikeness for deterministic and probabilistic laws

Alfonso García Lapeña



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Verosimilitud en las leyes deterministas y probabilistas

Doctoral thesis submitted by Alfonso García-Lapeña to the University of Barcelona for the degree of Doctor of Philosophy

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Every statement or theory is not only either true or false but has, independently of its truth value, some degree of verisimilitude.

Karl Popper, Conjectures and Refutations

To my parents

RESUMEN

La verosimilitud es una propiedad de una proposición o teoría que representa su cercanía, su aproximación, o su semejanza a la verdad. La noción permite defender una posición intermedia entre el infalibilismo y el escepticismo y proporciona una lectura optimista de un conjunto de ideas sobre la ciencia que parecen implicar, en un primer momento, una visión instrumentalista o pesimista de las teorías científicas. A este respecto, quizá todas las teorías científicas sean, estrictamente hablando, falsas, aunque algunas pueden estar más cerca de la verdad que otras; el progreso científico es posible gracias a un aumento de la cercanía a la verdad de las teorías; la verdad, aunque quizá inalcanzable, puede definirse como el objetivo final de la ciencia en el sentido de la búsqueda de una mejor aproximación a ella; nuestras mejores teorías científicas (incluyendo sus partes inobservables) funcionan porque son aproximadamente verdaderas; y uno puede abrazar el falibilismo y aun así ser capaz de estimar que algunas teorías están más cerca de la verdad que otras.

Tras el fracaso de Popper en proporcionar una definición satisfactoria sobre la verosimilitud, la noción se convirtió en un tema de intensa discusión por parte de los filósofos de la ciencia y los lógicos, lo que dio lugar a dos líneas de investigación principales, la del enfoque del *contenido-consecuencia* y la del enfoque de la *similaridad*. Esta tesis doctoral desarrolla un marco para definir la verosimilitud de las leyes deterministas y probabilistas dentro del enfoque de la similaridad tal y como lo ha desarrollado Ilkka Niiniluoto.

Según Niiniluoto, la verosimilitud de las leyes deterministas cuantitativas puede definirse a través de la métrica de Minkowski. La tesis presenta algunos contraejemplos a dicha definición y argumenta que falla debido al hecho de que considera que la verosimilitud de las leyes deterministas cuantitativas es solo una función de su "precisión", pero una ley altamente precisa puede errar respecto a la "estructura" o el "comportamiento" real del sistema que pretende describir. Desarrolla una modificación de la propuesta de Niiniluoto que define la verosimilitud de las leyes deterministas cuantitativas basándose en una función de dos variables: la "precisión" y la "nomicidad". Esta última representa los comportamientos cualitativos implicados por una ley que no pueden capturarse apelando únicamente a la comparación de sus valores predichos con los valores reales. Muestra que la nomicidad puede medirse mediante la similaridad de formas entre dos funciones, apelando a la distancia euclidiana entre las derivadas de las funciones

correspondientes. La propuesta final resuelve los contraejemplos presentados y define una nueva forma de entender el progreso científico.

El marco se amplía para cubrir las leyes probabilistas, que representan un subconjunto relevante de las leyes científicas actuales. Al desarrollarse esta investigación, la literatura sobre la verosimilitud apenas contenía propuestas sobre cómo tratar con las leyes probabilistas o con las verdades probabilistas en general. En este sentido, la investigación partió de la sugerencia de Niiniluoto de usar la divergencia Kullback-Leibler para definir la distancia entre una ley de probabilidad X y la verdadera ley de probabilidad T y argumenta que dicha divergencia parece ser la mejor de las distancias entre funciones de probabilidad disponibles para medir la precisión de las leyes probabilistas. Sin embargo, como en el caso de las leyes deterministas, argumenta que la precisión representa una condición necesaria pero no suficiente, ya que dos leyes probabilistas pueden ser igualmente precisas y aun así una implicar consecuencias, comportamientos o hechos probabilísticos más verdaderos o semejantes a la verdad que la otra. La propuesta final define, nuevamente, la verosimilitud de las leyes probabilistas como una función de su precisión y nomicidad, en íntima conexión con la propuesta desarrollada para las leyes deterministas.

Palabras clave

Truthlikeness · Verosimilitud · *Legisimilitude* · Leyes científicas · Leyes deterministas · Leyes probabilistas · Similaridad · Distancia · Precisión · Nomicidad · Filosofía de la ciencia

ABSTRACT

Truthlikeness is a property of a theory or a proposition that represents its closeness, similarity or likeness to the truth. The notion allows to defend a middle position between infallibilism and scepticism, providing an optimistic understanding of a set of ideas regarding science that may seem to imply, *prima facie*, an instrumentalist or pessimistic view of scientific theories. In this sense, perhaps all scientific theories are strictly speaking false, but some may be closer to the truth than others; scientific progress is possible because of an increase in truthlikeness; truth might be said to be the aim of science in the sense of pursuing a better approximation to it; our best developed theories (including the unobservable parts) work because they are close to the truth; and finally, one may embrace fallibilism and still be able to estimate that some theories are closer to the truth than others.

Since Popper's failure to provide a satisfactory definition of truthlikeness, the notion has become a topic of intense discussion by philosophers of science and logicians. This gave rise to two main perspectives, the content-consequence and the similarity approach. This dissertation proposes a framework to define truthlikeness for deterministic and probabilistic laws within Niiniluoto's version of the similarity approach.

According to Niiniluoto, truthlikeness for quantitative deterministic laws can be defined by the Minkowski metric. We present some counter-examples to the definition and argue that it fails because it considers truthlikeness for quantitative deterministic laws to be just a function of "accuracy", but an accurate law can be wrong about the actual "structure" or "behaviour" of the system it intends to describe. We develop a modification of Niiniluoto's proposal that defines truthlikeness for quantitative deterministic laws as a function of two factors: accuracy and nomicity. The latter represents the qualitative behaviours implied by a law that are not captured by value comparison and can be measured by shape similarity, appealing to the Euclidean distance between the corresponding derivative functions. The final proposal solves the presented counter-examples and defines a new way of understanding scientific progress.

The framework is expanded to cover probabilistic laws, which represent a relevant subset of actual scientific laws. When this research was developed, there were almost no proposals in the literature of truthlikeness to deal with probabilistic laws or probabilistic truths in general. In this way, we

followed Niiniluoto's suggestion to use the Kullback–Leibler divergence to define the distance between a probability law X and the true probability law T and we argue that the Kullback–Leibler divergence seems to be the best of the available probability distances to measure accuracy between probabilistic laws. However, as in the case of deterministic laws, we argue that accuracy represents a necessary but not sufficient condition, as two probabilistic laws may be equally accurate and still one may imply more true or truthlike probabilistic consequences, behaviours or facts about the system than the other. The final proposal defines truthlikeness for probabilistic laws again as a function of accuracy and nomicity, in intimate connexion with the proposal developed for deterministic laws.

Keywords

Truthlikeness · Verisimilitude · Legisimilitude · Scientific Laws · Deterministic Laws · Probabilistic Laws · Similarity · Distance · Accuracy · Nomicity · Philosophy of Science

CONTENTS

PREFACE

In a possible world, Aristotle, Newton and Einstein enter the stage. The audience applauds enthusiastically for the imminent debate that is going to take place. The tickets had been resold on eBay for more than a thousand euros. The three physicists take a seat and the moderator initiates the debate with the question: 'Why does a stone fall when we drop it near the surface of the Earth?'.

'All things have a natural place in the Universe' Aristotle begins. 'The earth element is situated on its centre and the stone is composed mainly of the earth element. Therefore, when we drop it, it has a *natural tendency* to go down, towards its *natural place*'.

'Your theory has been proven to be false' Newton replies. 'Objects exert a force of attraction among them. The stone falls because of the *gravitational force* that the Earth exerts on it'.

'No way' Einstein intervenes. 'There are no such forces. The Earth's mass curves the *space-time* near its surface. The stone just follows the shortest possible path'.

'Well' the moderator says. 'Your theory, Dr Einstein, is probably and strictly speaking false too'. She turns to the audience. 'You see! All past scientific theories are false and current scientific theories are probably and strictly speaking false too. How could someone talk about scientific progress? There is no progress in moving between falsehoods. We have better take a pragmatist attitude towards science. Truth is not something that science has or can achieve'.

A person in the audience stands up, visibly angry with the moderator's comment. 'Anything to add, Sir Karl?' the moderator asks. 'All scientific theories might be false' Karl claims. 'But they do not need to be on a par. Some may be *closer to the truth* than others'.

A second person stands up, shaking his head. 'Larry Laudan here' He says. 'Until someone explains to me what *closer to the truth* means, these sort of claims are just so much mumbo-jumbo'

* * *

My interest in the notion of truthlikeness appeared as a consequence of my previous interest on scientific realism. As a graduate student, I considered the appeals to the notion of *closeness to the*

truth as a very natural and elegant solution to many of the problems discussed in the literature on scientific realism. However, the concept was usually not elucidated and my philosophical curiosity remained unsatisfied. This changed with Ilkka Niiniluoto's *Truthlikeness* (1987), which opened an exciting and entirely new philosophical world to me.

I am deeply thankful to my supervisor, Jose Díez, for all his support and inestimable criticisms which have undoubtedly raise the quality of my research. Not only that, but he has particularly influenced my way of looking at philosophy, for which I will always be grateful.

I wish also to specially thank Ilkka Niiniluoto, who I had the honour to meet in Prague during CLMPST 2019. He gave me a very warm welcome into the truthlikeness family and some months later he offered me the opportunity to do a research stay in Helsinki. I have learned a lot from his works, our conversations and correspondence.

I shall also thank Gustavo Cevolani and Theo Kuipers for great comments on previous versions of the presented papers; Stathis Psillos, for being the first to comment a draft of this research and whose positive words meant a lot to me; members of LOGOS (Research Group in Analytic Philosophy, Barcelona) in general and in particular LOGOS' people of philosophy of science, all of them have had influenced on my philosophical training over the last four years; and Howard Holland, for kindly taking the time to correct my typing. Finally, I am thankful to all those who attended any of my talks and asked difficult questions that helped to improve the points developed in this research.

I would like to finish by deeply thanking my parents, Mª Ángeles and Carlos, and my family, for all the encouragement and support. Isabel, for her endless positive energy in the most frustrating moments. And Albert, Alex, Oriol and David, for stoically putting up with my long conversations about truthlikeness.

* * *

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OVERVIEW OF THE DOCTORAL PROJECT

1. Introduction to the notion of truthlikeness

1.1. Intuitions about truthlikeness

The notion of truthlikeness intuitively emerges from considering the fact that the different potential answers to a cognitive problem may not be on a par regarding the true answer. Theories and propositions seem to present a property, independent from their truth value, which locates them in a kind of topological space regarding their *closeness* to the true theory or proposition of a given cognitive problem.

Consider the problem regarding the number of planets in our Solar System. This problem has (2021) the true answer 'eight planets'. However, there are plenty (infinite perhaps) of other potential answers. Consider the following two possible answers:

a) The number of planets in our Solar System is ten.
b) The number of planets in our Solar System is ten billion.

Although a) and b) are false, they do not seem to be on a par. Intuitively, answer a) seems to "correspond better" to the facts or to present "less error" than answer b). The notions of 'better correspondence' and 'less error' might need philosophical clarification, but at the level of intuitions they seem to posit a difference between both potential answers.

Intuitions do not seem to be restricted to numerical examples. Consider the cognitive problem regarding the shape of the Earth. Two potentials answers might be:

c) The shape of the Earth is a sphere.
d) The shape of the Earth is a cube.

The actual shape is very similar to an oblate spheroid, a sphere flattened at the poles and a bit bulged at the equator. Again, answers c) and d), although being false, do not seem to be on a par regarding some property. Answer c) seems to be a better description of the actual shape of the Earth than answer d).

Moreover, intuitions do not seem to be restricted to false potential answers either. Consider the following two alternative true possible answers regarding the first cognitive problem:

e) The number of planets in our Solar System is between seven and ten inclusive.
f) The number of planets in our Solar System is between one and ten billion inclusive.

Once more, there seems to be some property regarding which e) and f) are not on a par. Answer e) seems to be "more accurate" or "closer" to properly capturing the actual number of planets than answer f).

Further, there seem to be cases where a true potential answer "corresponds better" to the facts than a false potential answer. In this sense, answer e) seems to be "more accurate" or "closer" to the actual number of planets than answer b).

Finally, there may be cases where a false potential answer might "correspond better" to the facts than a true one. Consider the following two possible answers regarding the first cognitive problem:

g) The number of planets in our Solar System is 7.
h) The number of planets in our Solar System is either 1, less than 1 or greater than 1.

As in the previous cases, answers g) and h) do not seem to be on a par. Answer g) seems, at least, more "informative" or "accurate" than answer h) regarding the cognitive problem in question. Note that the property with respect to that they are not on a par, is not that of "closeness to the truth *in general*" or "closeness to *being* true" (see section 1.2.). In that case, h) would obviously be closer to being true than g), as h) is indeed true.

What these cases seem to point out, is that different potential answers to a cognitive problem, besides the property of being true or false, present an independent property *P* related to their *degree of better or worse correspondence* to the facts. As in a Tarskian correspondence theory of truth the notion of 'correspondence to the facts' can be taken as a synonym for 'truth' (Popper, 1963, p. 303), we may take the notion of the 'degree of better or worse correspondence to the facts' as expressing the degree of closeness, likeness, or similarity to the truth, and naturally call this property *P* 'truthlikeness' or 'verisimilitude'. Then, truthlikeness, in a first rough characterization, can be defined as a property of a theory or a proposition that represents its closeness, similarity or likeness to the truth.

As an independent property from the truth value of potential answers, what the previous examples seem to show is that:

i. Some falsehoods can be closer to the truth (in question) than some falsehoods.
ii. Some truths can be closer to the truth (in question) than some truths.
iii. Some truths can be closer to the truth (in question) than some falsehoods.
iv. Some falsehoods can be closer to the truth (in question) than some truths.

Condition i) is probably the most interesting possibility for the philosophy of science and science in general. In this way, the notion of truthlikeness entered in the philosophical scene during the early sixties, relatively late in comparison to other relevant philosophical notions. One of the main reasons, according to Oddie (2016), was the abandoning, by mainstream philosophers during the latter half of the twentieth century, of the traditional Cartesian goal of infallible knowledge for a fallibilist conception of human inquiry. This left the philosophical landscape dangerously close to scepticism, pessimism or pragmatism: if all we can tell about theories and propositions is their truth value and we are forced to conclude that we may never have conclusive reasons to ascribe truth to a theory, then the only presumable available criteria to choose among theories seems to be related to pragmatic or instrumentalist considerations. However, the notion of truthlikeness emerged as a lifesaver: one may embrace fallibilism and still conclude that a theory A is preferable to a theory B *because* of considerations related to truth, as A may be closer to the truth than B.

Still at the intuitive level, we seem to postulate a truthlikeness relation regarding a large number of scientific theories and fields, so that the notion emerges as a natural explanation of theory success and scientific progress. Some examples are given in Figure 1 (where '$>_t$' stands for 'closer to the truth than').

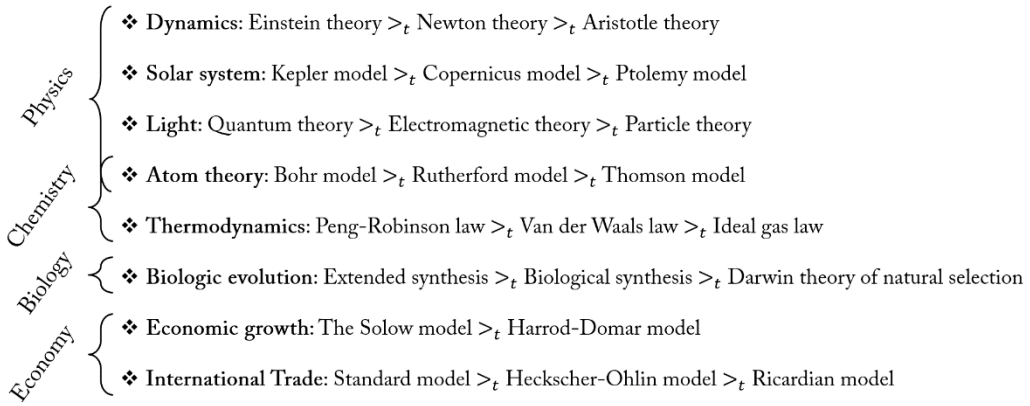

Figure 1. Intuitive truthlikeness relations between scientific theories.

Note that, if we don't move beyond the true/false dichotomy, all we can tell about all (or many) of the listed theories in Figure 1 is that they are false and simply false. Truthlikeness aims to overcome this limitation: although the dynamical laws described by Aristotle, Newton and Einstein are (probably and strictly speaking) false, we have the strong intuition that the Einsteinian world is closer to "how the world is" than the Newtonian world, and both seem to "correspond better to the facts" than the Aristotelian world. That was, precisely, one of Popper's original motivations to introduce the notion of truthlikeness: to make sense of the idea that

different scientific theories might be false and yet constitute progress towards the truth, where progress is explained in terms of increasing truthlikeness.

As a final and very illustrative example, in thermodynamics the Van der Waals law of real gases is usually presented as a clear example of scientific progress regarding the Ideal gas law, being the former closer to the actual behaviour of gases than the latter. Interestingly for our topic, Barnett describes in his classical *Chemical Engineering Thermodynamics* this scientific progress in terms of truthlikeness: "it is clear that this equation [Van der Waals] is a very much closer approximation to the truth than the ideal gas equation" (1944, p. 175).

1.2. What truthlikeness is not

Prima facie, the concept of truthlikeness is different from the notions of "closeness to being true", "being an approximate truth", "being a partial truth", "vagueness" and "epistemic probability". Not to beg the question, we will try to show that in each case there seems to be a (same) property P (truthlikeness) that captures different information than what is captured by the listed notions.

Regarding "closeness to the truth" (truthlikeness) and "closeness to being true", imagine that the actual speed of light was exactly $300.000\ km/s$. Then, the claim that the speed of light is between $300.001 km/s$ and $900.000\ km/s$ would be very "close to being true": just by increasing in a tiny amount the postulated interval ($+1 km/s$), the claim would turn out to be true. In comparison, the claim that the speed of light is $299.998\ km/s$ would be less "close to being true" than the former, as one would need to vary the claim by $2 km/s$ in order to make it true. However, the latter claim would be much better in another sense: it would be much more informative and closer to accurately describe the target fact. This latter property is precisely what truthlikeness aims to capture.

A statement with a high degree of "closeness to being true" might be said to be "approximately true", "almost true" or "nearly true", and the notion might be defined appealing to the minimum distance of a theory or a proposition from the actual world or the complete truth. However, an "approximately true" statement (in this sense) may not be very truthlike, as its information content might be low. In this sense, a "quasi-tautology" is very "approximately true", but not informative at all. Moreover, all truths have a maximal degree of "closeness to being true", as they are indeed true, but they might have different degrees of truthlikeness, as argued in section 1.1.

In this regard, the adjective "approximately true" might be said to correspond to the notion of "approximate truth", which may be taken as a synonym of the notion of "closeness to being true". However, the notion of "approximate truth" or "being an approximate truth" might have an alternative reading, if used to capture the idea of *true* claims that are not sharp or precise

(Niiniluoto, 1987, p. 177). Under this reading, the claim that the speed of light is between $298.000\ km/s$ and $302.000\ km/s$ has a high degree of "being an approximate truth", as it is true and highly accurate, whereas the claim that the speed of light is between $200.000\ km/s$ and $400.000\ km/s$ has a considerably lower degree of "being an approximate truth". In comparison, the claim that the speed of light is $299.999\ km/s$ is presumably very truthlike, but it is not an "approximate truth" in the explicated sense, as it is false.

To sum up, consider the following three claims:

a) The speed of light is $299.998\ km/s$
b) The speed of light is between $300.001 km/s$ and $300.100\ km/s$
c) The speed of light is between $299.997\ km/s$ and $300.003\ km/s$

As we will see, according to the developed theories of truthlikeness, a) is the most truthlike claim. In comparison, b) is the most "approximately true" claim and c) is the only "approximate truth", (if these notions are taken in the explicated sense).

In this way, as will be shown in section 1.3, a notion related to these concepts has been postulated by many authors as indispensable for a plausible defence of scientific realism. Roughly, the realist's idea is that our best developed theories (including the unobservable parts) work because they are "close to the truth", so that our best scientific theories give truthlike or "approximately true" descriptions of both the observable and the unobservable aspects of the world. However, in general, the literature on scientific realism seems to fail to distinguish the expounded notions. For example, in the entry about Scientific Realism of the Stanford Encyclopedia of Philosophy, Chakravartty (2010) claims:

> Realists regarding scientific knowledge… have special need of a notion of *approximate truth*… the realist requires some means of making sense of the claim that [theories] may be false and yet *close to the truth*, and increasingly so over time [our emphasis].

Based on what has been exposed, "approximate truth" and "close to the truth" are different notions (taking "close to the truth" as "being truthlike"). As argued, a theory might be "close to the truth" and yet not an "approximate truth", and a theory T_1 can be "closer to the truth" than a theory T_2 and yet T_2 may be more "approximately true" than T_1 (in the previous example, a) is more truthlike than b) whereas b) is more "approximately true" than a)).

Like this, scientific realism is sometimes formulated by claiming that our best developed theories are "approximately true", but if "approximately true" is understood in the sense of having a high degree of "closeness to being true", the notion would not work for the realist. As mentioned, a quasi-tautological theory would be very "approximately true", but this theory would not presumably instantiate the kind of property that the realist has in mind. What is presumably

meant by the claim that our best theories are "approximately true" is that they are "truthlike", in the sense of having a "sufficiently high degree of truthlikeness". Then, what the realist presumably needs is a notion of "being truthlike" (see section 1.4).

Furthermore, a possible sense of the notion of "partial truth" defines a claim which is true but does not express the "whole truth" in question. For example, a witness in a court may not lie but conceal some relevant facts, telling just "a part" of the whole truth (Niiniluoto, 1987, p. 175). Then, presumably, a claim may be very truthlike without being "partially true" in this explicated sense.

Moreover, regarding the difference between truthlikeness and "vagueness", suppose one analyses the latter notion in the many-valued logic style, where statements are assigned a truth value which lies between zero (full falsehood) and one (full truth). Consider the claim 'Isabel is exactly 30 years old'. The proposition 'Isabel is 29.9 years old' does not contain any vague elements, so it would be classified as "clearly false" with a truth value of zero. However, in truthlikeness terms it would presumably be classified as very close to the truth. Therefore, a proposition might be clearly false in vagueness terms and really close to the truth in truthlikeness terms, presenting a minimum degree of vagueness and a high degree of truthlikeness.

Finally, truthlikeness does not seem to match with "epistemic probability" either. Popper (1963, p. 295) was the first to point out a constitutive difference between both notions. Consider first the notion of the content of a theory or a proposition. Given two statements p and q, Popper claims that the informative content of the conjunction $(p \wedge q)$ will always be greater than (or at least equal to) the content of any of its components. If $Ct(p)$ stands for 'the content of p', we have that:

$$Ct(p) \leq Ct(p \wedge q\ ) \geq Ct(q)$$

This condition contrasts with one of the fundamental laws of the calculus of probabilities (being $P(p)$ the probability of p):

$$P(p) \geq P(p \wedge q\ ) \leq P(q)$$

Both conditions imply that, when content increases, probability decreases (and *vice versa*). This brought Popper to conclude that our search for better theories (theories which will imply a growth in our knowledge, an increase in content) goes hand in hand with the search of less probable theories. As content is one of the fundamental components that define Popperian truthlikeness, both conditions imply that when truthlikeness increases, probability decreases (and *vice versa*).

More generally, the divergence between both notions can be easily illustrated by considering how they treat false propositions. The probability of a known false proposition is always zero, but presumably a false proposition can have a very high degree of truthlikeness. Moreover, the

probability of a tautology is always 1, but in general tautologies do not seem to exhibit a high degree of truthlikeness, as they are not informative at all.

1.3. Philosophical virtues

The notion of truthlikeness enables an optimistic understanding of a set of ideas regarding science that one may want to accept individually but that may appear to imply, *prima facie*, a pessimistic or instrumentalist view of scientific theories.

(I) *The nature of scientific theories.* Many (or perhaps all) scientific theories involve abstractions (neglecting some properties) and idealizations (distorting some properties), therefore being strictly speaking false. However, appealing to the notion of truthlikeness one need not claim that they are all on a par: some may be closer to the truth than others.

(II) *Scientific progress.* A quick overview of the history of science seems to show a historical development in terms of *progress*. It seems natural and intuitive to claim that there has been epistemic progress from Aristotle to Newton and from the latter to Einstein regarding the understanding and the explanation of the physical world. However, accounting for the notion of scientific progress may appear challenging, particularly in the (many) cases where a false theory is replaced by another false theory (as the movement from Aristotle to Newton). How is it possible to advance in our knowledge of the world if we move between falsehoods? One possible answer is to claim that scientific progress from a false theory to another false theory is possible because of an increase in truthlikeness[1]. Newton's theory represents progress regarding Aristotle's theory because the former is closer to the truth than the latter.

(III) *Fallibilism.* Fallibilism is nowadays probably the most popular position among epistemologists, philosophers of science and scientists. The view holds that we might never have conclusive reasons to ascribe truth to a theory, so that scientific theories should be taken as hypothetical and always corrigible in principle. As pointed out in section 1.1, this may seem to imply epistemic pessimism, scepticism or pragmatism towards scientific theories. However, although we may not be able to ascribe truth to theories, we may be able to estimate their degrees of truthlikeness, in that, given some evidence, it may be rational to claim that a theory T_1 is more truthlike than a theory T_2 (see sections 1.5 and 4.3).

(IV) *Truth as the aim of science.* Many realist positions hold that truth (informative truth) is the aim (or at least one of the aims) of science. What scientists try to do is to offer complete true

[1] See Niiniluoto (1984) for a defence of scientific progress in truthlikeness terms; Bird (2007) for a critic to the explanation of progress appealing to the notion of truthlikeness; and Niiniluoto (2014) and Cevolani and Tambolo (2013) for replies.

descriptions of (fragments of) the world. However, because of (I) we may raise reasonable doubts that this aim has been or can be realized, which seems to imply that science has an unachievable end. Nevertheless, appealing to the notion of truthlikeness, we might say that the aim of science is achievable in the sense of pursuing a better approximation to it, by scientific theories that grow in closeness to the truth. On the other hand, because of (III) we might never know that science has realized its aim, but with a notion of *estimated truthlikeness* (see sections 1.5 and 4.3) we might be in a position to reasonably claim how that science is approaching it.

(V) *Scientific Realism.* Broadly, scientific realism defends a positive epistemic attitude toward our best developed scientific theories, recommending belief in both the observable and the unobservable parts described by those theories. However, one cannot claim that science works because our best developed theories (including the unobservable parts) are true (roughly, because of (I), (III) and the Pessimistic Induction, among others). Fortunately, the claim can nicely reformulated by suggesting that our best developed theories (including the unobservable parts) work because they are "close to the truth", such that they provide truthlike descriptions of both the observable and the unobservable aspects of the world.

In this sense, truthlikeness has been postulated by many authors as an indispensable notion for a plausible formulation of scientific realism. However, as mentioned in section 1.2, different notions have been used in the literature to presumably represent the same property. As a few examples, Smart (1963) appeals to "nearly true" and "approximately true", McMullin (1970) to "approximate correspondence", Putnam (1975) to "approximately true", Worrall (1982) to "approximately true" and "close to the truth", McAllister (1993) to "close to the truth", Leplin (1997) to "approximately true" and "partially true", Laudan (1981), Boyd (1983, 1990), Weston (1992) and Chakravartty (2007) to "approximately true" and "approximate truth", and Newton-Smith (1981), Niiniluoto (1999) and Psillos (1999) to "truthlikeness".

What the realist seems to have in mind when claiming that our best developed theories are "approximately true" is that, although being strictly speaking false, they provide a "good" description of both the observable and the unobservable aspects of the world, where a "good" description is one that represents "very closely" how the world is. Metaphorically, theories being "approximately true" are to be imagined as a realist painting of a landscape: although a painting does not "perfectly" represent all the shapes and colours, it may be a very "good" or "close" representation of the landscape. In this sense, the notions of "approximately true" and "approximate truth" understood as described in section 1.2 would not do the work for the realist's purposes. What the realist presumably needs is the specific notion of "closeness to the truth" or "truthlikeness" as detailed in this introduction.

The notion of truthlikeness naturally combines all the mentioned ideas, providing an optimistic interpretation related to truth: perhaps all scientific theories are strictly speaking false, but some may be closer to the truth than others; progress is possible because of an increase in truthlikeness; truth might be said to be the aim of science in the sense of pursuing a better approximation to it; our best developed theories (including the unobservable parts) work because they are close to the truth; and finally, one may embrace fallibilism and still be able to estimate than some theories are closer to the truth than others.

To be sure, for each point (I)-(V) other philosophical concepts might do the same job as truthlikeness in the sense of providing alternative explanations that avoid pessimistic or instrumentalist interpretations of science. However, we take as one of the main virtues of truthlikeness its ability to optimistically interpret (I)-(V) through a unique concept, in what we take in a very natural and intuitive way. In this sense, truthlikeness allows for an optimistic middle position between infallibilism and scepticism regarding scientific theories.

1.4. Types of truthlikeness statements

Being T_1 and T_2 two theories or propositions, one may find, at least, three different types of statements which might be interesting to frame in terms of truthlikeness:

(a) T_1 is closer to the truth than T_2.
(b) T_1 is close to the truth.
(c) The degree of truthlikeness of T_1 is x.

Claims of type (a) need a comparative condition which can be qualitative or quantitative and provide a possible explanation of scientific progress. Claims of type (b) need the introduction of a threshold which establishes the condition of possessing or not the property in question, which again can be done in a qualitative or quantitative way. Those types claims are of special interest for the scientific realist, who would like to claim that our best developed theories "are truthlike" (though being able to establish claims of type (a) might be already relevant for a modest realist). Finally, claims of type (c) need the introduction of a metric, being therefore only available to quantitative definitions of truthlikeness.

A metric on a set is just a function that defines the distance between each pair of elements of the set, being a numerical description of how distant objects are from each other. A set X with a defined metric d constitutes a metric space (X, d). Formally, a metric d on a set X is a function $[d\colon X \times X \to [0, \infty)]$, so that for all $x, y, z \in X$:

i. $d(x,y) \geq 0$
ii. $d(x,y) = d(y,x)$
iii. $d(x,y) = 0 \leftrightarrow x = y$
iv. $d(x,z) \leq d(x,y) + d(y,z)$

Although a quantitative definition is not necessary for a comprehensive definition, the introduction of a metric arguably provides a more fine-grained notion. However, it may involve some arbitrary conventions (see section 4.2).

1.5. The semantic and the epistemological problems

As with many other philosophical concepts, one must clearly distinguish between: (a) the logical or semantic problem and (b) the epistemological problem. Popper (1963, p. 317) elegantly formulated them for the case of truthlikeness as:

> …we have here again to distinguish between the question "What do you intend to say if you say that the theory t_2 has a higher degree of verisimilitude than the theory t_1?", and the question "How do you know that the theory t_2 has a higher degree of verisimilitude than the theory t_1?"

The former has to do with the philosophical definition of the notion and needs to show how it is meaningful to claim that a theory T_2 is more truthlike than a theory T_1. The latter has to do with our epistemic access to the conditions defining the concept and needs to show how, given some evidence, it is rational to claim that a theory T_2 is more truthlike than a theory T_1.

Keeping both notions clearly distinguished is crucial, as they obey different aims and require a different treatment. The most common (negative) reaction to the notion of truthlikeness we have encountered during our research, seems to focus exclusively on the epistemological problem, presumably overlooking the virtues of a satisfactory solution to the semantic problem.

This common reaction goes roughly as follows: in order to claim that "a): the number of planets in our Solar System is ten" is closer to the truth than "b): the number of planets in our Solar System is ten billion" (or any other comparative truthlike claim), we need to know *what* the truth actually is. But then:

(1) If we know what the truth is, then it seems of minor interest to know which theories or propositions are closer to the truth than others (we already know what the truth is).

(2) If we don't know what the truth is, then it seems impossible to make claims about the truthlikeness of theories or propositions, as we cannot establish their closeness to the truth. So for many (perhaps all) real scientific cases where the notion would be useful, it cannot be applied.

∴ Therefore, the notion of truthlikeness is either uninteresting or useless.

As we have encountered this reaction so many times, it may be worth discussing it in a bit of detail.

On the one hand, (1) may be disputable. Even if we knew the truth, establishing the degrees of truthlikeness of different theories could be of interest for the sake of studying or justifying the

historical evolution of science. For example, assume that we come to know that Einstein's theory is the true theory of our universe. Then, it might be an interesting task to calculate the degrees of truthlikeness of Newton's theory and Aristotle's theory in order to study the evolution of scientific theories regarding the physical world. A parallelism might help to illustrate the point: even knowing at some time t that the world's $100m$ record is x seconds (and even considering that, for some reasons, this record could not be overcome ever), it still might be an interesting task to graphically represent past world's $100m$ records in order to study the evolution of athletics.

In any case, even granting (1) as unproblematic, the main challenge and negative reaction to the notion of truthlikeness seems to mainly come from (2). As we do not generally know the truth with certainty, the notion seems inapplicable to real scientific theories and therefore epistemically useless. As mentioned, this criticism targets the epistemological problem. Let us assume that this problem was irresolvable in any sense[2], that is to say, that given some available evidence there were no possible rational procedure to estimate that a theory T_1 is more truthlike than a theory T_2. Even then, we take the semantic aspect of truthlikeness as providing enough philosophical utility to make the concept valuable: showing that the notion is meaningful justifies an optimistic philosophical interpretation of (I)-(V), as argued in section 1.3. Chakravartty (2007, p. 213) offers a similar consideration, claiming that being able to respond to the antirealist scepticism by proving that truthlikeness is a coherent idea is the main goal of a theory of truthlikeness.

Moreover, if one sets the possible interest of a philosophical concept on the possibility of its (infallible) applicability to real cases, our guess is that very few (or none) philosophical concepts would end up being valuable. In that sense, (2) seems to be an unfair reaction to truthlikeness, as the possible epistemic complications do not seem to undermine the philosophical value of many other concepts. Let us mention a couple of examples to illustrate the point.

On the one hand, the notion of knowledge might be defined appealing to "justified true belief" or to some other post-gettier set of necessary and sufficient conditions. However, many (perhaps all) of the proposed definitions cannot be infallibly applied to concrete cases, as they all involve truth as a necessary condition. Even so, providing a satisfactory definition of knowledge seems valuable *per se*.

On the other hand, in the debate about the nature of laws of nature, the realist proposals appeal to some sort of necessity, which is not logical but nomological. The realist intuition is that laws of nature *govern* the evolution of events that constitute our Universe. In this way, laws have been conceptualized as *necessitarian relations* between universals (Dretske 1977; Tooley 1977;

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[2] We take this assumption to be false, as we believe that there may be rational ways of estimating the degree of truthlikeness of a theory based on some available evidence. See section 4.3.

Armstrong 1983), as *powers* or *dispositional essences* (Woodward, 1992; Ellis, 2001; Bird, 2005), as *ontological primitives* (Maudlin, 2007); as *causal/explanatory regularities* (Carroll, 2008) or as *members of a counterfactually stable set of propositions* (Langue, 2009). All this proposals make laws epistemically inaccessible in the same way as (2), but still many philosophers find them valuable.

To sum up, if the negative reaction towards the notion of truthlikeness lies in the impossibility of its infallible application to real scientific cases, then we take the reaction as unfair. A solution to the semantic problem seems sufficient to make the concept valuable and there may be rational (and fallible) ways of estimating the degree of truthlikeness of a theory based on some available evidence (see section 4.3). For those reasons, we take the "negative-reaction argument" as an unsound criticism.

1.6. Historical development of the notion of truthlikeness

1.6.1. Popper

Popper (1963) was the first to take the concept of truthlikeness seriously and to give a formal definition of it as a function of two variables: *content* and *truth value*. Departing from Tarski's theory of truth, his purpose was to capture the idea of a *better or worse correspondence to the facts*. Take a theory T^A to be identified with the set A of all its logical consequences. If T^A is true, A will only contain true propositions, whereas if T^A is false, A will contain true and false propositions. Call the set of all true consequences of T^A its "truth content" (A_T) and the set of all false consequences its "falsity content" (A_F). Thus, $A = A_T \cup A_F$. Then, Popper's natural idea is that T^A is more truthlike than T^B if and only if (i) T^A has more truth content than T^B and at least the same falsity content or if (ii) T^A has at least the same truth content as T^B and less falsity content. As the sets A and B could have infinite elements, the comparison must be stated in terms of set inclusion. Then, T^A is more truthlike than T^B if and only if:

$$A_T \supset B_T \ \wedge \ A_F \subseteq B_F \qquad or \qquad A_T \supseteq B_T \ \wedge \ A_F \subset B_F$$

This would have been a virtuous solution to the semantic problem, but unfortunately Miller (1974) and Tichý (1974) proved independently that Popper's definition didn't work in the intended way. Their argument goes as follows. Consider that T^A and T^B are false and that $A_T \supset B_T \wedge A_F \subseteq B_F$. Then, T^A has at least a true consequence q such that $q \in A_T$ and $q \notin B_T$. Moreover, T^A and T^B will have at least one common false consequence r, such that $r \in A_F$ and $r \in B_F$. Now, as $q \in A$ and $r \in A$, then $(q \wedge r) \in A$. As r is a false consequence, then specifically $(q \wedge r) \in A_F$. But crucially, $(q \wedge r) \notin B_T$. Therefore, it is not true that $A_F \subseteq B_F$, as was assumed.

A similar case can be constructed if option two of the definition is assumed ($A_T \supseteq B_T \wedge A_F \subset B_F$). What the argument shows is that if T^A and T^B are false, then neither option of the *definiens* can be realized, so that Popper's proposal implies that no falsehood is closer to the truth than any other falsehood. This is a fatal flaw, as allowing for a truthlikeness ordering between falsehoods is probably the core motivation of the notion of truthlikeness.

1.6.2. Two approaches to truthlikeness

Since Popper's failure, the notion of truthlikeness has become a topic of intense discussion by philosophers of science and logicians. One may find in the literature two broad approaches to deal with the concept. On the one hand, what we may call the *official approach* emanates directly from Popper's analysis and can be considered the "institutionalized" approach to the notion. The proposals of the official approach agree on a basic general framework and principles, and tend to produce quantitative or logical definitions of the notion. On the other hand, what we may call the *unofficial approach* incorporates different definitions given by philosophers usually unrelated to the official approach. These proposals tend to be qualitative, appearing in the broader context of scientific realism and analysing truthlikeness informally, in terms of another notion that remains at an intuitive understanding. Let us emphasize that the chosen names are just labels for a possible classification, without any intended type of positive or negative judgements. The two approaches and the corresponding main authors are summarized in Figure 2.

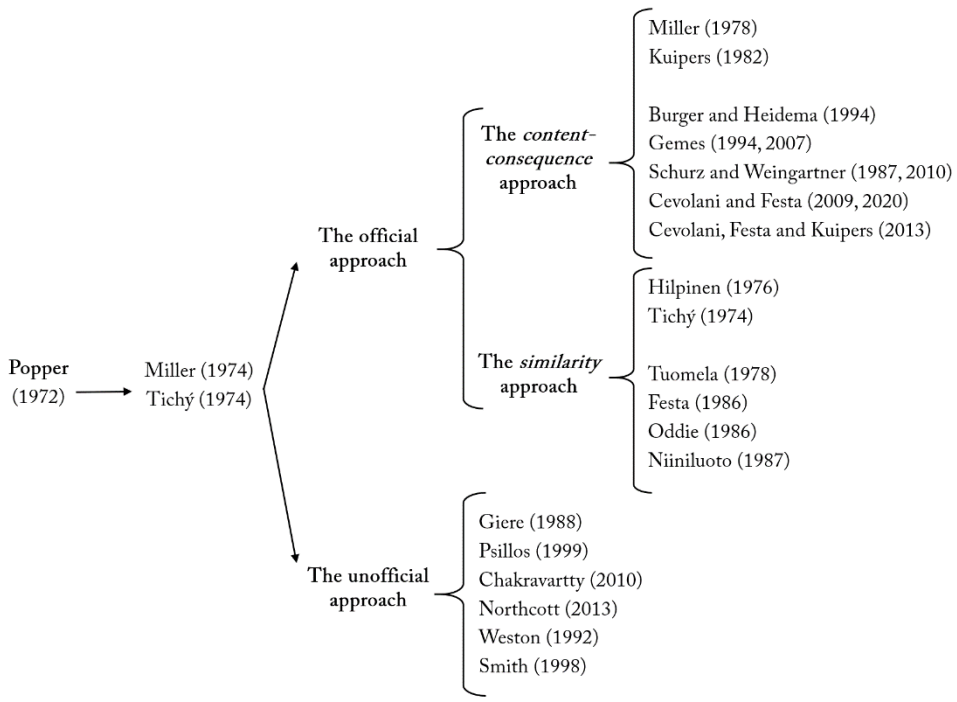

Figure 2. Historical approaches to the notion of truthlikeness.

1.6.3. The unofficial approach

The notion of "limiting case" has been suggested as a relation of truthlikeness between theories (Chakravartty, 2017). If theory T_1 is a limiting case of theory T_2, then T_2 is closer to the truth than T_1. The classical example is the degeneration of the Einsteinian equations into the Newtonian equations when velocity tend to zero. In that case, Einstein's theory might be said to be more truthlike than Newton's theory because the latter is a limiting case of the former. Such a definition, even if correct, considerably restricts the application of the concept, not allowing for truthlikeness comparisons between rival theories which do not instantiate inter-theory reductive relations.

Giere (1988, p. 106) argues that the notion of "approximation" for scientific theories should not be understood in terms of the "bastard semantical relationship" (p.106) idea of *approximation to the truth*. He seems to share Laudan's (1981) arguments that no one has presumably provided a successful definition of truthlikeness (though he just mentions Popper's (1963) and Newton-Smith's (1981) proposals). Moreover, he argues that truthlikeness is not a kind of truth, but a kind of falsehood, as he takes that "being close to the truth" implies being "not exactly true", which in turn means "being false". This is questionable, as shown in section 1.1, and false according to the official approach: a theory can be closer to the truth than another even when both are true, such that "being close to the truth" does not imply "being false".

Giere's proposal then defines the "approximation" of a (false) scientific theory in terms of the notion of "similarity", which is the basic relation he postulates between scientific models and real systems. Sadly, the notion remains undefined in his proposal, but he offers the following example: "the positions and velocities of the earth and moon in the earth-moon system are very close to those of a two-particle Newtonian model with an inverse square central force" (1988, p. 81). With that example in mind, we take Giere's anti-truthlikeness arguments as a kind of verbal confusion. If the earth-moon system represents the "real system" (a set of true facts), then similarity to the earth-moon system literally means similarity to the truth (similarity to the target set of true facts). Therefore, we take Giere as indeed defining a notion of qualitative truthlikeness[3] in terms of an intuitive notion of similarity.

Psillos (1999), after what we take as some unfair refusal of the similarity approach, defends what he calls an "intuitive approach", based on understanding truth as *fittingness* and therefore accounting for truthlikeness in terms of *approximate fittingness*. A description, a law or a theory is truthlike if it *fits* as *accurately* as possible the relevant facts. Based on these ideas, he proposes the following definition of "being truthlike" (p. 268):

[3] A similar criticism is given by Psillos (1999:265)

> A description D *approximately fits* a state S if there is another state S' such that S and S' are linked by specific conditions of approximation, and D *fits* S'.

As an example, he mentions that the Ideal gas law is truthlike since it is true for ideal gases and the behaviour of real gases *approximates* that of ideal gases under certain conditions. Note that this definition only allows for absolute truthlikeness claims (statement of type (ii), section 1.4). On personal communication (2018[4]) he stated a similar definition for comparative truthlikeness:

> A description D1 of the actual worldly state S is more truthlike than a description D2 if there are states S1, S2 such that D1 approximately fits S1, D2 approximately fits S2 and S1 is a better approximation to S than S2.

Then, a theory T^A is closer to the truth than a theory T^B if T^A approximately fits better the actual world state S than T^B. As in the case of Giere, a natural reaction might be to point out that one has substituted a notion that was in need of clarification (truthlikeness) for another notion ("approximate fittingness", "approximation") which remains vague and undefined. However, Psillos (1999, p. 268) considers that the notion of "approximation" is easier to make more precise and concrete than the notion of truthlikeness.

According to Chakravartty (2010), the official (quantitative) approaches have ignored the intrinsic qualitative dimension of truthlikeness and have not analysed the actual ways in which scientific theories diverge from truth. He concludes that truthlikeness is a concept that is *multiply realized* by means of the different ways in which scientific representation works: degrees of truthlikeness are represented by degrees of idealizations, by degrees of abstraction or by a combination of both factors.

Northcott (2013) defines truthlikeness according to how well a theory captures the relevant *causes* of the world. A theory is close to the truth if it accurately captures the strengths of the causes present in a given phenomenon. The better a theory correctly weighs the actual causal structure of reality –limited to some specific context- the more truthlike it is. "Closeness" between a theory's postulated causal weightings and the true causal weightings is defined by the absolute difference or Manhattan distance, which is one of the considered distance measures in Paper 1 and Paper 2.

Other proposals that could be included in this category but focus on the notion of 'approximate truth' include Weston (1992) and Smith (1998). Roughly, Weston defines 'approximate truth' in terms of accuracy: a statement is approximately true if it is true according to any method of measuring error and within any allowable margin of error. As error measuring and acceptable

[4] Psillos presented this proposal in the workshop *Explanation and realism in the physical and biomedical sciences* (University of Barcelona, 2018), during his comments on García-Lapeña's *Truthlikeness and scientific realism.*

limits are context dependent on each specific scientific field, he concludes that a neutral notion of 'approximate truth' is unattainable.

On the other hand, Smith (1998) departs from the basic schema that <'P' is approximately true if and only if approximately P>. He then takes scientific theories to be represented by the tuple $< M, A >$: an abstract mathematical structure M and its empirical applications A. As M is a perfect characterization of some geometrical structure, the approximation operation of the schema is transmitted only to A. Therefore, a theory being approximately true means that the geometric structure defined by the theory approximately replicates the corresponding structure of the target real-world phenomenon. From here, he concludes that being approximately true is nothing mysterious: it just requires us to define an appropriate geometrical closeness relation between structures. This final idea is very similar to Niiniluoto's proposal and to our own framework developed in Paper 1 and Paper 2.

1.6.4. The official approach

The official approach can be divided into two main perspectives, the (i) *content-consequence approach* and the (ii) *likeness or similarity approach* (being the *consequence approach* a special modification of the *content approach*). As there are already a number of excellent presentations of these proposals (Niiniluoto (1987), (1998), (2020) and Oddie (2013), (2016)), we will limit ourselves to a general exposition.

1.6.4.1. The content-consequence approach

Perhaps the core element of the *content approach* is the idea that truthlikeness supervenes on just two variables: content and truth value. In this sense, the main problem with Popper's proposal is that all consequences are given the same value, but this may be questionable. Take a theory T^A to imply the true consequence p and the false consequence $\sim q$. Popper's intuitive idea is that p should count in favour of T^A's degree of truthlikeness and $\sim q$ against it. However, once we have counted p and $\sim q$, it may seem *redundant* to also count T^A's false consequence $(p \wedge \sim q)$ against its degree of truthlikeness. Moreover, the classical notion of logical consequence implies that in the previous example $(p \vee \alpha)$, being α an arbitrary formula, would be a true consequence of T^A. But again, it may seem *irrelevant* to count these kinds of implications in favour of T^A's truthlikeness. These types of presumably irrelevant and redundant consequences are the core of the Tichý-Miller's argument.

This suggests that Popper's proposal may be saved by taking into account only the "relevant" consequences of a theory, restricting the elements that conform its truth content and its falsity

content to some special set of consequences. This is the core idea of the *consequence approach*, which implies the specification of a criterion R that differentiates between relevant and irrelevant consequences. As an important constraint, the conjunction of all the relevant consequences should be logically equivalent to the original theory, so no information is lost by the relevant criterion. Then, truthlikeness can be defined in the Popperian way: a theory T^A is more truthlike than a theory T^B if and only if T^A's *relevant* truth content is larger and its *relevant* falsity content is at least the same as those of T^B; or if T^A's *relevant* falsity content is smaller and its *relevant* truth content is at least the same than those of T^B. Proposals that follow this strategy include Burger and Heidema (1994), Gemes (2007), Schurz and Weingartner (1987, 2010), Schurz (2018, 2021), Cevolani, Festa and Kuipers (2013) and Cevolani and Festa (2020).

Miller (1978) and Kuipers (1982) offered a structuralist model-theoretic version of Popper's account. A theory T^A framed in language L is defined by its class of models $Mod(T^A)$ (the class of L-structures M where T^A is true). Then, T^A is at least as truthlike as T^B if and only if (where 'Δ' stands for the symmetric difference):

$$Mod(T^A)\Delta Mod(T) \subseteq Mod(T^B)\Delta Mod(T)$$

Kuipers' first "naive definition" can be considered as belonging to the consequence approach, as he takes the truth T to represent the "nomic truth", the special set of "real" possibilities (physical, chemical, biological, etc.) among all the conceptual possibilities of a given domain, and a theory T^A is framed as a conjunctive set of relevant consequences (i.e., nomic consequences). This proposal can lead to the Miller–Tichý's problem when the truth is a complete theory. Later (1987a, 1987b, 2000) he developed a "refined definition" which appeals to a qualitative notion of similarity in order to allow that not all false consequences are on a par. This later proposal can be considered as a structuralist version belonging to the similarity approach.

1.6.4.2. The similarity approach

The similarity approach to truthlikeness was first proposed by Hilpinen (1976), within possible world semantics, and Tichý (1974), within propositional logic, and rapidly expanded by Niiniluoto (1987), Oddie (1986), Tuomela (1978) and Festa (1986), among others. Tichý (1974) elegantly sketched his proposal at the end of the paper criticising Popper's account:

> For a simple language which, like L, is based on propositional logic only, this is easily done. The 'distance' between two constituents can be naturally defined as the number of primitive sentences negated in one of the constituents but not in the other. The verisimilitude of an arbitrary sentence a can then be defined as the arithmetical mean of the distances between the true constituent t and the constituents appearing in the disjunctive normal form of a. It is easily seen that such a definition meets all intuitive requirements.

The core idea of the similarity approach can be captured by the claim that the truthlikeness of a theory or a proposition rests on the *similarities* between the states of affairs it allows and the actual states of affairs of the world. The approach can be summarized as follows. Consider a phenomenon or system P and a language L to talk about P. One can construct a space of possibilities S_P^L which contains all the mutually exclusive and jointly exhaustive ways $(c_1, c_2, c_3 \ldots)$ P can be regarding L, all the possible complete descriptions of P given L[5]. Then, a theory or claim h of L will be expressible as a set of elements of S_P^L. The next step is to introduce a metric $d(c_i, c_j)$ which defines the distance (in terms of similarity) between the elements of S_P^L and an *extension* of d into another metric $d'(h, c_i)$ which defines the distance (in terms of similarity) from a set of elements of S_P^L (theories or claims) to a single element. Then, given a correspondence theory of truth, some element c_t^* of S_P^L will represent the truth in question (the actual world or the most informative true description of the world given L). Connecting all the above, the degree of truthlikeness of a theory or claim h, once d' is normalized, is defined by the similarity between h and c_t^*:

$$Tr(h) = 1 - d'(h, c_t^*)$$

The key element of the similarity approach is its appeal to a (quantitative) notion of likeness or similarity, which is defined according to metrics d and d'. A similarity measure is just the inverse of a given distance function (see section 1.4). However, usually it is not enough that d and d' satisfy the mathematical criteria for metric functions in order to be "good" distance to the truth functions. Additional restrictions may come from clear-cut intuitive cases and/or from general truthlikeness principles.

The distance function d has to be specified for each cognitive problem, but there are natural ways of doing so in most of the cases (Niiniluoto, 1998; p. 4). As a simple example, consider again the number of planets in the Solar System as the target cognitive problem. In that case, the space S_p^L will contain $\mathbb{N}$ elements of the form $c_x =$ <the number of planets in our Solar System is x> $\forall x \in \mathbb{N}$. One natural similarity metric d for this structure will be the absolute difference $d(c_x, c_y) = |x - y|$. Then, as $d(c_9, c_8) = 1 < d(c_{12}, c_8) = 4$, c_9 is closer to the truth (c_8) than c_{12}, which matches our truthlikeness intuitions regarding c_9 and c_{12}.

On the other hand, function d' is supposed to be universal. Given some similarity values defined by an appropriate metric d, the extension d' defines the best "similarity combination". In this way, there are two main proposals in the similarity approach to define function d': Oddie and Tichý have favoured the "average measure" (the average of the distances of the elements that

[5] In Oddie's proposal, the *elements* $c_1, c_2, c_3 \ldots$ represent *possible worlds*. Niiniluoto's framework is more flexible, allowing to represent *state descriptions*, *structure descriptions*, *monadic constituents* or *scientific laws*, depending on our cognitive interests.

constitute h to c_t^*), while Niiniluoto has defended the "min-sum measure" (the weighted average of the minimum distance plus the normalized sum of all the distances of the elements that constitute h to c_t^*). Both proposals are rival hypotheses, as in some cases and for some propositions they conclude in different truthlikeness orderings[6] (see section 1.6.5).

1.6.5. Agreements and disagreements in the official approach

Zwart (2001) proposes that the content-consequence and the similarity approaches can be distinguished by the theory T^{worst} that they consider as the least truthlike theory. To illustrate the difference, consider a simple propositional language where the complete truth is given by $(p_1 \wedge p_2 \wedge p_3)$. Zwart claims that the content-consequence approach takes the negation of the complete truth $(\sim p_1 \vee \sim p_2 \vee \sim p_3)$ as T^{worst}, whereas the similarity approach judges $(\sim p_1 \wedge \sim p_2 \wedge \sim p_3)$ as T^{worst}. Moreover, Zwart and Franssen (2007) argue, via Arrow's impossibility theorem, that both approaches cannot be combined in order to define a unifying truthlikeness ordering (for reactions see Schurz and Weingartner (2010) and Oddie (2013))

Minimally, the content ($Cont$), consequence ($Cons$) and similarity (Sim) approaches can be characterized as:

- $Tr(A)_{Cont} = f\big(truth\ value, content(A)\big) = f(A_T, A_F)$
- $Tr(A)_{Cons} = f(truth\ value, content(A), R) = f(A_T^R, A_F^R)$
- $Tr(A)_{Sim} = similarity(A, T) = f(d(A, T))$

It should be noted that most of the proposals agree on a large number of cases and that for concrete practical applications, almost all accounts would yield a very similar (or same) truthlikeness ordering regarding a selected group of historical scientific theories. Disagreements mainly come from the truthlikeness orderings of disjunctions and implications of theories and propositions. In this sense, the proposals by Niiniluoto (1987), Kuipers (2000), Oddie (2013), Schurz and Weingartner (S&W, 2010) and Cevolani and Festa (C&F, 2020) agree on the following properties[7]:

(P1) The true theory T is more truthlike than any other theory.

(P2) Some false theories may be more truthlike than some true theories.

(P3) A false theory may be more truthlike than another false theory.

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[6] For Niiniluoto's comparison between $average$ and $minsum$ see 1987, Chapter 6.6. See Oddie (2013) for a defence of $average$ based on three general principles.

[7] The chosen properties of agreement and disagreement are partly taken from Niiniluoto (2020).

Discrepancies emerge mainly from two principles. The former has been labelled by Oddie (2013, 2016) 'the strong value of content for truths' and was initially postulated by Popper as a desideratum for a satisfactory definition of truthlikeness:

(P4) Among true theories, truthlikeness increases with logical strength

Proposals by Niiniluoto, Kuipers and S&W satisfy it, while those of Oddie and C&F violate it. Within the similarity approach, this principle is the main difference between *minsum* and *average*. According to *average*:

(P4a) Among true theories, truthlikeness does not always increase with logical strength.

(P4b) Among true theories, truthlikeness does not always decrease with logical strength.

For a simple example, consider the following two propositions regarding the number of planets in our Solar System:

a) The number of planets is eight or twenty.
b) The number of planets is eight or thirteen or twenty.

Where a) is a stronger truth than b). According to Niiniluoto's *minsum* (and Kuipers and S&W) a) is more truthlike than b), while according to Tichý-Oddie's *average* (and C&F) b) is more truthlike than a).

A kind of correspondence principle to (P4) for false theories can be formulated as:

(P5) Among false theories, truthlikeness decreases with logical strength.

The content-consequence proposals (Kuipers, C&F, and S&W) satisfy it (which implies that the negation of the whole truth is considered as T^{worst}), while the similarity proposals (Niiniluoto and Oddie) violate it. According to *minsum* and *average*:

(P5a) Among false theories, truthlikeness does not always increase with logical strength.

(P5b) Among false theories, truthlikeness does not always decrease with logical strength.

For a simple example, consider the cognitive problem regarding the number of moons on the Earth (Niiniluoto, 2020):

c) The number of moons is two.
d) The number of moons is two or two thousand.

Where c) is a stronger falsity than d). According to the content-consequence proposals (Kuipers, C&F, and S&W) d) is more truthlike than c), while according to the similarity proposals (Niiniluoto and Oddie) c) is more truthlike than d). The satisfied principles by each proposal are summarized in Figure 3.

	(P4)	(P4a)	(P4b)	(P5)	(P5a)	(P5b)
Schurz and Weingartner (2010)	×			×		
Kuipers (2000)	×			×		
Niiniluoto (1987)	×				×	×
Cevolani and Festa (2018)		×	×	×		
Oddie (2013)		×	×		×	×

Figure 3. Satisfied principles by each truthlikeness proposal.

In this way, S&W and Kuipers presents the strongest "content" proposals, in the sense that they imply the most strict relations between <truth value, content> and the degree of truthlikeness. Oddie presents the weakest "content" proposal, in the sense of implying the most contingent relations between <truth value, content> and the degree of truthlikeness. Niiniluoto and C&F offer middle-point (non-equivalent) proposals.

Finally, let us briefly mention a presumable problem commonly associated with the official approach. One may read in various works about scientific realism that, up to now, no one has presumably provided a *satisfactory* or *coherent* definition of truthlikeness (e.g. Laudan (1981), Giere (1988), Psillos (1999), Chakravartty (2007)). The legitimacy of this claim seems to partly depend on how one understands the terms "satisfactory" and "coherent". All the presented proposals in section 1.6.5 provide coherent, strongly accurate and well-defined notions of truthlikeness. As stressed, they all share a great number of common principles and consequences, diverging only by some general principles and intuitive cases related to the truthlikeness orderings of disjunctions and implications.

In comparison, one may find less "accuracy" in the different rival philosophical proposals regarding other philosophical notions. As an example, Lewis' (1973, 1983, 1986, 1994) best system account (BSA) of laws of nature roughly defines laws as those regularities that figure out in the best systematization of the universe, where the best systematization (the Best System) is the true one that properly balances two main properties, strength and simplicity. "Strength" and "simplicity", however, still remain undefined in contemporary BSA proposals. And the most known alternative (realist) approach, the Universals account, seems to share the same fate. Under this account, laws are conceptualized as *necessitarian relations* between universals (Dretske 1977; Tooley 1977; Armstrong 1983), but the concrete nature of those kinds of relations remains a mystery. To be fair, both accounts are much richer than what this sketchy presentation suggests, but the point is to illustrate that, although there is not a universally accepted definition of truthlikeness, all the different accounts are extremely accurate in their definitions.

The syntactic and the semantic view of scientific theories, or the descriptivist and the causal theories of reference, are coherent, accurate and well-defined rival views. Our claim is that the different content-consequence and similarity approaches to truthlikeness instantiate this same kind of rivalry. Therefore, the claim that no one has presumably provided a *satisfactory* definition of truthlikeness should be understood in the sense that, up to now, there is no universally accepted definition by all the participants in the debate, mainly because they accept different desideratum and have different intuitions regarding some specific cases. However, this seems to happen with many (perhaps all) philosophical concepts.

2. Research methodology and goals

The similarity approach, particularly Niiniluoto's formulation, has been the main theoretical and methodological framework applied in our research. This has been so for several reasons.

On the one hand, historically the majority of truthlikeness definitions have focussed almost exclusively on qualitative languages (particularly the content-consequence approaches). Many of the examples of application involve very simple propositional languages, which seemed to build a large bridge between the notion of truthlikeness and the application to actual (quantitative) scientific theories. In this sense, a notable exception is Niiniluoto. Already in (1982) he developed a proposal within the similarity approach to define truthlikeness for singular quantitative statements, interval statements and simple quantitative laws. Later, in (1987), he showed how to transform a conceptual space defined by Carnapian Q-predicates into a mathematical *state space* generated by quantities. We take this as one of the main virtues of his approach, as state spaces are the most common conceptual apparatus used in the formulation of scientific theories. This implies a nice connexion between his truthlikeness theory and the actual scientific practice, making its application very natural.

In general, the content-consequence approaches have been silent on how to apply their proposals to cases of numerical approximation, which seem to posit a challenging application. Consider two theories, IG (Ideal gas law) and IG', where the only difference between them is that IG' postulates a more accurate universal gas constant R. Both theories would imply the same set of qualitative true consequences (roughly, that $V \propto 1/P$, $V \propto T$, $P \propto T$ and $V \propto n$) and all their quantitative consequences would be false. Presumably, a content-consequence approach would have to conclude that IG and IG' are equally truthlike, as their sets of true and false consequences would have an equal "size". On the contrary, the similarity approach could capture the intuitive result that IG' is closer to the truth than IG, appealing to the fact that IG' false quantitative consequences are more similar to the actual values of pressure, volume and temperature than the ones implied by IG.

Further, we see a philosophical connexion between many of the unofficial proposals (section 1.6.3) and the similarity approach. For example, we think that the qualitative notions that Giere and Psillos have in mind are in fact the ones captured by the similarity approach, framed in a more

accurate and rich conceptual apparatus. In this way, Popper himself conceived truthlikeness as "the idea of a degree… of greater (or less) likeness or similarity to truth" (1963:315). His mistake, perhaps, was to assume that "similarity" could be well captured appealing only to truth value and content. On the contrary, "similarity" is a well-defined and largely studied notion in mathematics and geometry, and the similarity approach naturally incorporates and applies all this knowledge into the concept of truthlikeness.

For all these reasons, our main goal was to make a contribution to the similarity approach. This took us to focus on the topic of truthlikeness for scientific laws, as the available definitions regarding function d for the case of scientific laws presented some relevant problems.

Typical scientific laws are formulated in quantitative languages. In those languages one can distinguish two main types of quantitative statements: singular quantitative statements and quantitative laws. Cases of singular quantitative statements within scientific theories involve the estimation of some real quantity θ^*, which can be a physical constant (like Avogadro's constant), a parameter (like a and b from the Van der Waals equation) or the prediction by a theory of the value of some quantity. Niiniluoto ((1982), (1986), (1987), (2018)) and Festa (1986) developed similar truthlikeness proposals for singular and interval quantitative statements, where function d is defined by the absolute difference or, in general, by any of the *Minkowski metrics* (see Paper 1). Kieseppä (1996a, 1996b), however, showed that these proposals can fail when the hypotheses compared have different dimensions, because they are based on Lebesgue integrals. For example, if θ^* is a point of $\mathbb{R}^2$, I_1 is a line and I_2 is a surface, then taking a measure of length to compare them assigns an infinite "size" to I_2, while taking a measure of surface assigns a 0 "size" to I_1. Kieseppä then considers possible generalizations of *average* and *minsum* using Hausdorff measures, his proposal being an improvement in the similarity approach regarding singular quantitative statements.

For quantitative (deterministic) laws, the most developed proposal when this research started was that of Niiniluoto, which defined the distance d between a law X and the true law T by either of the values p equal to 1, 2 or "∞" of the *Minkowski metric* (known as Manhattan, Euclidean and Chebyshev distances respectively). However, at least Thom (1975), Weston (1992) and Liu (1999) had offered counter-examples to the use of the Manhattan and the Euclidean distances to define truthlikeness between laws. Moreover, we were able to develop (presumably) more pushing counter-examples, and managed to expand them to the use of the Chebyshev distance.

In this sense, our first goal was to develop a satisfactory definition of truthlikeness for quantitative deterministic laws within Niiniluoto's version of the similarity approach, with the additional purpose to be a positive complement to Kieseppä's improvements over singular quantitative statements. This led to the publication of Paper 1.

Once the framework was developed, a natural move was to expand it to cover probabilistic laws, which represent a relevant subset of actual scientific laws. This expansion, and the aim to offer a general account for both deterministic and probabilistic laws, was the second goal of our research. In the literature of truthlikeness, however, there were almost no proposals of appropriate distances for probabilistic laws or probabilistic truths in general, so that the topic was both challenging and motivating. This changed in 2021 with the publication of the special issue by Synthese *Approaching Probabilistic Truths*, in which Paper 2 was published.

To sum up, our goal was to develop a satisfactory definition of truthlikeness for deterministic and probabilistic laws within Niiniluoto's version of the similarity approach. The research mainly focussed on the semantic problem of truthlikeness for scientific laws, leaving its epistemic application for future development (see section 4.3).

3. Research outcomes: Description of the two papers

The doctoral research gave rise to two papers, each focussed on one of the presented goals (see section 2). Paper 1 deals with the definition of truthlikeness for quantitative deterministic laws, whereas Paper 2 expands the proposal to cover probabilistic laws and offers a general treatment of both deterministic and probabilistic laws. Both papers are briefly described below.

3.1. Paper 1: 'Truthlikeness for Quantitative Deterministic Laws'.

Paper 1 appeared first online 21th May, 2021: https://doi.org/10.1086/714984.

The paper first applies the similarity approach to the case of quantitative deterministic laws (QDL) and presents Niiniluoto's proposal. It is shown that QDL can be naturally represented in a n-dimensional state space $\mathbb{S}^n$, being n the total sum of the dimensions of the quantities $(h_1, \ldots, h_m)$ that represent the (relevant) properties of a target system. A QDL is then a mathematical functional relation between the physical real-value quantities characterizing a system.

Assume the general form $h_m(x) = f^i(h_1(x), \ldots, h_{m-1}(x))$, which defines a possible continuous real-value function in $\mathbb{S}^n$. Niiniluoto (1982, 1987, 2018) proposes to define the distance between a deterministic law X and the true deterministic law T by the *Minkowski* or L_p *metric* for functions ($p \geq 1$):

$$d(X,T) = \left(\int |f^X(x) - f^T(x)|^p \, dx \right)^{\frac{1}{p}}$$

As special cases, when p is 1, 2 or tends to ∞, the resulting metrics are known as Manhattan, Euclidean and Chebyshev, respectively. Niiniluoto considers these three metrics as good candidates to define an appropriate similarity metric d regarding QDL. Although the proposal is attractive in a number of ways, it seems to yield an unintuitive result in some cases. The paper presents the (structurally very similar) counter-examples by Thom (1975), Weston (1992) and Liu (1999) and argues that they cannot be considered conclusive counter-examples. Then, it develops new counter-examples to the Manhattan, Euclidean and Chebyshev metrics which are taken to be conclusive.

Taking '$d(X,T)$' to refer to either the Manhattan or the Euclidean distance, the papers argues that the underlying problem with Niiniluoto's proposal lies in the fact that $d(X,T)$ measures the "accuracy" of a law, so that $d(X,T)$ defines truthlikeness for QDL as a function of accuracy. Relevantly, this idea is also at the core of Weston's (1992), Oddie's (2019) and Schurz's (2021) proposals. However, it is argued that accuracy represents a necessary but not sufficient condition to define truthlikeness for QDL, as two laws may be equally accurate and still one may imply more true or truthlike consequences, behaviours or facts about the system than the other. Then, it is argued that Niiniluoto's definition should be complemented with an additional factor, labelled as 'nomicity', which measures the similarity of some aspects of the "structures" or "behaviours" described by a law compared to the true "structures" or "behaviours" of the target system that are not captured by accuracy.

It is shown that the introduction of this new factor implies two issues that must be clarified: (i) the need of defining or quantifying the notion of 'nomicity' and (ii) the need of combining accuracy and nomicity into a single function in order to define a comparative notion of truthlikeness and the numerical degree of truthlikeness of a law.

For (i), the paper defines the degree of nomicity of a law X regarding the true law T by the distance between the derivative functions X' and T'. This can be thought of as a measure of the similarity of their shapes or the qualitative behaviours implied by X and T.

It is shown that problem (ii) is much challenging, as there are infinite ways to combine two variables into a single function. The only way to proceed is to establish a set of desideratum and/or intuitive cases that the combination function would have to respect. The paper argues for two main properties that a combination function between accuracy and nomicity should satisfy, which represent the idea that great failures in one of the factors should not be compensated by great success in the other:

(1) $(d^{eu}(X,T) \approx \infty \wedge d^{eu}(X',T') \approx 0) \rightarrow Tr(X) \approx 0$

(2) $(d^{eu}(X,T) \approx 0 \wedge d^{eu}(X',T') \approx \infty) \rightarrow Tr(X) \approx 0$

The final proposal presented in the paper defines accuracy by the Euclidean distance $d^{eu}(X,T)$, nomicity by the Euclidean distance of the derivative functions $d^{eu}(X',T')$ and the general degree of similarity $d_1^{an}(X,T)$ between a law X and the true law T by the combination of both factors via summation:

$$d_1^{an}(X,T) = d^{eu}(X,T) + (m-n)d^{eu}(X',T')$$

The parameter $(m-n)$, which represents the interval under consideration, is a constant introduced to equate the units of accuracy and nomicity and make the sum meaningful. The main

obvious drawback of the proposal is the arbitrariness of multiplying nomicity by a constant, which may change the distance to the truth of a law depending on the chosen value. That being said, every numerical definition involves some arbitrariness, so that it must be judged according to its consequences in intuitive cases and/or according to the principles it satisfies. In that sense, $d_1^{an}(X,T)$ satisfies properties (1) and (2) and provides the intuitive results in the counter-examples presented in the paper.

The final section of the paper applies the developed framework in two directions. First, it shows how to define absolute claims of the form 'law X is close to the truth' based on a truthlikeness definition grounded on $d_1^{an}(X,T)$. Second, it applies the framework to define scientific progress for QDL. Interestingly, the nomicity factor shows that progress from a law X to a law Y can be achieved even if Y is less or equally accurate than X. A small case study regarding the movement from the Ideal gas law to the Van der Waals law is presented to exemplify this idea.

3.2. Paper 2: 'Truthlikeness for Probabilistic Laws'

Paper 2 appeared first online 18th June, 2021: https://doi.org/10.1007/s11229-021-03206-4.

After a brief study of the similarities and differences between deterministic laws (DL) and probabilistic laws (PL), the paper first presents Niiniluoto's (1987) proposal of truthlikeness for DL and the expanded version developed on Paper 1. However, a relevant modification to the definition $d_1^{an}(X,T)$ developed in Paper 1 is introduced, via a new way $d_2^{an}(X,T)$ of combining accuracy and nomicity:

$$d_2^{an}(X,T) = (d^{eu}(X,T)+1)(d^{eu}(X',T')+1) - 1$$

Function $d_2^{an}(X,T)$ also satisfies properties (1) and (2) and provides the intuitive results in the relevant cases. As a main drawback, it is not a metric function (it does not satisfy the triangle inequality) as $d_1^{an}(X,T)$. However, $d_2^{an}(X,T)$ presents two additional virtues. First, truthlikeness defined by $d_2^{an}(X,T)$ does not depend on the arbitrary constant $(m-n)$. Second, normalizing $d_2^{an}(X,T)$ and rearranging, one can obtain the following definition of truthlikeness:

$$Tr_2(X) = \frac{1}{\left(1+d^{eu}(X,T)\right)}\frac{1}{\left(1+d^{eu}(X',T')\right)}$$

This final formulation appears to be more conceptually elegant than the one obtained by $d_1^{an}(X,T)$. $Tr_2(X)$ can be interpreted in the following way: we first normalize accuracy and nomicity and then truthlikeness is defined as the combination of both factors via multiplication, which is the most natural mathematical way of combining two different properties.

Moving to probabilistic laws, the paper shows that, according to the spirit of the similarity approach, what is needed to define truthlikeness for PL is an appropriate similarity metric between probability functions. The paper notices that there is a host of available distances in the literature of probability developed for this purpose, much more comparatively than in the case of distance metrics for geometrical functions that represent DL. However, it is shown that many can be classified in families and share common properties. In this sense, the paper rests on Cha's (2007) excellent analysis, which offers a categorization of fifty-six of the main and most used probability distance functions.

Cha's results are interpreted as offering two main clusters of probability distance functions, labelled as 'Geometric' and 'Divergence' families. As Cha's clusters present a similarity of behaviours, a small sample of representatives of each family can be taken for posterior analysis, reducing the host of available distances to a few. Particularly, the Manhattan (MA) and the Euclidean (EU) distances are taken as main representatives of the Geometrical family and the Kullback–Leibler (KL) and Jeffreys divergence (JD) are taken as main representatives of the Divergence family. In order to decide, the paper constructs four different scenarios. Appealing to intuitions, it is concluded that KL and JD seem to represent a better notion of similarity to the truth for probabilistic systems than MA and EU.

Then, the paper argues that what KL measures, can be taken as analogous to *accuracy* in the case of DL. This raises the natural question about whether nomicity is also a relevant factor for PL. Through a small case study regarding atom decay and the normal distribution, it is concluded that nomicity is also a necessary factor for probability density functions, which again can be measured by the comparison of the derivatives of the relevant density functions.

As in the case of DL, both factors need to be combined into a single function in order to make comparative judgements and to obtain the numerical degree of truthlikeness of a PL. Analogously to the case of DL, a version of $d_2^{qn}(X,T)$ is proposed for PL.

4. Final remarks and open questions for future development

4.1. Summary of the developed framework

The general framework developed to define truthlikeness for deterministic and probabilistic laws can be summarized as follows. In both deterministic and probabilistic laws it has been argued that accuracy represents a necessary but not sufficient condition to define closeness to the truth, as two deterministic or probabilistic laws may be equally accurate and still one may imply more true or truthlike consequences, behaviours or facts about the system than the other. The additional factor, labelled as 'nomicity', measures the qualitative behaviours implied by a law that are not captured by value comparison. The proposed method to measure nomicity appeals to shape similarity, captured by the Euclidean distance between the corresponding derivative functions.

In brief, the proposal defines the similarity $d^{an}(X,T)$ between a deterministic or probabilistic law X and the true deterministic or probabilistic law T as a function of two factors, accuracy and nomicity:

$$d^{an}(X,T) = F(accuracy,\ nomicity)$$

Specifically, we may call 'accuracy' the general concept defining "value similarity", the similarity between the values of a (deterministic or probabilistic) law X and those of the true law T; 'd-accuracy' (deterministic accuracy) the concept defining the similarity between the values of a DL X and those of the true DL T (a similarity between states); and 'p-accuracy' (probabilistic accuracy) the concept defining the similarity between the values of a PL X and those of the true PL T (a similarity between probabilities). Analogously, we may call 'nomicity' the general concept defining "shape similarity", the similarity between the shape of a (deterministic or probabilistic) law X and that of the true law T; 'd-nomicity' (deterministic nomicity) the concept defining the similarity between the shape of a DL X and that of the true DL T; and 'p-nomicity' (probabilistic nomicity) the concept defining the similarity between the shape of a PL X and that of the true PL T.

The developed framework is summarized in Figure 4.

	Value similarity	Shape similarity	Combination Function	Truthlikeness
Deterministic laws Measured by	*d-accuracy* $d^{eu}(X,T)$	*d-nomicity* $d^{eu}(X',T')$	$d_d^{an}(X,T)$	$Tr_d(X) = \frac{1}{(1+d^{eu}(X,T))}\frac{1}{(1+d^{eu}(X',T'))}$
Probabilistic laws Measured by	*p-accuracy* $d^{KL}(X,T)$	*p-nomicity* $d^{eu}(X',T')$	$d_p^{an}(X,T)$	$Tr_p(X) = \frac{1}{(1+d^{KL}(X,T))}\frac{1}{(1+d^{eu}(X',T'))}$

Figure 4. Summary of the developed framework.

4.2. The combination function $d^{an}(X,T)$

The research has produced two possible combination functions, $d_1^{an}(X,T)$ and $d_2^{an}(X,T)$. As they are not ordering-equivalent, they can conclude in different truthlikeness ordering for a same set of given laws.

This may induce two relevant and related worries. On the one hand, other more satisfactory combination functions may be possible. On the other hand, it seems that truthlikeness should be an "objective notion", in the sense that the distance to the truth of a scientific law X should not depend on the (partly arbitrary) mathematical way of combining two variables. As the notion of truth, the distance to the truth of a law X should be "an objective value".

Both are fair worries. Partly, it is true that every numerical formulation might involve some arbitrariness. Even in the choice of function d for very simple cases, this arbitrariness might be present. Remember the cognitive problem regarding the number of planets in the Solar System exposed in section 1.6.4.2. In that case, we defined d by the absolute difference, as it seemed the most "natural metric" for that particular structure. However, other possible distances d could conclude in different orderings, implying for instance that c_{18} is closer to the truth (c_8) than c_{10}.

In the similarity approach, this presumed arbitrariness comes particularity from the fact that any function that satisfies conditions $i - iv$ (see section 1.4) is a valid metric function. However, many possible metric functions "do not seem" valid similarity-truthlikeness functions, either because they give an unintuitive result for some cases or because they violate some appealing principles (for an example of an intuitively invalid similarity function, see Paper 1, section 2). As with almost (perhaps all) philosophical concepts, one needs to appeal to clear-cut intuitive cases and/or general principles in order to evaluate the "goodness" of a philosophical proposal. In that sense, and a bit metaphorically, "objective" or "pre-theoretical" truthlikeness comes from a combination of

intuitions and principles, and a proposed truthlikeness definition would be "objective" or "good" if it satisfies the presumed principles and provides the expected results in the intuitive cases. As with any other philosophical proposal, problems might appear when there is disagreement regarding some of the principles or the presumed clear-cut intuitive cases (see section 1.6.5).

For the case of $d^{an}(X,T)$, its "goodness" comes from implying the intuitive truthlikeness orderings in the presented cases and from satisfying the postulated properties (1) and (2). As noted, both $d_1^{an}(X,T)$ and $d_2^{an}(X,T)$ fulfil these requirements, but further reasons might bring us to prefer $d_2^{an}(X,T)$. In any case, it is an open possibility that: (a) new cases might posit counter-examples to $d_1^{an}(X,T)$ or $d_2^{an}(X,T)$; (b) that a more satisfactory combination function $d_i^{an}(X,T)$ is formulated, perhaps appealing to different principles; (c) or that further factors besides accuracy and nomicity are shown to be relevant to define truthlikeness for scientific laws.

4.3. The epistemological problem

As mentioned in section 2, the research has only focused on the semantic problem of truthlikeness. However, the application of $d^{an}(X,T)$ to real cases appears challenging, as we don't know the true law T (see section 1.5). What can be expected at most, is to define an *estimation of the degree of truthlikeness* of a law X based on some available evidence e. In this section we will sketch an epistemological application of the developed framework, open for future development. The focus will be only on deterministic laws, but the proposed methods may be naturally extended to cover probability density functions.

An estimation of $d^{an}(X,T)$ can be reduced to an estimation of the accuracy and the nomicity factors. For deterministic laws, take a simple case where the target system is represented according to two quantities X and Y and consider a set of n empirical observations $(x_1^T, y_1^T), \dots, (x_n^T, y_n^T)$[8]. Then, the *estimated degree of accuracy* of a law X, $E[d^{eu}(X,T)]$, can be naturally calculated by comparing, for each empirical point, the true value y_i^T with the predicted value y_i^X:

$$E[d^{eu}(X,T)] = \left(\frac{x_n^T - x_1^T}{n} * \sum_1^n (y_i^T - y_i^X)^2\right)^{\frac{1}{2}}$$

[8] Although technically these values come with a margin of error ε, and therefore the exact "true values" should be considered indeterminate and inside the error's range $[(x_i^T, y_j^T) = (x_i^T \pm \varepsilon, y_j^T \pm \varepsilon)]$, let us consider the simplification of assuming (x_i^T, y_j^T) as the "true values" (which is alike to take the mean value of the error's range). The described situation might be a bit unrealistic for some areas of science where empirical observations come with a large margin of error or present significant noise. In any case, we take this proposal as a starting point, which can then be modified and refined to incorporate these more complex scenarios.

This proposal matches with Niiniluoto's (1987, p. 288; 2018, p. 131), except for the factor $\frac{x_n^T - x_1^T}{n}$. This factor is introduced to approximate the summation to the *Riemann sum* of interval $x_n^T - x_1^T$, which in turn is an approximation to the definite integral of the interval[9]. Note that the value of the definite integral of the interval matches with the theoretical degree of accuracy of law X as defined in Paper 1. Therefore, in the limit, when $n \to \infty$, we obtain the result that our epistemological proposal equates with the theoretical definition:

$$\lim_{n\to\infty} E[d^{eu}(X,T)] = d^{eu}(X,T)$$

Nomicity, however, is harder to estimate, as the "true function" is unknown and the values of the derivative at the points (x_i^T, y_i^T) cannot be empirically observed. The only way to proceed is to rationally estimate the value of the derivative of the true function for each empirical point. This would allow to compare them with the values of the derivatives of each of the laws (which we can calculate), in the same way as in the case of accuracy. In what follows, we will sketch two possible ways of doing so, based on a philosophical assumption of "simplicity" (see below).

For Method 1, consider an empirical observation (x_i^T, y_i^T), where (x_{i-1}^T, y_{i-1}^T) and (x_{i+1}^T, y_{i+1}^T) represent the two closest previous and posterior observations as represented in some state space. The simplest path between (x_{i-1}^T, y_{i-1}^T) and (x_i^T, y_i^T) might be rationally claimed to be given by a linear equation, with slope:

$$m_{i-1,i} = \frac{y_i^T - y_{i-1}^T}{x_i^T - x_{i-1}^T}$$

If that were the "true function" connecting (x_{i-1}^T, y_{i-1}^T) and (x_i^T, y_i^T), then the value of the derivative of the "true function" at (x_i^T, y_i^T), $y_i^{T\prime}$, would correspond to the slope $m_{i-1,i}$.

If the empirical observations do not follow a perfectly linear path, then typically $m_{i-1,i} \neq m_{i,i+1}$. Therefore, on a first approximation it may seem rational to define the estimated slope $E[m_i]$ (i.e., the estimated value $y_i^{T\prime}$ of the true derivative at (x_i^T, y_i^T)) by taking some kind of average between $m_{i-1,i}$ and $m_{i,i+1}$, weighted by w_1 and w_2:

$$E[m_i] = w_1 m_{i-1,i} + w_2 m_{i,i+1}$$

[9] Without the factor, the more observations or points we had, the higher the value of $E[d^{eu}(X,T)]$ would be, so the law would be less accurate the more observations we add. That has the undesirable consequence that if we had a huge number of empirical observations, say 10^{10}, and each of the predicted values had a small difference regarding the overserved value, the final degree of estimated accuracy of the law would be incredibly high. Roughly, without the factor we are just summing "segments", differences, but with the factor we are summing rectangles, areas, so it is a better approximation to the theoretical value of the integral. In the limit of infinite equidistant observations, $\frac{x_n^T - x_1^T}{n}$ would transform into dx and we would get the value of the integral.

The estimation might be claimed to be more "grounded" if it includes a larger amount of previous and posterior observations, such as, for instance:

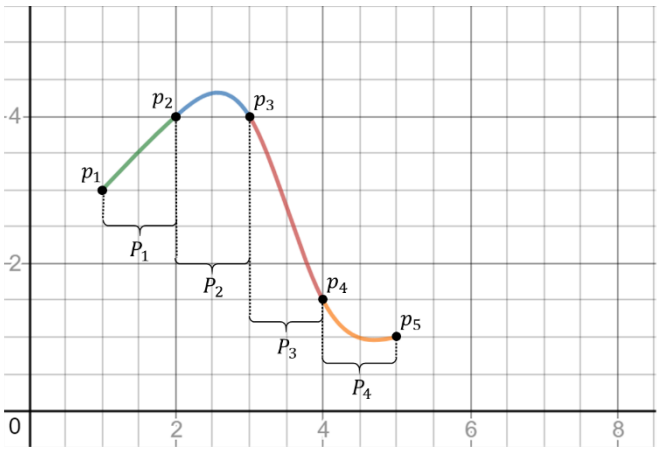
$$E[m_i] = w_1 m_{i-2,i} + w_2 m_{i-1,i} + w_3 m_{i+1,i} + w_4 m_{i+2,i}$$

This can be intuitively thought of as including more information regarding "where the function comes before (x_i^T, y_i^T)" and "where the function goes after (x_i^T, y_i^T)" in the estimation of the most probable slope of the "true function" at (x_i^T, y_i^T). In the limit, each estimated slope $E[m_i]$ of each empirical observation is to be calculated appealing to its linear connection to all the other empirical points, with proper weights.

$$E[m_i] = w_1 m_{i-r,i} + \cdots + w_f m_{i-1,i} + w_g m_{i+1,i} + \cdots + w_n m_{i+s,i}$$

Weights $w_1, \ldots, w_n$ could be defined by a function of the relative distance of each empirical point to the target point (x_i^T, y_i^T). In that way, closest points would have a higher impact in the determination of $E[m_i]$ than further points, as it may intuitively seem it should be.

A similar spirit is behind the proposed Method 2, which is a method of interpolation known as *cubic splines*. Roughly, a *cubic spline interpolation* results in a piecewise function with each of its pieces being a three order polynomial. Given a set of $p_1, \ldots, p_n$ points, a *cubic spline interpolation* produces $n - 1$ polynomials. The polynomials are defined by intervals according to pairs of points: P_1 is a cubic polynomial for the interval $[p_1, p_2]$, P_2 is a cubic polynomial for the interval $[p_2, p_3]$, etc. The resulting "total function" is then a piecewise function defined by the conjunction of all the three order polynomials. What is crucial, however, is that one imposes the condition that in each shared point p_i, both polynomials P_{i-1} and P_{i+1} must have the same value for the first and the second derivatives. For example, in point p_2 we require that $P_1' = P_2'$ and $P_1'' = P_2''$, in point p_3 we require that $P_2' = P_3'$ and $P_2'' = P_3''$, etc. With that we obtain a highly *smooth* transition from one polynomial to the following. As an example, given the following set of points (1, 3), (2, 4), (3, 4), (4, 1.5), (5, 1), we obtain the *cubic spline interpolation* represented in Figure 5.



$$f^{Spline}(x) \begin{cases} P_1 & if\ p_1 \le x \le p_2 \\ P_2 & if\ p_2 \le x \le p_3 \\ P_3 & if\ p_3 \le x \le p_4 \\ P_4 & if\ p_4 \le x \le p_5 \end{cases}$$

Figure 5. Example of cubic spline interpolation

What makes cubic splines philosophically interesting for the purpose of estimating nomicity is that they instantiate the *minimum curvature property* (Holladay, 1957), which can be stated as follows: among all possible curves that can pass through a set of given points, cubic splines represent the path which presents minimum curvature (Leslie *et al.*, 1996). Roughly, they can be understood as defining the "shortest" plus "smoothest" possible path between a set of known points: any shortest path will present some sharp changes, so it will be globally less smooth; and any smoothest path will necessarily be longer. In other words, among all possible smooth paths, cubic splines instantiate the shortest one.

Both Methods are fallible, in the sense that the "estimated" value of the true derivative at (x_i^T, y_i^T) could diverge from the "actual" true value, which is unobservable. The *True law* could, in principle, highly oscillate between the available empirical points. It is a known fact that any finite set of points is compatible with infinite functions and that a unique function is underdetermined by any finite set of point. At the end of the day, what we face here is the classical problem of induction, as Niiniluoto points out "the problem of estimating verisimilitude is neither more nor less difficult than the traditional problem of induction" (1987, p. 263). However, both Methods can be thought of as providing a rational estimation of the value of the true derivative, assuming a metaphysical principle of *simplicity* in nature. An extended discussion of this claim would require much more space, but let us point out some intuitive reasons.

Method 1 assumes a notion of simplicity related to the shortest possible path between two points. Then, it weighs the simplest possible paths (linear paths) between a target point (x_i^T, y_i^T) and each of the other empirical points $(x_{i\pm k}^T, y_{i\pm k}^T)$, $\forall i, k(1, \dots, n)$. Therefore, this method provides the expected value $y_i^{T'}$ of the derivative of the "true function" at (x_i^T, y_i^T) if nature followed the shortest possible path through a set of given points, properly balancing the contribution of each point $(x_{i\pm k}^T, y_{i\pm k}^T)$ by weights $w_1, \dots, w_n$, according to their distance to the target point.

Method 2 appeals to a notion of simplicity related to a kind of "balance" or "optimization" between total *length* and total *curvature*. On the one hand, a shortest path than the one provided by cubic splines will be simpler, but it will necessarily have some "sharp" changes that, on a possible reading, would make it, globally, more complex (sharp changes usually need larger amounts of energy than smooth changes). On the other hand, any other smooth path will present a higher curvature which, again, would add complexity to the path by increasing its length. In intuitive energetic terms, assume that one would like to go from point p_1 to point p_n, having to pass through a finite set of intermediate known points. Consider that one would like to minimize the energy needed for the travel and that, broadly, *short* paths require less energy than *long* paths and that *smooth* changes require less energy than *sharp* changes. Then, *cubic splines* would yield the path that minimizes energy consumption. This may be claimed to match, roughly, with what entropy tell us about the behaviour of physical systems.

For our purposes, we take that, assuming *simplicity* in any of the explicated senses, Method 1 or Method 2 estimate the "most probable path" (or one of the "most probable paths") of the *True law* in the proximities of each empirical point, and so they enable us to estimate the "most probable value" (or one of the "most probable values") of the *True derivative* for each empirical point.

Therefore, the *estimated degree of nomicity* of a law X, $E[d^{eu}(X',T')]$, can be calculated by comparing, for each empirical point, the value of the estimated true derivative ($y_i^{T'}$) by Method 1 or 2 with the value of the derivative of law X ($y_i^{X'}$):

$$E[d^{eu}(X',T')] = \left(\frac{x_n^T - x_1^T}{n} * \sum_1^n \left(y_i^{T'} - y_i^{X'}\right)^2\right)^{\frac{1}{2}}$$

With the estimation of both factors, the normalized *estimated degree of truthlikeness* of a law X, $E[Tr(X)]$, can be calculated by:

$$E[Tr(X)] = \frac{1}{(1 + E[d^{an}(X,T)])}$$

Given this proposal, a final relevant complication should be noted. We defined (Paper 1) a quantitative deterministic law as a mathematical functional relation between the physical real-value quantities $h_1, \dots, h_m$ characterizing the system, so that:

$$f(h_1, \dots, h_m) = 0$$

However, a detailed analysis of deterministic laws (see Paper 2, section 3.3) shows that they can present two "kinds" of quantities, variables $(v_1, \dots, v_i)$ and parameters $(k_1, \dots, k_j)$, and some constants $(c_1, \dots c_k)$, so that their structure may be more accurately formulated as:

$$f(v_1, \dots, v_i, k_1, \dots, k_j, c_1, \dots c_k) = 0$$

Parameters usually represent initial or boundary conditions and can take the form of variables or constants. When parameters are variables, we may call the resulting function the "general form" of a law. This is represented in the state space as a set of possible trajectories (for laws of succession) or possible regions (for laws of coexistence), each corresponding to some specification of the parameters. When parameters are specified, they become constants, representing some initial or boundary conditions. We may call the resulting function the "specific form" of a law. This is represented in the state space as singular trajectories or regions, which model a particular system.

For example, the general form of the Ideal gas law may be formulated as $PV = nRT$ (4-dimensional), where P (pressure), T (temperature) and V (volume) are variables, n (number of moles of the gas substance) is a parameter and R (universal gas constant) is a constant. A possible

specific form of the Ideal gas law is $PV = 2RT$ (3-dimensional), where it is assumed that the gas is composed by two moles of substance.

Function $d^{an}(X,T)$, and therefore the theoretical degree of truthlikeness of a law X, has been defined appealing to the general form of X and T. In this sense, one can think of $d^{an}(X,T)$ as measuring the similarities of each possible trajectory or region defined by X and T regarding each possible value of the parameters (roughly, regarding each possible initial condition).

However, for testing and making predictions, usually the specific form of a law is needed. This implies that $E[Tr(X)]$ would be calculated given some concrete specifications of the parameters and initial conditions, and for some range of the variables $(v_1, \dots, v_i)$. Relevantly, however, it is possible that for some specifications of $(k_1, \dots, k_j)$ and for some range of values of $(v_1, \dots, v_i)$, law X performs much better than for others, so that taking a small sample of different values of $(k_1, \dots, k_j)$ and $(v_1, \dots, v_i)$ can be a bad indicator of its estimated truthlikeness.

As an example, the Ideal gas law is quite accurate for gases at low densities, which correspond to values of "low" pressures and "high" temperatures (where the values of "low" pressures and "high" temperatures match with the ones usually found on Earth under normal conditions). At low densities the particles' volumes and the intermolecular forces are negligible, but at high densities those factors, among others, become relevant. This implies that the Ideal gas law results much more accurate for some ranges of P-V-T than for others. Additionally, from modern physics and chemistry we know that not all gas substances have the same "nature" or "properties". From a microscopic perspective, a substance is a combination of a number of atoms in different quantities. These combinations produce a series of properties with macroscopic impact. Among others, substances can be *polar* (e.g. H_2O) or *non-polar* (e.g. N_2), depending on whether they are asymmetric ("charged") or symmetric ("not-charged"). Regarding this taxonomy, it is known that the Ideal gas law laws perform well in *non-polar* and weak *polar* substances, but badly in *polar* ones. Therefore, estimating the degree of truthlikeness of the Ideal gas law by taking a sample of *non-polar* substances and a value range of P-V-T corresponding to "low" pressures and "high" temperatures can result in a distorted estimated degree of truthlikeness.

Being so, $E[Tr(X)]$ should be calculated for an adequate n sample of different values of $(k_1, \dots, k_j)$ and $(v_1, \dots, v_i)$, covering as many different situations i of the target system as possible. As a result, the expected-pondered degree of truthlikeness of a law X, $\bar{E}[Tr(X)]$, should be defined as:

$$\bar{E}[Tr(X)] = \frac{1}{n}\sum_{n} E[Tr(X)]_i$$

Where different weights may be included to average the possible different importance of the specifications of $(k_1, \dots, k_j)$ and $(v_1, \dots, v_i)$.

4.4. From laws to theories

The ultimate aim of a theory of truthlikeness for science is to define closeness to the truth for scientific theories. As in the case of the epistemological problem, in this final section we will sketch a possible expansion of the developed framework to cover scientific theories, open for future development.

The Sneedian-structuralist framework (Balzer *et al.*, 1987) provides two key ideas for this purpose. In the first place, structuralism conceives theories, very roughly, as sets of laws, but crucially not all laws are equally essential. For example, in classical mechanics Newton's Second Law might be claimed to be a central component, whereas the law of simple pendulum seems less central. This implies a representation of theories as nets (more precisely, as inverted tree-like nets) each knot being a "theory-element" which structuralism characterises as a pairs $< K, I >$ of laws and intended applications (the empirical systems to which the laws are intended to apply).

This idea of "degrees of relevance" might be a worthwhile component to incorporate into a theory of truthlikeness for theories. For a simple proposal, take a theory T to be constituted by $X_1, .., X_n$ laws with weights of importance $w_1, \dots, w_n$ ($\sum w_i = 1$). Then, the degree of truthlikeness Tr of a theory T could be defined as (where $Tr(X_i)$ is our proposed definition of the degree of truthlikeness of law X_i):

$$Tr(T) = w_1 Tr(X_1) + \cdots + w_n Tr(X_n)$$

How to quantify the different weights $w_1, \dots, w_n$, however, might be a challenging task. A possible way to proceed would be to appeal to the concept of 'strength' as developed in Lewis' Best System Account of laws of nature (see section 1.6.5). Roughly, in that context, strength is usually conceived in terms of the informativeness or logical content of the theorems of a given system with respect to the mosaic. The strength of a systematization lies in the number of consequences that are realized, on the amount of facts it accounts for. A strong system is then one that accounts for many facts of the mosaic.

With this idea, the strength of a law X_i may be defined by the number of types of phenomena and the number of instances of those types that it intends to describe, and its weight of importance w_i appealing to a function of its relative strength. Take s_i to represent the degree of strength of law X_i. Then, w_i of law X_i (given $X_1, .., X_n$ laws) could be defined as:

$$w_i = \frac{s_i}{\sum_n s_x}$$

This proposal would imply the intuitive result that Newton's law of universal gravitation would be considered much more relevant in classical mechanics than the law of simple pendulum, as the former applies to a greater number of types of phenomena and instances. As an example, consider

a theory to be constituted by three laws X_1, X_2, X_3, where X_1 and X_2 are intended to apply to k instances and X_3 to $k/2$ instances. Then, $w_1 = w_2 = 0.4$ and $w_3 = 0.2$.

In the second place, structuralism may provide a useful framework to define comparative truthlikeness between theories. When defining truthlikeness, the content-consequence and the similarity approaches appeal to a given language L in which all theories and propositions are formulated (see section 1.6.4). Then, closeness to the truth is defined, roughly, as the distance of the different *L-potential answers* to a given *L-true answer*. In this way, in the framework developed in Paper 1 and Paper 2 a given state space composed by quantities $(h_1, \dots, h_m)$ was assumed, and the different candidate laws were taken to be different mathematical functional relations between the quantities. In some cases (perhaps typically), however, rival scientific theories may not be formulated in the same language L.

This induces the problem on how to compare theories, in truthlikeness terms, which are formulated in different languages, i.e., on how to compare theories which postulate different physical quantities to represent the properties of the same system, generating different state-spaces.

Structuralism divides a theory's T vocabulary V_T into two components: (a) its T-non theoretical or T-empirical vocabulary and (b) its T-theoretical or T-explanatory vocabulary. The former is the part of V_T used in the description of the phenomena which T intends to explain (ex. 'spatial position', 'time' and 'particle' in classical mechanics). Crucially, T-non theoretical concepts can be measured/determined without presupposing any T-law (Díez, 2006). This implies that T's data is not theory-laden by T. On the other hand, T-theoretical concepts are those which cannot be measured/determined without presupposing some T-law (ex, 'mass' and 'force' in classical mechanics). This distinction does not coincide with the classical "observational/non observational" characterization of scientific concepts.

The structuralist distinction implies three conceptual possibilities for two rival theories T_1 and T_2 (Díez, 2006), that one may want to compare in truthlikeness terms:

(1) T_1 and T_2 share the same T-non theoretical and T-theoretical vocabulary.

(2) T_1 and T_2 share the same T-non theoretical vocabulary and differ, at least partly, in their T-theoretical vocabulary.

(3) T_1 and T_2 do not share their T-non theoretical and T-theoretical vocabulary.

In (1), T_1 and T_2 are formulated in the same language, so that the truthlikeness comparison may impose no challenge. The situation matches with the one assumed in Paper 1 and Paper 2: given

some shared quantities $(h_1, \ldots, h_m)$, T_1 and T_2 postulate different mathematical functional relations (i.e. different laws) between them.

This case, however, may be atypical in science. Usually, two rival theories diverge, at least partly, in their T-theoretical concepts, as they postulate different explanations of the same phenomena. Compared to the Ideal gas law, the Van der Waals law includes two new concepts in its theoretical machinery, a (representing the attraction between the gas particles) and b (representing the volume gas particles occupy).

A historical example of (1) may be found in the series of modifications that followed the publication of the Van der Waals law (VW) in 1873[10]:

$$P^{VW} = \frac{RT}{v-b} - \frac{a}{v^2}$$

Firstly, in 1899, Dieterici proposed an equation of state that replaced the squared volume factor of VW by an exponential function:

$$P^{DI} = \frac{RT}{v-b} * e^{-a/vRT}$$

Although Dieterici's law was similar in terms of accuracy to VW, its structural change didn't had a historical continuation. Some years after, in 1907, Berthelot tried to incorporate to VW Clasius' findings about the dependence (inverse proportionality) of parameter a with temperature:

$$P^{BE} = \frac{RT}{v-b} - \frac{a}{Tv^2}$$

His proposal, however, resulted in many cases much less accurate than VW and was rarely used. The VW equation became forgotten until 1949, when the Redlich-Kwong (RK) modification appeared, supposing a revival of van der Waals' ideas and attracting a great deal of interest within the scientific community (Valderrama, 2003):

$$P^{RK} = \frac{RT}{v-b} - \frac{a/\sqrt{T}}{v^2+vb}$$

These four models share the same T-non theoretical and T-theoretical vocabulary, so that their *general forms* (see section 4.3) can be represented in the same 6-dimensional state space. Their main difference is the functional relation they establish between the six quantities (the three variables and the three parameters).

[10] Where P is pressure, T temperature, v molar volume ($v = V/n$), V volume, n the number of moles, a represents the attraction between the gas particles, b represents the volume gas particles occupy and R is the universal gas constant.

Case (2) may be the most typical situation in science, where T_1 and T_2 share a common T-non theoretical vocabulary and differ, at least partly, in the theoretical concepts they postulate to explain the target phenomena. Examples include, among others, the Ideal gas law and the Van der Waals law, the Aristotelian and Ptolemaic models of astronomy and the phlogiston and oxygen theories of combustion (Díez, 2006). In such cases, the shared T-non theoretical vocabulary allows the construction of a common state space where the laws postulated by T_1 and T_2 can be compared in truthlikeness terms. In this sense, the Ideal gas law and the Van der Waals law can be represented in a same (P-V-T) three-dimensional state space $\mathbb{R}^3$ (which coincides with their *specific forms*). Moreover, the phlogiston and oxygen theories of combustion can be represented in a same two-dimensional state-space $\mathbb{R}^2$ with variables "time" and "weight of the combusted element". In such case, the phlogiston theory would be (for any element) a decreasing function and the oxygen theory would be a decreasing or increasing function depending on the element in question. For a similar example regarding the Aristotelian and Ptolemaic models of astronomy, see Díez (2006, p. 36).

Case (3) may imply two possible situations. In a *weak* scenario (which might be considered a special case of possibility (2)), T_1 and T_2 may share *some* of their T-non theoretical vocabulary. For example, take T_1-non theoretical vocabulary to be constituted by quantities $< M, N, O >$ and T_2-non theoretical vocabulary to be constituted by quantities $< M, N, P >$. Then, T_1 and T_2 may be compared in truthlikeness terms appealing to a common (M, N) two-dimensional state space.

A more challenging possibility, however, is a *strong* scenario where T_1 and T_2 do not share any T-non theoretical vocabulary at all. As an example, Díez (2006) mentions the case of Classical and Relativistic mechanics. Intuitively, both theories try to account for "kinematic trajectories", but the former conceptualizes these trajectories in a three-dimensional space and a one-dimensional time and the latter appeals to a four-dimensional space-time, being their T-non theoretical machineries conceptually different. Still, as Díez points out, there is a strong intuitive sense in which Classical and Relativistic mechanics "speak about the same things". In order to talk about rivalry and comparability in these cases, Díez concludes that we should look "outside" the conceptual machinery of both theories, and explores two possibilities for this purpose.

The former appeals to "instrumental data", things as "tables of astronomic data, spectroscopic images, galvanometer readings, (optic-, radio-) telescope images, (optic-, electronic-) microscope images, tables of changing characteristics in a species population, trajectories of particles in cloud chambers, tables codifying values in computer screens, etc…" (2006, p. 42). The "immediate" description of these data does not use T-non theoretical vocabulary, but a more "basic" pre-theoretic vocabulary. The relation between T and its "instrumental data" is that of representation: the intended applications of T represent, via T-non theoretical vocabulary, the "instrumental data".

Given that situation, we may define the *expected instrumental data* by theory T, $E[T(ID)]$, as the instrumental data one expects to obtain given T, for example, "such and such spectroscopic images" or "such and such trajectories in cloud chamber". Then, two theories T_1 and T_2 which instantiate the *strong* scenario (3) and share the same instrumental basis may be compared, in truthlikeness terms, appealing to their $E[T_1(ID)]$ and $E[T_2(ID)]$. Given an appropriate distance d for the kind of instrumental data in question, one may claim that T_1 is closer to the truth than T_2 if and only if $E[T_1(ID)]$ is more similar to the actual instrumental data obtained ID^* than $E[T_2(ID)]$:

$$d(E[T_1(ID)], ID^*) < d(E[T_2(ID)], ID^*)$$

Still, Díez argues that we may find cases of rival theories that do not share even an instrumental base. He convincingly exemplifies the point appealing to the case of Galileo's observations of Venus' phases using the telescope. In that historical episode, the Aristotelians rejected the instrumental basis of the Galileans, the fact that from the telescope's changing shapes one could infer changes in Venus itself, as according to the Aristotelians the sub-lunar and the supra-lunar worlds do not share the same physical laws, so one should not expect that the telescope works alike in both worlds. In such cases, Díez argues that the only possibility to account for rivalry is to look for something common "below" the instrumental data, less elaborated, which he labels as 'observational scenes' (2002, 2006). Things like "dots in relative movement in the sky, grey paths in a cloud chamber, angular displacement of a galvanometer-needle… or going back to our example, increasing/decreasing shapes in a telescope lens" (2006, p. 44). Crucially, there is no possible disagreement among normal human beings regarding observational scenes.

Then, analogously to the previous case, two theories T_1 and T_2 which instantiate the *strong* scenario (3) and do not share the same instrumental basis may be compared, in truthlikeness terms, appealing to their *expected observational scenes*, $E[T_1(OS)]$ and $E[T_2(OS)]$. Given an appropriate distance d for the kind of observational scenes in question, one may claim that T_1 is closer to the truth than T_2 if and only if $E[T_1(OS)]$ is more similar to the actual observational scene OS^* than $E[T_2(OS)]$:

$$d(E[T_1(OS)], OS^*) < d(E[T_2(OS)], OS^*)$$

For example, in the telescope's case heliocentrism might be claim to be closer to the truth than geocentrism since the expected increasing/decreasing shapes in a telescope lens by heliocentrism are more similar to the increasing/decreasing shapes observed than the ones expected by geocentrism.

To sum up:

- If T_1 and T_2 share the same T-non theoretical and T-theoretical vocabulary $(h_1, \dots, h_m)$, then T_1 is closer to the truth than T_2 if and only if:

$$d\big(f^1(h_1, \dots, h_m), f^*(h_1, \dots, h_m)\big) < d\big(f^2(h_1, \dots, h_m), f^*(h_1, \dots, h_m)\big)$$

- If T_1 and T_2 share the same T-non theoretical vocabulary $(h_1, \dots, h_i)$ and differ, at least partly, in their T-theoretical vocabulary, then T_1 is closer to the truth than T_2 if and only if:

$$d\big(f^1(h_1, \dots, h_i), f^*(h_1, \dots, h_i)\big) < d\big(f^2(h_1, \dots, h_i), f^*(h_1, \dots, h_i)\big)$$

- If T_1 and T_2 do not share their T-non theoretical and T-theoretical vocabulary but share a common instrumental base, then T_1 is closer to the truth than T_2 if and only if:

$$d(E[T_1(ID)], ID^*) < d(E[T_2(ID)], ID^*)$$

- If T_1 and T_2 do not share their T-non theoretical and T-theoretical vocabulary or their instrumental basis but share a common kind of observational scenes, then T_1 is closer to the truth than T_2 if and only if:

$$d(E[T_1(OS)], OS^*) < d(E[T_2(OS)], OS^*)$$

4.5. Final remarks

Larry Laudan famously stated that until a coherent account of truthlikeness was developed, claims such as 'actual scientific theories are close to the truth', 'actual scientific theories are closer to the truth than older theories' and 'a theory being explanatory successful implies it being close to the truth' are just "so much mumbo-jumbo" (1981, p. 32).

Since then, many philosophers of science and logicians have worked in developing a satisfactory notion of truthlikeness, so that the notion may no longer be claimed to be absurd or incomprehensible. With this research, we have tried to contribute to the development of the concept of truthlikeness, providing a framework in which claiming that a scientific law is closer to the true law than another law is a coherent, meaningful and accurate statement.

On a larger scale, we hope to have provided some additional tools for a realist view about science and for a fallibilist optimistic view of human knowledge.

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ANNEXES

PAPER 1

TRUTHLIKENESS FOR QUANTITATIVE DETERMINISTIC LAWS

Abstract

Truthlikeness is a property of a theory or a proposition that represents its closeness to the truth. According to Niiniluoto, truthlikeness for quantitative deterministic laws (QDL) can be defined by the Minkowski metric. We will present some counterexamples to the definition and argue that it fails because it considers truthlikeness for QDL to be just a function of accuracy, but an accurate law can be wrong about the actual 'structure' or 'behaviour' of the system it intends to describe. We will develop a modification of Niiniluoto's proposal that defines truthlikeness for QDL according to two parameters: accuracy and nomicity. The presented proposal solves the counterexamples and defines a new way of understanding scientific progress.

I. Preliminaries

Truthlikeness, in a first rough characterization, is a property of a theory or a proposition that represents its 'closeness', 'similarity' or 'likeness' to the truth. A classical example in the literature (Oddie [1986], [2016]) to intuitively introduce the concept invites us to consider the following propositions:

a) The number of planets in our Solar System is ten.
b) The number of planets in our Solar System is ten billion.

As the number of planets is actually eight, both propositions are false. However, intuitively, they don't seem to be 'equally false': a) seems, in some sense, 'less false' or 'more similar' to the truth, closer to how things are, closer to the actual number of planets, than b).

For a more qualitative case, consider:

c) The shape of the Earth is a sphere.
d) The shape of the Earth is a cube.

Where, again, as the actual shape of the Earth is something similar to an oblate spheroid (a bit flattened at the poles and bulged at the Equator), both propositions are false. However, intuitively, c) seems closer to the actual shape of the Earth, closer to the truth in question, than d).

Still at the intuitive level, we seem to postulate such a relation regarding a large number of scientific theories and fields (take '$>_t$' to mean 'closer to the truth than'):

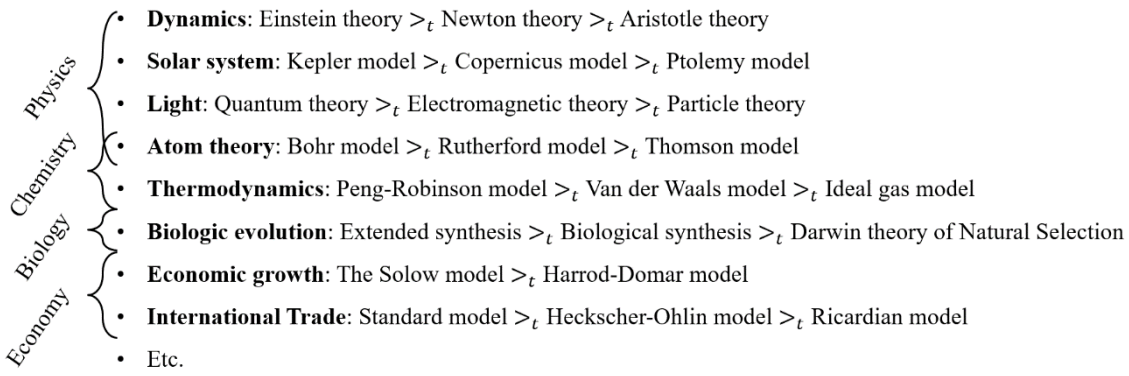

Figure 1

Why is the notion of truthlikeness philosophically interesting? First note that, if we don't move beyond the true/false dichotomy, all we can tell about all (or many) of the listed theories is that they are false and simply false. Truthlikeness aims to overcome this limitation and express some fundamental ideas: although the dynamical laws described by Aristotle, Newton and Einstein are (probably and strictly speaking) false, we have the strong intuition that the Einsteinian world is closer to 'how the world is' than the Newtonian, and much closer than the Aristotelian. That was, precisely, one of Popper's original motivations to introduce the concept of truthlikeness: to make sense of the idea that different scientific theories might be false and yet constitute progress towards the truth, where progress is explicated in terms of increasing truthlikeness.

The notion of truthlikeness enables us to make compatible, in a satisfactory and organic way, a large set of ideas regarding science: (1) many (or all) scientific theories involve abstractions and idealizations, an therefore are strictly speaking false, but not 'equally false': some may be closer to the truth than others; (2) scientific progress from a false theory to another false theory is possible because of an increase in truthlikeness; (3) truth (informative truth), although perhaps unachievable, might be said to be the aim of science in the sense of pursuing a better approximation to it; (4) we might never have conclusive reasons to ascribe truth to a theory (fallibilism), but we may be able to estimate degrees of truthlikeness; (5) we can't claim that

science works because our best developed theories (including the unobservable parts) are true (we know that many aren't, in the strict sense), but we can nicely reformulate the claim by postulating that our best developed theories (including the unobservable parts) work because they are 'close to the truth' (in this sense, truthlikeness is indispensable for a plausible formulation of scientific realism).

For truthlikeness to do such a nice job, a philosophical proposal must show (i) how it is meaningful to claim that T_2 is more truthlike than T_1 (logical or semantic problem); and (ii) how, given some evidence, it is rational to claim that T_2 is more truthlike than T_1 (epistemological problem). In the present paper we will try to answer the first problem regarding quantitative deterministic laws (QDL). Section II introduces the similarity approach to truthlikeness. Section III conceptualizes QDL within the framework of the similarity approach. Section IV presents Niiniluoto's definition of truthlikeness for singular quantitative statements and QDL. Section V develops some counterexamples to Niiniluoto's proposal. Section VI presents a new definition of truthlikeness for QDL as a combination of two factors, accuracy and nomicity. Finally, section VII presents a two dimensional space where scientific progress and the objective distance from laws to the truth can be visually represented and defines a new way of understanding scientific progress.

II. The similarity approach

Popper ([1963]) was the first to take the concept of truthlikeness seriously and to give a formal definition of it. Some years later, however, Miller ([1974]) and Tichý ([1974]) proved independently that Popper's definition didn't work in the intended way, as according to it no false theory or proposition is closer to the truth than any other. Since then, the notion has been a topic of intense discussion by philosophers of science and logicians. It is not the aim of this paper to provide an extensive survey of the different philosophical proposals developed to deal with the notion (see the excellent reviews by Niiniluoto [1987], [1998], [2018a] and Oddie [2013], [2016]). We will limit ourselves to outline the one that we take to be the most developed and satisfactory: the similarity approach.

The similarity approach was firstly proposed by Hilpinen ([1976]), within possible world semantics, and Tichý ([1974]), within propositional logic, and rapidly expanded by Niiniluoto ([1987]), Oddie ([1986]), Tuomela ([1978]) and Festa ([1986]), among others. Its core idea can be captured by the claim that the truthlikeness of a theory or a proposition rests on the 'similarities' between the state of affairs it allows and the actual state of affairs of the world. The approach can be outlined in five main points:

(1) Given a phenomenon or system P and a language L to talk about P we can construct a space of possibilities S_p^L which contains all the ways $(c_1, c_2, c_3 \ldots)$[1] the world, P, can be regarding L, all the possible descriptions of P given L.[2]

(2) Any theory or claim h of L will be expressible as a set of elements of S_p^L.

(3) We introduce a *metric* $d(c_i, c_j)$ which defines the distance (in terms of similarity) between elements of S_p^L and an extension of d into another metric $d'(h, c_i)$ which defines the distance (in terms of similarity) from theories or claims (set of elements of S_p^L) to a single element of S_p^L.

(4) Some element c_t^* will be the truth in question (the actual world or the most informative true description of the world given L)[3].

(5) Connecting all the above, we can finally define the degree of truthlikeness of a claim h, once d' is normalized, as the function:

$$Tr(h) = 1 - d'(h, c_t^*)$$

Where:

- $Tr(c_t^*) = 1 - d'(c_t^*, c_t^*) = 1$
- $Tr(h) > Tr(s) \leftrightarrow d'(h, c_t^*) < d'(s, c_t^*)$

Therefore, truthlikeness is defined by the tuple $<P, L, S_p^L, h_i, d, d', c_t^*, Tr>$. Within that framework we obtain an elegant solution to the semantic problem: 'T_2 is more truthlike than T_1' means that: (i) given a space S_p^L where T_1, T_2 and the truth in question c_t^* are represented and (ii) given some appropriate similarity metrics d and d' for S_p^L, then (iii) $d'(T_2, c_t^*) < d'(T_1, c_t^*)$.

Much of the work developed from the mid-seventies to the mid-eighties was focused on defining and detailing all the elements from the mentioned tuple. L was usually taken as a qualitative first order language, where S_p^L can be constructed via Carnapian predicates or possible worlds. However, for quantitative languages there are many fewer proposals. The most developed is Niiniluoto's ([1987]), where he suggests constructing S_p^L as a state-space (section III).

The key element in the similarity approach is its appeal to a notion of 'likeness' or 'similarity'. At the end of the day, similarity is the concept which explicates and defines truthlikeness: similarity to the truth. The notion is defined according to metrics d and d' on the space S_p^L. A metric on a

[1] Mutually exclusive and jointly exhaustive.

[2] In Oddie's proposal, each element $(c_1, c_2, c_3 \ldots)$ represents a possible world; Niiniluoto's framework is more flexible, allowing to represent state descriptions, structure descriptions, monadic constituents or scientific laws, depending on our cognitive interests.

[3] When the target c_t^* represents the true law T, it may no longer be conceived as the actual world or the most informative true description of the world given L. See footnote 8 for more details.

set is just a function that defines the distance between each pair of elements of the set, being a numerical description of how distant objects are from each other. A set X with a defined metric d constitutes a metric space (X, d). Formally, a metric d on a set X is a function $[d\colon X \times X \to [0, \infty)]$, such that for all $x, y, z \in X$:

i. $d(x,y) \geq 0$
ii. $d(x,y) = d(y,x)$
iii. $d(x,y) = 0 \leftrightarrow x = y$
iv. $d(x,z) \leq d(x,y) + d(y,z)$

However, in the similarity approach it is not enough that d and d' satisfy these mathematical criteria for metric functions in order to be 'good' similarity and truthlikeness functions. Besides, they must capture, precisely, the similarities between the elements of S_p^L. This can be easily exemplified by considering the following *trivial metric*:

$$d^{\oplus}(x,y) \begin{cases} 0 & if\ x = y \\ 1 & if\ x \neq y \end{cases}$$

According to $d^{\oplus}$, which satisfies $i - iv$, every element that is not the truth is at distance one from it. Therefore, 'the number of planets in our Solar System is ten' and 'the number of planets in our Solar System is ten billion' are at the same distance ('equally similar') from the truth. As this is obviously an unintuitive consequence, $d^{\oplus}$ is not a good candidate for a similarity function. We have to add some additional criteria to $i - iv$ such that its satisfaction would guarantee that the distance function is a good measure of likeness or similarity.

These additional criteria and the search for good similarity metrics is the core philosophical aspect of the similarity approach. They can be established according to three (partly incompatible) ways: (I) pointing out some clear-cut intuitive cases that any d and d' would have to respect; (II) postulating some general principles that any d and d' would have to respect; (III) taking an equilibrium between (I) and (II). So far, we find no consensus in the literature on how to proceed (although the problems, as we will see, have to do with the way of defining d', not d, where there is a general agreement). This, however, is nothing surprising, as choosing between (I), (II) and (III) is a general philosophical-methodological problem concerning any (or many) philosophical concept(s). Regarding truthlikeness, roughly, Tichý and Oddie[4] advocate for (I), Popper for (II) and Niiniluoto for (III).

A clear example of this issue raised in the development of truthlikeness for qualitative languages. There was, since the beginning, full agreement in defining d as the *Clifford measure* (the

[4] Oddie ([1986], pp. 5-10) initially argued in favour of low-level intuitive judgements for evaluating a theory of truthlikeness. However, in a more recent paper (Oddie [2013]) he argues for the need of some general principles to constrain the possibilities of the extension from d to d'.

symmetric difference between the elements of S_p^L), but the extension of d into d' is still a subject of dispute. This can be illustrated with a toy example. Suppose we are interested in Sahara's desert weather and we have a simple propositional language with three primitive propositions, h, d and s (for the primitive states 'hot', 'dry' and 'sunny') to describe it. Then, S_p^L will contain eight elements, eight possible complete descriptions of the world: $c_1(h \wedge d \wedge s), c_2(h \wedge d \wedge \neg s), \ldots, c_5(h \wedge \neg d \wedge \neg s), \ldots, c_8(\neg h \wedge \neg d \wedge \neg s)$. Take c_1 to be c_t^*, the truth, the actual weather. Then, according to the number of disagreements between c_i and c_1, we obtain: $d(c_1, c_1) = 0, d(c_2, c_1) = 1, \ldots, d(c_5, c_1) = 2, \ldots, d(c_8, c_1) = 3$. Now suppose a proposition p that claims that the world is hot and wet $(h \wedge \neg d)$, $p = \{c_3 \vee c_5\}$. We know the similarities of its components to the truth $(c_3 = 1, c_5 = 2)$, but how should we measure their combination? Oddie and Tichý have favoured the *average measure* (the average of the distances of the elements that constitute p to c_t^*), while Niiniluoto has argued for the *min-sum measure*[5] (the weighted average of the minimum distance plus the sum of all distances of the elements that constitute p to c_t^*). In our example, $d^{average}(p, c_1) = \frac{1}{2}(1 + 2) = 1.5$ and $d^{min-sum}(p, c_1) = \frac{1}{2}(1) + \frac{1}{2}(1 + 2) = 2$. The fact that they result in different values is not a fundamental problem. The problem arises because, in some cases and for some propositions, the two proposals result in different truthlikeness orderings. Their main difference appears regarding the Popperian general principle that Oddie has called 'the value of content for truth': the fact that among true theories truthlikeness should covary with logical strength. *Average* violates the principle, whereas *min-sum* satisfies it. Niiniluoto accepts the validity of the principle, while Oddie and Tichý reject it based on their intuitions regarding some specific cases. Here we find a conflict between methodologies (I) and (II). Which is more rationally justified is a complex (perhaps impossible) issue to determine[6].

As Niiniluoto has stated many times, the distance function d 'has to be specified for each cognitive problem B separately, but there are "canonical" ways of doing this for special types of problems' (Niiniluoto [1998], p. 4). For example, if our cognitive problem is the number of planets in the Solar System, then each element of S_p^L will be of the form 'the number of planets in our Solar System is x' $\forall x \in \mathbb{N}$. The natural metric d for this structure will not be the *Clifford measure* but the *absolute difference* $d(c_x, c_y) = |x - y|$. However, function d' is supposed to be universal regardless of the cognitive problem. Given some similarity values defined by an appropriate d, then the extension d' (*average* or *min-sum*) is supposed to define the best similarity combination

[5] Specifically, $d^{min-sum}(p, c_t^*) = \gamma * d^{min}(p, c_t^*) + \gamma' d^{sum}(p, c_t^*)$, where $0 < \gamma \leq 1$ and $0 < \gamma' \leq 1$. Parameters γ and γ' indicate the relative weights of both factors. For other measures that Niiniluoto considers, see Niiniluoto ([1987]).

[6] An extensive discussion of the general principles characterizing the different approaches to truthlikeness and the different implications of $average$ and $minsum$ can be found in (Oddie [2013]). Niiniluoto's comparison between $average$ and $minsum$ can be found in ([1987], Chapter 6.6).

of the elements regarding the truth. As we will see, our concerns have to do with how function d has been defined for QDL, so our proposal is, in a sense, neutral regarding function d'.

To sum up, what we need to define truthlikeness for scientific laws within the framework of the similarity approach is an appropriate space S_p^L to represent scientific laws and an appropriate similarity measure d for scientific laws as represented in S_p^L. In the next section we define the kind of space S_p^L where QDL can be represented. We follow Niiniluoto ([1987], [1990], [1994], [1998], [2018]) in considering that those spaces are best defined by the state-space. One of the strengths of this concept lies in the fact that it is actually used in the formulation of many scientific theories (see for instance Greiner *et al.*, [1997]).

III. The state-space and quantitative deterministic laws

Suppose we are studying some phenomenon or system S. In a simple description, S is composed of p individuals which exemplify some properties and interact in a certain way. What scientific theories usually do is: (i) postulate some physical real-value quantities $(h_1, \dots, h_m)$ that represent the (relevant) properties those individuals instantiate and (ii) explain the 'behaviour' of the system according to some mathematical relations among the quantities. The chosen quantities to represent the system can have different dimensions, depending on whether they are scalar, vector or tensor. A system of $(i_1, \dots, i_p)$ individuals with $(h_1, \dots, h_m)$ physical quantities with dimensions $(h_1^i, \dots, h_m^k)$ will generate a $n = p * \sum i, \dots, k$ dimensional state-space $\mathbb{S}^n$, where all the possible *states* of the system can be naturally represented. If we assume that $(h_1, \dots, h_m)$ are one dimensional, then each possible *state* of a system with $(i_1, \dots, i_p)$ individuals will correspond to the tuple $< i_1(h_1), \dots, i_1(h_m), \dots, i_p(h_1) \dots, i_p(h_m) >$ and to one and only one point in $\mathbb{S}^n$. Then, the behaviour of the system can be represented as the change from one state (point) to another.

Examples:

A. In classical mechanics, the state of a particle at each instant of time is fixed by position $r^3 = (r_x, r_y, r_z)$ and momentum $p^3 = (p_x, p_y, p_z)$. Therefore, $\mathbb{S}^n$ can be taken to be $\mathbb{R}^6$, being each state a 6-tuple of real numbers $(r_x, r_y, r_z, p_x, p_y, p_z)$.
B. In classical thermodynamics, the state of a gas is fixed by pressure, volume and temperature. Therefore, $\mathbb{S}^n$ can be taken to be $\mathbb{R}^3$ being each state a 3-tuple of real numbers (P, V, T).
C. In classical economics, the state of an economy is fixed by production, capital, labour and land. Therefore, $\mathbb{S}^n$ can be taken to be $\mathbb{R}^4$ being each state a 4-tuple of real numbers (Y, K, L, N).

A quantitative deterministic law of a system S is then a mathematical functional relation between the physical real-value quantities $h_1, \dots, h_m$ characterizing the system. They are normally formulated as functions $f\colon \mathbb{R}^n \to \mathbb{R}$ such that:

$$f(h_1, \dots, h_m) = 0$$

If for all $h_1, \dots, h_{m-1} \in \mathbb{R}$ there is only one value $h_m \in \mathbb{R}$ such that $f(h_1, \dots, h_m) = 0$, then we can write:

$$h_m = g(h_1, \dots, h_{m-1})$$

This is the archetypal form of many scientific laws, where usually $h_1, \dots, h_{m-1}$ represent observable or measurable quantities (independent values) and h_m the quantity we are interested in predicting (dependent value).

If time t is one of the physical quantities $(h_1, \dots, h_{m-1}, t)$, then $(h_1, \dots, h_{m-1})$ can have just one value for every t. The representation of $f(h_1, \dots, h_{m-1}, t)$ will correspond to a trajectory in $\mathbb{S}^n$. Those kinds of laws have been traditionally called *laws of succession*. A typical example are the Newtonian laws of motion.

If t is not one of the physical quantities then, generally, a concrete value of any quantity h_j is compatible with a set of values of the other quantities. In this case, the representation of $f(h_1, \dots, h_m)$ will correspond to a region in $\mathbb{S}^n$. Those kinds of laws have been traditionally called *laws of coexistence* and usually correspond to equilibrium states. A typical example is the Ideal gas law.

Scientific laws can present two kinds of quantities, variables and parameters, and some constants. Parameters take the form of variables in what we may call the 'general form' of the law, but transform into constants in what we may call the 'specific form' of the law, when it is applied to particular systems. For example, in the Van der Waals equation:

$$\left(P + \frac{n^2 N_A^2 a'}{V^2}\right)(V - nN_A b') = nN_A K_B T$$

We have:

- Variables: P (pressure), T (temperature), V (volume of the container).
- Parameters: n (number of moles), a' (average attraction between the particles), b' (volume excluded by a particle),
- Constants: N_A (Avogadro's constant), K_B (Boltzmann's constant).

Therefore, the state-space generated by the Van der Waals law is strictly speaking $\mathbb{R}^6$, which contains all the possible combinations of P, T, V, n, a' and b' for any gas. The general form of a law defines a set of possible trajectories (for laws of succession) or possible regions (for laws of

coexistence), each corresponding to some specification of the parameters. However, when a state-space is considered, usually only the variables are taken to conform its dimensions. Parameters usually represent initial or boundary conditions. Then, it is more accurate to say that a scientific law generates a state-space of the dimensions of its variables given some initial conditions (given some concrete values of the parameters). In the above example, given some values of n, a' and b' for a particular gas system, the van de Van der Waals equation generates a $\mathbb{R}^3$ state-space.

In the displayed framework, two different laws A and B of the same system S will have a real function representation $f^A(h_1, \dots, h_m)$ and $f^B(h_1, \dots, h_m)$ typically in the same state-space $\mathbb{S}^n$. The Ideal Gas law and the Van der Waals law exemplify this situation. In some cases, however, we might have to compare theories which postulate different physical quantities to represent the properties of the same system, generating different state-spaces. To compare them (in truthlikeness terms) we must search for a common state-space generated by their common physical quantities. For example, phlogiston and oxygen theories of combustion can be represented in a same two dimensional state-space $\mathbb{R}^2$ with variables 'time' and 'weight' of the combusted element. Then, the phlogiston theory would be (for any element) a decreasing function and the oxygen theory would be a decreasing or increasing function depending on the element in question. As another example, we can represent classical dependence between rest mass (m_0), relative mass (m) and velocity (v) as the constant function $m = m_0$ in state-space $\mathbb{R}^2$, where we can also represent the relativistic dependence $m = m_0/(1 - v^2/c^2)^{1/2}$ (Niiniluoto [1987], p. 392). In other cases, the comparison might be possible via intertheoretical connections of the quantities. For example, thermodynamic statistical mechanics can be represented in the state-space $\mathbb{R}^3$ generated by classical thermodynamics by considering pressure as the result of the collision of the particles, temperature as their mean kinetic energy and volume in the classical way.

IV. Niiniluoto's definition of truthlikeness for quantitative deterministic laws

In quantitative languages we can distinguish two main types of quantitative statements: singular quantitative statements and quantitative laws (interval statements can be treated as infinite disjunctions of singular quantitative statements). Our main focus lies on the latter, but we will say a few words about singular statements.

Typical cases of singular statements within scientific theories are those in which we are trying to estimate some real quantity θ^*, which can be a physical constant (like Avogadro's constant), a parameter (like a' and b' from the Van der Waals equation) or the prediction by a theory of the value of some quantity (like the prediction from Newtonian mechanics of the position where a cannonball will fall).

If θ is the estimate of the true value θ^*, then θ^* and each possible θ can be represented in a one dimensional state-space. In order to measure the distance (in terms of similarity) between θ^* and all the potential estimations θ, Niiniluoto ([1982], [1986], [1987], [2018b]) and Festa ([1986]) proposed to define d as the absolute difference:

$$d(\theta\,, \theta^*) = |\theta - \theta^*|$$

If the values θ and θ^* are n-dimensional, $\theta = (x_1, \ldots, x_n)$ and $\theta^* = (x_1^*, \ldots, x_n^*)$, then the most natural measure is the Euclidean:

$$d(\theta\,, \theta^*) = \sqrt{(x_1 - x_1^*)^2 +, \ldots, +(x_n - x_n^*)^2}$$

More generally, as shown by Niiniluoto, the distance between θ and θ^* can be defined by any of the *Minkowski metrics* or *p-norms*, which performs here an analogous role to the *Clifford measure* in qualitative languages. ($p \geq 1$):

$$d(\theta\,, \theta^*) = \left(\sum_{i=1}^{n} |x_i - x_i^*|^p\right)^{\frac{1}{p}}$$

Regarding the extension of d into d', Niiniluoto (1986) advocates again for its *min-sum measure*. For the case of measuring the distance from an interval I to θ^*, Niiniluoto formulates *min-sum* as follows[7]:

$$d'(I, \theta^*) = min_{x \in I}(x - \theta^*)^2 + \beta \int_I |x - \theta^*|\, dx$$

This can be generalized to situations where I is a finite union of intervals. Kieseppä ([1996a], [1996b]), however, showed that this proposal, together with Festa's approach, fails when the hypotheses compared have different dimensions, because they are based on Lebesgue integrals. Kieseppä then considers possible generalizations of *average* and *min-sum* using Hausdorff measures. His proposal can be considered as an improvement in the similarity approach regarding singular quantitative statements. Our goal is to make an analogous improvement regarding QDL.

Given a quantitative language to characterise a system, $\mathbb{S}^n$ will contain, as we saw, all the possible 'behaviours' (functions, laws) that the system can exhibit regarding the chosen physical quantities. Then, what we need to define truthlikeness for QDL is a metric d which defines the distance (in terms of similarity) between functions. Assume the general form $h_m(x) = f^i(h_1(x), \ldots, h_{m-1}(x))$, which defines a possible continuous real-value function in $\mathbb{S}^n$. Niiniluoto

[7] Where: (a) $\beta > 0$ is a constant; (b) in the case of a discrete set of points, the integral is substituted by a summation.

([1982], [1987], [2018b]) proposes to define the distance between two laws A and B with the *Minkowski metric* for functions ($p \geq 1$):

$$d(A,B) = \left(\int |f^A(x) - f^B(x)|^p \, dx \right)^{\frac{1}{p}}$$

As special cases, when p is 1, 2 or tends to ∞, the metrics are known as Manhattan, Euclidean and Chebyshev respectively:

- $d^{ma}(A,B) = \int |f^A(x) - f^B(x)|$
- $d^{eu}(A,B) = (\int |f^A(x) - f^B(x)|^2)^{\frac{1}{2}}$
- $d^{ch}(A,B) = sup\, |f^A(x) - f^B(x)|$

Where $d^{ma}(A,B)$ corresponds to the volume between the surfaces (the area when $m = 2$) and $d^{ch}(A,B)$ to the maximum distance between the surfaces; $d^{eu}(A,B)$ can be bigger or smaller than $d^{ma}(A,B)$ depending on the situation.

Niiniluoto considers that these three metrics are good candidates for a definition of truthlikeness regarding quantitative deterministic laws. Kieseppä ([1996a]) seems to agree with Niiniluoto's proposal. If we take $f^T(x)$ as the true law T, the true 'connexion' or 'relationship' between the quantities, and $d^i(A,T)$ with $[i = ma, eu, ch]$ as either of the three metrics, then a comparative judgement of truthlikeness can be defined by:

$$A >_t B \leftrightarrow d^i(A,T) < d^i(B,T)$$

And a normalized definition of the degree of truthlikeness of a law[8] can be defined by:

[8] The target c_t^* of a theory of truthlikeness for QDL is the true law T. This target, however, might be argued to be of a different kind than the actual world or the most informative true description of the world given L, which are the traditional representations of the truth in question in the literature of truthlikeness. In this regard, Cohen ([1980]) made a distinction between 'verisimilitude' (truthlikeness) and 'legisimilitude', claiming that the latter is the proper aim of science. Cohen defined legisimilitude as 'nearness to natural necessity', 'likeness to physically necessary truth' or 'lawlikeness' (likeness to law), such that legisimilitude aims to capture likeness to 'truth about other physically possible worlds as well as about the actual one' (Cohen [1980], p. 500). A similar distinction was made by Kuipers ([1982]) between 'descriptive' and 'theoretical' verisimilitude, and more recently ([2019]) between the 'actual truth' and the 'nomic truth', claiming that truthlikeness for laws and scientific theories must be evaluated in terms of their similarity to the 'nomic truth' (the set of 'really' possibilities –physical, chemical, biological, etc.- among all the conceptual possibilities of a domain). In both authors, roughly, the target T is conceived as a special set of possibilities. Our proposal is in concordance with these ideas. As we have formulated QDL in section III, what they define (either in general or in specific form) is a special set of possibilities of a given domain, and truthlikeness would be defined as the similarity between the possibilities defined by a law X and the possibilities defined by the true law T. Therefore, our concept of 'truthlikeness for QDL' can be taken as closely related to the concept of 'legisimiliude' or 'nomic truthlikeness'. See

$$Tr(A) = \frac{1}{\left(1 + d^{i}(A,T)\right)}$$

In the next section we will expose some counterexamples to the use of d^{ma}, d^{eu} and d^{ch} in defining truthlikeness for QDL. We will argue that d^{ch} is not a good measure of truthlikeness and that d^{ma} or d^{eu} represent a necessary but not sufficient condition.

V. Counterexamples

As we have seen, it is not enough that a metric d satisfies the mathematical criteria for metric functions, but it must also capture the similarities between the elements of S_p^L. Therefore, d^{ma}, d^{eu} and d^{ch} would be good candidates for the state-space of functions if they correctly capture the similarities between functions. We will argue that this is not the case.

5.1. Counterexamples to the Chebyshev metric

According to the *Chebyshev metric*, law A is more truthlike than law B if an only if $d^{ch}(A,T) < d^{ch}(B,T)$, that is to say, if and only if the maximum distance between $f^A(x)$ and $f^T(x)$ is smaller than the maximum distance between $f^B(x)$ and $f^T(x)$. Consider the following cases:

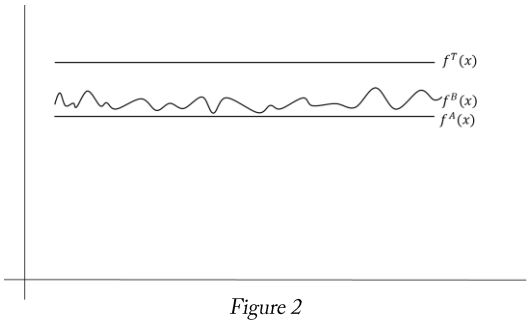

Figure 2

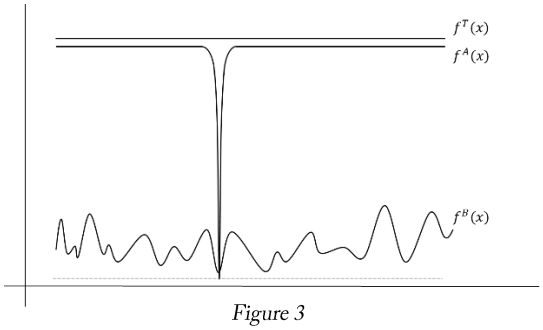

Figure 3

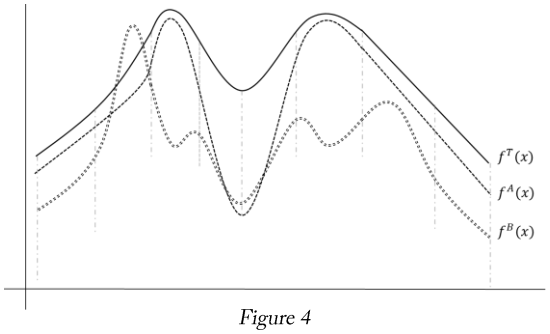

Figure 4

Oddie ([1982]) and Niiniluoto ([1987], Chapter 11; [2018b]) for a discussion of the concept of legisimilitude within their respective proposals.

In all three, it seems that $f^A(x)$ is more similar to $f^T(x)$, to the actual behaviour of the system, than $f^B(x)$. Therefore, intuitively $A >_t B$. However, according to the *Chebyshev metric*, as $d^{ch}(A,T) > d^{ch}(B,T)$, then $B >_t A$, contrary to our "similarity intuitions".

The *Chebyshev* metric is based on the idea of 'no large errors'. For example, if we want to ensure that a function $f^A(x)$ makes no errors larger than ε with respect to some function $f^B(x)$, then $d^{ch}(A,B) < \varepsilon$ guarantees this requirement. This might be a useful indicator for some purposes. However, we have tried to show that this property does not always go hand in hand with truthlikeness; that $f^A(x)$ can make an error larger than $f^B(x)$ with respect to $f^T(x)$ and even then $f^A(x)$ could be more truthlike or close to how $f^T(x)$ is than $f^B(x)$.

5.2. Counterexamples to the Manhattan and Euclidian metrics

In this section, $d(X,T)$ will refer to either $d^{ma}(X,T)$ or $d^{eu}(X,T)$. Thom ([1975]), Weston ([1992]) and Liu[9] ([1999]) present the following structurally similar cases:

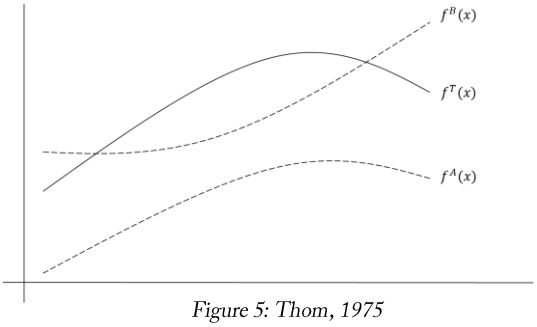

Figure 5: Thom, 1975

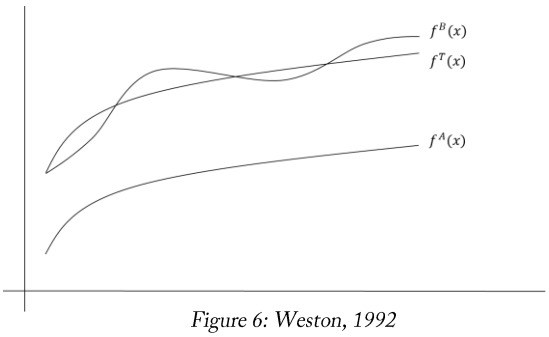

Figure 6: Weston, 1992

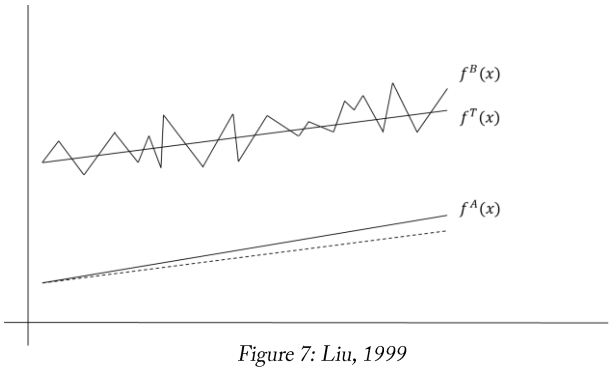

Figure 7: Liu, 1999

In all three, as $d(A,T) > d(B,T)$, Niiniluoto's proposal yields the result that $f^B(x)$ is closer to the truth $f^T(x)$ than $f^A(x)$.

[9] In Liu's case the dashed line represents the slope of the true function.

Thom (although he is not addressing directly the issue of truthlikeness) argues that in such a situation a theorist would prefer $f^A(x)$ rather than $f^B(x)$, even at the cost of a greater quantitative error, feeling that '$[f^A(x)]$, which gives rise to a graph of the same appearance as the experimental result, must be a better clue to the underlying mechanisms of $[f^T(x)]$ than the quantitatively more exact $[f^B(x)]$' ([1975], p. 4).

Weston, relying on Thom's (slightly modified) example, claims that it shows that 'we cannot compare curves in a space by simply subtracting them' ([1992], p. 59). To the question of which of the curves, $f^A(x)$ or $f^B(x)$, is more similar to $f^T(x)$, he answers that we can't give an *a priori* response. If our aim is to make numerical predictions, then $f^B(x)$ is more similar to $f^T(x)$; for other purposes, $f^A(x)$ might be consider more qualitatively similar to $f^T(x)$.

Liu talks of lawlikeness in reference to truthlikeness applied to laws. He claims that his example proves why truthlikeness definitions based on $d(X,Y)$ are wrong, because according to those definitions $B >_t A$ but, considered 'as laws', we would say that $A >_t B$, that $f^A(x)$ is closer to the true law than $f^B(x)$.

Niiniluoto answers Liu's objection suggesting that the counterexample might show that truthlikeness regarding laws is related to two different questions: '(i) what are the values of the true law? and (ii) what is the correct mathematical form of the true law?' ([2018], p. 131). Then, he points out that truthlikeness regarding laws should be considered as a 'balanced combination of them' ([2018], p. 131), but offers no formalization of the idea. We totally agree with Niiniluoto at that point and our proposal will try to integrate both properties into a formalized solution.

Further, Niiniluoto explores an alternative solution to the problem, which consists in fixing the possible mathematical forms of the functions either by some theoretical background knowledge or by considerations of simplicity. If in Liu's case we restrict the mathematical form of the functions to linear functions, then $f^B(x)$ is no longer a valid candidate.

This suggestion would not work for several reasons. First, the state-space where all the possible functions of a given system are represented cannot be constrained by theoretical background or functional considerations[10]. By definition, it must contain all the possible elements expressible in the chosen language. As Niiniluoto himself points out in relation to another counterexample of Liu, 'the state space Q should be a neutral framework for comparing various kinds of hypotheses independently of assumed background knowledge' ([2018], p. 130). Therefore, from a logical

[10] Sometimes the state-space is restricted by some theoretical basis in the form of upper and/or lower limits. However, these restrictions are minimal and in a sense derive from the definition of the quantities or the chosen language (if our language includes the term 'mass', then the axis representing this quantity can't be negative, by definition of 'mass').

point of view, Liu's oscillating function is a perfectly valid potential answer to the behaviour of the system.

Second, even if we restrict the mathematical form of the functions, we could find further counterexamples. Suppose that we introduce in the state-space a functional restriction to 'linear' mathematical forms:

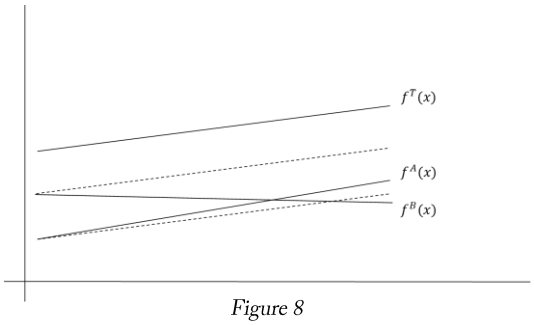

Figure 8

Then, according to $d(X,T)$, $f^B(x)$ is more truthlike than $f^A(x)$, as the surface between $f^B(x)$ and $f^T(x)$ is smaller (in the considered interval), but it intuitively seems that $f^A(x)$ should be considered more similar to $f^T(x)$ than $f^B(x)$. One could now increase the level of constraints and consider only 'positive linear' mathematical functions:

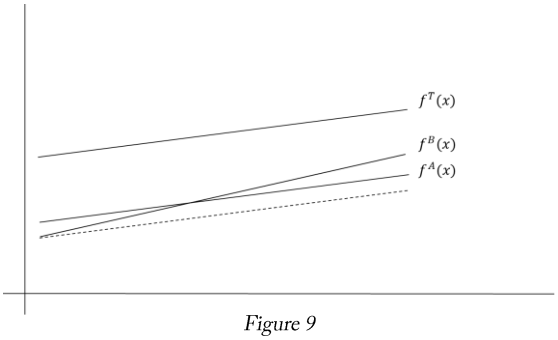

Figure 9

Where again, according to $d(X,T)$, $f^B(x)$ is more truthlike than $f^A(x)$, but it perhaps intuitively seems that $f^A(x)$ should be considered more truthlike than $f^B(x)$. Therefore, even if we admit background or simplicity constraints at the logical level, this would not constitute a solution to the counterexamples.

Having said that, we take the presented cases by Thom, Weston and Liu not to be conclusive counterexamples to Niiniluoto's proposal. Is it $f^A(x)$, in all cases, according to our intuition, clearly more similar to $f^T(x)$ than $f^B(x)$, clearly more similar to how the world is? Well, it depends. Similarity is a matter of respects. Regarding the shape of the functions, $f^A(x)$ is certainly more similar to $f^T(x)$ than $f^B(x)$. However, regarding the values of the quantities, which are also a (highly relevant) part of how the system really is, $f^B(x)$ is in all cases clearly more similar to $f^T(x)$ than $f^A(x)$.

To overcome this problem, consider the following case:

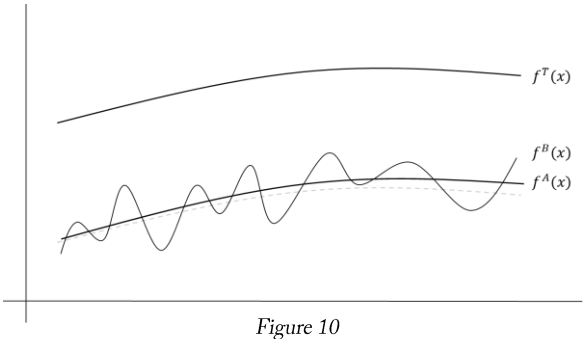

Figure 10

Consider that $f^A(x)$ and $f^B(x)$ are such that $d(A,T) = d(B,T)$. In that situation, according to Niiniluoto's proposal both laws would be equally truthlike, $A =_t B$. However, we take it as strongly intuitive that $f^A(x)$ appears to be more similar to $f^T(x)$ than $f^B(x)$, that $f^A(x)$ clearly seems to be closer to how the world is or more similar to the actual behaviour of the system than $f^B(x)$. Therefore, intuitively, $f^A(x)$ seems more truthlike than $f^B(x)$, so a satisfactory definition of truthlikeness should give the result that $A >_t B$.

Moreover, consider a slightly modified situation in which $d(A,T) = x$ and $d(B,T) = x - dx$ (meaning that $d(B,T)$ is just infinitesimally smaller than $d(A,T)$, while the shapes remain as in F10 [Figure 10]). In that case, Niiniluoto's proposal will yield the result that $B >_t A$. However, again, we take that the intuitive result should be that $A >_t B$, as $f^A(x)$ would still appear more similar to $f^T(x)$ than $f^B(x)$, even if $f^B(x)$ is infinitesimally closer to $f^T(x)$ than $f^A(x)$ in terms of $d^{ma}(X,T)$ or $d^{eu}(X,T)$.

Therefore, we take F10 to be a clear counterexample to a definition of truthlikeness for QDL relying just on the Minkowski distance between functions. And we take the presented cases by Thom, Weston and Liu and our own case as pointing to the idea that the shape of the functions seems to have some role in the definition of truthlikeness for QDL.

VI. A new definition of truthlikeness for quantitative deterministic laws

The problem lies in the fact that $d(X,T)$ measures the accuracy of a law, but an accurate law can be completely wrong about the 'actual way in which the quantities are related', and that 'structure' or 'behaviour' seem to have a weight in evaluating the truthlikeness of laws, as Niiniluoto himself points out. To see this in a less abstract way consider a car moving from Barcelona to Paris on a straight line at a constant speed of $1\ m/s$. We characterize the system by two quantities, position and time, generating a two dimensional state-space $\mathbb{R}^2$. The true function that describes the

movement of the car is $r^T = 1t$ (being r position and t time). Suppose we postulate law A as a description of the car's movement:

$$r^A = 1t + \frac{\sin(100t)}{10} + \frac{e}{(t+5)^{10}} - \pi \cdot 10^{-3}$$

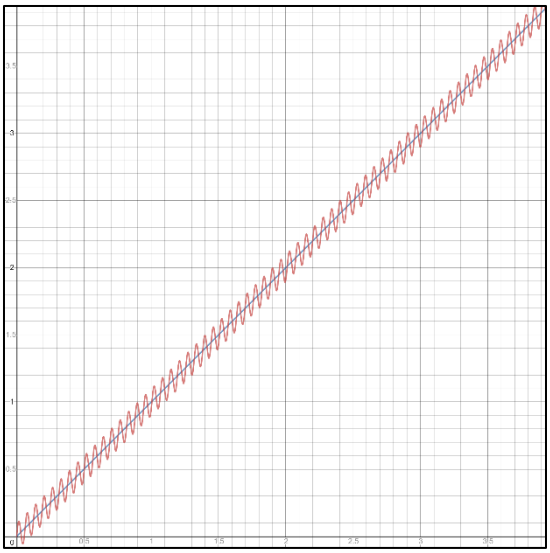

Figure 11

Law A (oscillating function in F11) is highly accurate regarding the position of the car at each instant of time. It has a maximum predictive error of less than $0.1\ m$. However, it is in a sense completely wrong about the actual true relation between position and time. According to law A: the movement of the car is not uniform; the car has a constantly oscillating acceleration; the position of the car depends to some degree on the irrational numbers e and π and is, also to some degree, inversely proportional to time; the net sum of the forces acting on the car is never zero neither constant; etc. All those facts are radically false regarding the actual 'way' in which the quantities are related. Law A, although being highly accurate, is in a sense completely wrong about some aspects of 'how the world is', about how position and time are actually related, about some relevant features of the real behaviour of the system under consideration.

Therefore, accuracy seems not enough to define truthlikeness for QDL. There seems to be another parameter, exhibited by the 'shape' of the law, which should also play a role. The shape describes the 'way' in which the quantities are related or connected, representing some relevant aspects of the 'structure' or the 'behaviour' of the system. It tells us, for instance, if the system presents a global increasing or decreasing behaviour, and the increasing or decreasing rate of such behaviour. If the shape includes some (local or global) minimums or maximums, this may imply some properties of the system (such as local or global stable or unstable states). An oscillating shape implies a continuous trade-off between the quantities, which points out to relevant

properties and behaviours that a non-oscillating shape does not represent. In all these situations, the behaviours implied by the shape of a law may or may not be obtained in the world.

To sum up, the shape of a law captures some aspects of the structure or behaviour of a system that are not captured by accuracy. What's more, two laws may exhibit very similar values but represent very different behaviours (as in F10, F11). Then, closeness or similarity to the true law should also take into account this parameter represented by the shape. We will call it 'nomicity'[11].

Obviously, this does not mean that accuracy does not play a role at all, as Liu seems to suggest. When two laws have the same or very similar shape:

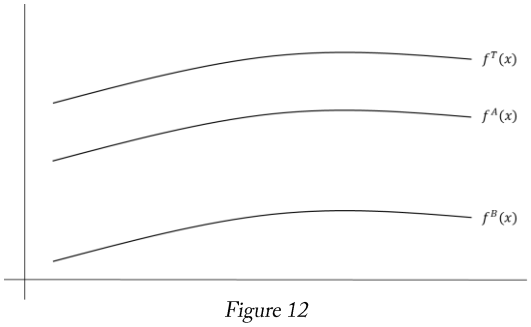

Figure 12

Then clearly the most accurate ($f^A(x)$ in F12) is the most truthlike. The numerical values of the quantities are also a relevant part of 'how the system is'. In this sense, accuracy, defined as $d^{ma}(X,T)$ or $d^{eu}(X,T)$, is a necessary but not sufficient condition to define truthlikeness for quantitative deterministic laws.

Our proposal, then, is to judge the distance (in terms of similarity) between a law X and the true law T as a function of two factors, accuracy and nomicity:

$$d^{an}(X,T) = F(accuracy,\ nomicity)$$

Accuracy measures the similarity of the quantities' values and is well captured by the Minkowski metric, either by $d^{ma}(X,T)$ or $d^{eu}(X,T)$. Nomicity measures the similarity in terms of how the quantities are related, according to the 'structures' or 'behaviours' described by the laws. Truthlikeness of quantitative deterministic laws, then, results from a combination of both factors.

[11] Under a standard account of laws of nature, nomic or natural necessity is what differentiates laws from accidental generalizations. Then, an accidental connection or relation between some quantities involves no nomicity and a lawful connection or relation between the quantities is nomic. Calling the parameter nomicity seems pertinent, as the shape of the true law represents the true way in which the quantities are connected and 'shape similarity' aims to measure the closeness between the way in which the quantities are connected in a law with respect to the true way in which they are connected in the true law. If the reader disagrees with the chosen name, just take into account that hereafter with 'nomicity' we will refer to the degree of similarity in shapes.

We propose that nomicity can be well measured by the derivative. Roughly, the derivative in a point measures the 'behaviour' of the function in the proximities or neighbourhoods of the point. It tells how the variables are related near the point, being an indicator of the 'structure' or 'behaviour' of the system at each point. In the simplest two dimensional case $y = f(x)$, if in the proximities of a point c we increase variable x by dx and variable y increases by $a \cdot dx$, a is precisely the value of the derivative in c. If two corresponding points of two functions have the same derivative then, in the proximities of these points, both functions behave alike. If two functions have a very similar derivative in each of its corresponding points across an interval I, then they define a very similar behaviour between the variables across I, and so their shapes will be very similar across I. If I is their full domain, then both functions will have a very similar shape across all their domain.

Therefore, we can measure how much two functions agree or disagree in shape (in nomicity) by calculating the distance between the derivative functions, where a small distance will indicate a high similarity in shapes and a big distance will correspond to a low similarity in shapes. The distance between derivative functions (representing the distance in terms of nomicity) is given again by any of the Minkowski metrics, where now the functions are the derivative functions[12]:

$$d(A', B') = \left(\int_{n}^{m} |f^{A'}(x) - f^{B'}(x)|^{p} \, dx \right)^{\frac{1}{p}}$$

Now, however, the Manhattan case ($p = 1$) could fail in capturing the idea of 'similarity of shapes' in some cases of symmetry[13]. The problem can be avoided by taking the Euclidean value ($p = 2$), which also has the nice property of giving more weight to large differences than to small differences.

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[12] For the purpose of measuring similarity / dissimilarity in shapes it is enough to consider the distance between the first derivatives. For functions with more than two variables, each point can be evaluated by the comparison of the total differentials, which is obtained by the sum of all the partial derivatives.

[13] Consider: $y^{T} = 2x$; $y^{A} = x$; $y^{B} = x + sinx$. For complete cycles of $f^{B}(x)$ $(2\pi, 4\pi, 6\pi \ldots)$, we obtain that $d^{ma}(A', T') = d^{ma}(B', T')$ (for $[0, 2\pi]$, $d^{ma}(A', T') = d^{ma}(B', T') = 2\pi$; for $[0, 4\pi]$, $d^{ma}(A', T') = d^{ma}(B', T') = 4\pi$; etc.). For not complete cycles of $f^{B}(x)$ we obtain that $d^{ma}(A', T') < d^{ma}(B', T')$. If one has the intuition that $f^{A}(x)$ is more similar in terms of nomicity to $f^{T}(x)$ than $f^{B}(x)$ even in complete cycles, then $d^{ma}(A', B')$ is not a good candidate for the measure of shape similarity. Note that this situation would also appear if we measure accuracy with $d^{ma}(X, T)$. In the previous example, for complete cycles of $f^{B}(x)$, $d^{ma}(A, T) = d^{ma}(B, T)$. However, this result could seem intuitively correct, as $f^{A}(x)$ and $f^{B}(x)$ are equally accurate in average. Of course, one could share the same intuition for the nomic case, and think that in complete cycles $f^{A}(x)$ and $f^{B}(x)$ are equally nomic in average. If that is the case, then $d^{ma}(A', B')$ is a good candidate for the measure of shape similarity between functions.

Therefore, the distance, in terms of similarity, between a law X and the true law T will be a function of $d^{eu}(X,T)$ and $d^{eu}(X',T')$[14]:

$$d^{an}(X,T) = F(accuracy, nomicity) = F(d^{eu}(X,T), d^{eu}(X',T'))$$

And a normalized definition of the degree of truthlikeness of a law X regarding the true law T can be achieved by:

$$Tr(X) = \frac{1}{(1 + d^{an}(X,T))}$$

The next step is how to combine both factors into a single function in order to obtain the numerical degree of truthlikeness of a law, as there are multiple ways of doing so. We have searched for a combination that satisfies at least the following properties:

(1) $d^{an}(X,T)\ is\ a\ metric\ function$
(2) $d^{an}(T,T) = 0 \rightarrow Tr(T) = 1$
(3) $Tr(X) = 1 \leftrightarrow X = T$
(4) $\Diamond(\ Tr(A) \neq Tr(B)\ )$
(5) $\Diamond(\ Tr(A) = Tr(B) \wedge A \neq B\ )$
(6) $if\ d^{eu}(X,T) = 0 \rightarrow Tr(X) = 1$
(7) $if\ d^{eu}(X',T') = 0 \rightarrow \Diamond(\ Tr(X) \neq 1\ )$
(8) $(\ if\ d^{eu}(X,T) \approx \infty\ \wedge\ d^{eu}(X',T') \approx 0) \rightarrow Tr(X) \approx 0$
(9) $(\ if\ d^{eu}(X,T) \approx 0\ \wedge d^{eu}(X',T') \approx \infty) \rightarrow Tr(X) \approx 0$

Property 2 implies that the truth has a degree of truthlikeness 1. Property 3 implies that just the truth can have a degree of truthlikeness 1. Property 4 implies that not all laws may have the same degree of truthlikeness. Property 5 implies that different laws may have the same degree of truthlikeness. Property 6 implies that just the truth can be completely accurate. Property 7 implies that inaccurate laws can be perfectly nomic, that inaccurate laws can have the same shape as the truth. Properties 8 and 9 imply that great failures in one of the factors cannot be compensated by great successes in the other. Property 8 implies that if a law is highly inaccurate and highly nomic, it cannot be considered to be truthlike. Property 9 implies that if a law is highly accurate and its

[14] Note that according to the developed reading, functions $d^{eu}(X,T)$ and $d^{eu}(X',T')$ measure accuracy and nomicity in an inverse way: the smaller the result of $d^{eu}(X,T)$, the more accurate law X is, and the bigger the result of $d^{eu}(X,T)$, the less accurate law X is (same for $d^{eu}(X',T')$ and nomicity). Under this reading, 'X is more accurate than Y' means that the result from $d^{eu}(X,T)$ is smaller than the result from $d^{eu}(Y,T)$. Alternatively, one can read $d^{eu}(X,T)$ as measuring inaccuracy (and $d^{eu}(X',T')$ as measuring anomicity). Then, the smaller the result of $d^{eu}(X,T)$, the less inaccurate law X is, and the bigger the result of $d^{eu}(X,T)$, the more inaccurate law X is (same for $d^{eu}(X',T')$ and anomicity). We will keep with the first reading.

shape highly diverges from the truth, it cannot be considered to be truthlike. With properties 8 and 9 we want to ensure that if a law X is close to the truth then, necessarily, neither factor involves large mistakes.

Summing both factors satisfies those properties[15], but a parameter must be added to *nomicity* in order to equate the units and make the sum meaningful[16]. We have opt for $(m-n)$, which represents the length of the interval under consideration[17]. Therefore[18]:

$$d^{an}(X,T) = d^{eu}(X,T) + (m-n)d^{eu}(X',T')$$

With expanded form:

$$d^{an}(X,T) = \left(\int_n^m (f^X(x) - f^T(x))^2\,dx\right)^{\frac{1}{2}} + (m-n)\left(\int_n^m (f^{X'}(x) - f^{T'}(x))^2\,dx\right)^{\frac{1}{2}}$$

Metric $d^{an}(X,T)$ fits our intuitions of the previous cases. In F10 it yields the intuitive result that $A >_t B$. In F11 it will ascribe to r^A a greater distance from the truth than its level of accuracy. In F12 the level of nomicity will play no role in comparing A and B, as it should be. In F5, F6 and F7 the result will depend on the actual structure of the functions. To see this, consider a concrete case which resembles F5 (for interval $[10 \leq x \leq 32]$):

- $f^T(x) = -0.001x^3 + 0.03x^2 + 0.2x + 10$
- $f^A(x) = -0.001x^3 + 0.03x^2 + 0.23x + 1$
- $f^B(x) = 0.001x^3 - 0.03x^2 - 0.25x + 23$

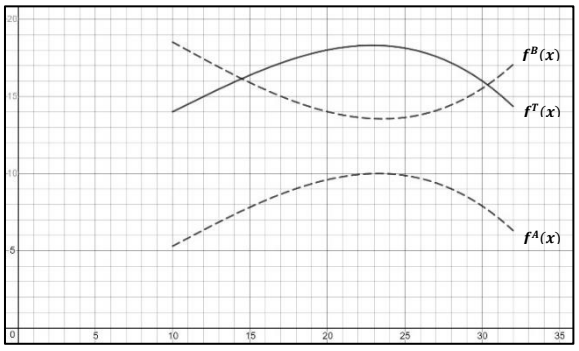

Figure 13

[15] Another natural way to combine both factors would be to multiply them, but then properties 1, 8 and 9 would be violated.

[16] In the two dimensional case, *accuracy* will have $y * x^{1/2}$ units and *nomicity* $y * x^{-1/2}$ units. Directly summing them would not make sense.

[17] After a number of mathematical considerations, this seemed to us the most rational way to equate the units.

[18] Two further parameters γ and γ', inspired in Niiniluoto, might be added: $d^{an}(X,T) = \gamma\, d^{eu}(X,T) + \gamma'\,(m-n)d^{eu}(X',T')$ $[0 < \gamma \leq 1, 0 < \gamma' \leq 1]$, in order to be able to weighthen the values of *accuracy* and *nomicity*.

Then:

- $d^{eu}(A,T) = 39.3$
- $d^{eu}(B,T) = 14.9$
- $d^{eu}(A',T') = 0.1407$
- $d^{eu}(B',T') = 4.0792$
- $d^{an}(A,T) = 39.3 + (32 - 10) * 0.1407 = 39.3 + 3.1 = 42.4$
- $d^{an}(B,T) = 14.9 + (32 - 10) * 4.0792 = 14.9 + 89.8 = 104.6$

Therefore, according to $d^{eu}(X,T)$ we obtain that $B >_t A$ $(14.9 < 39.3)$ and according to $d^{an}(X,T)$ we obtain that $A >_t B$ $(42.4 < 104.6)$.

Besides resolving the counterexamples and providing a presumably better definition of truthlikeness for quantitative deterministic laws, we can derive some additional interesting insights from $d^{an}(X,T)$. This is done in the next section.

VII. The *a-n space* and scientific progress

As we have seen, given a system S and some postulated quantities we can construct a state-space $\mathbb{S}^n$ which will contain all the possible laws regarding S. Each of these laws will have a degree of truthlikeness defined by $d^{an}(X,T)$. As $d^{an}(X,T)$ defines truthlikeness according to two variables, we can represent all the possible laws of $\mathbb{S}^n$ regarding their degrees of accuracy and nomicity in a two-dimensional space, the *a-n space*[19], where: the x-axis represents the degree of accuracy; the y-axis represents the degree of nomicity; point $(0,0)$ corresponds to the true law T; each law of $\mathbb{S}^n$ is represented by a point (a_x, n_y) with a_x degree of accuracy and n_y degree of nomicity; the closer a law is to $(0,0)$ regarding the x-axis the more accurate it is; the closer a law is to $(0,0)$ regarding the y-axis the more nomic it is.

Given a fixed degree of similarity k, such that $d^{an}(X,T) = k$, many combinations of accuracy and nomicity will have the same value k, so many different laws of $\mathbb{S}^n$ can have the same degree of truthlikeness. Each value k_i will generate a level line, which will contain a set of possible laws of $\mathbb{S}^n$ equally truthlike, with degree Tr_i defined by $\frac{1}{(1+k_i)}$:

[19] Niiniluoto ([2003]) constructs a similar two dimensional space to illustrate the truthlikeness of sentences of a propositional language given the two factors (*min* and *sum*) of his *min-sum* metric.

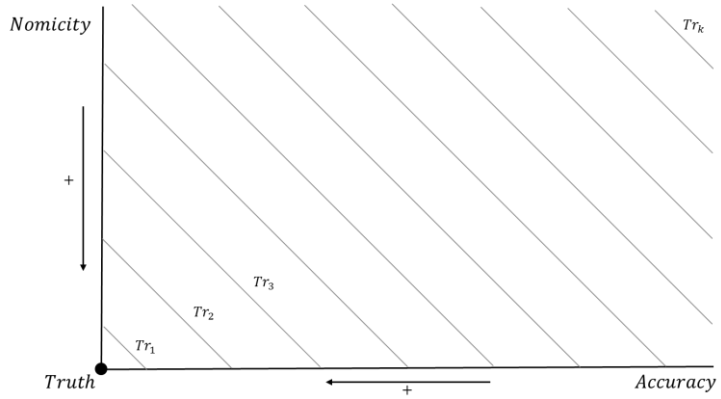

Figure 14

Note that the y-axis is empty, level lines do not cut it at the limit. This is because the only possibility for a law to have a zero degree of accuracy (perfectly accurate) is to be T itself, whereas several laws may have zero degree of nomicity (same 'shape' as T) and still different degrees of accuracy. If we understand scientific progress (regarding QDL) as increasing truthlikeness, then we can represent scientific progress in the *a-n space* as the movement from Tr_j to Tr_i, where $i < j$. A direct consequence of the framework is that scientific progress regarding laws can be achieve by two different sources: by improving accuracy, by improving nomicity or by a variation of both factors in different degrees. This means that progress can be achieved even if one of the factors gets worse, as long as the improvement of the other factor compensates and exceeds the worsening.

In order to provide a visual example, consider the dynamical laws of Einstein, Newton and Aristotle. The example does not pretend to be rigorous but merely intuitive and illustrative. Arguably, moving from Aristotle to Newton produces a great gain in accuracy and nomicty. Moving from Newton to Einstein does not increase too much the accuracy of the predictions of many phenomena, but, if the nature of space and time is very close to the ones described by Einstein, it represents a relatively higher increase in nomicity:

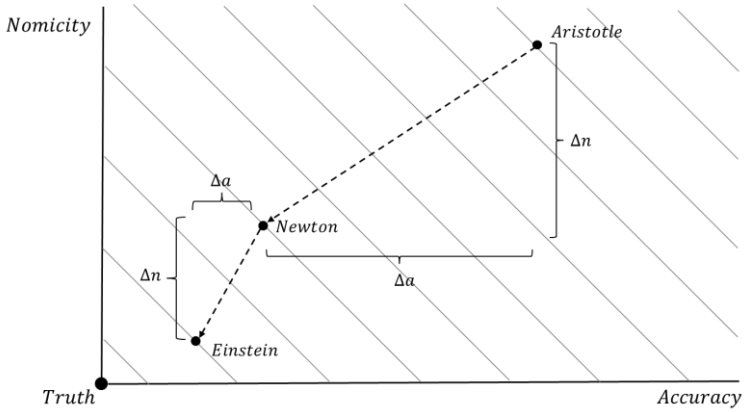

Figure 15

Function $y = x$ in the *a-n space* would represent the set of laws of $\mathbb{S}^n$ that linearly progress towards T with equal increments of accuracy and nomicity. This linear progress, however, is probably an exception in science. In general, scientific progress seems to be 'non-linear' in the explicated sense and would involve a variation of both factors in different degrees. This is because they are intimately connected, as nomicity results from the derivative function of accuracy. Any structural variation of a function in order to obtain more accurate predictions will usually vary, to some degree, its shape.

A clear historical case of 'mainly nomic scientific progress' is exemplified by the movement from the Ideal gas law (IG) to the Van der Waals law (VW). Historically, IG (formulated by Emil Clapeyron in 1834) was the first gas law developed to represent the behaviour of gas substances. Roughly, in classical thermodynamics the state of a gas substance is described by three main properties: pressure, volume and temperature. Opting for pressure as the dependent variable, as it is common, IG can be formulated as[20]:

$$P^{IG} = \frac{RT}{v}$$

Soon after it was revealed that IG is only accurate for gases at low densities (states of 'low pressures' and 'high temperatures'), being quite inaccurate for large regions of the P-V-T space. However, the main deficiency of IG was its lack of prediction of some qualitative properties of gases, such as 'phase transitions', 'phase boundaries', 'triple point', 'critical point', etc. That was partly solved by the formulation of VW in 1873 by Johannes van der Waals, which proposed a modification of IG incorporating two new parameters, a (representing the attraction between the particles) and b (representing the volume particles occupy):

$$P^{VW} = \frac{RT}{v - b} - \frac{a}{v^2}$$

Although VW increases, to some degree, the accuracy of IG, it is still quite inaccurate for large regions of the P-V-T space. However, it is widely considered as a clear case of scientific progress. Interestingly, Barnett describes this progress in truthlikeness terms: 'it is clear that this equation [VW] is a very much closer approximation to the truth than the ideal gas equation' ([1944], p. 175). This is mainly because van der Waals was the first to recognize the influence of the molecular size and the intermolecular forces of the particles of a gas, and as a result VW predicted some of the qualitative properties mentioned earlier (in other words, the 'shape' of VW approximates better to the shape of the observed behaviour of real gases). The main contribution of VW regarding scientific progress comes from its modified shape, which implied some of the

[20] Where v is the molar volume ($v = V/n$), n the number of moles and R is the universal gas constant.

observed behaviour of real gases that the IG shape did not predicted. As expressed by Gilbert in his classical *Physical Chemistry* ([1983],p. 39; p. 45):

> The van der Waals equation is a distinct improvement over the ideal gas law in that it gives qualitative reasons for the deviations from ideal behaviour.
>
> […] the van der Waals equation cannot be used for a precise calculation of the gas properties – although it is an improvement over the ideal gas law-. The great virtue of the van der Waals equation is that the study of its predictions gives an excellent insight into the behaviour of gases and their relation to liquids and the phenomenon of liquefaction. The important thing is that the equation does predict a critical state.

The tools developed so far enable us to make meaningful claims of the form 'law A is more truthlike than law B' (comparative claim) and 'the degree of truthlikeness of law A is k' (quantitative claim). If we would like to make absolute claims of the form 'law A *is* approximately true', in order to capture the idea that A has a 'sufficiently high degree of truthlikeness', then the introduction of a threshold is needed. This kind of claims is of special interest for the scientific realist, who, regarding laws, would like to claim that our best scientific laws are approximately true[21].

A natural way to proceed would be to claim that:

(D1) Law X is approximately true if and only if $d^{an}(X,T) \leq k_i$

Where k_i is some stipulated threshold. Then the region (triangle) below k_i would define the set of truthlike laws.

Alternatively, we can define some degrees of accuracy and nomicity that a law would have to satisfy independently in order to be considered truthlike. For example, as the values of a set of empirical observations always come with a margin of error ε, it may seem rational to claim that a law is accurate just in the case its predictions lie within ε or very close to it. Therefore, we can stipulate that:

Law X is sufficiently accurate if and only if $d^{eu}(X,T) \leq a_i$

Where a_i is some degree of accuracy defined by ε.

[21] Being able to establish comparative claims is already relevant for a modest realist.

Visually represented in F16:

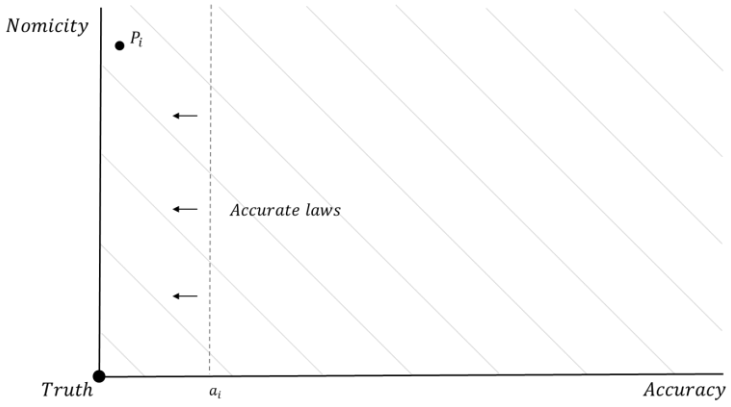

Figure 16

The region at the left of the dashed line defines the set of accurate laws. Those are the ones that Niiniluoto's proposal considers to be truthlike. F16 shows why in cases F5-F11 we were intuitively inclined (in different degrees) not to consider the accurate laws as being close to the truth. We can visually see that a law can be really accurate and still far away from T. Point P_i would be a good representation of $f^B(x)$ in F7 or of $f^A(x)$ in F11.

In the same way we can stipulate that:

Law X is sufficiently nomic if and only if $d^{eu}(X', T') \leq n_j$

Where n_j is some degree of nomicity. This will define a horizontal region which will contain the set of nomic laws. In this case, the counterpart to P_i would represent a law with a very similar shape to T but very far away in terms of $d^{eu}(X, T)$. Function $f^B(x)$ in F12 would be a close example.

How to determine a rational degree of nomicity n_j, however, is not as straightforward as in the case of accuracy. Usually, nomicity is empirically presented as qualitative observational relations between the quantities that any law regarding the phenomenon must satisfy, for example: objects dropped near the Earth's surface go down, pressure is proportional to temperature and inversely proportional to volume, a diminution in the number of preys decreases the number of predators, an increment of capital or labour increases the level of production, etc. All these empirical observations restrict the possible shapes of the laws, but in a very low degree. In some cases, it might be possible to introduce small perturbations in a system and measure its behaviour among some interval in order to define a more accurate degree of nomicity n_j.

Combining both conditions we can define that:

(D2) Law X is approximately true if and only if $a_X \leq a_i$ and $n_X \leq n_j$

That is to say, if and only if law X is sufficiently accurate and sufficiently nomic. Now a_i and n_j define a square that will contain the set of truthlike laws. If we represent D1 and D2 in the *a-n space*:

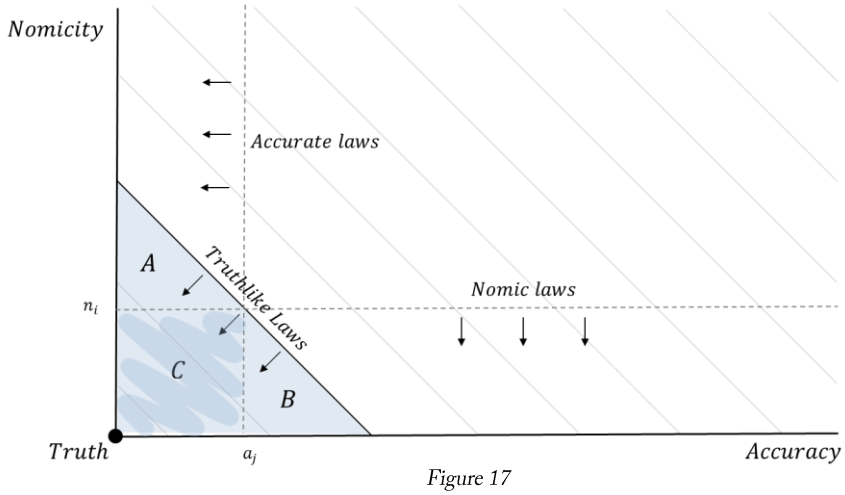

Figure 17

According to D1, truthlike laws are those belonging to ABC, while according to D2, truthlike laws are those belonging to C. In that sense, D2 might be a more restrictive definition, as it excludes some possible combinations of accuracy and nomicity allowed by D1. According to D2 'great' failures in one of the factors are not compensated by 'great' successes on the other. Therefore, laws belonging to C will have different degrees of truthlikeness, but they will be highly accurate and will present a very similar shape to T. According to D2, there is a single 'least truthlike law' Z among all the truthlike laws, corresponding to $d^{an}(Z,T) = a_i + n_j$.

Accepting D2 might conclude in apparently paradoxical situations if combined with our previous definition of progress. Suppose a law M situated in area A, a law N situated in area C, and that M lies in a level line closer to T than N. Then M will be more truthlike than N, M will constitute progress regarding N (according to our previous definition of progress), but M will not be considered truthlike whereas N will be. Accepting D2, then, leads to the following definition of progress:

(i) If $M, N \notin$ C, M constitutes progress regarding N if and only if $Tr(M) > Tr(N)$.

(ii) If $M \in$ C and $N \notin$ C, M constitutes progress regarding N.

(iii) If $M, N \in$ C, M constitutes progress regarding N if and only if $Tr(M) > Tr(N)$.

Once absolute claims regarding truthlikeness for quantitative deterministic laws have been defined by either D1 or D2, the basic scientific realist attitude regarding laws can be formulated as the claim that our best scientific laws belong to the region ABC (according to D1) or by the claim that our best scientific laws belong to the region C (according to D2).

VIII. Conclusions

We have shown that Niiniluoto's proposal defines the accuracy of a law and we have argued that an accurate law can be completely wrong about some aspects of the 'structure' or the 'behaviour' of the phenomenon it intends to describe, about the actual way in which the properties of a system interact. In that sense, accuracy turns out to be a necessary but not sufficient condition to define truthlikeness for QDL. We have argued that truthlikeness between a law X and the true law T should be understood as a function of two factors, accuracy and nomicity, where accuracy measures the similarity of the quantities' values and nomicity measures the *similarity* in terms of how the quantities are related, according to the 'behaviours' described by the laws. Our proposed function $d^{an}(X,T)$ to combine both factors satisfies a series of intuitive and desirable properties. In particular, it is based on the idea that great failures in one of the factors should not be compensated by great successes in the other. Besides, function $d^{an}(X,T)$ resolves the presented counterexamples to Niiniluoto's proposal and can be used to construct a two dimensional space, the *a-n space*, where scientific progress and the objective distance from laws to the truth can be visually represented. In particular, it shows that scientific progress understood as an increment of truthlikeness can be achieved from two different (but related) sources, by improving the accuracy factor, by improving the nomicity factor or by a variation of both factors in different degrees.

The proposal leaves open some topics for future development. On the one hand, *accuracy* and nomicity may not be the only factors that play a role in the determination of truthlikeness for laws and 'the shape of a law' may not be the only determinant of *nomicity*. Metric $d^{an}(X,T)$ is open to the addition of more parameters.

Moreover, the application of $d^{an}(X,T)$ to real cases (the epistemic problem) appears challenging, as we don't know the true law T. Given a set of empirical observations it seems natural to define an 'estimation of *accuracy*' by comparing, for each empirical observation, the observed value with the predicted value by a law. However, an 'estimation of *nomicity*' imposes additional complications, as we can't observe the derivative values of the true law.

In addition, an extension of the framework to cover probabilistic laws and probabilistic truths is desirable, as those laws conform an important family in today's science. In principle, $d^{an}(X,T)$ could be applied to cases in which X and T are probability density functions. However, one may rise doubts about the meaning of the derivative of a probability density function. The application of $d^{an}(X,T)$ to probabilistic laws needs further elaboration.

Finally, the ultimate aim of a theory of truthlikeness for science is to define the truthlikeness of scientific theories, so an expansion from laws to theories is needed. This move could be naturally done in the Sneedian-structuralist framework (Balzer *et al.*, [1987]), where, roughly, theories are conceive as structured nets whose knots are pairs of laws and intended applications, with different

weights of relevance. Take a theory M to be constituted by $L_1,..,L_n$ laws with weights $w_1,\ldots,w_n$ ($\sum w_i = 1$). Then, the degree of truthlikeness TR of theory M could be defined as $TR(M) = w_1Tr(L_1) + \cdots + w_nTr(L_n)$, where $Tr(L_i)$ is our proposed definition of the degree of truthlikeness of law L_i. For sure, this simple proposal will present many problems, particularly regarding the comparison between theories with different intended applications. It is just a starting working hypothesis for future development.

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PAPER 2

TRUTHLIKENESS FOR PROBABILISTIC LAWS

Abstract

Truthlikeness is a property of a theory or a proposition that represents its *closeness to the truth*. We start by summarizing Niiniluoto's (1987) proposal of truthlikeness for deterministic laws (DL), which defines truthlikeness as a function of accuracy, and García-Lapeña's (2021) expanded version, which defines truthlikeness for DL as a function of two factors, accuracy and nomicity. Then, we move to develop an appropriate definition of truthlikeness for probabilistic laws (PL) based on Niiniluoto's (1987) suggestion to use the Kullback–Leibler divergence to define the distance between a probability law X and the true probability law T. We argue that the Kullback–Leibler divergence seems to be the best of the available probability distances to measure accuracy between PL. However, as in the case of DL, we argue that accuracy represents a necessary but not sufficient condition, as two PL may be equally accurate and still one may imply more true or truthlike consequences, behaviours or true facts about the system than the other. The final proposal defines truthlikeness for PL as a function of two factors, p-accuracy and p-nomicity, in intimate connexion with García-Lapeña's proposal for DL.

1. Preliminaries

Truthlikeness is a property of a theory or a proposition that represents its "closeness", "similarity" or "likeness" to the truth. To intuitively introduce the notion, consider the following pair of propositions (Oddie, 1986, 2016):

a) The number of planets in our Solar System is ten.
b) The number of planets in our Solar System is ten billion.

Both *a)* and *b)* are false, as the number of planets in our Solar System is actually eight. Intuitively, however, they don't seem to be on a par: *a)* seems, in some sense, "less false" or more similar to the truth in question, closer to how things are, closer to the actual number of planets, than *b)*.

We seem to apply this intuitive relation to a large number of scientific theories and fields. For instance, regarding dynamics, Einstein's theory (probably and strictly speaking false) seems closer to the truth than Newton's theory, and both seem much closer to the truth than Aristotle's theory. Regarding the Solar System, Kepler's model seems closer to the actual behaviour of the planets than Copernicus' model, and both seem closer to the truth in question than Ptolemy's model. In thermodynamics, the van der Waals model of real gases is usually presented as an example of scientific progress regarding the ideal gas model, being the former closer to the actual behaviour of gases than the later. In fact, Barnett (1944) describes in his classical *Chemical Engineering Thermodycamics* this scientific progress in truthlikeness terms: "it is clear that this equation [van der Waals] is a very much closer approximation to the truth than the ideal gas equation".

If we remain at the true/false dichotomy level, all we can tell about all (or many) of the mentioned theories is that they are false and "simply false". With the notion of truthlikeness we aim to overcome this limitation and make compatible a set of desired ideas regarding science: (1) many (or all) scientific theories may be strictly speaking false, as they involve abstractions and idealizations, but some may be closer to the truth than others; (2) scientific progress from a false theory to another false theory can be explicated appealing to an increase in truthlikeness; (3) Although (informative) truth is perhaps unachievable, it might be said to be the aim of science in the sense of pursuing a better approximation to it; (4) we may be able to estimate the degree of truthlikeness of a theory, even if we ascribe to a fallibilist position where we will never have conclusive reasons to ascribe truth to it; (5) finally, as it is well known, truthlikeness is indispensable for many plausible formulations of scientific realism.

As with many other philosophical concepts, we must clearly distinguish between: (a) the logical or semantic problem (how it is meaningful to claim that T_2 is more truthlike than T_1) and (b) the epistemological problem (how, given some evidence, it is rational to claim that T_2 is more truthlike than T_1).

In the present paper we will primarily deal with the semantic problem of truthlikeness for probabilistic laws (PL henceforth), building on and expanding García-Lapeña's (2021) definition of truthlikeness for quantitative deterministic laws (DL henceforth). Section II introduces the similarity approach to truthlikeness, which will be later applied to DL and PL. Section III conceptualizes DL and PL within the framework of the similarity approach, showing their similarities and differences. Section IV summarizes Niiniluoto's definition of truthlikeness for DL and García-Lapeña's expanded version, which defines truthlikeness for DL as a function of

two factors, accuracy and nomicity. Section V applies the similarity approach to the case of PL. It will be argued that the Kullback–Leibler divergence seems to be the best of the available probability distances to measure accuracy between PL and that, as in the case of DL, accuracy represents a necessary but not sufficient condition to define truthlikeness for PL. The final proposal will define truthlikeness for PL as a function of two factors, p-accuracy and p-nomicity.

2. The similarity approach to truthlikeness

The similarity approach to truthlikeness was firstly proposed by Hilpinen (1976), within possible world semantics, and Tichý (1974), within propositional logic, and rapidly expanded by Niiniluoto (1987), Oddie (1986), Tuomela (1978) and Festa (1986), among others. Its core idea can be captured by the claim that the truthlikeness of a theory or a proposition can be defined appealing to the similarities between the state of affairs it allows and the actual state of affairs of the world.

The approach can be summarized as follows[1]. Consider a phenomenon or system P and a language L to talk about P. We can construct a space of possibilities S_P^L which contains all the mutually exclusive and jointly exhaustive ways $(c_1, c_2, c_3 \ldots)$ P can be regarding L, all the possible complete descriptions of P given L[2]. Then, a theory or claim h of L will be expressible as a set of elements of S_P^L. The next step is to introduce a metric $d(c_i, c_j)$ which defines the distance (in terms of similarity) between the elements of S_P^L and an *extension* of d into another metric $d'(h, c_i)$ which defines the distance (in terms of similarity) from a set of elements of S_P^L (theories or claims) to a single element. Now, given some kind of correspondence theory of truth, some element c_t^* of S_P^L will represent the truth in question (the actual world or the most informative true description of the world given L[3]). Then, connecting all the above, the degree of truthlikeness of a theory or claim h, once d' is normalized, is defined by the similarity between h and c_t^*:

$$Tr(h) = 1 - d'(h, c_t^*)$$

Where:

- $Tr(c_t^*) = 1$
- $Tr(h) > Tr(s) \leftrightarrow d'(h, c_t^*) < d'(s, c_t^*)$

[1] For an extensive review of the similarity approach and other approaches to truthlikeness see: Niiniluoto (1987, 1998, 2020); Oddie (2013, 2016); Kuipers (1987); Zwart and Franssen, M. (2007); Cevolani and Festa (2020).

[2] In Oddie's proposal, the *elements* $c_1, c_2, c_3 \ldots$ represent *possible worlds*. Niiniluoto's framework is more flexible, allowing to represent *state descriptions*, *structure descriptions*, *monadic constituents* or *scientific laws*, depending on our cognitive interests.

[3] When the target c_t^* represents the true law T, it may no longer be conceived as the actual world or the most informative true description of the world given L. See section 3.4.

This framework provides an elegant solution to the semantic problem of truthlikeness: 'T_2 is more truthlike than T_1' means that: (i) given a space S_P^L where T_1, T_2 and the truth in question c_t^* are represented and (ii) given some appropriate similarity metrics d and d' for S_p^L, then (iii) $d'(T_2, c_t^*) < d'(T_1, c_t^*)$.

As a simple example, consider the number of planets in the Solar System as our cognitive problem. In that case, the space S_p^L will contain $\mathbb{N}$ elements of the form $c_x =$ <the number of planets in our Solar System is x> $\forall x \in \mathbb{N}$. One natural similarity metric d for this structure will be the absolute difference $d(c_x, c_y) = |x - y|$. Then, if c_9 = <the number of planets in our Solar System is nine> and c_{12} = <the number of planets in our Solar System is twelve>, c_9 is closer to the truth (eight planets, c_8) than c_{12}, as $d(c_9, c_8) = 1 < d(c_{12}, c_8) = 4$, which matches our truthlikeness intuitions regarding c_9 and c_{12}.

The key element in the similarity approach is its appeal to a notion of likeness or similarity. The notion is defined according to metrics d and d' on the space S_P^L. The distance function d "has to be specified for each cognitive problem B separately, but there are 'canonical' ways of doing this for special types of problems" (Niiniluoto, 1998; p. 4). In the previous example, the absolute difference seems the most natural way to capture the similarities between the elements of S_P^L, which only differ regarding the number of planets they ascribe to the Solar System.

A similarity measure is the inverse of a given distance or metric: the less the distance between two elements is, the more similar they are. A metric d on a set X is a function $[d\colon X \times X \to [0, \infty)]$, such that for all $x, y, z \in X$:

i. $d(x, y) \geq 0$
ii. $d(x, y) = d(y, x)$
iii. $d(x, y) = 0 \leftrightarrow x = y$
iv. $d(x, z) \leq d(x, y) + d(y, z)$

However, in the similarity approach it is not enough that d and d' satisfy these mathematical criteria in order to be "good" distance to the truth functions. Given a cognitive problem, many metrics satisfying *i-iv* would result in unintuitive truthlikeness orderings. Additional restrictions for satisfactory distance to the truth functions may come from clear-cut intuitive cases and/or from general principles that any d and d' would have to respect. In the present paper we will appeal to both intuitive cases and general principles in order to define a satisfactory distance to the truth function for PL.

The literature on the similarity approach (and on truthlikeness in general) has focused mainly on qualitative languages, where d has been usually defined as the Clifford measure (the symmetric difference between the elements of S_P^L). For quantitative languages and scientific laws

(deterministic and probabilistic) there are many fewer proposals. We will outline Niiniluoto's proposal in sections IV and V.

Whereas function d must be defined for each specific cognitive problem and space S_P^L, function d' is supposed to be universal. Given some similarity values defined by an appropriate metric d, the extension d' defines the best "similarity combination". There are two main proposals in the literature to define function d'. Oddie and Tichý have favoured the "average measure" (the average of the distances of the elements that constitute h to c_t^*), while Niiniluoto has proposed the "min-sum measure"[4] (the weighted average of the minimum distance and the normalized sum of all the distances of the elements that constitute h to c_t^*). Both proposals are rival hypotheses, as in some cases and for some propositions they conclude in different truthlikeness orderings[5]. Our goal is to find an appropriate similarity metric d for PL and to establish a truthlikeness connection between DL and PL, so our proposal will be neutral regarding metric d' and compatible with both "average" and "min-sum".

To sum up, what we need to define truthlikeness for scientific laws within the framework of the similarity approach is an appropriate space S_P^L to represent scientific laws and an appropriate similarity measure d for scientific laws as represented in S_P^L. In the next section we define the kind of space S_P^L where DL and PL can be represented, showing their similarities and differences. In sections IV and V we turn to define appropriate similarity measures d for DL and PL.

3. Deterministic and probabilistic laws

3.1. Deterministic laws

One natural way to represent deterministic laws is the *state space*, used in many scientific fields[6] and philosophical proposals. We follow Niiniluoto (1987, 1990, 1994, 1998, 2018) in considering that the state space is the best way to construct the space S_P^L where scientific quantitative statements and scientific laws can be represented.

Given a phenomenon or system S, scientific theories usually (i) postulate some physical real-value quantities $(h_1, \dots, h_m)$ that represent the (relevant) properties of the system and (ii) explain its

[4] Specifically, $d^{min-sum}(h, c_t^*) = \gamma * d^{min}(h, c_t^*) + \gamma' d^{sum}(h, c_t^*)$, where $0 < \gamma \leq 1$ and $0 < \gamma' \leq 1$. Parameters γ and γ' indicate the relative weights of both factors. See Niiniluoto (1987) for other measures that Niiniluoto considers.

[5] See Oddie (2013) for an extensive discussion of the general principles characterizing the different approaches to truthlikeness and the different implications of "average" and "min-sum"; and Cevolani (2017) for a deep analysis of measure sensitive in truthlikeness measures. For Niiniluoto's comparison between $average$ and $minsum$ see 1987, Chapter 6.6. and Kuipers (1987) Chapter 1.

[6] See for instance Greiner et al., (1997).

behaviour according to some mathematical relations among the quantities. The state space is then a mathematical abstract space composed by the quantities $(h_1, \dots, h_m)$. If n is the total sum of the dimensions of the quantities, then each possible *state* of the system can be represented as a point in a n-dimensional state space $\mathbb{S}^n$. The evolution or behaviour of the system is given by the change from one point to another. For example, in classical thermodynamics the state of a gas is fixed by pressure, volume and temperature. Therefore, $\mathbb{S}^n$ can be represented by $\mathbb{R}^3$, being each possible state a 3-tuple of real numbers.

A quantitative DL is then a mathematical functional relation between the physical real-value quantities $h_1, \dots, h_m$ characterizing the system. If for all $h_1, \dots, h_{m-1} \in \mathbb{R}$ there is only one value $h_m \in \mathbb{R}$ such that $f(h_1, \dots, h_m) = 0$, then we can write:

$$h_m = g(h_1, \dots, h_{m-1})$$

This is the archetypal form of many scientific laws. Usually, $h_1, \dots, h_{m-1}$ represent observable or measurable quantities (independent values) and h_m the quantity we are interested in predicting (dependent value).

3.2. Probabilistic laws

When the target system is a random phenomenon, i.e., one in which different outcomes are possible and the one obtained depends on chance, its modelling is partly analogous to a deterministic system. We can represent all the possible outcomes of a system in a "state space" (Feller, 1971), which in probability is typically called "sample space", Ω. Each element in the sample space represents a possible outcome and each (considered) possible outcome is represented by an element in Ω. Then, a function P is introduced such that it assigns to each outcome a number between 0 and 1 (its probability), where the sum of all probabilities must equal 1. The tuple $< \Omega, P >$ defines then a probability space[7].

As in the case of deterministic systems, where some real-value quantities are postulated to represent the properties of the system, in a probabilistic system one postulates random variables to represent the possible outcomes. A random variable X is typically defined as a function $X: \Omega \longrightarrow \mathbb{R}$. If X is a discrete variable, a discrete probabilistic law $p(x)$ (probability mass function) can be defined as:

$$p(x) = P(X = x)$$

[7] Formally, a probability space is a 3-tuple $< \Omega, \mathcal{F}, P >$, where $\mathcal{F}$ is a collection of events (sets of simple or compound outcomes) and P is defined as a function $P: \mathcal{F} \longrightarrow [0,1]$. In order to simplify the exposition, we take the limiting case where $\mathcal{F} = \Omega$.

When X is a continuous variable, the probability of each single outcome is 0, such that we can only define the probabilities of intervals. Those probabilities are defined according to some probability density function. Take $f(x)$ as the associated probability density function to variable X, which can be intuitively thought as expressing the probability of each infinitesimal interval dx. Then, the probability of any interval can be defined as:

$$Pr[a \leq X \leq b] = \int_a^b f(x)dx$$

Therefore, the probability density function $f(x)$ can be taken as the representation of a PL when the system is characterized by continuous random variables[8].

Finally, let us note that the only difference between two probability laws of the same target system are the probabilities each assigns to the different possible outcomes (or intervals) of the system.

3.3. The general and the specific form of a law

A detailed analysis of DL shows that they can present two "kinds" of quantities, variables $(v_1, \ldots, v_i)$ and parameters $(k_1, \ldots, k_j)$, and some constants $(c_1, \ldots c_k)$, such that their structure may be formulated as:

$$f(v_1, \ldots, v_i, k_1, \ldots, k_j, c_1, \ldots c_k) = 0$$

Parameters usually represent initial or boundary conditions and can take the form of variables or constants. When parameters are variables, we may call the resulting function the "general form" of a law. This is represented in the state space as a set of possible trajectories (for laws of succession) or possible regions (for laws of coexistence), each corresponding to some specification of the parameters, to some initial or boundary conditions. We may say that the general form is "empirically empty" in the sense that it is not representing any particular system.

When parameters are specified, they become constants, representing some initial or boundary conditions. We may call the resulting function the "specific form" of a law. This is represented in the state space as singular trajectories or regions, which model a particular system. From now on we will use the term 'path' to refer to either a trajectory or a region. Therefore, the general form of a DL defines a set of possible paths, whereas the specific form of a DL defines a single path.

For example, the general form of the uniform linear motion law is $x = vt + x_o$ (4-dimensional), where x (position) and t (time) are variables, x_o (initial position) and v (velocity) are parameters,

[8] We will use '$f(x)$' to refer to either a (continuous) deterministic or probabilistic law. The context would make clear if we are representing a deterministic law or a probability density function.

and there are no constants. A possible specific form of the uniform linear motion law is $x = 2t + 5$ (2-dimensional). The general form of the ideal gas law may be formulated as $PV = nRT$ (4-dimensional), where P (pressure), T (temperature) and V (volume) are variables, n (number of mols of the gas substance) is a parameter and R (universal gas constant) is a constant. A possible specific form of the ideal gas law is $PV = 2RT$ (3-dimensional). As it can be appreciated, the state space corresponding to the general form of a DL is usually of a higher dimensionality than the state space corresponding to its specific form. Moreover, for testing and making predictions, usually the specific form is needed.

This distinction may be also applied to PL. Consider the "law" representing a fair coin. If a fair coin is tossed n times, the sample space will contain 2^n possible paths, each with probability 2^{-n} and the range of X will be $\{x_1, \dots, x_{2^n}\}$. Then, the general form of the "fair coin flipping" law can be represented as $p(x_i) = 2^{-n}$, where n is a parameter. Each specification of n will represent a concrete case of n fair coin flips, being a possible specific form of the law $p(x_i) = 0.25$ (corresponding to two-coin flips). One could also consider a more general coin flipping law, allowing for different basic probabilities of heads and tails. In that case the general form may be formulated as $p(x_i) = k^h(1-k)^{n-h}$, where h is the number of heads in path x_i and k is the chance of flipping heads. Now n, h and k are parameters, whose specification would represent again a concrete case of n (fair or unfair) coin flips.

The distinction between the general and the specific form of a law enables us to formulate some of the main similarities and differences between probabilistic and deterministic laws. On the one hand, both the general form of DL and PL represent a set of possible systems or models, each being specified by concrete values of the parameters. In that sense, both DL and PL specify a set of (physical) possibilities. Moreover, both DL and PL, in specific form, diminish the set of possibilities defined by their general forms[9].

On the other hand, the set of possible systems or models defined by the general form of a DL or a PL are structurally different. Each possibility defined by the general form of a DL produces a single path, whereas each possibility defined by the general form of a PL produces a set of possible paths. Accordingly, whereas the specific form of a DL represents a single path, the specific form of a PL represents a set of possible paths, associating a probability to each path (or to each subset of paths).

[9] This conclusion, argued from a different perspective (the general – specific forms of laws), is in consonance with Koslow's (2003) thesis regarding the relation between laws and possibilities and with Maudlin's (2007) conceptualization of deterministic and probabilistic laws.

3.4. The true law T

Finally, let us note that in the case of deterministic or probabilistic laws the target c_t^* of a theory of truthlikeness represents the true (deterministic or probabilistic) law T. This target may no longer be conceived as the actual world or the most informative true description of the world given L.

In this regard, Cohen (1980) distinguished between "verisimilitude" and "legisimilitude", defining the later as "nearness to natural necessity", "likeness to physically necessary truth" or "lawlikeness". Legisimilitude aims to capture likeness to "truth about other physically possible worlds as well as about the actual one" (Cohen, 1980; p. 500). A similar distinction was made by Kuipers (1982) between "descriptive" and "theoretical" verisimilitude and more recently (2019) between the "actual truth" and the "nomic truth". According to Kuipers, truthlikeness for laws and scientific theories should be evaluated according to their similarity to the "nomic truth", which is defined as the set of "real" possibilities (physical, chemical, biological, etc.) among all the conceptual possibilities of a given domain.

According to both authors, the target of a theory of truthlikeness for scientific laws or theories is, roughly, a "special" set of possibilities. Our proposal is in concordance with these ideas. As we have defined deterministic and probabilistic laws (either in general or in specific form) they represent a particular set of possibilities of a given domain. In the case of DL the special set of possibilities is represented by some determinate values defined on a state-space by the true deterministic law T. In the case of PL the special set of possibilities is represented by some determinate probabilities defined on a state-space by the true probabilistic law T. Then, in the spirit of the similarity approach, truthlikeness for deterministic or probabilistic laws will be defined as the similarity between the possibilities defined by a law X and the possibilities defined by the true law T. Therefore, our concept of truthlikeness for deterministic or probabilistic laws is closely related to the concepts of "legisimiliude" or "nomic truthlikeness"[10].

4. Truthlikeness for deterministic laws

4.1. Niiniluoto's definition of truthlikeness for DL

As we have seen in the previous section, given a quantitative language to characterise a system, $\mathbb{S}^n$ will contain all the possible behaviours (functions, laws) that the system can exhibit regarding the chosen physical quantities. Note that the possible functions cannot be constrained by theoretical background or functional considerations. As Niiniluoto states: "the state space Q

[10] See Oddie (1982) and Niiniluoto (1983; 1987, Chapter 11; 2018) for a discussion of the concept of legisimilitude within their respective proposals.

should be a neutral framework for comparing various kinds of hypotheses independently of assumed background knowledge" (2018, p. 130).

Then, given a space $\mathbb{S}^n$ and a set of possible functions, what we need to define truthlikeness for DL is an appropriate metric d which defines the distance (in terms of similarity) between functions.

Assume the general form $h_m(x) = f^i(h_1(x), \ldots, h_{m-1}(x))$, which defines a possible continuous real-value function in $\mathbb{S}^n$. Niiniluoto (1982, 1987, 2018) proposes to define the distance between a deterministic law X and the true deterministic law T (the true "connexion" between the quantities) by the *Minkowski* or L_P *metric* for functions ($p \geq 1$):

$$d(X,T) = \left(\int |f^X(x) - f^T(x)|^p \, dx \right)^{\frac{1}{p}}$$

As special cases, when p is 1, 2 or tends to ∞, the resulting metrics are known as Manhattan, Euclidean and Chebyshev respectively[11]:

- $d^{ma}(X,T) = \int |f^X(x) - f^T(x)| \, dx$
- $d^{eu}(X,T) = (\int |f^X(x) - f^T(x)|^2 \, dx)^{\frac{1}{2}}$
- $d^{ch}(X,T) = sup \, |f^X(x) - f^T(x)|$

Niiniluoto considers these three metrics as good candidates to define an appropriate similarity metric d regarding DL. Kieseppä (1996) agrees with Niiniluoto's proposal. If we take $d^i(X,T)$ with $[i = ma, eu, ch]$ as either of the three metrics, then a comparative judgement of truthlikeness between two laws, X and Y, can be defined as:

$$X >_t Y \leftrightarrow d^i(X,T) < d^i(Y,T)$$

And a normalized definition of the degree of truthlikeness of a law X can be defined by:

$$Tr(X) = \frac{1}{(1 + d^i(X,T))}$$

Niiniluoto's proposal, particularly $d^{ma}(X,T)$ and $d^{eu}(X,T)$, is attractive in a number of ways[12]. Both represent one of the most natural and intuitive ways of measuring the distances between points and functions. The L_P *metric* is one of the basic constituents of functional analysis (Deza,

[11] Where $d^{ma}(A,B)$ corresponds to the volume between the surfaces (the area in the two dimensional case) and $d^{ch}(A,B)$ to the maximum distance between the surfaces. $d^{eu}(A,B)$ can be bigger or smaller than $d^{ma}(A,B)$ depending on the situation.

[12] For counterexamples regarding the use of the Chebyshev metric to define truthlikeness, see García-Lapeña (2021).

2013). Moreover, both distances seem to match with our truthlikeness intuitions regarding a large number of cases, including historical cases. For instance, consider the situation depicted in Figure 1:

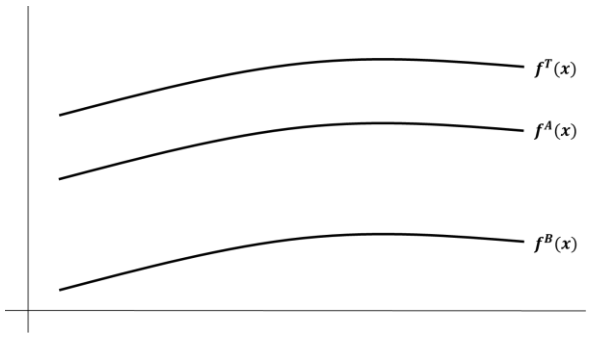

Figure 1

Then, according to $d^{ma}(X,T)$ and $d^{eu}(X,T)$, $f^A(x)$ is closer to the truth than $f^B(x)$, which fits well with our truthlikeness intuitions regarding Figure 1.

In other cases, however, Niiniluoto's proposal seem to yield an unintuitive result. Thom (1975), Weston (1992) and Liu (1999) provide three different but structurally very similar counterexamples. Consider Liu's case, shown in Figure 2[13]:

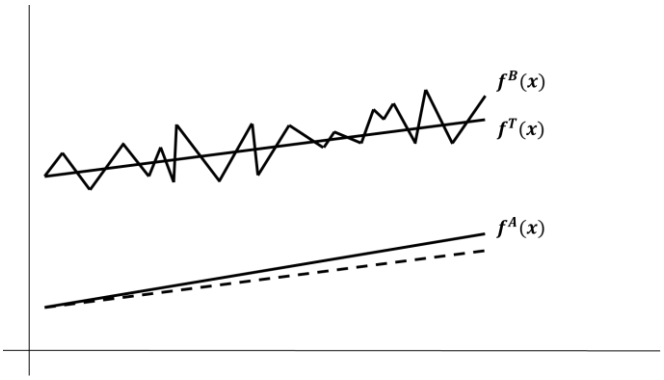

Figure 2: Liu (1999)

Take $d(X,T)$ to refer to either $d^{ma}(X,T)$ or $d^{eu}(X,T)$. In that situation, as $d(A,T) > d(B,T)$, according to Niiniluoto's proposal $f^B(x)$ is closer to the truth than $f^A(x)$. According to Liu, however, the expected result should be the opposite, because considered "as laws" it intuitively seems that $f^A(x)$ is closer to the truth than $f^B(x)$.

Liu's intuitive conclusion, however, might be disputable. It certainly seems that there is a sense in which $f^A(x)$ is more similar to $f^T(x)$ than $f^B(x)$. However, regarding the values of the

[13] The dashed line represents the slope of the true function.

quantities, $f^B(x)$ is clearly more similar to $f^T(x)$ than $f^A(x)$, and the values or states represent also a relevant aspect of how the system is.

Consider the following modified case (García-Lapeña, 2021) depicted in Figure 3:

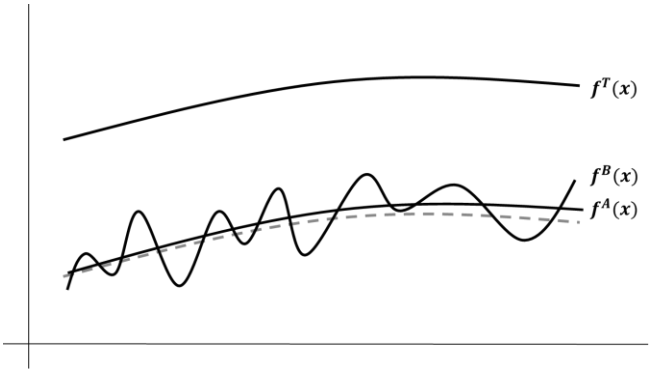

Figure 3

Consider that $f^A(x)$ and $f^B(x)$ are such that $d(A,T) = d(B,T)$, that is to say, that A and B are at the same distance from the truth (where distance is defined according to $d^{ma}(X,T)$ or $d^{eu}(X,T)$). In that situation, Niiniluoto's proposal yields the result that both laws are equally truthlike. However, we take it as strongly intuitive that $f^A(x)$ appears to be more similar to $f^T(x)$ than $f^B(x)$. Law $f^A(x)$ clearly seems to be closer to how the world is or more similar to the actual behaviour of the system than $f^B(x)$. Therefore, a satisfactory definition of truthlikeness should give the result that $A >_t B$.

This case, together with the ones presented by Thom, Weston and Liu, point to the idea that a definition of truthlikeness for DL can't rely just on the Minkowski distance between functions. The "shape" of the functions seems to play a role too regarding the similarity to the truth of DL[14].

4.2. Accuracy as a necessary but not sufficient condition

The underlying problem with Niiniluoto's proposal lies in the fact that metric $d(X,T)$ measures the accuracy of a law, as it compares each predicted value to the corresponding true value. Therefore, metric $d(X,T)$ can be taken as defining truthlikeness for DL as a function of accuracy. This idea is also at the core of Weston's proposal (1992) and Oddie's Proximity principle (2019).

[14] Niiniluoto seems to agree with this claim, as he points outs that Liu's case shows that truthlikeness regarding DL may be related to two different questions: "(i) what are the values of the true law? and (ii) what is the correct mathematical form of the true law?" (2018, p. 131), such that truthlikeness regarding DL should be considered as a "balanced combination of them" (*ibid.*).

However, Figure 3 shows that two laws may be equally accurate and still one may imply more true or truthlike consequences, behaviours or facts about the system than the other. For example, in Figure 3 according to the true law $f^T(x)$ it is a true fact about the behaviour of the system that whenever quantity x increases quantity y increases too. Law $f^A(x)$ implies this same true fact about the system but law $f^B(x)$ does not. According to law $f^B(x)$, there are many intervals in which quantity x increases and quantity y decreases. Therefore, accuracy seems to be a necessary but not sufficient condition for truthlikeness regarding DL, as an accurate law can fail to properly capture some aspects of the "structure" or "behaviour" of the system.

Consider a less abstract case to exemplify this idea, represented in Figure 4:

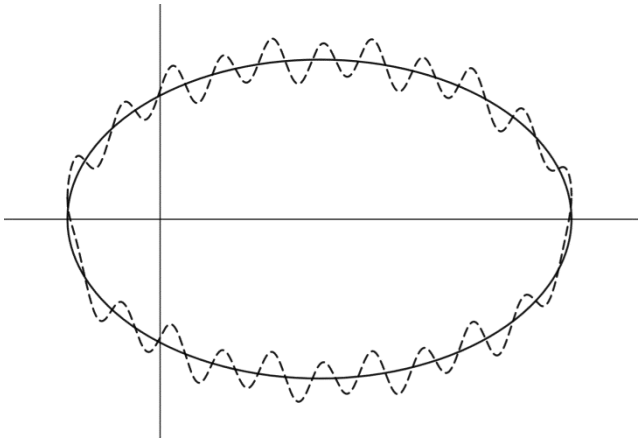

Figure 4

Suppose that the continuous curve describes the true orbit T of a planet and the dashed curve represents an orbit described by a law A. Law A can be made really accurate in describing the planet's position, but still would imply completely wrong behaviours regarding the planet's orbit. For example: according to T the planet's orbit is an ellipse, whereas this is not the case according to A; according to T, if the planet is in the first quadrant and moves in the positive sense, it always moves at a greater distance from point $(0,0)$, whereas this is not the case according to A; moreover, the acceleration and the forces acting on the planet described by T are radically different from the ones described by A.

Consider now a modified situation, shown in Figure 5, where A (dashed curve) represents again the oscillating orbit described by a law A and B (dotted curve) represents an elliptical orbit described by law B:

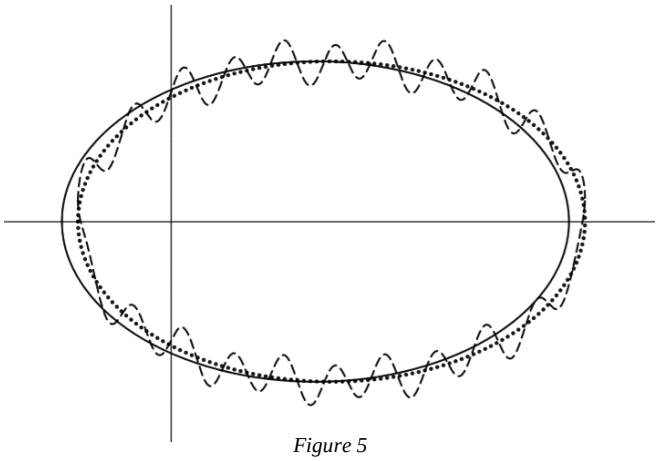

Figure 5

Consider that $f^A(x)$ and $f^B(x)$ are such that $d(A,T) = d(B,T)$, where again $d(X,T)$ refers to either $d^{ma}(X,T)$ or $d^{eu}(X,T)$). Therefore, Niniiluoto's proposal would yield the result that A and B are equally truthlike. But again, intuitively, it seems that B (dotted curve) should be considered closer to the truth than A (dashed curve). Although being equally accurate, the mentioned behaviours implied by T and not by A are now implied by B.

To sum up, the "shape" of a law seems to represent an additional factor which should play a role in defining truthlikeness for DL. It describes the way in which the quantities are related, representing some relevant aspects of the "structure" or "behaviour" of the system that are not captured by accuracy. The shape may include local or global minima or maxima, which imply relevant properties of the system, such as stable or unstable states or regions. It might represent an increasing or decreasing behaviour and its increasing or decreasing rate. An oscillating shape represents some properties that a non-oscillating shape does not imply. As a result, two laws may present very similar values but very different behaviours. Then, closeness or similarity to the true law should also take into account this factor represented by the shape. In García-Lapeña (2021) this factor is called 'nomicity'.

4.3. García-Lapeña's definition of truthlikeness for DL

Relying on the arguments and cases seen in section 4.2., García-Lapeña (2021) proposes to define the distance $d^{an}(X,T)$ between a law X and the true law T as:

$$d^{an}(X,T) = F(accuracy,\ nomicity)$$

Metric $d^{an}(X,T)$ is a natural expansion of Niiniluoto's proposal. Accuracy measures the similarity of the quantities' values and is well captured by either $d^{ma}(X,T)$ or $d^{eu}(X,T)$. Nomicity measures the similarity of some aspects of the "structures" or "behaviours" described by a law compared to the true "structures" or "behaviours" of the target system.

García-Lapeña proposes that nomicity can be captured by the distance between the derivative functions[15]. Opting in both cases for the Euclidean distance:

$$d^{an}(X,T) = F(d^{eu}(X,T), d^{eu}(X',T'))$$

Both factors need to be combined into a single function in order to make comparative judgements and to obtain the numerical degree of truthlikeness of a law. This can be done in infinitely many ways. García-Lapeña argues that $d^{an}(X,T)$ should satisfy, among others, the following two properties:

(1) $(if\ d^{eu}(X,T) \approx \infty \wedge d^{eu}(X',T') \approx 0) \rightarrow Tr(X) \approx 0$

(2) $(if\ d^{eu}(X,T) \approx 0 \wedge d^{eu}(X',T') \approx \infty) \rightarrow Tr(X) \approx 0$

These properties represent the idea that great failures in one of the factors should not be compensated by great successes in the other. According to property (1), a highly inaccurate law but with a very similar shape to the truth cannot be considered close to the truth. According to property (2), a highly accurate law but with a very diverging shape from the truth cannot be considered close to the truth. Liu's function $f^B(x)$ in Figure 2 would represent a law satisfying the antecedent of property (2). Then, according to property (2) Liu's function $f^B(x)$ cannot be considered close to the truth.

[15] The derivative in a point measures the "behaviour" of the function in the neighbourhoods of the point, describing how the variables are related near the point. Two functions with a very similar derivative across some interval will define a very similar behaviour between the variables across the interval, exhibiting a very similar "shape". Therefore, by calculating the distance between the derivative functions one can numerically define how much two functions agree or disagree in shape (in nomicity). Again, one natural metric to define the distance between the derivative functions is the *Minkowski* metric. In that case, however, $d^{eu}(X',T')$ might be preferred to $d^{ma}(X',T')$ (see García-Lapeña, 2021, for further details).

Properties (1) and (2) exclude some natural ways of combining both factors. For example, directly combining them via multiplication (as in classical mechanics, where momentum is the result of directly combining mass and velocity via multiplication) would violate both principles.

García-Lapeña proposes to combine accuracy and nomicity via summation:

$$d_1^{an}(X,T) = d^{eu}(X,T) + (m-n)d^{eu}(X',T')$$

The parameter $(m-n)$, which represents the interval under consideration, is a constant introduced to equate the units of accuracy and nomicity and make the sum meaningful. The factor $(m-n)d^{eu}(X',T')$ intends to scale shape differences into value differences in order to make them comparable. The main obvious drawback is the arbitrariness of multiplying nomicity by a constant, which may change the distance to the truth of a law depending on the chosen value. That being said, every numerical definition involves some arbitrariness, such that it must be judged according to its consequences in intuitive cases and/or according to the principles it satisfies.

In this regard, if we apply $d_1^{an}(X,T)$ to the case presented in Figure 3 we will obtain the intuitive result that law A is closer to the truth than law B, and applied to the case presented in Figure 5 we will obtain the intuitive result that law B is closer to the truth than law A. Moreover, $d_1^{an}(X,T)$ is a proper metric function and satisfies the properties (1) and (2). Finally, choosing as a constant $(m-n)$ may seem reasonable, as it represents the length of the interval under consideration.

Given $d_1^{an}(X,T)$, a normalized definition of the degree of truthlikeness of a deterministic law X regarding the true deterministic law T can be achieved by:

$$Tr_1(X) = \frac{1}{\left(1 + d_1^{an}(X,T)\right)}$$

An alternative way of combining accuracy and nomicity without the need of introducing a constant would be[16]:

$$d_2^{an}(X,T) = (d^{eu}(X,T)+1)(d^{eu}(X',T')+1) - 1$$

Function $d_2^{an}(X,T)$ also provides the intuitive results in Figure 3 and Figure 5 and satisfies properties (1) and (2). As a main drawback, it is not a metric function (it does not satisfy the triangle inequality). However, it can be normalized and used to define the degree of truthlikeness of a deterministic law X regarding the true deterministic law T as:

$$Tr_2(X) = \frac{1}{\left(1 + d_2^{an}(X,T)\right)}$$

[16] I am deeply grateful to an anonymous referee for some great suggestions that have led to this formulation.

Rearranging we can obtain:

$$Tr_2(X) = \frac{1}{\left(1 + d^{eu}(X,T)\right)} \frac{1}{\left(1 + d^{eu}(X',T')\right)}$$

This final formulation of $Tr_2(X)$ is more conceptually elegant than the one of $Tr_1(X)$. $Tr_2(X)$ can be interpreted in the following way: we first normalize accuracy and nomicity and then truthlikeness is defined as the combination of both factors via multiplication, which is the most natural mathematical way of combining two different properties. Although we lose the property of the triangle inequality, $Tr_2(X)$ does not depend on an arbitrary constant.

As an additional virtue, if it were the case that the second and further derivatives might represent further factors that might be relevant to define truthlikeness, they could be added to $Tr_2(X)$ in a very natural way:

$$Tr_2(X) = \frac{1}{\left(1 + d^{eu}(X,T)\right)} \frac{1}{\left(1 + d^{eu}(X',T')\right)} \frac{1}{\left(1 + d^{eu}(X'',T'')\right)} \ldots$$

However, the first derivative is enough to solve the difficulties presented in Figures 2-5 and to properly measure similarity and dissimilarity in shapes between functions.

5. Truthlikeness for probabilistic laws

According to the spirit of the similarity approach, what we need to define truthlikeness for PL is an appropriate similarity metric between probability functions. In the literature of probability there is a host of available distances. The Encyclopaedia of Distances (Deza, 2013) displays a total of forty-three and the list is not exhaustive. Choosing between them usually depends on the problem under consideration and the structure of the available data. For our purposes, we need to select a distance that seems appropriate to measure the closeness to the truth between a probability function X to the true probability function T.

In the literature of truthlikeness, there are almost no proposals of appropriate distances for probabilistic laws. The only known proposal to us is Niiniluoto's (1987). Inspired by Rosenkrantz's (1980) definition of truthlikeness as expected support, Niiniluoto suggests to use the Kullback–Leibler divergence (also known as relative entropy, divergence or information deviation) to define the distance between a probability law X and the true probability law T. Niiniluoto, however, does not provide an extensive argumentation on why the Kullback–Leibler divergence (KL henceforth) would be an appropriate similarity metric for PL. In what follows we will present the notion of relative entropy (5.1), confront it with other possible similarity distances

for PL (5.2), analyse if the notion of nomicity is also a relevant factor to define truthlikeness for PL (5.3) and propose a presumably satisfactory definition of truthlikeness for PL (5.4).

5.1. Relative entropy

Consider a discrete probabilistic system with n possible outcomes. Suppose a probabilistic law X with probability distribution function $p^X(x)$. Take the true law T as the one with probability distribution function $p^T(x)$. The KL distance between X and T is defined as:

$$d^{KL}(X,T) = KL(\, p^T(x) || \, p^X(x)) = \sum_n p^T(x) \log\left(\frac{p^T(x)}{p^X(x)}\right)$$

For a continuous probabilistic system, being $f^X(x)$ and $f^T(x)$ the associated probability density functions with range $m, \dots, n$:

$$d^{KL}(X,T) = KL(\, f^T(x) || \, f^X(x)) = \int_m^n f^T(x) \log\left(\frac{f^T(x)}{f^X(x)}\right) dx$$

KL was originally developed by Kullback and Leibler (1951) within the field of information theory, based on Shannon's notion of entropy (1948). It can be interpreted as measuring the average inefficiency of assuming the distribution $f^X(x)$ when the true distribution is $f^T(x)$ or alternatively as the amount of information that is lost by assuming $f^X(x)$ when the true distribution is $f^T(x)$ (Cover et. al., 2006). If base 2 logarithms are used, the resulting number can be interpreted in "bit" units. With those units in mind, KL measures the expected number of extra bits required to code a sample of $f^T(x)$ using a code based on $f^X(x)$.

In order to clarify the above interpretation, one must start from the concept of entropy. The entropy $H(p)$ of a (discrete) random variable with probability distribution $p(x)$ is defined as (Shannon, 1948):

$$H(p) = -\sum_n p(x) \log p(x)$$

Entropy can be thought as a measure of the uncertainty of the system or, alternatively, as the amount of information that is needed for removing the uncertainty of the system, such that $H(p)$ represents the average length of the shortest description of the random variable. For example, consider a random variable X with eight possible outcomes and a uniform probability distribution $p(x)$. In order to identify an outcome we would need a label that takes eight different values, which can be obtained with a string of three bits each: $\{000, 001, 010, \dots, 111\}$. If we calculate the entropy of X, the resulting number is precisely three "bits":

$$H(p) = -\sum_{1}^{8} \frac{1}{8} \log_2 \frac{1}{8} = 3 \text{ ("bits")}$$

In a similar way, the concept of cross-entropy between two probability distributions $p(x)$ and $q(x)$ is defined as:

$$H(p, q) = -\sum_{n} p(x) \log q(x)$$

Which resembles the definition of entropy, but now we are comparing a true distribution $p(x)$ to an assumed distribution $q(x)$, such that cross-entropy measures the number of "bits" on average to transmit an event governed by $p(x)$ by using a code based on $q(x)$.

For example, consider a random event with five possible outcomes and probability distribution $p(x) = \{0.25, 0.25, 0.25, 0.125, 0.125\}$. The entropy of $p(x)$ will be $H(p) = 2.25$, which represents the amount of information on average one gets from a sample drawn of $p(x)$. Consider now a probability distribution $q(x) = \{0.125, 0.125, 0.25, 0.25, 0.25\}$, with the same *entropy* $H(p) = 2.25$. If an event of $q(x)$ takes place every day and we have to transmit which one has been, an optimal code C would be $\{001, 010, 01, 10, 11\}$. This can be seen by calculating the expected number of bits to transmit the outcomes of $q(x)$ with the code C: $3bits * 0.125 + 3bits * 0.125 + \cdots + 2bits * 0.25 = 2.25\ bits$, which is precisely $q(x)$'s entropy. In other words, C implicitly assumes the correct probability distribution $q(x)$.

Now suppose we use C to transmit the outcomes of a system governed by the distribution $p(x)$. The expected number of bits to transmit the outcomes of $p(x)$ with C would be: $3bits * 0.25 + 3bits * 0.25 + \cdots + 2bits * 0.125 = 2.5\ bits$. As the average bits of the optimal code to transmit $p(x)$ is given by its *entropy* (2.25), by using C we are being $0.25\ bits$ inefficient. So, we can appreciate that using a code C based on $q(x)$ to transmit the information of a system governed by $p(x)$ is inefficient, as C assumes that the events are governed by $q(x)$. Actually, the expected number of bits to transmit the outcomes of $p(x)$ with C, $2.5\ bits$, is precisely the cross-entropy $H(p, q)$. Therefore, if we are interested in measuring the inefficiency of assuming a distribution $q(x)$ to represent a random phenomenon governed by the true distribution $p(x)$, the inefficiency can be calculated by $H(p, q) - H(p)$ ($2.5\ bits - 2.25\ bits = 0.25\ bits$). And that is precisely what KL or relative entropy measure, such that KL can be formulated as:

$$KL(p||q) = H(p, q) - H(p)$$

KL is always nonnegative and zero if $p_i^X = p_i^Y$. The usual convention is to take $0 \log \frac{0}{0} = 0$, $0 \log \frac{0}{q} = 0$ and $p \log \frac{p}{0} = \infty$. Although it is used as a distance measure in many fields, it is not a true metric. In particular, it is not symmetric and it does not satisfy the triangle inequality.

However, a symmetric version, also suggested by Niiniluoto (1987), can be easily obtained and is given by the Jeffreys divergence (JD):

$$d^{JD}(X,T) = KL(T||X) + KL(X||T)$$

5.2. Distance between probability distributions

Although, as we mentioned, there is a vast number of distances for probability functions, many can be classified in families and share common properties. Cha (2007) offers an excellent categorization based on syntactic and semantic similarities. Syntactically, he divides a total of fifty-six distances between probability functions into eight different groups, which include the most representative distances for probability functions (Gibbs et. al., 2002). Semantically, Cha develops a cluster categorization of the measures. This is done by generating thirty randomly generated probability functions, calculating the distances each of the fifty-six measures deliver between the thirty randomly generated probability functions and a target randomly generated probability function, and measuring the degree of correlation between the calculated resulting distances[17].

The results offer two main clusters of probability functions, which broadly match with the classification offered by Weller-Fahy et al. (2015) and Cha et. al. (2002). We will label them 'Geometric' and 'Divergence' families. For our purposes, the relevant point is that Cha's analysis shows a similarity of behaviours between the distance functions belonging to the same cluster, such that we can select a small sample of them as representatives of each family for our posterior analysis.

The main representative of the Geometrical cluster is, again, the *Minkowski metric*, now applied to discrete or continuous probability functions. For two discrete probability distributions $p(x)$ and $q(x)$:

$$d(p(x), q(x)) = \left(\sum_{i=1}^{n} |p(x) - q(x)|^p\right)^{\frac{1}{p}}$$

The Manhattan ($p = 1$) and the Euclidean ($p = 2$) distances are the core of subfamilies which belong to this same cluster. For instance, Manhattan is the departing point of the Sørensen and other related distances, particularly used in ecology to the comparison between species. The Euclidean distance (or some variation, as its weighted or standardized version) is also used in many fields, as data clustering or Network Intrusion Detection (Weller-Fahy et al.: 2015). The

[17] More precisely, the procedure is repeated thirty times and the final correlations are the resulting average.

Mahalanobis distance, largely used in statistics, emerges as a slight modification of the Euclidean, incorporating the correlation between the variables. Although it gives different values compared to the Euclidean, both exhibit in many cases a similar behaviour. In this cluster we also find the cosine family, another geometrical distance which roughly measures the distance between two vectors by the angle they form, not taking into consideration their weight or magnitude. This measure is particularly useful in text comparison, where one wants to get rid of the difference in length between two documents.

The second cluster, Divergence, contains mainly semimetrics, meaning that they do not satisfy some of the four requirements for metric functions. For this reason, they are usually called 'divergences', as they are "oriented distances" (not symmetric). Symmetry, however, can be easily obtained, as we showed in the case of KL. If $d^i(X,Y)$ is not symmetric, it can be transformed into a symmetric distance by the addition method: $d^{Sym(i)}(X,Y) = d^i(X,Y) + d^i(Y,X)$. Moreover, the vast majority of divergences do not satisfy the triangle inequality.

The "entropy family", as defined by Cha (2007), can be taken as the main representative of this cluster. Besides KL and JD, the Jensen-Shannon divergence and the Jensen difference are also used in some fields. Remarkably, the squared root of the Jensen-Shannon divergence does satisfy the triangle inequality. The Bhattacharyya distance and the closely related Hellinger and Matusita distances are also well-known distances, and present tight bounds with the entropy family (Gibbs et. al., 2002). Famously, Csiszár (1967) proved that many of those divergences are cases of *f-divergences*:

$$D_f(P||Q) = \int f\left(\frac{dP}{dQ}\right) dQ$$

Choosing different functions $f(t)$ one can obtain many of the mentioned distances that belong to the Divergence cluster, therefore exhibiting a similar behaviour. For instance, if $f(t) = t\,log(t)$, one obtains KL. It is interesting to point out that another more general family of divergences is given by the *Bregman Divergences*, where choosing different functions one can obtain the squared Euclidean, the squared Mahalanobis and the KL distances. Surprisingly, KL is the only distance that belongs to both *f-divergences* and *Bregman Divergences* (Amari, 2009).

From the developed analysis, we can appreciate that Niiniluoto's proposal of using KL to define the distance between a probability law X and the true probability law T is a reasonable and attractive option. KL is one of the most known and used representatives of the Divergence cluster. Both KL and its symmetric version, JD, seem good candidates to define truthlikeness between probability laws.

As an alternative, the Manhattan and the Euclidean distances (MA and EU henceforth), main representatives of the Geometrical cluster, seem reasonable options too. We take these four distances as representatives of the two main clusters of probability distance measures and as viable candidates to define truthlikeness for PL. As they can exhibit different behaviours in some circumstances, resulting in different truthlikeness orderings for PL, in order to decide which represent a better notion of closeness to the truth for PL we proceed to analyse their behaviours in some intuitive scenarios.

Scenario 1

Consider first the case S_1. Suppose a fair coin with true probability law $T = \{0.5, 0.5\}$ for the outcomes heads and tails. Suppose a law $A = \{0.4, 0.6\}$ and a law $B = \{0.6, 0.4\}$ that postulate the corresponding coin's probabilities of heads and tails. The resulting distances from A and B to T are shown in Figure 6:

	$d(A,T)$	$d(B,T)$
MA	0.2000	0.2000
EU	0.1414	0.1414
KL	0.0089	0.0089
JD	0.0176	0.0176

Figure 6

All four distances place A and B at the same distance from T (therefore, as equally truthlike). This seems to match with the intuitive result we would expect, as A and B appear to be symmetric regarding T.

Scenario 2

Consider now a modified case S_2. Suppose an unfair coin with $T = \{0.3, 0.7\}$ for the outcomes heads and tails. Suppose a law $A = \{0.35, 0.65\}$ and a law $B = \{0.25, 0.75\}$ that postulate the corresponding coin's probabilities of heads and tails. The resulting distances from A and B to T are shown in Figure 7:

	$d(A,T)$	$d(B,T)$
MA	0.1000	0.1000
EU	0.0707	0.0707
KL	0.0024	0.0028
JD	0.0050	0.0055

Figure 7

This case is crucial for our purposes, as MA and EU produce a different truthlikeness ordering than KL and JD. According to MA and EU, laws A and B are at the same distance from the truth, whereas according to KL and JD, A is closer to the truth than B. Intuitions, however, might be less clear than in S_1.

At a first glance, it may seem that A and B are symmetric regarding T, so that the correct assessment is the one given by MA and EU. If we analyse the cases individually, as "points", we have that $P^T(Head) = 0.3$, $P^A(Head) = 0.35$ and $P^B(Head) = 0.25$. The difference between $P^T(Head)$ and $P^A(Head)$ seems to be the same as the difference between $P^T(Head)$ and $P^B(Head)$, and the same is the case for the probabilities of tails. Therefore, it may seem that A and B should be considered equally truthlike, so that MA and EU yield the expected result. However, this intuitive symmetric appearance may be disputable.

On the one hand, if we represent A, B and T as vectors (Figure 8), B is not the reflection vector of A in T, meaning that the angle α between A and T is different from the angle β between T and B (their moduli are different too, as shown in Figure 6). The true reflection vector of A in T is $B' = \{0.246, 0.754\}$. In that case, if we calculate the new eight distances from A and B' to T (considering B' as the "symmetric in the vector sense" probability distribution of A regarding T), MA and EU no longer yield the result that A and B' at the same distance from T.

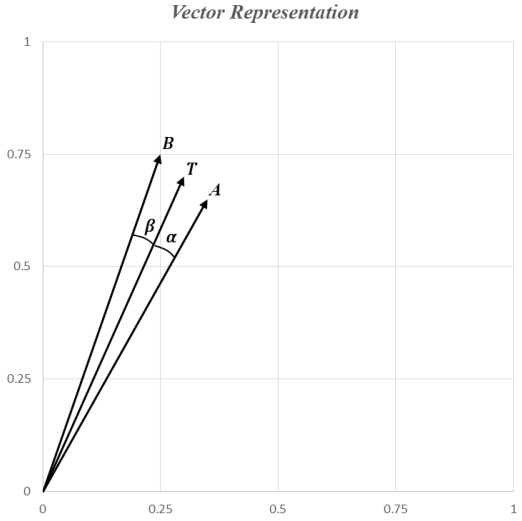

Figure 8

On the other hand, it can be argued that there is a crucial difference between S_1 and S_2 because of the probabilistic nature of the phenomenon under consideration. In S_1, there is a sense in which A and B postulate the same "probabilistic behaviour" of the coin. According to both laws, some outcome happens 40% of the time and the other happens 60% of the time. It is true that the outcomes (heads or tails) that each law ascribes to each percentage or frequency is the opposite. But the global "probabilistic behaviour" 40%-60% is the same and symmetric regarding the true

"probabilistic behaviour" defined by T. In S_2, however, the "probabilistic behaviour" represented by A and B regarding T is no longer the same, such that "symmetry of probabilistic behaviours" is broken and we should not expect that A and B are at the same distance from T, as KL and JD conclude. To reinforce this idea, consider a new scenario S_3.

Scenario 3

Consider that $T = \{0.3, 0.7\}$, law $A = \{0.599, 0.401\}$ and law $B = \{0.001, 0.999\}$. The resulting distances from A and B to T are shown in Figure 9:

	$d(A,T)$	$d(B,T)$
MA	0.5980	0.5980
EU	0.4228	0.4228
KL	0.0793	0.6350
JD	0.1621	0.7868

Figure 9

A and B are still symmetric in the same "individual" sense defined in S_2 and accordingly MA and EU locate them at the same distance from T. However, the "probabilistic behaviour" they ascribe to the system is radically different. Law A postulates a probabilistic behaviour close to 60%-40%. In contrast, law B postulates a behaviour that is almost deterministic. We could even consider the limiting scenario S_4 where $A = \{0.6, 0.4\}$ and $B = \{0, 1\}$. Again, MA and EU would locate them at the same distance from T. This may seem clearly wrong, as in S_4 the system under consideration is probabilistic and law B postulates a deterministic behaviour.

This difference in "probabilistic behaviour" and its relation to the true "probabilistic behaviour" defined by T is a strong sense in which A and B are not symmetric, either in S_2, S_3 or S_4, such that their distances to the truth should not be the same. What KL and JD actually measure is this difference between the "probabilistic behaviour" of X compared to the true "probabilistic behaviour" of T (measuring this "probabilistic behaviour" in terms of entropy or uncertainty).

From scenarios $S_1 - S_4$ we conclude that KL and JD seem to represent a better notion of similarity to the truth for probabilistic systems than MA and EU. Therefore, there are good reasons to take Niinilioto's suggestion of KL as an appropriate truthlikeness measure for PL.

5.3. Probabilistic laws, accuracy and nomicity

As we have seen, relative entropy compares each value of a given probability distribution X with the corresponding true value of the probability distribution T. In that sense, the concept can be taken as analogous to *accuracy* in DL. Both relative entropy and accuracy measure the similarity of the quantities' values between a law X a the true law T.

There is another sense, however, in which accuracy and relative entropy measure different things. As it has been defined, accuracy measures a similarity between states, whereas relative entropy measures a similarity between probabilities of states.

The issue here is partly terminological. We may now fix terminology in the following way. We may call 'accuracy' the general concept defining "value similarity", the similarity between the values of a (deterministic or probabilistic) law X and those of the true law T; 'd-accuracy' (deterministic accuracy) the concept defining the similarity between the values of a DL X and those of the true DL T (a similarity between states); and 'p-accuracy' (probabilistic accuracy) the concept defining the similarity between the values of a PL X and those of the true PL T (a similarity between probabilities).

This raises the natural question whether nomicity is also a relevant factor in PL. We will argue that indeed it is also a necessary condition for a satisfactory definition of truthlikeness for (continuous) PL. Then, we may call 'nomicity' the general concept defining "shape similarity", the similarity between the shape of a (deterministic or probabilistic) law X and that of the true law T; 'd-nomicity' (deterministic nomicity) the concept defining the similarity between the shape of a DL X and that of the true DL T; and 'p-nomicity' (probabilistic nomicity) the concept defining the similarity between the shape of a PL X and that of the true PL T.

The four introduced concepts are summarized in Figure 10:

	Value similarity	**Shape similarity**
Deterministic laws	***d-accuracy***	***d-nomicity***
Probabilistic laws	***p-accuracy***	***p-nomicity***

Figure 10

Following this terminology, we rename the presented $d_2^{an}(X,T)$ and $Tr_2(X)$ functions in section 4.3. as $d_d^{an}(X,T)$ and $Tr_d(X)$, and we will name $d_p^{an}(X,T)$ and $Tr_p(X)$ the corresponding similarity and truthlikeness definitions for probabilistic laws.

In order to argue that nomicity is also a necessary factor for PL, we will start by analysing a hypothetical situation regarding atom decay, with the additional purpose to exemplify the application of KL to a real scientific probabilistic law.

Radioactive decay is the process through which an atomic nucleus loses energy, usually by the emission of particles or photons, transmuting into another atom. Radioactivity was discovered in 1896 and soon after it was empirically observed that radioactive substances decay following an exponential law (Krane, 1988). Some years after scientists realized that it was impossible to predict the exact time when a given individual atom of a radioactive sample would decay, such that radioactivity represents a stochastic process. In fact, it is the probabilistic nature of decays which leads to the exponential observed behaviour.

Consider a substance formed by N radioactive nuclei at time t. After some time dt, the number of radioactive nuclei would have varied by dN. As the decay of an individual atom is a random phenomenon, the (expected) variation dN would be proportional to N and dt. Then, the variation of N nuclei per unit of time would be given by:

$$\frac{dN}{dt} = -\lambda N$$

Where λ is the so called 'decay constant' and represents the probability per unit of time for an individual atom to decay. Integrating and rearranging, we obtain the exponential law of radioactive decay (ED henceforth), where N_0 is the original number of nuclei in the substance at time t_0:

$$N(t) = N_0 e^{-\lambda t}$$

Note that ED, which we have derived from simple theoretical considerations, matches the exponential observed behaviour of radioactive substances. Note also, however, that its structure does not match the way we have defined PL in section III. ED does not assign a probability to each possible path a radioactive substance may follow. Actually, ED may seem deterministic, as it defines a unique path in a two-dimensional state space (defined by quantities N and t) and allows to make accurate predictions regarding the remaining quantity N after some time t. This is because ED represents the expected path a radioactive substance would follow, the average number of nuclei that would have not decayed after some time t, and the set of paths "close" to the expected one have a probability close to 1. Therefore, in order to apply KL to atom decay, we might consider its formulation for a single atom. Consider an atom with λ^T true probability of decay per unit of time. Consider a law A that postulates a probability λ^A of decay per unit of time. Then, we can calculate the distance from A to the truth as:

$$d^{KL}(A,T) = KL(T||A) = \lambda^T \log\left(\frac{\lambda^T}{\lambda^A}\right) + (1-\lambda^T)\log\left(\frac{(1-\lambda^T)}{(1-\lambda^A)}\right)$$

Consider now a concrete case where $\lambda^T = 0.05$ and $\lambda^A = 0.1$[18]. Suppose a law B that postulates a probability of decay that changes with time, defined by $\lambda^B = 0.1 + \frac{\sin(10t)}{100}$. In order to calculate the KL distance between B and T, we can either take λ^B's mean value or compare the divergence between A and T and B and T for a complete cycle of λ^B $(0, \dots, 2\pi)$. In both cases we will obtain the result that $d^{KL}(A,T) = d^{KL}(B,T)$, that is to say, that A and B are equally truthlike, which seems an unintuitive result. If we represented A (dashed curve), B (oscillating curve) and T (continuous curve) in the exponential law form, they would look something like the situation depicted in Figure 11:

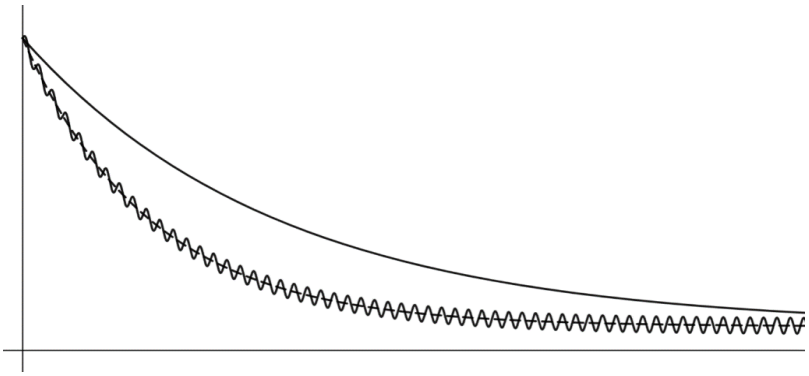

Figure 11

Where one may have the intuition that A seems closer to the truth than B, such that an appropriate similarity metric between PL should yield the result that $A >_t B$. As in the case of DL, it seems that some further factor needs to be added to p-accuracy.

As in the cases represented in Figures 3-5, this example shows that two different PL may be equally p-accurate and still one may imply more true or truthlike consequences, behaviours or facts about the system than the other. Although λ^A and λ^B are equally p-accurate, λ^A implies, among others, the true fact that the probability of decay per unit of time is constant. Therefore, p-accuracy seems to be a necessary but not sufficient condition for truthlikeness regarding PL, as a p-accurate law can fail to properly capture some aspects of the "structure" or "behaviour" of the system.

Consider another case to reinforce this idea. The normal distribution, one of the most relevant and used distributions in statistics, has a probability density function defined as (where μ is the mean and σ^2 is the variance of the random variable):

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

[18] Actual decay constants of real atoms are much lower. For instance, Cs^{137} has a λ around $4.4 * 10^{-8}$. For the clarity of the example we choose bigger values, without loss of generality.

Suppose a probabilistic system following the true normal distribution T (continuous curve) and two probabilistic laws, A (dashed curve) and B (oscillating curve), that postulate the following density functions, represented in Figure 12[19]:

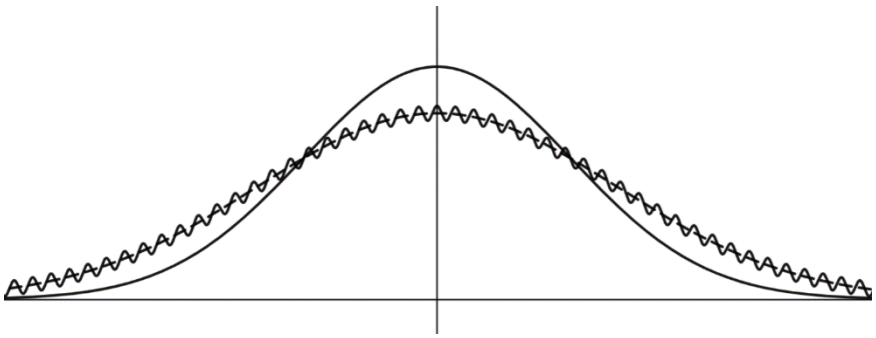

Figure 12

If A and B are properly chosen, it would be the case that $d^{KL}(A,T) = d^{KL}(B,T)$, that is to say, that A and B are equally truthlike if truthlikeness for PL is measured just as a function of p-accuracy. Again, this may seem an unintuitive result, such that an appropriate similarity metric between probability functions that measures closeness to the truth should yield the result that $A >_t B$.

The intuitive result can be achieved appealing to p-nomicity, now defined between the derivatives of the probability density functions. As in the case of DL, the Euclidean distance between the derivatives provides a satisfactory measure of shape similarity[20]:

$$d^{eu}(X',T') = \left(\int_{n}^{m} |f^{X'}(x) - f^{T'}(x)|^2 \, dx \right)^{\frac{1}{2}}$$

Where $f^{X'}(x)$ and $f^{T'}(x)$ are now the derivatives of the probability density functions. P-nomicity again, measures some aspects of the "structure" or "behaviour" of the probabilistic system that are not captured by p-accuracy. The comparison of the derivatives of two density functions $f^X(x)$ and $f^Y(x)$ for a point x_i (the possible outcome x_i) indicates the similarity between some aspects of the probabilistic behaviour ascribed to the system by $f^X(x)$ and $f^Y(x)$ near the outcome x_i. If, for some interval, $d^{eu}(X',T')$ is smaller than $d^{eu}(Y',T')$, then the shape of $f^X(x)$ is more similar to the shape of $f^T(x)$ than the shape of $f^Y(x)$. This implies that the way

[19] In order to avoid possible negative values of law B when x is large (due to its oscillations), suppose its oscillating shape is defined for the range $-x_m, \ldots, 0, \ldots, x_n$, such that a complete number of cycles are performed, and outside that range law B equals law A.

[20] KL is not used to measure the distance between derivative functions as entropy is not a defined property for derivatives of density functions. In fact, the "entropy" of the derivative of any normal distribution is zero.

probabilities behave in the interval (the way they change through the outcomes of the interval) is better captured by $f^X(x)$ than by $f^Y(x)$.

Consider the previous probabilistic system shown in Figure 13. Assume that the random variable X represents the possible positions of a particle. According to the true law T, if $x_i, x_j, x_m, x_n > 0$ and $x_m, x_n > x_i, x_j$ and $x_n - x_m = x_j - x_i$, then $P[x_i \leq X \leq x_j] > P[x_m \leq X \leq x_n]$. That is to say, the further a given interval of length l is from the origin, the less probable is to find the particle in that interval. Law A implies this same fact about the system. However, this is not true in the case of law B. According to law B, there are some positions $x_i, x_j, x_m, x_n > 0$ where $x_m, x_n > x_i, x_j$ and $x_n - x_m = x_j - x_i$, and $P[x_i \leq X \leq x_j] < P[x_m \leq X \leq x_n]$. This and many other implied behaviours by B that are not implied by T (nor by A), reinforces the idea that A and B should not be considered at the same distance from the truth, as p-accurcay concludes. Appealing to p-nomicity we can capture these differences.

5.4. Defining truthlikeness for probabilistic laws

Based on the previous cases, the similarity $d_p^{an}(X,T)$ between a (continuous) probabilistic law X and the true probabilistic law T should be a function of p-accuracy and p-nomicity:

$$d_p^{an}(X,T) = F(p\text{–}accuracy,\ p\text{–}nomicity) = F(d^{KL}(X,T), d^{eu}(X',T'))$$

If X and T are discrete probabilistic laws there is no sense of p-nomicity, as they describe no shapes. Then, for discrete probabilistic laws the similarity $d_p^{an}(X,T)$ between X and T is just a function of p-accuracy: $d_p^{an}(X,T) = d^{KL}(X,T)$.

As in the case of DL, both factors need to be combined into a single function in order to make comparative judgements and to obtain the numerical degree of truthlikeness of a PL. There are, however, some important differences between the possibilities of "value similarity" and "shape similarity" in the case of PL in comparison to the possibilities of both factors in DL.

On the one hand, in the case of DL it is possible for a law X to have the same shape as the true law T ($d^{eu}(X',T') = 0$) and different degrees of accuracy (see Figure 1). This situation is not possible in the case of PL because of the general constraint that the "summation" of the probabilities of all the possible outcomes must equal 1. The only possibility for a (continuous) PL X to have the same shape as T is to be T itself. This implies that two different (continuous) PL will always have different shapes, such that a non-zero value of KL necessarily implies a difference in shape.

On the other hand, the only possibility for a PL X to present a really high value in p-accuracy ($d^{KL}(X,T) \approx \infty$) is that X contains at least a point (or interval) with a probability really close to

zero and that T does not define a probability really close to zero in that same point or interval. However, in that situation it would not be possible that X and T present a very similar shape ($d^{eu}(X',T') \approx 0$). At least near the point or interval their shapes would largely differ. Therefore, if (1') is the property equivalent to property (1) for PL:

(1') $(if\ d^{KL}(X,T) \approx \infty \wedge d^{eu}(X',T') \approx 0) \rightarrow Tr(X) \approx 0$

The antecedent of property (1') cannot be realized in the case of PL, such that (1') is satisfied regardless of the way we combine p-accuracy and p-nomicity.

The property (2') equivalent to property (2) for PL would be:

(2') $(if\ d^{KL}(X,T) \approx 0 \wedge d^{eu}(X',T') \approx \infty) \rightarrow Tr(X) \approx 0$

As in the case of DL, we take (2') as a desirable property to be satisfied by $d_p^{an}(X,T)$. According to (2'), a highly p-accurate law but with a very diverging shape from the truth cannot be considered close to the truth.

The same combination proposal suggested in $d_d^{an}(X,T)$ works now for p-accuracy and p-nomicity:

$$d_p^{an}(X,T) = (d^{KL}(X,T)+1)(d^{eu}(X',T')+1) - 1$$

Function $d_p^{an}(X,T)$ satisfies properties (1') and (2') and implies the expected intuitive results in Figures 11 and 12. If we apply $d_p^{an}(X,T)$ to the case presented in Figure 11 regarding atom decay, we will obtain the intuitive result that λ^A is closer to the truth than λ^B. Applying it to the case of the normal distribution presented in Figure 12, we will obtain the intuitive result that A (dashed curve) is closer to the truth than B (oscillating curve).

As in the case of $d_d^{an}(X,T)$, $d_p^{an}(X,T)$ can be normalized and rearranged to define the degree of truthlikeness $Tr_p(X)$ of a probabilistic law X regarding the true probabilistic law T as:

$$Tr_p(X) = \frac{1}{\left(1+d^{KL}(X,T)\right)}\frac{1}{\left(1+d^{eu}(X',T')\right)}$$

Where:

$$X >_t Y \leftrightarrow d_p^{an}(X,T) < d_p^{an}(Y,T)$$

Functions $d_p^{an}(X,T)$ and $Tr_p(X)$ present some relevant properties:

(1) $d_p^{an}(X,T) \geq 0$

(2) $d_p^{an}(X,T) = 0 \leftrightarrow X = T$

(3) $d_p^{an}(T,T) = 0 \rightarrow Tr_p(T) = 1$

(4) $Tr_p(X) = 1 \leftrightarrow X = T$

(5) $\Diamond\,(\,Tr_p(A) \neq Tr_p(B)\,)$

(6) $\Diamond\,(\,Tr_p(A) = Tr_p(B) \wedge A \neq B\,)$

(7) $if\ d^{KL}(X,T) = 0 \rightarrow X = T$

(8) $if\ d^{eu}(X',T') = 0 \rightarrow X = T$

Property 1 implies that $d_p^{an}(X,T)$ is always nonnegative. Property 2 implies that just the truth is at distance 0 from it. Properties 3 and 4 expand property 2 to the degree of truthlikeness: the truth and just the truth has a degree of truthlikeness 1. Property 5 implies that not all PL may have the same degree of truthlikeness. Property 6 implies that different PL may have the same degree of truthlikeness. Property 7 implies that just the truth produces the most efficient encoding. Property 8 implies that no PL can have the same shape as the truth.

6. Conclusions

We have tried to develop a satisfactory answer to the semantic problem of truthlikeness for probabilistic laws within the framework of the similarity approach, in intimate connexion with García-Lapeña's (2021) proposal for deterministic laws.

In both deterministic and probabilistic laws it has been argued that accuracy represents a necessary but not sufficient condition to define closeness to the truth, as two deterministic or probabilistic laws may be equally accurate and still one may imply more true or truthlike consequences, behaviours or facts about the system than the other. The proposed method to measure this additional factor appeals to shape similarity, named 'nomicity', between a (deterministic or probabilistic) law X and the true law T. We have argued that nomicity can be well captured appealing to the Euclidean distance between the corresponding derivative functions.

Regarding deterministic laws, we proposed an alternative way of combining d-accuracy and d-nomicity which agrees with the intuitive results in the presented cases and satisfies properties (1) and (2). As its main virtues, it does not appeal to an arbitrary constant and provides a more elegant definition of truthlikeness for deterministic laws. For probabilistic laws, based on Niiniluoto's suggestion we have argued that KL seems to be the best of the available probability distances to measure p-accuracy. Then, p-accuracy and p-nomicity have been combined in the same proposed

way for deterministic laws. The final proposal agrees with the intuitive results in the presented probability cases and satisfies properties (1') and (2').

The developed framework of truthlikeness for deterministic and probabilistic laws is summarized in the following table (Figure 13):

	Value similarity	Shape similarity	Combination	Truthlikeness
Deterministic laws	***d-accuracy***	***d-nomicity***		
Measured by	$d^{eu}(X,T)$	$d^{eu}(X',T')$	$d^{an}(X,T)$	$Tr_d(X) = \frac{1}{(1+d^{eu}(X,T))}\frac{1}{(1+d^{eu}(X',T'))}$
Probabilistic laws	***p-accuracy***	***p-nomicity***		
Measured by	$d^{KL}(X,T)$	$d^{eu}(X',T')$	$d^{en}(X,T)$	$Tr_p(X) = \frac{1}{(1+d^{KL}(X,T))}\frac{1}{(1+d^{eu}(X',T'))}$

Figure 13

The proposal leaves open two main topics for future development. On the one hand, the epistemological problem has not been addressed. A direct application of the provided definitions to real cases is not possible, as we do not know the true law T. However, the semantic definitions are the base to develop ways of "estimating" accuracy and nomicity for deterministic and probabilistic laws. On the other hand, the ultimate aim of a theory of truthlikeness for science is to define truthlikeness for scientific theories, which is the notion a scientific realist is in need of. Therefore, an expansion from truthlikeness for laws to truthlikeness for theories is desirable.

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