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Global geometry of surfaces defined by non-positive and negative at infinity valuations.

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Memoria presentada por Carlos Jesús Moreno Ávila para optar al grado de doctor por la Universitat Jaume I

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"El buen paso, el regalo y el reposo allá se inventó para los blandos; mas el trabajo, la inquietud y las armas solo se inventaron e hicieron para aquellos que el mundo llama caballeros andantes, de los cuales yo, aunque indigno, soy el menor de todos."
(Miguel de Cervantes, *Quijote*, I, XIII).

"La Matemática es una hermosa y arisca dama que solo muestra sus encantos a quienes porfían con ella."
(Carlos Benítez, *Lección inaugural del curso académico 2007/08 de la UEx*).

Abstract

We introduce the concepts of non-positive and negative at infinity plane valuation of a Hirzebruch surface and determine nice global and local geometric properties of the surfaces given by those divisorial valuations.

Let $\mathbb{F}_\delta$ be a Hirzebruch surface over an algebraically closed field, where δ is a non-negative integer. The concepts of non-positive and negative at infinity divisorial valuation were firstly introduced in [60] for valuations considered over the projective plane $\mathbb{P}^2$. We extend these concepts to divisorial valuations of $\mathbb{F}_\delta$, giving easy to check conditions to decide whether a valuation is of these types, and study positivity properties of the surfaces that they define. We prove that non-positive at infinity divisorial valuations of $\mathbb{F}_\delta$ are those divisorial valuations of $\mathbb{F}_\delta$ such that cone of curves of the surface that they define is minimally generated. Our results extend those in [60].

Non-positivity and negativity at infinity are also extended to the class of real valuations of $\mathbb{P}^2$ and $\mathbb{F}_\delta$. Their dual graphs are studied and compared according to the valuations they come from.

Finally, given a flag $E_\bullet = \{Z \supset E \supset \{q\}\}$, where E is an exceptional divisor defining a non-positive at infinity divisorial valuation of $\mathbb{F}_\delta$ over $\mathbb{C}$ and Z the rational surface given by ν_E, we explicitly compute the Seshadri-type constant for pairs (ν_E, D), where D is a big divisor on $\mathbb{F}_\delta$, and obtain the vertices of the Newton-Okounkov bodies of pairs $(E_\bullet, D)$.

Resumen

Introducimos los conceptos de no positividad y negatividad en el infinito para valoraciones planas de una superficie de Hirzebruch y determinamos interesantes propiedades geométricas globales y locales de las superficies definidas por las valoraciones divisoriales que cumplen dicha condición.

Sea $\mathbb{F}_\delta$ una superficie de Hirzebruch sobre un cuerpo algebraicamente cerrado, donde δ es un entero no negativo. Los conceptos de valoración divisorial no positiva y negativa en el infinito fueron primeramente introducidos en [60] para valoraciones sobre $\mathbb{P}^2$. Nosotros extendemos estos conceptos a valoraciones divisoriales de $\mathbb{F}_\delta$, aportando una condición fácil de comprobar para decidir cuándo una valoración es de estos tipos, y estudiamos propiedades de positividad de las superficies que estas definen. En particular, probamos que las valoraciones divisoriales de $\mathbb{F}_\delta$ no positivas en el infinito son aquellas valoraciones divisoriales de $\mathbb{F}_\delta$ tales que el cono de curvas de la superficie que definen está generado por un número mínimo de generadores.

Los conceptos de no positividad y negatividad en el infinito también los extendemos para valoraciones reales de $\mathbb{P}^2$ y $\mathbb{F}_\delta$. Sus grafos duales son estudiados y comparados acorde a las valoraciones que les corresponden.

Sea una bandera $E_\bullet = \{Z \supset E \supset \{q\}\}$, donde E es un divisor excepcional que define una valoración divisorial ν_E de $\mathbb{F}_\delta$ (sobre $\mathbb{C}$) no positiva en el infinito y Z la superficie racional dada por ν_E. En la última parte de la tesis calculamos explícitamente la constante de tipo Seshadri de pares (ν_E, D), donde D es un divisor big en $\mathbb{F}_\delta$, y también obtenemos los vértices de los cuerpos de Newton-Okounkov de pares $(E_\bullet, D)$.

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Introduction

Let $A_1(X)$ (respectively, $A^1(X)$) be the $\mathbb{R}$-vector space of numerical equivalence classes of one-cycles (respectively, numerical equivalence classes of Cartier divisors) of an n-dimensional complete algebraic scheme X over an algebraically closed field k of any characteristic. Assuming that X is a smooth projective variety, $A^1(X)$ is a vector space of finite dimension. The dimension of both $A^1(X)$ and its dual $A_1(X)$ is called the Picard number of X and written $\rho = \rho(X)$.

Given X as above (projective and smooth), the *cone of curves* of X, $\mathrm{NE}(X)$, is the convex cone (see Subsection 1.4.1) of $A_1(X)$ spanned by the numerical equivalence classes of all effective 1-cycles of X (positive combinations of integral curves); its closure $\overline{\mathrm{NE}}(X)$ is a closed convex cone. In addition, the *numerically effective (nef) cone* of X, $\mathrm{Nef}(X)$, is the convex cone in $A^1(X)$ generated by the numerical equivalence classes of Cartier divisors whose intersection product with all irreducible curves is non-negative (and it is also the dual cone of $\overline{\mathrm{NE}}(X)$). The previously introduced cones are relevant in the study of the geometry of the variety X. When X is 2-dimensional, the case considered in this work, $A^1(X)$ is identified with $A_1(X)$ and the previously defined convex cones live in the same vector space of finite dimension. In addition, it holds that $\mathrm{Nef}(X) \subseteq \overline{\mathrm{NE}}(X)$. See [11, 94, 95, 21, 110] for some recent works with respect to $\overline{\mathrm{NE}}(X)$ and [23, 24, 29, 45] with respect to $\mathrm{Nef}(X)$.

Kleiman [76] also proved an important numerical criterion for ample divisors. It claims that the closure of the cone of curves $\overline{\mathrm{NE}}(X)$ except of the origin lies entirely in the positive halfspace of $A_1(X)$ determined by any ample Cartier divisor.

The cone $\overline{\mathrm{NE}}(X)$ plays a significant role in the minimal model program. This program aims a birational classification theory of higher-dimensional algebraic varieties and started at the end of the 20th century. The program pretends to extend (and include) the case of curves and the Enriques classification of algebraic surfaces. In the surfaces case, Enriques, Castelnuovo, Severi, Zariski, Mumford, Bombieri were important contributors. Some of the main mathematicians involved in the higher dimensional case are Mori, Kollar, Shokurov, Kawamata, Reid, Cascini and Birkar. Even when the program is in constant evolution, one can found an introduction in [77] and also in [88], where the Enriques classification of surfaces is included in the framework of the minimal model program. See [10] for a recent survey.

Let K_X be a canonical divisor on a smooth projective complex variety X. Set

D any divisor on X and write $\overline{\mathrm{NE}}(X)|_{D_{\geq 0}} = \overline{\mathrm{NE}}(X) \cap D_{\geq 0}$ the subset of $\overline{\mathrm{NE}}(X)$ which is in the non-negative half-space given by D. The closure of the cone of curves has a particular structure which helps to construct minimal models for varieties. This fact was proved by Mori [93] for the 3-dimensional case, and later generalized by Kawamata [75] for arbitrary dimension. Indeed, assuming that the canonical divisor K_X is not nef, the subset $\overline{\mathrm{NE}}(X)|_{K_X<0}$ is generated by countable many rational curves and they can only accumulate on the hyperplane defined by the elements of $A_1(X)$ whose intersection product with K_X vanishes. Even more, fixed an ample divisor H and given a positive number ε, there are only finitely many curves $C_1, C_2, \ldots, C_s$ whose classes lie in the region $\overline{\mathrm{NE}}(X)|_{K_X+\varepsilon H\leq 0}$ and then $\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)|_{K_X+\varepsilon H\geq 0} + \sum_{i=1}^{s} \mathbb{R}_{>0}[C_i]$. These curves C_i generate extremal rays R_i of $\overline{\mathrm{NE}}(X)$ and define maps which contract all the curves whose class lies in the extremal rays.

In general, even if X is a smooth projective surface, there is no characterization of the fact that the cone $\overline{\mathrm{NE}}(X)$ is finitely generated (i.e. it is finite polyhedral and then $\overline{\mathrm{NE}}(X) = \mathrm{NE}(X)$). It is known (see [71, Chapter V, Problem 4.15] and [85, Section 1.5.D]) that if Y is the surface obtained by blowing-up ten or more points of $\mathbb{P}^2$ in very general position, there are infinitely many smooth irreducible curves with self-intersection -1 (named (-1)-curves) and these curves span extremal rays of $\overline{\mathrm{NE}}(Y)|_{K_Y<0}$. In addition, it is conjectured that the region $\overline{\mathrm{NE}}(Y)|_{K_Y>0}$ is "almost circular" (i.e. is supported upon a spherical cone) [33, 34].

When X is a smooth projective surface, the literature contains several works related to the structure of $\overline{\mathrm{NE}}(X)$. In [106] it is proved that the unique smooth surfaces whose cone of curves could have infinitely many -1-curves are the rational surfaces. In a complementary way, many authors, using different methods and under specific assumptions on rational surfaces X, have given either sufficient or equivalent conditions to the fact that $\overline{\mathrm{NE}}(X)$ is finite polyhedral (see for example [69, 89, 17, 55, 70, 56, 60, 35, 50, 51]).

Valuations have been used in several areas of the mathematics to study different problems, including the classification of varieties. In 1882 Dedekind and Weber used valuations to treat Riemann surfaces, but the first axiomatic definition of valuation was given in 1912 by Kürschák. Valuations are important objects in the problem of resolution of singularities as one can see in the works of Zariski and Abhyankar [114, 115, 117, 1, 2]. Although Hironaka solved this problem in characteristic zero without using them, they seem to be a relevant tool for the case of positive characteristic [111]. Some recent works about valuations are [41, 112, 16, 96, 105, 97, 32, 6].

There is no classification of valuations. Nevertheless, valuations of the quotient field of regular two-dimensional local rings $(R, \mathfrak{m})$ centered at $\mathfrak{m}$ were classified by Zariski in terms of three invariants: the rank, the rational rank and the transcendence degree. These valuations are usually named plane valuations. A refinement of the previous classification was given by Spivakosky [109] in terms of dual graphs (see also [47] and [66]). Plane valuations have an interesting geometric point of view: they are in one-to-one correspondence with simple sequences of point blowups which start with the blowup of Spec$R$ at $\mathfrak{m}$. That is, sequences of point blowups where each center (except the first one $\mathfrak{m}$) is a closed point in the last created exceptional divisor. These sequences need not to be finite and those which are correspond to divisorial valuations: those defined by the order at the last appearing exceptional divisor in the corresponding sequence of blowups.

Spivakovsky's classification divides plane valuations in five types, being divisorial valuations one of them. In this dissertation we will be particularly interested in two other types: irrational and exceptional curve valuations (see Section 1.3).

In the last years several authors have considered a class of plane valuations ν called *non-positive at infinity*. These valuations provide nice information about geometric global objects as the cone of curves of surfaces given by ν [20, 60] or on parameters of good error-correcting codes linked to this class of valuations [57, 59]. Non-positive at infinity valuations form part of the so-called valuations centered at infinity used to study the dynamics of polynomial maps of the affine plane [47, 48, 49].

Recall that k stands for an algebraically closed field of arbitrary characteristic and $\mathbb{P}^2 := \mathbb{P}^2_k$ the projective plane over k. Denote by $(X : Y : Z)$ projective coordinates in $\mathbb{P}^2$, by L the projective line "at infinity" with equation $Z = 0$, and by p the point of $\mathbb{P}^2$ whose coordinates are $(1 : 0 : 0)$. Take affine coordinates $x = X/Z$ and $y = Y/Z$. Consider a plane valuation ν of the function field of $\mathbb{P}^2$ centered at $\mathcal{O}_{\mathbb{P}^2,p}$. The valuation ν is called non-positive at infinity if it holds that $\nu(f) \leq 0$ for all $f \in k[x, y] \setminus \{0\}$.

A sub-class of non-positive at infinity valuations was studied in [20]. Valuations in that sub-class are determined by those sequences of point blowups which remove the base points of pencils defined by the line at infinity L and curves with only one place at infinity (i.e., a projective curve C on $\mathbb{P}^2$ which is reduced and unibranched at p and such that $C \cap L = \{p\}$). The authors show that the cone of curves of the surfaces defined by the previous valuations ν is generated by the smallest possible set of curves, that is the classes of the strict transform of L and those of the strict transforms of the exceptional divisors created by the sequence of blowups given by ν (i.e. the cone of curves is regular). This result is extended in [60] proving that the divisorial valuations ν providing surfaces with regular cone of curves are exactly those which are non-positive at infinity. Even more, [60, Theorem 1] gives other two equivalent conditions to the result of the cone of curves: a numerical local property which is easy to check and the nefness of a certain divisor derived from ν. [60] also presents sufficient and necessary conditions to the fact that $\nu(f) < 0$ for all $f \in k[x, y] \setminus k$.

Non-positive at infinity valuations are also useful in coding theory. There is another sub-class of non-positive at infinity valuations which is very related to projective curves with only one place at infinity. Abhyankar and Moh [3, 4] studied these curves and proved, under certain conditions, a nice result called Abhyankar-Moh (semigroup) Theorem, which determines the generators of the so-called semigroup at infinity. In [58], the authors show an analogue result to Abhyankar-Moh Theorem for the semigroup at infinity of plane valuations of the former sub-class $S_{\nu,\infty} = \{-\nu(f) \mid f \in k[x,y] \setminus \{0\}\}$ considering adequate fields k. This semigroup is the key to construct and determine parameters of a large set of error-correcting codes [57, 59].

Assume now for a while that X is a smooth irreducible projective variety over $k = \mathbb{C}$ of dimension n.

Recall that a divisor D on X is big when $h^0(X, \mathcal{O}_X(mD))$ grows like m^n [85, Section 2.2]. The convex cone in $A^1(X)$ generated by the numerical equivalence classes of big divisors is called the big cone of X and denoted $\mathrm{Big}(X)$. It holds that $\mathrm{Big}(X) = \mathrm{Int}\big(\overline{\mathrm{NE}}(X)\big)$ [85, Theorem 2.2.26], where Int means topological interior, and, when X is a surface, there is a locally finite decomposition of $\mathrm{Big}(X)$ into rational locally polyhedral subcones called Zariski chambers (see Subsection 1.5.2 and [7]).

Newton-Okounkov bodies of big divisors D on varieties X as above are convex sets used to study the asymptotic behaviour of linear systems $|mD|$, for $m \gg 0$. Firstly, they were introduced by Okounkov [99, 100] and afterwards developed independently by Lazarsfeld and Mustaţă [86], and Kaveh and Khovanskii [74]. Many authors are interested in these convex sets since they also reveal interesting information about invariants and positivity properties of divisors on X [15, 104, 81, 79, 80, 82]. Other recent works about Newton-Okounkov bodies are [26, 27, 92, 12, 101].

Let us recall the definition of Newton-Okounkov body of a big divisor with respect to a flag of subvarieties of X. A flag of subvarieties $Y_\bullet$ of X is a sequence of smooth irreducible subvarieties Y_i of codimension i in X, $0 \leq i \leq n$:

$$Y_\bullet := \{X = Y_0 \supset Y_1 \supset \ldots \supset Y_n = \{q\}\}.$$

The flag $Y_\bullet$ defines a valuation $\nu_{Y_\bullet}$ of the function field $K(X)$ with rank n. Then the Newton-Okounkov body $\Delta_{\nu_{Y_\bullet}}(D)$ of a big divisor D on X with respect to $\nu_{Y_\bullet}$ (or $Y_\bullet$) is the closed convex hull of the set

$$\bigcup_{m \geq 1} \left\{ \frac{\nu_{Y_\bullet}(f)}{m} \mid f \in H^0(X, \mathcal{O}_X(mD)) \setminus \{0\} \right\}.$$

The Newton-Okounkov bodies $\Delta_{\nu_{Y_\bullet}}(D)$ are non-empty compact sets [86, 74, 13] and polygons when X is a surface [82]. In higher dimensions they are extremely complicated and could be non-polyhedral even if we suppose that X and D satisfy good properties [82].

The bodies $\Delta_{\nu_{Y_\bullet}}(D)$ satisfy the following property:

$$\mathrm{vol}_X(D) = n! \, \mathrm{vol}_{\mathbb{R}^n}(\Delta_{\nu_{Y_\bullet}}(D)),$$

where $\mathrm{vol}_{\mathbb{R}^n}$ means the standard volume and

$$\mathrm{vol}_X(D) := \limsup_{m\to\infty} \frac{h^0(X, \mathcal{O}_X(mD))}{m^n/n!},$$

called the volume of D. This volume is a birational invariant of big divisors (see [85, Proposition 2.2.43 and Example 2.2.49]) which gives information about the asymptotic behaviour of their linear systems. We cite some works that study the volume of a divisor: [85, Section 2.2], [7] and [42]. The volume of an ample divisor D coincides with its self-intersection number. Other recent works about the volume of big divisors are [14, 113, 98, 25].

Some authors have found a relation between the Newton-Okounkov bodies and Seshadri constants [79, 78], objects introduced by Demailly in order to study Fujita's conjecture [37]. One can see [39, 67, 87, 118, 68, 101] for recent works about Seshadri constants.

An explicit computation of the Newton-Okounkov bodies $\Delta_{\nu_{Y_\bullet}}(D)$ is a difficult task even if X is a surface. In this last case, Lazarsfeld and Mustaţă [86] provide a generic way to describe these bodies. This description depends on the Zariski decomposition of certain divisors (see Subsection 1.5.2). More explicitly and roughly speaking, the authors describe in [86] the Newton-Okounkov bodies $\Delta_{\nu_{Y_\bullet}}(D)$ as convex sets whose vertices are determined by the Zariski decomposition of divisors $D_t = D - tC$, where C is the smooth irreducible subvariety of codimension 1 in $Y_\bullet$ and t is a (real) number such that $a \leq t \leq \mu(D, C)$, a being the coefficient of C in the negative part of the Zariski decomposition of D and $\mu(D, C) := \sup\{s > 0 \,|\, D - sC \text{ is big}\}$. In particular, the first components of the vertices of the bodies $\Delta_{\nu_{Y_\bullet}}(D)$ are those values t where the ray $[D_t]$ crosses into a different Zariski chamber [82]. Furthermore, [65] considers surfaces X defined by divisorial valuations ν of the function field of $\mathbb{P}^2$ centered at some point $p \in \mathbb{P}^2$ and flags $E_\bullet = \{X \supset E \supset \{q\}\}$ where E is the exceptional divisor defining $\nu\,(= \nu_E)$. In that paper, the authors show that the exceptional curve valuations centered at p and valuations $\nu_{E_\bullet}$ defined by flags $E_\bullet$ are the same thing. Moreover, it is proved that the Newton-Okounkov body of the pullback of a general projective line H with respect to $E_\bullet$ is a triangle or a quadrilateral, characterizing when one gets triangle or not. Even more, these Newton-Okounkov bodies are explicitly and completely computed when the valuation ν_E is non-positive at infinity. Previously, these Newton-Okounkov bodies were only characterized for surfaces defined by certain family of divisorial valuations [28].

On the other hand, some authors [15, 38, 64] are interested in an invariant associated to the vanishing sequence of sections of a big line bundle L, $H^0(X, L)$, along a real valuation ν of the function field $K(X)$ centered at a local ring $(R, \mathfrak{m})$ (over an algebraically closed field k). This invariant is defined as

$$\hat{\mu}_L(\nu) := \lim_{m\to\infty} a_{\max}(mL, \nu)/m,$$

where $a_{\max}(mL, \nu)$ is the last value of the vanishing sequence. This object provides similar information for valuations as the Seshadri constant does for points. We call $\hat{\mu}_L(\nu)$ the Seshadri-type constant for the pair (L, ν) (see [31] for a similar invariant). In general, computing the value $\hat{\mu}_L(\nu)$ is also a hard task, although there exists the bound $\hat{\mu}_L(\nu) \geq \sqrt{\mathrm{vol}(L)/\mathrm{vol}(\nu)}$, where $\mathrm{vol}(\nu)$ means volume of the valuation ν, i.e.

$$\mathrm{vol}(\nu) = \lim_{\alpha \to \infty} \frac{\mathrm{length}(R/\mathcal{P}_\alpha)}{\alpha^n/n!},$$

$\mathcal{P}_\alpha$ being the set $\mathcal{P}_\alpha = \{f \in R \mid \nu(f) \geq \alpha\} \cup \{0\}$ (see [15]). Assuming $X = \mathbb{P}^2$, L a projective line which does not go through p and ν_E a divisorial valuation of the function field of $\mathbb{P}^2$ centered at $\mathcal{O}_{\mathbb{P}^2,p}$, the last bound is expressed as $\hat{\mu}_L(\nu_E) \geq \sqrt{1/\mathrm{vol}(\nu_E)}$ and, if the equality holds, the valuation ν_E is called minimal. This concept is strongly involved in a valuative conjecture formulated in [64] (see also [38]) which implies the well-known Nagata conjecture. In addition, $\hat{\mu}_L(\nu_E)$ can be geometrically understood as $\hat{\mu}_L(\nu_E) = \sup\{s > 0 \,|\, L^* - sE \text{ is big}\}$ (when k has characteristic zero), where L^* is the pull-back of L on the surface defined by ν_E, which establishes a relation with the Newton-Okounkov bodies [15, 28, 65]. When ν_E is non-positive at infinity, $\hat{\mu}_L(\nu_E)$ have been computed explicitly in [64].

Let $\mathbb{F}_\delta$ be the δth *Hirzebruch surface* over an algebraically closed field k, δ being a non-negative integer. The projective plane $\mathbb{P}^2$ and the Hirzebruch surfaces $\mathbb{F}_\delta$, $\delta \neq 1$, constitute the classical minimal models for rational surfaces [9]. In this dissertation we mainly consider divisorial valuations ν of the function field of $\mathbb{F}_\delta$ centered at $\mathcal{O}_{\mathbb{F}_\delta,p}$, where $\mathbb{F}_\delta$ is any Hirzebruch surface and p is a closed point of $\mathbb{F}_\delta$, which we name divisorial valuations of $\mathbb{F}_\delta$ for short. As algebraic and local objects, these valuations do not differ of those corresponding to quotient fields of bidimensional local rings; however our valuations will be considered over global objects giving rise to global information.

The main goals in this thesis are four:

1. To find adequate affine charts on $\mathbb{F}_\delta$ that allow us to establish a concept of non-positivity (or negativity) at infinity in a close way as it was done in the projective case. The main fact is that valuations ν which are non-positive (or negative) on non-zero regular functions on these charts give rise to surfaces Z with interesting geometrical global properties.

2. To compare non-positive at infinity divisorial and real valuations of $\mathbb{P}^2$ with those of $\mathbb{F}_\delta$ and provide algorithms determining their corresponding dual graphs.

3. To extend the concept of minimality of divisorial valuations of $\mathbb{P}^2$ to divisorial valuations of $\mathbb{F}_\delta$ and computing the Seshadri-type constant $\hat{\mu}_D(\nu)$ for any non-positive at infinity divisorial valuation of $\mathbb{F}_\delta$ and any big divisor D on $\mathbb{F}_\delta$.

4. Finally, to explicitly determine the Newton-Okounkov bodies with respect to flags $E_\bullet = \{Z \supset E \supset \{q\}\}$ of divisors D which are the pull-back of big

divisors and E is the last created exceptional divisor by the sequence of blowups corresponding to non-positive at infinity valuations ν of $\mathbb{F}_\delta$.

The main results of this thesis are stated and proved in the following papers accomplished jointly my advisors:

[63] C. Galindo, F. Monserrat and C.-J. Moreno-Ávila. Non-positive at infinity divisorial valuations of Hirzebruch surfaces. *Rev. Mat. Complut.*, 33:349-372, 2020.

[62] C. Galindo, F. Monserrat and C.-J. Moreno-Ávila. Seshadri-type constants and Newton-Okounkov bodies for non-positive at infinity valuations of Hirzebruch surfaces. *Arxivmath:1905.03531*, 2019.

[61] C. Galindo, F. Monserrat and C.-J. Moreno-Ávila. Discrete equivalence of non-positive at infinity plane valuations. *Arxivmath:1911.06661*, 2019.

We devote the remaining of this introduction to summarize the contents of this memoir.

Chapter 1 reviews concepts and results which we will use in the upcoming chapters mainly about Hirzebruch surfaces, divisors, blowups and plane valuations. Here we also establish several conventions and notation that we will follow in this work. We focus on the 2-dimensional case, but we provide several references which contain details in higher dimension.

In Chapter 2 we introduce non-positive and negative at infinity divisorial and irrational valuations of Hirzebruch surfaces $\mathbb{F}_\delta$. Valuations of $\mathbb{F}_\delta$ is a short name for valuations of the function field of $\mathbb{F}_\delta$ centered at $\mathcal{O}_{\mathbb{F}_\delta,p}$ for some $p \in \mathbb{F}_\delta$. We study global geometric properties associated to the surfaces defined by these valuations in the divisorial case. Finally, we compare the dual graphs which can correspond with non-positive at infinity divisorial and irrational valuations of $\mathbb{P}^2$ and $\mathbb{F}_\delta$ and provide an algorithm to obtain all dual graphs corresponding to the previous valuations. Here our ground fields are algebraically closed of arbitrary characteristic.

Being more specific, Section 2.1 recalls the concepts of non-positive and negative at infinity divisorial valuation of $\mathbb{P}^2$ and the main results that characterize these valuations.

In Section 2.2 we distinguish two types of divisorial valuations of $\mathbb{F}_\delta$, called special and non-special valuations (Definition 2.2.1). This distinction is due to the particular geometric structure of the Hirzebruch surfaces (Propositions 1.6.9 and 1.6.10) and the situation arising from considering a particular finite simple sequence of points blowups (Proposition 2.2.2). As a result, there exist two natural charts "at infinity". On the one hand, that one given by the points of $\mathbb{F}_\delta$ which belong to neither the fiber F_1 containing p nor the special section M_0 on $\mathbb{F}_\delta$ -the unique integral curve on $\mathbb{F}_\delta$ with non-positive self-intersection (Subsection 1.6.1). On the other hand, the chart given by the points which are neither in F_1 nor in a particular uniquely defined curve $M_1 \neq M_0$.

In Section 2.3 we focus on the study of global and local geometric properties of the rational surfaces Z defined by non-positive or negative at infinity special divisorial valuations ν_n of $\mathbb{F}_\delta$. We begin by considering certain divisors $\Lambda_i, 1 \leq i \leq n$, on Z (Proposition 2.3.1) and prove that they satisfy several nice properties. We also describe the relation of these divisors on $\mathbb{F}_1$ with those used in [60] for treating a similar problem for valuations of $\mathbb{P}^2$; notice that $\mathbb{F}_1$ can be obtained by blowing-up a point in $\mathbb{P}^2$. Our main result is Theorem 2.3.7 which proves [60, Theorem 1] as a particular case. Next, we state that result.

Theorem A (Theorem 2.3.7)**.** *Let ν_n be a special divisorial valuation of the function field of $\mathbb{F}_\delta$ centered at $\mathcal{O}_{\mathbb{F}_\delta,p}$, p being a closed point in $\mathbb{F}_\delta$. Set Z the surface that ν_n defines. Denote by φ_{F_1} (respectively, φ_{M_0}) the germ of the fiber F_1 (respectively, the special section M_0) at p. Consider the divisor Λ_n mentioned previously and the last maximal contact value of ν_n, $\overline{\beta}_{g+1}(\nu_n)$. Then the following conditions are equivalent:*

(a) The valuation ν_n is non-positive at infinity.

(b) The divisor Λ_n is nef.

(c) The inequality $2\nu_n(\varphi_{M_0})\nu_n(\varphi_{F_1}) + \delta\nu_n(\varphi_{F_1})^2 \geq \overline{\beta}_{g+1}(\nu_n)$ holds.

(d) The cone of curves $\mathrm{NE}(Z)$ is generated by the classes of the strict transforms on Z of F_1, M_0 and the irreducible exceptional divisors associated with the composition of point blowups $\pi : Z \to \mathbb{F}_\delta$ given by ν_n.

From our reasoning, we are able to prove that all the divisors $\Lambda_i, 1 \leq i \leq n$, are nef and effective when we suppose that ν_n is non-positive at infinity (Remark 2.3.9 and Corollary 2.3.11). Even more, each special divisorial valuation ν_i, defined by the exceptional divisor E_i created by the sequence of blowups π, is also non-positive at infinity (Corollary 2.3.10) and a non-positive at infinity special irrational valuation of $\mathbb{F}_\delta$ can be approached by suitable sequences of non-positive at infinity special divisorial valuations of $\mathbb{F}_\delta$ (Corollary 2.3.12).

Negative at infinity special divisorial valuations of $\mathbb{F}_\delta$ are also characterized in this memoir; next we state this result which corresponds to Theorem 2.3.14 and has algebraic and geometric connotations. It extends Theorem 2 of [60].

Theorem B (Theorem 2.3.14)**.** *Under the notations in Theorem A, the following conditions are equivalent:*

(a) The valuation ν_n is negative at infinity.

(b) It holds that either $2\nu_n(\varphi_{M_0})\nu_n(\varphi_{F_1}) + \delta\nu_n(\varphi_{F_1})^2 = \overline{\beta}_{g+1}(\nu_n)$ and the Iitaka dimension of the divisor Λ_n vanishes, or $2\nu_n(\varphi_{M_0})\nu_n(\varphi_{F_1}) + \delta\nu_n(\varphi_{F_1})^2 > \overline{\beta}_{g+1}(\nu_n)$.

(c) *The inequality* $\Lambda_n \cdot \tilde{C} > 0$ *holds for the strict transform on* Z, $\tilde{C}$, *of any curve* C *on* $\mathbb{F}_\delta$, $C \neq F_1, M_0$.

This dissertation also considers surfaces defined by non-special divisorial valuations ν_n of $\mathbb{F}_\delta$. We study algebraic and local and global geometric properties related to the above mentioned surfaces. In this case the family of divisors one needs to use is much bigger and more complex than in special case (Proposition 2.4.2). We show that it is enough to consider a set of divisors denoted $\Delta_i, \delta + 1 \leq i \leq n$, (Lemma 2.4.7) to prove the main results of this section (Theorems 2.4.8 and 2.4.14) and their consequences (Remark 2.4.9 and Corollaries 2.4.10, 2.4.11 and 2.4.12).

Notice that our results on the cone of curves do not assume any positivity property for the canonical divisor of the considered rational surfaces.

Continuing with our summary of Chapter 2, Section 2.5 is devoted to compare dual graphs corresponding to the three types of non-positive at infinity divisorial and irrational valuations considered in this dissertation. As mentioned, valuations are essentially algebraic objects and then the valuations of Hirzebruch surfaces do not differ from valuations centered at points of other smooth surfaces. Nevertheless, it is interesting to attach them to the surface where they are centered and study the three types of non-positive at infinity valuations which have been defined because they also contain geometric information. Thus, we would like to know better the surfaces which they define and to try to find a more purely algebraic definition for them. As a suitable tuple of rational or real numbers is an equivalent datum to the dual graph of a valuation as ours (Section 1.3), we say that two real valuations are discretely equivalent if they have the same tuple (Definition 2.5.1). We provide numerical conditions involving the tuples corresponding to real valuations of $\mathbb{P}^2$ or $\mathbb{F}_\delta$ to decide which dual graphs admit valuations of some of the above types of non-positive at infinity valuations (Theorem 2.5.4). Furthermore, we define the sets of discrete classes corresponding to each type and show inclusions among them (Theorem 2.5.5 and Remark 2.5.6). Finally, we develop an algorithm computing the discrete equivalence classes that admit non-positive at infinity divisorial and irrational valuations (Subsection 2.5.1).

Chapter 3 studies Seshadri-type constants and Newton-Okounkov bodies of non-positive at infinity valuations of the projective plane and Hirzebruch surfaces. Here we assume that $\mathbb{C}$ is our ground field.

We start by reviewing the concept of Seshadri-type constant $\hat{\mu}_D(\nu_n)$ for divisorial valuations ν_n of $\mathbb{P}^2$ or $\mathbb{F}_\delta$ and big divisors D on the above surfaces. We also extend the concept of minimal valuation of $\mathbb{P}^2_{\mathbb{C}} := \mathbb{P}^2$ (Definition 3.0.1); our extension works for valuations ν_n and divisors as above. Our definition for the value $\hat{\mu}_D(\nu_n)$ involves a limit, but it can be also interpreted geometrically as

$$\hat{\mu}_D(\nu_n) = \sup\{s > 0 \,|\, D^* - sE_n \text{ is big}\},$$

where E_n is the exceptional divisor defining ν_n and D^* is the pull-back of D on the surface that ν_n defines. We will prove in Section 3.1 that our extended definition of minimal valuation coincides with that given in [38, 64] when applying to a divisorial valuation of $\mathbb{P}^2$. We also show in this section, when ν_n is a non-minimal valuation of $\mathbb{P}^2$, that the so-called supraminimal curve generates an extremal ray of $\overline{\mathrm{NE}}(Z)$, Z being the surface defined by ν_n (Corollary 3.1.6). This supraminimal curve is defined by the unique monic irreducible polynomial f such that $\deg(f)\hat{\mu}_L(\nu_n) = \nu_n(f)$ ([38, Lemma 5.1] and [65, Lemma 3.10]).

Our Section 3.2 proves one of the main results of Chapter 3, which shows the value of $\hat{\mu}_D(\nu_n)$ for non-positive at infinity divisorial valuations of $\mathbb{F}_\delta$ and big divisors on $\mathbb{F}_\delta$:

Theorem C (Theorem 3.2.1)**.** *Let ν_n be a non-positive at infinity divisorial valuation of the function field of $\mathbb{F}_\delta$ centered at $\mathcal{O}_{\mathbb{F}_\delta,p}, p \in \mathbb{F}_\delta$. Let D be a big divisor that is linearly equivalent to $aF + bM$ on $\mathbb{F}_\delta$. Then*

(a) If the valuation ν_n is special, then $\hat{\mu}_D(\nu_n) = (a+b\delta)\nu_n(\varphi_{F_1}) + b\nu_n(\varphi_{M_0})$, where φ_{F_1} (respectively, φ_{M_0}) is the germ of F_1 (respectively, M_0) at p.

(b) Otherwise, $\hat{\mu}_D(\nu_n) = a\nu_n(\varphi_{F_1}) + b\nu_n(\varphi_{M_1})$, where φ_{F_1} (respectively, φ_{M_1}) is the germ of F_1 (respectively, M_1) at p.

Theorem C allows us to characterize those non-positive at infinity divisorial valuation of $\mathbb{F}_\delta$ which are minimal with respect to a big and nef divisor on $\mathbb{F}_\delta$ (Corollary 3.2.3).

Section 3.3 is devoted to study the Newton-Okounkov bodies of the pull-back on surfaces Z of big divisors D on either the projective plane $\mathbb{P}^2$ or Hirzebruch surfaces $\mathbb{F}_\delta$ with respect to flags $E_\bullet = \{Z \supset E_r \supset \{p_{r+1}\}\}$, where E_r are exceptional divisors defining divisorial valuations ν_r of $\mathbb{P}^2$ or $\mathbb{F}_\delta$ and Z the surface given by ν_r. For short, we will denote these Newton-Okounkov bodies by $\Delta_\nu(D)$ (see Definition 3.3.3). Notice that following [65, Section 3.2], it can be proved that exceptional curve valuations ν of $\mathbb{P}^2$ or $\mathbb{F}_\delta$ correspond to valuations $\nu_{E_\bullet}$ attached to the flags $E_\bullet$ as above. These exceptional curve valuations ν have two components $\nu = (v_1, v_2)$, where $v_1 = \nu_r$ is the divisorial valuation defined by E_r. We introduce concepts like special, non-special, non-positive at infinity and minimal exceptional curve valuation of a Hirzebruch surface which depend on the first component of the exceptional curve valuation (Definitions 3.3.1 and 3.3.2). Key objects in this section are the so-called maximal contact values of some divisorial valuations ν_i defined by the exceptional divisors E_i appearing in the sequence of point blowups corresponding to ν_r. These values are denoted $\{\overline{\beta}_j(\nu_i)\}_{j=0}^{g+1}$ and we are only interested in the indices $i = r$ and $i = \eta$ (Subsection 1.3.1). We set $g^* + 2$ the number of maximal contact values of the exceptional curve valuation ν. Lemma 3.3.6 considers maximal contact values and plays an important role in our explicit description of Newton-Okounkov bodies.

The first main result in Section 3.3 describes the Newton-Okounkov bodies $\Delta_\nu(D)$ of big divisors on $\mathbb{P}^2$ and $\mathbb{F}_\delta$ of minimal exceptional curve valuations ν.

Theorem D (Theorem 3.3.7). *Let $E_\bullet = \{Z \supset E_r \supset \{p_{r+1}\}\}$ be a flag and $\nu = \nu_\bullet$ its attached exceptional curve valuation. Set D a big divisor on either $\mathbb{P}^2$ or $\mathbb{F}_\delta$. Then, the valuation ν is minimal if and only if the Newton-Okounkov body $\Delta_\nu(D)$ is a triangle whose vertices are $(0,0)$, $\left(\hat{\mu}_D(\nu_r), \frac{\hat{\mu}_D(\nu_r)\overline{\beta}_0(\nu_\eta)}{\overline{\beta}_0(\nu_r)}\right)$ and $\left(\hat{\mu}_D(\nu_r), \frac{\hat{\mu}_D(\nu_r)\overline{\beta}_{g^*}(\nu_\eta)}{\overline{\beta}_{g^*}(\nu_r)}\right)$, when $p_{r+1} \in E_r \cap E_\eta$ with $\eta \neq r$. If p_{r+1} is a free point, the triangle is given by the vertices $(0,0), (\hat{\mu}_D(\nu_r), 0)$ and $\left(\hat{\mu}_D(\nu_r), \frac{\hat{\mu}_D(\nu_r)}{\overline{\beta}_{g^*+1}(\nu_r)}\right)$.*

An explicit description of the Newton-Okounkov bodies of flags as above for non-minimal valuations ν is an open problem. Section 3.3 solves it when the valuations ν are non-positive at infinity. Notice that [65] solves the case where ν is a non-minimal exceptional curve valuations of $\mathbb{P}^2$.

We divide our study in two subsections (Subsections 3.3.1 and 3.3.2) providing a description that depends on the fact that the exceptional curve valuation is either special or non-special.

Following ideas in [86] and [82], we start by looking for suitable values t_i which will be the first components of the vertices of the Newton-Okounkov bodies to be described (Lemmas 3.3.13 and 3.3.34). Then, we obtain the corresponding Zariski decompositions (Propositions 3.3.17 and 3.3.38). Recall that these values t_i are those where the ray $[D] - t[E_r]$ crosses into a different Zariski chamber and therefore they characterize locally the structure of the interior of the cone of curves $\mathrm{NE}(Z)$ according to the exceptional curve valuation ν is either minimal or non-minimal (Remarks 3.3.15, 3.3.18, 3.3.36 and 3.3.39). The ray $[D^*] - t[E_r]$ crosses a unique Zariski chamber when ν is minimal with respect to D, and the same holds in a particular case when ν is special and non-minimal with respect to D. The ray $[D^*] - t[E_r]$ crosses at least two Zariski chamber when ν is non-special and non-minimal with respect to D.

We conclude this introduction by explaining that the main results in these subsections are explicit descriptions of the Newton-Okounkov bodies $\Delta_\nu(D)$ of big divisors with respect to a non-minimal non-positive at infinity exceptional curve valuation ν of $\mathbb{F}_\delta$ (see Theorems 3.3.21, 3.3.24, 3.3.27, 3.3.42 and 3.3.45). As a consequence of these explicit calculations, we show that the vertices of our Newton-Okounkov bodies depend only on the expression of D, the volume of ν and the values of the germs at p of the fibre and sections on $\mathbb{F}_\delta$ whose strict transforms (together with those of the exceptional divisors) span the cone of curves. As a particular case of the results of the special exceptional curve valuations of $\mathbb{F}_\delta$, we obtain the Newton-Okounkov body computed in [65, Corollary 5.2] with respect to a non-positive at infinity exceptional curve valuation of $\mathbb{P}^2$ (Corollary 3.3.29).

Chapter 1

Preliminaries

In this chapter we introduce the concepts, notation and conventions needed to develop this dissertation. We restrict the considered objects to the 2-dimensional case since we will always work there, although almost all of them can be extended to higher dimension. The style of this first part is descriptive due to its compilatory nature. Unless otherwise stated, our ground fields are algebraically closed of arbitrary characteristic.

1.1 Basic concepts

We start by collecting some concepts of algebraic geometry. We mainly follow [71]. Other useful references for us are [102, 107] and [108].

Let k be an algebraically closed field of arbitrary characteristic and $k^* := k \setminus \{0\}$. A *variety* is an integral separated scheme of finite type over k. Varieties of dimension 2 are named *surfaces*.

Set Z a surface and p a closed point of Z (i.e., an irreducible 0-dimensional closed subscheme on Z). We denote by $\mathcal{O}_Z$ the structural sheaf of Z, by $\mathcal{O}_{Z,p}$ the stalk of $\mathcal{O}_Z$ at p and by $\mathfrak{m}_p$ the maximal ideal of the local ring $\mathcal{O}_{Z,p}$. The elements of the stalk $\mathcal{O}_{Z,p}$ are called *germs*. It is said that p is a *smooth point* of Z if the local ring $\mathcal{O}_{Z,p}$ is regular. Otherwise, p is a *singular point* (or a *singularity*) of Z. Moreover, the surface Z is called to be *smooth*, *non-singular* or *regular* if every point of Z is smooth. In this work, from now on, *surface* means 2-dimensional smooth projective variety and will be usually denoted by Z.

Let G be a prime divisor (i.e., a closed integral subscheme of codimension 1) on Z and η its generic point. Then, the local ring $\mathcal{O}_{Z,\eta}$ is a discrete valuation ring with quotient field $K(Z)$, the function field of Z. If ν_G denotes the corresponding discrete valuation (see Section 1.3) then, for all non-zero element f of $K(Z)$, the *order of f along G*, denoted $\mathrm{ord}_G(f)$, is defined as $\nu_G(f)$.

A *Weil divisor* D on Z is an element of the free $\mathbb{Z}$-module $\mathrm{Div}(Z)$ generated by the prime divisors on Z. Thus, D can be written

$$D = n_1G_1 + n_2G_2 + \cdots + n_rG_r,$$

where G_i are prime divisors on Z and n_i integer numbers. If $n_i = 0$ for all i, we write $D = 0$; if $n_i \geq 0$ for all i and some of them is strictly positive, then D is called to be *effective* or a *curve*. The union $\cup_{n_i \neq 0} G_i$ is called the *support* of D and it is denoted by $\mathrm{supp}(D)$. If C is a curve on Z passing through p, $\varphi_{C,p}$ represents the germ of the curve C at the point p. Often, we only use φ_C if no confusion arises.

Let f be a non-zero rational function of $K(Z)$. The Weil divisor of f, denoted by $\mathrm{div}(f)$, is defined as

$$\mathrm{div}(f) := \sum_G \mathrm{ord}_G(f)G,$$

where the sum runs over the prime divisors G on Z. A Weil divisor which equals the divisor of a rational function is named *principal divisor*. We say that two Weil divisors D and D' are *linearly equivalent*, written $D \sim D'$, if $D - D'$ is a principal divisor. This is an equivalence relation on $\mathrm{Div}(Z)$ and we denote by $\mathrm{Cl}(Z)$ the quotient group $\mathrm{Div}(Z)/\sim$.

Let $\mathcal{K}$ be the constant sheaf corresponding to the function field $K(Z)$ [71, Chapter II, Example 1.0.3]. Denote by $\mathcal{K}^*$ (respectively, $\mathcal{O}_Z^*$) the sheaf of invertible elements in the sheaf of rings $\mathcal{K}$ (respectively, $\mathcal{O}_Z$). A *Cartier divisor* D on Z is a global section of the sheaf $\mathcal{K}^*/\mathcal{O}_Z^*$. As a result, a Cartier divisor can be described by giving a family of pairs $\{(U_i, f_i)\}_{i\in I}$, where $\{U_i\}_{i\in I}$ is an open covering of Z and $f_i \in \Gamma(U_i, \mathcal{K}^*)$ is such that $f_i/f_j \in \Gamma(U_i \cap U_j, \mathcal{O}_Z^*)$ for all $i, j \in I$. A Cartier divisor $\{(U_i, f_i)\}_{i\in I}$ is *principal* if it is in the image of the natural map $\Gamma(Z, \mathcal{K}) \to \Gamma(Z, \mathcal{K}^*/\mathcal{O}_Z^*)$, i.e. all the functions f_i are restrictions of the same rational function $f \in K(Z)$. We say that a Cartier divisor $\{(U_i, f_i)\}_{i\in I}$ is *effective* when $f_i \in \Gamma(U_i, \mathcal{O}_{U_i})$ for all $i \in I$.

Set D_1 and D_2 two Cartier divisors defined by $\{(U_i, f_i)\}_{i\in I}$ and $\{(V_j, g_j)\}_{j\in J}$, respectively. We define the sum $D_1 + D_2$ as the Cartier divisor given by the set $\{(U_i \cap V_j, f_ig_j)\}_{i\in I, j\in J}$. With this operation the set of Cartier divisors is an abelian group. We say that D_1 and D_2 are *linearly equivalent*, $D_1 \sim D_2$, if their difference is a principal divisor. This is an equivalence relation on the set of Cartier divisors and we denote by $\mathrm{CaCl}(Z)$ its quotient group.

An invertible sheaf $\mathcal{L}$ on Z is a locally free $\mathcal{O}_Z$-module of rank 1. Set $\mathcal{L}_1$ and $\mathcal{L}_2$ two invertible sheaves on Z, then the tensor product $\mathcal{L}_1 \otimes_{\mathcal{O}_Z} \mathcal{L}_2$ ($\mathcal{L}_1 \otimes \mathcal{L}_2$ for short) is also an invertible sheaf. In addition, if $\mathcal{L}$ is an invertible sheaf on Z, there exists an invertible sheaf $\mathcal{L}^{-1}$ on Z such that $\mathcal{L} \otimes \mathcal{L}^{-1} \cong \mathcal{O}_Z$ ([71, Chapter II, Proposition 6.12]). The group of isomorphism classes of invertible sheaves on Z under the tensor product is called *Picard group* of Z and denoted $\mathrm{Pic}(Z)$.

Set D a Cartier divisor on a surface Z defined by $\{(U_i, f_i)\}_{i\in I}$. The *sheaf associated* to D, denoted $\mathcal{O}_Z(D)$, is the sub-sheaf of $\mathcal{K}$ defined by taking $\mathcal{O}_Z(D)$ to be the sub-$\mathcal{O}_Z$-module of $\mathcal{K}$ generated by f_i^{-1} on U_i. This is well-defined because f_i/f_j is invertible in $U_i \cap U_j$, so f_i^{-1} and f_j^{-1} generate the same $\mathcal{O}_Z$-module.

Proposition 1.1.1. [71, Chapter II, Proposition 6.13] *Let Z be a surface and $\mathcal{O}_Z$ its structural sheaf. Then:*

(a) for any Cartier divisor D, $\mathcal{O}_Z(D)$ is an invertible sheaf. The map $D \mapsto \mathcal{O}_Z(D)$ gives a 1-1 correspondence between Cartier divisors on Z and invertible subsheaves of $\mathcal{K}$.

(b) given two Cartier divisors, D_1 and D_2, the sheaf associated to $D_1 - D_2$ is isomorphic to $\mathcal{O}_Z(D_1) \otimes \mathcal{O}_Z(D_2)^{-1}$.

(c) two Cartier divisors, D_1 and D_2, are linearly equivalent if and only if its associated invertible sheaves $\mathcal{O}_Z(D_1)$ and $\mathcal{O}_Z(D_2)$ are isomorphic.

All the surfaces Z we are going to consider are noetherian, integral, separated locally factorial schemes. So, by [71, Chapter II, Proposition 6.11], the group of the Weil divisors, $\mathrm{Div}(Z)$, is isomorphic to the group of Cartier divisors and principal divisors correspond under this isomorphism. Therefore, we can say that a Cartier divisor is *principal* (respectively, *effective*) if it corresponds to a principal (respectively, effective) Weil divisor ([71, Chapter II, Remark 6.17.1]). As a consequence, in the future, we will use the word *divisor*, without specifying if it is a Weil or Cartier divisor, and also a curve on Z and its corresponding effective divisor will be denoted by the same letter. In addition, $\mathrm{Pic}(Z)$ is isomorphic to the groups $\mathrm{Cl}(Z)$ and $\mathrm{CaCl}(Z)$ ([71, Chapter II, Proposition 6.15 and Corollary 6.16]). From now we will use this isomorphism and we will identify $\mathrm{Pic}(Z)$ with the group of the divisors on Z modulo linear equivalence. The element of $\mathrm{Pic}(Z)$ defined by a divisor D will be denoted $[D]$.

Intersection multiplicity and intersection number on surfaces are important tools in this work. Let us recall them.

Let Z be a surface as above and suppose that C_1 and C_2 are two curves on Z that meet at a point $p \in Z$. The *intersection multiplicity* $(\varphi_{C_1}, \varphi_{C_2})_p$ of C_1 and C_2 at p is defined to be

$$(\varphi_{C_1}, \varphi_{C_2})_p := \dim_k \mathcal{O}_{Z,p}/\langle \varphi_{C_1}, \varphi_{C_2} \rangle.$$

Moreover, it holds that $(\varphi_{C_1}, \varphi_{C_2})_p = 1$ if and only if φ_{C_1} and φ_{C_2} generate the maximal ideal $\mathfrak{m}_p$ of $\mathcal{O}_{Z,p}$. In this case, it is said that C_1 and C_2 are *transversal* at p, or *meet transversally* at p ([9, Chapter I, Definition I.3]). We will understand that $(\varphi_{C_1}, \varphi_{C_2})_p = 0$ when C_1 or C_2 does not pass through p.

Theorem 1.1.2. [71, Chapter V, Theorem 1.1] *Keep the previous notation. There is a unique pairing* $\mathrm{Div}(Z) \times \mathrm{Div}(Z) \to \mathbb{Z}$, *denoted by $D_1 \cdot D_2$ for any two divisors D_1, D_2, such that*

(a) *if* D_1 *and* D_2 *are non-singular curves meeting transversally, then* $D_1 \cdot D_2 = \#(D_1 \cap D_2)$, *the number of points of* $D_1 \cap D_2$,

(b) *it is symmetric:* $D_1 \cdot D_2 = D_2 \cdot D_1$,

(c) *it is additive:* $(D_1 + D_2) \cdot D_3 = D_1 \cdot D_3 + D_2 \cdot D_3$, *for all* $D_3 \in \mathrm{Div}(Z)$,

(d) *it depends only on the linear equivalence classes: if* D_1 *and* D_2 *are two linearly equivalent divisors then* $D_1 \cdot D_3 = D_2 \cdot D_3$, *for all* $D_3 \in \mathrm{Div}(Z)$.

$D_1 \cdot D_2$ is usually called the *intersection number* of D_1 and D_2.

The following proposition connects intersection number and intersection multiplicity.

Proposition 1.1.3. [71, Chapter V, Proposition 1.4] *If* C_1 *and* C_2 *are curves on a surface* Z *having no common irreducible component then*

$$C_1 \cdot C_2 = \sum_{p \in C_1 \cap C_2} (\varphi_{C_1}, \varphi_{C_2})_p.$$

To compute the self-intersection of a divisor D on Z, that is, the intersection number $D^2 := D \cdot D$, linear equivalence must be used (see [71, Chapter V, Example 1.4.1]). In addition, using this concept we can define a numerical invariant on Z. Let $\Omega_{Z/k}$ be the sheaf of differentials of Z over k and $\omega_Z = \wedge^2 \Omega_{Z/k}$ the canonical sheaf. A *canonical divisor* on Z is any divisor whose associated invertible sheaf is ω_Z. K_Z will denote an arbitrary canonical divisor of Z and its self-intersection depends only on the surface Z.

The intersection number of two classes $[D_1]$ and $[D_2]$ of $\mathrm{Pic}(Z)$ is defined as intersection number $D_1 \cdot D_2$ of any two representatives D_1 of $[D_1]$ and D_2 of $[D_2]$. Moreover, we say that D_1 (respectively, $[D_1]$) is *orthogonal* to D_2 (respectively, $[D_2]$) whenever $D_1 \cdot D_2 = 0$ (respectively, $[D_1] \cdot [D_2] = 0$). We denote by $D^\perp$ (respectively, $[D]^\perp$) the set of divisors (respectively, the classes of divisors) on Z which are orthogonal to D (respectively, $[D]$).

To conclude this section, we introduce several more concepts.

Set $\mathcal{L}$ an invertible sheaf on Z. By [71, Chapter II, Theorem 5.19], $H^0(Z, \mathcal{L})$ is a finite-dimensional k-vector space. Fixed $s \in H^0(Z, \mathcal{L})$, we will denote by $\mathrm{div}(s)_0$ the *divisor of zeros* of s, that is, the divisor defined by $\{U_i, \phi_{U_i}(s|_{U_i})\}_{i \in \mathcal{I}}$, where $\{U_i\}_{i \in \mathcal{I}}$ is an open covering of Z and $\phi_{U_i} : \mathcal{L}(U_i) \to \mathcal{O}_Z(U_i)$ the isomorphism given by $\mathcal{L}$.

Proposition 1.1.4. [71, Chapter II, Proposition 7.7] *Let* Z *be a surface and* D *a divisor on* Z. *Set* $\mathcal{L} = \mathcal{O}_Z(D)$ *the corresponding invertible sheaf. Then*

(a) *For each non-zero* $s \in H^0(Z, \mathcal{L})$, *the divisor of zeros of* s *is an effective divisor linearly equivalent to* D.

(b) *Every effective divisor linearly equivalent to* D *is* $\mathrm{div}(s)_0$ *for some* $s \in H^0(Z, \mathcal{L})$.

(c) *Two sections $s, s' \in H^0(Z, \mathcal{L})$ have the same divisor of zeros if and only if there is a $\lambda \in k^*$ such that $s' = \lambda s$.*

A *complete linear system* on Z is the set (which might be empty) of effective divisors linearly equivalent to some given divisor D. We denote it by $|D|$. When $\mathcal{L}$ is isomorphic to the invertible sheaf $\mathcal{O}_Z(D)$ associated to D, by the above result, one has a one-to-one correspondence $(H^0(Z, \mathcal{L}) \setminus \{0\})/k^* \to |D|$ defined by $s \mapsto \mathrm{div}(s)_0$, which gives $|D|$ a structure of projective space over k.

Proposition 1.1.5. [77, Lemma 1.20] *If D is a divisor on a surface Z such that $D^2 > 0$, then either $|nD| \neq \emptyset$ or $|-nD| \neq \emptyset$ for n large enough.*

Any k-vector subspace $\mathfrak{d}$ of a complete linear system $|D|$ is called *linear system*. In the case of $\mathcal{L}$ being isomorphic to $\mathcal{O}_Z(D)$, the linear system $\mathfrak{d}$ corresponds to a k-vector subspace $W \subseteq H^0(Z, \mathcal{L})$, where $W = \{s \in H^0(Z, \mathcal{L}) \mid \mathrm{div}(s)_0 \in \mathfrak{d}\} \cup \{0\}$, and the dimension of $\mathfrak{d}$, denoted $\dim(\mathfrak{d})$, is defined as the dimension of $\mathfrak{d}$ as projective space, that is, $\dim(\mathfrak{d}) = \dim(W) - 1$. A point $p \in Z$ is said to be a *base point* of a linear system $\mathfrak{d}$ if $p \in \mathrm{supp}(D)$ for all $D \in \mathfrak{d}$. A linear system $\mathfrak{d}$ is *base-point-free* when $\mathfrak{d}$ does not have base points.

Proposition 1.1.6. [71, Chapter II, Lemma 7.8] *Let $\mathfrak{d}$ be a linear system on Z corresponding to a subspace $W \subseteq H^0(Z, \mathcal{L})$. Then, $\mathfrak{d}$ is base-point-free if and only if $\mathcal{L}$ is generated by its global sections in W.*

To finish, the *Iitaka dimension* [72] of a divisor D on Z is defined as $\kappa(D) := -\infty$ if $|nD| = \emptyset$ for all $n \in \mathbb{Z}_{>0}$ and, otherwise,

$$\kappa(D) := \kappa(Z, D) = \max\{\dim \phi_{|nD|}(Z)\},$$

where n runs over $\{m \in \mathbb{Z}_{>0} \mid H^0(Z, \mathcal{O}_Z(mD)) \neq 0\}$, dim means the projective dimension and, for each n, $\phi_{|nD|}(Z)$ is the closure of the image of the rational map defined by the complete linear system $|nD|$. When $|nD| \neq \emptyset$, for some n, the Iitaka dimension satisfies $0 \leq \kappa(D) \leq \dim(Z) = 2$. By definition, the *Kodaira dimension* of Z is the Iitaka dimension of a canonical divisor on Z.

1.2 Blowups and configurations

A well-known tool in algebraic geometry is the concept of *blowup*. In this section we recall its definition in the case of surfaces and some of its properties that will be applied in later sections and chapters. We have mainly followed [9], [18], [22], [91] and [19]. We will keep the notations and conventions of the above section.

Let Z_0 be a smooth projective surface and p a closed point of Z_0. Then, there exist a surface Z and a morphism $\pi : Z \to Z_0$, which are unique up to isomorphism, such that

(1) The restriction of π to $\pi^{-1}(Z_0 \setminus \{p\})$ is an isomorphism onto $Z_0 \setminus \{p\}$.

(2) $\pi^{-1}(p) = E$ is isomorphic to $\mathbb{P}^1$.

We say that π is the *blowup* of Z_0 at p and E is the *exceptional divisor* (or *exceptional curve*) of the blowup (see, for instance, [9, Chapter II, Section II.1]).

In the literature the above definition often corresponds to the concept of *monoidal transformation* (see [71, Chapter V, Section 3]) to distinguish it from other more general transformations, and the surface Z is denoted by $\mathrm{Bl}_p(Z_0)$.

Let D be an effective divisor on Z_0. We call *strict transform* of D on Z to the divisor defined by the closure of $\pi^{-1}(D \setminus \{p\})$ on Z and we denote it by $\tilde{D}$. The *total transform* of D on Z is the pull back π^*D of D. It will be usually denoted by D^*. If D_1 and D_2 are two effective divisors on Z_0 which are linearly equivalent, then D_1^* and D_2^* are also linearly equivalent (see [73, Proposition 2.15]).

Some properties related to the blowup concept are the following.

Proposition 1.2.1. [9, Chapter II, Lemma II.2 and Proposition II.3] *Let Z_0 be a surface, Z the surface created by a blowup π at a closed point $p \in Z_0$ and E the exceptional divisor.*

(a) Set D an effective divisor on Z_0. It holds that $D^* = \tilde{D} + \mathrm{mult}_p(\varphi_D)E$.

(b) There is an isomorphism $\mathrm{Pic}(Z_0) \oplus \mathbb{Z} \cong \mathrm{Pic}(Z)$ given by $(D, m) \mapsto D^* + mE$.

(c) Let D_1 and D_2 be divisors on Z_0. Then $D_1^* \cdot D_2^* = D_1 \cdot D_2, D_1^* \cdot E = 0$ and $E^2 = -1$.

(d) $K_Z \sim K_{Z_0}^* + E$, where K_Z and K_{Z_0} are canonical divisors on Z and Z_0, respectively.

Now we are going to consider finite sequences of blowups at closed points and recall some concepts associate to them. As we will see, the above results can be also extended to these sequences.

Let Z_0 be a surface and p a closed point of Z_0. The set of closed points in the exceptional divisor created after blowing up p is called the *first infinitesimal neighbourhood of p*. By induction, if $i > 1$, the *i-th infinitesimal neighbourhood of p* is defined to be the first infinitesimal neighbourhood of some point in the $(i-1)$-th infinitesimal neighbourhood of p. A point q is *infinitely near p* when q belongs to some i-th infinitesimal neighbourhood of p for $i \geq 1$. The points which are infinitely near some point of Z_0 are called *infinitely near Z_0*. Occasionally, the closed points of Z_0 will be named *proper* in order to distinguish from the infinitely near ones.

Let p and q be two points which are proper or infinitely near Z_0. It is said that *p precedes q*, denoted $p < q$, if and only if q is infinitely near p. We write $p \leq q$ if p equals or precedes q. The *level of p*, written $\mathrm{l}(p)$, is the number of proper or infinitely near points which precede p. The proper points of Z_0 are the points of level zero.

Consider a finite, or infinite, set $\mathcal{C}$ of proper and infinitely near Z_0 points. This set is said to be a *configuration of infinitely near points* (or *configuration* for short) *over* Z_0 if, for every point $q \in \mathcal{C}$, the points which precede q belong to $\mathcal{C}$. The points of level zero of $\mathcal{C}$ are called *origins* of $\mathcal{C}$. When there exists a unique origin p in $\mathcal{C}$, we will say that $\mathcal{C}$ is a *configuration with origin at* p. Unless otherwise stated, we only consider finite configurations.

The relation $\leq$ is a partial order for the set of elements of a configuration $\mathcal{C}$. Usually this relation is called *natural order*. If the natural order is a total order, then we say that the configuration $\mathcal{C}$ is a *chain*. Set $\preceq$ a total order on a configuration $\mathcal{C}$. The total order $\preceq$ is called *admissible* for any $p, q \in \mathcal{C}$, $p \leq q$ implies $p \preceq q$.

Fixed a configuration of infinitely near points $\mathcal{C}$, we can set $\mathcal{C} = \{p_i\}_{i=1}^n$ under an admissible total order such that $p_1 \preceq p_2 \preceq \ldots \preceq p_n$. Moreover, we can attach the pair $(\mathcal{C}, \preceq)$ a unique sequence of points blowups,

$$\pi_{\mathcal{C},\preceq} : Z = Z_n \xrightarrow{\pi_n} Z_{n-1} \to \cdots \to Z_1 \xrightarrow{\pi_1} Z_0.$$

such that $p_i \in Z_{i-1}$ and $Z_i = \mathrm{Bl}_{p_i}(Z_{i-1})$, for $1 \leq i \leq n$. The surface Z is called the *sky of the configuration.*

The sky of a configuration is eventually independent of the admissible total order as we see in the following result (see [22, Proposition 4.3.2] or [91, Proposition 1.2.4]).

Proposition 1.2.2. *Assume that $\preceq$ and $\preceq'$ are two admissible total orders of a same configuration of infinitely near points $\mathcal{C}$ over a surface Z_0. Set Z and Z' the skies of the configuration associated to $\preceq$ and $\preceq'$, respectively. Then there exists a unique isomorphism $\phi : Z \to Z'$.*

From now on, we consider configurations over a surface Z_0 endowed with an admissible total order and denote the corresponding sequence of blowups by π.

Let $\mathcal{C} = \{p_i\}_{i=1}^n$ be a configuration and consider

$$\pi : Z = Z_n \xrightarrow{\pi_n} Z_{n-1} \to \cdots \to Z_1 \xrightarrow{\pi_1} Z_0 \tag{1.1}$$

its sequence of blowups. The exceptional divisor created by blowing-up p_i will be denoted E_{p_i} (or simply E_i when no confusion arises).

Let D be a divisor on one of the surfaces in (1.1), Z_i. Abusing of notation, the strict (respectively, total) transform of D on Z_j, for $j \geq i$, is denoted by $\tilde{D}$ (respectively, D^*). Frequently, the strict transform of E_i will be simply written E_i. Also, we denote by $\mathrm{mult}_{p_j}(\varphi_D)$ the multiplicity of the strict transform of D at the point $p_i \in \mathcal{C}, 1 \leq i \leq n$.

A point p_i in $\mathcal{C}$ is *proximate to* a point $p_j \in \mathcal{C}, i > j$, when the point p_i belongs to the strict transform of E_j on Z_{i-1}. It is denoted by $p_i \to p_j$. A point $p_i \in \mathcal{C}$ is called *satellite* if p_i is proximate to two points of $\mathcal{C}$. Otherwise, p_i is named *free*. When each point p_i belongs to the exceptional divisor E_{i-1}, for $2 \leq i \leq n$, (that is, when $\mathcal{C}$ is a chain) the sequence π is said to be *simple*. Notice that, when $\mathcal{C}$ is a chain, a point $p_i \in \mathcal{C}$ is satellite if there exists an integer $j < i-1$ such that p_i is proximate to p_j.

Proposition 1.2.3. [18, Theorem 1.6] *Let $\mathcal{C}$ be a configuration of infinitely near points and p, q and r points in $\mathcal{C}$. The relation of proximity on $\mathcal{C}$ satisfies the following properties:*

(a) If $q \to p$ then $p < q$.

(b) If $p \leq q$ and $l(q) = l(p) + 1$ then $q \to p$.

(c) If $p < q \leq r$ and $r \to p$ then $q \to p$.

The following results considers a sequence of blowups as (1.1).

Proposition 1.2.4. [5, Proposition 1.1.26] *Keep the notation introduced in the last paragraphs, the following statements hold.*

(a) Set D an effective divisor on Z_0. Then $D^* = \tilde{D} + \sum_{i=1}^n \mathrm{mult}_{p_i}(\varphi_D) E_i^*$.

(b) There exists an isomorphism $\mathrm{Pic}(Z_0) \oplus \mathbb{Z}^n \cong \mathrm{Pic}(Z)$ given by

$$(D, m_1, m_2, \ldots, m_n) \mapsto D^* + \sum_{i=1}^n m_i E^*.$$

(c) $K_Z \sim K_{Z_0}^* + \sum_{i=1}^n E_i^*$, where K_Z and K_{Z_0} are canonical divisors on Z and Z_0, respectively.

(d) Set D_1 and D_2 two divisors on Z_0. Then it holds that

$$D_1^* \cdot D_2^* = D_1 \cdot D_2, \ D_1 \cdot E_i^* = 0 \text{ and } D_1^* \cdot E_i = 0, \text{ for } 1 \leq i \leq n.$$

(e) For all $i, j \in \{1, 2, \ldots, n\}$

$$E_i \cdot E_j = \begin{cases} -r_i - 1 & \text{if } i = j, \\ 1 & \text{if } i \neq j \text{ and } E_i \cap E_j \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

where r_i is the number of points that are proximate to p_i.

(f) For all $i, j \in \{1, 2, \ldots, n\}$

$$E_i^* \cdot E_j^* = \begin{cases} -1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

(g) *For all* $i, j \in \{1, 2, \ldots, n\}$

$$E_i \cdot E_j^* = \begin{cases} -1 & \text{if } i = j, \\ 1 & \text{if } p_i \to p_j, \\ 0 & \text{otherwise.} \end{cases}$$

(h) *If* D_1 *is a divisor on* Z_0 *and* D_2 *is a divisor on* Z *then* $D_1^* \cdot D_2 = D_1 \cdot \pi_* D_2$, *where* $\pi_* D_2$ *is the direct image of the divisor* D_2 *on* Z_0 *induced by* π.

Proposition 1.2.4 (a) gives the following expression with respect to the strict and total transforms of the exceptional divisors:

$$E_i = E_i^* - \sum_{p_j \to p_i} E_j^*.$$

As a consequence, the set $\{E_1^*, E_2^*, \ldots, E_n^*\}$ is a $\mathbb{Z}$-basis of the free abelian group $\mathbb{Z}E_1 \oplus \mathbb{Z}E_2 \oplus \cdots \oplus \mathbb{Z}E_n$ of the divisors with exceptional support. The matrix $\mathbf{P}_{\mathcal{C}} := (p_{ij})_{1 \leq i,j \leq n}$, where $p_{ij} = -E_i \cdot E_j^*$, is called *proximity matrix* of $\mathcal{C}$ and gives the change of basis between $\{E_i\}_{i=1}^n$ and $\{E_i^*\}_{i=1}^n$. In addition, for a curve C it holds that

$$\text{mult}_{p_i}(\varphi_C) = \sum_{p_j \to p_i} \text{mult}_{p_j}(\varphi_C). \tag{1.2}$$

See [22, Theorem 3.5.3 (Proximity equalities)].

We conclude the section stating the concept of dual graph of a configuration over a surface Z_0.

A *vertex-weighted graph* is a pair (Γ, ℓ_Γ), where Γ is a non-directed graph whose set of vertices is V_Γ and $\ell_\Gamma : V_\Gamma \to \mathbb{Z}$ is a map. For simplicity, we will denote it by Γ. Set Γ_1 and Γ_2 two vertex-weighted graphs. A *graph homomorphism* $(\Gamma_1, \ell_{\Gamma_1}) \to (\Gamma_2, \ell_{\Gamma_2})$ of vertex-weighted graphs is a morphism $\phi : \Gamma_1 \to \Gamma_2$ such that $\ell_{\Gamma_2}(\phi(\mathbf{v})) = \ell_{\Gamma_1}(\mathbf{v})$ for all $\mathbf{v} \in V_{\Gamma_1}$.

Let Ω be a set $\{C_1, C_2, \ldots, C_m\}$ of integral curves on a surface Z. The *dual graph* of Ω, denoted by $\Gamma(\Omega)$, is a vertex-weighted graph $(\Gamma(\Omega), \ell_{\Gamma(\Omega)})$ obtained as follows. Its set of vertices corresponds one-to-one to the curves of Ω (that is, one assigns a vertex $\mathbf{v}(C)$ of $\Gamma(\Omega)$ to each curve $C \in \Omega$); two vertices $\mathbf{v}(C_i)$ and $\mathbf{v}(C_j)$ are connected by an edge when $i \neq j$ and $C_i \cap C_j \neq \emptyset$; and $\ell_{\Gamma(\Omega)}(\mathbf{v}(C_i)) := C_i^2$.

Set $\mathcal{C}$ a configuration over a surface Z_0. The *dual graph* $\Gamma_{\mathcal{C}}$ of $\mathcal{C}$ is defined to be the dual graph of $\Omega := \{E_p\}_{p \in \mathcal{C}}$.

1.3 Plane valuations

In this section we describe some properties and objects related to valuations of quotient fields of 2-dimensional regular local rings centered at them. As we will see, these valuations are directly connected to simple sequences of points blowups. Moreover, we recall some of their invariants, which are key objects in this dissertation. We have developed this section following [117], [109] and [36]. We will keep the notation as above.

Let K be the quotient field of a 2-dimensional regular local ring $(R,\mathfrak{m})$ and set $K^* = K \setminus \{0\}$. Assume that k is the residue field $R/\mathfrak{m}$ and it is algebraically closed and contained in R. Set G a totally ordered abelian group. A *valuation* of K is a surjective map $\nu : K^* \to G$ that satisfies:

$$\nu(f+g) \geq \min\{\nu(f), \nu(g)\} \text{ and } \nu(fg) = \nu(f) + \nu(g), \text{ for } f, g \in K^*.$$

The group G is called the *value group* of ν. Given two valuations, $\nu_1 : K^* \to G_1$ and $\nu_2 : K^* \to G_2$, ν_1 and ν_2 are *equivalent* if there exists an order-preserving isomorphism $h : G_1 \to G_2$ of their groups of values such that $h \circ \nu_1 = \nu_2$ (see [117, Chapter VI, Section 8]).

The *valuation ring* of ν is the local regular ring $R_\nu := \{f \in K^* \mid \nu(f) \geq 0\} \cup \{0\}$. Its maximal ideal is $\mathfrak{m}_\nu = \{f \in K^* \mid \nu(f) > 0\} \cup \{0\}$. A valuation ν is said to be *centered* at R when $R \cap \mathfrak{m}_\nu = \mathfrak{m}$. Two valuations centered at R are equivalent if and only if their valuation rings are isomorphic [117, Chapter VI, Section 8].

The valuations of K centered at R are usually called *plane valuations*. Valuations (not only plane valuations) were central objects in some works of the middle of the last century by Abhyankar and Zariski which aim to solve problems of resolution of singularities in algebraic geometry (see [114, 115, 117, 1, 2] for example). Zariski introduced three classical invariants for valuations which, in the plane case, help us to classify them: the *rank* of ν, which is denoted by $\mathrm{rk}(\nu)$ and defined as the Krull dimension of R_ν; the *rational rank* $\mathrm{r.rk}(\nu)$ of ν, which is the dimension of the $\mathbb{Q}$-vector space $G \otimes_{\mathbb{Z}} \mathbb{Q}$; and the *transcendence degree* of ν, written $\mathrm{tr.deg}(\nu)$ and defined as the transcendence degree of the field extension k_ν/k, where k_ν is the residue field $k_\nu := R_\nu/\mathfrak{m}_\nu$. Abhyankar in [2] showed that (in our plane case) the above invariants satisfy the following inequalities:

$$\mathrm{rk}(\nu) + \mathrm{tr.deg}(\nu) \leq \mathrm{r.rk}(\nu) + \mathrm{tr.deg}(\nu) \leq \dim(R) = 2.$$

An important geometrical property of the plane valuations is given in the next theorem (see [117] and [109]).

Keep the above notation and let ν be a plane valuation. Set π the (non-necessarily finite) simple sequence of point blowups:

$$\pi : \cdots \to Z_n \xrightarrow{\pi_n} Z_{n-1} \to \cdots \to Z_1 \xrightarrow{\pi_1} Z_0 = \mathrm{Spec} R, \tag{1.3}$$

where π_1 is the blowup at the closed point $p := p_1 \in Z_0$ defined by the maximal ideal $\mathfrak{m}$ and $\pi_{i+1}, i \geq 1$, is the blowup at the unique closed point p_{i+1} that belongs to the exceptional divisor E_i created by π_i and such that the plane valuation ν is centered at $\mathcal{O}_{Z_i, p_{i+1}}$.

The above assignation gives rise to the following result.

Theorem 1.3.1. *Plane valuations (up to equivalence) of K centered at R correspond one-to-one with sequences of point blowups as* (1.3).

Let ν be a plane valuation. The *semigroup* of ν, S_ν, is defined to be the monoid $\nu(R \setminus \{0\}) \cup \{0\}$, and the set $\mathcal{C}_\nu = \{p_1, p_2, \dots\}$ is called the *configuration of infinitely near points of* ν. Following Section 1.2, notice that $\mathcal{C}_\nu$ is a configuration over Z_0 with origin at p whose cardinality might be infinite. From now on, an exceptional divisor E_i is called *satellite* (respectively, *free*) if the point p_i is satellite (respectively, free).

Another useful object related to a plane valuation ν is its *dual graph* Γ_ν, which is the dual graph of the configuration $\mathcal{C}_\nu$. Here, for simplicity, the weight attached to each vertex will be the positive integer i which represents the exceptional divisor E_i, instead of the self-intersection of E_i. Notice that both ways of attaching weights to the graph are equivalent. A vertex of Γ_ν is called to be *satellite* (respectively, *free*) when it represents a satellite (respectively, free) exceptional divisor. As the sequence of blowups corresponding to ν is simple, the dual graph Γ_ν is a tree and the degree of the vertices of Γ_ν may be $0, 1, 2$ or 3.

Let Γ_ν be the dual graph of a plane valuation ν and $\mathcal{C}_\nu$ the configuration of ν. Next we introduce some notation. Set $st_0 := \mathbf{1}$ and let us define $g \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ and an increasing sequence of g weights of Γ_ν $st_1, st_2, \dots$ distinguishing three cases: (1) $\mathcal{C}_\nu$ is infinite and Γ_ν has infinitely many vertices of degree 3, (2) $\mathcal{C}_\nu$ is finite and its last point p_n is satellite, and (3) otherwise. In case (1) we define $g := +\infty$ and we set $st_1 < st_2 < \cdots$ the weights of the vertices of degree 3. In case (2) $g-1$ is defined to be the number of vertices of degree 3 in Γ_ν and we set $st_1 < st_2 < \cdots < st_{g-1}$ the weights of those vertices (if there are any) and st_g the weight of the vertex associated to E_{p_n}. In case (3) g is the number of vertices of degree 3 in Γ_ν and set $st_1 < st_2 < \cdots < st_g$ the weights of those vertices (if there are any). The vertices with weight st_i before introduced are usually called *star vertices* or *stars*.

In the above cases (2) and (3), we distinguish two subgraphs of $\Gamma_\nu : \hat{\Gamma}_\nu$ and Γ_ν^{g+1}. The subgraph $\hat{\Gamma}_\nu$ is a connected graph formed by the vertices α such that $\alpha \leq st_g$ and the edges which join them. The subgraph Γ_ν^{g+1} contains those vertices α such that $\alpha \geq st_g$ and their attached edges. In case (1) we set $\hat{\Gamma}_\nu := \Gamma_\nu$

Notice that $\hat{\Gamma}_\nu$ is just the vertex $\mathbf{1}$ when $g = 0$. Otherwise, one can find more than one vertex with degree 1 in $\hat{\Gamma}_\nu$. These vertices are named *dead ends* and denoted in a ordered way by $\ell_i, 0 \leq i \leq g$. Thus, $\ell_0 := \mathbf{1}$.

In this work, we will use the following partial order on the set of the vertices of $\hat{\Gamma}_\nu$: given two vertices α and β, we define $\alpha \preccurlyeq \beta$ if there exists a path from $\mathbf{1}$ to β going through α. This ordering allows us to divide $\hat{\Gamma}_\nu$ into subgraphs Γ_ν^i, $1 \leq i \leq g$, where Γ_ν^i contains the vertices α such that $st_{i-1} \preccurlyeq \alpha \preccurlyeq \ell_i$ and the attached edges.

Plane valuations have been classified by Spivakovsky in five types according to their dual graphs (see [109]). Next we recall this classification. The names of the different types of valuations come from [47].

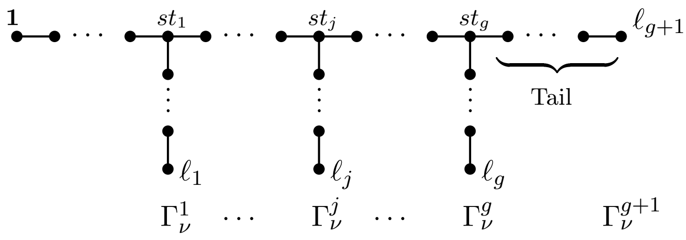


Figure 1.1: Dual graph of a divisorial valuation

A plane valuation ν is called *divisorial* if its configuration of infinitely near points is finite. We set ℓ_{g+1} the weight the last vertex. The subgraph of Γ_ν whose vertices (together with the edges joining them) are those whose weights α satisfy $st_g \preccurlyeq \alpha \preccurlyeq \ell_{g+1}$ is denoted Γ_ν^{g+1} and named *tail*. Note that $st_g = \ell_{g+1}$ and it has degree 0 or 2 when the tail has a unique vertex. If $g = 0$, every vertex of Γ_ν^{g+1} is free. Otherwise, all the vertices of Γ_ν^{g+1} are free with the exception of st_g. Figure 1.1 shows the shape of Γ_ν.

Up to isomorphism, the group of values of a divisorial valuation ν is $\mathbb{Z}$. When we consider this group embedded into $\mathbb{R}$, then there exists $c \in \mathbb{R} \setminus \{0\}$ such that $\nu(f) = c \cdot \mathrm{ord}_E(f)$ for all $f \in K^*$, where E is the last exceptional divisor created by the sequence of point blowups associated to ν [109, Remark 2.7]. Different constants c give rise to equivalent divisorial valuations defined by the same exceptional divisor. Throughout this work, we usually assume $c = 1$. The equivalent valuation to ν given by $c = 1/\nu(\mathfrak{m})$, where $\nu(\mathfrak{m}) = \min\{\nu(f) \mid f \in \mathfrak{m} \setminus \{0\}\}$, is called the *normalization* of ν and it is denoted ν^N.

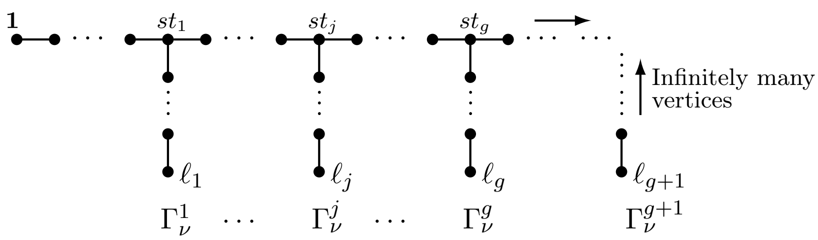


Figure 1.2: Dual graph of an irrational valuation

A valuation ν is named *irrational* when its configuration is infinite and there is a positive integer r such that the points p_i, for $i \geq r$, are satellite and they are not proximate to a same point. Its group of values is a subgroup of $\mathbb{R}$. The dual graph Γ_ν is like that in Figure 1.2. The subgraph Γ_ν^{g+1} contains infinitely many vertices, the exceptional divisors corresponding to those vertices with weight $st_g + 1 \leq i \leq l_{g+1}$ are free and the remaining ones are satellite. Here, the vertex with weight st_g is always a star vertex if $g \neq 0$.

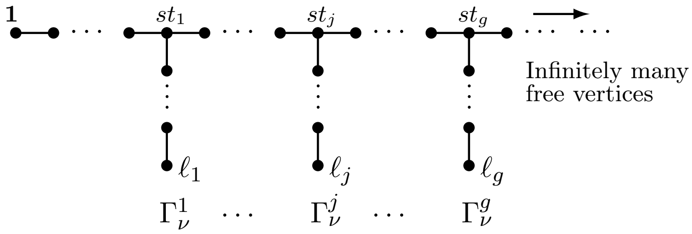


Figure 1.3: Dual graph of a non-exceptional curve valuation

A plane valuation is a *non-exceptional curve* valuation if its configuration contains infinitely many points and all of them are free after a point p_r. The group of values is isomorphic to $\mathbb{Z}^2_{\text{lex}}$ (lexicographically ordered). The dual graph associated to this type of valuations is depicted in Figure 1.3. In this case Γ_ν^{g+1} has infinitely many free vertices with the exception of st_g when $g \neq 0$.

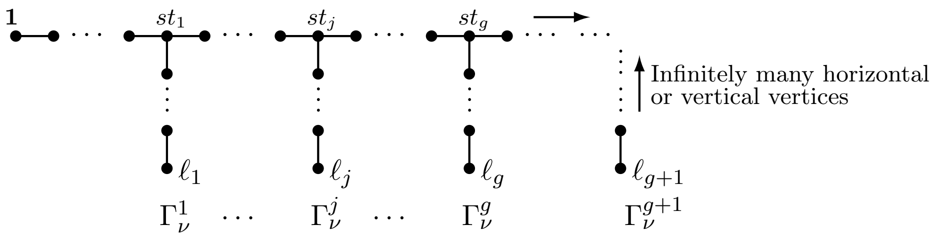


Figure 1.4: Dual graph of an exceptional curve valuation

An *exceptional curve valuation* is a plane valuation whose configuration has infinitely many points and there exists a point p_r such that all the points p_i, where $i > r$, are proximate to p_r. $\mathbb{Z}^2_{\text{lex}}$ is the corresponding group of values. Its dual graph can be seen in Figure 1.4. The subgraph Γ_ν^{g+1} contains infinitely many vertices such that at the beginning a finite number of them are free and the remaining ones are satellite. Here, st_g is also the weight of a star vertex if $g \neq 0$ and ℓ_{g+1} is the last free vertex in Γ_ν^{g+1}.

Finally, a plane valuation is called to be *infinitely singular* if its associated configuration is infinite and there are infinitely many points which give rise to divisors E_{st_i}. That is, its dual graph (Figure 1.5) has an infinite number of star vertices. Its group of values is a subgroup of $\mathbb{R}$.

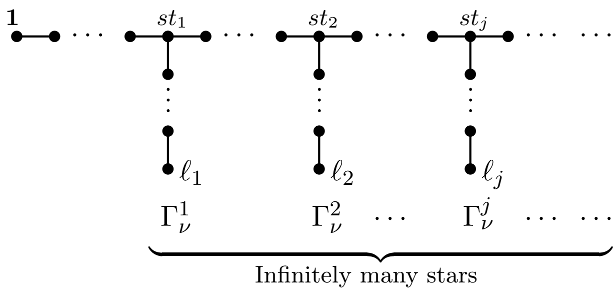


Figure 1.5: Dual graph of infinitely singular valuation

The following result is useful to perform computations with non-divisorial valuations. More details can be found in [54] and [65, Section 3.2].

Theorem 1.3.2. *Let $\mathcal{O}_{Z_0,p}$ be a regular local ring.*

(a) Let ν be a non-exceptional curve valuation centered at $\mathcal{O}_{Z_0,p}$. There exists an analytically irreducible germ of curve φ_C passing through p such that

$$\nu(f) = \begin{cases} (0, (\varphi_C, f)_p), & \text{if } \varphi_C \nmid f \\ (\alpha, (\varphi_C, f)_p), & \text{if } f = \varphi_C^\alpha h; \varphi_C \nmid h \end{cases}$$

where $f, h \in \mathcal{O}_{Z_0,p} \setminus \{0\}$.

(b) Let ν be an exceptional curve valuation centered at $\mathcal{O}_{Z_0,p}$, and $\mathcal{C}_\nu = \{p_1, p_2, \ldots\}$ its configuration where all points p_i are proximate to a point p_r for $i > r \geq 1$. Then, up to equivalence,

$$\nu(f) = (v_1(f), v_2(f)), \text{for all } f \in \mathcal{O}_{Z_0,p} \setminus \{0\},$$

where $v_1(f) := \nu_r(f)$ is the divisorial valuation defined by the exceptional divisors E_r and $v_2(f) = \nu_\eta(f) + \sum_{p_i \to p_r} \mathrm{mult}_{p_i}(f)$, ν_η being the divisorial valuation defined by E_η and η the index different from r such that $p_{r+1} \in E_\eta$ when p_{r+1} is satellite; otherwise, $\nu_\eta(f) = 0$.

(c) Let ν be an irrational valuation centered at $\mathcal{O}_{Z_0,p}$. Then, up to equivalence,

$$\nu(f) = \lim_{i \to \infty} \nu_i^N(f), \text{ for all } f \in \mathcal{O}_{Z_0,p} \setminus \{0\},$$

where ν_i^N is the normalization of the divisorial valuation defined by the exceptional divisor E_i created by the sequence of points blowups corresponding to ν.

(d) Let ν be an infinitely singular valuation centered at $\mathcal{O}_{Z_0,p}$. Then, up to equivalence,

$$\nu(f) = \lim_{i \to \infty} \nu_i^N(f), \text{ for all } f \in \mathcal{O}_{Z_0,p} \setminus \{0\},$$

where ν_i^N is defined as above.

Notice that the irrational and infinitely singular valuations of the above theorem are normalized. In this dissertation we always suppose that condition.

To conclude, in Figure 1.6 we show the relation of the above classification of the plane valuations with the Zariski invariants.

Type	$\mathrm{rk}(\nu)$	$\mathrm{r.rk}(\nu)$	$\mathrm{tr.deg}(\nu)$
Divisorial valuation	1	1	1
Irrational valuation	1	2	0
Non-exceptional curve valuation	2	2	0
Exceptional curve valuation	2	2	0
Infinitely singular valuation	1	1	0

Figure 1.6: Table of the types of plane valuations

1.3.1 More invariants of plane valuations

Several invariants have been considered to study plane valuations. We define some of them for divisorial, irrational and exceptional curve valuations, since we will only consider these valuations in the following chapters (see [109, 36] for further information).

Let ν be a divisorial, irrational or exceptional curve valuation and $\mathcal{C}_\nu = \{p_i\}_{i\geq 1}$ ($p := p_1$) its configuration of infinitely near points. Write $\mathfrak{m}_i$ the maximal ideal corresponding to the point p_i for $i \geq 1$. We call *sequence of values of* ν to the set $\{\nu(\mathfrak{m}_i)\}_{i\geq 1}$, where $\nu(\mathfrak{m}_i) = \min\{\nu(f) \mid f \in \mathfrak{m}_i \setminus \{0\}\}$. The sequence of values satisfies the proximities equalities [22, Theorem 8.1.7]:

$$\nu(\mathfrak{m}_i) = \sum_{p_j \to p_i} \nu(\mathfrak{m}_j),\ i \geq 1,$$

when the set $\{p_j \in \mathcal{C}_\nu \mid p_j \to p_i\}$ is not empty. If ν is an exceptional curve valuation and $p_i \to p_r$ for every $i > r$, then $\nu(\mathfrak{m}_r) = (a, b)$ and $\nu(\mathfrak{m}_i) = (0, c)$, for some $a, b, c \in \mathbb{Z}$, $a, c > 0$.

Denote by π the simple sequence of blowups associated to ν as showed in (1.3). Following [109, Section 7] and [47, Chapter 6, Section 6, Subsection 1], there exists an analytically irreducible germ φ_i at p such that its strict transform on Z_i is transversal to E_i at any previously fixed non-singular point of the exceptional locus. In the divisorial case, it holds that $\nu(\mathfrak{m}_i) = \mathrm{mult}_{p_i}(\varphi_n)$, for $i \geq 1$.

Along this work we will often use the so-called Noether formula for valuations. One can find a proof in [22, Theorem 8.1.6]:

Lemma 1.3.3. *Let ν be a divisorial, irrational or exceptional curve valuation whose configuration (of infinitely near points) is $\mathcal{C}_\nu = \{p_i\}_{i\geq 1}$ ($p := p_1$). Let C be a curve* *on* Z_0*. Then*

$$\nu(\varphi_C) = \sum_{i\geq 1} \nu(\mathfrak{m}_i)\cdot \mathrm{mult}_{p_i}(\varphi_C).$$

As a consequence of the above result, if ν is a divisorial valuation defined by an exceptional divisor E_n, one has that, under a suitable choice of φ_n, $\nu(\varphi_C) = (\varphi_n, \varphi_C)_p$.

The *sequence of Puiseux exponents of* ν is defined to be the ordered set $\{\beta'_i(\nu)\}_{i=0}^{g+1}$ where $\beta'_0(\nu) = 1$ and, for $1 \leq i \leq g+1$, $\beta'_i(\nu)$ is the continued fraction

$$\beta'_i(\nu) = a_1^i + \cfrac{1}{a_2^i + \cfrac{1}{\ddots + \cfrac{1}{a_{r_i}^i}}} = \langle a_1^i; a_2^i, \ldots, a_{r_i}^i\rangle,$$

where a_k^i, $1 \leq k < r_i + 1$, successively counts the number of vertices in Γ_ν^i with the same value $\nu(\mathfrak{m}_j)$. It holds that $\beta'_i(\nu) \in \mathbb{Q}_{>0}\setminus\mathbb{Z}_{>0}$, for $1 \leq i < g$, and they are bigger than 1. When ν is divisorial (respectively, irrational), then $r_{g+1} = 1$ (respectively, $r_{g+1} = \infty$) and thus $\beta'_{g+1}(\nu)$ is a positive integer (respectively, $\beta'_{g+1}(\nu) \in \mathbb{R}_{>0}\setminus\mathbb{Q}_{>0}$). When ν is an exceptional curve valuation, then $a_{r_{g+1}}^{g+1} = \infty$ and so $\beta'_{g+1}(\nu)$ does not exist (see [36] for more information).

Finally, the sequence of *maximal contact values of* ν is the set $\{\overline{\beta}_i(\nu)\}_{i=0}^{g+1}$, where each element $\overline{\beta}_i(\nu)$ is the value $\nu(\varphi_{\ell_i})$ for $0 \leq i \leq g+1$. This family of invariants generates the semigroup of values S_ν of ν. Moreover, if ν is a divisorial valuation defined by the divisor E_n, $\overline{\beta}_{g+1}(\nu)$ can be obtained as a combination of the remaining maximal contact values and satisfies

$$\overline{\beta}_{g+1}(\nu) = \sum_{i=1}^{n} \nu(\mathfrak{m}_i)^2, \tag{1.4}$$

by the Noether formula.

Before stating an useful result, following [46, Section 5.2], we introduce an algorithm that extends the Euclidean division and the greatest common divisor for some values in the additive semigroup $\mathbb{R}^n_{\geq 0}$, where $n \geq 1$, under the lexicographical order. We denote the elements of $\mathbb{R}^n$ by $\overline{x} = (x_1, x_2, \ldots, x_n)$.

Proposition 1.3.4. [46, Proposition 5.13] *Let* $\overline{y} \leq \overline{x} \in \mathbb{R}^n_{\geq 0}$ *be such that there exists an index* t $(1 \leq t \leq n)$ *satisfying* $x_j = y_j = 0$ *for* $j < t$ *and* $y_t > 0$*. Then there exists a unique positive integer* m *such that* $\overline{x} = m\,\overline{y} + \overline{z}$ *and* $(0, 0, \ldots, 0) =: \overline{0} \leq \overline{z} < \overline{y}$.

As a consequence, given two elements $\overline{x}_0$ and $\overline{x}_1$ in $\mathbb{R}^n_{\geq 0}$ such that $\overline{x}_1 \leq \overline{x}_0$, performing when possible "Euclidean divisions" as described in Proposition 1.3.4, one gets the following algorithm:

$$\begin{array}{rclc} \overline{x}_0 & = & m_0\,\overline{x}_1 + \overline{x}_2; & \overline{0} < \overline{x}_2 < \overline{x}_1 \\ \overline{x}_1 & = & m_1\,\overline{x}_2 + \overline{x}_3; & \overline{0} < \overline{x}_3 < \overline{x}_2 \\ \vdots & \vdots & \vdots & \vdots \\ \overline{x}_{l-1} & = & m_{l-1}\,\overline{x}_l + \overline{x}_{l+1}; & \overline{0} < \overline{x}_{l+1} < \overline{x}_l \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

The next possibilities for the algorithm can happen:

(1) It stops and $\overline{x}_k = m_k\,\overline{x}_{k+1} + \overline{0}$ holds for some index k.

(2) It never stops and one obtains an infinite sequence of non-negative integers $m_l, l \geq 0$.

(3) It stops and there exists, for some index k, another index t $(1 \leq t \leq n)$ such that the first t components $x_{k+1,j}, 1 \leq j \leq t$, vanish, but $x_{k,t} \neq 0$, being $x_{k,1} = \ldots = x_{k,t-1} = 0$, that is, $m_k = \infty$.

When Item (1) holds, it is said that $\overline{x}_{k+1}$ is the *greatest common divisor* of $\overline{x}_0$ and $\overline{x}_1$, denoted by $\gcd(\overline{x}_0, \overline{x}_1)$.

Consider now the sequences of Puiseux exponents $\{\beta'_i(\nu)\}_{i=0}^{g+1}$ and the maximal contact values $\{\overline{\beta}_i(\nu)\}_{i=0}^{g+1}$ of a divisorial, irrational or exceptional curve valuation. Then both sequences satisfy the following result [36, Lemma 1.8].

Proposition 1.3.5. *Under the above notation, for* $0 \leq i < g$,

$$\overline{\beta}_{i+1}(\nu) = e_i(\nu)(\beta'_{i+1}(\nu) - 1) + n_i(\nu)\overline{\beta}_i(\nu) \tag{1.5}$$

where $e_i(\nu) := \gcd(\overline{\beta}_0(\nu), \overline{\beta}_1(\nu), \ldots, \overline{\beta}_i(\nu)), n_0(\nu) = 1$ *and* $n_i(\nu) = e_{i-1}(\nu)/e_i(\nu)$. *The above formula is also true for* $i = g$ *when* ν *is a divisorial or irrational valuation.*

As a result, it holds that $\overline{\beta}_i(\nu) \in \mathbb{Z}_{>0}, \overline{\beta}_{i+1}(\nu) \notin e_i(\nu)\mathbb{Z}$ and $\overline{\beta}_{i+1}(\nu) > n_i(\nu)\overline{\beta}_i(\nu)$, for $0 \leq i < g$. In addition, $\overline{\beta}_{g+1}(\nu) \geq n_g(\nu)\overline{\beta}_g(\nu)$ and, if ν is divisorial, $\overline{\beta}_{g+1}(\nu)$ is a positive integer.

The following results will be useful.

Corollary 1.3.6. *Let* ν *be a divisorial valuation,* $\{\beta'_i(\nu)\}_{i=0}^{g+1}$ *its sequence of Puiseux exponents and* $\{\overline{\beta}_i(\nu)\}_{i=0}^{g+1}$ *its sequence of maximal contact values. Define* $e_g(\nu) = 1$, $e_i(\nu) = \gcd(\overline{\beta}_0(\nu), \overline{\beta}_1(\nu), \ldots, \overline{\beta}_i(\nu)), n_0(\nu) = 1, n_i(\nu) = e_{i-1}(\nu)/e_i(\nu)$, *for* $0 \leq i \leq g$, *as in Proposition 1.3.5. Then*

$$\beta'_i(\nu) = \frac{q_i(\nu)}{n_i(\nu)}, \text{ for } 1 \leq i < g+1,$$

for some $q_i(\nu) \in \mathbb{Z}_{>0}$ *such that* $\gcd(q_i(\nu), n_i(\nu)) = 1$, *and*

$$e_i(\nu) = \prod_{k=i+1}^{g} n_k(\nu), \text{ for } 0 \leq i < g.$$

Proof. Suppose $g = 1$. It follows easily that $e_0(\nu) = \overline{\beta}_0(\nu)$ and $e_1(\nu) = 1$. By Condition (1.5),

$$\beta'_1(\nu) = \frac{\overline{\beta}_1(\nu)}{e_0(\nu)} = \frac{\overline{\beta}_1(\nu)/e_1(\nu)}{e_0(\nu)/e_1(\nu)} = \frac{q_1(\nu)}{n_1(\nu)},$$

where $q_1(\nu)$ and $n_1(\nu)$ are positive integers. Also, it holds that

$$\gcd(q_1(\nu), n_1(\nu)) = \gcd(\overline{\beta}_1(\nu), \overline{\beta}_0(\nu)) = 1,$$

which completes the proof for the case $g = 1$.

Now consider $g > 1$. As above, by (1.5), one gets that

$$\begin{aligned} \beta'_i(\nu) &= \frac{\overline{\beta}_i(\nu) - n_{i-1}(\nu)\overline{\beta}_{i-1}(\nu) + e_{i-1}(\nu)}{e_{i-1}(\nu)} \\ &= \frac{\big(\overline{\beta}_i(\nu) - n_{i-1}(\nu)\overline{\beta}_{i-1}(\nu) + e_{i-1}(\nu)\big)/e_i(\nu)}{e_{i-1}(\nu)/e_i(\nu)} = \frac{q_i(\nu)}{n_i(\nu)}, \end{aligned}$$

for $i \in \{1, 2, \ldots, g\}$, where $q_i(\nu)$ and $n_i(\nu)$ are positive integers satisfying that $\gcd(q_i(\nu), n_i(\nu)) = 1$. Indeed, as $\overline{\beta}_{i-1}(\nu) \in e_{i-1}(\nu)\mathbb{Z}$ and $e_{i-2}(\nu), e_{i-1}(\nu), \overline{\beta}_{i-1}(\nu)$ and $\overline{\beta}_i(\nu) \in e_i(\nu)\mathbb{Z}$, it holds that $q_i(\nu)$ and $n_i(\nu)$ are positive integers. Now we prove that $\gcd(q_i(\nu), n_i(\nu)) = 1$ reasoning by contradiction. Suppose that there exists a positive integer $m_i > 1$ such that $\gcd(q_i(\nu), n_i(\nu)) = m_i$. Consequently, one obtains $\gcd(\overline{\beta}_0(\nu), \overline{\beta}_1(\nu), \ldots, \overline{\beta}_i(\nu)) = m_i e_i(\nu)$, which leads to a contradiction.

Finally, it holds that $e_i(\nu) = \prod_{k=i+1}^{g} n_k(\nu)$, where $n_k(\nu) = e_{k-1}(\nu)/e_k(\nu)$ and $0 \leq i < g$, since $e_g(\nu) = 1$. This concludes the proof. □

As a consequence of Proposition 1.3.5 and Corollary 1.3.6, the sequence of maximal contact values of a divisorial valuation ν might be computed inductively using exclusively its sequence of Puiseux exponents and taking $\overline{\beta}_0(\nu) = \prod_{i=1}^{g} n_i(\nu)$. Another immediate consequence of the above mentioned results is the following corollary.

Corollary 1.3.7. *Let ν (respectively, ν') be a divisorial valuation, $\{\beta'_i(\nu)\}_{i=0}^{g+1}$ (respectively, $\{\beta'_i(\nu')\}_{i=0}^{\hat{g}+1}$) its sequence of Puiseux exponents and $\{\overline{\beta}_i(\nu)\}_{i=0}^{g+1}$ (respectively, $\{\overline{\beta}_i(\nu')\}_{i=0}^{\hat{g}+1}$) its sequence of maximal contact values. Assume that $\hat{g} \leq g$. Consider the value $e_i(\nu)$ (respectively, $e_i(\nu')$) defined as in Corollary 1.3.6.*

(a) If $\beta'_i(\nu) = \beta'_i(\nu')$ for $0 \leq i \leq k$, where $0 \leq k < \hat{g}$, then

$$e_i(\nu') = e \cdot e_i(\nu) \text{ and } \overline{\beta}_i(\nu') = e \cdot \overline{\beta}_i(\nu), \text{ for } 0 \leq i \leq k,$$

where $0 \leq k < \hat{g}$ and $e = e_k(\nu')/e_k(\nu)$.

(b) If $\beta'_i(\nu) = \beta'_i(\nu')$ for $0 \leq i \leq \hat{g} < g$, then

$$e_i(\nu') = e \cdot e_i(\nu) \text{ and } \overline{\beta}_i(\nu') = e \cdot \overline{\beta}_i(\nu), \text{ for } 0 \leq i \leq \hat{g} < g,$$

where $e = e_{\hat{g}}(\nu)^{-1}$.

(c) If $\beta'_i(\nu) = \beta'_i(\nu')$ for $0 \leq i \leq \hat{g} = g$, then

$$e_i(\nu') = e_i(\nu) \text{ and } \overline{\beta}_i(\nu') = \overline{\beta}_i(\nu), \text{ for } 0 \leq i \leq \hat{g} = g.$$

1.4 Cones associated to a surface

This section contains some basic concepts in convex analysis needed to study the cone of curves of a surface, one of the most important objects in this dissertation. In addition, we recall the definitions of several types of divisors, the convex cones that they generate and the connections among them. For the first part we have followed [53], [103] and [91] and for the second one [71] and [85]. We keep the notation established in the above sections.

1.4.1 Convex cones

Let $M \cong \mathbb{Z}^m$ be a free module of rank m over $\mathbb{Z}$ and $N := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ its dual module. Consider the $\mathbb{Z}$-bilinear pairing

$$\langle \, , \rangle : N \times M \to \mathbb{Z},$$

where $\langle \bar{y}, \bar{x} \rangle$ is the *value* of $\bar{y}$ in $\bar{x}$, for $\bar{x} \in M$ and $\bar{y} \in N$.

Set the vector space $V := M \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^m$ over $\mathbb{R}$ and its dual vector space $V^* := N \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^m$. Each element $\bar{x}$ of M (respectively, $\bar{y}$ of N) can be identified with its image $\bar{x} \otimes 1$ in V (respectively, $\bar{y} \otimes 1$ in V^*). We regard $M \subseteq V$ and $N \subseteq V^*$ and notice that the above pairing can be extended as follows:

$$\langle \, , \rangle : V^* \times V \to \mathbb{R}.$$

Consider on V and V^* the induced usual topology of $\mathbb{R}^m$ and set $\|\cdot\|$ the associated metric on V and V^*. Given A, B subsets of V (or V^*), denote by $\bar{A}$ the topological closure (with respect to the usual topology of $\mathbb{R}^m$), $\mathrm{Int}(A)$ the topological interior, $A^{\perp}$ the orthogonal complement of A with respect to the bilinear pairing $\langle \, , \rangle$, $-A$ the set $-A = \{-a \mid a \in A\}$ and $A + B$ the set $A + B = \{a + b \mid a \in A \text{ and } b \in B\}$.

We also consider the vector spaces $V_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q}$ and $V^*_{\mathbb{Q}} = N \otimes \mathbb{Q}$ over $\mathbb{Q}$. They can be identified as subsets of V and V^*, respectively. We will denote by $A_{\mathbb{Q}}$ the intersection of a subset A of V (respectively, V^*) and $V_{\mathbb{Q}}$ (respectively, $V^*_{\mathbb{Q}}$).

Let A be a subset of V. A is said to be *convex* if $\lambda \, \bar{x} + (1 - \lambda) \, \bar{y} \in A$, for $\bar{x}, \bar{y} \in A$ and $\lambda \in \mathbb{R}$ such that $0 \leq \lambda \leq 1$. The intersection of all convex sets containing A is called the *convex hull* of A and denoted by $\mathrm{conv}(A)$. Notice that this convex set is the smallest one which contains A.

Let S be a non-empty subset of V. We say that S is a *convex cone* (or simply a *cone*) if it satisfies the following conditions:

$$\bar{x} + \bar{y} \in S \text{ and } \alpha \, \bar{x} \in S, \text{ for } \bar{x}, \bar{y} \in S \text{ and } \alpha \in \mathbb{R}_{\geq 0}.$$

The vector subspace of V generated by S is denoted by $\mathrm{lin}(S)$ and the dimension of $\mathrm{lin}(S)$ by $\dim(S)$. If $S = \{0\}$, the dimension is 0.

Notice that $\text{Int}(S) \cup \{0\}$, $\bar{S}$ and the intersection of non-empty cones are cones. Indeed, set $\bar{x}, \bar{y} \in \text{Int}(S) \cup \{0\}$ and we are going to show that $\bar{x}+\bar{y} \in \text{Int}(S) \cup \{0\}$. We can suppose that $\bar{x}$ and $\bar{y}$ are different from 0. There exist two neighbourhoods of 0 in $\text{lin}(S)$, U and V, such that $\bar{x}+U \subset S$ and $\bar{y}+V \subset S$. Reasoning by contradiction, suppose that $\bar{x}+\bar{y} \notin \text{Int}(S)$. Therefore, we can find an element $\bar{z} \in U \cap V$ such that $\bar{x}+\bar{y}+\bar{z} \notin S$. That is a contradiction, since S is a cone. Set $\bar{x} \in \text{Int}(S) \cup \{0\}$ and $\alpha \in \mathbb{R}_{\geq 0}$, let us show that $\alpha\bar{x} \in \text{Int}(S) \cup \{0\}$. We can suppose that both α and $\bar{x}$ do not vanish. Then there exists a neighbourhood U of 0 in $\text{lin}(S)$ such that $\bar{x}+U \subset S$. Thus one has that $\alpha\bar{x} \in \text{Int}(S)$ since otherwise, as αU is also a neighbourhood of 0 in $\text{lin}(S)$, $\alpha\bar{x}+\alpha\bar{z} \notin \text{Int}(S)$ for some $\bar{z} \in U$, but $\alpha\bar{x}+\alpha\bar{z} = \alpha(\bar{x}+\bar{z}) \in S$ because S is a cone and $\bar{x}+\bar{z} \in S$ and then one gets a contradiction. Similar arguments prove that $\bar{S}$ and the intersection of non-empty cones are cones.

Let A be a non-empty subset of V. The family of cones which contain A is non-empty, since V is a cone. The intersection of all cones that contain A is the smallest cone that contains A and is denoted by $\text{con}(A)$. When a cone S of V can be expressed as $S = \text{con}(A)$, where A is a non-empty subset of V, it is said that S is *generated* by A.

A subset A of V is said to be *polyhedral* if it is the intersection of a finite number of closed semi-spaces. That is, if there exists $\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_k \in V^*$ and $\beta_1, \beta_2, \ldots, \beta_k \in \mathbb{R}$ such that

$$A = \{\bar{x} \in V \mid \langle \bar{b}_i, \bar{x} \rangle \geq \beta_i \text{ for all } i = 1, 2, \ldots, k\}.$$

If A admits an expression like the previous one such that $\beta_i = 0$ for all $1 \leq i \leq k$, it is said that A is a *polyhedral cone*. Notice that every polyhedral set is a closed convex set.

A convex set A in V is said to be *finitely generated* if it can be expressed as $A = \text{conv}(A_1) + \text{con}(A_2)$, where A_1 and A_2 are two finite subsets of V.

One has the following property whose proof can be found in [103, Theorem 19.1].

Theorem 1.4.1. *A set $A \subseteq V$ is polyhedral if and only if A is a finitely generated convex set.*

As a consequence of the above theorem, a polyhedral cone is a set S of V which can be defined as $S = \text{con}(A)$, where A is a finite subset of V.

Let S be a cone in V. The cone S is named *rational* if it is generated by elements in M. A rational cone $S \subset V$ is called *regular* when it is generated by a subset of a $\mathbb{Z}$-basis of M. If S is a regular cone generated by a subset $\{\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_k\}$ of a $\mathbb{Z}$-basis of M, any element of $S \cap M$ can be expressed as a unique positive linear combination of $\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_k$.

The *dual cone* of a cone S is defined to be the closed cone

$$S^\vee := \{\bar{y} \in V^* \mid \langle \bar{y}, \bar{x} \rangle \geq 0 \text{ for all } \bar{x} \in S\}.$$

If S is a polyhedral cone, its dual cone is polyhedral (see [103, (Farkas' theorem) Chapter 19 and 22]. Moreover, $S = S^{\vee\vee}$ (see [103, Theorem 14.1]).

Proposition 1.4.2. *Let $S \subset V$ be a regular cone generated by the $\mathbb{Z}$-basis $B = \{\bar{e}_i\}_{i=1}^m$ of $\mathbb{Z}^m$. Then, $S^\vee$ is generated by the dual $\mathbb{Z}$-basis of B, $B^* = \{\bar{e}_i^*\}_{i=1}^m$.*

Proof. Set C^* the cone generated by B^*. It is clear that $C^* \subseteq C^\vee$ by the definition of C^*. We are going to prove that $C^\vee \subseteq C^*$. Let $\bar{x}$ be an element of $C^\vee \cap \mathbb{Z}^m$. As B^* is the dual $\mathbb{Z}$-basis of B, one has that $\bar{x} = \sum_{i=1}^m \alpha_i \bar{e}_i^*$, where $\alpha_i \in \mathbb{Z}$, for $1 \leq i \leq m$. Then

$$\alpha_i = \langle \bar{x}, \bar{e}_i \rangle \geq 0, \text{ for } 1 \leq i \leq m,$$

since $\bar{x}$ is an element of $C^\vee$. As a consequence, $\bar{x}$ can be expressed as a positive combination of the generators of C^* and then it belongs to C^*. □

Let S be a polyhedral cone in V. The vector subspace of V given by $S \cap (-S)$ is called the *linearity space* of S. By definition, the cone S is *strongly convex* if it is closed and its linearity space is $\{0\}$.

There are some useful equivalent conditions to the fact of that a cone is strongly convex (see [53, Section 1.2]).

Proposition 1.4.3. *Let S be a polyhedral cone. Then, the following conditions are equivalent:*

(a) S is strongly convex.

(b) S contains no non-zero linear subspace.

(c) There is an element $\bar{q} \in S^\vee$ such that $S \cap \{\bar{q}\}^\perp = \{0\}$.

A subset $\mathcal{F}$ of a cone S of V is called a *face* of S whenever $\mathcal{F} := S \cap \{\bar{z}\}^\perp$, for some $\bar{z} \in S^\vee$. It holds that S is a face of S, since $S = S \cap \{0\}^\perp$. If S is a strongly convex cone, $\{0\}$ is also a face of S. Every face different from S and $\{0\}$ is named *proper face*.

Some properties of the faces of a cone S are shown in the next result.

Proposition 1.4.4. *Let S be a cone in V. The following conditions are satisfied:*

(a) The faces of S are cones in V.

(b) If S is strongly convex, its faces are also strongly convex.

(c) If S is polyhedral, its proper faces are also polyhedral cones.

(d) If S is polyhedral, S has a finite number of faces.

(e) If $\mathcal{F}_1$ and $\mathcal{F}_2$ are faces of S such that $\mathcal{F}_1 \subseteq \mathcal{F}_2$, then $\mathcal{F}_1$ is a face of $\mathcal{F}_2$.

(f) *If S is polyhedral, $\mathcal{F}_2$ is a face of S and $\mathcal{F}_1$ is a face of $\mathcal{F}_2$, then $\mathcal{F}_1$ is a face of S.*

(g) *If S is polyhedral, then the map $\mathcal{F} \mapsto S^\vee \cap \mathcal{F}^\perp$ is a bijection between the set of the faces of S and $S^\vee$. Moreover, if $\mathcal{F}_1$ and $\mathcal{F}_2$ are faces of S, it holds that $dim(\mathcal{F}) + dim(S^\vee \cap \mathcal{F}^\perp) = dim(V)$ and*

$$\mathcal{F}_1 \subseteq \mathcal{F}_2 \text{ if and only if } S^\vee \cap \mathcal{F}_2^\perp \subseteq S^\vee \cap \mathcal{F}_1^\perp.$$

Proof. One obtains (a) as consequence of the definition of face.

To get (b) notice that, if $\mathcal{F}$ is a face of S, then $\mathcal{F} \cap (-\mathcal{F}) \subseteq S \cap (-S)$. As a result, if S is strongly convex, the face $\mathcal{F}$ is strongly convex.

We are going to prove (c) and (d). As S is a polyhedral cone, S is the cone generated by a subset $\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k\}$ of V. Moreover, by definition, a face $\mathcal{F}$ of S satisfies that $\mathcal{F} = S \cap \{\bar{q}\}^\perp$, where $\bar{q} \in S^\vee$, then $\mathcal{F}$ is the cone generated by those $\bar{v}_i$ such that $\langle \bar{q}, \bar{v}_i \rangle = 0$. This fact completes the proof.

We also have (e) because if $\mathcal{F}_1 = S \cap \{\bar{q}\}^\perp$, where $\bar{q} \in S^\vee$, then $\bar{q}$ belong to $\mathcal{F}_2^\vee$ by the definition of dual cone. Thus, $\mathcal{F}_1$ is a face of $\mathcal{F}_2$.

Finally, for proving (f) and (g) it sufficient to see [44, Proposition II.1.7] and [53, Section 1.2], respectively. □

A face $\mathcal{F}$ of a cone S is said to be *extremal* if, given two elements $\bar{z}_1, \bar{z}_2 \in S \setminus \{0\}$ such that $\bar{z}_1 + \bar{z}_2 \in \mathcal{F}$, it holds that $\bar{z}_1, \bar{z}_2 \in \mathcal{F}$. A 1-dimensional extremal face is an *extremal ray*. In addition, any polyhedral cone is the convex hull of its extremal rays [77, Definition 1.15].

We finish this first part of the section with the following result which allows us to compute the extremal rays of the dual cone of a cone in V. We only show a sketch of the proof which can be found in [91, Proposition A.3.22].

Proposition 1.4.5. *Let V be a vector space of dimension m and S a strongly convex polyhedral cone in V such that* $\dim(S) = m$*. Set $R = \mathbb{R}_{\geq 0}\,\bar{r} = \{\alpha\,\bar{r} \mid \alpha \in \mathbb{R}_{\geq 0}\} \subset S^\vee$, where $\bar{r} \in V^*$. Then, R is a extremal ray of $S^\vee$ if and only if there exist $m-1$ extremal rays $\mathcal{R}_1 = \mathbb{R}_{\geq 0}\,\bar{r}_1, \mathcal{R}_2 = \mathbb{R}_{\geq 0}\,\bar{r}_2, \ldots, \mathcal{R}_{m-1} = \mathbb{R}_{\geq 0}\,\bar{r}_{m-1}$ of S such that $\{\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_{m-1}\}$ are linearly independent vectors of V and*

$$R = S^\vee \cap \mathcal{R}_1^\perp \cap \mathcal{R}_2^\perp \cap \cdots \cap \mathcal{R}_{m-1}^\perp.$$

Proof. Let $\mathcal{R}$ be an extremal ray of $S^\vee$. By Proposition 1.4.4, there exists a unique $(m-1)$-dimensional face $\mathcal{F}$ of S such that $\mathcal{R} = S^\vee \cap \mathcal{F}^\perp \setminus \{0\}$, $\mathcal{F}$ being a strongly convex polyhedral cone generated by extremal rays of S contained in $\mathcal{F}$. Denote by $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_s$ these rays. Since $\dim(\mathcal{F}) = m-1$, there exist $i_1, i_2, \ldots, i_{m-1} \in \{1, 2, \ldots, s\}$ such that $\{\mathcal{R}_{i_1}, \mathcal{R}_{i_2}, \ldots, \mathcal{R}_{i_{m-1}}\}$ is a basis of $\mathrm{lin}(\mathcal{F})$. Therefore,

$$\mathcal{F}^\perp = \mathrm{lin}(F)^\perp = \mathcal{R}_{i_1}^\perp \cap \mathcal{R}_{i_2}^\perp \cap \cdots \cap \mathcal{R}_{i_{m-1}}^\perp \text{ and } \mathcal{R} = S^\vee \cap \mathcal{R}_{i_1}^\perp \cap \mathcal{R}_{i_1}^\perp \cap \cdots \cap \mathcal{R}_{i_{m-1}}^\perp.$$

Now suppose that $R = \mathbb{R}_{\geq 0}\,\bar{r}$ is a subset of $S^\vee$ ($\bar{r} \in V^*$) such that

$$\mathcal{R} = S^\vee \cap \mathcal{R}_1^\perp \cap \mathcal{R}_2^\perp \cap \cdots \cap \mathcal{R}_{m-1}^\perp \setminus \{0\},$$

where $\mathcal{R}_1 = \mathbb{R}_{\geq 0}\,\bar{r}_1, \mathcal{R}_2 = \mathbb{R}_{\geq 0}\,\bar{r}_2, \ldots, \mathcal{R}_{m-1} = \mathbb{R}_{\geq 0}\,\bar{r}_{m-1}$ are extremal rays of S and $\{\bar{r}_1, \bar{r}_2, ..., \bar{r}_{m-1}\}$ linearly independent vectors of V. Set A the vector subspace of V generated by $\{\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_{m-1}\}$ and $\mathcal{F} := S \cap A$. Given $\bar{z} \in \mathcal{R}$, it holds that $A = \{\bar{z}\}^\perp$ and $\mathcal{F} := S \cap \{\bar{z}\}^\perp$. By definition of $\mathcal{F}$, $\bar{r}_i$ belongs to $\mathcal{F}$, for $1 \leq i \leq m-1$, and thus $\mathcal{F}$ is a $(m-1)$-dimensional face of S. As a result, $\mathcal{R} = S^\vee \cap \mathcal{F}^\perp$, since $\mathcal{F}^\perp = A^\perp = \mathcal{R}_1^\perp \cap \mathcal{R}_2^\perp \cap \cdots \cap \mathcal{R}_{m-1}^\perp$. Moreover, by Proposition 1.4.4 (e), $\mathcal{R}$ is a 1-dimensional face of $S^\vee$ and, consequently, it is an extremal ray of $S^\vee$. □

1.4.2 Cone of curves of a surface

In this subsection we apply the above concepts and their properties to some geometric objects which we desire to study. We keep the notation introduced before.

Let Z be a smooth projective surface over an algebraically closed field k (surface for short). A divisor D on Z is *numerically equivalent to zero*, $D \equiv 0$, if $D \cdot C = 0$ for every curve C on Z. Two divisors D_1 and D_2 are numerically equivalent if $D_1 - D_2 \equiv 0$.

Recall that $\mathrm{Pic}(Z)$ is the Picard group of Z and $\cdot$ the bilinear pairing associated to $\mathrm{Pic}(Z)$ (see Section 1.1 for more information). Inside $\mathrm{Pic}(Z)$ we can find the subgoup $\mathrm{Pic}^\tau(Z)$ given by the classes of the divisors numerically equivalent to zero and define $\mathrm{Num}(Z)$ as the set of classes of $\mathrm{Pic}(Z)$ modulo numerical equivalence, that is, $\mathrm{Num}(Z) = \mathrm{Pic}(Z)/\mathrm{Pic}^\tau(Z)$. As Z is a smooth projective surface, the dual $\mathbb{Z}$-module $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Num}(Z), \mathbb{Z})$ of $\mathrm{Num}(Z)$ is isomorphic to $\mathrm{Num}(Z)$. Therefore, we have a bilinear pairing, $\mathrm{Num}(Z) \times \mathrm{Num}(Z) \to \mathbb{Z}$, associated to $\mathrm{Num}(Z)$, induced by the bilinear pairing associated to $\mathrm{Pic}(Z)$ (see [71, Chapter V, Remark 1.9.1]). As a consequence of the Nerón-Severi theorem [83], it holds that $\mathrm{Num}(Z)$ is a free $\mathbb{Z}$-module of finite rank. The rank of $\mathrm{Num}(Z)$ is named the *Picard number* of Z.

An element D of the $\mathbb{Q}$-vector space $\mathrm{Div}_{\mathbb{Q}}(Z) := \mathrm{Div}(Z) \otimes \mathbb{Q}$ ($\mathbb{R}$-vector space $\mathrm{Div}_{\mathbb{R}}(Z) := \mathrm{Div}(Z) \otimes \mathbb{R}$) is called $\mathbb{Q}$-*divisor* (respectively, $\mathbb{R}$-*divisor*). That is, D can be expressed as

$$D = n_1 G_1 + n_2 G_2 + \cdots + n_r G_r,$$

where G_i is a divisor on Z and $n_i \in \mathbb{Q}$ (respectively, $n_i \in \mathbb{R}$) for $1 \leq i \leq r$. A $\mathbb{Q}$-divisor (respectively, $\mathbb{R}$-divisor) is said to be *effective* when G_i is effective and $n_i \geq 0$ for all i and some of them is positive.

A $\mathbb{Q}$-divisor (respectively, $\mathbb{R}$-divisor) D is *numerically equivalent to zero*, $D \equiv 0$, if $D \cdot C = 0$ for every curve C on Z. Two $\mathbb{Q}$-divisors (respectively, $\mathbb{R}$-divisors) D_1 and D_2 are numerically equivalent if $D_1 - D_2 \equiv 0$. The resulting vector space of numerical equivalence classes is denoted by $\mathrm{Num}_{\mathbb{Q}}(Z)$ (respectively, $\mathrm{Num}_{\mathbb{R}}(Z)$) and one has the isomorphism $\mathrm{Num}_{\mathbb{Q}}(Z) = \mathrm{Num}(Z) \otimes_{\mathbb{Z}} \mathbb{Q}$ [85, Remark 1.3.4] (respectively, $\mathrm{Num}_{\mathbb{R}}(Z) = \mathrm{Num}(Z) \otimes_{\mathbb{Z}} \mathbb{R}$ [85, Example 1.3.10]). The vector space $\mathrm{Num}(Z)_{\mathbb{R}}$ has finite dimension by Nerón-Severi theorem. Moreover, we can extend the bilinear pairing $\mathrm{Num}(Z) \times \mathrm{Num}(Z) \to \mathbb{Z}$ to a bilinear pairing $\mathrm{Num}_{\mathbb{R}}(Z) \times \mathrm{Num}_{\mathbb{R}}(Z) \to \mathbb{R}$ that, abusing of the notation, will also be denoted by $\cdot$. That is, given $[D_1], [D_2] \in \mathrm{Num}_{\mathbb{R}}(Z)$, $[D_1] \cdot [D_2] \in \mathbb{R}$. As a result, we have similar conditions to those introduced at the beginning of this section and then we can consider the concepts and results of convex cones within $\mathrm{Num}_{\mathbb{R}}(Z)$.

By definition, the *cone of curves* of Z, denoted by $\mathrm{NE}(Z)$, is the following convex cone in $\mathrm{Num}_{\mathbb{R}}(Z)$:

$$\mathrm{NE}(Z) = \left\{ [D] \in \mathrm{Num}_{\mathbb{R}}(Z) \mid [D] = \sum_i a_i [C_i], \begin{array}{l} \text{where } C_i \text{ is an integral curve} \\ \text{on } Z \text{ and } a_i \in \mathbb{R}_{\geq 0}. \end{array} \right\}$$

Its topological closure for the usual topology is denoted by $\overline{\mathrm{NE}}(Z)$ and, as Z is a surface, the $\mathbb{R}$-divisors whose class belongs $\overline{\mathrm{NE}}(Z)$ are called *pseudoeffective* [85, Definition 2.2.25 and Remark 2.2.27].

Proposition 1.4.6. [77, Lemma 1.22] *Let C be an irreducible curve on a surface Z. If $C^2 \leq 0$, then $[C]$ is in the boundary of $\overline{\mathrm{NE}}(Z)$. If $C^2 < 0$, then $[C]$ generates an extremal ray in $\mathrm{NE}(Z)$ and $\overline{\mathrm{NE}}(Z)$.*

A divisor (respectively, $\mathbb{Q}$-divisor, $\mathbb{R}$-divisor) on Z is said to be *nef* if

$$D \cdot C \geq 0, \text{ for every irreducible curve } C \text{ on } Z.$$

The convex cone containing the numerical equivalence classes of these $\mathbb{R}$-divisors is denoted by $\mathrm{Nef}(Z)$. The dual cone of the nef cone $\mathrm{Nef}(Z)$ is the convex cone $\overline{\mathrm{NE}}(Z)$ (see [85, Proposition 1.4.28]) and then a divisor D is pseudoeffective if and only if $D \cdot D' \geq 0$ for every nef divisor D'. Moreover, it holds that $\mathrm{Nef}(Z) \subset \overline{\mathrm{NE}}(Z)$ [85, Example 1.4.33 (i)], since Z is a smooth projective surface.

We will also use *ample* and *big divisors*. A divisor D which satisfies Nakai-Moishezon criterion [71, Chapter V, Theorem 1.10], that is, $D^2 > 0$ and $D \cdot C > 0$, for every irreducible curve C on Z, is called to be *ample*. A divisor D is named *big* if $\kappa(D) = \dim Z$. In [85, Corollary 2.2.7] one can find the following characterization of bigness for a divisor: A divisor D is big if and only if there exist an ample divisor H, a positive integer n and an effective divisor G such that $nD \equiv H + G$.

One can find in [85, Theorem 2.2.16] the following characterization of bigness for a nef divisor.

Theorem 1.4.7. *Let D be a nef divisor on a surface Z. Then D is big if and only if its self-intersection is strictly positive, i.e. $D^2 > 0$.*

A $\mathbb{Q}$-divisor (respectively, $\mathbb{R}$-divisor) D is said to be *ample* if it can be written as

$$D = a_1H_1 + a_2H_2 + \cdots + a_rH_r,$$

where H_i is an ample divisor and $a_i \in \mathbb{Q}_{>0}$ (respectively, $a_i \in \mathbb{R}_{>0}$), for $1 \leq i \leq r$. A $\mathbb{Q}$-divisor D is *big* if there is a positive integer $m > 0$ such that mD is integral and big. A $\mathbb{R}$-divisor D is *big* if it can be written as

$$D = a_1B_1 + a_2B_2 + \cdots + a_rB_r,$$

where, for all i, B_i is a big integral divisor and $a_i \in \mathbb{R}_{>0}$. The convex cone generated by the numerical equivalence classes of ample (respectively, big) $\mathbb{R}$-divisors is denoted by $\mathrm{Amp}(Z)$ (respectively, $\mathrm{Big}(Z)$).

The following result gives us a relation among the convex cones above introduced. A proof can be found in [76] and [85, Theorem 1.4.23 and Theorem 2.2.26].

Theorem 1.4.8. *Keeping the notation used before, it holds that:*

(a) The nef cone $\mathrm{Nef}(Z)$ *is the closure of the ample cone* $\mathrm{Amp}(Z)$ *and* $\mathrm{Amp}(Z)$ *is the interior of* $\mathrm{Nef}(Z)$*.*

(b) The big cone $\mathrm{Big}(Z)$ *is the interior of* $\overline{\mathrm{NE}}(Z)$ *and* $\overline{\mathrm{NE}}(Z)$ *is the closure of* $\mathrm{Big}(Z)$*.*

Now we are going to introduce a last convex cone named *characterisitic cone* (see [76] for further information). The *characteristic cone* of a surface Z is the convex cone of $\mathrm{Num}_{\mathbb{R}}(Z)$ generated by the numerical equivalence classes of the *semiample* divisors on Z, that is, those divisors D such that the complete linear system $|mD|$ is base-point-free for some positive integer m. This convex cone is denoted $\tilde{P}(Z)$ and satisfies the following property.

Proposition 1.4.9. *[76] With the previous notation, it holds that* $\tilde{P}(Z) \subseteq \mathrm{Nef}(Z)$ *and their interiors coincide.*

To conclude this section, we present several results which we will use in the following chapters.

Theorem 1.4.10 (Kleiman's Ampleness Criterion)**.** *Let Z be a surface. A divisor D on Z is ample if and only if* $[D] \cdot [D'] > 0$ *for all* $[D'] \in \overline{\mathrm{NE}}(Z) \setminus \{0\}$*.*

The above result was proved by Kleiman [76] and shows that $\overline{\mathrm{NE}}(Z)$ is a strongly convex cone.

Theorem 1.4.11. [71, Chapter V, Theorem 1.9 (Hodge index theorem)] *Let H be an ample divisor on a surface Z and D a divisor on Z such that $D \not\equiv 0$ and $D \cdot H = 0$. Then, $D^2 < 0$.*

A well-known consequence of Sylvester theorem (see [84, Chapter XV, Section 4, Theorem 4.1]) is that the bilinear form defined on the vector space $\mathrm{Num}_{\mathbb{R}}(Z)$ can be diagonalized with only ± 1's on the diagonal. In addition, by Hodge index theorem (see [71, Chapter V, Remark 1.9.1]), the diagonalized intersection pairing only has one $+1$ corresponding to a multiple of an ample divisor H, and the remaining values on the diagonal are -1. This allows us to state and prove the following result.

Proposition 1.4.12. *Let D be a big and nef divisor on a surface Z. Let $\Omega = \{C_1, C_2, \ldots, C_m\}$ be a set of integral curves on Z such that $D \cdot C_i = 0$ for all $i = 1, 2, \ldots, m$. Then the intersection matrix $(C_i \cdot C_j)$ is negative definite.*

Proof. Let M be the $m \times m$ matrix whose entries are $C_i \cdot C_j$. First notice that for every $\overline{y} = (y_1, y_2, \ldots, y_m) \in \mathbb{R}^m \setminus \{0\}$,

$$\overline{y}\, M\, \overline{y}^t = G(\overline{y})^2, \tag{1.6}$$

where $G(\overline{y})$ is the $\mathbb{R}$-divisor $\sum_{i=1}^m y_i C_i$. Hence, it is enough to see that the self-intersection of any non-zero class in the hyperplane

$$[D]^{\perp} := \{\overline{x} \in \mathrm{Num}_{\mathbb{R}}(Z) \mid [D] \cdot \overline{x} = 0\}$$

of the space $\mathrm{Num}_{\mathbb{R}}(Z)$ is strictly negative. Next we are going to prove it.

Set an ample divisor H on Z. By Hodge index theorem (Theorem 1.4.11) there exists a basis $B = \{\mathbf{h}_0, \mathbf{h}_1, \ldots, \mathbf{h}_n\}$ of $\mathrm{Num}_{\mathbb{R}}(Z)$ such that $\mathbf{h}_0$ is a (real) multiple of $[H]$, $\mathbf{h}_0^2 = 1$, $\mathbf{h}_0 \cdot \mathbf{h}_i = 0$ and $\mathbf{h}_i \cdot \mathbf{h}_j = -\delta_{ij}$ for all $i, j \in \{1, 2, \ldots, n\}$, where δ_{ij} denotes the Kronecker delta. Since the intersection matrix whose entries are $(\mathbf{h}_i \cdot \mathbf{h}_j)_{1 \leq i,j \leq n}$ is negative definite, we can restrict our proof to those points of the hyperplane $[D]^{\perp}$ of $\mathrm{Num}_{\mathbb{R}}(Z)$ whose first coordinate with respect to B is 1. Let $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$ such that $(1, \alpha_1, \ldots, \alpha_n)$ are the coordinates of the vector $\frac{1}{D \cdot H}[D]$ with respect to the basis B. Since D is nef and big, $D^2 > 0$ by Theorem 1.4.7 and therefore

$$\sum_{i=1}^{n} \alpha_i^2 < 1. \tag{1.7}$$

Now, an arbitrary element of $\mathrm{Num}_{\mathbb{R}}(Z)$ whose coordinates with respect to B are

$$(1, x_1, \ldots, x_n)$$

belongs to $[D]^{\perp}$ if and only if

$$\sum_{i=1}^{n} \alpha_i x_i = 1, \tag{1.8}$$

and has negative self-intersection if and only if $\sum_{i=1}^n x_i^2 > 1$.

Then our statement on the classes in $[D]^{\perp}$ follows because the map

$$f : \mathbb{R}^n \to \mathbb{R}, \quad f(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} x_i^2,$$

subject to the restriction (1.8), has an absolute minimum at the point

$$\overline{p} = \frac{1}{\sum_{i=1}^n \alpha_i^2}(\alpha_1, \alpha_2, \dots, \alpha_n) \text{ and } f(\overline{p}) = \frac{1}{\sum_{i=1}^n \alpha_i^2} > 1.$$

Notice that $\overline{p}$ is the closest to the origin point of the hyperplane defined by the equation (1.8) and the last inequality follows from (1.7). This concludes the proof. □

Proposition 1.4.13. [77, Corollary 1.21] *Let H be an ample divisor on a surface Z. The set $Q(Z) := \{[D] \in \mathrm{Num}_{\mathbb{R}}(Z) \mid [D]^2 > 0\}$ has two connected components*

$$Q^+(Z) := \{[D] \in Q(Z) \mid [D] \cdot [H] > 0\} \text{ and } Q^-(Z) := \{[D] \in Q(Z) \mid [D] \cdot [H] < 0\}.$$

In particular, $Q^+(Z) \subset \overline{\mathrm{NE}}(Z)$.

Proposition 1.4.14. *Let Z be a surface. Set H an ample divisor on Z and write*

$$A(Z) := \{[D] \in \mathrm{Num}_{\mathbb{R}}(Z) \mid [D]^2 \geq 0 \text{ and } [D] \cdot [H] \geq 0\}.$$

Then, it holds that $A(Z) = A(Z)^\vee$, where $A(Z)^\vee$ is the dual cone of $A(Z)$.

Proof. As the proof of Proposition 1.4.12, assume that $B = \{\mathbf{h}_0, \mathbf{h}_1, \dots, \mathbf{h}_n\}$ is a basis of $\mathrm{Num}_{\mathbb{R}}(Z)$ such that $\mathbf{h}_0^2 = 1$, $\mathbf{h}_0 \cdot \mathbf{h}_i = 0$ and $\mathbf{h}_i \cdot \mathbf{h}_j = -\delta_{ij}$, for $1 \leq i, j \leq n$, where δ_{ij} is the Kronecker delta. Denote by $\langle \cdot, \cdot \rangle$ and $||\cdot||$ the standard inner product in $\mathbb{R}^n$ and the Euclidean norm which it defines.

We start by showing that $A(Z) \subseteq A(Z)^\vee$. Fix $[D] \in A(Z) \setminus \{0\}$ whose coordinates in the basis B are $(d_0, d_1, \dots, d_n)$. Since $[D] \in A(Z) \setminus \{0\}$, it holds that $d_0 > 0$. Otherwise, $d_0 = [D] \cdot \mathbf{h}_0 = 0$, $-\sum_{i=1}^n d_i^2 = [D]^2 \geq 0$ and then d_i vanishes for all i. Set $\overline{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ such that $x_i = d_i/d_0$, for $1 \leq i \leq n$. By definition, one has that $||\overline{x}|| \leq 1$. Assume, reasoning by contradiction, that $[D] \notin A(Z)^\vee$. Consequently, one can find $[D'] \in A(Z)$ with coordinates $(d'_0, d'_1, \dots, d'_n)$ in the basis B such that $[D] \cdot [D'] < 0$. Write $\overline{y} = (y_1, y_2 \dots, y_n) \in \mathbb{R}^n$ such that $y_i = d'_i/d'_0$, for $1 \leq i \leq n$. One obtains that

$$1 < \sum_{i=1}^n x_i y_i = \langle \overline{x}, \overline{y} \rangle = |\langle \overline{x}, \overline{y} \rangle| \leq ||\overline{x}|| \, ||\overline{y}|| \leq 1 \cdot 1 = 1,$$

where $|\cdot|$ is the absolute value and the second inequality is satisfied by Cauchy-Schwarz inequality. That is a contradiction and then $[D] \in A(Z)^\vee$.

Now we are going to show that $A(Z)^\vee \subseteq A(Z)$, which will conclude the proof. Set $[D] \in A(Z)^\vee \setminus \{0\}$ with coordinates $(d_0, d_1, \dots, d_n)$ in the basis B. Notice that $\mathbf{h}_0, \mathbf{h}_0 + \mathbf{h}_i$ and $\mathbf{h}_0 - \mathbf{h}_i$ belong to $A(Z)$ for all i. Hence, $d_0 > 0$ because, otherwise, $d_0 = [D] \cdot \mathbf{h}_0 = 0, d_i = [D] \cdot (\mathbf{h}_0 + \mathbf{h}_i) \geq 0, -d_i = [D] \cdot (\mathbf{h}_0 - \mathbf{h}_i) \geq 0$ and then d_i vanishes for all i. Reasoning by contradiction, suppose that $[D] \notin A(Z)$. Therefore, $[D]^2 < 0$. Set an element $\overline{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, where $x_i = d_i/d_0$ for $i \in$

$\{1, 2, \ldots, n\}$. Consequently, $||\overline{x}|| > 1$. Take a divisor D' whose class has coordinates $(1, x_1/||\overline{x}||, \ldots, x_n/||\overline{x}||)$ (with respect to B) and a vector $\overline{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ such that $y_i = x_i/||\overline{x}||$, for $1 \leq i \leq n$. It holds that $||\overline{y}|| = 1$, $[D']^2 = 0$ and so the class of D' belongs to $A(Z)$. Then

$$[D] \cdot [D'] \geq 0 \text{ and } \langle \overline{x}, \overline{y} \rangle \leq 1.$$

On the other hand, taking into account that $\overline{y} = \overline{x}/||\overline{x}||$,

$$||\overline{x}|| = \frac{1}{||\overline{x}||}\langle \overline{x}, \overline{x} \rangle = \langle \overline{x}, \overline{y} \rangle \leq 1$$

holds, which is a contradiction. □

1.5 Seshadri-type constants and Newton-Okounkov bodies

In this section we briefly introduce Seshadri-type constants and show their relation to Newton-Okounkov bodies corresponding to certain flags on surfaces. These bodies will be defined later and are convex sets of $\mathbb{R}^2$. Our development is supported on [85, Chapter 5], [8],[86], [82] [15], [40] and [38]. Here, we use the notation established before, although in this section $k = \mathbb{C}$.

1.5.1 Seshadri-type constants of divisorial valuations

Let Z_0 be a (complex smooth irreducible projective) surface and D a nef divisor on Z_0. Let $\pi : Z \to Z_0$ be the blowup at a point $p \in Z_0$ with exceptional divisor E. The *Seshadri constant of D at p*, denoted by $\varepsilon(Z_0, D; p)$ (or $\varepsilon(D; p)$ for short), is the non-negative real number

$$\varepsilon(D; p) = \sup\{t \in \mathbb{R}_{\geq 0} \mid D^* - tE \text{ is nef on } Z\},$$

where D^* denotes the pull back $\pi^* D$.

This constant depends only on the numerical equivalence class of D and satisfies the homogeneity property: $\varepsilon(nD; p) = n\varepsilon(D; p)$, for every positive integer n and point $p \in Z_0$ [85, Examples 5.1.3 and 5.1.4].

Notice that the ray $[D^*] - t[E]$ meets the boundary of the nef cone of Z for the value $t = \varepsilon(D; p)$. Then, the Seshadri constant $\varepsilon(D; p)$ provides information about the positivity at p of the divisor D [85, Remark 5.1.2].

An equivalent definition of the Seshadri constant is given in the following result. A proof can be found in [85, Proposition 5.1.5].

Proposition 1.5.1. *Keep the notation introduced before. Then*

$$\varepsilon(D; p) = \inf_{p \in C \subsetneq Z_0} \left\{ \frac{D \cdot C}{\mathrm{mult}_p(\varphi_C)} \right\},$$

where the infimum is taken over all integral curves $C \subsetneq Z_0$ going through p.

The explicit computation of the Seshadri constants is very hard. However, there exist some upper and lower bounds given by several authors. Let us show some of them.

According [8], if D is an ample divisor on a smooth projective surface Z_0, then

$$\varepsilon(D,p) \leq \sqrt{D^2}.$$

Even more, if the Seshadri constant $\varepsilon(D,p)$ is irrational, then the equality holds and, when the strict inequality is satisfied, $\varepsilon(D,p)$ is rational. In addition, it holds that $\varepsilon(D,p) \geq 1$, for all except countably many points $p \in Z_0$ [85, Proposition 5.2.3].

The Seshadri constants have not only been used to study deeply the positivity of divisors on surfaces, but also on varieties and, particularly, on projective spaces, see [85, Chapter 5] and [8] for more information.

Some invariants which contain similar information to Seshadri constants have been also introduced. Next we give some information about them.

Let $\mathcal{I} \subseteq \mathcal{O}_{Z_0}$ be an ideal sheaf and set D an ample divisor on Z_0. Let $\pi : Z \to Z_0$ be the blowup of Z_0 along $\mathcal{I}$ whose exceptional divisor is denoted by E. The *s-invariant of $\mathcal{I}$ with respect to D* is the positive real number

$$s_D(\mathcal{I}) = \inf\{t \in \mathbb{R} \mid tD^* - E \text{ is nef on } Z\}.$$

Notice that if we set the closed point $p \in Z_0$ defined by a maximal ideal $\mathfrak{m} \subseteq \mathcal{O}_{Z_0}$, then

$$\varepsilon(D;p) = 1/s_D(\mathfrak{m}).$$

The s-invariant of $\mathcal{I}$ with respect to D is introduced in [31] to study the complexity of a polynomial ideal in terms of the degrees of its generators from a geometrical perspective.

Another important invariant related to the Seshadri constants is given in [40]. Let D be an big divisor on Z_0. The value $\mu(D;p)$ is defined to be

$$\mu(D;p) := \sup\{t \in \mathbb{R} \mid D^* - tE \text{ is big on } Z\}.$$

Note that the ray $[D^*] - t[E]$ meets the boundary of the pseudoeffective cone $\overline{\mathrm{NE}}(Z)$ for the value $t = \mu(D;p)$.

In [40, Remark 2.1] a connection between the invariant $\mu(D;p)$ and the Seshadri constant $\epsilon(D;p)$ is established. Assume that D is an ample divisor. If $\varepsilon(D;p)$ is irrational, $\mu(D;p) = \varepsilon(D;p)$ holds and, when $\mu(D;p)$ is rational, then so is $\varepsilon(D;p)$. In addition, it is proved (see [40, Proposition 2.2]) that

$$\mu(D;p) = \limsup_{m\to\infty} \frac{\max\{\mathrm{ord}_p(f) \mid f \in H^0(Z_0, \mathcal{O}_{Z_0}(mD))\}}{m}.$$

The authors of [15] (see also [38] and [64]) extend the previous value $\mu(D;p)$ from a point p to a real valuation ν of the function field of Z_0 centered at the local ring $\mathcal{O}_{Z_0,p}$ and a big divisor D on Z_0 giving rise to the value

$$\hat{\mu}_D(\nu) := \lim_{m\to\infty} \frac{\max\{\nu(f) \mid f \in H^0(Z_0, \mathcal{O}_{Z_0}(mD))\}}{m}.$$

Furthermore, they prove that

$$\hat{\mu}_D(\nu) = \sup\{t \in \mathbb{R} \mid D^* - tE \text{ is big on } Z\}, \tag{1.9}$$

when ν is the divisorial valuation defined by an exceptional divisor E on Z [15, Theorem 2.24], and provide the following lower bound for a real valuation of the function field of Z_0:

$$\hat{\mu}_D(\nu) \geq \sqrt{\frac{\mathrm{vol}_{Z_0}(D)}{\mathrm{vol}(\nu)}}, \tag{1.10}$$

where, by [43], the value

$$\mathrm{vol}(\nu) := \lim_{\alpha\to\infty} \frac{\mathrm{length}(\mathcal{O}_{Z_0,p}/\mathcal{P}_\alpha)}{\alpha^2/2}, \tag{1.11}$$

$\mathcal{P}_\alpha$ being the ideal $\mathcal{P}_\alpha = \{f \in \mathcal{O}_{Z_0,p} \mid \nu(f) \geq \alpha\} \cup \{0\}$ of the local ring $\mathcal{O}_{Z_0,p}$ where ν is centered, is called the *volume of* ν and

$$\mathrm{vol}_{Z_0}(D) := \limsup_{m\to\infty} \frac{h^0(Z_0, mD)}{m^2/2} \tag{1.12}$$

is named *the volume of* D.

The case when $Z_0 = \mathbb{P}^2$ and $D = L$ a general projective line on $\mathbb{P}^2$ is studied in [38] and [64]. This value $\hat{\mu}_L(\nu)$ can be expressed as

$$\hat{\mu}_L(\nu) = \lim_{m\to\infty} \frac{\max\{\nu(f) \mid f \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(mL))\}}{m}$$

and also can been seen geometrically as $\hat{\mu}_L(\nu) = \sup\{t \in \mathbb{R} \mid L^* - tE \text{ is big on } Z\}$, where Z is the surface given by ν and L^* the pull-back of L on Z.

It is important to mention that $\hat{\mu}_L(\nu)$ is involved in a variation of the Nagata Conjecture in the valuative setting, which implies that conjecture (see [38] and [64] for more information).

1.5.2 Newton-Okounkov bodies of big divisors on a surface

In this subsection we introduce the Newton-Okounkov bodies corresponding to flags on surfaces. A general description of these bodies can be seen in the introduction of this dissertation (see page 4).

Let Z be a smooth complex projective surface and $K(Z)$ its function field. A *flag* on Z is a sequence

$$C_\bullet := \{Z \supset C \supset \{q\}\},$$

where C is a smooth irreducible curve on Z and q a closed point of C named *center* of $C_\bullet$.

A flag $C_\bullet$ as above comes with a discrete valuation $\nu_{C_\bullet}$ of rank 2 of the function field $K(Z)$ (see [28] for further information). Set $g = 0$ the local equation at q of C on Z. This valuation $\nu_{C_\bullet} : K(Z) \to \mathbb{Z}^2_{\text{lex}}$ can be computed as follows:

$$\nu_{C_\bullet}(f) = (v_1(f), v_2(f)), \text{ where } v_1(f) := \text{ord}_C(f) \text{ and } v_2(f) := \text{ord}_q\left(\frac{f}{g^{v_1(f)}}\right),$$

for $f \in K(Z)$.

Let $C_\bullet$ be a flag and D a big divisor on Z. The *Newton-Okounkov body of D with respect to $C_\bullet$* is the subset of $\mathbb{R}^2$

$$\Delta_{C_\bullet}(D) = \Delta_{\nu_{C_\bullet}}(D) := \overline{\bigcup_{m \geq 1} \left\{ \frac{\nu_{C_\bullet}(f)}{m} \mid f \in H^0(Z, mD) \setminus \{0\} \right\}},$$

where the upper line means the closed convex hull in $\mathbb{R}^2$. This definition can be extended to big $\mathbb{Q}$-divisors and big $\mathbb{R}$-divisors (see [86]). In our case, Z is a surface, the Newton-Okounkov body is a polygon (see [82]) and

$$\text{vol}_Z(D) = 2\,\text{vol}_{\mathbb{R}^2}(\Delta_\nu(D)),$$

where $\text{vol}_{\mathbb{R}^2}$ means Euclidean area (see [86]) and $\text{vol}_Z(D) > 0$ since D is big [85, Subsection 2.2.C]. In addition, Newton-Okounkov bodies satisfy the homothetic property, that is, $\Delta_\nu(dD) = d \cdot \Delta_\nu(D)$ for any integer $d > 0$ (see [86, Proposition 4.1]).

Newton-Okounkov bodies for big $\mathbb{R}$-divisors on a surface can be described making use of the Zariski decomposition of these divisors.

Set D a pseudoeffective (see page 35) $\mathbb{R}$-divisor on a surface Z. Then the divisor D can be written as

$$D = P_D + N_D,$$

where N_D is an effective $\mathbb{R}$-divisor such that, if $N_D \neq 0$, their irreducible components generate a negative definite intersection matrix and P_D is a nef $\mathbb{R}$-divisor orthogonal to N_D, i.e. $P_D \cdot N_D = 0$ (see [116] and [52]). The above expression is called the *Zariski decomposition of* D, and P_D (respectively, N_D) is the *positive* (respectively, *negative*) *part of* D.

An important consequence of the Zariski decomposition is that $\text{vol}(D) = P_D^2$ [85, Corollary 2.3.22].

Let D be a big $\mathbb{R}$-divisor on a surface Z and $C \subset Z$ an integral curve. Write

$$\mu(D, C) = \sup\{t > 0 \mid D - tC \text{ is big}\}.$$

Theorem 1.5.2. [86, Theorem 6.4] and [82, Theorem B] *Keep the notation considered before. The Newton-Okounkov body $\Delta_{\nu_{C_\bullet}}(D)$ of D with respect to $\nu_{C_\bullet}$ is the region*

$$\Delta_{\nu_{C_\bullet}}(D) = \{(t, y) \mid 0 \leq t \leq \mu(D, C) \text{ and } \alpha(t) \leq y \leq \beta(t)\},$$

where $\alpha(t) := \mathrm{ord}_q(N_{D_t}|_C)$ *and* $\beta(t) := \alpha(t) + P_{D_t} \cdot D$. *Here,* P_{D_t} *(respectively,* N_{D_t}*) is the positive (respectively, negative) part of the divisor* $D_t = D - tC$.

Let $\pi : Z \to Z_0$ be the blowup of a point $p \in Z_0$ with exceptional divisor E. Consider a big divisor D on Z_0 such that p does not belong to its support. Denote by D^* the total transform (or the pull-back) of D on Z. Then, by Equality (1.9),

$$\mu(D^*, E) = \hat{\mu}_D(\nu),$$

where ν is the divisorial valuation defined by E.

The Zariski decomposition of a big and nef divisor also allows us to provide a description of the big cone $\mathrm{Big}(Z)$ of a smooth projective surface Z (see [7] for more information).

Let D be a $\mathbb{R}$-divisor on Z and $D = P_D + N_D$ its Zariski decomposition. We define the sets

$$\mathrm{Null}(D) = \{C \mid C \text{ is a irreducible curve with } D \cdot C = 0\}$$

and

$$\mathrm{Neg}(D) = \{C \mid C \text{ is a irreducible component of } N_D\}.$$

Notice that $\mathrm{Neg}(D) \subseteq \mathrm{Null}(P_D)$ holds since $P_D \cdot N_D = 0$.

Let P be a big and nef divisor. The set Σ_P defined as

$$\Sigma_P := \{[D] \in \mathrm{Big}(Z) \mid \mathrm{Neg}(D) = \mathrm{Null}(P)\}$$

is called *Zariski chamber*. This set is a convex cone which, in general, is neither open nor closed. By [7, Proposition 1.6], the interior of Σ_P is given by the set

$$\{[D] \in \mathrm{Big}(Z) \mid \mathrm{Neg}(D) = \mathrm{Null}(P) = \mathrm{Null}(P_D)\}.$$

In addition, a big divisor D on Z is in the boundary of some Σ_P if and only if $\mathrm{Neg}(D) \neq \mathrm{Null}(P_D)$ (see [7, Proposition 1.5]).

Theorem 1.5.3. *[7, Theorem 1.11] Let* Z *be a smooth projective surface. Then there is a locally finite decomposition of the big cone of* Z *into rational locally polyhedral subcones such that in each subcone the support of the negative part of the Zariski decomposition of the divisors is constant.*

1.6 Rational surfaces

In this section we review the definition and some properties of Hirzebruch surfaces which will be fundamental in the next chapters. Our main references are [71], [9], [102] and [90]. We also show some other results about rational surfaces that can be found in [71, Chapter V, Section 5] and [9, Chapter II]. We maintain the notation of the previous sections.

Let k be an algebraically closed field of arbitrary characteristic and Z_1 and Z_2 two smooth projective surfaces over k (surfaces for short). Recall that a Zariski open subset of a surface is dense and, given two morphism f and g from Z_1 to Z_2 such that there is a non-empty open subset $U \subset Z_1$ satisfying $f|_U = g|_U$, then $f = g$ [71, Chapter I, Lemma 4.1]. A *rational map* $f : Z_1 \dashrightarrow Z_2$ is a morphism $f : U \to Z_2$ from a non-empty open subset $U \subset Z_1$ to Y. Two rational maps $f : U \to Z_2$ and $g : V \to Z_2$, where $U, V \subset Z_1$, are the same if f and g coincide on a non-empty open subset of $U \cap V$. A rational map $f : Z_1 \dashrightarrow Z_2$ is said to be *dominant* if the image of f contains a non-empty open subset W of Z_2. In this case, if $g : Z_2 \dashrightarrow Z_3$ is a rational map between surfaces defined on a non-empty open subset V of Z_2, the composition $g \circ f : Z_1 \dashrightarrow Z_3$ is defined on the non-empty subset $f^{-1}(V \cap W)$. A rational map f is called to be *birational* when it is dominant and there exists another dominant rational map $g : Z_2 \dashrightarrow Z_1$ such that $g \circ f = \mathrm{Id}_{Z_1}$ and $f \circ g = \mathrm{Id}_{Z_2}$ as birational morphism. In this last case, it is said that Z_1 and Z_2 are *birational*, or *birationally equivalent*. Notice that, by definition, two surfaces Z_1 and Z_2 are birational if and only if they contain isomorphic open subsets.

Let Z be a surface. We will denote by $B(Z)$ the set of isomorphism classes of surfaces which are birationally equivalent to Z. A surface Z is *relatively minimal* if its class in $B(Z)$ is minimal in the following sense: each birational morphism $Z \to Z'$ is an isomorphism, where Z' is a surface which belongs to $B(Z)$.

Proposition 1.6.1. [9, Chapter II, Proposition II.16] *For any surface Z, there is a birational morphism $Z \to Z'$, where Z' is a relatively minimal surface.*

Theorem 1.6.2. [9, Chapter II, Theorem II.11] *Let $f : Z \to Z_0$ be a birational morphism of surfaces. Then, there is a sequence of blowups $\pi_k : Z_k \to Z_{k-1}$, where $1 \leq k \leq n$, and an isomorphism $u : Z \to Z_n$ such that $f = \pi_1 \circ \ldots \circ \pi_n \circ u$.*

These results can also be found in [71, Chapter V, Theorem 5.8 and Theorem 5.5, respectively]. As a consequence, a smooth projective surface can be obtained by a finite sequence of point blowups where the first one is centered at a point of a relatively minimal surface. We are going to show another characterization of these surfaces.

Let C be a curve on a surface Z. C is a (-1)-*curve* (or a *exceptional curve of the first kind*) whenever C is isomorphic to $\mathbb{P}^1$ and $C^2 = -1$. Notice that the exceptional divisor created by a blowup is a (-1)-curve.

The following result, called *Castelnuovo's contractibility criterion*, shows that a (-1)-curve on a surface is the exceptional divisor of a blowup.

Theorem 1.6.3. [71, Chapter V, Theorem 5.7] *Let Z be a surface containing a (-1)-curve C. Then, there exist a morphism $f : Z \to Z'$ between surfaces, a closed* *point* $p \in Z'$ *and an isomorphism* $u : Z \to Bl_p(Z')$ *such that* $u(C)$ *is the exceptional divisor of the blowup* $\pi : Bl_p(Z') \to Z'$ *and* $f = \pi \circ u$.

As a consequence, one has the following result.

Corollary 1.6.4. *A surface* Z *is relatively minimal if and only if* Z *contains no* (-1)*-curve.*

We are interested in a particular class of surfaces called *rational*.

Let $\mathbb{P}^1 := \mathbb{P}^1_k$ be the projective line over field k. A surface Z is called *birationally ruled* if it is birationally equivalent to $C \times \mathbb{P}^1$ for some curve C. If $C = \mathbb{P}^1$, Z is said to be a *rational surface*. A *geometrically ruled surface*, or simply a *ruled surface*, is a surface Z, together with a surjective projective morphism $\pi : Z \to C$ to a (non-singular) curve C, such that the fibre $Z_p := \pi^{-1}(p)$ is isomorphic to $\mathbb{P}^1$ for every point $p \in C$, and such that π admits a section (i.e., a morphism $\sigma : C \to X$ such that $\pi \circ \sigma = \mathrm{Id}_C$). It holds that every ruled surface is birationally ruled [71, Chapter V, Proposition 2.2].

An example of rational surface is the projective plane $\mathbb{P}^2 := \mathbb{P}^2_k$ over k. Other examples are the surfaces we next define.

Definition 1.6.5. Let δ be a non-negative integer. The *δ-th Hirzebruch surface* is the projective space $\mathbb{F}_\delta := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-\delta))$.

The Hirzebruch surfaces are ruled surfaces over $\mathbb{P}^1$ and every ruled surface Z over $\mathbb{P}^1$ is isomorphic to $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-\delta))$ for some non-negative integer δ [9, Chapter III, Proposition III.15]. As a result, the Hirzebruch surfaces are rational surfaces. In addition, we have the following consequence of [9, Chapter V, Theorem V.10] and Corollary 1.6.4.

Theorem 1.6.6. *Z is a relatively minimal rational surface if and only if Z is isomorphic to* $\mathbb{P}^2$ *or to one of the surfaces* $\mathbb{F}_\delta$*, for* $\delta = 0$ *or* $\delta \geq 2$.

The above result allows us to conclude that every rational surface can be seen as a surface created by a finite sequence of point blowups over $\mathbb{P}^2$ or $\mathbb{F}_\delta$, where δ is a non-negative integer different from 1.

To finish this first part of this section, we notice that two divisors on $\mathbb{P}^2$ or $\mathbb{F}_\delta$ are linearly equivalent if and only if they are numerically equivalent. As a result, it holds that

$$\mathrm{Pic}(\mathbb{P}^2) = \mathrm{Num}(\mathbb{P}^2) \text{ and } \mathrm{Pic}(\mathbb{F}_\delta) = \mathrm{Num}(\mathbb{F}_\delta).$$

1.6.1 Hirzebruch surfaces

We devote this subsection to give an extended description of the Hirzebruch surfaces. Recall that k is an algebraically closed field of arbitrary characteristic.

Let $\mathbb{F}_\delta$ be a Hirzebruch surface over k and $pr : \mathbb{F}_\delta \to \mathbb{P}^1$ the projective morphism associated to $\mathbb{F}_\delta$ (since $\mathbb{F}_\delta$ is a ruled surface). Denote by $\mathrm{Pic}(\mathbb{F}_\delta)$ the Picard group of $\mathbb{F}_\delta$ and by $[D]$ the linear equivalence class of a divisor D. It holds that $\mathrm{Pic}(\mathbb{F}_\delta)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ and it is generated by the set $\{[F], [M]\}$, where F is a fiber of pr and M a section of pr whose self-intersection is δ. The matrix of the bilinear pairing of $\mathrm{Pic}(\mathbb{F}_\delta)$ with respect to that basis is

$$\begin{pmatrix} 0 & 1 \\ 1 & \delta \end{pmatrix}.$$

When δ is positive, there exists a unique irreducible curve on $\mathbb{F}_\delta$ with negative self-intersection. That curve is linearly equivalent to $-\delta F + M$ and its self-intersection equals $-\delta$ (see [9, Chapter IV, Proposition IV.1]); it is denoted by M_0 and is usually called the *special section*. Notice that, if $\delta = 0$, M_0 will denote a section such that $M_0^2 = 0$.

The following result will be useful.

Proposition 1.6.7. *Let $\mathbb{F}_\delta$ be a Hirzebruch surface.*

(a) Let C be an irreducible curve $C \neq F, M_0$. Then $C \sim aF + bM$, where $a \geq 0$ and $b > 0$.

(b) The cone of curves $\mathrm{NE}(\mathbb{F}_\delta)$ is generated by the class of a fiber F and that of the section M_0.

(c) A divisor $D \sim aF + bM$ on $\mathbb{F}_\delta$ is nef if and only if $a \geq 0$ and $b \geq 0$.

(d) A divisor $D \sim aF + bM$ on $\mathbb{F}_\delta$ is big if and only if $b > 0$ and $a > -\delta b$.

Proof. Item (a) follows from [71, Chapter V, Proposition 2.20].

Item (b) holds by (a) and by the fact that M_0 is the curve with non-positive self-intersection on $\mathbb{F}_\delta$.

Now we are going to prove (c). By (b), one has that a divisor $D \sim aF + bM$ is nef if and only if $D \cdot F \geq 0$ and $D \cdot M_0 \geq 0$. Then, a divisor D is nef if and only if $b \geq 0$ and $a \geq 0$.

Finally we show (d). A divisor $D \sim aF + bM$ is big if and only if $D \cdot D' > 0$ for every nef divisor D' on $\mathbb{F}_\delta$. Taking into account (c), one obtains that the nef divisors are generated by the class of a fiber F and that of a section M. As a consequence, a divisor D is big if and only if the conditions $b = D \cdot F > 0$ and $a + b\delta = D \cdot M > 0$ hold. □

Remark 1.6.8. Let $\mathbb{F}_\delta$ be a Hirzebruch surface. When δ is a positive integer, we can find big divisors D on $\mathbb{F}_\delta$ which are not nef. In fact, these divisors are linearly equivalent to $aF + bM$, where $b \in \mathbb{Z}_{>0}$ and $-\delta b < a < 0$. In addition, it is easy to check that the Zariski decomposition of D, $D = P_D + N_D$, is $P_D \sim (b + a/\delta)M$ and $N_D \sim (-a/\delta)M_0$.

Let $\mathbb{A}^2 = \mathbb{A}^2_k$ be the affine plane over k. A Hirzebruch surface $\mathbb{F}_\delta$ can be obtained as the quotient of $(\mathbb{A}^2 \setminus \{(0,0)\}) \times (\mathbb{A}^2 \setminus \{(0,0)\})$ by an action of the product of multiplicative groups, $k^* \times k^*$, where $k^* = k \setminus \{0\}$ (see [102, Chapter 2]). Let X_0, X_1 be coordinates in the first factor and Y_0, Y_1 coordinates in the second one. For each $(\lambda, \mu) \in k^* \times k^*$ the action is defined as follows:

$$\begin{array}{rcll} (\lambda,1): & (X_0,X_1;Y_0,Y_1) & \to & (\lambda X_0, \lambda X_1; Y_0, \lambda^{-\delta} Y_1) \\ (1,\mu): & (X_0,X_1;Y_0,Y_1) & \to & (X_0, X_1; \mu Y_0, \mu Y_1) \end{array}. \tag{1.13}$$

Note that the action preserves the ratio $(X_0 : X_1)$ and then the projective morphism $pr : \mathbb{F}_\delta \to \mathbb{P}^1$ is just the projection onto the first factor.

On account of [30] and [90, Section 1.2], the homogeneous coordinate ring of $\mathbb{F}_\delta$ is $S_\delta := k[X_0, X_1, Y_0, Y_1]$, where each variable is graded on $\mathbb{Z} \times \mathbb{Z}_{\geq 0}$ as follows

$$\deg X_0 = (1,0), \ \deg X_1 = (1,0), \ \deg Y_0 = (0,1), \ \deg Y_1 = (-\delta, 1).$$

The set of homogeneous elements of degree $(a,b) \in \mathbb{Z} \times \mathbb{Z}_{\geq 0}$ in S_δ is described as

$$S_\delta(a,b) := \bigoplus_{\alpha_0+\alpha_1=\delta\beta_1+a,\, \beta_0+\beta_1=b} k X_0^{\alpha_0} X_1^{\alpha_1} Y_0^{\beta_0} Y_1^{\beta_1}$$

and so, it is said that an integral curve C on $\mathbb{F}_\delta$ *has degree* (a,b) if it is defined by an irreducible and reduced polynomial $H \in S_\delta(a,b)$. Recall that, by Proposition 1.6.7(a), any irreducible curve C of degree (a,b), different from a fiber F and M_0, is linearly equivalent to $aF + bM$, where $a \geq 0$ and $b > 0$.

Assume that δ is positive. The above development shows that an irreducible curve on $\mathbb{F}_\delta$ of degree $(1,0)$ is linearly equivalent to a fiber F and it is defined by the equation $aX_0 + bX_1 = 0$, where $a, b \in k$ and at least one of them is not equal to zero. Similarly, an irreducible curve of degree $(0,1)$ on $\mathbb{F}_\delta$ is linearly equivalent to M and its equation is $aY_0 + \sum_{i=0}^{\delta} b_i X_0^i X_1^{\delta-i} Y_1 = 0$, where $a \in k^*$ and $b_i \in k$. Finally, one can see that M_0 is defined by the equation $Y_1 = 0$.

Notice that a point $p \in \mathbb{F}_\delta$ determines a fiber F and p cannot belong to the intersection of the special section M_0 and an irreducible curve linearly equivalent to M. Next, we distinguish two types of points in $\mathbb{F}_\delta$ which will be considered along this work. A point $p \in \mathbb{F}_\delta$ is called a *special point* if p belongs to the special section. Otherwise, p is named a *general point*.

A useful result about the geometry of a Hirzebruch surface $\mathbb{F}_\delta$, where $\delta \geq 1$, is the following one stated in [90].

Proposition 1.6.9. *Assume that $\delta \geq 1$. Then, through any general point of $\mathbb{F}_\delta$ a δ-dimensional family of irreducible curves linearly equivalent to M goes. Furthermore, an irreducible curve of degree $(1,0)$ and an irreducible curve of degree $(0,1)$ meet at a general point.*

Proof. Let $(p_{x_0}\colon p_{x_1}; p_{y_0}, p_{y_1})$ be the coordinates of a general point $p \in \mathbb{F}_\delta$; this means $p_{y_1} \neq 0$. In addition p determines a fiber F and so either p_{x_0} or p_{x_1} does not vanish. Consequently, with the previous notation, one obtains that

$$b_\delta = \frac{ap_{y_0} + \sum_{i=0}^{\delta-1} b_i (p_{x_0})^i (p_{x_1})^{\delta-i} p_{y_1}}{(p_{x_0})^\delta p_{y_1}} \text{ or } b_0 = \frac{ap_{y_0} + \sum_{i=1}^{\delta} b_i (p_{x_0})^i (p_{x_1})^{\delta-i} p_{y_1}}{(p_{x_1})^\delta p_{y_1}}$$

which finishes the proof of our first statement taking into account $a \in k^*$.

The second one holds since an irreducible curve of degree $(1,0)$ (respectively, an irreducible curve of degree $(0,1)$) is linearly equivalent to F (respectively, M) and $F \cdot M = 1$. □

When $\delta = 0$, the Hirzebruch surface is $\mathbb{F}_0 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1})$ and this surface is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ by (1.13). This gives rise to two morphisms to $\mathbb{P}^1$, obtaining a double ruling structure. Note that, in this case, there is no special section ($M_0 \sim M$) and the matrix of the bilinear pairing is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The homogeneous coordinate ring of $\mathbb{F}_0$ is $S_0 := k[X_0, X_1, Y_0, Y_1]$ whose variables have the following degrees:

$$\deg X_0 = (1,0) = \deg X_1 \quad \text{and} \quad \deg Y_0 = (0,1) = \deg Y_1.$$

The homogeneous part of degree $(a,b) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ of S_0 equals

$$S_0(a,b) = \bigoplus_{\alpha_0+\alpha_1=a,\, \beta_0+\beta_1=b} kX_0^{\alpha_0} X_1^{\alpha_1} Y_0^{\beta_0} Y_1^{\beta_1}$$

and an irreducible curve C of degree (a,b) on $\mathbb{F}_0$ is linearly equivalent to $aF + bM$, where $b > 0$ and $a \geq 0$. In addition, when studying $\mathbb{F}_0$, there is no need of distinguishing between special and general points.

Proposition 1.6.10. [90, Lemma 1.2.3] *A point p in $\mathbb{F}_0$ determines and it is determined by a unique curve F of degree $(1,0)$ and a unique curve M of degree $(0,1)$.*

Notice that a point $p = (p_{x_0}\colon p_{x_1}; p_{y_0}, p_{y_1})$ in $\mathbb{F}_0$ defines and is defined by an irreducible curve F of degree $(1,0)$ with equation $p_{x_1}X_0 - p_{x_0}X_1 = 0$ and an irreducible curve M of degree $(0,1)$ with equation $p_{y_1}Y_0 - p_{y_0}Y_1 = 0$.

Remark 1.6.11. When considering either a point $p \in \mathbb{F}_0$ or a special point in $\mathbb{F}_\delta$, $\delta \geq 1$, we get the same behaviour as in Proposition 1.6.10: a point in $\mathbb{F}_0$ determines a unique fiber and a unique section M, and a special point belongs to the special section and it determines a unique fiber F.

To finish this section, we show some details about the local structure of a Hirzebruch surface.

Let δ be a non-negative integer and $\mathbb{F}_\delta$ a Hirzebruch surface. $\mathbb{F}_\delta$ is covered by four open sets, all of them isomorphic to $\mathbb{A}^2$. These open sets will be denoted by $U_{i,j} := \mathbb{F}_\delta \setminus \mathbf{V}(X_iY_j)$, for $0 \leq i, j \leq 1$, where $\mathbf{V}(X_iY_j)$ is the closed set in $\mathbb{F}_\delta$ described by the points which satisfy $X_iY_j = 0$.

Indeed, when $i = j = 0$, taking $\lambda = 1/X_0$, it holds that

$$(X_0 : X_1; Y_0, Y_1) \equiv (1 : X_1/X_0; Y_0, X_0^\delta Y_1).$$

Now, if we pick $\mu = 1/Y_0$, one obtains that

$$(1 : X_1/X_0; Y_0, X_0^\delta Y_1) \equiv (1 : X_1/X_0; 1, (X_0^\delta Y_1)/Y_0).$$

As X_0 and Y_0 are not equal to zero, and X_1 and Y_1 can take any value in k, we get affine coordinates $(1 : X_1/X_0; 1, (X_0^\delta Y_1)/Y_0) \cong (u, v)$, where $(u, v) \in \mathbb{A}^2$. Analogous descriptions can be done for the remaining open sets.

The following result provides some changes of coordinates for the previous open sets.

Proposition 1.6.12. *Let $U_{i,j}$ be the open sets of the surface $\mathbb{F}_\delta$ above defined. Consider, as described, affine coordinates (u, v) for $U_{i,j}$ and (u', v') for $U_{k,l}$, where $i, j, k, l \in \{0, 1\}$. Then*

(a)

$$\begin{cases} u' = \dfrac{1}{u} \\ v' = \dfrac{1}{u^\delta v} \end{cases}, \text{ if } i = j = 0 \text{ and } k = l = 1. \text{ And}$$

(b)

$$\begin{cases} u' = \dfrac{1}{u} \\ v' = \dfrac{v}{u^\delta} \end{cases}, \text{ if } i = 0 \text{ and } j = k = l = 1.$$

Proof. We prove (a). A proof for (b) runs similarly. Consider a point $p \in U_{0,0} \cap U_{1,1}$. Then the coordinates of p in $U_{0,0}$ are $(p_{x_0} : p_{x_1}; p_{y_0}, p_{y_1})$, where $p_{x_0} \neq 0$ and $p_{y_0} \neq 0$, and

$$(p_{x_0} : p_{x_1}; p_{y_0}, p_{y_1}) \equiv \left(1 : \frac{p_{x_1}}{p_{x_0}}; 1, \frac{(p_{x_0})^\delta p_{y_1}}{p_{y_0}}\right).$$

Moreover, $p \in U_{1,1}$, therefore

$$(p_{x_0} : p_{x_1}; p_{y_0}, p_{y_1}) \equiv \left(\frac{p_{x_0}}{p_{x_1}} : 1; \frac{p_{y_0}}{(p_{x_1})^\delta p_{y_1}}, 1\right).$$

Denote $\frac{p_{x_1}}{p_{x_0}}$ and $\frac{(p_{x_0})^\delta p_{y_1}}{p_{y_0}}$ by u and v, respectively; and $\frac{p_{x_0}}{p_{x_1}}$ and $\frac{p_{y_0}}{(p_{x_1})^\delta p_{y_1}}$ by u' and v', respectively. As a consequence, it holds that

$$u' = \frac{p_{x_0}}{p_{x_1}} = \frac{1}{u}, \text{ and}$$
$$v' = \frac{p_{y_0}}{(p_{x_1})^\delta p_{y_1}} = \frac{p_{y_0}}{(p_{x_0})^\delta p_{y_1}} \frac{(p_{x_0})^\delta}{(p_{x_1})^\delta} = \frac{1}{u^\delta v},$$

which finishes the proof. □

Chapter 2

Non-positive and negative at infinity valuations of a rational surface

In this chapter we introduce the concepts of *non-positivity and negativity at infinity* of divisorial and irrational valuations of the projective plane $\mathbb{P}^2$ and of a Hirzebruch surface $\mathbb{F}_\delta, \delta \geq 0$. We will show that rational surfaces defined by divisorial valuations of the above types enjoy good geometrical properties. Asymptotic results could be deduced from real valuations. For our study, we divide the above considered valuations into three types. Since these valuations are essentially characterized by their dual graphs (as we will explain), we study their dual graphs and provide an algorithm to generate them. Our main results and proofs concerning these objects can be found in [63] and [61].

To start we introduce the notation that we use in this chapter.

Let k be an arbitrary algebraically closed field. Assume that the surface Z_0 is either the projective plane $\mathbb{P}^2$ or a Hirzebruch surface $\mathbb{F}_\delta, \delta \geq 0$, over k. Consider a finite or infinite simple sequence of blowups

$$\pi : \cdots \to Z_n \xrightarrow{\pi_n} Z_{n-1} \to \cdots \to Z_1 \xrightarrow{\pi_1} Z_0, \tag{2.1}$$

where each blowup $\pi_i : Z_i \to Z_{i-1}$ is centered at a closed point $p_i \in Z_{i-1}$ such that $p_1 = p \in Z_0$ and, otherwise, p_i belongs to the exceptional divisor created by π_{i-1}. Abusing of notation, for each surface Z_n (with $n \geq 1$), we denote by E_i (respectively, E_i^*), $i \leq n$, the strict (respectively, total) transform on Z_n of the exceptional divisor E_i created after blowing-up p_i and by $\tilde{D}$ (respectively, D^*) the strict (respectively, total) transform on Z_n of a divisor D on Z_i. Write φ_C (respectively, φ_i) the germ of a curve C at p (respectively, an analytically irreducible germ at p whose strict transform on Z_i is transversal to E_i at a non-singular point of the exceptional locus).

In this chapter we only consider divisorial and irrational valuations. Set ν the divisorial or irrational valuation defined by a simple sequence of blowups as (2.1). Frequently and for short, ν is usually called a *divisorial or irrational valuation of* Z_0. Recall that if π is finite, then ν is a divisorial valuation and this valuation is defined by the last exceptional divisor E_n. Often, when the situation requires it, the valuation ν is denoted ν_n. $\mathcal{C}_\nu = \{p_i\}_{i\geq 1}$ stands for the configuration of infinitely near points of ν, the dual graph of ν is denoted by Γ_ν and its corresponding subgraphs by Γ_ν^i, for $i \in \{1, 2, \ldots, g+1\}$. We also denote by $\{\overline{\beta}_i(\nu)\}_{i=0}^{g+1}$ (respectively, $\{\beta_i'(\nu)\}_{i=0}^{g+1}$) the sequence of maximal contact values (respectively, the sequence of Puiseux exponents) associated to ν.

Let ν_n be the divisorial valuation of the function field of Z_0 centered at $\mathcal{O}_{Z_0,p}$ defined by the exceptional divisor E_n. Set $Z := Z_n$ the surface generated by ν_n. According the previous chapter, it holds that

$$\mathrm{Pic}(Z) \cong \mathrm{Pic}(Z_0) \oplus \mathbb{Z}E_1^* \oplus \mathbb{Z}E_2^* \oplus \cdots \oplus \mathbb{Z}E_n^*.$$

The corresponding bilinear paring is denoted by $\cdot$. Write $\mathrm{Pic}_{\mathbb{R}}(Z)$ the tensor product $\mathrm{Pic}(Z) \otimes \mathbb{R}$ and, by abuse of notation, $\cdot$ its bilinear pairing. Notice that $\mathrm{Pic}_{\mathbb{R}}(Z) = \mathrm{Num}_{\mathbb{R}}(Z)$.

2.1 Non-positive at infinity valuations of the projective plane

In this section we assume that $Z_0 = \mathbb{P}^2$. Let ν be a divisorial or irrational valuation of the function field of $\mathbb{P}^2$ centered at $\mathcal{O}_{\mathbb{P}^2,p}$ and π the simple sequence of point blowups that it defines. We always suppose that the number of blowups is at least 2. Set L the projective line (called the *line at infinity*) containing the point p and whose strict transform passes through p_2.

Definition 2.1.1. Under the above assumptions, the valuation ν is called to be *non-positive at infinity* (NPI) when $\nu(f) \leq 0$, for all $f \in \mathcal{O}_{\mathbb{P}^2}(\mathbb{P}^2 \setminus L)$. If $\nu(f) < 0$ holds, for all $f \in \mathcal{O}_{\mathbb{P}^2}(\mathbb{P}^2 \setminus L) \setminus k$, then ν is said to be *negative at infinity* (NI).

From a geometric point of view, NPI and NI divisorial valuations of $\mathbb{P}^2$ are interesting since they give us information about the cone of curves of the surfaces that they define, as we will see in the following results. These theorems can be found in [60] and our future Theorems 2.3.7 and 2.3.14 will extend them to the case $Z_0 = \mathbb{F}_\delta$.

Theorem 2.1.2. *Let ν_n be a divisorial valuation of $\mathbb{P}^2$ and L the line at infinity. Set $\overline{\beta}_{g+1}(\nu_n)$ (respectively, $\{\nu_n(\mathfrak{m}_i)\}_{i=1}^n$) the last maximal contact value (respectively, the sequence of values) of ν_n. Then, the following conditions are equivalent:*

(a) The valuation ν_n is non-positive at infinity.

(b) $\nu_n(\varphi_L)^2 \geq \overline{\beta}_{g+1}(\nu_n)$.

(c) The divisor $\nu_n(\varphi_L)E_0^* - \sum_{i=1}^n \nu_n(\mathfrak{m}_i)E_i^*$, where E_0^* is the total transform of a projective line E_0 which does not pass through p, is nef.

(d) The cone of curves $\mathrm{NE}(Z)$ is generated by the classes of $\tilde{L}, E_1, E_2, \ldots, E_n$ (i.e., it is regular).

Theorem 2.1.3. *Keeping the assumptions and notation of Theorem 2.1.2, the following conditions are equivalent:*

(a) The valuation ν_n is negative at infinity.

(b) Either $\nu_n(\varphi_L)^2 > \overline{\beta}_{g+1}(\nu_n)$, or $\nu_n(\varphi_L)^2 = \overline{\beta}_{g+1}(\nu_n)$ and the Iitaka dimension of $\nu_n(\varphi_L)E_0^* + \sum_{i=1}^n \nu_n(\mathfrak{m}_i)E_i^*$ vanishes.

(c) The intersection product of $\nu_n(\varphi_L)E_0^* + \sum_{i=1}^n \nu_n(\mathfrak{m}_i)E_i^*$ and the strict transform of any integral curve C on $\mathbb{P}^2$ different from the projective line L is positive.

2.2 Valuations of a Hirzebruch surface

This section considers divisorial and irrational valuations of a Hirzebruch surface and divides them into two types named special and non-special.

In this section we assume that the surface Z_0 of the sequence π (2.1) is a Hirzebruch surface $\mathbb{F}_\delta, \delta \geq 0$, and p a closed point in $\mathbb{F}_\delta$. Recall that a basic introduction about Hirzebruch surfaces was provided in Subsection 1.6.1. Let ν_n be a divisorial valuation of the function field of $\mathbb{F}_\delta$ centered at $\mathcal{O}_{\mathbb{F}_\delta,p}$.

Definition 2.2.1. A divisorial valuation ν_n as above is called to be *special* (with respect to $\mathbb{F}_\delta$ and p) when one of the following conditions holds:

1. $\delta = 0$.

2. $\delta > 0$ and p is a special point.

3. $\delta > 0$, p is a general point and there is no integral curve in the complete linear system $|M|$ whose strict transform on Z has negative self-intersection.

Otherwise, ν_n will be named *non-special*.

We start with a property of non-special divisorial valuations.

Proposition 2.2.2. *Let ν_n be a non-special divisorial valuation and $\mathcal{C}_{\nu_n} = \{p_i\}_{i=1}^n$ its configuration of infinitely near points. Suppose also that there exists an integral curve M', linearly equivalent to M, going through $p = p_1$ and whose strict transform passes through p_2. Then, there exists a unique integral curve $M_1 \in |M|$ such that M_1 goes through p and their strict transforms pass through the points $p_2, p_3, \ldots, p_{\delta+1}$.*

Proof. We start by showing the existence of M_1. Without loss of generality suppose that the general point p has coordinates $(0,1;0,1) \in U_{11}$. Pick affine coordinates $\{u,v\} = \left\{\frac{X_0}{X_1}, \frac{Y_0}{X_1^\delta Y_1}\right\}$ for which $p=(0,0)$. By Subsection 1.6.1, an integral curve of degree $(0,1)$ passing through p is defined by the equation $aY_0+\sum_{i=1}^{\delta} b_i X_0^i X_1^{\delta-i} Y_1 = 0$, where $a \in k^*$ and $b_i \in k$, $1 \leq i \leq \delta$. One can assume $a=1$ with local equation (in U_{11}) $v + \sum_{i=1}^{\delta} b_i u^i = 0$.

Blowing up the point p and looking at the strict transform of curves as M, we notice the existence of a $(\delta-1)$-dimensional family of curves linearly equivalent to $\tilde{M} = M^* - E_1^*$ which goes through each point of the exceptional divisor E_1 with the exception of the point at infinity. If now we blow any point different from the infinity, we obtain a similar situation appearing a $(\delta-2)$-dimensional family of curves linearly equivalent to $\tilde{M} = M^* - E_1^* - E_2^*$ going through each free point of E_2. Repeating this procedure and noticing that every point p_i, $i \leq \delta+1$, is free, we conclude the existence of a section M_1 as in the statement.

To finish let us show the uniqueness. Suppose that the strict transforms $\tilde{M}_1$ and $\tilde{M}_2$ of two curves M_1 and M_2 of degree $(0,1)$ go through $p_{\delta+1}$. Then $\tilde{M}_1$ and $\tilde{M}_2$ also pass through p_i, $1 \leq i \leq \delta$. This fact fixes the values b_i in the above given equation for M_1 and M_2. As a consequence, the equations of M_1 and M_2 are the same and we conclude the proof. □

Remark 2.2.3. By the above proposition, another way of stating the condition (3) of Definition 2.2.1 is to say: $\delta > 0$, p is a general point and, either $p_2 \in \mathcal{C}_{\nu_n}$ belongs to strict transform of the fiber of pr passing through p on Z_1, or there does not exist any $j \geq \delta+1$ such that the points p_i, $1 \leq i \leq j$, of $\mathcal{C}_{\nu_n}$ are free.

Let us see how our types of divisorial valuations can be extended to irrational ones.

Definition 2.2.4. Let ν be an irrational valuation of $\mathbb{F}_\delta$. Set ν_i^N the normalized divisorial valuations of $\mathbb{F}_\delta$ which satisfy $\nu(f) = \lim_{i\to\infty} \nu_i^N(f)$, for all $f \in \mathcal{O}_{\mathbb{F}_\delta,p}$. Then ν is said *special* (respectively, *non-special*) when so are ν_i^N, $i \gg 1$.

Let ν be a divisorial or irrational valuation of $\mathbb{F}_\delta$ centered at $\mathcal{O}_{\mathbb{F}_\delta,p}$. Denote by F_1 the fiber of pr which contains the point p. When ν is non-special, M_1 is the curve provided by Proposition 2.2.2 whose strict transform has negative self-intersection on the surface defined by ν (if ν is divisorial), or by ν_i^N, $i \gg 1$ (otherwise).

Definition 2.2.5. Under the above notation, a special (respectively, non-special) divisorial or irrational valuation ν of $\mathbb{F}_\delta$ is called *non-positive at infinity* (NPI) when $\nu(f) \leq 0$, for $f \in \mathcal{O}_{\mathbb{F}_\delta}(\mathbb{F}_\delta \setminus (F_1 \cup M_0))$ (respectively, $f \in \mathcal{O}_{\mathbb{F}_\delta}(\mathbb{F}_\delta \setminus (F_1 \cup M_1))$). And, ν is said to be *negative at infinity* (NI) if $\nu(f) < 0$, for $f \in \mathcal{O}_{\mathbb{F}_\delta}(\mathbb{F}_\delta \setminus (F_1 \cup M_0))$, $f \notin k$ (respectively, $f \in \mathcal{O}_{\mathbb{F}_\delta}(\mathbb{F}_\delta \setminus (F_1 \cup M_1)), f \notin k$).

2.3 The cone of curves of a surface defined by an NPI special divisorial valuation of a Hirzebruch surface

Our goal in this section is to give a geometric characterization to the fact that a special divisorial valuation of a Hirzebruch surface $\mathbb{F}_\delta, \delta \geq 0$, is non-positive, or negative, at infinity.

Let p be a point of a Hirzebruch surface $\mathbb{F}_\delta, \delta \geq 0$, and ν_n a special divisorial valuation of the function field of $\mathbb{F}_\delta$ centered at $\mathcal{O}_{\mathbb{F}_\delta,p}$. Keeping the notation of the above sections, denote by F_1 the fiber of pr going through the point p and M_0 the special section, or the section of pr containing p when $\delta = 0$. Recall that $Z := Z_n$ is the rational surface defined by the finite simple sequence of blowups

$$\pi : Z := Z_n \xrightarrow{\pi_n} Z_{n-1} \to \cdots \to Z_1 \xrightarrow{\pi_1} Z_0 = \mathbb{F}_\delta \tag{2.2}$$

corresponding to the divisorial valuation ν_n.

Notice that $\tilde{F}_1$ and $\tilde{M}_0$ have negative self-intersection on Z since $F_1^2 = 0$ and $M_0^2 = -\delta$. Therefore, by Proposition 1.4.6, the classes of $\tilde{F}_1$ and $\tilde{M}_0$ generate extremal rays of the cones $\mathrm{NE}(Z)$ and $\overline{\mathrm{NE}}(Z)$. For our purposes, we need to consider the strongly convex cone $S_1(Z)$ of $\mathrm{Pic}_{\mathbb{R}}(Z)$ generated by the set $\{[\tilde{F}_1], [\tilde{M}_0]\} \cup \{[E_i]\}_{i=1}^n$ and also its dual cone

$$S_1^\vee(Z) = \{[D] \in \mathrm{Pic}_{\mathbb{R}}(Z) \mid [D] \cdot [C] \geq 0, \text{ for all } [C] \in S_1(Z)\}.$$

The following proposition provides generators for the cone $S_1^\vee(Z)$.

Proposition 2.3.1. *The dual cone $S_1^\vee(Z)$ is generated by $[F^*], [M^*]$ and the classes $\{[\Lambda_i]\}_{i=1}^n$ of the divisors*

$$\Lambda_i := a_i F^* + b_i M^* - \sum_{j=1}^{i} \mathrm{mult}_{p_j}(\varphi_i) E_j^*, \tag{2.3}$$

where $a_i := (\varphi_i, \varphi_{M_0})_p$ and $b_i := (\varphi_i, \varphi_{F_1})_p$.

Proof. By Proposition 1.4.2, it suffices to show that $\{[F^*], [M^*]\} \cup \{[\Lambda_i]\}_{i=1}^n$ is the dual basis of the basis $\{[\tilde{F}_1], [\tilde{M}_0]\} \cup \{[E_i]\}_{i=1}^n$ of $\mathrm{Pic}(Z)$ with respect to the intersection product.

Let $p_{i_{F_1}}$ be the last point in the configuration $\mathcal{C}_{\nu_n}$ of the valuation ν_n giving rise to Z through which the strict transform of F_1 passes. Also, if p belongs to M_0, we define i_{M_0} such that $p_{i_{M_0}}$ is the last point of $\mathcal{C}_\nu$ through which the strict transform of M_0 passes; otherwise we define $i_{M_0} := 0$. Taking into account that φ_i is analytically irreducible, the proximity equalities (Equation (1.2)) show that $\Lambda_i \cdot E_j = \delta_{ij}$, where δ_{ij} denotes the Kronecker delta. Also, for each $i \in \{1, 2, \ldots, n\}$, it holds

$$\Lambda_i \cdot \tilde{F}_1 = b_i - \sum_{j=1}^{\min\{i, i_{F_1}\}} \mathrm{mult}_{p_j}(\varphi_i) = 0,$$

and

$$\Lambda_i \cdot \tilde{M}_0 = a_i - \sum_{j=1}^{\min\{i, i_{M_0}\}} \text{mult}_{p_j}(\varphi_i) = 0,$$

where the summations with upper index equal to 0 are defined to be 0. Finally notice that $F^* \cdot \tilde{F}_1 = 0$, $F^* \cdot \tilde{M}_0 = 1$, $M^* \cdot \tilde{F}_1 = 1$, $M^* \cdot \tilde{M}_0 = 0$ and $F^* \cdot E_i = M^* \cdot E_i = 0$ for all $i = 1, 2, \ldots, n$. This concludes the proof. □

Remark 2.3.2. By Subsection 1.3.1, the divisor Λ_i defined in (2.3) can also be written as

$$\Lambda_i = \nu_i(\varphi_{M_0})F^* + \nu_i(\varphi_{F_1})M^* - \sum_{j=1}^{i} \nu_i(\mathfrak{m}_j)E_j^*, \tag{2.4}$$

where $\nu_i, 1 \leq i \leq n$, is the divisorial valuation of $\mathbb{F}_\delta$ defined by the divisor E_i created in the sequence (2.2) and $\{\nu_i(\mathfrak{m}_j)\}_{j=1}^{i}$ its sequence of values. In addition, the self-intersection of Λ_i is

$$\Lambda_i^2 = 2a_ib_i + \delta b_i^2 - \overline{\beta}_{g+1}(\nu_i) = 2\nu_i(\varphi_{M_0})\nu_i(\varphi_{F_1}) + \delta\nu_i(\varphi_{F_1})^2 - \overline{\beta}_{g+1}(\nu_i),$$

where $\overline{\beta}_{g+1}(\nu_i)$ is the last maximal contact value of ν_i.

From now on the expression shown in the right hand of Equation (2.4) will be denoted $\Lambda(\nu_i)$.

The following two results show that a divisorial valuation of $\mathbb{P}^2$ determines a special divisorial valuation of the Hirzebruch surface $\mathbb{F}_1$ providing the same surface Z. These results also relate some useful divisors and invariants corresponding to both valuations.

Proposition 2.3.3. *Let ν_n be a divisorial valuation of $\mathbb{P}^2$ and L the line at infinity as in Subsection 2.1. Set $\mathcal{C}_{\nu_n} = \{p_i\}_{i=1}^n$ (respectively, $\{\nu_n(\mathfrak{m}_i)\}_{i=1}^n$) the configuration of infinitely near points (respectively, the sequence of values) of ν_n and consider $\nu_n(\varphi_L)E_0^* - \sum_{i=1}^n \nu_n(\mathfrak{m}_i)E_i^*$ the divisor defined in Theorem 2.1.2(c). Then there exists a special divisorial valuation ν of $\mathbb{F}_1$ defining the same surface Z as ν_n such that*

$$E_0^* - E_1^* \sim F^* \textit{ and } \nu_n(\varphi_L)E_0^* - \sum_{i=1}^{n} \nu_n(\mathfrak{m}_i)E_i^* \sim \Lambda(\nu), \textit{for } n > 1.$$

Proof. Consider the finite sequence defined by ν_n (finishing at $Z = Z_n$) given in (2.2). For $1 \leq i \leq n$, denote by E_{p_i} the exceptional divisor created after blowing-up p_i. By [9, Chapter IV, Proposition IV.1], $\mathbb{F}_1$ can be regarded as the projective plane $\mathbb{P}^2$ with the point p blown up. Since we are considering that $p = p_1 \in L$ (see Subsection 2.1), any fiber F of pr is a curve which belongs to $|\tilde{L}|$, where $\tilde{L}$ is the strict transform of L on $Z_1 = \mathbb{F}_1$, and $M_0 \sim E_{p_1}$ on $\mathbb{F}_1$. Now, take $\mathcal{C}_\nu = \{q_i\}_{i=1}^{n-1}$ such that $q_i = p_{i+1}$, for $1 \leq i \leq n-1$; it is the configuration of infinitely near points of a special divisorial valuation ν of $\mathbb{F}_1$ and defines the same surface Z as ν_n. Set E_{q_i} the exceptional divisor created after blowing-up q_i, for $1 \leq i \leq n-1$. Write $\{\nu(\mathfrak{n}_i)\}_{i=1}^{n-1}$ the sequence of values of ν. It holds that

$$\nu(\varphi_{M_0}) = \nu_n(\mathfrak{m}_1) \text{ and } \nu(\mathfrak{n}_i) = \nu_n(\mathfrak{m}_{i+1}), \text{ for } 1 \leq i \leq n-1.$$

Suppose now that p_{i_L} is the last point of $\mathcal{C}_{\nu_n}$ through which $\tilde{L}$ goes and set F_1 the fiber which contains $q_1 = p_2$ on $\mathbb{F}_1$ whose strict transforms pass through $q_2, q_3, \ldots, q_{i_{F_1}} = p_{i_L}$. Then

$$\begin{aligned}\nu_n(\varphi_L) = \sum_{j=1}^{i_L} \nu_n(\mathfrak{m}_j) \cdot \mathrm{mult}_{p_j}(\varphi_L) &= \sum_{j=1}^{i_{F_1}} \nu(\mathfrak{n}_j) \cdot \mathrm{mult}_{q_j}(\varphi_{F_1}) + \nu_n(\mathfrak{m}_1)\\ &= \nu(\varphi_{F_1}) + \nu(\varphi_{M_0}).\end{aligned}$$

As a result, one has that

$$\begin{aligned}\Lambda(\nu) &= \nu(\varphi_{M_0})F^* + \nu(\varphi_{F_1})M^* - \sum_{i=1}^{n-1} \nu(\mathfrak{n}_i) E_{q_i}^*\\ &\sim (\nu(\varphi_{M_0}) + \nu(\varphi_{F_1}))F^* + \nu(\varphi_{F_1})M_0^* - \sum_{i=1}^{n-1} \nu(\mathfrak{n}_i) E_{q_i}^*\\ &\sim \nu_n(\varphi_L)E_0^* - \nu_n(\mathfrak{m}_1)E_{p_1}^* - \sum_{i=2}^{n} \nu_n(\mathfrak{m}_i) E_{p_i}^*\\ &= \nu_n(\varphi_L)E_0^* - \sum_{i=1}^{n} \nu_n(\mathfrak{m}_i) E_{p_i}^*,\end{aligned}$$

which completes the proof. □

Corollary 2.3.4. *Let ν_n be a divisorial valuation of $\mathbb{P}^2$ and $\mathcal{C}_{\nu_n} = \{p_i\}_{i=1}^n$ (respectively, $\{\nu_n(\mathfrak{m}_i)\}_{i=0}^{g+1}$, $\{\beta_i'(\nu_n)\}_{i=0}^{g+1}$, $\{\overline{\beta}_i(\nu_n)\}_{i=0}^{g+1}$) its configuration of infinitely near points (respectively, its sequence of values, its sequence of Puiseux exponents, its sequence of maximal contact values). Set ν the special divisorial valuation of $\mathbb{F}_1$ defined by ν as described in Proposition 2.3.3 and denote by $\{\beta_i'(\nu)\}_{i=0}^{\hat{g}+1}$ (respectively, $\{\overline{\beta}_i(\nu)\}_{i=0}^{\hat{g}+1}$) its sequence of Puiseux exponents (respectively, its sequence of maximal contact values). Then,*

(a) If $\beta_1'(\nu_n) \geq 2$,

- $\beta_1'(\nu) = \beta_1'(\nu_n) - 1$ *and* $\beta_i'(\nu) = \beta_i'(\nu_n)$, *for* $2 \leq i < g+2$;
- $\overline{\beta}_0(\nu) = \overline{\beta}_0(\nu_n)$ *and* $\overline{\beta}_i(\nu) = \overline{\beta}_i(\nu_n) - \nu_n(\mathfrak{m}_1) \cdot \mathrm{mult}_{p_1}(\varphi_{l_i})$, *for* $1 \leq i \leq g+1$.

(b) If $1 < \beta_1'(\nu_n) < 2$ and $\hat{g} = g$,

- $\beta_1'(\nu) = \dfrac{1}{\beta_1'(\nu_n) - 1}$ *and* $\beta_i'(\nu) = \beta_i'(\nu_n)$, *for* $2 \leq i \leq g+1$;
- $\overline{\beta}_0(\nu) = \overline{\beta}_1(\nu_n) - \overline{\beta}_0(\nu_n), \overline{\beta}_1(\nu) = \overline{\beta}_0(\nu_n)$ *and*
  $\overline{\beta}_i(\nu) = \overline{\beta}_i(\nu_n) - \nu_n(\mathfrak{m}_1) \cdot \mathrm{mult}_{p_1}(\varphi_{\ell_i})$, *for* $2 \leq i \leq g+1$.

(c) If $1 < \beta'_1(\nu_n) < 2$ *and* $\hat{g} = g - 1$,

- $\beta'_1(\nu) = \dfrac{1}{\beta'_1(\nu_n) - 1} + \beta'_2(\nu_n) - 1$ *and* $\beta'_i(\nu) = \beta'_{i+1}(\nu_n)$, *for* $2 \leq i < g+1$;
- $\overline{\beta}_0(\nu) = \overline{\beta}_1(\nu_n) - \overline{\beta}_0(\nu_n)$ *and* $\overline{\beta}_i(\nu) = \overline{\beta}_{i+1}(\nu_n) - \nu_n(\mathfrak{m}_1) \cdot \mathrm{mult}_{p_1}(\varphi_{\ell_{i+1}})$, *for* $1 \leq i \leq g$.

Proof. Following the proof of Proposition 2.3.3, recall that $\mathcal{C}_\nu = \{q_i\}_{i=1}^{n-1}$ (respectively, $\{\nu(\mathfrak{n}_i)\}_{i=1}^{n-1}$) denotes the configuration of infinitely near points (respectively, the sequence of values) of ν which satisfies $q_i = p_{i+1}$ (respectively, $\nu(\mathfrak{n}_i) = \nu(\mathfrak{m}_{i+1})$), for $1 \leq i \leq n-1$. In addition, E_{p_i} (respectively, E_{q_i}) is the exceptional divisor created after blowing-up p_i (respectively, q_i).

We begin by proving (a). By assumption one has $\nu_n(\mathfrak{m}_1) = \nu_n(\mathfrak{m}_2)$ and consequently $\hat{g} = g, \overline{\beta}_0(\nu) = \overline{\beta}_0(\nu_n)$,

$$\beta'_1(\nu) = \beta'_1(\nu_n) - 1 \text{ and } \beta'_i(\nu) = \beta'_i(\nu_n), \text{ for } 2 \leq i < g+2.$$

Moreover, by definition $\overline{\beta}_i(\nu_n) = \nu_n(\varphi_{p_{\ell_i}})$ (respectively, $\overline{\beta}_i(\nu) = \nu_n(\varphi_{q_{k_i}})$) where $\varphi_{p_{\ell_i}}$ (respectively, $\varphi_{q_{k_i}}$) is an analytically irreducible germ at p_1 (respectively, q_1) whose strict transform is transversal to $E_{p_{\ell_i}}$ (respectively, $E_{q_{k_i}}$), being these last two exceptional divisors those corresponding with the dead ends of $\Gamma^i_{\nu_n}$ and Γ^i_ν, respectively. Moreover, it holds that

$$q_{k_i} = q_{\ell_i - 1} = p_{\ell_i} \text{ and } \mathrm{mult}_{q_j}(\varphi_{q_{k_i}}) = \mathrm{mult}_{p_{j+1}}(\varphi_{p_{\ell_i}}), \text{ for } 1 \leq j \leq n-1,$$

since $\hat{g} = g$. Therefore,

$$\begin{aligned} \overline{\beta}_i(\nu) = \nu(\varphi_{q_{k_i}}) &= \sum_{j=1}^{n-1} \nu(\mathfrak{n}_j) \cdot \mathrm{mult}_{q_j}(\varphi_{q_{k_i}}) \\ &= \sum_{j=2}^{n} \nu_n(\mathfrak{m}_j) \cdot \mathrm{mult}_{p_j}(\varphi_{p_{\ell_i}}) \\ &= \nu_n(\varphi_{p_{\ell_i}}) - \nu_n(\mathfrak{m}_1) \cdot \mathrm{mult}_{p_1}(\varphi_{p_{\ell_i}}), \end{aligned}$$

which proves (a).

Let us show (b) recall that $\hat{g} = g$. As $1 < \beta'_1(\nu_n) < 2$, $\nu_n(\mathfrak{m}_1) > \nu_n(\mathfrak{m}_2)$ and then $\overline{\beta}_0(\nu) = \nu_n(\mathfrak{m}_2) = \overline{\beta}_1(\nu_n) - \overline{\beta}_0(\nu_n), \overline{\beta}_1(\nu) = \overline{\beta}_0(\nu_n)$,

$$\beta'_1(\nu) = \frac{1}{\beta'_1(\nu_n) - 1} \text{ and } \beta'_i(\nu) = \beta'_i(\nu_n), \text{ for } 2 \leq i \leq g+1,$$

since $\hat{g} = g$. This proves the statement after noticing that the relation among the maximal contact values of the valuations follows from a similar argument to that developed before.

Finally, we prove (c). In this case, note that $\beta'_1(\nu_n) = 1 + 1/a^1_2$, where $a^1_2 \in \mathbb{Z}_{>0}$. Thus,

$$\beta'_1(\nu) = \frac{1}{\beta'_1(\nu_n) - 1} + \beta'_2(\nu_n) - 1 \text{ and } \beta'_i(\nu) = \beta'_{i+1}(\nu_n), \text{ for } 2 \leq i < g+1.$$

In addition, $\overline{\beta}_0(\nu) = \nu_n(\mathfrak{m}_2) = \overline{\beta}_1(\nu_n) - \overline{\beta}_0(\nu_n)$ and, taking into account that

$$q_{k_i} = p_{\ell_{i+1}}, \text{ for } 1 \leq i \leq g, \text{ and } \mathrm{mult}_{q_j}(\varphi_{q_{k_i}}) = \mathrm{mult}_{p_{j+1}}(\varphi_{p_{\ell_{i+1}}}), \text{ for } 1 \leq j \leq n-1,$$

the relation among the maximal contact values of the valuations follows by using an analogous reasoning to that shown previously. This completes the proof. □

Remark 2.3.5. Corollary 2.3.4 provides the following condition

$$\overline{\beta}_{\hat{g}+1}(\nu) = \overline{\beta}_{g+1}(\nu_n) - \nu_n(\mathfrak{m}_1)^2.$$

As a consequence, the following chain of equalities holds:

$$\begin{aligned}\Lambda(\nu)^2 &= 2\nu(\varphi_{M_0})\nu(\varphi_{F_1}) + \nu(\varphi_{F_1})^2 - \overline{\beta}_{\hat{g}+1}(\nu)\\ &= (\nu(\varphi_{F_1}) + \nu(\varphi_{M_0}))^2 - (\overline{\beta}_{\hat{g}+1}(\nu) + \nu(\varphi_{M_0})^2)\\ &= \left(\nu_n(\varphi_L)E_0^* - \sum_{i=1}^{n} \nu_n(\mathfrak{m}_i)E_i^*\right)^2,\end{aligned}$$

where we have used the fact that $\nu_n(\varphi_L) = \nu(\varphi_{F_1}) + \nu(\varphi_{M_0})$ and $\nu_n(\mathfrak{m}_1) = \nu(\varphi_{M_0})$.

Now we return to our study of surfaces given by special divisorial valuations of Hirzebruch surfaces $\mathbb{F}_\delta$ where δ need not be one. The family of divisors $\{\Lambda_i\}_{i=1}^n$ defined in (2.3) is important in this section because it has interesting properties which are suitable to achieve our propose, as we will see in the following result.

Lemma 2.3.6. *Let ν_n be a special divisorial valuation of $\mathbb{F}_\delta$. Then, under the above notation, it holds that $\Lambda_1^2 \geq 0$, and the inequality $\Lambda_i^2 \geq 0$ for some index $i \in \{2, 3, \ldots, n\}$ implies:*

(a) $\Lambda_i^2 > 0$, if p_i is a satellite point of the configuration $\mathcal{C}_{\nu_n}$.

(b) $\Lambda_{i-1}^2 \geq 0$ and, when $\Lambda_{i-1}^2 = 0$, the point p_i is satellite and the point p_{i-1} is free.

Proof. The self-intersection of the divisor Λ_1 satisfies $\Lambda_1^2 = 1 + \delta$ when p_1 is a special point and also when $\delta = 0$. Otherwise, $\Lambda_1^2 = \delta - 1$.

For proving the remaining statements, we can assume, without loss of generality, that $i = n \geq 2$.

We are going to prove the result when p_1 is a special point. Otherwise, the proof is the same after setting $\delta = 0$ or $a_n = 0$.

We start with the proof of (a) for which we will use some properties of the set of maximal contact values of ν_n, $\{\overline{\beta}_j(\nu_n)\}_{j=0}^{g+1}$. We divide this proof in two cases.

Case 1(a): $g > 1$. Reasoning by contradiction and taking into account that the point p_n is satellite, we get that

$$0 = \Lambda_n^2 = 2a_nb_n + \delta b_n^2 - e_{g-1}(\nu_n)\overline{\beta}_g(\nu_n) = e_{g-1}(\nu_n)\left[\frac{2a_nb_n + \delta b_n^2}{e_{g-1}(\nu_n)} - \overline{\beta}_g(\nu_n)\right],$$

where $e_{g-1}(\nu_n) = \gcd(\overline{\beta}_0(\nu_n), \overline{\beta}_1(\nu_n), \ldots, \overline{\beta}_{g-1}(\nu_n))$ (see Proposition 1.3.5). Since both a_n and b_n are either a multiple of $\overline{\beta}_0(\nu_n)$ or $\overline{\beta}_1(\nu_n)$, the first addend in the brackets is a multiple of $e_{g-1}(\nu_n)$, which gives a contradiction because $\gcd(e_{g-1}(\nu_n), \overline{\beta}_g(\nu_n))$ equals 1.

Case 2(a): $g = 1$. We distinguish three sub-cases: *The first one when the values a_n and b_n are divisible by $\overline{\beta}_0(\nu_n)$.* Then $e_{g-1}(\nu_n) = e_0(\nu_n) = \overline{\beta}_0(\nu_n)$ and the proof follows as above. *The second one when the value a_n satisfies $a_n = \overline{\beta}_1(\nu_n)$;* then $\Lambda_n^2 = \overline{\beta}_0(\nu_n)(2\overline{\beta}_1(\nu_n) + \overline{\beta}_0(\nu_n)\delta - \overline{\beta}_1(\nu_n)) > 0$. *Otherwise*, it holds that

$$\Lambda_n^2 = \overline{\beta}_1(\nu_n)(2\overline{\beta}_0(\nu_n) + \overline{\beta}_1(\nu_n)\delta - \overline{\beta}_0(\nu_n)) > 0,$$

which concludes the proof of (a).

Now we prove (b). Again we can suppose that $i = n$. We also assume that the point p_n is satellite (otherwise $\Lambda_{n-1}^2 > 0$ by the Noether formula). Denote by ν_{n-1} the divisorial valuation defined by the divisor E_{n-1}. Let $\{\overline{\beta}_j(\nu_{n-1})\}_{j=0}^{\hat{g}+1}$ be the sequence of maximal contact values of ν_{n-1},

$$e_{g-1}(\nu_{n-1}) = \gcd(\overline{\beta}_0(\nu_{n-1}), \overline{\beta}_1(\nu_{n-1}), \ldots, \overline{\beta}_{g-1}(\nu_{n-1}))$$

and $e := e_{\hat{g}-1}(\nu_{n-1})/e_{\hat{g}-1}(\nu_n)$. Consider two cases with two sub-cases.

Case 1(b): $g = \hat{g}$. Assume first that $g > 1$. From the following equality, which is proved in [60, Lemma 2],

$$|\overline{\beta}_g(\nu_{n-1}) - e\overline{\beta}_g(\nu_n)| = \frac{1}{e_{g-1}(\nu_n)}, \tag{2.5}$$

one can deduce that

$$-\frac{e_{g-1}(\nu_n)\overline{\beta}_g(\nu_{n-1})}{e} \geq -\frac{1}{e} - e_{g-1}(\nu_n)\overline{\beta}_g(\nu_n). \tag{2.6}$$

In this case both valuations ν_n and ν_{n-1} are defined by satellite points, therefore $a_{n-1} = ea_n, b_{n-1} = eb_n$, $\overline{\beta}_{g+1}(\nu_n) = e_{g-1}(\nu_n)\overline{\beta}_g(\nu_n)$ and

$$\overline{\beta}_{g+1}(\nu_{n-1}) = e_{g-1}(\nu_{n-1})\overline{\beta}_g(\nu_{n-1})$$

by Equality (1.5) and Corollary 1.3.7. As a consequence

$$\begin{aligned}\Lambda_{n-1}^2 &= e^2\left[2a_nb_n + \delta b_n^2 - \frac{e_{g-1}(\nu_n)\overline{\beta}_g(\nu_{n-1})}{e}\right]\\ &\geq e^2\left[2a_nb_n + \delta b_n^2 - \frac{1}{e} - e_{g-1}(\nu_n)\overline{\beta}_g(\nu_n)\right]\\ &= e^2\left[\Lambda_n^2 - \frac{1}{e}\right] > 0,\end{aligned}$$

where the first inequality is deduced from (2.6) and the last one holds since $\Lambda_n^2 > e_{g-1}(\nu_n) > 1/e$.

To conclude the proof in this case, it remains to study what happens when $g = 1$. We consider the same subcases as above. The first one where the values a_n and b_n are both divisible by $\overline{\beta}_0(\nu_n)$ and the fact $\Lambda_{n-1}^2 > 0$ can be proved as before. The second one where the value a_n equals $\overline{\beta}_1(\nu_n)$, then one has that

$$\begin{aligned}\Lambda_{n-1}^2 &= 2\overline{\beta}_0(\nu_{n-1})\overline{\beta}_1(\nu_{n-1}) + \delta\overline{\beta}_0(\nu_{n-1})^2 - \overline{\beta}_2(\nu_{n-1})\\ &= \overline{\beta}_0(\nu_{n-1})(2\overline{\beta}_1(\nu_{n-1}) + \delta\overline{\beta}_0(\nu_{n-1}) - \overline{\beta}_1(\nu_{n-1}))\\ &= \overline{\beta}_0(\nu_{n-1})(\overline{\beta}_1(\nu_{n-1}) + \delta\overline{\beta}_0(\nu_{n-1})) > 0.\end{aligned}$$

Otherwise, one has $\Lambda_{n-1}^2 = \overline{\beta}_1(\nu_{n-1})(\overline{\beta}_0(\nu_{n-1}) + \delta\overline{\beta}_1(\nu_{n-1})) > 0$ and the proof ends.

Case 2(b): $\widehat{g} = g - 1$. When $g > 1$, it holds

$$\overline{\beta}_{\hat{g}+1}(\nu_{n-1}) = \frac{\overline{\beta}_{g+1}(\nu_n) + 2}{4}$$

and thus

$$\Lambda_{n-1}^2 = \frac{1}{4}\left(2a_nb_n + \delta b_n^2 - \overline{\beta}_{g+1}(\nu_n) - 2\right) = \frac{1}{4}\Lambda_n^2 - \frac{1}{2} \geq 0,$$

because $\Lambda_n^2 \geq 2$.

Finally we must assume that $g = 1$ and, as above, when the values a_n and b_n are divisible by $\overline{\beta}_0(\nu_n) = 2$, $\Lambda_{n-1}^2 \geq 0$. When $a_n = \overline{\beta}_1(\nu_n)$, $\Lambda_{n-1}^2 = \overline{\beta}_1(\nu_{n-1}) + \delta \geq 0$, and otherwise,

$$\Lambda_{n-1}^2 = 2\overline{\beta}_1(\nu_{n-1}) + \delta\overline{\beta}_1(\nu_{n-1})^2 - \overline{\beta}_1(\nu_{n-1}) \geq 0,$$

which concludes the proof. □

Recall that ν_n is a special divisorial valuation of $\mathbb{F}_\delta$ and Z is the surface defined by ν_n. The next theorem provides equivalent conditions to the fact that ν_n is non-positive at infinity. This result uses the values a_n and b_n and the divisor Λ_n introduced in Proposition 2.3.1.

Theorem 2.3.7. *Let ν_n be a special divisorial valuation of the function field of $\mathbb{F}_\delta$ centered at $\mathcal{O}_{\mathbb{F}_\delta,p}$. Set Z the surface that ν_n defines. Consider the divisor Λ_n given in (2.3) and the last maximal contact value of ν_n, $\overline{\beta}_{g+1}(\nu_n)$. Then the following conditions are equivalent:*

(a) The valuation ν_n is non-positive at infinity.

(b) The divisor Λ_n is nef.

(c) The inequality $2a_nb_n + \delta b_n^2 \geq \overline{\beta}_{g+1}(\nu_n)$ holds.

(d) The cone of curves $\mathrm{NE}(Z)$ is generated by the classes of the strict transforms on Z of the fiber passing through p, the section M_0 and the irreducible exceptional divisors associated with the map π given by ν_n.

Proof. Our first step is to prove the equivalence between (a) and (b), and we start by proving that (b) implies (a). We assume firstly that $\delta > 0$ and $p = p_1$ is a special point. We are going to use the notation introduced in Subsection 1.6.1. Without loss of generality, suppose that the special point p has coordinates $(1:0;1,0)$. The point p belongs to the fiber F_1 whose equation is $X_1 = 0$, and the special section M_0 is defined by the equation $Y_1 = 0$. Set U_{00} the affine open set of $\mathbb{F}_\delta$ given by $X_0 \neq 0$ and $Y_0 \neq 0$, whose associated affine coordinates are $\{u, v\} = \left\{\frac{X_1}{X_0}, \frac{X_0^\delta Y_1}{Y_0}\right\}$. Consider also the affine open set of $\mathbb{F}_\delta$, U_{11}, defined by $X_1 \neq 0$ and $Y_1 \neq 0$, with coordinates $\{x, y\} = \left\{\frac{X_0}{X_1}, \frac{Y_0}{X_1^\delta Y_1}\right\}$. It holds that $p \in U_{00}$ and F_1 and M_0 have local equations $u = 0$ and $v = 0$, respectively. Denote by $\mathcal{P}$ the set of non-constant functions in $\mathcal{O}_{\mathbb{F}_\delta}(U_{11})$ (up to multiplication by a nonzero element of k) such that neither x nor y divide them. In terms of the coordinates $\{u, v\}$, $f \in \mathcal{P}$ can be expressed as

$$f(x, y) = f(1/u, 1/u^\delta v) = \frac{h_f(u, v)}{u^{\deg_1(h_f)+\delta \deg_2(h_f)} v^{\deg_2(h_f)}}, \tag{2.7}$$

where $h_f(u, v) \in \mathcal{O}_{\mathbb{F}_\delta}(U_{00})$. The bi-homogeneous polynomial

$$X_0^{\deg_1(h_f)} Y_0^{\deg_2(h_f)} \cdot h_f\left(\frac{X_1}{X_0}, \frac{X_0^\delta Y_1}{Y_0}\right)$$

defines a curve C_f on the surface $\mathbb{F}_\delta$ of degree $(\deg_1(h_f), \deg_2(h_f))$ and, if F' and M' are the fiber and the section on $\mathbb{F}_\delta$ with equations $X_0 = 0$ and $Y_0 = 0$, it holds that the map $f \to C_f$ defines a one-to-one correspondence between $\mathcal{P}$ and the set of the curves on $\mathbb{F}_\delta$ containing no curve in $\{F_1, F', M_0, M'\}$ as a component. Now, the conditions Λ_n nef and (2.7) show that

$$\begin{aligned} 0 \leq \Lambda_n \cdot C_f = \Lambda_n \cdot & \left[\deg_1(h_f) F^* + \deg_2(h_f) M^* - \sum_{i=1}^{n} \mathrm{mult}_{p_i}(h_f) E_i^*\right] \\ = - & \left[-(\deg_1(h_f) + \deg_2(h_f)\delta)\nu_n(u) - \deg_2(h_f)\nu_n(v) + \nu_n(h_f)\right] = -\nu_n(f). \end{aligned}$$

Thus, to finish the proof of (a) in this case (p is a special point), it only remains to assume that either x or y or both are factors of f. Then the proof follows from the existence of non-negative integers α, β with $\alpha + \beta \neq 0$ and $f_1 \in \mathcal{P}$ such that

$$\nu_n(f) = \nu_n(x^\alpha y^\beta f_1) = -(\alpha + \beta\delta)\nu_n(u) - \beta\nu_n(v) + \nu_n(f_1) \leq 0.$$

If $\delta = 0$ the proof is analogous, and the non-positivity of ν_n for the case when p is a general point can be proved in a similar way after assuming that p has coordinates $(0:1;0,1)$ and considering local coordinates $\{u, v\} = \left\{\frac{X_0}{X_1}, \frac{Y_0}{X_1^\delta Y_1}\right\}$ in the affine open set U_{11} and $\{x, y\} = \left\{\frac{X_1}{X_0}, \frac{Y_0}{X_0^\delta Y_1}\right\}$ in U_{01}.

Now we are going to prove that (a) implies (b). Assume by contradiction that the divisor Λ_n is not nef and, therefore, that there exists an effective divisor C such that $\Lambda_n \cdot C < 0$. This implies that, with the above notation, if p is a special point

–or $p \in \mathbb{F}_0$–, (respectively, p is a general point), then there exists $f \in \mathcal{O}_{\mathbb{F}_\delta}(U_{11})$ (respectively, $f \in \mathcal{O}_{\mathbb{F}_\delta}(U_{01})$) such that $-\nu_n(f) = \Lambda_n \cdot C < 0$, a contradiction.

The fact that (b) implies (c) follows easily from previous computations given in Remark 2.3.2.

Let us prove that (d) can be deduced from (c). Fix any ample divisor H on the surface Z and consider the set

$$A(Z) := \{[D] \in \mathrm{Pic}_{\mathbb{R}}(Z) \mid [D]^2 \geq 0 \text{ and } [H] \cdot [D] \geq 0\}.$$

Recall that the above defined cone $S_1(Z)$ is generated by the classes $[\tilde{F}_1]$, $[\tilde{M}_0]$ and $[E_i]$, $1 \leq i \leq n$, and we are going to prove that

$$\overline{\mathrm{NE}}(Z) = S_1(Z) + S_1^\vee(Z) = \mathrm{NE}(Z) \tag{2.8}$$

and

$$S_1^\vee(Z) \subseteq A(Z) \subseteq S_1(Z) \tag{2.9}$$

hold, which shows (d). Our Condition (c) means that $\Lambda_n^2 \geq 0$ and by Lemma 2.3.6, one has that $\Lambda_i^2 \geq 0$, $1 \leq i \leq n-1$. Proposition 2.3.1 proves the first inclusion in (2.9) and the last one follows from the first one and the equality $A(Z)^\vee = A(Z)$ given in Proposition 1.4.14.

It remains to prove the chain of equalities (2.8). First, notice that $A(Z) \subseteq \overline{\mathrm{NE}}(Z)$ by Proposition 1.4.13. Thus $S_1^\vee(Z) \subseteq \overline{\mathrm{NE}}(Z)$. Now, if $[C]$ is the class of an irreducible curve on Z and it is not one of the given generators of $S_1(Z)$, then $[C] \in S_1^\vee(Z)$ because otherwise $[C] \cdot [D] < 0$ for $[D] \in S_1(Z)$ and C and D would have a common component. Therefore we have proved (2.8) with inclusions $\supseteq$ instead of equalities. Taking topological closures we deduce that (2.8) holds.

Finally (d) implies (b) by Proposition 2.3.1, which concludes the proof. $\square$

Remark 2.3.8. Taking into account the proof of Proposition 2.3.3 and Remark 2.3.5, one can state Theorem 2.1.2 as a particular case of Theorem 2.3.7. Moreover, a divisorial valuation of $\mathbb{P}^2$ is NPI if and only if the corresponding special divisorial valuation of $\mathbb{F}_1$ is NPI.

Remark 2.3.9. By Proposition 2.3.1, Proposition 2.3.3, Theorem 2.3.7 and Remark 2.3.8, it holds that the nef cone $\mathrm{Nef}(Z)$ of Z is generated by $[F^*], [M^*]$ and $[\Lambda_i]$, $1 \leq i \leq n$, when ν_n is a NPI special divisorial valuation of $\mathbb{F}_\delta$, and by $[E_0^*]$ and $[D_i]$, where $D_i = \nu_n(\varphi_L)E_0^* - \sum_{i=1}^n \nu_n(\mathfrak{m}_i)E_i^*$ and $1 \leq i \leq n$, if ν_n is a NPI divisorial valuation of $\mathbb{P}^2$.

Corollary 2.3.10. *Under the notation in Theorem 2.3.7, assume that ν_n is NPI and, for $1 \leq i \leq n-1$, denote by ν_i the special divisorial valuation of $\mathbb{F}_\delta$ defined by the divisor E_i created by each map π_i appearing in the sequence (2.2) defined by ν_n. Then the valuation ν_i is also non-positive at infinity.*

Proof. By hypothesis and Lemma 2.3.6, one has that $\Lambda_i^2 \geq 0$, for $i \in \{1, 2 \ldots, n-1\}$. As a result, following Remark 2.3.2 and Theorem 2.3.7, ν_i is NPI for $1 \leq i \leq n-1$. □

Corollary 2.3.11. *Keeping the notation as in Theorem 2.3.7, if Λ_n is a nef divisor, then every Λ_i is effective, for $1 \leq i \leq n$. In particular,*

$$\Lambda_i = (\Lambda_i \cdot M^*)\tilde{F}_1 + (\Lambda_i \cdot F^*)\tilde{M}_0 + \sum_{j=1}^{n} (\Lambda_i \cdot \Lambda_j) E_j.$$

Proof. It is clear that Λ_i is nef and its self-intersection is non-negative by Corollary 2.3.10. Moreover, the set $\{[F^*], [M^*]\} \cup \{[\Lambda_i]\}_{i=1}^n$ is the dual basis of the basis $\{[\tilde{F}_1], [\tilde{M}_0]\} \cup \{[E_i]\}_{i=1}^n$ of $\mathrm{Pic}_{\mathbb{R}}(Z)$ by Proposition 2.3.1 and the classes of the divisors in the set $\{F^*, M^*\} \cup \{\Lambda_i\}_{i=1}^n$ belong to $A(Z) \subset \overline{\mathrm{NE}}(Z)$. As a consequence, the divisor Λ_i can be written as

$$\Lambda_i = \alpha_{i1}\tilde{F} + \alpha_{i2}\tilde{M}_0 + \sum_{j=1}^{n} \beta_{ij} E_j,$$

where $\alpha_{i1} = \Lambda_i \cdot M^* \geq 0, \alpha_{i2} = \Lambda_i \cdot F^* \geq 0$ and $\beta_{ij} = \Lambda_i \cdot \Lambda_j \geq 0$, for $1 \leq j \leq n$, which concludes the proof. □

Now we state a result which allows us to define a sequence of non-positive at infinity special valuations of $\mathbb{F}_\delta, \delta \geq 0$, which approaches a non-positive at infinity special irrational valuation of $\mathbb{F}_\delta$.

Corollary 2.3.12. *Let ν_n be a non-positive at infinity special divisorial valuation of $\mathbb{F}_\delta$ and $\mathcal{C}_{\nu_n} = \{p_i\}_{i=1}^n$ its configuration of infinitely near points. Set a special divisorial valuation ν_m such that its configuration $\mathcal{C}_{\nu_m} = \{q_i\}_{i=1}^m, n < m$, satisfies that $\mathcal{C}_{\nu_n} \subset \mathcal{C}_{\nu_m}$ and the points $q_i, n+1 \leq i \leq m$, are satellite. Then, the valuation ν_m is non-positive at infinity.*

Proof. Denote by $\{\overline{\beta}_i(\nu_n)\}_{i=1}^{g+1}$ (respectively, $\{\overline{\beta}_i(\nu_m)\}_{i=1}^{\hat{g}+1}$) the sequence of maximal contact values of ν_n (respectively, ν_m). Define $e_{g-1}(\nu_n) := \gcd(\overline{\beta}_1(\nu_n), \overline{\beta}_2(\nu_n), \ldots, \overline{\beta}_{g-1}(\nu_n))$ and $e_{\hat{g}-1}(\nu_m) = \gcd(\overline{\beta}_1(\nu_m), \overline{\beta}_2(\nu_m), \ldots, \overline{\beta}_{\hat{g}-1}(\nu_m))$.

Notice that it holds that

$$e_{\hat{g}-1}(\nu_m)\overline{\beta}_{g+1}(\nu_n) \geq \overline{\beta}_{\hat{g}}(\nu_m). \tag{2.10}$$

In fact, consider that p_n is a free point and then the inequality holds by the Noether formula and the inequality $e_{\hat{g}-1}(\nu_m)\mathrm{mult}_{p_n}(\varphi_n) \geq \mathrm{mult}_{p_n}(\varphi_m)$. The satellite case follows from the previous one.

By Theorem 2.3.7 and Remark 2.3.2, the valuation ν_m is non-positive at infinity if and only if $2\nu_m(\varphi_{M_0})\nu_m(\varphi_{F_1}) + \delta\nu_m(\varphi_{F_1})^2 - \overline{\beta}_{\hat{g}+1}(\nu_m) \geq 0$. It is easy to check that the last condition holds since

$$\begin{aligned} 2\nu_m(\varphi_{M_0})\nu_m(\varphi_{F_1}) + \delta\nu_m(\varphi_{F_1})^2 &= e_{\hat{g}-1}^2(\nu_m)\left(2\nu_n(\varphi_{M_0})\nu_n(\varphi_{F_1}) + \delta\nu_n(\varphi_{F_1})^2\right) \\ &\geq e_{\hat{g}-1}^2(\nu_m)\overline{\beta}_{g+1}(\nu_n) \geq \overline{\beta}_{\hat{g}+1}(\nu_m), \end{aligned}$$

by Corollary 1.3.7, Equation (2.10) and the fact that ν_n is non-positive at infinity, which completes the proof. □

Notice that one gets a similar property for non-positive at infinity divisorial valuation of $\mathbb{P}^2$ by Remark 2.3.8.

Next we show an example of the previously stated results.

Example 2.3.13. Let ν be a special divisorial valuation of $\mathbb{F}_2$ whose sequence of maximal contact values is $\{16, 24, 60, 131, 524\}$. Γ_ν (Figure 2.1) is its dual graph. Set $\mathcal{C}_\nu = \{p_i\}_{i=1}^{12}$ (where $p = p_1$) its configuration of infinitely near points and assume that p is a special point, F_1 is the fiber going through p and the strict

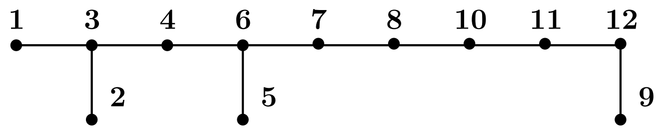


Figure 2.1: Dual graph of ν in Example 2.3.13

transform of M_0 passes through p_2. Therefore, it holds that $a_{12} = (\varphi_{M_0}, \varphi_{12})_p = 24$, $b_{12} = (\varphi_{F_1}, \varphi_{12})_p = 16$ and $2a_{12}b_{12} + \delta b_{12}^2 = 1280 > 524 = \overline{\beta}_{g+1}(\nu)$. Consequently, by Theorem 2.3.7, the cone of curves $\mathrm{NE}(Z)$ of Z defined by ν is generated by $[\tilde{F}_1], [\tilde{M}_0]$ and $\{[E_i]\}_{i=1}^{12}$ and the divisors $\Lambda_i, 1 \leq i \leq 12$, defined in Proposition 2.3.1 are nef.

To finish this section we provide a result that characterizes the fact that a special divisorial valuation of $\mathbb{F}_\delta$ is negative at infinity. Also, by Proposition 2.3.3, Theorem 2.1.3 is a particular case of that result.

Theorem 2.3.14. *Under the notation in Theorem 2.3.7, the following conditions are equivalent:*

(a) The valuation ν_n is negative at infinity.

(b) It holds that either $2a_nb_n + b_n^2\delta > \overline{\beta}_{g+1}(\nu_n)$, or $2a_nb_n + b_n^2\delta = \overline{\beta}_{g+1}(\nu_n)$ and the Iitaka dimension of the divisor Λ_n vanishes.

(c) The inequality $\Lambda_n \cdot \tilde{C} > 0$ holds for the strict transform on Z, $\tilde{C}$, of any curve C on $\mathbb{F}_\delta$, $C \neq F_1, M_0$.

Proof. For a start, we recall that the Iitaka dimension [72] of a divisor D on Z is the maximum of the projective dimensions of the closures of the images of the rational maps defined by the complete linear systems $|nD|$, when n runs over those positive integers m such that $H^0(Z, \mathcal{O}_Z(mD)) \neq 0$.

We assume that p_1 is a special point. The other cases can be proved similarly. Consider the same notations as in the proof of Theorem 2.3.7. Assume also, without loss of generality, that p_1 has coordinates $(1:0;1,0)$.

We start by proving by contradiction that (b) can be deduced from (a). Hence, assume that (a) holds but (b) is false (what means, taking into account Theorem 2.3.7, that $\Lambda_n^2 = 0$ and $\dim|m\Lambda_n| > 0$ for m large enough). Therefore, there exists $f \in \mathcal{P}$ such that the class of $m\Lambda_n - \tilde{C}_f$ is effective for m large enough. This implies that

$$0 \leq \Lambda_n \cdot (m\Lambda_n - \tilde{C}_f) = m\Lambda_n^2 - \Lambda_n \cdot \tilde{C}_f = -\Lambda_n \cdot \tilde{C}_f.$$

Hence $0 = \Lambda_n \cdot \tilde{C}_f = -\nu_n(f)$ because Λ_n is nef (by Theorem 2.3.7), and this fact contradicts (a).

To prove that (b) implies (c), we reason again by contradiction and consider C an integral curve on $\mathbb{F}_\delta$ different from F_1 and M_0, and such that $\Lambda_n \cdot \tilde{C} \leq 0$. In fact $\Lambda_n \cdot \tilde{C} = 0$ because, by Theorem 2.3.7, Λ_n is nef. Let $\mathcal{F}$ be the face of the cone of curves of Z spanned by the classes $[\tilde{F}_1], [\tilde{M}_0], [E_1], \ldots, [E_{n-1}]$, that is, $\mathcal{F} = [\Lambda_n]^\perp \cap \mathrm{NE}(Z)$. It is clear that $[\tilde{C}] \in \mathcal{F}$ and, since the extremal rays of $\mathrm{NE}(Z)$ are generated by classes of irreducible curves with negative self-intersection, $\tilde{C}^2 = 0$. $\tilde{C}$ is nef, so $[\tilde{C}]^\perp \cap \mathrm{NE}(Z)$ is a face of $\mathrm{NE}(Z)$ which contains $[\tilde{C}]$ and, then, it must coincide with $[\Lambda_n]^\perp \cap \mathrm{NE}(Z)$. Indeed, this is a consequence of the fact that, in suitable coordinates, $A(Z)$ is the projective cone over an Euclidean ball B (by the Hodge index theorem (Theorem 1.4.11)) and B is strictly convex. Then, $\tilde{C}$ is linearly equivalent to a multiple of Λ_n and, by Corollary 2.3.11, we get a contradiction.

To finish, the fact that (c) implies (a) can be proved as in Theorem 2.3.7 when proving that (b) implies (a).

□

2.4 The cone of curves of a surface defined by an NPI non-special divisorial valuation of a Hirzebruch surface

In this section we only consider non-special divisorial valuations ν_n of $\mathbb{F}_\delta, \delta > 0$. As in Section 2.3, we are going to provide several equivalent conditions to the fact that ν_n is non-positive or negative at infinity.

Notice that, since our divisorial valuations are non-special, by Proposition 2.2.2, one has that two important properties hold: On the one hand, at least the points $p = p_1, p_2, \ldots, p_{\delta+1}$ of the configuration of infinitely near points are free. On the other hand, there exists a unique integral curve of degree $(0, 1)$ which is linearly equivalent to M whose strict transform on Z has negative self-intersection (see Proposition 2.2.2). This curve will be denoted by M_1. Consequently, $[\tilde{M}_1]$ generates an extremal ray of $\mathrm{NE}(Z)$ and $\overline{\mathrm{NE}}(Z)$ as well as $[\tilde{F}_1]$ and $[\tilde{M}_0]$, where F_1 is the fiber passing through the point p and M_0 the special section.

Following Section 2.3, denote by $S_2(Z)$ the strongly convex cone of $\mathrm{Pic}_{\mathbb{R}}(Z)$ generated by the classes of the divisors of the set $\{\tilde{F}_1, \tilde{M}_0, \tilde{M}_1\} \cup \{E_1\}_{i=1}^n$. We will describe its dual cone $S_2^\vee(Z)$ in Proposition 2.4.2. Before stating this result, we prove a useful lemma.

Lemma 2.4.1. *The class of the strict transform of M_1 on Z, $[\tilde{M}_1]$, can be written as*

$$[\tilde{M}_1] = \delta[\tilde{F}_1] + [\tilde{M}_0] + (\delta-1)[E_1] + (\delta-2)[E_2] + \cdots$$
$$+[E_{\delta-1}] + d_{\delta+1}[E_{\delta+1}] + \cdots + d_n[E_n],$$

where $d_i \in \mathbb{Z}$ and $d_i \leq -1$ for all $i = \delta+1, \delta+2, \ldots, n$.

Proof. It is clear that we can write $[\tilde{M}_1]$ as

$$[\tilde{M}_1] = d_{01}[\tilde{F}_1] + d_{02}[\tilde{M}_0] + d_1[E_1] + \cdots + d_n[E_n],$$

for some values $d_{01}, d_{02}, d_i \in \mathbb{Z}$, $1 \leq i \leq n$. Now, using the equalities

$$[\tilde{F}_1] = [F^*] - [E_1^*],\ [\tilde{M}_0] = \delta[F^*] + [M^*] \text{ and } [E_i] = [E_i^*] - \sum_{p_j \to p_i} [E_j^*],$$

we can compare the above expression with the equality $[\tilde{M}_1] = [M^*] - \sum_{j=1}^{i_{M_1}} [E_j^*]$, where i_{M_1} is the index of the last point of the configuration of infinitely near points given by ν, $\mathcal{C}_\nu$, through which $\tilde{M}_1$ goes. This gives rise to a system of linear equations in the variables $d_{01}, d_{02}, \{d_i\}_{i=1}^n$, whose first equations are

$$d_{01} - \delta d_{02} = 0,\ d_{02} = 1,\ d_1 - d_{01} = -1,\ d_2 - d_1 = -1, \ldots, d_{\delta-1} - d_{\delta-2} = -1,$$
$$d_\delta - d_{\delta-1} = -1.$$

These equations determine the values of d_{01}, d_{02}, d_i for $i \in \{1, 2, \ldots, \delta\}$, that coincide with those given in the statement. The fact that $d_i \leq -1$ for $i \geq \delta + 1$ follows from considering the remaining equations and recalling that non-free points can only appear when $j > \delta + 1$. □

Proposition 2.4.2. *Let Z be the surface given by a non-special divisorial valuation and let $S_2(Z)$ be the cone of $\mathrm{Pic}_{\mathbb{R}}(Z)$ defined before Lemma 2.4.1. Then the dual cone of $S_2^\vee(Z)$, $S_2^\vee(Z)$, is generated by the following classes of divisors: $[F^*], [M^*]$, $\{[\Theta_i]\}_{i=1}^{\delta}$, $\{[\Delta_i]\}_{i=\delta+1}^{n}$, $\{[\Gamma_i]\}_{i=\delta+1}^{n}$ and $\{[\Upsilon_{ik}]\}_{i=\delta+1,k=1,\ldots,\delta-1}^{n}$, where*

$$\Theta_i := b_i M^* - \sum_{j=1}^{i} \mathrm{mult}_{p_j}(\varphi_i) E_j^*,$$

$$\Delta_i := (-\delta b_i + c_i)F^* + b_i M^* - \sum_{j=1}^{i} \mathrm{mult}_{p_j}(\varphi_i) E_j^*,$$

$$\Gamma_i := c_i M^* - \sum_{j=1}^{i} \left(\delta\, \mathrm{mult}_{p_j}(\varphi_i)\right) E_j^*, \text{ and}$$

$$\Upsilon_{ik} := (c_i - kb_i)M^* - \sum_{j=1}^{k} (c_i - kb_i)E_j^* - \sum_{j=k+1}^{i} \left((\delta-k)\mathrm{mult}_{p_j}(\varphi_i)\right) E_j^*,$$

and where

$$b_i := (\varphi_{F_1}, \varphi_i)_p, 1 \leq i \leq n, \quad \textit{and} \quad c_i := (\varphi_{M_1}, \varphi_i)_p, \delta + 1 \leq i \leq n.$$

Proof. Proposition 1.4.5 shows that it suffices to consider every $(n+1)$-dimensional linear subspace H generated by elements of $S_2(Z)$ and check whether $H^{\perp}$ is generated by an element of $S_2^{\vee}(Z)$. We will see that these generators are those in the statement.

Denote by $\mathrm{lin}(S)$ the linear subspace generated by a set $S \subseteq \mathrm{Pic}_{\mathbb{R}}(Z)$. Then

$$\mathrm{lin}\left(\{[\tilde{F}_1]\} \cup \{[E_i]\}_{i=1}^n\right)^{\perp} = \mathrm{lin}\Big([F^*]\Big), \quad \mathrm{lin}\left(\{[\tilde{M}_0]\} \cup \{[E_i]\}_{i=1}^n\right)^{\perp} = \mathrm{lin}\Big([M^*]\Big),$$

and $[F^*], [M^*] \in S_2^{\vee}(Z)$. Moreover, $\mathrm{lin}\left(\{[\tilde{M}_1]\} \cup \{[E_i]\}_{i=1}^n\right)^{\perp}$ is not generated by an element in $S_2^{\vee}(Z)$.

We have studied subspaces H generated by, at least, all the classes $[E_i]$. Now we are going to treat the cases where a class $[E_i]$, $1 \leq i \leq n$, is not considered. Let us start with the linear space $\mathrm{lin}\Big(\{[\tilde{F}_1], [\tilde{M}_0]\} \cup \{[E_j]\}_{1 \leq j \leq n, j \neq i}\Big)$, set

$$[D_i] = d_{i01}[F^*] + d_{i02}[M^*] + d_{i1}[E_1^*] + \cdots + d_{in}[E_n^*] \in \mathrm{Pic}_{\mathbb{R}}(Z)$$

with arbitrary coefficients and impose the conditions:

$$[D_i] \cdot [\tilde{F}_1] = 0, [D_i] \cdot [\tilde{M}_0] = 0, [D_i] \cdot [E_j] = 0, [D_i] \cdot [\tilde{M}_1] \geq 0 \text{ and } [D_i] \cdot [E_i] \geq 0.$$

Then we obtain the system of equalities and inequalities:

$$d_{i02} + d_{i1} = 0,\ d_{i01} + \delta d_{i02} - \delta d_{i02} = 0,\ -d_{ij} + \sum_{p_s \to p_j} d_{is} = 0,$$
$$d_{i01} + \delta d_{i02} + \sum_{j=1}^{\min\{i, i_{M_1}\}} d_{ij} \geq 0 \text{ and } -d_{ii} + \sum_{p_s \to p_i} d_{is} \geq 0,$$

where i_{M_1} is the index defined in the proof of Lemma 2.4.1. Solving the above system, we obtain $d_{ii} < 0; d_{ij} = \sum_{p_s \to p_j} d_{is}$, $1 \leq j \leq i-1$; $d_{ij} = 0$, $i+1 \leq j \leq n$; $d_{i01} = 0$; and $d_{i02} = -d_{i1}$. This proves that $d_{ij} = -\mathrm{mult}_{p_j}(\varphi_i)$ holds and also

$$\delta \ \mathrm{mult}_{p_1}(\varphi_i) - \sum_{j=1}^{\min\{i, i_{M_1}\}} \mathrm{mult}_{p_j}(\varphi_i) \geq 0$$

by our first inequality, which shows that the classes $\{[\Theta_i]\}_{1 \leq i \leq \delta}$ in the statement give generators of the dual cone $S_2^{\vee}(Z)$.

Reasoning as above for the subspace $\mathrm{lin}\Big(\{[\tilde{F}_1], [\tilde{M}_1]\} \cup \{[E_j]\}_{1 \leq j \leq n, j \neq i}\Big)$ and with the same notation, we get the system of equalities and inequalities:

$$d_{i02} + d_{i1} = 0,\ d_{i01} + \delta d_{i02} + \sum_{j=1}^{\min\{i, i_{M_1}\}} d_{ij} = 0,\ -d_{ij} + \sum_{p_s \to p_j} d_{is} = 0,$$
$$d_{i01} + \delta d_{i02} - \delta d_{i02} \geq 0 \text{ and } -d_{ii} + \sum_{p_s \to p_i} d_{is} \geq 0.$$

Here, the equality $d_{ij} = -\text{mult}_{p_j}(\varphi_i)$ is again true and the first inequality means that

$$\sum_{j=1}^{\min\{i,i_{M_1}\}} \text{mult}_{p_j}(\varphi_i) - \delta\ \text{mult}_{p_1}(\varphi_i) \geq 0$$

must hold. As a consequence, we have proved that the classes $\{[\Delta_i]\}_{\delta+1\leq i\leq n}$ in the statement give extremal rays of $S_2^\vee(Z)$.

Repeating the procedure with $\text{lin}\Big(\{[\tilde{M}_0],[\tilde{M}_1]\}\cup\{[E_j]\}_{1\leq j\leq n, j\neq i}\Big)$, the obtained system is

$$d_{i01}+\delta d_{i02}-\delta d_{i02}=0,\ d_{i01}+\delta d_{i02}+\sum_{j=1}^{\min\{i,i_{M_1}\}} d_{ij}=0,\ -d_{ij}+\sum_{p_s\to p_j} d_{is}=0,$$
$$d_{i02}+d_{i1}\geq 0 \text{ and } -d_{ii}+\sum_{p_s\to p_i} d_{is}\geq 0.$$

This proves, on the one hand, that $d_{ii}<0; d_{ij}=\sum_{p_s\to p_j} d_{is}$, $1\leq j\leq i-1$; $d_{ij}=0$, $i+1\leq j\leq n$; $d_{i01}=0$; and $d_{i02}=(1/\delta)\sum_{j=1}^{\min\{i,i_{M_1}\}}-d_{ij}$. On the other hand, reasoning as above, $d_{ij}=\delta\ \text{mult}_{p_j}(\varphi_i)$ and then

$$\sum_{j=1}^{\min\{i,i_{M_1}\}} \text{mult}_{p_j}(\varphi_i) - \delta\ \text{mult}_{p_1}(\varphi_i) \geq 0,$$

which shows that the set of classes $\{[\Gamma_i]\}_{\delta+1\leq i\leq n}$ gives generators of $S_2^\vee(Z)$.

It only remains to consider those subspaces

$$\text{lin}\Big(\{[\tilde{F}_1],[\tilde{M}_0],[\tilde{M}_1]\}\cup\{[E_j]\}_{j\in\{1,2,\ldots,n\}\setminus\{k,i\}}\Big)$$

attached to pairs of indices k,i, $1\leq k<i\leq n$. Lemma 2.4.1 proves the $(n+1)$-dimensionality of these subspaces. Our computations depend on two indices i and k. So, we write

$$[D_{ik}] = d_{ik01}[F^*]+d_{ik02}[M^*]+d_{ik1}[E_1^*]+d_{ik2}[E_2^*]+\cdots+d_{ikn}[E_n^*].$$

We must impose the following conditions:

$$[D_{ik}]\cdot[\tilde{F}_1]=0, [D_{ik}]\cdot[\tilde{M}_0]=0, [D_{ik}]\cdot[\tilde{M}_1]=0, [D_{ik}]\cdot[E_j]=0, [D_{ik}]\cdot[E_k]\geq 0$$
$$\text{and } [D_{ik}]\cdot[E_i]\geq 0,$$

which give the equivalent system

$$d_{ik02}+d_{ik1}=0,\ d_{ik01}=0,\ d_{ik01}+\delta d_{ik02}+\sum_{j=1}^{\min\{i,i_{M_1}\}} d_{ikj}=0,$$
$$-d_{ikj}+\sum_{p_s\to p_j} d_{iks}=0, -d_{ikk}+\sum_{p_s\to p_k} d_{iks}\geq 0 \text{ and } -d_{iki}+\sum_{p_s\to p_i} d_{iks}\geq 0.$$

To solve it we can assume that the inequalities are strict because, otherwise, we would obtain that $[D_{ik}]$ either vanishes or it gives the class $[\Theta_\delta]$. Indeed, if both inequalities are equalities, then $[D_{ik}] = 0$. Otherwise, taking into account that the first $\delta+1$ points in $\mathcal{C}_\nu$ are free, by considering the third equality and $\delta+1 \leq i_{M_1}$, it holds that one of the indices i or k equals δ. This shows that we obtain $[\Theta_\delta]$ as a solution.

The solutions of the system satisfy that $d_{iki} < 0$; $-d_{ikk} > -\sum_{p_s \to p_k} d_{iks}$; $d_{ikj} = \sum_{p_s \to p_j} d_{iks}$, $1 \leq j \neq k \leq i-1$; $d_{ikj} = 0$, $i+1 \leq j \leq n$; $d_{ik01} = 0$; $d_{ik02} = -d_{ik1}$; and it must hold that

$$-\delta d_{ik1} = -\sum_{j=1}^{\min\{i,i_{M_1}\}} d_{ikj}, \tag{2.11}$$

by the third equation. Note that, for $k+1 \leq j \leq i$, $d_{ikj} = -\mathrm{mult}_{p_j}(\varphi_i)$ up to a positive factor, and also that $-d_{ikk} > -\sum_{p_s \to p_k} d_{iks} \geq 0$.

The indices i and k must satisfy that $1 \leq k \leq \delta-1$ and $\delta+1 \leq i \leq n$. Indeed, with respect to k and reasoning by contradiction, suppose that $k \geq \delta$. By hypothesis, $k < i$, $\delta+1 \leq i_{M_1}$, and $d_{ikj} = d_{ik\delta}$ for $1 \leq j \leq \delta-1$, because the first $\delta+1$ points in $\mathcal{C}_\nu$ are free, then

$$-\sum_{j=1}^{\min\{i,i_{M_1}\}} d_{ikj} = -\delta d_{ik1} - \sum_{j=\delta+1}^{\min\{i,i_{M_1}\}} d_{ikj},$$

where $-\sum_{j=\delta+1}^{\min\{i,i_{M_1}\}} d_{ikj} > 0$, which does not hold by (2.11). Notice that this equality is true by our imposed equalities. With respect to the index i, again reasoning by contradiction, suppose that $i \leq \delta$. As $1 \leq k \leq \delta-1$, Equality (2.11) is equivalent to

$$-(\delta-k)d_{ikk} = -\sum_{j=k+1}^{\min\{i,i_{M_1}\}} d_{ikj}, \tag{2.12}$$

because $d_{ikj} = d_{ikk}$ for $1 \leq j \leq k$. This implies that $-(\delta-k)d_{ikk} = -(i-k)d_{ikk+1}$, which is a contradiction since $-d_{ikk} > -d_{ikk+1}$.

Notice that (2.11) also gives us the value of d_{ikk}, which can be obtained from the following chain of equalities:

$$d_{ikk} = \delta d_{ik1} - \sum_{j=1,\, j\neq k}^{\min\{i,i_{M_1}\}} d_{ikj} = \delta d_{ikk} - (k-1)d_{ikk} - \sum_{j=k+1}^{\min\{i,i_{M_1}\}} d_{ikj}.$$

Thus, if we take $d_{ikj} = -(\delta-k)\mathrm{mult}_{p_j}(\varphi_i)$, $k+1 \leq j \leq i$, one gets that

$$d_{ik1} = \cdots = d_{ikk} = \frac{-(\delta-k)\sum_{j=k+1}^{\min\{i,i_{M_1}\}} \mathrm{mult}_{p_j}(\varphi_i)}{(\delta-k)} = -\sum_{j=k+1}^{\min\{i,i_{M_1}\}} \mathrm{mult}_{p_j}(\varphi_i),$$

and the coefficient of $[M^*]$ is $d_{ik02} = -d_{ik1}$.

As a result, we have that $[D_{ik}] = [\Upsilon_{ik}]$, where

$$\Upsilon_{ik} := \left(\sum_{j=k+1}^{\min\{i, i_{M_1}\}} \mathrm{mult}_{p_j}(\varphi_i) \right) M^* - \sum_{j=1}^{k} \left(\sum_{s=k+1}^{\min\{i, i_{M_1}\}} \mathrm{mult}_{p_s}(\varphi_i) \right) E_j^* - \sum_{j=k+1}^{i} \left((\delta - k) \mathrm{mult}_{p_j}(\varphi_i) \right) E_j^*,$$

where $\delta + 1 \leq i \leq n$ and $1 \leq k \leq \delta - 1$. This finishes the proof. □

Remark 2.4.3. From the above proof, it can be deduced that, when considering the surface $\mathbb{F}_1$ and a non-special valuation ν, no class $[\Upsilon_{ik}]$ appears as a generator of $S_2^\vee(Z)$.

Remark 2.4.4. The divisors $\{\Theta_i\}_{i=1}^{\delta}$, $\{\Delta_i\}_{i=\delta+1}^{n}$, $\{\Gamma_i\}_{i=\delta+1}^{n}$ and $\{\Upsilon_{ik}\}_{i=\delta+1,k=1,\ldots,\delta-1}^{n}$ defined in Proposition 2.4.2 can be written as

$$\Theta_i = \Theta(\nu_i) := \nu_i(\varphi_{F_1}) M^* - \sum_{j=1}^{i} \nu_i(\mathfrak{m}_j) E_j^*,$$

$$\Delta_i = \Delta(\nu_i) := (-\delta \nu_i(\varphi_{F_1}) + \nu_i(\varphi_{M_1})) F^* + \nu_i(\varphi_{F_1}) M^* - \sum_{j=1}^{i} \nu_i(\mathfrak{m}_j) E_j^*,$$

$$\Gamma_i = \Gamma(\nu_i) := \nu_i(\varphi_{M_1}) M^* - \sum_{j=1}^{i} (\delta\, \nu_i(\mathfrak{m}_j)) E_j^*, \text{ and}$$

$$\Upsilon_{ik} = \Upsilon_k(\nu_i) := (\nu_i(\varphi_{M_1}) - k\nu_i(\varphi_{F_1})) M^* - \sum_{j=1}^{k} (\nu_i(\varphi_{M_1}) - k\nu_i(\varphi_{F_1})) E_j^* - \sum_{j=k+1}^{i} ((\delta - k)\nu_i(\mathfrak{m}_j)) E_j^*,$$

where ν_i is the non-special divisorial valuation defined by the exceptional divisor E_i and $\{\nu_i(\mathfrak{m}_j)\}_{j=1}^{i}$ its sequence of values. Moreover, their self-intersections satisfy

$$\begin{aligned}
\Theta_i^2 &= \delta b_i^2 - \overline{\beta}_{g+1}(\nu_i) = \delta \nu_i(\varphi_{F_1})^2 - \overline{\beta}_{g+1}(\nu_i), \\
\Delta_i^2 &= 2(-\delta b_i + c_i) b_i + \delta b_i^2 - \overline{\beta}_{g+1}(\nu_i) \\
&= 2 b_i c_i - \delta b_i^2 - \overline{\beta}_{g+1}(\nu_i) \\
&= 2\nu_i(\varphi_{F_1})\nu_i(\varphi_{M_1}) - \delta \nu_i(\varphi_{F_1})^2 - \overline{\beta}_{g+1}(\nu_i), \\
\Gamma_i^2 &= \delta c_i^2 - \delta^2 \overline{\beta}_{g+1}(\nu_i) = \delta \nu_i(\varphi_{M_1})^2 - \delta^2 \overline{\beta}_{g+1}(\nu_i), \\
\Upsilon_{ik}^2 &= \delta (c_i - k b_i)^2 - k (c_i - k b_i)^2 - (\delta - k)^2 \sum_{j=k+1}^{n} \mathrm{mult}_{p_j}^2(\varphi_i)) \\
&= (\delta - k)[(c_i - k b_i)^2 - (\delta - k)(\overline{\beta}_{g+1}(\nu_i) - k b_i^2)] \\
&= (\delta - k)[c_i^2 - 2k c_i b_i + \delta k b_i^2 - (\delta - k)\overline{\beta}_{g+1}(\nu_i)]
\end{aligned}$$

$$= (\delta - k)[c_i^2 - k(2c_ib_i - \delta b_i^2) - (\delta - k)\overline{\beta}_{g+1}(\nu_i)]$$
$$= (\delta - k)[\nu_i(\varphi_{M_1})^2 - k(2\nu_i(\varphi_{M_1})\nu_i(\varphi_{F_1}) - \delta\nu_i(\varphi_{F_1})^2) - (\delta - k)\overline{\beta}_{g+1}(\nu_i)],$$

where $\overline{\beta}_{g+1}(\nu_i)$ is the last maximal contact value of ν_i. Note that these divisors have a close behaviour to the divisors $\{\Lambda_i\}_{i=1}^n$ defined in Proposition 2.3.1,

The before mentioned divisors introduced in Proposition 2.4.2 are essential to characterize the cone of curves $\mathrm{NE}(Z)$ of a surface Z defined by a non-special divisorial valuation of a Hirzebruch surface. Some of their properties are stated in the forthcoming lemma.

Lemma 2.4.5. *Let Z (respectively, ν_n) be a rational surface (respectively, a valuation) as in Proposition 2.4.2. Consider the set of divisors there defined. Then $\Delta_{\delta+1}^2 > 0$, $\Gamma_{\delta+1}^2 > 0$ and $\Upsilon_{\delta+1k}^2 > 0$ for all $k \in \{1, 2, \ldots, \delta - 1\}$. In addition, for any index $i \in \{\delta + 2, \delta + 3, \ldots, n\}$ such that $\Delta_i^2 \geq 0$ (respectively, $\Gamma_i^2 \geq 0$, $\Upsilon_{ik}^2 \geq 0$), the following properties are satisfied:*

(a) If p_i is a satellite point of the configuration $\mathcal{C}_{\nu_n}$ that ν_n defines, it holds $\Delta_i^2 > 0$ (respectively, $\Gamma_i^2 > 0$, $\Upsilon_{ik}^2 > 0$).

(b) $\Delta_{i-1}^2 \geq 0$ (respectively, $\Gamma_{i-1}^2 \geq 0, \Upsilon_{i-1k}^2 \geq 0$) and, moreover, if $\Delta_{i-1}^2 = 0$ (respectively, $\Gamma_{i-1}^2 = 0$, $\Upsilon_{i-1k}^2 = 0$) then p_i is a satellite point and p_{i-1} is free.

Proof. To prove our first assertion, it suffices to notice that the following three equalities hold:

$$\Delta_{\delta+1}^2 = 2 + \delta - (\delta + 1) = 1 > 0,$$
$$\Gamma_{\delta+1}^2 = \delta(\delta + 1)^2 - \delta^2(\delta + 1) = \delta(\delta + 1)(\delta + 1 - \delta) = \delta(\delta + 1) > 0,$$
$$\begin{aligned}\Upsilon_{\delta+1k}^2 &= \delta(\delta + 1 - k)^2 - k(\delta + 1 - k)^2 - (\delta + 1 - k)(\delta - k)^2\\ &= (\delta - k)(\delta + 1 - k)^2 - (\delta - k)^2(\delta + 1 - k)\\ &= (\delta - k)(\delta + 1 - k)[\delta - k + 1 - (\delta - k)]\\ &= (\delta - k)(\delta + 1 - k) > 0.\end{aligned}$$

Finally we remark that (a) and (b) can be proved reasoning as in the proof of Lemma 2.3.6. Indeed, recalling that $g + 2$ is the cardinality of the set of maximal contact values of ν_n, the case $g = 1$ follows as in that proof, and, when $g > 1$, with notations as in that lemma and in Proposition 2.4.2, it suffices to consider the following equalities and to reason again as we did in the mentioned Lemma 2.3.6.

$$\Delta_n^2 = 2b_nc_n - \delta b_n^2 - \overline{\beta}_{g+1}(\nu_n) = e_{g-1}(\nu_n)\left[\frac{2b_nc_n - \delta b_n^2}{e_{g-1}(\nu_n)} - \overline{\beta}_g(\nu_n)\right],$$
$$\Gamma_n^2 = c_n^2\delta - \delta^2\overline{\beta}_{g+1}(\nu_n), \quad \text{and}$$
$$\Upsilon_{nk}^2 = (\delta - k)[c_n^2 - k(2c_nb_n - \delta b_n^2) - (\delta - k)\overline{\beta}_{g+1}(\nu_n)].$$

□

Remark 2.4.6. The non-negativity of the self-intersection of the divisors introduced in Proposition 2.4.2 can be numerically determined. By Remark 2.4.4, it holds that $\Theta_i^2 = \delta - i \geq 0$, $1 \leq i \leq \delta$, since the points $p_1, p_2 \ldots, p_\delta$ are free. For the remaining divisors, these conditions follow from Remark 2.4.4 and Lemma 2.4.5.

Indeed, the inequality $\Delta_n^2 \geq 0$ (or, equivalently, $2b_n c_n - \delta b_n^2 \geq \overline{\beta}_{g+1}(\nu_n)$) implies that $\Delta_i^2 \geq 0$, that is, $2b_i c_i - \delta b_i^2 \geq \overline{\beta}_{g+1}(\nu_i)$ for $1 \leq i \leq n$, where ν_i is the divisorial valuation defined as in Remark 2.4.4. Likewise, when $\Gamma_n^2 \geq 0$, it holds that $\Gamma_i^2 \geq 0$, or equivalently that $c_i^2 \geq \delta\overline{\beta}_{g+1}(\nu_i)$, for $1 \leq i \leq n$.

To finish, given a positive integer k, $1 \leq k \leq \delta - 1$, the inequality

$$c_n^2 - k(2c_n b_n - \delta b_n^2) \geq (\delta - k)\overline{\beta}_{g+1}(\nu_n),$$

allows us to conclude that $\Upsilon_{ik}^2 \geq 0$, for $1 \leq i \leq n$, which numerically can be expressed as $c_i^2 - k(2c_i b_i - \delta b_i^2) \geq (\delta - k)\overline{\beta}_{g+1}(\nu_i)$.

The next lemma will be useful for proving the main result in this section.

Lemma 2.4.7. *Let ν_n be a non-special divisorial valuation of a Hirzebruch surface and Z the surface that it defines. Consider the divisors Δ_i, Γ_i and Υ_{ik}, $\delta + 1 \leq i \leq n$; $1 \leq k \leq \delta - 1$, given in Proposition 2.4.2. Then, for each index i, $\Delta_i^2 \geq 0$ implies $\Gamma_i^2 \geq 0$ and $\Upsilon_{ik}^2 \geq 0$ for all $k \in \{1, 2, \ldots, \delta - 1\}$.*

Proof. Our proof is consequence of the following two properties:

Property 1: If the self-intersections of the divisors Δ_i and $\Upsilon_{i\delta-1}$ are non-negative, then the same property holds for the divisors Γ_i and Υ_{ik}, $1 \leq k \leq \delta - 1$.

Property 2: If the self-intersection of the divisor Δ_i is non-negative, so is the self-intersection of $\Upsilon_{i\delta-1}$.

For proving Property 1, our hypotheses are, by Remark 2.4.6,

$$\overline{\beta}_{g+1}(\nu_i) \leq 2c_i b_i - \delta b_i^2 \quad \text{and} \tag{2.13}$$

$$(\delta - 1)(2c_i b_i - \delta b_i^2 - \overline{\beta}_{g+1}(\nu_i)) \leq c_i^2 - \delta\overline{\beta}_{g+1}(\nu_i). \tag{2.14}$$

The inequality in (2.14) and the following one

$$\overline{\beta}_{g+1}(\nu_i) \leq c_i^2 - (\delta - 1)(2c_i b_i - \delta b_i^2)$$

are equivalent. From the last inequality and the one in (2.13), we get that $c_i^2 \geq \delta\overline{\beta}_{g+1}(\nu_i)$ and then $\Gamma_i^2 \geq 0$. Finally, $\Upsilon_{ik}^2 \geq 0$, $1 \leq k \leq \delta - 1$, if and only if the inequality

$$k(2c_i b_i - \delta b_i^2 - \overline{\beta}_{g+1}(\nu_i)) \leq c_i^2 - \delta\overline{\beta}_{g+1}(\nu_i)$$

holds, fact that follows straightforwardly from (2.13) and (2.14).

To conclude we prove Property 2. It suffices to check that the following inequalities

$$\overline{\beta}_{g+1}(\nu_i) \leq 2c_i b_i - \delta b_i^2 < c_i^2 - (\delta - 1)(2c_i b_i - \delta b_i^2) \tag{2.15}$$

are true. In fact, the first inequality comes from our hypothesis $\Delta_i^2 \geq 0$ and the inequality given by the first and the last sides in (2.15) allows us to show $\Upsilon_{i\delta-1}^2 \geq 0$. To prove the second inequality in (2.15), set $c_i = x$ and $b_i = b$ for simplicity. We are considering non-special valuations, which means that $x > \delta b$. In our new notation we want to prove that

$$2bx - \delta b^2 < x^2 - (\delta - 1)(2bx - \delta b^2).$$

This inequality is equivalent to

$$0 < x^2 - (2b\delta)x + \delta^2 b^2,$$

and it holds for all $x \neq \delta b$ since the point $(\delta b, 0)$ is the vertex of the parabola given by the right-hand side of the inequality. □

In the same way that Theorem 2.3.7 gives equivalent conditions for the non-positivity of special divisorial valuations, the forthcoming theorem provides two interesting geometrical conditions characterizing the non-positivity at infinity of non-special divisorial valuations. It also includes a numerical and local expression that can be easily checked.

Theorem 2.4.8. *Let ν_n be a non-special divisorial valuation of the function field of $\mathbb{F}_\delta$ centered at $\mathcal{O}_{\mathbb{F}_\delta,p}$ and $\mathcal{C}_{\nu_n} = \{p_i\}_{i=1}^n$ its configuration of infinitely near points. Let Z be the surface that ν_n defines and consider the divisor Δ_n on Z defined in Proposition 2.4.2. Then the following conditions are equivalent:*

(a) The valuation ν_n is non-positive at infinity.

(b) The divisor Δ_n is nef.

(c) It holds the following inequality: $2c_n b_n - \delta b_n^2 \geq \overline{\beta}_{g+1}(\nu_n)$.

(d) The cone of curves of Z is generated by $[\tilde{F}_1], [\tilde{M}_0], [\tilde{M}_1], [E_1], [E_2], \ldots, [E_n]$.

Proof. Our proof uses a close reasoning to that of Theorem 2.3.7. Keeping the notation as in that theorem, we are going to give a sketch of the proof emphasizing only the main differences.

To prove that (a) can be deduced from (b), we can suppose that p is a general point of $\mathbb{F}_\delta$ with coordinates $(0 : 1; 0, 1)$. Consider local coordinates $\{x, y\} = \left\{\frac{X_1}{X_0}, \frac{X_0^\delta Y_1}{Y_0}\right\}$ in the affine open set U_{00} and $\{u, v\} = \left\{\frac{X_0}{X_1}, \frac{Y_0}{X_1^\delta Y_1}\right\}$ in U_{11}. Notice that, with our notation, F_1 and M_1 are defined by the equations $X_0 = 0$ and $Y_0 = 0$, $p \in U_{11}$, and F_1 and M_1 have local equations $u = 0$ and $v = 0$, respectively.

If now $\mathcal{S}$ denotes the set of non-constant polynomials in $\mathcal{O}_{\mathbb{F}_\delta}(U_{00})$ (up to multiplication by a nonzero element of k) such that neither x nor y divide them, $f \in \mathcal{S}$ satisfies

$$f(x, y) = f(1/u, u^\delta/v) = \frac{h_f(u, v)}{u^{\deg_1(h_f)} v^{\deg_2(h_f)}}, \tag{2.16}$$

where $h_f(u,v) \in \mathcal{O}_{\mathbb{F}_\delta}(U_{11})$.

The bi-homogeneous polynomial $X_1^{\deg_1(h_f)+\delta\deg_2(h_f)}Y_1^{\deg_2(h_f)}h_f(\frac{X_0}{X_1}, \frac{Y_0}{X_1^\delta Y_1})$ defines a curve C_f on $\mathbb{F}_\delta$ of degree $(\deg_1(h_f), \deg_2(h_f))$ and $f \mapsto C_f$ is a one-to-one correspondence between $\mathcal{S}$ and the set of curves on $\mathbb{F}_\delta$ containing no curve F_1, F', M_0, M_1 as a component, where F' and M_0 are defined by the equations $X_0 = 0$ and $Y_1 = 0$. Then $\Delta_n \cdot C_f = -\nu(f)$ and by (b), $-\nu(f) \geq 0$. The case when $f \in \mathcal{O}_{\mathbb{F}_\delta}(U_{00})$ and x or y or both are factors of f follows as in Theorem 2.3.7 and (a) is proved.

A proof of the fact that (a) implies (b), (b) implies (c) and (d) implies (b) can be done as in Theorem 2.3.7.

To see that (c) implies (d), it suffices to notice that, by Lemmas 2.4.5 and 2.4.7,

$$S_2^\vee(Z) \subseteq \{[D] \in \mathrm{Pic}_{\mathbb{R}}(Z) \mid [D]^2 \geq 0 \text{ and } [H]\cdot[D] \geq 0\} =: A(Z),$$

where $S_2^\vee(Z)$ is the dual cone defined in Proposition 2.4.2 and H an ample divisor on Z. Finally, the fact

$$S_2^\vee(Z) \subseteq A(Z) \subseteq (S_2^\vee(Z))^\vee = S_2(Z)$$

and a reasoning as in Theorem 2.3.7 completes our proof. □

Remark 2.4.9. The extremal rays of the nef cone $\mathrm{Nef}(Z)$ of Z are the elements of the set $\{[F^*],[M^*]\} \cup \{[\Theta_i]\}_{i=1}^{\delta} \cup \{[\Delta_i]\}_{i=\delta+1}^{n} \cup \{[\Gamma_i]\}_{i=\delta+1}^{n} \cup \{[\Upsilon_{i\ell}]\}_{\delta+1\leq i\leq n}^{1\leq k\leq \delta-1}$, by Proposition 2.4.2 and Theorem 2.4.8.

As in the special case, one has the following two corollaries as a consequence of the above theorem.

Corollary 2.4.10. *Let ν_n be a non-positive at infinity non-special divisorial valuation of $\mathbb{F}_\delta$. Consider the divisorial valuations ν_i defined by the divisors E_i associated to the simple sequence of point blowups that ν_n defines. Then the valuations ν_i, $\delta+1 \leq i \leq n-1$, are non-positive at infinity (non-special of $\mathbb{F}_\delta$).*

Proof. This result follows from the same reasoning that Corollary 2.3.10 using Theorem 2.4.8, Lemmas 2.4.5 and 2.4.7 and Remark 2.4.6. □

Corollary 2.4.11. *Let Z be a surface as in Theorem 2.4.8 defined by a non-positive at infinity non-special valuation. Then all the divisors Θ_i, $i = 1, 2, \ldots, \delta$; Δ_i, Γ_i and Υ_{ik}, $i = \delta+1, \delta+2, \ldots, n$ and $k = 1, 2, \ldots, \delta-1$, defined in Proposition 2.4.2, are effective. In particular, it holds that*

$$\Theta_i = (\Theta_i \cdot F^*)\tilde{M}_1 + \sum_{j=1}^{n}(\Theta_i \cdot \Delta_j)E_j,$$

$$\Delta_i = (\Delta_i \cdot M_0^*)\tilde{F}_1 + (\Delta_i \cdot F^*)\tilde{M}_1 + \sum_{j=1}^{n}(\Delta_i \cdot \Delta_j)E_j,$$

$$\Gamma_i = (\Gamma_i \cdot F^*)\tilde{M}_1 + \sum_{j=1}^{n}(\Gamma_i \cdot \Delta_j)E_j,$$

$$\Upsilon_{ik} = (\Upsilon_{ik} \cdot F^*)\tilde{M}_1 + \sum_{j=1}^{n} (\Upsilon_{ik} \cdot \Delta_j) E_j.$$

Proof. Notice that the set $\{F^*, M^*\} \cup \{\Delta_i\}_{i=1}^n$ is the dual basis of the basis $\{\tilde{F}_1, \tilde{M}_1\} \cup \{E_i\}_{i=1}^n$ of $\mathrm{Pic}_{\mathbb{R}}(Z)$ by Proposition 2.4.2, and each divisor of the first set is nef and has non-negative self-intersection by Corollary 2.4.10 and then its class belongs to $A(Z) \subset \overline{\mathrm{NE}}(Z)$. Finally, arguing as in Corollary 2.3.11, the result is proved. □

As in the non-positive at infinity special case, we can set a sequence of non-positive at infinity non-special divisorial valuations of $\mathbb{F}_\delta$ which approaches a non-positive at infinity non-special irrational valuation. The proof follows from the arguments used to prove Corollary 2.3.12.

Corollary 2.4.12. *Let ν_n be a non-positive at infinity non-special divisorial valuation of $\mathbb{F}_\delta$ and $\mathcal{C}_{\nu_n} = \{p_i\}_{i=1}^n$ its configuration of infinitely near points. Set a non-special divisorial valuation ν_m such that its configuration $\mathcal{C}_{\nu_m} = \{q_i\}_{i=1}^m, n < m$, satisfies that $\mathcal{C}_{\nu_n} \subset \mathcal{C}_{\nu_m}$ and the points $q_i, n+1 \leq i \leq m$, are satellite. Then, the valuation ν_m is non-positive at infinity.*

Example 2.4.13. Let ν be a non-special divisorial valuation of the Hirzebruch surface $\mathbb{F}_2$ whose sequence of maximal contact values is $\{15, 51, 262, 786\}$. Set $\mathcal{C}_\nu = \{p_i\}_{i=1}^{12}$ the configuration of infinitely near points of ν and Γ_n (Figure 2.2) its dual graph. Denote by F_1 the fiber of $\mathbb{F}_2$ that goes through p_1, by M_0 the special section

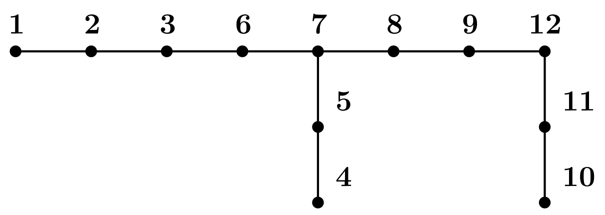


Figure 2.2: Dual graph of ν in Example 2.4.13.

and by M_1 the integral curve that is linearly equivalent to M whose its strict transform pass through p_1, p_2 and p_3. Then $b_{12} = 15, c_{12} = 45$ and $\overline{\beta}_{g+1}(\nu) = 786$, and so, Theorem 2.4.8 (c) is satisfied. Therefore, the cone of curves of the surface Z defined by ν is generated by $\{[\tilde{F}_1], [\tilde{M}_0], [\tilde{M}_1]\} \cup \{E_i\}_{i=1}^{12}$ and the divisors Δ_i, $1 \leq i \leq 12$, defined in Proposition 2.4.2 are nef.

We conclude the section stating a result that provides two equivalent conditions characterizing the negativity at infinity of non-special divisorial valuations. It can be proved as we did in Theorem 2.3.14.

Theorem 2.4.14. *Keeping the same assumptions and notations as in Theorem 2.4.8, the following conditions are equivalent:*

(a) The valuation ν_n is negative at infinity.

(b) It holds that either $2c_nb_n - b_n^2\delta > \overline{\beta}_{g+1}(\nu_n)$, or $2c_nb_n - b_n^2\delta = \overline{\beta}_{g+1}(\nu_n)$ and the Iitaka dimension of the divisor Δ_n vanishes.

(c) The inequality $\Delta_n \cdot \tilde{C} > 0$ is satisfied for the strict transform on Z, $\tilde{C}$, of any curve C on $\mathbb{F}_\delta$, $C \neq F_1, M_1$.

2.5 Discrete equivalence of NPI valuations

In this section we study the dual graphs of NPI divisorial and irrational valuations of the projective plane and the Hirzebruch surfaces. Moreover, we will provide an algorithm to compute explicitly those dual graphs which admit NPI valuations. We keep the notation of the above sections. As at the beginning of the chapter, Z_0 denotes either the projective plane $\mathbb{P}^2$ or a Hirzebruch surface $\mathbb{F}_\delta, \delta \geq 0$.

As we have seen in Subsection 1.3.1, the sequence of Puiseux exponents of a plane valuation and its dual graph are very close data. The mentioned sequence allows us to work easily with the dual graph. For this reason we introduce the following definition.

Definition 2.5.1. Let ν be a divisorial or irrational valuation of Z_0. Denote by $\{\beta'_j(\nu)\}_{j=0}^{g+1}$ the sequence of Puiseux exponents of ν. We call *discrete class of* ν to the tuple

$$\mathbf{d}(\nu) = (g, \beta'_0(\nu), \beta'_1(\nu), \ldots, \beta'_{g+1}(\nu)).$$

Two valuations as before which have the same discrete class are named *discretely equivalent*.

The next result is a consequence of the definition of sequence of Puiseux exponents of a plane valuation.

Proposition 2.5.2. *Let ν, ν' be two divisorial or irrational valuations. Then the dual graphs of ν and ν' coincide if and only if ν and ν' have the same discrete class.*

Before stating a characterization of the discrete classes of NPI valuations, we present a lemma explaining how to get the last contact value of a plane valuation from its sequence of Puiseux exponents.

Lemma 2.5.3. *Let ν be a divisorial or irrational valuation of Z_0 and $\mathbf{d}(\nu) = (g, \beta'_0(\nu), \beta'_1(\nu), \ldots, \beta'_{g+1}(\nu))$ its discrete class. Then the last maximal contact value of ν satisfies*

$$\overline{\beta}_{g+1}(\nu) = \sum_{j=0}^{g} e_j(\nu)^2(\beta'_{j+1}(\nu) - 1) + e_0(\nu)^2\beta'_0(\nu).$$

Proof. By Equality (1.5) and since $e_g(\nu) = \gcd(\overline{\beta}_0(\nu), \overline{\beta}_1(\nu), \ldots, \overline{\beta}_g(\nu)) = 1$, it holds that

$$\overline{\beta}_{g+1}(\nu) = e_g(\nu)^2(\beta'_{g+1}(\nu) - 1) + e_{g-1}(\nu)\overline{\beta}_g(\nu).$$

Now, again by Condition (1.5), one has

$$e_j(\nu)\overline{\beta}_{j+1}(\nu) = e_j(\nu)^2(\beta'_{j+1}(\nu) - 1) + e_{j-1}(\nu)\overline{\beta}_j(\nu),$$

for $1 \leq j \leq g-1$. This concludes the proof after noticing that $e_0(\nu) = \overline{\beta}_0(\nu)$. □

Recall that the coordinates of a discrete class $\mathbf{d} = (g, \beta'_0, \beta'_1, \ldots, \beta'_{g+1})$ of a divisorial valuation satisfy that g and β'_{g+1} are non-negative integers, $\beta'_0 = 1$ and (when $g > 0$) $\beta'_j = q_j/n_j \in \mathbb{Q}_{>0} \setminus \mathbb{Z}$, where $\gcd(q_j, n_j) = 1$. Moreover, for each $\mathbf{d}$ we define $e_g = 1$, $e_j = \prod_{k=j+1}^{g} n_k$, $0 \leq j < g$, and $\tau_j = e_j/e_0$, for $0 \leq j \leq g$. When we consider discrete classes of irrational valuations, their coordinates are defined similarly but $\beta'_{g+1} \in \mathbb{R}_{>0} \setminus \mathbb{Q}$. We gave more information about these values in Subsection 1.3.1.

In what follows, we denote by $\mathbb{D}$ the set of tuples $\mathbf{d}$ as before.

Theorem 2.5.4. *Set a class $\mathbf{d} \in \mathbb{D}$. Then*

(a) There exists an NPI divisorial or irrational valuation of $\mathbb{P}^2$ with discrete class $\mathbf{d}$ if, and only if, the following inequality holds:

$$\beta_1'^2 \geq \sum_{j=0}^{g} \tau_j^2(\beta'_{j+1} - 1) + \beta'_0.$$

(b) There exists an NPI special divisorial or irrational valuation of $\mathbb{F}_\delta$ with discrete class $\mathbf{d}$ if, and only if, δ is a non-negative integer and the following inequality

$$\beta'_1(\delta\beta'_1 + 2) \geq \sum_{j=0}^{g} \tau_j^2(\beta'_{j+1} - 1) + \beta'_0$$

holds.

(c) There exists an NPI non-special divisorial or irrational valuation of $\mathbb{F}_\delta$ with discrete class $\mathbf{d}$ if, and only if, it holds that

$$2\beta'_1 - \delta \geq \sum_{j=0}^{g} \tau_j^2(\beta'_{j+1} - 1) + \beta'_0.$$

Proof. We only need to show the result for the set of discrete classes of divisorial valuations because the irrational case follows from the divisorial one and Theorem 1.3.2(c).

Let $\mathbf{d} = (g, \beta'_0, \beta'_1, \ldots, \beta'_{g+1})$ be a discrete class where β'_{g+1} is a positive integer. Set $\overline{\beta}_0, \overline{\beta}_1$ and $\overline{\beta}_{g+1}$ the values which can be computed from the components of $\mathbf{d}$ following the formulas given in Equality (1.5) and Lemma 2.5.3.

We first prove (a). By Lemma 2.5.3 and the equality $\beta'_1 = \overline{\beta}_1/\overline{\beta}_0$, it suffices to show that $\mathbf{d}$ is the discrete class of an NPI divisorial valuation of $\mathbb{P}^2$ if and only if $\overline{\beta}_1^2 \geq \overline{\beta}_{g+1}$. Assume first that ν_n is an NPI divisorial valuation of $\mathbb{P}^2$ whose discrete class is $\mathbf{d}$. Following Theorem 2.1.2, one has

$$(\varphi_L, \varphi_n)_p^2 = \nu_n(\varphi_L)^2 \geq \overline{\beta}_{g+1}.$$

Now, if the strict transform of L goes through all initial free points in $\mathcal{C}_{\nu_n}$, then the equality $\nu_n(\varphi_L) = \overline{\beta}_1$ holds. Otherwise, $\nu_n(\varphi_L) = s_L\overline{\beta}_0$, where $1 \leq s_L \leq \lfloor \overline{\beta}_1/\overline{\beta}_0 \rfloor$. As a result one gets that $\overline{\beta}_1^2 \geq \overline{\beta}_{g+1}$ and this implication is proved. Conversely, if we set a discrete class $\mathbf{d} \in \mathbb{D}$ (where $\beta'_{g+1} \in \mathbb{Z}_{>0}$) such that $\overline{\beta}_1^2 \geq \overline{\beta}_{g+1}$, it suffices to take a divisorial valuation ν of $\mathbb{P}^2$ whose discrete class is $\mathbf{d}$ and whose first free points in $\mathcal{C}_\nu$ are determined the projective line L. Consequently the equality $\nu(\varphi_L) = \overline{\beta}_1$ is satisfied and the proof is over by Theorem 2.1.2.

As in the proof of (a), to show the equivalence claimed in (b) we are going to use the fact that the inequality given there is equivalent to the following one:

$$2\overline{\beta}_1\overline{\beta}_0 + \delta\overline{\beta}_1^2 \geq \overline{\beta}_{g+1}.$$

Now suppose that $\mathbf{d}$ is the discrete class of a special divisorial valuation ν_n of $\mathbb{F}_\delta$. By Theorem 2.3.7, the inequality

$$2(\varphi_{F_1}, \varphi_n)_p(\varphi_{M_0}, \varphi_n)_p + \delta(\varphi_{F_1}, \varphi_n)_p^2 = 2\nu_n(\varphi_{F_1})\nu_n(\varphi_{M_0}) + \delta\nu_n(\varphi_{F_1})^2 \geq \overline{\beta}_{g+1} \quad (2.17)$$

holds. Now $\nu_n(\varphi_{F_1})$ is equal either to $\overline{\beta}_1$ if $\tilde{F}_1$ goes through all initial free points in $\mathcal{C}_{\nu_n}$ or to $s_{F_1}\overline{\beta}_0$, where $1 \leq s_{F_1} \leq \lfloor \overline{\beta}_1/\overline{\beta}_0 \rfloor$, otherwise. The section M_0 has a behaviour like F_1 and then $\nu_n(\varphi_{M_0})$ equals either $\overline{\beta}_1$ or $s_{M_0}\overline{\beta}_0$, where $0 \leq s_{M_0} \leq \lfloor \overline{\beta}_1/\overline{\beta}_0 \rfloor$. This proves that $2\overline{\beta}_1\overline{\beta}_0 + \delta\overline{\beta}_1^2 \geq \overline{\beta}_{g+1}$ since F_1 and M_0 pass both through p but at most one of their strict transforms goes through p_2. Conversely, set a class $\mathbf{d} \in \mathbb{D}$ (where $\beta'_{g+1} \in \mathbb{Z}_{>0}$) such that the inequality $2\overline{\beta}_1\overline{\beta}_0 + \delta\overline{\beta}_1^2 \geq \overline{\beta}_{g+1}$ is satisfied, then taking the special divisorial valuation ν of $\mathbb{F}_\delta$ with discrete class $\mathbf{d}$ whose first free points in $\mathcal{C}_\nu$ coincide with those through which $\tilde{F}_1$ goes, by Theorem 2.3.7, one obtains an NPI special divisorial valuation of $\mathbb{F}_\delta$ with discrete class $\mathbf{d}$.

Finally arguing as before one can give a proof of (c) which is supported in the next two facts. First, by Theorem 2.4.8 the inequality in the statement is equivalent to the following one

$$2(\varphi_{F_1}, \varphi_n)_p(\varphi_{M_1}, \varphi_n)_p - \delta(\varphi_{F_1}, \varphi_n)_p^2 = 2\nu_n(\varphi_{M_1})\nu_n(\varphi_{F_1}) - \delta\nu_n(\varphi_{F_1})^2 \geq \overline{\beta}_{g+1}.$$

Second, the fiber F_1 and the section M_1 satisfy that $\nu_n(\varphi_{F_1}) = \overline{\beta}_0$ and $\nu_n(\varphi_{M_1})$ equals either $\overline{\beta}_1$ or $s_{M_1}\overline{\beta}_0$, where $\delta + 1 \leq s_{M_1} \leq \lfloor \overline{\beta}_1/\overline{\beta}_0 \rfloor$. This completes the proof. □

The following result explains the relation among the dual graphs of the different types of NPI valuations using their discrete classes and the above theorem. Before stating the result, denote by $\mathbf{D}_{\mathbb{P}^2}$ (respectively, $\mathbf{D}^1_{\mathbb{F}_\delta}, \mathbf{D}^2_{\mathbb{F}_\delta}$) the set of discrete classes in $\mathbb{D}$ which represent to some (respectively, special, non-special) NPI divisorial or irrational valuation of $\mathbb{P}^2$ (respectively, $\mathbb{F}_\delta$).

Theorem 2.5.5. *Let $\mathbf{D}_{\mathbb{P}^2}, \mathbf{D}^1_{\mathbb{F}_\delta}, \mathbf{D}^2_{\mathbb{F}_\delta}$ be the sets introduced before. Write $\mathbf{D}^{\leq 2}_{\mathbb{P}^2}$ (respectively, $\mathbf{D}^{\geq 2}_{\mathbb{P}^2}$) the set of discrete classes in $\mathbf{D}_{\mathbb{P}^2}$ whose coordinate β'_1 satisfies $\beta'_1 \leq 2$ (respectively, $\beta'_1 \geq 2$). Set similarly $\mathbf{D}^{1,\leq 2}_{\mathbb{F}_0}$ and $\mathbf{D}^{1,\geq 2}_{\mathbb{F}_0}$. Then*

(a) $\mathbf{D}_{\mathbb{P}^2} \subseteq \mathbf{D}^1_{\mathbb{F}_\delta}$ *for all* $\delta > 0$.

(b) $\mathbf{D}^{\leq 2}_{\mathbb{P}^2} \subseteq \mathbf{D}^{1,\leq 2}_{\mathbb{F}_0}$ *and* $\mathbf{D}^{1,\geq 2}_{\mathbb{F}_0} \subseteq \mathbf{D}^{\geq 2}_{\mathbb{P}^2}$.

(c) $\mathbf{D}^2_{\mathbb{F}_\delta} \subseteq \mathbf{D}_{\mathbb{P}^2}$ *for all* $\delta > 0$.

Proof. According to the proof of Theorem 2.5.4, for any $\delta > 0$ it is satisfied that $\mathbf{d} \in \mathbf{D}_{\mathbb{P}^2}$ (respectively, $\mathbf{d} \in \mathbf{D}^1_{\mathbb{F}_\delta}$) if and only if $\beta'^2_1 \geq \overline{\beta}_{g+1}/\overline{\beta}^2_0$ (respectively, $2\beta'_1 + \delta\beta'^2_1 \geq \overline{\beta}_{g+1}/\overline{\beta}^2_0$). As we are supposing $\beta'^2_1 \geq \overline{\beta}_{g+1}/\overline{\beta}^2_0$, the inequality $2\beta'_1 + \delta\beta'^2_1 \geq \beta'^2_1$ is true when δ is a positive integer and also if $\delta = 0$ and $\beta'_1(\nu) \leq 2$, which proves (a) and the first inclusion in (b).

The second inclusion in (b) follows from an analogous argument which yields the opposite inequality.

To finish, using again Theorem 2.5.4, a class $\mathbf{d}$ is an element of $\mathbf{D}^2_{\mathbb{F}_\delta}$, $\delta > 0$, if and only if $2\beta'_1 - \delta \geq \overline{\beta}_{g+1}/\overline{\beta}^2_0$. Consequently, the fact that $\beta'^2_1 \geq 2\beta'_1 - \delta$ concludes the proof. □

Remark 2.5.6. The inclusions introduced in Theorem 2.5.5 are strict. Let us show this fact with some examples.

(a) Fix the discrete class $\mathbf{d} = (2, 1, 4/3, 17/3, 1)$. Therefore $e_0 = 9, e_1 = 3$ and $e_2 = 1$. Moreover,

$$\beta'_1 = \frac{48}{27} \text{ and } \sum_{i=0}^{g} \tau_i^2(\beta'_{i+1}(\nu) - 1) + \beta'_0 = \frac{50}{27}$$

which shows that $\mathbf{d} \notin \mathbf{D}_{\mathbb{P}^2}$. Nevertheless,

$$\beta'_1(\delta\beta'_1 + 2) = \frac{48\delta + 72}{27}.$$

This proves that $\mathbf{d} \in \mathbf{D}^1_{\mathbb{F}_\delta}$ for all non-negative integer δ. Consequently the inclusion in Theorem 2.5.5 (a) and the first one of Theorem 2.5.5 (b) are strict.

(b) Consider $\delta = 0$ and the class $\mathbf{d} = (3, 1, 7/3, 43/2, 14/3, 1)$. Thus $e_0 = 18, e_1 = 6$, $e_2 = 3$ and $e_3 = 1$ and one obtains

$$2\beta'_1 = \frac{504}{108}, \quad \sum_{j=0}^{g} \tau_j^2(\beta'_{j+1} - 1) + \beta'_0 = \frac{509}{108} \text{ and } \beta'^2_1 = \frac{588}{108},$$

which implies that $\mathbf{d} \in \mathbf{D}^{\geq 2}_{\mathbb{P}^2}$ but $\mathbf{d} \notin \mathbf{D}^{1,\geq 2}_{\mathbb{F}_0}$.

(c) To conclude, set the class $\mathbf{d} = (2, 1, 5/2, 57/5, \Phi)$, where Φ denotes the golden ratio. One has $e_0 = 10, e_1 = 5$ and $e_2 = 1$. Then

$$2\beta_1' - \delta = \frac{100}{20} - \delta \text{ and } \sum_{j=0}^{g} \tau_j^2(\beta_{j+1}' - 1) + \beta_0' = \frac{102}{20} + \frac{1}{100}(\Phi - 1).$$

This shows that $\mathbf{d} \notin \mathbf{D}^2_{\mathbb{F}_\delta}$ for any positive integer δ, although $\mathbf{d} \in \mathbf{D}_{\mathbb{P}^2}$ since $\beta_1'^2 = 125/20$.

2.5.1 An algorithm for obtaining the dual graphs of NPI valuations

The aim of this subsection is to describe a procedure in order to generate those discrete classes (that is, dual graphs) admitting some NPI divisorial or irrational valuation. We assume $g > 0$ because any discrete class $(0, 1, \beta_1')$ admits an NPI (respectively, NPI special) valuation of $\mathbb{P}^2$ (respectively, $\mathbb{F}_\delta$) and by Theorem 2.5.4 the discrete classes $(0, 1, \beta_1' \geq \delta)$ are those that admit NPI non-special valuations of $\mathbb{F}_\delta$.

Our algorithm starts with an *input*

$$\mathbf{d}(\nu_I) = (g, \beta_0'(\nu_I), \beta_1'(\nu_I), \ldots, \beta_{g+1}'(\nu_I) = 1), \tag{2.18}$$

which is a discrete class belonging to $\mathbf{D}_{\mathbb{P}^2}$, $\mathbf{D}^1_{\mathbb{F}_\delta}$ or $\mathbf{D}^2_{\mathbb{F}_\delta}$, simply written by $\mathbf{D}$. It provides two outputs:

Output 1. Another discrete class of the same set $\mathbf{D}$ of the input of the form:

$$\mathbf{d}(\nu_{O_1}) = \big(g + 1, \beta_0'(\nu_{O_1}) = \beta_0'(\nu_I), \beta_1'(\nu_{O_1}) = \beta_1'(\nu_I), \\ \ldots, \beta_g'(\nu_{O_1}) = \beta_g'(\nu_I), \beta_{g+1}'(\nu_{O_1}), \beta_{g+2}'(\nu_{O_1}) = 1\big). \tag{2.19}$$

Output 2. In fact, it is a double output. The first one is a discrete class in the same set $\mathbf{D}$ of the input as follows:

$$\mathbf{d}(\nu_{O_2^1}) = \big(g, \beta_0'(\nu_{O_2^1}) = \beta_0'(\nu_{O_1}), \beta_1'(\nu_{O_2^1}) = \beta_1'(\nu_{O_1}), \\ \ldots, \beta_g'(\nu_{O_2^1}) = \beta_g'(\nu_{O_1}), \beta_{g+1}'(\nu_{O_2^1})\big), \tag{2.20}$$

where $\beta_{g+1}'(\nu_{O_2^1}) \in \mathbb{R}_{>0} \setminus \mathbb{Q}$.

And the second one is a discrete class in the same set $\mathbf{D}$ of the input with the following shape

$$\mathbf{d}(\nu_{O_2^2}) = \big(g + 1, \beta_0'(\nu_{O_2^2}) = \beta_0'(\nu_{O_1}), \beta_1'(\nu_{O_2^2}) = \beta_1'(\nu_{O_1}), \\ \ldots, \beta_{g+1}'(\nu_{O_2^2}) = \beta_{g+1}'(\nu_{O_1}), \beta_{g+2}'(\nu_{O_1^2})\big), \tag{2.21}$$

$\beta_{g+2}'(\nu_{O_1^2})$ being a positive integer different from 1.

Note that the outputs are not unique. In fact, one can obtain infinitely tuples as Output 1 and also as the first one in Output 2.

It is easily seen that the tuple of the input corresponds to a discrete class of an NPI divisorial valuation which is defined by a satellite (exceptional) divisor whose dual graph Γ has subgraphs $\Gamma^j, 1 \leq j \leq g$, and Γ^{g+1} that contains only the vertex st_g. The algorithm gives us dual graphs admitting the same type of NPI valuations which maintain the subgraphs $\Gamma^j, 1 \leq j \leq g$, and have a new subgraph Γ^{g+1}; the obtained graphs correspond to divisorial valuations defined by a satellite divisor or to irrational valuations. In addition, our algorithm also provides dual graphs of NPI divisorial valuations defined by free divisors which preserve the subgraphs $\Gamma^j, 1 \leq j \leq g$, and add two subgraphs more, Γ^{g+1} and (a tail) Γ^{g+2}.

It becomes clear that selecting appropriate tuples $(1, \beta'_0 = 1, \beta'_1, 1)$ at the beginning, we are able to give the dual graph of any NPI valuation of any desired type with g as large as we want.

Let us show our algorithm. For a start and under the notation stated before, set

$$q(\mathbf{d}(\nu)) := \begin{cases} \beta'_1(\nu)^2, & \text{if } Z_0 = \mathbb{P}^2, \\ \beta'_1(\nu)(\delta\beta'_1(\nu) + 2), & \text{if } Z_0 = \mathbb{F}_\delta \text{ and } \nu \text{ is special}, \\ 2\beta'_1(\nu) - \delta, & \text{if } Z_0 = \mathbb{F}_\delta \text{ and } \nu \text{ is non-special}, \end{cases}$$

$\beta'_1(\nu)$ being the third coordinate of $\mathbf{d}(\nu)$.

Input: A discrete class $\mathbf{d}(\nu_I)$ as in (2.18) that satisfies

$$q\left(\mathbf{d}(\nu_I)\right) > \sum_{j=0}^{g-1} e_j \left(\nu_I^N\right)^2 \left(\beta'_{j+1}(\nu_I) - 1\right) + \beta'_0(\nu_I). \tag{2.22}$$

This condition must be imposed since we start with the discrete class of an NPI divisorial valuation and one needs to have some degree of freedom to add Puiseux exponents.

Output 1: It will be a tuple $\mathbf{d}(\nu_{O_1})$ as in (2.19) computed as follows. Set $\beta'_{g+1}(\nu_{O_1}) = q_{g+1}/n_{g+1}$, where $q_{g+1}, n_{g+1} \in \mathbb{Z}_{>0}$ are such that $\gcd(q_{g+1}, n_{g+1}) = 1$ and $q_{g+1} > n_{g+1}$. The output must satisfy

$$\beta'_j(\nu_{O_1}) = \beta'_j(\nu_I) \text{ and } e_j(\nu_{O_1}^N) = e_j(\nu_I^N), \text{ for } 0 \leq j \leq g,$$

and $\overline{\beta}_0(\nu_{O_1}) = n_{g+1}\overline{\beta}_0(\nu_I)$. Then

$$e_{g+1}(\nu_{O_1}^N) = \frac{1}{\overline{\beta}_0(\nu_{O_1})} = \frac{1}{n_{g+1}\overline{\beta}_0(\nu_I)} = \frac{e_g(\nu_I^N)}{n_{g+1}}.$$

As we desire that the tuple $\mathbf{d}(\nu_{O_1})$ corresponds to a discrete class of an NPI valuation, *for obtaining a suitable Output* 1 *it suffices to take a pair* q_{g+1} *and* n_{g+1} *(which defines* $\beta'_{g+1}(\nu_{O_1})$*) satisfying the following inequality:*

$$\frac{q(\mathbf{d}(\nu_I)) - \sum_{j=0}^{g-1} e_j(\nu_I^N)^2(\beta'_{j+1}(\nu_I) - 1) - \beta'_0(\nu_I)}{e_g(\nu_I^N)^2} + 1 \geq \beta'_{g+1}(\nu_{O_1}) = \frac{q_{g+1}}{n_{g+1}} > 1. \tag{2.23}$$

Note that the algorithm must make a choice since there are infinitely many options.
Output 2: To obtain a tuple $\mathbf{d}(\nu_{O_2^1})$ as (2.20), it suffices to search an irrational number $\beta'_{g+1}(\nu_{O_2^1})$ which satisfies Inequality (2.23) when we replace the rational number $\beta'_{g+1}(\nu_{O_1})$ with $\beta'_{g+1}(\nu_{O_2^1})$. Notice that we have to make a choice again.

Finally, for getting a discrete class $\mathbf{d}(\nu_{O_2^2})$ as (2.21), it suffices to find a positive integer $\beta'_{g+2}(\nu_{O_2^2}) > 1$ such that

$$\frac{q(\mathbf{d}(\nu_{O_2^2})) - \sum_{j=0}^{g} e_j(\nu_{O_1}^N)^2(\beta'_{j+1}(\nu_{O_1}) - 1) - \beta'_0(\nu_{O_1})}{e_{g+1}(\nu_{O_1}^N)^2} \geq \beta'_{g+2}(\nu_{O_2^2}) - 1. \tag{2.24}$$

We remark that the biggest non-negative integer $\beta'_{g+2}(\nu_{O_2^2}) - 1$ satisfying Inequality (2.24) provides the maximum number of free vertices one can obtain in the tail of the dual graph.

To conclude this subsection, we give two examples which show how our algorithm runs.

Example 2.5.7. Consider $Z_0 = \mathbb{P}^2$ and the input $\mathbf{d}(\nu_I) = (2, 1, 5/2, 7/5, 1)$. Therefore $\{e_i(\nu_I)\}_{i=0}^2 = \{10, 5, 1\}$. As required, our input satisfied Inequality (2.22) since

$$\frac{25}{4} = \left(\frac{5}{2}\right)^2 > \left(\frac{5}{2} - 1\right) + \left(\frac{1}{2}\right)^2 \left(\frac{7}{5} - 1\right) + 1 = \frac{52}{20}.$$

Now suitable values for our purposes $\beta'_3(\nu_{O_1})$ are those satisfying Inequality (2.23), that is those $\beta'_3(\nu_{O_1})$ such that

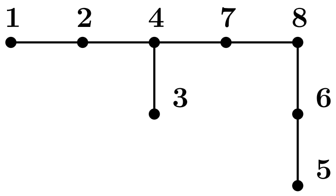
$$366 = \frac{\beta'_1(\nu_I)^2 - e_0(\nu_I^N)^2(\beta'_1(\nu_I) - 1) - e_1(\nu_I^N)^2(\beta'_2(\nu_I) - 1) - \beta'_0(\nu_I)}{e_2(\nu_I^N)^2} + 1 \geq$$
$$\beta'_3(\nu_{O_1}) > 1.$$

If, for example, we select $\mathbf{d}(\nu_{O_1}) = (3, 1, 5/2, 7/5, 8/3, 1)$, we get a valid Output 1. Our algorithm also searches values $\beta'_4(\nu_{O_2^2})$, which must satisfy Inequality (2.24), that is $3270 \geq \beta'_4(\nu_{O_2^2}) - 1 \geq 0$. Consequently a possible Output 2 would be $\mathbf{d}(\nu_{O_2^2}) = (3, 1, 5/2, 7/5, 8/3, 3200)$. The dual graphs of ν_I, ν_{O_1} and $\nu_{O_2^2}$ can be seen in Figure 2.3.

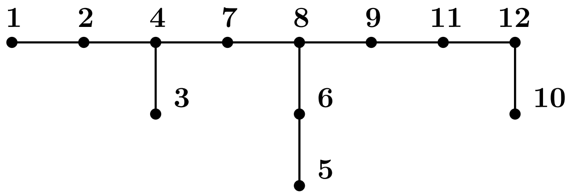
Example 2.5.8. Assume now that $Z_0 = \mathbb{F}_2$ and $\mathbf{d}(\nu_I) = (3, 1, 5/3, 12/5, 5/2, 1)$. This input is as the algorithm requires because $\{e_i(\nu_I)\}_{i=0}^2 = \{30, 10, 2, 1\}$,

$$\frac{80}{9} = \frac{5}{3}\left(2\frac{5}{3} + 2\right) > \left(\frac{5}{3} - 1\right) + \left(\frac{1}{3}\right)^2 \left(\frac{12}{5} - 1\right) + \left(\frac{1}{15}\right)^2 \left(\frac{5}{2} - 1\right) + 1 = \frac{823}{450}$$

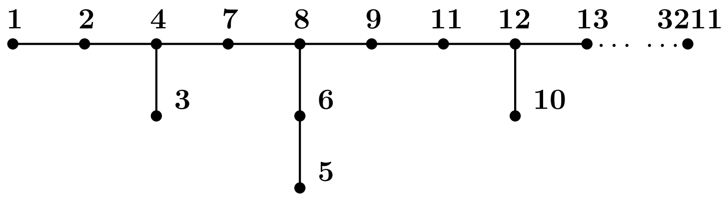
and, then, Inequality (2.22) holds. Thus, if we desire an output $\mathbf{d}(\nu_{O_2^1})$, we need to find some irrational number such that it satisfies Inequality (2.23). That is, we can use any value $\beta'_4(\nu_{O_2^1})$ such that $6355 \geq \beta'_4(\nu_{O_2^1}) > 0$. Therefore, $\mathbf{d}(\nu_{O_2^1}) = (3, 1, 5/3, 12/5, 5/2, \pi)$ is a suitable output. One can see the dual graphs of ν_I and $\nu_{O_2^1}$ in Figure 2.4.



(a) Dual graph of ν_I

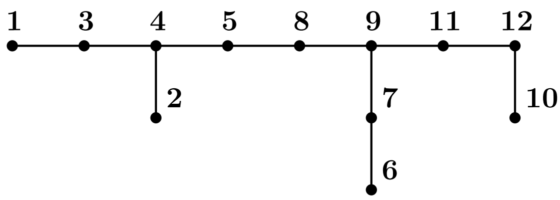


(b) Dual graph of ν_{O_1}

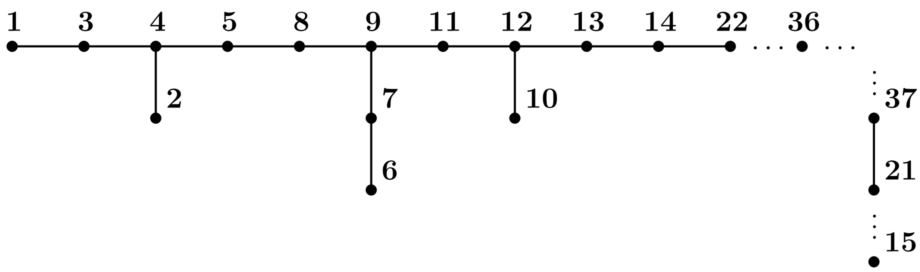


(c) Dual graph of $\nu_{O_2^2}$

Figure 2.3: Dual graphs in Example 2.5.7.



(a) Dual graph of ν_I



(b) Dual graph of $\nu_{O_2^1}$

Figure 2.4: Dual graphs in Example 2.5.8.

Chapter 3

Seshadri-type constants and Newton-Okounkov bodies for non-positive at infinity valuations

The aim of this chapter is to describe the Seshadri-type constants and Newton-Okounkov bodies with respect to non-positive at infinity valuations of Hirzebruch surfaces. Our main results and proofs concerning these objects can be found in [62]. This chapter also contains the study of the same constants and bodies when the valuations are of the projective plane. As a reference, we have used the articles [64] and [65]. Recall that a basic introduction about Seshadri-type constants and Newton-Okounkov bodies for surfaces was provided in Section 1.5. We keep the notation of the previous chapters. In this chapter k will be the complex field $\mathbb{C}$.

Denote by Z_0 the projective plane $\mathbb{P}^2$ or a Hirzebruch surface $\mathbb{F}_\delta, \delta \geq 0$, over $\mathbb{C}$ and p a point of Z_0. Let ν be an exceptional curve valuation of the function field of Z_0 centered at $\mathcal{O}_{Z_0,p}$. Write ν_n the divisorial valuation corresponding to a finite simple sequence of blowups as follows:

$$\pi : Z := Z_n \xrightarrow{\pi_n} Z_{n-1} \to \ldots \to Z_1 \xrightarrow{\pi_1} Z_0.$$

The valuation ν has a distinguished point p_r such that $p_i \to p_r$ for $i > r$ and its first component, which is a divisorial valuation (Theorem 1.3.2), is defined by the divisor E_r, thus it is usually denoted ν_r (see Section 1.3). As in Chapter 2, for simplicity, we denote by E_i (respectively, E_i^*) the strict (respectively, total) transform on Z_n of the exceptional divisor E_i created after blowing-up p_i and by $\tilde{D}$ (respectively, D^*) the strict (respectively, total) transform on Z_n of a divisor D on Z_i, for $i \leq n$. Write φ_C (respectively, φ_i) the germ of a curve C at p (respectively, an analytically irreducible germ at p whose strict transform on Z_i is transversal to E_i at a non-singular point of the exceptional locus).

Keeping the conventions of the above chapters, ν (respectively, ν_n) will be often called exceptional curve (respectively, divisorial) valuations of Z_0. If $Z_0 = \mathbb{P}^2$, and one considers a divisorial valuation ν_n, L denotes the line at infinity (see Section 2.1). In the case $Z_0 = \mathbb{F}_\delta$, F_1 denotes the fiber containing p, M_0 denotes the special section and, when ν_n is non-special, M_1 denotes the irreducible curve of degree $(0, 1)$ going through p whose strict transform has negative self-intersection on the surface defined by ν_n (see Proposition 2.2.2).

Set again ν_n a divisorial valuation of Z_0 and consider a big divisor D on Z_0. Recall that the value $\hat{\mu}_D(\nu_n)$ introduced in Subsection 1.5.1 is

$$\hat{\mu}_D(\nu_n) = \lim_{m\to\infty} \frac{\max\{\nu_n(f) \mid f \in H^0(Z_0, \mathcal{O}_{Z_0}(mD))\}}{m}. \tag{3.1}$$

In addition, this value $\hat{\mu}_D(\nu_n)$ has a lower bound (1.10) which can be written as

$$\hat{\mu}_D(\nu_n) \geq \sqrt{\mathrm{vol}_{Z_0}(D)\, \overline{\beta}_{g+1}(\nu_n)}, \tag{3.2}$$

because $\overline{\beta}_{g+1}(\nu_n) = \mathrm{vol}(\nu_n)^{-1}$ by [64, Section 2.5].

We conclude this brief introduction with the concept of minimal valuation (of $\mathbb{P}^2$ o $\mathbb{F}_\delta$) with respect to a big divisor.

Definition 3.0.1. Let D be a big divisor on a surface Z_0 as defined above. A divisorial valuation ν_n of Z_0 is called to be *minimal with respect to D* if

$$\hat{\mu}_D(\nu_n) = \sqrt{\mathrm{vol}_{Z_0}(D)\, \overline{\beta}_{g+1}(\nu_n)}.$$

When the above condition is not satisfied, the valuation ν_n is named *non-minimal with respect to D*.

3.1 Seshadri-type constants for divisorial valuations of the projective plane

In this section we are going to give a definition of minimal divisorial valuation of $\mathbb{P}^2 := \mathbb{P}^2_{\mathbb{C}}$ which does not depend on a divisor. It can be found in [15, 38] and [64]. We will prove in Proposition 3.1.3 that this definition is equivalent to Definition 3.0.1 in the projective case and the reference to a divisor can be deleted. Moreover, we will show interesting results related to non-minimal valuations of $\mathbb{P}^2$.

Assume that $Z_0 = \mathbb{P}^2 := \mathbb{P}^2_{\mathbb{C}}$. Consider projective coordinates $(X : Y : Z)$ in $\mathbb{P}^2$, a point $p \in \mathbb{P}^2$ with coordinates $(1 : 0 : 0)$ and the affine coordinates $u = Y/X$ and $v = Z/X$ around p. Let ν_n be a divisorial valuation of the function field of $\mathbb{P}^2$ centered at $\mathcal{O}_{\mathbb{P}^2,p}$ and set E_0 a general projective line on $\mathbb{P}^2$. Following [38] and [64], the value $\hat{\mu}(\nu_n)$ is defined as

$$\hat{\mu}(\nu_n) := \lim_{m\to\infty} \frac{\mu_m(\nu_n)}{m}, \tag{3.3}$$

where

$$\begin{aligned}\mu_m(\nu_n) &= \max\{\nu_n(f) \mid f \in H^0(Z_0, \mathcal{O}_{Z_0}(mE_0))\}\\ &= \max\{\nu_n(f) \,|\, f \in \mathbb{C}[u,v], \deg(f) \le m\}.\end{aligned}$$

Taking into account (3.1) and (3.2), it holds that $\hat{\mu}(\nu_n) \geq \sqrt{\overline{\beta}_{g+1}(\nu_n)}$.

Now we introduce the above mentioned concept of minimal valuation of $\mathbb{P}^2$.

Definition 3.1.1. [38, 64] A divisorial valuation ν_n of $\mathbb{P}^2$ is called to be *minimal* if $\hat{\mu}(\nu_n) = \sqrt{\overline{\beta}_{g+1}(\nu_n)}$.

Notice that a divisorial valuation of $\mathbb{P}^2$ is minimal in the sense of the above definition if and only if it is minimal with respect to E_0 in the sense of Definition 3.0.1.

The following lemma will allow us to show the relation between our definitions 3.0.1 and 3.1.1 in the projective case.

Lemma 3.1.2. *Let ν_n be a divisorial valuation of $\mathbb{P}^2$ and d a positive integer. Then*

$$\hat{\mu}_{dE_0}(\nu_n) = d\hat{\mu}(\nu_n).$$

Proof. Taking into account (3.1), it holds that

$$\begin{aligned}\hat{\mu}_{dE_0}(\nu_n) = \lim_{m\to\infty} &\frac{\max\{\nu_n(f) \,|\, f \in \mathbb{C}[u,v], \deg(f) \le dm\}}{m}\\ =&d \lim_{m'\to\infty} \frac{\max\{\nu_n(f) \,|\, f \in \mathbb{C}[u,v], \deg(f) \le m'\}}{m'}\\ =&d\hat{\mu}(\nu_n),\end{aligned}$$

where the second equality follows from the replacement $m' = dm$ and the third one from (3.3). □

Proposition 3.1.3. *Let ν be a divisorial valuation of $\mathbb{P}^2$. Then ν_n is minimal in the sense of Definition 3.1.1 if and only if it is minimal with respect to any big divisor D on $\mathbb{P}^2$ in the sense of Definition 3.0.1.*

Proof. By Definition 3.1.1, a minimal divisorial valuation of $\mathbb{P}^2$ satisfies

$$\hat{\mu}(\nu_n)^2 = \overline{\beta}_{g+1}(\nu_n).$$

Consequently, it holds that $d^2\hat{\mu}(\nu_n)^2 = d^2\overline{\beta}_{g+1}(\nu_n)$, for all positive integers d. In addition, by Lemma 3.1.2, the equality $\hat{\mu}_D(\nu_n)^2 = D^2\overline{\beta}_{g+1}(\nu_n)$ holds, where D is a big divisor linearly equivalent to dE_0, which completes the proof. □

We finish this section giving two results concerning non-minimal divisorial valuations of $\mathbb{P}^2$.

Proposition 3.1.4. [[38, Lemma 5.1] and [65, Lemma 3.10]] *Let ν_n be a divisorial valuation of $\mathbb{P}^2$. Suppose the existence of an irreducible polynomial $f \in \mathbb{C}[u,v]$ such that $\nu_n(f) > \deg(f)\sqrt{\overline{\beta}_{g+1}(\nu_n)}$. Then*

$$\hat{\mu}(\nu_n) = \frac{\nu_n(f)}{\deg(f)}.$$

In addition, if ν_n is a non-minimal valuation, then there exists such an irreducible polynomial f and it is the unique irreducible polynomial (up to product by a non-zero constant) satisfying the above condition.

Definition 3.1.5. Let ν_n be a non-minimal divisorial valuation of $\mathbb{P}^2$. A curve C is called *supraminimal* of ν_n if it is defined by an irreducible polynomial $f \in \mathbb{C}[u,v]$ satisfying $\nu_n(f)/\deg(f) = \hat{\mu}(\nu_n)$. This curve is unique by the above proposition.

Corollary 3.1.6. *Let ν_n be a non-minimal divisorial valuation of $\mathbb{P}^2$ and Z the surface which ν_n defines. Then the class of the strict transform of the supraminimal curve of ν_n generates an extremal ray of the cone of curves $\mathrm{NE}(Z)$ of Z.*

Proof. To prove the result we show that the strict transform of the supraminimal curve of ν_n has negative self-intersection and, by Proposition 1.4.6, we conclude the proof.

Consider the divisor

$$D = E_0^* - \frac{1}{\hat{\mu}(\nu_n)} \sum_{i=1}^{n} \nu_n(\mathfrak{m}_i) E_i^*.$$

This divisor is nef and big. Indeed, suppose that $\tilde{C}_h$ is the strict transform on Z of a curve C_h on $\mathbb{P}^2$ defined by a polynomial $h \in \mathbb{C}[u,v]$. Then

$$D \cdot \tilde{C}_h = \deg(h) - \frac{1}{\hat{\mu}(\nu_n)} \sum_{i=1}^{n} \nu_n(\mathfrak{m}_i) \cdot \mathrm{mult}_{p_i}(h) = \deg(h) - \frac{\nu_n(h)}{\hat{\mu}(\nu_n)} \geq 0,$$

where the second equality holds by Noether's formula and the inequality by the fact that ν_n is non-minimal. In addition, by the proximities equalities, $D \cdot E_i = 0$, for all $i \in \{1, 2, \ldots, n\}$. Therefore, we have just proved that D is nef. The self-intersection of the divisor D satisfies

$$D^2 = (E_0^*)^2 - \frac{\overline{\beta}_{g+1}(\nu_n)}{\hat{\mu}(\nu_n)^2} > 1 - 1 = 0$$

by assumption, and by Theorem 1.4.7, D is big.

Now, denote by $\tilde{C}_f$ the strict transform of the supraminimal curve of ν_n C_f. C_f is defined by an irreducible polynomial $f \in \mathbb{C}[u,v]$. Consequently, by Proposition 3.1.4,

$$D \cdot \tilde{C} = \deg(f) - \frac{\nu_n(\varphi_C)}{\hat{\mu}(\nu_n)} = \deg(f) - \deg(f) = 0,$$

and so $\tilde{C}_f$ is orthogonal to D. This implies that $\tilde{C}_f$ has negative self-intersection by Proposition 1.4.12, which completes the proof. □

Remark 3.1.7. Let ν_n be a non-positive at infinity divisorial valuation of $\mathbb{P}^2$. By [64, Proposition 5.4], $\hat{\mu}(\nu_n) = \nu_n(\varphi_L)$ and the line at infinity L is the supraminimal curve of ν_n when ν_n is non-minimal.

3.2 Seshadri-type constants for NPI divisorial valuations of Hirzebruch surfaces

We devote this subsection to study the value $\hat{\mu}_D(\nu_n)$ for any non-positive at infinity divisorial valuation ν_n of a Hirzebruch surface $\mathbb{F}_\delta$ over $\mathbb{C}$ and any big divisor D on $\mathbb{F}_\delta$. We will also show some consequences of our study.

Our main result is the following one:

Theorem 3.2.1. *Let ν_n be an NPI divisorial valuation of the function field of $\mathbb{F}_\delta$ centered at $\mathcal{O}_{\mathbb{F}_\delta,p}$. Set $D \sim aF + bM$ a big divisor on $\mathbb{F}_\delta$. Then:*

(a) If ν_n is special, then $\hat{\mu}_D(\nu_n) = (a + b\delta)\nu_n(\varphi_{F_1}) + b\nu_n(\varphi_{M_0})$.

(b) Otherwise, $\hat{\mu}_D(\nu_n) = a\nu_n(\varphi_{F_1}) + b\nu_n(\varphi_{M_1})$.

Proof. To show (a) we assume that p is a special point. When p is a point of $\mathbb{F}_0$ (respectively, p is a general point), the proof is similar and holds assuming $\delta = 0$ (respectively, $\nu_n(\varphi_{M_0}) = 0$). Set C a curve on $\mathbb{F}_\delta$ such that $C \in |mD|$, where $m \in \mathbb{Z}_{>0}$, and $\tilde{C}$ its strict transform on Z. The divisor on Z

$$\Lambda(\nu_n) = \nu_n(\varphi_{M_0})F^* + \nu_n(\varphi_{F_1})M^* - \sum_{i=1}^{n} \nu_n(\mathfrak{m}_i)E_i^*$$

is nef by Theorem 2.3.7 and so $\Lambda(\nu_n) \cdot \tilde{C} \geq 0$. This implies that

$$(a + b\delta)\nu_n(\varphi_{F_1}) + b\nu_n(\varphi_{M_0}) \geq \frac{\nu_n(\varphi_C)}{m}$$

and consequently we have obtained an upper bound for $\nu_n(\varphi_C)/m$, where C belongs to $|mD|$ and $m \in \mathbb{Z}_{>0}$. Now fix the curve $C_1 = m(a + \delta b)F_1 + mbM_0$ and then

$$C_1 \in |mD| \text{ and } \frac{\nu_n(\varphi_{C_1})}{m} = (a + \delta b)\nu_n(\varphi_{F_1}) + b\nu_n(\varphi_{M_0}),$$

which proves (a) since we have just seen that the bound can be reached.

Likewise a proof for (b) follows from considering the divisor

$$\Delta(\nu_n) = (\nu_n(\varphi_{M_1}) - \delta\nu_n(\varphi_{F_1}))F^* + \nu_n(\varphi_{F_1})M^* - \sum_{i=1}^{n} \nu_n(\mathfrak{m}_i)E_i^*,$$

which is nef by Theorem 2.4.8, and the curve $C_1 = maF_1 + mbM_1$. □

Remark 3.2.2. Let ν_n be an NPI divisorial valuation of $\mathbb{P}^2$ and ν the corresponding NPI special divisorial valuation of $\mathbb{F}_1$ described in the proof of Proposition 2.3.3. Set E_0 a general projective line on $\mathbb{P}^2$ and M an irreducible curve of degree $(0,1)$ on $\mathbb{F}_1$. The proofs of Proposition 2.3.3 and Theorem 3.2.1 prove that

$$\hat{\mu}(\nu_n) = \hat{\mu}_{E_0}(\nu_n) = \nu_n(\varphi_L) = \nu(\varphi_{F_1}) + \nu(\varphi_{M_0}) = \hat{\mu}_M(\nu).$$

Even more, by Remark 2.3.5, if ν_n is minimal then ν is non-minimal with respect to M since

$$\hat{\mu}_M(\nu) = \hat{\mu}(\nu_n) = \sqrt{\overline{\beta}_{g+1}(\nu_n)} = \sqrt{\overline{\beta}_{\hat{g}+1}(\nu) + \nu(\varphi_{M_0})^2} > \sqrt{\overline{\beta}_{\hat{g}+1}(\nu)},$$

where $\overline{\beta}_{g+1}(\nu_n)$ (respectively, $\overline{\beta}_{\hat{g}+1}(\nu)$) is the last maximal contact value of ν_n (respectively, ν).

Corollary 3.2.3. *Let ν_n be an NPI divisorial valuation of $\mathbb{F}_\delta$ and set $D \sim aF + bM$ a big and nef divisor on $\mathbb{F}_\delta$. Then*

(a) When ν_n is special, it is minimal with respect to D if and only if

$$2\nu_n(\varphi_{M_0})\nu_n(\varphi_{F_1}) + \delta\nu_n(\varphi_{F_1})^2 = \mathrm{vol}(\nu_n)^{-1}$$

and $a = b\nu_n(\varphi_{M_0})/\nu_n(\varphi_{F_1})$.

(b) Otherwise, ν_n is minimal with respect to D if and only if

$$2\nu_n(\varphi_{M_1})\nu_n(\varphi_{F_1}) - \delta\nu_n(\varphi_{F_1})^2 = \mathrm{vol}(\nu_n)^{-1}$$

and $a = b(\nu_n(\varphi_{M_1}) - \delta\nu_n(\varphi_{F_1}))/\nu_n(\varphi_{F_1})$.

Proof. We will see (a). Applying a similar argument, one can prove (b).

We begin by proving that ν_n is minimal with respect to D under the assumptions mentioned in the statement. As the equalities $2\nu_n(\varphi_{M_0})\nu_n(\varphi_{F_1}) + \delta\nu_n(\varphi_{F_1})^2 = \mathrm{vol}(\nu_n)^{-1}$ and $D^2 = \mathrm{vol}(D)$ are satisfied, one gets

$$\begin{aligned}\frac{\mathrm{vol}(D)}{\mathrm{vol}(\nu_n)} =&(2ab\delta + b^2\delta^2)\nu_n(\varphi_{F_1})^2 + 2b(a+b\delta)\nu_n(\varphi_{F_1})\nu_n(\varphi_{M_0}) + 2ab\nu_n(\varphi_{F_1})\nu_n(\varphi_{M_0})\\ =&(a+b\delta)^2\nu_n(\varphi_{F_1})^2 + 2b(a+b\delta)\nu_n(\varphi_{F_1})\nu_n(\varphi_{M_0}) + b^2\nu_n(\varphi_{M_0})^2\\ =&\hat{\mu}_D(\nu_n)^2,\end{aligned}$$

where the second equality is obtained as consequence of the condition $(a\nu_n(\varphi_{F_1}) - b\nu_n(\varphi_{M_0}))^2 = 0$, which holds by hypothesis. This shows that ν_n is minimal with respect to D.

Conversely suppose that ν_n is minimal with respect to D. Theorem 3.2.1 shows that

$$((a+b\delta)\nu_n(\varphi_{F_1}) + b\nu_n(\varphi_{M_0}))^2 = b(2a+\delta b)\mathrm{vol}(\nu_n)^{-1}. \tag{3.4}$$

In addition, it holds that

$$\begin{aligned}((a+b\delta)\nu_n(\varphi_{F_1})+b\nu_n(\varphi_{M_0}))^2 &= a^2\nu_n(\varphi_{F_1})^2+b^2\nu_n(\varphi_{M_0})^2+b(2a+b\delta)\delta\nu_n(\varphi_{F_1})^2\\ &\quad +b(a+\delta b)(2\nu_n(\varphi_{F_1})\nu_n(\varphi_{M_0}))\\ &=(a\nu_n(\varphi_{F_1})-b\nu_n(\varphi_{M_0}))^2\\ &\quad +b(2a+\delta b)(2\nu_n(\varphi_{F_1})\nu_n(\varphi_{M_0})+\delta\nu_n(\varphi_{F_1})^2),\end{aligned}$$

which, together with Equality (3.4), gives rise to

$$(a\nu_n(\varphi_{F_1})-b\nu_n(\varphi_{M_0}))^2+b(2a+\delta b)(2\nu_n(\varphi_{F_1})\nu_n(\varphi_{M_0})+\delta\nu_n(\varphi_{F_1})^2-\mathrm{vol}(\nu_n)^{-1})=0.$$

Both summands in the above expression are not negative and so they must vanish. This concludes the proof. □

Corollary 3.2.4. *Let ν_n be an NPI divisorial valuation of $\mathbb{F}_\delta$. Then ν_n is non-minimal with respect to any big divisor D on $\mathbb{F}_\delta$ whenever some of the following conditions holds:*

(a) ν_n is special and $2\nu_n(\varphi_{M_0})\nu_n(\varphi_{F_1})+\delta\nu_n(\varphi_{F_1})^2>\mathrm{vol}(\nu_n)^{-1}$.

(b) ν_n is non-special and $2\nu_n(\varphi_{M_1})\nu_n(\varphi_{F_1})-\delta\nu_n(\varphi_{F_1})^2>\mathrm{vol}(\nu_n)^{-1}$.

Proof. Let us first prove (a). It suffices to check that any big divisor $D\sim aF+bM$ satisfies

$$\hat{\mu}_D(\nu_n)^2/P_D^2>\overline{\beta}_{g+1}(\nu_n),$$

where P_D is the positive part of the Zariski decomposition of D (see Remark 1.6.8). Define the map $q_1:(-\delta,\infty)\to\mathbb{R}_{>0}$ as

$$q_1(x):=\begin{cases}\dfrac{((x+\delta)\nu_n(\varphi_{F_1})+\nu_n(\varphi_{M_0}))^2}{((1/\delta)x+1)^2\delta} & \text{if } x\in(-\delta,0),\\[2ex] \dfrac{((x+\delta)\nu_n(\varphi_{F_1})+\nu_n(\varphi_{M_0}))^2}{2x+\delta} & \text{if } x\in[0,\infty).\end{cases}$$

It follows easily that q_1 has an absolute minimum at the point $(x_1,q_1(x_1))$, where

$$x_1=\frac{\nu_n(\varphi_{M_0})}{\nu_n(\varphi_{F_1})}\ \text{and}\ q_1(x_1)=2\nu_n(\varphi_{M_0})\nu_n(\varphi_{F_1})+\nu_n(\varphi_{F_1})^2\delta.$$

The equality $q_1(a/b)=\hat{\mu}_D(\nu_n)^2/P_D^2$, the assumption

$$2\nu_n(\varphi_{M_0})\nu_n(\varphi_{F_1})+\delta\nu_n(\varphi_{F_1})^2>\mathrm{vol}(\nu_n)^{-1}$$

and Theorem 3.2.1 complete the proof.

Finally, we can proceed analogously as above to prove (b). Here we have to consider the map $q_2:(-\delta,\infty)\to\mathbb{R}_{>0}$,

$$q_2(x):=\begin{cases}\dfrac{(\nu_n(\varphi_{F_1})x+\nu_n(\varphi_{M_1}))^2}{((1/\delta)x+1)^2\delta} & \text{if } x\in(-\delta,0),\\[2ex] \dfrac{(\nu_n(\varphi_{F_1})x+\nu_n(\varphi_{M_1}))^2}{2x+\delta} & \text{if } x\in[0,\infty),\end{cases}$$

instead of q_1, which has an absolute minimum at the point $(x_2, q_2(x_2))$, where

$$x_2 = \frac{\nu_n(\varphi_{M_1}) - \delta\nu_n(\varphi_{F_1})}{\nu_n(\varphi_{F_1})} \text{ and } q_2(x_2) = 2\nu_n(\varphi_{M_1})\nu_n(\varphi_{F_1}) - \delta\nu_n(\varphi_{F_1})^2.$$

□

3.3 Newton-Okounkov bodies of non-positive at infinity valuations

Newton-Okounkov bodies are convex sets which provide interesting geometric information [86, 74, 15]. However, giving an explicit description of them is very difficult. As we have seen in Subsection 1.5.2, these bodies can be described; however the involved objects for that description are also very hard to compute. In this section we give a much more simple and explicit description of the Newton-Okounkov bodies of big divisors on a surface Z_0, which is the projective plane $\mathbb{P}^2 := \mathbb{P}^2_{\mathbb{C}}$ or a Hirzebruch surface $\mathbb{F}_\delta$ over $\mathbb{C}, \delta \geq 0$, with respect to flags defined by exceptional divisors associated to NPI divisorial valuations.

Let E_r be the last exceptional divisor created by a finite simple sequence of blowups

$$\pi : Z := Z_r \xrightarrow{\pi_r} Z_{r-1} \to \ldots \to Z_1 \xrightarrow{\pi_1} Z_0 \tag{3.5}$$

and p_{i+1}, $0 \leq i \leq r-1$, the closed point of Z_i where the blowup $\pi_{i+1} : Z_{i+1} \to Z_i$ is centered. Denote by $E_\bullet$ the flag of Z (see Subsection 1.5.2) defined as

$$E_\bullet := \{Z = Z_r \supset E_r \supset \{p_{r+1}\}\}, \tag{3.6}$$

where the closed point $p_{r+1} \in E_r$ is the center of $E_\bullet$. The point p_{r+1} could belong to another exceptional divisor. In this case, this divisor is denoted by E_η, that is, η is a positive integer $\eta < r$ such that $p_{r+1} \in E_\eta \cap E_r$.

As we have mentioned in Subsection 1.5.2, a flag of a smooth surface comes with a discrete valuation of rank 2; in fact, they are equivalent objects. In this chapter, following [65, Section 3.2] and our Section 1.3, the discrete valuation $\nu := \nu_{E_\bullet}$ attached to $E_\bullet$ corresponds with an exceptional curve valuation whose value group is $\mathbb{Z}^2$ and $\nu(\mathfrak{m}_r) = (1, 0)$ and $\nu(\mathfrak{m}_{r+1}) = (0, 1)$, up to equivalence. Recall that ν can be computed as follows: $\nu(f) = (v_1(f), v_2(f))$, where

$$v_1(f) := \nu_r(f) \text{ and } v_2(f) := \nu_\eta(f) + \sum_{p_i \to p_r} \operatorname{mult}_{p_i}(f), \text{ for } f \in \mathcal{O}_{Z_0,p},$$

ν_r (respectively, ν_η) being the divisorial valuation defined by E_r (respectively, E_η).

Set $\mathcal{C}_\nu = \{p_i\}_{i\geq 1}$ the configuration of infinitely near points of ν, where $p_i \to p_r$ for all $i > r$, and $\mathcal{C}_{\nu_r} = \{p_i\}_{i=1}^r$ the configuration of infinitely near points of ν_r. Write Γ_ν (respectively, Γ_{ν_r}) the dual graph of ν (respectively, ν_r). We consider the following useful ordering on the set of vertices of Γ_{ν_r} (Section 1.3): given two vertices α and β, we say that $\alpha \preccurlyeq \beta$ if there exists a simple path on the graph Γ_{ν_r} from $\mathbf{1}$ to β passing through α.

Denote by $\{\overline{\beta}_i(\nu_n)\}_{i=0}^{g+1}$ the sequence of maximal contact values of the divisorial valuation ν_n for $n = r, \eta$. Write S_ν the semigroup of values of ν. It is generated by $\{\overline{\beta}_i(\nu)\}_{i=0}^{g^*+1}$, where $\overline{\beta}_i(\nu) = (\overline{\beta}_i(\nu_r), \overline{\beta}_i(\nu_\eta))$ (respectively, $\overline{\beta}_i(\nu) = (\overline{\beta}_i(\nu_r), 0)$ and $\overline{\beta}_{g^*+1}(\nu) = (\overline{\beta}_{g^*+1}(\nu_r), 1)$) if the point p_{r+1} is satellite (respectively, free) (see [65]). Notice that it holds that $g = g^* + 1$ if p_{r+1} and p_r are satellite points. Otherwise, $g = g^*$.

The distinction between special and non-special valuations and the definition of minimal valuation given for divisorial valuations can be easily extended to the exceptional curve case.

Definition 3.3.1. Let ν be an exceptional curve valuation of Z_0. We say that ν is *special* when $Z_0 = \mathbb{F}_\delta, \delta \geq 0$, and its first component ν_r is special, and is *non-special* if $Z_0 = \mathbb{F}_\delta, \delta > 0$, and its first component ν_r is non-special. Likewise, the valuation ν is called to be *non-positive at infinity (NPI)* when its first component ν_r is NPI.

Definition 3.3.2. Let D be a big divisor on a surface Z_0. An exceptional curve valuation of Z_0 is said to be *minimal with respect to* D if its first component ν_r is minimal with respect to D.

Notice that, in the above definition, is enough to say *minimal* when $Z_0 = \mathbb{P}^2$ by Proposition 3.1.3.

The aim of this section is to explicitly compute Newton-Okounkov bodies of divisors D^* with respect to $\nu := \nu_{E_\bullet}$, where ν is a exceptional curve valuation of Z_0, $E_\bullet$ is a flag as (3.6), and D^* is the total transform of a big divisor D on Z_0. Under the above conditions, $H^0(Z_0, \mathcal{O}_{Z_0}(D)) \cong H^0(Z, \mathcal{O}_Z(D^*))$ holds and, without restriction of generality, we can use the following definition about Newton-Okounkov bodies.

Definition 3.3.3. Let ν be an exceptional curve valuation of Z_0 and D a big divisor on Z_0. The *Newton-Okounkov body of D with respect to ν* is defined as

$$\Delta_\nu(D) := \overline{\bigcup_{m\geq 1} \left\{ \frac{\nu(f)}{m} \mid f \in H^0(Z_0, mD) \setminus \{0\} \right\}},$$

where the upper line means closed convex hull in $\mathbb{R}^2$.

Remark 3.3.4. When $Z_0 = \mathbb{P}^2$, it suffices to study the case where D is a projective line since $\Delta_\nu(dD) = d \cdot \Delta_\nu(D)$ for all integer $d > 0$.

Set $\mathfrak{C}(\nu)$ the convex cone of $\mathbb{R}^2$ generated by the semigroup of values S_ν of ν and $\mathfrak{H}_D(\nu)$ the half-plane $\{(x, y) \in \mathbb{R}^2 \mid x \leq \hat{\mu}_D(\nu_r)\}$. We explicitly describe the set $\mathfrak{C}(\nu) \cap \mathfrak{H}_D(\nu)$ in the following result.

Proposition 3.3.5. *Keep the notation introduced before. The Newton-Okounkov body $\Delta_\nu(D)$ of D with respect to ν is contained in the triangle $\mathfrak{C}(\nu) \cap \mathfrak{H}_D(\nu)$ whose vertices are*

$$(0,0), \quad \left(\hat{\mu}_D(\nu_r), \frac{\hat{\mu}_D(\nu_r)\overline{\beta}_0(\nu_\eta)}{\overline{\beta}_0(\nu_r)}\right) \quad \text{and} \quad \left(\hat{\mu}_D(\nu_r), \frac{\hat{\mu}_D(\nu_r)\overline{\beta}_{g^*+1}(\nu_\eta)}{\overline{\beta}_{g^*+1}(\nu_r)}\right)$$

if $p_{r+1} \in E_\eta \cap E_r$ (with $\eta \neq r$); and

$$(0,0), \quad (\hat{\mu}_D(\nu_r), 0) \quad \text{and} \quad \left(\hat{\mu}_D(\nu_r), \frac{\hat{\mu}_D(\nu_r)}{\overline{\beta}_{g+1}(\nu_r)}\right),$$

otherwise.

Proof. To start, we suppose that p_{r+1} is the satellite point $E_\eta \cap E_r$. Following Subsection 1.3.1 and Corollary 1.3.7, the jth maximal contact value of ν, $0 \leq j \leq g^*$, is $\overline{\beta}_j(\nu) = (\overline{\beta}_j(\nu_r), \overline{\beta}_j(\nu_\eta))$. All these values belong to the line passing through the origin with slope $\overline{\beta}_0(\nu_\eta)/\overline{\beta}_0(\nu_r)$. This line gives a ray of the cone $\mathfrak{C}(\nu)$ which is defined by another line going through the origin with slope $\overline{\beta}_{g^*+1}(\nu_\eta)/\overline{\beta}_{g^*+1}(\nu_r)$. Both lines together with the line $x = \hat{\mu}_D(\nu_r)$ determine the triangle $\mathfrak{C}(\nu) \cap \mathfrak{H}_D(\nu)$ giving rise to the displayed vertices. Finally, Equality (3.1) completes the proof of (a).

Let us assume that p_{r+1} is a free point. It holds that $\overline{\beta}_j(\nu) = (\overline{\beta}_j(\nu_r), 0)$ for $j \in \{0, 1, \ldots, g^*\}$ and $\overline{\beta}_{g^*+1}(\nu) = (\overline{\beta}_{g+1}(\nu_r), 1)$, and consequently one obtains the triangle $\mathfrak{C}(\nu) \cap \mathfrak{H}_D(\nu)$. Finally, by Equality (3.1), we conclude the proof. □

Next, we provide a description of the Newton-Okounkov bodies with respect to a minimal exceptional curve valuation of Z_0. Before stating the result, we show a useful lemma.

Lemma 3.3.6. *Keep the notation introduced before. Assume that p_{r+1} is the satellite point $E_\eta \cap E_r, \eta \neq r$. Then,*

(a) It holds that

$$\overline{\beta}_{g+1}(\nu_r) = \left| \frac{\overline{\beta}_{g^*+1}(\nu_\eta)}{\overline{\beta}_{g^*+1}(\nu_r)} - \frac{\overline{\beta}_0(\nu_\eta)}{\overline{\beta}_0(\nu_r)} \right|^{-1}.$$

(b) If $\eta \preccurlyeq r$, then

$$\nu_r(\varphi_\eta) = \overline{\beta}_{g+1}(\nu_r) \cdot \frac{\overline{\beta}_{g^*+1}(\nu_\eta)}{\overline{\beta}_{g^*+1}(\nu_r)} \text{ and } \nu_r(\varphi_\eta) + 1 = \overline{\beta}_{g+1}(\nu_r) \cdot \frac{\overline{\beta}_0(\nu_\eta)}{\overline{\beta}_0(\nu_r)}.$$

(c) If $\eta \not\preccurlyeq r$, then

$$\nu_r(\varphi_\eta) = \overline{\beta}_{g+1}(\nu_r) \cdot \frac{\overline{\beta}_0(\nu_\eta)}{\overline{\beta}_0(\nu_r)} \text{ and } \nu_r(\varphi_\eta) + 1 = \overline{\beta}_{g+1}(\nu_r) \cdot \frac{\overline{\beta}_{g^*+1}(\nu_\eta)}{\overline{\beta}_{g^*+1}(\nu_r)}.$$

Proof. The first item is proved in [65, Lemma 3.9]. We are going to show Item (b). We can distinguish three cases:

Case 1: Assume that p_r is a free point. That is, $\eta = r-1, \eta \preccurlyeq r$ and $g = g^*$. By Corollary 1.3.7,

$$\nu_r(\varphi_\eta) + 1 = \overline{\beta}_{g+1}(\nu_r) = \overline{\beta}_{g+1}(\nu_r) \cdot \frac{\overline{\beta}_0(\nu_\eta)}{\overline{\beta}_0(\nu_r)}.$$

Finally, it holds that

$$\nu_r(\varphi_\eta) = \overline{\beta}_{g+1}(\nu_\eta) = \overline{\beta}_{g+1}(\nu_\eta) \cdot \frac{\overline{\beta}_{g+1}(\nu_r)}{\overline{\beta}_{g+1}(\nu_r)} = \overline{\beta}_{g+1}(\nu_r) \cdot \frac{\overline{\beta}_{g^*+1}(\nu_\eta)}{\overline{\beta}_{g^*+1}(\nu_r)},$$

which proves the result in this case.

Case 2: Let us show the case when p_r is a satellite point, p_η is a free point and $\eta < \ell_g$. That is, $\eta = \ell_g - 1, \eta \preccurlyeq r$ and $g = g^* + 1$. In addition, it holds that $\beta_i'(\nu_r) = \beta_i'(\nu_\eta)$ for $0 \leq i \leq g-1$ and $\nu_r(\varphi_\eta) = e_{g-1}(\nu_r)\nu_\eta(\varphi_\eta)$. As a consequence, it holds that

$$\nu_r(\varphi_\eta) = e_{g-1}(\nu_r)\nu_\eta(\varphi_\eta) = \overline{\beta}_{g+1}(\nu_r) \cdot \frac{\overline{\beta}_g(\nu_\eta)}{\overline{\beta}_g(\nu_r)} = \overline{\beta}_{g+1}(\nu_r) \cdot \frac{\overline{\beta}_{g^*+1}(\nu_\eta)}{\overline{\beta}_{g^*+1}(\nu_r)}$$

and

$$\nu_r(\varphi_\eta) + 1 = \overline{\beta}_g(\nu_r) = \frac{\overline{\beta}_{g+1}(\nu_r)}{e_{g-1}(\nu_r)} = \overline{\beta}_{g+1}(\nu_r) \cdot \frac{\overline{\beta}_0(\nu_\eta)}{\overline{\beta}_0(\nu_r)},$$

by Corollary 1.3.7 and the equality $\overline{\beta}_{g+1}(\nu_r) = e_{g-1}(\nu_r)\overline{\beta}_g(\nu_r)$.

Case 3: Suppose now that $\eta \preccurlyeq r$ and neither Case 1 nor Case 2 hold. Thus, $g = g^* + 1$ holds since p_r and p_{r+1} are satellite points. In this situation, one has

$$e_{g-1}(\nu_\eta)\overline{\beta}_g(\nu_r) > e_{g-1}(\nu_r)\overline{\beta}_g(\nu_\eta)) = \nu_r(\varphi_\eta),$$

by [65, Proposition 2.5]. Thus,

$$\nu_r(\varphi_\eta) = \overline{\beta}_{g+1}(\nu_r) \cdot \frac{\overline{\beta}_g(\nu_\eta)}{\overline{\beta}_g(\nu_r)} = \overline{\beta}_{g+1}(\nu_r) \cdot \frac{\overline{\beta}_{g^*+1}(\nu_\eta)}{\overline{\beta}_{g^*+1}(\nu_r)}.$$

Moreover, by [65, Lemma 3.9], $e_{g-1}(\nu_\eta)\overline{\beta}_g(\nu_r) - e_{g-1}(\nu_r)\overline{\beta}_g(\nu_\eta) = 1$ and then, by Corollary 1.3.7,

$$\nu_r(\varphi_\eta) + 1 = e_{g-1}(\nu_\eta)\overline{\beta}_g(\nu_r) = e_{g-1}(\nu_r)\overline{\beta}_g(\nu_r) \cdot \frac{\overline{\beta}_0(\nu_\eta)}{\overline{\beta}_0(\nu_r)} = \overline{\beta}_{g+1}(\nu_r) \cdot \frac{\overline{\beta}_0(\nu_\eta)}{\overline{\beta}_0(\nu_r)},$$

which completes the proof of (b).

Finally, Item (c) can be proved arguing as in the proof of Case 3 of (b). □

Theorem 3.3.7. *Let ν be an exceptional curve valuation of Z_0 and set D a big divisor on Z_0. Then, the Newton-Okounkov body $\Delta_\nu(D)$ of D with respect to ν coincides with the triangle $\mathfrak{C}(\nu) \cap \mathfrak{H}_D(\nu)$ if and only if ν is minimal with respect to D.*

Proof. We start by recalling that the triangle $\mathfrak{C}(\nu) \cap \mathfrak{H}_D(\nu)$ contains the Newton-Okounkov body $\Delta_\nu(D)$ by Proposition 3.3.5. In addition, the area of this triangle is

$$\frac{\hat{\mu}_D(\nu_r)^2}{2} \left| \frac{\overline{\beta}_{g^*+1}(\nu_\eta)}{\overline{\beta}_{g^*+1}(\nu_r)} - \frac{\overline{\beta}_0(\nu_\eta)}{\overline{\beta}_0(\nu_r)} \right| \left(\text{respectively, } \frac{\hat{\mu}_D(\nu_r)^2}{2\overline{\beta}_{g+1}(\nu_r)} \right)$$

when p_{r+1} is a satellite point (respectively, p_{r+1} is a free point). By Lemma 3.3.6, the area of the triangle $\mathfrak{C}(\nu) \cap \mathfrak{H}_D(\nu)$ in the satellite case becomes $\hat{\mu}_D(\nu_r)^2/2\overline{\beta}_{g+1}(\nu_r)$ and then, in both cases, the triangle $\mathfrak{C}(\nu) \cap \mathfrak{H}_D(\nu)$ has the same area. From Subsection 1.5.2 and (3.2), one can deduce that

$$\frac{\hat{\mu}_D(\nu_r)^2}{2\overline{\beta}_{g+1}(\nu_r)} \geq \frac{\operatorname{vol}(D)}{2} = \operatorname{vol}_{\mathbb{R}^2}(\Delta_\nu(D))$$

and therefore the triangle $\mathfrak{C}(\nu) \cap \mathfrak{H}_D(\nu)$ and the body Δ_ν coincide if and only if they have the same area, or equivalently, the valuation ν is minimal with respect to D. □

Remark 3.3.8. When $Z_0 = \mathbb{P}^2$, the above theorem has been proved for any big divisor D on $\mathbb{P}^2$. Notice that the Newton-Okounkov bodies $\Delta_\nu(D)$ described in the minimal case satisfy the homothetic property by Lemma 3.1.2.

Notice that, if we consider NPI exceptional curve valuations, then we can compute explicitly the values $\hat{\mu}_D(\nu_r)$ and therefore Newton-Okounkov bodies in the minimal case. Let us show and example:

Example 3.3.9. Let p be a special point of the Hirzebruch surface $\mathbb{F}_2$ and ν_r a special divisorial valuation centered at $\mathcal{O}_{\mathbb{F}_2,p}$ whose maximal contact values are $\{\overline{\beta}_i(\nu_r)\}_{i=0}^4 = \{12, 18, 117, 239, 720\}$. Set $\mathcal{C}_{\nu_r} = \{p_i\}_{i=0}^{25}$ (where $p = p_1$) its configuration of infinitely near points, F_1 the fiber containing p and M_0 the special section whose strict transform goes through p_2. Therefore

$$\nu_r(\varphi_{F_1}) = 12, \nu_r(\varphi_{M_0}) = 18 \text{ and } 2\nu_r(\varphi_{M_0})\nu_r(\varphi_{F_1}) + \delta\nu_r(\varphi_{F_1})^2 = 720.$$

Consider the divisor $D = 18F + 12M$. By Theorem 3.2.1 and Corollary 3.2.3, the value $\hat{\mu}_D(\nu_r)$ is equal to 720 and then ν_r is minimal with respect to D.

Let $\nu = \nu_{E_\bullet}$ be the valuation defined by the flag $E_\bullet = \{Z_{25} \supset E_{25} \supset \{p_{26}\}\}$, where $p_{26} \in E_{25} \cap E_{24}$, and whose first component is the previous valuation ν_r. The semigroup of values S_ν of ν is generated by

$$\left\{\overline{\beta}_j(\nu)\right\}_{i=0}^4 = \{(12, 12), (18, 18), (117, 117), (239, 239), (720, 719)\}$$

and, by Theorem 3.3.7, the coordinates of the Newton-Okounkov body $\Delta_\nu(D)$ (Figure 3.1(a)) of D with respect to ν are

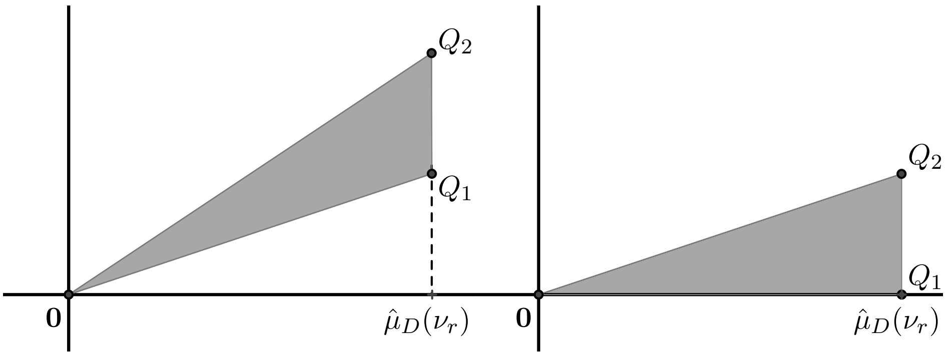
$$\mathbf{0} = (0, 0), Q_1 = (720, 720) \text{ and } Q_2 = (720, 719)\,.$$

If now we assume that p_{26} is a free point, we get another example. Here, the semigroup of values S_ν of ν is generated by

$$\{\overline{\beta}_j(\nu)\}_{i=0}^{4} = \{(12,0),(18,0),(117,0),(239,0),(720,1)\}$$

and the coordinates of the vertices of $\Delta_\nu(D)$ (Figure 3.1(b)) are

$$\mathbf{0} = (0,0), Q_1 = (720,0) \text{ and } Q_2 = (720,1).$$



(a) The point p_{r+1} is satellite. (b) The point p_{r+1} is free.

Figure 3.1: $\Delta_\nu(18F + 12M)$ in Example 3.3.9.

Explicitly computing Newton-Okounkov bodies with respect to non-minimal valuation is a hard task. Whenever $Z_0 = \mathbb{P}^2$, an explicit description of the vertices of these bodies can be found in [65, Theorems 3.12 and 3.14]. When considering a non-positive at infinity exceptional curve valuation ν of $\mathbb{P}^2$, the Newton-Okounkov body $\Delta_\nu(E_0)$ of a general projective line E_0 can be completely computed as we show in the next result, which will be proved later. The proof is a consequence of further results (see Corollary 3.3.29).

Theorem 3.3.10. *Let $E_\bullet = \{Z = Z_r \supset E_r \supset \{p_{r+1}\}\}$ be a flag and $\nu = \nu_{E_\bullet}$ its attached exceptional curve valuation. Assume that the first component ν_r of ν is an NPI divisorial valuation of $\mathbb{P}^2$. Denote by $\overline{\beta}(\nu)_{i=0}^{g^*+1}$ (respectively, $\overline{\beta}(\nu_r)_{i=0}^{g^*+1}$) the sequence of maximal contact values of ν (respectively, ν_r). Consider a general projective line E_0 and the projective line at infinity L as in Subsection 2.1. Then the Newton-Okounkov body of E_0 with respect to $\nu = \nu_{E_\bullet}$ is a triangle whose vertices have the following coordinates:*

(a)

$$\mathbf{0} = (0,0), Q_1 = \left(\frac{\overline{\beta}_{g+1}(\nu_r)}{\nu_r(\varphi_L)}, 0\right) \text{ and } Q_2 = (\nu_r(\varphi_L), 1),$$

when p_{r+1} is a free point and $\nu(\varphi_L) = \overline{\beta}_1(\nu)$.

(b)

$$\mathbf{0} = (0,0), Q_1 = \left(\frac{\overline{\beta}_{g+1}(\nu_r)}{\nu_r(\varphi_L)}, \frac{1}{\nu_r(\varphi_L)}\right) \text{ and } Q_2 = (\nu_r(\varphi_L), 0),$$

when p_{r+1} is a free point and $\nu(\varphi_L) \neq \overline{\beta}_1(\nu)$.

(c)

$$\mathbf{0} = (0,0), Q_1 = \left(\frac{\overline{\beta}_{g+1}(\nu_r)}{\nu_r(\varphi_L)}, \frac{\nu_r(\varphi_\eta)}{\nu_r(\varphi_L)}\right) \text{ and } Q_2 = (\nu_r(\varphi_L), \nu_\eta(\varphi_L)),$$

when $p_{r+1} \in E_\eta \cap E_r$ $(\eta \neq r)$, $\eta \preccurlyeq r$ and, either $g^* > 0$, or $g^* = 0$ and $\nu(\varphi_L) \neq \overline{\beta}_1(\nu)$. The latter points also describe the Newton-Okounkov body of E_0 if p_{r+1} is the satellite point $E_\eta \cap E_r$ $(\eta \neq r)$, $\eta \not\preccurlyeq r, g^* = 0$ and $\nu(\varphi_L) = \overline{\beta}_1(\nu)$.

(d)

$$\mathbf{0} = (0,0), Q_1 = \left(\frac{\overline{\beta}_{g+1}(\nu_r)}{\nu_r(\varphi_L)}, \frac{\nu_r(\varphi_\eta)+1}{\nu_r(\varphi_L)}\right) \text{ and } Q_2 = (\nu_r(\varphi_L), \nu_\eta(\varphi_L)),$$

when $p_{r+1} \in E_\eta \cap E_r$ $(\eta \neq r)$, $\eta \not\preccurlyeq r$ and, either $g^* > 0$, or $g^* = 0$ and $\nu(\varphi_L) \neq \overline{\beta}_1(\nu)$. The latter points also describe the Newton-Okounkov body of E_0 if p_{r+1} is the satellite point $E_\eta \cap E_r$ $(\eta \neq r)$, $\eta \preccurlyeq r, g^* = 0$ and $\nu(\varphi_L) = \overline{\beta}_1(\nu)$.

Let us see an example which corresponds to Theorem 3.3.10.

Example 3.3.11. Let ν_n be a divisorial valuation of $\mathbb{P}^2$ whose configuration of infinitely near points is $\mathcal{C}_{\nu_n} = \{p_i\}_{i=1}^{17}$ and whose sequence of maximal contact values is $\{8, 20, 63, 256\}$. Firstly, assume that the strict transforms of the projective line L pass through p_1 and p_2. Then, by Remark 3.1.7,

$$\hat{\mu}(\nu_n) = \nu_n(\varphi_L) = 16 \text{ and } \hat{\mu}(\nu_n)^2 = 256 = \overline{\beta}_{g+1}(\nu_n).$$

Consequently, the valuation ν_n is non-positive at infinity and minimal.

Set $\nu := \nu_{E_\bullet}$ the exceptional curve valuation of $\mathbb{P}^2$ which is defined by the flag $\{Z_{17} \supset E_{17} \supset \{p_{18}\}\}$. Its first coordinate is the above divisorial valuation ν_n. If we assume that p_{18} is satellite, then $p_{18} \in E_{16} \cap E_{17}$ and the Newton-Okounkov body $\Delta_\nu(E_0)$ of a general projective line E_0 with respect to ν is a triangle whose coordinates are

$$\mathbf{0} = (0,0),\ (16,16) \text{ and } \left(16, \frac{255}{16}\right).$$

If we suppose that p_{18} is a free point, then $\Delta_\nu(E_0)$ is the triangle with coordinates

$$\mathbf{0} = (0,0),\ (16,0) \text{ and } \left(16, \frac{1}{16}\right).$$

Now consider another configuration of infinitely near points $\mathcal{C}_{\nu_n}$ for which the strict transforms of the projective line L pass through p_1, p_2 and p_3. Then, ν_n is negative at infinity and non-minimal, since

$$\hat{\mu}(\nu_n) = \nu_n(\varphi_L) = 20 \text{ and } \hat{\mu}(\nu_n)^2 = 400 > 256 = \overline{\beta}_{g+1}(\nu_n).$$

Considering the exceptional curve valuation corresponding to a flag

$$\{Z_{17} \supset E_{17} \supset \{p_{18}\}\}$$

corresponding to this new configuration $\mathcal{C}_{\nu_n}$, it holds that the Newton-Okounkov body $\Delta_\nu(E_0)$ is determined by the vertices of coordinates

$$\mathbf{0} = (0,0), \ \left(\frac{256}{20}, \frac{255}{20}\right) \ \text{and} \ (20,20)$$

when $p_{18} \in E_{16} \cap E_{17}$, since $16 = \eta \preccurlyeq r = 17$. Otherwise, these vertices are

$$\mathbf{0} = (0,0), \ \left(\frac{256}{20}, \frac{1}{20}\right) \ \text{and} \ (20,0)\,.$$

In what follows, we assume that ν is an exceptional curve valuation of $\mathbb{F}_\delta, \delta \geq 0$, which is non-minimal with respect to a big divisor $D \sim aF + bM$ on $\mathbb{F}_\delta$. Its first component will be the divisorial valuation ν_r of $\mathbb{F}_\delta$.

The divisor D is also nef when $\delta = 0$. Otherwise ($\delta \neq 0$), D can be big and not nef. In this last case, the Zariski decomposition of the total transform D^* on $Z = Z_r$ of D is

$$P_{D^*} \sim \left(b + \frac{a}{\delta}\right) M^* \ \text{and} \ N_{D^*} = \frac{-a}{\delta}\tilde{M}_0 + \sum_{i=0}^{i_{M_0}} \frac{-a\nu_i(\varphi_{M_0})}{\delta} E_i,$$

where P_{D^*} (respectively, N_{D^*}) is the positive (respectively, negative) part of D^* and i_{M_0} indicates the last point in C_{ν_r} through which the strict transform of M_0 passes. In the following subsections, in virtue of Theorem 1.5.2 and [81, Lemma 1.10], we will distinguish two situations to compute $\Delta_\nu(D)$.

The first one corresponds to the case when the point p_{r+1} is not in the support of the divisor N_{D^*}, denoted by $\operatorname{supp}(N_{D^*})$. Here, we can also assume that the divisor D is nef. Indeed, when $D \sim aF + bM$ is big and not nef, then $b > 0$ and $-b\delta < a < 0$ and, by [81, Lemma 1.10], the Newton-Okounkov body $\Delta_\nu(D)$ satisfies

$$\Delta_\nu(D) = \Delta_\nu(P_D) = \left(b + \frac{a}{\delta}\right) \Delta_\nu(M).$$

Otherwise, the point p_{r+1} belongs to $\operatorname{supp}(N_{D^*})$. This fact happens if and only if $g^* = 0, p_1$ is a special point, all the points in $\{p_i\}_{i=1}^{r+1}$ are free, D is big but not nef, $i_{M_0} = r$ and $p_{r+1} \in \operatorname{supp}(\tilde{M}_0)$. Notice that we are in the first situation whenever $\delta = 0$.

We conclude by noting that, when $p_{r+1} \notin \operatorname{supp}(N_{D^*})$, one can write (3.2) as

$$\hat{\mu}_D(\nu_r) \geq \sqrt{D^2 \overline{\beta}_{g+1}(\nu_r)}, \tag{3.7}$$

since $\operatorname{vol}(D) = D^2$. Otherwise, we will replace D by P_D.

In the forthcoming subsections we will explicitly describe the Newton-Okounkov body of a divisor D as above with respect to an NPI exceptional curve valuation ν of $\mathbb{F}_\delta$. We begin with special valuations, where the first situation explained before could happen.

3.3.1 Newton-Okounkov bodies with respect to non-positive at infinity special valuations

Along this subsection we consider a big divisor $D \sim aF + bM$ on $\mathbb{F}_\delta, \delta \geq 0$, and an NPI special exceptional curve valuation ν of $\mathbb{F}_\delta$ which is non-minimal with respect to D. Also, we assume that ν_r is the first component of ν.

Set $Z = Z_r$ the rational surface that ν_r defines and $E_i, 1 \leq i \leq r$, the exceptional divisors obtained in the sequence of point blowups defined by ν_r. We denote by E_i^* (respectively, D^*) the total transform of the exceptional divisor E_i (respectively, the total transform of the divisor D) on Z.

For simplicity, the symbol $\theta_1^r(D)$ stands for the expression $a\nu_r(\varphi_{F_1}) - b\nu_r(\varphi_{M_0})$, where F_1 is the fiber which contains p and M_0 the special section; both on $\mathbb{F}_\delta$. If $\theta_1^r(D)$ vanishes, then either $a = b\nu_r(\varphi_{M_0})/\nu_r(\varphi_{F_1})$; or $\nu_r(\varphi_{M_0}) = 0$ and $a = 0$. Note that, under the second condition, some of the expressions we are going to introduce are not defined and they will not used when $\theta_1^r(D) = 0$. In addition, when $p_{r+1} \in \operatorname{supp}(N_{D^*})$, $\theta_1^r(D)$ is negative.

Firstly, we state (and prove) three lemmas which will help us to obtain the Zariski decomposition of some key divisors for our goal.

Lemma 3.3.12. *Let D be a big and nef divisor on $\mathbb{F}_\delta$ and ν_r an NPI special divisorial valuation of $\mathbb{F}_\delta$. Set $\theta_1^r(D)$ the value above defined. Then the divisor on Z*

$$D_1 = D^* - \frac{b}{\nu_r(\varphi_{F_1})} \sum_{i=1}^{r} \nu_r(\mathfrak{m}_i)E_i^* \left(\textit{respectively, } D_2 = D^* - \frac{a}{\nu_r(\varphi_{M_0})} \sum_{i=1}^{r} \nu_r(\mathfrak{m}_i)E_i^* \right)$$

is nef if $\theta_1^r(D) \geq 0$ (respectively, $\theta_1^r(D) < 0$).

Proof. We only show that D_1 is nef. The proof for D_2 runs similarly. Taking into account that b is positive, it holds that

$$\begin{aligned} D_1 &= D^* - \frac{b}{\nu_r(\varphi_{F_1})} \sum_{i=1}^{r} \nu_r(\mathfrak{m}_i)E_i^* \\ &\sim \frac{b}{\nu_r(\varphi_{F_1})} \left(\frac{a\nu_r(\varphi_{F_1})}{b} F^* + \nu_r(\varphi_{F_1})M^* - \sum_{i=1}^{r} \nu_r(\mathfrak{m}_i)E_i^* \right) \\ &= \frac{b}{\nu_r(\varphi_{F_1})} \left(\frac{\theta_1^r(D)}{b} F^* + \Lambda_r \right), \end{aligned}$$

where $\Lambda_r = \nu_r(\varphi_{M_0})F^* + \nu_r(\varphi_{F_1})M^* - \sum_{i=1}^{r} \nu_r(\mathfrak{m}_i)E_i^*$. As $\theta_1^r(D)$ is non-negative and F^* and Λ_r are nef divisors, by Theorem 2.3.7, D_1 is also nef. □

Lemma 3.3.13. *Let ν_r be an NPI special divisorial valuation of $\mathbb{F}_\delta$ and Z the surface defined by ν_r. Set $D \sim aF + bM$ a big divisor on $\mathbb{F}_\delta$ and, as above, denote by $\theta_1^r(D)$ the expression $a\nu_r(\varphi_{F_1}) - b\nu_r(\varphi_{M_0})$. Let t_1, t_2, t_3 and t_4 be the following four rational numbers:*

$$t_1 = \frac{b}{\nu_r(\varphi_{F_1})} \overline{\beta}_{g+1}(\nu_r), \quad t_2 = \frac{b}{\nu_r(\varphi_{F_1})} \overline{\beta}_{g+1}(\nu_r) + \theta_1^r(D),$$

$$t_3 = \frac{a}{\nu_r(\varphi_{M_0})}\overline{\beta}_{g+1}(\nu_r) \text{ and } t_4 = \frac{(a+b\delta)\overline{\beta}_{g+1}(\nu_r) - \theta_1^r(D)\nu_r(\varphi_{M_0})}{\nu_r(\varphi_{M_0}) + \delta\nu_r(\varphi_{F_1})}.$$

(1) Assume that the divisor D is nef. The values t_1 and t_2 (respectively, t_3 and t_4) belong to the set

$$T_{D,\nu_r} := \{t \in \mathbb{Q} \mid 0 \leq t \leq \hat{\mu}_D(\nu_r)\}$$

if $\theta_1^r(D) \geq 0$ (respectively, $\theta_1^r(D) < 0$).

(2) Assume that $p_{r+1} \in \operatorname{supp}(N_{D^*})$. Then, the value t_4 satisfies

$$0 < -a\nu_r(\varphi_{M_0})/\delta < t_4 \leq \hat{\mu}_D(\nu_r).$$

Proof. We only prove that $t_1, t_2 \leq \hat{\mu}_D(\nu_r)$ for the first part. A proof for the values t_3 and t_4 in our second assertion follows from an analogous reasoning.

We start by proving that $t_1 \leq \hat{\mu}_D(\nu_r)$ when $\theta_1^r(D)$ is non-negative. Consider the nef divisor on Z, $D_1 = D^* - \frac{b}{\nu_r(\varphi_{F_1})}\sum_{i=1}^r \nu_r(\mathfrak{m}_i)E_i^*$, defined in Lemma 3.3.12. For any curve $C \in |mD|, m \in \mathbb{Z}_{>0}$, it holds that

$$m(2ab + b^2\delta) - \frac{b}{\nu_r(\varphi_{F_1})}\nu_r(\varphi_C) = D_1 \cdot \tilde{C} \geq 0,$$

where $\tilde{C}$ is the strict transform of C under the birational map that ν_r defines. Consequently, one has

$$2ab + b^2\delta \geq \frac{b}{\nu_r(\varphi_{F_1})}\hat{\mu}_D(\nu_r),$$

and together with (3.7) we obtain

$$\hat{\mu}_D(\nu_r) \geq \frac{D^2\overline{\beta}_{g+1}(\nu_r)}{\hat{\mu}_D(\nu_r)} = \frac{(2ab + b^2\delta)\overline{\beta}_{g+1}(\nu_r)}{\hat{\mu}_D(\nu_r)} \geq \frac{b\overline{\beta}_{g+1}(\nu_r)}{\nu_r(\varphi_{F_1})}, \tag{3.8}$$

which show our claim.

Now we are going to prove that $t_2 \leq \hat{\mu}_D(\nu_r)$ if $\theta_1^r(D) \geq 0$. Using Theorem 3.2.1, it is sufficient to see

$$b\,\overline{\beta}_{g+1}(\nu_r) \leq b\left(2\nu_r(\varphi_{M_0})\nu_r(\varphi_{F_1}) + \delta\nu_r(\varphi_{F_1})^2\right),$$

which is true by Theorem 2.3.7 and the fact that b is positive.

To conclude, we show (2), that is, $0 < -a\nu_r(\varphi_{M_0})/\delta < t_4$ when $D \sim aF + bM$ is big and $p_{r+1} \in \operatorname{supp}(N_{D^*})$. It is easy to check that

$$t_4 + \frac{a\nu_r(\varphi_{M_0})}{\delta} = \frac{(a+b\delta)(\nu_r(\varphi_{M_0})^2 + \delta\overline{\beta}_{g+1}(\nu_r))}{\delta(\nu_r(\varphi_{M_0}) + \delta\nu_r(\varphi_{F_1}))} > 0,$$

where the inequality holds since, as mentioned at the end of the last subsection, D is big but not nef and then $-b\delta < a < 0$, which completes the proof. □

A consequence of the above lemma and Theorem 3.2.1 is the following result.

Corollary 3.3.14. *Let D be a big divisor on $\mathbb{F}_\delta, \delta \geq 0$, and ν_r an NPI special divisorial valuation of $\mathbb{F}_\delta$. Consider the values $\theta_1^r(D)$, t_2 and t_4 given in Lemma 3.3.13. Then,*

(a) *Assume that $\theta_1^r(D) \geq 0$, then $2\nu_r(\varphi_{M_0})\nu_r(\varphi_{F_1}) + \delta\nu_r(\varphi_{F_1})^2 = \overline{\beta}_{g+1}(\nu_r)$ if and only if $t_2 = \hat{\mu}_D(\nu_r)$.*

(b) *Otherwise ($\theta_1^r(D) < 0$), the equality $2\nu_r(\varphi_{M_0})\nu_r(\varphi_{F_1}) + \delta\nu_r(\varphi_{F_1})^2 = \overline{\beta}_{g+1}(\nu_r)$ holds if and only if $t_4 = \hat{\mu}_D(\nu_r)$.*

Proof. We only show a proof for Item (a). A similar argument shows Item (b). We start assuming that $2\nu_r(\varphi_{M_0})\nu_r(\varphi_{F_1}) + \delta\nu_r(\varphi_{F_1})^2 = \overline{\beta}_{g+1}(\nu_r)$. Applying the above equality to the expression of t_2 in Lemma 3.3.13, the next equalities follow:

$$\begin{aligned} t_2 &= 2b\nu_r(\varphi_{M_0}) + \delta b\nu_r(\varphi_{F_1}) + a\nu_r(\varphi_{F_1}) - b\nu_r(\varphi_{M_0}) \\ &= (a + b\delta)\nu_r(\varphi_{F_1}) + b\nu_r(\varphi_{M_0}) = \hat{\mu}_D(\nu_r), \end{aligned}$$

which show one implication. For the other one, assume that $t_2 = \hat{\mu}_D(\nu_r)$. Then,

$$0 = \hat{\mu}_D(\nu_r) - t_2 = b\delta\nu_r(\varphi_{F_1}) + 2b\nu_r(\varphi_{M_0}) - \frac{b}{\nu_r(\varphi_{F_1})}\overline{\beta}_{g+1}(\nu_r),$$

and, as b and $\nu_r(\varphi_{F_1})$ are positive, the result follows by multiplying $\nu_r(\varphi_{F_1})/b$. □

Remark 3.3.15. Some extra information can be given on the values $t_i, 1 \leq i \leq 4$.

(a) The valuation ν_r is minimal with respect to a big and nef divisor D on $\mathbb{F}_\delta$ if and only if

$$\hat{\mu}_D(\nu_r) = \frac{b\overline{\beta}_{g+1}(\nu_r)}{\nu_r(\varphi_{F_1})} = t_1 = t_2 \left(= \frac{a\overline{\beta}_{g+1}(\nu_r)}{\nu_r(\varphi_{M_0})} = t_3 = t_4, \text{ when } \nu_r(\varphi_{M_0}) \neq 0 \right).$$

In fact, the equalities $t_1 = t_2$ and $\hat{\mu}_D(\nu_r) = t_2$ are equivalent to those given in Corollary 3.2.3 by Corollary 3.3.14.

(b) Assume that ν_r is non-minimal with respect to a big and nef divisor D on $\mathbb{F}_\delta$. Then,

(b.1) The value $\theta_1^r(D)$ vanishes if and only if $\hat{\mu}_D(\nu_r) > t_1 = t_2 (= t_3 = t_4$, when $\nu_r(\varphi_{M_0}) \neq 0)$.

(b.2) Consider that $\theta_1^r(D) > 0$. The inequality

$$2\nu_r(\varphi_{M_0})\nu_r(\varphi_{F_1}) + \delta\nu_r(\varphi_{F_1})^2 \geq \overline{\beta}_{g+1}(\nu_r)$$

holds if and only if $\hat{\mu}_D(\nu_r) \geq t_2 > t_1 > 0$.

(b.3) Otherwise ($\theta_1^r(D) < 0$). The conditions

$$2\nu_r(\varphi_{M_0})\nu_r(\varphi_{F_1}) + \delta\nu_r(\varphi_{F_1})^2 \geq \overline{\beta}_{g+1}(\nu_r) \text{ and } a \geq 0$$

hold if and only if $\hat{\mu}_D(\nu_r) \geq t_4 > t_3 \geq 0$.

Lemma 3.3.16. *Keep the notation of Lemma 3.3.13. Assume that ν_r is non-minimal with respect to a big divisor $D \sim aF + bM$ on $\mathbb{F}_\delta$.*

(1) Suppose that D is also a nef divisor. Then, the intersection matrices defined by the sets $\{\tilde{F}_1, E_1, E_2, \ldots, E_{r-1}\}$ and $\{\tilde{M}_0, E_1, E_2, \ldots, E_{r-1}\}$ are negative definite.

(2) Suppose that $p_{r+1} \in \operatorname{supp}(N_{D^*})$, it holds that the set $\{\tilde{M}_0, E_1, E_2, \ldots, E_{r-1}\}$ determines a negative definite intersection matrix.

Proof. Set D_1 the divisor given in Lemma 3.3.12. There we proved that D_1 is nef. Now we are going to show that it is also a big divisor. Indeed, as ν_r is non-minimal with respect to D and taking into account (3.8), one gets the following inequalities

$$\frac{D^2\overline{\beta}_{g+1}(\nu_r)}{\hat{\mu}_D(\nu_r)} \geq \frac{b\overline{\beta}_{g+1}(\nu_r)}{\nu_r(\varphi_{F_1})} \text{ and } 1 > \frac{D^2\overline{\beta}_{g+1}(\nu_r)}{\hat{\mu}_D(\nu_r)^2},$$

which allow us to show that $D_1^2 > 0$ since

$$D_1^2 = D^2 - \frac{b^2\overline{\beta}_{g+1}(\nu_r)}{\nu_r(\varphi_{F_1})^2} \geq D^2 - \frac{b\hat{\mu}_D(\nu_r)}{\nu_r(\varphi_{F_1})}\left(\frac{D^2\overline{\beta}_{g+1}(\nu_r)}{\hat{\mu}_D(\nu_r)^2}\right) > D^2 - \frac{b\hat{\mu}_D(\nu_r)}{\nu_r(\varphi_{F_1})} \geq 0.$$

As a consequence, using Theorem 1.4.7, it holds that D_1 is big. In addition, D_1 is orthogonal to $\tilde{F}_1$ and to E_i, for $1 \leq i \leq r-1$, and consequently the set $\{\tilde{F}_1, E_1, E_2, \ldots, E_{r-1}\}$ generates a negative definite intersection matrix by Proposition 1.4.12.

The remaining cases follow from a similar argument using, either the divisor D_2 defined in Lemma 3.3.12 or the big and nef divisor $(b + a/\delta)M^*$. □

The following proposition provides the positive and negative parts of the Zariski decomposition of certain divisors on Z which we will use to describe the Newton-Okounkov body of a big divisor on $\mathbb{F}_\delta, \delta \geq 0$.

Proposition 3.3.17. *Let ν_r be an NPI special divisorial valuation of $\mathbb{F}_\delta$ and $Z = Z_r$ the rational surface that ν_r defines. Set ν_i the NPI special divisorial valuation which the divisor E_i defines, for $i \in \{1, 2, \ldots, r-1\}$. Consider a big divisor $D \sim aF + bM$ on $\mathbb{F}_\delta$ and assume that ν_r is non-minimal with respect to D. Set $\theta_1^r(D) = a\nu_r(\varphi_{F_1}) - b\nu_r(\varphi_{M_0})$ and the divisor on Z, $\Lambda_r = \nu_r(\varphi_{M_0})F^* + \nu_r(\varphi_{F_1})M^* - \sum_{i=1}^r \nu_r(\mathfrak{m}_i)E_i^*$. Let D_1 and D_2 be the divisors given in Lemma 3.3.12 and t_1, t_2, t_3 and t_4 the rational numbers defined in Lemma 3.3.13.*

(1) Assume that D is also a nef divisor.

(a) Suppose that $\theta_1^r(D) \geq 0$. The positive and negative parts of the Zariski decomposition of the divisors on Z $D_{t_1} := D^* - t_1E_r$, and $D_{t_2} := D^* - t_2E_r$

are

$$P_{D_{t_1}} \sim D_1 \quad and \quad N_{D_{t_1}} = \frac{b}{\nu_r(\varphi_{F_1})} \sum_{i=1}^{r-1} \nu_r(\varphi_i) E_i,$$

$$and\ P_{D_{t_2}} \sim \frac{b}{\nu_r(\varphi_{F_1})} \Lambda_r \ and$$

$$N_{D_{t_2}} = \frac{\theta_1^r(D)}{\nu_r(\varphi_{F_1})} \tilde{F}_1 + \sum_{i=1}^{r-1} \frac{b\nu_r(\varphi_i) + \theta_1^r(D)\nu_i(\varphi_{F_1})}{\nu_r(\varphi_{F_1})} E_i.$$

(b) *Otherwise, the positive and negative parts of the Zariski decomposition of the divisors $D_{t_3} := D^* - t_3 E_r$, and $D_{t_4} := D^* - t_4 E_r$ are*

$$P_{D_{t_3}} \sim D_2 \quad and \quad N_{D_{t_3}} = \frac{a}{\nu_r(\varphi_{M_0})} \sum_{i=1}^{r-1} \nu_r(\varphi_i) E_i,$$

$$and\ P_{D_{t_4}} \sim \frac{a + b\delta}{\nu_r(\varphi_{M_0}) + \delta\nu_r(\varphi_{F_1})} \Lambda_r \ and$$

$$N_{D_{t_4}} = \left(\frac{-\theta_1^r(D)}{\nu_r(\varphi_{M_0}) + \delta\nu_r(\varphi_{F_1})} \right) \tilde{M}_0 + \sum_{i=1}^{r-1} \frac{(a + b\delta)\nu_r(\varphi_i) - \theta_1^r(D)\nu_i(\varphi_{M_0})}{\nu_r(\varphi_{M_0}) + \delta\nu_r(\varphi_{F_1})} E_i.$$

(2) *Assume that $p_{r+1} \in \operatorname{supp}(N_{D^*})$. Then the previous divisors $P_{D_{t_4}}$ and $N_{D_{t_4}}$ are the positive and negative parts of D_{t_4}.*

Proof. We only show a proof for (a); a proof for the remaining cases follows from a similar argument. We begin with the Zariski decomposition of D_{t_1}. It is immediate that $P_{D_{t_1}} + N_{D_{t_1}} \sim D_{t_1}$. In addition, the divisor $P_{D_{t_1}}$ is nef by Lemma 3.3.12. Finally, each component of $N_{D_{t_1}}$ is orthogonal to $P_{D_{t_1}}$ by the proximity equalities, and they determine an intersection matrix which is negative definite.

To conclude, we will prove the result for D_{t_2}. The divisor $P_{D_{t_2}}$ is nef and orthogonal to each component of $N_{D_{t_2}}$ by Proposition 2.3.1 and Theorem 2.3.7. Moreover, it is clear that the intersection matrix given by the components of $N_{D_{t_2}}$ is negative definite by Lemma 3.3.16. Finally, the fact that $D_{t_2} \sim P_{D_{t_2}} + N_{D_{t_2}}$ follows from summing the next two expressions

$$D - \frac{b}{\nu_r(\varphi_{F_1})} \overline{\beta}_{g+1}(\nu_r) E_r \sim \frac{b}{\nu_r(\varphi_{F_1})} \Lambda_r + \frac{\theta_1^r(D)}{\nu_r(\varphi_{F_1})} F^* + \frac{b}{\nu_r(\varphi_{F_1})} \sum_{i=1}^{r-1} \nu_r(\varphi_i) E_i$$

and

$$-\theta_1^r(D) E_r = \frac{\theta_1^r(D)}{\nu_r(\varphi_{F_1})} \left(\sum_{i=1}^{r-1} \nu_i(\varphi_{F_1}) E_i - \sum_{i=1}^{i_{F_1}} E_i^* \right),$$

and from considering that $\tilde{F}_1 \sim F^* - \sum_{i=1}^{i_{F_1}} E_i^*$, where i_{F_1} indicates the last point in the configuration of infinitely near points $\mathcal{C}_{\nu_r}$ of the valuation ν_r through which the strict transform of F_1 goes, which proves the result. □

Remark 3.3.18. Keep the notation used at the end of Subsection 1.5.2, in Lemmas 3.3.12, 3.3.13 and 3.3.16 and in Proposition 3.3.17. The ray $[D_t] := [D^*]-t[E_r]$, where $0 \leq t \leq \hat{\mu}_D(\nu_r)$, crosses the interior of $\mathrm{NE}(Z)$ heading towards the face of $\mathrm{NE}(Z)$ spanned by the classes of the divisors $\tilde{F}_1, \tilde{M}_0, E_1, E_2, \dots,\ E_{r-1}$. Indeed, notice that, when t is a small enough value, it holds that

$$\mathrm{Null}(D_t) = \{E_1, E_2, \dots, E_{r-1}\} = \mathrm{Neg}(D_t).$$

When $\theta_1^r(D) > 0$, the ray $[D^*] - t[E_r]$ is contained in the boundary of a Zariski chamber for the values t_1 and t_2 since

$$\mathrm{Neg}(D_{t_1}) = \{E_1, E_2, \dots, E_{r-1}\} \text{ and } \mathrm{Null}(P_{D_{t_1}}) = \{\tilde{F}_1, E_1, E_2, \dots, E_{r-1}\},$$

and

$$\mathrm{Neg}(D_{t_2}) = \{\tilde{F}_1, E_1, E_2, \dots, E_{r-1}\} \text{ and } \mathrm{Null}(P_{D_{t_2}}) = \{\tilde{F}_1, \tilde{M}_0, E_1, E_2, \dots, E_{r-1}\}.$$

An analogous result happens if $\theta_1^r(D) < 0$. In this case the ray $[D^*] - t[E_r]$ is in the boundary of a Zariski chamber for the values t_3 and t_4 because

$$\mathrm{Neg}(D_{t_3}) = \{E_1, E_2, \dots, E_{r-1}\} \text{ and } \mathrm{Null}(P_{D_{t_3}}) = \{\tilde{M}_0, E_1, E_2, \dots, E_{r-1}\},$$

and

$$\mathrm{Neg}(D_{t_4}) = \{\tilde{M}_0, E_1, E_2, \dots, E_{r-1}\} \text{ and } \mathrm{Null}(P_{D_{t_4}}) = \{\tilde{F}_1, \tilde{M}_0, E_1, E_2, \dots, E_{r-1}\}.$$

Finally, for $\theta_1^r(D) = 0$, the ray is in the boundary of several Zariski chambers for the value $t_1 = t_2$, since

$$\mathrm{Neg}(D_{t_1}) = \{E_1, E_2, \dots, E_{r-1}\} \text{ and } \mathrm{Null}(P_{D_{t_1}}) = \{\tilde{F}_1, \tilde{M}_0, E_1, E_2, \dots, E_{r-1}\}.$$

Figure 3.2 depicts the above situations, where each case has a different color; the vertical line, denoted $[\Lambda_n]^\perp$, represents the face of $\mathrm{NE}(Z)$ spanned by the classes of the divisors $\tilde{F}_1, \tilde{M}_0, E_1, E_2, \dots,\ E_{r-1}$; and the Zariski chambers $\Sigma_1, \Sigma_2, \Sigma_3$ and Σ_4 below defined are delimited by discontinuous lines.

$$\Sigma_1 := \big\{[\xi] \in \mathrm{Big}(Z) \,|\, \mathrm{Neg}(\xi) = \mathrm{Null}(\xi) = \{[E_1], [E_2], \dots, [E_{r-1}]\}\big\},$$
$$\Sigma_2 := \big\{[\xi] \in \mathrm{Big}(Z) \,|\, \mathrm{Neg}(\xi) = \mathrm{Null}(\xi) = \{[\tilde{F}_1], [E_1], [E_2], \dots, [E_{r-1}]\}\big\},$$
$$\Sigma_3 := \big\{[\xi] \in \mathrm{Big}(Z) \,|\, \mathrm{Neg}(\xi) = \mathrm{Null}(\xi) = \{[\tilde{M}_0], [E_1], [E_2], \dots, [E_{r-1}]\}\big\} \text{ and}$$
$$\Sigma_4 := \big\{[\xi] \in \mathrm{Big}(Z) \,|\, \mathrm{Neg}(\xi) = \mathrm{Null}(\xi) = \{[\tilde{F}_1], [\tilde{M}_0], [E_1], [E_2], \dots, [E_{r-1}]\}\big\}.$$

Now we are going to state the three main results in this subsection. Recall that $D \sim aF + bM$ is a big divisor on $\mathbb{F}_\delta$ and ν an NPI special exceptional curve valuation which is non-minimal with respect to D and whose first component is ν_r.

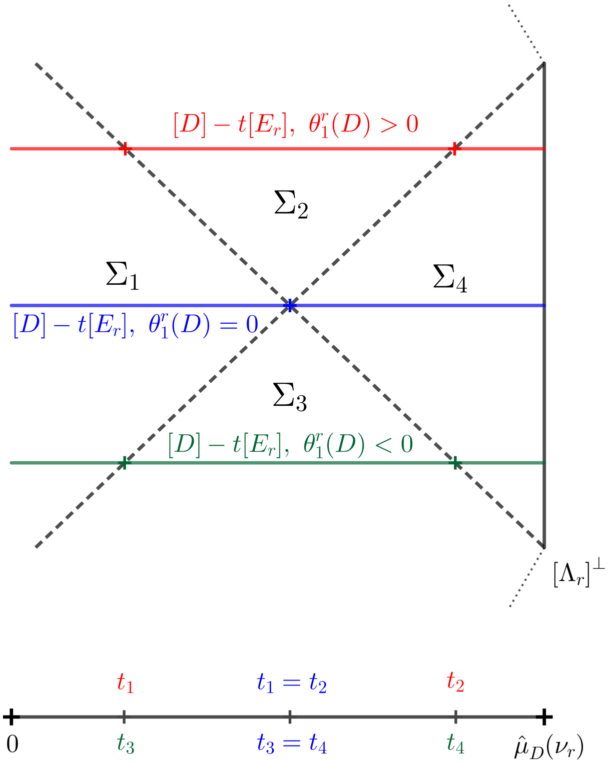


Figure 3.2: Local description of the cone of curves $\mathrm{NE}(Z)$ of a rational surface Z given by a non-positive at infinity special divisorial valuation of $\mathbb{F}_\delta$.

The results explicitly describe the Newton-Okounkov bodies $\Delta_\nu(D)$. We distinguish three cases:

Case A: Either $g^*>0$, or $g^*=0, \nu(\varphi_{F_1})\neq\overline{\beta}_1(\nu)$ and $\nu(\varphi_{M_0})\neq\overline{\beta}_1(\nu)$.

Case B: The integer g^* equals 0 and $\nu(\varphi_{F_1})=\overline{\beta}_1(\nu)$.

Case C: The integer g^* equals 0 and $\nu(\varphi_{M_0})=\overline{\beta}_1(\nu)$.

Before starting our description for Case A, we give a useful property.

Lemma 3.3.19. *Let ν be an NPI special exceptional curve valuation of $\mathbb{F}_\delta$ and $\mathcal{C}_\nu=\{p_i\}_{i\geq 1}$ its configuration of infinitely near points such that $p_i\to p_r$ for all $i>r$. Consider the cases described before.*

(a) Suppose that p_{r+1} is the satellite point $E_\eta\cap E_r, \eta\neq r$.

(a.1) Assume we are in Case A. Then,

$$\nu_\eta(\varphi_{F_1})=\nu_r(\varphi_{F_1})\cdot\frac{\overline{\beta}_0(\nu_\eta)}{\overline{\beta}_0(\nu_r)}\ \text{and}\ \nu_\eta(\varphi_{M_0})=\nu_r(\varphi_{M_0})\cdot\frac{\overline{\beta}_0(\nu_\eta)}{\overline{\beta}_0(\nu_r)}.$$

(a.2) Assume we are in Case B. Then,

$$\nu_\eta(\varphi_{F_1})=\nu_r(\varphi_{F_1})\cdot\frac{\overline{\beta}_{g^*+1}(\nu_\eta)}{\overline{\beta}_{g^*+1}(\nu_r)}\ \text{and}\ \nu_\eta(\varphi_{M_0})=\nu_r(\varphi_{M_0})\cdot\frac{\overline{\beta}_0(\nu_\eta)}{\overline{\beta}_0(\nu_r)}.$$

(a.3) Assume we are in Case C. Then,

$$\nu_\eta(\varphi_{F_1}) = \nu_r(\varphi_{F_1}) \cdot \frac{\overline{\beta}_0(\nu_\eta)}{\overline{\beta}_0(\nu_r)} \text{ and } \nu_\eta(\varphi_{M_0}) = \nu_r(\varphi_{M_0}) \cdot \frac{\overline{\beta}_{g^*+1}(\nu_\eta)}{\overline{\beta}_{g^*+1}(\nu_r)}.$$

(b) Otherwise (p_{r+1} is a free point).

(b.1) Assume we are in Case A. Then,

$$\nu(\varphi_{F_1}) = (\nu_r(\varphi_{F_1}), 0) \text{ and } \nu(\varphi_{M_0}) = (\nu_r(\varphi_{M_0}), 0).$$

(b.2) Assume we are in Case B. Then,

$$\nu(\varphi_{F_1}) = (\nu_r(\varphi_{F_1}), 1) \text{ and } \nu(\varphi_{M_0}) = (\nu_r(\varphi_{M_0}), 0).$$

(b.3) Assume we are in Case C. Then,

$$\nu(\varphi_{F_1}) = (\nu_r(\varphi_{F_1}), 0) \text{ and } \nu(\varphi_{M_0}) = (\nu_r(\varphi_{M_0}), 1).$$

Proof. The result easily follows from the next observations. The value $\nu(\varphi_{F_1})$ equals $\overline{\beta}_1(\nu)$ when the strict transform of the fiber F_1 goes through all initial free points of $\mathcal{C}_\nu$. Otherwise, $\nu(\varphi_{F_1}) = s_{F_1}\overline{\beta}_0(\nu)$ for a positive integer s_{F_1}. Similarly, the value $\nu(\varphi_{M_0})$ is equals to either $\overline{\beta}_1(\nu)$, or $s_{M_0}\overline{\beta}_0(\nu)$, for a non-negative integer s_{M_0}. Finally at most one of the strict transforms of F_1 and M_0 pass through p_2. □

We begin with Case A. *Here D can be also considered a nef divisor without loss of generality, since $\Delta_\nu(D)$ with D big and not nef can be obtained as we said in the paragraphs before Subsection 3.3.1*. Following Subsection 1.5.2, the Newton-Okounkov body $\Delta_\nu(D)$ can be seen as the set

$$\{(t, y) \in \mathbb{R}^2 \mid 0 \leq t \leq \hat{\mu}_D(\nu_r) \text{ and } \alpha(t) \leq y \leq \beta(t)\}, \tag{3.9}$$

where $\alpha(t) := \mathrm{ord}_{p_{r+1}}(N_{D_t}|_{E_r})$ and $\beta(t) := \alpha(t) + P_{D_t} \cdot E_r$ for all $t \in [0, \hat{\mu}_D(\nu_r)]$; P_{D_t} and N_{D_t} being the positive part and the negative part of the divisor on Z $D_t = D^* - tE_r$, respectively. As a result, using Proposition 3.3.17, some points which belong to $\Delta_\nu(D)$ are

$$Q_1 = \left(\frac{b\overline{\beta}_{g+1}(\nu_r)}{\nu_r(\varphi_{F_1})}, \frac{b\nu_r(\varphi_\eta)}{\nu_r(\varphi_{F_1})}\right) \left(\text{respectively, } Q_1 = \left(\frac{b\overline{\beta}_{g+1}(\nu_r)}{\nu_r(\varphi_{F_1})}, 0\right)\right),$$
$$Q_2 = Q_1 + \left(0, \frac{b}{\nu_r(\varphi_{F_1})}\right),$$
$$Q_3 = \left(\frac{b\overline{\beta}_{g+1}(\nu_r)}{\nu_r(\varphi_{F_1})} + \theta_1^r(D), \frac{b\nu_r(\varphi_\eta) + \theta_1^r(D)\nu_\eta(\varphi_{F_1})}{\nu_r(\varphi_{F_1})}\right)$$
$$\left(\text{respectively, } Q_3 = \left(\frac{b\overline{\beta}_{g+1}(\nu_r)}{\nu_r(\varphi_{F_1})} + \theta_1^r(D), 0\right)\right) \text{ and } Q_4 = Q_3 + \left(0, \frac{b}{\nu_r(\varphi_{F_1})}\right) \tag{3.10}$$

when $\theta_1^r(D) \geq 0$ and p_{r+1} is the satellite point $E_\eta \cap E_r$ (respectively, a free point). If $\theta_1^r(D) < 0$ and p_{r+1} is the satellite point $E_\eta \cap E_r$ (respectively, a free point), then the points which are contained in $\Delta_\nu(D)$ are

$$Q_5 = \left(\frac{a\overline{\beta}_{g+1}(\nu_r)}{\nu_r(\varphi_{M_0})}, \frac{a\nu_r(\varphi_\eta)}{\nu_r(\varphi_{M_0})}\right) \left(\text{respectively, } Q_5 = \left(\frac{a\overline{\beta}_{g+1}(\nu_r)}{\nu_r(\varphi_{M_0})}, 0\right)\right),$$
$$Q_6 = Q_5 + \left(0, \frac{a}{\nu_r(\varphi_{M_0})}\right),$$
$$Q_7 = \left(\frac{(a+b\delta)\overline{\beta}_{g+1}(\nu_r) - \theta_1^r(D)\nu_r(\varphi_{M_0})}{\nu_r(\varphi_{M_0}) + \delta\nu_r(\varphi_{F_1})}, \frac{(a+b\delta)\nu_r(\varphi_\eta) - \theta_1^r(D)\nu_\eta(\varphi_{M_0})}{\nu_r(\varphi_{M_0}) + \delta\nu_r(\varphi_{F_1})}\right)$$
$$\left(\text{respectively, } Q_7 = \left(\frac{(a+b\delta)\overline{\beta}_{g+1}(\nu_r) - \theta_1^r(D)\nu_r(\varphi_{M_0})}{\nu_r(\varphi_{M_0}) + \delta\nu_r(\varphi_{F_1})}, 0\right)\right)$$
$$\text{and } Q_8 = Q_7 + \left(0, \frac{a+b\delta}{\nu_r(\varphi_{M_0}) + \delta\nu_r(\varphi_{F_1})}\right). \tag{3.11}$$

The point $Q_9 = (\hat{\mu}_D(\nu_r), \hat{\mu}_D(\nu_\eta))$ (respectively, $Q_9 = (\hat{\mu}_D(\nu_r), 0)$) if p_{r+1} is satellite (respectively, free) is also contained in $\Delta_\nu(D)$ by definition. This last point can be explicitly computed using Theorem 3.2.1.

Remark 3.3.20. In this remark, we give some observations about the previous points Q_i. They are the following ones.

(a) The value $\theta_1^r(D)$ vanishes if and only if $Q_1 = Q_3 (= Q_5 = Q_7$ when $\nu_r(\varphi_{M_0}) \neq 0)$ and $Q_2 = Q_4 (= Q_6 = Q_8$ when $\nu_r(\varphi_{M_0}) \neq 0)$. Moreover, if $\theta_1^r(D) < 0$, it holds that $\delta > 0$ and $a = 0$ if and only if $Q_5 = (0,0) = Q_6$.

(b) Some of the above points Q_i are collinear by Lemma 3.3.6 and Lemma 3.3.19: Indeed,

(b.1) Assume that p_{r+1} is the satellite point $E_\eta \cap E_r, \eta \neq r$. The points $(0,0), Q_2, Q_4$ (respectively, Q_6, Q_8) and Q_9 are in the line $T_1 \equiv \overline{\beta}_0(\nu_r)y = \overline{\beta}_0(\nu_\eta)x$ when $\eta \preccurlyeq r$ and $\theta_1^r(D) \geq 0$ (respectively, $\theta_1^r(D) < 0$). If $\eta \not\preccurlyeq r$ and $\theta_1^r(D) \geq 0$ (respectively, $\theta_1^r(D) < 0$), then $(0,0), Q_1, Q_3$ (respectively, Q_5, Q_7) and Q_9 are in T_1.

(b.2) Suppose that p_{r+1} is a free point. The points $(0,0), Q_1, Q_3$ (respectively, Q_5, Q_7) and Q_9 are in the line $y = 0$.

Let us state our main result for the Case A.

Theorem 3.3.21. *Let ν be an exceptional curve valuation. Assume that ν belongs to Case A defined before Lemma 3.3.19. Following the notations of the above paragraphs, the Newton-Okounkov body $\Delta_\nu(D)$ of a big and nef divisor $D \sim aF + bM$ on $\mathbb{F}_\delta$ with respect to ν is a quadrilateral if and only if $a \neq 0$ and $\theta_1^r(D) \neq 0$. Otherwise, it is a triangle (because one of the conditions of Remark 3.3.20(a) happens).*

The vertices of $\Delta_\nu(D)$ are

(a) $(0,0), Q_1, Q_3$ *(respectively,* Q_5, Q_7*) and* Q_9 *if* p_{r+1} *is the satellite point* $E_\eta \cap E_r$, $\eta \neq r$, $\eta \preccurlyeq r$ *and* $\theta_1^r(D) \geq 0$ *(respectively,* $\theta_1^r(D) < 0$*).*

(b) $(0,0), Q_2, Q_4$ *(respectively,* Q_6, Q_8*) and* Q_9 *if* p_{r+1} *is the satellite point* $E_\eta \cap E_r$, $\eta \neq r$, $\eta \not\preccurlyeq r$ *and* $\theta_1^r(D) \geq 0$ *(respectively,* $\theta_1^r(D) < 0$*).*

(c) $(0,0), Q_2, Q_4$ *(respectively,* Q_6, Q_8*) and* Q_9 *if* p_{r+1} *is a free point and* $\theta_1^r(D) \geq 0$ *(respectively,* $\theta_1^r(D) < 0$*).*

Proof. We first prove that the sets of points $\{(0,0), Q_1, Q_2, Q_3, Q_4, Q_9\}$ and $\{(0,0), Q_5, Q_6, Q_7, Q_8, Q_9\}$ generate convex hulls, denoted by Δ and Δ' respectively, whose area is equal to $D^2/2$.

We begin with Δ. Consider the triangle $(0,0), Q_1$ and Q_2 (respectively, Q_3, Q_4 and Q_9). Its area is

$$\frac{b^2\overline{\beta}_{g+1}(\nu_r)}{2\nu_r(\varphi_{F_1})^2}\left(\text{respectively, } \frac{b}{2\nu_r(\varphi_{F_1})}\left(\hat{\mu}_D(\nu_r) - \left(\frac{b}{\nu_r(\varphi_{F_1})}\overline{\beta}_{g+1}(\nu_r) + \theta_1^r(D)\right)\right)\right).$$

Now take the parallelogram Q_1, Q_2, Q_3 and Q_4. It is immediate that its area is $\frac{b}{\nu_r(\varphi_{F_1})}\theta_1^r(D)$. Therefore, summing the previous areas yields that of Δ which equals

$$\frac{2ab + b^2\delta}{2} = \frac{D^2}{2}.$$

Proceeding analogously with Δ', we sum the area of the triangles with vertices $(0,0), Q_5$ and Q_6, and Q_7, Q_8 and Q_9 together with that of the trapezium with vertices Q_5, Q_6, Q_7 and Q_8. It is a simple matter to see that the areas of the triangles are $\frac{a^2}{2\nu_r(\varphi_{M_0})^2}\overline{\beta}_{g+1}(\nu_r)$ and

$$\frac{a + b\delta}{2(\nu_r(\varphi_{M_0}) + \delta\nu_r(\varphi_{F_1}))}\left(\hat{\mu}_D(\nu_r) - \left(\frac{(a + b\delta)\overline{\beta}_{g+1}(\nu_r) - \theta_1^r(D)\nu_r(\varphi_{M_0})}{\nu_r(\varphi_{M_0}) + \delta\nu_r(\varphi_{F_1})}\right)\right).$$

Now we consider the trapezium. The length of its parallel sides (generated by Q_5 and Q_6, and Q_7 and Q_8) and the distance between them are

$$\frac{a}{\nu_r(\varphi_{M_0})}, \quad \frac{a + b\delta}{\nu_r(\varphi_{M_0}) + \delta\nu_r(\varphi_{F_1})} \text{ and}$$
$$\frac{-\theta_1^r(D)(\delta\overline{\beta}_{g+1}(\nu_r) + \nu_r(\varphi_{M_0})^2)}{\nu_r(\varphi_{M_0})(\nu_r(\varphi_{M_0}) + \delta\nu_r(\varphi_{F_1}))}.$$

Then the area of the trapezium is

$$\frac{-\theta_1^r(D)\left((2a + b\delta)\nu_r(\varphi_{M_0}) + a\delta\nu_r(\varphi_{F_1})\right)\left(\delta\overline{\beta}_{g+1}(\nu_r) + \nu_r(\varphi_{M_0})^2\right)}{2\nu_r(\varphi_{M_0})^2(\nu_r(\varphi_{M_0}) + \delta\nu_r(\varphi_{F_1}))^2}.$$

Continuing the process, when we sum the previous areas, the first observation is that the coefficients of $\overline{\beta}_{g+1}(\nu_r)$ are cancelled. Therefore, it suffices to sum the following

fractions
$$\frac{(a+b\delta)\hat{\mu}_D(\nu_r)}{2(\nu_r(\varphi_{M_0})+\delta\nu_r(\varphi_{F_1}))}, \frac{\theta_1^r(D)(a+b\delta)\nu_r(\varphi_{M_0})}{2(\nu_r(\varphi_{M_0})+\delta\nu_r(\varphi_{F_1}))^2} \text{ and}$$
$$\frac{-\theta_1^r(D)\nu_r(\varphi_{M_0})^2((2a+b\delta b)\nu_r(\varphi_{M_0})+a\delta\nu_r(\varphi_{F_1}))}{2\nu_r(\varphi_{M_0})^2(\nu_r(\varphi_{M_0})+\delta\nu_r(\varphi_{F_1}))^2},$$
which provides the desired value $D^2/2$.

If we look at the vertices of Δ and Δ', some of the defining points of Δ and Δ' can be removed depending on the assumptions by Remark 3.3.20(b). As a result, the Newton-Okounkov body $\Delta_\nu(D)$ is a triangle or a quadrilateral.

To conclude the proof we are going to see that $\Delta_\nu(D)$ is a triangle if and only if one of the situations of Remark 3.3.20 (a) happens.

Suppose, for instance, that p_{r+1} is a satellite point and $\eta \not\preccurlyeq r$. The remaining cases run similarly. Take $\theta_1^r(D) \geq 0$. Here the Newton-Okounkov body $\Delta_\nu(D)$ is a triangle if and only if one of the following conditions holds: either the line going through Q_2 and Q_9 contains Q_4, or the point Q_4 is in the line $T_2 \equiv \overline{\beta}_{g^*+1}(\nu_\eta)x = \overline{\beta}_{g^*+1}(\nu_r)y$. Both options are true if and only if $Q_2 = Q_4$, or equivalently, $\theta_1^r(D) = 0$. Now consider $\theta_1^r(D) < 0$. In this situation, the Newton-Okounkov body $\Delta_\nu(D)$ is a triangle if and only if one of the following conditions happens: the line passing through Q_6 and Q_9 contains Q_8; the point Q_8 is in the line T_2; or $Q_5 = (0,0) = Q_6$. As above, the first and second conditions are satisfied if and only if $\theta_1^r(D) = 0$, which contradicts our assumption ($\theta_1^r(D) < 0$). By Remark 3.3.20, the third one holds if and only if $\delta > 0$ and $a = 0$, which completes the proof. □

Let us show an example of Newton-Okounkov body $\Delta_\nu(D)$ which corresponds to Theorem 3.3.21 (a).

Example 3.3.22. Let ν_r be a special divisorial valuation of $\mathbb{F}_\delta$ (centered at $\mathcal{O}_{\mathbb{F}_2,p}$), where p is a special point of $\mathbb{F}_2$, and $\{\overline{\beta}_i(\nu_r)\}_{i=0}^3 = \{20, 28, 153,\ 612\}$ its sequence of maximal contact values. Set $\mathcal{C}_{\nu_r} = \{p_i\}_{i=1}^{12}$ (with $p = p_1$) the configuration of infinitely near points of ν and write F_1 the fiber containing p. Assume that the strict transform of M_0 only passes through p_2. Therefore, $\nu_r(\varphi_{F_1}) = 20$, $\nu_r(\varphi_{M_0}) = 28$ and $2\nu_r(\varphi_{F_1})\nu_r(\varphi_{M_0}) + \nu_r(\varphi_{F_1})^2\delta = 1920 > 612 = \overline{\beta}_{g+1}(\nu_r)$. That is, ν_r is non-positive at infinity by Theorem 2.3.7.

Set $\nu = \nu_{E_\bullet}$ the valuation defined by the flag $E_\bullet = \{Z = Z_{12} \supset E_{12} \supset \{p_{13}\}\}$, where $p_{13} \in E_8 \cap E_{12}$, and whose first coordinate is the above divisorial valuation ν_r. Following Theorem 3.3.21, the Newton-Okounkov body $\Delta_\nu(F+2M)$ is a quadrilateral and its vertices are
$$\mathbf{0} = (0,0), Q_5 = \left(\frac{612}{28}, \frac{152}{28}\right), Q_7 = \left(\frac{4068}{68}, \frac{1012}{68}\right) \text{ and } Q_9 = (156, 39),$$
since ν_r is non-minimal with respect to $F + 2M$ by Corollary 3.2.4, $\theta_1^r(F+2M) < 0$ and $8 \preccurlyeq 12$. Figure 3.3 shows the Newton-Okounkov body $\Delta_\nu(F+2M)$ (in dark) and the triangle $\mathfrak{C}(\nu) \cap \mathfrak{H}_{F+2M}(\nu)$ given in Proposition 3.3.5.

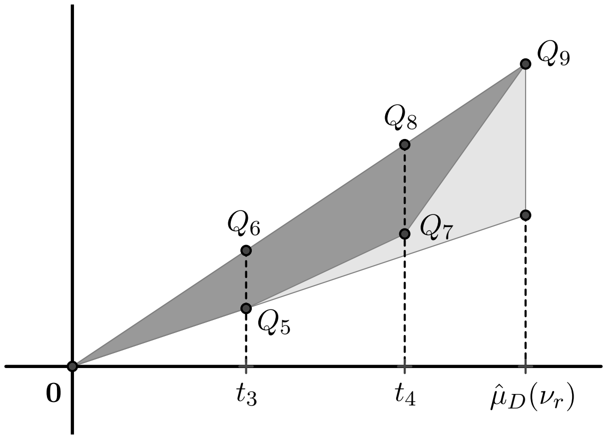


Figure 3.3: $\Delta_\nu(F+2M)$ and $\mathfrak{C}(\nu) \cap \mathfrak{H}_{F+2M}(\nu)$ in Example 3.3.22.

Next we formulate a similar result to Theorem 3.3.21 for Case B described before Lemma 3.3.19. That is, we assume that ν is a valuation in Case B and then $g^* = 0$ and $\nu(\varphi_{F_1}) = \overline{\beta}_1(\nu)$. As the first point in $\mathbb{F}_\delta$ where we blow up can be special or general, it may happen that $\nu(\varphi_{M_0}) = (0,0)$. In this last case the value $\theta_1^r(D) = a\nu_r(\varphi_{F_1})$ is non-negative. *Moreover, we can suppose that D is big and nef, since $\Delta_\nu(D)$ with D big and not nef can be obtained as we said in the paragraphs before Subsection 3.3.1.*

Notice that $\Delta_\nu(D)$ can also be described as the convex set (3.9). So, using Proposition 3.3.17, when p_{r+1} is the satellite point $E_\eta \cap E_r$ and $\theta_1^r(D) \geq 0$ (respectively, $\theta_1^r(D) < 0$), the points Q_1, Q_2, Q_3, Q_4 (respectively, Q_5, Q_6, Q_7, Q_8) and Q_9 given in (3.10) (respectively, (3.11)) for the satellite case belong to $\Delta_\nu(D)$. Otherwise (p_{r+1} is free), the points

$$Q_1 = \left(\frac{b\overline{\beta}_{g+1}(\nu_r)}{\nu_r(\varphi_{F_1})}, 0 \right), Q_2 = Q_1 + \left(0, \frac{b}{\nu_r(\varphi_{F_1})}\right),$$

$$Q_3 = \left(\frac{b\overline{\beta}_{g+1}(\nu_r)}{\nu_r(\varphi_{F_1})} + \theta_1^r(D), \frac{\theta_1^r(D)}{\nu_r(\varphi_{F_1})} \right) \text{ and } Q_4 = Q_3 + \left(0, \frac{b}{\nu_r(\varphi_{F_1})}\right)$$

(respectively, Q_5, Q_6, Q_7, Q_8 given in (3.11) for the free case) and

$$Q_9 = (\hat{\mu}_D(\nu_r), a + b\delta)$$

are in $\Delta_\nu(D)$ if $\theta_1^r(D) \geq 0$ (respectively, $\theta_1^r(D) < 0$).

Before determining the vertices of the Newton-Okounkov body in Case B, we show some situations where the previous points are aligned.

Remark 3.3.23. As in Case A, the points Q_i described in the last but one paragraph satisfy the following properties.

(a) The statement in Remark 3.3.20(a) remains true.

(b) Assume that $\nu(\varphi_{M_0}) = (0,0)$. It follows from Lemma 3.3.6 and Lemma 3.3.19 that the points $(0,0), Q_1, Q_3$ (respectively, Q_2, Q_4) and Q_9 are in the line $\overline{\beta}_{g^*+1}(\nu_\eta)x = \overline{\beta}_{g^*+1}(\nu_r)y$ when p_{r+1} is the satellite point $E_\eta \cap E_r, \eta \neq r$, and $\eta \preccurlyeq r$ (respectively, $\eta \not\preccurlyeq r$), since $\hat{\mu}_D(\nu_r) = a\nu_r(\varphi_{F_1})$ and we are in Case B. If p_{r+1} is a free point, then $(0,0), Q_2, Q_4$ and Q_9 belong to the line $y = x/\overline{\beta}_{g+1}(\nu_r)$.

(c) Assume $\nu(\varphi_{M_0}) \neq (0,0)$. As a consequence of Lemma 3.3.6 and Lemma 3.3.19, some of the before described points Q_i are collinear. Indeed,

(c.1) Suppose that p_{r+1} is the satellite point $E_\eta \cap E_r, \eta \neq r$. The points $(0,0), Q_1$ and Q_3 (respectively, $(0,0), Q_6$ and Q_8) belong to the line $T_2 \equiv \overline{\beta}_{g^*+1}(\nu_r)y = \overline{\beta}_{g^*+1}(\nu_\eta)x$ (respectively, $T_1 \equiv \overline{\beta}_0(\nu_r)y = \overline{\beta}_0(\nu_\eta)x$), and the point Q_4 (respectively, Q_7) is contained in the line which passes through Q_2 and Q_9 (respectively, Q_5 and Q_9), when $\eta \preccurlyeq r$ and $\theta_1^r(D) \geq 0$ (respectively, $\theta_1^r(D) < 0$).

(c.2) If $p_{r+1} \in E_\eta \cap E_r, \eta \neq r$, $\eta \not\preccurlyeq r$ and $\theta_1^r(D) \geq 0$ (respectively, $\theta_1^r(D) < 0$), then the points $(0,0), Q_2$ and Q_4 (respectively, $(0,0), Q_5$ and Q_7) are in the previous line T_2 (respectively, T_1), and Q_3 (respectively, Q_8) belongs to the line which goes through Q_1 and Q_9 (respectively, Q_6 and Q_9).

(c.3) Otherwise (p_{r+1} is a free point), the points $(0,0), Q_2$ and Q_4 (respectively, $(0,0), Q_5$ and Q_7) are in the line $y = x/\overline{\beta}_{g+1}(\nu_r)$ (respectively, $y = 0$), and Q_3 (respectively, Q_8) is contained in the line which passes through Q_1 and Q_9 (respectively, Q_6 and Q_9) when $\theta_1^r(D) \geq 0$ (respectively, $\theta_1^r(D) < 0$).

Theorem 3.3.24. *Let ν be an exceptional curve valuation in Case B described before Lemma 3.3.19. With notations as in the previous paragraphs, the Newton-Okounkov body $\Delta_\nu(D)$ of a big and nef divisor $D \sim aF + bM$ on $\mathbb{F}_\delta$ with respect to ν is a quadrilateral if and only if $a \neq 0$. Otherwise, it is a triangle (see Remark 3.3.23).*

(a) When $\nu(\varphi_{M_0}) = (0,0)$, the vertices of the quadrilateral are

(a.1) $(0,0), Q_2, Q_4$ (respectively, Q_1, Q_3) and Q_9 if p_{r+1} is the satellite point $E_\eta \cap E_r, \eta \neq r$, and $\eta \preccurlyeq r$, (respectively, $\eta \not\preccurlyeq r$).

(a.2) $(0,0), Q_1, Q_3$ and Q_9 whenever p_{r+1} is a free point.

In addition, if $\delta > 0$ and $a = 0$, then the vertices of the triangle $\Delta_\nu(D)$ are the above ones, where $Q_1 = Q_3$ and $Q_2 = Q_4$.

(b) When $\nu(\varphi_{M_0}) \neq (0,0)$, the vertices of the quadrilateral are

(b.1) $(0,0), Q_2, Q_3$ (respectively, Q_5, Q_8) and Q_9 if $\theta_1^r(D) \geq 0$ (respectively, $\theta_1^r(D) < 0$), p_{r+1} is the satellite point $E_\eta \cap E_r, \eta \neq r$, and $\eta \preccurlyeq r$.

(b.2) $(0,0), Q_1, Q_4$ *(respectively,* Q_6, Q_7*) and* Q_9 *if* $\theta_1^r(D) \geq 0$ *(respectively,* $\theta_1^r(D) < 0$*),* p_{r+1} *is the satellite point* $E_\eta \cap E_r, \eta \neq r$*, and* $\eta \not\preccurlyeq r$*.*

(b.3) $(0,0), Q_1, Q_4$ *(respectively,* Q_6, Q_7*) and* Q_9 *if* $\theta_1^r(D) \geq 0$ *(respectively,* $\theta_1^r(D) < 0$*) and* p_{r+1} *is a free point.*

Moreover, if $\delta > 0$ *and* $a = 0$*, the vertices of the triangle* $\Delta_\nu(D)$ *are the above ones where* $Q_5 = (0,0) = Q_6$*.*

Proof. Take the convex hulls defined by the points $\{(0,0), Q_1, Q_2, Q_3, Q_4, Q_9\}$ and $\{(0,0), Q_5, Q_6, Q_7, Q_8, Q_9\}$. Reasoning as in the proof of Theorem 3.3.21, we deduce that the area of both sets is $D^2/2$. In addition, taking into account Remark 3.3.23, one gets (a). Finally, checking that Q_9 does not belong to neither the line $T_1 \equiv \overline{\beta}_0(\nu_r)y = \overline{\beta}_0(\nu_\eta)x$ nor the line $T_2 \equiv \overline{\beta}_{g^*+1}(\nu_r)y = \overline{\beta}_{g^*+1}(\nu_\eta)x$, we obtain (b) by Remark 3.3.23, which proves the result. □

Example 3.3.25. Let ν_r be a special divisorial valuation of $\mathbb{F}_2$ and $\{\overline{\beta}_i(\nu_r)\}_{i=0}^2 = \{2, 5, 10\}$ its sequence of maximal contact values. Set $\mathcal{C}_{\nu_r} = \{p_i\}_{i=1}^4$ the configuration of infinitely near points of ν_r such that p_1 is a special point. Moreover, assume that the strict transforms of F_1 (the fiber passing through p_1) go through p_2 and p_3. Therefore, $\nu_r(\varphi_{F_1}) = 5$ and $\nu_r(\varphi_{M_0}) = 2$ and then

$$2\nu_r(\varphi_{M_0})\nu_r(\varphi_{F_1}) + \delta\nu_r(\varphi_{F_1})^2 = 70 > 10 = \overline{\beta}_{g+1}(\nu_r).$$

As a result, the divisorial valuation ν_r is non-positive at infinite.

Let $\nu = \nu_{E_\bullet}$ be the exceptional valuation associated to the flag $E_\bullet = \{Z = Z_4 \supset E_4 \supset \{p_{r+1}\}\}$ and whose first component is the last divisorial valuation ν_r. In addition, suppose that p_{r+1} is the satellite point $E_3 \cap E_4$. Then, its sequence of maximal contact values is $\{\overline{\beta}_0(\nu), \overline{\beta}_1(\nu)\} = \{(2,1), (5,3)\}$, $\nu(\varphi_{F_1}) = (5,3) = \overline{\beta}_1(\nu)$ and $\nu(\varphi_{M_0}) = (2,1)$. As a result, we are in Case B. Assume that $D = F + M$. By Theorem 3.2.1, $\hat{\mu}_D(\nu_r) = (a + b\delta)\nu_r(\varphi_{F_1}) + b\nu_r(\varphi_{M_0}) = 17$ and so ν is non-minimal respect to D (since $\hat{\mu}_D(\nu_r)^2 > D^2\overline{\beta}_{g+1}(\nu_r)$). Therefore, by Theorem 3.3.24, the Newton-Okounkov body $\Delta_\nu(D)$ is the convex hull generated by

$$(0,0), Q_1 = (2,1), Q_4 = (5,3) \text{ and } Q_9 = (17,10),$$

because $\theta_1^r(D) = a\nu_r(\varphi_{F_1}) - b\nu_r(\varphi_{M_0}) = 3$ and $3 = \eta \not\preccurlyeq r = 4$. One can see in Figure 3.4 the Newton-Okounkov $\Delta_\nu(F+M)$ in dark and the triangle $\mathfrak{C}(\nu) \cap \mathfrak{H}_{F+M}(\nu)$ given in Proposition 3.3.5.

To conclude this subsection, we describe the Newton-Okounkov body $\Delta_\nu(D)$ in Case C introduced before Lemma 3.3.19. Therefore, assume that $g^* = 0$ and $\nu(\varphi_{M_0}) = \overline{\beta}_1(\nu)$. *Here, we can suppose that* D *is a big and nef divisor except for the case when all the points* $\{p_i\}_{i=1}^{r+1}$ *are free. In this last situation,* $p_{r+1} \in \operatorname{supp}(N_{D^*})$ *if and only if* D *is big and not nef.*

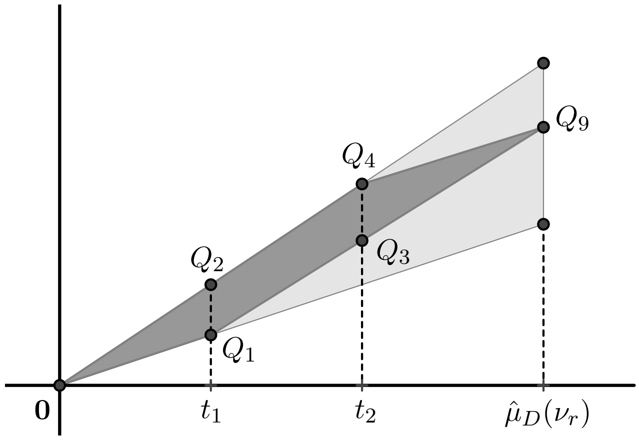


Figure 3.4: $\Delta_\nu(F+M)$ and $\mathfrak{C}(\nu)\cap\mathfrak{H}_{F+M}(\nu)$ in Example 3.3.25.

Let us start assuming that D is big and nef. As above in Case B, $\Delta_\nu(D)$ can be described as in (3.9). So, by Proposition 3.3.17, when $\theta_1^r(D)\geq 0$ (respectively, $\theta_1^r(D)<0$) and p_{r+1} is the satellite point $E_\eta\cap E_r$, the points Q_1,Q_2,Q_3,Q_4 (respectively, Q_5,Q_6, Q_7,Q_8) and Q_9 provided in (3.10) (respectively, (3.11)) for the satellite situation belong to $\Delta_\nu(D)$.

If p_{r+1} is a free point and $\theta_1^r(D)<0$ (respectively, $\theta_1^r(D)\geq 0$), the points

$$Q_5=\left(\frac{a\overline{\beta}_{g+1}(\nu_r)}{\nu_r(\varphi_{M_0})},0\right), Q_6=Q_5+\left(0,\frac{a}{\nu_r(\varphi_{M_0})}\right),$$
$$Q_7=\left(\frac{(a+b\delta)\overline{\beta}_{g+1}(\nu_r)-\theta_1^r(D)\nu_r(\varphi_{M_0})}{\nu_r(\varphi_{M_0})+\delta\nu_r(\varphi_{F_1})},\frac{-\theta_1^r(D)}{\nu_r(\varphi_{M_0})+\delta\nu_r(\varphi_{F_1})}\right),$$
$$Q_8=Q_7+\left(0,\frac{a+b\delta}{\nu_r(\varphi_{M_0})+\delta\nu_r(\varphi_{F_1})}\right)$$

(respectively, Q_1,Q_2,Q_3,Q_4 given in (3.10) for the free case) and $Q_9=(\hat{\mu}_D(\nu_r),b)$ are in $\Delta_\nu(D)$.

Finally, assume that D is big and not nef and all the points in $\{p_i\}_{i=1}^{r+1}$ are free. Recall that these hypothesis are equivalent to the fact that $p_{r+1}\in\operatorname{supp}(N_{D^*})$ (see the paragraphs before Subsection 3.3.1). Therefore, the points

$$P_1=\left(\frac{-a\nu_r(\varphi_{M_0})}{\delta},\frac{-a}{\delta}\right),$$
$$P_2=\left(\frac{(a+b\delta)\overline{\beta}_{g+1}(\nu_r)-\theta_1^r(D)\nu_r(\varphi_{M_0})}{\nu_r(\varphi_{M_0})+\delta\nu_r(\varphi_{F_1})},\frac{-\theta_1^r(D)}{\nu_r(\varphi_{M_0})+\delta\nu_r(\varphi_{F_1})}\right),$$
$$P_3=P_2+\left(0,\frac{a+b\delta}{\nu_r(\varphi_{M_0})+\delta\nu_r(\varphi_{F_1})}\right)\text{ and }P_4=(\hat{\mu}_D(\nu_r),b)$$

belong to $\Delta_\nu(D)$.

Remark 3.3.26. As the above cases, the points $Q_i, 1 \leq i \leq 9$, and $P_j, 1 \leq j \leq 4$, satisfy the following properties.

(a) The statement in Remark 3.3.20(a) holds if D is a big and nef divisor.

(b) When D is big and nef, some points Q_i are collinear by Lemma 3.3.6 and Lemma 3.3.19. Indeed,

(b.1) If p_{r+1} is the satellite point $E_\eta \cap E_r, \eta \neq r, \eta \preccurlyeq r$ and $\theta_1^r(D) \geq 0$ (respectively, $\theta_1^r(D)$), then the points $(0,0), Q_2$ and Q_4 (respectively, $(0,0), Q_5$ and Q_7) are in the line $T_1 \equiv \overline{\beta}_0(\nu_r)y = \overline{\beta}_0(\nu_\eta)x$ (respectively, $T_2 \equiv \overline{\beta}_{g^*+1}(\nu_r)y = \overline{\beta}_{g^*+1}(\nu_\eta)x$), and the point Q_3 (respectively, Q_8) belongs to the line which goes through Q_1 and Q_9 (respectively, Q_6 and Q_9).

(b.2) When $p_{r+1} \in E_\eta \cap E_r, \eta \neq r, \eta \not\preccurlyeq r$ and $\theta_1^r(D) \geq 0$ (respectively, $\theta_1^r(D) < 0$), the points $(0,0), Q_1$ and Q_3 (respectively, $(0,0), Q_6$ and Q_8) are in the line T_1 (respectively, T_2) and the point Q_4 (respectively, Q_7) is contained in the line which passes through Q_2 and Q_9 (respectively, Q_5 and Q_9).

(b.3) Otherwise (p_{r+1} is a free point), the points $(0,0), Q_1$ and Q_3 (respectively, $(0,0), Q_6$ and Q_8) are in the line $y = 0$ (respectively, $y = x/\overline{\beta}_{g+1}(\nu_r)$) and the point Q_4 (respectively, Q_7) belongs to the line which passes through Q_2 and Q_9 (respectively, Q_5 and Q_9).

(c) If $p_{r+1} \in \mathrm{supp}(N_{D^*})$, then $\theta_1^r(D) < 0$ and the point P_2 is contained in the line which goes through P_1 and P_4, and P_1 and P_3 belong to the line $y = x/\overline{\beta}_{g+1}(\nu_r)$.

Now, we formulate our result for Case C describing the vertices of the Newton-Okounkov body $\Delta_\nu(D)$. Notice that, as said, D is big and nef except when p_{r+1} is in $\mathrm{supp}(N_{D^*})$. Recall that the Newton-Okounkov bodies $\Delta_\nu(D)$ for the remaining cases where D is big but not nef can be reduced to the big and nef situation (see the paragraphs before Subsection 3.3.1).

Theorem 3.3.27. *Let ν be an exceptional curve valuation as described in Case C before Lemma 3.3.19. With assumptions and notations as in the previous paragraphs, the Newton-Okounkov body $\Delta_\nu(D)$ of a big divisor $D \sim aF + bM$ on $\mathbb{F}_\delta$ with respect to ν is a quadrilateral if and only if $a \neq 0$ and D is nef. Otherwise, it is a triangle.*

(a) When D is a big and nef divisor, then the vertices of the quadrilateral are

(a.1) $(0,0), Q_1, Q_4$ (respectively, Q_6, Q_7) and Q_9 if $\theta_1^r(D) \geq 0$ (respectively, $\theta_1^r(D) < 0$), p_{r+1} is the satellite point $E_\eta \cap E_r, \eta \neq r$, and $\eta \preccurlyeq r$.

(a.2) $(0,0), Q_2, Q_3$ (respectively, Q_5, Q_8) and Q_9 if $\theta_1^r(D) \geq 0$ (respectively, $\theta_1^r(D) < 0$), p_{r+1} is the satellite point $E_\eta \cap E_r, \eta \neq r$, and $\eta \not\preccurlyeq r$.

(a.3) $(0,0), Q_2, Q_3$ (respectively, Q_5, Q_8) and Q_9 if $\theta_1^r(D) \geq 0$ (respectively, $\theta_1^r(D) < 0$) and p_{r+1} is a free point.

Moreover, if $\delta > 0$ and $a = 0$, the vertices of the triangle $\Delta_\nu(D)$ are the above ones where $Q_5 = (0,0) = Q_6$.

(b) If D is big but not nef and all the points in $\{p_i\}_{i=1}^{r+1}$ are free, then the vertices of the triangle $\Delta_\nu(D)$ are P_1, P_3 and P_4.

Proof. It easy see that (a) follows as in Theorem 3.3.24 (b) using Lemma 3.3.6 and Lemma 3.3.19.

Let us prove (b). Firstly, we are going to see that the area of the convex hull Δ generated by P_1, P_2, P_3 and P_4 is $P_{D^*}^2/2$. Indeed, the area of the triangle generated by P_1, P_2 and P_3 (respectively, P_2, P_3 and P_4) is

$$\frac{(a+b\delta)\left(\frac{(a+b\delta)\overline{\beta}_{g+1}(\nu_r)-\theta_1^r(D)\nu_r(\varphi_{M_0})}{\nu_r(\varphi_{M_0})+\delta\nu_r(\varphi_{F_1})} - \frac{-a\nu_r(\varphi_{M_0})}{\delta}\right)}{2(\nu_r(\varphi_{M_0})+\delta\nu_r(\varphi_{F_1}))}$$

$$\left(\text{respectively, } \frac{(a+b\delta)\left(\hat{\mu}_D(\nu_r) - \frac{(a+b\delta)\overline{\beta}_{g+1}(\nu_r)-\theta_1^r(D)\nu_r(\varphi_{M_0})}{\nu_r(\varphi_{M_0})+\delta\nu_r(\varphi_{F_1})}\right)}{2(\nu_r(\varphi_{M_0})+\delta\nu_r(\varphi_{F_1}))}\right).$$

Summing both areas, we have

$$\begin{aligned}\frac{(a+b\delta)\left(\hat{\mu}_D(\nu_r) - \frac{-a\nu_r(\varphi_{M_0})}{\delta}\right)}{2(\nu_r(\varphi_{M_0})+\delta\nu_r(\varphi_{F_1}))} &= \frac{(a+b\delta)\left(b+\frac{a}{\delta}\right)(\nu_r(\varphi_{M_0})+\delta\nu_r(\varphi_{F_1}))}{2(\nu_r(\varphi_{M_0})+\delta\nu_r(\varphi_{F_1}))}\\ &= \frac{\left(\left(b+\frac{a}{\delta}\right)M^*\right)^2}{2} = \frac{P_{D^*}^2}{2},\end{aligned}$$

which is the desired value. After taking account Remark 3.3.26(c), the proof is concluded. □

Example 3.3.28. Let ν_r be a special divisorial valuation of the Hirzebruch surface $\mathbb{F}_3$ and $\mathcal{C}_{\nu_r} = \{p_i\}_{i=1}^3$ its configuration of infinitely near point, where all points are free and p_1 is special. Assume that the strict transforms of the special section M_0 pass through p_2 and p_3. Consequently, $\nu_r(\varphi_{F_1}) = 1, \nu_r(\varphi_{M_0}) = 3$ and

$$2\nu_r(\varphi_{M_0})\nu_r(\varphi_{F_1}) + \delta\nu_r(\varphi_{F_1})^2 = 9 > 3 = \overline{\beta}_{g+1}(\nu_r).$$

Thus ν_r is non-positive at infinity.

Set $\nu = \nu_{E_\bullet}$ the valuation associated to the flag $E_\bullet := \{Z = Z_3 \supset E_3 \supset \{p_{r+1}\}\}$, where p_{r+1} is a free point, such that its first component is the above divisorial valuation ν_r. The sequence of maximal contact values of ν is $\{\overline{\beta}_0(\nu_r), \overline{\beta}_1(\nu_r)\} = \{(1,0),(3,1)\}$. Assume that the strict transform of M_0 passes through p_{r+1}. Consider the divisor $D = -2F + M$, which is big and not nef. Therefore, we are in Case C. By Theorem 3.2.1, $\hat{\mu}_D(\nu_r) = (a+b\delta)\nu_r(\varphi_{F_1}) + b\nu_r(\varphi_{M_0}) = 4$ and then ν is non-minimal respect to D (because $\hat{\mu}_D(\nu_r)^2 > D^2\overline{\beta}_{g+1}(\nu_r)$). In addition, $\theta_1^r(D) = a\nu_r(\varphi_{F_1}) - b\nu_r(\varphi_{M_0}) = -5$.

Then, by Theorem 3.3.27, the Newton-Okounkov body $\Delta_\nu(D)$ is the convex set whose vertices are

$$P_1 = \left(2, \frac{2}{3}\right), P_3 = (3,1) \text{ and } P_4 = (4,1).$$

Figure 3.5 shows the Newton-Okounkov body $\Delta_\nu(-2F+M)$ in dark and the triangle $\mathfrak{C}(\nu) \cap \mathfrak{H}_{F+M}(\nu)$ described in Proposition 3.3.5.

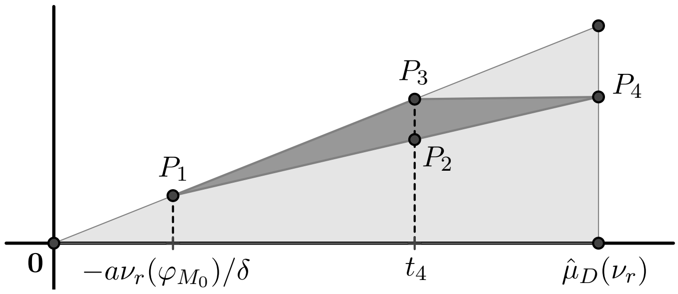


Figure 3.5: $\Delta_\nu(-2F+M)$ and $\mathfrak{C}(\nu) \cap \mathfrak{H}_{-2F+M}(\nu)$ in Example 3.3.28.

Corollary 3.3.29. *Let $E_\bullet = \{Z = Z_r \supset E_r \supset \{p_{r+1}\}\}$ be a flag and $\nu = \nu_{E_\bullet}$ its attached exceptional curve valuation. Assume that its first component ν_r is an NPI divisorial valuation of $\mathbb{P}^2$ and thus Z is obtained from a sequence of points blowups starting at $\mathbb{P}^2$. Then there exists an exceptional curve valuation $\hat{\nu}$ whose first component $\hat{\nu}_r$ is an NPI special divisorial valuation of $\mathbb{F}_1$ which corresponds to the same flag. Moreover, if E_0 is a general projective line on $\mathbb{P}^2$ and M a curve of degree $(0,1)$ on $\mathbb{F}_1$, the Newton-Okounkov body $\Delta_\nu(E_0)$ of E_0 coincides with the Newton-Okounkov body $\Delta_{\hat{\nu}}(M)$ of M.*

Proof. The first part is a consequence of Theorem 1.3.2(b), Proposition 2.3.3 and Corollary 2.3.4.

Now we are going to prove the second part of the result. Set $\{\overline{\beta}_i(\nu)\}_{i=0}^{g^*+1}$ $\Big($respectively, $\{\overline{\beta}_i(\nu_r)\}_{i=0}^{g+1}, \{\overline{\beta}_i(\hat{\nu})\}_{i=0}^{\hat{g}^*+1}, \{\overline{\beta}_i(\hat{\nu}_r)\}_{i=0}^{\hat{g}+1}\Big)$ the sequence of maximal contact values of ν (respectively, $\nu_r, \hat{\nu}, \hat{\nu}_r$) and ν_η (respectively, $\hat{\nu}_{\hat{\eta}}$) the divisorial valuation which appears in the second coordinate of ν (respectively, $\hat{\nu}$) when p_{r+1} is a satellite point. Notice that $\hat{\eta} = \eta - 1$.

Note that a projective line on $\mathbb{P}^2$ corresponds with a curve of degree $(0,1)$ $D = M$ on $\mathbb{F}_1$. Therefore one always has that $\theta_1^r(D) = -\hat{\nu}_r(\varphi_{M_0}) < 0$. Denote by L the line at infinity on $\mathbb{P}^2$ (see Subsection 2.1). By the Noether formula, Proposition 2.3.3, Corollary 2.3.4, Remark 2.3.5 and Remark 3.2.2 the following equalities hold

$$\overline{\beta}_{g+1}(\nu_r) = \overline{\beta}_{\hat{g}+1}(\hat{\nu}_r) + \hat{\nu}_r(\varphi_{M_0})^2, \nu(\mathfrak{m}_1) = \hat{\nu}(\varphi_{M_0}), \nu(\varphi_L) = \hat{\nu}(\varphi_{F_1}) + \hat{\nu}(\varphi_{M_0}),$$
$$\nu_r(\varphi_\eta) = \hat{\nu}_r(\varphi_{\hat{\eta}}) + \hat{\nu}_r(\varphi_{M_0}) \cdot \hat{\nu}_{\hat{\eta}}(\varphi_{M_0}) \text{ and } \hat{\mu}(\nu_r) = \hat{\mu}_D(\hat{\nu}_r). \tag{3.12}$$

We distinguish three situations to show the result. *Case 1*: either $g^* \neq 0$, or $g^* = 0$ and $\nu(\varphi_L) \neq \overline{\beta}_1(\nu)$; *Case 2*: $g^* = 0, \nu(\varphi_L) = \overline{\beta}_1(\nu)$ and $i_L > 2$; and *Case 3*: $g^* = 0, \nu(\varphi_L) = \overline{\beta}_1(\nu)$ and $i_L = 2$, where i_L indicates the last point in the configuration of infinitely near points $\mathcal{C}_{\nu_r}$ of ν_r through which the strict transform of L passes.

Let us see Case 1. Consider the points given in (3.11) for Case A corresponding to the divisor $D = M$, the value $\theta_1^r(D) < 0$ and $\delta = 1$. That is, the points

$$Q_5 = (0,0) = Q_6 \text{ (respectively, } Q_5 = (0,0) = Q_6\text{)},$$

$$Q_7 = \left(\frac{\overline{\beta}_{g+1}(\hat{\nu}_r) + \hat{\nu}_r(\varphi_{M_0})^2}{\hat{\nu}_r(\varphi_{M_0}) + \hat{\nu}_r(\varphi_{F_1})}, \frac{\hat{\nu}_r(\varphi_{\hat{\eta}}) + \hat{\nu}_r(\varphi_{M_0})\hat{\nu}_{\hat{\eta}}(\varphi_{M_0})}{\hat{\nu}_r(\varphi_{M_0}) + \hat{\nu}_r(\varphi_{F_1})} \right)$$

$$\left(\text{respectively, } Q_7 = \left(\frac{\overline{\beta}_{g+1}(\hat{\nu}_r) + \hat{\nu}_r(\varphi_{M_0})^2}{\hat{\nu}_r(\varphi_{M_0}) + \hat{\nu}_r(\varphi_{F_1})}, 0 \right) \right),$$

$$Q_8 = Q_7 + \left(0, \frac{1}{\hat{\nu}_r(\varphi_{M_0}) + \hat{\nu}_r(\varphi_{F_1})}\right) \text{ and } Q_9 = (\hat{\mu}_D(\hat{\nu}_r), \hat{\mu}_D(\hat{\nu}_{\hat{\eta}}))$$
$$\text{(respectively, } Q_9 = (\hat{\mu}_D(\hat{\nu}_r), 0)\text{)}$$

if p_{r+1} is a satellite (respectively, free) point. Taking into account (3.12) and Theorems 3.3.10 and 3.3.21, we obtain the result for this case.

Notice that Case 2 and Case 3 can be proved with a similar reasoning and using the cases B and C (described before Lemma 3.3.19) and Theorems 3.3.24 and 3.3.27, respectively, which completes the proof. □

Remark 3.3.30. The previous theorem provides two ways to describe the same Newton-Okounkov body, but the triangles given in Proposition 3.3.5 where it is contained are different. Notice that they correspond to distinct exceptional curve valuations.

Remark 3.3.31. Following the assumptions and notations of Corollary 3.3.29, the Newton-Okounkov body $\Delta_\nu(E_0)$ of a projective line E_0 on $\mathbb{P}^2$, where the valuation ν of $\mathbb{P}^2$ is minimal and non-positive at infinity, is described as the Newton-Okounkov body $\Delta_{\hat{\nu}}(M)$ of a curve of degree $(0,1)$ M, where $\hat{\nu}$ is a non-minimal with respect to M and special non-positive at infinity valuation of $\mathbb{F}_1$ (see Remark 3.2.2). Even more, this Newton-Okounkov body is the set given in Theorem 3.3.21 when $\delta = 1, a = 0$ and $\theta_1^r(D) < 0$, since ν is minimal.

Example 3.3.32. Consider the first divisorial valuation ν_n of $\mathbb{P}^2$ described in Example 3.3.11. Recall that its configuration of infinitely near points is $\mathcal{C}_{\nu_n} = \{p_i\}_{i=1}^{17}$

and its sequence of maximal contact values is $\{\overline{\beta}_i(\nu_n)\}_{i=0}^3 = \{8, 20, 63, 256\}$. By Proposition 2.3.3 and Corollary 2.3.4, we can see the above divisorial valuation of $\mathbb{P}^2$ as the divisorial valuation ν_r of $\mathbb{F}_1$ whose sequence of maximal contact values is $\{\overline{\beta}_i(\nu_r)\}_{i=0}^3 = \{8, 12, 47, 192\}$.

In the first case of Example 3.3.11, we had that $\nu_n(\varphi_L) = 16 = \hat{\mu}(\nu_n)$ and then ν_n is non-positive at infinity and minimal. By the proof of Proposition 2.3.3, Theorem 3.2.1 and Remark 2.3.5, one gets that $\nu_r(\varphi_{F_1}) = 8, \nu_r(\varphi_{M_0}) = 8$ and $\hat{\mu}_M(\nu_r) = 16$ and so ν_r is non-positive at infinity (by Remark 2.3.5) and non-minimal with respect to M (since $256 = \hat{\mu}_D(\nu_r)^2 > M^2\overline{\beta}_{g+1}(\nu_r) = 192$). Recall that M is a curve of degree $(0,1)$ on $\mathbb{F}_1$. Taking into account Corollary 3.3.29, we can assume that ν_r is the first component of the valuation $\nu = \nu_{E_\bullet}$, where $E_\bullet$ is the flag $E_\bullet = \{Z_{17} \supset E_{17} \supset \{p_{r+1}\}\}$ defined in Example 3.3.11. If p_{r+1} is the satellite point $E_{16} \cap E_{17}$ (with the notation of Example 3.3.11), the Newton-Okounkov body $\Delta_\nu(M)$ is the convex hull determined by the vertices

$$(0,0), Q_7 = \left(16, \frac{255}{16}\right) \text{ and } Q_9 = (16, 16),$$

since $\theta_1^r(M) = -8$ and $16 \preccurlyeq 17$. Otherwise (p_{r+1} is a free point), $\Delta_\nu(D)$ is triangle with vertices

$$(0,0), Q_8 = \left(16, \frac{1}{16}\right) \text{ and } Q_9 = (16, 0).$$

Now assume that ν_n is the second divisorial valuation of $\mathbb{P}^2$ considered in Example 3.3.11. Here, we recall that $\nu_n(\varphi_L) = 20 = \hat{\mu}(\nu_n)$ and then ν_n is non-positive at infinity and non-minimal. With respect to the corresponding divisorial valuation ν_r of $\mathbb{F}_1$, one has $\nu_r(\varphi_{F_1}) = 12, \nu_r(\varphi_{M_0}) = 8$ and $\hat{\mu}_M(\nu_r) = 20$ and then ν_r is non-positive at infinity and non-minimal with respect to M. In this case, $\Delta_\nu(M)$ is described by the vertices

$$(0,0), Q_7 = \left(\frac{256}{20}, \frac{255}{20}\right) \text{ and } Q_9 = (20, 20),$$

since p_{r+1} is the satellite point $E_{16} \cap E_{17}, 16 \preccurlyeq 17$ and $\theta_1^r(M) = -8$. Finally, if p_{r+1} is a free point, the coordinates of $\Delta_\nu(M)$ are

$$(0,0), Q_8 = \left(\frac{256}{20}, \frac{1}{20}\right) \text{ and } Q_9 = (20, 20).$$

3.3.2 Newton-Okounkov bodies with respect to non-positive at infinity non-special valuations

To finish this chapter we complete our description of the Newton-Okounkov bodies associated to NPI valuations. It remains to study the case of non-special valuations which are non-minimal with respect to a big divisor on $\mathbb{F}_\delta$. We continue with the same notation introduced at the beginning of Section 3.3.

Let ν be an NPI non-special exceptional curve valuation of $\mathbb{F}_\delta$ whose first component is ν_r. Set D a big divisor on $\mathbb{F}_\delta$. We can assume that D is also nef, since p_1

is general (see Definition 2.2.1) and then $p_{r+1} \notin \operatorname{supp}(N_{D^*})$. Denote by $\theta_2^r(D)$ the expression $a\nu_r(\varphi_{F_1}) - b\big(\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{F_1})\big)$, where F_1 is the fiber containing the point p_1 and M_1 the section described in Proposition 2.2.2. Notice that $\theta_2^r(D) = 0$ if and only if $a = b(\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{F_1}))/\nu_r(\varphi_{F_1})$.

Lemma 3.3.33. *Let ν_r be an NPI non-special divisorial valuation of $\mathbb{F}_\delta$. Consider a divisor D and the value $\theta_2^r(D)$ as above. Then the divisor on the surface Z defined by ν_r*

$$D_3 = D^* - \frac{b}{\nu_r(\varphi_{F_1})}\sum_{i=1}^{r}\nu_r(\mathfrak{m}_i)E_i^* \left(\text{respectively, } D_4 = D^* - \frac{a+b\delta}{\nu_r(\varphi_{M_1})}\sum_{i=1}^{r}\nu_r(\mathfrak{m}_i)E_i^*\right)$$

is nef if $\theta_2^r(D) \geq 0$ (respectively, $\theta_2^r(D) < 0$).

Proof. Let us see that D_4 is a nef divisor if $\theta_2^r(D) < 0$. The remaining case follows from a similar reasoning to that developed in Lemma 3.3.12.

Define

$$\Delta_r := \big(\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{F_1})\big)F^* + \nu_r(\varphi_{F_1})M^* - \sum_{i=1}^{r}\nu_r(\mathfrak{m}_i)E_i^* \text{ and}$$

$$\Gamma_r := \nu_r(\varphi_{M_1})M^* - \delta\sum_{i=1}^{r}\nu_r(\mathfrak{m}_i)E_i^*.$$

Both divisors are nef by Theorem 2.3.7 and this finishes the proof because

$$D_4 \sim \frac{a}{\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{F_1})}\Delta_r + \frac{-\theta_2^r(D)}{\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{F_1})}\Gamma_r$$

and $-\theta_2^r(D) > 0$. □

The next result can be proved as we did for Lemma 3.3.13. Recall that the non-positivity at infinity of a non-special divisorial valuation can be checked with the inequality $2\nu_r(\varphi_{M_1})\nu_r(\varphi_{F_1}) - \delta\nu_r(\varphi_{F_1}) \geq \overline{\beta}_{g+1}(\nu_r)$ (see Theorem 2.3.7). Besides, $D \sim aF + bM$ is a big and nef divisor on $\mathbb{F}_\delta$ and we also use the value $\theta_2^r(D)$.

Lemma 3.3.34. *Let ν_r be an NPI non-special divisorial valuation of $\mathbb{F}_\delta$. Then the rational numbers*

$$t_5 = \frac{b}{\nu_r(\varphi_{F_1})}\overline{\beta}_{g+1}(\nu_r) \text{ and } t_6 = \frac{b}{\nu_r(\varphi_{F_1})}\overline{\beta}_{g+1}(\nu_r) + \theta_2^r(D)$$

$$\left(\text{respectively, } t_7 = \frac{a+b\delta}{\nu_r(\varphi_{M_1})}\overline{\beta}_{g+1}(\nu_r) \text{ and } t_8 = \frac{a\overline{\beta}_{g+1}(\nu_r) - \nu_r(\varphi_{M_1})\theta_2^r(D)}{\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{F_1})}\right)$$

belong to the set $T_{D,\nu_r} := \{t \in \mathbb{Q} \mid 0 \leq t \leq \hat{\mu}_D(\nu_r)\}$ when $\theta_2^r(D) \geq 0$ (respectively, $\theta_2^r(D) < 0$).

Corollary 3.3.35. *Let ν_r be an NPI non-special divisorial valuation of $\mathbb{F}_\delta$. Set $D \sim aF + bM$ a big and nef divisor on $\mathbb{F}_\delta$. Consider the value $\theta_2^r(D)$ given before Lemma 3.3.33 and the rational numbers t_6 and t_8 provided in Lemma 3.3.34. Then* *(a) When* $\theta_2^r(D) \geq 0$*, it holds* $2\nu_r(\varphi_{M_1})\nu_r(\varphi_{F_1}) - \delta\nu_r(\varphi_{F_1})^2 = \overline{\beta}_{g+1}(\nu_r)$ *if and only if* $t_6 = \hat{\mu}_D(\nu_r)$*.*

(b) Otherwise ($\theta_2^r(D) < 0$)*, it holds either* $2\nu_r(\varphi_{M_1})\nu_r(\varphi_{F_1}) - \delta\nu_r(\varphi_{F_1})^2 = \overline{\beta}_{g+1}(\nu_r)$*, or* $a = 0$*, if and only if* $t_8 = \hat{\mu}_D(\nu_r)$*.*

Proof. We only show (b), because (a) follows easily from the proof of Lemma 3.3.14. Let us start by considering that $2\nu_r(\varphi_{M_1})\nu_r(\varphi_{F_1}) + \delta\nu_r(\varphi_{F_1})^2 = \overline{\beta}_{g+1}(\nu_r)$. Applying the last condition to t_8, it holds

$$\begin{aligned} t_8 &= \frac{a(2\nu_r(\varphi_{M_1})\nu_r(\varphi_{F_1}) - \delta\nu_r(\varphi_{F_1})^2) - \nu_r(\varphi_{M_1})\theta_2^r(D)}{\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{F_1})} \\ &= \frac{(a\nu_r(\varphi_{F_1}) + b\nu_r(\varphi_{M_1}))(\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{F_1}))}{\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{F_1})} = \hat{\mu}_D(\nu_r). \end{aligned}$$

In addition, it is easy to check that t_8 equals $\hat{\mu}_D(\nu_r)$ when $a = 0$.

Conversely, suppose that $t_8 = \hat{\mu}_D(\nu_r)$. Then,

$$\begin{aligned} 0 = \hat{\mu}_D(\nu_r) - t_8 &= \frac{\hat{\mu}_D(\nu_r)(\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{F_1})) - a\overline{\beta}_{g+1}(\nu_r) + \nu_r(\varphi_{M_1})\theta_2^r(D)}{\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{F_1})} \\ &= \frac{a(2\nu_r(\varphi_{M_1})\nu_r(\varphi_{F_1}) + \delta\nu_r(\varphi_{F_1})^2 - \overline{\beta}_{g+1}(\nu_r))}{\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{F_1})}, \end{aligned}$$

which completes the proof. □

Remark 3.3.36. As in Remark 3.3.15, one can obtain information from the values t_i, $5 \leq i \leq 8$.

(a) ν_r is minimal with respect to a big and nef divisor D if and only if

$$\hat{\mu}_D(\nu_r) = \frac{b\overline{\beta}_{g+1}(\nu_r)}{\nu_r(\varphi_{F_1})} = t_5 = t_6 = \frac{(a + b\delta)\overline{\beta}_{g+1}(\nu_r)}{\nu_r(\varphi_{M_1})} = t_7 = t_8.$$

In fact, the equalities $t_5 = t_6$ and $\hat{\mu}_D(\nu_r) = t_6$ are equivalents to those provide in Corollary 3.2.3 by Corollary 3.3.35.

(b) Assume that ν_r is non-minimal with respect to D. Then,

(b.1) $\theta_2^r(D) = 0$ if and only if $\hat{\mu}_D(\nu_r) > t_5 = t_6 = t_7 = t_8 > 0$.

(b.2) If $\theta_2^r(D) > 0$, $2\nu_r(\varphi_{M_1})\nu_r(\varphi_{F_1}) - \delta\nu_r(\varphi_{F_1})^2 \geq \overline{\beta}_{g+1}(\nu_r)$ if and only if $\hat{\mu}_D(\nu_r) \geq t_6 > t_5 > 0$.

(b.3) If $\theta_2^r(D) < 0$, $2\nu_r(\varphi_{M_1})\nu_r(\varphi_{F_1}) - \delta\nu_r(\varphi_{F_1})^2 \geq \overline{\beta}_{g+1}(\nu_r)$ and $a \geq 0$ if and only if $\hat{\mu}_D(\nu_r) \geq t_8 > t_7 > 0$.

Arguing with the divisors D_3 and D_4 from Lemma 3.3.33 as we did with those from Lemma 3.3.16, one can show that D_3 and D_4 are also big. In addition,

$$D_3 \cdot \tilde{F}_1 = 0, D_4 \cdot \tilde{M}_1 = 0 \text{ and } D_3 \cdot E_i = 0 \text{ and } D_4 \cdot E_i = 0,$$

for $1 \leq i \leq r - 1$. As a result, one obtains the next lemma.

Lemma 3.3.37. *Let ν_r be a divisorial valuation and D a divisor as in Corollary 3.3.35. Assume also that ν_r is non-minimal with respect to D. The intersection matrix determined by the set of divisors $\{\tilde{F}_1, E_1, E_2, \ldots, E_{r-1}\}$ (respectively, $\{\tilde{M}_1, E_1, E_2, \ldots, E_{r-1}\}$) is negative definite.*

Let ν_r be a divisorial valuation and D a divisor as stated in Corollary 3.3.35. Consider the rational numbers t_i, $5 \leq i \leq 8$, defined in Lemma 3.3.34. The following proposition determines the Zariski decomposition of the divisors $D^* - t_i E_r$, which will use to describe the Newton-Okounkov bodies $\Delta_\nu(D)$ of exceptional curve valuations as before Lemma 3.3.33. Our determination depends on the above defined value $\theta_2^r(D)$ and the divisors D_3, D_4 and

$$\Delta_r = (\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{F_1}))F^* + \nu_r(\varphi_{F_1})M^* - \sum_{i=1}^{r} \nu_r(\mathfrak{m}_i)E_i^*$$

provided in Lemma 3.3.33 and its proof.

Proposition 3.3.38. *The following statements hold.*

(a) The positive and negative parts of the Zariski decomposition of the divisor $D_{t_5} = D^* - t_5 E_r$ (respectively, $D_{t_6} = D^* - t_6 E_r$) are

$$P_{D_{t_5}} \sim D_3 \quad \text{and} \quad N_{D_{t_5}} = \frac{b}{\nu_r(\varphi_{F_1})} \sum_{i=1}^{r-1} \nu_r(\varphi_i) E_i$$

$$\left(\text{respectively, } P_{D_{t_6}} \sim \frac{b}{\nu_r(\varphi_{F_1})} \Delta_r \text{ and} \right.$$

$$\left. N_{D_{t_6}} = \frac{\theta_2^r(D)}{\nu_r(\varphi_{F_1})} \tilde{F}_1 + \sum_{i=1}^{r-1} \frac{b\nu_r(\varphi_i) + \theta_2^r(D)\nu_i(\varphi_{F_1})}{\nu_r(\varphi_{F_1})} E_i \right),$$

when $\theta_2^r(D) \geq 0$.

(b) The positive and negative parts of the Zariski decomposition of $D_{t_7} = D^* - t_7 E_r$ (respectively, $D_{t_8} = D^* - t_8 E_r$) are

$$P_{D_{t_7}} \sim D_4 \quad \text{and} \quad N_{D_{t_7}} = \frac{a + b\delta}{\nu_r(\varphi_{M_1})} \sum_{i=1}^{r-1} \nu_r(\varphi_i) E_i$$

$$\left(\text{respectively, } P_{D_{t_8}} \sim \frac{a}{\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{F_1})} \Delta_r \text{ and} \right.$$

$$\left. N_{D_{t_8}} = \frac{-\theta_2^r(D)}{\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{F_1})} \tilde{M}_1 + \sum_{i=1}^{r-1} \frac{a\nu_r(\varphi_i) - \theta_2^r(D)\nu_i(\varphi_{M_1})}{\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{F_1})} E_i \right),$$

when $\theta_2^r(D) < 0$.

Proof. We are going to show (b). A proof for (a) follows similarly. On the one hand, the components of the divisor $N_{D_{t_7}}$ determine a negative definite intersection matrix.

On the other hand, $P_{D_{t_7}}$ is a nef divisor by Lemma 3.3.33 and orthogonal to each component of $N_{D_{t_7}}$ by the proximity equalities. Therefore, $P_{D_{t_7}} + N_{D_{t_7}}$ is the Zariski decomposition of D_{t_7}.

Let us prove our statement for D_{t_8}. The components of $N_{D_{t_8}}$ determine a negative definite intersection matrix by Lemma 3.3.37 and $P_{D_{t_8}}$ is a nef divisor and orthogonal to each component of $N_{D_{t_8}}$ by Proposition 2.4.2 and Theorem 2.4.8. To finish, let us show that $P_{D_{t_8}} + N_{D_{t_8}} \sim D_{t_8}$, which finishes the proof. Indeed, set $p_{i_{M_1}}$ the last point in the configuration of infinitely near points $\mathcal{C}_{\nu_r}$ of the valuation ν_r through which the strict transform of M_1 passes. From the fact that $\tilde{M}_1 \sim M^* - \sum_{i=1}^{i_{M_1}} E_i^*$, one deduces that

$$\frac{a(\Delta_r + \sum_{i=1}^{r-1} \nu_r(\varphi_i)E_i) + \theta_2^r(D)M^*}{\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{F_1})} \sim D - \frac{a\overline{\beta}_{g+1}(\nu_r)}{\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{F_1})}E_r.$$

Besides,

$$\frac{-\theta_2^r(D)}{\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{M_1})}\left(\sum_{i=1}^{r-1}\nu_i(\varphi_{M_1})E_i - \sum_{i=1}^{i_{M_1}} E_i^*\right) = \frac{-\theta_2^r(D)\nu_r(\varphi_{M_1})}{\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{F_1})}E_r,$$

and the proof concludes after summing both expressions. □

Remark 3.3.39. As in Remark 3.3.18, we can deduce from the above results that the ray $[D_t] := [D^*] - t[E_r]$, where $0 \leq t \leq \hat{\mu}_D(\nu_r)$, crosses the interior of $\mathrm{NE}(Z)$ heading towards the face of $\mathrm{NE}(Z)$ spanned by the classes of the divisors $\tilde{F}_1, \tilde{M}_1, E_1, E_2, \ldots,$ E_{r-1}. Figure 3.6 shows the situation in this case, where $[\Delta_n]^\perp$ represents the mentioned face of $\mathrm{NE}(Z)$ and the Zariski chambers $\Sigma_1, \Sigma_2, \Sigma_3$ and Σ_4 defined below are delimited by discontinuous lines.

$$\Sigma_1 := \big\{[\xi] \in \mathrm{Big}(Z) \,|\, \mathrm{Neg}(\xi) = \mathrm{Null}(\xi) = \{[E_1], [E_2], \ldots, [E_{r-1}]\}\big\},$$
$$\Sigma_2 := \big\{[\xi] \in \mathrm{Big}(Z) \,|\, \mathrm{Neg}(\xi) = \mathrm{Null}(\xi) = \{[\tilde{F}_1], [E_1], [E_2], \ldots, [E_{r-1}]\}\big\},$$
$$\Sigma_3 := \big\{[\xi] \in \mathrm{Big}(Z) \,|\, \mathrm{Neg}(\xi) = \mathrm{Null}(\xi) = \{[\tilde{M}_1], [E_1], [E_2], \ldots, [E_{r-1}]\}\big\} \text{ and}$$
$$\Sigma_4 := \big\{[\xi] \in \mathrm{Big}(Z) \,|\, \mathrm{Neg}(\xi) = \mathrm{Null}(\xi) = \{[\tilde{F}_1], [\tilde{M}_1], [E_1], [E_2], \ldots, [E_{r-1}]\}\big\}.$$

We finish this subsection describing the Newton-Okounkov bodies $\Delta_\nu(D)$, where ν and D are as in the paragraph before Lemma 3.3.33. Recall that the first component of ν is the divisorial valuation ν_r. As in the previous subsection, we divide our description in two cases.

Case D: Either $g^* > 0$ or $g^* = 0$ and $\nu(\varphi_{M_1}) \neq \overline{\beta}_1(\nu)$.

Case E: The value g^* equals 0 and $\nu(\varphi_{M_1}) = \overline{\beta}_1(\nu)$.

Lemma 3.3.40. *Let ν be an NPI non-special exceptional valuation of $\mathbb{F}_\delta$ and $\mathcal{C}_\nu = \{p_i\}_{i\geq 1}$ its configuration of infinitely near points such that $p_i \to p_r$ for all $i > r$. Consider the cases described before.*

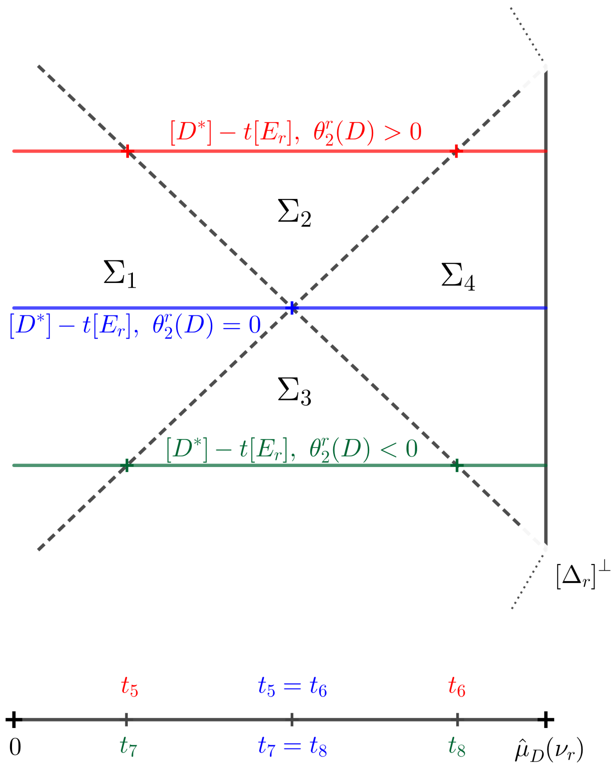


Figure 3.6: Local description of the cone of curves NE(Z) of a rational surface Z given by a non-positive at infinity non-special divisorial valuation of $\mathbb{F}_\delta$.

(a) Suppose that p_{r+1} is the satellite point $E_\eta \cap E_r, \eta \neq r$.

(a.1) Assume we are in Case D. Then,

$$\nu_\eta(\varphi_{F_1}) = \nu_r(\varphi_{F_1}) \cdot \frac{\overline{\beta}_0(\nu_\eta)}{\overline{\beta}_0(\nu_r)} \text{ and } \nu_\eta(\varphi_{M_1}) = \nu_r(\varphi_{M_1}) \cdot \frac{\overline{\beta}_0(\nu_\eta)}{\overline{\beta}_0(\nu_r)}.$$

(a.2) Assume we are in Case E. Then,

$$\nu_\eta(\varphi_{F_1}) = \nu_r(\varphi_{F_1}) \cdot \frac{\overline{\beta}_0(\nu_\eta)}{\overline{\beta}_0(\nu_r)} \text{ and } \nu_\eta(\varphi_{M_1}) = \nu_r(\varphi_{M_1}) \cdot \frac{\overline{\beta}_{g^*+1}(\nu_\eta)}{\overline{\beta}_{g^*+1}(\nu_r)}.$$

(b) Otherwise (p_{r+1} is a free point).

(b.1) Assume we are in Case D. Then,

$$\nu(\varphi_{F_1}) = (\nu_r(\varphi_{F_1}), 0) \text{ and } \nu(\varphi_{M_0}) = (\nu_r(\varphi_{M_0}), 0).$$

(b.2) Assume we are in Case E. Then,

$$\nu(\varphi_{F_1}) = (\nu_r(\varphi_{F_1}), 0) \text{ and } \nu(\varphi_{M_0}) = (\nu_r(\varphi_{M_1}), 1).$$

Proof. The result is immediate using the next properties: The value $\nu(\varphi_{F_1})$ equals $\overline{\beta}_0(\nu)$ since the strict transform of F_1 does not go through p_2. Moreover, the value $\nu(\varphi_{M_1})$ equals either $\overline{\beta}_1(\nu)$ when the strict transforms of M_1 pass through all initial free points of $\mathcal{C}_\nu$, or $s_{M_1}\overline{\beta}_0(\nu)$, where s_{M_1} is a positive integer greater than δ. This concludes the proof. □

We begin our study with Case D. Reasoning as in the previous subsection, the Newton-Okounkov body $\Delta_\nu(D)$ can be described as in (3.9) and, by Proposition 3.3.38, the points

$$Q_{10} = \left(\frac{b\overline{\beta}_{g+1}(\nu_r)}{\nu_r(\varphi_{F_1})}, \frac{b\nu_r(\varphi_\eta)}{\nu_r(\varphi_{F_1})}\right) \left(\text{respectively, } Q_{10} = \left(\frac{b\overline{\beta}_{g+1}(\nu_r)}{\nu_r(\varphi_{F_1})}, 0\right)\right),$$
$$Q_{11} = Q_{10} + \left(0, \frac{b}{\nu_r(\varphi_{F_1})}\right),$$
$$Q_{12} = \left(\frac{b\overline{\beta}_{g+1}(\nu_r)}{\nu_r(\varphi_{F_1})} + \theta_2^r(D), \frac{b\nu_r(\varphi_\eta) + \theta_2^r(D)\nu_\eta(\varphi_{F_1})}{\nu_r(\varphi_{F_1})}\right)$$
$$\left(\text{respectively } Q_{12} = \left(\frac{b\overline{\beta}_{g+1}(\nu_r)}{\nu_r(\varphi_{F_1})} + \theta_2^r(D), 0\right)\right) \text{ and } Q_{13} = Q_{12} + \left(0, \frac{b}{\nu_r(\varphi_{F_1})}\right) \tag{3.13}$$

are in $\Delta_\nu(D)$ if $\theta_2^r(D) \geq 0$ and p_{r+1} is the satellite point $E_\eta \cap E_r$ (respectively, a free point).

When $\theta_2^r(D) < 0$ and $p_{r+1} \in E_\eta \cap E_r$ (respectively, p_{r+1} is a free point), the points in $\Delta_\nu(D)$ are

$$Q_{14} = \left(\frac{(a+b\delta)\overline{\beta}_{g+1}(\nu_r)}{\nu_r(\varphi_{M_1})}, \frac{(a+b\delta)\nu_r(\varphi_\eta)}{\nu_r(\varphi_{M_1})}\right)$$
$$\left(\text{respectively, } Q_{14} = \left(\frac{(a+b\delta)\overline{\beta}_{g+1}(\nu_r)}{\nu_r(\varphi_{M_1})}, 0\right)\right), Q_{15} = Q_{14} + \left(0, \frac{a+b\delta}{\nu_r(\varphi_{M_1})}\right),$$
$$Q_{16} = \left(\frac{a\overline{\beta}_{g+1}(\nu_r) - \theta_2^r(D)\nu_r(\varphi_{M_1})}{\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{F_1})}, \frac{a\nu_r(\varphi_\eta) - \theta_2^r(D)\nu_\eta(\varphi_{M_1})}{\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{F_1})}\right)$$
$$\left(\text{respectively, } Q_{16} = \left(\frac{a\overline{\beta}_{g+1}(\nu_r) - \theta_2^r(D)\nu_r(\varphi_{M_1})}{\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{F_1})}, 0\right)\right)$$
$$\text{and } Q_{17} = Q_{16} + \left(0, \frac{a}{\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{F_1})}\right). \tag{3.14}$$

In addition, the point $Q_{18} = (\hat{\mu}_D(\nu_r), \hat{\mu}_D(\nu_\eta))$ (respectively, $Q_{18} = (\hat{\mu}_D(\nu_r), 0)$) belongs to $\Delta_\nu(D)$ when p_{r+1} is satellite (respectively, free), by Theorem 3.2.1.

Remark 3.3.41. The latter points $Q_i, 10 \leq i \leq 18$, satisfy the following nice properties:

(a) $\theta_2^r(D) = 0$ if and only if $Q_{10} = Q_{12} = Q_{14} = Q_{16}$ and $Q_{11} = Q_{13} = Q_{15} = Q_{17}$. Moreover, when $\theta_2^r(D) < 0$, it holds that $a = 0$ if and only if $Q_{16} = Q_{17} = Q_{18}$.

(b) Some of the previous points Q_i are collinear. Indeed,

(b.1) Consider p_{r+1} the satellite point $E_\eta \cap E_r, \eta \neq r$, and $\eta \preccurlyeq r$. When $\theta_2^r(D) \geq 0$ (respectively, $\theta_2^r(D) < 0$) the points $(0,0), Q_{11}, Q_{13}$ (respectively, Q_{15}, Q_{17}) and Q_{18} are in the line $T_1 \equiv \overline{\beta}_0(\nu_r)y = \overline{\beta}_0(\nu_\eta)x$. If $p_{r+1} \in E_\eta \cap E_r, \eta \not\preccurlyeq r$, and $\theta_2^r(D) \geq 0$ (respectively, $\theta_2^r(D) < 0$), the points $(0,0), Q_{10}, Q_{12}$ (respectively, Q_{14}, Q_{16}) and Q_{18} also belong to T_1.

(b.2) Assume that p_{r+1} is a free point. If $\theta_2^r(D) \geq 0$ (respectively, $\theta_2^r(D) < 0$), the points $(0,0), Q_{10}, Q_{12}$ (respectively, Q_{14}, Q_{16}) and Q_{18} are contained in the line $y = 0$.

Theorem 3.3.42. *Let ν be an exceptional valuation in Case D. Keeping the notation in the previous paragraphs, the Newton-Okounkov body $\Delta_\nu(D)$ of D with respect to ν is a quadrilateral if and only if $a \neq 0$ and $\theta_2^r(D) \neq 0$. Otherwise, it is a triangle (see Remark 3.3.41).*

The vertices of $\Delta_\nu(D)$ are

(a) $(0,0), Q_{10}, Q_{12}$ (respectively, Q_{14}, Q_{16}) and Q_{18} when $\theta_2^r(D) \geq 0$ (respectively, $\theta_2^r(D) < 0$), p_{r+1} is the satellite point $E_\eta \cap E_r, \eta \neq r$, and $\eta \preccurlyeq r$.

(b) $(0,0), Q_{11}, Q_{13}$ (respectively, Q_{15}, Q_{17}) and Q_{18} when $\theta_2^r(D) \geq 0$ (respectively, $\theta_2^r(D) < 0$), p_{r+1} is the satellite point $E_\eta \cap E_r, \eta \neq r$, and $\eta \not\preccurlyeq r$.

(c) $(0,0), Q_{11}, Q_{13}$ (respectively, Q_{15}, Q_{17}) and Q_{18} when $\theta_2^r(D) \geq 0$ (respectively, $\theta_2^r(D) < 0$) and p_{r+1} is a free point.

Proof. Let us see that $D^2/2$ is the area of the convex hull Δ generated by the points $(0,0), Q_{14}, Q_{15}, Q_{16}, Q_{17}$ and Q_{18}. By Remark 3.3.41, the case concerning the points $(0,0), Q_{10}, Q_{11}, Q_{12}, Q_{13}$ and Q_{18} and the fact of being a quadrilateral or a triangle follow as in the proof of Theorem 3.3.21.

The area of the triangle with vertices $(0,0), Q_{14}$ and Q_{15} (respectively, Q_{16}, Q_{17} and Q_{18}) is

$$\frac{(a+b\delta)^2}{2\nu_r(\varphi_{M_1})^2}\overline{\beta}_{g+1}(\nu_r)\Bigg(\text{respectively,}$$

$$\frac{a}{2(\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{F_1}))}\left(\hat{\mu}_D(\nu_r) - \left(\frac{a\overline{\beta}_{g+1}(\nu_r) - \theta_2^r(D)\nu_r(\varphi_{M_1})}{\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{F_1})}\right)\right)\Bigg).$$

The area of the trapezium provided by Q_{14}, Q_{15}, Q_{16} and Q_{17} is

$$\frac{-\theta_2^r(D)\left((a+b\delta)(\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{F_1})) + a\nu_r(\varphi_{F_1})\right)\left(\nu_r(\varphi_{M_0})^2 - \delta\overline{\beta}_{g+1}(\nu_r)\right)}{2\nu_r(\varphi_{M_1})^2(\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{F_1}))^2}.$$

Summing the areas of that trapezium and the two previous triangles, we note that the coefficients of $\overline{\beta}_{g+1}(\nu_r)$ vanish and it is sufficient to sum the next three fractions

$$\frac{a\hat{\mu}_D(\nu_r)}{2(\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{F_1}))}, \frac{a\theta_2^r(D)\nu_r(\varphi_{M_1})}{2(\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{F_1}))^2} \text{ and}$$
$$\frac{-\theta_2^r(D)\nu_r(\varphi_{M_1})^2((a+b\delta)(\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{F_1})) + a\nu_r(\varphi_{F_1}))}{2\nu_r(\varphi_{M_1})^2(\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{F_1}))^2}.$$

After computing, one obtains $(2ab + \delta b^2)/2$, which concludes the proof. □

Example 3.3.43. Let p be a general point of the Hirzebruch surface $\mathbb{F}_2$ and ν_r a non-special divisorial valuation centered at $\mathcal{O}_{\mathbb{F}_2,p}$, whose sequence of maximal contact values is $\{\overline{\beta}_i(\nu_r)\}_{i=0}^3 = \{15, 51, 262, 786\}$. Set $\mathcal{C}_{\nu_r} = \{p_i\}_{i=1}^{12}$ (with $p = p_1$) the configuration of infinitely near points of ν_r, F_1 the fiber going through p and M_1 the integral curve described in Proposition 2.2.2 whose strict transforms pass through p_2 and p_3. That is, the self-intersection of $\tilde{M}_1$ is negative. Therefore, $\nu_r(\varphi_{F_1}) = 15$ and $\nu_r(\varphi_{M_1}) = 45$ and then

$$2\nu_r(\varphi_{F_1})\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{F_1})^2 = 900 > 786 = \overline{\beta}_{g+1}(\nu_r).$$

As a result, ν_r is non-positive at infinity by [63, Theorem 4.8].

Let $\nu = \nu_{E_\bullet}$ be the valuation defined by the flag

$$E_\bullet = \{Z = Z_{12} \supset E_{12} \supset \{p_{13}\}\},$$

where $p_{13} \in E_9 \cap E_{12}$, whose first component is the above divisorial valuation ν_r. By Theorem 3.3.42, the coordinates of the vertices of the Newton-Okounkov body $\Delta_\nu(2F + 5M)$ are

$$(0,0), Q_{14} = \left(\frac{9432}{45}, \frac{3132}{45}\right), Q_{16} = \left(\frac{3597}{15}, \frac{1197}{15}\right) \text{ and } Q_{18} = (255, 85),$$

because ν_r is non-minimal with respect to $2F + 5M$ by Corollary 3.2.4, $\theta_2^r(D) < 0$ and $9 = \eta \preccurlyeq r = 12$.

To finish, we study $\Delta_\nu(D)$ when ν is in Case E. As in Case D, $\Delta_\nu(D)$ can be seen as the set in (3.9) and then, by Proposition 3.3.38, the points $Q_{10}, Q_{11}, Q_{12}, Q_{13}$ (respectively, $Q_{14}, Q_{15}, Q_{16}, Q_{17}$) and Q_{18} provided in (3.13) (respectively, (3.14)) for the satellite case are in $\Delta_\nu(D)$ when p_{r+1} is the satellite point $E_\eta \cap E_r, \eta \neq r$, and $\theta_2^r(D) \geq 0$ (respectively, $\theta_2^r(D) < 0$). If p_{r+1} is a free point and $\theta_2^r(D) < 0$ (respectively, $\theta_2^r(D) \geq 0$), the points

$$Q_{14} = \left(\frac{(a+b\delta)\overline{\beta}_{g+1}(\nu_r)}{\nu_r(\varphi_{M_1})}, 0\right), Q_{15} = Q_{14} + \left(0, \frac{a+b\delta}{\nu_r(\varphi_{M_1})}\right),$$
$$Q_{16} = \left(\frac{a\overline{\beta}_{g+1}(\nu_r) - \theta_2^r(D)\nu_r(\varphi_{M_1})}{\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{F_1})}, \frac{-\theta_2^r(D)}{\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{F_1})}\right),$$
$$Q_{17} = Q_{16} + \left(0, \frac{a}{\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{F_1})}\right)$$

(respectively, $Q_{10}, Q_{11}, Q_{12}, Q_{13}$ given in (3.13) for the free case) and $Q_{18} = (\hat{\mu}_D(\nu_r)), b)$ belong to $\Delta_\nu(D)$.

Remark 3.3.44. The previous points Q_i, $1 \leq i \leq 18$, satisfy the following conditions.

(a) Remark 3.3.41(a) is also true for these points.

(b) Some of the above points Q_i are collinear. Indeed,

(b.1) Assume that p_{r+1} is the satellite point $E_\eta \cap E_r, \eta \neq r$, and $\eta \preccurlyeq r$. When $\theta_2^r(D) \geq 0$ (respectively, $\theta_2^r(D)$), the points $(0,0), Q_{11}$ and Q_{13} (respectively, $(0,0)$, Q_{14} and Q_{16}) are in the line $T_1 \equiv \overline{\beta}_0(\nu_r)y = \overline{\beta}_0(\nu_\eta)x$ (respectively, $T_2 \equiv \overline{\beta}_{g^*+1}(\nu_r)y = \overline{\beta}_{g^*+1}(\nu_\eta)x$) and the point Q_{12} (respectively, Q_{17}) belongs to the line which goes through Q_{10} and Q_{18} (respectively, Q_{15} and Q_{18}).

(b.2) Suppose now that $p_{r+1} \in E_\eta \cap E_r, \eta \neq r$, and $\eta \not\preccurlyeq r$. If $\theta_2^r(D) \geq 0$ (respectively, $\theta_2^r(D)$), the points $(0,0), Q_{10}$ and Q_{12} (respectively, $(0,0)$, Q_{15} and Q_{17}) are contained in the line $T_1 \equiv \overline{\beta}_0(\nu_r)y = \overline{\beta}_0(\nu_\eta)x$ (respectively, $T_2 \equiv \overline{\beta}_{g^*+1}(\nu_r)y = \overline{\beta}_{g^*+1}(\nu_\eta)x$) and the point Q_{13} (respectively, Q_{16}) is in the line which passes through Q_{11} and Q_{18} (respectively, Q_{14} and Q_{18}).

(b.3) Assume that p_{r+1} is a free point. If $\theta_2^r(D) \geq 0$ (respectively, $\theta_2^r(D) < 0$), the points $(0,0), Q_{10}$ and Q_{12} (respectively, $(0,0)$, Q_{15} and Q_{17}) belong to the line $y = 0$ (respectively, $y = x/\overline{\beta}_{g+1}(\nu_r)$) and the point Q_{13} (respectively, Q_{16}) is contained in the line going through Q_{11} and Q_{18} (respectively, Q_{14} and Q_{18}).

Theorem 3.3.45. *Let ν be an exceptional curve valuation in Case E described before Lemma 3.3.40. Under the previous notations, the Newton-Okounkov body $\Delta_\nu(D)$ of D with respect to ν is a quadrilateral if and only if $a \neq 0$. Otherwise, it is a triangle (see Remark 3.3.44).*

The vertices of $\Delta_\nu(D)$ are

(a) $(0,0), Q_{10}, Q_{13}$ (respectively, Q_{15}, Q_{16}) and Q_{18} if $\theta_1^r(D) \geq 0$ (respectively, $\theta_1^r(D) < 0$), p_{r+1} is the satellite point $E_\eta \cap E_r, \eta \neq r$, and $\eta \preccurlyeq r$.

(b) $(0,0), Q_{11}, Q_{12}$ (respectively, Q_{14}, Q_{17}) and Q_{18} if $\theta_1^r(D) \geq 0$ (respectively, $\theta_1^r(D) < 0$), p_{r+1} is the satellite point $E_\eta \cap E_r, \eta \neq r$, and $\eta \not\preccurlyeq r$.

(c) $(0,0), Q_{11}, Q_{12}$ (respectively, Q_{14}, Q_{17}) and Q_{18} if $\theta_1^r(D) \geq 0$ (respectively, $\theta_1^r(D) < 0$) and p_{r+1} is a free point.

Proof. The result follows as in the proof of Theorem 3.3.42 to compute the area of the convex hulls generated by the points given in the statement, and as in Theorem 3.3.24 (b) after taking into account Remark 3.3.44. □

Example 3.3.46. Let ν_r be a non-special divisorial valuation of the Hirzebruch surface $\mathbb{F}_4$ and $\mathcal{C}_{\nu_r} = \{p_i\}_{i=1}^5$ the configuration of infinitely near points of ν_r where all the points are free. Assume that the strict transforms of M_1 go through p_2, p_3, p_4 and p_5. Then, $\nu_r(\varphi_{F_1}) = 1, \nu_r(\varphi_{M_0}) = 5$ and $2\nu_r(\varphi_{M_1})\nu_r(\varphi_{F_1}) - \delta\nu_r(\varphi_{F_1})^2 = 6 > 5 = \overline{\beta}_{g+1}(\nu_r)$, and then ν_r is non-positive at infinity.

Set $\nu = \nu_{E_\bullet}$ the exceptional curve valuation associated to the flag $E_\bullet = \{Z = Z_5 \supset E_5 \supset \{p_{r+1}\}\}$, whose first component is the previous divisorial valuation ν_r and p_{r+1} is a free point. The sequence of maximal contact values of ν is $\{\overline{\beta}_0(\nu), \overline{\beta}_1(\nu)\} = \{(1,0),(5,1)\}$. Assume that the strict transform of M_1 goes through p_{r+1}, therefore we are in Case E. Consider the big and nef divisor $D = 2F + 3M$. By Theorem 3.2.1, $\hat{\mu}_D(\nu_r) = a\nu_r(\varphi_{F_1}) + b\nu_r(\varphi_{M_0}) = 17$ and then ν is non-minimal with respect to D (since $\hat{\mu}_D(\nu_r) > D^2\overline{\beta}_{g+1}(\nu_r)$). Consequently, by Theorem 3.3.45, the vertices of the Newton-Okounkov body $\Delta_\nu(D)$ are

$$(0,0), Q_{14} = (14,0), Q_{17} = (15,3) \text{ and } Q_{18} = (17,3),$$

because $\theta_2^r(D) = a\nu_r(\varphi_{F_1}) - b(\nu_r(\varphi_{M_1}) - \delta\nu_r(\varphi_{F_1})) = -1 < 0$. We depict the Newton-Okounkov body $\Delta_\nu(2F+3M)$ and the triangle $\mathfrak{C}(\nu) \cap \mathfrak{H}_{2F+3M}(\nu)$ in Figure 3.7.

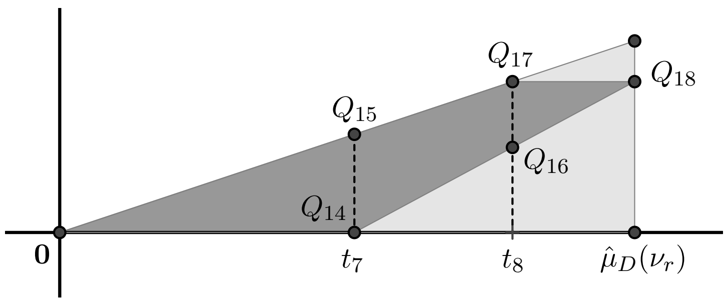


Figure 3.7: $\Delta_\nu(2F+3M)$ and $\mathfrak{C}(\nu) \cap \mathfrak{H}_{2F+3M}(\nu)$ in Example 3.3.46.

Conclusions

Let $\mathbb{F}_\delta$ be a Hirzebruch surface, δ a non-negative integer, and ν_r a divisorial valuation of the function field of $\mathbb{F}_\delta$ centered at $\mathcal{O}_{\mathbb{F}_\delta,p}$, where p is a point in $\mathbb{F}_\delta$ (we say for short that ν_r is a divisorial valuation of $\mathbb{F}_\delta$).

Due to the particular geometric structure of $\mathbb{F}_\delta$, in this thesis we distinguish two types of divisorial valuations of $\mathbb{F}_\delta$: special and non-special valuations (see Definition 2.2.1). We introduce the concepts of non-positivity and negativity at infinity of a divisorial valuation of $\mathbb{F}_\delta$, concepts which we have adapted to special and non-special valuations by using adequate affine charts on $\mathbb{F}_\delta$. One of these charts is formed by the points in $\mathbb{F}_\delta$ which do not belong to the fiber F_1 containing p nor the special section M_0; the second one is determined by the points which are neither in F_1 nor in a specific curve $M_1 \neq M_0$.

Theorem 2.3.7 (respectively, Theorem 2.4.8) shows interesting global and local geometrical properties which are equivalent to the fact that a special (respectively, non-special) divisorial valuation ν_r of $\mathbb{F}_\delta$ is non-positive at infinity. In particular, the cone of curves $\mathrm{NE}(Z)$ of the rational surface Z defined by a non-positive at infinity special (respectively, non-special) divisorial valuation ν_r is generated by the classes of the strict transforms of F_1 and M_0 (respectively, F_1, M_0 and M_1) and the classes of the strict transforms of the exceptional divisors associated with the composition of point blowups $\pi : Z \to \mathbb{F}_\delta$ given by ν_r. This property also characterizes the above mentioned divisorial valuations.

Consider the flag $E_\bullet = \{Z \supset E_r \supset \{q\}\}$, where E_r is the exceptional divisor defining a non-positive at infinity special, or non-special, divisorial valuation ν_r of $\mathbb{F}_\delta$ over $\mathbb{C}$ and Z the rational surface given by ν_r. In Theorem 3.2.1 we explicitly compute a Seshadri-type constant for pairs (ν_r, D), where D is a big divisor on $\mathbb{F}_\delta$, denoted $\hat{\mu}_D(\nu_r)$. In addition, Theorems 3.3.7, 3.3.21, 3.3.24, 3.3.27, 3.3.42 and 3.3.45 determine the coordinates of the vertices of the Newton-Okounkov bodies of pairs $(E_\bullet, D)$. Our description considers two cases: ν_r is minimal with respect to D and ν_r is non-minimal with respect to D (see Definition 3.0.1).

The first components of the coordinates of the vertices of the Newton-Okounkov bodies of pairs $(E_\bullet, D)$, denoted t_i, are the values t where the ray $[D] - t[E_r]$ crosses into a different Zariski chamber (see Subsection 1.5.2). Consequently, when ν_r is special (respectively, non-special), we obtain a local description of the cone of curves

NE(Z) of Z near the face $[\Lambda_r]^\perp$ (respectively, $[\Delta_r]^\perp$) generated by the classes of the strict transforms of F_1, M_0 (respectively, F_1, M_1) and the classes of the divisors $E_1, E_2, \ldots, E_{r-1}$, where $E_1, E_2, \ldots, E_{r-1}$ are the strict transforms of the exceptional divisors associated to π (see Remark 3.3.18 and 3.3.39).

In [60] the authors consider divisorial valuations of $\mathbb{P}^2$ and study global and local geometry properties associated to non-positive and negative at infinity divisorial valuations ν of $\mathbb{P}^2$. In Section 2.3 we reprove [60, Theorems 1 and 2] as a particular case. The same happens with Corollary 3.3.29 where we obtain [65, Corollary 5.2] as a particular case. Notice that Corollary 5.2 in [65] computes the Newton-Okounkov bodies of pairs $(E_\bullet, D)$, where D is a general projective line on $\mathbb{P}^2$ and $E_\bullet$ is a flag as $\{X \supset E \supset \{q\}\}$, E being an exceptional divisor defining a non-positive at infinity divisorial valuation ν_E of $\mathbb{P}^2$ and X the rational surface defined by ν_E.

Finally, Section 2.5 studies and compares the dual graphs of non-positive at infinity divisorial and irrational valuations of $\mathbb{P}^2$ and $\mathbb{F}_\delta$ and provides an algorithm which inductively determines all dual graphs admitting non-positive at infinity valuations.

Conclusiones

Sea $\mathbb{F}_\delta$ una superficie de Hirzebruch, δ un entero no negativo, y ν_r una valoración divisorial del cuerpo de funciones de $\mathbb{F}_\delta$ centrada en $\mathcal{O}_{\mathbb{F}_\delta,p}$, donde p es un punto en $\mathbb{F}_\delta$ (abreviadamente, una valoración divisorial de $\mathbb{F}_\delta$).

En esta tesis, debido a la particular estructura geométrica de $\mathbb{F}_\delta$, distinguimos dos tipos de valoraciones divisoriales de $\mathbb{F}_\delta$, valoraciones especiales y no especiales (véase Definición 2.2.1). Introducimos el concepto de no positividad y negatividad en el infinito para valoraciones divisoriales de $\mathbb{F}_\delta$, conceptos adaptados a valoraciones especiales y no especiales. Para ello utilizamos cartas afines adecuadas en $\mathbb{F}_\delta$. Una de ellas está formada por los puntos de $\mathbb{F}_\delta$ que no pertenecen a la fibra F_1 que contiene a p, ni a la sección especial M_0. La segunda está determinada por los puntos que no están en F_1 ni en una curva particular $M_1 \neq M_0$.

En el Teorema 2.3.7 (respectivemente, Teorema 2.4.8) mostramos interesantes propiedades geométricas globales y locales que son equivalentes al hecho de que una valoración especial (respectivamente, no especial) de $\mathbb{F}_\delta$ sea no positiva en el infinito. En particular, el cono de curvas $\mathrm{NE}(Z)$ de la superficie racional Z definida por una valoración ν_r divisorial especial (respectivamente, no especial) no positiva en el infinito está generado por las clases de los transformados estrictos de F_1 y M_0 (respectivamente, F_1, M_0 y M_1) y las clases de los transformados estrictos de los divisores excepcionales asociados a la composición de explosiones $\pi : Z \to \mathbb{F}_\delta$ definidas por ν_r. Esta propiedad también caracteriza las valoraciones divisoriales antes mencionadas.

Consideremos la bandera $E_\bullet = \{Z \supset E_r \supset \{q\}\}$, donde E_r es un divisor excepcional que define una valoración divisorial ν_r especial, o no especial, de $\mathbb{F}_\delta$ (sobre $\mathbb{C}$) no positiva en el infinito y Z la superficie racional dada por ν_r. En el Teorema 3.2.1 calculamos explícitamente una constante de tipo Seshadri para pares (ν_r, D), donde D es un divisor big en $\mathbb{F}_\delta$, denotada $\hat{\mu}_D(\nu_r)$. Además, en los Teoremas 3.3.7, 3.3.21, 3.3.24, 3.3.27, 3.3.42 and 3.3.45, expresamos las coordenadas de los vértices de los cuerpos de Newton-Okounkov de pares $(E_\bullet, D)$. Nuestra descripción considera dos casos: aquel donde ν_r es minimal respecto D y el caso donde ν_r es no minimal respecto D (véase Definición 3.0.1).

Las primeras componentes de las coordenadas de los vértices de los cuerpos de Newton-Okounkov de pares $(E_\bullet, D)$, denotadas t_i, son los valores t donde el rayo

$[D] - t[E_r]$ cruza a una cámara de Zariski diferente (véase Subsección 1.5.2). Consecuéntemente, cuando ν_r es especial (respectivamente, no especial), obtenemos una descripción local del cono de curvas $\mathrm{NE}(Z)$ de Z cerca de la cara $[\Lambda_r]^\perp$ (respectivamente, $[\Delta_r]^\perp$) generado por las clases de los transformados estrictos de los divisores F_1, M_0 (respectivamente, F_1, M_1) y las clases de los divisores $E_1, E_2, \ldots, E_{r-1}$, donde $E_1, E_2, \ldots, E_{r-1}$ son los transformados estrictos de los divisores excepcionales asociados a π (véase Notas 3.3.18 y 3.3.39).

En [60] los autores consideran valoraciones divisoriales de $\mathbb{P}^2$ y estudian propiedades geométricas globales y locales asociadas a valoraciones divisoriales de $\mathbb{P}^2$ no positivas y negativas en el infinito. En la Sección 2.3 probamos [60, Teoremas 1 y 2] como un caso particular. Lo mismo ocurre con el Corolario 3.3.29 donde obtenemos [65, Corolario 5.2] como un caso particular. Nótese que el Corolario 5.2 en [65] calcula explícitamente los cuerpos de Newton-Okounkov de pares $(E_\bullet, D)$, donde D es una recta proyectiva general en $\mathbb{P}^2$ y $E_\bullet$ es una bandera como $\{X \supset E \supset \{q\}\}$, siendo E un divisor excepcional que define una valoración divisorial ν_E de $\mathbb{P}^2$ (sobre $\mathbb{C}$) no positiva en el infinito y X la superficie racional definida por ν_E.

Finalmente, en la Sección 2.5 estudiamos y comparamos los grafos duales de valoraciones divisoriales e irracionales de $\mathbb{P}^2$ y $\mathbb{F}_\delta$ que son no positivas en el infinito y aportamos un algoritmo que determina inductivamente todos los grafos duales que admiten valoraciones no positivas en el infinito.

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