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Expectation, Information Friction and Macro-Finance

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ABSTRACT

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Agents' belief and information friction are crucial for asset price and macroeconomy. This thesis applies "Internal Rationality" learning approach to explain some interesting facts in the financial market and investigate the implication of information friction for social welfare. The first chapter documents the high, volatile and persistent AH premium in China stock market. We show that various standard RE and Bayesian RE asset pricing models cannot explain the AH premium, but a model of internally rational learning where agents learn about stock prices provides a natural explanation. This emphasizes the importance of modeling investors who learn about equity prices. The second chapter show that survey data suggests stock price forecasts are not anchored by forecasts of fundamentals and rejects this aspect of the formation of stock price expectations in a wide range of asset pricing models. The evidence casts some doubt on the modeling of expectation formation in the asset pricing models which assume agents possess the knowledge of the equilibrium pricing function as in Rational Expectations and Bayesian Rational Expectations models. Relaxing this knowledge appears necessary for models to reconcile the survey evidence and potential resolutions are discussed. The third chapter establishes that the dispersed belief as a result of information friction creates a novel channel through which the welfare cost of inflation in a sector is increasing in its price flexibility and alters the optimal inflation index. A monetary policy that stabilizes the optimal inflation index mitigates the paradox.

Abstracte: La creencia de los agentes y la fricción de información son cruciales para el precio de los activos y la macroeconomía. Esta tesis aplica el enfoque de aprendizaje de "racionalidad interna" para explicar algunos hechos interesantes en el mercado financiero e investigar las implicaciones de la fricción de la información para el bienestar social. El primer capítulo documenta la prima AH alta, volátil y persistente en el mercado de valores de China. Mostramos que varios modelos estándar de precios de activos RE y Bayesianos no pueden explicar la prima AH, pero un modelo de aprendizaje internamente racional donde los agentes aprenden sobre los precios de las acciones proporciona una explicación natural. Esto enfatiza la importancia de modelar a los inversores que aprenden sobre los precios de las acciones. El segundo capítulo muestra que los datos de la encuesta sugieren que los pronósticos de precios de acciones no están anclados por pronósticos de fundamentos y rechaza este aspecto de la formación de expectativas de precios de acciones en una amplia gama de modelos de precios de activos. La evidencia arroja algunas

dudas sobre el modelo de formación de expectativas en los modelos de fijación de precios de activos que suponen que los agentes poseen el conocimiento de la función de fijación de precios de equilibrio como en los modelos de Expectativas racionales y Expectativas racionales bayesianas. Relajar este conocimiento parece necesario para que los modelos concilien la evidencia de la encuesta y se discuten las posibles resoluciones. El tercer capítulo establece que la creencia dispersa como resultado de la fricción de la información crea un nuevo canal a través del cual el costo de bienestar de la inflación en un sector está aumentando en su flexibilidad de precios y altera el índice de inflación óptimo. Una política monetaria que estabilice el índice de inflación óptimo mitiga la paradoja.

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Preface

Agents' belief and information friction are crucial for asset price and macro-economy. This thesis applies "Internal Rationality" learning approach to explain some interesting facts in the financial market and investigate the implication of information friction for social welfare.

The first chapter is *AH Premium: A Natural Experiment* with Tongbin Zhang aims to explain the AH premium which also serves as a natural experiment to test asset pricing models.

The shares of many important Chinese companies trade in both the Shanghai and the Hong Kong markets. Even though these twin shares pay the same dividends and have the same voting rights, they are distinct shares labelled "A-shares" and "H-shares" respectively. The two markets were connected in November 2014, since then investors in either market can buy both A- and H- shares for all companies participating in the connecting program. Surprisingly, A-shares trade at a substantial premium relative to the H-shares of the same companies in the AH-market. This premium increased greatly after the connection date. It has been fluctuating with a mean of 30 percent and a maximum of 50 percent since the connection.

We know of no formal paper explaining the AH premium. Some explanations based on "segmented markets" with agents that are heterogeneous across markets have been provided. But the AH-market is not segmented since November 2014, and the premium became very large precisely after the connection date, therefore standard asset pricing theories where agents know the pricing mapping from fundamental to prices imply that market forces should equalize prices even if market participants are heterogeneous. Other informal explanations appeal to transaction costs, but as we show these would explain only a small part of the premium. Ad-hoc explanations based on Chinese traders being less sophisticated (granny trading) or lack of competition in the Chinese stock market are even harder to sustain.

By relaxing the strong information assumption about agents' knowledge about pricing mapping, agents' expected price becomes an additional factor to determine the equity price on top of the information given by fundamentals. We find that models of learning about stock prices provide a very natural explanation for this premium. Recent papers based on Internal Rationality show that learning about stock prices is compatible with rational investors and that model explains high volatility of stock prices empirically. Investors' subjective beliefs about stock prices are given by a model that is a good description of actual prices and this perceived model is hard to reject given actual stock prices. Under this framework, agents realize that an A-share is actually a different security from an H-share and that it can be purchased or sold next period at a possibly different price. Even if the

A-, H- dividends are (nearly) the same, just because investors expect A- and H- prices to differ these prices will actually differ in equilibrium, even when investors are (internally) rational. Our explanation for AH premium would be that some factors increased either the actual price and/or the expected capital gains in A-shares, shortly after November 2014 (perhaps the more liquid market, a change in transaction costs, perhaps new market participants, a change in expectations, or perhaps some special events in China around those dates). This increased A-prices, leading to high expected prices and so on. We show formally that learning about prices can generate persistent and volatile AH premium and we match the key moments of the observed AH premium data.

After Shanghai and Hong Kong markets connected, the widening AH premium motivated many investors who bet on the convergence of prices of A- and H-shares to implement a convergence trading strategy: short selling A-shares and long buying H-shares. Chinese finance newspaper, however, reported in June 2015 that these investors encountered great losses in the end. We show, by simulation, that when people learn about prices, AH premium may not shrink in the short term and convergence traders probably incur high losses.

This explanation of the AH puzzle implies that agents' expectations could be a key element determining the outcome of financial liberalization. Our paper also contributes to the debate on how to model investors' expectations in stock markets. We think of AH premium as a natural experiment rejecting Bayesian/RE models in favor of learning about prices.

The second Chapter with Pei Kuang and Tongbin Zhang *New Tests of Expectation Formation with Applications to Asset Pricing Models* is related to my job market paper and provides some empirical facts to support the lack of common knowledge about pricing function.

We develop new tests of expectation formation which are generally applicable in asset pricing models with various informational assumptions. We show these models typically impose a large number of cointegration restrictions between forecasts of economic variables, and these cointegration restrictions imply that agents have strong information set when they form their expectations. Our tests utilize these restrictions. Researchers can apply these tests to study the cointegration between forecasts of exogenous variables and forecasts of endogenous variables in their model as well as the cointegration between forecasts of different endogenous variables. Moreover, these models impose cointegration restrictions between forecasts of the same variable (e.g., stock prices) over different forecasting horizons.

The essence of our new tests is demonstrating that agents' knowledge of the equilibrium pricing function in asset pricing models imposes cointegration restrictions between their forecasts of stock prices and forecasts of fundamentals, i.e., aggregate consumption or dividends. Intuitively, the long-run component of stock price forecasts in these models is anchored by consumption forecasts via this knowledge of agents. We show this is a robust feature across a wide range of RE or Bayesian RE asset pricing models that assume this knowledge of agents either explicitly or implicitly.

Yet a central new piece of evidence from expectations data uncovered by the paper is that the median (or mean) survey forecasts of aggregate stock price index are not cointegrated with the median (or mean) forecasts of aggregate consumption. Put differently, the long-run or trend component of stock price forecasts is not anchored by consumption forecasts. This evidence is robust to different sources of expectations data (Livingston Survey and Shiller Survey for stock price forecasts and Survey of Professional Forecasters and Greenbook forecasts for consumption forecasts), forecasting horizons (1-, 2-, 4-quarter and 10-year ahead forecasts), statistical tests (Phillips Perron test, ADF-GLS test, KPSS test with correcting small sample problems and Johansen test), using median or mean forecasts for testing, and using stock price forecasts data which is made at the same or different dates from consumption forecasts.

The evidence casts some doubt on the modeling of expectation formation in asset pricing models which assume agents possess the knowledge of the equilibrium pricing function. Irrespective of agents having rational or non-rational (or extrapolative) expectations about fundamentals (consumption), stock price forecasts are cointegrated with aggregate consumption forecasts in these models, appearing inconsistent with the survey evidence. Reconciling the new survey evidence appears to call for relaxing agents' knowledge of the equilibrium pricing function as is present in e.g., adaptive learning models.

The third chapter *Information Frictions and the Paradox of Price Flexibility* with Shengliang Ou and Donghai Zhang is trying to understand the implication of information friction where agents have imperfect-common-knowledge.

Electronic shelf labels (ESL) permits retailers to set prices digitally without any costs that would otherwise occur using paper price tags. Over the past decade, we have witnessed an expansion in the usage of digital price tags thanks to the growing affordability of ESL.1 The introduction of digital price tags may facilitate price adjustment and reduce the degree of nominal rigidity in the economy. Is such technological progress welfare improving?

We address this research question in a multi-sector New Keynesian (NK) model with information frictions and dispersed beliefs. We highlight a new channel—dispersed belief channel that is relevant to understand the welfare consequence of a reduction in nominal rigidity. The multi-sector feature of the model permits us to analyze the consequence of both an economy-wide and a sectoral reduction in nominal rigidities. The latter is relevant as the technological progress, such as the expansion of the ESL, might be merely a sectoral phenomenon rather than an economy-wide change. Moreover, the multi-sector model that we build allows us to revisit the optimal inflation index stabilization policy.

To fully understand the welfare consequence of a change in price flexibility, we derive the welfare loss function around the perfect information steady-state and decompose it into four components. We derive the following results. First, in a static and symmetric two-sector model, we derive analytically the conditions under which an economy-wide increase in price flexibility is welfare-deteriorating—the paradox of price flexibility. With perfect information, such a reduction in nominal friction is welfare-improving. However, in the presence of information frictions, the dispersed beliefs channel might dominate. Consequently, the paradox of price flexibility arises. Second, the paradox is more severe if the reduction in nominal frictions is merely a sectoral phenomenon. In our baseline analysis, we focus on an inflation-targeting central bank that fully stabilizes the Consumer Price Index (CPI), which is the principal mandate among many central banks in the world. Given this policy, increased sectoral price flexibility is detrimental to social welfare, even in the absence of information frictions.

We study the design of optimal inflation index stabilization policy, emphasizing the role of the dispersed beliefs. The objective is to find the price/inflation index, constructed by the weighted average of sectoral prices/inflations, associated with the lowest welfare losses. A sector with a relatively more flexible price deserves a smaller weight as the welfare cost of inflation in this sector, which arises through the Calvo channel, is lower. The dispersed belief channel that we emphasize dampens the results arising from the Calvo channel. Through the former, the same level of inflation causes much more price dispersions (among price resetting firms) in that sector if its nominal rigidity is reduced. Therefore, ignoring the dispersed beliefs channel would lead to a policy recommendation that under-reacts to inflation of a relatively more flexible price sector. Moreover, due to dispersed beliefs, the optimal weight that a fully flexible price sector receives is nontrivial.

Does an optimal inflation stabilization policy resolve the paradox of price flexibility? An economy-wide change in nominal frictions does not alter the optimal inflation index, and therefore, the optimal inflation index stabilization policy does not play a role in this case. When a reduction in nominal frictions is asymmetric across sectors, the optimal price index stabilization policy mitigates the paradox of price flexibility. After a reduction in nominal frictions in one sector, the central bank reacts optimally by assigning a higher weight on the price in the second sector at the cost of higher volatility in the first sector. With perfect information, the benefit strongly dominates the cost, and aggregate welfare losses are reduced. Therefore, the policy mitigates the paradox substantially. However, in the presence of information frictions, an additional source of cost arises. A more volatile price in the first sector is associated with a higher price dispersion among price resetting firms due to the dispersed beliefs. This adds additional challenge for a central bank to resolve the paradox/improve social welfare.

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Introduction

In the field of asset pricing, there exists many interesting asset market facts in addition to the well-known equity premium and stock prices volatility. This thesis has grounded on reconciling empirical facts in asset market and asset pricing models, especially the models with the agents who don't have rational expectations.

Chapter 1 shows that A large portion of the Chinese twin stocks is traded both in the Shanghai (A-share) and Hong-Kong (H-share) markets. The A- and H-shares are different assets since they can not be exchanged one-to-one. A-shares have sold at a premium: the AH premium. This premium is large (20-50%) and persistent, it has been present since the two markets were connected in Nov 2014 until now. Since both shares pay the same dividends and traders can operate on both markets this provides a natural experiment to test asset pricing models. We show that various standard RE and Bayesian RE asset pricing models cannot explain the AH premium, but a model of internally rational learning where agents learn about stock prices provides a natural explanation. This emphasizes the importance of modeling investors who learn about equity prices. The premium survives the introduction of convergence traders: those who bet on the AH premium going to zero are highly likely to suffer big losses.

Chapter 2 develops new tests for expectation formation in financial and macroeconomic models under various informational assumptions. Survey data suggests stock price forecasts are not anchored by forecasts of fundamentals and rejects this aspect of the formation of stock price expectations in a wide range of asset pricing models. The evidence casts some doubt on the modeling of expectation formation in the asset pricing models which assume agents possess the knowledge of the equilibrium pricing function as in Rational Expectations and Bayesian Rational Expectations models. Relaxing this knowledge appears necessary for models to reconcile the survey evidence and potential resolutions are discussed.

In chapter 3, we ask whether the introduction of digital price tags that facilitate price adjustments and reduce the degree of nominal rigidity in is welfare-improving. To address this question, we build a multi-sector New Keynesian model with information frictions and dispersed beliefs. We show analytically in a static model and quantitatively in a dynamic model that increased price flexibility is welfare-deteriorating—the paradox of price flexibility. The presence of information frictions aggravates the paradox. Moreover, we study the optimal inflation index stabilization policy within our framework. The dispersed belief creates a novel channel through which the welfare cost of inflation in a sector is increasing in its price flexibility, and alters the optimal inflation index. A monetary policy that stabilizes the optimal inflation index mitigates the paradox.

CHAPTER 1

AH Premium: A Natural Experiment

1.1. Introduction

The shares of many important Chinese companies trade in both the Shanghai and the Hong Kong markets. Even though these twin shares pay the same dividends and have the same voting rights, they are distinct shares labelled "A-shares" and "H-shares" respectively. They are different assets, can not be traded one-to-one anywhere. The two markets were connected in November 2014, since then investors in either market can buy both A- and H-shares of all companies participating in the connecting program, we refer to this connected market of dual-listed companies as the "AH-market".

Surprisingly, A-shares trade at a substantial premium relative to the H-shares of the same companies in the AH-market. This premium increased greatly precisely after the connection date. It has been fluctuating with a mean of 27 percent and a maximum of 50 percent since the connection.[1]

The aim of this paper is to explain the AH premium. Traditional asset pricing theories are unable to explain such a large premium, including theories based on

[1] See Figure 2.

Bayesian learning about fundamentals. However, we find that models of learning about stock prices provide a very natural explanation.

Understanding the AH premium is important, first, because of the huge size of the AH-market: the market value of AH-share reached 2.7 trillion U.S. dollars in 2015, which was about 70% of the market capitalization of the London stock exchange;[2] second, because it speaks to the consequences of financial liberalization; third, because it matters for investors in this market; and fourth, because it provides a rare opportunity for economists to test asset pricing theories (in terms of stock price expectation formation) with a natural experiment.

We have not found a satisfactory explanation of the AH premium in the literature. Some explanations based on "segmented markets" with agents that are heterogeneous across markets have been provided. But the AH-market is not segmented since November 2014, standard asset pricing theories imply that market forces should equalize prices even if market participants are heterogeneous. Other informal explanations appeal to transaction costs and dividend taxes, but as we show these would explain only a small part of the premium.[3] Ad-hoc explanations based on Chinese traders being less sophisticated (granny trading) or lack of competition in the Chinese stock market are even harder to sustain as only sufficiently wealthy traders can participate in the AH-market,[4] and therefore most Chinese

[2]See section 2.2 for descriptive details of the AH-market.
[3]See section 2.3 for a more detailed discussion of the literature and section 3 for a detailed analysis of how these explanations do not work for the AH premium.
[4]We discuss this in section 2.2.

grannies have a much harder time trading stocks than American grannies.[5] Only systematic research can discover if US or Chinese stock markets are sufficiently well described by the competitive market assumption, along with an appropriate assumption about how agents form expectations.

We show that learning about stock prices provides a natural explanation. Recent papers based on Internal Rationality (IR) such as Adam and Marcet (2011), Adam, Marcet and Nicolini (2016) (AMN) and Adam, Marcet and Beutel (2017) show that learning about stock prices is compatible with rational investors. They show that learning about prices explains observed volatility of stock prices. Investors' subjective beliefs about stock prices are given by a model that is a good description of actual prices, and this perceived model is hard to reject given actual stock prices. Under this framework agents realize that an A-share is actually a different security from an H-share and that it can be purchased or sold next period at a possibly different price. Even if the A-, H- dividends are (nearly) the same, just because investors expect A- and H- prices to differ these prices will actually differ in equilibrium, even when investors are (internally) rational.

When agents learn about stock prices high expected capital gains generate high prices, this leads to higher expected capital gains and so on in a self-referential fashion. Our explanation for Figure 2 would be that some factors increased either

[5]Another informal explanation has been lack of competitiveness in the Chinese stock market. But this is difficult to square with the fact that A-shares have a higher price. In any case, lack of competition is always a possible explanation for almost anything. The US stock market is not a textbook example of competitiveness either and yet a large asset pricing literature entertains the assumption of competitive markets in American stocks.

the actual prices and/or the expected capital gains in A-shares, shortly after July 2014 (perhaps the more liquid market, new market participants, the widely spread optimistic media reports on the A-share market, or perhaps some special events in China around those dates). This increased A-prices, leading to high expected prices and so on. We show formally that learning about prices can generate a large, volatile, and persistent AH premium and we match the key moments of the observed AH premium data.

This explanation of the AH premium implies that agents' expectations could be a key element determining the outcomes of financial liberalization. The average and the variation of the AH premium were relatively small a couple of years before November 2014, and the market connection took place in part to promote price convergence. But as seen in Figure 2, the result was the opposite, and the AH premium became much larger after connecting the markets. To the extent that the AH premium is seen as undesirable, as it injects uncertainty in the system and it may promote unproductive speculation, our observation should be relevant for the recent connection of the Shanghai and London stock markets.[6]

Learning about prices by internally rational agents also matters for arbitrage. After Shanghai and Hong Kong markets connected, the widening AH premium motivated some arbitrageurs to bet on the convergence of prices of A- and H-

[6] Shanghai-London Stock Connect is a mechanism that connects the London Stock Exchange and the Shanghai Stock Exchange. Eligible companies listed on the two stock exchanges can issue, list and trade depositary receipts on the counterpart's stock market in accordance with the corresponding laws and regulations. The connection took place on June 17, 2019

shares, short selling A-shares and long buying H-shares. Chinese finance newspapers, however, reported in June 2015 that these investors encountered great losses in the end.[7] We show, by simulation, that AH premium caused by investors learning about prices may not shrink in the short term and arbitrageurs are likely to incur high losses.[8]

Our paper also contributes to the debate on how to model investors' expectations in stock markets. As is well known, it is very difficult to explain stock price volatility and the behavior of survey expectations under rational expectations (RE). A large literature deviates from RE by assuming agents imperfectly know the distribution of fundamental shocks, but this literature assumes that investors understand the equilibrium pricing function that prices equal the present-value of dividends. We dub this literature Bayesian RE,[9] it includes models of Bayesian learning, "agreeing to disagree", robustness and behavioral economics.[10] Although Bayesian RE can not explain quantitatively stock price volatility, many authors claim that it could explain survey behavior, just as learning about prices can.[11]

Some researchers have designed lab experiments to distinguish these theories, where subjects trading stocks are told the dividend process and the experiment is

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[7]See the news in Chinese in the link: http://finance.jrj.com.cn/2015/06/11082819343903.shtml

[8]See the literature on the limit of arbitrage in section 2.3.

[9]Adam, Marcet and Beutel (2017) dub this literature Bayesian RE. They point out that this literature makes a very asymmetric assumption about fundamentals and prices: investors are assumed not to understand the behavior of fundamentals (say, dividends) but investors understand perfectly well how current stock prices relate to future fundamentals, so that investors are assumed to have RE about the pricing function.

[10]See section 2.3 on related literature for more details.

[11]See Adam, Marcet and Beutel (2017).

designed to understand if agents will quickly converge to a common knowledge of the pricing function (as it is assumed in the RE and Bayesian RE literature but not in the learning about prices literature). These papers tend to show that laboratory subjects in the same group still differ in their price expectations and hold different positions accordingly. We contend that the AH-market tests the same issue: to the extent that traders' stakes are many times higher in the true AH-market than in the lab, we think of this as a natural experiment rejecting RE and Bayesian RE models in favor of learning about prices.

The paper is structured as follows. Section 2 discusses institutional issues of the Chinese stock market and the literature review. Section 3 argues that the standard asset pricing models where agents know the equilibrium pricing function have difficulties in generating data-like AH premium. We show that learning about prices can explain the AH premium naturally in section 4. Section 5 shows that convergence traders are likely to suffer big losses. Section 6 discusses some extensions. Section 7 discusses other possible explanations. Section 8 concludes this paper.

1.2. Overview of the Chinese Stock Markets and Literature Review

This section introduces the mainland Chinese stock market briefly, describes some key features of AH-market and literature related to our paper. We argue, first, that markets for twin shares were segmented before November 2014, but the AH-market was connected after this date. We also find that the AH premium before November 2014 is easy to understand using segmented market models with heterogeneous agents found in the literature. However, the large AH premium after that date is a puzzle, specially because the premium increased so much precisely after the connection.

1.2.1. Mainland Chinese Stock Market

Mainland Chinese stock market is relatively young; it opened in 1990 with the establishment of the Shanghai Stock Exchange (SSE) and the Shenzhen Stock Exchange (SZSE). Stocks traded on these two exchanges with RMB (Chinese currency) are called A-shares. The Chinese economy grew at 10% annually on average from 1990 to 2018, and this led to the rapid development of equity markets. The market capitalization of the A-shares is the second-largest in the world. It reached about $6.3 trillion at the end of 2018 with the turnover of $13 trillion in 2018.[12]

Figure 1.1 plots the Shanghai stock price index (Shanghai Composite Index) from 1995 to 2018 in real terms. There were two boom-bust episodes: the stock price index reached its historical peak in 2007, having risen from a trough in 2005, and then quickly busted. Stock prices boomed again in the second half of 2014 and almost doubled by mid-2015, but still quickly trended down.

Some economists and market participants have expressed the view that the Chinese central government directly and frequently intervenes in the mainland

[12]See data sources in Appendix 9.5.

stock market. However, it is not clear that it does more so than the US government. Since 2005, based on public information, it has only intervened once when the A-share bubble burst at the end of June 2015. The Chinese government, out of concerns about the high leverage taken by many Chinese investors, required a national team of state-owned security companies to support stock prices to avoid a severe financial crisis.[13] It is common worldwide that governments directly intervene to stabilize the financial market turmoils. The U.S. government intervention (Troubled Asset Relief Program) in 2008 is a prominent example (Veronesi and Zingales, 2010). The intervention of the Hong Kong government in the stock market during the 1998 Asian financial crisis is also well-documented (Goodhart and Lu 2003).

A unique feature in the mainland Chinese stock market is that several dozen companies have issued twin shares listed on the same exchanges. Twin shares dubbed A- and B-shares have existed since 1993. They have identical dividend and voting rights, but they were traded by different investors, so it is not surprising that they had different prices. A-shares traded with RMB used to be restricted to mainland investors before November 2014. In addition, many companies issued B-shares traded with US dollars were strictly confined to international investors. Mainland investors were allowed to trade B-shares using US dollars since February 2001, but, as discussed in Mei, Scheinkman and Xiong (2009), the difficulties for Chinese citizens to acquire US dollars still serve as severe restrictions for mainland

[13]See Huang, Miao and Wang (2019).

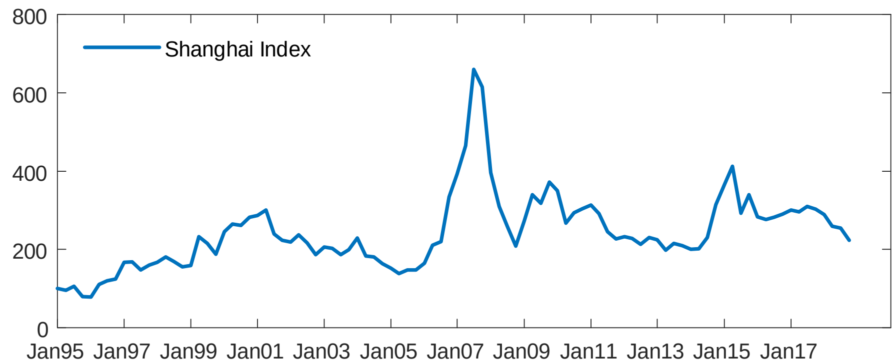

Figure 1.1. Quarterly Shanghai Composite Index in Real Terms

investors to participate in the B-share market. Therefore, A- and B-share markets were strictly segmented before February 2001 and are still segmented to a large extent for mainland investors. The literature documented that A-shares traded at higher prices than B-shares despite their identical rights, e.g., Fernald and Rogers (2002) and Mei, Scheinkman and Xiong (2009). This premium has been explained in the light of models of segmented markets and heterogeneous agents.

1.2.2. AH-shares and Shanghai-Hong Kong Stock Connect

The shares of companies incorporated in mainland China but traded on the Hong Kong Stock Exchange (HKSE) with Hong Kong dollars (HKD) are called H-shares. Currently, about one hundred mainland companies dual-list in both A-share (most on the SSE) and H-share markets. Twin shares of the same companies have the same fundamentals as they are identical with respect to shareholder rights, such as voting and profit-sharing. The information clarifying that the two classes of shares entertaining the same fundamental is widely available. However, they are different stocks because they have different international identification codes which serve for the uniform identification of securities. For instance, the A-share of Air China is identified as CNE000001NN0, but its H-share is identified as CNE1000001S0. It is impossible to sell one class of shares (e.g., CNE000001NN0) in the market of another class of shares (e.g., CNE1000001S0).

Many Chinese companies with large market capitalization are included in the AH-market: the market value of A-shares of these dual-listed companies accounts for 20% of the total A-share market capitalization, while the market value of H-shares of these companies accounts for 35% of the market capitalization of HKSE.[14]

Before November 17, 2014, the Shanghai and Hong Kong markets were segmented: mainland investors were not allowed to trade in the H-share market while Hong Kong and international investors were barred from investing in the A-share market. Therefore an AH premium before November 2014 can be explained by appealing to segmented markets with heterogeneous agents. However, the Shanghai and Hong Kong markets became connected after the Shanghai-Hong Kong Stock Connect program initialized on November 17, 2014.[15] All of the Shanghai

[14]Refer to Appendix 9.5 for data sources. Also see an analysis artical by the famous investment bank economist Shanwen Gao. (http://www.thfr.com.cn/post.php?id=34627)

[15]Detailed information of connect problem can be found on the website of HKSE. (www.hkex.com.hk/-/media/HKEX-Market/Mutual-Market/Stock-Connect/Getting-Started/Information-Booklet-and-FAQ/Information-Book-for-Investors/Investor_Book_En.pdf)

and Hong Kong twin stocks are included in the connect program.[16] Mainland investors can participate in the H-share market (southbound trading) very easily through the trading and clearing facilities of SSE, and Hong Kong and international investors are allowed to trade in the Shanghai A-share market (northbound trading).[17] Transaction costs are low and same for local and foreign investors in each market, as we quantify in section 7. Trading and settlement currency through Shanghai-Hong Kong stock connect is RMB. Mainland investors do not need HKD to buy H-shares via the connect program.[18] Meanwhile, Hong Kong and international investors can acquire RMB easily in the Hong Kong offshore RMB market to buy A-shares.[19]

Surprisingly, the AH premium increased markedly after the connection date. The Hang Seng China AH Premium Index plotted in Figure 1.2 measures the weighted average price ratio (A-share prices over H-share prices of same companies)

[16]Through the connect program, mainland investors can trade more than 300 stocks in the HKSE, and foreign investors can trade more than 500 stocks in the SSE. See Ma, Rogers and Zhou (2019).

[17]Precisely, all Hong Kong and international investors are allowed to trade A-shares listed in Shanghai. All mainland institutional investors and individual investors who have at least RMB 500,000 in their accounts are eligible to trade H-shares. According to the Financial Times report in July 2015, small retail investors hold less than 5% of the overall market value of A-shares (https://www.ft.com/content/f3d94f92-2715-11e5-9c4e-a775d2b173ca).

[18]China Securities Depository and Clearing Company provides unlimited currency conversion service automatically in the background for the mainland investors to trade in the H-share market. After mainland investors sell H-share stocks, they get RMB rather than HKD.

[19]Northbound trading and Southbound trading are respectively subject to a separate set of Daily Quota. The Northbound Daily Quota is set at RMB 52 billion, and the Southbound Daily Quota is set at RMB 42 billion. The Daily Quota is applied on a "net buy" basis. Under that principle, investors are always allowed to sell their cross-boundary securities regardless of the quota balance. These quota balance in the data almost never bind. See data sources in Appendix 9.5.

in percentage for these AH-twin shares.[20] The index of 100 means that A-shares are trading at par, an index larger than 100 indicates A-shares trade at a premium. After the market connection, according to standard asset pricing theories, the AH premium index should have converged to 100 (or to somewhere close to 100 because of the different dividend taxes and transaction costs). However, it diverged dramatically to almost 150 as a peak in June 2015, and fluctuated between 120 and 150 since then. Note that AH premium index had been rising for 4 months before the connection from July 2014 and the rising momentum was kept before it got stabilized. As for the individual stocks, most of them have larger premium for the A-shares after the connection. Hence, the AH premium does not only come from a few dominating companies with large weights.[21] However A- and H-share prices are highly positively correlated, as shown in Figure 1.3: from November 2014 to June 2015, both A- and H-share prices rose, but A-share prices rose faster than H-share contributing to the divergence in prices. We observe a similar pattern of different speeds of adjustment when the price indexes fell after July 2015.

Note: A is the price index of A-shares of the dual listed companies and H is the price index of H-shares of the dual listed companies.

[20] Let P^A be the weighted average price of A-shares and P^H the weighted average price of H-shares, the index is $P^A/P^H * 100$. The exchange rate is taken into account when constructing this index. The index methodology refers to https://www.hsi.com.hk/static/uploads/contents/en/dl_centre/methodologies/IM_chinaahe.pdf

[21] The weight of each firm in the AH Premium Index can be found in https://www.hsi.com.hk/static/uploads/contents/en/dl_centre/factsheets/chinaahe.pdf.

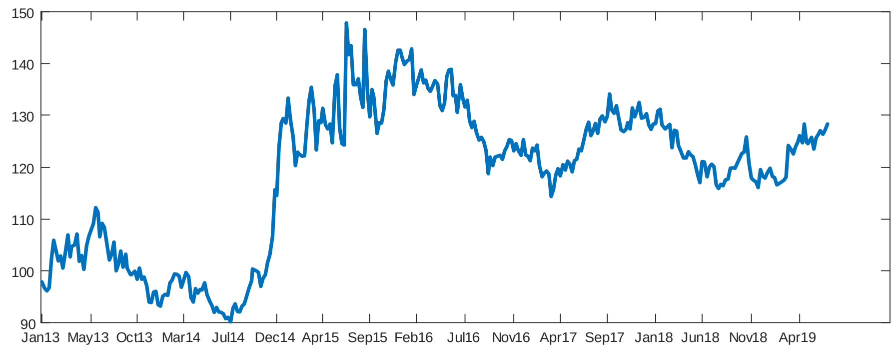

Figure 1.2. Hang Seng China AH Premium Index

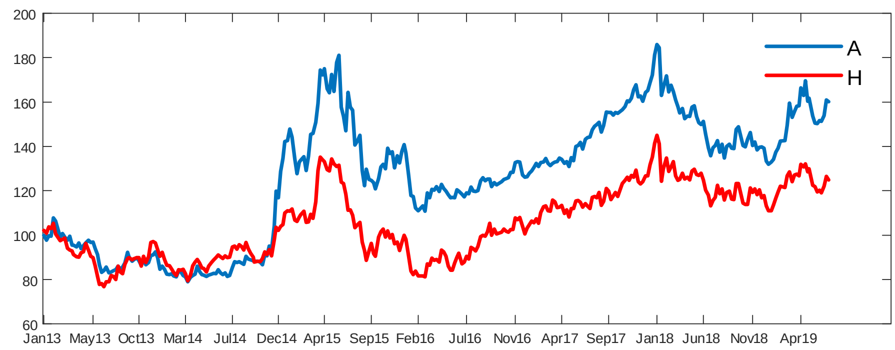

Figure 1.3. Hang Seng China AH-A and AH-H Index

1.2.3. Related Literature

This paper is related to the literature on the price differences of twin stocks. First, Froot and Dabora (1999), Fernald and Rogers (2002) and Mei, Scheinkman and Xiong (2009) show that transaction costs are not the main reason for price differences. Second, many papers focus on explaining the price differences in the segmented market environment, where AB premium is a prominent example. Heterogeneities among different participants across segmented markets such as agents' different stochastic discount factors (Fernald and Rogers 2002), and diverse beliefs about fundamentals (Chan, Menkveld and Yang 2008; Mei, Scheinkman and Xiong 2009; Jia, Wang and Xiong 2017) are argued to explain the price differences. However, heterogeneities fail in explaining the price differences after market connection in November 2014.

The literature also widely document the difficulties of explaining price differences in connected markets. For instance, Froot and Dabora (1999) show the price differences of three twin stocks in the connected U.S. and European markets but did not find satisfactory explanations.[22] Lamont and Thaler (2003) think this is an anomaly in violation of the law of one price and difficult to get rationalized.[23] The AH premium is even more puzzling because of the much larger size of the market compared with the cases considered in these papers. The exogenous "non-fundamental" demand shocks of irrational noise traders, which can drive stock prices away from their fundamental values, as in De Long et al. (1990), Gromb and Vayanos (2010) and others, also have the potential to produce price differences, despite that these paper do not explicitly address this issue. To our knowledge, our paper is the first one to propose a micro-founded asset pricing model with rational agents that can rationalize and quantitatively explain the premium in twin stocks and, in particular, the AH premium.

[22] They are Royal Dutch Petroleum and Shell Transport and Trading, PLC; Unilever N.V. and Unilever PLC; and SmithKline Beecham.

[23] They think that one partial answer to the premium of Royal Dutch over Shell is that Royal Dutch was a member of the S&P 500 index, but Shell was not.

As is well known, external habit (Campbell and Cochrane 1999) and long-run risk (Bansal and Yaron 2004) can explain the stock market regularities under RE, but some difficulties of these theories have been pointed out, for example, they, in any case, can not explain survey behavior. An extensive literature deviates from RE by assuming agents imperfectly know the fundamentals, but this Bayesian RE literature assumes that investors, like investors in RE literature, know the equilibrium pricing function, the mapping from fundamentals to prices. It includes models of Bayesian learning (e.g., Timmermann 1996, Collin-Dufresne, Johannes and Lochstoer 2016), agreeing to disagree (e.g., Scheinkman and Xiong 2003, Ehling, Graniero and Heyerdahl-Larsen 2018), robustness (e.g., Cogley and Sargent 2008), behavioral economics (e.g., Barberis, Shleifer and Vishny 1998 and Barberis, Greenwood, Jin and Shleifer 2015). However, these can not explain the AH premium, since equilibrium prices of stocks paying the same dividends are equal, even in the presence of heterogeneous agents.

Recent papers propose various strategies to test these asset pricing theories. Greenwood and Shleifer (2014), Adam, Marcet and Beutel (2017) and Kuang, Zhang and Zhang (2019) use survey expectations data to test the expectation formations in many asset pricing models. Bansal, Gallant, and Tauchen (2007), Beeler and Campbell (2012), Bansal, Kiku, and Yaron (2012), and Barro and Jin (2016) apply moment matching methods to compare the empirical performance of the habit, the long run risks, and the rare disasters models. Aldrich and Gallant (2011) utilize a Bayesian framework to compare habit, long run risks, and prospect theory. Our paper instead uses the AH premium in connected markets as a natural experiment to test various asset pricing models in terms of expectation formation directly.

To rationalize the AH premium in connected markets (also the price differences of twin stocks in general), relaxing agents' knowledge of the equilibrium pricing function may be necessary, as is done in adaptive learning models. Examples of this type of models include Lansing (2010), Branch and Evans (2011), Boswijk, Hommes and Manza (2007), Carceles-Poveda and Giannitsarou (2008), all of which build asset pricing models where agents learn adaptively and do not have the perfect knowledge about the true stochastic process for payoff relevant variables beyond their control. Adam and Marcet (2011) develop IR to provide microfoundations for models of adaptive learning. Adam, Kuang and Marcet (2012) show an application to housing prices, Adam, Marcet and Nicolini (2016) explain stock price volatility, Winkler (2019) matches both asset price and business cycle moments.

The experimental literature also provides evidence supporting the presence of subjective price beliefs. Hirota and Sunder (2007) and Asparouhova et al. (2016) design a Lucas asset pricing laboratory experiment, they find that stock prices display excess volatility unaccounted for by fundamentals and it is most likely attributed to participants' subjective price expectations. Crockett et al. (2018) suggest that speculative behavior in the lab causes the emergence of price bubbles.

Finally, our paper is related to the literature about the limits of arbitrage. Shleifer and Vishny (1997) think the limits of arbitrage are the reason why arbitrage fails to eliminate anomalies in financial markets. Xiong (2001) offers a formal model on the convergence trading strategy, and he finds that convergence traders would amplify the unfavorable shock by liquidating their positions when the shock of noise traders causes them to suffer substantial capital losses. In De Jong, Rosenthal and Van Dijk (2009), the uncertainty faced by convergence traders arises from the absence of an identifiable date at which dual-listed stock prices will converge. Our results are in line with them, and show that learning about prices caused by internally rational agents rather than noise traders also hinders arbitrage.

1.3. AH Premium with Knowledge of Pricing Function

There has not been a theoretical study of the features that can generate a premium in twin stocks. In this section, we discuss various models to elicit whether available theories where agents know the equilibrium pricing function (that is, RE or Bayesian RE) can explain the observations. From now on, we take it as given in the paper that post-Nov 2014, the AH-market is not segmented; traders in both markets can purchase stocks in the other market. We focus on the economy without the rational bubbles throughout this section.[24]

[24]We discuss the rational bubbles in section 7.

We argue that it is challenging for the standard RE and Bayesian RE paradigm to explain the premium in connected markets. Essentially, the knowledge of equilibrium pricing function places a strict restriction on agents' expectations about prices. A- and H-share prices are the same in the equilibrium as agents expect them to be.

It is clear that the homogeneous agent model with RE or Bayesian RE can't explain the premium in twin stocks as the agent is marginal for both shares in each period and the twin shares enjoy the same payoff in each state. However, it might be less clear how heterogeneous agents in the standard RE and Bayesian RE paradigm are related to the premium in connected markets. Some authors have argued that heterogeneities across agents can explain price differences of twin shares across segmented markets, in particular the AB premium in Chinese market. Heterogeneity can be due to features such as different stochastic discount factors (Fernald and Rogers 2002), and diverse beliefs about fundamentals (Chan, Menkveld and Yang 2008; Mei, Scheinkman and Xiong 2009; Jia, Wang and Xiong 2017) between mainland investors and foreign investors. In addition, informal explanations of the AH premium in connected markets are still often based only on heterogeneity.[25] We find it useful to explain that heterogeneity with RE or

[25]For example, see articles in several financial newspapers (in Chinese): https://finance.sina.cn/stock/relnews/hk/2019-10-17/detail-iicezuev2751039.d.html?from=wap, http://www.xinhuanet.com/finance/2017-02/22/c_129489122.htm.

Bayesian RE does not generate a premium in twin shares with the same fundamentals; it can only explain different consumption allocations and bond positions, but cannot explain price differences.

We illustrate it with several models for which there is a well-developed theory and where we can use well-known results. The common setup for all models to show in section 3.1 is that the economy is populated with two types of infinitely-lived agents, with type 1 agents standing for mainland investors and type 2 agents for Hong Kong and international investors. Within each group, the agents are homogeneous, with the same preference satisfying Inada conditions. As is common in the RE and Bayesian RE literature, agents understand the equilibrium pricing function mapping from fundamentals into prices, but agents understand fundamentals imperfectly. Moreover, it is common knowledge that A- and H-shares pay the same dividends. All agents can participate in the market of each share but are confronted with short-selling constraints for stocks. Finally, Ponzi schemes and rational bubbles are ruled out.

Literature has suggested that transaction costs levied on trading values can but only explain a very small part of the premium in twin stocks. This holds for the AH premium case. We find that dividend taxes can bring about some price differences as well. We consider dividends taxes in section 3.2 by embedding them to the models discussed in section 3.1 and argue that they can not explain the observed AH premium quantitatively.

1.3.1. Models with Pricing Knowledge

This section presents results that follow directly from the literature on heterogeneous agents and "agreeing-to-disagree". We first show analytically how heterogeneity in RE and Bayesian RE fails to produce any price difference in the complete markets with two types of agents. The two types of agents differ in discount factors and utilities in the first model and in beliefs about fundamentals in the second model. These two simple models carry the basic logic and have analytical expressions.

In the first case, type i agents of μ^i fraction have utility u^i and subjective discount factor δ^i, for $i \in \{1, 2\}$. Preference u^i satisfy the Inada condition.[26] Investors consume goods C_t^i, receive a deterministic endowment Y_t and choose portfolios.[27] Investors' portfolios consist of A-shares $S_t^{i,A}$, H-shares $S_t^{i,H}$, and Arrow securities (state-contingent bonds) $B_t^i(D)$. The two classes of equities share the same dividend payments *i.e.* $D_t^A = D_t^H = D_t$. The exogenous process of dividend D_t is assumed to be $i.i.d$ with probability $prob(D)$ taking values of $D \in D$ over the set of states D. The state-contingent bonds $B_t(D)$ purchased in period t delivers 1 unit of consumption if the realized dividend in period $t + 1$ is D.

[26]Preference can be CRRA utility, habit utility or Epstein-Zin utility.

[27]Y_t is assuemd to be deterministic without loss of generality, it is straightforward to extend it to the case with stochastic Y_t.

The aggregate supply of each share is normalized to be 1 such that market clearing conditions of equities read as

$$\mu^1 S_t^{1,j} + \mu^2 S_t^{2,j} = 1, \quad j \in \{A, H\}$$

The market clearing condition of A-share separates from that of H-share, which indicates that the twin shares are distinct assets. It is forbidden to sell A-share in the market of H-share and vice versa.

The state-contingent bond market clears with

$$\mu^1 B_t^1(D) + \mu^2 B_t^2(D) = 0 \ \forall D \in \mathcal{D}.$$

Denote C_t the aggregate consumption supply. The feasibility constraint $C_t = Y_t + D_t = \mu^1 C_t^1 + \mu^1 C_t^2$ is satisfied by Walras law.

Type i agents maximizes expected lifetime utility

$$\max \sum_{t=0}^{\infty} E_0 (\delta^i)^t u^i(C_t^i)$$

subject to the budget constraint

$$\begin{aligned} & S_t^{i,A} P_t^A + S_t^{i,H} P_t^H + C_t^i + \sum_{D \in \mathcal{D}} B_t^i(D) Q_t(D) \\ = \quad & S_{t-1}^{i,A}(P_t^A + D_t) + S_{t-1}^{i,H}(P_t^H + D_t) + B_{t-1}^i(D_t) + Y_t, \ \forall t \end{aligned}$$

and bounds on shareholding

$$\begin{aligned} \underline{S} &\leq S_t^{A,i} \leq \overline{S} \\ \underline{S} &\leq S_t^{H,i} \leq \overline{S}. \end{aligned} \tag{1.1}$$

Type i agents in period t consume C_t^i amount of goods, purchase A-shares $S_t^{i,A}$ at price P_t^A, H-shares $S_t^{i,H}$ at price P_t^H, Arrow securities $B_t^i(D)$ at price $Q_t(D)$, and receive payments from the outstanding shares $S_{t-1}^{i,A}(P_t^A + D_t)$ and state-contingent bonds $B_{t-1}^i(D_t)$ as well as the endowment Y_t. The upper and lower bounds serve to avoid the Ponzi scheme and will be common to the remainder of the paper.[28]

The standard optimal full-insurance conditions in equilibrium read as

$$\delta^1 \frac{u_c^1(C_{t+1}^1)}{u_c^1(C_t^1)} = \delta^2 \frac{u_c^2(C_{t+1}^2)}{u_c^2(C_t^2)} \ \forall D \in \mathcal{D} \tag{1.2}$$

where u_c^i is the marginal utility with respect to C_t^i. The stochastic discount factors (SDF) of the two types of agents are identical in each state. This condition implies that there exists a unique market SDF. Since agents are assumed to know the pricing mapping from the fundamentals to prices — the present-value form of equity prices, using type i agent's SDF, we have the following analytical expression

$$P_t^A = P_t^H = E_t[\sum_{j=1}^{\infty} (\delta^i)^j \frac{u_c^i(C_{t+j}^i)}{u_c^i(C_t^i)} D_{t+j}]$$

[28]As is usually assumed in the literature, the lower bound is more stringent than the upper bound.

This equation clearly delivers zero AH premium, even with heterogeneous agents.

Besides the heterogeneities in preferences, investors could differ in their beliefs about fundamentals. Some authors have documented that foreign investors tend to be pessimistic about the Chinese economy. In contrast, mainland Chinese citizens have more optimistic views (e.g., Jia, Wang and Xiong 2017) or have an informational advantage about fundamentals than foreigners (e.g., Chan, Menkveld and Yang 2008). Some stockbrokers and market analysts tend to propagate this kind of story as a way to rationalize the AH premium.

To analyze the potential for this explanation to explain the premium, we turn to the second case of Bayesian RE in which agents do not know the objective probability $prob(D)$ of dividend. Specifically, we assume type i agents have subjective beliefs about dividend $Prob^i(D)$. These heterogeneous beliefs are common knowledge for all agents, but they agree to disagree. To isolate the effects of diverse beliefs, we abstract away heterogeneous preferences.

In this case, the first-order conditions with respect to the contingent bonds now read as

$$\delta \frac{u_c(C^1_{t+1}(D))}{u_c(C^1_t)} Prob^1(D) = \delta \frac{u_c(C^2_{t+1}(D))}{u_c(C^2_t)} Prob^2(D) \ \forall D \in \mathcal{D} \tag{1.3}$$

Although the agents' SDF are not equal because of the heterogeneous beliefs on fundamentals, the products of agents' SDF and their subjective probability are identical in each state. Equation (1.3) immediately implies that the prices evaluated with agent 1's SDF and subjective probability equal to those evaluated with agent 2's SDF and subjective probability *i.e.*

$$\begin{aligned} P_t^A &= P_t^H = E_t^1[\sum_{j=1}^{\infty}(\delta)^j \frac{u_c(C_{t+j}^1)}{u_c(C_t^1)} D_{t+j}] \\ &= E_t^2[\sum_{j=1}^{\infty}(\delta)^j \frac{u_c(C_{t+j}^2)}{u_c(C_t^2)} D_{t+j}]. \end{aligned}$$

Therefore the conclusion of no premium still applies. This equilibrium pricing function dictates that stock prices are determined by fundamentals in the equilibrium. Prices of twins shares are the same as a result of the same fundamentals.

Even though heterogeneity cannot produce AH premium, it affects the allocation of consumption and bond positions. For instance, a higher risk aversion induces smoother consumption; agents with a larger subjective discount factor and more precise subjective beliefs about fundamentals will accumulate assets and achieve dominance in the economy gradually. A illustration of this by simulation is deferred to Appendix 9.1.

These models with the Arrow securities served to provide the basic logic for why the heterogeneities across investors are not relevant for AH premium in connected markets. However, despite the latest ongoing innovations in financial and insurance markets, markets in the real world are still arguably incomplete. We want to show that the conclusion of no premium is carried over to the incomplete market environment when agents have the pricing knowledge. The following example illustrates this.

Let's turn to a variant of the model discussed above but without access to the Arrow securities. For the incomplete market, consider a typical process for the endowment Y_t and dividend payment D_t as in Campbell and Cochrane (1999). This setup will also be valid in the learning model in section 4. Specifically, we directly impose restrictions on the dividend and the aggregate consumption supply processes

$$\frac{D_t}{D_{t-1}} = a\epsilon_t^d, \log \epsilon_t^d \sim iiN(-\frac{s_d^2}{2}, s_d^2) \tag{1.4}$$

$$\frac{C_t}{C_{t-1}} = a\epsilon_t^c, \log \epsilon_t^c \sim iiN(-\frac{s_c^2}{2}, s_c^2) \tag{1.5}$$

The dividend and consumption growth rates share the same mean a, and ($\log \epsilon_t^d$, $\log \epsilon_t^c$) are jointly normally distributed with a correlation of $\rho_{c,d}$, and the standard deviations of s_d and s_c. The endowment Y_t in the economy is given by the feasibility constraint $C_t = Y_t + D_t$.

Agents might have imperfect knowledge about (1.4) and (1.5). They maximize their expected lifetime utility subject to the budget constraint with no access to the Arrow securities

$$S_t^{i,A}P_t^A + S_t^{i,H}P_t^H + C_t^i = S_{t-1}^{i,A}(P_t^A + D_t) + S_{t-1}^{i,H}(P_t^H + D_t) + Y_t, \ \forall t \tag{1.6}$$

as well as the shareholding constraints (1.1).

In this case, the twin shares are perfect substitutes for any agent as she realizes that the twin shares delivering the same payments in each state play an identical role in the economy and expects the twin shares to have the same prices through the equilibrium pricing function.

In general, we provide the following argument for a lack of premium in the incomplete market models when agents know the equilibrium pricing function. Although the marginal persons can switch across the two types of agents over time, the present value formula of prices still holds as agents know the pricing function.[29] Therefore there exists one equilibrium where prices of twin shares equal the present value of dividends, and prices are equal.[30]

The intuition is that the prices are the same in equilibrium as each agent expects them to be via the knowledge of equilibrium pricing function. The knowledge of equilibrium pricing function places a strict restriction on agents' expectations

[29]With the common knowledge about each agent's preference and belief, agents know who is the marginal person in the current period and have the right expectations about who are going to be the marginal persons in the future.

[30]It is well known that there exists an SDF that price all payoffs in the incomplete market even. Refer to Ross (1978) and Luttmer (1996).

about prices. Agents' price expectations are governed by their expectations about fundamentals through the equilibrium pricing function. There is no room for investors to entertain different expected prices across the twin shares to yield price differences. When agents know the equilibrium pricing function, the heterogeneity among agents is not relevant.

As is discussed below, the dividend taxes resulting in some differences in the fundamentals across the two classes of shares is the key to produce price differences in the paradigm of RE and Bayesian RE. However, they are not large enough to explain the premium quantitatively.

1.3.2. Dividend Taxes

The real word is not free of dividend taxes, transaction costs, and the exchange rate risk.[31] We highlight the dividend taxes here and defer the discussion of transaction costs and exchange rate risk to section 7 as they appear not to be as remarkable as dividend taxes.

According to the stipulated financial regulations, mainland investors pay a constant 20% dividend taxes for holding H-shares while Hong Kong and international investors pay a 10% dividend tax for both shares. Dividend taxes for mainland investors to hold A-shares depend on the holding period. The dividend tax is 20%,

[31]According to the regulations, there are no capital gain taxes in both SSE and HKSE.

10%, and 5% respectively for investors holding the shares for less than one month, between one month and one year, and more than one year.[32]

Let $\tau^{i,A}$ $\tau^{i,H}$ represent the dividend taxes confronted by type i agents in A- and H-share markets respectively. For type 1 agents, the after-tax dividend payments per share are $(1-\tau^{1,A})D_t$ for A-share and $(1-\tau^{1,H})D_t$ for H-share respectively, while they are $(1-\tau^{2,A})D_t$ and $(1-\tau^{2,H})D_t$ for type 2 agents where $\tau^{1,H}=20\%$, $\tau^{2,A}=\tau^{2,H}=10\%$. In particular, we deliberately chose the 5% dividend tax for type 1 agents to hold A-shares *i.e.* $\tau^{1,A}=5\%$ to give the model below the best chance as it will becomes later that this induces the largest wedge between dividend taxes confronting the marginal persons across the two markets. Given these dividend taxes, type 1 agents have more after-tax dividend from A-shares, while type 2 agents obtain more after-tax dividend from H-shares *i.e.* $(1-\tau^{1,A})D_t > (1-\tau^{2,A})D_t$ and $(1-\tau^{2,H})D_t > (1-\tau^{1,H})D_t$.

We explore the effects of dividend taxes on the price premium by incorporating them into the models discussed in section 3.1. Although different dividend taxes across the two markets can give rise to some price differences, we argue that they can not explain some basic features of the observed AH premium.

To begin with, we embed dividend taxes to the first model in section 3.1. One can easily prove that, with the scheme of dividend taxes, an equilibrium is as follows: agent 1(2) buys all A(H)-shares in the first period, and she is against the short-selling constraint on the other shares (corner solutions), each agent keeps

[32]Refer to Appendix 9.5 for the data sources of transaction costs and dividend taxes.

this holding position forever. This is the equilibrium because of the short-selling constraint and the same SDF across the two agents (due to complete markets). The two agents' Euler equations for A-shares (H-shares) can't hold with equality at the same time. Due to the tax advantage of agent 1(2) for A(H)-shares, it is the agent 1(2) that is marginal for A-shares (H-shares). Therefore, we have

$$P_t^A = E_t \sum_{j=1}^{\infty} (\delta^1)^j \frac{u_c^1(C_{t+j}^1)}{u_c^1(C_t^1)} (1 - \tau^{1,A}) D_{t+j} \tag{1.7}$$

$$P_t^H = E_t \sum_{j=1}^{\infty} (\delta^2)^j \frac{u_c^1(C_{t+j}^2)}{u_c^1(C_t^2)} (1 - \tau^{2,H}) D_{t+j}. \tag{1.8}$$

Given that the $SDFs$ of the two types of agents are identical and the constant $(1 - \tau^{1,A})$ and $(1 - \tau^{2,H})$ can be factored out, (1.7) and (1.8) lead to

$$\frac{P_t^A}{P_t^H} = \frac{1 - \tau^{1,A}}{1 - \tau^{2,H}}. \tag{1.9}$$

Therefore price differences are driven by different dividend taxes confronting the marginal agents across the two markets. This price ratio is constant over time at 105.6% according to equation (1.9), which is inconsistent with the size and fluctuation of the observed AH premium.[33] This equation still holds if agents have heterogeneous beliefs on dividends because of the equation (1.3).

[33]One can show with the similar arguments that there would be no premium if $\tau^{1,A} = 10\%$ or 20%.

The argument so far in this section relied on the existence of state-contingent bonds, consider now the case without access to state-contingent bonds. In this case type 1(2) will hold H(A)-shares with positive probability, in other words the agents may invest in both markets in order to accumulate precautionary savings. The exact dynamics of this incomplete market model will depend on exactly the process for income, the type of heterogeneity, the assets available etc. However, in many setups, due to the dividend tax advantage, it will still be the case that with probability one agent 1(2) holds A(H)-shares in all periods in the stationary equilibrium, that is, each agent is marginal in "her" market with probability one, although she may also be marginal in the "other" market in some periods. Under these circumstances equations (1.7) or (1.8) still hold, since the Euler equation for the "own" market always holds.

Now, as is well known the $SDFs$ are not equal across agents under incomplete markets in all periods, so we can not derive (1.9) exactly as we did before. However, many papers in the incomplete market literature report that equilibrium prices are set so as to make $SDFs$ across agents almost equal in incomplete market equilibria, so that we have $\frac{u_c(C^1_{t+1})}{u_c(C^1_t)} \approx \frac{u_c(C^2_{t+1})}{u_c(C^2_t)}$.[34] Combining these two observations we have

[34]An early paper reporting a high correlation of SDFs in a two agent model with two assets and heterogeneous incomes is Marcet and Singleton (1999) (see their Table 2 reporting a high correlation of individual consumption and aggregate income for calibrations of Models 1-3 with high serial correlation of income). Levine and Zame (2001) argue that for high discount factors incomplete market equilibria nearly completes the markets. In large OLG models young agents are against a borrowing constraint for a long time so in that model their SDF is not highly correlated with that of old agents, but large differences in age do not seem to be an issue in the AH-market. Some recent models argue that different liquidity can also drive a wedge between

that (1.9) holds approximately around the steady state with corner solutions. However, as we argued above, (1.9) is so far from explaining the AH premium.

Along with the previous argument, we also simulate a calibrated transitional economy where both shares are split equally among the two agents in the initial period as we assume two types of agents have the same fractions *i.e.* $\mu^1 = \mu^2 = 0.5$, and agents trade shares with each other actively afterwards. There is one main difference from the previous incomplete market model in section 3.1 in addition to the introduction of dividend taxes. That is, we assume each agent to have the same logarithm utility to isolate the effects of dividend taxes. We also take away endowment Y_t for simplicity. The calibration for dividend process is the same as that in the following section 5.

We find that over the transition, both agents are marginal for A- and H-shares, and assets are traded in equilibrium such that the twin shares have the same prices.[35] The dynamics of share-holding, as in Figure 1.4, displays that over the transition type 1 agents accumulate A-shares, while type 2 agents accumulate H-shares. This is because agent 1 has the tax advantage for A-shares, and agent 2 has the tax advantage for H-shares. However, we find no price differences. Therefore, we conclude that the observed premium is incompatible with incomplete markets as the price differences are close to (1.9) or even less when agents trade with each other actively.

agents' SDFs. We do not address issues of liquidity in this paper except to point out that liquidity is very high in both A- and H- markets, see our comment in section 7.3.

[35]The simulation method follows Marcet and Singleton (1999).

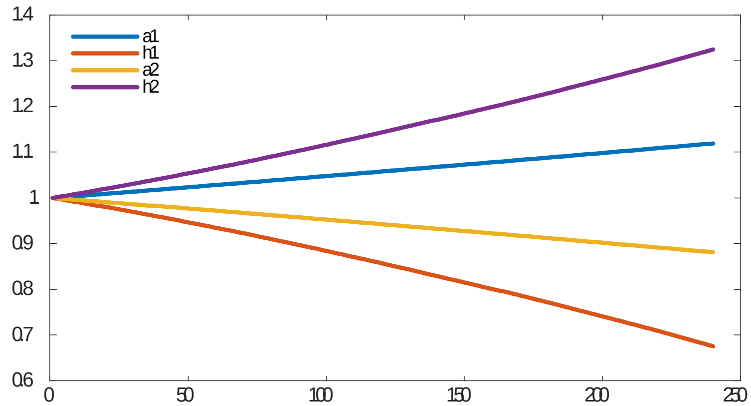

Figure 1.4. Dynamics of Share-holding

Note: a1 is type 1 agents' holding of A-shares, a2 is type 2's holding of A-shares, h1 is type 1's holding of H-shares and h2 is type 2's holding of H-shares.

1.4. An Internal Rationality Learning Model

Our conclusion from section 3 is that staying within the standard paradigm of agents who know the equilibrium pricing function has difficulty explaining quantitatively the observed behavior of the AH premium. This section shows that a parsimonious "Internal Rationality" learning about prices model, in which rational agents do not know the equilibrium pricing function, can explain the AH premium quantitatively. As in Adam and Marcet (2011), when the preferences and beliefs of agents are not common knowledge, they cannot deduce the equilibrium pricing function from their own optimization conditions;[36] their subjective expectations

[36]In Adam, Marcet and Niconoli (2016) and Adam, Marcet and Beutel (2017), despite agents are homogeneous, homogeneity is not the common knowledge. Each agent probably believes that others have very different beliefs from them.

about future prices are not restricted by their expectations about fundamentals; their expectations about prices becomes crucial in pricing an equity.

Agents have subjective beliefs about prices. Investors' subjective beliefs about stock prices are described by a model that is a good description of actual prices, and this perceived model is hard to reject given actual stock prices. Under this framework agents realize that an A-share is actually a different security from an H-share and that it can be purchased or sold next period at a possibly different price. Agents learn about prices and, in that way, a difference in expected prices across the twin shares feeds on itself to generate an AH premium. Even if the A-, H-dividends are (nearly) the same, A- and H- prices will actually differ in equilibrium just because investors expect them to differ. This behavior is compatible with rationality.

1.4.1. Model Environment

The environment is almost the same to the incomplete market case of section 3.1. The consumption process and dividend process (1.4) and (1.5) still hold. Agents face the budget constraint (1.6) and shareholding constraints (1.1).

Here we abstract from the agents heterogeneity to isolate the role of learning about prices in generating high, volatile, and persistent AH premium. [37] The

[37] Heterogeneity may be an interesting way to complement the main story of this section. But we think learning about prices is the main story.

economy is populated by a unit mass of infinite-horizon agents. They are homogeneous agents. This, however, is not common knowledge among agents, which differs from the previous models.

Agents' preference is specified with the standard CRRA utility with a relative risk aversion of γ for quantitative exercise

$$E_0^{\mathcal{P}} \sum_{t=0}^{\infty} \delta^t \frac{(C_t^i)^{1-\gamma}}{1-\gamma}.$$

Importantly, instead of the objective probability measure, expectations are formed using the subjective probability measure $\mathcal{P}$, which describes the probability distributions for all "external" variables. Section 4.2 contains more details about the probability space.

1.4.2. Probability Space

This section explicitly describes the general joint-probability space of the variables that agents take as given ("external" variables). Stock prices $\{P_t^A, P_t^H\}$ under competitive stock market assumption and exogenous endowment and dividend processes $\{Y_t, D_t\}$ are external to agents' decision problem. In section 3, agents know the equilibrium pricing function as in RE and Bayesian RE models, therefore stock prices only carry redundant information. However, when agents do not know the equilibrium pricing function as a result of no common knowledge about agents' preferences and beliefs, equilibrium stock prices should be included in the underlying state space of agents' decision problem. Formally, we define the probability space as $(\mathcal{P}, \mathcal{B}, \Omega)$ with $\mathcal{B}$ denoting the corresponding σ-Algebra of the Borel subsets of Ω and $\mathcal{P}$ denoting the agents' subjective probability measure over $(\mathcal{B}, \Omega)$. The state space Ω for the realized exogenous variables is

$$\Omega = \Omega_Y \times \Omega_D \times \Omega_{P^A} \times \Omega_{P^H}$$

where Ω_X represents the state space for all possible infinite sequences of the variable $X \in \{Y, D, P^A, P^H\}$. Thereby, a specific element in the set Ω represents an infinite sequence $\omega = \{Y_t, D_t, P_t^A, P_t^H\}_{t=0}^{\infty}$. Agent i chooses plans for the endogenous variables C_t^i, $S_t^{A,i}$, $S_t^{H,i}$ contingent on future realizations of Ω^t, where Ω^t represents the set of histories from period zero up to period t, that is

$$(C_t^i, S_t^{A,i}, S_t^{H,i}) : \Omega^t \rightarrow R^3.$$

The expected utility with the probability measure $\mathcal{P}$ is defined as

$$E_0^{\mathcal{P}} \sum_{t=0}^{\infty} \delta^t \frac{(C_t^i)^{1-\gamma}}{1-\gamma} \equiv \int_{\Omega} \sum_{t=0}^{\infty} \delta^t \frac{C_t^i(\omega^t)^{1-\gamma}}{1-\gamma} d\mathcal{P}(\omega^t).$$

We assume agents know the processes of dividend and consumption supply (1.4) (1.5) to differentiate from Bayesian RE. Section 4.4 will specify the subjective probability distribution for prices.

1.4.3. Optimality Conditions

Since the objective function is concave and the feasible set is convex, the agent's optimal plan is characterized by the first order conditions

$$(C_t^i)^{-\gamma} P_t^j = \delta E_t^{\mathcal{P}}((C_{t+1}^i)^{-\gamma}(P_{t+1}^j + D_{t+1})) \quad j \in \{A, H\} \tag{1.10}$$

Using standard arguments, if agents have rational expectation instead, the RE solution is

$$P_t^{j.RE} = \frac{\delta a^{1-\gamma}\rho_\epsilon}{1-\delta a^{1-\gamma}\rho_\epsilon} D_t \quad j \in \{A, H\} \tag{1.11}$$

where $\rho_\epsilon = E[(\epsilon_{t+1}^c)^{1-\gamma}\epsilon_{t+1}^d] = e^{\gamma(1+\gamma)\frac{s_c^2}{2}} e^{-\gamma\rho_{c,d}s_c s_d}$. Prices are governed by the dividends and are equalized.

We now characterize the equilibrium outcomes under learning. Following AMN, we make the technical assumption that the endowment is large enough (see Appendix 9.3). Define subjective expectations of the (risk-adjusted) capital gains β_t^j as

$$\beta_t^j \equiv E_t^{\mathcal{P}}[(\frac{C_{t+1}}{C_t})^{-\gamma}(\frac{P_{t+1}^j}{P_t^j})] \quad j \in \{A, H\}. \tag{1.12}$$

(1.12) and (1.10) give rise to the asset pricing equations in equilibrium

$$P_t^j = \frac{\delta a^{1-\gamma}\rho_\epsilon}{1-\delta\beta_t^j} D_t \quad j \in \{A, H\}. \tag{1.13}$$

We see that price P_t^j is larger when agents expect a higher capital gains β_t^j, and learning model can generate price differences when $\beta_t^A \neq \beta_t^H$, even if two stocks share the same dividends D_t. As $\delta\beta_t^j$ in the denominator is close to 1, a small difference in expected capital gains across the twin shares can be amplified into a big difference in prices. Despite knowing that payoffs of the two shares are identical, agents could have different expected (risk-adjusted) capital gains β_t^j across shares as they don't know the equilibrium pricing function. The expected (risk-adjusted) returns $E_t^{\mathcal{P}}[(\frac{C_{t+1}}{C_t})^{-\gamma}\frac{P_{t+1}^j+D_{t+1}}{P_t^j}]$ need to be equalized across the twin shares as in equation (1.10). Given different β_t^j, current prices of A- and H-shares need to differ in a way such that the expected (risk-adjusted) dividend yields $E_t^{\mathcal{P}}[(\frac{C_{t+1}}{C_t})^{-\gamma}\frac{D_{t+1}}{P_t^j}]$ achieve this equalization.

Generating different prices hinges on $\beta_t^A \neq \beta_t^H$. This appears to be natural in light of the fact that actual capital gains have in fact differed across these twin shares even before the connect program. We now turn to show a system of beliefs about prices that justifies this formally.

1.4.4. Belief-Updating Rule

This section fully specifies the subjective probability distribution for prices in $\mathcal{P}$, and the optimal belief-updating rule for the subjective beliefs β_t^j. We start with the popular assumption in the adaptive learning/IR literature, namely that agents believe risk-adjusted capital gains are the sum of a persistent and a transitory component motivated by the existence of periods in which price dividend ratio increases or falls persistently

$$(1.14) \qquad \begin{aligned} \left(\frac{C_t}{C_{t-1}}\right)^{-\gamma} \frac{P_t^j}{P_{t-1}^j} &= b_t^j + \epsilon_t^j, \ \epsilon_t^j \sim iiN(0, \sigma^2_{\epsilon,j}) \quad j \in \{A, H\} \\ b_t^j &= b_{t-1}^j + \xi_t^j, \ \xi_t^j \sim iiN(0, \sigma^2_{\xi,j}) \end{aligned}$$

where b_t^j are persistent components and ϵ_t^j are transitory components. This section assumes all innovations are independent from each other, in section 7.1 we consider a more general case where agents believe that the prices of two shares are correlated.

Along with other arguments in the following section 4.6, one way to justify this belief system is that it is compatible with RE: the setup (1.14) encompasses the RE beliefs (1.11) as a special case. Specifically, when agents believe that $\sigma^2_{\xi^j} = 0$, and $b_0^j = a^{1-\gamma}\rho_\epsilon$, prices are as given by RE equilibrium prices in all periods. Therefore, this gives a sense in which the beliefs of agents are "close" to RE, as long as we consider small values for $\sigma^2_{\xi^j}$ and β_0^j are close to $a^{1-\gamma}\rho_\epsilon$ (as we will).

In the following we allow for nonzero variances $\sigma^2_{\xi,j}$, that is, for the presence of persistent time-varying components b^j_t. Agents observe the realizations of the risk-adjusted capital gains, but not the persistent and transitory components separately. By construction, agents' expected capital gains β^j_t are their estimation of b^j_t using the information up to period t. Forecasting the future capital gains by estimating the persistent components b^j_t engenders a filtering problem. For normally distributed initial priors $b^j_0 \sim N(\beta^j_0, \sigma^2_{0,j})$, the optimal steady state Kalman filter gives rise to a belief-updating rule as

$$\beta^j_t = \beta^j_{t-1} + \alpha^j \left[(\frac{C_{t-1}}{C_{t-2}})^{-\gamma} \frac{P^j_{t-1}}{P^j_{t-2}} - \beta^j_{t-1} \right] \qquad j \in \{A, H\} \tag{1.15}$$

where α^j are optimal (Kalman) gains which measure the speed at which the agents update their beliefs with new information. It is well known that α^j increases in the signal-noise ratio $\frac{\sigma_{\xi^j}}{\sigma_{\epsilon^j}}$. We estimate α^j to be small as in section 4.5, which will be compatible with small $\sigma^2_{\xi^j}$ implying a small deviation from RE. Iterating (1.15) backward shows that β^j_t is a geometric weighted average of the history of actual capital gains.

The belief-updating scheme (1.15) and the pricing equation (1.13) generate rich price dynamics arising from a self-referential channel between the stock price P^j_t and the subjective beliefs β^j_t. According to equation (1.13), a high (low) β^j_t leads to a high (low) realized stock price P^j_t. High P^j_t reinforces the subjective beliefs, induce an even higher (lower) β^j_{t+1} through equation (1.15), thus lead to a higher (lower) P_{t+1}^j and so on. The difference between β_t^A and β_t^H can originate from a small difference in initial beliefs, or in the Kalman gains α^j, or in past realized actual capital gains, or in the addition of expectation shocks (see section 6.2 below). Because of the self-referential property of the model, a difference in beliefs is able to produce persistent price differences between A- and H-shares.[38]

1.4.5. Quantitative Performance

This section evaluates the quantitative performance of the model in which agents learn about prices. The model is simulated at the weekly frequency.[39] We set the relative risk-aversion $\gamma = 5$ following AMN, then calibrate the mean and standard deviation of the dividend growth a, $\sigma_{\Delta D/D}$, the standard deviation of the consumption growth $\sigma_{\Delta C/C}$, and the correlation between consumption growth and dividend growth $\rho_{c,d}$ using the financial data of SSE and Chinese macroeconomic data. The data sources are described in the Appendix 9.6. In particular, we follow the method as in Campbell (2003) to construct the quarterly dividend. Then, we use quarterly dividend and consumption data to calibrate weekly parameters following Campbell and Cochrane (1999).[40] We also calibrate δ to match the averaged

[38]As is standard in this literature agents' subjective beliefs β_t^j are truncated by a projection facility, this is detailed in Appendix 9.3.

[39]We use weekly frequency because first we have the weekly price; second, the data sample becomes much smaller if we aggregate it to monthly or quarterly frequency since the sample only starts at November, 2014. High frequency financial data is widely used in the literature, for example Lucca and Moench (2015) find that there is a pre-FOMC announcement excess return using daily and hourly data.

[40]Campbell and Cochrane (1999) use annual data to calibrate monthly parameters. We divide the average quarterly growth rate of dividend (consumption) by 13 to get the weekly mean and

4% annually deposit interest rate[41], that is, we get δ such that the corresponding weekly interest rate matches annually deposit interest rate according to the no-arbitrage condition. We set initial subjective expectations β_0^A, β_0^H and β_1^H at their RE values, and calibrate β_1^A slightly larger than β_1^H such that we have 3% premium initially as is observed in the data when connect program was initialized.[42]

We apply the method of simulated moments (MSM) to estimate α^A and α^H, matching the mean, standard deviation, and persistence of the AH premium. Table 1.1 contains the parameter values. The estimation results show that $\alpha^A > \alpha^H$, which implies agents perceive that the signal-noise ratio of A-share is larger than that of H-share *i.e.* $\frac{\sigma_{\xi^A}}{\sigma_{\epsilon^A}} > \frac{\sigma_{\xi^H}}{\sigma_{\epsilon^H}}$. [43] Intuitively, agents tend to adjust their beliefs about A-shares more quickly if they believe that the variation in the trend component contributes relatively more to the fluctuation of A-share prices than that of H-share prices.

Table 1.2 reports the quantitative results. Column 1 is the targeted moments. Column 2 shows the moments of the actual data from November 2014 to June 2019, while Column 3 reports the 95% intervals of the model's simulated moments. We find that the mean and standard deviation of the data are located within the

obtain the weekly standard deviation of dividend (consumption) growth by dividing its quarterly counterpart by $\sqrt{13}$.

[41]We don't model deposit explicitly in the model since it is not our concern.

[42]As discussed in section 6.2, higher beliefs for A-shares can come from a positive expectation shock. In the Appendix 9.4, we also use the actual stock prices to recover initial beliefs and find similar ones.

[43]Suppose that processes (1.14) are the data generating process (in fact, as is in section 4.6, (1.14) are compatible with actual data), the maximum likelihood estimation (MLE) of (1.14) using the actual data also supports this inequality.

Parameters	Value
γ	5
$\sigma_{\Delta D/D}$	0.0197
$\sigma_{\Delta C/C}$	0.0025
a	1.0015
$\rho_{c,d}$	-0.02
δ	0.999
α^A	0.0033
α^H	0.0014

Table 1.1. Parameters Values for Learning Model

intervals, although the model generates a slightly more persistent AH premium than the data.[44] Therefore, the learning model is broadly consistent with the data in the sense that it can produce data-like behavior.

The fact that the A- and H-share prices usually move in the same direction but A-share prices often adjust more quickly than H-share prices is displayed graphically in Figure 3. The formal statistics capturing these characteristics are the correlation $corr(P_t^A, P_t^H)$, $corr(P_t^A - P_t^H, P_t^H)$ and standard deviation ratio $\sigma(P_t^A)/\sigma(P_t^H)$. Column 2 of Table 1.3 reports the empirical statistics. We find a high positive correlation $corr(P_t^A, P_t^H)$. And the positive $corr(P_t^A - P_t^H, P_t^H)$ indicates that A-share prices tend to rise (fall) more as H-share prices rise (fall), while $\sigma(P_t^A)/\sigma(P_t^H) > 1$ implies the A-share prices are more volatile. The learning model reported in column 3 of Table 1.3 quantitatively replicates these moments as data moments are within 95% intervals of the simulated moments even though they are not targeted when estimating the model.

[44]By introducing the preference shock to δ, the persistence could be reduced.

Targeted Moments	**Data**	**Model** (95% interval)
$E(\frac{P_t^A}{P_t^H} * 100)$	126.68	[93.54 128.74]
$\sigma(\frac{P_t^A}{P_t^H} * 100)$	6.98	[3.79 26.01]
$\rho(\frac{P_t^A}{P_t^H})$	0.88	[0.96 0.99]

Table 1.2. Simulated Targeted Moments

Non-targeted Moments	**Data**	**Model** (95% interval)
$corr(P_t^A, P_t^H)$	0.91	[0.67 0.99]
$corr(P_t^A - P_t^H, P_t^H)$	0.18	[-0.14 0.94]
$\sigma(P_t^A)/\sigma(P_t^H)$	1.20	[1.07 2.37]

Table 1.3. Simulated Non-targeted Moments

Observing that learning model generates similar targeted and non-targeted moments as in the data, while RE and Bayesian RE models discussed in section 3 have difficulty in producing the data-like AH premium, these quantitative results confirm the key role of learning about prices in equity pricing.

1.4.6. Testing for the Rationality of Price Expectations

The idea of IR is that even though agents do not know the equilibrium pricing function, they entertain beliefs that are not large deviations from rational expectations, in the sense that agents do not make large, easily detected mistakes. As we explained in section 4.4, by choosing small $\sigma^2_{\xi^j}$ and initial conditions β_0^j close to $a^{1-\gamma}\rho_\epsilon$ gives gives a precise sense of "closeness" of beliefs to RE beliefs. In addition, agents can test their belief system against the actual data and will reject their subjective model if it leads to big mistakes. In this section, we examine whether agents would be able to reject their beliefs if they do test their beliefs.

For this purpose, we use a set of testable restrictions implied by the agents' belief system (1.4) (1.5) (1.15) developed by AMN to test the belief system against the actual data.[45] Denote $x_t \equiv (e_t, D_t/D_{t-1}, C_t/C_{t-1})$, where $e_t \equiv \Delta(\frac{C_t}{C_{t-1}})^{-\gamma}\frac{P_t}{P_{t-1}}$, with Δ representing the first difference operator. These restrictions are listed as follows:

Restriction 1: $E(x_{t-i}e_t) = 0$ for all $i \geqslant 2$,

Restriction 2: $E((\frac{D_t}{D_{t-1}} + \frac{D_{t-1}}{D_{t-2}}, \frac{C_t}{C_{t-1}} + \frac{C_{t-1}}{C_{t-2}})e_t) = 0$,

Restriction 3: $b'_{DC}\sum_{DC} b_{DC} + E(e_t e_{t-1}) < 0$,

Restriction 4: $E(e_t) = 0$,

where $\sum_{DC} \equiv var(\frac{D_t}{D_{t-1}}, \frac{C_t}{C_{t-1}})$ and $b_{DC} \equiv \sum_{DC}^{-1} E((\frac{D_t}{D_{t-1}}, \frac{C_t}{C_{t-1}})' e_t)$.

These four restrictions are necessary and sufficient conditions for the agents' beliefs to be compatible with $\{x_t\}$ in terms of second-order moments.[46] Under standard assumptions, any process satisfying these testable restrictions can - in terms of its autocovariance function - be generated by the postulated system of beliefs.

Table 1.4 reports the test statistics associated with Restrictions 1 to 4.[47] The second column is the values of the test statistics while the last column reports the

[45] AMN show that the postulated belief system is consistent with the simulated data.

[46] See AMN for the proof and details.

[47] Given the availability of consumption and dividend data, here we use monthly frequency data starting from January 2006.

	Test Statistics for A (H)	**5% Critical Value**
Restriction 1 using $\frac{D_t}{D_{t-i-1}}$	2.81 (0.76)	9.48
Restriction 1 using $\frac{C_t}{C_{t-i-1}}$	4.02 (4.77)	9.48
Restriction 1 using $\Delta(\frac{C_{t-i}}{C_{t-i-1}})^{-\gamma}\frac{P_{t-i}}{P_{t-i-1}}$	2.13 (2.55)	9.48
Restriction 2	0.04 (0.15)	5.99
Restriction 3	-3.55 (-3.60)	1.64
Restriction 4	0.002 (0.001)	3.84

Table 1.4. Testing Subjective Beliefs against Actual Data

5% critical values of the statistics. It shows that the test statistics in all cases are below their critical values and often by a wide margin. Agents would find the observed stock prices compatible with their belief system as they would not reject their subjective model if they test it against the actual data. We conclude that the belief system is reasonable given the behavior of data.

1.5. Convergence Trading

Lamont and Thaler (2003) think that the different prices of twin shares are a violation of law of one price. With regard to this issue, Lamont and Thaler (2003) put that "First, some agents have to believe falsely that there are real differences between two identical goods, and second, there have to be some impediments to prevent rational arbitrageurs from restoring the equality of prices that rationality predicts." In section 1.4, we answer the first question of how investors could believe A- and H-shares with identical payoffs are different. Now, we switch to what prevents arbitrageurs from bringing stock prices to their fundamental values in the context of agents learning about prices.

Arbitrageurs with superior information expect that the prices of two assets with the same or similar fundamentals will converge in the future. In fact, when agents learn prices, the prices in equilibrium will converge in the long run as well. According to the results in Chapter 7 of Evans and Honkapoha (2001), learning agents' price expectations $\beta_t^{A(H)}$ in section 4 should converge in distribution as

$$\lim_{t\to\infty} \beta_t^j \sim N(\beta^{RE}, \alpha^j Z^j), j = A, H$$

where α^j and Z^j are numbers given in their Theorem 7.9. Hence, the difference in price expectations *i.e.* $\beta_t^A - \beta_t^H$ converges in distribution with a mean of zero. Given AH premium is caused by different price expectations, the initialization of Shanghai-Hong Kong stock connect can promote price convergence in the long run. However, the learning agents don't know $\beta_t^{A(H)}$ converges in the long run and already take the optimal strategy compatible with their beliefs. Hence, they will not act as the arbitrageurs.

Arbitrageurs typically implement the convergence trading strategy that takes long positions in an asset believed to be undervalued and short positions in the other one believed to be overvalued. A well-known example is the convergence trading of the hedge fund Long-Term Capital Management (LTCM) betting on the convergence of prices of the Royal dutch and the Shell.[48] However, the convergence trading is not always profitable, especially in a "short-term world".[49] LTCM incurred a big loss from this strategy. As is mentioned in the introduction, news media reported that some arbitrageurs in the AH-market also lost heavily and quickly abandoned the convergence trading. This is due to the limits of arbitrage as follows.

Among others, we list several prominent limits. First, Shleifer and Vishny (1997) argue that arbitrage is conducted by a relatively small number of professional and highly specialized investors. Second, Edwards (1999) and Shleifer and Vishny (1997) among others have pointed out that the naked short-selling (long buying) not allowed in U.S. market. The Regulation T requires that arbitrageurs who were to short sell (long buy) an security must have the initial margin requirement of 50% of the market value of the security in their credit accounts. Effectively, this means that it requires short (long) trades to have 150% of the value of position at the time the position is created. The same situation holds in China.[50] Third, arbitrageurs also have to meet the maintenance ratio regulation over time,

[48]Edwards (1999) and Lamont and Thaler (2003) document that LTCM expected the convergence of bond yields in emerging market countries and the US. Hence, they bought bonds from emerging markets and short-sold US government bonds in 1997. At the same time, LTCM also bet that share prices of the dual-listed company Royal Dutch and Shell would converge because of the same fundamentals. However, the spread of bond yields and price differences of stocks widened unexpectedly, which made LTCM incur a large loss from this strategy and led to the near-collapse of LTCM.

[49]See the section with the title of Long-Term Capital in a Short-Term World in Lamont and Thaler (2003)

[50]The detailed regulations of short selling can be found in the official website of China Securities Regulatory Commission. (http://www.csrc.gov.cn/pub/xizang/xxfw/tzzsyd/201003/t20100324_178727.htm)

which says the ratio of the market value of asset (initial margin plus cash from short-selling and the asset value from long-buying) to the market value of debt in the credit accounts has to be at least 125%.[51] In the case of China, the minimum maintenance ratio is 130%. Arbitrageurs would receive margin calls and be forced to remargin (to provide additional collateral) if the ratio is below the threshold. Involuntary liquidation could occur due to a lack of enough capital to remargin. Fourth, Shleifer and Vishny (1997) propose that arbitrageurs are very likely to confront liquidation issue as the external funding to them is usually performance-based. Fifth, arbitrageurs have an additional restriction of the 1-year maximal duration of margin in mainland market, while there is no such regulation in U.S. markets. Sixth, it is very difficult (may be impossible) for arbitragers to design a strategy to short sell a stock index.

Moreover, we find learning about prices by internally rational agents also serves as a impediment to arbitrage since the price differences induced by learning may not converge and could even diverge in the short run, which might incur arbitrageurs' losses and a high odds of margin calls in the short run. To illustrate this, we study the risk of convergence trading by a simple simulation experiment as follows. We assume only a small number of arbitrageurs in line with Shleifer and Vishny (1997) who basically take the prices as given start to take arbitrage in period T. Given that the A-share prices are higher than H-share prices in period T, arbitrageurs

[51]Note that this maintenance ratio has to be at least 150% at the time the position is created as a result of the initial margin requirement.

wish to short sell A-shares and long buy H-shares. We assume arbitragers have enough cash in hand as the initial margin. To short sell one unit of A-share, they deposit cash of $50\% * P_T^A$ as the initial margin required, borrow one unit of A-share from brokers and then keep the cash of P_T^A from the short position in the credit account as collateral. Meanwhile, we assume arbitrageurs borrow P_T^A amount of money with the same amount of initial margin of $50\% * P_T^A$ to buy P_T^A/P_T^H units of H-share as the long position. We ignore the broker's charged fee, interest rate, other transaction costs, and dividend payments here as they are relatively small. In this case, the assets in arbitrageurs' account includes total initial margin of P_T^A, cash received from short position P_T^A, and market value of the long position $P_T^A/P_T^H * P_t^H$. Hence, the maintenance ratio mr at period $T+t$, the ratio of the market value of assets to that of debt, is

$$mr_{T+t} \equiv \frac{2P_T^A + P_T^A/P_T^H * P_t^H}{P_t^A + P_T^A} \tag{1.16}$$

We do not model arbitrageurs' decision making about remargining except to point out later that the odds of margin calls are high. In this case, the profit π_t by terminating the convergence trading in period $T+t$ is

$$\begin{aligned} \pi_{T+t} &\equiv (P_{T+t}^H * P_T^A/P_T^H - P_T^A) + (P_T^A - P_{T+t}^A) \\ &= P_{T+t}^H * P_T^A/P_T^H - P_{T+t}^A \end{aligned}$$

where $P^H_{T+t} * P^A_T / P^H_T - P^A_T$ is profit from long position, and $P^A_T - P^A_{T+t}$ is profit from short position.

Consider a scenario where the arbitrageurs observing the AH premium start to implement their margin trading in period $T = 100$ (the first period is the time when connect program is initialized). We Monte-Carlo simulate the model with each path covering 52 periods (1 year) from period T. The actual A- and H-share prices are substituted into the belief updating scheme (1.15) to recover the learning agents' capital gain expectations in period T as the initial beliefs for simulation.[52]

Our simulation shows the convergence trading probably experiences marginal calls as A- and H-share prices can diverge in the short-run caused by learning about prices. The probability of receiving a margin call in each period is obtained by calculating the fraction of paths with $mr_t < 130\%$ among all the simulated paths over each period. Although we assume arbitrageurs already have large amount of cash as initial margin requirement, the maximal probability of margin calls over all periods still reaches as high as 9.6%.

Furthermore, we explore whether arbitrageurs will make positive profits through convergence trading. Table 1.5 displays the means, standard deviations of π_{T+t} and probabilities of negative profits *i.e.* $\pi_{T+t} < 0$ for $t = 13, 26, 39$ and 52, corresponding to 3 months, 6 months, 9 months, and 1 year after the creation of positions. As apparent in Table 1.5, the means of π_t are negative, and the standard deviations

[52]We keep the consumption growth at its mean value because weekly consumption data doesn't exist.

	Mean	**Std**	$\Pr(\boldsymbol{\pi}_{T+t} < \mathbf{0})$
3m	-0.798	16.050	0.425
6m	-3.223	29.584	0.412
9m	-4.432	37.645	0.388
1y	-4.910	41.626	0.382

Table 1.5. The Statistics of Profits from Convergence Trading Strategy

are large implying the convergence trading is risky; the probabilities of negative profits are high. Figure 1.5 plots the distribution of profits for $t = 52$, the left-fat-tail of which indicates a high probability of a great loss.[53] This left-tail distribution together with the high odds of margin calls induced by agents learning about prices prevent a small number of arbitrageurs from taking the risky convergence trading.

Our results are in line with a large amount of literature on arbitrage, including Shleifer and Vishny (1997), Xiong (2001), and De Jong, Rosenthal and Van Dijk (2009). Although arbitrage in the long-run is profitable[54], the limits of arbitrage along with the short-run price divergence cause arbitrageurs to fail in “a short-term world”.

1.6. Extensions

In this section, we discuss some extensions to the benchmark learning model in section 4.

[53]If we take maintenance ratio requirements into account, convergence trading would be riskier. This table can be regarded as giving the lower bound of the loss.

[54]Because prices converge in the long-run, and AH premium is stationary by ADF test.

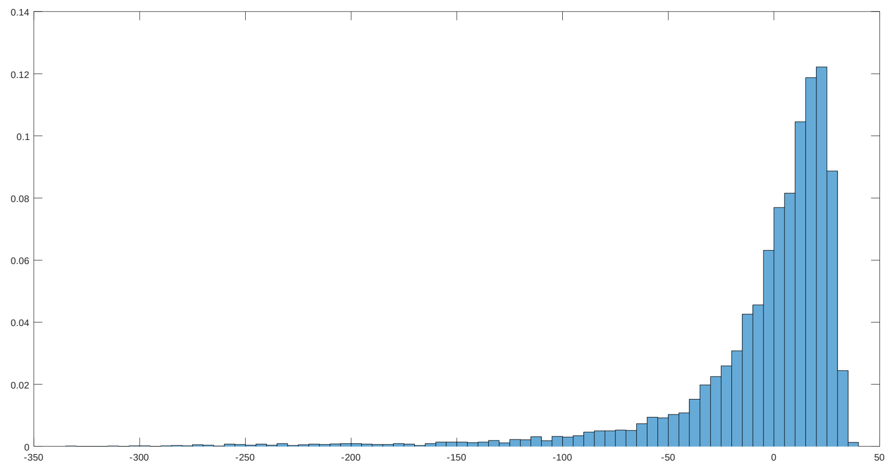

Figure 1.5. The Distribution of Profits by Implementing Convergence Trading (1 year)

1.6.1. Cross-Learning Scheme

Figure 1.3 displays that A- and H-share prices are highly positively correlated. By observing this high correlation, agents probably think that the prices of one share can provide information for the other share. Therefore, instead of assuming independent price processes (1.14), it is natural to model agents' subjective belief about prices as the cross-learning scheme

$$\begin{bmatrix} (\frac{C_t}{C_{t-1}})^{-\gamma}\frac{P_t^A}{P_{t-1}^A} \\ (\frac{C_t}{C_{t-1}})^{-\gamma}\frac{P_t^H}{P_{t-1}^H} \end{bmatrix} = \begin{bmatrix} b_t^A \\ b_t^H \end{bmatrix} + \begin{bmatrix} \epsilon_t^A \\ \epsilon_t^H \end{bmatrix}$$

$$\begin{bmatrix} b_t^A \\ b_t^H \end{bmatrix} = \begin{bmatrix} b_{t-1}^A \\ b_{t-1}^H \end{bmatrix} + \begin{bmatrix} \xi_t^A \\ \xi_t^H \end{bmatrix}$$

where ϵ_t^A, ϵ_t^H are jointly-normal $(\epsilon_t^A\ \epsilon_t^H)' \sim N(0, R)$ and ξ_t^A, ξ_t^H are jointly-normal $(\xi_t^A, \xi_t^H)' \sim N(0, Q)$, R and Q are variance-covariance matrices. The independent subjective belief about A- and H-share prices is a special case of the cross-learning scheme when R and Q are reduced to be diagonal matrices.

Agents optimally update their beliefs according to:

$$\begin{bmatrix} \beta_t^A \\ \beta_t^H \end{bmatrix} = \begin{bmatrix} \beta_t^A \\ \beta_t^H \end{bmatrix} + \begin{bmatrix} \alpha^A & \alpha^C \\ \alpha^C & \alpha^H \end{bmatrix} \begin{bmatrix} (\frac{C_{t-1}}{C_{t-2}})^{-\gamma} \frac{P_{t-1}^A}{P_{t-2}^A} - \beta_{t-1}^A \\ (\frac{C_{t-1}}{C_{t-2}})^{-\gamma} \frac{P_{t-1}^H}{P_{t-2}^H} - \beta_{t-1}^H \end{bmatrix}.$$

They use not only the newly observed A-share prices (H-share prices) but also the H-share prices (A-share prices) to update their beliefs about future capital gains of A-shares (H-shares).[55]

The pricing equation is still (1.15). We use MSM to estimate the parameters $[\alpha^A\ \alpha^H\ \alpha^C]$ to match the targeted moments in section 4 as well as the correlation of A- and H-share prices. α^A and α^H are estimated to be 0.0033 and 0.0015 respectively similar to the estimation in section 4, while $\alpha^C = 0.0002$ is smaller than them with one order of magnitude. Meanwhile, we find that the 95% intervals of model moments in Table 1.6 are very close to those in section 4.

[55]Define X such that $X(X+R)^{-1}X = Q$. The Kalman gain vector $K \equiv \begin{bmatrix} \alpha^A & \alpha^C \\ \alpha^C & \alpha^H \end{bmatrix}$ is obtained by $K = X(X+R)^{-1}$.

Moments	**Data**	**Model** (95% interval)
$E(\frac{P_t^A}{P_t^H} * 100)$	126.68	[94.31 127.06]
$\sigma(\frac{P_t^A}{P_t^H} * 100)$	6.98	[4.31 26.48]
$\rho(\frac{P_t^A}{P_t^H} * 100)$	0.88	[0.96 0.99]
$corr(P_t^A, P_t^H)$	0.91	[0.59 0.99]

Table 1.6. Model Simulated Moments

1.6.2. Learning with Expectation Shocks

Expectation shocks are shown to be important for business cycle and asset prices, e.g. Bullard, Evans and Honkapohja (2008) and Milani (2011, 2017). They are identified as the exogenous component of expectations not accounted for by the learning model.[56] We incorporate expectation shocks into the learning model by modeling the belief updating scheme as follows

$$\beta_t^j = \beta_{t-1}^j + \alpha^j((\frac{C_{t-1}}{C_{t-2}})^{-\gamma}\frac{P_{t-1}^j}{P_{t-2}^j} - \beta_{t-1}^j) + \epsilon_t^{\beta^j} \quad j \in \{A, H\} \tag{1.17}$$

where $\epsilon_t^{\beta^j} \sim i.i.N(0, (\sigma^j)^2)$ are the expectation shocks. We can regard expectation shocks as the shocks to agents' information set (Adam, Marcet and Beutel 2017). Alternatively, expectation shocks can be understood as the small deviations from IR – waves of optimism and pessimism. If agents are fully internally rational, their optimal beliefs will follow the belief updating scheme (1.15) strictly. However, agents' expectations might deviate from the optimal forecast: they can be either over-optimistic – by believing that future capital gains will be higher than those

[56]Bullard, Evans and Honkapohja (2008) call this exogenous component "judgment'.

predicted by their learning model – or over-pessimistic. Expectation shocks, in this way, are close to the misperception shocks of noise traders, which is widely assumed in the behavioral finance literature, e.g., De Long et al. (1990). The different realized expectation shocks give rise to the difference in β_t^A and β_t^H. The initial higher belief for A-share $\beta_1^A > \beta_1^H$ can come from $\epsilon_1^{\beta^A} > \epsilon_1^{\beta^H}$, stemming from the widely spread optimistic media reports on the A-share market in 2014 as a result of the narrative of a reform-based bull market.[57]

We simulate the model to understand the effects of expectation shocks on the AH premium and convergence trading. Instead of estimating parameter values of expectation shocks as in Milani (2011), we set the standard deviations σ^j such that standard deviations of β^j are ten percent larger than the β^j in section 4. Table 1.7 contains the model moments. We find that expectation shocks lead to higher and more volatile premiums; the model moments are still broadly compatible with data. The probability of receiving marginal calls ($mr_t < 130\%$) becomes larger and can reach as high as 14.4% as A- and H-share prices are more likely to diverge in the short-run. Furthermore, as apparent in Table 1.8, we see a more negative mean and larger standard deviation of convergence trading profit π_{T+t}, as well as a higher probability of negative profit, illustrated by a more-skewed left-fat-tailed distribution of profits in Figure 1.6.

[57]See a discussion in Huang, Miao and Wang (2019).

Moments	Data	**Model** (95% interval)
$E(\frac{P_t^A}{P_t^H} * 100)$	126.68	[91.91 145.07]
$\sigma(\frac{P_t^A}{P_t^H} * 100)$	6.98	[3.96 32.36]
$\rho(\frac{P_t^A}{P_t^H} * 100)$	0.88	[0.90 0.99]

Table 1.7. Model Simulated Moments

	Mean	**Std**	Pr ($\boldsymbol{\pi}_{T+t} < \mathbf{0}$)
3m	-2.455	20.581	0.472
6m	-7.891	39.116	0.469
9m	-10.938	47.600	0.465
1y	-13.356	54.247	0.468

Table 1.8. The Statistics of Profits from Convergence Trading Strategy

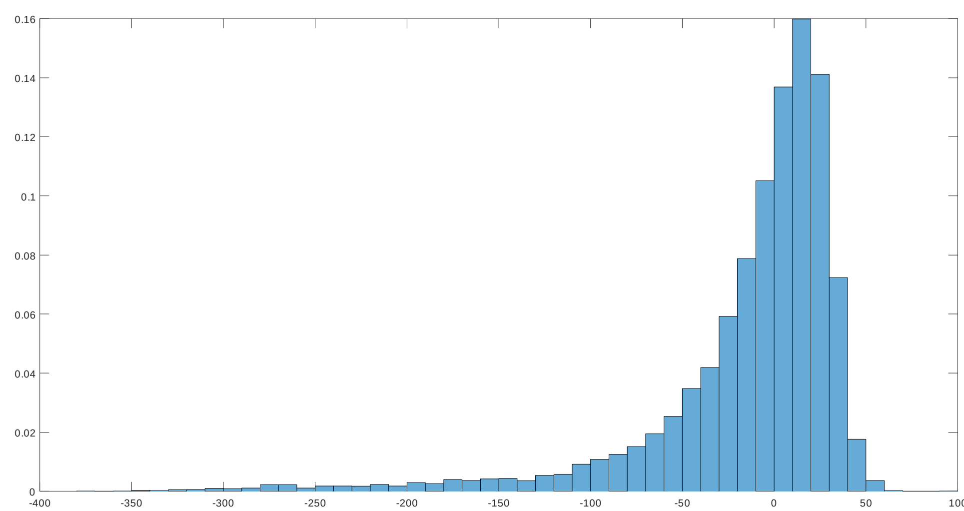

Figure 1.6. The Distribution of Profit by Implementing Convergence Trading Stragtegy (1 year)

1.7. A Discussion

In sections 3 we considered RE and Bayesian RE models with heterogenous agents who have pricing knowledge. Obviously the analysis in section 3 does not cover all possible theories where the knowledge of the equilibrium pricing function is preserved. In this section, we now discuss informally other possible explanations for the AH premium. While more research may show that these can explain some of the differences in stock prices, it seems that it is unlikely that they can make up for the large and persistent premium that is observed.

Exchange rate risk appears to play a role in explaining AH premium because A-shares trade in RMB while H-shares trade in HKD. We conjecture that a no-arbitrage condition in the stock market similar to the interest rate parity would hold, that is, the expected return differential between A-shares in RMB and H-shares in HKD should equal the expected change of the exchange rate. However, the Hong Kong Dollar was expected to appreciate against the RMB by an average of 0.02%, as measured by the 1-week exchange rate forward from November 2014 to June 2019.[58] According to the no-arbitrage condition, the expected appreciation of Hong Kong Dollar should have implied an averaged higher H-share prices.

We have not formally modeled twin shares with different liquidity. It is widely believed that the higher the liquidity, the higher the asset price. One popular measure of liquidity for a stock (index) is the proportion of price-change days over a certain period (Mei, Scheinkman and Xiong 2009). Based on daily data for the period 2006-2019, the proportion of trading days with price changes for A-shares

[58]The forward rate can reflect investors' expectations about future exchange rate. The expected change is calculated by dividing the forward rate by the spot exchange rate and taking the average over the sample period. Hong Kong dollar was expected to appreciate, on average, by 0.15%, 0.50%, and 0.72% measured by 1-month, 3-month, and 1-year forward rates.

is 98.69%, while this proportion for H-shares is 97.28%. It is 96.43% for the U.S. S&P 500 over the same sample period. These numbers suggest that both A- and H-shares are quite liquid, and A-shares are just marginally more liquid than H-shares. Besides, there is no well-accepted model of liquidity in asset markets; exploring this issue formally would be a massive undertaking that goes beyond the scope of this paper.

Moreover, transaction costs levied on the trading value in SSE and HKSE are small, including the stamping duty, the security management fee, the transfer fee, and the handling fee. The total transaction cost in the HKSE is about 0.118% according to regulations, while it is about 0.169% in SSE.[59] This tiny difference throws doubt on the potential of transaction costs in producing the data-like premium. And the lower transaction costs in HKSE might implies slightly higher H-share price.

There has been a recent interest in rational bubbles of stock markets, for example Martin and Ventura (2012) and Miao and Wang (2018). The rational bubbles in this literature facilitate liquidity and relax borrowing constraints. The work of Miao and Wang is for infinitely lived agent models, and it has the potential of generating quantitatively relevant bubbles when agents borrow against the market value of stocks. But we know of no papers studying rational bubbles in two assets sharing the same fundamentals in infinitely-lived agent models. In principle, there could be equilibria where there is a rational bubble in one stock but not in the

[59]See the sources of transaction costs in Appendix 9.5.

other, thus justifying the AH premium. However, since both A- and H- shares are allowed to be used as collateral through the share pledging to secure loans[60], it seems challenging to have such a equilibria. In addition, we conjecture that it is likely that if such an equilibrium was found, it would display an explosive AH premium, unlike the stable one found in the data, but a more formal study of the issue would be of interest.

Additionally, there also exist some market microstructure differences between SSE and HKSE.[61] First, SSE imposes daily price limits of 10% on regular stocks and 5% on special treatment (ST) stocks, but HKSE has no such limits. Second, the trading system in SSE is "T+1" and in HKSE is "T+0".[62] Third, there are other differences in the regular trading sessions, pre-market opening sessions, and the block trade.[63] All of these microstructure differences might affect stock prices. Indeed, Chen et al. (2019) find that the daily price limit scheme amplifies rather than stabilizes the market volatility because of the destructive strategy of large speculators. The larger volatility of A-shares than H-shares could be partially attributed to the daily price limit. The question is how these microstructure differences would generate a large positive AH premium.

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[60]See Li et al. (2019) for A-share pledging and the link below for H-share pledging. (https://en-rules.hkex.com.hk/node/2505)

[61]Refer to the official document "http://www.sse.com.cn/aboutus/research/research/c/3996072.pdf".

[62]T+1 only allows investors to get ownership and sell the shares the next day after they buy the shares. By contrast, T+0 trading system allows investors to buy and sell shares on the same day.

[63]The pre-market is the period of trading activity that occurs before the regular trading session. A block trade is a permissible, noncompetitive, privately negotiated transaction either at or exceeding an exchange determined minimum threshold quantity of shares.

This paper presents a natural "Internal Rationality" learning theory to quantitatively rationalize AH premium and emphasizes the critical role of agents' price expectations. Nevertheless, we cannot rule out other potential explanations, including the liquidity, the rational bubble, and the market microstructure discussed above. The formal studies leave for future research.

1.8. Conclusion

This paper explains the AH premium. Since AH-market is a large market, understanding it is essential for market investors, and it also provides a natural experiment to test expectation formation in asset pricing models. We show that many versions of RE and Bayesian RE asset equilibria where agents know the equilibrium pricing function fails to generate a data-like AH premium, including models with agents' heterogeneities and dividend taxes. We also discuss some other possible explanations.

A model of learning about prices explains the price differences of dual-listed companies in a very natural way. The reason is that small differences in beliefs about the stocks in different markets provide a "nearly" self-fulfilling prophecy that prices are different. When the internally rational agents learn about stock prices, A- and H-share prices will differ in equilibrium just because investors them to differ. Therefore, we think of AH premium as a natural experiment supporting learning about prices in terms of expectation formation of stock prices. Of course, some factors uncovered by RE and Bayesian RE models could explain part of the story but they seem not to be the main characters in the story. Transaction costs, the liquidity and the rational bubble may act as good supporting actors for this story in future research, but the main actor seems to be learning about prices.

Although agents do not know precisely the equilibrium pricing function, they hold beliefs that are close to the actual behavior of the data. Agents make "mistakes" that would be undetectable statistically and, therefore, are not really "mistakes". We also show that convergence trading is highly likely to suffer a significant loss because the prices might diverge in the short run. Hence, even if arbitrageurs have superior information, short-run price divergences prevent them from bringing stock prices to their fundamental values.

As we follow the literature to point out in section 5, the limits of arbitrage are strict, and only a small number of professional investors conduct arbitrage. However, as the classical literature shows (e.g., Fama 1965, Sharpe 1964), if the number of arbitrageurs is large, their collective actions should force stock prices to converge to their fundamental values. While we assume a zero measure of arbitrageurs as a shortcut in Section 5, it would be interesting to investigate whether a sizeable positive measure of them with rational expectation trading with learning agents will ride the wave or lean against the bubble. The conclusion might not be as obvious as it seems, as many papers such as Blume and Easly (2006), Cogley and Sargent (2009) and Dumas, Kurshev and Uppal (2009) point out that it could be that the agents with more information will be driven out of the market in the incomplete market or that it takes a very long time for the learning agents to be driven out of the market.

1.9. Appendix

1.9.1. Algorithms and Simulations for State-Contingent Bond Positions

1.9.1.1. Rational Expectations. We assume the dividend payment takes value of D^h (high) with probability $prob(D^h) = (1-\pi)$ and D^l (low) with probability $prob(D^l) = \pi$ in each period. The dividend payment is assumed to be high D^h in the first period. The two types of agents with rational expectation share the same subjective discount factors $i.e.$ $\delta^1 = \delta^2$ and the endowment is constant $i.e.$ $Y_t = Y$. In this case, the policy function is time-invariant and only depends on the current realization of dividend payment.

Step 1:Draw N series of M periods each, of dividends $\{\{D_t^n\}_{t=0}^M\}_{n=1}^N$ using a $i.i.d$ random number generator with $D_{n,0} = D^h$. Guess a value for λ representing the constant ratio of marginal utility in the complete market $i.e.$ $\lambda = \frac{u'(C_t^{1,n})}{u'(C_t^{2,n})}$ and then simulate for consumption $\{\{C_t^{1,n}, C_t^{2,n}\}_{t=0}^M\}_{n=1}^N$ using commodity goods market clearing condition given the guess of λ. Solve for λ by iteration to satisfy the following intertemporal budget constraint of type 1 agent:

$$\frac{1}{N}\sum_{n=1}^{N}\sum_{t=0}^{M}\delta^t\frac{u'(C_t^{1,n})}{u'(C_0^{1,n})}(C_{t+j}^{1,n} - D_{t+j}^n - Y) = B_{-1}^1$$

where B^1_{-1} is the initial inherited holding of the state-contingent bond. Then we obtain the state-contingent bond price once we get $c(D^l)$ and $c(D^h)$.

Step 2: Let $Z \equiv u^{'}(c(D^l))(c(D^l)-D^l-Y^1)\pi+u^{'}(c(D^h))(c(D^h)-D^h-Y^1)(1-\pi)$. We obtain $B^1_{t-1}(D^l)$ by

$$\begin{aligned} B^1_{t-1}(D^l) &= E_t \sum_{j=0}^{\infty} \delta^j \frac{u'(C^{1,n}_{t+j})}{u'(C^{1,n}_t)}(C^{1,n}_{t+j} - D^n_{t+j} - Y) \\ &= (C^1(D=D^l) - D^l) + \frac{\delta}{u'(C^1(D=D^l))(1-\delta)} Z \quad \forall t. \end{aligned}$$

and $B^1_{t-1}(D^h) = B^1_{-1}$ because of the time-invariant policy functions and the assumption of $D_0 = D^h$. The bond positions of type 2 agents are the opposite to those of type 1 agents as a result of the bond market clearing condition. Similarly, we obtain the stock price.

In our simulation, we let $\delta = 0.99$, $\gamma^2 = 1, Y = 0$, $D^h = 1$, $D^l = 0.5$ and $\pi = 0.5$. We alter γ^1 while fixing γ^2 to illustrate how the degree of risk aversion affects consumption and bond allocations.[64] When $\gamma^1 = \gamma^2$, the two types of agents share the same consumption and don't hold the state-contingent bonds. As type 1 agents become more risk averse, they want a more smooth consumption profile by holding more $B^1(D^l)$ seeking to be compensated in the bad states. While varying γ^1 does affect the price of equities, it is consistent with the equalization of P^A and

[64]With state-contingent bonds, holdings of A-shares and H-shares are not uniquely determined because the shares are 'redundant' assets. Yet, the value for consumption, bond positions and price are uniquely determined. In the simulation, we keep agents' shareholdings of the two assets fixed over time so that they don't trade stocks with each other.

	$\gamma^1 = 1$	$\gamma^1 = 2$	$\gamma^1 = 3$	$\gamma^1 = 4$
$C^1(D^h)$	1	0.8736	0.8045	0.7629
$C^1(D^h)$	0.5	0.5514	0.5717	0.5818
$B^1(D^l)$	0	0.1017	0.1416	0.1619
$P^A(D^h)$	100	112.64	119.55	123.71
$P^H(D^h)$	100	112.64	119.55	123.71
$P^A(D^l)$	50	44.97	42.99	41.99
$P^H(D^l)$	50	44.97	42.99	41.99

Table 1.9. Consumption and State-Contingent-Bond Holdings in Two States

P^H. These results are shown in Table 1.9, where $x(D^h)$ means the value of x for $D_t = D^h$.

When agents have different subjective discount factors, say $\delta^1 < \delta^2$, then $\frac{u'(c^1_{t+1})}{u'(c^2_{t+1})} = \frac{\delta^2}{\delta^1}\frac{u'(c^1_t)}{u'(c^2_t)}$. In this case, the marginal rate of substitution MRS_{12} is not constant and increases over time. Impatient type 1 agents consume less and less over time. In the limit, $\frac{u'(c^1_M)}{u'(c^2_M)} \to \infty$ as $M \to \infty$, this means that in the limit type 2 agents consume all the dividends $c^2_M \to y_M$ while type 1 agents consume nothing $c^1_M \to 0$. Given δ^2, the smaller δ^1 is, the faster the economy converges to the limit.

1.9.1.2. Diverse Beliefs. Now suppose the two types of agents have diverse beliefs on fundamentals. Type 1 (2) agents' subjective probability is $prob^1(D_t)$ ($prob^1(D_t)$).

Step 1: The first step 1 is to solve the MRS_{12} in the first period λ_0. It is the same as the step 1 above but the change of measure by multiplying the ratio of the subjective probability to the objective probability should be taken into account.

Step 2 : Draw one long series of M_L periods of dividends using the random number generator. Simulate the time-varying MRS_{12} $\{\lambda_t\}$ following:

$$\lambda_t = \alpha_{t-1}(D_t)\lambda_{t-1}$$

where $\alpha_{t-1}(D_t) = \frac{prob^2(D_t)}{prob^1(D_t)}$. Find the corresponding consumption $\{C_t^1, C_t^2\}_{t=0}^{T_L}$ by the market-clearing condition given $\{\lambda_t\}_{t=0}^{M_L}$. Calculate the present value of primary deficits adjusted by the change of measure for agent 1, $Dd_t^1 \equiv \sum_{j=0}^{M_l} \delta^{1,j} \frac{u'(C_{t+1}^1)}{u'(C_t^1)}(C_{t+j}^1 - D_{t+j})\frac{prob(D_t)}{prob^1(D_t)}$, which can be solved backward assuming $Dd_{M_L}^1 = 0$.

Step 3: Run a regression of $\{Dd_t^1\}_{t=1}^{M_L-O}$ on $\{D_t\}_{t=1}^{M_L-O}$ and $\{\lambda_{t-1}\}_{t=1}^{M_L-O}$ with the last O periods omitted.[65] The time-varying bond positions of type 1 agents are the fitting values of the regression. The bond positions of type 2 agents are the opposite of those of type 1 agents.

We assume $prob^1(D_t = D^h) = 0.51$, $prob^1(D_t = D^h) = 0.54$ and $\gamma^1 = \gamma^2 = 1$. Other parameters are the same with those in the Appendix 9.2. Type 1 agents are more accurate in terms of the "distance" of the subjective probability from the true probability. As apparent in Figure 1.7 for one simulation, they accumulate assets and consume more goods over time while type 2 agents behave in the opposite way. In the limit, type 1 agents will consume the total resources while type 2 agents get nothing. It is the relative correctness of the perceived beliefs that drives this bond-trading pattern rather than their degree of optimism. While type 2 agents are

[65] In this case, the state-contingent-bond positions in period t are not only the function of D_t but also a function of λ_{t-1} *i.e.* $B_{t-1}^1(D^h) = E(Dd_t^1|D_t = D^h, \lambda_{t-1})$ and $B_{t-1}^1(D^l) = E(Dd_t^l|D_t = D^l, \lambda_{t-1})$.

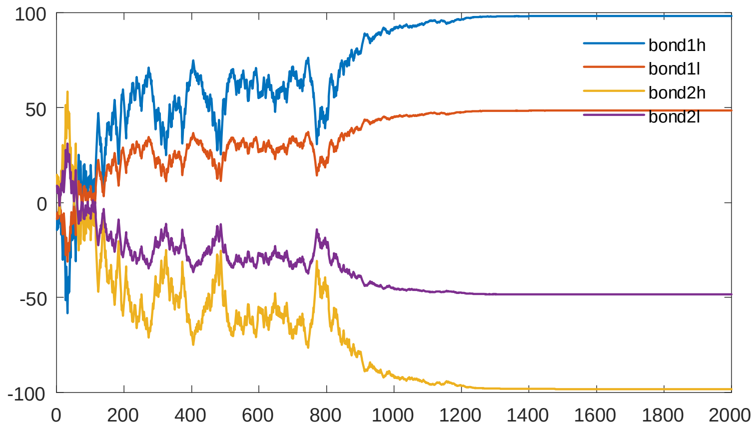

Figure 1.7. Agents' State-Contingent-Bond Positions

more optimistic than type 1 agents, it is the type 2 agents who want to buy more state-contingent bonds. However, although the two agents have heterogeneous perspectives about the economic fundamentals, the prices of the two shares are identical for all states.

1.9.2. An Assumption for Learning

According to the arguments described by AMN, without strict rational expectations we may obtain $E^{\mathcal{P}}[C^i_{t+1}] \neq E^{\mathcal{P}}_t[C_{t+1}]$, even if $C^i_{t+1} = C_{t+1}$ holds ex-post in the equilibrium. We invoke a similar approximation, as follows:

$$E^{\mathcal{P}}_t[(\frac{C^i_{t+1}}{C^i_t})^{-\gamma}(P^j_{t+1} + D_{t+1})] \simeq E^{\mathcal{P}}_t[(\frac{C_{t+1}}{C_t})^{-\gamma}(P^j_{t+1} + D_{t+1})], j = A, H \tag{1.18}$$

The following assumption as in AMN provides sufficient conditions for this to be the case:

Assumption We assume that Y_t is sufficiently large and that $E_t^{\mathcal{P}} P_{t+1}^j / D_t < \overline{M}$, $j = A, H$ for some $\overline{M} < \infty$ so that, given finite asset bounds $\underline{S}$ and $\overline{S}$, the approximation (1.18) holds with sufficient accuracy.

1.9.3. Differentiable Projection Facility

The function ω for the differentiable projection facility is:

$$\omega(\beta) = \left\{ \begin{array}{cc} \beta & \text{if } x \leq \beta^L \\ \beta^L + \frac{\beta - \beta^L}{\beta + \beta^U - 2\beta^L}(\beta^U - \beta^L) & \text{if } \beta^L < x \leq \beta^U \end{array} \right\}$$

In our numerical exercise, we choose β^U such that the implied price-dividend ratio never exceeds $U^{PD} = 600$ and set $\beta^L = \delta^{-1} - 2(\delta^{-1} - \beta^U)$.

1.9.4. New Initial Conditions of Price Beliefs

We can recover the initial beliefs investors hold in November 2014 by substituting the actual A- and H-share prices into the belief updating scheme (1.15), in the same way as we did in section 5. The parameter values in (1.15) are the same as section 4.5. We use the actual stock prices data from April 2014 since this is the time the Chinese government announced the approval of the connect program. Investors are assumed to have RE beliefs at the beginning and then update their beliefs according to (1.15) after the announcement. The new set of beliefs recovered

Targeted Moments	**Data**	**Model** (95% interval)
$E(\frac{P_t^A}{P_t^H} * 100)$	126.68	[94.84 130.78]
$\sigma(\frac{P_t^A}{P_t^H} * 100)$	6.98	[4.41 26.98]
$\rho(\frac{P_t^A}{P_t^H} * 100)$	0.88	[0.96 0.99]
$corr(P_t^A, P_t^H)$	0.91	[0.57 0.99]
$corr(P_t^A - P_t^H, P_t^H)$	0.18	[-0.16 0.93]
$\sigma(P_t^A)/\sigma(P_t^H)$	1.20	[1.06 2.63]

Table 1.10. Model Simulated Moments Using New Initial Beliefs

are very close to the ones calibrated in section 4.5. Hence, we have the similar simulated moments reported in Table 1.10.

1.9.5. Data Sources

We have used financial and macroeconomic data. The financial data including Hang Seng China AH Premium Index, Hang Seng China A(H) index, Shanghai Composite Index, dividend yield, market capitalization, and Northbound (Southbound) trading quota balance are downloaded from the Wind Financial Database (http://www.wind.com.cn). The sample period of Hang Seng China AH Premium, A and H Index is from Jan 2006 to June 2019. The daily (monthly) price series has been transformed into a weekly series by extracting the value of the last trading day of the corresponding week (month). The dividend yield is a moving average of the dividends in the previous 11 months and the current month divided by the Shanghai Composite Index. The monthly dividends on the index portfolio are obtained from the monthly dividend yield under the assumption that dividends have been approximately constant during the last 12 months. We have quarterly dividends by adding up the corresponding monthly series. Following Campbell (2003), we deseasonalize dividends by taking averages of the actual dividend payments over the current and preceding three quarters. The sample period of dividend yield is from 1995 Q1 to 2018 Q4. Stock market turnover data are from CEIC (www.ceicdata.com). The U.S. stock market data are from Shiller's website (http://www.econ.yale.edu/~shiller/) and CRSP. The spot exchange rate and the forward rates are also from Wind.

The transaction costs and the dividend taxes can be found on the official websites of regulatory authorities and stock exchanges. The transaction costs and the dividend taxes in SSE are found on "http://www.sse.com.cn" while the transaction costs and the dividend taxes in HKSE are in "https://www.hkex.com.hk".

The macroeconomic data including consumption, 1-year deposit interest rate and CPI in quarterly frequency are downloaded from CQER Fed Atlanta, which are used in Chang et al. (2016). The sample period is from 1995 Q1 to 2018 Q4. To obtain real values, nominal variables are deflated using China CPI.

CHAPTER 2

New Tests of Expectation Formation with Applications to Asset Pricing Models

2.1. Introduction

Asset prices are crucially determined by investors' expectations about the future. Yet asset pricing models are usually silent about to what extent model-implied asset price forecasts resemble forecasts made by agents in reality. Recent research employs survey expectations data to test and guide the modeling of expectation formation in financial markets and/or to discipline the modeling of asset price dynamics, examples are Greenwood and Shleifer (2014), Barberis, Greenwood, Jin and Shleifer (2015), Adam, Marcet and Beutel (2017), Adam, Matveev and Nagel (2018), and Nagel and Xu (2018).

Along this line, our paper develops new tests of expectation formation which are generally applicable in asset pricing models with various informational assumptions. We show these models typically impose a large number of cointegration restrictions between forecasts of economic variables, and these cointegration restrictions imply that agents have strong information set when they form their expectations. Our tests utilize these restrictions. Researchers can apply these tests to study the cointegration between forecasts of exogenous variables and forecasts of endogenous variables in their model as well as the cointegration between forecasts of different endogenous variables. Moreover, these models impose cointegration restrictions between forecasts of the same variable (e.g., stock prices) over different forecasting horizons.

In asset pricing literature, according to expectation formation we have (1) rational expectations (RE) modeling (e.g., Campbell and Cochrane (1999), Bansal, Kiku and Yaron (2012), Boldrin, Christiano and Fisher (2001), Croce (2014)), (2) Bayesian RE modeling,[1] including models of consumption learning (e.g., Collin-Dufresne, Johannes and Lochstoer (2016)), consumption sentiment (e.g., Jin and Sui (2018)), and "agree to disagree" heterogeneous beliefs (e.g., Ehling, Graniero, and Heyerdahl-Larsen (2018)). Both RE and Bayesian RE models assume agents possess the knowledge of the equilibrium pricing function, i.e., the equilibrium mapping from fundamentals (or beliefs about fundamentals) to stock prices. This knowledge typically requires strong informational assumptions on agents.[2]

The essence of our new tests is demonstrating that agents' knowledge of the equilibrium pricing function in asset pricing models imposes cointegration restrictions between their forecasts of stock prices and forecasts of fundamentals, i.e., aggregate consumption or dividends. Intuitively, the long-run component of stock

[1]Many models make an asymmetric assumption about fundamentals and prices: investors are assumed not to understand the behavior of fundamentals (say, consumption) but investors understand perfectly well how current stock prices relate to future fundamentals, so that investors are assumed to have RE about the pricing function. Adam, Marcet and Beutel (2017) dub this literature "Bayesian RE."

[2]See Adam and Marcet (2011).

price forecasts in these models is anchored by consumption forecasts via this knowledge of agents. We show this is a robust feature across a wide range of RE or Bayesian RE asset pricing models which assume this knowledge of agents either explicitly or implicitly.

Yet a central new piece of evidence from expectations data uncovered by the paper is that the median (or mean) survey forecasts of aggregate stock price index are *not* cointegrated with the median (or mean) forecasts of aggregate consumption.[3] Put differently, the long-run or trend component of stock price forecasts is not anchored by consumption forecasts. This evidence is robust to different sources of expectations data (Livingston Survey and Shiller Survey for stock price forecasts and Survey of Professional Forecasters and Greenbook forecasts for consumption forecasts), forecasting horizons (1-, 2-, 4-quarter and 10-year ahead forecasts), statistical tests (Phillips Perron test, ADF-GLS test, KPSS test with correcting small sample problems and Johansen test), using median or mean forecasts for testing, and using stock price forecasts data which is made at the same or different dates from consumption forecasts.[4]

[3] While it would be interesting to study if forecasts of aggregate stock price index are cointegrated with forecasts of aggregate dividend index in the data, we do not pursue this analysis due to lack of high-quality data on aggregate dividend forecasts with a sufficiently long sample. A recent example is De La O and Myers (2018) which constructs analysts' forecasts of aggregate dividend index. Yet this data may be inappropriate for our purpose as their constructed data is relatively short (i.e., from 2003 to 2015).

[4] The paper also finds that, for instance, forecasts of stock prices (or consumption) over different horizons in the data are cointegrated with each other, consistent with the stock pricing models we consider. See section 3 for details.

What does the survey evidence, especially the no cointegration between stock price forecasts and consumption forecasts, tell us about modeling expectation formation in asset pricing models? First, the evidence casts some doubt on the modeling of expectation formation in asset pricing models (discussed above) which assume agents possess the knowledge of the equilibrium pricing function. Irrespective of agents having rational or non-rational (or extrapolative) expectations about fundamentals (consumption), stock price forecasts are cointegrated with aggregate consumption forecasts in these models, appearing inconsistent with the survey evidence.

Second, reconciling the new survey evidence appears to call for relaxing agents' knowledge of the equilibrium pricing function as is present in e.g., adaptive learning models; examples include Lansing (2010, 2019), Carceles-Poveda and Giannitsarou (2008), Branch and Evans (2010, 2011), Boswijk, Hommes and Manza (2007), Adam, Marcet and Beutel (2017, henceforth AMB).

Third, the survey evidence suggests that major asset pricing models may have missed an important persistent component which independently drives stock price forecasts but not consumption forecasts. One way to reconcile the evidence is to modify the belief specification in AMB by adding a non-stationary sentiment or judgment component (which is not cointegrated with consumption forecasts) *directly* to stock price forecasts together with assuming the agent's lack of knowledge of the equilibrium pricing function.[5]

2.1.1. Related literature

This paper relates to recent work on utilizing survey expectations data to test expectation formation and discipline modeling. First, there is a large literature about testing RE hypothesis. Greenwood and Shleifer (2014) and AMB reject the RE hypothesis using stock return survey expectations data. Branch (2004) and Coibion and Gorodnichenko (2015) study the survey data on inflationary expectation and find that (full-information) RE is difficult to explain it. Our tests contribute by focusing on the the discrepancy between realized and forecasts of stock price consumption ratio. On the one hand, the literature generally finds that the realized stock price consumption ratios are stationary. On the other hand, we show the forecasts of stock price consumption ratios are non-stationary. This discrepancy can be interpreted as a rejection of the RE hypothesis.

Beyond RE test, Malmendier and Nagel (2011) show that investors' experience of macroeconomic outcomes affects their financial risk-taking decisions. Adam, Matveev and Nagel (2018) empirically reject that survey return expectations are formed by risk-neutral investors or ambiguity averse/robust investors. Based on

[5]In our proposed resolution, investors' stock price forecasts are assumed to contain two components. The first component is generated from an econometric forecasting model. On top of this, investors make some judgmental adjustment, defined as "sentiment."

survey data findings (Malmendier and Nagel 2011, 2016), Nagel and Xu (2018) build an asset pricing model with learning with fading memory about dividend process to replicate several stock market facts such as counter-cyclical risk premium. Bordalo, Gennaioli, Ma and Shleifer (2018) find that forecasters typically over-react to their individual news, while consensus forecasts under-react to average news in terms of the forecasts of macroeconomic and financial variables. To reconcile the findings, they combine a diagnostic expectation model of belief formation with a noisy information model of belief dispersion. We contribute to the literature by focusing on the cointegration restrictions as a new dimension of survey expectation and deeply exploring which information is missing when agents make forecasts.

To reconcile the new survey evidence from our tests, it may be necessary to relax agents' knowledge of the equilibrium pricing function, as is done in e.g., adaptive learning models. Examples of this type of models include Lansing (2010), Branch and Evans (2011), Boswijk, Hommes and Manza (2007), Carceles-Poveda and Giannitsarou (2008), all of which build asset-pricing models where agents learn adaptively and do not have the perfect knowledge about the true stochastic process for payoff relevant variables beyond their control. Adam and Marcet (2011) develop 'internal rationality' to provide micro-foundations for models of adaptive learning. Adam, Kuang and Marcet (2012) show an open economy asset pricing model with internally rational agents who learn prices can replicate the dynamics of the house price and current account. Adam, Marcet and Nicolini (2016) study an asset pricing model in which agents learn about risk-adjusted price growth and replicate a set of standard asset-pricing moments describing stock price volatility. Winkler (2019) presents a learning-based business cycle model with financial frictions to match both asset price and business cycle moments. Zhang and Zhang (2019) show that relaxing agents' knowlege of the pricing function can rationalize the price differences of twin stocks in connected markets.

Section 2.2 develops new tests of expectation formation in full-information RE models. New evidence on the formation of stock price expectations is provided in Section 2.3. Section 2.4 shows stock price forecasts are cointegrated with consumption forecasts in major RE asset pricing models. Cointegration tests of expectation formation are provided for Bayesian RE models in Section 2.5. Section 2.6 discusses potential resolutions to reproduce the new evidence. Section 1.8 concludes.

2.2. New tests of expectation formation in RE models

This section develops new tests of expectation formation in RE models (and sections 4 and 5 provide tests in several classes of Bayesian/RE models). These models impose a large number of cointegration restrictions among forecasts of model variables and the tests utilize the restrictions. These tests can be generally applied by researchers to guide the modeling of expectation formation in financial and macroeconomic models. Our tests make the most of survey expectations data as they can be applied even if researchers have only limited data. For instance, forecasts of different variables in surveys may be made over different horizons or made at different dates. Or surveys only provide data on the forecast of the average value of model variables over a number of periods (e.g., average GDP growth over the next five years).

Consider a variable $\{X_t\}$ from a RE model, such as (log) stock price or aggregate consumption, which is generally represented by

$$X_t = X_t^P + X_t^C, \quad (2.1)$$

$$X_t^P = \mu + X_{t-1}^P + \sigma_{\epsilon,t}\epsilon_t, \quad (2.2)$$

$$(1-\phi(L))X_t^C = (1+\psi(L))\,\sigma_{\eta,t}\eta_t, \quad (2.3)$$

$$(1-\widetilde{\phi}(L))\left(\sigma_{\epsilon,t}^2 - \overline{\sigma}_\epsilon^2\right) = \left(1+\widetilde{\psi}(L)\right)\widetilde{\epsilon}_t, \quad (2.4)$$

$$(1-\widehat{\phi}(L))\left(\sigma_{\eta,t}^2 - \overline{\sigma}_\eta^2\right) = \left(1+\widehat{\psi}(L)\right)\widetilde{\eta}_t. \quad (2.5)$$

This variable contains a unit root. The superscripts P and C stand for the permanent and cyclical component. ϵ_t, η_t, $\widetilde{\epsilon}_t$ and $\widetilde{\eta}_t$ are $i.i.d$ innovations and independent to each other. The variance of ϵ_t and η_t are normalized to 1. $\sigma_{\epsilon,t}$ and $\sigma_{\eta,t}$ are allowed to be time-varying and their mean is constant and positive, i.e., $\overline{\sigma}_\epsilon^2$ and $\overline{\sigma}_\eta^2$. $\phi(L) = \phi_1 L + \phi_2 L^2 + ... + \phi_p L^p$ and $\psi(L) = \psi_1 L + \psi_2 L^2 + ... + \psi_q L^q$ where L is the lag operator. $\widetilde{\phi}(L)$, $\widetilde{\psi}(L)$, $\widehat{\phi}(L)$ and $\widehat{\psi}(L)$ are similarly defined.[6] The roots of $1-\phi(z)=0$, $1-\widetilde{\phi}(z)=0$, and $1-\widehat{\phi}(z)=0$ are within the unit circle, so X_t^C is a stationary process.

[6]Specifically, $\widetilde{\phi}(L) = \widetilde{\phi}_1 L + \widetilde{\phi}_2 L^2 + ... + \widetilde{\phi}_{\widetilde{p}} L^{\widetilde{p}}$, $\widetilde{\psi}(L) = \widetilde{\psi}_1 L + \widetilde{\psi}_2 L^2 + ... + \widetilde{\psi}_{\widetilde{q}} L^{\widetilde{q}}$, $\widehat{\phi}(L) = \widehat{\phi}_1 L + \widehat{\phi}_2 L^2 + ... + \widehat{\phi}_{\widehat{p}} L^{\widehat{p}}$ and $\widehat{\psi}(L) = \widehat{\psi}_1 L + \widehat{\psi}_2 L^2 + ... + \widehat{\psi}_{\widehat{q}} L^{\widehat{q}}$.

2.2.1. Integration property of conditional forecasts

Given the assumption of RE and full information, agents know the law of motion of X_t (equation (2.1)-(2.5)) and make use of this knowledge to make forecasts. The following lemma shows that if the variable X_t is integrated of order 1 ($X_t \sim I(1)$), conditional forecasts of this variable over arbitrary forecasting horizons i (i.e., $E_t X_{t+i}$) contain a unit root. For instance, if stock prices is an I(1) process, 1-year ahead forecasts of stock prices also contain a unit root.

Lemma 1. *If X_t follows (2.1)-(2.5) (i.e., $X_t \sim I(1)$), $E_t X_{t+i} \sim I(1)$ for $i > 0$.*

Proof. Given (2.1)-(2.5), we have $E_t X_{t+i} = E_t X_{t+i}^P + E_t X_{t+i}^C = \mu i + X_t^P + E_t X_{t+i}^C$. $E_t X_{t+i}$ is the sum of a unit root process and a stationary process and hence a unit root process. □

2.2.2. Cointegration among forecasts of different variables

This section establishes the cointegration relationship among forecasts of different variables when their realizations are cointegrated. Researchers can apply these results to test the cointegration between forecasts of exogenous variables and forecasts of endogenous variables in their model. Moreover, they can study the cointegration between forecasts of different endogenous variables. Suppose $y_t = (y_{1,t}\ y_{2,t}\ ...y_{n,t})'$ is a $1 \times n$ vector which is cointegrated with cointegrating vector $a = (a_1\ a_2\ ...\ a_n)'$ and $a'y_t$ is a stationary process (with possibly time-varying volatility). Mathematically,

$$\begin{aligned}(1-\phi(L))a'y_t &= (1+\psi(L))\,\sigma_{\eta,t}\eta_t,\\ (1-\widehat{\phi}(L))\left(\sigma^2_{\eta,t}-\overline{\sigma}^2_{\eta}\right) &= \left(1+\widehat{\psi}(L)\right)\widetilde{\eta}_t,\end{aligned}$$

where the roots of $1-\phi(z)=0$ and $1-\widehat{\phi}(z)=0$ are within the unit circle. We firstly establish a preliminary result which says the forecasts of an I(1) variable X made at date t over an arbitrary horizon i (i.e., E_tX_{t+i}) are cointegrated with X_k with cointegrating vector $(1,-1)$, where k can be identical to or different from t.

Lemma 2. *If X_t follows (2.1)-(2.5) (i.e., $X_t \sim I(1)$), $E_tX_{t+i}-X_k \sim I(0)$ for $i>0$.*

Proof. Let

$$\begin{aligned}E_tX_{t+i}-X_k &= \left(E_tX^P_{t+i}+E_tX^C_{t+i}\right)-X_t+(X_t-X_k)\\ &= \left(E_tX^P_{t+i}-X^P_t\right)+\left(E_tX^C_{t+i}-X^C_t\right)+(X_t-X_k)\\ &= \mu i+\left(E_tX^C_{t+i}-X^C_t\right)+(X_t-X_k)\,.\end{aligned}$$

$(E_tX_{t+i}-X_k)$ is stationary as $E_tX^C_{t+i}$, X^C_t and (X_t-X_k) are stationary. Note a special case is $t=k$. □

Denote by $E_{i_1}y_{1,i_1+j_1}$ j_1-period ahead expectation of variable y_1 made at date i_1.

Theorem 3. *If $a'y_t$ is a stationary process, $a_1 E_{i_1} y_{1,i_1+j_1} + a_2 E_{i_2} y_{2,i_2+j_2} + ... + a_n E_{i_n} y_{n,i_n+j_n}$ is stationary for arbitrary $i_1, i_2, ...i_n, j_1, j_2, ..., j_n > 0$.*

Proof. Let

$$
\begin{aligned}
& \left[a_1 E_{i_1} y_{1,i_1+j_1} + a_2 E_{i_2} y_{2,i_2+j_2} + ... + a_n E_{i_n} y_{n,i_n+j_n}\right] \\
= & \left[\sum_{k=1}^{n} a_k \left(E_{i_k} y_{k,i_k+j_k} - y_{k,i_k}\right) + \sum_{k=1}^{n} a_k y_{k,i_k}\right] \\
= & \left[\sum_{k=1}^{n} a_k \left(E_{i_k} y_{k,i_k+j_k} - y_{k,i_k}\right) + \sum_{k=1}^{n} a_k y_{k,i_1} + \sum_{k=2}^{n} a_k \left(y_{k,i_k} - y_{k,i_1}\right)\right].
\end{aligned}
$$

Note Lemma 2 implies $(E_{i_k} y_{k,i_k+j_k} - y_{k,i_k})$ is stationary for $k = 1, 2, ...n$. In addition, the cointegration of the vector y_t yields $\sum_{k=1}^{n} a_k y_{k,i_1}$ is stationary and $y_{k,t} \sim I(1)$ gives $(y_{k,i_k} - y_{k,i_1})$ is stationary. Thus, we have $a_1 E_{i_1} y_{1,i_1+j_1} + a_2 E_{i_2} y_{2,i_2+j_2} + ... + a_n E_{i_n} y_{n,i_n+j_n}$ is stationary. $\square$

The theorem contains a rich set of testable implications for expectation formation. For illustration, consider the asset pricing models discussed later (e.g., the long-run risks model and habit model) in which realized stock prices and consumption are cointegrated with cointegrating vector $(1, -1)$. First, a special case of the theorem is that forecasts of stock prices and consumption made at the same date (i.e., $i_1 = i_2 = ... = i_n$) and over the same forecasting horizons (i.e., $j_1 = j_2 = ... = j_n$) are cointegrated. And forecasts of stock price consumption ratio, i.e., $(E_t \log P_{t+j} - E_t \log C_{t+j})$ is stationary. This means, for example, 1-year ahead forecasts of stock prices and 1-year ahead forecasts of consumption (made at the same date) are cointegrated with cointegrating vector $(1, -1)$.

Second, the cointegration relation holds for forecasts of different variables *over different forecasting horizons* (i.e., $j's$ need not to be identical) as $(E_t \log P_{t+j_1} - E_t \log C_{t+j_2})$ is stationary for $j_1 \neq j_2$. This means, for instance, 10-year ahead forecast of stock prices and 1-year ahead forecast of consumption made at the same date are cointegrated. This result is particularly useful when the forecasting horizons of expectation data available to researchers are different across different variables. For instance, researchers may have data on 10-year ahead forecasts of stock prices and 1-year ahead (but not 10-year ahead) forecasts of consumption.

Third, the cointegration relation also holds for forecasts of different variables *made at different dates* (i.e., $i's$ need not to be identical) as $(E_{i_1} \log P_{i_1+j1} - E_{i_2} \log C_{i_2+j_2})$ is stationary for $i_1 \neq i_2$. This means, for instance, stock price forecasts made during $1960 - 1990$ (over an arbitrary forecasting horizon) are cointegrated with consumption forecasts made during $1970 - 2000$ (over an arbitrary forecasting horizon). This result is useful when the sample period of expectation data available to researchers is different (or do not exactly overlap) across different variables.

Section 2.3.3 provides empirical evidence on the formation of stock price forecasts by applying all these tests to data on forecasts of stock prices and consumption. Perhaps surprisingly, all testable implications (i.e. cointegration restrictions) are also present in various learning and sentiment-based models (as well as models with heterogeneous beliefs), as is shown later.

2.2.3. Cointegration among forecasts of the same variable

The following theorem shows that the forecasts of the same I(1) variable made at the same date i over two arbitrary and different horizons $j \neq l$ are cointegrated with cointegrating vector $(1\ \ -1)$. This means, for instance, 1-year ahead and 10-year ahead forecasts of stock prices made at the same date are cointegrated. In addition, the forecasts of the same variable made at two different dates (i.e., $i \neq k$) over two arbitrary horizons ($i.e.$, j and l) are cointegrated with cointegrating vector $(1\ \ -1)$.

Theorem 4. *If X_t follows (2.1)-(2.5) (i.e., $X_t \sim I(1)$), $E_i X_{i+j} - E_k X_{k+l} \sim I(0)$ for (a) $i = k$, $j \neq l$ or (b) $i \neq k$, $j > 0$, $l > 0$.*[7]

Proof. First, consider case (a) when $i = k$ and $j \neq l$. Let $E_i X_{i+j} - E_i X_{i+l} = (\mu j + X_i^P + E_i X_{i+j}^C) - \left(\mu l + X_i^P + E_i X_{i+l}^C\right) = \mu (j - l) + \left(E_i X_{i+j}^C - E_i X_{i+l}^C\right)$. $(E_i X_{i+j} - E_i X_{i+l})$ is stationary because $(E_i X_{i+j}^C - E_i X_{i+l}^C)$ is stationary. Turning to case (b) when $i \neq k$. Let $E_i X_{i+j} - E_k X_{k+l} = (E_i X_{i+j} - X_i) - (E_k X_{k+l} - X_k) + (X_i - X_k)$. Lemma 2 yields that $(E_i X_{i+j} - X_i)$ and $(E_k X_{k+l} - X_k)$ are stationary. Moreover, given $X_t \sim I(1)$, $(X_i - X_k)$ is stationary. Thus, $(E_i X_{i+j} - E_k X_{k+l})$ is stationary. □

[7] Note if $i = k$ and $j = l$, $E_i X_{i+j}$ and $E_k X_{k+l}$ are identical to each other.

2.2.4. Tests using average forecasts over many periods

Economic surveys often ask participants their forecast of the average value of economic variables X_t over the next m periods, for instance, the average unemployment rate over the next five years. This section provides testable implications for average forecasts when $X_t \sim I(1)$. The tests are useful when researchers have data on average expectations over a number of periods.[8]

Define the average forecast $\overline{X}_t^m = \frac{1}{m}\sum_{i=1}^{m} E_t X_{t+i}$ where the average is calculated over a number of time periods (rather than across different survey participants). Part (1) of the following Lemma shows that the average forecasts $\overline{X}_t^m$ contain a unit root. Part (2) shows that the average forecasts of X over the next m periods made at an arbitrary date h ($\overline{X}_t^m$) are cointegrated with conditional forecasts of X over horizon l made at an arbitrary date j ($E_j X_{j+l}$) with cointegrating vector $(1, -1)$. Part (3) shows that $\overline{X}_t^m - X_j \sim I(0)$.

Lemma 5. *If X_t follows (2.1)-(2.5) (i.e., $X_t \sim I(1)$), then (1) $\overline{X}_t^m \sim I(1)$ for $m > 0$; (2) $\overline{X}_h^m - E_j X_{j+l} \sim I(0)$ for arbitrary h, j, m and $l > 0$; (3) $\overline{X}_t^m - X_j \sim I(0)$ for arbitrary t, j, $m > 0$.*

Proof. (1) Let $\overline{X}_t^m - \overline{X}_{t-1}^m = \frac{1}{m}\sum_{i=1}^{m}(E_t X_{t+i} - E_{t-1}X_{t-1+i})$. Lemma 1 implies that $E_t X_{t+i} \sim I(1)$ and hence $(E_t X_{t+i} - E_{t-1}X_{t-1+i})$ is stationary. Thus, $\left(\overline{X}_t^m - \overline{X}_{t-1}^m\right)$ is stationary.

[8]Note they are not used in empirical testing of the paper as we do not have average expectations data in the current context.

(2) For arbitrary h, j, m and l, let $\overline{X}_h^m - E_j X_{j+l} = \frac{1}{m}\sum_{i=1}^{m} E_h X_{h+i} - E_j X_{j+l} = \frac{1}{m}\sum_{i=1}^{m}(E_h X_{h+i} - E_j X_{j+l})$. Theorem 4 shows that $(E_h X_{h+i} - E_j X_{j+l})$ is stationary. Thus, $\overline{X}_h^m - E_j X_{j+l}$ is stationary.

(3) Let $\overline{X}_t^m - X_j = \frac{1}{m}\sum_{i=1}^{m} E_t X_{t+i} - X_j = \frac{1}{m}\sum_{i=1}^{m}(E_t X_{t+i} - X_j)$. Lemma 2 implies that $(E_t X_{t+i} - X_j)$ is stationary. Thus, we have $\left(\overline{X}_t^m - X_j\right)$ is stationary. □

Let $y_t = (y_{1,t}\ y_{2,t}\ ...y_{n,t})'$ be a $1 \times n$ vector which is cointegrated with cointegrating vector $a = (a_1\ a_2\ ...\ a_n)'$ and denote $Z_{k,i_k}^{j_k} = E_{i_k} y_{k,i_k+j_k}$ or $\overline{y}_{k,i_k}^{j_k}$ where $\overline{y}_{k,i_k}^{j_k} = \frac{1}{j_k}\sum_{l=1}^{j_k} E_{i_k} y_{k,i_k+l}$.

Theorem 6. *$a_1 Z_{1,i_1}^{j_1} + a_2 Z_{2,i_2}^{j_2} + ... + a_n Z_{n,i_n}^{j_n}$ is stationary for arbitrary $i_1, i_2, ...i_n$, $j_1,\ j_2,\ ...,\ j_n > 0$.*

Proof. Let $a_1 Z_{1,i_1}^{j_1} + a_2 Z_{2,i_2}^{j_2} + ... + a_n Z_{n,i_n}^{j_n} = \left[\sum_{k=1}^{n} a_k \left(Z_{k,i_k}^{j_k} - y_{k,i_k}\right) + \sum_{k=1}^{n} a_k y_{k,i_k}\right]$. It is stationary for two reasons. First, Lemma 2 and part (3) of Lemma 5 imply that $\left(Z_{k,i_k}^{j_k} - y_{k,i_k}\right)$ is stationary for $Z_{k,i_k}^{j_k} = E_{i_k} y_{k,i_k+j_k}$ or $\overline{y}_{k,i_k}^{j_k}$. Second, $\sum_{k=1}^{n} a_k y_{k,i_k}$ is stationary as is shown in the proof of Theorem 3. □

Theorem 6 shows that if a vector of variables are cointegrated, a linear combination of the average and conditional forecasts of these variables (with the same cointegrating vector) is stationary. Note a special case is when the forecast of all variables are made at the same date $(i_1 = i_2 = ... = i_n)$.

2.3. New evidence on the formation of stock price expectations

Using the tests developed from the previous section, this section presents new evidence on the formation of stock price expectations. A central piece of evidence from survey expectations data is that forecasts of aggregate stock price index are not cointegrated with forecasts of aggregate consumption, as opposed to RE asset pricing models considered in the next section (e.g., the long-run risks model in Bansal, Kiku and Yaron, 2012 or the habit model of Campbell and Cochrane, 1999).[9] Put differently, the long-run or trend component of stock price forecasts is *not* anchored by consumption forecasts.

This evidence is robust to different sources of expectations data (Livingston Survey and Shiller Survey for stock price forecasts and Survey of Professional Forecasters and Greenbook forecasts for consumption forecasts), different forecasting horizons (1-, 2-, 4-quarter and 10-year ahead forecasts), and using stock price forecasts data which is made at the same or different dates from consumption forecasts. Moreover, the evidence is robust to different statistical tests (Phillips Perron test, ADF-GLS test, KPSS test with correcting potential small sample problems and Johansen test) and using the median or mean forecasts for testing. In subsequent sections, we show several classes of Bayesian RE asset pricing models with incomplete information or heterogeneous beliefs appear inconsistent with this evidence too.

[9]Other evidence includes that forecasts of stock prices (or consumption) over different forecasting horizons are cointegrated with each other, consistent with all asset pricing models considered in the paper.

2.3.1. Data

Two sources of survey forecasts of US stock prices are used. One is the Livingston Survey managed by the Federal Reserve Bank of Philadelphia. The survey contains forecasts of S&P 500 index made by professional economists from industry, government, banking, and academia. The stock price forecast data is semi-annual and covers from 1952 to the second half of 2017.[10] Two forecasting horizons are available: 2- and 4-quarter ahead. The other source is Robert Shiller's survey of individual investors. This forecast of stock prices is measured by forecasts of the Dow Jones index and available at a quarterly frequency. The data covers from the first quarter of 1999 to the second quarter of 2015. Four forecasting horizons are available: 1-quarter, 2-quarter, 4-quarter and 10-year ahead. Both survey forecasts of stock prices are deflated by forecasts of inflation rate obtained from the Survey of Professional Forecasters (SPF) conducted by the Philadelphia Fed. The forecasting horizons of inflation forecast data are 1- to 4- quarter ahead as well as 10-year ahead.[11]

Two sources of US aggregate consumption forecasts are used. One is SPF forecasts of the chain-weighted real personal consumption expenditures. It is available at a quarterly frequency and from 1981 Q3 onwards. SPF consumption forecasts data is provided with varying base years. Appendix A explains the rebasing of

[10]In all cases (with one exception, i.e., Table 4), we use the data from 1981 onwards which corresponds to the longest sample of consumption forecasts in the Survey of Professional Forecasters.

[11]For robustness analysis, we also deflate 1-year ahead stock price forecasts using 1-year ahead inflation forecasts using the Michigan Survey of Consumers. Our results are robust to this alternative measure of inflation expectation.

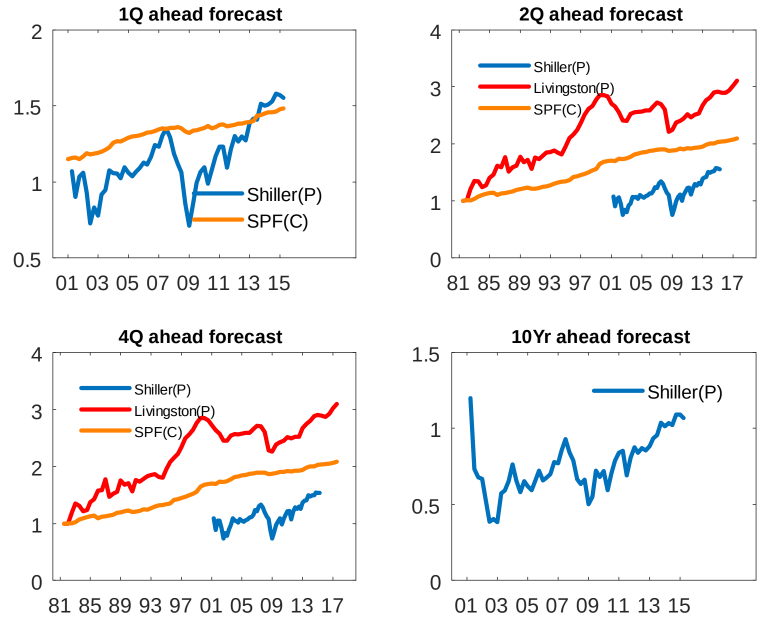

Figure 2.1. Median Forecasts of (log) Stock Price and Consumption

consumption forecast data. As an alternative, consumption forecasts from the US Federal Reserve Board's Greenbook datasets are employed. In the text, we report testing results using SPF consumption forecasts. Most results reported in the text use median survey forecasts. Appendix B (Appendix C) shows our results are robust to using the mean forecasts (Greenbook consumption forecasts). Figure 2.1 plots the (normalized) median forecasts of (log) stock prices and rebased aggregate consumption for all available forecasting horizons.

Before proceeding to the test results, we discuss several issues. First, the paper employs several widely used datasets about expectations data (e.g., SPF, Shiller and Livingston Survey) and follows common practices in the literature to interpret the data. Specifically, the median (or mean) forecasts from these expectations surveys is interpreted as a proxy for the forecasts of the representative agent in models and hence these surveys are interpreted as (reasonably) representative; see e.g., Piazzesi, Salomao and Schneider (2015), Adam, Marcet and Beutel (2017), Coibion and Gorodnichenko (2015), Eusepi and Preston (2011) and Kuang and Mitra (2016). With this view, the median (or mean) stock price forecasts from e.g., the Shiller Survey is a proxy for stock price expectations of the representative agent. And the median (or mean) consumption forecasts from the Survey of Professional Forecasters (SPF) is a proxy for consumption forecasts of the representative agent. And to test heterogeneous agents asset pricing models, we use the mean survey forecasts as a proxy for the average forecasts across different agents in the models.

We use the median (or mean) stock price forecasts from one survey and the median (or mean) consumption forecasts from another survey to test the cointegration between the forecasts of the two variables made by the representative agent in asset pricing models, despite that the set of forecasters are not the same in different surveys.[12] This is also a quite common practice and in line with the literature. For instance, Adam, Marcet and Beutel (2017) uses the median (or mean) nominal stock price forecasts from the Shiller Survey and inflation forecasts from SPF and Michigan Survey of Consumers to compute the median (or mean) forecasts of

[12]Carroll (2003) develops a model of expectation formation in which households derive their expectations from professional forecasters and provides evidence supporting this model using inflation expectations data. If this model is generally true (for other variables), forecasts of a variable (e.g., stock prices or consumption) made by professional forecasters will be cointegrated with forecasts of the same variable made by households. This may also help to justify our use of different surveys where forecasts of stock prices and consumption are not made by the same set of forecasters.

real stock capital gains which are then used as a proxy for forecasts of real stock capital gains made by the representative agent in their model. Another example is Malmendier and Nagel (2011) which documents heterogeneity in inflation expectations using the Michigan Survey of Consumers and then studies empirically the implications of this heterogeneity for borrowing and lending decisions using data from the Survey of Consumer Finances (SCF); the forecasters from the Michigan survey are different from the decision makers from the SCF.

Second, survey data on expected stock returns are often criticized as being noisy and thus meaningless, or that people do not mean what they say, or that survey responses are strongly dependent on framing and language. Greenwood and Shleifer (2014) discusses and addresses these criticisms; see their Section 1.8. They show stock return forecasts from different surveys are highly correlated and provide evidence that investors act in line with their reported expectations. Giglio, Maggiori, Stroebel and Utkus (2019) also provides evidence addressing these criticisms and strongly supports the use of survey expectations data in macro-finance models. Moreover, we think, in our context, as long as the noises or measurement errors in survey forecast data are $i.i.d$ or stationary (which is commonly assumed in the literature), they do not affect the integration and cointegration properties of the forecast data as well as our empirical findings.

Third, the power of the standard Dickey-Fuller class of unit root tests was frequently criticized in the 1980s and 1990s, e.g., Cochrane (1991, 1994). Subsequent work has made great advances in improving the power of the tests. Ng and Perron

(2001) and Haldrup and Jansson (2006) argue some subsequently developed tests have much improved or excellent power. The paper uses some of the most powerful tests like the DF-GLS tests. Also, the earlier criticisms were concerned about the power of testing the null hypothesis of a non-stationary process against the alternative of a stationary process. Another way to address this issue is applying the KPSS test which tests the null hypothesis of a stationary process against the alternative of a unit root. Our conclusion still holds with this test even after we take account of potential small sample problems using Monte Carlo simulation; see Appendix D. For instance, we simulate the distribution of the KPSS test statistic under the null hypothesis that the long-run risks model of Bansal, Kiku, and Yaron (2012) is true (and hence forecasts of $log(P/C)$ over different horizons are stationary). Survey forecasts data rejects this null hypothesis using the critical values obtained from the Monte Carlo study. Finally, the Johansen test is also applied without imposing the $(1, -1)$ cointegration restriction between stock price forecasts and consumption forecasts; we find no cointegration between stock price forecasts and consumption forecasts using the longer Livingston survey data.[13]

2.3.2. Integration properties of the forecasts

[13]The size of the sample in our testing may not be too small, e.g., about 90 for the Livingston survey forecasts data. One way to mitigate this issue is using forecasts for different horizons which provide some independent information. Our Monte Carlo study addresses this issue too. Also, note the tests used in the paper like the PP test, DF-GLS test and the Monte-Carlo simulation using KPSS test are suitable for addressing potential heteroskedasticity in the testing.

	1Q ahead	2Q ahead	4Q ahead	10-yr ahead
Panel A: I(1) test				
Shiller (PP Z_t stat.)	−2.231	−2.183	−2.242	−1.997
10% critical value	−3.172	−3.172	−3.172	−3.172
Shiller (DF-GLS)	−2.236	−2.150	−2.183	−1.317
10% critical value	−2.851	−2.851	−2.851	−2.851
Livingston (PP Z_t stat.)	*n.a.*	−2.195	−2.118	*n.a.*
10% critical value	*n.a.*	−3.167	−3.167	*n.a.*
Livingston (DF-GLS)	*n.a.*	−1.714	−1.756	*n.a.*
10% critical value	*n.a.*	−2.818	−2.818	*n.a.*
Panel B: I(2) test				
Shiller (PP Z_t stat.)	−7.618	−7.596	−8.012	−9.351
1% critical value	−2.615	−2.615	−2.615	−2.615
Shiller (DF-GLS)	−3.607	−3.505	−3.311	−1.518
1% critical value	−2.615	−2.615	−2.615	−2.615
Livingston (PP Z_t stat.)	*n.a.*	−7.206	−6.950	*n.a.*
1% critical value	*n.a.*	−2.612	−2.612	*n.a.*
Livingston (DF-GLS)	*n.a.*	−4.528	−3.084	*n.a.*
1% critical value	*n.a.*	−2.611	−2.611	*n.a.*

Table 2.1. Integration Properties: Forecasts of $logP$

This section studies the integration properties of forecasts of the aggregate stock price index and aggregate consumption. Table 1 reports the test statistics and critical value of the Phillips-Perron (PP) test (see Phillips and Perron (1988)) and the Augmented Dickey-Fuller Generalized Least Squares (DF-GLS) test for median forecasts of stock prices. Panel A shows that for both surveys and all forecast horizons, both tests cannot reject that stock price forecasts are I(1) at a 10% significance level.[14] Panel B shows that for both surveys and all forecasting

[14]DF-GLS test gives all test statistics for a series of models that include 1 to k lags of the first differenced, detrended variable, where k is set by default. We report the statistics produced with the number of lags leading to the lowest mean squared errors. And the results are robust to alternative lags.

	1Q ahead	2Q ahead	4Q ahead
Panel A: I(1) test			
PP (Z_t stat.)	−1.324	−1.327	−1.340
10% critical value	−3.167	−3.167	−3.167
DF-GLS	−1.223	−1.227	−1.238
10% critical value	−2.818	−2.818	−2.818
Panel B: I(2) test			
PP (Z_t stat.)	−4.725	−4.805	−4.837
1% critical value	−2.612	−2.612	−2.612
DF-GLS	−3.809	−4.082	−3.862
1% critical value	−2.611	−2.611	−2.611

Table 2.2. Integration Properties: Forecasts of $logC$ (SPF)

horizons (with one exception), stock price forecasts are not integrated of order 2, i.e., I(2).[15] Table 2.2 reports the test statistic value and critical value of the unit root tests for aggregate consumption forecasts. Similarly, for all forecasting horizons, both tests suggest that consumption forecasts are an I(1) but not I(2) process. Lemma 1 suggests that forecasts of stock prices and consumption from full-information RE stock pricing models (e.g., the long-run risks model) are I(1) but not I(2) processes, consistent with the evidence here.

[15]The only exception is the DF-GLS test cannot reject that 10-year ahead median forecast of stock prices from the Shiller Survey follows an I(2) process. Yet we show that it is rejected using the mean forecast at 1% significance level, see column 4 of Table A1 in Appendix B.

2.3.3. No cointegration between stock price forecasts and consumption forecasts

Using data on forecasts of stock price and consumption, this section examines the three sets of testable implications discussed after Theorem 3.

2.3.3.1. No cointegration between stock price forecasts and consumption forecasts made at the same dates and over the same or different horizons. Figure 2.2 displays the difference between the median forecasts of $logP$ and $logC$ made at the same date using both stock price surveys. "1Q ahead forecast" corresponds to (normalized) 1-year ahead stock price forecasts minus 1-year ahead SPF consumption forecasts; similarly for 2Q and 4Q ahead forecast. The exception is "10 Yr ahead forecast" which corresponds to 10-year ahead stock price forecasts minus 1-year ahead consumption forecasts, given the unavailability of 10-year ahead consumption forecasts in the SPF.

Recall Theorem 3 implies that in RE asset pricing models, stock prices forecasts and consumption forecasts made at the same dates (and over possibly different horizons) are cointegrated with cointegrating vector $(1, -1)$. These models imply, for instance, 1-quarter ahead forecasts of stock prices are cointegrated with 1-quarter ahead forecast of aggregate consumption and 10-year ahead forecast of stock prices are cointegrated with 1-year ahead forecast of consumption.

Note: the bottom right panel plots the (normalized) difference between 10-year ahead stock price forecasts and 1-year ahead consumption forecasts.

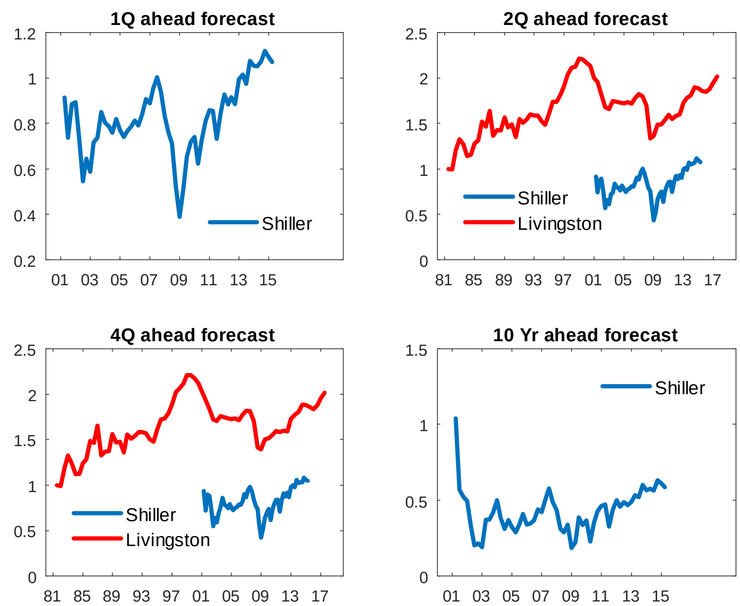

Figure 2.2. Median Forecast of (log) Price Consumption ratio

Table 3 reports the test results of whether median forecasts of aggregate consumption are cointegrated with median forecasts of stock prices made at the same date and over the same forecasting horizon (with cointegrating vector $(1, -1)$);[16] the only exception is that the column "10-yr ahead" is the test results on the cointegration between 10-year ahead forecasts of stock prices and 1-year ahead forecasts of consumption, given the unavailability of 10-year ahead consumption forecasts data. Both PP and DF-GLS tests show that we cannot reject the null hypothesis that stock price forecasts are not cointegrated with consumption forecasts with cointegrating vector (1, −1), robust to different data sources and forecasting

[16]For DF-GLS test, we report the test statistics produced with the number of lags leading to the lowest mean squared errors. The results are robust to different choices of lags.

horizons. The same conclusion is reached with mean forecasts, see Table A3 of the Appendix. We also applied the KPSS test which tests the null hypothesis of a stationary process against the alternative of a unit root. The tests yield the same conclusion that forecasts of stock price to consumption ratio is non-stationary, robust to different sources of forecasts data, different forecasting horizons, correcting potential small sample problems, etc (see Appendix 2.8.4).

Using the Johansen test, we test if stock price forecasts are cointegrated with consumption forecasts without imposing the restriction of the cointegrating vector $(1, -1)$. With the Livingston Survey data, we find that the null hypothesis that stock price forecasts are not cointegrated with consumption forecasts for any forecasting horizons cannot be rejected at the conventional significance level (e.g., 5%). Using the shorter Shiller Survey data, we reject the null hypothesis that stock price forecasts are not cointegrated with consumption forecasts at 5% significance level. The estimated cointegrating vector is $(1, -2.55)$ for 1-year ahead forecasts of stock prices and consumption; similar cointegrating vectors are found for other forecasting horizons. This also rejects the formation of stock price expectations in the asset pricing models studied in the paper which has $(1, -1)$ as the cointegration vector.

*Note: for the first three columns, forecasts of stock prices and consumption are made at the same date and over the same forecasting horizons. For the fourth column, the forecasts of stock prices and consumption are made at the same dates but different

	1Q ahead	2Q ahead	4Q ahead	10-yr ahead*
I(1) test				
Shiller (PP Z_t stat.)	−2.039	−1.962	−2.097	−2.304
10% critical value	−2.595	−2.595	−2.595	−2.595
Shiller (DF-GLS)	−1.665	−1.569	−1.602	−0.814
10% critical value	−1.929	−1.929	−1.929	−1.929
Livingston (PP Z_t stat.)	*n.a.*	−2.234	−2.161	*n.a.*
10% critical value	*n.a.*	−2.591	−2.591	*n.a.*
Livingston (DF-GLS)	*n.a.*	−0.246	−0.227	*n.a.*
10% critical value	*n.a.*	−1.895	−1.895	*n.a.*

Table 2.3. No Cointegration between Forecasts of $logP$ and $logC$

forecasting horizons (i.e.,10-year ahead stock price forecasts vs 1-year ahead consumption forecasts).

2.3.3.2. No cointegration between stock price forecasts and consumption forecasts made at different dates. Theorem 3 suggests stock price forecasts and consumption forecasts are cointegrated with cointegrating vector $(1, -1)$ even if stock price forecasts are *made at different dates* from consumption forecasts. Denote by $E_{i_1} \log P_{i_1+j_1}$ stock price forecasts made at period i_1 over horizon j_1 and similarly, $E_{i_2} \log C_{i_2+j_2}$ consumption forecasts made at period i_2 over horizon j_2. A period means a quarter here. For illustration, we conduct two exercises. In the first exercise, we use 1-year ahead Livingston median stock price forecasts ($j_1 = 4$) where $i_1 =$ 1978Q2, 1978Q4, 1979Q2, ..., 2014Q2. And we take 1-year ahead SPF median consumption forecasts ($j_2 = 4$) where $i_2 =$ 1981Q4, 1982Q2, 1982Q4, ..., 2017Q4.[17] Note that stock price forecasts and consumption forecasts are made at different dates, i.e., the set of i_1 is different from the set of i_2. Yet the sample size of stock price forecasts and consumption forecasts are the same. In the second exercise, we use 1-year ahead Livingston median stock price forecasts ($j_1 = 4$) where i_1 = 1978Q2, 1978Q4, 1979Q2, ...2017Q4 and 1-year ahead SPF median consumption forecasts ($j_2 = 4$) where i_2 = 1998Q2, 1998Q3, 1998Q4, ..., 2018Q1. Using PP and DF-GLS tests, Panel A (or B) of Table 2.4 reports the testing results of the first (or second) exercise. Both panels suggest no cointegration between stock price forecasts and consumption forecasts made at different dates.[18]

The long-run or trend component of stock price forecasts made by agents in reality is not anchored by consumption forecasts. The survey evidence rejects this aspect of the formation of stock price expectations in full-information RE asset pricing models and in various learning or sentiment-based models and models with heterogeneous beliefs, as is shown later.

[17] Note the Livingston survey is a semi-annual survey and conducted close to the end of June and December each year. Since SPF is a quarterly survey, some data is discarded for the test, i.e., those made in the first and third quarter of each year.

[18] Of course there are many possible ways of utilizing the survey expectations data. One may choose different sets of i_1 and i_2 from our choice here as long as the two forecasts have the same sample size. We find other choices also lead to the same conclusion but it may be impractical and unnecessary to report all results here.

Panel A: i_1 = 78Q2, 78Q4, ..., 14Q2 and $j_1 = 4$; i_2 = 81Q4, 82Q2, ..., 17Q4 and $j_2 = 4$		
	Median forecasts	Mean forecasts
I(1) test		
PP (Z_t statistics)	-1.628	-1.603
10% critical value	-2.591	-2.591
DF-GLS	-1.080	-0.964
10% critical value	-1.895	-1.895
Panel B: i_1 = 78Q2, 78Q4, ..., 17Q4 and $j_1 = 4$; i_2 = 98Q2, 98Q3, ..., 18Q1 and $j_2 = 4$		
	Median forecasts	Mean forecasts
PP (Z_t stat.)	-1.501	-0.680
10% critical value	-2.588	-2.588
DF-GLS	-0.167	-0.137
10% critical value	-1.871	-1.825

Table 2.4. No cointegration between $E_{i_1} logP_{i_1+j_1}$ & $E_{i_2} logC_{i_2+j_2}$

2.3.4. Cointegration among stock price (or consumption) forecasts

In RE models (e.g., the long-run risks model), realized stock prices (or consumption) contains a unit root. Theorem 4 implies that forecasts of stock prices (or consumption) over two different horizons should be cointegrated with cointegrating vector $(1, -1)$. This section shows that this aspect of expectation formation in these models is broadly consistent with survey forecasts of stock prices and consumption.

Table 2.5 reports the p-value of the PP test on whether forecasts of stock prices (or aggregate consumption) over two different horizons are cointegrated with each other with cointegrating vector $(1, -1)$. Using the median and mean forecasts data, PP test shows that at 1% significance level, we can reject the null hypothesis that

PP test	1 & 2Q	1 & 4Q	2 & 4Q
Stock prices (Shiller median)	0.0019	0.0005	0.0047
Stock prices (Shiller mean)	0.0019	0.0005	0.0047
Consumption (SPF median)	0.0004	0.0228	0.0011
Consumption (SPF mean)	0.0006	0.0035	0.0000

Table 2.5. P-value of Testing the Stationarity of $E_t \log(X_{t+i}) - E_t \log(X_{t+j})$

the difference between the forecasts of stock prices, i.e., $E_t \log(p_{t+i}) - E_t \log(p_{t+j})$, contains a unit root, for various pairs of forecasting horizons $(i,\ j) = (1,2)$, $(i,\ j) = (1,4)$ and $(i,\ j) = (2,4)$.[19] Similarly, forecasts of consumption over two different horizons are cointegrated with cointegrating vector $(1,-1)$.

Note: X stands for consumption or stock price

What does the survey evidence, especially the no cointegration between stock price forecasts and consumption forecasts, tell us about modeling expectation formation in asset pricing models? In the next two sections, we study if the new survey evidence can be reconciled in models with rational expectations, or non-rational (or extrapolative) consumption forecasts, or deviation from the representative agent assumption, or incorporating financial and labor market frictions, or structural break.

[19] DF-GLS test rejects the null hypothesis that aggregate price index forecasts across different horizons are not cointegrated at 1 percent level. It also accepts the cointegration between 1 quarter ahead and 2 quarter ahead consumption forecast. However, it fails to reject that 1- (or 2-) quarter ahead consumption forecast is not cointegrated with 4 quarter ahead consumption forecast.

2.4. RE asset pricing models

This section demonstrates that stock prices and aggregate consumption are cointegrated in several major RE asset pricing models. The tests in Section 2.2 can be applied. Particularly, Theorem 3 implies that stock price forecasts are cointegrated with aggregate consumption forecasts in these models. Thus, the survey evidence in Section 2.3.3 appears incompatible with these models.

2.4.1. The long-run risks model

Consider the long-run risks model studied in Bansal, Kiku and Yaron (2012). The representative agent with recursive preferences maximizes his life-time utility given by

$$V_t = [(1-\delta)C_t^{\frac{1-\gamma}{\theta}} + \delta(E_t[V_{t+1}^{1-\gamma}])^{\frac{1}{\theta}}]^{\frac{\theta}{1-\gamma}}. \tag{2.6}$$

The variable θ is defined as $\theta \equiv \frac{1-\gamma}{1-1/\psi}$ where the parameters γ and ψ represent relative risk aversion and the elasticity of intertemporal substitution. Log consumption c_t and dividend d_t have the following joint dynamics

$$\Delta c_{t+1} = \mu_c + x_t + \sigma_t \eta_{t+1}, \tag{2.7}$$

$$x_{t+1} = \rho x_t + \varphi_e \sigma_t e_{t+1}, \tag{2.8}$$

$$\sigma_{t+1}^2 = \overline{\sigma}^2 + \nu(\sigma_t^2 - \overline{\sigma}^2) + \sigma_w w_{t+1}, \tag{2.9}$$

$$\Delta d_{t+1} = \mu_d + \phi x_t + \pi \sigma_t \eta_{t+1} + \varphi \sigma_t u_{d,t+1}. \tag{2.10}$$

I(1) test (Long-run risks)		*I(1) test (Habit)*	
PP (Z_t statistics)	-5.962	PP (Z_t statistics)	-26.752
1% critical value	-3.455	1% critical value	-3.430
DF-GLS	-3.737	DF-GLS	-3.560
1% critical value	-3.480	1% critical value	-2.580

Table 2.6. Stationarity of log price consumption ratio

$\mu_c + x_t$ is the conditional expectation of the growth rate of aggregate consumption. x_t is a persistent component which captures long run risks in consumption and drives both the consumption and dividend process. ϕ captures a levered exposure of dividend to x_t. In addition, the i.i.d consumption shock η_{t+1} is allowed to influence the dividend process. It serves as an additional source of risk premia and π governs the magnitude of this influence.

Their paper provides the analytical solution for (log) price-consumption ratio

$$\log(\frac{P_t}{C_t}) = A_0 + A_1 x_t + A_2 \sigma_t^2, \tag{2.11}$$

where A_0, A_1, A_2 are all constants and functions of model parameters, see their p. 189. Stock prices and aggregate consumption are cointegrated as the right-hand side of equation (2.11) is stationary. The following proposition summarizes the result.

Proposition 7. *In the model of Bansal, Kiku and Yaron (2012), stock prices and aggregate consumption are cointegrated with cointegrating vector* $(1, -1)$ *and realized (log) stock price consumption ratio is a stationary process.*

We also simulate the long-run risks model for 948 periods (months) as in Bansal, Kiku and Yaron (2012) to confirm the stationarity. The left panel of Table 2.6 shows the unit root testing results by applying the PP test and the DF-GLS test to (log) price consumption ratio. Both test statistics are smaller than the corresponding 1% critical value, suggesting that realized stock price consumption ratios pass the unit root tests.

2.4.2. The habit model

Consider the habit model of Campbell and Cochrane (1999). Readers are referred to their paper on the details. Aggregate consumption contains a unit root. There is no analytical solution for the habit model. We simulate the habit formation model for 120, 000 months (as in Campbell and Cochrane (1999)) and then test the cointegration between realized quarterly (log) stock prices and (log) aggregate consumption. Using the PP and DF-GLS test, the right panel of Table 6 shows that realized price consumption ratios pass both tests as the null hypothesis is rejected.[20]

Here we interpret stocks as a claim to the consumption stream. This interpretation is present in Campbell and Cochrane (1999) as it is common in the equity premium literature; see Section D, p. 216 as well as their Table 2. The price of

[20]Figure 3 of Campbell and Cochrane (1999) shows that $\log(P_t) - \log(C_t)$ is approximately linear in the stationary state variable, i.e., consumption surplus ratio s_t. This also suggests the stationarity of $\log(P_t) - \log(C_t)$.

a claim to the consumption stream is cointegrated with aggregate consumption. One may prefer to interpret stocks as a claim to the dividend stream. In some asset pricing models, aggregate consumption is assumed not to be cointegrated with aggregate dividends. Thus, in these models, the price of a claim to the dividend stream is not necessarily cointegrated with aggregate consumption.

However, the assumption of no cointegration between consumption and dividends is not made based on empirical evidence but typically for the purpose of simplifying analysis. In the basic habit formation model of Campbell and Cocharane (1999), aggregate consumption and aggregate dividends are not cointegrated. They, nevertheless, say "*It would be better to make dividends and consumption cointegrated. We have explored a model in which the log dividend/consumption ratio is i.i.d. and the correlation of one-period dividend and consumption growth rates is low as in the data... A cointegrated model with a persistent log dividend/consumption ratio would be more realistic, but this modification would require an additional state variable*" (p. 217). Using the postwar US data, we indeed find that aggregate consumption and aggregate dividends are cointegrated. In stock pricing models that assume aggregate consumption is cointegrated with aggregate dividends (e.g., Adam, Marcet and Beutel (2017)), the price of a claim to the dividend stream will also be cointegrated with consumption. With this view, we tend to think that allowing no cointegration between consumption and dividends in models may not be a promising avenue to reconcile the new survey evidence.[21]

2.5. Bayesian RE Models

The Bayesian RE literature assumes that agents find it difficult to discover the process for fundamentals (agents hold subjective consumption beliefs), but understand perfectly how the equity price is mapped from the history of observed fundamentals. Their forecast about future price is pinned down by their forecast of the stream of future payments. No other beliefs about price consistent with optimal conditions exist.

Can models with incorporating non-rational (or extrapolative) consumption forecasts reconcile our survey evidence in Section 2.3.3? We firstly show consumption growth forecasts do not pass the conventional rationality test. This preliminary testing result lends some supports to asset pricing models with non-rational (or extrapolative) consumption forecasts. Yet we show that the evidence in Section 2.3.3 cannot be reconciled by these models which *deviate from the assumption of full information* and assume agents having the knowledge of the equilibrium pricing function.

[21]In production-based models, both stock prices and aggregate consumption are endogenous. They typically share a common trend with a productivity process and cointegrate with each other, appearing inconsistent with our survey evidence too.

2.5.1. RE tests of consumption growth forecasts

Using data covering 1981Q1 to 2017Q4, actual consumption growth rates (g_{ct}) is regressed on the 1-quarter ahead SPF median consumption growth forecast (g_{ct}^{e}) . We obtain

$$g_{ct} = \underset{(0.565)}{-0.423} + \underset{(0.214)}{1.340} g_{ct}^{e}, \qquad R^2 = 0.294, \tag{2.12}$$

where the numbers in the parenthesis are standard deviations. An F-test of the joint hypothesis that the intercept of (2.12) equals 0 and slope coefficient equals 1 yield a highly significant F-statistics of 8.36. The results reject that 1-quarter ahead SPF consumption growth forecasts are (full-information) rational forecasts. The same conclusion is reached by using a different forecasting horizon (i.e., 2-quarter ahead) or using the mean price forecasts. However, as illustrated below, incorporating non-rational (or extrapolative) consumption growth forecasts alone cannot reconcile equity pricing models with the new evidence we documented. In what follows, we consider two types of incomplete information models in both of which agents possess the knowledge of the equilibrium pricing function.

2.5.2. Sentiment-based models

Some papers introduce sentiment into asset pricing models, such as Jin and Sui (2018) and the exchange rate model of Yu (2013). As an example, suppose the representative agent's preferences are represented by the Epstein-Zin utility (2.6) and the actual exogenous driving processes are

$$\Delta c_{t+1} = \mu_c + \sigma_t \eta_{t+1}, \tag{2.13}$$

$$\sigma_{t+1}^2 = \overline{\sigma}^2 + \nu(\sigma_t^2 - \overline{\sigma}^2) + \sigma_w w_{t+1}, \tag{2.14}$$

$$\Delta d_{t+1} = \mu_d + \pi \sigma_t \eta_{t+1} + \varphi \sigma_t u_{d,t+1}. \tag{2.15}$$

Comparing (2.13) - (2.15) with (2.7) - (2.10) in the long-run risks model of Bansal, Kiku and Yaron (2012), we drop the persistent component x_t in the actual exogenous driving processes (because sentiment plays the role of the persistent component x_t).[22]

Now assume agents have a misperception about the exogenous consumption and dividend process. They perceive consumption and dividend processes as

$$\Delta c_{t+1} = \mu_c + a_t + \sigma_t \widehat{\eta}_{t+1}, \tag{2.16}$$

$$a_{t+1} = \rho a_t + \varphi_e \sigma_t e_{t+1}, \tag{2.17}$$

$$\sigma_{t+1}^2 = \overline{\sigma}^2 + \nu(\sigma_t^2 - \overline{\sigma}^2) + \sigma_w w_{t+1}, \tag{2.18}$$

$$\Delta d_{t+1} = \mu_d + \phi a_t + \pi \sigma_t \eta_{t+1} + \varphi \sigma_t u_{d,t+1}, \tag{2.19}$$

where $(\widehat{\eta}_{t+1}, e_{t+1})$ are i.i.d joint standard normal under agents' belief. a_t is an AR(1) process and does not appear in the true driving processes (called "sentiment").

[22]Our proposition below is not affected by adding x_t in the exogenous driving processes, as is argued later.

When a_t is positive (negative), agents are optimistic (pessimistic). Assuming $0 < \rho < 1$. If $\rho = 1$, $\log(P_t/C_t)$ will have unbounded volatility (see equation (2.20) later and discussion in section 8.1), which is rejected by the data. Thus, we require $\rho < 1$. Note in this type of models, despite agents' misperception of the exogenous driving process, they know the equilibrium pricing function. The following result provides testable implications for the formation of stock price expectations.

Proposition 8. *Given agents' beliefs (2.16) - (2.19), agents' stock price forecasts* $E_i \log P_{i+j}$ *are cointegrated with their forecasts of aggregate consumption* $E_k \log C_{k+l}$ *with cointegrating vector* $(1, -1)$ *for arbitrary* i, j, k, $l > 0$.

Proof. Following Bansal, Kiku and Yaron (2012), the (approximate) analytical solution for price consumption ratio can be derived as

$$\log(\frac{P_t}{C_t}) = A_0 + A_1 a_t + A_2 \sigma_t^2, \tag{2.20}$$

where A_0, A_1, A_2 remain the same constant as in Bansal, Kiku and Yaron (2012). The trend growth rate of both stock prices forecasts and consumption forecasts are identical to μ_c (noting sentiment is a stationary process). Thus, stock price forecasts and consumption forecasts can be expressed as $E_i \log P_{i+j} = \log P_i + j\mu_c + s(i,j)$ and $E_k \log C_{k+l} = \log C_k + l\mu_c + \widetilde{s}(k,l)$ where $s(i,j)$ and $\widetilde{s}(k,l)$ are stationary terms and omitted. Let $E_i \log P_{i+j} - E_k \log C_{k+l} = (\log P_t - \log C_k) + (j-l)\,\mu_c + s(i,j) - \widetilde{s}(k,l)$. It is stationary because $(\log P_t - \log C_k)$ is stationary.[23] $\square$

Stock price forecasts depend on current stock prices and beliefs about the growth rate of stock prices. Similarly, consumption forecasts depend on current consumption and beliefs about the growth rate of consumption. Stock price forecasts and consumption forecasts are cointegrated because (1) realized stock prices and consumption are cointegrated and (2) beliefs about consumption growth and stock price growth rates are mean-reverting. If beliefs about consumption growth rates are not mean-reverting ($\rho = 1$), price consumption ratio will have unbounded volatility given agents' knowledge of the equilibrium pricing function.

If the consumption driving process (2.13) - (2.15) contains a persistent and stationary component x_t as in Bansal, Kiku and Yaron (2012), (log) price consumption ratio will be a linear function of x_t, a_t and σ_t^2 with constant coefficients. The above proposition will still hold because realized price consumption ratios are stationary and the trend growth rate of both stock prices forecasts and consumption forecasts are identical to μ_c.

2.5.3. Learning about consumption dynamics

Many asset pricing models maintain the assumption of RE but assume agents have incomplete information and learn about the exogenous consumption process;

[23] $(\log P_t - \log C_k) = (\log P_t - \log C_t) + (\log C_t - \log C_k)$ is stationary because $(\log P_t - \log C_t)$ is stationary (see equation (2.20)) and $(\log C_t - \log C_k)$ is stationary (since $\log C_t$ is I(1)).

an example is Collin-Dufresne, Johannes and Lochstoer (2016). In this type of learning models, agents know the equilibrium pricing mapping. Suppose the representative agent's preferences are represented by the Epstein-Zin utility (2.6). For illustration, the consumption process is

$$\Delta c_{t+1} = \mu_c + \overline{\sigma}\eta_{t+1}, \tag{2.21}$$

where η_{t+1} is an i.i.d process. Agents do not know the consumption growth rate but know the constant variance $\overline{\sigma}^2$. Agents learn μ_c over time and beliefs about μ_c is updated by

$$\mu_{c,t} = \mu_{c,t-1} + g_t \left(\Delta c_t - \mu_{c,t-1}\right). \tag{2.22}$$

Assuming constant gain or Kalman filter learning (under steady state variance ratio) is used, i.e., $g_t = g \in (0,1)$. Substituting (2.21) into (2.22) yields $\mu_{c,t} = (1-g)\mu_{c,t-1} + g\left(\mu_c + \overline{\sigma}\eta_t\right)$ which is a stationary process.

Proposition 9. *Given the beliefs (2.21), agents' forecasts of stock prices* $E_i \log P_{i+j}$ *are cointegrated with their forecasts of aggregate consumption* $E_k \log C_{k+l}$ *with cointegrating vector* $(1,-1)$ *for arbitrary* i, j, k, $l > 0$.

Proof. The RE version of the model here is a special case of Bansal, Kiku and Yaron (2012) with setting $\rho = 0$, $\nu = 0$, $x_t = 0$, $\sigma_t = \overline{\sigma}$, $\sigma_w = 0, \varphi_e = 0$. From the RE version to the learning model, we replace the actual growth rate of consumption

μ_c by agents' beliefs about consumption growth rate $\mu_{c,t}$ in the analytical solution. The (analytical) solution for log price consumption ratio in the learning model is

$$\log(\frac{P_t}{C_t}) = \widetilde{A}_{0,t} + \widetilde{A}_2 \overline{\sigma}^2, \tag{2.23}$$

where $\widetilde{A}_{0,t} = \frac{1}{1-\kappa_{1,t}} \left(\log \delta + \kappa_{0,t} + \left(1 - \frac{1}{\psi}\right) \mu_{c,t} + \kappa_{1,t} \widetilde{A}_2 \overline{\sigma}^2 \right)$, $\widetilde{A}_2 = -\frac{(\gamma-1)\left(1-\frac{1}{\psi}\right)}{2}$, $\kappa_{0,t} = \log(1+\exp(\overline{z}_t)) - \kappa_{1,t}\overline{z}_t$, $\kappa_{1,t} = \frac{\exp(\overline{z}_t)}{1+\exp(\overline{z}_t)}$, $\overline{z}_t = \widetilde{A}_{0,t}\left(\overline{z}_t\right) + \widetilde{A}_2 \overline{\sigma}^2$. Note $\widetilde{A}_{0,t}$ is a non-linear function of $\mu_{c,t}$. Using Taylor expansion, the right hand side of the process (2.23) can be well approximated by a polynomial function of the AR(1) process $\mu_{c,t}$ and is again a stationary process.[24] Again, $(E_i \log P_{i+j} - E_k \log C_{k+l})$ is stationary because realized stock price and consumption are cointegrated and the trend growth rate of stock price forecasts and consumption forecasts are identical to each other. □

Notice Proposition 8 and 9 hold even if stock price forecasts and consumption forecasts are made at different dates and/or over different forecasting horizons. While survey consumption growth forecasts appear to be non-rational, incorporating this feature alone into asset pricing models cannot break the cointegration relationship between stock price forecasts and consumption forecasts. This aspect of the formation of stock price expectations in those models appears inconsistent with the evidence in Section 2.3.3.

[24]If agents learn use least squares (i.e., $g_t = 1/t$), $\mu_{c,t}$ will converge to μ_c and $\widetilde{A}_{0,t}$ will converge to a constant.

2.5.4. Heterogeneous beliefs

Can *deviation from the representative agent assumption* reconcile the new survey evidence in Section 2.3.3? This section shows models with heterogeneous beliefs and a willingness to "agree to disagree" do not necessarily reproduce this evidence. In these models, all agents can deduce and agree on the equilibrium pricing function and state-contingent stock prices. This knowledge usually requires strong informational assumptions. For instance, all agents' beliefs about fundamentals and preferences, etc are common knowledge.

Consider the model of Ehling,Graniero and Heyerdahl-Larsen (2018) (EGH henceforth) which is a continuous-time overlapping generations economy. They study asset prices and portfolio choice by incorporating agents' learning from own experience about output process in a dynamic complete market setting. There are different cohorts who are born at different times and have heterogeneous beliefs about fundamentals, i.e., exogenous aggregate output process Y_t. The true process for Y_t is $dY_t/Y_t = \mu_Y dt + \sigma_Y dz_t$ where z_t is a standard Brownian motion. Agents disagree on this process and perceive that

$$dY_t/Y_t = \hat{\mu}_{s,t} dt + \sigma_Y dz_{s,t},$$

where the subscript s represents the cohort born at time s, $\hat{\mu}_{s,t}$ agents' perceived output growth rate, $z_{s,t}$ denotes a Brownian motion under the belief of an agent born at time s. Agents know the standard deviation of output σ_Y.

Denote by $E_i^s \log P_{i+j}$ stock price forecasts made by cohort s at time i and over horizon j; similarly for the forecast of aggregate consumption $E_k^s \log C_{k+l}$.[25] Define $\overline{E}_i \log P_{i+j}$ the average of stock price forecasts across all agents made at time i and over horizon j; similarly for average forecasts of aggregate consumption $\overline{E}_k \log C_{k+l}$.

Proposition 10. *In the EGH model, the average stock price forecasts across all agents* $\overline{E}_i \log P_{i+j}$ *is cointegrated with the average consumption forecasts* $\overline{E}_k \log C_{k+l}$ *with cointegrating vector* $(1, -1)$ *for arbitrary* i, j, k, $l > 0$.

Proof. All agents know (1) aggregate consumption C_t equals to Y_t ($C_t = Y_t$) each period and that consumption process is identical to output process and (2) the mapping from output to equilibrium stock prices, i.e., $P_t = \frac{1-\omega}{\rho+\nu(1-\beta)} Y_t$ (equation (B60) in EGH), and hence $P_t = \frac{1-\omega}{\rho+\nu(1-\beta)} C_t$. Thus, the perceived consumption process for cohort s is $dC_t/C_t = \hat{\mu}_{s,t} dt + \sigma_Y dz_{s,t}$. Agents' heterogeneous beliefs about fundamentals Y_t (or C_t) are carried over to stock prices P_t, which reads as $dP_t/P_t = \hat{\mu}_{s,t} dt + \sigma_Y dz_{s,t}$. Thus, every cohort perceives that the growth rate of C_t and S_t are the same stochastic process despite heterogeneous beliefs about the growth rate across cohorts.

For cohort s, stock price forecasts $E_i^s \log P_{i+j}$ depends on realized stock prices P_i and perceived stock price growth rates $\hat{\mu}_{s,i}$. Similarly, consumption forecasts $E_k^s \log C_{k+l}$ depend on realized consumption C_k and perceived consumption growth

[25] In EGH, S_t stands for stock prices. Here we use a different notation P_t.

rates $\hat{\mu}_{s,k}$. First, note $P_i = \frac{1-\omega}{\rho+\nu(1-\beta)} Y_i$ and $C_k = Y_k$. Clearly, $logP_i$ is cointegrated with $logC_k$ with cointegrating vector $(1,-1)$ because $logY_i$ and $logY_k$ is cointegrated with cointegrating vector $(1,-1)$. Second, Proposition 1 of EGH implies that as $t \to \infty$, $\hat{\mu}_{s,t}$ and $\hat{\mu}_{s,k}$ will converge almost surely to the true output growth rate μ_Y. So $E_i^s \log P_{i+j}$ is cointegrated with $E_k^s \log C_{k+l}$ with cointegrating vector $(1,-1)$ for every cohort s. Taking the two features together, the average stock price forecasts $\overline{E}_i \log P_{i+j}$ is cointegrated with $\overline{E}_k \log C_{k+l}$ with cointegrating vector $(1,-1)$. □

Although agents have heterogeneous beliefs about consumption (or output), the knowledge of the equilibrium pricing function implies that stock price forecasts made by every individual is cointegrated with its aggregate consumption forecasts. Thus, the average stock price forecasts across agents is cointegrated with the average consumption forecasts. This aspect of the formation of stock price forecasts in this model appears inconsistent with our survey evidence in Section 2.3.3 or Appendix B.

2.6. Reconciling models with the new survey evidence

How can we break the tight link between the long-run component of stock price forecasts and consumption forecasts in asset pricing models? This section discusses potential resolutions for reconciling models with the new survey evidence. In particular, it shows one potential resolution in representative agent asset pricing models contains two elements: (1) relaxing investors' knowledge of the equilibrium pricing function and (2) adding a non-stationary sentiment shock to investors' subjective stock price forecasts.

As is shown in Section 2.5 and 2.5.4, incorporating non-rational (or extrapolative) consumption expectations or allowing heterogeneous beliefs with a willingness to "agree to disagree" among agents does not break the cointegration between stock price forecasts and consumption forecasts. The long-run component of stock price forecasts is anchored by consumption forecasts in these models via agents' knowledge of the equilibrium pricing function. Thus, our survey evidence casts some doubt on the modeling of expectation formation in these models. The discrepancy between stationary realized price consumption ratios and the corresponding non-stationary forecasts poses a challenge for the asset pricing models which assume agents possess this knowledge.

The knowledge of the equilibrium pricing function usually requires strong informational assumptions on agents, such as other agents' beliefs and preferences, etc; see e.g., Adam and Marcet (2011). One way to reconcile the new evidence appears to allow the representative agent not having knowledge of the equilibrium pricing function or a sufficiently large fraction of investors not having this knowledge in models with heterogeneous beliefs. This assumption is made in e.g., adaptive learning models; examples include Lansing (2010), Branch and Evans (2011), Boswijk, Hommes and Manza, (2007), Adam, Marcet and Beutel (2017). Due to agents' lack of this knowledge, it is possible for an asset pricing model to simultaneously produce stationary realizations and non-stationary forecasts of price to consumption ratios.

Yet relaxing agents' knowledge of the equilibrium pricing function alone is insufficient to reconcile the survey evidence, which is shown in Section 7 of an earlier draft, i.e., Kuang, Zhang and Zhang (2019).[26] This is because the typical specifications of stock price beliefs in adaptive learning models imply that stock price forecasts are cointegrated with consumption forecasts. The survey evidence suggests that major asset pricing models may have missed an important persistent component which independently drives stock price forecasts but not consumption forecasts.

Below we outline an asset pricing model with learning and sentiment in a representative agent setting and show that it reproduces the survey evidence. We modify the belief specification in the AMB model by adding a non-stationary sentiment or judgment component (which is not cointegrated with consumption forecasts) *directly* to stock price forecasts and assume the agent does not have the knowledge of the equilibrium pricing function (as in AMB). Later we also provide some discussions on reconciling the survey evidence in models with heterogeneous beliefs.

[26] Unit root econometrics can also be applied to test the expectation formation in adaptive learning models. Deriving the testable implications requires using agents' perceived and actual law of motion for model variables.

2.6.1. A potential resolution

The model builds on the stock pricing model of AMB. The discussion here focuses on introducing the process of fundamentals and subjective stock price beliefs which may be sufficient for our purpose. Readers are referred to the AMB paper for other details. It is an infinite horizon endowment economy model where agents receive dividends and exogenous wage income. Agents decide on stock holding and consumption for each period. Dividends grow at a constant rate

$$\log D_t = \log \beta^D + \log D_{t-1} + \log \varepsilon_t^D, \tag{2.24}$$

where $\beta^D \geq 1$ stands for the mean growth rate and $\ln \varepsilon_t^D$ an $i.i.d.$ growth innovation described further below. The exogenous wage income process W_t is

$$\log\left(1+\frac{W_t}{D_t}\right) = (1-p)\log(1+\rho) + p\log\left(1+\frac{W_{t-1}}{D_{t-1}}\right) + \ln \varepsilon_t^W, \tag{2.25}$$

where $1+\rho$ is the average consumption-dividend ratio and $p \in [0,1)$ its quarterly persistence. The innovations are given by

$$\begin{pmatrix} \log \varepsilon_t^D \\ \log \varepsilon_t^W \end{pmatrix} \sim iiN\left(-\frac{1}{2}\begin{pmatrix} \sigma_D^2 \\ \sigma_W^2 \end{pmatrix}, \begin{pmatrix} \sigma_D^2 & \sigma_{DW} \\ \sigma_{DW} & \sigma_W^2 \end{pmatrix}\right), \tag{2.26}$$

with $E\varepsilon_t^D = E\varepsilon_t^W = 1$. Aggregate consumption is

$$C_t = W_t + D_t. \tag{2.27}$$

Agents' beliefs and preferences etc are not common knowledge. They cannot correctly deduce the equilibrium law of motion for asset prices. Instead, they form subjective price belief and learn from market prices. They are assumed to know the exogenous driving processes, (2.24), (2.25) and (2.27). These equations give that

$$\log C_t = \log(D_t + W_t) = \log D_t + \log(1 + \frac{W_t}{D_t}). \tag{2.28}$$

Denote by $\log P_{t+1}^e$ the agent's forecast of stock prices in period $t+1$ made at period t. The representative agent's subjective stock price forecasts ($\log P_{t+1}^e$) consist of two components. Mathematically, the agent's stock price forecasts are

$$\log P_{t+1}^e = E_t \log P_{t+1} + \log \gamma_t, \tag{2.29}$$

$$\log \gamma_t = \log \gamma_{t-1} + \log \xi_t. \tag{2.30}$$

$\log \xi_t$ is assumed as an i.i.d process. The first component of stock price forecasts ($E_t \log P_{t+i}$) is generated by learning from past stock prices as in AMB. Mathematically, the agent has the following forecasting model

$$\Delta \log P_t = \log \beta_t + \log \epsilon_t, \quad (2.31)$$

$$\log \beta_t = \left(1 - \eta_\beta\right) \log \beta^D + \eta_\beta \log \beta_{t-1} + \log \nu_t, \quad (2.32)$$

where $\log \epsilon_t$ and $\log \nu_t$ are i.i.d. innovations. $\log \xi_t$, $\log \epsilon_t$, and $\log \nu_t$ are independent to each other. The agent is uncertain about the parameter of the model ($\log \beta_t$) over time.[27] She needs to learn about it using historical stock price data and then makes forecasts using the estimated model. The agent's belief about $\log \beta_t$ is updated by

$$\begin{aligned} \log m_t &= \left(1 - \eta_\beta\right) \log \beta^D + \eta_\beta \log m_{t-1} \\ &\quad + g \left(\log P_{t-1} - \log P_{t-2} - \log m_{t-1}\right), \end{aligned} \quad (2.33)$$

where g is the Kalman gain parameter.

The second component is new (relative to e.g., AMB) and denoted by $\log \gamma_t$. An interpretation of the sentiment variable $\log \gamma_t$ is (the "guesswork" component of) judgment made by forecasters. This is known as "add-factoring" the forecast in the forecasting community. On top of the forecasts generated from an econometric

[27] We assume agents know η_B and β^D as is typical in the literature; this is an unimportant assumption for our purpose.

forecasting model, investors make some judgmental adjustment.[28] When $\log \gamma_t$ is positive (or negative), the agent is more optimistic (or pessimistic) than the stock price forecasts produced by the econometric model.

Reconciling our survey evidence requires two crucial assumptions on $\log \gamma_t$. **Assumption 1**: $\log \gamma_t$ is an I(1) process. **Assumption 2**: $\log \gamma_t$ is independent of or at least not cointegrated with the agent's aggregate consumption forecasts. If either of the two assumptions is violated, that is, if $\log \gamma_t$ is stationary or cointegrated with consumption forecasts, stock price forecasts will be cointegrated with consumption forecasts, which is inconsistent with our survey evidence. Appendix 2.8.5 tests and confirms the two assumptions.

Two remarks are as follows. First, $\log \gamma_t$ in equation (2.29) is assumed to be a random walk process. It can be specified as a more general $I(1)$ process and the proposition below will not be affected. Second, for the purpose of replicating the evidence, assuming $\log \gamma_t$ is private information or common knowledge does not matter.

Proposition 11. *Given the agent's price belief (2.29) - (2.32), we have (1) the agent's stock price forecasts* $\log P^e_{i+j}$ *are I(1), consistent with the evidence in Section 2.3.2; (2) stock prices forecasts* $\log P^e_{i+j}$ *and consumption forecasts* $\log C^e_{k+l}$ *are not cointegrated for arbitrary* i, j, k, $l > 0$, *consistent with the survey evidence*

[28] For instance, Bullard, Evans and Honkapohja (2008) examines the role of agents' judgmental adjustment to forecasts in learning models. They show this may lead to self-fulfilling fluctuations in New Keynesian models. Another example is that monetary poliymakers frequently add judgment into their forecasts. Alternatively, $\log \gamma_t$ may be called "expectation shocks", see Milani (2011) for an estimated New Keynesian model with learning and expectation shocks.

in Section 2.3.3; (3) forecasts of stock prices, i.e., $\log P^e_{i+j}$ *and* $\log P^e_{k+l}$*, are cointegrated for (a)* $i = k$, $j \neq l$ *or (b)* $i \neq k$, $j > 0$, $l > 0$*, consistent with the evidence in Section 2.3.4.*

Proof. (1) Given (2.29) - (2.32), stock price forecasts are $\log P^e_{i+j} = E_i \log P_{i+j} + \log \gamma_i = \log P_i + j \log \beta^D + \widetilde{s}\,(i,j) + \log \gamma_i$, where $\widetilde{s}\,(i,j)$ is a stationary term and omitted because it is irrelevant for the proof. Denote by L the lag operator. Taking the difference of $\log P^e_{i+j}$ yields $(1-L)\log P^e_{i+j} = (E_i \log P_{i+j} + \log \gamma_i)$ $-\left(E_{i-1} \log P_{i-1+j} + \log \gamma_{i-1}\right) = (\log P_i + j \log \beta^D + \widetilde{s}\,(i,j) + \log \gamma_i) - \left(\log P_{i-1} + j \log \beta^D + \widetilde{s}\,(i-1, j\right.$ $\Delta \log P_i + \log \xi_i + \Delta \widetilde{s}\,(i,j)$. Given $\Delta \log P_i$ is stationary (as in the data) and $\Delta \widetilde{s}\,(i,j)$ is stationary, we have shown stock price forecasts are I(1), consistent with the evidence in Section 2.3.2.

(2) Let $\log P^e_{i+j} - \log C^e_{k+l} = (\log P_i + j \log \beta^D + \widetilde{s}(i,j) + \log \gamma_i) - (\log D_k + l \log \beta^D + E_k(\log(1 + \frac{W_{k+l}}{D_{k+l}}))) = (\log P_i - \log D_k) + \log \gamma_i + (j-l)\log \beta^D + \widetilde{s}(i,j) - E_k(\log(1 + \frac{W_{k+l}}{D_{k+l}}))$. Because $\log \gamma_i$ is I(1) (Assumption 1) and not cointegrated with consumption forecasts (Assumption 2), we have shown that $\left(\log P^e_{i+j} - \log C^e_{k+l}\right)$ is I(1), consistent with the evidence in Section 2.3.3.

(3) Let $\log P^e_{i+j} - \log P^e_{k+l} = (E_i \log P_{i+j} + \log \gamma_i) - (E_k \log P_{k+l} + \log \gamma_k) = (\log P_i + j \log \beta^D + \widetilde{s}\,(i,j)) - (\log P_k + l \log \beta^D + \widetilde{s}\,(k,l))$. It is stationary because $(\log P_i - \log P_k)$, $\widetilde{s}\,(i,j)$ and $\widetilde{s}\,(i,k)$ are stationary. □

In a representative agent setting, introducing a non-stationary sentiment $\log \gamma_t$ – which is not cointegrated with consumption forecasts – directly to stock price forecasts is the key to break the tight link between the trend component of stock price forecasts and consumption forecasts and reconcile the new survey evidence. Appendix 2.8.5 provides an illustration on how to construct a time series of this sentiment shock and confirms the two assumptions made earlier.

The literature generally finds that realized price to consumption (or dividend) ratios are stationary. For instance, Golez and Koudijs (2018) combine three countries' stock market data together to construct dividend yields in the last four centuries and show they are stationary. Moreover, there is evidence suggesting that the wealth to consumption ratio is stationary if human wealth is included (Lettau and Ludvigson 2004, Lustig, Van Nieuwerburgh, and Verdelhan 2013). This suggests a stationary stock price consumption ratio, despite that stock market value is a small fraction of total wealth.[29]

Can the model still produce a stationary realized price to consumption (or dividend) ratio when we add a non-stationary sentiment variable into stock price forecasts? Due to agents' lack of knowledge of the equilibrium pricing function, this type of models can simultaneously produce non-stationary forecasts of stock price to consumption (or dividend) ratios as well as stationary realized stock price to consumption (or dividend) ratios. In the AMB model, when expectations about

[29]Cochrane (2007) also argues realized stock price to dividend ratios are stationary. Campbell and Yogo (2006), nevertheless, find that the stationarity of price dividend ratio is sensitive to sample period and frequency. Note whether realized price consumption (or dividend) ratio is stationary or not does not affect our new survey evidence and our derived testable implications for the formation of stock price expectations in various full- and incomplete-information asset pricing models.

stock price appreciation are sufficiently high, the wealth effect becomes as strong as (or even stronger than) the substitution effect. Investors' stock demand and stock prices will start to decline which leads to mean-reversion of the stock price to consumption (or dividend) ratio. This mechanism will help to produce a stationary price to consumption ratio after adding the sentiment variable. An alternative way is imposing a projection facility as in Adam, Marcet and Nicolini (2016) which will lead to mean-reversion in stock price to consumption or dividend ratio. Kuang and Tang (2019) estimates a stock pricing model with learning and sentiment which replicates the joint dynamics of survey forecasts of stock prices and realized stock prices. In particular, investors' stock price forecasts in the model are not anchored by forecasts of aggregate consumption, consistent with the survey evidence here. In the estimated model, the interaction between sentiment shocks and learning accounts for about two-thirds of the volatility of US stock price to dividend ratios.[30]

2.6.2. Discussions

Several other possible resolutions may deserve some discussions. First, for models with heterogeneous beliefs, Section 2.5.4 shows if all agents have the knowledge of the equilibrium pricing function and "agree to disagree" about the fundamentals, the average forecast of stock prices across agents is still cointegrated with the average forecast of consumption. Thus, reconciling the survey evidence in a heterogeneous agents setting would require no cointegration between stock price forecasts

[30] A draft of the paper is available upon request.

and consumption forecasts for a sufficiently large fraction of investors. This may in turn require (1) assuming these investors do not have the knowledge of the equilibrium pricing function and (2) for each of these investors, a non-stationary sentiment (or judgment) variable independently drives her stock price forecasts but not consumption forecasts. With the availability of surveys containing individual stock price forecasts and consumption forecasts, applying our tests at the level of individual investors can help to understand better the degree and nature of belief heterogeneity among investors, such as the relationship between the forecasts made by investors with different characteristics (e.g., education, income) and if a common (or different) non-stationary sentiment variable drives stock price forecasts made by different investors.

Second, can the survey evidence be reconciled by adding a non-stationary sentiment component to consumption forecasts (but not stock price forecasts) in the AMB model and assuming investors do not have the knowledge of the equilibrium pricing function? Consider firstly the models in Section 2.5.2 which assume investors have the knowledge of the equilibrium pricing function but misperception about the consumption process. Adding a non-stationary sentiment component to consumption (like setting $\rho = 1$ in equations 2.16 - 2.17) will yield price consumption ratios with unbounded volatility; see equation (2.20). The AMB model with this alternative belief specification – together with assuming lack of knowledge of the equilibrium pricing function – will amplify the fluctuations of price consumption ratios as the agent's price growth beliefs will fluctuate around the corresponding value under the case with exact knowledge. In this case, the price consumption ratios will have unbounded volatility too. Thus, this avenue appears not promising to reconcile the evidence.

Third, can stock pricing models with features like limited stock market participation, or other financial and labor market frictions replicate the survey evidence? We argue these features *per se* are not helpful. Even if one builds a model with these frictions, this model needs to generate a cointegration relation between the aggregate price index and aggregate consumption, and a cointegration relation between aggregate consumption and aggregate dividend as in the data (see the discussion in Section 2.4.2). Agents' knowledge of the equilibrium pricing function will again impose cointegration restrictions between forecasts of the aggregate stock price index and aggregate consumption (or dividend) forecasts.

Fourth, one may think structural breaks is a potential reason why stock price forecasts and consumption forecasts are not cointegrated. The focus of our paper is testing expectation formation in major asset pricing models which do not usually include structural breaks. Yet we think our testing results here are robust to considering some types of structural breaks. For instance, the 2007 - 2008 Global Financial Crisis may be viewed as a structural break that the trend growth rate of consumption is reduced. Rational agents will observe this decrease and the trend growth rate of equilibrium stock prices will have a one-to-one decline. The cointegration relation between stock prices and consumption (including the cointegrating vector) is unaltered. Thus, our evidence on the no cointegration between stock price forecasts and consumption forecasts may not be reconciled by allowing this type of structural break and assuming agents possess the knowledge of the equilibrium pricing function. Similarly, this conclusion would still hold if the Financial Crisis is viewed as, for example, a shift in the level of the trend of consumption.

Lastly, some models of stock prices, e.g., De Long, Shleifer, Summers and Waldmann (1990) and Scheinkman and Xiong (2003), avoid modelling consumption so that consumption is not a component for understanding stock price movement. Similarly, consumption is not an important driving force of stock prices in the X-CAPM model of Barberis, Greenwood, Jin and Shleifer (2015). We note these models are not inconsistent with the new survey evidence here. However, these models are developed mainly for theoretical purposes and the feature that aggregate consumption does not influence (or influence little) on aggregate stock prices may be viewed by some economists as a little extreme.

2.7. Conclusion

The paper demonstrates the usefulness of time series econometrics in analyzing expectations data and testing expectation formation in financial and macroeconomic models with various informational assumptions. Models usually impose a large number of cointegration restrictions, e.g., between forecasts of endogenous and exogenous variables or among forecasts of different endogenous variables. The tests developed in the paper utilize these restrictions.

We show the median (or mean) survey stock price forecasts are not cointegrated with the median (or mean) consumption forecasts. This evidence is robust to different sources of expectations data, forecasting horizons, statistical tests, using median or mean forecasts for testing, and using stock price forecasts data which is made at the same or different dates from consumption forecasts.

The evidence casts some doubt on the modeling of expectation formation in a wide range of RE and Bayesian RE asset pricing models which assume agents possess the knowledge of equilibrium pricing function. Obtaining this knowledge by agents within the models usually requires strong informational assumptions on them, e.g., all agents' beliefs and preferences, etc need to be common knowledge (see e.g., Adam and Marcet, 2011). Due to this exact knowledge, the long-run component of stock price forecasts is anchored by consumption forecasts in these models.

Relaxing agents' knowledge of the equilibrium pricing function (as in e.g., Lansing, 2010, 2019, Branch and Evans, 2010, 2011, Adam, Marcet and Beutel, 2017, and Boswijk, Hommes and Manza, 2007) appears necessary to reconcile the new evidence. Moreover, the evidence suggests major asset pricing models may miss an important persistent component which independently drives stock price forecasts but not consumption forecasts. In a representative agent setting, one way to reproduce the new evidence is incorporating a non-stationary sentiment (or judgment) component directly to subjective stock price forecasts together with assuming the agent's lack of knowledge of the equilibrium pricing function.

The formation of stock price expectations can be further explored in two dimensions. First, aggregate dividend forecasts and stock price forecasts are cointegrated in various asset pricing models considered in the paper. These can be tested with the availability of high-quality data on aggregate dividend forecasts with a sufficiently long sample.[31] Second, the degree and nature of belief heterogeneity among investors can be studied further by applying the tests at the level of individual investors with the availability of surveys containing individual stock price forecasts and aggregate consumption forecasts (or dividend forecasts).

The paper employs several widely used datasets on expectations data and adopts a common way to interpret the median (or mean) survey expectations data in this literature.[32] And some state-of-the-art statistical tests (with Monte Carlo studies) are employed. As the quality of survey expectations data continue improving and with the advancement of statistical tests, applying unit root econometrics to expectations data – as is done in the paper – may be a promising avenue to provide useful guidance on the modeling of expectation formation.

The tests of expectation formation developed in the paper can be applied in or adapted to other settings. First, while the tests are applied using the mean or median survey forecasts, they can be applied using individual-level data (where

[31] We do not pursue this analysis for the reason discussed in the Introduction.

[32] In line with the literature, we use the median (or mean) expectations from these widely used expectations surveys as a proxy for the expectations of the representative agent.

available) or experimental data too. The tests can be employed to study the relationship between forecasts made by economic agents with different characteristics (e.g., age, education, income, occupation) and to guide the modeling of expectation formation in models with heterogeneous beliefs. Second, in the presence of structural breaks, financial and macroeconomic models impose cointegration restrictions among forecasts of model variables. Our tests can be adapted to test and guide the modeling of expectation formation in this setting. Third, they can be implemented in other models, such as stochastic growth models and exchange rate models. We leave these for future work.

2.8. Appendix

2.8.1. Rebasing consumption forecasts data

Since the Survey of Professional Forecasters (SPF) began, there have been a number of changes of the base year in the national income and product accounts (NIPA). The forecasts for levels of consumption (SPF variable name: RCONSUM) use the base year that was in effect when the forecasters received the survey questionnaire. This Appendix explains how consumption forecasts data are rebased.

Table 2.7 provides the base year in effect for NIPA variables (including consumption expenditures), reproduced from Table 4 of the documentation of the

Range of Survey Dates	Base Year	Ratio
1976:Q1 to 1985:Q4	1972	3.31
1986:Q1 to 1991:Q4	1982	1.48
1992:Q1 to 1995:Q4	1987	1.23
1996:Q1 to 1999:Q3	1992	1.04
1999:Q4 to 2003:Q4	1996	1
2004:Q1 to 2009:Q2	2000	0.94
2009:Q3 to 2013:Q2	2005	0.84
2013:Q3 to present	2009	0.79

Table 2.7. Base Years and Ratios for Rebasing

Survey of Professional Forecasters (p. 23). For rebasing, we use real consumption expenditures data of different vintages from the Real-Time Data Set for Macroeconomists managed by the Federal Reserve Bank of Philadelphia. The year 1996 is used as the common base year for all consumption forecast data. The data in each window needs to be rebased by multiplying a base ratio. For instance the 1959:Q4 real consumption in the window from 1996:Q1 to 1999:Q3 is 1409.5 while it is 1469.5 in 1999:Q4 to 2003:Q4 window and hence the ratio is 1469.5/1409.5.

2.8.2. Results using mean forecasts

Table 2.8 and 2.9 test the integration properties of the mean forecasts of (log) stock prices and aggregate consumption, respectively. We consider all sources of forecasts, different forecasting horizons, and tests. The results suggest the mean forecasts of stock prices and aggregate consumption are I(1) and not I(2) at the 10% significance level.

	1Q ahead	2Q ahead	4Q ahead	10-yr ahead
Panel A: I(1) test statistics				
Shiller (PP Z_t stat.)	−2.196	−2.225	−2.215	-2.400
10% critical value	−3.172	−3.172	−3.172	−3.172
Shiller (DF-GLS)	−2.218	−2.247	−2.163	−1.683
10% critical value	−2.851	−2.851	−2.851	−2.851
Livingston (PP Z_t stat.)	$n.a.$	−2.297	−2.086	$n.a.$
10% critical value	$n.a.$	−3.167	−3.167	$n.a.$
Livingston (DF-GLS)	$n.a.$	−1.698	−1.717	$n.a.$
10% critical value	$n.a.$	−2.818	−2.818	$n.a.$
Panel B: I(2) test (p-value)				
Shiller (PP Z_t stat.)	−7.576	−7.714	−7.681	−9.262
1% critical value	−2.615	−2.615	−2.615	−2.615
Shiller (DF-GLS)	−3.737	−3.724	−3.785	−3.212
1% critical value	−2.615	−2.615	−2.615	−2.615
Livingston (PP Z_t stat.)	$n.a.$	−8.506	−6.590	$n.a.$
1% critical value	$n.a.$	−2.612	−2.612	$n.a.$
Livingston (DF-GLS)	$n.a.$	−5.933	−3.154	$n.a.$
1% critical value	$n.a.$	−2.611	−2.611	$n.a.$

Table 2.8. Integration properties: mean forecasts of $logP$

Table A3 shows that the mean forecasts of stock prices are not cointegrated with the mean forecasts of aggregate consumption with cointegrating vector $(1, -1)$ when testing at 10% significance level with one exception. That is, for 10-year ahead stock price forecasts from the Shiller survey, the null hypothesis that forecasts of 10-year ahead stock price forecast are not cointegrated with forecasts of 1-year ahead consumption with cointegrated vector $(1, -1)$ is rejected by the PP test at 10% significance level but not at 5% significance level (because the 5% critical value is -2.920).

	1Q ahead	2Q ahead	4Q ahead
Panel A: I(1) test			
PP (Z_t stat.)	−1.316	−1.323	−1.323
10% critical value	−3.167	−3.167	−3.167
DF-GLS	−1.225	−1.226	−1.188
10% critical value	−2.818	−2.818	−2.818
Panel B: I(2) test			
PP (Z_t stat.)	−4.696	−4.769	−4.747
1% critical value	−2.612	−2.612	−2.612
DF-GLS	−3.844	−4.006	−4.215
1% critical value	−2.611	−2.611	−2.611

Table 2.9. Integration properties: forecasts of $logC$

Mean	1Q ahead	2Q ahead	4Q ahead	10-yr ahead
I(1) test				
Shiller (PP Z_t stat.)	−2.430	−2.444	−2.389	−2.673
10% critical value	−2.595	−2.595	−2.595	−2.595
Shiller (DF-GLS)	−1.653	−1.664	−1.596	−0.917
10% critical value	−1.929	−1.929	−1.929	−1.929
Livingston (PP Z_t stat.)	*n.a.*	−2.234	−2.213	*n.a.*
10% critical value	*n.a.*	−2.591	−2.591	*n.a.*
Livingston (DF-GLS)	*n.a.*	−0.233	−0.185	*n.a.*
10% critical value	*n.a.*	−1.895	−1.895	*n.a.*

Table 2.10. No cointegration between forecasts of $logP$ and $logC$

2.8.3. Testing using Greenbook consumption forecasts

This Appendix shows the result of no cointegration between forecasts of stock prices and consumption still holds when we use consumption forecasts from the Greenbook data sets instead of SPF data. Our test results in the main text are robust to this alternative consumption forecast.

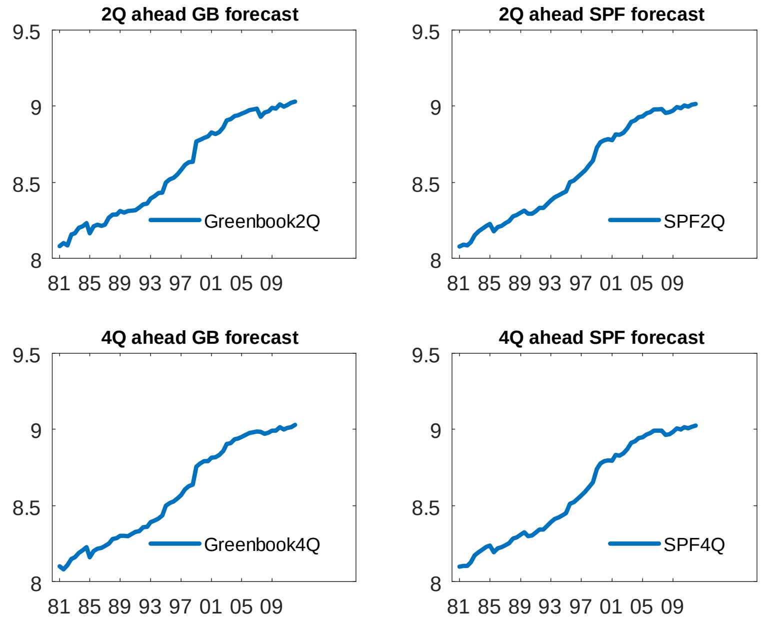

Figure 2.3. Greenbook and median SPF consumption forecasts

The Greenbook contains projections on the US economy in future quarters and is produced before each meeting of the Federal Open Market Committee. It includes projections for a large number of macroeconomic variables including real consumption growth. Four forecasting horizons are reported in each projection: 1- to 4-quarter ahead (while more horizons are issued from time to time). The dataset is published with a five-year lag. The sample of Greenbook consumption growth forecast is from 1967 to 2012. We obtain real consumption level forecast by multiplying the consumption growth forecast by (rebased) consumption level; the latter is obtained from real-time datasets for the US economy maintained by the Philadelphia Fed. To conduct the tests, we use the vintage of Greenbook forecasts

	2Q ahead	4Q ahead
Panel A: I(1) test statistics		
PP (Z_t stat.)	-1.320	-1.275
10% critical value	-3.173	-3.173
DF-GLS	-1.195	-1.091
10% critical value	-2.825	-2.825
Panel B: I(2) test statistics		
PP (Z_t stat.)	-9.340	-8.416
1% critical value	-3.565	-3.565
DF-GLS	-5.469	-2.995
1% critical value	-2.615	-2.615

Table 2.11. Integration Properties: Forecasts of $logC$

in the way that the corresponding FOMC meeting date is closest to the date of the Livingston survey.

The tests are conducted using Livingston survey stock price forecasts and Greenbook consumption forecasts.[33] Figure 2.3 displays 2Q- and 4Q- ahead forecast of (log) consumption from the Greenbook (GB) datasets and the SPF. "Greenbook2Q" and SPF2Q"correspond to the 2-Quarter ahead Greenbook and median SPF consumption forecast respectively; similarly for 4Q ahead forecast. The forecasts from the two sources look quite similar.

Table 2.11 reports the test statistics value and critical value for the unit root tests of forecasts of (log) aggregate consumption. For all forecasting horizons, both tests suggest that consumption forecasts are an I(1) but not I(2) process.

[33]The sample period which the Shiller Survey and the Greenbook datasets overlap is relatively short. Thus, we do not conduct the test using the Shiller survey data.

	Median 2Q ahead	Median 4Q ahead	Mean 2Q ahead	Mean 4Q ahead
I(1) test statistics				
PP (Z_t stat.)	-2.370	-2.328	-2.442	-2.321
10% critical value	-2.595	-2.595	-2.595	-2.595
DF-GLS	-0.794	-0.788	-0.808	-0.754
10% critical value	-1.903	-1.903	-1.903	-1.903

Table 2.12. No Cointegration between Forecasts of $logP$ and $logC$

Table 2.12 shows that both PP and DF-GLS tests suggest that we cannot reject the null hypothesis that forecasts of $\log(p)$ are not cointegrated with forecasts of $\log(c)$ with cointegrating vector $(1, -1)$. This is robust to both forecasting horizons (2Q-ahead and 4Q-ahead) and using the median or mean stock price forecasts.

2.8.4. KPSS testing results (with correcting potential small sample problems)

This Appendix shows that the KPSS test rejects the null hypothesis that stock prices forecasts and consumption forecasts are cointegrated with vector $(1, -1)$. The KPSS test tests the null hypothesis of a stationary process against the alternative of a unit root. We find this test basically leads to the same conclusion as other unit root tests. Using the Livingston Survey data, the KPSS test rejects the stationarity of forecasts of stock price to consumption ratios at 1% level (see Table 2.13). The test results are robust to up to 11 lags for Livingston data. Using the shorter Shiller Survey data, the KPSS test rejects the null hypothesis at 5% level

	1Q ahead	2Q ahead	4Q ahead
KPSS test			
Livingston	$n.a.$	1.43	1.57
1% critical value	$n.a.$	0.739	0.739
Shiller	0.484	0.512	0.454
5% critical value	0.463	0.463	0.463

Table 2.13. No Cointegration between Median Forecasts of $logP$ and $logC$

	1Q ahead	2Q ahead	4Q ahead
KPSS test			
Livingston	$n.a.$	1.56	1.62
1% critical value	$n.a.$	0.739	0.739
Shiller	0.487	0.465	0.428
5% critical value	0.463	0.463	0.463
10% critical value	0.347	0.347	0.347

Table 2.14. No Cointegration between Mean Forecasts of $logP$ and $logC$

for 1− and 2-quarter ahead forecasts and at 10% for 4-quarter forecasts (see Table 2.14). The results in both tables are based on KPSS tests with one lag.

The KPSS testing results might suffer from small sample problems. The following Monte-Carlo study is conducted to compute the critical values of the KPSS test in the small sample case. We simulate the distribution of the KPSS test statistics under the null hypothesis that the long-run risks model presented in Section 2.4.1 is true (and hence forecasts of $\log(P/C)$ over different horizons are stationary as analyzed in the text). First, we obtain analytical expressions for 1-, 2- and 4Q ahead forecasts of $\log(P/C)$. Second, we simulate the long-run risk model with $N = 1,000$ repetitions and each sample matching the sample size and frequency

	2Q ahead	4Q ahead
Livingston (Median)	1.43	1.57
1% critical value	0.889	0.891
Livingston (Mean)	1.56	1.62
1% critical value	0.889	0.891

Table 2.15. No Cointegration between Forecasts of $logP$ and $logC$ (KPSS Test)

of the Livingston survey data (longer than the Shiller Survey). We then obtain the KPSS statistics from each repetition and hence the distribution of statistics. The critical values in Table 2.15 are obtained from our Monte-Carlo simulation and differ from those in Table 2.13 and 2.14. The simulated critical values do not change much. Using 2- and 4-quarter ahead Livingston survey forecasts data, the KPSS test can still reject the null hypothesis of stationarity at 1% significance level, robust to using median or mean forecasts data.

Note: Critical values based on monte-carlo simulation of the Long-run risks model

2.8.5. Constructing and testing the sentiment shock: an illustration

We illustrate how to empirically construct a time series of this new component of stock price forecasts $\log \gamma_t$ (in a representative agent setting). We show the sentiment shock series does not pass standard unit root tests and is not cointegrated with consumption forecasts, confirming the two assumptions we made earlier. This exercise isolates the component of stock price forecasts which contains a unit root and is not cointegrated with consumption forecasts.

Assuming $\eta_\beta = 0$ which corresponds to the benchmark specification of AMB. Equation (2.33) gives a simple belief updating equation for stock price growth rates

$$\log m_t = \log m_{t-1} + g\left(\log P_{t-1} - \log P_{t-2} - \log m_{t-1}\right), \tag{2.34}$$

where g is the gain parameter.[34] This gives, for instance, the first component of 1-year ahead stock forecasts as $\log P_t + 2\log m_t$; note we consider a semi-annual frequency. On top of this, the agent adds a nonstationary new component (i.e., judgment or sentiment) $\log \gamma_t$ which follows a random walk process. Thus, the agent's 1-year ahead stock price forecast is $\log P_t + 2\log m_t +\log \gamma_t$, which will be proxied by survey expectations data.

The agent's initial belief about semi-annual stock price growth rates is set to 4.5% (which is the average value in the historical stock price data taken from Adam, Marcet and Nicolini (2016)). We then substitute actual stock prices data over the period 1981Q2 - 2017Q2 into the belief updating rule (2.34). The gain value is set to 0.0262 taken from AMB.[35] With these, we obtain a time series of the first component of stock price forecasts. The sentiment shocks can be constructed as the difference between 1-year ahead Livingston median stock price forecasts

[34]The isolated sentiment shock series may be different if investors use a different econometric model to produce the first component of stock price forecasts. We do not explore this here as our exercise aims to provide an illustration (using the AMB specification which has good quantitative performance).

[35]The choice of the gain parameter can be alternatively determined by estimating stock pricing models with utilizing stock price expectations data. The constructed sentiment shock may be different if investors use a different gain parameter. But we find the testing results are robust to alternative choices of the gain parameter.

PP Z_t stat.	10% critical value	DF-GLS	10% critical value
-1.5577	-1.6129	-1.701	-1.903

Table 2.16. Unit root tests of the sentiment shock

and the first component of price forecasts (up to a stationary or *i.i.d* measurement error).[36]

Two properties of the sentiment shock are established. First, Table 2.16 reports the statistics of PP and DF-GLS testing of the constructed sentiment shock along with 10% critical values. Both tests suggest the null hypothesis that this series contains a unit root cannot be rejected at 10% level. Second, using the Johansen test, we find no cointegration between the sentiment shock and consumption forecasts (the results are omitted here and available upon request).

Note sentiment here is incorporated into price forecast, while in the literature it is usually added to fundamentals (e.g., consumption or dividend). It is the difference between survey expectation forecasts and the forecasts generated by an econometric model made by agents.[37] Thus, this sentiment shock may be substantially different from e.g., the Consumer Sentiment Index in the Michigan Survey of Consumers or the Consumer Confidence Index provided by the Conference Board.

[36]Note the integration and cointegration property of the constructed sentiment shock is not affected by measurement errors as long as the measurement errors are stationary or *i.i.d.* over time.

[37]Further research on the formation of stock price expectations would help to understand better the nature of this sentiment shock which drives stock price forecasts but not consumption forecasts.

CHAPTER 3

Information Frictions and the Paradox of Price Flexibility

3.1. Introduction

Electronic shelf labels (ESL) permits retailers to set prices digitally without any costs that would otherwise occur using paper price tags. Over the past decade, we have witnessed an expansion in the usage of digital price tags thanks to the growing affordability of ESL.[1] The introduction of digital price tags may facilitate price adjustment and reduce the degree of nominal rigidity in the economy. Is such technological progress welfare-improving? The thought experiment of a similar nature was first exploited by Keynes (1936) and later formalized by Long and Summers (1986). They postulated that an increase in price flexibility might increase output volatility.[2] While the recent trend of technological progress in price-setting has made this thought experiment likely to occur in the near future, its welfare consequence and implications for the conduct of monetary policy has received limited attention in the academic circle.

We address this research question in a multi-sector New Keynesian (NK) model with information frictions and dispersed beliefs. Empirical evidence supporting

[1] See the report by Global Market Insights: https://www.globenewswire.com/news-release/2018/09/18/1572161/0/en/Electronic-Shelf-Label-ESL-Market-to-hit-1bn-by-2024-Global-Market-Insights-Inc.html

[2] Bhattarai et al. (2018) and Gali (2013) revisited this issue recently.

the presence of information frictions and dispersed beliefs are abundant, see, e.g., Coibion and Gordodnichenko (2012, 2015), Andrade et al. (2016) and Coibion et al. (2018) for a recent survey. We highlight a new channel—*dispersed belief channel* that is relevant to understand the welfare consequence of a reduction in nominal rigidity. The multi-sector feature of the model permits us to analyze the consequence of both an economy-wide and a sectoral reduction in nominal rigidities. The latter is relevant as the technological progress, such as the expansion of the ESL, might be merely a sectoral phenomenon rather than an economy-wide change. Moreover, the multi-sector model that we build allows us to revisit the optimal inflation index stabilization policy (Aoki 2001, Mankiw and Reis 2003, Benigno 2004). In particular, we show how monetary policy contributes to the welfare consequence of increased price flexibility, and draw policy recommendations in case such a thought experiment materializes.

To fully understand the welfare consequence of a change in price flexibility, we derive the welfare loss function around the perfect information steady-state and decompose it into four components. The first two components are proportional to inefficient dispersions in price and quantities within sectors arise due to price stickiness and dispersed beliefs. Such dispersions are inefficient because goods matter for households' utility symmetrically, and production technologies are identical. First, due to nominal rigidity such as staggered prices *à la* Calvo (1983), price dispersion arises across those that can reset prices and those that are staggered with old ones. We denote this as the *Calvo channel*. The welfare losses associated with this component are a hump shape function of price rigidity. Second, dispersed information that gives birth to imperfect common knowledge or dispersed beliefs creates another channel through which price dispersion arises. Firms have different assessments about the state of the economy due to information frictions. Therefore, in contrast to a standard model with perfect information, price dispersion emerges among those firms that reset prices. We denote this as the *dispersed beliefs channel*. The associated welfare losses increase monotonically in the degree of price flexibility. The third component is proportional to the output gap volatility, and the fourth is proportional to the relative price gap across sectors. Both tend to increase as the nominal friction increases. The aggregate effect of increased price flexibility on welfare is thus ambiguous ex-ante.

We derive the following results. First, in a static and symmetric two-sector model, we derive analytically the conditions under which an economy-wide increase in price flexibility is welfare-deteriorating—the paradox of price flexibility. With perfect information, such a reduction in nominal friction is welfare-improving. However, in the presence of information frictions, the dispersed beliefs channel might dominate. Consequently, the paradox of price flexibility arises. The relative importance of the dispersed beliefs component as compared to other elements of the welfare loss function depends on three key parameters: the signal-noise ratio that characterizes how dispersed the agents' beliefs are, and the elasticities of substitution across goods both within a sector and across sectors. Those elasticities determine, respectively, the magnitude of welfare losses generated by price dispersion within a sector and relative price gap across sectors.

Second, the paradox is more severe if the reduction in nominal frictions is merely a sectoral phenomenon. In our baseline analysis, we focus on an inflation-targeting central bank that fully stabilizes the Consumer Price Index (CPI), which is the principal mandate among many central banks in the world. Given this policy, increased sectoral price flexibility is detrimental to social welfare, even in the absence of information frictions. The result is driven by the across-sector spillover effect: increased flexibility in one sector, conditional on the same monetary policy, makes price in the other sector more volatile. The introduction of information frictions renders the paradox of price flexibility aggravated due to the dispersed beliefs channel. Those results also hint on the sub-optimality of a CPI stabilization policy.

We study the design of optimal inflation index stabilization policy, emphasizing the role of the dispersed beliefs. The objective is to find the price/inflation index, constructed by the weighted average of sectoral prices/inflations, associated with the lowest welfare losses. The presence of information frictions brings a new insight into the design of an optimal price index stabilization policy. Aoki (2001) and Benigno (2004) stress the importance of constructing an optimal price/inflation index that is proportional to sectoral nominal frictions. A sector with a relatively more flexible price deserves a smaller weight as the welfare cost of inflation in this sector, which arises through the Calvo channel, is lower. The dispersed belief channel that we emphasize dampens the results arising from the Calvo channel. Through the former, the same level of inflation causes much more price dispersions (among price resetting firms) in that sector if its nominal rigidity is reduced. Therefore, ignoring the dispersed beliefs channel would lead to a policy recommendation that under-reacts to inflation of a relatively more flexible price sector. Moreover, due to dispersed beliefs, the optimal weight that a fully flexible price sector receives is nontrivial. In other words, in contrast to Aoki (2001)'s advice, our results suggest that the central bank should not focus on core inflation solely.

Does an optimal inflation stabilization policy resolve the paradox of price flexibility? An economy-wide change in nominal frictions does not alter the optimal inflation index, and therefore, the optimal inflation index stabilization policy does not play a role in this case.[3] When a reduction in nominal frictions is asymmetric across sectors, the optimal price index stabilization policy mitigates the paradox of price flexibility. After a reduction in nominal frictions in one sector, the central bank reacts optimally by assigning a higher weight on the price in the second sector at the cost of higher volatility in the first sector. With perfect information, the benefit strongly dominates the cost, and aggregate welfare losses are reduced.

[3]In fact, in the symmetric case where nominal rigidities in both sectors are the same, the optimal inflation index coincides with the CPI, as it is shown by Mankiw and Reis (2003) and Benigno (2004).

Therefore, the policy mitigates the paradox substantially. However, in the presence of information frictions, an additional source of cost arises. A more volatile price in the first sector is associated with a higher price dispersion among price resetting firms due to the dispersed beliefs. This adds additional challenge for a central bank to resolve the paradox/improve social welfare.

We extend our static model into a dynamic stochastic general equilibrium model with nominal rigidity and dispersed beliefs. Our findings survive in a dynamic model. Under reasonable calibrations, with perfect information, an economy-wide reduction in nominal rigidity is welfare-improving only if the reduction is sufficiently large: a drop of average price duration by two quarters (the average duration of a price drops from four quarters to two quarters). Because of information frictions, the paradox of price rigidity arises irrespective of the magnitude of a reduction in nominal rigidity.[4] This result holds under an optimal inflation index stabilization policy as the optimal inflation index coincides with the CPI. When the reduction in price rigidity only occurs in one sector, a strong paradox arises if the central bank does not adjust its policy optimally. An optimal inflation index stabilization policy would mitigate the paradox. In this case, for it to be welfare-improving, the cut in nominal rigidity needs to surpass more than 1.5 quarters of reduction in average price duration if the information were perfect, and more than 2.5 quarters in the presence of information frictions.

[4] We do not consider the extreme case when price becomes fully flexible.

3.2. Literature Review

The theoretical framework is related to the literature that incorporates dispersed beliefs into business cycle models (e.g. Lucas (1972), Woodford (2001), Nimark (2008), Lorenzoni (2009), Angeletos and Jennifer (2009), Hellwig and Venkateswaran (2009), Angeletos and La O (2013), Huo and Takayama (2015a,b), Melosi (2016), Angeletos et al. (2016), Angeletos and Lian (2018), Huo and Pedroni (2019)).[5] We contribute to this literature by extending a NK model with dispersed beliefs to a multi-sector framework and derive different components of welfare loss function explicitly. Moreover, we focus on the implications of interactions of nominal rigidity and information frictions for social welfare.

Our paper contributes to the literature that studies the implications of a reduction in nominal rigidity. The idea that an increase in price flexibility may increase output volatility dates back to Keynes (1936), was formalized by Long and Summers (1986), and recently it is revisited by Bhattarai et al. (2018). Gali (2013) concludes that a reduction in wage rigidity improves welfare only if the central bank reacts to inflation sufficiently aggressive in a closed-economy New Keynesian model featuring both price rigidity and wage rigidity. Following similar reasoning, in an open economy with the fixed exchange rate (Gali and Monacelli 2016) or a closed economy when the monetary policy is constrained by the Zero Lower Bound (Amano and Gnocchi (2017) and Billi et al. (2018)), a labor market reform that

[5]See Mankiw and Reis (2002), Sims (2003), Mackowiak and Wiederholt (2009) for other applications of information frictions for business cycle analysis.

results in a more flexible wage is not necessarily welfare-improving. Our contribution with respect to this literature is to emphasize the role of information frictions or dispersed beliefs in the welfare analysis of a changing nominal rigidity. With this, we introduce a new channel that was previously ignored in the literature: the dispersed belief channel. We show that a more flexible price is welfare-deteriorating even if i) it does not increase output volatility ii) monetary policy strongly reacts to the inflation.

Lastly, this paper contributes to the literature that studies the optimal inflation index: e.g., Erceg, Henderson, and Levin (2000), Aoki (2001), Mankiw and Reis (2003), Benigno (2004), Huang and Liu (2005), Anand,Prasad and Zhang (2015), Basu and Leo (2016) and Zhang (2018). Most of those papers rely on nominal rigidities, except for Anand et al. (2015) who study the role of financial frictions; and Zhang (2018) who studies the interaction between markup and nominal rigidity and its implication for the optimal inflation index stabilization policy. We contribute to this line of research by investigating the role of information frictions.

The remainder of the paper is organized as follows. Section 2 presents a static model and derive the analytical solution to shed light on the role of dispersed beliefs for the implications of a reduction in the nominal rigidity for social welfare and the optimal inflation index stabilization policy. Section 3 builds a dynamic model. Section 4 conducts quantitative analysis. Section 5 concludes.

3.3. Static Model

In this section, we present a static model to illustrate our mechanism, and we derive analytical solutions.

3.3.1. The Economy

The static model is a two-sector economy with imperfect common knowledge among firms. Sectors differ in the degree of nominal rigidity and are subject to sector-specific shocks.

Household. We assume a representative household with complete information. A household is composed of two types of workers. Each type of them is endowed with a sector-specific skill and supply their labor in that sector. There is no labor mobility across sectors. The household makes sectoral labor supply and aggregate consumption decision to maximize the expected utility with the following utility function:

$$U(C, L) = \frac{C^{1-\sigma}}{1-\sigma} - \sum_{k=1}^{2} L_k,$$

subject to the budget constraint:

$$PC = \sum_{k=1}^{2} W_k L_k + \sum_{k=1}^{2} \Pi_k + T,$$

where C is the consumption, P is the price of the consumption goods, L_k denotes the amount of labor that a household member supplies to sector k, W_k is the nominal wage in sector k, π_k is the profits generated by firms in sector k and T

captures the net transfer from government. The final good that enters household's utility function is defined as:

$$C = \left[\sum_{k=1}^{2} n_k^{1/\eta} C_k^{(\eta-1)/\eta} \right]^{\eta/(\eta-1)}$$

where n_k is the size of the sector k with n_k summing up to one and η measures the degree of substitution among goods across sectors. The sector k's output is a CES aggregate of different varieties C_{ki} :

$$C_k = \left[n_k^{-1/\epsilon} \int_0^1 C_{ki}^{(\epsilon-1)/\epsilon} di \right]^{\epsilon/(\epsilon-1)},$$

where ϵ measures the degree of substitution among varieties within a sector k. The implied aggregate prices are:

$$P = \left[\sum_{k=1}^{2} n_k P_k^{1-\eta} \right]^{1/(1-\eta)} \qquad P_k = \left[\int_0^1 p_{k,i}^{1-\epsilon} di \right]^{1/(1-\epsilon)}$$

where P denotes the economy-wise aggregate price and P_k denotes the aggregate price at sector k.

Solving the consumer's problem yields the following demand functions:

$$C_k = \left(\frac{P_k}{P} \right)^{-\eta} n_k C \qquad C_{ki} = \left(\frac{p_{k,i}}{P_k} \right)^{-\epsilon} \frac{1}{n_k} C_k. \tag{3.1}$$

Firms. There is a continuum of monopolistic competitive firms in each sector. Both sectors share the production function of the same functional form:

$$Y_{ki} = e^{a_k} L_{ki}, \tag{3.2}$$

where the (log) productivity a_k is unknown. We assume a fraction $(1-\theta_k)$ of firms in sector k are free to adjust prices, and the remaining θ_k of firms are not allowed to reset price. The reoptimizing firms would choose the optimal price to maximize the expected profits:

$$\max_{P^*_{k,i}} E\left\{ P^*_{k,i} Y_{ki} - \frac{W_k Y_{ki}}{e^{a_k}} | \mathcal{I}_{ki} \right\} \tag{3.3}$$

subject to the demand functions 3.1, production function 3.2 and market clearing conditions $C_{ki} = Y_{ki}$, $C_k = Y_k$, $C = Y$. $\mathcal{I}_{ki}$ is the information set of firm i in sector k. We will explain it carefully later.

Solving the profit maximization problem of a firm i in sector k yields the following loglinearized optimal price setting rule:

$$p^*_{k,i} = E[p + \sigma(y - y^N) + u_k | \mathcal{I}_{ki}] \tag{3.4}$$

All variables are in log deviations from their initial values. p is the aggregate price defined as $p = \sum_1^k p_k$. p_k is the sector level price denoted as $p_k = \int_i p_{k,i}$. As a fraction θ_k of firms are staggered to (log) price at initial level zero in sector k, therefore $p_k = (1-\theta_k)p_k^*$ where $p_k^* = \int_i p_{k,i}^*$. y^N is the natural output under the assumption that all firms are allowed to adjust their prices with perfect information, which is linear in the productivities a_k[6], u_k is proportional to the relative sectoral productivity, in particular: $u_1 = n_2(a_2 - a_1)$ and $u_2 = n_1(a_1 - a_2)$. Let x denotes the output gap $(y - y^N)$ and we can rewrite equation (3.4) as:

$$p_{k,i}^* = E[p + \sigma x + u_k | \mathcal{I}_{ki}] \tag{3.5}$$

We focus on the case in which the size of two sectors is equal, *i.e.*, $n_1 = n_2$.[7] In this case:

$$u_1 = -u_2 = u \tag{3.6}$$

Therefore, u summarizes the exogenous unknown aggregate state variables that matter for firms' optimization problem. The aggregate state u is normally distributed according to:

$$u \sim N(0, \sigma_u^2) \tag{3.7}$$

Each firm i in sector k receives a signal $s_{k,i}$ about the unobserved state u:

$$s_{k,i} = u + e_{k,i} \quad with \quad e_{k,i} \sim N(0, \sigma_e^2) \tag{3.8}$$

---

[6]In particular, $y^N \equiv \frac{1}{\sigma}(n_1 a_1 + n_2 a_2)$.

[7]When the size of the two sectors is not equal, $u_1 + \frac{n_1}{n_2} u_2 = 0$ and our following results are robust to it.

Each firm draws a signal with the same precision, but realizations are different, and therefore, beliefs are dispersed across firms. Firms apply Bayes' theorem to form beliefs about the unobserved state:

$$E(u|\mathcal{I}_{ki}) = Ks_{k,i}, \tag{3.9}$$

with $K = \frac{\sigma_u^2}{\sigma_e^2+\sigma_u^2}$. Firms have the same belief updating rule governed by the signal-noise ratio. However, beliefs are dispersed due to the different signals that firms receive.

The Central Bank. The central bank conducts a price index stabilization policy. At the beginning of the world, before the realization of shocks, the central bank is perfectly informed about the structure of the economy. The central bank chooses the price index that she commits to stabilize when shocks hit the economy. That is, ex-ante the central chooses the weight, ω, on sector 1's price, such that ex-post the price index is constant.

$$\omega p_1 + (1-\omega)p_2 = 0, \tag{3.10}$$

In a special case, when $\omega_k = n_k$, the price index that the central bank commits to stabilize coincides with the CPI.

We will analyze two types of monetary policy. The first is that the central bank commits to stabilize the CPI. In the second scenario, the central chooses the price index optimally. In particular, optimal price index stabilization policy minimizes welfare loss function subject to the equilibrium conditions (3.5) with the definition of aggregate price p, (3.10) and firms' beliefs updating rule (3.9). The welfare loss function will be derived later.

The Timing of the Model. The model consists of four stages. In stage one, the central bank chooses the inflation index that she commits to stabilize. In stage two, nature draws a shock u, and each firm i sector k receives a private signal $s_{k,i}$ about u. In stage three, each firm i forms a belief and decides a price-setting plan. In the last stage, the representative household observes the state of the economy and makes the consumption and labor decision. Simultaneously, the goods and labor markets clear.

Model Solution. The solution of the model is characterized by (3.5) with the definition of aggregate price p, (3.10) and the firms' beliefs updating rule (3.9).

Solving those equations, we obtain the following aggregate allocations:

$$p_1^* = 2(1-\omega)Ku, \tag{3.11}$$

$$p_2^* = -2\omega Ku, \tag{3.12}$$

$$x = -(1-K+\theta K)(2\omega-1)u \tag{3.13}$$

where x denotes the output gap.

3.3.2. Welfare analysis

Our objective is to evaluate the consequence of a change in nominal rigidity θ_k in terms of welfare. To this end, we derive the welfare loss function as the second order approximation of the household's utility function:

$$L = \sum_{k=1}^{K} \left(\epsilon n_k var_i\{p_{k,i}\} \right) + \sigma x^2 + \eta \sum_{k=1}^{K} n_k \widetilde{p}_{R,k}^2$$

By combining the welfare loss function with the solution of the model, we show that the unconditional welfare losses can be rewritten as:

$$E(L) = \epsilon \underbrace{\sum_{i=1}^{2} \left\{ \underbrace{2(1-\theta_i)\theta_i(1-\omega)^2 K^2 \sigma_u^2}_{\text{Calvo}} + \underbrace{2(1-\theta_i)(1-\omega)^2 K^2 \sigma_e^2}_{\text{Dispersed belief}} \right\}}_{\text{Price dispersions}}$$

$$+ \underbrace{\sigma \left\{ (1-K+\theta K)^2 (2\omega-1)^2 \sigma_u^2 \right\}}_{\text{Output gap volatility}} + \underbrace{\eta (1-K+\theta K)^2 \sigma_u^2}_{\text{Relative price gap across sectors}} \tag{3.14}$$

The welfare loss function consists of three components: i) price dispersions within a sector, ii) output gap volatility and iii) relative price gap across sectors (or relative output gap). Such dispersions and volatilities reflect the degree of inefficiency of the economy because goods matter for households' utility symmetrically and production technologies are identical.

The novelty, in a dispersed belief model, as compared to the perfect information case, is that two channels drive price dispersions. The first is the standard *Calvo* *channel* that leads to price dispersions across the group of firms who can reset prices with those who cannot. The second is the *dispersed belief channel* that generates price dispersions *within* the group of price resetting firms. Firms form different beliefs about the state of the economy. Therefore, in contrast to a standard model with perfect information, price dispersion emerges among those firms that can reset prices.

Symmetric Reduction in Nominal Frictions. We are now ready to discuss the welfare implication of an increase in price flexibility. We begin with the analysis of a special case in which the two sectors are symmetric with the same degree of nominal rigidity i.e., $\theta_1 = \theta_2 = \theta$, but are subject asymmetric shocks. We study how a change in the degree of nominal rigidity in the entire economy, i.e., θ, affects welfare. We will derive an analytical solution for the welfare losses as a function of the Calvo parameter θ under the assumption that the central bank fully stabilizes the CPI. It is worth noting that our results do not depend on this assumption. As long as zero welfare losses are not achievable, which is always the case in a multi-sector model due to asymmetric shocks (Woodford, 2011), our results will always survive. Moreover, within the family of full price index stabilization policy, the full stabilization of the CPI is the optimal one in a multi-sector model with symmetric sectoral characteristics.

We begin with the discussion of the case in which there is no information imperfection, i.e., $\sigma_e = 0$.

Proposition 12. *Consider a two-sector model with symmetric sectoral characteristics. Under CPI index stabilization policy, and with perfect information, when $\eta \geq \frac{1}{4}\epsilon$, the welfare losses are increasing in $\theta \in (0,1)$; when $\eta < \frac{1}{4}\epsilon$, the welfare loss function is concave in θ, and it attains the maximum at $\theta^{max} = \frac{\epsilon}{2\epsilon - 4\eta}$.*

Note that η determines the relative importance of the relative price gap in the welfare loss function. The welfare losses from the relative price gap are monotonically increasing in θ. The higher is the nominal rigidity (a higher θ), the bigger is the gap between the relative price (across sectors) and the desired one, hence bigger welfare losses. Therefore, if η is relatively large $\eta \geq \frac{1}{4}\epsilon$, the relative price gap component is dominating in the welfare loss function, which explains the first half of the Proposition (12). The standard calibration in the literature is to set η equal to 1. Although researchers disagree on the values for ϵ, they are normally bigger or equal to 6. Hence, the condition $\eta \geq \frac{1}{4}\epsilon$ is violated and we proceed to the discussion of the second half of the proposition.

Now consider the Calvo price dispersion component and the output gap volatility component. In general, the output gap volatility component is weakly increasing in the degree of price rigidity θ. However, in the symmetric case that we consider here, the stabilization of the CPI is equivalent to the full stabilization of the output gap, as it can be inferred from equation (3.5). Therefore, the output gap is always equal to zero across different values of θ. Price dispersion originating from the Calvo pricing friction is a hump-shaped function of nominal rigidity.

To understand this, consider the two extreme cases: one with full price flexibility ($\theta = 0$) and one with full price rigidity ($\theta = 1$). In both cases, firms either all reset the same optimal price or have to keep the initial price (steady-state price) that is the same across all firms. Hence, there is no price dispersion in those two extreme cases. When $\theta \in (0, 1)$, the dispersion in prices arises among the group of firms who can reset prices with those who cannot change their prices. Given that the loss function is well-behaved, it must be a concave function in $\theta \in (0, 1)$. Moreover, the welfare losses peak at the point when $\theta^{max} = \frac{\epsilon}{2\epsilon - 4\eta}$. Naturally, θ^{max} depends on ϵ and η as those parameters capture the relative importance of price dispersion within a sector and relative price gap across sectors for welfare. This hump shape loss function thus suggests that if a decrease in price rigidity (θ) is welfare-improving if it takes place in the parameter region $\theta < \theta^{max}$. To give a sense of how big this parameter space is, let's take the popular calibration described above: $\epsilon = 6$, $\eta = 1$. θ^{max} is equal to 0.75, a value that coincides with the popular calibration of the Calvo parameter in the literature and it is of a similar magnitude with the empirical estimate provided by Nakamura and Steinsson (2008). Therefore, a model with full information predicts that given the current nominal rigidity in the economy, an increase in price flexibility is welfare-improving. Figure (3.1) provides a visual representation of those results discussed above by plotting the components of the welfare losses as a function of price rigidity using the calibration described above.

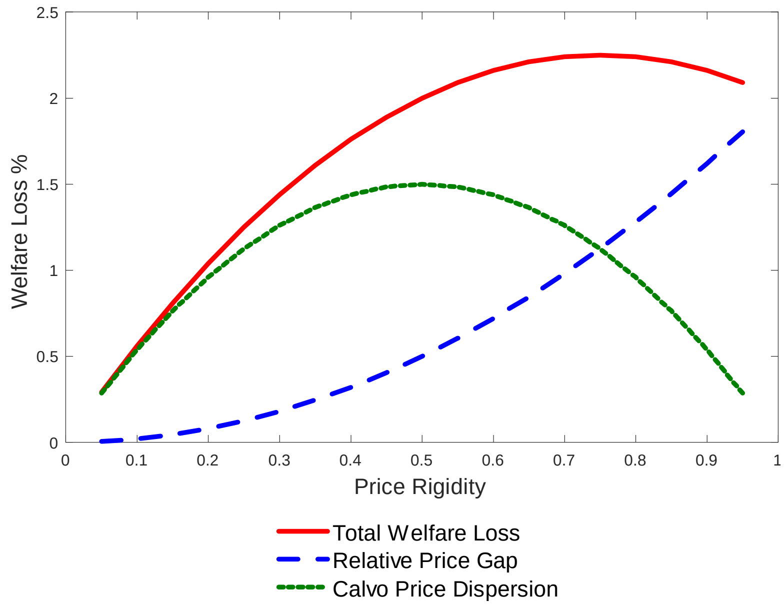

Figure 3.1. Decomposition of the Welfare Losses: Perfect Information

In the following discussion, we allow for information frictions and dispersed beliefs.

Proposition 13. *Consider a two-sector model with symmetric sectoral characteristics. Under CPI stabilization policy, when* $\eta \geqslant \frac{1}{4}\epsilon$*, the welfare losses are increasing in* $\theta \in (0,1)$*; when* $\eta < \frac{1}{4}\epsilon$*, the welfare loss function is concave in* θ*, and it attains the maximum at* $\theta^{max} = \frac{\epsilon}{2\epsilon - 4\eta} - \frac{\epsilon - 4\eta}{2\epsilon - 4\eta}\frac{\sigma_e^2}{\sigma_u^2}$*, and* θ^{max} *is increasing in the signal-to-noise ratio* $\frac{\sigma_a}{\sigma_e}$*. When* $\frac{\sigma_e^2}{\sigma_u^2} > \frac{\epsilon}{\epsilon - 4\eta}$*, the welfare losses are decreasing in* $\theta \in (0,1)$*.*

Figure (3.2) demonstrates proposition 13 visually. It plots the welfare losses as a function of price rigidity under different scenarios. The dotted green line

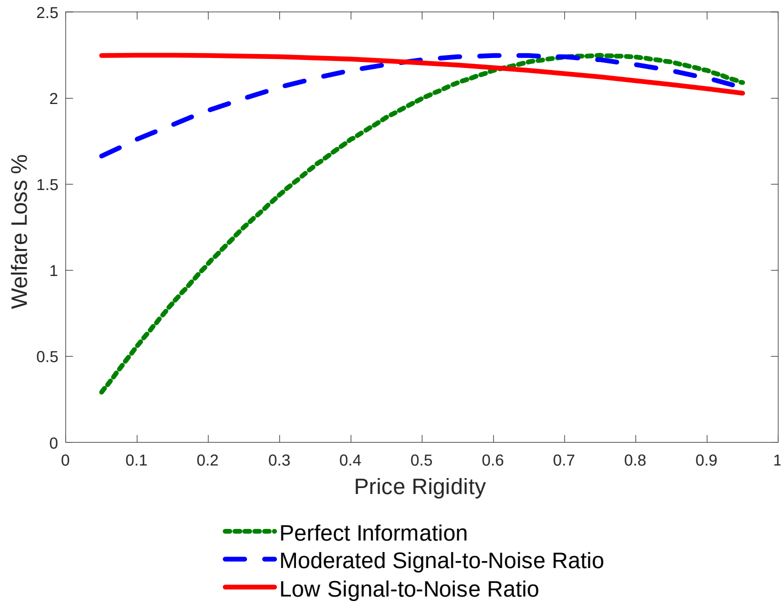

Figure 3.2. Welfare Losses under Different Level of Information Frictions

presents the perfect information case (the same as the solid red line in Figure 3.1). The dashed blue line plots the prediction of the model with dispersed beliefs and moderated signal-to-noise ratio, i.e., moderated degree of information friction. The solid red line shows the result with a low level of the signal-to-noise ratio. If the degree of information friction is sufficiently strong, in contrast to the case with perfect information (solid black line), the paradox of price flexibility arises — reducing price rigidity is no longer welfare-improving!

The above result is entirely driven by the dispersed beliefs channel. With information imperfection and dispersed beliefs, dispersion in prices arises within the group of firms who can reset prices. The dispersed belief component of the welfare losses is strictly increasing in the price flexibility $(1-\theta)$. This is verified

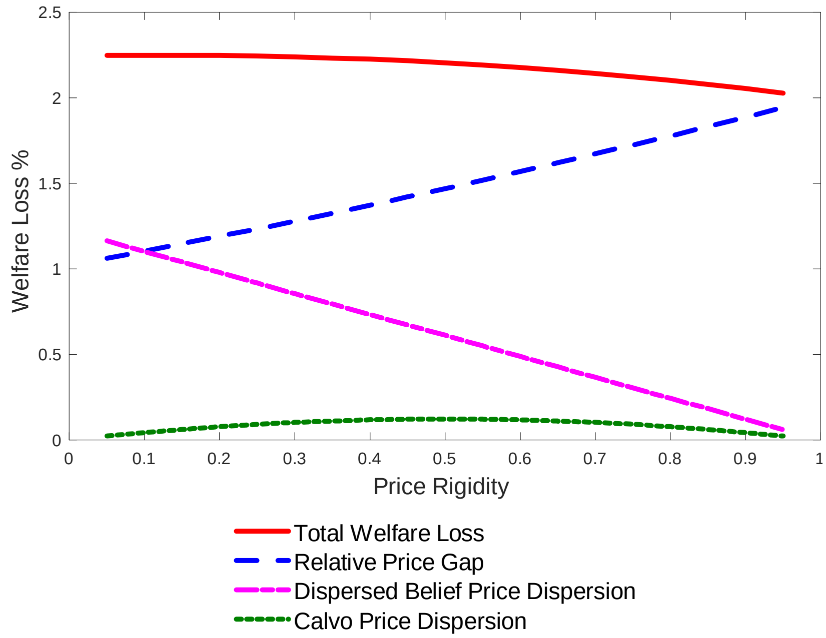

Figure 3.3. Welfare Losses Decomposition: Information Frictions

in Figure (3.3), which plots each component of the welfare losses as a function of price rigidity for the case with dispersed beliefs. The dash-dotted pink line is strictly decreasing in price rigidity.

Asymmetric Reduction in Nominal Frictions. So far, we have analyzed the consequence of an economy-wide reduction in θ. We now turn to the scenario that only one sector benefits from the technology that facilitate price adjustment, and therefore price flexibility increases in that sector. The remaining sectoral characteristics such as the sector size and the volatility of shocks are assumed to be the same.

Figure 3.4 shows how the welfare losses change when the degree of nominal rigidity in sector 1 varies. The dashed blue line in the left panel plots the result for the model with perfect information and with a central bank that commits to stabilize the CPI. As one can see, a decease in θ_1 leads to higher welfare losses. To further investigate which component of the welfare loss function drives the result, we plot welfare decomposition under different calibration of θ_1 in Figure 3.8. Again, the dashed blue line corresponds to the case where the central bank stabilize the CPI. Figure 3.8 shows that the main driver of the paradox is the price dispersion in sector 2: it increases substantially as prices in sector 1 becomes more flexible. The intuition is the following. Consider a positive productive shock that occurs in sector 1, output in this sector increases although not as much as the rise in its natural level due to nominal rigidity. This generates an increase in aggregate demand, which results in a positive output gap in sector 2 hence price in the second sector rises. Now if the nominal rigidity in sector 1 were lower, everything else equal, the initial increase in sector 1's output would be higher that would ultimately result into an even higher price in Sector 2. Therefore, a bigger price dispersion arises in sector 2.

The dashed blue line in the right panel of Figure 3.4 shows that the paradox of price flexibility holds for a model with information frictions under the CPI stabilization policy. However, a detailed inspection shows that the main source of the paradox is the price dispersion originated from dispersed beliefs. as it is evidenced by dashed blues in Figure 3.9.

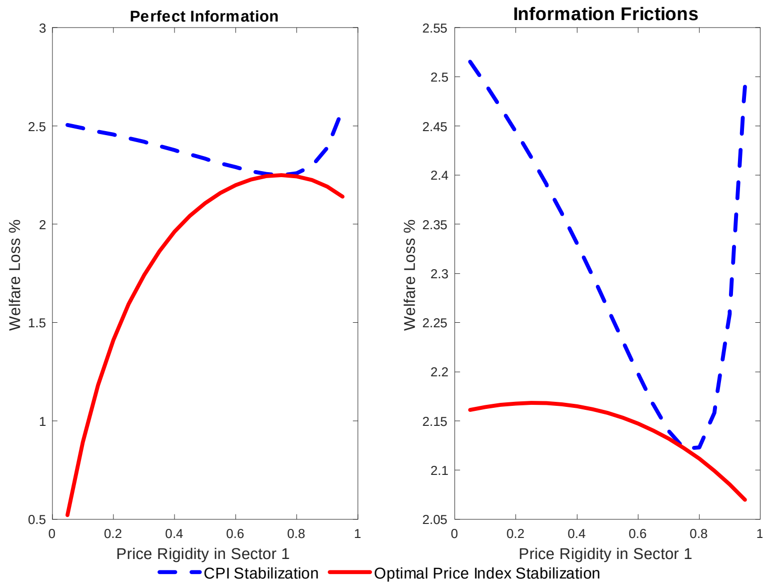

Figure 3.4. Welfare Losses and Nominal Rigidity in Sector 1

3.3.3. Optimal Inflation Index Stabilization Policy

We have looked at the welfare consequence of a reduction in θ_1, holding everything else constant including the CPI stabilization policy. Clearly, as it is hinted by the substantial increase of price dispersion in sector 2, such a policy is suboptimal. We now turn to the discussion of the optimal inflation index stabilization policy and to check whether an optimal policy can resolve the paradox.

The solid red lines in Figure 3.4 plots the result of our thought experiment: how welfare losses vary with price rigidity in sector 1, conditional on a central bank that conducts the optimal price index stabilization policy. Under such a policy, for each value of θ_1, the central bank recalculates the optimal price index—the one associated with the lowest welfare losses. Interestingly, the optimal inflation index

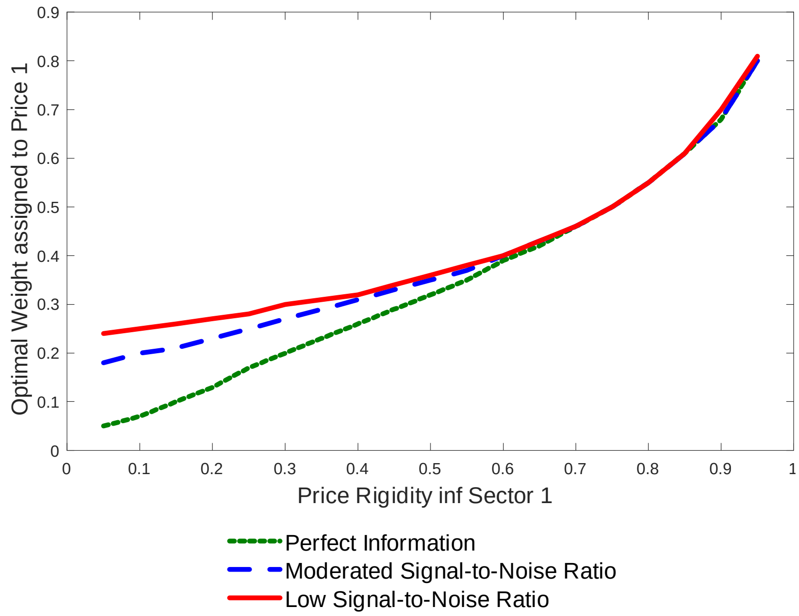

Figure 3.5. Optimal Price Index

stabilization policy resolves the paradox of price flexibility in a model with perfect information, as it is shown in the left panel. Under the optimal policy, the central bank is aware of the risk of high volatility of price in sector 2 when price in sector 1 becomes more flexible. As a response, the central bank assigns a higher weight to p_2 in the price index that it aims to stabilize. The dotted green line in Figure 3.5 plots this result: the optimal weight assigned to p_1 (p_2) is strictly increasing (decreasing) in θ_1, consistent with the prediction of Benigno (2004). The success of this policy is shown in Figure 3.8. Under the optimal price index stabilization policy (red solid lines), the price dispersion in sector 2 is dampened as compared to the case with the stabilization of the CPI, at the cost of a higher price dispersion in sector 1.

In contrast to the model with perfect information, in the presence of information frictions and dispersed beliefs, the paradox of price flexibility survives even under the optimal price index stabilization policy, as it is shown in the right panel of Figure 3.4. This is due to the emergence of price dispersions originated from dispersed beliefs, see Figure 3.9. The optimal price index stabilization policy is unable to attenuate the welfare losses resulting from belief dispersions. As discussed above, a reduction in price rigidity in sector 1 leads to a bigger price adjustment in sector 2, and therefore the central bank reacts to assign a higher weight on p_2 at the cost of a more volatile price in sector 1. However, the welfare cost of price volatility in sector 1 decreases as its nominal friction declines. This is Benigno (2004)'s result. In contrast, in the presence of dispersed beliefs, price dispersion among price resetting firms arises. A more volatile price in sector 1 results into a larger price dispersions due to dispersed beliefs, which is costly in terms of welfare. Moreover, as the fraction of price resting firms increase, the dispersed beliefs price dispersion channel becomes more dominant. As a result, it offsets the Calvo channel highlighted in Benigno (2004). This is shown in Figure 3.5, in a model with information frictions, the optimal weight attached to p_1 decreases as θ_1 drops, however the decline is more moderated as compared to the case with perfect information.

This result hints on the importance o the consideration of information frictions in the design of optimal inflation index stabilization policy. To the best of our knowledge, this is not yet addressed in the literature. We postpone the discussion about the quantitative importance of a such consideration to the dynamic version of the model that we present in the next section.

Summing up. To sum up what we have derived so far in a static model. First, in a symmetric two-sector model, an economy-wide increase in price flexibility is likely to be welfare-improving in the absence of information frictions. Second, with information frictions and dispersed beliefs, the paradox of price flexibility arises. Third, this paradox arises both in the case with or without information frictions if the reduction in nominal frictions is merely a sectoral phenomenon and the central bank stabilizes CPI. Fourth, the optimal price index stabilization policy is capable of resolving the paradox but only in a model with perfect information. Fifth, price dispersion from dispersed beliefs offsets the Calvo channel highlighted in Benigno (2004), and therefore affects the design of the optimal price index.

In order to evaluate the quantitative importance of those results, we extend our model to a dynamic version.

3.4. Dynamic model

In this section, we present a dynamic multi-sector model with dispersed beliefs. The two-sector economy is populated by a continuum of (0,1) of households, a fraction of n_k of monopolistic competitive firms in sector k for $k = 1, 2$, and a fiscal-monetary authority. A fraction of θ_k of firms (Calvo probability) in sector k are allowed to reset their prices in each period. Households consume the composite goods, buy a one-period risk-free government bond, supply labor to the sectoral competitive labor market, receive dividends (profits) from firms and pay taxes or receive transfers from the fiscal-monetary authority. Firms demand labor to produce and sell goods to households. The fiscal-monetary authority issues government bonds, collect lump-sum taxes or pay transfers to households , sets the nominal interest by following a simple rule and chooses the inflation target weight to maximize social welfare.

3.4.1. The economy

3.4.1.1. Households. Households have complete information. The representative household chooses consumption, bond holding, and labor supply to maximize her lifetime utility:

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \Big(\frac{C_t^{1-\sigma}}{1-\sigma} - \sum_{k=1}^{K} \frac{L_{k,t}^{1+\varphi}}{1+\varphi} \Big)$$

where β is the discount factor, σ is the relative risk aversion and φ governs the elasticity of labor supply. The budget constraint of the household in period t reads as:

$$P_t C_t + Q_t B_{t+1} = B_t + \sum_{k=1}^{K} W_{kt} L_{kt} + \sum_{k=1}^{K} \Pi_{kt} + T_t \tag{3.15}$$

where P_t is the price level of the composite good, Q_t is the nominal one-period risk-free bond price, and W_{kt} is the sectoral nominal wage, Π_{kt} stands for sectoral profit, and T_t stands for lump-sum taxes/transfers. The composite good that enters into household's utility function is a Dixit-Stiglitz aggregator of the following form:

$$C_t = \left[\sum_{k=1}^{K} n_k^{1/\eta} C_{k,t}^{(\eta-1)/\eta} \right]^{\eta/(\eta-1)},$$

where n_k is the size of the sector k and the weights n_k sum up to one and $C_{k,t}$ is sectoral output which is a Dixit-Stiglitz aggregator of the following form:

$$C_{k,t} = \left[n_k^{-1/\varepsilon} \int_0^{n_k} C_{ki,t}^{(\varepsilon-1)/\varepsilon} di \right]^{\varepsilon/(\varepsilon-1)}.$$

The bond holdings satisfy a no-Ponzi-scheme condition.

Solving the consumer's problem yields the following demand functions:

$$(3.16) \qquad C_{k,t} = \left(\frac{P_{k,t}}{P_t} \right)^{-\eta} n_k C_t \qquad C_{ki,t} = \left(\frac{P_{ki,t}}{P_{kt}} \right)^{-\varepsilon} \frac{1}{n_k} C_{kt}$$

where the implied aggregate price and sectoral prices level are:

$$(3.17) \qquad P_t = \left[\sum_{k=1}^{K} n_k P_{k,t}^{1-\eta} \right]^{1/(1-\eta)} \qquad P_{k,t} = \left[n_k^{-1} \int_0^{n_k} P_{ki,t}^{1-\varepsilon} di \right]^{1/(1-\varepsilon)}.$$

3.4.1.2. Firms. There is a continuum of monopolistic competitive firms in each sector. Each firm i in a sector k is endowed with a common linear production function:

$$Y_{ki,t} = e^{a_{k,t}} L_{ki,t},$$

where the log productivity $a_{k,t}$ is homogenous within a sector and it evolves according to an autoregressive process:

$$a_{k,t} = \rho_k a_{k,t-1} + \epsilon_{k,t}$$

where the sectoral technology shock follows a normal distribution with mean zero and standard error σ_{a_k}.[8]

Firms are subject to nominal rigidity à la Calvo (1983): each firm in a sector k may reset its price with probability $(1-\theta_k)$. Hence, the log level price at sector k, p_{kt}, evolves as the following:

$$p_{kt} = \theta_k p_{kt-1} + (1-\theta_k)p^*_{kt},$$

where p^*_{kt} is the average of the optimal price that a re-optimizing firm at sector k would set to, the latter is the solution to the following problem:

$$\max_{p^*_{ki,t}} \sum_{h=0}^{\infty} \theta_k^h E_t \left\{ Q_{t,t+h}(P^*_{ki,t} Y_{ki,t+h|t} - \Psi_{ki,t+h}(Y_{ki,t+h|t})) \right\}$$

subject to the demand schedule specified in (3.16). $Q_{t,t+h} \equiv \beta^h (C_{t+h}/C_t)^{-\sigma}(P_t/P_{t+h})$ denotes the stochastic discount factor, $\Psi_{ki,t+h}$ is the nominal cost function and $Y_{ki,t+h|t}$ is the output for a firm i in sector k that last reset its price in period t.

[8] We abstract from firm level productivity shocks. In the presence of idiosyncratic shocks, the optimal target inflation rate will not be equal to zero: see Blanco (2016) and Adam and Weber (2019) for a detailed discussion. However, incorporating idiosyncratic shocks will not affect the main message of our paper: core inflation is not the optimal choice of inflation index.

The optimality condition implied by the firm's problem is:

$$\sum_{h=0}^{\infty} \theta_k^h E_t \left\{ Q_{t,t+h} Y_{ki,t+h|t} (P^*_{ki,t} - \frac{\varepsilon}{\varepsilon - 1} \Psi'_{ki,t+h}(Y_{ki,t+h|t})) \right\} = 0.$$

3.4.1.3. The monetary policy. The central bank has a unique mandate of inflation stabilization. in particular, it commits to stabilize the following inflation index:

$$\omega_1 \pi_1 + (1 - \omega_1)\pi_2 = 0.$$

Similar to the static model, we analyze two types of policies: i) a CPI stabilization policy, and ii) the central bank chooses ω_1 optimally to minimize the welfare losses.

3.4.1.4. Firm's information set. Both firms and households are rational and know the structure of the economy. However, firms can not observe the true sectoral TPF shocks. Instead, firms receive signals about sectoral TFP shocks. Specifically, a firm i in sector l receives a signal $s_{k,l,t}(i)$ about sectoral shock a_{kt} for all k in period t:

$$s_{k,l,t}(i) = a_{kt} + e_{k,l,t}(i) \quad with \quad e_{k,l,t}(i) \sim N(0, \sigma^2_{e_k}) \tag{3.18}$$

where $e_{k,l,t}(i)$ is a white noise. The precision so signals are common across firms in the entire economy. That is, we abstract from heterogeneity in the degree of information frictions among firms. Imperfect signals capture the idea that the firms might not pay full attention to economic indicators. The information set of a firm i in sector l is:

$$\mathcal{I}_{l,t}(i) \equiv \{s_{k,l,\tau}(i), \hat{y}_{R,kt-1}, P_{ki,t} : \forall k, \forall l, \forall \tau \leq t\}.$$

Note that we have made the following simplifying assumptions that are common in the literature.[9] First, firms are allowed to observe the lagged state variable, i.e., $\hat{y}_{R,kt-1}$ the lagged relative output. Otherwise, firms would need to form high-order beliefs about $\hat{y}_{R,kt-1}$ that increases the number of state variables in the model exponentially. Second, $s_{k,l,t}(i)$ summarizes all remaining information that firms possess and conceptually we exclude the possibility that firms observe output and labor perfectly and simultaneously. That is, they cannot back up the underlining unobserved productivity shocks perfectly.

It is worth mentioning that a firm's action, $P_{ki,t}$, is a redundant piece of information as its action is based on its posterior belief. Therefore, it does not reflect any further information.

3.4.2. Linearized Equilibrium Conditions

We log-linearize the household's and firm's problem around the perfect information deterministic steady state and obtain the dynamic IS equation and sectoral New Keynesian Phillips curve (or NKPC for short). Throughout this paper, a variable with tilde denotes this variable in deviation from its natural level. And a variable

[9]See e.g., Nimark (2008) and Melosi (2016).

with hat denotes this variable in deviation from its steady-state. The imperfect-common-knowledge sectoral NKPC are obtained as the following:

(3.19)

$$\pi_{kt} = (1-\theta_k)(1-\beta\theta_k)\sum_{j=1}^{\infty}(1-\theta)^{j-1}(\overset{\wedge}{mc}{}^{(j)}_{kt|t} - \hat{p}{}^{(j)}_{R,kt|t}) + \beta\theta_k\sum_{j=1}^{\infty}(1-\theta_k)^{j-1}\overset{\wedge}{\pi}{}^{(j)}_{kt+1|t}$$

where

$$\overset{\wedge}{mc}{}^{(j)}_{kt|t} = \sigma(y_t - y^N)^{(j)} + \varphi(y_{kt|t} - y^N_{kt|t})^{(j)} + \eta^{-1}(y^N_{t|t} - y^N_{kt|t})^{(j)}. \quad (3.20)$$

Regarding notations, for a variable z_t the following notation for high-order expectations are used:

$$\begin{aligned} z^{(0)}_{t|t} &\equiv z_t \\ z^{(1)}_{t|t} &\equiv \int\int E(z_t|\mathcal{I}_{l,t}(i))didl \\ z^{(2)}_{t|t} &\equiv \int\int E(z^{(1)}_{t|t}|\mathcal{I}_{l,t}(i))didl \\ z^{(j)}_{t|t} &\equiv \int\int E(z^{(j-1)}_{t|t}|\mathcal{I}_{l,t}(i))didl. \end{aligned}$$

From the goods demand equation, the relative price relates to the relative output according to:

$$\hat{p}{}^{(j)}_{R,kt|t} = -\eta^{-1}(y_{kt|t} - y_{t|t})^{(j)} \quad (3.21)$$

so that the sectoral NKPC can be rewritten as:

$$\pi_{kt} = (1-\theta_k)(1-\beta\theta_k)\Theta\sum_{j=1}^{\infty}(1-\theta_k)^{j-1}\Big[(\sigma+\varphi)\tilde{y}^{(j)}_{t|t}+$$

$$(\eta^{-1}+\varphi)(\hat{y}^{(j)}_{R,kt|t}-\Phi(a^{(j)}_{kt|t}-\sum_{k=1}^{K}n_k a^{(j)}_{kt|t}))\Big]+\beta\theta_k\sum_{j=1}^{\infty}(1-\theta_k)^{j-1}\overset{\wedge}{\pi}^{(j)}_{kt+1|t}. \tag{3.22}$$

where $\Phi \equiv \frac{1+\varphi}{\eta^{-1}+\varphi}$, and we have used the fact that $\hat{y}^{N,(j)}_{R,kt|t} = \Phi(a^{(j)}_{kt|t} - \sum_{k=1}^{K} n_k a^{(j)}_{kt|t})$. In (3.22), the high-order expectations are weighted by $(1-\theta_k)^{j-1}$ and the impact of expectations is decreasing as the order of expectation increases. The smaller the Calvo parameter θ_k, the more important high-order expectations are.

The law of motion for $\hat{y}_{R,kt}$ is:

$$\hat{y}_{R,kt} = -\eta(\pi_{kt} - \sum_{k=1}^{K} n_k\pi_{kt}) + \hat{y}_{R,kt-1}, \tag{3.23}$$

where we have used the definition of π_{kt}, π_t, and equation (3.21) for $j = 0$. Therefor, one can easily see that $\hat{y}_{R,kt-1}$ is a state variable in period t.

The Dynamic IS equation is obtained by log-linearizing the Euler equation:

$$\tilde{y}_t = E_t\tilde{y}_{t+1} - \frac{1}{\sigma}(i_t - E_t\sum_{k=1}^{K} n_k\pi_{kt+1} - r^N_t) \tag{3.24}$$

where $r^N_t = \rho + \sigma\psi^a \sum_{k=1}^{K} n_k E \bigtriangleup a_{kt+1}$ with $\psi^a = \frac{1+\varphi}{\sigma+\varphi}$.

Finally recall the monetary authority's policy rule that completes the description of the economy:

$$\omega_1\pi_1 + (1-\omega_1)\pi_2 = 0. \tag{3.25}$$

3.4.3. Solution

We follow the method described in Nimark (2008) and Melosi (2016) to solve the two-sector model with imperfect-common-knowledge. We leave the detailed description of the algorithm to the Appendix 3.7.3. When the model is solved, the policy function for endogenous variables $s_t \equiv [\pi_{kt}\ \tilde{y}_t \hat{y}_{R,kt}]'$ reads as:

$$s_t = v_0 X_{t|t}^{(0:J)} \tag{3.26}$$

where the state vector $X_{t|t}^{(0:J)} \equiv [\ a_{1,t|t}^{(s)}, a_{2,t|t}^{(s)}, \hat{y}_{R,1t-1} : 0 \leq s \leq J]'$ contains the hierarchy of average high-order expectations about the exogenous state variables $X_t \equiv [a_{1t}\ a_{2t}]'$. As a standard way to tackle the infinite dimension of the state vector, we truncated it up to order J to keep the dimension of state vector finite. The state equation can be summarized as:

$$X_{t|t}^{(0:J)} = M X_{t-1|t-1}^{(0:J)} + N\boldsymbol{\epsilon}_t. \tag{3.27}$$

To keep notation simple, denote $\mathbf{X}_t \equiv X_{t|t}^{(0:J)}$ and denote $Z_{lt}(i) \equiv [s_{1,l\tau}(i)\ s_{2,l\tau}(i)\ y_{R,1t-1}]'$ as signals received by firm i in sector l in period t. Then the observation equation becomes:

$$Z_{lt}(i) = L\mathbf{X}_t + Q\mathbf{e}_{lt}(i). \tag{3.28}$$

Equation (3.27) and (3.28) consist of a linear state space, which calls for the Kalman filter to estimate the unobserved state vector that contains high-order expectations. Simultaneously, the Kalman filter also determines the law of motion of the high-order beliefs, that is, determines the matrices M and N in (3.28). Hence the problem of solving the model boils down to finding a fixed point over the parameters v_0 that characterizes the policy function (3.26) and M and N that governs the law of motion of the state vector (3.27).

Note that so far we have solved the economy for any given the inflation index that the central bank targets in the Taylor rule.

3.4.4. Welfare loss function and policy

Following Rotemberg and Woodford (1997, 1999)and Woodford (2002), the second order approximation of the representative consumer's period welfare loss expressed in consumption equivalent variation (CEV) is:

$$\mathbb{L} = E_0 \sum_{k=1}^{K} \frac{\varepsilon n_k}{\Theta} var_i\{p_{ki,t}\} + (\sigma + \varphi) var(\widetilde{y}_t) + \sum_{k=1}^{K} (\eta^{-1} + \varphi) n_k var(\widetilde{y}_{R,kt}).$$

Similar to a multi-sector model with complete information, the welfare loss function consists of i) price dispersions; ii) the volatility of aggregate output gap; and iii) the volatility of relative output gap across sectors. The novelty in the presence of information frictions is that the price dispersions can be further decomposed into:

$$var_i\{p_{ki,t}\} = \frac{1}{1-\beta\theta_k}[(1-\theta_k)\int_i (p^*_{ki,t} - p^*_{k,t})^2 di + \frac{\theta_k}{1-\theta_k} var(\pi_{k,t})].$$

In contrast to the case with complete information, dispersion across price resetting firms $\int_i (p^*_{ki,t} - p^*_{k,t})^2 di$ arises because firms have different assessments about the state of the economy. The detailed derivation for the welfare loss function can be found in the Appendix.

As discussed earlier, under an optimal price index stabilization policy, the benevolent monetary authority chooses inflation target weight $\{\omega_k\}$ to minimize the unconditional welfare losses. The problem for the monetary authority can be formalized as follows:

$$\min_{\{\omega_k\}} E_0 \sum_{k=1}^{K} \frac{\varepsilon n_k}{\Theta} \frac{1}{1-\beta\theta_k}[(1-\theta_k)\int_i (p^*_{ki,t} - p^*_{k,t})^2 di + \frac{\theta_k}{1-\theta_k} var(\pi_{kt})]$$
$$+ (\sigma + \varphi) var(\widetilde{y}_t) + \sum_{k=1}^{K} (\eta^{-1} + \varphi) n_k var(\widetilde{y}_{R,kt})$$

subject to equilibrium conditions discussed in the previous section.

This welfare loss function nests the standard welfare loss function with perfect information as a special case. Without dispersed beliefs, each firm i in a sector k would set the same price, i.e., $\int_i (p^*_{ki,t} - p^*_{k,t})^2 di = 0$. In the absence of information frictions, the inflation of a flexible price sector ($\theta_k = 0$) does not enter in the welfare loss function. In other words, inflation in this sector is costless in terms of welfare. This is the key result derived by Aoki (2001), and Woodford (2011) provides a textbook treatment.

However, in the presence of dispersed beliefs, the flexible price sector's inflation generates a welfare loss as it creates price dispersion across re-optimizing firms. Targeting the core inflation is no longer optimal.

3.5. Results

3.5.1. Calibration

The calibrations of parameters are listed in Table (3.1) and Table (3.2). The utility function is log-linear in consumption and linear in labor. To highlight the new channel introduced by information frictions, we consider the two-sector economy in which sectors are identical in the following characteristics: the size of the sector, the process for productivity shocks and elasticity of substitution. However, sectors differ in the degree of nominal rigidity. In particular, we allow nominal rigidity in sector 1 to vary and fix the nominal rigidity in sector 2 such that $\theta_2 = 0.75$. We estimate the variance of TFP shock a, its persistence and the variance of noise by matching the standard deviation of inflation, the persistence of inflation and inflation forecast dispersion.

Panel A	Fixed Parameters
Utility function	$\sigma = 1, \varphi = 0$
Discount factor	$\beta = 0.99$
Size of sectors	$n_1 = 0.5, n_2 = 0.5$
Nominal rigidity	$\theta_2 = 0.75$
Elasticity of substitution	$\eta = 1, \varepsilon_1 = \varepsilon_2 = 6$
Panel B	Calibrated Parameters
Exogenous shocks	$\sigma_{a_1}^2 = \sigma_{a_2}^2 = (0.078)^2$
Signal-to-Noise ratio	$\sigma_a/\sigma_e = 2.04$
Persistent of shocks	$\rho_1 = \rho_2 = 0.54$

Table 3.1. Calibration

Description	Data Moments	Model Moments
S.D. of inflation	0.0024	0.0024
Average inflation forecast dispersion	0.0018	0.0018
Inflation AR coefficient	0.65	0.60

Table 3.2. Model Fit

3.5.2. Symmetric Reduction in Nominal Frictions

Figure 3.6 plots the welfare losses against an economy-wide change in nominal rigidity, both for a model with perfect information (the left panel) and with information frictions (the right panel). Note that, we conducted this thought experiment both under the CPI stabilization and an optimal price index stabilization policy. Since sectors are symmetric, the optimal price index coincides with the CPI and therefore the two curves overlap. Without information frictions, given the current level of nominal rigidity in the U.S., i.e., $\theta = 0.75$ that corresponds to an average price duration of four quarters, it would be welfare-deteriorating if the price is more flexible. However, a significant reduction in nominal friction, that

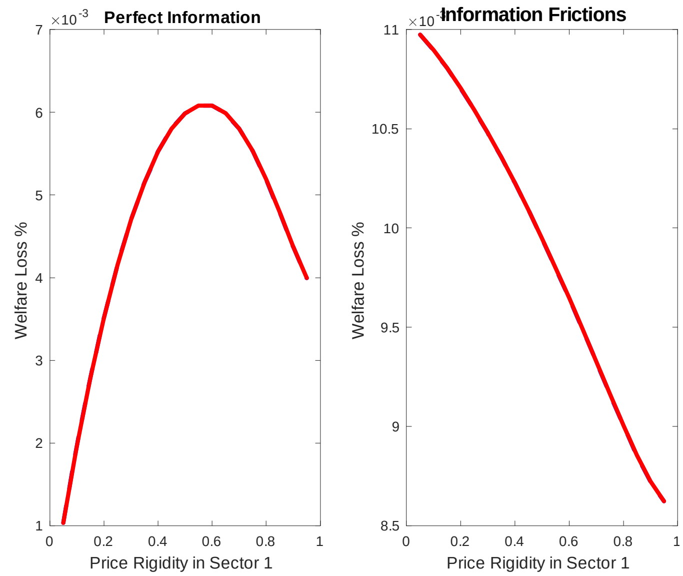

Figure 3.6. Welfare Losses: Economy-wide changes in Nominal Frictions

lowers the average price duration to two quarters (θ smaller than 0.5), would improve social welfare. In contrast, the presence of information frictions aggravates the paradox substantially: an economy-wide improvement in nominal frictions is always welfare-deteriorating! Furthermore, the central bank, by setting the inflation index optimally, cannot resolve this paradox.

3.5.3. Asymmetric Reduction in Nominal Frictions

We now consider the thought experiment when only nominal rigidity in sector 1 varies. Figure 3.7 plots the result, both for a model with perfect information (the left panel) and with information frictions (the right panel). As explained above in the static model, the paradox of price flexibility arises even in a model

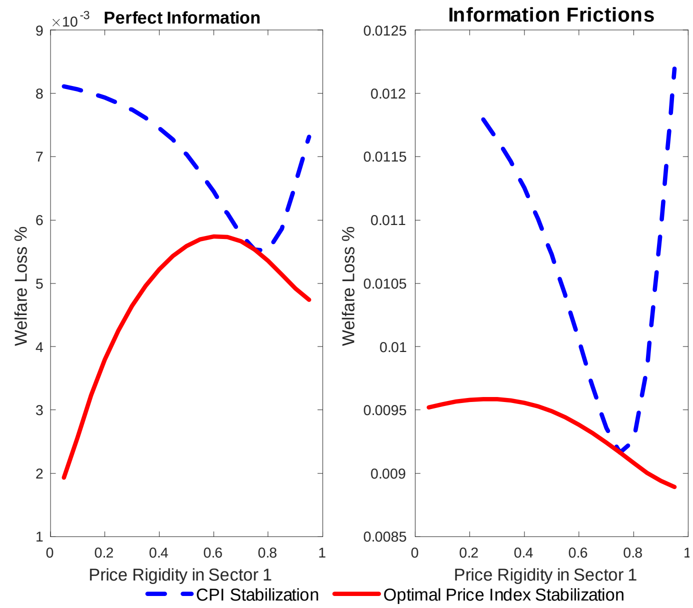

Figure 3.7. Welfare Losses: asymmetric changes in Nominal Frictions

with perfect information if the inflation index that the central bank stabilizes is a sub-optimal one (CPI). An increased price flexibility in sector 1 brings about a more volatile price in sector 2, which is very costly if the central bank does not adjust its policy properly. Similar to the static economy, the optimal inflation index stabilization policy greatly mitigates the paradox in the case of perfect information. However, the paradox still remains to a large extent when it comes to the case with information frictions.

3.6. Conclusion

We have derived the following results, based on a multi-sector NK model with dispersed beliefs. First, in a symmetric two-sector model, an economy-wide increase in price flexibility is likely to be welfare-improving in the absence of information frictions. Second, with information frictions and dispersed beliefs, the paradox of price flexibility arises. Third, this paradox arises independent of information frictions if the reduction in nominal frictions is merely a sectoral phenomenon and the central bank stabilizes CPI. Fourth, the optimal price index stabilization policy is capable of resolving the paradox but only in a model with perfect information. Fifth, with price dispersion originated from dispersed beliefs offsets the Calvo channel highlighted in Benigno (2004), and therefore affects the design of the optimal price index.

Increased price flexibility might not be far away until it reaches our society, given the current trend of technological progress. Such progress that is beneficial from an individual firm's perspective might be harmful to social welfare. Of course, the first best solution to resolve the paradox of price flexibility is to reduce information frictions. However, this might be challenging. For example, if information imperfection arises from a human's limited ability of reasoning or rational inattention, as it is emphasized by Sims (2003) and Mackowiak and Wiederholt (2009), there is little hope to reduce such friction. Our analysis suggests the important role that the central bank has to play to mitigate or resolve the paradox of price flexibility when the reduced nominal frictions is a sectoral phenomenon.

Our results point to a market inefficiency that deserves a further investigation in future research. In our model, from an individual firm's perspective, it would be better off if the firm was allowed to adjust its price more frequently as they would be able to respond to shocks on a timely manner. Therefore, firms would pay for new pricing technologies. However, if all individual firms adopt the new technology to facilitate the price adjustment, the aggregate welfare losses may increase, as we discussed above. Thus, the constrained social planner allowing for the presence of dispersed information would not introduce the new technology.

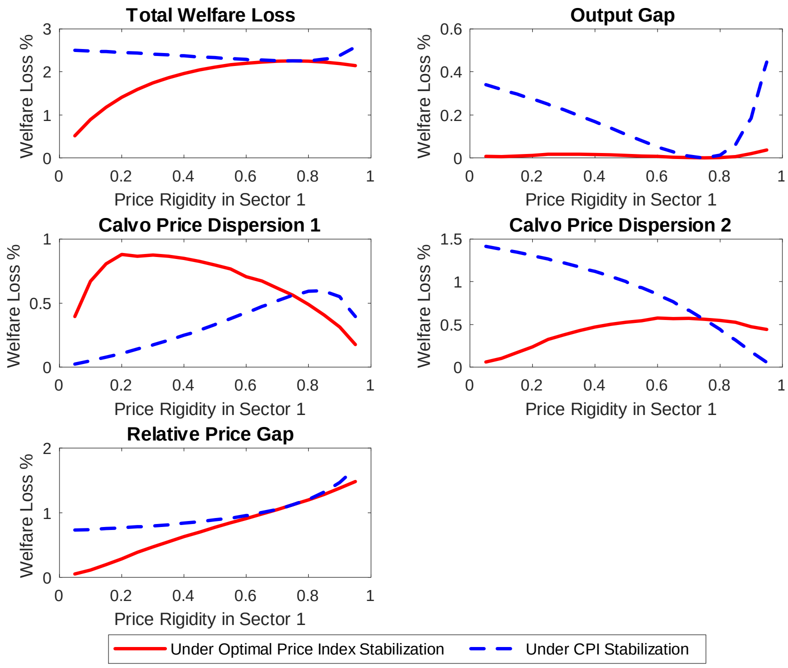

Figure 3.8. Welfare Losses Decomposition: Perfect Information

3.7. Appendix

3.7.1. Figures

3.7.2. Welfare Loss function

The second order Taylor expansion of the representative household's utility U_t around a steady-state (C, L) in terms of log deviations can be written as:

$$U_t - U \approx U_c C\left(\widehat{y}_t + \frac{1-\sigma}{2}\widehat{y}_t^2\right) + \sum_{k=1}^{K} U_{L_k} L_k \left(\widehat{l}_{kt} + \frac{1+\varphi}{2}\widehat{l}_{kt}^2\right) di.$$

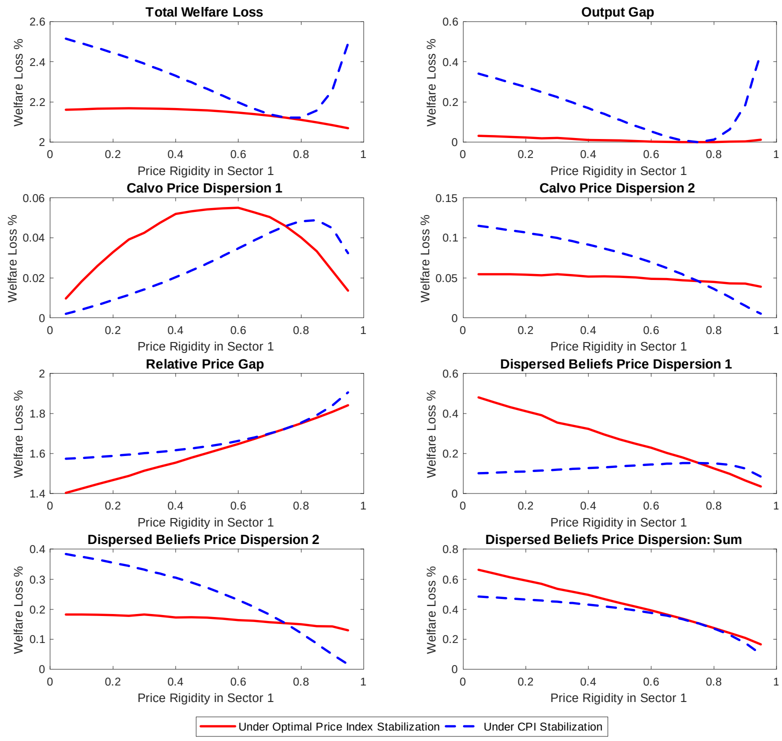

Figure 3.9. Welfare Losses Decomposition: Dispersed Beliefs

Note that

$$\widehat{l}_{kt} = \widehat{y}_{kt} - a_{kt} + d_{kt}$$

where $d_{kt} \equiv log(\frac{P_{kt}(i)}{P_{kt}})^{-\epsilon}$.

Lemma 14. $d_{kt} = \frac{\epsilon}{2} var_i\{p_{kt}(i)\}$

Proof: Gali (2008, chapter 4)

Therefore,

$$U_t - U \approx U_c C\Big(\widehat{y}_t + \frac{1-\sigma}{2}\widehat{y}_t^2\Big) + \sum_{k=1}^{K} U_{L_k} L_k \Big(\widehat{y}_{kt} + \frac{\epsilon}{2} var_i\{p_{kt}(i)\} + \frac{1+\varphi}{2}(\widehat{y}_{kt} - a_{kt})^2\Big) + t.i.p.$$

where $t.i.p$ denotes the terms independent of policy. Under the assumption that cost of employment is subsidized optimally at sectorial level to eliminate distortions originate from monopolistic competition, the steady-state is efficient and $-\frac{U_{L_k}}{U_c} = MPN$.

Approximate the aggregate C_t around $c_k = c + log(n_k)$:

$$\sum_{k=1}^{K} n_k \widehat{y}_{kt} \approx \widehat{y}_t - \frac{1-\eta^{-1}}{2} \sum_{k=1}^{K} n_k \widehat{y}_{R,kt}^2$$

with $\sum_{k=1}^{K} n_k \widehat{y}_{R,kt}^2 \equiv \sum_{k=1}^{K} n_k (\widehat{y}_{kt} - \widehat{y}_t)^2$. Using the fact that $MPN = (Y_k/L_k), Y = C$, it follows that:

$$\frac{U_t - U}{U_c C} \approx -\frac{1}{2}\Big[\sum_{k=1}^{K}\Big(\epsilon n_k var_i\{p_{kt}(i)\}\Big) - (1-\sigma)\widehat{y}_t^2 - (1-\eta^{-1})\sum_{k=1}^{K} n_k \widehat{y}_{R,kt}^2$$

$$+ (1+\varphi)\sum_{k=1}^{K} n_k(\widehat{y}_{kt} - a_{kt})^2\Big] + t.i.p$$

$$= -\frac{1}{2}\Big[\sum_{k=1}^{K}\Big(\epsilon n_k var_i\{p_{kt}(i)\}\Big) - (1-\sigma)\widehat{y}_t^2 - (1-\eta^{-1})\sum_{k=1}^{K} n_k \widehat{y}_{R,kt}^2$$

$$+ (1+\varphi)\sum_{k=1}^{K} n_k(\widehat{y}_{kt}^2 - 2\widehat{y}_{kt} a_{kt})\Big] + t.i.p$$

$$= -\frac{1}{2}\Big[\sum_{k=1}^{K}\Big(\epsilon n_k var_i\{p_{kt}(i)\}\Big) + (\sigma+\varphi)\widehat{y}_t^2 + (\eta^{-1}+\varphi)\sum_{k=1}^{K} n_k \widehat{y}_{R,kt}^2$$

$$- 2(1+\varphi)\sum_{k=1}^{K} n_k \widehat{y}_{kt} a_{kt}\Big] + t.i.p$$

$$= -\frac{1}{2}\Big[\sum_{k=1}^{K}\Big(\epsilon n_k var_i\{p_{kt}(i)\}\Big) + (\sigma+\varphi)\widehat{y}_t^2 + (\eta^{-1}+\varphi)\sum_{k=1}^{K} n_k \widehat{y}_{R,kt}^2$$

$$- 2(1+\varphi)\sum_{k=1}^{K} n_k(\widehat{y}_t + \widehat{y}_{kt} - \widehat{y}_t)a_{kt})\Big]$$

$$= -\frac{1}{2}\Big[\sum_{k=1}^{K}\Big(\epsilon n_k var_i\{p_{kt}(i)\}\Big) + (\sigma+\varphi)\widehat{y}_t^2 + (\eta^{-1}+\varphi)\sum_{k=1}^{K} n_k \widehat{y}_{R,kt}^2$$

$$- 2(\sigma+\varphi)\widehat{y}_t y^N - 2(\eta^{-1}+\varphi)\sum_{k=1}^{K} n_k(\widehat{y}_{kt} - y_t)(\widehat{y}_{kt}^N - y_t^N)\Big] + t.i.p$$

$$= -\frac{1}{2}\Big[\sum_{k=1}^{K}\Big(\epsilon n_k var_i\{p_{kt}(i)\}\Big) + (\sigma+\varphi)\widetilde{y}_t^2 + (\eta^{-1}+\varphi)\sum_{k=1}^{K} n_k \widetilde{y}_{R,kt}^2\Big] + t.i.p$$

where $\widetilde{y}_t \equiv y_t - y_t^N$. From line 2 to line 3, we have used the fact that$\sum_{k=1}^K n_k \widehat{y}_{kt}^2 = \sum_{k=1}^K n_k \widehat{y}_{R,kt}^2 + (\sum_{k=1}^K n_k \widehat{y}_{kt})^2 \approx \sum_{k=1}^K n_k \widehat{y}_{R,kt}^2 + \widehat{y}_t^2$. From line 4 to line 5, we have used the fact that $\sum_{k=1}^K n_k a_{kt} = \frac{\sigma+\varphi}{1+\varphi} y_t^N$ and $a_{kt} - \sum_{k=1}^K n_k a_{kt} = \frac{\eta^{-1}+\varphi}{1+\varphi}(y_{kt}^N - y_t^N)$.

The discounted life-time welfare loss is

$$\mathbb{W} = E_0 \sum_{t=0}^{\infty} \beta^t \sum_{k=1}^{K} \Big(\epsilon n_k var_i\{p_{kt}(i)\} \Big) + (\sigma+\varphi)\widetilde{y}_t^2 + (\eta^{-1}+\varphi) \sum_{k=1}^{K} n_k \widetilde{y}_{R,kt}^2.$$

$\widetilde{y}_t$ and $\widetilde{y}_{R,kt}$ are the endogenous variables we solve for. We hereby derive for $\Big(\epsilon n_k var_i\{p_{kt}(i)\}\Big)$. Let $\Delta_{k,t} \equiv var_i\{p_{ki,t}\}$

$$\begin{aligned}
\Delta_{k,t} &= \theta_k \Delta_{k,t-1} + (1-\theta_k) \int_i (p_{ki,t}^* - p_{k,t}^* + p_{k,t}^* - p_{k,t-1})^2 di - \pi_{k,t}^2 \\
&= \theta_k \Delta_{k,t-1} + (1-\theta_k) \int_i (p_{ki,t}^* - p_{k,t}^*)^2 di + (1-\theta_k) \int_i (p_{k,t}^* - p_{k,t-1})^2 di - \pi_{k,t}^2 \\
&= \theta_k \Delta_{k,t-1} + (1-\theta_k) \int_i (p_{ki,t}^* - p_{k,t}^*)^2 di + \frac{\theta_k}{1-\theta_k} \pi_{k,t}^2.
\end{aligned}$$

Therefore,

$$E_0 \sum_{t=0}^{\infty} \beta^t var_i\{p_{ki,t}\} = E_0 \frac{1}{1-\beta\theta_k} \sum_{t=0}^{\infty} \beta^t [(1-\theta_k) \int_i (p_{ki,t}^* - p_{k,t}^*)^2 di + \frac{\theta_k}{1-\theta_k} \pi_{k,t}^2].$$

To summarize, the second order approximation of the representative consumer's period welfare loss as a fraction of steady-state consumption is:

$$\mathbb{L} = E_0 \sum_{k=1}^{K} \Big(\epsilon n_k var_i\{p_{kt}(i)\} \Big) + (\sigma+\varphi) var(\widetilde{y}_t) + (\eta^{-1}+\varphi) \sum_{k=1}^{K} n_k var(\widetilde{y}_{R,kt})$$

$$= E_0 \sum_{k=1}^{K} \Big(\epsilon n_k var_i\{p_{kt}(i)\} \Big) + (\sigma+\varphi) var(\widetilde{y}_t) + (\eta^{-1}+\varphi)\eta^2 \sum_{k=1}^{K} n_k var(\widetilde{p}_{R,kt})$$

(3.29)

$$= E_0 \sum_{k=1}^{K} \epsilon n \frac{1}{1-\beta\theta_k} [(1-\theta_k) \int_i (p^*_{ki,t} - p^*_{k,t})^2 di + \frac{\theta_k}{1-\theta_k} \pi^2_{k,t}]$$

$$+ (\sigma+\varphi) var(\widetilde{y}_t) + (\eta^{-1}+\varphi)\eta^2 \sum_{k=1}^{K} n_k var(\widetilde{p}_{R,kt})$$

As long as we solve for $\int_i (p^*_{ki,t} - p^*_{k,t})^2 di$, we obtain a quantitative assessment of the welfare loss. The following is for $\int_i (p^*_{ki,t} - p^*_{k,t})^2 di$.

$$p^*_{ki,t} - p_{k,t-1} = \frac{1}{1-\theta_k} H_k X^0_{t|t}(ki)$$

$$p^*_{k,t} - p_{k,t-1} = \frac{1}{1-\theta_k} H_k X^0_{t|t}$$

where $X^0_{t|t}(ki) = [s_{1,k,t}(i);\, s_{2,k,t}(i); E_{ki} X^0_{t|t}] \equiv [s_{1,k,t}(i); s_{2,k,t}(i)\ ; X^1_{t|t}(ki)]$ and $X^0_{t|t} = [a_{1t}; a_{2t}\ ; E X^0_{t|t}] \equiv [a_{1t}; a_{2t}\ ; X^1_{t|t}]$ is the state variable for aggregate endogenous variables in the economy. Note that the lag relative output is included in $X^0_{t|t}$.

Hence

$$p^*_{ki,t} - p^*_{k,t} = \frac{1}{1-\theta_k} H_k (X^0_{t|t}(ki) - X^0_{t|t}).$$

Then

$$\int \left(p^*_{ki,t} - p^*_{kt}\right)^2 di = \frac{1}{(1-\theta_k)^2} H_k \int (X^0_{t|t}(ki) - X^0_{t|t})(X^0_{t|t}(ki) - X^0_{t|t})' di H_k'.$$

Our aim is to find a solution for $\int (X^0_{t|t}(ki) - X^0_{t|t})(X^0_{t|t}(ki) - X^0_{t|t})' di$.

By the Kalman filter which will be discussed later, we have

$$X^1_{t|t}(ki) = (M - KDM) X^1_{t-1|t-1}(ki) + K[DMX_{t-1} + DN\epsilon_t + Qe_{k,t}(i)]$$

$$X^1_{t|t} = (M - KDM) X^1_{t-1|t-1} + K[DMX_{t-1} + DN\epsilon_t].$$

where $e_{k,t}(i) = \begin{bmatrix} e_{1,k,t}(i) \\ e_{2,k,t}(i) \end{bmatrix}$.

From the above two equations, we have the law of motion for $X^1_{t|t}(ki) - X^1_{t|t}$ which follows

$$X^1_{t|t}(ki) - X^1_{t|t} = (M - KDM)(X^1_{t-1|t-1}(ki) - X^1_{t-1|t-1}) + KQe_{k,t}(i)$$

We also know that

$$\begin{bmatrix} s_{1,k,t}(i) \\ s_{2,k,t}(i) \end{bmatrix} - \begin{bmatrix} a_{1,t} \\ a_{2,t} \end{bmatrix} = Q \begin{bmatrix} e_{1,k,t}(i) \\ e_{2,k,t}(i) \end{bmatrix}$$

Combine the above two equations, we obtain

$$X^0_{t|t}(ki) - X^0_{t|t} = \bar{M}(X^0_{t-1|t-1}(ki) - X^0_{t-1|t-1}) + \bar{N}e_{i,t}. \tag{3.30}$$

where

$$\bar{M} = \begin{bmatrix} 0 & 0 \\ 0 & M - KDM \end{bmatrix}$$

$$\bar{N} = \begin{bmatrix} Q \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ KQ \end{bmatrix}.$$

Using equation (3.30), we can obtain a solution for $\int (X^0_{t|t}(ki) - X^0_{t|t})(X^0_{t|t}(ki) - X^0_{t|t})' di$.

Welfare loss function in the static model. The counterpart of equation 3.29 in the static model is:

$$\mathbb{L} = \sum_{k=1}^{K} \left(\epsilon n_k var_i\{p_{k,i}\} \right) + \sigma x^2 + \eta \sum_{k=1}^{K} n_k \widetilde{p}^2_{R,k}$$

where x denotes output gap in the static model.

$$
\begin{aligned}
var_i p_{k,i} &= \int_i (p_{k,i} - p_k)^2 di \\
&= (1-\theta_k) \int_i (p^*_{ki} - p_k)^2 di + \theta_k p_k^2 \\
&= (1-\theta_k) \left(\int_i (p^*_{ki} - p^*_k)^2 di + (p^*_k - p_k)^2 \right) + \theta_k p_k^2 \\
&= (1-\theta_k)\phi_k^2 \sigma_e^2 + (1-\theta_k)\theta_k p_k^{*2}.
\end{aligned}
$$

Note that in the two-sector model $\sum_{k=1}^2 n_k \widetilde{p}_{R,k}^2 = n_1 n_2 (\widetilde{p}_1 - \widetilde{p}_2)^2$, therefore, the period welfare loss function in the static model is:

$$
\mathbb{L} = \epsilon \sum_{k=1}^{2} n_k \Big((1-\theta_k)\phi_k^2\sigma_e^2 + (1-\theta_k)\theta_k p_k^{*2} \} \Big) + \sigma x^2 + \eta n_1 n_2 (\widetilde{p}_1 - \widetilde{p}_2)^2
$$

3.7.3. Solution Method for the Static Model

In this section, we solve the static model with heterogeneous and imperfect information using the method of the undetermined coefficient. The method is shown in a general way, which allows u_1 and u_2 not to be perfectly linearly dependent on each other and each firm receives signals for both u_1 and u_2.

3.7.3.1. Solution method. We follow the solution first discussed by Morris and Shin (2002), and Veldkamp (2011) provides a textbook treatment. Given the quadratic loss function and linear constraints, we conjecture that the output gap will take the following form (our initial guess),

$$x = \beta_1 u_1 + \beta_2 u_2. \tag{3.31}$$

Step1: solving the firm's belief updating problem. A firm i in sector 1 updates her belief based on the Bayes' law,

$$\mathbb{E}_{1i}\begin{pmatrix} x \\ u_1 \\ u_2 \end{pmatrix} = G\begin{pmatrix} s_{u1,1}(i) \\ s_{u2,1}(i) \end{pmatrix} = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \\ G_{31} & G_{32} \end{pmatrix}\begin{pmatrix} s_{u1,1}(i) \\ s_{u2,1}(i) \end{pmatrix}, \tag{3.32}$$

with each entry of G defined as $\frac{cov(s,x)}{var(s)}$, where $cov(s,x)$ is the covariance of the unobserved fundamental and the observed signal, and $var(s)$ is the variance of the signal. In particular,

$$\begin{aligned} G_{11} &= \frac{cov(x, s_{u1,1}(i))}{var(s_{u1,1}(i))} = \frac{\beta_1 \sigma^2_{u1}}{\sigma^2_{u1} + \sigma^2_{u1,1}} \\ G_{12} &= \frac{cov(x, s_{u2,1}(i))}{var(s_{u2,1}(i))} = \frac{\beta_2 \sigma^2_{u2}}{\sigma^2_{u2} + \sigma^2_{u2,1}}. \end{aligned}$$

Similarly, a firm i in sector 2 updates her belief according to:

$$\mathbb{E}_{2i}\begin{pmatrix} x \\ u_1 \\ u_2 \end{pmatrix} = H\begin{pmatrix} s_{u1,2}(i) \\ s_{u2,2}(i) \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \\ H_{31} & H_{32} \end{pmatrix}\begin{pmatrix} s_{u1,2}(i) \\ s_{u2,2}(i) \end{pmatrix} \tag{3.33}$$

where

$$H_{11} = \frac{cov(x, s_{u1,2}(i))}{var(s_{u1,2}(i))} = \frac{\beta_1 \sigma_{u1}^2}{\sigma_{u1}^2 + \sigma_{u1,2}^2}$$

$$H_{12} = \frac{cov(x, s_{u2,2}(i))}{var(s_{u2,2}(i))} = \frac{\beta_2 \sigma_{u2}^2}{\sigma_{u2}^2 + \sigma_{u2,2}^2}.$$

Define $Z \equiv \begin{pmatrix} x \\ u_1 \\ u_2 \end{pmatrix}$, $S_1 \equiv \begin{pmatrix} s_{u1,1}(i) \\ s_{u2,1}(i) \end{pmatrix}$, $S_2 \equiv \begin{pmatrix} s_{u1,2}(i) \\ s_{u2,2}(i) \end{pmatrix}$, and denote the rows of G and H by G_1, G_2, G_3, H_1, H_2, H_3 respectively. Then equation (3.32) and (3.33) can be rewritten more compactly as:

$$\mathbb{E}_{1i} Z = G S_1$$

$$\mathbb{E}_{2i} Z = H S_2.$$

Define $g \equiv \begin{pmatrix} G_{21} & G_{22} \\ G_{31} & G_{32} \end{pmatrix}$, $h \equiv \begin{pmatrix} H_{21} & H_{22} \\ H_{31} & H_{32} \end{pmatrix}$, and $U \equiv \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ that will be used below to simplify notations.

Step 2: Recall that the individual optimal price-setting decision follows

$$p_{1,i}^* = \mathbb{E}_{1i}[n_1 p_1 + n_2 p_2 + \alpha x + u_1], \tag{3.34}$$

$$p_{2,i}^* = \mathbb{E}_{2i}[n_1 p_1 + n_2 p_2 + \alpha x + u_2] \tag{3.35}$$

Hence, we should obtain the representations for $\mathbb{E}_{ki}p_1$ $\mathbb{E}_{ki}p_2$ $\mathbb{E}_{ki}x$ $\mathbb{E}_{ki}u_1$ for $k = 1, 2$, which is what step 3 presents.

Step 3: Conjecture that

$$\begin{aligned} p^*_{1,i} &= \phi_{11} s_{u1,1}(i) + \phi_{12} s_{u2,1}(i) \\ &= \Phi_1 S_1 \\ p^*_{2,i} &= \phi_{21} s_{u1,2}(i) + \phi_{22} s_{u2,2}(i) \\ &= \Phi_2 S_2 \end{aligned}$$

where Φ_1 stands for $\begin{pmatrix} \phi_{11} \\ \phi_{12} \end{pmatrix}'$, Φ_2 for $\begin{pmatrix} \phi_{21} \\ \phi_{22} \end{pmatrix}'$. Since the means of signals are the true states, our conjectures imply the following aggregate price-setting behavior,

$$\begin{aligned} p^*_1 &= \phi_{11} u_1 + \phi_{12} u_2 \\ &= \Phi_1 U \\ p^*_2 &= \phi_{21} u_1 + \phi_{22} u_2 \\ &= \Phi_2 U \end{aligned}$$

Let $\lambda_k = (1-\theta_k)$ to simplify notation. Thus, the individual firm's expectations of average sectoral prices are

$$E_{1i}(p_1) = \lambda_1 E_{1i}(p_1^*) = \lambda_1\big[\phi_{11}E_{1i}(u_1) + \phi_{12}E_{1i}(u_2)\big] \tag{3.36}$$
$$= \lambda_1\Phi_1 E_{1i}(U) = \lambda_1\Phi_1 g S_1$$

$$E_{1i}(p_2) = \lambda_2 E_{1i}(p_2^*) = \lambda_2\big[\phi_{21}E_{1i}(u_1) + \phi_{22}E_{1i}(u_2)\big] \tag{3.37}$$
$$= \lambda_2\Phi_2 E_{1i}(U) = \lambda_2\Phi_2 g S_1$$

$$E_{2i}(p_1) = \lambda_1 E_{2i}(p_1^*) = \lambda_1\big[\phi_{11}E_{1i}(u_1) + \phi_{12}E_{1i}(u_2)\big] \tag{3.38}$$
$$= \lambda_1\Phi_1 E_{2i}(U) = \lambda_1\Phi_1 h S_2$$

$$E_{2i}(p_2) = \lambda_2 E_{2i}(p_2^*) = \lambda_1\big[\phi_{23}E_{1i}(u_1) + \phi_{24}E_{1i}(u_2)\big] \tag{3.39}$$
$$= \lambda_2\Phi_2 E_{2i}(U) = \lambda_2\Phi_2 h S_2$$

Similarly, we have

$$E_{1i}(x) = G_1 S_1 \tag{3.40}$$

$$E_{1i}(u_1) = G_2 S_1 \tag{3.41}$$

$$E_{2i}(x) = H_1 S_2 \tag{3.42}$$

$$E_{2i}(u_2) = H_3 S_2. \tag{3.43}$$

Step 4: Plugging equation (3.36) to (3.43) to the right hand side of equation (3.34) and equation (3.35) gives rise to

$$\Phi_1 S_1 = n_1\lambda_1\Phi_1 g S_1 + n_2\lambda_2\Phi_2 g S_1 + \alpha G_1 S_1 + G_2 S_1 \tag{3.44}$$

$$\Phi_2 S_2 = n_1\lambda_1\Phi_1 h S_2 + n_2\lambda_2\Phi_2 h S_2 + \alpha H_1 S_2 + H_3 S_2. \tag{3.45}$$

The above two equations need to hold for all S_1 and S_2 respectively. Collecting terms involving same factors leads to

$$\Phi_1 = n_1\lambda_1\Phi_1 g + n_2\lambda_2\Phi_2 g + \alpha G_1 + G_2 \tag{3.46}$$

$$\Phi_2 = n_1\lambda_1\Phi_1 h + n_2\lambda_2\Phi_2 h + \alpha H_1 + H_3 \tag{3.47}$$

And we have

$$\omega\lambda_1\Phi_1 = -(1-\omega)\lambda_2\Phi_2$$

led by $0 = \omega p_1 + (1-\omega)p_2$.

Note that G_1 and H_1 are actually a function of policy coefficients β_1 and β_2 while G_2 and H_3 only consist of parameters. Implicitly, we have 6 equations for 6 unknowns ϕ_{11} ϕ_{12} ϕ_{21} ϕ_{22} β_1 β_2. We further simplify the above system of equations to the follows equations

$$\Phi_1 = [n_2\lambda_2(\alpha H_1 + \alpha H_3)(I - n_2\lambda_2 h)^{-1} g + \alpha G_1 + G_2] \cdot [I - \theta_2\lambda_2 n_1\lambda_1 h(I - \theta_2\lambda_2 h)^{-1} g - n_1\lambda_1 g]^{-1} \tag{3.48}$$

$$\Phi_2 = [n_1\lambda_1(\alpha G_1 + \alpha G_2)(I - n_1\lambda_1 g)^{-1} h + \alpha H_1 + H_3] \cdot [I - n_1\lambda_1 n_2\lambda_2 g(I - n_1\lambda_1 g)^{-1} h - n_2\lambda_2 h]^{-1} \tag{3.49}$$

$$\omega\lambda_1\Phi_1 = -(1-\omega)\lambda_2\Phi_2 \tag{3.50}$$

where I is a 2 by 2 identity matrix. Thus we have verified that our initial guesses are correct and show how to get the solution for the pricing function. Moreover, it is shown by Morris and Shin (2002) that the solution in this type of model is unique. For readers who wonder how high-order beliefs are simplified in this framework, this is due to two reasons: first, the optimal price-setting function of individual firms are common knowledge and second expectations of the other's beliefs rely solely on the expectations of the fundamentals.

step 5: the optimal ω. The optimal price target is achieved by choosing the optimal weight ω to minimize the welfare loss function.

3.7.4. Solution Method for Dynamic Model

We follow the method developed by Nimark (2008) and Melosi (2016) to solve the dynamic model.

3.7.4.1. The Undetermined Coefficients Method. As is in Nimark (2008) and Melosi (2016), the law of motion of the endogenous variables $\mathbf{s}_t \equiv [\widetilde{y}_t, \{\hat{y}_{R,kt}\}_{k=1}^{K}, \{\pi_{kt}\}_{k=1}^{K}]'$ reads as follows:

$$\mathbf{s}_t = \mathbf{v}_0 X_{t|t}^{(0:J)}$$

where $X_{t|t}^{(0:J)} \equiv [\ \left\{a_{k,t|t}^{(s)}\right\}_{k=1}^{K}, \left\{\hat{y}_{R,kt-1}\right\}_{k=1}^{K-1} : 0 \leq s \leq J\ ]$ are the truncated state space as a $K(J+2)-1$ by 1 vector. Note that due to collinearity $\hat{y}_{R,Kt} = -\sum_{k=1}^{K-1} n_k \hat{y}_{R,kt}$, we have deleted $\hat{y}_{R,Kt-1}$ from the state variables space and deleted $\hat{y}_{R,Kt}$ from choice variables. We move directly to the truncated state space approximation rather than showing the original infinite high-order expectation case to facilitate the description of the solution algorithm. Specifically,

$$\begin{aligned}
\widetilde{y}_t &= \mathbf{a} X_{t|t}^{(0:J)} \\
\hat{y}_{R,kt} &= \mathbf{b}_k X_{t|t}^{(0:J)} \\
\pi_{kt} &= \mathbf{c}_k X_{t|t}^{(0:J)}
\end{aligned}$$

and

$$\mathbf{v}_0 = [\mathbf{a};\ \mathbf{b}_1; ...; \mathbf{b}_{K-1};\ \mathbf{c}_1; ...; \mathbf{c}_K]$$

where ; represents another row and these policy function coefficients are not determined yet.

As in Nimark (2008) and Melosi (2016), the law of motion of state variables follows

$$X_{t|t}^{(0:J)} = MX_{t-1|t-1}^{(0:J)} + N\boldsymbol{\epsilon}_t$$

where $\boldsymbol{\epsilon}_t = [\epsilon_{1t}...\epsilon_{Kt}]'$, M is $K(J+2)-1$ by $K(J+2)-1$ matrix and N is $K(J+2)-1$ by K matrix. M and N will be determined by the Kalman filter. Define matrix $T^{(s)}$ as follows:

$$T^{(s)} = \begin{bmatrix} \mathbf{0}_{K(J-s+1)\times Ks} & I_{K(J-s+1)} & \mathbf{0}_{K(J-s+1)\times(K-1)} \\ \mathbf{0}_{Ks\times Ks} & \mathbf{0}_{Ks\times K(J-s+1)} & \mathbf{0}_{Ks\times(K-1)} \\ \mathbf{0}_{(K-1)\times Ks} & \mathbf{0}_{(K-1)\times K(J-s+1)} & I_{K-1} \end{bmatrix}$$

where $I_{K(J-s+1)}$ represents $K(J-s+1)$ by $K(J-s+1)$ identity matrix. Then we have the following important results

$$\begin{aligned} \mathbf{s}_{t|t}^{(s)} &= \mathbf{v}_0 T^{(s)} X_{t|t}^{(0:J)} \\ \mathbf{s}_{t+h|t}^{(s)} &= \mathbf{v}_0 M^h T^{(s)} X_{t|t}^{(0:J)} \end{aligned}$$

for any $0 \leq s \leq J$.

Define $\gamma_k^{(s)} \equiv [\ \mathbf{0}_{1\times Ks},\ \{-n_m\}_{m=1}^{k-1},\ (1-n_k), \{-n_m\}_{m=k+1}^{K},\ \mathbf{0}_{1\times K(J+1-s)-1}\ ]$. Substituting the policy function to the (3.22) after the truncation leads to

$$\begin{aligned}\mathbf{c}_k X_{t|t}^{(0:J)} &= (1-\theta)(1-\beta\theta)\Theta\sum_{j=1}^{\infty}(1-\theta)^{j-1}[(\sigma+\varphi)\mathbf{a}T^{(j)}X_{t|t}^{(0:J)} \\ &\quad +(\eta^{-1}+\varphi)(\mathbf{b}_k T^{(j)}X_{t|t}^{(0:J)} - \Psi\epsilon_k^{(j)}X_{t|t}^{(0:J)})] + \beta\theta\sum_{j=1}^{\infty}(1-\theta)^{j-1}\mathbf{c}_k M T^{(j)}X_{t|t}^{(0:J)}.\end{aligned}$$

which implies a restriction on the policy function coefficients

$$\begin{aligned}\mathbf{c}_k &= (1-\theta)(1-\beta\theta)\Theta\sum_{j=1}^{J}(1-\theta)^{j-1}[(\sigma+\varphi)\mathbf{a}T^{(j)} + (\eta^{-1}+\varphi)(\mathbf{b}_k T^{(j)} - \Psi\gamma_k^{(j)})] \\ &\quad +\beta\theta\sum_{j=1}^{J}(1-\theta)^{j-1}\mathbf{c}_k M T^{(j)} \qquad (3.51)\end{aligned}$$

Define $A_k \equiv [1\ (1-\theta_k)\ (1-\theta_k)^2 ... (1-\theta_k)^{J-1}]$ and let

$$V \equiv \begin{bmatrix} (\sigma+\varphi)\mathbf{a}T^{(1)} + (\eta^{-1}+\varphi)(\mathbf{b}_k T^{(1)} - \Psi\gamma_k^{(1)}) \\ \vdots \\ (\sigma+\varphi)\mathbf{a}T^{(J)} + (\eta^{-1}+\varphi)(\mathbf{b}_k T^{(J)} - \Psi\gamma_k^{(J)}) \end{bmatrix}$$

and

$$H \equiv \begin{bmatrix} \mathbf{c}_k M T^{(1)} \\ \vdots \\ \mathbf{c}_k M T^{(J)} \end{bmatrix}.$$

Then equation (3.51) can be rewritten more compactly as

$$\mathbf{c}_k = (1-\theta)(1-\beta\theta)\Theta A_k V + \beta\theta A H$$

Similarly, we have

(3.52)
$$\mathbf{a}X_{t|t}^{(0:J)} = \mathbf{a}MT^{(1)}X_{t|t}^{(0:J)} - \frac{1}{\sigma}(\phi_\pi(\sum_{k=1}^{K}\omega_k\mathbf{c}_kX_{t|t}^{(0:J)}) - \sum_{k=1}^{K}n_k\mathbf{c}_kMT^{(1)}X_{t|t}^{(0:J)} + \sigma\psi^a\sum_{k=1}^{K}n_k(1-\rho_k)a_{kt})$$

by substituting the policy function to IS curve.

Define matrix G by $G \equiv [\{n_k(1-\rho_k)\}_{k=1}^{K}\ \ \mathbf{0}_{1\times K(J+1)-1}]$. Then we have another restriction to the policy function coefficients,

$$\mathbf{a} = \mathbf{a}MT^{(1)} - \frac{1}{\sigma}(\phi_\pi(\sum_{k=1}^{K}\omega_k\mathbf{c}_k) - \sum_{k=1}^{K}n_k\mathbf{c}_kMT^{(1)} + \sigma\psi^aG). \tag{3.53}$$

Define $\chi_k \equiv [\mathbf{0}_{1\times K(J+1)}, \mathbf{0}_{1\times k-1}, 1\ , \mathbf{0}_{1\times(K-k)-1}]$. Similarly, for the law of motion of relative output, we have

$$\mathbf{b}_kX_{t|t}^{(0:J)} = -\eta(\mathbf{c}_kX_{t|t}^{(0:J)} - \sum_{k=1}^{K}n_k\mathbf{c}_kX_{t|t}^{(0:J)}) + \chi_kX_{t|t}^{(0:J)} \tag{3.54}$$

which leads to the third restriction to the policy function coefficients

$$\mathbf{b}_k = -\eta(\mathbf{c}_k - \sum_{k=1}^{K} n_k \mathbf{c}_k) + \chi_k \tag{3.55}$$

for $k \neq K$.

In summary, (3.51) (3.53) and (3.55) are the three equations that characterize the restrictions on policy functions given the law of motion for the state which in turn are governed by the Kalman filter and the policy functions described in a moment. Given the law of motion for the state, the policy functions can be solved by iteration easily.

3.7.4.2. Signal Extraction. Now we describe how the firms update their beliefs about the state and how the law of motion for the state are pinned down. Basically, given the law of motion for the state, we can apply the Kalman filter to derive a new expression for the law of motion for the state. Equating the new one with the previous one gives an updating rule to update the law of motion for the state. Moreover, the law of motion for state intervenes with the policy functions, which calls for a fixed point problem to solve for the law of motion for the state and the policy functions simultaneously. Let's proceed with some notation to facilitate exposition.

Define $Z_{lt}(i) \equiv [\{\hat{y}_{R,k\tau-1}\}_{k=1}^{K-1}, \{s_{k,l\tau}(i)\}_{k=1}^{K}]'$ as signals received by firm i in sector l in period t, $\mathbf{c} \equiv [\mathbf{c}_1; \mathbf{c}_2; ...; \mathbf{c}_K]$ as K by $K(J+2)-1$ matrix, $\mathbf{b} \equiv [\mathbf{b}_1; \mathbf{b}_2; ...; \mathbf{b}_{K-1}]$ as $K-1$ by $K(J+2)-1$ matrix, and $\mathbf{e}_{lt}(i) \equiv [\{e_{k,l,t}(i)\}_{k=1}^{K}]'$ as K by 1 vector.

The observation equation with the lagged relative output as signals is

$$Z_{lt}(i) = L \begin{bmatrix} \mathbf{X}_t \\ \mathbf{X}_{t-1} \end{bmatrix} + Q\mathbf{e}_{lt}(i) \tag{3.56}$$

where

$$\begin{aligned} L &\equiv \begin{bmatrix} \mathbf{0}_{K-1\times K(J+1)} & I_{K-1} & \mathbf{0}_{K-1\times K(J+2)-1} \\ \mathbf{0}_{K\times K(J+1)} & \mathbf{0}_{K\times K-1} & \mathbf{c} \\ I_K & \mathbf{0}_{K\times K-1} & \mathbf{0}_{K\times K(2J+2)-1} \end{bmatrix} \\ Q &\equiv \begin{bmatrix} \mathbf{0}_{K-1\times K} \\ \mathbf{0}_{K\times K} \\ I_K \end{bmatrix} \\ \mathbf{X}_t &\equiv \begin{bmatrix} X_{t|t}^{(0:J)} \end{bmatrix} \end{aligned}$$

Recall that the law of motion for the state is

$$\mathbf{X}_t = M\mathbf{X}_{t-1} + N\boldsymbol{\epsilon}_t.$$

However, to implement the Kalman filter with the lagged state, we need to expand the state space as (3.56) hints. The law of motion for the expanded state follows

$$\mathcal{X}_t = W\mathcal{X}_{t-1} + \mathcal{N}\boldsymbol{\epsilon}_t \tag{3.57}$$

where

$$\begin{aligned}
W &= \begin{bmatrix} M_{K(J+2)-1\times K(J+2)-1} & \mathbf{0}_{K(J+2)-1\times K(J+2)-1} \\ I_{K(J+2)-1} & \mathbf{0}_{K(J+2)-1\times K(J+2)-1} \end{bmatrix} \\
\mathcal{N} &= \begin{bmatrix} N_{K(J+2)-1\times K} \\ \mathbf{0}_{K(J+2)-1\times K} \end{bmatrix} \\
\mathcal{X}_t &= \begin{bmatrix} \mathbf{X}_t \\ \mathbf{X}_{t-1} \end{bmatrix}.
\end{aligned}$$

Σ_{eel} and $\Sigma_{\epsilon\epsilon}$ represent the variance-covariance matrix for the noise and for the sectoral TFP shock respectively. In summary, (3.56) and (3.57) consist of a linear space problem where the Kalman filter applies directly.

We use $\bar{K}$ to denote the stationary Kalman gain to differentiate from the sector size K. The steady-state Kalman filter gives

$$\begin{aligned}
\mathcal{X}^{(1)}_{t|t,l}(i) &= (W - \bar{K}LW)\mathcal{X}^{(1)}_{t-1|t-1,l}(i) + \bar{K}Z(i) \qquad (3.58)\\
&= (W - \bar{K}LW)\mathcal{X}^{(1)}_{t-1|t-1,l}(i) + \bar{K}[LW\mathcal{X}_{t-1} + L\mathcal{N}\boldsymbol{\epsilon}_t + Q\mathbf{e}_{lt}(i)] \qquad (3.59)
\end{aligned}$$

where

$$\bar{K} = PL'(LPL' + \Sigma_{eel})^{-1}$$

with

$$P = W(P - PL'(LPL' + \Sigma_{eel})^{-1}LP)W' + \Sigma_{\epsilon\epsilon}.$$

Denote the average of signals by Z_t such that $Z_t \equiv [\{\hat{y}_{R,k\tau-1}\}_{k=1}^{K-1}, \{a_k\}_{k=1}^{K}]' = \int_{i\in l} Z_{lt}(i)di$ since the noise within each sector cancels out.

Taking average across (3.58) gives

$$\begin{aligned}
\mathcal{X}_{t|t}^{(1)} &= W\mathcal{X}_{t-1|t-1}^{(1)} + \bar{K}\left[Z_t - LW\mathcal{X}_{t-1|t-1}^{(1)}\right] \\
&= (W - \bar{K}LW)\mathcal{X}_{t-1|t-1}^{(1)} + \bar{K}[LW\mathcal{X}_{t-1} + L\mathcal{N}\boldsymbol{\epsilon}_t]. \qquad (3.60)
\end{aligned}$$

Now we want to insert the law of motion of the TFP shock into (3.60) to have a new expression for the law of motion for the state since (3.60) has excluded the zero-order expectation.

We can rewrite equation(3.60) as

$$\begin{bmatrix} \mathbf{X}^{(1)}_{t|t} \\ \mathbf{X}^{(1)}_{t-1|t} \end{bmatrix} = \begin{bmatrix} (W - KLW)_{11} & (W - \bar{K}LW)_{12} \\ (W - \bar{K}LW)_{21} & (W - \bar{K}LW)_{22} \end{bmatrix} \begin{bmatrix} \mathbf{X}^{(1)}_{t-1|t-1} \\ \mathbf{X}^{(1)}_{t-2|t-1} \end{bmatrix}$$
$$+ \begin{bmatrix} \bar{K}LW_{11} & \bar{K}LW_{12} \\ \bar{K}LW_{21} & \bar{K}LW_{22} \end{bmatrix} \begin{bmatrix} \mathbf{X}_{t-1} \\ \mathbf{X}_{t-2} \end{bmatrix}$$
$$+ \begin{bmatrix} \bar{K}L\mathcal{N}_1 \\ \bar{K}L\mathcal{N}_2 \end{bmatrix} \boldsymbol{\epsilon}_t$$

where $(W - \bar{K}LW)_{12} = (W - \bar{K}LW)_{22} = 0$, $\bar{K}LW_{12} = \bar{K}LW_{22} = 0$ because of the construction of W.

Hence

$$\mathbf{X}^{(1)}_{t|t} = (W - \bar{K}LW)_{11}\mathbf{X}^{(1)}_{t-1|t-1} + \bar{K}LW_{11}\mathbf{X}_{t-1} + \bar{K}L\mathcal{N}_1\boldsymbol{\epsilon}_t.$$

The $(J+1)^{th}$ order of the average expectation which should be truncated by construction. Thus

$$\mathbf{X}^{(1)}_{t|t} \quad | \quad _{\sim(KJ+1:(K+1)J)} = (W - \bar{K}LW)_{11} \,|_{\sim(KJ+1:(K+1)J)}\, \mathbf{X}^{(1)}_{t-1|t-1} \,|_{\sim(KJ+1:(K+1)J)}$$
$$+\bar{K}LW_{11} \quad | \quad _{\sim(KJ+1:(K+1)J)}\mathbf{X}_{t-1} + \bar{K}L\mathcal{N}_1 \,|_{\sim(KJ+1:(K+1)J)}\, \boldsymbol{\epsilon}_t.$$

where $|_{\sim(KJ+1:(K+1)J)}$ with a bit of abuse of notation means deleting rows from row $KJ+1$ to row $(K+1)J$ for $\mathbf{X}^{(1)}_{t|t}$, $\mathbf{X}^{(1)}_{t-1|t-1}$ and $\bar{K}L\mathcal{N}_1$ while deleting both rows and columns for $(W-\bar{K}LW)_{11}$.

In order to simplify notation, we define $(W-\bar{K}LW)^*_{11} \equiv (W-\bar{K}LW)_{11} |_{\sim(KJ+1:(K+1)J)}$, $\bar{K}LW^*_{11}$ $\bar{K}LW_{11} |_{\sim(KJ+1:(K+1)J)}$ and $\bar{K}L\mathcal{N}^*_1 \equiv \bar{K}L\mathcal{N}_1 |_{\sim(KJ+1:(K+1)J)}$.

And recall that the sectoral TFP shocks evolve as

$$\begin{bmatrix} a_{1t} \\ \vdots \\ a_{Kt} \end{bmatrix} = \boldsymbol{\rho} \begin{bmatrix} a_{1t-1} \\ \vdots \\ a_{Kt-1} \end{bmatrix} + I_K \boldsymbol{\epsilon}_t$$

where $\boldsymbol{\rho} \equiv \begin{bmatrix} \rho_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \rho_K \end{bmatrix}$ and I_K is a K by K matrix identity matrix.

Therefore, we can fully characterize the matrices M and N.

$$M = \begin{bmatrix} \boldsymbol{\rho} & \mathbf{0}_{K\times K(J+1)-1} \\ \mathbf{0}_{K(J+1)-1\times K} & (W-\bar{K}LW)^*_{11} \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{K\times K(J+2)-1} \\ \bar{K}LW^*_{11} \end{bmatrix}$$

and

$$N = \begin{bmatrix} I_K \\ \mathbf{0}_{K(J+1)-1\times K} \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{K\times K} \\ \bar{K}L\mathcal{N}^*_1 \end{bmatrix}.$$

Note that the law of motion for the state depends on the policy functions and in turn, the policy functions depends on the law of motion for the state. In the end, we need to solve for a fixed point problem where the policy function and law of motion are jointly determined.

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