Flatness, tangent systems and flat outputs

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To Zael and Carlota

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Chapter 1

Introduction

Feedback linearization of nonlinear control systems is a problem on which several scientists have been working during the last twenty years. Its importance lies in the fact that it enables us to transfer the properties of a linear system to a nonlinear one, as well as to propose a simple solution to one of the main problems of automatica, which is stabilization around a given trajectory of the system.
Feedback linearization is in general an open problem. There are solutions for specific cases, such as linearization by static feedback, the equivalence between static feedback and dynamic feedback linearization for single input systems, systems with m inputs and $m+1$ state variables, some cases of systems without drift,....
For many years differential geometric tools have been used in order to solve the problem of feedback linearization. Notions such as Lie brackets and involutive fields or distributions are the most common tools in this context. But to solve partial differential equations is also needed.
In the nineties, a new way of tackling the problem was proposed. This method, related to the notion of flatness, was introduced in the differential algebraic setting. This setting led to new concepts, and it has implied the introduction of new concepts for linear and nonlinear systems. It allows us to deal with a greater number of problems than the classic framework. Some years after the first works on the differential algebraic setting were done, two new versions of flatness appeared, one using the differential geometry of infinite jets, and the other in the exterior differential systems.

1.1 Linear systems

In 1963, Kalman [36] introduced a new method for describing linear control systems. In this first work the foundations for a good comprehension and a revision of the results known at that time can be found. Keywords introduced by Kalman are *state variables, controllability, observability, realization, minimal realization* ....
At the end of the sixties and the beginning of the seventies, the algebraic theory of linear control systems in an arbitrary field was developed . Roucheleau [55] and Roucheleau, Wyman and

Kalman [56] studied the realization problem over commutative rings. The situation in the non commutative case, treated in [60] by Sontag in 1976, is different because the Cayley-Hamilton fails to hold.
Willem's papers [64] - [67] must be mentioned here, because the geometric concept of trajectories introduced therein plays a crucial role, and allows us to deal with many questions without distinguishing between inputs, outputs, states and other variables. Also important was the work of Brunovsky [5], who gave a classification of linear controllable systems. Since then, the researchers in this area have referred to the Brunovsky canonical form, both those who work in linear systems and those who try to linearize nonlinear control systems.
At the beginning of the nineties, Michel Fliess suggested a new algebraic treatment for linear control systems ([10], [11], [15], [16], [20]). The cornerstone of his work resides in the fact that it enables us to put linear control theory in an algebraic setting which utilizes module theory in a more general manner than that commonly employed since Kalman. According to Fliess, these papers sketch an attempt to rehabilitate Kalman's point of view in the new context of module theory. His work is based on a state variable representation, where the dynamics is strictly in the Kalman form, but where the output map not only involves the state but also the control variables and their derivatives. This is the frame we wish to use in Chapter 2.

1.2 Static feedback linearization

Since 1973, the problem of linearization of continuous nonlinear control systems has been extensively studied. Krener [41] found conditions for linearizing a system by means of state space diffeomorphisms. A particular type of state feedback transformation was first introduced by Brockett [4]. This was later generalized for single input systems by Su [61], who also related his results to the notion of relative degree. The problem for multi input systems was finally solved by Hunt, Su and Meyer [31] and Jakubczyk and Respondek [34]. Their works used mathematical tools such as Lie brackets and involutive distributions. In fact, they proved the equivalence between the static feedback linearization and the rank and involutivity of certain distributions. The Kronecker indices [51] were also a fundamental to this procedure.
For non static feedback linearizable systems, some authors have considered partial linearizations ([42],[44]), as well as approximate feedbacks ([29], [32]).

1.3 Dynamic feedback linearization

Partial feedback linearization is related to input-output decoupling. Necessary and sufficient conditions are available for this problem. For linear systems it is known that those conditions can be weakened if one allows for a dynamic compensator. This motivated the introduction of a nonlinear dynamic state feedback transformation, which is a generalization of the static state feedback transformation. In [6] and [7], the problem of dynamic feedback linearization was studied by Charlet, Levine and Marino. Approaching the problem from the differential geometric point of view, they showed that single input systems that are dynamically feedback

linearizable are also statically feedback linearizable, and two very special cases of dynamically feedback linearizable multi input systems are also given in [6]. In [7] they presented fairly general sufficient conditions for a system to be dynamic feedback linearizable by prolongations, as well as a necessary conditions. Unfortunately, as they also showed with examples, neither are the sufficient conditions necessary nor are the necessary conditions sufficient.
Aranda-Bricaire, Moog and Pomet gave a different approach in [1] and [2]. They characterized the flat or linearizing outputs in their framework, the so-called infinitesimal Brunovsky form. Again, although their result establishes a sufficient condition for the existence of a dynamic feedback transformation that linearizes the system, this condition is not necessary in general.
Sluis and Tilbury [59] gave an upper bound on the number of integrators needed to linearize a control system, but they proved only the sharpness of the bound for systems with two inputs. Their work was based on exterior differential systems. In the same framework, Rathinam and Sluis [53] obtained a test for dynamic feedback linearization by reduction to single input systems.

1.4 Flatness in the differential algebraic setting

Differential algebra was established by Ritt [54], Kaplansky [39] and Kolchin [40]. What interests us most about this theory is the differential field extensions. Fliess was the first to introduce differential algebra into control theory for linear and nonlinear systems. One of the chief features of the utilization of differential algebra is the avoidance of explicit equations. This enables us to deal with a greater number of problems.
Using the differential version of the theorem of the primitive element, Fliess proposed a generalized canonical form in [10]. This was followed by a series of papers as a result of his joint work with Levine, Martin and Rouchon. See, for instance, [13], [14], [17], [18], [19], [22], [45]. In these papers some concepts such as flatness and defect were introduced. One major property of flatness is the existence of what the authors called flat or linearizing outputs. The system is flat, if and only if, without integrating any differential equation, the state and input variables can be directly expressed in terms of the flat outputs and a finite number of their derivatives. Flatness is best defined by not distinguishing between input, state, output and other variables. This standpoint matches Willems' approach in [64] well. He did not make distinctions among the different types of variables.
Flatness might be seen as another nonlinear extension of Kalman's controllability. In fact, any flat nonlinear system is controllable. In addition, for linear systems, flatness is equivalent to controllability. A set of flat outputs is the nonlinear analogue of a basis of a free module. It must be emphasized that from trajectories of the flat outputs, trajectories for the states and the inputs are immediately deduced.
The relationship between the nonlinear theory (using differential field extensions) and the linear theory, which utilizes modules, is given by what is called the Kähler differential [35]. This mathematical tool is used in this context to compute the associated tangent system to a nonlinear one. This tangent system is linear. Therefore, one strategy to obtain the flat outputs could be to compute the tangent system, and to find out an integrable basis of this tangent system.

1.5 Contents and contributions

The aim of Chapter 2 is to present the state of the art on linear control systems within the framework of module algebraic theory. Fliess' papers are collected, although some proofs and examples are new. Among new proofs, we would like to emphasize the proof of proposition 1, which gives the equivalence between a linear control system in state-space representation and modules over a ring of differential operators. The proofs of section 2.4 are extensions of known proofs, including all the details required to make such proofs more clear. This chapter has been submitted as a survey to the journal *Linear algebra and its applications.*
In Chapter 3 some background necessary for understanding the main results of this work is given. The different types of linearization are presented: namely, static feedback linearization, linearization by prolongations, dynamic feedback linearization, and flatness. Some known results in this field are stated, with appropriate references to locate the proofs.
Chapter 4 deals with the problem of flatness in a nonlinear multi input (m inputs) system. In the framework of differential algebra, the tangent system is used in order to find out the m^{th} flat output when $m-1$ flat outputs have been guessed. The quotient of modules is crucial in this procedure. The contributions in this Chapter include:

1. A new proof of the well known fact that linearization by static and dynamic feedback are equivalent for single-input systems.

2. A new algorithm to linearize single-input systems, as well as an algorithm to linearize multi-input systems by static feedback.

3. A theoretic procedure to linearize any multi-input systems, together with a software package to carry out the computations. Once the system is linearized, a condition to check whether or not the system can be linearized via prolongations is also derived.

4. An application of the procedure to a vertical take off and landing (VTOL) aircraft. Two new flat outputs have been obtained, and it is proven that these flat outputs can be found just by using prolongations.

These results have been published in two conferences. In 1997 SAAEI [25], which refers to static feedback linearization, and in 1998 ACC [26], which is related to dynamic feedback linearization. Some parts of this Chapter, together with a part of Chapter 5, have been submitted to *Automatica.*
In Chapter 5, the problem of linearization by prolongations of systems with two inputs is studied. A bound on the number of integrators needed to linearize a control system is obtained, using the most elementary tools of differential geometry, such as Lie brackets and involutive distributions. An algorithm derived from this result is applied to some examples, some of which were thought until now to be not linearizable by prolongations. For instance, the VTOL and the planar ducted fan. A part of this Chapter will appear in 1999 SAAEI [28].
Chapter 7 generalizes the results of Chapter 5 to an arbitrary number of inputs, improving the existent bounds in the literature when the number of inputs is greater than or equal to four. It

also contains a new proof of the fact that, when linearization by prolongations are considered, not all the inputs must be prolonged. This Chapter has been submitted to *Systems and Control Letters.*
In the case of three inputs, better results are given in Chapter 6. These results have appeared in 1999 IFAC [27].
This work ends with the conclusions and some suggestions for future research.

Chapter 2

Linear systems

This Chapter is organized as follows: Section 2.1 is devoted to comparing two definitions of linear control systems in order to show their equivalence, and examples are given at the end. In Section 2.3, controllability and observability are presented in the module formalism. Modes, poles and zeros are treated in section 2.4. Some examples clarify the work. Finally, some applications to sliding control and linear systems interconnections are explained.

2.1 Linear control systems

This section deals with two definitions of linear control systems, the classical one in state-variables and a new one using left modules ([10]). The equivalence between both definitions is shown and some examples are given.

Definition 1 *A linear control system in state-space representation is a system described by:*

$$\dot{X} = A(t)X + B(t)U$$

$$Y = C(t)X$$

where $U = (u_1, ..., u_m) \in R^m$, $A(t) \in \mathcal{M}_{n\times n}$, $B(t) \in \mathcal{M}_{m\times n}$, $X = (x_1, ..., x_n) \in R^n$ *and* $Y = (y_1, ..., y_p) \in R^p$

X are called state variables, U are input variables and Y are output variables.

Definition 2 *A linear control system using left modules is a left finitely-generated* $K[\frac{d}{dt}]$*-module* Λ. *(* K *is supposed to be a field* [1] *)*

Definition 3 *A linear dynamic with input* $U = (u_1, ..., u_n)$ *is a linear control system* Λ *(that is to say a left finitely-generated* $K[\frac{d}{dt}]$*-module) which contains* $< U >$ *and such that* $\Lambda / < U >$ *is a torsion module. An output* $Y = (y_1, ..., y_p)$ *is a finite set of elements of the system.*

[1] $K = R$ or C for constant linear control systems, otherwise K is a field of meromorphic functions

Proposition 1 *The above definitions are equivalent in the sense that if a system as in definition 1 is given, the left module of the definition 2 can be built, fulfilling the desired conditions. Conversely, if a system as in definition 2 is given, a realization as in 1 can be obtained.*

Proof. First we see that definition 1 implies definition 2.
Let

$$\dot{X} = A(t)X + B(t)U \tag{2.1}$$

be a linear system as in definition 1. Consider the left $K[\frac{d}{dt}]$-module generated by X and U. That is to say

$$M = K[\frac{d}{dt}] < X, U >$$

On the other hand, let N be the submodule generated by the relations of 2.1. Consider

$$\Lambda = M/N$$

the quotient submodule. As M is finitely generated (a finite number of X and U), Λ will also be finitely generated. So, it remains to be shown that $\Lambda/ < U >$ is torsion. Let $z \in M$. Then,

$$z = a_1x_1 + ... + a_nx_n + b_1u_1 + ...b_mu_m$$

where $a_i, b_i \in K[\frac{d}{dt}]$. Consider the natural projection over Λ,

$$\overline{z} = \overline{a_1x_1 + ... + a_nx_n + b_1u_1 + ...b_mu_m}$$

By construction, any element of Λ has this form. Making the quotient $\Lambda/ < U >$ we get:

$$\overline{z} = \overline{a_1x_1 + ... + a_nx_n}$$

If the torsion elements make up a submodule, it is only necessary to show that $\overline{x_i}$ is torsion $\forall i$. In order to end the proof the following lemmas are stated (the proof will be performed later):

Lemma 1 $\forall x_i\ \exists P_i \in K[\frac{d}{dt}]$ *such that*

$$P_ix_i \in K[\frac{d}{dt}] < U >$$

This lemma states that x_i is torsion in $\Lambda/ < U >$.

Lemma 2 *The torsion elements make up a submodule.*

Now, it must be proven that definition 3 implies definition 1. As will be seen in lemma 4, $\Lambda = T \ominus F$, where T is a torsion submodule and F is a free submodule.
Let be $\{x_i\}_{i=1}^{n_1}$ a set of generators of F and $\{z_j\}_{j=1}^{m_1}$ a set of generators of T. z_j are torsion elements, so there exists $Q_j(\frac{d}{dt})$ such that

$$Q_j(\frac{d}{dt})z_j = 0$$

On the other hand, there exists a submodule U such that Λ/U is torsion. That is to say:

$$\forall x_i \exists P_i(\frac{d}{dt}) \mid P_i(\frac{d}{dt})x_i \in U$$

An output y is an element of Λ. So

$$y = \sum_{i=1}^{n_1} R_i(\frac{d}{dt})x_i + \sum_{j=1}^{m_1} S_j(\frac{d}{dt})z_j$$

For all i, j let the next integers be defined by:

$$d_i = max\{degree(P_i), degree(R_i) + 1\}$$

$$e_j = max\{degree(Q_j), degree(S_j) + 1\}$$

Then, for any i we have a system of the form:

$$\begin{aligned} \dot{x}_i^0 &= x_i^1 \\ \dot{x}_i^1 &= x_i^2 \\ &\vdots \\ \dot{x}_i^{d_i-1} &= \sum_{k=0}^{d_i-1} a_k x_i^k + u_i \end{aligned}$$

where $x_i^k = x_i^{(k)}$, the coeffiecients a_k come from the equation $P_i(\frac{d}{dt})x_i \in U$ or the $(d_i - degree(P_i))^{th}$ derivative of this equation, if necessary, and $u_i = P_i(\frac{d}{dt})$ or also the $(d_i - degree(P_i))^{th}$ if necessary.
The same can be done for z_j:

$$\begin{aligned} \dot{z}_j^0 &= z_j^1 \\ \dot{z}_j^1 &= z_j^2 \\ &\vdots \\ \dot{z}_j^{e_j-1} &= \sum_{k=0}^{e_j-1} b_k z_j^k \end{aligned}$$

where $z_j^k = z_j^{(k)}$ and the coeffiecients b_k come from the equation $Q_j(\frac{d}{dt})z_j = 0$ or the $(e_j - degree(Q_j))$-th derivative of this equation, if necessary.
With all these variables each output y can be written as a linear combination of x_i^k and z_j^h $(i = 1, \ldots, n_1,\ j = 1, \ldots, m_1,\ k = 0, \ldots, d_i - 1,\ h = 0, \ldots, e_i - 1)$, which will be the state variables. So a system in the state-space form is obtained. Notice that the module generated by the new state-variables is Λ. And also

$$\frac{\Lambda}{< u_1, ..., u_{n_1} >} = \Lambda/U$$

That is to say, the dynamic generated by the realization obtained is the dynamic Λ/U. ∎

Now we are going to prove the two lemmas previously stated.

Proof of lemma 1: First case: A is a matrix with constant coefficients. Then, P_A (the characteristical polynomial of A) accomplishes

$$P_A(\frac{d}{dt})x_i \in K[\frac{d}{dt}] < U >$$

Indeed, in $\Lambda/ < U >$ the system is only

$$\dot{X} = AX$$

The solution of this system is $X(t) = e^{At}X_0$. If P_A is applied to this system we obtain

$$P_A(\frac{d}{dt})X = P_A(\frac{d}{dt})e^{At}X_0 = P_A(A)e^{At}X_0 = 0$$

Where the Cayley-Hamilton theorem has been applied to the last equality. So, each component of X, labelled x_i fulfills $P_A(\frac{d}{dt})x_i = 0$.
Since $X(t) = e^{At}X_0$ in the non constants coefficients case cannot be assured, this demonstration does not hold. The corresponding equation to x_i must be derived, replacing the other variables by their corresponding equations. If this method is iterated until the n^{th} derivative of x_i, we obtain:

$$\begin{pmatrix} x_i \\ \dot{x}_i \\ \vdots \\ x_i^{(n)} \end{pmatrix} = \begin{pmatrix} 0 \\ a_{i1}^1 \\ \vdots \\ a_{i1}^n \end{pmatrix} x_1 + \cdots + \begin{pmatrix} 1 \\ a_{ii}^1 \\ \vdots \\ a_{ii}^n \end{pmatrix} x_i + \cdots + \begin{pmatrix} 0 \\ a_{in}^1 \\ \vdots \\ a_{in}^n \end{pmatrix} + x_n$$

This is a (n+1)-vector which is a linear combination of n vectors. For this reason these vectors are linearly dependent. Hence their determinant is vanishing and this determinant yields a polynomial with indeterminate $\frac{d}{dt}$ such that when it is applied to x_i, it is zero.
Obviously this can be done for any state-variable. Thus we can state that

$$\forall i \; \exists P_i \mid P_i x_i = 0$$

in $\Lambda/ < U >$. In other words, x_i is torsion in the quotient submodule. ∎

Example: Consider the system

$$\dot{X} = \begin{pmatrix} t & 1 \\ 1 & t \end{pmatrix} X + \begin{pmatrix} 0 \\ 1 \end{pmatrix} U$$

Let Λ be

$$\Lambda = \frac{< x_1, x_2, u >}{< \dot{x}_1 - tx_1 - x_2, \dot{x}_2 - x_1 - tx_2 - u} >$$

Following the above algorithm, torsion of x_1 in $\Lambda / < U >$ will be proven. x_1 fulfills the following equations (the same notation is used for x_1 in Λ or in Λ/U):

$$\begin{aligned} x_1 &= x_1 \\ \dot{x}_1 &= tx_1 + x_2 \end{aligned}$$

Deriving this equation: $\ddot{x}_1 = x_1 + t\dot{x}_1 + \dot{x}_2$. And making the substitution for $\dot{x}_1$ and $\dot{x}_2$, it becomes

$$\ddot{x}_1 = (t^2 + 2)x_1 + 2tx_2 \tag{2.2}$$

The following system can be written

$$\begin{pmatrix} x_1 \\ \dot{x}_1 \\ \ddot{x}_1 \end{pmatrix} = \begin{pmatrix} 1 \\ t \\ t^2 + 2 \end{pmatrix} x_1 + \begin{pmatrix} 0 \\ 1 \\ 2t \end{pmatrix} x_2$$

So the following determinant is vanishing:

$$\begin{vmatrix} x_1 & 1 & 0 \\ \dot{x}_1 & t & 1 \\ \ddot{x}_1 & t^2 + 2 & 2t \end{vmatrix}$$

That is

$$\ddot{x}_1 - 2t\dot{x}_1 + (t^2 - 2)x_1 = 0$$

in Λ/U. So there exists a polynomial with inderminate $\frac{d}{dt}$ that voids x_1 in Λ/U. The same can be done for x_2.

Proof of lemma 2: The torsion elements must make up a submodule. Let T be the set of torsion elements. Two conditions must be proven:

$$\forall x, y \in T \Rightarrow x + y \in T \quad (1)$$

$$\forall x \in T, k \in K[\frac{d}{dt}] \Rightarrow kx \in T \quad (2)$$

First of all a property of the ring of differential operators $A = K[\frac{d}{dt}]$ will be proven:

Lemma 3

$$\forall a, b \in A, a \neq 0, b \neq 0, \exists a', b' \mid 0 \neq b'a = a'b$$

Proof: Let be

$$a = \sum_{i=0}^{n} a_i (\frac{d}{dt})^i, \quad b = \sum_{j=0}^{m} b_j (\frac{d}{dt})^j$$

If there exist,

$$a' = \sum_{k=0}^{n} a'_k (\frac{d}{dt})^k, \quad b' = \sum_{l=0}^{m} b'_l (\frac{d}{dt})^l$$

The equality $a'b = b'a$ must be verified; that is to say:

$$(\sum_{k=0}^{n} a'_k (\frac{d}{dt})^k)(\sum_{j=0}^{m} b_j (\frac{d}{dt})^j) = (\sum_{l=0}^{m} b'_l (\frac{d}{dt})^l)(\sum_{i=0}^{n} a_i (\frac{d}{dt})^i)$$

Equaling term to term, a system with $n+m+1$ homogeneous equations and $n+m+2$ unknowns is obtained. So it has a non trivial solution, and hence the existence of a' and b' fulfilling the required conditions can be deduced. ∎

Once this fact has been proven, it is not difficult to prove lemma 2: As $x \in T$, $\exists a \in A \mid ax = 0$ Analogously, as $y \in T$, $\exists b \in A \mid by = 0$. Using the property just proved $\exists a', b' \mid a'b = b'a \neq 0$. Then $b'a(x + y) = b'ax + b'ay = b'ay = a'by = 0$. So $x + y \in T$.
On the other hand, $a'kx = k'ax = 0 \Rightarrow kx \in T$. This fact finishes the proof of the lemma 2. ∎

Now, another lemma will be stated and proven. This lemma will be useful in order to decompose the submodule Λ into a direct sum of a torsion submodule and a free submodule.

Lemma 4 $\Lambda = T \oplus F$, *where T is a torsion submodule and F is a free submodule (that is to say, without torsion elements).*

Proof: Consider the canonical morphism:

$$\Pi : \Lambda \longrightarrow \Lambda/T$$

By definition this morphism is linear and exhaustive. It is clear that the kernel of this morphism is T. So it must must ensured that Λ/T is free. Let $\overline{y} \neq 0$ be a torsion element of Λ/T. Such element is the image by the morphism Π of an element $y \in \Lambda$. As $\overline{y}$ is a torsion element $\exists p_1 \in K[\frac{d}{dt}] \mid p_1\overline{y} = 0$ Thus $p_1 y \in T$ and, consequently, $\exists p_2 \in K[\frac{d}{dt}] \mid p_2 p_1 y = 0$. This is the same as y is torsion or, in other words, $\overline{y} = 0$, which contradicts our initial assumptions. Thus, there is no torsion element in Λ/T; that is to say Λ/T is a free module. F will be generated by the one element of each subset of inverse images of the generators of the free module Λ/T. ∎

2.2 Examples

1.

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{cases}$$

In this example, $M =< x_1, x_2, u >$. N is generated by the above equations. So,

$$\Lambda = \frac{M}{N} = \frac{< x_1, x_2, u >}{< \dot{x}_1 - x_2, \dot{x}_2 - u >} =< x_1 >$$

When the quotient $\Lambda/ < u >$ is done, it can be seen that $\frac{d^2}{dt^2}x_1 = 0$. Then an element of the ring of differential operators that cancels x_1 is obtained. So $\Lambda/ < u >$ is torsion because its only generator is torsion.

2.

$$\begin{cases} \dot{x}_1 = x_1 + u \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = u \end{cases}$$

In this example, $M =< x_1, x_2, x_3, u >$. N is also generated by the above equations. So

$$\Lambda = \frac{M}{N} = \frac{< x_1, x_2, x_3, u >}{< \dot{x}_1 - x_1 + u, \dot{x}_2 - x_3, \dot{x_3} - u >} = \frac{< x_1, x_2 >}{< \dot{x}_1 - x_1 + u >}$$

In $\Lambda/ < u >$ the relations $(\frac{d}{dt} - I)x_1 = 0$ and $\frac{d^2}{dt^2}x_2 = 0$ are satisfied. Thus, the quotient module is torsion again.

3.

$$\dot{X} = AX + BU$$

$$A = \begin{pmatrix} 0 & 1 & & & & & & & \\ & 0 & 1 & & & & & & \\ & & 0 & & & & & & \\ & & & 0 & 1 & & & & \\ & & & & 0 & 1 & & & \\ & & & & & 0 & & & \\ & & & & & & 0 & 1 & \\ & & & & & & & 0 & 1 \\ & & & & & & & & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Notice that $\frac{d}{dt}x_3 = u_1$, $\frac{d^2}{dt^2}x_2 = u_1$ and $\frac{d^3}{dt^3}x_1 = u_1$. The same happens to x_4, x_5 ,x_6 and u_2. Furthermore, x_7, x_8, x_9 are torsion elements. Therefore, it can be written

$$\Lambda = \frac{< x_1, ..., x_9, u_1, u_2 >}{< \dot{x}_1 - x_2, \dot{x}_2 - x_3, \dot{x}_3 - u_1, ... >} = \frac{< x_1, x_4, x_7 >}{< x_7^{(3)} >} =< x_1, x_4 > \oplus \frac{< x_7 >}{< x_7^{(3)} >}$$

So the system has a torsion submodule, generated by x_7 and a free submodule generated by x_1 and x_4. Notwithstanding, $\Lambda/ < u_1, u_2 >$ is a torsion submodule.

4.

$$\dot{X} = \begin{pmatrix} t & 1 \\ 1 & t \end{pmatrix} X + \begin{pmatrix} 0 \\ 1 \end{pmatrix} U$$

This is an example of a non-constant coefficient. The ring over which is defined the system is $R(t)[\frac{d}{dt}]$, where $R(t)$ is the field of fractions of real polynomials. The system is rewritten in the following way:

$$\Lambda = \frac{< x_1, x_2, u >}{< \dot{x}_1 - tx_1 - x_2, \dot{x}_2 - x_1 - tx_2 - u >}$$

The relation

$$x_2 = (\frac{d}{dt} - tI)x_1$$

can be deduced from the first equation. And from the second equation

$$u = (\frac{d}{dt} - tI)x_2 - Ix_1 = ((\frac{d}{dt})^2 - 2t\frac{d}{dt} + (t^2 - 2)I)x_1$$

So it can be said that

$$\Lambda = < x_1 >$$

So, It is clear that, using the latest relations written, $\Lambda / < u >$ is a torsion module.

5.

$$\dot{x} = x + u^{(2)}$$

The derivatives of the input are not considered in the state-space representation. Let be $y = x - u - \dot{u}$. y is a generator of the module and the module is free. The equation in the variable y is:

$$\dot{y} = y + u$$

2.3 Controllability and observability

In this section, simple characterizations of controllability and observability based on module theory techniques are given. See [10] again. The equivalence between control systems in state-space form and control systems in module theory will be used.

Theorem 1 *A system Λ is controllable if and only if it is a free module.*

Proof. First assume that Λ is free. If the system is uncontrollable, as in [36], we have a Kalman realization as follows:

$$\begin{pmatrix} \dot{X}_1 \\ \dot{X}_2 \end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ A_2 & A_3 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} + \begin{pmatrix} 0 \\ B_2 \end{pmatrix} U$$

where A_1 is a square matrix and belongs to the uncontrollable part. Now lemma1 can be applied. Therefore, the elements of X_1 are torsion. So there exists a contradiction with the freeness of Λ.
Now assume that the system is controllable. Let $\Lambda = F \oplus T$ be a descomposition in a direct sum of a free left module and a torsion left module. If Λ was not free, $T \neq 0$. Let be

$$T =< x_1, ..., x_k >$$

Thus $\forall x_i \ \exists P_i \in K[\frac{d}{dt}] \mid P_i x_i = 0$, because x_i are torsion elements. Thus a system with k equations of order n_k (polynomial degree) is obtained. This can be transformed into a system with order 1 equations where the state variables are $x_{1,1}, ..., x_{1,n_1}, ..., x_{k,1}, ..., x_{k,n_k}$ This is an expression of the form:

$$\dot{X}_1 = A_1 X_1$$

Here there is a further contradiction because this is an uncontrollable Kalman realization. The contradiction comes from the assumption of non-freeness of the module. ∎

The above proof also shows an equivalence between the torsion submodule and the uncontrollable part of the Kalman realization.
It can also be seen that, if the system is controllable, each element of Λ is related, directly or indirectly, to the inputs: directly, if it can be expressed as a linear combination of the inputs; or indirectly, because each state-variable accomplishes a differential equation where there are inputs.
Next the relation between observability and module theory is shown. In the classical theory, a system:

$$\dot{X} = AX + BU$$
$$Y = CX$$

is called observable if and only if

$$rank < C^t, (A^t + \frac{d}{dt})C^t, ..., (A^t + \frac{d}{dt})^{n-1} C^t >= n$$

where n is the dimension of the state-variable vector.

Theorem 2 *A system is observable if and only if*

$$\Lambda =< U, Y >$$

That is to say, if and only if each variable of Λ can be written as a linear combination of inputs, outputs and their derivatives.

Proof: Let $Y = (y_1, \ldots, y_p)$. There is no loss of generality in assuming that $y_1, \ldots, y_p$ are linearly independent. First of all, we suppose that $\Lambda =< U, Y >$ or, in other words, $< U, Y >=< U, X >$. This is also equivalent to

$$\frac{< U, Y >}{< U >} = \frac{< U, X >}{< U >}$$

So, the Kalman realization is written in the quotient, where the variables are overlined:

$$\dot{\overline{X}} = A\overline{X}$$

$$\overline{Y} = C\overline{X}$$

Deriving k times:

$$\frac{d^k}{dt^k}\overline{Y} = (\overline{X}^t((A^t + \frac{d}{dt})^k)C^t)^t$$

Consider the linear system obtained by gathering the former equation for $k = 0, \ldots, n-1$. Since we have assumed that $y_1, \ldots, y_p$ are linearly independent, the system has a unique solution if, and only if,

$$rank(C^t, (A^t + \frac{d}{dt})C^t, ..., (A^t + \frac{d}{dt})^{n-1}C^t) = n$$

which is the classical observability condition. Therefore,

$$rank(C^t, (A^t + \frac{d}{dt})C^t, ..., (A^t + \frac{d}{dt})^{n-1}C^t) = n$$

if, and only if, $\overline{X}$ are written as a unique linear combination of $\overline{Y}$ and their derivatives; if, and only if, X are written as a unique linear combination of U, Y and their derivatives.
Notice that, from this proof, an equivalence between the observability part of the Kalman realization and the submodule $< U, Y >$ follows. ∎

2.4 Modes, poles and zeros

In this section we attempt to give an algebraic interpretation of the hidden modes, poles and zeros of the constant linear systems. It follows [11], although some proofs have been extended. Let us recall the Kalman realizations in the uncontrollable and in the unobservable cases. In the uncontrollable case the Kalman realization is:

$$\begin{pmatrix} \dot{X}_1 \\ \dot{X}_2 \end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ A_2 & A_3 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} + \begin{pmatrix} 0 \\ B_2 \end{pmatrix} U$$

where A_1 is the uncontrollable part matrix. And in the unobservable case:

$$\begin{pmatrix} \dot{X}_1 \\ \dot{X}_2 \end{pmatrix} = \begin{pmatrix} A^1 & 0 \\ A^2 & A^3 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} + \begin{pmatrix} B^1 \\ B^2 \end{pmatrix} U$$

$$Y = \begin{pmatrix} C^1 \\ 0 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

where A^3 is the unobservable part matrix.

Classically, the hidden modes were the eigenvalues of A_1 (input-decoupling zeros) and A_3 (output-decoupling zeros). What is the interpretation of these matrix in the module theory? Let us begin with A_1:
As this is the uncontrollable part matrix, Λ is not free. So the module can be decomposed in a direct sum of its free part and its torsion part: $\Lambda = F \oplus T$. Let us denote the linear mapping induced by the derivative $\frac{d}{dt}$ by

$$\tau : T \longrightarrow T$$

This mapping is well defined because $\frac{d}{dt}$ is an element of the ring over which the module Λ has been defined, and T is a submodule of Λ. Recall also the equivalence between the uncontrollable part and the torsion submodule T. So A_1 is the matrix of τ, and therefore the input-decoupling zeros are the eigenvalues of the mapping τ.

Analogously, note that there is an equivalence between the observable part and $< U, Y >$. Consider the quotient submodule $S = \Lambda / < U, Y >$. Obviously there is an equivalence between this quotient submodule and the unobservable part. Denote the linear mapping induced by $\frac{d}{dt}$ by:

$$\sigma : S \longrightarrow S$$

This mapping is again well-defined and its matrix will be A^3. Therefore the output-decoupling zeros are the eigenvalues of this mapping.
Now, an interpretation for poles is looked for. Let be

$$\Delta = \frac{\Lambda/T}{< U >} = \frac{< X_1, X_2, U > / < X_1 >}{< U >}$$

which is torsion, because $\Lambda / < U >$ is also torsion. As in the above interpretations, denote the linear mapping induced by $\frac{d}{dt}$ by:

$$\delta : \Delta \longrightarrow \Delta$$

Again remember that poles are the eigenvalues of A_3 (the controllable part matrix). Then, since $x \in \Delta$, the equation

$$\dot{x} = A_3 x$$

is satisfied, because when the quotient is done, the U part vanishes. Thus we obtain the proof that poles are the eigenvalues of δ.

Lastly, let us look for an interpretation of zeros. Consider the greatest torsion submodule in $< U, Y >$ and call it T_1. Let

$$J = \frac{< U, Y >}{< Y, T_1 >}$$

be a quotient module. Notice that $< U, Y > / < T_1 >$ is free because the torsion part has been removed. If when the quotient by $< Y >$ is done, a torsion submodule appears, then the quotient submodule J will be a dynamic where Y are now the inputs. More precisely, there exist two polynomial matrices $P(\frac{d}{dt})$ and $Q(\frac{d}{dt})$, in such a way that

$$P(\frac{d}{dt})U = Q(\frac{d}{dt})Y$$

Now, it is straightforward procedure to obtain a realization of the system such as

$$\begin{aligned} \dot{Z} &= \overline{A}Z + \overline{B}Y \\ U &= \overline{C}Z \end{aligned}$$

This is the inverse system, where U are the outputs and Y the inputs. We know that zeros are the poles of the inverse system. On the other hand denote the linear mapping induced by $\frac{d}{dt}$ by:

$$\epsilon : J \longrightarrow J$$

The poles of the inverse system are the eigenvalues of this mapping and, therefore, the zeros of the initial system are the eigenvalues of ϵ.

2.5 Examples

1. Consider the linear control system described by

$$\dot{X} = \begin{pmatrix} -4 & -4 & 0 & -1 & -2 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & -5 & -2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} X + \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ -1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} U$$

$$Y = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \end{pmatrix} X$$

The module description of the system is

$$\Lambda = \frac{< x_1, x_2, x_3, x_4, x_5, u_1, u_2 >}{< equations >} =< x_2, x_5 >$$

This is a free module and the system is therefore controllable. On the other hand it is clear that $\Lambda / < U >$ is torsion. The observability of the system can also be checked: $x_1 = y_1$, $x_3 = y_1 + y_2$, $x_4 = (\frac{d^2}{dt^2} + 4\frac{d}{dt} + 4I)y_1 + (y_1 + y_2) - \frac{d^2}{dt^2}u_2$, $x_5 = -1/2\frac{d}{dt}(y_1 + y_2) - 4(y_1 + y_2) - 5x_4 - u_1$ and $x_2 = -1/4(\frac{d}{dt}x_1 + 4x_1 + x_4 + x_5 + u_2)$. In short $< X, U >=< Y, U >$. So, the system is observable.

Now, the derivatives of u_1 and u_2 are written as functions of the outputs:

$$\frac{d^2}{dt^2}u_1 = -(\frac{d^3}{dt^3} + 4\frac{d^2}{dt^2} + 5\frac{d}{dt} + 2I)(y_1 + y_2)$$

$$\frac{d^2}{dt^2}u_2 = (\frac{d^3}{dt^3} + 4\frac{d^2}{dt^2} + 4\frac{d}{dt})y_1 + (\frac{d}{dt} + I)(y_1 + y_2)$$

The submodule J such that $J =< U, Y > / < U, T_1 >$ is, in this example, the same as $J =< U, Y > / < Y >$. By the above equalities it can be affirmed that J is a torsion module. And the matrix of the mapping ϵ is

$$\overline{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus, it is clear that there is only one eigenvalue, which is zero. Moreover, it can be checked that this is the unique zero of the system. See [38] for more details.

2. Consider the linear control system described by an aircraft altitude dynamics ([58]).

$$\dot{X} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -4 & -4 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 6 & 0 & 0 & 0 \end{pmatrix} X + \begin{pmatrix} 0 \\ 3 \\ 0 \\ -1 \end{pmatrix} u$$

$$y = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}$$

Now the module of the system is

$$\Lambda = \frac{< x_1, x_2, x_3, x_4, u >}{< equations >}$$

In order to find a generator for this module, the next procedure is shown:
Write $z = ax_1 + bx_2 + cx_3 + dx_4$. Then, the following condition must be imposed: derivatives up to order three cannot contain the control variable u. With this condition a generator of the system is $z = 15x_1 + 2x_2 + +21x_3 + 6x_4$, because the following equalities are verified:

$$x_1 = \frac{z^{(2)}}{98}$$

$$x_2 = \frac{z^{(3)}}{98}$$

$$x_3 = \frac{14z - 4\dot{z} - z^{(2)}}{294}$$

$$x_4 = \frac{14\dot{z} - 4z^{(2)} - z^{(3)}}{294}$$

So, z is a generator of Λ. Moreover, $\Lambda =< z >$ is a free module or, in other words, the system is controllable.

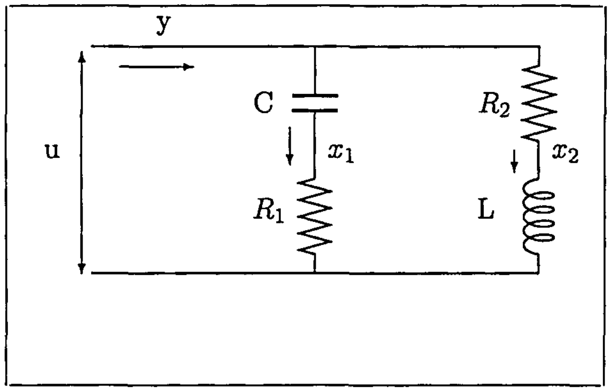

Figure 2.1: Elementary RLC-circuit worked in example 3

Let us recall that $\Lambda =< u, y >$ is the observability condition. This is an easy computation and can be left to the reader. Looking for poles and zeros is equivalent to finding a relation like:

$$p(\frac{d}{dt})y = q(\frac{d}{dt})u$$

And in this case we have:

$$((\frac{d}{dt})^4 + 4(\frac{d}{dt})^3 + 4(\frac{d}{dt})^2)y = (-(\frac{d}{dt})^2 - 4(\frac{d}{dt}) + 14I)u$$

Therefore, poles are the zeros of the polynomial $p(x) = x^4 + 4x^3 + 4x^2$, that is to say, $x = 0$ and $x = 2$.
On the other hand, zeros are the zeros of the polynomial $q(x) = x^2 + 4x - 14$ and, therefore, are $x = -2 \pm 3\sqrt{2}$.

3. Consider the circuit shown in figure 2.1. The equations are:

$$\dot{X} = \begin{pmatrix} -1/R_1C & 0 \\ 0 & -R_2L \end{pmatrix} X + \begin{pmatrix} 1/R_1C \\ 1/L \end{pmatrix} u$$

$$y = \begin{pmatrix} -1/R_1 & 1 \end{pmatrix} X + 1/R_1 u$$

Using the same method as in the last example, a generator of the module is found: $z = \frac{-R_1C}{L}x_1 + x_2$. So $\Lambda =< z >$. This module is free in all cases except $L = R_1R_2C$. In this case, $(L\frac{d}{dt} + R_2I)z = 0$. That is to say, there are torsion elements in Λ, and therefore the system is not controllable.

Let us consider the observability condition. It is necessary to check whether or not Λ is equal to $< y, u >$. The state variables are involved in the following equations:

$$y - 1/R_1u = -1/R_1x_1 + x_2$$

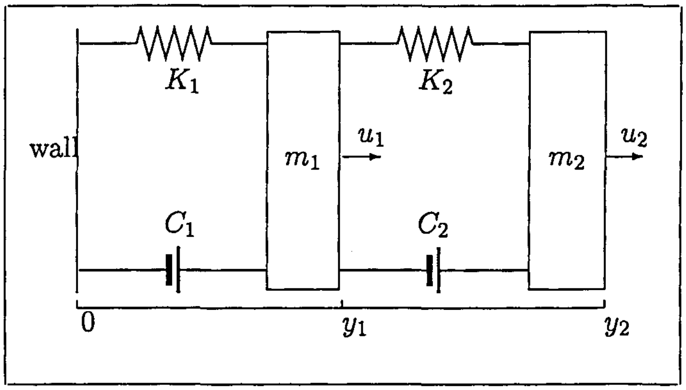

Figure 2.2: Mechanical system with springs and dampers corresponding to example 4

$$\dot{y} - 1/R_1\dot{u} + (\frac{1}{R_1^2C} - 1/L)u = \frac{1}{R_1^2C}x_1 - R_2/Lx_2$$

This system has a solution if, and only if, $L \neq R_1R_2C$. Therefore the system is observable if and only if $L \neq R_1R_2C$.

In the case $L = R_1R_2C$ there exist decoupling zeros. Remember that the input-decoupling zeros are the eigenvalues of the linear mapping induced by $\frac{d}{dt}$ in the torsion submodule T. The generator of this submodule is z and it fulfills the following equation:

$$\dot{z} + \frac{1}{R_1C}z = 0$$

So, $\frac{-1}{R_1C}$ is the eigenvalue sought.

The output-decoupling zeros are the eigenvalues of the linear mapping induced by $\frac{d}{dt}$ in $S = \frac{\Lambda}{<u,y>}$ A generator of this submodule is x_1. In this submodule the equation is

$$\dot{x}_1 = -\frac{1}{R_1C}x_1$$

So, again, the output-decoupling zero is $\frac{-1}{R_1C}$

4. Another example is drawn in figure 2.2 and modelled by the linear system:

$$\dot{X} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{k_1+k_2}{m_1} & \frac{k_2}{m_1} & -\frac{c_1+c_2}{m_1} & \frac{c_2}{m_1} \\ \dfrac{k_2}{m_2} & -\frac{k_2}{m_2} & \frac{c_2}{m_2} & -\frac{c_2}{m_2} \end{pmatrix} X + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1/m_1 & 0 \\ 0 & 1/m_2 \end{pmatrix} U$$

$$Y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

It can be shown that $\Lambda =< x_1, x_2 >$. This module is free and, for this reason, the system is controllable. It is also very easy to check the observability condition $\Lambda =< u_1, u_2, y_1, y_2 >$.

It is possible to find the following relation between the input and the output variables:

If

$$p(\tfrac{d}{dt}) = \frac{m_1m_2(\frac{d}{dt})^4+(m_1c_2+m_2c_1+m_2c_2)(\frac{d}{dt})^3+(c_1c_2+k_2m_1+k_1m_2+k_2m_2)(\frac{d}{dt})^2}{m_1m_2} + \frac{(k_2c_1+k_1c_2)(\frac{d}{dt})+k_1k_2I}{m_1m_2}$$

Then:

$$p(\frac{d}{d})\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} Y = \begin{pmatrix} m_2(\frac{d}{dt})^2 + c_2(\frac{d}{dt}) + k_2 & c_2(\frac{d}{dt}) + k_2 \\ c_2(\frac{d}{dt}) + k_2 & m_1(\frac{d}{dt})^2 + (c_1 + c_2)(\frac{d}{dt}) + (k_1 + k_2) \end{pmatrix} U$$

For this reason the zeros are the zeros of the determinant of the last matrix, and the poles are the zeros of $p(\dfrac{d}{dt})$.

2.6 Final remarks

A formal Laplace tranform and the transfer function matrix are naturally defined in the module formalism in [20], where the relationship between left(right) coprime matrix decomposition and controllability(observability) is also studied.

Most of the concepts and results in linear control systems have been presented within the framework of the new algebraic formalism introduced by M. Fliess. In addition to the concision, clarity and stylishness of the concepts, this approach is specially appropriate for problems involving tracking of references and generation of signals. Moreover, this algebraic framework enables the classical results to be improved. On the other hand, the concision, clarity and stylishness make it easier to consider some phenomena which have sometimes been ignored in the control literature and seem difficult to explain in any classical framework, as will be seen in the next subsections:

2.6.1 The matching condition in sliding control mode

Let

$$\begin{cases} \dot{X} &= AX + Bu \\ y &= CX \end{cases}$$

be a linear single-input system, and assume that we want $y = 0$ to be achieved as steady state; that is to say, a sliding regime on the sliding surface $CX = 0$. In classical references ([62],[16]) the existence of a sliding regime and the description of the ideal sliding dynamics is closely related to the equivalent control (u_{eq}) , which is derived from $\dot{y} = C\dot{X} = 0$. A necessary

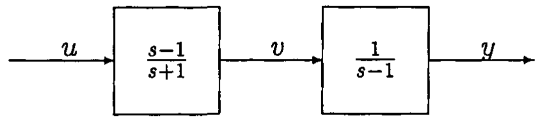

Figure 2.3: Not stabilizable example

condition for obtaining u_{eq} from $\dot{y} = 0$ is that the relative degree of y is 1. Otherwise u_{eq} cannot be well defined.
In the framework of module theory, the module over $K[\frac{d}{dt}]$ spanned by y is considered instead of the sliding surface equations. In this module an element

$$\sigma(y) := (\frac{d^n}{dt^n} + a_1 \frac{d^{n-1}}{dt^{n-1}} + \ldots + a_{n-1} \frac{d}{dt} + a_n I) y \tag{2.3}$$

where $n = reld^0(y, u) - 1$ (relative degree minus one) and $p(z) = z^n + a_1 z^{n-1} + \ldots + a_{n-1} z + a_n$ is a Hurwitz polynomial, can be chosen.
Thus, equation 2.3 determines a well defined equivalent control u_{eq}. The control policy

$$u(y) = \begin{cases} u^+(y) & \text{if } \frac{\partial \sigma}{\partial X} B > 0 \\ u^-(y) & \text{if } \frac{\partial \sigma}{\partial X} B < 0 \end{cases}$$

with $u^-(y) < u_{eq}(y) < u^+(y)$ guarantees the achievement of equation 2.3. Finally $y = 0$ is the assymptotically stable equilibrium solution of equation 2.3. Hence $y = 0$ is obtained as steady state and the control objective is attained.

2.6.2 The linear system interconnections

Let us consider a motivating example. The system:

$$y^{(2)} - y = \dot{u} - u \tag{2.4}$$

whose transfer function is

$$\frac{s-1}{s^2-1} = \frac{1}{s+1}$$

corresponds to the block diagram in figure 2.3, since $\dot{u} - u = \dot{v} + v$, $v = \dot{y} - y$.
Let $z = y + \dot{y} - u$. Then $\dot{z} - z = 0$. So, z satisfies an unstable equation. This implies that system 2.4 is not stabilizable.
The reverse block diagram is figure 2.4.
It corresponds to $u = \dot{w} - w = y + \dot{y}$. That is to say,

$$\dot{y} + y = u \tag{2.5}$$

Its transfer function is also $\frac{1}{s+1}$; but this is input-output stable.
Finally, let us consider the feedback system in figure 2.5.

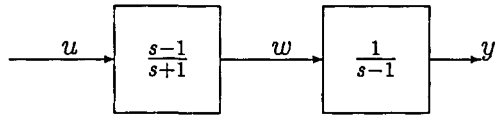

Figure 2.4: Reverse block diagram of 2.3

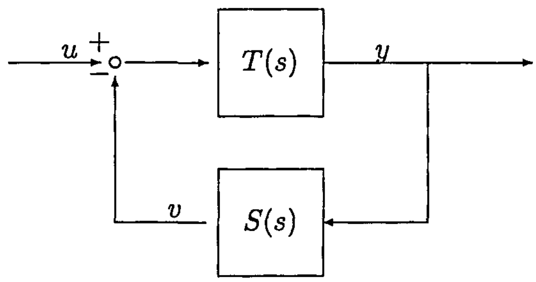

Figure 2.5: Ill-posed system

Its transfer function is $\frac{T}{1+TS}$. If $TS = -1$, then the system is "ill-posed" in the sense of Willems [68].
It is difficult to explain these phenomena in any classic framework. There is no difference between systems 2.4 and 2.5 in the transfer function approach. But in the module framework we notice that system 2.4 is not free torsion because z is a torsion element. On the other hand, system 2.5 is free and y is a generator.
These kinds of systems, called interconnections, have been examined by M. Fliess and H. Bourlès [21] via a standard algebraic tool, coproducts of modules ([8],[43]). They confirm Willem' standpoint [64]:
"It is often misleading to distinguish between systems variables".
Consider a family of modules $\{M_\alpha, \alpha \in A\}$. Let E be a given module such that, for any $\alpha \in A$, there exists a morphism:

$$h_\alpha : E \longrightarrow M_\alpha$$

Define the submodule ϵ of the cartesian product $\times_{\alpha \in A} M_\alpha$ spanned by the elements of the form

$$(\ldots, 0, \ldots, h_{\alpha_1}(e), \ldots, 0, \ldots, -h_{\alpha_2}(e), \ldots, 0, \ldots)$$

where $e \in E$ and $\alpha_1 \neq \alpha_2$. The quotient module $\times_{\alpha \in A} M_\alpha / E$ is called the coproduct (or the fibered sum, or the amalgamated sum) of the M_α's(referencies). It is written $\sqcup_{\alpha \in A, E} M_\alpha$.
When the modules M_α's are viewed as linear systems, the above coproduct is called a system interconnection. These interconnections are defined without distinguishing between system variables. Some examples are studied below. Let D^i be a dynamic with inputs $u^i = \{u_1^i, \ldots, u_m^i\}$ and outputs $y^i = \{y_1^i, \ldots, y_p^i\}$ for $i = 1, 2$.

1. If $m := m^1 = m^2$ and $p := p^1 = p^2$, consider the parallel interconnection from figure 2.6.

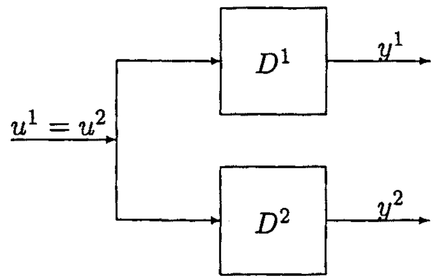

Figure 2.6: Parallel interconnection

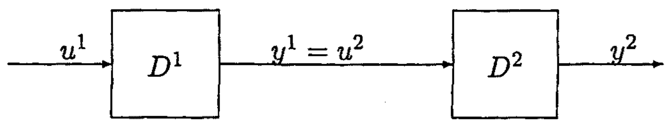

Figure 2.7: Series interconnection

Consider the free module $[\delta] = [\delta_1, \dots, \delta_m]$ of rank m, and the two canonical isomorphisms:

$$\begin{array}{rccc} \varphi^i : & [\delta] & \longrightarrow & [u^i] \\ & \delta_s & \longrightarrow & u^i_s \end{array}$$

ε is the submodule of $D^1 \times D^2$ spanned by the elements of the form $\{(\varphi^1(\delta_s), -\varphi^2(\delta_s), s = 1, \dots, m\}$. The interconnection is represented by $D^1 \sqcup_{u^1=u^2} D^2$, which is defined, in practice, by the sets of equations of each module plus the equation $u^1 = u^2$. An output of this parallel interconnection will be any K-linear combination of the components of y^1 and y^2.

2. Assume that $p^1 = m^2$ and $y^1 = u^2$. Consider the series interconnection 2.7.

   Consider the free module $[\epsilon] = [\epsilon_1, \dots, \epsilon_{p^1}]$ of rank p^1 and the canonical isomorphisms:

$$\begin{array}{rccc} \phi^i : & [\epsilon] & \longrightarrow & [u^i] \\ & \epsilon_s & \longrightarrow & u^i_s \end{array}$$

   The interconnection is $D^1 \sqcup_{y^1=u^2} D^2$, which is defined by the equations of each module plus the equation $y^1 = u^2$.

   The first two cases of the motivating examples are examples of this type of interconnection.

3. Let D^3 be a third dynamic with input $u^3 = \{u^3_1, \dots, u^3_{m^3}\}$ and output $y^3 = \{y^3_1, \dots, y^3_{p^3}\}$. Consider the feedback interconnection whose block diagram is figure 2.8.

   The input $u^3 = v \cup w$ is divided into two parts. Set $y^3 = u^1$, $y^1 = u^2$, $y^2 = w$. Therefore, the above block diagram corresponds to the coproduct $\sqcup_{y^3=u^1, y^1=u^2, y^2=w}(D^1, D^2, D^3)$.

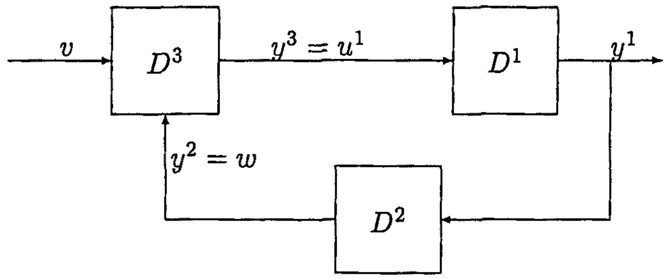

Figure 2.8: Feedback interconnection

A frequent phenomena is the lack of controllability or observability; that is to say, interconnecting controllable (or observable) linear systems may give rise to an uncontrollable (or unobservable) one. This cannot be detected by transfer functions. Moreover, when K is a field of constants, the hidden modes corresponding to the lack of controllability (or observability) may exhibit positive real parts which imply unstability. Let us see the examples stated at the beginning of this subsection. As has been said, example 2.4 is not torsion free. In other words, it is not controllable. The corresponding input decoupling zero, which is 1, is unstable.
In system 2.5, w cannot be expressed as a linear combination of u, y and a finite number of their derivatives. So, the system is unobservable.
Consider the third example. Write $T(s) = \frac{a(s)}{b(s)}$ and $S(s) = \frac{c(s)}{d(s)}$, $a, b, c, d \in R[s]$, $abcd \neq 0$, a, b (resp. c, d) coprime. The system is governed by the equations:

$$a(\frac{d}{dt})(u-v) = b(\frac{d}{dt})y \tag{2.6}$$
$$d(\frac{d}{dt})v = \cdot\, c(\frac{d}{dt})y \tag{2.7}$$

There are two possible situations:

1. If $ac + bd \neq 0$ (i.e. $ST \neq -1$), then $(b(s)d(s) + a(s)c(s))y(s) = a(s)d(s)u(s)$. Therefore, y can be obtained from u; and v can also be obtained from y, and, consequently, from u.

2. If $ac + bd = 0$, then u must satisfy $a(\frac{d}{dt})u = 0$. So u becomes a torsion element. The remaining variables y, v span a free module of rank 1. Here, the lack of controllability concerns the control variable.

Another strange phenomena is the change of rank. Generally speaking the rank is the maximum number of independent channels, but may change in some interconnections. Let us consider an example:
The system in figure 2.9 is governed by the equations:

$$-a(\frac{d}{dt})v = b(\frac{d}{dt})y \tag{2.8}$$
$$d(\frac{d}{dt})v = c(\frac{d}{dt})y \tag{2.9}$$

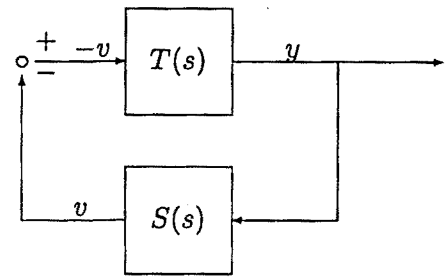

Figure 2.9: Example illustrating possible changes of rank

There are also two possible situations:

1. If $ac + bd \neq 0$ (i.e. $ST \neq -1$) then $(ac + bd)v = 0$. This implies that v, and consequently y, are torsion. Therefore, the rank is zero.

2. If $ac + bd = 0$ (i.e. $ST = -1$) the module is free of rank 1.

Chapter 3

Linearization of nonlinear systems and flatness

This chapter serves as an introduction to different types of linearizations for nonlinear control systems: static feedback linearization, dynamic feedback linearization and linearization by prolongations, which is a particular case of dynamic feedback linearization. The concept of flatness will be also introduced, as well as the concept of flat outputs. The tools and concepts of the two different frameworks used throughout this thesis will be stated in this chapter. These frameworks are: differential geometry and differential algebra.

3.1 Different types of linearizations

Definition 4 *A nonlinear system*

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i \quad x \in R^n$$

is said to be static feedback linearizable if it is possible to find a feedback

$$u = \alpha(z) + \beta(z)v \quad u \in R^m \quad v \in R^m \quad z \in R^n$$

and a diffeomorphism

$$z = \phi(x)$$

such that the original sistem is transformad into a linear controllable system

$$\dot{z} = Az + Bv$$

where A and B are matrices of appropiate size.

The next theorem is a characterization of static fedback linearizability in the differential geometry framework. A proof can be found in ([34], [31], [51]).

Theorem 3 *Let*

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i$$

be a nonlinear system with m inputs. This system is static feedback linearizable if and only if the following distributions have constant rank and are involutive:

$$\begin{array}{ll} D_0 = & < g_1, \ldots, g_m > \\ D_i = & < D_{i-1}, ad_f^i g_1, \ldots, ad_f^i g_m > \end{array} \quad i = 1, \ldots, n-1$$

and the rank of D_{n-1} is n

In the case that the above system is static feedback linearizable, there exists a change of variables and a feedback such that the system is written in the following way

$$\begin{array}{rcll} \dot{y}_i & = & y_{i+1} \quad \forall i = 1, \ldots, n & i \neq k_j,\ j = 1, \ldots, m \\ \dot{y}_{k_j} & = & v_j & j = 1, \ldots, m \end{array}$$

k_j are the so called Brunovsky indices [5],[51]. The definition of these indices is as follows: Define

$$\begin{array}{rcl} \rho_0 & = & dim D_0 \\ \rho_i & = & dim D_i - dim D_{i-1} \quad i \geq 1 \end{array}$$

Then,

$$k_j = \#\{\rho_i \geq j,\ i \geq 0\}$$

A generalization of the static feedback is a dynamic feedback transformation.

Definition 5 *A nonlinear system*

$$\dot{x} = f(x, u) \qquad x \in R^n \qquad u \in R^m \tag{3.1}$$

is said to be dynamic feedback linearizable if there exists:

1. *A regular dynamic compensator*

$$\left\{ \begin{array}{l} \dot{z} = a(x, z, v) \\ u = b(x, z, v) \end{array} \right. \tag{3.2}$$

   with $z \in R^q$ and $v \in R^m$. The regularity assumption implies the invertibility of 3.2 with input v and output u.

2. *A diffeomorphism*

$$\psi = \Psi(x, z) \tag{3.3}$$

   with $\psi \in R^{n+q}$, such that the original system 3.1 with the dynamic compensator 3.2, after applying 3.3, becomes a constant linear controllable system:

$$\dot{\psi} = A\psi + BV$$

This linear system may be written in Brunovsky canonical form ([5],[51]) by means of a static state feedback and a linear invertible change of coordinates:

$$y_i^{(k_i)} = v_i \qquad \forall i = 1, \ldots, m$$

where $\{k_i\}_{i=1}^m$ are the Kronecker indices. Therefore, setting

$$y = (y_1, \ldots, y_1^{(k_1-1)}, \ldots, y_m, \ldots, y_m^{(k_m-1)})$$

it is possible to write $y = T\psi$, where T is an invertible matrix. This can be transforned, using the invertibility of the diffeomorphism 3.3, into

$$\begin{pmatrix} x \\ z \end{pmatrix} = \Psi^{-1}(T^{-1}y)$$

And from 3.2, $u = b(\Psi^{-1}(T^{-1}y), v)$. That is to say, x and u can be expressed as real-analytic functions of the components of $(y_1, \ldots, y_m)$ and their derivatives. The dynamic feedback 3.2 is called *endogenous* if, and only if, the converse holds; that is to say, if, and only if, $(y_1, \ldots, y_m)$ can be expressed as real-analytic functions of x, u and a finite number of their derivatives.

Definition 6 *A dynamics 3.1 is called (differentially) flat if, and only if, is linearizable via dynamic endogenous feedback. The variables $(y_1, \ldots, y_m)$ are called flat or linearizing outputs.*

Therefore, a flat system is not only linearizable, but is also a system where x and u trajectories can be deduced immediately from $(y_1, \ldots, y_m)$ trajectories. In fact, this is the power of flatness. Once the flatness of a system is known, it does not imply that one intends to transform the system into a single linear one. When a system is flat, it is an indication that the nonlinear structure of the system is well characterized, and one can exploit that structure by designing control algorythms for motion planning, trajectory generation, and stabilization. Indeed, the flat outputs are the nonlinear analogue of a basis of the free module for linear controllable systems. Flatness was first introduced by Fliess and coworkers in [13],[14],[19],[22] using the formalism of differential algebra. In differential algebra, a system is viewed as a differential field generated by a set of variables (states and inputs). Recently, flatness has been defined in a more geometric context. One approach is to use exterior differential systems, and to regard a nonlinear control system as a Pfaffian system on an appropiate space (see, for instance [49] and referencies therein). A somewhat different geometric point of view is to consider a Lie-Bäcklund framework as the underlying mathematical structure ([23],[24]). In this context, a system is a smooth vector field on a smooth manifold, possibly of infinite dimension.

3.2 Flatness and differential algebra

For an introduction to differential algebra see [39],[40],[54].

Definition 7 *Let k be a given differential field. A system is a finitely generated differential extension D/k. This corresponds to a finite number of quantities which are related by a finite number of algebraic differential equations over k. The differential order of the system D/k is the differential transcendence degree of the extension D/k.*

Let $k < u >$ the differential field generated by k and a finite set $u = (u_1, \dots, u_m)$ of differential k-indeterminates. Assume $u_1, \dots, u_m$ differentially k-algebraically independent; that is to say, diff tr $d^0 k < u > /k = m$. A dynamics with input u is a finitely generated differentially algebraic extension $D/k < u >$. Note that the number of independent inputs is equal to the differential order of the system D/k as was proven in [63]. An output $y = (y_1, \dots, y_p)$ is a finite set of differential quantities in D.
According to theorem 6, there exists a finite trascendence basis $x = (x_1, \dots, x_n)$ of $D/k < u >$. Therefore, since $\dot{x}_i, y_j \in D$, $\dot{x}_i, y_j$ are $k < u >$-algebraically dependent on x. That is to say, there exist A_i and B_j, polynomials over k, such that

$$\left\{ \begin{array}{ll} A_i(\dot{x}_i, x, u, \dot{u}, \dots, u^{(r_i)}) = 0 & \forall i = 1, \dots, n \\ B_j(\dot{y}_j, x, u, \dot{u}, \dots, u^{(s_j)}) = 0 & \forall j = 1, \dots, p \end{array} \right.$$

x_i are called generalized states and n is the dimension of the dynamics $D/k < u >$.
As was stated in the former section, linear systems are viewed as finitely generated modules over principal ideal rings. The relation between these two approaches (field extensions and modules) is established by what is called *Kähler differential* ([35]). See appendix A for an introduction to differential algebra and details on the Kähler differential. To a finitely generated differential field extension L/K, associate a mapping (the Kähler differential)

$$d_{L/K} : L \longrightarrow \Omega_{L/K}$$

where $\Omega_{L/K}$ is a finitely generated left $L[\frac{d}{dt}]$-module, such that:

$$\begin{array}{llcl} \forall a \in L & d_{L/K}\frac{da}{dt} & = & \frac{d}{dt}(d_{L/K}a) \\ \forall a, b \in L & d_{L/K}(a+b) & = & d_{L/K}a + d_{L/K}b \\ \forall a, b \in L & d_{L/K}(ab) & = & bd_{L/K}a + ad_{L/K}b \\ \forall c \in K & d_{L/K}c & = & 0 \end{array}$$

As was seen in the previous section, a module like this corresponds to a linear system. In this case, this system is called the *tangent or variational system.* The inputs of this tangent system are $(d_{L/K}u_1, \dots, d_{L/K}u_m)$. Properties of the extension L/K can be translated into the linear module theoretic framework:

- A set $\psi = (\psi_1, \dots, \psi_m)$ is a differential transcendence basis of L/K if, and only if, their respective Kähler differentials $(d_{L/K}\psi_1, \dots, d_{L/K}\psi_m)$ make up a maximal set of $L[\frac{d}{dt}]$-linearly independent elements in $\Omega_{L/K}$. In other words, if, and only if, $(d_{L/K}\psi_1, \dots, d_{L/K}$ are a basis of $\Omega_{L/K}$. Thus, diff tr $d^0 L/K = rk\ \Omega_{L/K}$.

- The extension L/K is differentially algebraic if, and only if, the module $\Omega_{L/K}$ is torsion. And a set $X = (x_1, \ldots, x_n)$ is a transcendence basis of L/K (not differential) if, and only if, $(d_{L/K}X = d_{L/K}x_1, \ldots, d_{L/K}x_n)$ is a basis of $\Omega_{L/K}$ as a L-vector space.

- The extension L/K is algebraic if, and only if, $\Omega_{L/K} = \{0\}$.

The following definition states precisely what dynamic endogenous feedback means in this framework.

Definition 8 *Two systems D_1/k and D_2/k are said to be equivalent (by endogenous feedback) if, and only if, there exist two algebraic extensions (not differential algebraic) $\overline{D}_1/D_1$ and $\overline{D}_2/D_2$ and a differential k-automorphism between $\overline{D}_1/k$ and $\overline{D}_2/k$. In other words (identifying the systems with their respective images in the bigger fields), D_1/k and D_2/k are equivalent if, and only if, any element of D_1 (respectively D_2) is algebraic over D_2 (respectively D_1). Two dynamics $D_1/k < U >$ and $D_2/k < V >$ are said to be equivalent if, and only if, their corresponding systems D_1/k and D_2/k are equivalent.*

Proposition 2 *Two equivalent systems have the same differential order (and, therefore, the same number of inputs). And the same happens to the dynamics.*

Proof: Let K be the differential field generated by D_1 and D_2. Since D_1 and D_2 are equivalent, K/D_1 and K/D_2 are algebraic extensions. So

$$\text{diff tr } d^0 D_1/k = \text{diff tr } d^0 K/k = \text{diff tr } d^0 D_2/k$$

∎

Remark: Consider two equivalent dynamics, $D_1/k < u >$ and $D_2/k < v >$. Let n_1 and n_2 be the dimension of $D_1/k < u >$ and $D_2/k < v >$ respectively. Write the generalized state variable representations of both dynamics:

$$A_i^1(\dot{x}_i, x, u, \dot{u}, \ldots, u^{(r_i^1)}) = 0 \quad i = 1, \ldots, n_1$$

$$C_i^2(\dot{z}_i, z, v, \dot{v}, \ldots, v^{(r_i^2)}) = 0 \quad i = 1, \ldots, n_2$$

On the other hand, since any element of D_1 is algebraic over D_2 and viceversa,

$$\left\{ \begin{array}{ll} p_j^1(u_j, z, v, \dot{v}, \ldots, v^{(l_j^1)}) = 0 & j = 1, \ldots, m \\ q_i^1(x_i, z, v, \dot{v}, \ldots, v^{(h_i^1)}) = 0 & i = 1, \ldots, r_i^1 \\ p_j^2(v_j, x, u, \dot{u}, \ldots, u^{(l_j^2)}) = 0 & j = 1, \ldots, m \\ q_i^2(z_i, x, u, \dot{u}, \ldots, u^{(h_i^2)}) = 0 & i = 1, \ldots, r_i^2 \end{array} \right. \tag{3.4}$$

where $p_j^1, p_j^2, q_i^1, q_i^2$ are polynomials over k. 3.4 corresponds to two endogenous dynamic feedbacks because they do not use any variable trascendental over D_1 and D_2.

3.3 Linearization by prolongations

Definition 9 *Let*

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i$$

be a nonlinear system with m inputs. A prolongation of this system is

$$\begin{array}{rcl} \dot{x} & = & f(x) + \sum_{i=1}^{m} g_i(x)u_i^0 \\ \dot{u}_i^0 & = & u_i^1 \\ \vdots & & \\ \dot{u}_i^{k_i-1} & = & v_i \end{array} \qquad \forall i = 1 \dots m$$

where u_i^j, which corresponds to $u_i^{(j)}$, are new state variables $\forall i = 1 \dots m \quad j = 0 \dots k_i - 1$. And the new inputs are v_i.

Definition 10 *Let*

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i$$

be a nonlinear system with m inputs. This system is said to be linearizable by prolongations if there exists a prolongation of the original system which is static feedback linearizable.

In fact, a system which is linearizable by prolongations is dynamic feedback linearizable. That is to say, a linearization by prolongations is a particular case of dynamic feedback linearization. Let us see the relationship between these two types of linearizations. Consider a dynamic feedback compensator, affine respect to the inputs:

$$\left\{ \begin{array}{l} \dot{z} = a^0(x,z) + a^1(x,z)v \\ u = b^0(x,z) + b^1(x,z)v \end{array} \right. \tag{3.5}$$

with $z \in R^q$ and $v \in R^m$. A dynamic feedback compensator is a prolongation if, and only if,

$$u = \begin{pmatrix} z_1 & & & \\ & z_{k_1+1} & & \\ & & \ddots & \\ & & & z_{1+\sum_{i=1}^{m-1} k_i} \end{pmatrix} v$$

$$\dot{z}_i = \left\{ \begin{array}{ll} z_{i+1} & \text{if} \quad i \neq k_j,\ j = 1, \dots, m \\ v_j & \text{if} \quad i = k_j,\ j = 1, \dots, m \end{array} \right.$$

where $1 \leq i \leq q$.

That is to say,

$$b^0(x,z) = \begin{pmatrix} z_1 & & & \\ & z_{k_1+1} & & \\ & & \ddots & \\ & & & z_{1+\sum_{i=1}^{m-1} k_i} \end{pmatrix}$$

$$b^1(x,z) = 0$$

$$a_i^0(x,z) = \begin{cases} z_{i+1} & \text{if } \; i \neq k_j, \; j = 1,\ldots,m \\ 0 & \text{if } \; i = k_j, \; j = 1,\ldots,m \end{cases}$$

$$a_i^1(x,z) = \begin{cases} 0 & \text{if } \; i \neq k_j, \; j = 1,\ldots,m \\ 1 & \text{if } \; i = k_j, \; j = 1,\ldots,m \end{cases}$$

The following lemma will be used in some proofs related to linearization by prolongations in chapters 4, 5 and 6.

Lemma 5 *Let*

$$\frac{\partial}{\partial y_i}, \; i \in I$$

a family of coordinate vector fields in R^N*;*

$$\begin{aligned} D_1 &= < \alpha_1, \ldots, \alpha_j; \; \{\tfrac{\partial}{\partial y_k}\}_{k=n+1}^{n+l} > \\ D_2 &= < \{\tfrac{\partial}{\partial y_h}\}_{k=n+l+1}^{n+s} > \end{aligned}$$

where

$$\alpha_1, \ldots, \alpha_j \in < \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n} >$$

and not depending on the variables

$$y_{n+l+1}, \ldots, y_{n+s}$$

Then $D_1 \oplus D_2$ *is involutive if, and only if,* D_1 *is involutive.*

Proof: It is straightforward.

Chapter 4

Linearization using differential algebra

4.1 Introduction

This chapter deals with the problem of linearization of nonlinear control systems; that is to say, with the problem of flatness. Using the methods of differential algebra, we explain the conditions which in this framework must be satisfied in order for a nonlinear control system to be linearizable. First of all, the tangent system is computed using the Kähler differential. Then, for a single-input system, we give a new proof of the fact that a single-input system is static feedback linearizable if, and only if, it is dynamic feedback linearizable. We also tackle the static feedback linearizability problem for a multi-input system, thereby obtaining the conditions that a system must fulfill in order to be transformed into a linear one in this context. The problem of dynamic feedback linearizability is solved by trying to guess $m-1$ flat outputs and computing the last one. The quotient of modules appears to be a cornerstone in this procedure. Finally, without computing the tangent system, a procedure based also on guessing $m-1$ flat outputs is be designed. The main tool here will be the intermediate differential field extensions. A helpful software package in Maple V is created in order to simplify the computations required. The listing of this program can be found in appendix B.

4.2 Single-input systems

Consider the single-input system:

$$\dot{x} = f(x, u) \tag{4.1}$$

where the state $x = (x_1, \ldots, x_n) \in R^n$ and the control $u \in R$. Assume that 4.1 is controllable. Thanks to the property

$$d(\frac{d}{dt}a) = \frac{d}{dt}(da) \quad \forall a \in L$$

where d is the Kähler differential and L is the field extension corresponding to 4.1, the tangent system is:

$$\dot{dx} = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial u}du \tag{4.2}$$

Therefore, the basis of the corresponding module Ω contains just one element. Let w be such an element. In this context, a characterization of flatness for a single-input system is shown, and consequently the well known equivalence between dynamic and static feedback linearization for single input systems can be deduced (see [7], [2], [49] for other proofs of the same result):

Proposition 3 *: System 4.1 is static feedback linearizable if and only if the module Ω characterizing 4.2 is generated by an integrable one form w.*

Proof: The necessity is obvious.
Sufficiency: Since w is assumed to be a generator:

$$dx_i = q_i^{l_i}(\frac{d}{dt})w \quad \forall i = 1, \ldots, n \tag{4.3}$$

where $q_i^{l_i}(\frac{d}{dt})$ are elements of the ring $K[\frac{d}{dt}]$, and l_i are the respective degrees. Therefore, 4.3 can be written as follows:

$$dx_i = \sum_{h=0}^{l_i} b_h^i w^{(h)} \tag{4.4}$$

Note that w and its derivatives of any order are L-independents. If there exists a combination among them such as $w^{(r)} = \sum_{k=0}^{r-1} a_k w^{(k)}$ then, w is a torsion element, which contradicts the hypothesis of controllability (recall that the controllobality of a linear system is equivalent to the freeness of the module).
On the other hand, $dx_1, \ldots, dx_n$ are L-indepents too (there are no algebraic relations between the state variables). Since n independent elements can not be written as a combination of a set of l independent elements, with $l < n$, there exists l_i greater than or equal to $n-1$. Let us take

$$l_j = \max_{1 \leq j \leq n} \{l_i\} \geq n - 1$$

and substitute the expression of dx_{l_j} from 4.4 in the j^{th} equation of 4.2.

$$\frac{d}{dt}\left(\sum_{h=0}^{l_j} b_h^j w^{(h)}\right) = \left(\frac{\partial f}{\partial x}\right)_j \left(\sum_{h=0}^{l_j} b_h^j w^{(h)}\right) + \left(\frac{\partial f}{\partial u}\right)_j du$$

Thus, in the left hand side we have $\dot{dx_j}$, which is a polynomial in the indeterminate $\frac{d}{dt}$ of degree $l_j + 1$. And in the right hand side of the above equation we have dx_j, which is a polynomial in the indeterminate $\frac{d}{dt}$ of degree smaller than or equal to l_j. Therefore,

$$\left(\frac{\partial f}{\partial u}\right)_j du \neq 0$$

because, if it is zero, the above equation becomes

$$\frac{d}{dt}\left(\sum_{h=0}^{l_j} b_h^j w^{(h)}\right) - \left(\frac{\partial f}{\partial x}\right)_j \left(\sum_{h=0}^{l_j} b_h^j w^{(h)}\right)$$

which implies that w is a torsion element, in contradiction with the controllability of the system.
Now, since

$$\left(\frac{\partial f}{\partial u}\right)_j du \neq 0$$

it is possible to isolate du:

$$du = \frac{1}{\left(\frac{\partial f}{\partial u}\right)_j}\left(\frac{d}{dt}\left(\sum_{h=0}^{l_j} b_h^j w^{(h)}\right) - \left(\frac{\partial f}{\partial x}\right)_j \left(\sum_{h=0}^{l_j} b_h^j w^{(h)}\right)\right)$$

So,

$$du = p^{l_j+1}\left(\frac{d}{dt}\right)w \tag{4.5}$$

where $p^{l_j+1}(\frac{d}{dt})$ is a polynomial in the indeterminate $\frac{d}{dt}$ of degree $l_j + 1 \geq n$.
Note that it is no possible that

$$du = p^{s}\left(\frac{d}{dt}\right)w \tag{4.6}$$

with degree $s < n$, because if 4.6 holds, then, equalling 4.6 and 4.5:

$$p^{l_j+1}\left(\frac{d}{dt}\right)w = du = p^{s}\left(\frac{d}{dt}\right)w$$

which implies that w is a torsion element, in contradiction with the controllability hypothesis.
Summarizing, the relative degree of w with respect to du is $l_j + 1 \geq n$. But, it is well-known that the relative degree is always smaller than or equal to the dimension of the state. Therefore, the relative degree of w with respect to du is n.
Now, using the integrability condition of w, a variable y (the flat output) such that $dy = w$ is obtained. And since the Kähler differential commutes with the time derivative, this variable y, the flat output, fulfills the relative degree condition with respect to u . ■

Corollary 1 : *Linearization by static and dynamic feedback are equivalent for single-input systems.*

Proof: As already stated in chapter 2, flatness (or linearization by dynamic feedback) is equivalent to the existence of an integrable basis of the tangent system. From the proposition this last fact is equivalent to linearization by static feedback. Hence, dynamic and static feedback linearization are equivalent for single-input systems. ■

Now, imposing the former relative degree condition, w is computed as a solution of an homogenous linear system. Since w lies in Ω the expression of w must be:

$$w = a_1 dx_1 + \ldots + a_n dx_n := a \cdot dX$$

And an expression for $\dot{w}$ is obtained:

$$\begin{aligned} \dot{w} &= \dot{a}dx + a \cdot \left(\tfrac{\partial f}{\partial x} dx + \tfrac{\partial f}{\partial u} du\right) \\ &= \left(\dot{a} + a\tfrac{\partial f}{\partial x}\right) \cdot dx + a \cdot \tfrac{\partial f}{\partial u} du \end{aligned}$$

Therefore, the condition of the relative degree implies:

$$a \cdot \frac{\partial f}{\partial u} = 0 \tag{4.7}$$

Deriving 4.7, it is possible to compute another condition, useful in the following steps:

$$\dot{a} \cdot \frac{\partial f}{\partial u} + a \cdot \frac{d}{dt}\left(\frac{\partial f}{\partial u}\right) = 0 \tag{4.8}$$

The value of $\ddot{w}$ is calculated:

$$\begin{aligned} \ddot{w} &= \left(\ddot{a} + 2\dot{a}\tfrac{\partial f}{\partial x} + a\tfrac{d}{dt}\tfrac{\partial f}{\partial x} + a\left(\tfrac{\partial f}{\partial x}\right)^2\right) \cdot dx \\ &+ \left(\dot{a} + a\tfrac{\partial f}{\partial x}\right)\tfrac{\partial f}{\partial u} du \end{aligned}$$

The condition of the relative degree leads to:

$$\left(\dot{a} + a\frac{\partial f}{\partial x}\right) \cdot \frac{\partial f}{\partial u} = 0$$

which thanks to equation 4.8, becomes:

$$a \cdot \left(\frac{\partial f}{\partial x} - \frac{d}{dt} I\right)\frac{\partial f}{\partial u} = 0$$

Here it is also possible to obtain another useful expression:

$$a^{(2)} \cdot \frac{\partial f}{\partial u} + 2\dot{a} \cdot \frac{d}{dt}\left(\frac{\partial f}{\partial u}\right) = -a \cdot \frac{d^2}{dt^2}\left(\frac{\partial f}{\partial u}\right) \tag{4.9}$$

Imposing once again the relative degree condition, now to $w^{(3)}$:

$$\left(a^{(2)} + 2\dot{a}\frac{\partial f}{\partial x} + a\frac{d}{dt}\frac{\partial f}{\partial x} + a\left(\frac{\partial f}{\partial x}\right)^2\right) \cdot \frac{\partial f}{\partial u} = 0$$

Using 4.8 and 4.9, this last equation becomes:

$$a\left(\frac{\partial f}{\partial x}-\frac{d}{dt}I\right)^2\cdot\frac{\partial f}{\partial u}=0$$

Iterating this process, a system of equations is obtained:

$$a\left(\frac{\partial f}{\partial x}-\frac{d}{dt}I\right)^k\frac{\partial f}{\partial u}=0 \qquad \forall k=0,\ldots,n-2$$

This is an homogeneous system with $n-1$ equations and n unknowns ($a_1,\ldots,a_n$). So, the solution w will depend upon a certain function λ. Now, a variable y such that $dy=w$ must be obtained. The role of λ is the role of an integrant factor.

4.3 Static feedback linearization of multi-input systems

Once again the intention is to compute the basis of the module corresponding to the tangent system. As was stated in chapter 2, this basis contains the same number of elements as the number of the inputs. Let m be this number. And let $w_1,\ldots,w_m$ be the elements of this basis. And again, as in the single-input case, the conditions that must be imposed to find the basis are the relative degree conditions. The following lemmas will translate the relative degree conditions of a nonlinear system to its tangent system.

Lemma 6 *For all f, g vector fields, and for any w differential one form, the next equality is satisfied:*

$$L_f < w,g > = < L_f w,g > + < w,[f,g] > \tag{4.10}$$

Lemma 7 *The Lie derivative with respect to a vector field f commutes with the exterior derivative:*

$$L_f(dh)=dL_f(h) \tag{4.11}$$

The proof of these two lemmas can be found in any elementary text on differential geometry.

Lemma 8 *For any C^∞ function y and any vector fields f and h, it follows the equality:*

$$< dL_f^r y,h > = \sum_{k=0}^{r}(-1)^k\binom{r}{k}L_f^{r-k} < dy,ad_f^k h > \tag{4.12}$$

Proof: It will be proven by induction. For $r=1$, and thanks to 4.11

$$< dL_f y,h > = < L_f dy,h >$$

Applying now 4.10, we have

$$< L_f dy,h > = L_f < dy,h > - < dy,[f,h] >$$

which is the desired equality for $r = 1$.
Assuming the trueness of the statement for the case r, the case $r + 1$ will be proven. Applying 4.11 and 4.10:

$$< dL_f^{r+1}y, h >=< L_f dL_f^r y, h >= L_f < dL_f^r y, h > - < dL_f^r y, [f, h] >$$

Using the induction hypothesis, the latter expression becomes

$$L_f \left(\sum_{k=0}^{k=r} (-1)^k \binom{r}{k} L_f^{r-k} < dy, ad_f^k h > \right) - \left(\sum_{k=0}^{k=r} (-1)^k \binom{r}{k} L_f^{r-k} < dy, ad_f^k ad_f h > \right)$$

which, in turn, is equal to

$$\left(\sum_{k=0}^{k=r} (-1)^k \binom{r}{k} L_f^{r+1-k} < dy, ad_f^k h > \right) - \left(\sum_{k=0}^{k=r} (-1)^k \binom{r}{k} L_f^{r-k} < dy, ad_f^k + 1h > \right)$$

Finally, the equality

$$\binom{r}{k} + \binom{r}{k-1} = \binom{r+1}{k}$$

applied to the former expression, leads to

$$\left(\sum_{k=0}^{k=r+1} (-1)^k \binom{r+1}{k} L_f^{r+1-k} < dy, ad_f^k h > \right)$$

This is the desired expression for the case $r + 1$. ■

Proposition 4 *When a system*

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i$$

is linearizable by static feedback, then

$$\frac{d^r}{dt^r} y_i = L_f^r y_i \quad \forall r \leq k_i - 2$$

where y_i and its derivatives up to order $k_i - 1$ are the coordinates of the change of variables that linearizes the nonlinear system; and k_i are the Kronecker indices.

Proof: Again it will be proven by induction. For $r = 1$

$$\frac{d}{dt} y_j =< dy_j, f(x) + \sum_{i=1}^{n} g_i(x) u_i >=< dy_j, f >$$

since the static feedback linearization conditions imply

$$< dy_j, ad_f^r g_i >= 0 \quad \forall r \leq k_j - 2 \tag{4.13}$$

Assuming that the statement is true up to r, the case $r+1$ will be studied. Using the induction hypothesis we have

$$\frac{d^{r+1}}{dt^{r+1}} y_j = \frac{d}{dt}(L_f^r y_j) =< dL_f^r y_j, f(x) + \sum_{i=1}^{n} g_i(x) u_i >$$

Applying now the equation 4.12, we get:

$$\sum_{k=0}^{r} (-1)^k \binom{r}{k} L_f^{r-k} < dy_j, ad_f^k (f(x) + \sum_{i=1}^{n} g_i(x) u_i) >$$

which thanks to 4.13 can be written

$$L_f^r < dy_j, f > + \sum_{k=1}^{r} (-1)^k \binom{r}{k} L_f^{r-k} \sum_{i=1}^{n} u_i < dy_j, ad_f^k g_i >$$

Again, because of 4.13,

$$\sum_{k=1}^{r} (-1)^k \binom{r}{k} L_f^{r-k} \sum_{i=1}^{n} u_i < dy_j, ad_f^k g_i >= 0$$

Therefore,

$$\frac{d^{r+1}}{dt^{r+1}} y_j = L_f^{r+1} y_j$$

■

Corollary 2 *The Kähler differential of y_j^r does not depend on the inputs of the tangent system du_i $i = 1, \ldots, m$*

Proof: In the previous proposition it has been proven that $\frac{d^{r+1}}{dt^{r+1}} y_j$ does not depend on the input variables u_i $\forall i = 1, \ldots, m$. The commutativity between the time derivative and the Kähler differential was commented on in chapter 2. Putting them all together, the statement of the corollary is derived.
So, the relative degree of $w_j = dy_j$ with respect to du must be k_i, where k_i are the Kronecker indices ([51]).
Let

$$\dot{x} = f(x, u) \quad x \in R^n \quad u \in R^m$$

a multi-input system. Its Kähler differential is

$$\dot{dx} = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial u} du \tag{4.14}$$

The basis can be written

$$w_i = a_1^i dx_1 + \ldots + a_n^i dx_n = a^i \cdot dx \quad \forall i = 1, \ldots, m$$

Leading the relative degree condition to

$$a^i \left(\frac{\partial f}{\partial x} - \frac{d}{dt} I \right)^j \frac{\partial f}{\partial u_l} = 0 \quad \forall j = 0, \ldots, k_i - 2 \quad \forall l = 1, \ldots, m$$

The solution of this system will be a basis of Ω. Note that, if the Kronecker indices are such that $k_1 \geq k_2 \geq \ldots$, then w_1 depends upon, at least, $n - (k_1 - 1)m$ parameters; w_2 depends at least upon $n - (k_2 - 1)m$ parameters; and so on. These parameters act as integrant factors in order to find $y_i \in L$ such that $dy_i = w_i$.
Note that even though the original system was not affine in the inputs, this procedure can still be applied
Example: Let us consider the following system:

$$\begin{cases} \dot{x}_1 &= x_2 + x_1 u_2 \\ \dot{x}_2 &= x_3^2 u_1 u_2 + x_2 u_2 \\ \dot{x}_3 &= u_2 x_3 \end{cases}$$

Its tangent system is:

$$\begin{cases} \dot{dx_1} &= u_2 dx_1 + dx_2 + x_1 du_2 \\ \dot{dx_2} &= u_2 dx_2 + 2x_3 u_1 u_2 dx_3 + x_3^2 u_2 du_1 + (x_2 + x_3^2 u_1) du_2 \\ \dot{dx_3} &= u_2 dx_3 + x_3 du_2 \end{cases}$$

The Kronecker indices are $k_1 = 2$ y $k_2 = 1$.
Denoting

$$w_i = a_1^i dx_1 + a_2^i dx_2 + a_3^i dx_3 \quad i = 1, 2$$

Applying the conditions explained above, the next equations for w_1 are obtained:

$$a_2^1 = 0 \quad \text{and} \quad a_1^1 x_1 + a_3^1 x_3 = 0$$

and no conditions for w_2. Therefore

$$w_1 = \lambda(x_3 dx_1 - x_1 dx_3) \quad \lambda \in L$$

and w_2 can be choosen freely, taking into account that w_2 has to be differentially independent with respect to w_1. As λ is an integrant factor for w_1, it can be chosen in such a way that w_1 is an exact one form. A possibility is to choose

$$\lambda = \frac{1}{x_1 x_3}$$

Hence:

$$w_1 = \frac{1}{x_1}dx_1 - \frac{1}{x_3}dx_3$$

and, for the sake of simplicity,

$$w_2 = dx_2$$

It can be checked that

$$y_1 = \ln\frac{x_1}{x_3} \qquad y_2 = x_2$$

are such that $w_i = dy_i$, $i = 1, 2$. In other words, the flat outputs of the system are y_1, y_2.

4.4 Dynamic feedback linearization for multi-input systems

Let us consider a controllable multi-input system:

$$\dot{x} = f(x, u) \tag{4.15}$$

where $x \in R^n$ and $u \in R^2$. Recall that the tangent system is:

$$\dot{dx} = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial u}du \tag{4.16}$$

Let Ω be the module characterization of 4.16. Suppose that a flat output can be guessed (for example y_1). In order to obtain the second flat output y_2, a quotient of modules is made. Let us consider the quotient module:

$$\Omega / < dy_1 > \tag{4.17}$$

This is a controllable single-input system, and by application of the algorythm explained in the former section a basis of this module can be found. Let $\overline{w}_2$ be this basis. Therefore, a basis of Ω will be:

$$w_1 = dy_1, \qquad w_2 = \overline{w}_2 + p(\frac{d}{dt})dy_1$$

$p(\frac{d}{dt})$ must be chosen in such a way that w_2 is a K-integrable one form. So a system of partial differential equations must be solved.
This procedure can be generalized for systems with m inputs. In that case, $m-1$ flat outputs must be guessed, but the algorythm to get the last one is the same.
Once the flat outputs $y_1, \ldots, y_m$ have been obtained, it is possible to know if the original system is dynamic feedback linearizable by derivation of the inputs [7]. To get such a condition, let us define the parameters $\bar{k}_{ji}, r_p, n_j$ as follows:
let

$$w_j = dy_j = A_j dx + p_{j1}^{k_{j1}}(\frac{d}{dt})du_1 + \ldots + p_{jm}^{k_{jm}}(\frac{d}{dt})du_m \tag{4.18}$$

On the other hand, the system variables can be written using $\omega_1, \ldots, \omega_m$:

$$dx = R_1^{l_1}(\frac{d}{dt})dy_1 + \ldots + R_m^{l_m}(\frac{d}{dt})dy_m$$

$$du_i = q_{i1}^{h_{i1}}(\frac{d}{dt})dy_1 + \ldots + q_{im}^{h_{im}}(\frac{d}{dt})dy_m$$

where $R_1^{l_1}(\frac{d}{dt}),\ldots,R_m^{l_m}(\frac{d}{dt})$ are vectors with coefficients in $K[(\frac{d}{dt})]$ and $l_1,\ldots,l_m$ are the maximum degrees of the indeterminate in each vector. $q_{i1}^{h_{i1}}(\frac{d}{dt}) + \ldots + q_{im}^{h_{im}}(\frac{d}{dt})$ are polynomials in the indeterminate $(\frac{d}{dt})$ and $h_{i1},\ldots,h_{im}$ are the degrees of the respective polynomials.
Now

$$\bar{k}_{ji} := \begin{cases} k_{ji} & \text{if } p_{ji} \neq 0 \\ -\text{rel } d^o(dy_j, du_i) & \text{if } p_{ji} = 0 \end{cases}$$

$$r_p := \max\{\{l_j + \bar{k}_{jp}, \forall j = 1\ldots m\}, \{h_{ij} + \bar{k}_{jp}, \forall j = 1\ldots m, \forall i = 1\ldots m\}\}$$

$$n_j := \max\{l_j, h_{ij}, \forall i = 1,\ldots,m\}$$

It must be noticed that $l_j := -\infty$ (respectively $h_{ij} = -\infty$) if $R_j = 0$ (respectively $q_{ij} = 0$).

Corollary 3 *The system is dynamic feedback linearizable by prolongations if and only if the sets V and W have the same number of variables, where $W =$:*

$$\{dx_1,\ldots,dx_n,du_1,\ldots,du_1^{(r_1)},\ldots,du_m,\ldots,du_m^{(r_m)}\}$$

and

$$V = \{dy_1,\ldots,dy_1^{(n_1)},\ldots,dy_m,\ldots,dy_m^{(n_m)}\}$$

That is to say

$$n + \sum_{p=1}^{m} r_p = \sum_{j=1}^{m} n_j$$

Proof: Note that

$$dx_1,\ldots,dx_n,du_1,\ldots,du_1^{(r_1)},\ldots,du_m,\ldots,du_m^{(r_m)}$$

can be written as linear functions of

$$dy_1,\ldots,dy_1^{(n_1)},\ldots,dy_m,\ldots,dy_m^{(n_m)}$$

And conversely,

$$dy_1,\ldots,dy_1^{(n_1)},\ldots,dy_m,\ldots,dy_m^{(n_m)}$$

are linear functions of

$$dx_1,\ldots,dx_n,du_1,\ldots,du_1^{(r_1)},\ldots,du_m,\ldots,du_m^{(r_m)}$$

In other words, there exists a linear change of variables between the variables in V and W. This change of variables is, in fact, thanks to the integrability conditon, the jacobian of the change of variables between

$$x_1,\ldots,x_n,u_1,\ldots,u_1^{(r_1)},u_m,\ldots,u_m^{(r_m)}$$

and

$$y_1, \ldots, y_1^{(n_1)}, \ldots, y_m, \ldots, y_m^{(n_m)}$$

So, as the change of variables exists, the system

$$\begin{aligned} \dot{x} &= f(x,u) \\ \tfrac{d}{dt}u_i^{(j)} &= u_i^{(j+1)} \quad \forall i = 1, \ldots, m \quad \forall j = 0, \ldots, r_{i-1} \end{aligned}$$

is static feedback equivalent to

$$\frac{d}{dt}y_i^{(j)} = y_i^{(j+1)} \quad \forall i = 1, \ldots, m \quad \forall j = 0, \ldots, n_i - 1$$

In other words, the original system

$$\dot{x} = f(x,u)$$

is linearizable by prolongations.

4.5 Software

In order to simplify the calculations needed in the method presented above, a software package has been developed. It consists in many `Maple V` functions that allow us to use a computer to perform the basic operations -such as the computation of the tangent system- and those that are harder. For example, given a single input system, a basis of the module Ω can be determined and quotients in this module with arbitrary expressions can also be made. Given a two-input dynamic feedback linearizable system, it can be reduced to a single input one through a quotient, and, in this single input system, a basis can be computed.
The integrability of the basis is automatically tested, and, if it holds, the integrals will be the flat outputs.

4.6 Examples

1. This example has been borrowed from [45].

   Let:

   $$\begin{aligned} \dot{x_1} &= u_1 \\ \dot{x_2} &= u_2 \\ \dot{x_3} &= u_1 u_2 \end{aligned}$$

   be a nonlinear system.

   The tangent system associated to it is a quotient $K[\frac{d}{dt}]$-module Λ defined by the generators

   $$\{dx_1, dx_2, dx_3, du_1, du_2\}$$

and the relations:

$$\begin{aligned} \dot{dx_1} &= du_1 \\ \dot{dx_2} &= du_2 \\ \dot{dx_3} &= u_2 du_1 + u_1 du_2 \end{aligned}$$

dx_1 can be guessed as one of the two generators of the free module. Let us consider the quotient module $\Omega_1 = \frac{\Omega}{<dx_1>}$. In the state space representation, Ω_1 is given by the equations:

$$\begin{aligned} \dot{\bar{dx_2}} &= \bar{du_2} \\ \dot{\bar{dx_3}} &= u_1 \bar{du_2} \end{aligned}$$

Clearly, a basis of Ω_1 is given by $\overline{\omega_2} = u_1 \bar{dx_2} - \bar{dx_3}$.

Going back to $\omega_2 = u_1 dx_2 - dx_3 + p(\frac{d}{dt})dx_1$; where $p(\frac{d}{dt}) \in K[\frac{d}{dt}]$. In order for ω_2 to be K-integrable, an appropriate choice for $p(\frac{d}{dt})$ is

$$p(\frac{d}{dt}) = x_2 \frac{d}{dt}$$

Hence, an K-integrable basis of Ω is

$$\begin{aligned} \omega_1 &= dx_1 \\ \omega_2 &= u_1 dx_2 - dx_3 + x_2 du_1 \end{aligned}$$

and the flat outputs are:

$$\begin{aligned} y_1 &= x_1 \\ y_2 &= u_1 x_2 - x_3 \end{aligned}$$

The relationship between the state variables of the tangent system and the basis of the tangent module is:

$$\begin{aligned} dx_1 &= \omega_1 \\ dx_2 &= \frac{-x_2\omega_1^{(2)} + \dot{\omega}_2}{\dot{u}_1} \\ dx_3 &= \frac{x_2\dot{u}_1\dot{\omega}_1 - u_1x_2\omega_1^{(2)} - \dot{u}_1\omega_2 + u_1\dot{\omega}_2}{\dot{u}_1} \\ du_1 &= \dot{\omega}_1 \\ du_2 &= \text{involves third derivatives of } \omega_1 \\ & \text{ and second derivatives of } \omega_2 \end{aligned}$$

Therefore

$$
\begin{aligned}
r_1 &= \max\{1,1,0,-\infty,2,2\} = 2\\
r_2 &= \max\{-\infty,-1,-\infty,-\infty,-\infty,0\} = 0\\
n_1 &= \max\{2,1,3\} = 3\\
n_2 &= \max\{1,-\infty,2\} = 2
\end{aligned}
$$

Summarizing, the nonlinear system defined by

$$
\begin{aligned}
\dot{x_1} &= u_{11}\\
\dot{x_2} &= v_2\\
\dot{x_3} &= u_{11}v_2\\
\dot{u_{11}} &= u_{12}\\
\dot{u_{12}} &= v_1
\end{aligned}
$$

is static feedback equivalent to the system

$$
\begin{aligned}
\dot{y_{11}} &= y_{12}\\
\dot{y_{12}} &= y_{13}\\
\dot{y_{13}} &= y_{14}\\
\dot{y_{21}} &= y_{22}\\
\dot{y_{22}} &= y_{23}
\end{aligned}
$$

which is linear.

2. This example has been borrowed from [53].

$$
\begin{aligned}
x_2\dot{x_1} - x_1\dot{x_2} &= x_3\\
x_1\dot{x_3} &= x_4
\end{aligned}
$$

This system can be written down in the following way:

$$
\begin{aligned}
\dot{x_1} &= \tfrac{x_3+x_1u_1}{x_2}\\
\dot{x_2} &= u_1\\
\dot{x_3} &= \tfrac{u_2}{x_1}
\end{aligned}
$$

Its tangent system is:

$$dx = \begin{pmatrix} u_1 & -\frac{x_3+x_1u_1}{x_2^2} & \frac{1}{x_2} \\ 0 & 0 & 0 \\ -\frac{u_2}{x_1^2} & 0 & 0 \end{pmatrix} dx +$$

$$\begin{pmatrix} \frac{x_1}{x_2} & 0 \\ 1 & 0 \\ 0 & \frac{1}{x_1} \end{pmatrix} du$$

Making quotient by $dx_1 - \lambda du_2$ (that is to say, $x_1 - \lambda x_4$ in the original system) we get:

$$\dot{\bar{dx}} = \begin{pmatrix} u_1 & -\frac{x_3+x_1u_1}{x_2^2} & \frac{1}{x_2} \\ 0 & 0 & 0 \\ -\frac{u_2\lambda - x_1}{x_1^2\lambda} & 0 & 0 \end{pmatrix} \bar{dx} +$$

$$\begin{pmatrix} \frac{x_1}{x_2} \\ 1 \\ 0 \end{pmatrix} \bar{du}$$

Applying the algorithm for single-input systems, the basis $\overline{w_2}$ is obtained:

$$\overline{w_2} = a_1\left(-\frac{x_2(u_2\lambda - x_1)}{2\lambda x_1 x_3}\bar{dx_1} + \frac{u_2\lambda - x_1}{2\lambda x_3} + \bar{dx_3}\right)$$

where a_1 is a function depending upon the variables $x's$ and $u's$. Therefore:

$$w_2 = \overline{w_2} + p\left(\frac{d}{dt}\right)(dx_1 - \lambda du_2)$$

But this one form is not integrable. So, $x_1 - \lambda x_4$ in the original system cannot be a flat output. If λ is zero (that is to say, guessing x_1 as a flat output), the quotient is:

$$\begin{pmatrix} \dot{\bar{dx_2}} \\ \dot{\bar{dx_3}} \end{pmatrix} = \begin{pmatrix} \frac{x_3+x_1u_1}{x_1x_2} & -\frac{x_2}{x_1x_2} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{dx_2} \\ \bar{dx_3} \end{pmatrix}$$

$$+ \begin{pmatrix} 0 \\ \frac{1}{x_1} \end{pmatrix} \bar{du_2}$$

Clearly, dx_2 is a basis of this module. Therefore, the flat outputs are x_1 and x_2.

Comparing this method with the method used in [53], note that our algorithm reduces to:

(a) Making a quotient of modules.

(b) Solving an homogeneous linear system.

(c) Checking the integrability condition of just one form.

Therefore, it seems simpler than the procedure appearing in [53] .

3. The following example is related to a vertical take off and landing aircraft (VTOL). This aircraft was assumed to be not linearizable by prolongations since [23], but it was known to be flat . In our framework we prove not only that is flat, but that it is also linearizable by prolongations. Here are the equations of the system:

$$\begin{array}{rlll} \ddot{x} = & u_1 \sin\theta - & u_2\varepsilon\cos\theta & \\ \ddot{y} = & u_1 \cos\theta + & u_2\varepsilon\sin\theta & -1 \\ \ddot{\theta} = & & & u_2 \end{array}$$

Reducing the system to order 1 in order to be able to apply our algorithm:

$$\begin{array}{rlll} \dot{x_1} = & x_2 & & \\ \dot{x_2} = & & u_1 \sin x_5 - & u_2\varepsilon\cos x_5 \\ \dot{x_3} = & x_4 & & \\ \dot{x_4} = & -1+ & u_1\cos x_5 + & u_2\varepsilon\sin x_5 \\ \dot{x_5} = & x_6 & & \\ \dot{x_6} = & & & u_2 \end{array}$$

where

$$x_1 = x \quad x_2 = \dot{x} \quad x_3 = y \quad x_4 = \dot{y} \quad x_5 = \theta \quad x_6 = \dot{\theta}$$

As in the preceding examples, its Kähler differential is computed:

$$\begin{array}{rllll} d\dot{x_1} = & dx_2 & & & \\ d\dot{x_2} = & (u_1\cos x_5 + u_2\varepsilon\sin x_5)dx_5 + & \sin x_5 du_1 - & \varepsilon\cos x_5 du_2 & \\ d\dot{x_3} = & dx_4 & & & \\ d\dot{x_4} = & (-\sin x_5 + u_2\varepsilon\cos x_5)dx_5 + & \cos x_5 du_1 + & \varepsilon\sin x_5 du_2 & \\ d\dot{x_5} = & dx_6 & & & \\ d\dot{x_6} = & & & & du_2 \end{array}$$

Guessing $y_1 = x_5$ as a flat output, the quotient is

$$\begin{array}{rll} d\dot{x_1} = & dx_2 & \\ d\dot{x_2} = & & \sin x_5 du_1 \\ d\dot{x_3} = & dx_4 & \\ d\dot{x_4} = & & \cos x_5 du_1 \end{array}$$

A basis of this quotient module is

$$\overline{w_2} = \frac{2x_6^2 \sin x_5 + u_2 \cos x_5}{2x_6 \cos x_5}\lambda dx_1 + \lambda dx_2 + \frac{2x_6^2 \cos x_5 - u_2 \sin x_5}{2x_6 \cos x_5}\lambda dx_3 - \frac{\sin x_5}{\cos x_5}\lambda dx_4$$

When

$$\lambda = x_6 \cos x_5$$

and $p(\frac{d}{dt})$ is the appropiate polynomial,

$$y_2 = x_1 \left(\frac{2x_6^2 \sin x_5 + u_2 \cos x_5}{2}\right) + x_2 x_6 \cos x_5 + x_3 \frac{2x_6^2 \cos x_5 - u_2 \sin x_5}{2} + x_4 x_6 \sin x_5$$

Once the flat outputs have been obtained, we are able to decide whether or not this system is linearizable by prolongations. For this purpose the parameters previously defined are computed:

$$l_1 = 3 \qquad l_2 = 5$$

$$h_{11} = 4 \qquad h_{12} = 6 \qquad h_{21} = -\infty \qquad h_{22} = 2$$

$$k_{11} = -4 \qquad k_{12} = 0 \qquad k_{21} = -\infty \qquad k_{22} = -2$$

Therefore,

$$r_1 = 0 \qquad r_2 = 4$$

and

$$n_1 = 4 \qquad n_2 = 6$$

That is to say,

$$n + r_1 + r_2 = 6 + 0 + 4 = 10$$

While

$$n_1 + n_2 = 10$$

Therefore, following the corollary, the VTOL is linearizable by prolongations.

4.7 An analogous procedure using field extensions

The procedure explained in section 3.3 admits a nice counterpart using only field extensions. Let us consider a nonlinear control system

$$\dot{x} = f(x, u) \quad x \in R^n \quad u \in R^m$$

As was stated in chapter 2, this is a differential field extension L/R, where L is the minimum field containing the variables x, u and where the equations of the system are satisfied. If $m-1$ flat outputs $y_1, \ldots, y_m$ are guessed, let us consider the intermediate differential field extension $L/R < y_1, \ldots, y_{m-1} > /R$. Then, the extension $L/R < y_1, \ldots, y_{m-1} >$ is a single-input system. In this single-input system, the relative degree condition can be applied in order to find its flat output, which will be the remaining flat output y_n of L/R. This procedure has a disadvantage with respect to the one explained in section 3.3. The procedure in section 3.3 uses quotient of modules, and because of this, some equations and $m-1$ inputs are eliminated. In the field extension procedure, neither equations nor variables can be eliminated. Another difficulty is that there is no software package to apply the procedure, and neither is there a sistematic way to obtain the last flat output.

Example: Let us consider again the VTOL. Let us also recall that the system equations can be written as follows:

$$\begin{array}{rlll} \dot{x_1} = & x_2 & & \\ \dot{x_2} = & & u_1 \sin x_5 - & u_2 \varepsilon \cos x_5 \\ \dot{x_3} = & x_4 & & \\ \dot{x_4} = & -1+ & u_1 \cos x_5 + & u_2 \varepsilon \sin x_5 \\ \dot{x_5} = & x_6 & & \\ \dot{x_6} = & & & u_2 \end{array}$$

If $y_1 = x_5$ is guessed as one of the flat outputs, the extension $R < y_1, y_2 > /R < y_1 >$ is represented by the following system

$$\begin{array}{rll} \dot{x_1} = & x_2 & \\ \dot{x_2} = & u_1 \sin x_5 - & u_2 \varepsilon \cos x_5 \\ \dot{x_3} = & x_4 & \\ \dot{x_4} = & -1 + u_1 \cos x_5 + & u_2 \varepsilon \sin x_5 \end{array}$$

which is a single-input system. Applying the usual static feedback linearizability conditions to this single-input system, the following distributions have to be involutive:

$$D_1 = < \begin{pmatrix} 0 \\ \sin x_5 \\ 0 \\ \cos x_5 \end{pmatrix} >$$

is trivially involutive.

$$D_2 = < \begin{pmatrix} 0 \\ \sin x_5 \\ 0 \\ \cos x_5 \end{pmatrix}, \begin{pmatrix} -\sin x_5 \\ x_6 \cos x_5 \\ -\cos x_5 \\ -x_6 \sin x_5 \end{pmatrix} >$$

is involutive because the Lie bracket between its two elements is zero.

$$D_3 =< D_2, \begin{pmatrix} -2x_6 \cos x_5 \\ -x_6^2 \sin x_5 + u_2 \cos x_5 \\ 2x_6 \sin x_5 \\ -x_6^2 \cos x_5 - u_2 \sin x_5 \end{pmatrix} >$$

is involutive because the Lie brackets between its elements are zero. And, finally, D_4 is also involutive because its dimension is 4. Therefore the system is static feedback linearizable and its flat output y_2 can be computed from the ortogonality between ∇y_2 and D_3. The solution of this linear system leads to a system of partial differential equations. One solution of this system is

$$\begin{aligned} y_2 = \; & x_1 \left(\frac{2x_6^2 \sin x_5 + u_2 \cos x_5}{2} \right) + x_2 x_6 \cos x_5 + \\ & x_3 \frac{2x_6^2 \cos x_5 - u_2 \sin x_5}{2} + x_4 x_6 \sin x_5 \end{aligned}$$

which coincides with the solution found using the quotient module algorithm.

Chapter 5

Linearization by prolongations of 2-input systems

This chapter deals with the problem of linearization by prolongations for 2-input systems. A necessary and sufficient condition for a two input system to be linearizable is derived. As an application, two flat outputs for the VTOL and a planar model of a ducted fan are obtained. These examples were thought to be not linearizable by prolongations in the existent literature ([59], [23]). Another example, which shows the sharpness of the bound obtained, is also given.

5.1 Prolongations of m inputs are not necessary

The following proposition will be helpful for establishing conditions so that a two input system may be linearizable by means of prolongations. Thanks to this proposition, it suffices to prolong the system by derivatives of just one input. In other words, prolongations by derivatives of both inputs are not necessary. This proposition has been already proven in [59], but our proof uses a different approach.

Proposition 5 *If the system:*

$$\dot{x} = f(x) + g_1(x)u_1 + g_2(x)u_2 \qquad x \in R^m$$

is linearizable by derivation of u_1 n_1 times and u_2 n_2 times (with $n_1 \geq n_2 \geq 1$), then the system is linearizable by derivation of u_1 $n_1 - 1$ times and u_2 $n_2 - 1$ times.

Proof: Let $\Sigma_{n_1}^{n_2}$ and $\Sigma_{n_1-1}^{n_2-1}$ be the systems obtained by prolongation of u_1 n_1 times and u_2 n_2 times, and u_1 $n_1 - 1$ times and u_2 $n_2 - 1$ times respectively. The drifts associated to these systems are

$$f^1 = f + g_1 y_{n+1} + g_2 y_{n+n_1+1} + \sum_{j=1}^{n_1-1} y_{n+j+1}\frac{\partial}{\partial y_{n+j}} + \sum_{j=1}^{n_2-1} y_{n+n_1+j+1}\frac{\partial}{\partial y_{n+n_1+j}}$$

for system $\Sigma_{n_1}^{n_2}$. And

$$f^2 = f + g_1 y_{n+1} + g_2 y_{n+n_1+1} + \sum_{j=1}^{n_1-2} y_{n+j+1}\frac{\partial}{\partial y_{n+j}} + \sum_{j=1}^{n_2-2} y_{n+n_1+j+1}\frac{\partial}{\partial y_{n+n_1+j}}$$

for system $\Sigma_{n_1-1}^{n_2-1}$. While the input fields are

$$g_1^1 = \frac{\partial}{\partial y_{n+n_1}} \quad g_2^1 = \frac{\partial}{\partial y_{n+n_1+n_2}}$$

and

$$g_1^2 = \frac{\partial}{\partial y_{n+n_1-1}} \quad g_2^2 = \frac{\partial}{\partial y_{n+n_1+n_2-1}}$$

for the systems $\Sigma_{n_1}^{n_2}$ and $\Sigma_{n_1-1}^{n_2-1}$ respectively. By hypothesis, the system $\Sigma_{n_1}^{n_2}$ is static feedback linearizable. Therefore, the following distributions are involutive and constant rank:

$$D_i^1 =< \frac{\partial}{\partial y_{n+n_1}}, \frac{\partial}{\partial y_{n+n_1+n_2}}, \dots, ad_{f^1}^i \frac{\partial}{\partial y_{n+n_1}}, ad_{f^1}^i \frac{\partial}{\partial y_{n+n_1+n_2}} >$$

It is a straightforward computation to check the following equalities

$$D_i^1 =< \frac{\partial}{\partial y_{n+n_1}}, \dots, \frac{\partial}{\partial y_{n+n_1-i}}, \frac{\partial}{\partial y_{n+n_1+n_2}}, \dots, \frac{\partial}{\partial y_{n+n_1+n_2-i}} >$$

for all $i \leq n_2 - 1$. And

$$D_{n_2}^1 =< \frac{\partial}{\partial y_{n+n_1}}, \dots, \frac{\partial}{\partial y_{n+n_1-n_2}}, \frac{\partial}{\partial y_{n+n_1+n_2}}, \dots, \frac{\partial}{\partial y_{n+n_1+1}}, g_2 >$$

While

$$D_i^2 =< \frac{\partial}{\partial y_{n+n_1-1}}, \dots, \frac{\partial}{\partial y_{n+n_1-1-i}}, \frac{\partial}{\partial y_{n+n_1+n_2-1}}, \dots, \frac{\partial}{\partial y_{n+n_1+n_2-1-i}} >$$

for all $i \leq n_2 - 2$. And

$$D_{n_2-1}^2 =< \frac{\partial}{\partial y_{n+n_1-1}}, \dots, \frac{\partial}{\partial y_{n+n_1-n_2}}, \frac{\partial}{\partial y_{n+n_1+n_2-1}}, \dots, \frac{\partial}{\partial y_{n+n_1+1}}, g_2 >$$

Therefore, for all $i \neq n_2$,

$$D_i^1 = D_{i-1}^2 \oplus < \frac{\partial}{\partial y_{n+n_1}}, \frac{\partial}{\partial y_{n+n_1+n_2}}$$

For $i > n_2$, some computationals lemmas are required.

Lemma 9

$$D_{n_2+i}^1 =< \frac{\partial}{\partial y_{n+n_1}}, \dots, \frac{\partial}{\partial y_{n+n_1-n_2-i}}, \frac{\partial}{\partial y_{n+n_1+n_2}}, \dots, \frac{\partial}{\partial y_{n+n_1+1}}, ad_{\overline{f}^1}^i g_2 >$$

where

$$\overline{f}^1 = f + g_1 y_{n+1} + \sum_{j=1}^{n_1-1} y_{n+j+1}\frac{\partial}{\partial y_{n+j}}$$

Proof: The proof will be done by induction.
For $i = 1$,

$$ad_{f^1} g_2 = [f^1, g_2] = [\overline{f}^1, g_2] + y_{n+n_1+1}[g_2, g_2] = [\overline{f}^1, g_2]$$

Assuming the equality is true up to order i,

$$ad_{f^1}^{i+1} g_2 = [f^1, ad_{f^1}^i g_2]$$

can be replaced, through application of the induction hypothesis, by

$$[f^1, ad_{\overline{f}^1}^i g_2]$$

which is equal to

$$ad_{\overline{f}^1}^{i+1} g_2 + [g_2 y_{n+n_1+1} + \sum_{j=1}^{n_2-2} y_{n+n_1+j+1} \frac{\partial}{\partial y_{n+n_1+j}}, ad_{\overline{f}^1}^i g_2]$$

However, due to the involutivity of the distributions and the next lemma, the last summand belongs to $D^1_{n_2+i}$. Therefore, $ad_{\overline{f}^1}^{i+1} g_2$ can replace $ad_{f^1}^{i+1} g_2$ in $D^1_{n_2+i+1}$.

Lemma 10

$$ad_{\overline{f}^1}^i g_2 = h_i \frac{\partial}{\partial x} + y_{n+i}[g_1, g_2]$$

where h_i is a function depending on the variables $(x, y_{n+1}, \cdots, y_{n+i-1})$

Proof: It is done by induction. When $i = 1$,

$$ad_{\overline{f}^1} g_2 = [f, g_2] + y_{n+1}[g_1, g_2]$$

Assuming the trueness of the equality up to order i,

$$ad_{\overline{f}^1}^{i+1} g_2 = [\overline{f}, ad_{\overline{f}^1}^i g_2] =$$

$$[\overline{f}, h_i \frac{\partial}{\partial x}] + [\overline{f}, y_{n+i}[g_1, g_2]] =$$

$$[\overline{f}, h_i \frac{\partial}{\partial x}] + [f + g_1 y_{n+1} + \sum_{j=1}^{i} y_{n+j+1} \frac{\partial}{\partial y_{n+j}}, y_{n+i}[g_1, g_2]] =$$

$$h_{i+1} \frac{\partial}{\partial x} + y_{n+i+1}[g_1, g_2]$$

where

$$h_{i+1} \frac{\partial}{\partial x} = [\overline{f}, h_i \frac{\partial}{\partial x}] + y_{n+i}[f, [g_1, g_2]] + y_{n+1} y_{n+i}[g_1, [g_1, g_2]]$$

Lemma 11

$$D^1_{n_2+i} =< \frac{\partial}{\partial y_{n+n_1}}, \dots, \frac{\partial}{\partial y_{n+n_1-n_2-i}}, \frac{\partial}{\partial y_{n+n_1+n_2}}, \dots, \frac{\partial}{\partial y_{n+n_1+1}}, ad^i_{\overline{f}^2} g_2 >$$

where

$$\overline{f}^2 = f + g_1 y_{n+1} + \sum_{j=1}^{n_1-2} y_{n+j+1} \frac{\partial}{\partial y_{n+j}}$$

Proof: The statement is clear for all $i < n_1$ since

$$ad^i_{\overline{f}^1} g_2 = ad^i_{\overline{f}^2} g_2$$

because of the former lemma. When $i = n_1$,

$$ad^{n_1}_{\overline{f}^1} g_2 = [\overline{f}^1, ad^{n_1-1}_{\overline{f}^1} g_2] = [\overline{f}^1, ad^{n_1-1}_{\overline{f}^2} g_2] =$$

$$= ad^{n_1}_{\overline{f}^2} g_2 + y_{n+n_1} [\frac{\partial}{\partial y_{n+n_1-1}}, ad^{n_1-1}_{\overline{f}^2} g_2]$$

which, thanks to the former lemma, is equal to

$$ad^{n_1}_{\overline{f}^2} g_2 + y_{n+n_1} [g_1, g_2]$$

Notice that $g_1, g_2 \in D^1_{n_1} \subset D^1_{n_1+n_2-1}$. Therefore, due to the involutivity of this distribution, $ad^{n_1}_{\overline{f}^1} g_2$ can be replaced by $ad^{n_1}_{\overline{f}^2} g_2$ in $D^1_{n_1+n_2}$.

Lemma 12

$$D^2_{n_2+i} =< \frac{\partial}{\partial y_{n+n_1-1}}, \dots, \frac{\partial}{\partial y_{n+n_1-n_2-i-1}}, \frac{\partial}{\partial y_{n+n_1+n_2-1}}, \dots, \frac{\partial}{\partial y_{n+n_1+1}}, ad^i_{\overline{f}^2} g_2 >$$

Proof: This proof is equal to the one made for the first of these lemmas.
Summarizing,

$$D^2_i \oplus < \frac{\partial}{\partial y_{n+n_1}}, \frac{\partial}{\partial y_{n+n_1+n_2}} >= D^1_{i-1} \quad \forall i > n_2$$

Since the former equality has also been proven in the case $i \neq n_2$,

$$D^2_i \oplus < \frac{\partial}{\partial y_{n+n_1}}, \frac{\partial}{\partial y_{n+n_1+n_2}} >= D^1_{i-1} \quad \forall i$$

Therefore, since the hypotheses of lemma 5 are satisfied, the static feedback linearizability conditions are the same for both systems. ■

Corollary 4 *If a system is linearizable by prolongation of u_1 n_1 times and u_2 n_2 times (with $n_1 \geq n_2$), then the system is linearizable by prolongation of u_1 $n_1 - n_2$ times.*

Proof: It is the result of applying the former proposition n_2 times. ■

Thus, in the following, prolongations by derivatives of just one input will be taken into consideration for two input systems. The same proof can be done for a system with m inputs, where only $m - 1$ have to be prolonged.

5.2 Main result

The following theorem, which establishes a necessary and sufficient condition for the existence of prolongations, also provides a finite algorithm to decide whether or not a system is linearizable by prolongations.

Theorem 4 *The system*

$$\dot{x} = f(x) + g_1(x)u_1 + g_2(x)u_2 \qquad x \in R^n$$

is linearizable by prolongations if and only if one of the following systems is static feedback linearizable:

$$\Sigma_k : \begin{cases} \dot{x} &= f(x) + g_1(x)y_{n+1} + g_2(x)w_2 \\ \dot{y}_{n+j} &= y_{n+j+1} \qquad\qquad \forall j = 1,\dots,k-1 \\ \dot{y}_{n+k} &= w_1 \end{cases}$$

or

$$\left\{ \left(\overline{\Sigma}_k : \begin{array}{rcl} \dot{x} &=& f(x) + g_2(x)y_{n+1} + g_1(x)w_1 \\ \dot{y}_{n+j} &=& y_{n+j+1} \qquad\qquad \forall j = 1,\dots,k-1 \\ \dot{y}_{n+k} &=& w_2 \end{array} \right) \right.$$

where $k = 1,\dots,2n-3$ *and*

$$y_{n+j} = u_1^{(j-1)} \quad j = 1,\dots,k$$

$$(\textit{or, respectively})\; y_{n+j} = u_2^{(j-1)} \quad j = 1,\dots,k$$

are the new state variables and

$$w_1 = u_1^{(k)} \qquad w_2 = u_2 \qquad \left(\textit{respectively } w_1 = u_1 \qquad w_2 = u_2^{(k)}\right)$$

are the new inputs.

Proof: It will be proven that the static feedback linearizability conditions for the system:

$$\Sigma_{2n-3} : \begin{cases} \dot{x} &= f(x) + g_1(x)y_{n+1} + g_2(x)w_2 \\ \dot{y}_{n+j} &= y_{n+j+1} \qquad\qquad \forall j = 1,\dots,2n-4 \\ \dot{y}_{3n-3} &= w_1 \end{cases}$$

and the static feedback linearizability conditions for the system:

$$\Sigma_l : \begin{cases} \dot{x} &= f(x) + g_1(x)y_{n+1} + g_2(x)w_2 \\ \dot{y}_{n+j} &= y_{n+j+1} \qquad\qquad \forall j = 1,\dots,l-1 \\ \dot{y}_{n+l} &= w_1 \end{cases}$$

with $l > 2n-3$ are equivalent. So, this being proven, if a system is linearizable by prolongations adding l derivatives of u_1 ($l > 2n-3$), then it will also be linearizable by prolongations adding

only $2n-3$ derivatives of u_1, and, obviously, the same fact occurs with u_2. Therefore, a finite algorithm for checking the linearization by prolongations will be to check the static feedback linearizability conditions for Σ_k and $\overline{\Sigma}_k$, $\forall k = 1, \ldots, 2n-3$. As the same proof is valid for both types of prolongations (resulting from adding derivatives of u_1 or u_2), it will be proven just once, in this case for prolongations of u_1. The details for the prolongations by derivation of u_2 can be rewritten in the following proof by changing u_1 by u_2, and viceversa.
So, let

$$f^{2n-3}(x,y) = f(x) + g_1(x)y_{n+1} + \sum_{j=1}^{2n-4} y_{n+j+1}\frac{\partial}{\partial y_{n+j}}$$

be the drift associated with the system Σ_{2n-3}, and let

$$g_1^{2n-3}(x,y) = \frac{\partial}{\partial y_{3n-3}} \quad g_2^{2n-3}(x,y) = g_2(x)$$

be the its control fields.
Σ_{2n-3} is static feedback linearizable if and only if the following distributions are involutive and constant rank:

$$\begin{aligned} D_0^{2n-3} &= < \tfrac{\partial}{\partial y_{3n-3}}, g_2 > \\ D_j^{2n-3} &= < \tfrac{\partial}{\partial y_{3n-3}}, \ldots, ad^j_{f^{2n-3}}\tfrac{\partial}{\partial y_{3n-3}}, g_2, \ldots, ad^j_{f^{2n-3}} g_2 > \end{aligned}$$

Denote

$$\eta_k = ad^k_{f^{2n-3}} g_2 \quad \forall k \geq 0$$

The following lemmas clarify what $\{\eta_k \;\; k \geq 0\}$ are.

Lemma 13 $\eta_k \in S =< \frac{\partial}{\partial x} >$

Proof: It is proven by induction. For $i = 1$,

$$\eta_1 = [f + g_1 y_{n+1} + \sum_{j=1}^{2n-4} y_{n+j+1}\frac{\partial}{\partial y_{n+j}}, g_2] = [f, g_2] + y_{n+1}[g_1, g_2] \in S$$

Assuming $\eta_i \in S$, $i \leq k$,

$$\eta_{k+1} = [f + g_1 y_{n+1} + \sum_{j=1}^{2n-4} y_{n+j+1}\frac{\partial}{\partial y_{n+j}}, \eta_k] =$$

$$= \left(f + y_{n+1}g_1 + \sum_{j=1}^{2n-4} y_{n+j+1}\frac{\partial}{\partial y_{n+j}} \right)(\eta_k) - \eta_k(f + y_{n+1}g_1) \in S$$

thanks to the induction hypothesis. ∎

Therefore, the maximum independent number of these functions is n. In other words

$$\eta_n \in < \eta_0, \ldots, \eta_{n-1} >$$

Lemma 14 *η_k depends only on the variables $x, y_{n+1}, \ldots, y_{n+k}$.*

Proof: As in the previous lemma,

$$\eta_1 = [f, g_2] + y_{n+1}[g_1, g_2]$$

On the other hand,

$$\eta_{k+1} = [f^{2n-3}, \eta_k(x, y_{n+1}, \ldots, y_{n+k}]$$

where the induction hypothesis has been applied. Therefore, it is clear that

$$\eta_{k+1} = [f + y_{n+1}g_1 + \sum_{j=1}^{k-1} y_{n+j+1}\frac{\partial}{\partial y_{n+j}}, \eta_k] + [y_{n+k+1}\frac{\partial}{\partial y_{n+k}}, \eta_k]$$

which depends only on the variables $x, y_{n+1}, \ldots, y_{n+k+1}$. ∎

Let us now compute $ad^j_{f^{2n-3}}\frac{\partial}{\partial y_{3n-3}}$.

Lemma 15 *1.*

$$ad^j_{f^{2n-3}}\frac{\partial}{\partial y_{3n-3}} = (-1)^j\frac{\partial}{\partial y_{3n-3-j}} \quad \forall j \leq 2n-4$$

2.

$$ad^{2n-3}_{f^{2n-3}}\frac{\partial}{\partial y_{3n-3}} = -g_1$$

Proof:

1. For $j = 1$,

$$ad_{f^{2n-3}}\frac{\partial}{\partial y_{3n-3}} = [y_{3n-3}\frac{\partial}{\partial y_{3n-2}}, \frac{\partial}{\partial y_{3n-3}}] = -\frac{\partial}{\partial y_{3n-2}}$$

Assuming the equality is true up to j,

$$ad^{j+1}_{f^{2n-3}}\frac{\partial}{\partial y_{3n-3}} = [f^{2n-3}, ad^j_{f^{2n-3}}\frac{\partial}{\partial y_{3n-3}}] = [f^{2n-3}, (-1)^j\frac{\partial}{\partial y_{3n-3-j}}]$$

where the induction hypothesis has been applied. This Lie bracket is equal to

$$[f + g_1y_{n+1} + \sum_{j=1}^{2n-4} y_{n+j+1}\frac{\partial}{\partial y_{n+j}}, (-1)^j\frac{\partial}{\partial y_{3n-3-j}}] = (-1)^{j+1}\frac{\partial}{\partial y_{3n-3-j-1}}]$$

2. From the former proof we have

$$ad^{2n-4}_{f^{2n-3}}\frac{\partial}{\partial y_{3n-3}} = \frac{\partial}{\partial y_{n+1}}$$

Therefore,

$$ad^{2n-3}_{f^{2n-3}}\frac{\partial}{\partial y_{3n-3}} = [f^{2n-3}, \frac{\partial}{\partial y_{n+1}}] = -g_1$$

■

It has been pointed out that $\eta_n \in <\eta_0,\dots,\eta_{n-1}>$. But n may not be is not the least integer satisfying such property. So, let us define

$$r = \min\{k|\eta_k \in <\eta_0,\dots,\eta_{k-1}>\}$$

There are two possibilities for these distributions:

1. $r = n$
2. $r < n$

1. If $r = n$, the distribution D_{n-1}^{2n-3} is equal to

$$<\eta_0,\dots,\eta_{n-1},\frac{\partial}{\partial y_{3n-3}},\dots,\frac{\partial}{\partial y_{2n-2}}>$$

   Thus, this distribution being involutive, D_k^{2n-3} are involutive for all $k \geq n$. The reason for this fact is that

$$D_k^{2n-3} = <\eta_0,\dots,\eta_{n-1},\frac{\partial}{\partial y_{3n-3}},\dots,\frac{\partial}{\partial y_{3n-3-k}}>$$

   and the Lie brackets

$$[\frac{\partial}{\partial y_{3n-3-i}},\eta_j] \in S = <\eta_0,\dots,\eta_{n-1}> \quad \forall i \leq k, \quad \forall j \leq n-1$$

   So, system Σ_{2n-3} is linearizable by static feedback (in the case $r = n$) if, and only if,

$$[\eta_j,\eta_k] \in <\eta_0,\dots,\eta_k> \quad \forall j < k, \quad \forall k = 1,\dots,n-2$$

   Note that

$$[\eta_j,\frac{\partial}{\partial y_{3n-3-k}}] = 0 \quad \forall j < k, \quad \forall k = 1,\dots,n-2$$

   because η_j depends upon the variables $x, y_{n+1},\dots,y_{n+j}$ and $n+j \leq 2n-2$, while $3n-3-k \geq 2n-1$.

2. If $r < n$, the conditions for the distributions of the system Σ_{2n-3} to be involutive are

   - $\forall k \leq r$,

$$D_k^{2n-3} = <\eta_0,\dots,\eta_k,\frac{\partial}{\partial y_{3n-3}},\dots,\frac{\partial}{\partial y_{3n-3-k}}>$$

     Conditions: $[\eta_j,\eta_k] \in <\eta_0,\dots,\eta_k>$, $\forall j < k$. The reason is the same as in the former case.

- For $k = r+1, \ldots, 2n-r-4$,

$$D_k^{2n-3} =< \eta_0, \ldots, \eta_r, \frac{\partial}{\partial y_{3n-3}}, \ldots, \frac{\partial}{\partial y_{3n-3-k}} >$$

have no conditions to check. Again, the reason is the dependence of η_j on

$$x, y_{n+1}, \ldots, y_{n+j}$$

and the fact

$$n+j \leq n+r < n+r+1 \leq 3n-3-(2n-r-4)$$

- The following distributions to be studied are

$$D_{2n-r-3+k}^{2n-3} =< D_{2n-r-4}^{2n-3}, \frac{\partial}{\partial y_{n+r}}, \ldots, \frac{\partial}{\partial y_{n+r-k}} >$$

with $k = 0, \ldots, r-1$. The involutivity conditions are

$$[\frac{\partial}{\partial y_{n+r-k}}, \eta_j] \in < \eta_0, \ldots, \eta_r > \quad \forall r \geq j \geq r-k$$

Particularizing for $k = r-1$,

$$D_{2n-4}^{2n-3} =< \eta_0, \ldots, \eta_r, \frac{\partial}{\partial y_{3n-3}}, \ldots, \frac{\partial}{\partial y_{n+1}} <$$

- In order to establish the elements belonging to D_{2n-3+i}^{2n-3}, with $i \geq 0$, we need the following lemma:

Lemma 16

$$D_{2n-3+i}^{2n-3} =< D_{2n-4+i}^{2n-3}, ad_f^i g_1 > \quad \forall i \geq 0$$

Proof: when $i = 0$,

$$[f^{2n-3}, \frac{\partial}{\partial y_{n+1}}] = g_1 = ad_f^0 g_1$$

Thus,

$$D_{2n-3}^{2n-3} =< D_{2n-4}^{2n-3}, ad_f^0 g_1 >$$

When $i \geq 1$, in fact we will prove, by induction, the following equality

$$ad_{f^{2n-3}}^i g_1 = ad_f^i g_1 + D_{2n+i-4}^{2n-3} \tag{5.1}$$

where $ad_f^i g_1 + D_{2n+i-4}^{2n-3}$ means

$$ad_f^i g_1 + h \quad h \in D_{2n+i-4}^{2n-3}$$

For $i = 1$,

$$[f^{2n-3}, g_1] = [f, g_1] + y_{n+1}[g_1, g_1] = [f, g_1]$$

For $i > 1$,

$$ad^i_{f^{2n-3}} g_1 = [f + g_1 y_{n+1} + y_{n+2}\frac{\partial}{\partial y_{n+1}} + \ldots + y_{3n-3}\frac{\partial}{\partial y_{3n-4}}, ad_{f^{2n-3}} g_1]$$

Using the induction hypothesis, this expression becomes:

$$[f + g_1 y_{n+1} + y_{n+2}\frac{\partial}{\partial y_{n+1}} + \ldots + y_{3n-3}\frac{\partial}{\partial y_{3n-4}}, ad_f^{i-1} + D^{2n-3}_{2n+i-5}] =$$

$$[f + g_1 y_{n+1} + y_{n+2}\frac{\partial}{\partial y_{n+1}} + \ldots + y_{3n-3}\frac{\partial}{\partial y_{3n-4}}, ad_f^{i-1}] + D^{2n-3}_{2n+i-4}$$

by construction of the distributions D^{2n-3}_{2n+i-5} and D^{2n-3}_{2n+i-4}. Moreover, the elements

$$g_1, \frac{\partial}{\partial y_{n+1}}, \ldots, \frac{\partial}{\partial y_{3n-4}}$$

belong to D^{2n-3}_{2n+i-4} (in fact, they belong to D^{2n-3}_{2n-3}). Therefore, because of the involutivity of D^{2n-3}_{2n+i-4},

$$[g_1 y_{n+1} + y_{n+2}\frac{\partial}{\partial y_{n+1}} + \ldots + y_{3n-3}\frac{\partial}{\partial y_{3n-4}}, ad_f^{i-1} g_1] \in D^{2n-3}_{2n+i-4}$$

The proof of 5.1 ends by using this fact and $[f, ad_f^{i-1} g_1] = ad^i_f g_1$. ∎

This concludes the study of system Σ_{2n-3}.

Now, the system Σ_{2n-2} will be studied. Let

$$f^{2n-2} = f + g_1 y_{n+1} + \sum_{j=1}^{2n-3} y_{n+j+1}\frac{\partial}{\partial y_{n+j}}$$

be the drift associated with the system Σ_{2n-2}, and let

$$g_1^{2n-2} = \frac{\partial}{\partial y_{3n-2}} \quad g_2^{2n-2} = g_2$$

be its control fields. The distributions associated with this system are

- $\forall k \leq r$,

$$D_k^{2n-2} =< \eta_0, \ldots, \eta_k, \frac{\partial}{\partial y_{3n-2}}, \ldots, \frac{\partial}{\partial y_{3n-2-k}} >$$

So, the equality

$$D_k^{2n-2} \oplus \frac{\partial}{\partial y_{3n-3-k}} = D_k^{2n-3} \oplus < \frac{\partial}{\partial y_{3n-2}} >$$

holds, and the hypotheses of lemma 5 are fulfilled. Therefore, the involutivity conditions are the same for D_k^{2n-2} and for D_k^{2n-3}, with $k \leq r$.

- For $k = r, \ldots, 2n-r-4$,

$$D_{k+1}^{2n-2} =< \eta_0, \ldots, \eta_r, \frac{\partial}{\partial y_{3n-2}}, \ldots, \frac{\partial}{\partial y_{3n-3-k}} >$$

  satisfies the equality

$$D_{k+1}^{2n-2} = D_k^{2n-3} \oplus < \frac{\partial}{\partial y_{3n-2}} >$$

  As the hypotheses of lemma 5 are also fulfilled, the involutivity conditions are the same for D_{k+1}^{2n-2} and for D_k^{2n-3}, with $2n - r + 4 \geq k \geq r$.

- In fact, the former equality

$$D_{k+1}^{2n-2} = D_k^{2n-3} \oplus < \frac{\partial}{\partial y_{3n-2}} >$$

  is also valid for any k greater or equal than r. Thus, the static feedback linearizability conditions are the same for both systems when $r < n$.

If $r = n$, the distribution D_{n-1}^{2n-2} is equal to

$$< \eta_0, \ldots, \eta_{n-1}, \frac{\partial}{\partial y_{3n-2}}, \ldots, \frac{\partial}{\partial y_{2n-1}} >$$

Thus, this distribution being involutive, D_k^{2n-2} are involutive for all $k \geq n$. The reason for this fact is that

$$D_k^{2n-2} =< \eta_0, \ldots, \eta_{n-1}, \frac{\partial}{\partial y_{3n-2}}, \ldots, \frac{\partial}{\partial y_{3n-2-k}} >$$

and the Lie brackets

$$[\frac{\partial}{\partial y_{3n-2-i}}, \eta_j] \in S =< \eta_0, \ldots, \eta_{n-1} > \quad \forall i \leq k, \quad \forall j \leq n-1$$

So, system Σ_{2n-2} is linearizable by static feedback (in the case $r = n$) if, and only if,

$$[\eta_j, \eta_k] \in < \eta_0, \ldots, \eta_k > \quad \forall j < k, \quad \forall k = 1, \ldots, n-2$$

which are the same conditions as those for Σ_{2n-3}. ∎

REMARKS

1. The proof required in order to show that Σ_{2n-3} is static feedback linearizable if, and only if, Σ_l ($\forall l > 2n-3$) is static feedback linearizable, is the same using the fact

$$D_{k+i}^{l} = D_{k+i-h}^{2n-3} \oplus < \frac{\partial}{\partial y_{3n-2}}, \ldots, \frac{\partial}{\partial y_{3n-3+h}} > \quad \forall i \geq r$$

   where $h = l - (2n-3)$, and the lemma 5. This means that it is not necessary to add more than $2n-3$ derivatives of the input in order to check whether or not a system is linearizable by prolongations. In other words, it is sufficient to check the static feedback linearizability conditions of Σ_1,...,Σ_{2n-3} (and $\overline{\Sigma}_1$,...,$\overline{\Sigma}_{2n-3}$) in order to check the linearizability by prolongations of a certain system.

2. Let us compare the static feedback linearizability conditions of Σ_{2n-3} and Σ_{2n-4}. In order to be involutive, the distribution

$$D_{n-2}^{2n-4} =< \frac{\partial}{\partial y_{3n-4}}, \eta_0, \ldots, \frac{\partial}{\partial y_{2n-2}}, \eta_{n-2} >$$

needs an extra condition (which is not necessary for the system Σ_{2n-3}):

$$[\frac{\partial}{\partial y_{2n-2}}, \eta_{n-2}] \in D_{n-2}^{2n-4}$$

or, in other words,

$$[\frac{\partial}{\partial y_{2n-2}}, \eta_{n-2}] \in < \eta_0, \ldots, \eta_{n-2} >$$

This is the reason why, in general, $2n-4$ or any smaller number cannot be a bound of the number of derivatives added to the original system.

5.3 Examples

1. As an application of the former theorem, a static feedback linearizable prolongation will be sought in the following system:

$$\left\{ \begin{array}{lcl} \dot{x}_1 &=& x_2 \\ \dot{x}_2 &=& u_1 \sin x_5 - u_2 \epsilon \cos x_5 \\ \dot{x}_3 &=& x_4 \\ \dot{x}_4 &=& -1 + u_1 \cos x_5 + u_2 \epsilon \sin x_5 \\ \dot{x}_5 &=& x_6 \\ \dot{x}_6 &=& u_2 \end{array} \right. \tag{5.2}$$

This system comes from the Vertical Take Off and Landing (VTOL) aircraft model ([30], [23]), a model of a mechanical system with two inputs, whose evolution is restricted in the vertical plane. The original equations are:

$$\left\{ \begin{array}{l} \ddot{x} = u_1 \sin\theta - u_2 \epsilon \cos\theta \\ \ddot{y} = u_1 \cos\theta + u_2 \epsilon \sin\theta - 1 \\ \ddot{\theta} = u_2 \end{array} \right. \tag{5.3}$$

The changes made in 5.3 to obtain 5.2 are

$$x_1 = x \quad x_2 = \dot{x} \quad x_3 = y \quad x_4 = \dot{y} \quad x_5 = \theta \quad x_6 = \dot{\theta}$$

To apply the algorythm explained above, the input u_2 is derivated up to $2n = 12$ times. And the static feedback linearizability of the systems

$$\Sigma_k : \begin{cases} \dot{y}_1 &= & y_2 & & \\ \dot{y}_2 &= & -y_7\epsilon\cos y_5 & + & v_1 \sin y_5 \\ \dot{y}_3 &= & y_4 & & \\ \dot{y}_4 &= & -1 + y_7\epsilon\sin y_5 & + & v_1\cos y_5 \\ \dot{y}_5 &= & y_6 & & \\ \dot{y}_6 &= & y_7 & & \\ \vdots & & & & \\ \dot{y}_{6+k} &= & & & \qquad v_2 \end{cases}$$

is checked ($\forall k = 1, \ldots, 12$). A straightforward computation shows that Σ_1, Σ_2, Σ_3 does not satisfy the static feedback linearizability conditions.

System Σ_4. The distributions

$$D_0^4 =< \begin{pmatrix} 0 \\ \sin y_5 \\ 0 \\ \cos y_5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} >$$

$$D_1^4 =< \begin{pmatrix} 0 \\ \sin y_5 \\ 0 \\ \cos y_5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -\sin y_5 \\ y_6\cos y_5 \\ -\cos y_5 \\ -y_6\sin y_5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} >$$

and $D_2^4 =$

$$< \begin{pmatrix} 0 \\ \sin y_5 \\ 0 \\ \cos y_5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -\sin y_5 \\ y_6 \cos y_5 \\ -\cos y_5 \\ -y_6 \sin y_5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2y_6 \cos y_5 \\ -y_6^2 \sin y_5 + y_7 \cos y_5 \\ 2y_6 \sin y_5 \\ -y_6^2 \cos y_5 - y_7 \sin y_5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} >$$

are involutive because the Lie brackets between two of their elements are zero. The following distribution is

$$D_3^4 =< D_2^4, \begin{pmatrix} 3y_6^2 \sin y_5 - 3y_7 \cos y_5 \\ (-y_6^2 \cos y_5 - y_7 \sin y_5)y_6 - 2y_6 y_7 \sin y_5 + y_8 \cos y_5 \\ 3y_6^2 \cos y_5 + 3y_7 \sin y_5 \\ (y_6^2 \sin y_5 - y_7 \cos y_5)y_6 - 2y_6 y_7 \cos y_5 - y_8 \sin y_5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} >$$

It is also involutive because

$$S =< \eta_0 = \begin{pmatrix} 0 \\ \sin y_5 \\ 0 \\ \cos y_5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \eta_1 = \begin{pmatrix} -\sin y_5 \\ y_6 \cos y_5 \\ -\cos y_5 \\ -y_6 \sin y_5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \eta_2 = \begin{pmatrix} -2y_6 \cos y_5 \\ -y_6^2 \sin y_5 + y_7 \cos y_5 \\ 2y_6 \sin y_5 \\ -y_6^2 \cos y_5 - y_7 \sin y_5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\eta_3 = \begin{pmatrix} 3y_6^2 \sin y_5 - 3y_7 \cos y_5 \\ (-y_6^2 \cos y_5 - y_7 \sin y_5)y_6 - 2y_6 y_7 \sin y_5 + y_8 \cos y_5 \\ 3y_6^2 \cos y_5 + 3y_7 \sin y_5 \\ (y_6^2 \sin y_5 - y_7 \cos y_5)y_6 - 2y_6 y_7 \cos y_5 - y_8 \sin y_5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} >=$$

$$\{(a_1, a_2, a_3, a_4, 0, 0, 0, 0, 0, 0) \quad \text{where } a_i \text{ are functions of} \quad y_1, \ldots, y_{10}\}$$

and all the Lie brackets between two elements of D_3^4 belong to S. The next distribution is

$$D_4^4 =< D_3^4, \begin{pmatrix} 0 \\ -\epsilon \cos y_5 \\ 0 \\ \epsilon \sin y_5 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} >$$

because

$$\left[\begin{pmatrix} y_2 \\ -\epsilon y_7 \cos y_5 \\ y_4 \\ -1 + \epsilon y_7 \sin y_5 \\ y_6 \\ y_7 \\ y_8 \\ y_9 \\ y_{10} \\ 0 \end{pmatrix}, \eta_3 \right] \in S$$

and also the Lie brackets between two of its elements are in S.
Finally

$$D_5^4 =< D_4^4, \begin{pmatrix} \epsilon \cos y_5 \\ \epsilon y_6 \sin y_5 \\ -\epsilon \sin y_5 \\ \epsilon y_6 \cos y_5 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} >= R^{10}$$

which is obviously involutive.

2. Another interesting example, very similar to the VTOL, is exhibited by a planar model for the ducted fan ([46],[47] (veure ST)), given by:

$$\begin{array}{rcr} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & \frac{u_1}{m}\cos x_5 - \frac{u_2}{m}\sin x_5 \\ \dot{x}_3 & = & x_4 \\ \dot{x}_4 & = & \frac{u_1}{m}\sin x_5 + \frac{u_2}{m}\cos x_5 - mg \\ \dot{x}_5 & = & x_6 \\ \dot{x}_6 & = & \frac{r}{J}u_1 \end{array}$$

where m, J and r are constants. This system was thought to be not linearizable by prolongations ([59]). However, we are in fact able to prove that it is linearizable by prolongations. Take the prolongation coming from adding four derivatives of u_1 and zero derivatives of u_2:

$$\begin{array}{rcr} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & \frac{x_7}{m}\cos x_5 - \frac{v_2}{m}\sin x_5 \\ \dot{x}_3 & = & x_4 \\ \dot{x}_4 & = & \frac{x_7}{m}\sin x_5 + \frac{v_2}{m}\cos x_5 - mg \\ \dot{x}_5 & = & x_6 \\ \dot{x}_6 & = & \frac{r}{J}x_7 \\ \dot{x}_7 & = & x_8 \\ \dot{x}_8 & = & x_9 \\ \dot{x}_9 & = & v_1 \end{array}$$

The distributions associated with this prolonged system are

$$D_0 =< \frac{\partial}{\partial x_10}, \eta_0 >$$

$$D_1 =< D_0, \frac{\partial}{\partial x_9}, \eta_1 >$$

$$D_2 =< D_1, \frac{\partial}{\partial x_8}, \eta_2 >$$

where

$$\eta_0 = -\frac{\sin x_5}{m}\frac{\partial}{\partial x_2} + \frac{\cos x_5}{m}\frac{\partial}{\partial x_4}$$

$$\eta_1 = \frac{\sin x_5}{m}\frac{\partial}{\partial x_1} - \frac{x_6 \cos x_5}{m}\frac{\partial}{\partial x_2} - \frac{\cos x_5}{m}\frac{\partial}{\partial x_3} - \frac{x_6 \sin x_5}{m}\frac{\partial}{\partial x_4}$$

$$\eta_2 = \frac{2x_6 \cos x_5}{m}\frac{\partial}{\partial x_1} + (\frac{x_6^2 \sin x_5}{m} - \frac{r x_7 \cos x_5}{mJ})\frac{\partial}{\partial x_2} + \frac{2x_6 \sin x_5}{m}\frac{\partial}{\partial x_3} - (\frac{x_6^2 \cos x_5}{m} + \frac{r x_7 \sin x_5}{mJ})\frac{\partial}{\partial x_4}$$

Remark that all the Lie brackets between their elements are zero. Therefore, D_0, D_1, D_2 are involutive.

$$D_3 =< D_2, \frac{\partial}{\partial x_7}, \eta_3 >$$

where η_3 is such that

$$< \eta_0, \eta_1, \eta_2, \eta_3 >=< \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4} >$$

Notice that the Lie brackets between the elements in D_3 belong to $< eta_0, \eta_1, \eta_2, \eta_3 >$. So, D_3 is also involutive.

$$D_4 =< D_3, \frac{\cos x_5}{m}\frac{\partial}{\partial x_2} + \frac{\sin x_5}{m}\frac{\partial}{\partial x_4} + \frac{r}{J}\frac{\partial}{\partial x_6} >$$

And, because of

$$[\eta_i, \frac{\cos x_5}{m}\frac{\partial}{\partial x_2} + \frac{\sin x_5}{m}\frac{\partial}{\partial x_4} + \frac{r}{J}\frac{\partial}{\partial x_6}] \in < \eta_0, \eta_1, \eta_2, \eta_3 > \quad \forall i = 0, 1, 2, 3$$

D_4 is also involutive. Finally, $D_5 = R^{10}$. That is to say, the static feedback linearizability conditions are fulfilled for the prolonged system, or, in other words, the planar ducted fan is linearizable by prolongations.

3. This example shows the sharpness of the bound $2n - 3$.

$$\left\{ \begin{array}{lllll} \dot{x}_1 & = & x_2 & x_2 u_1 & \\ \dot{x}_2 & = & x_3 & x_1 u_1 & \\ \dot{x}_k & = & x_{k+1} & & \qquad k = 3, \ldots, n-2 \\ \dot{x}_{n-1} & = & & \quad u_2 & \\ \dot{x}_n & = & & x_{n-1} u_1 & \end{array} \right.$$

The system is clearly not static feedback linearizable since

$$D_0 =< x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} + x_{n-1} \frac{\partial}{\partial x_n}, \frac{\partial}{\partial x_{n-1}} >$$

is not involutive.

It is also easy to check that the system is not linearizable by prolongation of u_2: let

$$\overline{f} = \sum_{i=1}^{n-1} x_{i+1}\frac{\partial}{\partial x_i} + y_{n+1}\frac{\partial}{\partial x_{n-1}} + \sum_{j=1}^{r-1} y_{n+j+1}\frac{\partial}{\partial y_{n+j}}$$

be the drift of the system 3 prolonged with r derivations of u_2, and let

$$\overline{g}_1 = x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2} + x_{n-1}\frac{\partial}{\partial x_n} \quad \overline{g}_2 = \frac{\partial}{\partial y_{n+r}}$$

be its corresponding input fields. Then,

$$\overline{D}_1 =< \frac{\partial}{\partial y_{n+r}}, \frac{\partial}{\partial y_{n+r-1}}, \eta_0, \eta_1 >$$

where

$$\eta_0 = \overline{g}_1 = x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2} + x_{n-1}\frac{\partial}{\partial x_n}$$

and

$$\eta_1 = [\overline{f}, \overline{g}_1] = x_2\frac{\partial}{\partial x_2} + (x_3 - x_1)\frac{\partial}{\partial x_1} + y_{n+1}\frac{\partial}{\partial x_n}$$

Therefore,

$$[\eta_0, \eta_1] = -2x_2\frac{\partial}{\partial x_1} + (2x_1 - x_3)\frac{\partial}{\partial x_2} \notin \overline{D}_1$$

Hence, $\overline{D}_1$ is not involutive.

Finally, let us try with a r-prolongation of u_1. Let

$$\overline{f} = \sum_{i=1}^{n-1} x_{i+1}\frac{\partial}{\partial x_i} + y_{n+1}x_2\frac{\partial}{\partial x_1} + y_{n+1}x_1\frac{\partial}{\partial x_2} + y_{n+1}x_{n-1}\frac{\partial}{\partial x_n} + \sum_{j=1}^{r-1} y_{n+j+1}\frac{\partial}{\partial y_{n+j}}$$

be the drift of such a prolongation, and

$$\overline{g}_1 = \frac{\partial}{\partial y_{n+r}} \quad \overline{g}_2 = \frac{\partial}{\partial x_{n-1}}$$

its corresponding input fields. The distributions associated with this system are:

$$\overline{D}_0 =< \frac{\partial}{\partial y_{n+r}}, \frac{\partial}{\partial x_{n-1}} >$$

which is involutive.

$1 \leq i \leq n-3$,

$$\overline{D}_i =< \frac{\partial}{\partial y_{n+r}}, \ldots, \frac{\partial}{\partial y_{n+r-i}}, \frac{\partial}{\partial x_{n-1}}, \frac{\partial}{\partial x_{n-2}} + y_{n+1}\frac{\partial}{\partial x_n}, \ldots, \frac{\partial}{\partial x_{n-i-1}} + (-1)^{i-1}y_{n+i}\frac{\partial}{\partial x_n} >$$

are involutive if, and only if, $n+r-i > n+i$ or, equivalently, if, and only if, $r > 2i$.

$$\overline{D}_{n-2} = < \overline{D}_{n-3}, \frac{\partial}{\partial y_{r+2}}, (1+y_{n+1})\frac{\partial}{\partial x_1} + (-1)^{n-3} y_{2n-2}\frac{\partial}{\partial x_n} > \quad 1 \leq i \leq n-3$$

is involutive if, and only if, $r+2 > 2n-2$; that is to say, if, and only if, $r > 2n-4$ $(r \geq 2n-3)$. On the other hand, the main result of this chapter states that $r \leq 2n-3$. Therefore, since

$$< \frac{\partial}{\partial x} > \subset \overline{D}_{n-1}$$

there are no more conditions to check. Hence, the system 3 is linearizable by adding exactly $2n-3$ derivatives of the input u_1. So, in general, the bound $2n-3$ cannot be improved.

Chapter 6

Improvement of the bounds for 3-input systems

In Chapter 7, a bound in the number of integrators needed for the linearization of a nonlinear control system will be obtained. This bound is an improvement on the bounds existent in the literature ([59]) for systems with 4 or more inputs. However, for systems with 3 inputs the bound still needs to be better. This chapter, therefore, is devoted to the search for an inprovement on the bound for three-input systems obtained in Chapter 7.

6.1 Main Result

The following theorem, which states a necessary and sufficient condition for the linearizability by prolongations of a three input system, also provides an algorithm for determining whether or not a 3-input system is linearizable by prolongations, and improves on the bounds that have appeared before in the literature.
Theorem: The system Σ defined by

$$\dot{x} = f(x) + \sum_{i=1}^{3} g_i(x)u_i$$

is linearizable by prolongations if, and only if, at least one of the following systems $\Sigma_{k_1}^{k_2}$ is static feedback linearizable. $\Sigma_{k_1}^{k_2}$ is given by

$$\dot{x} = f(x) + g_{h_1}(x)y_{n+1} + g_{h_2}(x)y_{n+k_1+1} + g_{h_3}(x)w_3$$

$$\begin{array}{lll} \dot{y}_{n+j} & = y_{n+j+1}; & j = 1, \ldots, k_1 - 1 \\ \dot{y}_{n+k_1} & = w_1 & \\ \dot{y}_{n+k_1+l} & = y_{n+k_1+l+1}; & l = 1, \ldots, k_2 - 1 \\ \dot{y}_{n+k_1+k_2} & = w_2 & \end{array}$$

$k_1 \in \{1, \dots, 2r+1\}$, $k_2 \in \{1, \dots, 2n-2+r\}$. The new state variables y_s and inputs w_j are related to the old ones by

$$\begin{array}{lll} y_{n+j} & = u_{h_1}^{(j-1)} & j = 1, \dots, k_1 \\ y_{n+k_1+l} & = u_{h_2}^{(l-1)} & l = 1, \dots, k_2 \\ w_1 & = u_{h_1}^{(k_1)} & \\ w_2 & = u_{h_2}^{(k_2)} & \\ w_3 & = u_{h_3} & \end{array}$$

being $\{h_1, h_2, h_3\} = \{1, 2, 3\}$. And, finally, r is defined in the following way: let consider the prolongation Σ_n^n. Compute the Lie brackets

$$\eta_i = ad_{\overline{f}}^i g_{h_3}$$

where $\overline{f}$ is the drift associated with the prolonged system, and define

$$r = \min\{i \in \{0, n-1\} \mid \eta_{i+1} \in < \eta_0, \dots, \eta_i >\}$$

Proof: There is no loss of generality in assuming that $h_i = i$, $i = 1, 2, 3$. The proof is based on the lemma 5 from Chapter 2, and on the justification of the following items:

1. The static feedback linearizability conditions for the system Σ_{2r+1}^{2n-2+r} and $\Sigma_{r_1}^{r_2}$ (with $r_1 \geq 2r+1$ and $r_2 \geq 2n-2+r$) are the same.
2. The static feedback linearizability conditions for the system $\Sigma_{k_1}^{2n-2+r}$ and $\Sigma_{k_1}^{r_2}$ (with $k_1 < 2r+1$ and $r_2 > 2n-2+r$) are the same.
3. The static feedback linearizability conditions for the system $\Sigma_{2r+1}^{k_2}$ and $\Sigma_{r_1}^{k_2}$ (with $k_2 < 2n-2+r$ and $r_1 > 2r+1$) are the same.

This being proven, and in order to check if a system is linearizable by prolongations, a finite algorithm can be applied: it is only necessary to check if any of the systems $\Sigma_{k_1}^{k_2}$ (with $k_1 \leq 2r+1$ and $k_2 \leq 2n-2+r$) is static feedback linearizable. Obviously, this fact must be checked for any permutation of the inputs.

1. First of all, let us study the static feedback linearizability conditions for the system Σ_{2r+1}^{2n-2+r}.
   Let
   $$\begin{array}{rcl} \overline{f} & = & f + g_1 y_{n+1} + g_2 y_{n+2r+2} + \\ & + & \sum_{j=1}^{2r} y_{n+j+1} \frac{\partial}{\partial y_{n+j}} + \sum_{l=1}^{2n-3+r} y_{n+2r+1+l+1} \frac{\partial}{\partial y_{n+2r+1+l}} \end{array}$$
   be the drift of the system Σ_{2r+1}^{2n-2+r} (x must be understood as $(y_1, \dots, y_n)$). And let
   $$\overline{g_1} = \frac{\partial}{\partial y_{n+2r+1}} \quad \overline{g_2} = \frac{\partial}{\partial y_{3n+3r-1}} \quad \overline{g_3} = g_3$$

be the vector fields associated with the new inputs.
Using the conditions stated in Chapter 2 to check whether or not a system is static feedback linearizable, the following distributions of Σ_{2r+1}^{2n-2+r} must be involutive and constant rank:

$$D_0^{2r+1,2n-2+r} =< \frac{\partial}{\partial y_{n+2r+1}}, \frac{\partial}{\partial y_{3n+3r-1}}, g_3 >$$

$$D_{i+1}^{2r+1,2n-2+r} =< D_i^{2r+1,2n-2+r}, [\overline{f}, D_i^{2r+1,2n-2+r}] >$$

for all i such that $dim\ D_i^{2r+1,2n-2+r} < 3n+3r-1$.
Let us examine these distributions in some detail. First of all, a computational lemma is stated and proven.

Lemma 17 *(a)*

$$ad_{\overline{f}}^i\left(\frac{\partial}{\partial y_{n+2r+1}}\right) = (-1)^i \frac{\partial}{\partial y_{n+2r+1-i}} \quad \forall i \leq 2r$$

(b)

$$ad_{\overline{f}}^{2r+1}\left(\frac{\partial}{\partial y_{n+2r+1}}\right) = -g_1$$

(c)

$$ad_{\overline{f}}^i\left(\frac{\partial}{\partial y_{3n+3r-1}}\right) = (-1)^i \frac{\partial}{\partial y_{3n+3r-1-i}} \quad \forall i \leq 2n-3+r$$

(d)

$$ad_{\overline{f}}^{2n-2+r}\left(\frac{\partial}{\partial y_{3n+3r-1}}\right) = (-1)^{2n-2+r} g_2$$

Proof:

(a) This part will be proven by induction. The case $i=1$:

$$ad_{\overline{f}}\left(\frac{\partial}{\partial y_{n+2r+1}}\right) = [\overline{f}, \frac{\partial}{\partial y_{n+2r+1}}] = [y_{n+2r+1}\frac{\partial}{\partial y_{n+2r}}, \frac{\partial}{\partial y_{n+2r+1}}] = -\frac{\partial}{\partial y_{n+2r}}$$

Assuming the equality is true up to i, the case $i+1$ (with $i+1 \leq 2r$):

$$ad_{\overline{f}}^{i+1}\left(\frac{\partial}{\partial y_{n+2r+1}}\right) = [\overline{f}, ad_{\overline{f}}^i\left(\frac{\partial}{\partial y_{n+2r+1}}\right)]$$

Applying the induction hypothesis, this becomes

$$[\overline{f}, (-1)^i\left(\frac{\partial}{\partial y_{n+2r+1-i}}\right)] = [y_{n+2r-i+1}\left(\frac{\partial}{\partial y_{n+2r-i}}\right), (-1)^i\left(\frac{\partial}{\partial y_{n+2r+1-i}}\right)] =$$

$$(-1)^{i+1}\left(\frac{\partial}{\partial y_{n+2r-i}}\right)$$

(b) Using the former equality

$$ad_{\overline{f}}^{2r+1}\left(\frac{\partial}{\partial y_{n+2r+1}}\right) = [\overline{f}, ad_{\overline{f}}^{2r}\left(\frac{\partial}{\partial y_{n+2r+1}}\right)] = [\overline{f}, (-1)^{2r}\left(\frac{\partial}{\partial y_{n+2r+1-2r}}\right)] =$$

$$= [\overline{f}, \frac{\partial}{\partial y_{n+1}}] = [g_1 y_{n+1}, \frac{\partial}{\partial y_{n+1}}] = -g_1$$

(c) Again, it will be proven by induction. When $i = 1$

$$ad_{\overline{f}}\left(\frac{\partial}{\partial y_{3n+3r-1}}\right) = [\overline{f}, \frac{\partial}{\partial y_{3n+3r-1}}] = [y_{3n+3r-1}\frac{\partial}{\partial y_{3n+3r-2}}, \frac{\partial}{\partial y_{3n+3r-1}}] =$$

$$-\frac{\partial}{\partial y_{3n+3r-2}}$$

Assuming the trueness of the statement up to i, the equality will be proven for $i+1$ (with $i+1 \leq 2n-3+r$).

$$ad_{\overline{f}}^{i+1}\left(\frac{\partial}{\partial y_{3n+3r-1}}\right) = [\overline{f}, ad_{\overline{f}}^{i}\left(\frac{\partial}{\partial y_{3n+3r-1}}\right)]$$

which becomes, by application of the induction hypothesis,

$$[\overline{f}, (-1)^i\left(\frac{\partial}{\partial y_{3n+3r-1-i}}\right)] = [y_{3n+3r-1-i}\left(\frac{\partial}{\partial y_{3n+3r-i-2}}\right), (-1)^i\left(\frac{\partial}{\partial y_{3n+3r-1-i}}\right)] =$$

$$(-1)^{i+1}\left(\frac{\partial}{\partial y_{3n+3r-i-2}}\right)$$

(d) Using the former equality

$$ad_{\overline{f}}^{2n-2+r}\left(\frac{\partial}{\partial y_{3n+3r-1}}\right) = [\overline{f}, ad_{\overline{f}}^{2n-3+r}\left(\frac{\partial}{\partial y_{3n+3r-1}}\right)] =$$

$$= [\overline{f}, (-1)^{2n-3+r}\left(\frac{\partial}{\partial y_{3n+3r-1-2n+3-r}}\right)] = (-1)^{r-1}[\overline{f}, \frac{\partial}{\partial y_{n+2r+2}}] =$$

$$= (-1)^{r-1}[g_2 y_{n+2r+2}, \frac{\partial}{\partial y_{n+2r+2}}] = (-1)^r g_2$$

■

Remarks:

(a) Denote $\eta_k = ad_{\overline{f}}^k g_3 \quad \forall k \leq r$. Note that

$$\eta_k \in S = < \frac{\partial}{\partial x} >$$

Proof:

$$\eta_1 = [\overline{f}, g_3] = [(f + g_1 y_{n+1} + g_2 y_{n+2r+2}), g_3] \in < \frac{\partial}{\partial x} >$$

$$\eta_{k+1} = [\overline{f}, \eta_k] =$$

$$[(f + g_1 y_{n+1} + g_2 y_{n+2r+2}) + \sum_{j=1}^{2r} y_{n+j+1}\frac{\partial}{\partial y_{n+j}} + \sum_{l=1}^{2n-3+r} y_{n+2r+1+l+1}\frac{\partial}{\partial y_{n+2r+1+l}}, \eta_k]$$

Let us recall that, by the induction hypothesis, $\eta_k \in < \frac{\partial}{\partial x} >$. Therefore,

$$\eta_{k+1} = \overline{f}(\eta_k) - \eta_k\left((f + g_1 y_{n+1} + g_2 y_{n+2r+2})\right)$$

which belongs to $< \frac{\partial}{\partial x} >$. ■

(b) η_k depends upon the variables

$$x, y_{n+1}, \dots, y_{n+k}, y_{n+2r+2}, \dots, y_{n+2r+1+k}$$

Proof:

$$\eta_1 = [\overline{f}, g_3] = [(f + g_1 y_{n+1} + g_2 y_{n+2r+2}), g_3] =$$
$$[f, g_3] + y_{n+1}[g_1, g_3] + y_{n+2r+2}[g_2, g_3]$$

which depends on $x, y_{n+1}, y_{n+2r+1+1}$. ■

$$\eta_{k+1} = [\overline{f}, H(x, y_{n+1}, \dots, y_{n+k}, y_{n+2r+2}, \dots, y_{n+2r+1+k})\frac{\partial}{\partial x}]$$

where the induction hypothesis has been applied. Therefore, it is clear that $\eta_{k+1} =$

$$[f + g_1 y_{n+1} + g_2 y_{n+2r+2} + \sum_{j=1}^{k} y_{n+j+1}\frac{\partial}{\partial y_{n+j}} + \sum_{l=1}^{k} y_{n+2r+1+l+1}\frac{\partial}{\partial y_{n+2r+1+l}}, H\frac{\partial}{\partial x}]$$

depends upon the variables $x, y_{n+1}, \dots, y_{n+k+1}, y_{n+2r+2}, \dots, y_{n+2r+1+k+1}$. ■

Let us enumerate the conditions to be checked in order for the corresponding distribution to be involutive. Henceforth, when the involutivity of D_j^h is studied, we will assume the involutivity of D_l^h, $l \leq j-1$.

(a) Distributions $D_k^{2r+1,2n-2+r}$, $k \leq r$. $D_k^{2r+1,2n-2+r} =$

$$= < \eta_0, \dots, \eta_k, \frac{\partial}{\partial y_{n+2r+1}}, \dots, \frac{\partial}{\partial y_{n+2r+1-k}}, \frac{\partial}{\partial y_{3n+3r-1}}, \dots, \frac{\partial}{\partial y_{3n+3r-1-k}} >$$

Taking into account that η_k depends on the variables

$$x, y_{n+1}, \dots, y_{n+k}, y_{n+2r+2}, \dots, y_{n+2r+1+k}$$

note that $n+k < n+2r+1-k$ and $n+2r+1+k < 3n+3r-1-k$. So, the only involutivity conditions are

$$[\eta_i, \eta_k] \in < \eta_0, \dots, \eta_k >$$

(b) Distributions $D^{2r+1,2n-2+r}_{r+k}$, $1 \leq k \leq r$. The condition $\eta_{r+1} \in <\eta_0, \ldots, \eta_r>$ implies $D^{2r+1,2n-2+r}_{r+k} =$

$$=<\eta_0, \ldots, \eta_r, \frac{\partial}{\partial y_{n+2r+1}}, \ldots, \frac{\partial}{\partial y_{n+r+1-k}}, \frac{\partial}{\partial y_{3n+3r-1}}, \ldots, \frac{\partial}{\partial y_{3n+2r-1-k}}>$$

The additional involutivity conditions are:

$$\left[\frac{\partial}{\partial y_{n+r+1-k}}, \eta_{r+1-j}\right] \in <\eta_0, \ldots, \eta_r>$$

$\forall 1 \leq j \leq k$, because η_{r+1-j} depends on, among other variables, $y_{n+r+1-k}$ (take $j = k$). Note also that $3n + 2r - 1 - k > n + 2r + 1 + k$. So, there are no more conditions except those explained above.

(c) The distribution $D^{2r+1,2n-2+r}_{2r+1}$ is spanned by

$$<\eta_0, \ldots, \eta_r, \frac{\partial}{\partial y_{n+2r+1}}, \ldots, \frac{\partial}{\partial y_{n+1}}, g_1, \frac{\partial}{\partial y_{3n+3r-1}}, \ldots, \frac{\partial}{\partial y_{3n+r-2}}>$$

If $r = n - 1$, then

$$<\eta_0, \ldots, \eta_r> = S$$

Therefore,

$$D^{2r+1,2n-2+r}_{2r+1} =<\eta_0, \ldots, \eta_r, \frac{\partial}{\partial y_{n+2r+1}}, \ldots, \frac{\partial}{\partial y_{n+1}}, \frac{\partial}{\partial y_{3n+3r-1}}, \ldots, \frac{\partial}{\partial y_{3n+r-2}}>$$

is involutive since

$$[\eta_j, \frac{\partial}{\partial y_l}] \in S \subset D^{2r+1,2n-2+r}_{2r+1} \quad \forall l > n$$

And, for the same reason, any distribution with subindex greater than $2r+1$ will also be involutive.

Thus, it can be assumed that $r \leq n - 2$. This implies

$$3n + r - 2 > n + 3r + 1 \tag{6.1}$$

Proof: The above equality is equivalent to the following:

$$2n - 2 > 2r + 1$$

which, in turn, is equivalent to $r < n - 3/2$. The facts $r \leq n - 2$ and r, n integers imply $r < n - 3/2$. ■

6.1 and the dependance of η_r (see the above remark (b)) on the variables

$$x, y_{n+1}, \ldots, y_{n+r}, y_{n+2r+2}, \ldots, y_{n+3r+1}$$

lead to the following involutivity conditions:

$$[\eta_k, g_1] \in <\eta_0, \ldots, \eta_r, g_1> \quad \forall 0 \leq k \leq r$$

since

$$[\eta_k, \frac{\partial}{\partial y_{3n+3r-1-j}}] = 0 \quad \forall 0 \leq j \leq 2r+1$$

and

$$[g_1, \frac{\partial}{\partial y_{3n+3r-1-j}}] = 0 \quad \forall 0 \leq j \leq 2r+1$$

$$[g_1, \frac{\partial}{\partial y_{n+2r+1-j}}] = 0 \quad \forall 0 \leq j \leq 2r$$

In order to proceed, another computational lemma is required

Lemma 18 $D^{2r+1,2n-2+r}_{2r+1+j} =$

$$< D^{2r+1,2n-2+r}_{2r+j}, \frac{\partial}{\partial y_{3n+r-2-j}}, ad^j_{\tilde{f}} g_1 >=$$

$$< D^{2r+1,2n-2+r}_{2r+j}, \frac{\partial}{\partial y_{3n+r-2-j}}, ad^j_{\tilde{f}} g_1 >$$

where

$$\tilde{f} = f + g_2 y_{n+2r+2} + \sum_{l=1}^{2n-2+r} y_{n+2r+1+l+1} \frac{\partial}{\partial y_{n+2r+1+l}}$$

Proof: It will be proven by induction. For $j = 1$,

$$ad_{\overline{f}} g_1 = [\overline{f}, g_1] =$$

$$[f + g_1 y_{n+1} + g_2 y_{n+2r+2} + \sum_{j=1}^{2r} y_{n+j+1} \frac{\partial}{\partial y_{n+j}} + \sum_{l=1}^{2n-3+r} y_{n+2r+1+l+1} \frac{\partial}{\partial y_{n+2r+1+l}}, g_1] =$$

$$= [f, g_1] + y_{n+1}[g_1, g_1] + y_{n+2r+2}[g_2, g_1] =$$

$$= [f + g_2 y_{n+2r+2} + \sum_{l=1}^{2n-2+r} y_{n+2r+1+l+1} \frac{\partial}{\partial y_{n+2r+1+l}}, g_1] = ad_{\tilde{f}} g_1$$

because $[g_1, g_1] = 0$.

Assuming that the statement is true up to j,

$$ad^{j+1}_{\overline{f}} g_1 = [\overline{f}, ad^j_{\overline{f}} g_1]$$

span in $D_{2r+1+j+1}^{2r+1,2n-2+r}$, by application of the induction hypothesis, the same as

$$\begin{aligned}[\overline{f}, ad_{\bar{f}}^{j} g_1] &= [f, ad_{\bar{f}}^{j} g_1] + y_{n+1}[g_1, ad_{\bar{f}}^{j} g_1] + y_{n+2r+2}[g_2, ad_{\bar{f}}^{j} g_1] + \\ &+ [\textstyle\sum_{l=1}^{2n-2+r} y_{n+2r+1+l+1} \frac{\partial}{\partial y_{n+2r+1+l}}, ad_{\bar{f}}^{j} g_1]\end{aligned}$$

However, due to the involutivity of the former distributions,

$$[g_1, ad_{\bar{f}}^{j} g_1] \in D_{2r+1+j}^{2r+1,2n-2+r}$$

So, $ad_{\overline{f}}^{j+1} g_1$ can be replaced in $D_{2r+1+j+1}^{2r+1,2n-2+r}$ by $ad_{\bar{f}}^{j+1} g_1$. ■

So, let us define $\delta_j = ad_{\bar{f}}^{j} g_1$.
Remarks:

i.
$$\delta_j \in S$$

Proof: It will be performed by induction on j.

$$\delta_1 = [\tilde{f}, g_1] = [f, g_1] + y_{n+2r+2}[g_2, g_1]$$

since g_1 depends only on the x variables.

$$\delta_{j+1} = [\tilde{f}, \delta_j] = \tilde{f}(\delta_j) - \delta_j(\tilde{f})$$

The induction hypothesis implies that $\delta_j \in S$. Therefore $\delta_{j+1} \in S$, because the only members in $\tilde{f}$ depending on x are

$$f(x) + y_{n+2r+2} g_2$$

ii. δ_j depends upon the variables $x, y_{n+2r+2}, \ldots, y_{n+2r+1+j}$.
Proof: As has been said,

$$\delta_1 = [f, g_1] + y_{n+2r+2}[g_2, g_1]$$

which proves the statement for $j = 1$. By application of the induction hypothesis and the preceding remark

$$\delta_{j+1} = [\tilde{f}, \delta_j]$$

is equal to

$$[\tilde{f}, H(x, y_{n+2r+2}, \ldots, y_{n+2r+1+j}) \frac{\partial}{\partial x}] =$$

$$[f, H\frac{\partial}{\partial x}] + y_{n+2r+2}[g_2, H\frac{\partial}{\partial x}] + \sum_{l=1}^{j} [y_{n+2r+1+l+1} \frac{\partial}{\partial y_{n+2r+1+l}}, H\frac{\partial}{\partial x}]$$

which will depend on the same variables as δ_j, plus the variable $y_{n+2r+1+j+1}$. This last variable comes from

$$[y_{n+2r+1+j+1}\frac{\partial}{\partial y_{n+2r+1+j}}, H\frac{\partial}{\partial x}]$$

■

Define also

$$h = \min\{j \in \{0, n-r-2\} | \delta_{j+1} \in < \eta_0, \ldots, \eta_r, \delta_0, \ldots, \delta_j >\}$$

(d) For all $0 \leq j \leq h$, the distributions

$$D_{2r+1+j}^{2r+1,2n-2+r} = < D_{2r}^{2r+1,2n-2+r}, \delta_0, \ldots, \delta_j, \frac{\partial}{\partial y_{3n+r-2}}, \ldots, \frac{\partial}{\partial y_{3n+r-2-j}} >$$

must fulfill the following conditions in order to be involutive:

$$\left.\begin{array}{l}[\delta_j, \eta_k] \\ [\delta_j, \delta_i]\end{array}\right\} \in < \eta_0, \ldots, \eta_r, \delta_0, \ldots, \delta_j > \quad \forall 0 \leq k \leq r \quad \forall 0 \leq i \leq j-1$$

Proof: The definition of h implies $j \leq h \leq n-r-2$. This is equivalent to $n-j \geq r+2$. So, $3n+r-2-j \geq 2n+r-2+r+2 = 2n+2r$. Note that $r \leq n-2$. Therefore,

$$3n+r-2-j \geq 2n+2r \geq n+2r+r+2 > n+3r+1$$

Observe that $n+3r+1$ is the maximum subindex of the variables y appearing in η_r, while $3n+r-2-j$ is the minimum subindex of the coordinate charts in $D_{2r+1+j}^{2r+1,2n-2+r}$. So, the Lie brackets between η_k $(k \leq r)$ and $\frac{\partial}{\partial y_{3n+r-2-j}} >$ are vanishing.
On the other hand, $3n+r-2-2j \geq n+r-2+2(r+2) = n+3r+2 > n+3r+1$. Or, equivalently,

$$3n+r-2-j > n+3r+1+j$$

Note that $n+3r+1+j$ is the maximum subindex that appears in δ_j. ■

Now, the static feedback linearizability conditions for $\Sigma_{r_1}^{r_2}$ (with $r_1 \geq 2r+1$ and $r_2 \geq 2n-2+r$) will be studied. More precisely, the systems Σ_{2r+2}^{2n-2+r}, Σ_{2r+1}^{2n-1+r} and Σ_{2r+2}^{2n-1+r} will be detailed (that is to say, systems where an extra derivative of u_1 and/or u_2 have been added).

(a) System Σ_{2r+1}^{2n-1+r}.
Note that for all $j \leq r$

$$D_j^{2r+1,2n+r-1} = < \eta_0, \ldots, \eta_j, \frac{\partial}{\partial y_{n+2r+1}}, \ldots, \frac{\partial}{\partial y_{n+2r+1-j}}, \frac{\partial}{\partial y_{3n+3r}}, \ldots, \frac{\partial}{\partial y_{3n+3r-j}} >$$

Comparing this distribution with $D_j^{2r+1,2n+r-2}$, the following equality can be written:

$$D_j^{2r+1,2n-1+r} \oplus < \frac{\partial}{\partial y_{3n+3r-1-j}} >= D_j^{2r+1,2n-2+r} \oplus < \frac{\partial}{\partial y_{3n+3r}} >$$

When $r \leq j \leq 2r$

$$D_j^{2r+1,2n+r-1} =< \eta_0, \dots, \eta_r, \frac{\partial}{\partial y_{n+2r+1}}, \dots, \frac{\partial}{\partial y_{n+2r+1-j}}, \frac{\partial}{\partial y_{3n+3r}}, \dots, \frac{\partial}{\partial y_{3n+3r-j}} >$$

and the same equality as above is fulfilled:

$$D_j^{2r+1,2n-1+r} \oplus < \frac{\partial}{\partial y_{3n+3r-1-j}} >= D_j^{2r+1,2n-2+r} \oplus < \frac{\partial}{\partial y_{3n+3r}} >$$

Furthermore, this equality is also true for $2r+1 \leq j \leq 2r+1+h$. Moreover, in all three cases, the hypothesis of lemma 5 are satisfied. Thus, one may be sure that the involutivity conditions are the same for $D_j^{2r+1,2n-1+r}$ and $D_j^{2r+1,2n-2+r}$, with $0 \leq j \leq 2r+1+h$.
When $j \geq 2r+1+h$, $D_{j+1}^{2r+1,2n+r-1} =$

$$=< \eta_0, \dots, \eta_r, \frac{\partial}{\partial y_{n+2r+1}}, \dots, \frac{\partial}{\partial y_{n+1}}, \delta_0, \dots, \delta_h, \frac{\partial}{\partial y_{3n+3r}}, \dots, \frac{\partial}{\partial y_{3n+3r-j-1}} >=$$

$$D_j^{2r+1,2n-2+r} \oplus < \frac{\partial}{\partial y_{3n+3r}} >$$

and the technical lemma 5 is also fulfilled. Thus, the conditions for both distributions to be involutive are the same. Hence, both systems satisfy the same static feedback linearizability conditions. ∎

(b) System Σ_{2r+2}^{2n-2+r}.
Denoting

$$y_{3n+3r} = u_1^{(2r+1)} \quad \dot{y}_{3n+3r} = w_1$$

which appears in the equation

$$\dot{y}_{n+2r+1} = y_{3n+3r}$$

the distributions associated with this system are

$$D_0^{2r+2,2n-2+r} =< \eta_0, \frac{\partial}{\partial y_{3n+3r}}, \frac{\partial}{\partial y_{3n+3r-1}} >$$

So

$$D_0^{2r+2,2n-2+r} \oplus < \frac{\partial}{\partial y_{n+2r+1}} >= D_0^{2r+1,2n-2+r} \oplus < \frac{\partial}{\partial y_{3n+3r}} >$$

If $1 \leq j \leq r$, $D_j^{2r+2,2n-2+r} =$

$$=< \eta_0, \dots, \eta_j, \frac{\partial}{\partial y_{3n+3r}}, \frac{\partial}{\partial y_{n+2r+1}}, \frac{\partial}{\partial y_{n+2r+2-j}}, \frac{\partial}{\partial y_{3n+3r-1}}, \frac{\partial}{\partial y_{3n+3r-1-j}} >$$

Then

$$D_j^{2r+2,2n-2+r} \oplus < \frac{\partial}{\partial y_{n+2r+1-j}} >= D_j^{2r+1,2n-2+r} \oplus < \frac{\partial}{\partial y_{3n+3r}} >$$

For $r \leq j \leq 2r$, $D_{j+1}^{2r+2,2n-2+r} =$

$$=< \eta_0, \dots, \eta_r, \frac{\partial}{\partial y_{3n+3r}}, \frac{\partial}{\partial y_{n+2r+1}}, \frac{\partial}{\partial y_{n+2r+1-j}}, \frac{\partial}{\partial y_{3n+3r-1}}, \frac{\partial}{\partial y_{3n+3r-2-j}} >$$

satisfies the equality

$$D_{j+1}^{2r+2,2n-2+r} = D_j^{2r+1,2n-2+r} \oplus < \frac{\partial}{\partial y_{3n+3r}}, \frac{\partial}{\partial y_{3n+3r-2-j}} >$$

Moreover, $D_{j+1}^{2r+2,2n-2+r} =$

$$\begin{aligned} = \quad & < \eta_0, \dots, \eta_r, \tfrac{\partial}{\partial y_{3n+3r}}, \tfrac{\partial}{\partial y_{n+2r+1}}, \dots, \tfrac{\partial}{\partial y_{n+1}}, \\ & \delta_0, \dots, \delta_{j-2r-1}, \tfrac{\partial}{\partial y_{3n+3r-1}}, \tfrac{\partial}{\partial y_{3n+3r-2-j}} > \end{aligned}$$

also fulfills the above equality for $2r + 1 \leq j \leq 2r + 1 + h$.
Finally, if $j > 2r + 1 + h$, $D_j^{2r+2,2n-2+r} =$

$$\begin{aligned} = \quad & < \eta_0, \dots, \eta_r, \tfrac{\partial}{\partial y_{3n+3r}}, \tfrac{\partial}{\partial y_{n+2r+1}}, \dots, \tfrac{\partial}{\partial y_{n+1}}, \\ & \delta_0, \dots, \delta_h, \tfrac{\partial}{\partial y_{3n+3r-1}}, \tfrac{\partial}{\partial y_{3n+3r-1-j}} >= \\ & = D_j^{2r+1,2n-2+r} \oplus < \frac{\partial}{\partial y_{3n+3r}} > \end{aligned}$$

Therefore, taking into account the lemma 5, both systems fulfill the same conditions. ∎

(c) Finally, the relationship between system Σ_{2r+2}^{2n-2+r} and system Σ_{2r+2}^{2n-1+r} is the same as that one between system Σ_{2r+1}^{2n-2+r} and system Σ_{2r+1}^{2n-1+r}.
Proof: The equality

$$D_j^{2r+2,2n-1+r} \oplus < \frac{\partial}{\partial y_{3n+3r-1-j}} >= D_j^{2r+2,2n-2+r} \oplus < \frac{\partial}{\partial y_{3n+3r+1}} >$$

is satisfied for all $0 \leq j \leq 2r+2+h$. And for $j \geq 2r+2+h$,

$$D_{j+1}^{2r+2,2n+r-1} D_j^{2r+2,2n-2+r} \oplus < \frac{\partial}{\partial y_{3n+3r+1}} >$$

In addition, the hypotheses of the lemma 5 are fulfilled. Therefore, the static feedback linearizability conditions are the same for both systems.

To summariza, it is proven that the static feedback linearizability conditions for system Σ_{2r+1}^{2n-2+r} and the system where one more derivative of u_1 and/or u_2 have been added are the same. The same proof is also valid for addition of more derivatives of the inputs.

2. Systems $\Sigma_{k_1}^{2n-2+r}$ and $\Sigma_{k_1}^{r_2}$ (with $k_1 < 2r+1$ and $r_2 > 2n-2+r$) are compared. In fact, in order to clarify the proof, systems $\Sigma_{k_1}^{2n-2+r}$ and $\Sigma_{k_1}^{2n-1+r}$ are studied. Extra additions of u_2 does not affect the proof. Using the same notations as before, let us define:

$$h_1 = min\{i \in \{0,r\} \mid \eta_{i+1} \in < \eta_0, \dots, \eta_i >\}$$

The reason for this definition is that h_1 could be smaller than r. There are two possibilities, namely $h_1 < k_1$ and $h_1 \geq k_1$. Note that the drift of $\Sigma_{k_1}^{2n-2+r}$ is

$$\begin{array}{rcl} \overline{f} & = & f + g_1 y_{n+1} + g_2 y_{n+k_1+1} + \\ & + & \sum_{j=1}^{k_1-1} y_{n+j+1} \frac{\partial}{\partial y_{n+j}} + \sum_{l=1}^{2n-3+r} y_{n+k_1+l+1} \frac{\partial}{\partial y_{n+2r+1+l}} \end{array}$$

while the input vector fields are

$$\overline{g}_1 = \frac{\partial}{\partial y_{n+k_1}} \quad \overline{g}_2 = \frac{\partial}{\partial y_{3n+k_1+r-2}} \quad \overline{g}_3 = g_3$$

The distributions for the case $h_1 < k_1$ are:

- $$D_j^{k_1,2n-2+r} = < \eta_0, \dots, \eta_j, \frac{\partial}{\partial y_{n+k_1}}, \dots, \frac{\partial}{\partial y_{n+k_1-j}}, \frac{\partial}{\partial y_{3n+k_1+r-2}}, \dots, \frac{\partial}{\partial y_{3n+k_1+r-2-j}} >$$

  if $j \leq h_1$.
  On the other hand,

  $$D_j^{k_1,2n-1+r} = < \eta_0, \dots, \eta_j, \frac{\partial}{\partial y_{n+k_1}}, \dots, \frac{\partial}{\partial y_{n+k_1-j}}, \frac{\partial}{\partial y_{3n+k_1+r-1}}, \dots, \frac{\partial}{\partial y_{3n+k_1+r-1-j}} >$$

  They satisfy the same involutivity conditions because η_j does not depend on the variables $y_{3n+k_1+r-j-2}$. Note that η_j depends on the variables

  $$x, y_{n+1}, \dots, y_{n+j}, y_{n+k_1+1}, \dots, y_{n+k_1+j}$$

- $\forall h_1 < j < k_1$, $D_j^{k_1,2n-2+r} =$

$$=< \frac{\partial}{\partial y_{n+k_1}}, \ldots, \frac{\partial}{\partial y_{n+k_1-j}}, \frac{\partial}{\partial y_{3n+k_1+r-2}}, \ldots, \frac{\partial}{\partial y_{3n+k_1+r-2-j}}, \eta_0, \ldots, \eta_{h_1} >$$

and $D_j^{k_1,2n-1+r} =$

$$=< \frac{\partial}{\partial y_{n+k_1}}, \ldots, \frac{\partial}{\partial y_{n+k_1-j}}, \frac{\partial}{\partial y_{3n+k_1+r-1}}, \ldots, \frac{\partial}{\partial y_{3n+k_1+r-1-j}}, \eta_0, \ldots, \eta_{h_1} >$$

Therefore, the equality

$$D_j^{k_1,2n-2+r} \oplus < \frac{\partial}{\partial y_{3n+k_1+r-1}} >= D_j^{k_1,2n-1+r} \oplus < \frac{\partial}{\partial y_{3n+k_1+r-2-j}} >$$

holds, and because of the lemma 5 they provide the same conditions.

- Let us define

$$h_2 = \min\{i | \delta_{i+1} \in < \eta_0, \ldots, \eta_{h_1}, \delta_0, \ldots, \delta_i >\}$$

and let $D_{k_1+i}^{k_1,2n-2+r} =$

$$=< \frac{\partial}{\partial y_{n+k_1}}, \ldots, \frac{\partial}{\partial y_{n+1}}, \frac{\partial}{\partial y_{3n+k_1+r-2}}, \ldots, \frac{\partial}{\partial y_{3n+r-2-i}}, \eta_0, \ldots, \eta_{h_1}, \delta_0, \ldots, \delta_i >$$

be the following distributions to be studied, $\forall i \leq h_2$. Note that $h_2 \leq n - h_1 - 3$, because when $h_2 = n - h_1 - 2$, then

$$< \eta_0, \ldots, \eta_{h_1}, \delta_0, \ldots, \delta_{n-h_1-2} >= S$$

and $D_{k_1+n-h_1-2}^{k_1,2n-2+r}$ is trivially involutive, as well as all the distributions after it. Since $D_{k_1+i}^{k_1,2n-1+r} =$

$$=< \frac{\partial}{\partial y_{n+k_1}}, \ldots, \frac{\partial}{\partial y_{n+1}}, \frac{\partial}{\partial y_{3n+k_1+r-1}}, \ldots, \frac{\partial}{\partial y_{3n+r-1-i}}, \eta_0, \ldots, \eta_{h_1}, \delta_0, \ldots, \delta_i >$$

Then

$$D_{k_1+i}^{k_1,2n-2+r} \oplus < \frac{\partial}{\partial y_{3n+k_1+r-1}} >= D_{k_1+i}^{k_1,2n-1+r} \oplus < \frac{\partial}{\partial y_{3n+r-2-i}} >$$

However, to assure that the hypotheses of lemma 5 are fulfilled, one must prove that $3n+r-2-i > n+k_1+i$, which is the maximum subindex of the variables on which δ_i depends. It is enough to check this inequality when $i = n - h_1 - 3$, which is the highest possible value for i. Then,

$$2n + r + 1 + h_1 > 2n + k_1 - h_1 - 3$$

must be proven or, equivalently, $r - k_1 + 4 > -2h_1$. This is obvious when $k_1 \leq r$. If $k_1 > r$, note that $h_1 = r$. In this case, the inequality to be proven becomes $r - k_1 + 4 > -2r$, also trivial.

- Finally, for all $i \geq k_1 + h_2$, $D_{i+1}^{k_1,2n-1+r} =$

$$=< \frac{\partial}{\partial y_{n+k_1}}, \dots, \frac{\partial}{\partial y_{n+1}}, \frac{\partial}{\partial y_{3n+k_1+r-1}}, \dots, \frac{\partial}{\partial y_{3n+k_1+r-2-i}}, \eta_0, \dots, \eta_{h_1}, \delta_0, \dots, \delta_{h_2} >$$

and $D_{i+1}^{k_1,2n-2+r} =$

$$=< \frac{\partial}{\partial y_{n+k_1}}, \dots, \frac{\partial}{\partial y_{n+1}}, \frac{\partial}{\partial y_{3n+k_1+r-2}}, \dots, \frac{\partial}{\partial y_{3n+k_1+r-2-i}}, \eta_0, \dots, \eta_{h_1}, \delta_0, \dots, \delta_{h_2} >$$

So,

$$D_{i+1}^{k_1,2n-1+r} = D_i^{k_1,2n-2+r} \oplus \frac{\partial}{\partial y_{3n+k_1+r-1}}$$

which, thanks to lemma 5, assures that both distributions satisfy the same involutivity conditions. This ends the proof for the case $h_1 < k_1$.

Now, for the case $h_1 \geq k_1$,

- $\forall j < k_1$

$$D_j^{k_1,2n-2+r} =< \eta_0, \dots, \eta_j, \frac{\partial}{\partial y_{n+k_1}}, \dots, \frac{\partial}{\partial y_{n+k_1-j}}, \frac{\partial}{\partial y_{3n+k_1+r-2}}, \dots, \frac{\partial}{\partial y_{3n+k_1+r-2-j}} >$$

and

$$D_j^{k_1,2n-1+r} =< \eta_0, \dots, \eta_j, \frac{\partial}{\partial y_{n+k_1}}, \dots, \frac{\partial}{\partial y_{n+k_1-j}}, \frac{\partial}{\partial y_{3n+k_1+r-1}}, \dots, \frac{\partial}{\partial y_{3n+k_1+r-1-j}} >$$

- $\forall k_1 \leq j < h_1$ (if $h_1 < k_1 + h_2$. The case $h_1 \geq k_1 + h_2$ is treated analogously and is left to the reader), $D_j^{k_1,2n-2+r} =$

$$=< \eta_0, \dots, \eta_j, \frac{\partial}{\partial y_{n+k_1}}, \dots, \frac{\partial}{\partial y_{n+1}}, \delta_0, \dots, \delta_{j-k_1}, \frac{\partial}{\partial y_{3n+k_1+r-2}}, \dots, \frac{\partial}{\partial y_{3n+k_1+r-2-j}} >$$

and $D_j^{k_1,2n-1+r} =$

$$=< \eta_0, \dots, \eta_j, \frac{\partial}{\partial y_{n+k_1}}, \dots, \frac{\partial}{\partial y_{n+1}}, \delta_0, \dots, \delta_{j-k_1}, \frac{\partial}{\partial y_{3n+k_1+r-1}}, \dots, \frac{\partial}{\partial y_{3n+k_1+r-1-j}} >$$

- $\forall h_1 \leq j < k_1 + h_2$, $D_j^{k_1,2n-2+r} =$

$$=< \eta_0, \dots, \eta_{h_1}, \frac{\partial}{\partial y_{n+k_1}}, \dots, \frac{\partial}{\partial y_{n+1}}, \delta_0, \dots, \delta_{j-k_1}, \frac{\partial}{\partial y_{3n+k_1+r-2}}, \dots, \frac{\partial}{\partial y_{3n+k_1+r-2-j}} >$$

and $D_j^{k_1,2n-1+r} =$

$$=< \eta_0, \dots, \eta_j, \frac{\partial}{\partial y_{n+k_1}}, \dots, \frac{\partial}{\partial y_{n+1}}, \delta_0, \dots, \delta_{j-k_1}, \frac{\partial}{\partial y_{3n+k_1+r-1}}, \dots, \frac{\partial}{\partial y_{3n+k_1+r-1-j}} >$$

Therefore, in any of the three cases, the following equality holds

$$D_j^{k_1,2n-2+r} \oplus \frac{\partial}{\partial y_{3n+k_1+r-1}} = D_j^{k_1,2n-1+r} \oplus \frac{\partial}{\partial y_{3n+k_1+r-2-j}}$$

and the following inequalities must be fulfilled in order to satisfy the hypotheses of lemma 5 (just studying the maximum possible value for j, $j = n + k_1 - h_1 - 3$):

$$2n + r + h_1 + 1 > n + h_1 \quad 2n + r + h_1 + 1 > n + k_1 + n - h_1 - 3$$

or, equivalently, the trivial inequalities

$$n + r + 1 > 0 \quad r + 2h_1 + 4 > k_1$$

- Finally, $\forall j \geq k_1 + h_2$,

$$\begin{aligned} D_{j+1}^{k_1,2n-1+r} &=< \eta_0, \ldots, \eta_{h_1}, \tfrac{\partial}{\partial y_{n+k_1}}, \ldots, \tfrac{\partial}{\partial y_{n+1}}, \\ &\quad \delta_0, \ldots, \delta_{h_2}, \tfrac{\partial}{\partial y_{3n+k_1+r-1}}, \ldots, \tfrac{\partial}{\partial y_{3n+k_1+r-2-j}} > \end{aligned}$$

and

$$\begin{aligned} D_j^{k_1,2n-2+r} &=< \eta_0, \ldots, \eta_{h_1}, \tfrac{\partial}{\partial y_{n+k_1}}, \ldots, \tfrac{\partial}{\partial y_{n+1}}, \\ &\quad \delta_0, \ldots, \delta_{h_2}, \tfrac{\partial}{\partial y_{3n+k_1+r-1}}, \ldots, \tfrac{\partial}{\partial y_{3n+k_1+r-2-j}} > \end{aligned}$$

So,

$$D_{j+1}^{k_1,2n-1+r} = D_j^{k_1,2n-2+r} \oplus \frac{\partial}{\partial y_{3n+k_1+r-1}}$$

Therefore, both systems satisfy the same static feedback linearizability conditions.

3. The third case to compare is the static feedback linearizability conditions for the systems $\Sigma_{2r+1}^{k_2}$ and $\Sigma_{r_1}^{k_2}$ (with $2r < k_2 < 2n - 2 + r$ and $r_1 > 2r + 1$). In fact, systems $\Sigma_{2r+1}^{k_2}$ and $\Sigma_{2r+2}^{k_2+1}$ are compared (which still satisfies the restriction $k_2 + 1 \leq 2n - 2 + r$). Denoting

$$y_{n+2r+1+k_2+1} = u_1^{(n+2r+1)} \quad y_{n+2r+1+k_2+2} = u_2^{(k_2)}$$

which appear in the equations

$$\dot{y}_{n+2r+1+k_2+1} = u_1^{(n+2r+2)} = w_1 \quad \dot{y}_{n+2r+1+k_2+2} = u_2^{(k_2+1)} = w_2$$

- If $j \leq r$, $D_j^{2r+1,k_2} =$

$$=< \eta_0, \ldots, \eta_j, \frac{\partial}{\partial y_{n+2r+1}}, \ldots, \frac{\partial}{\partial y_{n+2r+1-j}}, \frac{\partial}{\partial y_{n+2r+1+k_2}}, \ldots, \frac{\partial}{\partial y_{n+2r+1+k_2-j}} >$$

and
$$D_j^{2r+2,k_2+1} =< \eta_0,\ldots,\eta_j, \frac{\partial}{\partial y_{n+2r+1+k_2+1}} \frac{\partial}{\partial y_{n+2r+1}},\ldots,\frac{\partial}{\partial y_{n+2r+2-j}},$$
$$\frac{\partial}{\partial y_{n+2r+3+k_2}},\frac{\partial}{\partial y_{n+2r+1+k_2}},\ldots,\frac{\partial}{\partial y_{n+2r+2+k_2-j}} >$$
then the following equality holds:
$$D_j^{2r+2,k_2+1} \oplus < \frac{\partial}{\partial y_{n+2r+1-j}},\frac{\partial}{\partial y_{n+2r+1+k_2-j}} >=$$
$$D_j^{2r+1,k_2} \oplus < \frac{\partial}{\partial y_{n+2r+1+k_2+1}},\frac{\partial}{\partial y_{n+2r+1+k_2+2}} >$$

- For $r < j \leq 2r$, $D_j^{2r+1,k_2} =$

$$=< \eta_0,\ldots,\eta_r, \frac{\partial}{\partial y_{n+2r+1}},\ldots,\frac{\partial}{\partial y_{n+2r+1-j}},\frac{\partial}{\partial y_{n+2r+1+k_2}},\ldots,\frac{\partial}{\partial y_{n+2r+1+k_2-j}} >$$

and
$$D_{j+1}^{2r+2,k_2+1} =< \eta_0,\ldots,\eta_r, \frac{\partial}{\partial y_{n+2r+1+k_2+1}} \frac{\partial}{\partial y_{n+2r+1}},\ldots,\frac{\partial}{\partial y_{n+2r+1-j}},$$
$$\frac{\partial}{\partial y_{n+2r+3+k_2}},\frac{\partial}{\partial y_{n+2r+1+k_2}},\ldots,\frac{\partial}{\partial y_{n+2r+1+k_2-j}} >$$

Therefore
$$D_{j+1}^{2r+2,k_2+1} = D_j^{2r+1,k_2} \oplus < \frac{\partial}{\partial y_{n+2r+1+k_2+1}},\frac{\partial}{\partial y_{n+2r+1+k_2+2}} >$$

Note that this last equality is still valid when $j > 2r$. Moreover, all these distributions satisfy the hypotheses of lemma 5. Thus, one system is linearizable by static feedback if, and only if, the other one is also.

The three different cases stated at the beginning of the proof having been proven, the proof is concluded. ∎

6.2 Where do the bounds $2r+1$ and $2n-2+r$ come from?

If the bound $2r+1$ is relaxed to $2r$, then the distribution $D_r^{2r,2n-2+r}$ of the system Σ_{2r}^{2n-2+r} is

$$D_r^{2r,2n-2+r} =< \eta_0,\ldots,\eta_r, \frac{\partial}{\partial y_{n+2r}},\ldots,\frac{\partial}{\partial y_{n+r}},\frac{\partial}{\partial y_{3n+3r-2}},\ldots,\frac{\partial}{\partial y_{3n+2r-2}} >$$

Therefore, a new involutivity condition appears, which is different from those appearing in $D_r^{2r+1,2n-2+r}$:

$$[\eta_r, \frac{\partial}{\partial y_{n+r}}] \in < \eta_0,\ldots,\eta_r >$$

Therefore, it is not possible to relax that bound.
In the same way, the purpose of the bound $2n-2+r$ is to avoid extra involutivity conditions among η_j ($j \leq r$), δ_i ($i \leq h$) and the other elements of D_{2r+1+i}. Nevertheless, it remains to be seen whether or not this last bound can be improved.

Chapter 7

Linearization by prolongations of m-input systems

This chapter gives a bound on the number of integrators needed to linearize a control system with an arbitrary number of inputs. Although some work have been done in this direction in [59], our bound improves the existing results for systems with four or more inputs. The bound for two input systems is the same as the one that appeared in [59], and has already been studied in Chapter 5. The bound for three input systems is further improved in Chapter 6. Nevertheless, the sharpness of our bound remains an open question.

7.1 Main results

First of all, we state and prove a proposition useful in the proof of the main result.

Proposition 6 *If a system with m inputs is linearizable by prolongation of u_i k_i times (with $k_i \geq 1$ for all i), then the system is also linearizable by prolongation of u_i $k_i - 1$ times.*

Proof: Note that there is no loss of generality in assuming that $k_1 \leq k_2 \leq \ldots \leq k_m$. Let Σ_k be the system obtained by prolongation of u_i k_i times (with $k_i \geq 1$ for all i) and $\Sigma_{k'}$ be the system obtained by prolongation of u_i $k_i - 1$ times. And consider

$$f^k = f + \sum_{i=1}^{m} y_1^i g_i + \sum_{i=1}^{m} \sum_{l=1}^{k_i-1} y_{l+1}^i \frac{\partial}{\partial y_l^i}$$

and

$$f^{k'} = f + \sum_{i=1}^{m} y_1^i g_i + \sum_{i=1}^{m} \sum_{l=1}^{k_i-2} y_{l+1}^i \frac{\partial}{\partial y_l^i}$$

the drifts associated to Σ_k and $\Sigma_{k'}$ respectively. And let

$$g_i^k = \frac{\partial}{\partial y_{k_i}^i}$$

$$g_i^{k'} = \frac{\partial}{\partial y^i_{k_i-1}}$$

the input vector fields of Σ_k and $\Sigma_{k'}$. The main idea of this proof is to see the equalities

$$D_l^k = D_{l-1}^{k'} \oplus < \{\frac{\partial}{\partial y^i_{k_i}},\ i = 1, \dots, m\} > \tag{7.1}$$

for all $l \geq 1$. Note that D_l^k and $D_l^{k'}$ are the distributions associated to system Σ_k and $\Sigma_{k'}$ respectively. We will also check to see if lemma 5 can be applied. Some lemmas are needed for this proof.

Lemma 19 *1.*

$$ad^l_{f^{k,1}} g_1$$

can replace $ad^l_{f^k} g_1$ in $D^k_{k_1+l}$, where

$$f^{k,1} = f + \sum_{i=2}^m y_1^i g_i + \sum_{i=2}^m \sum_{l=1}^{k_i-1} y^i_{l+1} \frac{\partial}{\partial y^i_l}$$

2. And, in general,

$$ad^l_{f^{k,j}} g_j$$

can replace $ad^l_{f^k} g_j$ in $D^k_{k_j+l}$, where

$$f^{k,j} = f + \sum_{i=j+1}^m y_1^i g_i + \sum_{i=j+1}^m \sum_{l=1}^{k_i-1} y^i_{l+1} \frac{\partial}{\partial y^i_l}$$

Lemma 20

$$ad^l_{f^{k,j}} g_j$$

belongs to $< \frac{\partial}{\partial x} >$ and depends on the variables

$$x, \{y_h^p \mid h = 1, \dots, l;\ p = j+1, \dots, m\}$$

Lemma 21

$$ad^l_{f^{k,j}} g_j$$

can be replaced by

$$ad^l_{f^{k'},j} g_j$$

in $D^k_{k_j+l}$.

Lemma 22 *1.*

$$ad^l_{f^{k'},1} g_1$$

can replace $ad^l_{f^{k'}} g_1$ *in* $D^{k'}_{k_1+l}$, *where*

$$f^{k',1} = f + \sum_{i=2}^{m} y_1^i g_i + \sum_{i=2}^{m} \sum_{l=1}^{k_i-2} y_{l+1}^i \frac{\partial}{\partial y_l^i}$$

2. And, in general,

$$ad^l_{f^{k'},j} g_j$$

can replace $ad^l_{f^{k'}} g_j$ *in* $D^{k'}_{k_j+l}$, *where*

$$f^{k',j} = f + \sum_{i=j+1}^{m} y_1^i g_i + \sum_{i=j+1}^{m} \sum_{l=1}^{k_i-2} y_{l+1}^i \frac{\partial}{\partial y_l^i}$$

The proof of these lemmas is exactly the same as that carried out for lemmas 9,10,11 and 12 in Chapter 5. Thanks to these lemmas, the equality 7.1 holds, and lemma 5 can be applied. ∎

Corollary 5 *If a system with m inputs is linearizable by prolongation of* u_i n_i *times (with* $n_i \geq 1$ *for all i), then the system is also linearizable by prolongation of* u_i $n_i - \min_{1 \leq i \leq m}\{n_i\}$ *times.*

Proof: This is straightforward after the application of the former proposition $\min_{1 \leq i \leq m}\{n_i\}$ times. ∎

Thanks to this corollary, we will only consider prolongations where one input is not prolonged. When a prolongation is considered, the inputs are ordered by the number of derivatives added to each one. So, the number of derivatives of u_1 is zero, and the number of derivatives of u_i is less than or equal to the number of derivatives of u_{i+1}, for all i from one to $m-1$. Our main result is established in the following theorem:

Theorem 5 *Let*

$$\Sigma: \quad \dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i \quad x \in R^n$$

be a m-input control system not static feedback linearizable, and such that the prolongation Σ_k *is static feedback linearizable, where* Σ_k *is given by*

$$\begin{array}{lll}
\dot{x} & = & f(x) + g_1(x)v_1 + \sum_{i=2}^{m} y_1^i g_i(x) & \\
\dot{y}_l^j & = & y_{l+1}^j & l = 1, \ldots, c_j, \textit{ for certain } j \in \{2, \ldots, m\} \\
\dot{y}_l^i & = & y_{l+1}^i & i = 2, \ldots, m, \ i \neq j \ l = 1, \ldots, k_i - 1 \\
\dot{y}_{c_j+1}^j & = & v_j & \\
\dot{y}_{k_i}^i & = & v_i \quad . & i = 2, \ldots, m
\end{array}$$

k_i being $k_i \leq c_i = 2(i-1)n - (i-1)(i+4)/2$, $i = 2, \ldots, m$. The new state variables and inputs are related to the old ones by

$$\begin{array}{rcll} y_l^j &=& u_j^{(l-1)} & l = 1, \ldots, c_j + 1 \\ y_l^i &=& u_i^{(l-1)} & l = 1, \ldots, k_i \\ v_j &=& u_j^{(c_j+1)} & \\ v_i &=& u_i^{(k_i)} & i = 2, \ldots, m \; i \neq j \\ v_1 &=& u_1 & \end{array}$$

Then, $\Sigma_{k'}$ is static feedback linearizable, where $\Sigma_{k'}$ is given by

$$\begin{array}{rcll} \dot{x} &=& f(x) + g_1(x)v_1 + \sum_{i=2}^m y_1^i g_i(x) & \\ \dot{y}_l^j &=& y_{l+1}^j & l = 1, \ldots, c_j - 1 \\ \dot{y}_l^i &=& y_{l+1}^i & i = 2, \ldots, m, \; i < j \; l = 1, \ldots, k_i - 1 \\ \dot{y}_l^i &=& y_{l+1}^i & i = 2, \ldots, m, \; i > j \; l = 1, \ldots, k_i - 2 \\ \dot{y}_{c_j}^j &=& v_j & \\ \dot{y}_{k_i}^i &=& v_i & i < j \\ \dot{y}_{k_i-1}^i &=& v_i & i > j \end{array}$$

Proof: Denote by f^k, g_i^k (resp. $f^{k'}$, $g_i^{k'}$) the drift and the input fields of Σ^k (resp. $\Sigma^{k'}$). Then

$$f^k = f + \sum_{i=2}^m y_1^i g_i + \sum_{i=2, i\neq j}^m \sum_{l=1}^{k_i-1} y_{l+1}^i \frac{\partial}{\partial y_l^i} + \sum_{l=1}^{c_j} y_{l+1}^j \frac{\partial}{\partial y_l^j}$$

$$g_1^k = g_1 \quad g_i^k = \frac{\partial}{\partial y_{k_i}^i} \quad g_j^k = \frac{\partial}{\partial y_{c_j+1}^j}$$

and

$$f^{k'} = f + \sum_{i=2}^m y_1^i g_i + \sum_{i=2, i\neq j}^m \sum_{l=1} k_i - 1 y_{l+1}^i \frac{\partial}{\partial y_l^i} + \sum_{l=1} c_j - 1 y_{l+1}^j \frac{\partial}{\partial y_l^i}$$

$$g_1^{k'} = g_1 \quad g_i^{k'} = \frac{\partial}{\partial y_{k_i}^i} \quad g_j^{k'} = \frac{\partial}{\partial y_{c_j}^j}$$

The main idea of the proof is to see the equalities

1.

$$D_l^k \oplus < \frac{\partial}{\partial y_{c_j-l}^j}, \{\frac{\partial}{\partial y_{k_i-1-l}^i}, \; i > j\} >= D_l^{k'} \oplus < \frac{\partial}{\partial y_{c_j+1}^j}, \{\frac{\partial}{\partial y_{k_i}^i}, \; i > j\} > \quad l \leq k_{j-1} + r_{j-1}$$

2.

$$D_{l+1}^k = D_l^{k'} \oplus < \frac{\partial}{\partial y_{c_j+1}^j}, \{\frac{\partial}{\partial y_{k_i}^i}, \; i > j\} > \quad l \geq k_{j-1} + r_{j-1}$$

and to verify the hypotheses of lemma 5, where

$$r_i = \min\{s | \eta^i_{s+1} \in <\eta^1_0, \ldots, \eta^1_{r_1}, \ldots, \eta^{i-1}_0, \ldots, \eta^{i-1}_{r_{i-1}}, \eta^i_0, \ldots, \eta^i_s>\}$$

and

$$\eta^i_l = ad^l_{f^k} g_i \quad i = 2, \ldots, m;\ l \geq 0$$

A series of lemmas lead us to the proof.

Lemma 23 *For $2 \leq i \leq m$,*

1.

$$ad^l_{f_k}\left(\frac{\partial}{\partial y^i_{k_i}}\right) = (-1)^l \frac{\partial}{\partial y^i_{k_i - l}} \quad \forall l < k_i$$

2.

$$ad^{k_i}_{f^k}\left(\frac{\partial}{\partial y^i_{k_i}}\right) = (-1)^k_i g_i$$

Proof:

1. It is proven by induction. For $l = 1$,

$$ad^1_{f_k}\left(\frac{\partial}{\partial y^i_{k_i}}\right) = [f^k, \frac{\partial}{\partial y^i_{k_i}}] = [y^i_{k_i} \frac{\partial}{\partial y^i_{k_i-1}}, \frac{\partial}{\partial y^i_{k_i}}] = -\frac{\partial}{\partial y^i_{k_i-1}}$$

   Assuming the trueness of the equality up to l,

$$ad^{l+1}_{f_k}\left(\frac{\partial}{\partial y^i_{k_i}}\right) = [f^k, ad^l_{f_k}\left(\frac{\partial}{\partial y^i_{k_i}}\right)]$$

   which becomes, by application of the induction hypothesis,

$$\begin{aligned} [f^k, (-1)^l \tfrac{\partial}{\partial y^i_{k_i-l}}] &= [y^i_{k_i-l} \tfrac{\partial}{\partial y^i_{k_i-l-1}}, (-1)^l \tfrac{\partial}{\partial y^i_{k_i-l}}] = \\ &= (-1)^{l+1} \tfrac{\partial}{\partial y^i_{k_i-l-1}}] \end{aligned}$$

2.

$$ad^{k_i}_{f^k}\left(\frac{\partial}{\partial y^i_{k_i}}\right) = [f^k, ad^{k_i-1}_{f^k}\left(\frac{\partial}{\partial y^i_{k_i}}\right)$$

   is equal, using the first part of the lemma, to

$$[f^k, (-1)^{k_i-1} \frac{\partial}{\partial y^i_{k_i-(k_i-1)}}] = [y^i_1 g_i, (-1)^{k_i-1} \frac{\partial}{\partial y^i_{1)}}] = (-1)^k_i g_i$$

∎

Lemma 24 *1.*

$$\eta^i_{s_i} \in S =< \frac{\partial}{\partial x} > \quad \forall i = 2, \ldots, m;\ s_i \leq r_i$$

2. η^i_s *only depends upon the variables* $(x, \{y^h_l,\ h = 2, \ldots, m;\ l = 1, \ldots, s\})$.

Proof: Both results are proven by induction.

1. For $s_i = 1$,

$$\eta^i_1 = [f^k, g_i] = [f + \sum_{h=2}^{m} y^h_1 g_h, g_i]$$

which clearly belongs to S.

Assuming that the statement is true up to s,

$$\eta^i_{s+1} = [f^k, \eta^i_s] = f^k(\eta^i_s) - \eta^i_s(f^k) \in S$$

because $\eta^i_s \in S$ and because the part of f^k depending on x is

$$f + \sum_{h=2}^{m} y^h_1 g_h$$

2.

$$[\eta^i_1 = [f^k, g_i] = [f + \sum_{h=2}^{m} y^h_1 g_h, g_i]$$

only depends on y^h_1.

$$[\eta^i_{s+1} = [f^k, \eta^i_s]$$

where it can be assured, thanks to the induction hypothesis and the previous lemma, that $\eta^i_s \in S$ and depends on the variables $(x, \{y^h_l,\ h = 2, \ldots, m;\ l = 1, \ldots, s\})$. Therefore,

$$[\eta^i_{s+1} = [f + \sum_{h=2}^{m} y^h_1 g_h + \sum_{h=2}^{m}\sum_{l=1}^{s} y^h_{l+1}\frac{\partial}{\partial y^h_l}, \eta^i_s]$$

will depend only on the variables $(x, \{y^h_l,\ h = 2, \ldots, m;\ l = 1, \ldots, s+1\})$. ∎

Lemma 25 η^i_l, $l \geq 0$, $i = 2, \ldots, m$, *can be replaced by*

$$ad^l_{f^{k,i}} g_i$$

in all the distributions where they appear, where

$$f^{k,i} = f + \sum_{h=i}^{m} y^h_1 g_h + \sum_{h=i}^{m}\sum_{l=1}^{k_h - 1} y^h_{l+1}\frac{\partial}{\partial y^h_l}$$

Proof: Again, the proof uses induction. For $l = 1$

$$\eta_1^i = [f^k, g_i]$$

appears in the distribution $D^k_{k_i+1}$ for the first time. Notice that $D^k_{k_i}$ contains $g_1, \ldots, g_i$ and is involutive since the system is static feedback linearizable. Therefore, all the Lie brackets between them are in $D^k_{k_i}$. So,

$$\eta_1^i = [f + \sum_{h=2}^{m} y_1^h g_h, g_i]$$

can be replaced by

$$\eta_1^i = [f + \sum_{h=i}^{m} y_1^h g_h, g_i]$$

in $D^k_{k_i+1}$ and all the distributions which follow.
If the statement is assumed to be true up to l,

$$\eta_{l+1}^i = [f^k, \eta_l^i]$$

can be replaced, by using the induction hypothesis, by

$$[f^k, ad^l_{f^{k,i}} g_i] = [f, ad^l_{f^{k,i}} g_i] + \sum_{h=1}^{m} [y_1^h g_h, ad^l_{f^{k,i}} g_i] + \sum_{h=i}^{m} \sum_{l=1}^{k_h-1} [y_{l+1}^h \frac{\partial}{\partial y_l^h}, ad^l_{f^{k,i}} g_i]$$

And due to the involutivity of all the previous distributions,

$$\sum_{h=1}^{i-1} [y_1^h g_h, ad^l_{f^{k,i}} g_i] + \sum_{h=1}^{i-1} \sum_{l=1}^{k_h-1} [y_{l+1}^h \frac{\partial}{\partial y_l^h}, ad^l_{f^{k,i}} g_i] \in D^k_{k_i+l}$$

Therefore, η_{l+1}^i can be replaced by

$$ad^l_{f^{k,i}} g_i$$

in $D^k_{k_i+l+1}$ and all the distributions which follow. ■

Thanks to this lemma, we are able to assume

$$\eta_l^i = ad^l_{f^{k,i}} g_i \quad l \geq 0,\ i = 2, \ldots, m$$

Now it is possible to write the distributions associated to both systems, Σ^k and $\Sigma^{k'}$. In order to do that, first we assume $r_i < k_{i+1} - k_i$, $\forall i < j$. Then:

$$D_0^k = < \eta_0^1 = g_1, \frac{\partial}{\partial y^j_{c_j+1}}, \{\frac{\partial}{\partial y^i_{k_i}}, i = 2, \ldots, m; i \neq j\} >$$

$$D_l^k = < \eta_0^1, \ldots, \eta_l^1, \frac{\partial}{\partial y^j_{c_j+1}}, \ldots, \frac{\partial}{\partial y^j_{c_j+1-l}}, \{\frac{\partial}{\partial y^i_{k_i}}, \ldots, \frac{\partial}{\partial y^i_{k_i-l}}, i = 2, \ldots, m; i \neq j\} > \quad l \leq r_1$$

$$D^k_{k_2-1} =< \eta^1_0, \ldots, \eta^1_{r_1}, \frac{\partial}{\partial y^j_{c_j+1}}, \ldots, \frac{\partial}{\partial y^j_{c_j+1-k_2+1}}, \{\frac{\partial}{\partial y^i_{k_i}}, \ldots, \frac{\partial}{\partial y^i_{k_i-k_2+1}}, i = 2, \ldots, m; i \neq j\} >$$

$$\begin{aligned} D^k_{k_2} \;&=\; < \eta^1_0, \ldots, \eta^1_{r_1}, \eta^2_0 = g_2, \tfrac{\partial}{\partial y^j_{c_j+1}}, \ldots, \tfrac{\partial}{\partial y^j_{c_j+1-k_2}}, \tfrac{\partial}{\partial y^2_{k_2}}, \ldots, \tfrac{\partial}{\partial y^2_{k_2-k_2+1}}, \\ &\quad \{\tfrac{\partial}{\partial y^i_{k_i}}, \ldots, \tfrac{\partial}{\partial y^i_{k_i-k_2}}, i = 3, \ldots, m; i \neq j\} > \end{aligned}$$

And, in general, if $p < j$, $D^k_{k_p+l} =$

$$\begin{aligned} &= \; < \eta^1_0, \ldots, \eta^1_{r_1}, \ldots, \eta^{p-1}_0, \ldots, \eta^{p-1}_{r_{p-1}}, \eta^p_0, \ldots, \eta^p_l, \tfrac{\partial}{\partial y^j_{c_j+1}}, \ldots, \tfrac{\partial}{\partial y^j_{c_j+1-k_p-l}}, \\ &\quad \{\tfrac{\partial}{\partial y^i_{k_i}}, \ldots, \tfrac{\partial}{\partial y^i_1}, i = 2, \ldots, p\}, \{\tfrac{\partial}{\partial y^i_{k_i}}, \ldots, \tfrac{\partial}{\partial y^i_{k_i-k_p-l}}, i = p+1, \ldots, m; i \neq j\} > \quad l \leq r_p \end{aligned}$$

and $D^k_{k_p+l} =$

$$\begin{aligned} &= \; < \eta^1_0, \ldots, \eta^1_{r_1}, \ldots, \eta^{p-1}_0, \ldots, \eta^{p-1}_{r_{p-1}}, \eta^p_0, \ldots, \eta^p_{r_p}, \tfrac{\partial}{\partial y^j_{c_j+1}}, \ldots, \tfrac{\partial}{\partial y^j_{c_j+1-k_p-l}}, \\ &\quad \{\tfrac{\partial}{\partial y^i_{k_i}}, \ldots, \tfrac{\partial}{\partial y^i_1}, i = 2, \ldots, p\}, \{\tfrac{\partial}{\partial y^i_{k_i}}, \ldots, \tfrac{\partial}{\partial y^i_{k_i-k_p-l}}, i = p+1, \ldots, m; i \neq j\} > \quad r_p < l < k_{p+1} - k_p \end{aligned}$$

On the other hand, for $\Sigma^{k'}$,

$$\begin{aligned} D^{k'}_{k'_p+l} \;&=\; < \eta^1_0, \ldots, \eta^1_{r_1}, \ldots, \eta^{p-1}_0, \ldots, \eta^{p-1}_{r_{p-1}}, \eta^p_0, \ldots, \eta^p_l, \tfrac{\partial}{\partial y^j_{c_j}}, \ldots, \tfrac{\partial}{\partial y^j_{c_j-k_p-l}}, \\ &\quad \{\tfrac{\partial}{\partial y^i_{k_i}}, \ldots, \tfrac{\partial}{\partial y^i_1}, i = 2, \ldots, p\}, \{\tfrac{\partial}{\partial y^i_{k_i}}, \ldots, \tfrac{\partial}{\partial y^i_{k_i-k_p-l}}, i = p+1, \ldots, j-1\}, \\ &\quad \{\tfrac{\partial}{\partial y^i_{k_i-1}}, \ldots, \tfrac{\partial}{\partial y^i_{k_i-1-k_p-l}}, i > j\} > \quad l \leq r_p \end{aligned}$$

and

$$\begin{aligned} D^{k'}_{k'_p+l} \;&=\; < \eta^1_0, \ldots, \eta^1_{r_1}, \ldots, \eta^{p-1}_0, \ldots, \eta^{p-1}_{r_{p-1}}, \eta^p_0, \ldots, \eta^p_{r_p}, \tfrac{\partial}{\partial y^j_{c_j}}, \ldots, \tfrac{\partial}{\partial y^j_{c_j-k_p-l}}, \\ &\quad \{\tfrac{\partial}{\partial y^i_{k_i}}, \ldots, \tfrac{\partial}{\partial y^i_1}, i = 2, \ldots, p\}, \{\tfrac{\partial}{\partial y^i_{k_i}}, \ldots, \tfrac{\partial}{\partial y^i_{k_i-k_p-l}}, i = p+1, \ldots, j-1\}, \\ &\quad \{\tfrac{\partial}{\partial y^i_{k_i-1}}, \ldots, \tfrac{\partial}{\partial y^i_{k_i-1-k_p-l}}, i > j\} > \quad r_p < l < k_{p+1} - k_p \end{aligned}$$

Therefore,

$$D^k_l \oplus < \frac{\partial}{\partial y^j_{c_j-l}}, \{\frac{\partial}{\partial y^i_{k_i-1-l}}, \; i > j\} >= D^{k'}_l \oplus < \frac{\partial}{\partial y^j_{c_j+1}}, \{\frac{\partial}{\partial y^i_{k_i}}, \; i > j\} > \quad l \leq k_{j-1} + r_{j-1}$$

It remains to be seen that lemma 5 can be applied. Indeed, it can be applied since

$$r_p < c_j - k_{j-1} - r_{j-1} \quad \forall p \leq j-1 \tag{7.2}$$

In fact, it must also be checked

$$r_p < k_i - 1 - k_{j-1} - r_{j-1} \quad \forall p \leq j-1;\ \forall i > j \tag{7.3}$$

But if 7.2 holds, 7.3 also holds because $k_i - 1 > c_j \quad \forall i > j$ since the inputs are ordered following the number of derivatives added to each one. To prove inequality 7.2, remark that

$$c_j - k_{j-1} \geq c_j - c_{j-1} = 2(j-1)n - (j-1)(j+4)/2 - 2(j-2) + (j-2)(j+3)/2 = 2n - j - 1$$

On the other hand $r_p \leq n-(p+1)$ for all $p \leq j-1$. Then,

$$r_p + r_{j-1} \leq n - (p+1) + n - j \leq 2n - j - 2 < 2n - j - 1 \leq c_j - k_{j-1} \quad \forall p \leq j-1$$

which is 7.2. In conclusion,

$$D_l^k \oplus < \frac{\partial}{\partial y^j_{c_j - l}}, \{\frac{\partial}{\partial y^i_{k_i-1-l}},\ i > j\} >= D_l^{k'} \oplus < \frac{\partial}{\partial y^j_{c_j+1}}, \{\frac{\partial}{\partial y^i_{k_i}},\ i > j\} > \quad l \leq k_{j-1} + r_{j-1}$$

and lemma 5 can be applied. This implies that the involutivity conditions are the same for D_l^k and $D_l^{k'}$ for $l \leq k_{j-1} + r_{j-1}$.
Now, we are going to prove the equality

$$D_{l+1}^k = D_l^{k'} \oplus < \frac{\partial}{\partial y^j_{c_j+1}}, \{\frac{\partial}{\partial y^i_{k_i}},\ i > j\} > \quad l \geq k_{j-1} + r_{j-1}$$

also checking that lemma 5 can be applied.

$$\begin{aligned} D^k_{k_{j-1}+r_{j-1}+1} \quad = \quad & < \eta_0^1, \ldots, \eta_{r_1}^1, \ldots, \eta_0^{j-1}, \ldots, \eta_{r_{j-1}}^{j-1}, \frac{\partial}{\partial y^j_{c_j+1}}, \ldots, \frac{\partial}{\partial y^j_{c_j+1-k_{j-1}-r_{j-1}-1}}, \\ & \{\frac{\partial}{\partial y^i_{k_i}}, \ldots, \frac{\partial}{\partial y^i_1}, i < j\}, \{\frac{\partial}{\partial y^i_{k_i}}, \ldots, \frac{\partial}{\partial y^i_{k_i - k_{j-1}-r_{j-1}-1}}, i > j\} \end{aligned}$$

implies

$$D^k_{k_{j-1}+r_{j-1}+1} = D^{k'}_{k_{j-1}+r_{j-1}} \oplus < \frac{\partial}{\partial y^j_{c_j+1}}, \{\frac{\partial}{\partial y^i_{k_i}},\ i > j\} >$$

And, in general, using induction,

$$D^k_{l+2} = D^k_{l+1} + < \{[f^k, \delta],\ \delta \in D^k_{l+1}\} >=< \frac{\partial}{\partial y^j_{c_j+1}}, \{\frac{\partial}{\partial y^i_{k_i}},\ i > j\} > + D_l^{k'} + < \{[f^k, \delta],\ \delta \in D_l^{k'}\} >=$$

$$=< \frac{\partial}{\partial y^j_{c_j+1}}, \{\frac{\partial}{\partial y^i_{k_i}},\ i > j\} > + D^{k'}_{l+1}$$

Moreover, lemma 5 can be applied since η_h^i does not depend on $y^j_{c_j+1}, \ldots, y^m_{k_m}$. So, the involutivity conditions are the same for $D_l^{k'}$ and D_{l+1}^k, for all $l \geq k_{j-1} + r_{j-1}$. This, together with the fact that the involutivity conditions for $D_l^{k'}$ and D_l^k are the same for all $l \leq k_{j-1} + r_{j-1}$, implies that $\Sigma_{k'}$ is static feedback linearizable providing that Σ_k is too.
Remarks:

1. If for some i, $r_i \geq k_{i+1} - k_i$, it is not difficult to check that the above equalities between distributions of Σ_k and $\Sigma_{k'}$ also hold.

2. Let Σ_k be a static feedback linearizable prolongation of Σ with $k = (0, k_2, \ldots, k_m$. Assume $k_{i_s} > c_{i_s}$ for some i_s. Then, $\Sigma_{k'}$ is static feedback linearizable with $k' = (0, k_2', \ldots, k_m')$, where

$$k_i' = \begin{cases} k_i & \text{if } i < i_1 \\ k_i - 1 & \text{if } i \geq i_1 \end{cases}$$

The proof is exactly the same as the one previouly made. We have chosen to do the above proof in order to clarify the notation.

∎

7.2 About the bounds

The bounds $c_i = 2(i-1) - (i-1)(i+4)/2$ have been chosen to enable us to apply lemma 5. Notice that the equality $c_j - c_{j-1} = 2n - j - 1$ has become fundamental for proving 7.2. This does not mean that these bounds are sharp. In order to prove the sharpness of these bounds, an example has to be constructed. This example must satisfy that the only static feedback linearizable prolongation was that with $k_i = c_i$, for all i. To date, the authors have been unable to find such an example, except that mentioned in Chapter 5.
In any case, the bounds obtained here improve the bounds existent in the literature when the number of inputs is greater than or equal to four. The two and three input cases have been treated in previous chapters. In [59], the bound for the number of integrators is

$$3^{m-2}(2n - \frac{9}{2}) + \frac{3}{2}$$

while the bound obtained here is

$$\sum_{i=1}^{m} c_i = \sum_{i=1}^{m} 2(i-1)n - \frac{(i-1)(i+4)}{2} = m(m-1)n - \frac{m^3 + 6m^2 - 7m}{6}$$

This equality is obvious for $m = 1$. Assuming the equality for the case m, we prove the case $m + 1$ using induction:

$$\sum_{i=1}^{m+1} 2(i-1)n - \frac{(i-1)(i+4)}{2} = \sum_{i=1}^{m} 2(i-1)n - \frac{(i-1)(i+4)}{2} + 2mn - \frac{m(m+5)}{2} =$$

$$= m(m-1)n - \frac{m^3+6m^2-7m}{6} + 2mn - \frac{m(m+5)}{2} =$$

$$= m(m+1)n - \frac{m^3+9m^2+8m}{6} = m(m+1)n - \frac{(m+1)^3+6(m+1)^2-7(m+1)}{6}$$

Proposition 7 *For $m \geq 4$,*

$$3^{m-2}(2n - \frac{9}{2}) + \frac{3}{2} > m(m-1)n - \frac{m^3+6m^2-7m}{6}$$

Proof: Note that $n \geq 5$ because $n > m \geq 4$. Then $2n - 9/2 > n$. Therefore, for $m \geq 5$,

$$3^{m-2}(2n - \frac{9}{2}) + \frac{3}{2} > 3^{m-2}n + \frac{3}{2} > m(m-1)n + \frac{3}{2}$$

On the other hand, since $m^3 + 6m^2 - 7m \geq 0$ (if $m \geq 1$),

$$\frac{3}{2} > 0 \geq -\frac{m^3+6m^2-7m}{6}$$

Putting everything together,

$$3^{m-2}(2n - \frac{9}{2}) + \frac{3}{2} > m(m-1)n - \frac{m^3+6m^2-7m}{6}$$

The case $m = 4$ reduces to prove

$$9(2n - \frac{9}{2}) + \frac{3}{2} > 12n - 22$$

which is equivalent to see $6n > 17$. This is true since $n \geq 5$. ∎

Note that for the two inputs case, the bounds are equal, and also equal to $2n-3$, which is the bound obtained in Chapter 5. For the three inputs case, our general bound is worse than that in [59]. However, the different approach we adopted in Chapter 6 has improved the bound in [59]. Let us recall that the bound obtained in Chapter 6 is $2n-2+r+2r+1$, while the bound in [59] is $6n-12$.

1. If $r = n-1$, there is no need to consider derivations of u_3 greater than $2r+1$. Therefore, the number of integrators is

   $$2(2r+1) = 4r+2 = 4n-2$$

   which is smaller than $6n-12$ for all $n > 5$ (and equal if $n = 5$).

2. In the case $r \leq n-2$, the maximum number of integrators is

   $$2n-2+r+2r+1 = 2n+3r-1 \leq 2n+3n-6-1 = 5n-7$$

   also smaller than $6n-12$ if $n > 5$ (and also equal for $n = 5$).

Chapter 8

Conclusions and suggestions for further research

In this dissertation we have presented different methods for linearizing nonlinear control systems or for studying differential flatness. These have been carried out using two different frameworks, namely: differential algebra and differential geometry. Since our approaches apply only to special classes of systems, the general problem remains open.

8.1 The differential algebraic approach

There are two Chapters in which the differential algebra setting for control systems has been used. In Chapter 2, a survey on linear control systems from the module theory has been presented, while Chapter 4 deals with nonlinear control systems. By means of the tangent system, obtained by application of the Kähler differential, a procedure for finding the last flat output has been designed, providing that the first $m-1$ flat outputs have been guessed. In this context, an easy new proof is given of the well known fact that dynamic and static feedback linearization are equivalent for single-input systems. This has been used to make a new algorithm to linearize single-input systems, working with the concept of relative degree.
Another algorithm for linearizing multi-input systems by static feedback has been created from the translation of the meaning of relative degree into the differential algebraic framework. For not static feedback linearizable systems, a procedure for reducing to single-input systems has been carried out. This procedure is based on guessing the first $m-1$ flat outputs and making a quotient of modules. Here, the results of Chapter two have been crucial.
However, a way of obtaining the first flat output is still a problem. Sometimes, when working with a concrete problem, some variables with physical meaning (center of oscillation [23], center of mass, ...) can be flat outputs. In other problems, one can guess some flat outputs from the structure of the system (backstepping, variables not appearing in any equation, ...). Unfortunately, there is no general method which is good for all systems. One possibility could be to make quotients in the tangent system by the input variables, until a single-input system

is obtained. We have applied this procedure to some examples with good results, but we have not obtained any general solution for all systems. Another difficulty in such a procedure is to struggle with modules with torsion elements, which implies that there exists no basis that generates all the elements of the module.
Another concept strongly related to dynamic feedback linearization and flatness is the concept of defect. The defect can measure how far a system is from being flat. Thus, a system with defect zero is, indeed, a flat system. Let us recall that a system is flat if, and only if, there exists an integrable basis of the module associate with the tangent system. One may suspect that the defect is the minimum number of not integrable elements of a basis, taking into account that there are infinite different basis for a module.

8.2 Linearization by prolongations and possible extensions

In Chapters 5, 6 and 7, a necessary and sufficient condition for a system to be linearizable by means of prolongations is given. This condition states that the involutivity of a finite number of distributions must be checked. The upper bounds on the number of derivatives of the controls added to the original system have been improved. For two-input systems, it has been shown that the bound is sharp, and the results have been applied to some systems that hitherto were thought to be not linearizable by prolongations. This procedure can be applied to other concrete examples. The first case that arises is a driftless system affine in the inputs. We have performed initial explorations in this direction for two-input systems. Let us write the equations of such a system:

$$\Sigma : \ \dot{x} = g_1(x)u_1 + g_2(x)u_2 \quad x \in R^n$$

Let Σ_r be a prolongation of Σ based on adding r derivatives of u_2 (recall that it is only necessary to add derivatives of just one input), the drift of the prolonged system becomes:

$$f^r = g_2(x)y_{n+1} + \sum_{i=1}^{r-1} y_{n+i+1}\frac{\partial}{\partial y_{n+i}}$$

and the input vector fields are

$$g_1^r = g_1(x) \quad g_2^r = \frac{\partial}{\partial y_{n+r}}$$

where

$$y_{n+i} = u_2^{(i-1)}$$

Remark that

$$[f^r, g_1] = [g_2, g_1]$$

and, in general,

$$ad_{f^r}^i g_1 = ad_{g_2}^i g_1$$

The conditions for Σ_r to be static feedback linearizable are the involutivity of the distributions

$$D_i = \langle \frac{\partial}{\partial y_{n+r}}, \dots, \frac{\partial}{\partial y_{n+r-i}}, g_1, \dots, ad_{g_2}^i g_1 \rangle \quad \forall i = 0, \dots, r-1$$

and

$$D_r = \langle \frac{\partial}{\partial y_{n+r}}, \dots, \frac{\partial}{\partial y_{n+1}}, g_1, \dots, ad_{g_2}^r g_1, g_2 \rangle$$

Since D_r must be involutive,

$$ad_{g_2}^{r+1} g_1 = [g_2, ad_{g_2}^r g_1] \in D_r$$

while

$$[f^r, g_2] = 0$$

Therefore, $D_{r+1} = D_r$. Thus, it is not necessary to check the involutivity of more distributions. Furthermore, since the rank of D_r must be $n+r$, then $r+2 \geq n$. Summarizing, Σ is linearizable by prolongations if, and only if,

$$\langle g_1, \dots, ad_{g_2}^i g_1 \rangle$$

are involutive for all $i \leq r-2$ and the rank of

$$\langle g_1, \dots, ad_{g_2}^{r-1} g_1, g_2 \rangle$$

is n. Or, exchanging g_1 by g_2 and viceversa,

$$\langle g_2, \dots, ad_{g_1}^i g_2 \rangle$$

are involutive for all $i \leq r-2$ and the rank of

$$\langle g_2, \dots, ad_{g_1}^{r-1} g_2, g_1 \rangle$$

is n.

These good results encourage us to tackle systems with more inputs using this technique.

As already stated in Chapter 3, a linearization by prolongation is a particular type of dynamic feedback linearization. Let us recall that a dynamic feedback linearization requires the existence of a dynamic compensator

$$\begin{cases} \dot{z} = a^0(x,z) + a^1(x,z)v \\ u = b^0(x,z) + b^1(x,z)v \end{cases} \tag{8.1}$$

with $z \in R^q$ and $v \in R^m$. A dynamic feedback compensator is a prolongation if, and only if,

$$b^0(x,z) = \begin{pmatrix} z_1 & & & \\ & z_{k_1+1} & & \\ & & \ddots & \\ & & & z_{1+\sum_{i=1}^{m-1} k_i} \end{pmatrix}$$

$$b^1(x,z) = 0$$

$$a_i^0(x,z) = \begin{cases} z_{i+1} & \text{if } \; i \neq k_j, \; j = 1, \ldots, m \\ 0 & \text{if } \; i = k_j, \; j = 1, \ldots, m \end{cases}$$

$$a_i^1(x,z) = \begin{cases} 0 & \text{if } \; i \neq k_j, \; j = 1, \ldots, m \\ 1 & \text{if } \; i = k_j, \; j = 1, \ldots, m \end{cases}$$

In order to proceed with the research on dynamic feedback linearization, the authors suggest studying dynamic compensators in an increasing order of difficulty.

Since the sharpness of the bounds obtained here is not clear, new work could well be done with the purpose of finding better bounds. Let us recall that the procedure used in the proofs has been the comparison between distributions of differents systems. However, to ensure the involutivity of a certain distribution of a given system, only some distributions of the other system have been considered. Taking into account all the distributions could lead to more restrictive bounds.

Considering the great number of systems and distributions involved in the application of the procedure in Chapters 5, 6 and 7, a software package for carrying out all the computations should definitively be done. This software package must be programmed carefully, to avoid repetitions of the same calculations.

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Appendix A

Introduction to differential algebra

This appendix is written in order to this thesis be self-contained. It follows [22] and [35].

A.1 Basics on differential algebra

Definition 11 *An ordinary differential ring A is a commutative ring equipped with a single derivation $\frac{d}{dt}$ such that*

- $$\forall a \in A, \quad \dot{a} = \frac{da}{dt} \in A$$

- $$\forall a, b \in A, \quad \frac{d}{dt}(a+b) = \frac{da}{dt} + \frac{db}{dt}$$

- $$\forall a, b \in A, \quad \frac{d}{dt}(ab) = \frac{da}{dt}b + a\frac{db}{dt}$$

A constant $c \in A$ is an element such that $\dot{c} = 0$.

An ordinary field is an ordinary ring ring which is a field. A differential field extension L/K is given by two fields, L and K, such that the restriction to K of the derivation of L coincides with the derivation of K.

Definition 12 *An element $x \in L$ is said to be differentially algebraic over K if, and only if, it satisfies an algebraic differential equation with coefficients on K. The extension L/K is said to be differentially algebraic if, and only if, any element of L is differentially algebraic over K.*

Definition 13 *An element $x \in L$ is said to be differentially K-trascendental if, and only if, it is not differentially algebraic over K. And the extension L/K is said to be differentially transcendental if, and only if, there exists at least one element $x \in L$ that is differentially K-transcendental.*

Definition 14 *A set $\{x_i \mid i \in I\}$ of elements in L is said to be differentially K-algebraically independent if, and only if, the set of derivatives of any order $\{x_i^{(j)} \mid i \in I,\ j \geq 0\}$ is K-algebraically independent. Such an independent set which is maximal with respect to the inclusion is called a differential transcendence basis of the extension L/K.*

Two different transcendence basis of an extension L/K have the same number of elements. This cardinality is called the differential transcendence degree of L/K, and it is denoted $diff\ tr\ d^0 L/K$. The following theorem establishes a relation between differential algebraic extension and transcendental extensions.

Theorem 6 *For a finitely generated differential extension L/K, the next two properties are equivalent:*

1. *L/K is differentially algebraic.*
2. *The transcendence degree (not the differential transcendence degree) of the extension L/K is finite.*

Let K a given differential field. The ring of differential operators over K is denoted by $K[\frac{d}{dt}]$, and it contains all the elements of the form

$$\sum_{i=0}^{n} a_i \frac{d^i}{dt^i}$$

This ring is commutative if, and only if, K is a field of constants. In the non-commutative case is always a principal ideal ring. Thus, the most important properties of the modules over commutative rings are fulfilled also by the left modules over $K[\frac{d}{dt}]$. Let M be a left module over $K[\frac{d}{dt}]$.

Definition 15 *An element $m \in M$ is said to be a torsion element if, and only if, there exists $p \in K[\frac{d}{dt}]$ such that $pm = 0$. A torsion module is a module in which all the elements are torsion elements.*

The following proposition relates a torsion module with a vector space:

Proposition 8 *For a finitely generated left $K[\frac{d}{dt}]$-module M, the next two properties are equivalent:*

1. *M is a torsion module.*
2. *The dimension of M as a K-vector space is finite.*

Definition 16 *A finitely generated module over a principal ideal ring is free if, and only if, there does not exist any torsion element.*

A.2 The Kähler differential

Let A be a differential ring and let B be a differential A-algebra (in our case, A is a differential field and B is a differential field extension of A). Let

$$p: \; B \otimes_A B \longrightarrow B$$

the canonical A-algebra homomorphism such that $p(b \otimes b') = bb'$. Let I be the kernel of p. Then,

$$(b \otimes 1 - 1 \otimes b) \in I$$

Since p is exhaustive and I an ideal of B, applying the theorem of isomorphism we have

$$(B \otimes_A B)/I \cong B$$

Let us recall that I/I^2 is a differential $(B \otimes_A B)/I$-module, because it is also a differential $(B \otimes_A B)$-module. Therefore, I/I^2 is a differential B-module. Let us define now the differential B-module

$$\Omega_{B/A} = I/I^2$$

and the application (Kähler differential):

$$d = d_{B/A} : \; B \longrightarrow \Omega_{B/A}$$

defined by $d(b) = (b \otimes 1 - 1 \otimes b) + I^2$.

Proposition 9 *$\Omega_{B/A}$ has a canonical structure of differential module over B such that, for any derivation δ, $\delta(d(b)) = d(\delta(b))$.*

Proof: The uniqueness of the differential structure is clear since the differential B-module is generated by the elements $d(b)$, for $b \in B$. The existence of the differential structure comes from the above construction. And, for $b \in B$,

$$\delta(d(b)) = \delta((b \otimes 1 - 1 \otimes b) + I^2) = (\delta(b) \otimes 1 - 1 \otimes \delta(b)) + I^2 = d(\delta(b))$$

∎

More details on the Kähler differential can be found on [35] and referencies therein.

Appendix B

Software package for Chapter 4

```
##################################################
#                                                #
# MAPLE V PROGRAMS TO LINEARIZE CONTROL SYSTEMS  #
# USING THE KHLER DIFFERENTIAL                  #
#                                                #
##################################################


###################################################################
#                                                                 #
# Function afegir_temps: it transforms expressions of the form    #
# x_i, u_j, dx_i, du_j in x_i(t), u_j(t), dx_i(t), du_j(t)        #
#                                                                 #
###################################################################

afegir_temps := # Ok!

  proc(expr)

    local res,i,j;
    global F,x,u,dx,du,n,m;

    res := expr;
    for i to n do
      res := subs({x[i] = x[i](t), dx[i] = dx[i](t)},eval(res));
    od;
```

```
    for j to m do
      res := subs({u[j] = u[j](t), du[j] = du[j](t)},eval(res));
    od;

    eval(res);
  end;


#####################################################################
#                                                                   #
# Function treure_temps: it transforms expressions of the form     #
# x_i(t), u_j(t), dx_i(t), du_j(t) in x_i, u_j, dx_i, du_j         #
#                                                                   #
#####################################################################

treure_temps := # Ok!

  proc(expr)

    local res,i,j;
    global F,x,u,dx,du,n,m;

    res := expr;
    for i to n do
      res := subs({x[i](t) = x[i],dx[i](t) = dx[i]},eval(res));
    od;
    for j to m do
      res := subs({u[j](t) = u[j],du[j](t) = du[j]},eval(res));
    od;

    eval(res);
  end;


#####################################################
#                                                   #
# Function d_dt: it computes temporal derivatives  #
# of expressions of the form x_i, dx_i, doing the  #
# substitutions given by the systems  sys and lin. #
#                                                   #
#####################################################
```

```
d_dt := # Ok!

  proc(expr,sys,lin)

    local res,i,j;
    global F,x,u,dx,du,n,m;

    res := expr;
    res := afegir_temps(res);
    res := diff(res,t);
    for i to n do
      res := subs({sys[i],lin[i]},eval(res));
    od;
    for j to m do
      res := subs(diff(u[j](t),t)=Diff(u[j],t),eval(res));
    od;

    res := treure_temps(res);

    eval(res);
  end;


############################################################
#                                                          #
# Function Kahler: it computes the tangent system of a     #
# systema x'=F(x,u). Writing down this tangent system as   #
# dx'=Adx+Bdu, the outputs are the matrixes A and B, and   #
# the right hand side of the equations.                    #
#                                                          #
############################################################

Kahler:= # Ok!

  proc()

    local A,B,lin,i,j;
    global F,x,dx,u,du,n,m;

    A:=matrix(n,n); B:=matrix(n,m);
    lin:=vector(n); i:=0;
```

```
    A:=jacobian(F,x); B:=jacobian(F,u);

    for i to n do
      lin[i]:=dotprod(row(A,i),dx)+dotprod(row(B,i),du);
    od;

    eval([evalm(A),evalm(B),eval(lin)]);
  end;


##############################################################
#                                                            #
# Function EqSys: it computes the equations of the original #
# system in the form x'(t)=F(x(t),u(t)).                     #
#                                                            #
##############################################################

EqSys:= # Ok!

  proc()

    global F,n,m,x,dx,u,du;
    local i,sortida;

    sortida:=vector(n); i:=1;

    for i to n do
      sortida[i]:= diff(x[i](t),t)=afegir_temps(F[i]);
    od;

    eval(sortida);
  end;


####################################################################
#                                                                  #
# Function EqLin: it computes the equations of the tangent system #
# in the form dx'(t)=A(x(t),u(t))dx(t)+B(x(t),u(t))du(t).          #
# The third output of the function Kahler must be given as a       #
# parameter.                                                       #
#                                                                  #
####################################################################
```

```
EqLin:= # Ok!

  proc(kah3)

    global F,n,m,x,dx,u,du;
    local i,sortida;

    sortida:=vector(n); i:=1;

    sortida:=afegir_temps(eval(kah3));
    for i to n do
      sortida[i]:=diff(dx[i](t),t)=sortida[i];
    od;

    eval(sortida);
  end;

############################################################
#                                                          #
# Function equacions: it gives the result of the former    #
# two equations, EqSys i EqLin.                            #
#                                                          #
############################################################

equacions:= # Ok!

  proc(kah3)
    global F,n,m,x,dx,u,du;
     eval([EqSys(),EqLin(kah3)]);
  end;

############################################################
#                                                          #
# Function integrable: the result is 1 or 0, depending     #
# whether or not the 1-form given to the function is       #
# integrable (exact 1-form) or not. It is done using       #
# Schwarz's conditions of integrability:                   #
# w=a_1 dx_1+...+a_n dx_n integrable if, and only if,      #
#      diff(a_i,x_j)==diff(a_j,dx_i), for all i,j          #
#                                                          #
############################################################
```

```
integrable:= # Ok!

  proc(w)

    local i,j,sortida;
    global F,x,dx,u,du,n,m;

    sortida:=1;

    for i to n do
      for j to n do
        if diff(coeff(w,dx[i]),x[j])<>diff(coeff(w,dx[j]),x[i]) then sortida:=0; fi;
        if sortida=0 then j:=n; fi;
      od;
      if sortida=0 then i:=n; fi;
    od;

    eval(sortida);
  end;


#############################################################
#                                                           #
# Function SingleInput: it computes the coefficients of     #
# the basis w for single-input systems                      #
#                                                           #
#############################################################

SingleInput:= # Ok!

  proc()

    local sol,kah,eq;
    global F,n,m,x,dx,u,du;

    kah:=vector(3); eq:=vector(2);
    kah[1]:=matrix(n,n); kah[2]:=matrix(n,m);

    kah:=Kahler();
    eq:=equacions(kah[3]);
    sol:=trobar_base(kah[1],kah[2],eq,n,1);
```

```
    eval([eval(sol),eval(dotprod(sol,dx))]);
  end;


##############################################################
#                                                            #
# Function SingleInput2: it computes the coefficients of     #
# the basis w for single-input systems coming from           #
# quotients done in dynamic feedback linearizable            #
# multiple-input systems.                                    #
#                                                            #
##############################################################

SingleInput2:= # Ok!

  proc(lin2,zeros)

    local i,j,compt,A,B,eq,sol,variable,uns,auxA,auxxA,auxB,auxdx;
    global F,n,m,x,dx,u,du;

    uns:=sum('zeros[i]','i'=1..n);

    auxA:=matrix(n,n); auxxA:=matrix(n,n-uns); auxB:=matrix(n,1);
    A:=matrix(n-uns,n-uns); B:=matrix(n-uns,1);
    eq:=vector(2); eq[1]:=vector(n); eq[2]:=vector(n);
    sol:=vector(n-uns); auxdx:=vector(n-uns);

    variable:=0;
    for i to n do
      for j to m do
        if coeff(lin2[i],du[j])<>0 then
          variable:=j;
          j:=m+1;
        fi;
      od;
      if variable<>0 then i:=n+1; fi;
    od;

    eq:=equacions(lin2);
    auxA:=jacobian(lin2,dx);
    auxB:=jacobian(lin2,[du[variable]]);
```

```
    compt:=1;
    for i to n do
      if zeros[i]=0 then
        copy_vec_col(n,col(auxA,i),auxxA,compt);
        compt:=compt+1;
      fi;
    od;
    compt:=1;
    for i to n do
      if zeros[i]=0 then
        copy_vec_row(n-uns,row(auxxA,i),A,compt);
        copy_vec_row(1,row(auxB,i),B,compt);
        compt:=compt+1;
      fi;
    od;

    sol:=trobar_base(A,B,eq,n-uns,2);

    compt:=1;
    for i to n do
      if zeros[i]=0 then
        auxdx[compt]:=dx[i];
        compt:=compt+1;
      fi;
    od;

    eval([eval(sol),eval(dotprod(sol,auxdx))]);
  end;


#############################################################
#                                                           #
# Function trobar_base: it is an auxiliar function for      #
# SingleInput and SingleInput2. It solves a linear system   #
# in order to find the coefficients of the basis w.         #
#                                                           #
#############################################################

trobar_base:= # Ok!

  proc(A,col,eq,dim,flag)
```

```
    local aux,M,i,o,a,oldm,oldn;
    global F,n,m,x,dx,u,du;

    oldm:=m; oldn:=n;

    if flag=2 then
      m:=1;
      n:=dim;
    fi;

    a:=vector(n); M:=matrix((dim-1)*m,n);
    o:=vector((dim-1)*m,t->0); aux:=matrix(n,1);

    aux:=col;
    M:=transpose(aux);
    for i from 2 to dim-1 do
      aux:=evalm(multiply(A,aux)-map(d_dt,aux,eq[1],eq[2]));
      M:=stack(M,transpose(aux));
    od;

    m:=oldm; n:=oldn;

    eval(linsolve(M,o,'r',a));
  end;

#############################################################
#                                                           #
# Function vec_col: It is an auxiliar function that         #
# transforms a vector in a column matrix.                   #
#                                                           #
#############################################################

vec_col:= # Ok!

  proc(dim,v)

    local i, M;
    global F,n,m,x,dx,u,du;

    M:=linalg[matrix](dim,1);
```

```
    for i to dim do M[i,1]:=v[i]; od;

    evalm(M);
  end;

##############################################################
#                                                            #
# Function col_vec: it is an auxiliar function that          #
# transforms a column matrix in a vector.                    #
#                                                            #
##############################################################

col_vec:= # Ok!

  proc(dim,B)

    local i, v;
    global F,n,m,x,dx,u,du;

    v:=linalg[vector](dim);

    for i to dim do v[i]:=B[i,1]; od;

    eval(v);
  end;

##############################################################
#                                                            #
# Function copy_vec_col: it is an auxiliar function that     #
# writes a vector 'vec' of dimension 'dim' in the column     #
# 'j' of the matrix 'Mat'.                                   #
#                                                            #
##############################################################

copy_vec_col:= # Ok!

  proc(dim,vec,Mat,j)

    local i;
    global F,n,m,x,dx,u,du;
```

```
    for i to dim do Mat[i,j]:=vec[i]; od;

    evalm(Mat);
  end;


#############################################################
#                                                           #
# Function copy_vec_row: it is an auxiliar function that    #
# writes a vector 'vec' of dimension 'dim' in the row 'i'   #
# of the matrix 'Mat'                                       #
#                                                           #
#############################################################

copy_vec_row:= # Ok!

  proc(dim,vec,Mat,i)

    local j;
    global F,n,m,x,dx,u,du;

    for j to dim do Mat[i,j]:=vec[j]; od;

    evalm(Mat);
  end;


#############################################################
#                                                           #
# Function Kronecker: it is an auxiliar function that       #
# computes the Kronecker indices for static feedback        #
# linearizable multi-input systems                          #
#                                                           #
#############################################################

Kronecker:= # Ok!

  proc(kah,eq)

    local D,d,j,k,rho,K,Aux;
    global F,n,m,x,dx,u,du;
```

```
    D:=matrix(n,m); Aux:=matrix(n,m);
    d:=vector(n); rho:=vector(n); K:=vector(m);

    Aux:=evalm(kah[2]);
    D:=evalm(Aux);
    d[1]:=rank(Aux);
    for k from 2 to n do
      Aux:=evalm(multiply(kah[1],Aux)-map(d_dt,Aux,eq[1],eq[2]));
      D:=augment(D,Aux);
      d[k]:=rank(D);
    od;

    rho[1]:=d[1];
    for k from 2 to n do
      rho[k]:=d[k]-d[k-1];
    od;

    for j from 1 to m do
      K[j]:=0;
      for k from 1 to n do
        if (rho[k]>=j) then
          K[j]:=K[j]+1;
        fi;
      od;
    od;

    eval(K);
  end;


##############################################################
#                                                            #
# Function MultiInputSFL: it computes the coefficients       #
# of the 1-forms w_1,...,w_m that make a basis for           #
# static feedback linearizable multi-input systems.          #
# It is analagous to the function SingleInput.               #
#                                                            #
##############################################################

MultiInputSFL:=  # Ok!

  proc()
```

```
local i,l,j,k,eq,kah,aux,aux2,K,W,a,o,M,formes;
global F,n,m,x,dx,u,du;

kah:=vector(3);
kah[1]:=matrix(n,n); kah[2]:=matrix(n,m);
W:=matrix(n,m); K:=vector(m); formes:=vector(m);

kah:=Kahler();
eq:=equacions(kah[3]);

K:=Kronecker(kah,eq);

for l from 1 to m do
  if (K[l] > 1) then
    a:=vector(n); M:=matrix((K[l]-1)*m,n);
    o:=vector((K[l]-1)*m,t->0);
    aux:=vector(n); aux2:=vector(n);

    aux:=vec_col(n,col(kah[2],1));
    M:=transpose(aux);
    for i from 2 to (K[l]-1) do
      aux:=evalm(multiply(kah[1],aux)-map(d_dt,aux,eq[1],eq[2]));
      M:=stack(M,transpose(aux));
    od;

    for j from 2 to m do
      aux:='aux'; aux:=vec_col(n,col(kah[2],j));
      M:=stack(M,transpose(aux));
      for i from 2 to (K[l]-1) do
        aux:=evalm(multiply(kah[1],aux)-map(d_dt,aux,eq[1],eq[2]));
        M:=stack(M,transpose(aux));
      od;
    od;

    aux2:=linsolve(M,o,'r',a);
    copy_vec_col(n,aux2,W,l);
  else

    a:=vector(n);
    copy_vec_col(n,a,W,l);
  fi;
```

```
   od;
   for j to m do
     formes[j]:=dotprod(col(W,j),dx);
   od;

   eval([evalm(W),eval(formes)]);
 end;


###########################################################
#                                                         #
# Function quocient: it makes the quotient of a certain   #
# system by a general expression 'w'.                     #
#                                                         #
###########################################################

quocient:= # Ok!

  proc(w)

    global F,n,m,x,dx,u,du;
    local i,final,currenteq,expr,kah,sys,lin,lin2,k,var,zeros;

    final:=0; expr:=w;

    kah:=Kahler();
    sys:=EqSys(); lin:=EqLin(kah[3]);
    lin2:=kah[3]; zeros:=vector(n);

    for i to n do zeros[i]:=0; od;

    while final<>1 do
      if coeff(expr,du[1])<>0 then
        currenteq:=du[1]=solve(expr,du[1]);
        for k to n do
          lin2[k]:=subs(currenteq,lin2[k]);
        od;
        final:=1;
      elif coeff(expr,du[2])<>0 then
        currenteq:=du[2]=solve(expr,du[2]);
        for k to n do
          lin2[k]:=subs(currenteq,lin2[k]);
```

```
      od;
      final:=1;
    else
      var:=1;
      while coeff(expr,dx[var])=0 do var:=var+1; od;
      zeros[var]:=1;
      currenteq:=dx[var]=solve(expr,dx[var]);
      for k to n do
        lin2[k]:=subs(currenteq,lin2[k]);
      od;
      lin:=EqLin(lin2);
      expr:=d_dt(expr,sys,lin);
      expr:=subs(currenteq,expr);
    fi;
  od;

  eval([map(simplify,eval(lin2)),eval(zeros)]);
end;
```