Universitat Autònoma de Barcelona

Approximation in the Zygmund Class and Distortion under Inner Functions

Odí SOLER I GIBERT

UNIVERSITAT AUTÒNOMA DE BARCELONA

FACULTAT DE CIÈNCIES

DEPARTAMENT DE MATEMÀTIQUES

Approximation in the Zygmund Class and Distortion under Inner Functions

Autor:
Odí SOLER I GIBERT

Director:
Dr. Artur NICOLAU NOS

Memòria presentada per obtenir el grau de Doctor en Matemàtiques

2020

"L'art és una còpia d'una còpia"

Agraïments

Aquesta memòria és un recull de la feina que he fet els últims quatre anys. De fet, aquesta feina va començar molt abans i espero que segueixi per molt de temps. I durant tot aquest temps m'ha ajudat i acompanyat molta gent, gent a qui vull donar les gràcies per tot. Aquí esmento algunes d'aquestes persones, però no totes ni de bon tros, perquè agrair tot el que cal a tothom podria ser un capítol sencer.

Moltes gràcies als amics que han fet aquest camí més divertit. Entre moltes altres persones que segur que em deixo: Oriol Marquínez, Diana Toboso, Álvaro Piedrafita, Víctor Montagud, Sergi Mourón, David López, Alba Carrión, Burkhard Hoppenstedt, Katharina Chvatal, Christian Springer, Johannes van Hauth, Paweł Pożarowszczyk, Evaldo Junior, Laura Sivet, Alex Mercille, David Hahn, Christoph Franzen, Anna Mackie, Sam Pinder, Jam, Usman Shakeel, Stephen Murphy, Johannes Kassel, Tim Beattie.

Moltes gràcies a la meva família i, en particular, als meus pares. Gràcies per recolzar-me sempre per poder seguir estudiant sense haver-me de preocupar massa d'altres coses.

Moltes gràcies també a l'Artur Nicolau, que m'ha ensenyat a fer matemàtiques de debò, que es diu aviat, i m'ha guiat en tot moment durant aquests anys.

Thanks to Eero Saksman for hosting me at Helsingin Yliopisto and for improving my comprehension of so many aspects of Analysis.

Moltes gràcies a tot el personal del Departament de Matemàtiques de la UAB. Gràcies al professorat per tot el que m'han ensenyat. I gràcies al PDI del departament, que tant m'han ajudat fent fàcils tots aquells problemes que no són de matemàtiques.

Moltes gràcies també a tots els doctorands del departament i de la universitat per totes les bones estones i pel suport mutu en els moments de desesperació. Moltes gràcies als companys de despatx Joan Claramunt i Louis Carlier, als companys de cafè Amanda Fernández, Walter Ortiz, Juan Carlos Cantero, als companys de dinars Elies i Juan Luis Durán, i als companys de pis Laura Brustenga i Carmelo Puliatti.

I sobretot moltes gràcies a la Eli, per fer tots els dies millors. Fins i tot els dies de confinament i redacció d'aquesta memòria. Tots els dies sense excepció, com si fos un teorema.

Contents

A la Eli, la meva persona preferida

Introduction

This work presents a study on two different topics in the field of Function Theory. The first of these topics concerns the Zygmund class Λ_*, which is a space of continuous functions defined by a regularity condition. If one considers the scale of spaces Λ_s of Hölder continuous functions with parameter $0 < s \leq 1$, the Zygmund class plays a similar role in it as that of the space BMO of functions with bounded mean oscillation in the scale of L^p spaces, for $1 \leq p \leq \infty$. Briefly, many results in Analysis that hold for the Hölder classes with parameter $0 < s < 1$, also hold for the Zygmund class, while they fail for the class of Lipschitz functions Λ_1. Moreover, at the local scale, the regularity condition defining the Zygmund class is more restrictive than the Hölder continuity. For this reason, the Zygmund class can be regarded as a limiting space for the scale of Hölder classes as the parameter s approaches 1. This reminds of what happens with the space BMO, which turns out to be the correct limiting space for the L^p scale in many problems, when p tends to infinity. This comparison between these two scales of function spaces motivates us to study certain aspects of the Zygmund class in a similar way as it has already been done for the space BMO.

The second topic covered in this dissertation is the study of a property of inner functions in the unit disk $\mathbb{D}$. Since an inner function has well-defined boundary values almost everywhere in the unit circle $\partial\mathbb{D}$, and because these values also belong to $\partial\mathbb{D}$ itself almost everywhere, such a function induces a map $\partial\mathbb{D} \to \partial\mathbb{D}$ that is well-defined at almost every point. In particular, one can consider the iterates of this induced map. These iterates define a discrete dynamical system on $\partial\mathbb{D}$ whose dynamic properties are very well understood in the case of inner functions with a fixed point in $\mathbb{D}$. Our starting motivation to work on this topic is to extend some of these properties to the case of inner functions that might only have fixed points in $\partial\mathbb{D}$.

Along this introductory chapter, we explain the main concepts that are studied in this dissertation. We also summarise the results obtained during this research and explain briefly their context. The proofs of these results appear throughout the following chapters, as well as the necessary background to understand them.

Chapters 1 and 2 are devoted to the Zygmund class and the Hölder classes. In particular, in Chapter 1, which is based on [NS20], we focus on the Zygmund class of functions of a single variable. On the other hand, in Chapter 2, based on [SS20], we extend the previous results to functions in the Hölder continuous classes and to any number of variables. Chapter 3, based on [LNS19], focuses on inner functions and how they distort sets in the unit circle. The articles [NS20] and [LNS19] have been done in collaboration with A. Nicolau and under his supervision. Moreover, the latter was also done in collaboration with M. Levi. The paper [SS20] has been done in collaboration with E. Saksman and under his supervision.

Notation and Conventions

Along this dissertation, we use $\mathbb{N}$ to denote the set of non-negative integers, that is, we consider $\mathbb{N}$ to include 0. We use $\mathbb{Z}$ to refer to the set of all integers, $\mathbb{R}$ for the real line and $\mathbb{C}$ for the set of complex numbers. For an integer $n \geq 1$, we denote the n-dimensional euclidean space by $\mathbb{R}^n$ and, given an element $x \in \mathbb{R}^n$, we will denote by $|x|$ its euclidean norm. Given a measurable set $A \subset \mathbb{R}^n$, we denote by $|A|$ its n-dimensional Lebesgue measure. Also, given a set A, we denote by χ_A its indicator function. If we consider a cube $Q \subset \mathbb{R}^n$, then we will denote by $l(Q)$ its side length.

The upper half-space whose boundary is $\mathbb{R}^n$ will be denoted by $\mathbb{R}^{n+1}_+$, that is

$$\mathbb{R}^{n+1}_+ := \{(x,y)\colon x \in \mathbb{R}^n,\, y > 0\}.$$

We will denote the unit disk in the complex plane by $\mathbb{D} := \{z \in \mathbb{C}\colon |z| < 1\}$. The boundary of $\mathbb{D}$, often referred to as the unit circle, will be denoted by $\partial\mathbb{D}$, and the closure of the unit disk by $\overline{\mathbb{D}}$. The Lebesgue measure on $\partial\mathbb{D}$ will be denoted by λ, in opposition to the notation used for the Lebesgue measure on $\mathbb{R}$.

Given a function f belonging to the space of square integrable functions $\mathrm{L}^2(\mathbb{R}^n)$, we define its Fourier transform, denoted by $\mathcal{F}[f]$, using the convention

$$\mathcal{F}[f](\xi) := \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi}\, dx, \qquad \xi \in \mathbb{R}^n,$$

where $x \cdot \xi$ denotes the inner product in the euclidean space $\mathbb{R}^n$.

Finally, we use the standard notation $a \lesssim b$ (respectively $a \gtrsim b$) if there exists an absolute constant $C > 0$ such that $a \leq Cb$ (resp. $a \geq Cb$). We will also denote $a \simeq b$ if $a \lesssim b$ and $a \gtrsim b$.

The Hölder Continuous Classes and the Zygmund Class

Consider a continuous function f defined on $\mathbb{R}^n$. We say that f belongs to the *homogeneous Hölder class* of regularity s, with $0 < s \leq 1$, denoted by $f \in \dot{\Lambda}_s$, if it satisfies the *Hölder condition*

$$\|f\|_{\dot{\Lambda}_s} := \sup_{x,\,|y|>0} \frac{|f(x+y) - f(x)|}{|y|^s} < \infty. \tag{I.1}$$

We will also use the notation $f \in \dot{\Lambda}_s(\mathbb{R}^n)$ whenever it is necessary to avoid ambiguity. In particular, for $s = 1$ we have the class of Lipschitz functions. Note that the quantity $\|\cdot\|_{\dot{\Lambda}_s}$ is a semi-norm, since for any constant function f it happens that $\|f\|_{\dot{\Lambda}_s} = 0$. Nonetheless, the set of functions satisfying (I.1) modulo constant functions is a Banach space endowed with the norm $\|\cdot\|_{\dot{\Lambda}_s}$. This definition can be extended to $s > 1$ as follows. We say that an n-tuple $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ is a multi-index of length $|\alpha| = \sum_{i=1}^n \alpha_i$. Given a multi-index α we denote by ∂^α the differential monomial $(\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n}$. If $s = m + t$, with $m \in \mathbb{Z}$ and $0 < t \leq 1$, a continuous function f is in $\dot{\Lambda}_s$ if it is m times continuously differentiable and $\partial^\alpha f \in \dot{\Lambda}_t$ for any multi-index with $|\alpha| \leq m$. On the other hand, we say that f belongs to the *homogeneous Zygmund class*, denoted by $f \in \dot{\Lambda}_*$, if it satisfies the *Zygmund condition*

$$\|f\|_{\dot{\Lambda}_*} := \sup_{x,\,|y|>0} \frac{|f(x+y) - 2f(x) + f(x-y)|}{|y|} < \infty. \tag{I.2}$$

Also in this case, we will use the notation $f \in \dot{\Lambda}_*(\mathbb{R}^n)$ to avoid ambiguity. Note that from this expression, it is clear that $\dot{\Lambda}_1 \subset \dot{\Lambda}_*$, and this inclusion is strict. For instance, one can easily check that $f(x) = x_1 \log|x_1|$ is in $\dot{\Lambda}_*$, but it is not a Lipschitz function because its derivative $\partial f/\partial x_1$ is unbounded. As it was the case for the Hölder classes, the quantity $\|\cdot\|_{\dot{\Lambda}_*}$ is also a semi-norm since for any polynomial f of degree at most 1, one has that $\|f\|_{\dot{\Lambda}_*} = 0$. However, the set of functions satisfying (I.2) modulo polynomials of degree at most 1 is a Banach space endowed with the norm $\|\cdot\|_{\dot{\Lambda}_*}$. In addition, in the same way that Hölder classes can be defined for regularities $s > 1$, one can also define higher regularity Zygmund classes. Namely, for an integer $k \geq 1$, we say that a continuous function f is in the Zygmund class of regularity k, denoted by $f \in \dot{\Lambda}_*^k$, if it is $k-1$ times continuously differentiable and $\partial^\alpha f \in \dot{\Lambda}_*$ for any multi-index with $|\alpha| \leq k-1$. Since a function is in the Hölder or Zygmund classes of regularity $s > 1$ if its derivatives are in $\dot{\Lambda}_t$, for some $0 < t \leq 1$ or in $\dot{\Lambda}_*^1$, we will focus on $\dot{\Lambda}_s$ for $0 < s \leq 1$ and on $\dot{\Lambda}_*^1$. Moreover, we will simply write $\dot{\Lambda}_* = \dot{\Lambda}_*^1$ to avoid an excess of indices. One final remark about the Hölder classes and their corresponding norms is that, given $f \in \dot{\Lambda}_s$ with $0 < s < 1$, it holds that

$$\|f\|_{\dot{\Lambda}_s} \simeq \sup_{x, |y|>0} \frac{|f(x+y) - 2f(x) + f(x-y)|}{|y|^s} \tag{I.3}$$

(see [Zyg45] or [Ste70, p. 146]). The same cannot happen for $s = 1$, as it would imply that $\dot{\Lambda}_1 = \dot{\Lambda}_*$, which we know is not true.

Note that the spaces $\dot{\Lambda}_s$, for $0 < s \leq 1$, and $\dot{\Lambda}_*$ are actually quotient spaces instead of function spaces. Thus, in order to study them, one usually has to construct a lifting operator that assigns a representative function to each equivalence class. For instance, such an operator is used to check that these are in fact Banach spaces. As we shall see, the results discussed here are approximation results of a local nature. For this reason, we will restrict ourselves to compactly supported functions in the spaces $\dot{\Lambda}_s$, for $0 < s \leq 1$, and $\dot{\Lambda}_*$. To be more precise, we will show our results for equivalence classes $f \in \dot{\Lambda}_s$ or $f \in \dot{\Lambda}_*$ for which there is a representative with compact support. Observe that, since these are quotient spaces modulo polynomials of a certain degree, this compactly supported representative is unique. Hence, we will just talk about *compactly supported functions*, either in $\dot{\Lambda}_s$ or in $\dot{\Lambda}_*$, understanding their support to be that of this privileged representative. Also, without loss of generality, we can take the additional assumption that the support of our functions is contained in the unit cube $Q_0 = [0,1]^n$. Any other case follows from this one by a rescaling and a translation.

All these spaces of functions can be studied in a unified manner. More concretely, the Hölder class $\dot{\Lambda}_s$ corresponds to the homogeneous Besov space $\dot{B}^s_{\infty,\infty}$ when $s \notin \mathbb{Z}$, while the Zygmund class $\dot{\Lambda}_*^k$ corresponds to $\dot{B}^k_{\infty,\infty}$. Thus, in the Besov scale $\dot{B}^s_{\infty,\infty}$, the Zygmund classes become the right replacement for the Hölder classes when $s \in \mathbb{Z}$. Furthermore, this replacement turns out to be the appropriate one in many different contexts, such as general theory of function spaces, polynomial approximation, potential theory or Calderón-Zygmund theory. Observe that a particular example where this replacement becomes natural is the norm equivalence (I.3), which naturally extends to $0 < s \leq 1$ if we substitute $\dot{\Lambda}_1$ by $\dot{\Lambda}_*$. For further information on these topics, see [Zyg45], [Ste70, Chapter V], [Mak89] and [DLN14], while for an exposition on general Besov spaces, which fall out of the scope of this dissertation, see [Tri10, Chapters 2, 5].

As mentioned before, both the Hölder classes and the Zygmund class have been largely studied. Later on, in Chapters 1 and 2, we shall see more properties and some equivalent characterisations for them, that will be necessary for our results. For a detailed expositions on these spaces, see [Mey92, Chapter 6], [Ste70, Chapter V] or [Tri10, Chapter 2].

An Open Problem about Λ_*

In this section, we focus on functions defined on $\mathbb{R}$, and denote the unit interval by $I_0 = [0,1]$. Moreover, we momentarily turn our attention to the inhomogeneous Hölder and Zygmund classes.

We say that a continuous function f defined on $\mathbb{R}$ is in the *inhomogeneous Hölder class* of regularity s, with $0 < s \leq 1$, denoted by $f \in \Lambda_s$, if it is uniformly bounded and it satisfies that

$$\|f\|_{\Lambda_s} := \sup_{x,\, 0<|y|<1} \frac{|f(x+y) - f(x)|}{|y|^s} < \infty.$$

Note that $\|\cdot\|_{\Lambda_s}$ is a semi-norm for the same reason as its homogeneous counterpart. For $0 < s \leq 1$, the space Λ_s is a Banach space endowed with the norm $\|f\|_{L^\infty} + \|f\|_{\Lambda_s}$ for $f \in \Lambda_s$. Furthermore, as in the homogeneous case, for $0 < s < 1$ we have the semi-norm equivalence

$$\|f\|_{\Lambda_s} \simeq \sup_{x,\, 0<|y|<1} \frac{|f(x+y) - 2f(x) + f(x-y)|}{|y|^s}, \qquad \text{for } f \in \Lambda_s. \tag{I.4}$$

We say that a continuous function f defined on $\mathbb{R}$ is in the *inhomogeneous Zygmund class*, $f \in \Lambda_*$, if it is uniformly bounded and

$$\|f\|_{\Lambda_*} := \sup_{x,\, 0<|y|<1} \frac{|f(x+y) - 2f(x) + f(x-y)|}{|y|} < \infty.$$

This last quantity is also a semi-norm, but the space Λ_* is a Banach space endowed with the norm $\|f\|_{L^\infty} + \|f\|_{\Lambda_*}$ for $f \in \Lambda_*$. These spaces can also be defined for higher regularities as it was done for the homogeneous spaces. In addition, even though we are restricting ourselves to functions defined on $\mathbb{R}$, one could define these spaces for functions on $\mathbb{R}^n$ for any $n \geq 1$ in the same way. Observe that a function $f \in \Lambda_s$ or $f \in \Lambda_*$ can also be considered as an element of the corresponding homogeneous space. Furthermore, a compactly supported function $f \in \dot{\Lambda}_s$ or $f \in \dot{\Lambda}_*$ is also in the respective inhomogeneous space and, under the assumption that f is supported on the unit interval I_0, it holds that $\|f\|_{\Lambda_s} = \|f\|_{\dot{\Lambda}_s}$ (or $\|f\|_{\Lambda_*} = \|f\|_{\dot{\Lambda}_*}$, depending on the case).

The spaces Λ_s and Λ_* satisfy the strict nested inclusions

$$\Lambda_s \supset \Lambda_{s'} \supset \Lambda_* \supset \Lambda_1 \qquad \text{for } 0 < s < s' < 1. \tag{I.5}$$

While the leftmost and rightmost inclusions are clear from the definition, the middle one follows from the semi-norm equivalence (I.4). Note that these inclusions do not hold in the case of the homogeneous Hölder and Zygmund classes, because of the growth at infinity of their functions (see [Mey92, pp. 180–181]). For the homogeneous Hölder and Zygmund classes, we only have the strict inclusion $\dot{\Lambda}_1 \subset \dot{\Lambda}_*$. The nested inclusions (I.5) resemble those of the Lebesgue spaces L^p. Recall that we say that a measurable function f defined on I_0 is in $\mathrm{L}^p(I_0)$, for $1 \leq p < \infty$, if

$$\|f\|_{\mathrm{L}^p} := \left(\int_{I_0} |f(x)|^p \, dx \right)^{1/p} < \infty,$$

and f is in $\mathrm{L}^\infty(I_0)$ if

$$\|f\|_{\mathrm{L}^\infty} := \sup_{x \in I_0} |f(x)| < \infty.$$

In addition, we say that a function f defined on I_0 has *bounded mean oscillation*, denoted by $f \in \mathrm{BMO}(I_0)$, if it is integrable and such that

$$\|f\|_{\mathrm{BMO}} := \sup_I \left(\frac{1}{|I|} \int_I |f(x) - f_I|^2 \, dx \right)^{1/2} < \infty,$$

where the supremum ranges over all intervals $I \subseteq I_0$ and

$$f_I := \frac{1}{|I|} \int_I f(x) \, dx$$

denotes the average of f on I. These spaces satisfy the nested inclusions

$$\mathrm{L}^1(I_0) \supset \mathrm{L}^p(I_0) \supset \mathrm{L}^{p'}(I_0) \supset \mathrm{BMO}(I_0) \supset \mathrm{L}^\infty(I_0), \qquad \text{for } 1 < p < p' < \infty.$$

In addition, even though for a function $f \in \mathrm{L}^\infty(I_0)$ the quantity $\|f\|_{\mathrm{L}^\infty}$ is the limit of $\|f\|_{\mathrm{L}^p}$ as p tends to infinity, it turns out that the space $\mathrm{BMO}(I_0)$ is the right replacement for $\mathrm{L}^\infty(I_0)$ in function theory, in a similar fashion to what happens with Λ_* and Λ_1.

To illustrate this statement with an example, let us focus momentarily on the Hilbert transform, although everything that follows holds for any other Calderón-Zygmund operator. Recall that for a function $f \in \mathrm{L}^1(\mathbb{R})$, its Hilbert transform $\mathrm{H}[f]$ is defined as its convolution with the kernel $1/(\pi x)$ in the principal value sense, that is

$$\mathrm{H}[f](x) := \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|x-t|>\varepsilon} \frac{f(t)}{x-t} \, dt,$$

which is well-defined almost everywhere. One can also define the Hilbert transform for functions in $\mathrm{L}^\infty(\mathbb{R})$ with a slight modification. Namely, for a function $f \in \mathrm{L}^\infty(\mathbb{R})$, its Hilbert transform is defined as

$$\mathrm{H}[f](x) := \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|x-t|>\varepsilon} \left(\frac{1}{x-t} - \frac{t}{1+t^2} \right) f(t) \, dt,$$

which is also well-defined almost everywhere. It is a classical fact that the Hilbert transform is a bounded operator on the spaces $\mathrm{L}^p(\mathbb{R})$ for $1 < p < \infty$ (see [Ste93, Chapter I]), as well as on the space $\mathrm{BMO}(\mathbb{R})$ (see [Ste93, Chapter IV]). However, it is not bounded neither on $\mathrm{L}^1(\mathbb{R})$ nor on $\mathrm{L}^\infty(\mathbb{R})$. In particular, for a function $f \in \mathrm{L}^1(\mathbb{R})$, its Hilbert transform $\mathrm{H}[f]$ belongs to the space *weak* $\mathrm{L}^1(\mathbb{R})$, or in other words, it satisfies the weaker condition

$$\sup_{t>0} \frac{|\{x \in \mathbb{R} \colon |\mathrm{H}[f](x)| > t\}|}{t} < \infty.$$

On the other hand, if a function f is in $\mathrm{L}^\infty(\mathbb{R})$, then $\mathrm{H}[f] \in \mathrm{BMO}(\mathbb{R})$. For further reference on these facts and other properties of the space BMO see [FS72], [Ste93, Chapter IV] and [Gar07, Chapter VI]. Similarly, the Hilbert transform is a bounded operator on the spaces Λ_s with $0 < s < 1$ (see, for instance, [Zyg02, pp. 121–122]), and on the space Λ_* (see [Zyg45] or [Zyg02, pp. 121–122]). Nonetheless, while it is not a bounded operator on Λ_1, for a function $f \in \Lambda_1$ it holds that $\mathrm{H}[f] \in \Lambda_*$. These analogies between the scale of $\mathrm{L}^p(I_0)$ spaces (including BMO) and the scale Λ_s (including Λ_*) motivate us to study aspects in the latter that are well established for the former.

Given a function $f \in \mathrm{BMO}(\mathbb{R})$ and a subspace $X \subseteq \mathrm{BMO}(\mathbb{R})$, the distance of f to X is

$$\mathrm{dist}_{\mathrm{BMO}}(f, X) := \inf_{g \in X} \|f - g\|_{\mathrm{BMO}}.$$

In [GJ78], J. Garnett and P. Jones estimate the distance for a function $f \in \mathrm{BMO}(\mathbb{R})$ to the subspace $\mathrm{L}^\infty(\mathbb{R})$. They do so by means of the John-Nirenberg Theorem (see [JN61], [Gar07, Chapter VI]). Therefore, it is natural to ask for a similar estimate for the space Λ_* and the subspace Λ_1 in terms of the norm $\|\cdot\|_{\Lambda_*}$. More concretely, given a function f in the Zygmund class and a subspace $X \subseteq \Lambda_*$, we define the distance of f to X as

$$\mathrm{dist}_{\Lambda_*}(f, X) := \inf_{g \in X} \|f - g\|_{\Lambda_*}.$$

The problem now is to characterise, given $f \in \Lambda_*$, the distance $\mathrm{dist}_{\Lambda_*}(f, \Lambda_1)$.

This question is related to another one in spaces of analytic functions posed by J. Anderson, J. Clunie and C. Pommerenke in [ACP74]. Namely, we say that a holomorphic function f defined on the unit disk $\mathbb{D}$ is in the Bloch space, denoted by $f \in \mathcal{B}(\mathbb{D})$, if

$$\sup_{z \in \mathbb{D}} (1 - |z|^2)|f'(z)| < \infty,$$

and $\mathcal{B}(\mathbb{D})$ is a Banach space endowed with the norm

$$\|f\|_{\mathcal{B}} := |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2)|f'(z)|.$$

For a holomorphic function f on $\mathbb{D}$, consider its primitive

$$F(z) := \int_0^z f(\zeta)\, d\zeta, \qquad z \in \mathbb{D}. \tag{I.6}$$

Clearly, one has that F is holomorphic in $\mathbb{D}$ and continuous in $\overline{\mathbb{D}}$ if $f \in \mathcal{B}(\mathbb{D})$. Moreover, it holds that f is in the Bloch space if and only if $F(e^{it})$, $t \in \mathbb{R}$, is in the Zygmund class (see [Zyg45] and [ACP74]). On the other hand, if f belongs to the space of bounded holomorphic functions on $\mathbb{D}$, denoted by $f \in \mathbb{H}^\infty(\mathbb{D})$, it is immediate to see that $F(e^{it})$, $t \in \mathbb{R}$, is in the Lipschitz class, where F is defined by (I.6). Furthermore, by Schwarz's Lemma, one can see that $\mathbb{H}^\infty(\mathbb{D}) \subset \mathcal{B}(\mathbb{D})$. Then, the corresponding problem is to give an estimate for the distance (in terms of the norm $\|\cdot\|_{\mathcal{B}}$) of a given function $f \in \mathcal{B}(\mathbb{D})$ to the subspace $\mathbb{H}^\infty(\mathbb{D})$. The characterisation of the closure of $\mathbb{H}^\infty(\mathbb{D})$ in the Bloch space norm is still an open problem. It is worth mentioning the recent work [LM20] of A. Limani and B. Malman, in which they find a necessary condition for a function $f \in \mathcal{B}(\mathbb{D})$ to be in the closure of $\mathbb{H}^\infty(\mathbb{D})$.

The BMO-Sobolev Subspace

In this dissertation we consider a simpler, yet non trivial, version of the previous problem. The reason to take this approach is that, apart from the interest that the corresponding results have on their own, the tools and techniques studied in the process might help in the future to shed some light on the more complicated previous problem.

To know, given $s \in \mathbb{R}$, we say that a tempered distribution f defined on $\mathbb{R}^n$ belongs to the space $\mathrm{I}_s(\mathrm{BMO})$ if there exists a function $g \in \mathrm{BMO}$ such that $f = \mathcal{F}^{-1}\left[|\xi|^{-s}\mathcal{F}[g]\right]$, where the Fourier transform and its inverse must be understood in the sense of tempered distributions modulo polynomials. This is the same to say that the space $\mathrm{I}_s(\mathrm{BMO})$ is the image of BMO under the Riesz potential I_s, defined by

$$I_s(f) := \mathcal{F}^{-1}\left[|\xi|^{-s}\mathcal{F}[f]\right]$$

for any tempered distribution f modulo polynomials, which is also the same as to say that the fractional laplacian $(-\Delta)^{s/2}f$ is in BMO. For this reason, these spaces are sometimes called BMO-*Sobolev spaces*. As before, we will also use the notation $\mathrm{I}_s(\mathrm{BMO})(\mathbb{R}^n)$ whenever it is necessary to avoid ambiguity. It is known that $\mathrm{I}_s(\mathrm{BMO})$ is a space of functions for any $s \geq 0$, and that for $s > 0$ it is actually a space of continuous functions. Moreover, for a given $s \in \mathbb{R}$, if k is a positive integer, then $f \in \mathrm{I}_s(\mathrm{BMO})$ if and only if $\partial^\alpha f \in \mathrm{I}_{s-k}(\mathrm{BMO})$ for every multi-index α of length $|\alpha| = k$. Due to this fact, we will restrict ourselves to the spaces $\mathrm{I}_s(\mathrm{BMO})$ with $0 < s \leq 1$ (the particular case $s = 0$ simply corresponds to BMO). This last fact implies, in particular, that $\mathrm{I}_1(\mathrm{BMO})$ can also be understood as the space of continuous functions whose partial derivatives, in the distributional sense, belong to the space BMO. Applying this characterisation to the particular case $\mathrm{I}_1(\mathrm{BMO})(\mathbb{R})$ and using basic properties of functions in $\mathrm{BMO}(\mathbb{R})$, one can easily see that $\mathrm{I}_1(\mathrm{BMO})(\mathbb{R}) \subset \dot{\Lambda}_*(\mathbb{R})$, and this inclusion is strict because there exist functions in $\dot{\Lambda}_*$ that are nowhere differentiable (see [Zyg45]). Actually, the more general fact that $\mathrm{I}_s(\mathrm{BMO})(\mathbb{R}^n) \subset \dot{\Lambda}_s(\mathbb{R}^n)$ is true, again with strict inclusions. However, the easiest way to see this is by means of equivalent characterisations of the spaces at hand that will be presented in Chapter 2. Observe as well that one clearly has that $\dot{\Lambda}_1(\mathbb{R}^n) \subset \mathrm{I}_1(\mathrm{BMO})(\mathbb{R}^n)$. An extensive study on the BMO-Sobolev spaces, including the properties stated here, can be found in [Str80].

We are now in situation to state the problem presented here in relation to these spaces. From now on, to simplify the exposition, when we talk about $\dot{\Lambda}_s$ for $0 < s \leq 1$, we implicitly replace the Lipschitz class $\dot{\Lambda}_1$ by the Zygmund class $\dot{\Lambda}_*$, which is justified by the theory of Besov spaces and the previous discussion. Given a function $f \in \dot{\Lambda}_s$ and a subspace $X \subseteq \dot{\Lambda}_s$, we define the distance of f to X as

$$\mathrm{dist}_{\dot{\Lambda}_s}(f, X) := \inf_{g \in X} \|f - g\|_{\dot{\Lambda}_s}.$$

Then, for a function $f \in \dot{\Lambda}_s$, with $0 < s \leq 1$, supported on Q_0, we want to estimate its distance to the subspace $\mathrm{I}_s(\mathrm{BMO})$. Moreover, we are particularly interested in estimates involving the ratios $|f(x+y) - 2f(x) + f(x-y)|/|y|^s$, since they naturally characterise both the Hölder continuity and the Zygmund condition.

In the particular case of the Zygmund class in a single variable, this problem has a complex variable version. The space of analytic functions on $\mathbb{D}$ with BMO boundary values, denoted by $\mathrm{BMOA}(\mathbb{D})$, is also a subspace of $\mathcal{B}(\mathbb{D})$. Then, it is not difficult to see that the problem treated here is analogous to estimating the distance of a function $f \in \mathcal{B}(\mathbb{D})$ to $\mathrm{BMOA}(\mathbb{D})$. P. G. Ghatage and D. C. Zheng give an answer to this problem in [GZ93]. However, their techniques rely on reproducing formulas for analytic functions, which do not apply to our case.

Approximation using Dyadic Martingales

In Chapter 1, based on [NS20], we address the case of $\mathrm{I}_1(\mathrm{BMO})(\mathbb{R}) \subset \dot{\Lambda}_*(\mathbb{R})$. The starting point is a characterisation of functions in $\mathrm{I}_1(\mathrm{BMO})$ in terms of the ratios $|f(x+y)-2f(x)+f(x-y)|/y$ due to R. Strichartz in [Str80], which are the same that are measured by $\|\cdot\|_{\dot{\Lambda}_*}$. This allows us to quantify how close a function $f \in \dot{\Lambda}_*$ is from being an $\mathrm{I}_1(\mathrm{BMO})$ function.

To simplify the notation, given a continuous function f defined on $\mathbb{R}$, we define the *second difference* of f at point x and scale y as

$$\Delta_2 f(x,y) := f(x+y) - 2f(x) + f(x-y), \qquad x \in \mathbb{R},\, y > 0.$$

It will often be the case that, for a continuous function f, we will consider $\Delta_2 f(x,y)$ as a function defined on the upper half-plane $\mathbb{R}^2_+ := \{(x,y) \in \mathbb{R}^2 \colon y > 0\}$. Our main result here is the following.

Theorem 1.2. *Given a function $f \in \dot{\Lambda}_*$ with compact support in I_0, for each $\varepsilon > 0$ consider the set*

$$A(f,\varepsilon) = \left\{(x,y) \in \mathbb{R}^2_+ \colon |\Delta_2 f(x,y)| > \varepsilon y\right\},$$

and the quantity

$$M(f,\varepsilon) = \sup_I \frac{1}{|I|} \int_I \int_0^{|I|} \chi_{A(f,\varepsilon)}(x,y)\, \frac{dy\, dx}{y},$$

where the supremum ranges over all intervals I. Then,

$$\operatorname{dist}_{\dot{\Lambda}_*}(f, \mathrm{I}_1(\mathrm{BMO})) \simeq \inf\{\varepsilon > 0 \colon M(f,\varepsilon) < \infty\}. \tag{1.3}$$

Observe that the set $A(f,\varepsilon)$ indicates at which points and scales the quotients $|\Delta_2 f(x,y)|/y$ are too large. Moreover, since $f \in \dot{\Lambda}_*$, these ratios are uniformly bounded by $\|f\|_{\dot{\Lambda}_*}$ and, hence, $M(f, \|f\|_{\dot{\Lambda}_*}) = 0$.

Let us denote by ε_0 the infimum appearing in (1.3). Assuming that $\varepsilon_0 > 0$, it is not difficult to see that there is no function $g \in \mathrm{I}_1(\mathrm{BMO})$ such that $\|f-g\|_{\dot{\Lambda}_*} \leq \varepsilon$, for any $0 < \varepsilon < \varepsilon_0$. If we consider $\varepsilon > \varepsilon_0$ (with no restriction on ε_0), then we will construct a function $g \in \mathrm{I}_1(\mathrm{BMO})$ satisfying the approximation $\|f-g\|_{\dot{\Lambda}_*} \lesssim \varepsilon$. The method used here is based on dyadic martingales, which will be properly defined in Chapter 1, and consists in constructing the derivative $g' \in \mathrm{BMO}$ of a function g with the desired approximating property.

Just observing the schematic idea behind the proof, one can already notice that this will not be an effective method to generalise Theorem 1.2 to functions in $\dot{\Lambda}_*(\mathbb{R}^n)$. If we wanted to repeat naively this process on $\mathbb{R}^n$, we would expect to construct n functions that should correspond to the n first order partial derivatives of a function $g \in \mathrm{BMO}(\mathbb{R}^n)$. However, it is not clear at all how to construct these n functions and, at the same time, ensure that they will be the gradient of a function on $\mathbb{R}^n$.

The tools presented in Chapter 1, though, can be applied to generalise Theorem 1.2 to Zygmund measures on $\mathbb{R}^n$. These measures are defined in Section 1.4, and the corresponding result is stated in Theorem 1.6.

As another application of Martingale Theory to this context, we consider the Sobolev spaces $\mathrm{W}^{1,p}$, with $1 < p < \infty$, and the subspaces of the Zygmund class $\dot{\Lambda}^p_* := \mathrm{W}^{1,p} \cap \dot{\Lambda}_*$. In Theorem 1.5 we present an estimate for the distance $\operatorname{dist}_{\dot{\Lambda}_*}(f, \dot{\Lambda}^p_*)$ for a function $f \in \dot{\Lambda}_*$ with compact support in I_0. This result is based on a result of A. Nicolau in [Nic18]. Moreover, it also has a complex variable version, which is to estimate the distance of a function $f \in \mathcal{B}(\mathbb{D})$ to the subspace $\mathbb{H}^p(\mathbb{D}) \cap \mathcal{B}(\mathbb{D})$, where $\mathbb{H}^p(\mathbb{D})$ is the space of analytic functions on $\mathbb{D}$ with boundary values in L^p. The closure of $\mathbb{H}^p(\mathbb{D}) \cap \mathcal{B}(\mathbb{D})$ in the Bloch space is studied by N. Monreal Galán and A. Nicolau in [MN11] and by P. Galanopoulos, N. Monreal Galán and J. Pau in [GMP15].

Approximation using Wavelets

In Chapter 2, based on [SS20], we tackle the previous problem with a different approach based on the use of wavelet bases. A wavelet basis is an orthonormal basis of $L^2(\mathbb{R}^n)$ that is generated by dyadic dilations and translations of a small number of initial functions. To be more precise, one can construct functions ψ_l, for $1 \leq l \leq 2^n - 1$, of a certain regularity such that their dyadic dilations and translations $\psi_{l,j,k}(x) := 2^{jn/2}\psi_l(2^j x - k)$, with $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$, form an orthonormal basis of $L^2(\mathbb{R}^n)$. A more detailed exposition on wavelets will be given in that chapter, and a complete explanation on their properties and how to construct them can be found in [Mey92].

The key point here is that wavelet bases satisfying a mild regularity condition are unconditional bases of the spaces $\dot{\Lambda}_s(\mathbb{R}^n)$ for $0 < s < 1$, and also for $\dot{\Lambda}_*(\mathbb{R}^n)$. P. G. Lemarié and Y. Meyer give in [LM86] a characterisation for these spaces in terms of the coefficients in such bases (see also [AB97] and [Mey92, p. 185]). Furthermore, in the same paper, the authors also give a similar characterisation for the space $\mathrm{BMO}(\mathbb{R}^n)$ (see also [AB97] and [Mey92, pp. 154–156]). We will also make use of the wavelet characterisation for the spaces $I_s(\mathrm{BMO})(\mathbb{R}^n)$. For completeness, in Chapter 2 we show how to modify the arguments used in [Mey92, pp.154–156] to extend the corresponding result for the space $\mathrm{BMO}(\mathbb{R}^n)$ to the scale of spaces $I_s(\mathrm{BMO})(\mathbb{R}^n)$. This wavelet characterisation of the spaces $I_s(\mathrm{BMO})(\mathbb{R}^n)$, at least for smooth wavelet bases, appears as a particular case of a theorem due to M. Frazier and B. Jawerth (see [FJ90, Sections 2 and 5], and Theorem 2.2 in that article in particular). The general result for either smooth or compactly supported wavelet bases can be found in [YSY10, pp. 255–260] or [Tri20, p. 16].

Given a continuous function f on $\mathbb{R}^n$, we define its *second difference* at point x and scale y by

$$\Delta_2 f(x,y) := \sup_{|t|=y} |f(x+t) - 2f(x) + f(x-t)|, \qquad x \in \mathbb{R}^n,\ y > 0.$$

Note that this definition is slightly different from that of the previous section. However, there will be no ambiguity since we will keep each convention to the corresponding chapter. For a continuous function f on $\mathbb{R}^n$ we will consider $\Delta_2 f(x,y)$ as a function defined on the upper half-space $\mathbb{R}^{n+1}_+ := \{(x,y) \colon x \in \mathbb{R}^n,\ y > 0\}$. Recall as well that, using this notation, it holds that

$$\|f\|_{\dot{\Lambda}_s} \simeq \sup_{x,y>0} \frac{\Delta_2 f(x,y)}{y^s},$$

for $0 < s \leq 1$ (with equality in the case $s = 1$, since we are replacing $\dot{\Lambda}_1$ by $\dot{\Lambda}_*$). The main result of Chapter 2 is the following generalisation of Theorem 1.2.

Theorem 2.2. *Let $0 < s \leq 1$, and consider a function $f \in \dot{\Lambda}_s$ with compact support in Q_0. For each $\varepsilon > 0$ consider the set*

$$S(s,f,\varepsilon) = \left\{ (x,y) \in \mathbb{R}^{n+1}_+ \colon \Delta_2 f(x,y) > \varepsilon y^s \right\},$$

and the quantity

$$M(S(s,f,\varepsilon)) = \sup_{Q\in\mathcal{D}} \frac{1}{|Q|}\int_Q \int_0^{l(Q)} \chi_{S(s,f,\varepsilon)}(x,y)\,\frac{dy\,dx}{y}.$$

Then,

$$\mathrm{dist}_s(f, \mathrm{I}_s(\mathrm{BMO})) \simeq \inf\{\varepsilon > 0\colon M(S(s,f,\varepsilon)) < \infty\}. \tag{2.6}$$

We also find an analogous estimate in terms of the size of hyperbolic derivatives. More concretely, given a uniformly bounded function f on $\mathbb{R}^n$, denote its Poisson extension to the upper half-space $\mathbb{R}^{n+1}_+$ by $u(x,y) = \mathrm{P}\,[f]\,(x,y)$. It is a well known fact that such a function f belongs to the inhomogeneous space Λ_s if and only if

$$\sup_{x\in\mathbb{R}^n} y^{2-s}\left|\frac{\partial^2 u}{\partial y^2}(x,y)\right| < \infty,$$

and this supremum is equivalent to the semi-norm $\|f\|_{\Lambda_s}$ (see for instance [Ste70, pp. 141–147]). The following result gives the distance estimate in terms of these magnitudes.

Theorem 2.7. *Let $0 < s \leq 1$, and consider a function $f \in \dot{\Lambda}_s$ with compact support in Q_0. Denote its Poisson extension by $u(x,y) = \mathrm{P}\,[f]\,(x,y)$. For each $\varepsilon > 0$ consider the set*

$$D(s,f,\varepsilon) = \left\{(x,y)\in\mathbb{R}^{n+1}_+ \colon y^2\left|\frac{\partial^2 u}{\partial y^2}(x,y)\right| > \varepsilon y^s\right\},$$

and the quantity

$$M(D(s,f,\varepsilon)) = \sup_{Q\in\mathcal{D}} \frac{1}{|Q|}\int_Q \int_0^{l(Q)} \chi_{D(s,f,\varepsilon)}(x,y)\,\frac{dy\,dx}{y}.$$

Then,

$$\mathrm{dist}_s(f, \mathrm{I}_s(\mathrm{BMO})) \simeq \inf\{\varepsilon > 0\colon M(D(s,f,\varepsilon)) < \infty\}. \tag{2.12}$$

The proof of both Theorems 2.2 and 2.7 is built on the use of wavelet bases. In Chapter 2 we will say that a cube $Q \subset \mathbb{R}^n$ is a *dyadic cube* if it is of the form

$$Q = \{x\in\mathbb{R}^n\colon 2^j x - k \in [0,1]^n\}$$

for some $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$. We relate the wavelet indices to the dyadic cubes by $\psi_{(l,j,k)} = \psi_{(l,Q)}$, where Q is as before. Given a function $f \in \dot{\Lambda}_s$, with $0 < s \leq 1$, and a wavelet basis $\psi_{(l,Q)}$, we denote the wavelet coefficients of f by

$$c_{(l,Q)}(f) = c_{(l,j,k)}(f) := \int_{\mathbb{R}^n} f(x)\psi_{l,j,k}(x)\,dx.$$

As we will see, the properties of the wavelet functions ensure that these coefficients are well-defined. Then, we get the following distance estimate in terms of the wavelet coefficients.

Theorem 2.6. *Let $0 < s \leq 1$, and consider a function $f \in \dot{\Lambda}_s$. For each $\varepsilon > 0$ consider the set*

$$C(s,f,\varepsilon) = \left\{Q\in\mathcal{D}\colon \sup_{1\leq l\leq 2^n-1} |c_{(l,Q)}(f)| > \varepsilon (l(Q))^s |Q|^{1/2}\right\},$$

and the quantity

$$M(C(s,f,\varepsilon)) = \sup_{Q\in\mathcal{D}} \frac{1}{|Q|} \sum_{\substack{P\in C(s,f,\varepsilon)\\ P\subseteq Q}} |P|.$$

Then,

$$\operatorname{dist}_s(f, \mathrm{I}_s(\mathrm{BMO})) \simeq \inf\{\varepsilon > 0\colon M(C(s,f,\varepsilon)) < \infty\}. \tag{2.9}$$

The proof of this theorem is straightforward using the wavelet characterisations of the spaces $\dot{\Lambda}_s$ and $\mathrm{I}_s(\mathrm{BMO})$. The proofs of Theorems 2.2 and 2.7 will be via Theorem 2.6. Given a cube $Q \subset \mathbb{R}^n$, we denote by $T(Q)$ its top half-cube in the upper half-space $\mathbb{R}^{n+1}_+$, that is

$$T(Q) := \{(x,y) \in \mathbb{R}^{n+1}_+ \colon x \in Q,\ l(Q)/2 \le y \le l(Q)\}.$$

We will use careful comparisons of the sets $S(s,f,\varepsilon)$, $D(s,f,\varepsilon)$ and the analogous set obtained from the wavelet coefficients

$$W(s,f,\varepsilon) := \bigcup_{Q\in C(s,f,\varepsilon)} T(Q).$$

Observe that $W(s,f,\varepsilon)$ is a subset of $\mathbb{R}^{n+1}_+$ such that, in Theorem 2.6, we have the equivalence

$$M(C(s,f,\varepsilon)) \simeq \sup_{Q\in\mathcal{D}} \frac{1}{|Q|} \int_Q \int_0^{l(Q)} \chi_{W(s,f,\varepsilon)}(x,y)\, \frac{dy\, dx}{y}.$$

These comparisons will then yield that the infimums appearing in (2.9), (2.6) and (2.12) are equivalent for functions with compact support.

Distortion and Distribution of Sets under Inner Functions

Chapter 3, based on [LNS19], is devoted to the study of a property of inner functions. An *inner function* is an analytic mapping $f\colon \mathbb{D} \to \overline{\mathbb{D}}$ such that the limit

$$f^*(\xi) := \lim_{r\to 1} f(r\xi)$$

exists and $|f^*(\xi)| = 1$ for almost every $\xi \in \partial\mathbb{D}$. Given a sequence $\{z_n\} \subset \mathbb{D}$, which can be finite or not, such that it satisfies

$$\sum_n \left(1 - |z_n|^2\right) < \infty,$$

we define the *Blaschke product* associated to $\{z_n\}$, denoted by $B_{\{z_n\}}$, as

$$B_{\{z_n\}}(z) := \prod_n \frac{\overline{z_n}}{|z_n|} \frac{z_n - z}{1 - \overline{z_n} z}, \qquad z \in \mathbb{D}.$$

For a finite positive singular measure μ on $\partial\mathbb{D}$, we define the *singular function* associated to μ by

$$S_\mu(z) := \exp\left(- \int \frac{\xi + z}{\xi - z}\, d\mu(\xi) \right).$$

Both Blaschke products and singular functions are inner functions and, in fact, any inner function f on the unit disk $\mathbb{D}$ can be factored as

$$f(z) = \alpha B(z)S(z),$$

where B is a Blaschke product, S a singular function and α a unimodular constant. For a detailed exposition on inner functions and some of their applications, see [Gar07, Chapter II].

Observe that, if we ignore constant functions, an inner function is an analytic self-map of the unit disk $\mathbb{D}$ that induces a self-map on the unit circle $\partial\mathbb{D}$ due to its defining property. However, despite the regularity of f, the map f^* is discontinuous at every point $\xi \in \partial\mathbb{D}$ at which f does not extend analytically. More precisely, if f does not extend analytically at $\xi \in \partial\mathbb{D}$, then $f(I) = \partial\mathbb{D}$ for any open arc I containing ξ. This fact can be proved easily using the Aleksandrov-Clark measures of f. Information on this topic, which falls out of the scope of this dissertation, can be found in [PS06] and [Sak07]. From now on, to avoid an excess of notation, we will also denote by f the map f^* whenever there is no ambiguity.

Our motivation for the work presented in Chapter 3 is the study of dynamic properties of iterates of inner functions. The Denjoy-Wolff Theorem (see [Sha93, p. 77]) states that any holomorphic function $g\colon \mathbb{D} \to \mathbb{D}$ which is not an elliptic automorphism has a fixed point $p \in \overline{\mathbb{D}}$ such that the iterates g^n of g converge to p uniformly on compact sets. When $p \in \partial\mathbb{D}$, we understand that it is fixed in the sense that

$$\lim_{r\to 1} g(rp) = p.$$

The fixed point of a function g whose existence asserts the Denjoy-Wolff Theorem is called the *Denjoy-Wolff fixed point* of g. In particular, this result describes the dynamics of iterates of inner functions in $\mathbb{D}$. On the other hand, if we focus on iterates of the induced map on $\partial\mathbb{D}$, they turn out to have much richer dynamics, and they have been extensively studied. A complete exposition on this topic can be found in [DM91]. A line of research in this area is the study of the distortion of measures and Hausdorff contents of sets under the action of inner functions, both in $\mathbb{D}$ and $\partial\mathbb{D}$. Recall that we denote by λ the Lebesgue measure on $\partial\mathbb{D}$ and, for $z \in \mathbb{D}$, let us denote by λ_z the harmonic measure from the point z, given by

$$\lambda_z(E) = \int_E \frac{1 - |z|^2}{|\xi - z|^2}\, d\lambda(\xi)$$

for any measurable set $E \subseteq \partial\mathbb{D}$. It is a classical result due to Löwner that, for an inner function f such that $f(0) = 0$, then

$$\lambda(f^{-1}(E)) = \lambda(E) \tag{I.7}$$

for any measurable set $E \subseteq \partial\mathbb{D}$ (see [Ahl73, p. 12]). In particular, this implies that for an inner function f with Denjoy-Wolff fixed point $p \in \mathbb{D}$, the harmonic measure λ_p remains invariant. J. L. Fernández and D. Pestana show in [FP92] a similar result for Hausdorff contents in $\partial\mathbb{D}$, again under the assumption that $f(0) = 0$. More concretely, for a fixed $0 < \alpha < 1$, we define the *Hausdorff content* of a Borel set $E \subseteq \partial\mathbb{D}$, denoted by $M_\alpha(E)$, as

$$M_\alpha(E) := \inf \sum_j \lambda(I_j)^\alpha,$$

where the infimum is taken over all collections of arcs $\{I_j\}$ of the unit circle such that

$E \subseteq \bigcup I_j$. Fernández and Pestana show that, given $0 < \alpha < 1$, there exists a constant $C_\alpha > 0$ such that if $f : \mathbb{D} \to \mathbb{D}$ is an inner function with $f(0) = 0$, then

$$M_\alpha(f^{-1}(E)) \geq C_\alpha M_\alpha(E)$$

for any Borel set $E \subseteq \partial\mathbb{D}$. The authors also give an example to show that the reverse inequality does not hold. In addition to the previous result, there is also a similar one for Hausdorff contents in $\mathbb{D}$ due to D.-H. Hamilton (see [Ham93]).

We say that a point $p \in \partial\mathbb{D}$ is a *boundary Fatou point* of f if $f(p) = \lim_{r\to 1} f(rp)$ exists and $f(p) \in \partial\mathbb{D}$. By definition, if f is an inner function, then almost every point $p \in \partial\mathbb{D}$ is a boundary Fatou point of f. For a point $p \in \partial\mathbb{D}$ and $\beta > 1$, we define the *Stolz angle* with opening β and vertex at p to be the set

$$\Gamma_\beta(p) := \{z \in \mathbb{D} \colon |z - p| < \beta(1 - |z|)\}.$$

We say that a holomorphic self-mapping $f : \mathbb{D} \to \mathbb{D}$ has finite angular derivative at $p \in \mathbb{D}$ if there exists $\eta \in \partial\mathbb{D}$ and $\beta > 1$ such that the non-tangential limit

$$f'(p) := \lim_{\Gamma_\beta(p) \ni z \to p} \frac{\eta - f(z)}{p - z}$$

exists and is finite. In particular, if it is the case, then $f(p) = \eta$. Otherwise, if f has no finite angular derivative at p, then we set $f'(p) = +\infty$. Since the limit $f'(p)$ will not depend on β, we will also use the notation $z \angle p$ to say that $\Gamma_\beta(p) \ni z \to p$ for any $\beta > 1$. The Julia-Carathéodory Theorem states that

$$\liminf_{z \to p} \frac{1 - |f(z)|}{1 - |z|} = |f'(p)| > 0$$

in the sense that either both quantities are finite, equal and positive, or both are infinite. Moreover, it also states that f has finite angular derivative at p if and only if

$$\lim_{z \angle p} f'(z)$$

exists and

$$\lim_{z \angle p} f(z) = \eta \in \partial\mathbb{D}.$$

This justifies the notation $f'(p)$ for the angular derivative at p. It is also a well known fact that, if f is an analytic self-map of $\mathbb{D}$ and $p \in \partial\mathbb{D}$ its Denjoy-Wolff fixed point, then $|f'(p)| \leq 1$. For more details on angular derivatives, the Julia-Carathéodory Theorem and their relation with the Denjoy-Wolff Theorem and iterations of analytic self-maps on $\mathbb{D}$, see [Sha93, Chapters 4–5].

In Chapter 3, we show results similar to those of Fernández and Pestana that also hold for inner functions without any fixed point in $\mathbb{D}$. Doering and Mañé introduced an infinite measure on $\partial\mathbb{D}$ that has the property of being quasi-invariant for inner functions with the Denjoy-Wolff fixed point at the boundary. To be more precise, consider a point $p \in \partial\mathbb{D}$ and the infinite measure μ_p on $\partial\mathbb{D}$ defined by

$$\mu_p(E) := \int_E \frac{1}{|\xi - p|^2}\, d\lambda(\xi)$$

for any measurable set $E \subseteq \partial\mathbb{D}$. Observe that, roughly speaking, this measure gives information both on the size of E and on its distribution around point p. Next, consider an inner function f with Denjoy-Wolff fixed point $p \in \partial\mathbb{D}$. These authors show in [DM91] that, in this situation, we have

$$\mu_p(f^{-1}(E)) = |f'(p)|\mu_p(E)$$

for any measurable set $E \subseteq \partial\mathbb{D}$. This result is analogous to (I.7) when f has its Denjoy-Wolff fixed point on the unit circle. The first result presented in Chapter 3 is the following generalisation of this fact.

Theorem 3.2. *Consider an inner function $f\colon \mathbb{D} \to \mathbb{D}$ and a boundary Fatou point $p \in \partial\mathbb{D}$ of f.*

(a) Assume $|f'(p)| < \infty$. Then

$$\mu_p(f^{-1}(E)) = |f'(p)|\mu_{f(p)}(E)$$

for any measurable set $E \subseteq \partial\mathbb{D}$.

(b) If $|f'(p)| = \infty$ and $E \subseteq \mathbb{D}$ is a measurable set, then $\mu_p(f^{-1}(E)) = \infty$ if $\mu_{f(p)}(E) > 0$ and $\mu_p(f^{-1}(E)) = 0$ if $\mu_{f(p)}(E) = 0$.

This still gives a general relation between the measure of a set and its preimage under f which does not depend on the set. However, in this case there appears a distortion factor, which is qualitatively significant in the case that $|f'(p)| = +\infty$. Next, we present an analogous result to that of Fernández and Pestana in [FP92], based on the use of the measure μ_p previously defined. Namely, given $0 < \alpha < 1$ and $p \in \partial\mathbb{D}$, we define the (p, α)-*Hausdorff content* of a Borel set $E \subseteq \partial\mathbb{D} \setminus \{p\}$ as

$$M_\alpha(\mu_p)(E) := \inf \sum_j \mu_p(I_j)^\alpha,$$

where the infimum is taken over all collections of arcs $\{I_j\}$ such that $E \subseteq \bigcup I_j$. Observe that the (p, α)-Hausdorff content is monotonous and subadditive, which can be seen applying the same arguments as for the usual Hausdorff content. Also, note that we decided to exclude point p in the definition of $M_\alpha(\mu_p)$ for technical reasons. Nonetheless, if we were to assign $M_\alpha(\mu_p)(\{p\}) = +\infty$, the results presented here would hold trivially for any set E containing p. Note as well that, for any set $E \subseteq \partial\mathbb{D} \setminus \{p\}$, we have that $M_\alpha(\mu_p)(E) > 0$ if and only if $M_\alpha(E) > 0$. Our result regarding the (p, α)-Hausdorff content of sets is stated in the following theorem.

Theorem 3.4. *Consider an inner function $f\colon \mathbb{D} \to \mathbb{D}$ and a boundary Fatou point $p \in \partial\mathbb{D}$ of f.*

(a) Assume $|f'(p)| < \infty$. Then for any $0 < \alpha < 1$ there exists a constant $C_\alpha > 0$, independent of f, such that

$$M_\alpha(\mu_p)(f^{-1}(E)) \geq C_\alpha |f'(p)|^\alpha M_\alpha(\mu_{f(p)})(E)$$

for any Borel set $E \subseteq \partial\mathbb{D} \setminus \{f(p)\}$.

(b) Assume $|f'(p)| = \infty$. Then we have that $M_\alpha(\mu_p)(f^{-1}(E)) = \infty$ for any Borel set $E \subseteq \partial\mathbb{D} \setminus \{f(p)\}$ such that $M_\alpha(\mu_{f(p)})(E) > 0$.

We conclude Chapter 3 with two applications of these results. The first one concerns a smoothness property of inner functions which omit large sets of the unit disc, and it is inspired by a nice result in [FP92]. In the second application we obtain analogous results on distortion of sets in the real line under inner mappings of the upper half-plane.

Chapter 1

Approximation in the Zygmund Class using Dyadic Martingales

As mentioned in the Introduction, in this chapter we focus on functions in the Zygmund class $\dot{\Lambda}_*$ on the real line, restricting to those having compact support contained in $I_0 = [0,1]$. In this case, we define the second differences of f by

$$\Delta_2 f(x,y) := f(x+y) - 2f(x) + f(x-y), \qquad x \in \mathbb{R},\ y > 0.$$

For future convenience, if $I = (x-y, x+y)$, we will also use the notation $\Delta_2 f(I) = \Delta_2 f(x,y)$. Recall also that a continuous real valued function f on the real line belongs to the *Zygmund class* $\dot{\Lambda}_*$ if

$$\|f\|_{\dot{\Lambda}_*} := \sup_{x,y>0} \frac{|\Delta_2 f(x,y)|}{y} < \infty.$$

A locally integrable function f on the real line is said to have *bounded mean oscillation*, $f \in$ BMO, if

$$\|f\|_{\mathrm{BMO}} := \sup_I \left(\frac{1}{|I|} \int_I |f(x) - f_I|^2 \, dx \right)^{1/2} < \infty, \tag{1.1}$$

where I ranges over all finite intervals in $\mathbb{R}$ and where

$$f_I := \frac{1}{|I|} \int_I f(x) \, dx,$$

is the average of f on I. The space $\mathrm{I}_1(\mathrm{BMO})$ can be defined as the space of continuous functions such that their derivatives, in the sense of distributions, are BMO functions. It is easy to check that $\mathrm{I}_1(\mathrm{BMO}) \subsetneq \dot{\Lambda}_*$. Indeed, if a function f is in $\mathrm{I}_1(\mathrm{BMO})$, then $f' \in$ BMO. This implies that, if $I_- = (x-y, x)$ and $I_+ = (x, x+y)$, it holds that

$$\frac{1}{y} \left| \int_{I_+} f'(x) \, dx - \int_{I_-} f'(x) \, dx \right| \lesssim \|f'\|_{\mathrm{BMO}}$$

(see [Gar07, p. 216]), which is equivalent to say that $|\Delta_2 f(x,y)|/y \lesssim \|f'\|_{\mathrm{BMO}}$. In [Str80], R. Strichartz found a characterisation for functions in $\mathrm{I}_1(\mathrm{BMO})$ in terms of their second differences. We state it below for compactly supported functions.

Theorem 1.1 (R. Strichartz)**.** *A compactly supported function f is in* $\mathrm{I}_1(\mathrm{BMO})$ *if and only if*

$$\sup_I \frac{1}{|I|} \int_I \int_0^{|I|} \frac{|\Delta_2 f(x,y)|^2}{y^2} \frac{dy\, dx}{y} < \infty, \tag{1.2}$$

where I ranges over all finite intervals on $\mathbb{R}$.

One of the goals in this chapter is to give an analogue of Theorem 1.1 for functions in the closure $\overline{\mathrm{I}_1(\mathrm{BMO})}$ in the Zygmund semi-norm $\|\cdot\|_{\dot{\Lambda}_*}$. This will follow from an estimate of the distance $\mathrm{dist}_{\dot{\Lambda}_*}(f, \mathrm{I}_1(\mathrm{BMO})) := \inf_{g\in \mathrm{I}_1(\mathrm{BMO})} \|f-g\|_{\dot{\Lambda}_*}$ of a function $f \in \dot{\Lambda}_*$ to the subspace $\mathrm{I}_1(\mathrm{BMO})$. In the rest of this chapter, given a function $f \in \dot{\Lambda}_*$ and $\varepsilon > 0$, consider the set

$$A(f,\varepsilon) := \{(x,y) \in \mathbb{R}^2_+ : |\Delta_2 f(x,y)| > \varepsilon y\},$$

where we use $\mathbb{R}^2_+$ to denote the upper half-plane $\mathbb{R}^2_+ := \{(x,y) : x \in \mathbb{R}, y > 0\}$.

Theorem 1.2. *Let f be a compactly supported function in $\dot{\Lambda}_*$. For each $\varepsilon > 0$, consider*

$$M(f,\varepsilon) = \sup_I \frac{1}{|I|} \int_I \int_0^{|I|} \chi_{A(f,\varepsilon)}(x,y)\, \frac{dy\, dx}{y},$$

where I ranges over all finite intervals. Then,

$$\mathrm{dist}_{\dot{\Lambda}_*}(f, \mathrm{I}_1(\mathrm{BMO})) \simeq \inf\{\varepsilon > 0 : M(f,\varepsilon) < \infty\}. \tag{1.3}$$

We deduce the following description of $\overline{\mathrm{I}_1(\mathrm{BMO})}$.

Corollary 1.1. *Let f be a compactly supported function in $\dot{\Lambda}_*$. Then $f \in \overline{\mathrm{I}_1(\mathrm{BMO})}$ if and only if for every $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ such that*

$$\frac{1}{|I|} \int_I \int_0^{|I|} \chi_{A(f,\varepsilon)}(x,y)\, \frac{dy\, dx}{y} \leq C(\varepsilon),$$

for every finite interval I.

Observe that Theorem 1.2 is actually a local result, and in this sense it can still be applied to functions that are not compactly supported by restricting to a finite interval. Hence, these results also hold for functions defined on the unit circle. We say that a continuous function f is in the small Zygmund class, $f \in \dot{\lambda}_*$, if

$$\lim_{t\to 0} \sup_{x, 0<y<t} \frac{|\Delta_2 f(x,y)|}{y} = 0.$$

It is worth mentioning that, for functions defined on the unit circle, the closure of the trigonometric polynomials in the Zygmund semi-norm is the small Zygmund class (see [Zyg45]). Observe as well that Theorem 1.2 also implies uniform approximation locally in the following sense. It is a well known fact (see for instance [JW84]) that for any function $f \in \dot{\Lambda}_*$, and for any finite interval $I \subseteq \mathbb{R}$, there exists a polynomial p_I of degree 1 such that

$$|f(x) - p_I(x)| \lesssim |I|\, \|f\|_{\dot{\Lambda}_*}, \qquad x \in I.$$

Thus, if $f \in \dot{\Lambda}_*$ is compactly supported on the interval I_0, there is $g \in \mathrm{I}_1(\mathrm{BMO})$ such that for any interval $I \subseteq I_0$ there exists a linear polynomial p_I with

$$|f(x) - (g + p_I)(x)| \lesssim |I| \mathrm{dist}_{\dot{\Lambda}_*}(f, \mathrm{I}_1(\mathrm{BMO})), \qquad x \in I.$$

The lower bound in (1.3) is easy, and the main part of the chapter is devoted to prove the upper bound. We will first introduce a dyadic version of the Zygmund class, BMO and $\mathrm{I}_1(\mathrm{BMO})$, and the corresponding notion for dyadic martingales. Then we state and prove a discrete version of (1.3). Afterwards, an averaging argument of J. Garnett and P. Jones (see [GJ82]) is used to prove the continuous result from the dyadic one. To this end, certain technical estimates are needed, which we have collected in Section 1.1.

For an integer $m \geq 0$, let $\mathcal{D}_m = \{[k2^{-m}, (k+1)2^{-m}) \colon k \in \mathbb{Z}\}$ be the collection of dyadic intervals of length 2^{-m}. For $m < 0$, consider the integer p such that $m = -2p+1$, when m is odd, or $m = -2p$, when m is even, and let $t_m = (4^p - 1)/3$. In this case, define $\mathcal{D}_m = \{[k2^{-m} - t_m, (k+1)2^{-m} - t_m) \colon k \in \mathbb{Z}\}$. Denote by $\mathcal{D} = \bigcup_{m\in\mathbb{Z}} \mathcal{D}_m$. We will call the intervals in $\mathcal{D}$ *dyadic intervals.* This definition, which might look unnecessarily complicated for the dyadic intervals with $m < 0$, where we add a translation by t_m units with respect to the previous ones, will turn out to be convenient later on. The reason is that with this choice any finite interval $I \subset \mathbb{R}$ is contained in some interval of $\mathcal{D}$, which is not true if we do not include any such translations.

A locally integrable function f has *dyadic bounded mean oscillation,* $f \in \mathrm{BMO}_d$, if condition (1.1) is required only for dyadic intervals, that is, if

$$\|f\|_{\mathrm{BMO}d} := \sup_{I\in\mathcal{D}} \left(\frac{1}{|I|} \int_I |f(x) - f_I|^2 \, dx \right)^{1/2} < \infty.$$

Note that $\mathrm{BMO} \subset \mathrm{BMO}_d$. The space BMO_d has been studied as a natural discrete substitute of BMO (see, for instance, [GJ82], [Mei03] and [Con13]). The following result, stated in [GJ82], summarises the averaging technique previously mentioned, and an explicit proof will be given in Section 1.3 for completeness.

Theorem 1.3 (J. Garnett, P. Jones)**.** *Suppose that $\alpha \mapsto b^{(\alpha)}$ is a measurable mapping from $\mathbb{R}$ to BMO_d such that all $b^{(\alpha)}$ are supported on I_0, and such that for every α, $\left\| b^{(\alpha)} \right\|_{\mathrm{BMO}d} \leq 1$ and*

$$\int_{\mathbb{R}} b^{(\alpha)}(x) \, dx = 0.$$

Then

$$b_R(x) = \frac{1}{2R} \int_{-R}^{R} b^{(\alpha)}(x+\alpha) \, d\alpha$$

is in BMO and there is a constant $C > 0$ such that $\|b_R\|_{\mathrm{BMO}} \leq C$ for any $R \geq 1$.

We shall need an analogous result for the Zygmund class. We say that a continuous function f belongs to the *dyadic Zygmund class,* $f \in \dot{\Lambda}_{*d}$, if

$$\|f\|_{\dot{\Lambda}_{*}d} := \sup_{I\in\mathcal{D}} \frac{|\Delta_2 f(I)|}{|I|} < +\infty.$$

Observe as well that $\dot{\Lambda}_* \subsetneq \dot{\Lambda}_{*d}$.

Theorem 1.4. *Suppose that $\alpha \mapsto t^{(\alpha)}$ is a measurable mapping from $\mathbb{R}$ to $\dot{\Lambda}_{*d}$ such that all $t^{(\alpha)}$ are supported on I_0, and such that for every α, $\left\| t^{(\alpha)} \right\|_{\dot{\Lambda}_{*}d} \leq 1$. Then, the function*

$$t_R(x) = \frac{1}{2R} \int_{-R}^{R} t^{(\alpha)}(x+\alpha) \, d\alpha, \qquad x \in \mathbb{R}$$

is in $\dot{\Lambda}_*$ and there is a constant $C > 0$ such that $\|t_R\|_{\dot{\Lambda}_*} \leq C$ for any $R \geq 1$.

As an application of the techniques exposed here, we also show a result similar to Theorem 1.2 for Sobolev spaces. For $1 < p < \infty$, we consider the Sobolev space

$W^{1,p}$ of functions $f \in L^p$ whose derivative f' in the distributional sense is also in L^p. Take then the subspace of the Zygmund class $\dot{\Lambda}_*^p := W^{1,p} \cap \dot{\Lambda}_*$. The next theorem gives estimates for distances to this subspace. Here, for $x \in \mathbb{R}$, $\Gamma(x)$ denotes the truncated cone defined as $\Gamma(x) := \{(t,y) \in \mathbb{R}^2_+ : |x-t| < y,\, 0 < y < 1\}$.

Theorem 1.5. *Let f be a compactly supported function in $\dot{\Lambda}_*$. For each $\varepsilon > 0$, define the function*

$$M(f,\varepsilon)(x) = \left(\int_{\Gamma(x)} \chi_{A(f,\varepsilon)}(t,y) \frac{dt\,dy}{y^2} \right)^{1/2}, \qquad x \in \mathbb{R}.$$

Then,

$$\operatorname{dist}_{\dot{\Lambda}_*}(f, \dot{\Lambda}_*^p) \simeq \inf\{\varepsilon > 0 \colon M(f,\varepsilon) \in L^p\}. \tag{1.4}$$

Finally, we find a higher dimensional analogue of Theorem 1.2 for Zygmund measures in $\mathbb{R}^n$. A signed Borel measure μ on $\mathbb{R}^n$ is called a *Zygmund measure* if

$$\|\mu\|_{\dot{\Lambda}_*} := \sup_Q \left| \frac{\mu(Q)}{|Q|} - \frac{\mu(Q^*)}{|Q^*|} \right| < \infty,$$

where Q ranges over all finite cubes in $\mathbb{R}^n$ with edges parallel to the axis, and where Q^* denotes the cube with the same centre as Q but double side length. In the case $n = 1$ it is obvious that μ is a Zygmund measure if and only if its primitive $f(x) = \mu([0,x])$ is in the Zygmund class. Note that there exist Zygmund measures that are singular with respect to the Lebesgue measure, as J. P. Kahane showed [Kah69]. More information on Zygmund measures can be found in [Mak89], [AP89] and [AAN99]. We consider the space of absolutely continuous measures ν such that $d\nu(x) = f(x)\,dx$ for some $f \in \mathrm{BMO}(\mathbb{R}^n)$. We call this the space of $I_1(\mathrm{BMO})$ measures. It is clear that a measure in $I_1(\mathrm{BMO})$ is a Zygmund measure as well. As before, given a Zygmund measure μ on $\mathbb{R}^n$, we want to describe the distance

$$\operatorname{dist}_{\dot{\Lambda}_*}(\mu, I_1(\mathrm{BMO})) := \inf\{\|\mu - \sigma\|_{\dot{\Lambda}_*} : \sigma \in I_1(\mathrm{BMO})\}.$$

For $x \in \mathbb{R}^n$ and $y > 0$, let $Q(x,y)$ be the cube centred at x of side length y. For a given Zygmund measure μ and for $\varepsilon > 0$, consider the set

$$A(\mu,\varepsilon) = \left\{ (x,y) \in \mathbb{R}^{n+1}_+ \colon \left| \frac{\mu(Q(x,y))}{|Q(x,y)|} - \frac{\mu(Q(x,2y))}{|Q(x,2y)|} \right| > \varepsilon \right\},$$

where we use $\mathbb{R}^{n+1}_+ = \{(x,y) \colon x \in \mathbb{R}^n, y > 0\}$ to denote the upper half-space.

Theorem 1.6. *Let μ be a compactly supported Zygmund measure on $\mathbb{R}^n$. For each $\varepsilon > 0$, consider*

$$M(\mu,\varepsilon) = \sup_Q \frac{1}{|Q|} \int_Q \int_0^{l(Q)} \chi_{A(\mu,\varepsilon)}(x,y)\, \frac{dy\,dx}{y},$$

where Q ranges over all finite cubes. Then,

$$\operatorname{dist}_{\dot{\Lambda}_*}(\mu, I_1(\mathrm{BMO})) \simeq \inf\{\varepsilon > 0 \colon M(\mu,\varepsilon) < \infty\}.$$

This chapter is organised in the following manner. In Section 1.1, we expose the technical estimates that we need in order to apply the averaging argument previously mentioned. We then state and prove the dyadic analogue of Theorem 1.2 in Section 1.2. In Section 1.3, we explain the averaging argument that yields both Theorem 1.3 and Theorem 1.4, and then we use it to prove Theorem 1.2. Next, we explain in Section 1.4 the variations in the previous construction that allow us to prove Theorem 1.6. Finally, we devote Section 1.5 to the application of our methods, showing Theorem 1.5.

1.1 Preliminaries

We need an auxiliary result that estimates the oscillation of the second divided differences when changing their centre and step size. For a continuous function f we define its *first difference* at $x \in \mathbb{R}$ with step size $y > 0$ as

$$\Delta_1 f(x,y) = f(x+y) - f(x).$$

For convenience, we may also denote $\Delta_1 f(x,y) = \Delta_1 f(I)$, where $I = (x, x+y)$.

Lemma 1.1. *Let $f \in \dot{\Lambda}_*$ with compact support and assume that $y' > y > y'/2 > 0$ and $|x - x'| < y'/2$. Then*

$$\left|\frac{\Delta_2 f(x,y)}{y} - \frac{\Delta_2 f(x',y')}{y'}\right| \lesssim \|f\|_{\dot{\Lambda}_*}\left(\frac{y'-y}{y'}\left(1+\log\frac{y'}{y'-y}\right) + \frac{|x-x'|}{y'}\log\left(\frac{y'}{|x-x'|}+1\right)\right). \quad (1.5)$$

Proof. We split the proof in two steps. First, we find an estimate for the case $y = y'$ and then another one for $x = x'$. We start showing that, for $y > 0$, when $|x - x'| < y/2$, then

$$\left|\frac{\Delta_2 f(x,y)}{y} - \frac{\Delta_2 f(x',y)}{y}\right| \lesssim \|f\|_{\dot{\Lambda}_*}\frac{|x-x'|}{y}\log\left(\frac{y}{|x-x'|}+1\right). \quad (1.6)$$

We claim that, if $|x - x'| > y/2$, then

$$\left|\frac{\Delta_1 f(x,y)}{y} - \frac{\Delta_1 f(x',y)}{y}\right| \lesssim \frac{\|f\|_{\dot{\Lambda}_*}}{y}\log\left(\frac{|x-x'|}{y}+1\right). \quad (1.7)$$

Indeed, let u be the harmonic extension of f on the upper half-plane $\mathbb{R}^2_+$. It is a well known fact (see [Ste70, pp. 141–147] or [Llo02]) that

$$\left|\frac{f(x+y)-f(x)}{y} - u_x(x,y)\right| \lesssim \|f\|_{\dot{\Lambda}_*},$$

and that

$$\sup_{(x,y)\in\mathbb{R}^2_+} y|\nabla u_x(x,y)| \lesssim \|f\|_{\dot{\Lambda}_*}.$$

Thus, if we denote by $\rho(a,b)$ the hyperbolic distance between two points $a, b \in \mathbb{R}^2_+$, we get

$$|u_x(x,y) - u_x(x',y)| \lesssim \|f\|_{\dot{\Lambda}_*}\rho((x,y),(x',y)).$$

Using the estimate

$$\rho((x,y),(x',y)) \lesssim \log\left(\frac{|x-x'|}{y}+1\right),$$

we get (1.7).

Now, assume $x > x'$ without loss of generality, and $x - x' < y/2$. Write

$$\begin{aligned}\Delta_2 f(x,y) - \Delta_2 f(x',y) = &\big(f(x+y) - f(x'+y)\big) - \big(f(x) - f(x')\big)\\ &+ \big(f(x-y) - f(x'-y)\big) - \big(f(x) - f(x')\big)\end{aligned}$$

and apply (1.7) to the first two terms taking $\Delta_1 f(x'+y, x-x')$ and $\Delta_1 f(x', x-x')$, and to the last two taking $\Delta_1 f(x'-y, x-x')$ and $\Delta_1 f(x', x-x')$. This shows (1.6).

Assume now that $y' > y > y'/2 > 0$. We want to see that

$$\left|\frac{\Delta_2 f(x,y')}{y'} - \frac{\Delta_2 f(x,y)}{y}\right| \lesssim \|f\|_{\dot\Lambda_*} \frac{y'-y}{y'}\left(1 + \log\frac{y'}{y'-y}\right). \tag{1.8}$$

First note the following identity

$$\begin{aligned}&\frac{\Delta_2 f(x,y)}{y} - \frac{\Delta_2 f(x,y')}{y'} =\\ &\qquad \frac{y'-y}{y'}\left[\frac{\Delta_2 f(x,y)}{y} - \left(\frac{\Delta_1 f(x+y, y'-y)}{y'-y} - \frac{\Delta_1 f(x-y', y'-y)}{y'-y}\right)\right].\end{aligned}$$

Using (1.7) on the last two terms, we get (1.8). Finally, (1.5) is a direct consequence of (1.6) and (1.8). □

1.2 The Dyadic Results

A *dyadic rational* is a number of the form $k2^{-m}$ with $k, m \in \mathbb{Z}$. For an integer $m \geq 0$, let $\mathcal{D}_m = \{[k2^{-m}, (k+1)2^{-m})\colon k \in \mathbb{Z}\}$. For $m < 0$, consider the integer p such that $m = -2p+1$, when m is odd, or $m = -2p$, when m is even, and let $t_m = (4^p - 1)/3$. In this case, define $\mathcal{D}_m = \{[k2^{-m} - t_m, (k+1)2^{-m} - t_m)\colon k \in \mathbb{Z}\}$. A *dyadic interval* I is an interval such that $I \in \mathcal{D}_m$ for some $m \in \mathbb{Z}$, and in this case we say that I is a dyadic interval *of generation m*. Denote by $\mathcal{D} = \bigcup_{m\in\mathbb{Z}} \mathcal{D}_m$ the set of all dyadic intervals. Note that, given $I \in \mathcal{D}_m$ for $m \in \mathbb{Z}$, there is a unique interval I^* in $\mathcal{D}_{m-1}$ that contains I, which we call the *predecessor of I*. If J is an arbitrary interval, we will use the notation $\mathcal{D}(J) = \{I \in \mathcal{D}\colon I \subseteq J\}$. Recall that a continuous real valued function f on $\mathbb{R}$ belongs to the *dyadic Zygmund class*, denoted $f \in \dot\Lambda_{*d}$, if

$$\|f\|_{\dot\Lambda_{*d}} = \sup_{I\in\mathcal{D}} \frac{|\Delta_2 f(I)|}{|I|} < \infty.$$

In a similar fashion, we say that a locally integrable function f has *bounded dyadic mean oscillation*, $f \in \mathrm{BMO}_d$, if

$$\|f\|_{\mathrm{BMO}d} = \sup_{I\in\mathcal{D}} \left(\frac{1}{|I|}\int_I |f(x) - f_I|^2\,dx\right)^{1/2} < \infty,$$

and we consider the *dyadic* $\mathrm{I}_1(\mathrm{BMO})$ space to be the space of continuous real valued functions on $\mathbb{R}$ whose distributional derivatives belong to BMO_d, that is

$$\mathrm{I}_1(\mathrm{BMO})_d := \{f \in \mathcal{C}(\mathbb{R})\colon f' \in \mathrm{BMO}_d\}.$$

It is easy to see that each dyadic space contains its corresponding homogeneous space, that is $\mathrm{BMO} \subseteq \mathrm{BMO}_d$ and $\dot\Lambda_* \subseteq \dot\Lambda_{*d}$. It is important to remark, as well, that none of these pairs are equal. More information on the relation between BMO and BMO_d can be found in [GJ82], [Mei03] and [Con13].

The spaces $\dot{\Lambda}_{*d}$ and $\mathrm{I}_1(\mathrm{BMO})_d$ can be regarded as well as spaces of dyadic martingales. We say that a sequence of functions $S = \{S_m\}$ is a *dyadic martingale* if for all $m \geq 0$ the following conditions are satisfied:

(i) S_m is constant on any $I \in \mathcal{D}_m$,

(ii) $S_m|I = \frac{1}{2}\left(S_{m+1}|I^{(1)} + S_{m+1}|I^{(2)}\right)$ for all $I \in \mathcal{D}_m$, where $I^{(1)}, I^{(2)}$ are the intervals in $\mathcal{D}_{m+1}$ contained in I.

We will denote the value of S_m at $I \in \mathcal{D}_{m'}$, $m' \geq m$, by $S_m(I)$, and, if there is no ambiguity, when $I \in \mathcal{D}_m$ we will just write $S(I)$. For $x \in \mathbb{R}$ and $m \geq 0$, let $I \in \mathcal{D}_m$ be such that $x \in I$. Then, we have that $S_m(x) = S(I)$, and we will denote $S(x) = \lim_{m\to\infty} S_m(x)$ when this limit exists. For $m \geq 1$, we call *jump of S at generation m* the function $\Delta S_m(x) = S_m(x) - S_{m-1}(x)$, and if $I \in \mathcal{D}_m$, we use the notation $\Delta S_m(I) = S_m(I) - S_{m-1}(I^*)$, where I^* is the predecessor of I. One can easily check that for a dyadic martingale S the jumps ΔS_j and ΔS_k are orthogonal in $\mathrm{L}^2(I)$ for any $I \in \mathcal{D}_0$ when $j \neq k$.

With these concepts at hand, we can associate to each function $f \in \dot{\Lambda}_{*d}$ a dyadic martingale S, which we shall call *the average growth martingale of f*, as follows. For a dyadic interval $I = [a, b) \in \mathcal{D}_m$, set

$$S_m(I) = \frac{f(b) - f(a)}{b - a} = 2^m (f(b) - f(a)). \tag{1.9}$$

Now, observe that the ratios defining $\|f\|_{\dot{\Lambda}_{*d}}$ can be expressed in terms of the jumps of S; that is, for $I \in \mathcal{D}_m$, we have the relation

$$\frac{|\Delta_2 f(I^*)|}{|I^*|} = 2|\Delta S(I)|.$$

Now it is obvious that any dyadic martingale S is related to a function $f \in \dot{\Lambda}_{*d}$ (up to a linear term that we will ignore) through the relation (1.9) if and only if

$$\|S\|_{\dot{\Lambda}_*} = \sup_{I \in \mathcal{D}} |\Delta S(I)| < \infty.$$

To get the corresponding description of martingales associated with $\mathrm{I}_1(\mathrm{BMO})_d$ functions, we will discretise (1.1). Note that for $f \in \mathrm{I}_1(\mathrm{BMO})_d$, with average growth martingale S, and $I \in \mathcal{D}_M$, using that the jumps $\{\Delta S_m\}_{m \geq M}$ restricted to I of the martingale S are orthogonal in L^2, one can express

$$\int_I |f'(x) - f_I'|^2 \, dx = \int_I \sum_{m > M} |\Delta S_m(x)|^2 \, dx.$$

Thus, a martingale S is related to a function $f \in \mathrm{I}_1(\mathrm{BMO})_d$ through the relation (1.9) if and only if

$$\|S\|_{\mathrm{BMO}} = \sup_{I \in \mathcal{D}} \left(\frac{1}{|I|} \sum_{J \in \mathcal{D}(I)} |\Delta S(J)|^2 |J| \right)^{1/2} < \infty. \tag{1.10}$$

The analogue of Theorem 1.2 for this setting is the following result, in which we use the distance $\mathrm{dist}_{\dot{\Lambda}_{*d}}(f, g) := \|f - g\|_{\dot{\Lambda}_{*d}}$ for any pair of functions $f, g \in \dot{\Lambda}_{*d}$.

Theorem 1.7. *Let f be a compactly supported function in $\dot{\Lambda}_{*d}$. For a fixed $\varepsilon > 0$, define $D(f,\varepsilon)$ by*

$$D(f,\varepsilon) = \sup_{I\in\mathcal{D}} \frac{1}{|I|} \sum_{\substack{J\in\mathcal{D}(I)\\ |\Delta_2 f(J)|>\varepsilon}} |J|. \tag{1.11}$$

Then,

$$\operatorname{dist}_{\dot{\Lambda}_{*d}}(f, \mathrm{I}_1(\mathrm{BMO})_d) = \inf\{\varepsilon > 0\colon D(f,\varepsilon) < \infty\}. \tag{1.12}$$

Note that we can rewrite this result in terms of martingales. Let $f \in \dot{\Lambda}_{*d}$ be compactly supported on a dyadic interval I_0, and consider its average growth martingale S defined by (1.9). In this way, $D(f,\varepsilon)$ in (1.11) can be expressed as

$$D(f,\varepsilon) = \sup_{I\in\mathcal{D}(I_0)} \frac{1}{|I|} \sum_{\substack{J\in\mathcal{D}(I)\\ |\Delta S(J)|>\varepsilon/2}} |J|. \tag{1.13}$$

Proof of Theorem 1.7. Without loss of generality, let us assume that f is supported on the dyadic interval $I_0 = [0,1]$. We need to prove that, for a given $\varepsilon > 0$, there exists a function $b \in \mathrm{I}_1(\mathrm{BMO})_d$ satisfying $\|f-b\|_{\dot{\Lambda}_{*d}} \le \varepsilon$ if and only if $D(f,\varepsilon) < \infty$. Denote by ε_0 the infimum in the right-hand side of (1.12).

Given $\varepsilon > \varepsilon_0$, we will construct a function $b \in \mathrm{I}_1(\mathrm{BMO})_d$ such that $\|f-b\|_{\dot{\Lambda}_{*d}} \le \varepsilon$. Consider the average growth martingale S for function f, defined by (1.9). First, we approximate the martingale S by a martingale B related to an $\mathrm{I}_1(\mathrm{BMO})_d$ function, that is satisfying (1.10). Take $B(I_0) = S(I_0)$ and construct B by setting $\Delta B(J) = \Delta S(J)$ whenever $|\Delta S(J)| > \varepsilon/2$ and $\Delta B(J) = 0$ otherwise, for $J \in \mathcal{D}(I_0)$.

By construction, it is clear that $\|S-B\|_{\dot{\Lambda}_*} \le \varepsilon/2$. Moreover, for any $I \in \mathcal{D}$, we have

$$\sum_{J\in\mathcal{D}(I)} |\Delta B(J)|^2 |J| \le \|S\|^2_{\dot{\Lambda}_*} \sum_{\substack{J\in\mathcal{D}(I)\\ |\Delta S(J)|>\varepsilon/2}} |J| \le |I|\, \|S\|^2_{\dot{\Lambda}_*} D(f,\varepsilon),$$

showing that B satisfies (1.10).

Now, using that the jumps ΔB_j and ΔB_k are orthogonal in L^2, we have

$$\int_{I_0} \left(\sum_{m=1}^{\infty} \Delta B_m(x) \right)^2 dx = \int_{I_0} \sum_{m=1}^{\infty} |\Delta B_m(x)|^2\, dx = \sum_{J\in\mathcal{D}(I_0)} |\Delta B(J)|^2 |J| < \infty.$$

This gives that $\lim_{m\to\infty} B_m(x)$ exists at almost every point $x \in I_0$ and it is actually a square integrable function. Hence, we can integrate it to get $b(x) = \int_0^x \lim_m B_m(s)\, ds$, which will be a function in $\mathrm{I}_1(\mathrm{BMO})_d$ such that $\|f-b\|_{\dot{\Lambda}_{*d}} \le \varepsilon$.

Finally, in the case that $\varepsilon_0 > 0$, if $0 < \varepsilon < \varepsilon_0$, we show that no function $b \in \mathrm{I}_1(\mathrm{BMO})_d$ satisfies $\|f-b\|_{\dot{\Lambda}_{*d}} \le \varepsilon$. To that end, take $\varepsilon < \varepsilon_1 < \varepsilon_0$, assume that there is $b \in \mathrm{I}_1(\mathrm{BMO})_d$ satisfying $\|f-b\|_{\dot{\Lambda}_{*d}} \le \varepsilon$, and let S and B be the respective average growth martingales for f and b. For any $I \in \mathcal{D}$ such that $|\Delta S(I)| > \varepsilon_1/2$, we have that $|\Delta B(I)| > (\varepsilon_1 - \varepsilon)/2 = \delta > 0$. Thus

$$\frac{1}{|I|} \sum_{J\in\mathcal{D}(I)} |\Delta B(J)|^2 |J| > \frac{\delta^2}{|I|} \sum_{\substack{J\in\mathcal{D}(I)\\ |\Delta S(J)|>\varepsilon_1/2}} |J|.$$

The supremum of this quantity when I ranges over all dyadic intervals is $\delta^2 D(f,\varepsilon_1)$, which is infinite because $\varepsilon_1 < \varepsilon_0$. This contradicts condition (1.10) for martingale B

and, hence, that function b is in $\mathrm{I}_1(\mathrm{BMO})_d$, concluding the proof of the theorem. □

1.3 From the Dyadic to the Continuous Setting

Before proving Theorem 1.4, let us make some observations. Consider the measurable mapping $\alpha \mapsto t^{(\alpha)}$ from $\mathbb{R}$ to $\dot{\Lambda}_{*d}$ such that all $t^{(\alpha)}$ are supported on $I_0 = [0,1]$ and such that $\left\|t^{(\alpha)}\right\|_{\dot{\Lambda}_{*d}} \leq 1$, and let $R \geq 1$. We will denote by $\mathcal{D}^0 = \mathcal{D}$ the standard dyadic filtration and by $\mathcal{D}^\beta$ the translated filtration by $(-\beta)$ units. We also extend this notation to denote by $\mathcal{D}^0_m$ the set of intervals of size 2^{-m} in $\mathcal{D}^0$ and by $\mathcal{D}^\beta_m$ the set of intervals of the same size in $\mathcal{D}^\beta$. Similarly, we denote by $\dot{\Lambda}^0_{*d}$ the dyadic Zygmund class with respect to the filtration $\mathcal{D}^0$ and $\dot{\Lambda}^\beta_{*d}$ the dyadic Zygmund class with respect to $\mathcal{D}^\beta$. With this notation, if $f(x) \in \dot{\Lambda}^0_{*d}$, then $f(x+\beta) \in \dot{\Lambda}^\beta_{*d}$.

Now consider an arbitrary interval I and the adjacent interval $\tilde{I} = I - |I|$ of the same size. Fix $R \geq 1$ and $\alpha \in [-R,R]$ and let m be the minimum integer such that I contains an interval of $\mathcal{D}^\alpha_m$, and let $\mathcal{F}_m(I)$ be the set of all such intervals. For each $p > m$, let $\mathcal{F}_p(I)$ be the set of intervals $J \in \mathcal{D}^\alpha_p$ such that $J \subset I \setminus \cup_{j=m}^{p-1} \mathcal{F}_j(I)$. Then, $\mathcal{F}(I) = \cup_{j \geq m} \mathcal{F}_j(I)$ is a covering of I by intervals of $\mathcal{D}^\alpha$. The covering $\mathcal{F}(\tilde{I})$ of $\tilde{I}$ is constructed in the exact same way.

Let us say that $\mathcal{F}(I) = \{I_j\}_{j=1}^\infty$. We may assume that the intervals I_j are ordered in the following way. Whenever $j > k$, $|I_j| \leq |I_k|$, and we may take I_k to be to the left of I_j if $|I_j| = |I_k|$. That is, we order the intervals decreasing in size and left to right for those that have the same length. We consider the covering $\mathcal{F}(\tilde{I}) = \{\tilde{I}_j\}_{j=1}^\infty$ to be ordered in the same way.

Lemma 1.2. *Let $I \subseteq \mathbb{R}$ be a finite interval and $\mathcal{F}(I)$ its covering by intervals of $\mathcal{D}^\alpha$ constructed and ordered as previously explained. Then, the intervals of $\mathcal{F}(I)$ have disjoint interiors and, for $j \geq 1$, they satisfy that $|I_{j+2}| \leq |I_j|/2$.*

Proof. The intervals in $\mathcal{F}(I)$ have disjoint interiors by construction. Moreover, these intervals are maximal in the sense that if $J \in \mathcal{F}(I)$ and $J \subsetneq J' \in \mathcal{D}^\alpha$, then $J' \not\subset I$. Thus, it is clear that for each $m \geq 1$ there are at most two intervals in $\mathcal{F}(I)$ of size $2^{-m}|I|$. This yields that, for $j \geq 1$, the intervals in $\mathcal{F}(I)$ satisfy $|I_{j+2}| \leq |I_j|/2$. □

When $|I| = 2^{-m}$ for some $m \in \mathbb{Z}$, the covering $\mathcal{F}(\tilde{I}) = \{\tilde{I}_j\}_{j=1}^\infty$ is a translation of $\mathcal{F}(I) = \{I_j\}_{j=1}^\infty$. More precisely, if we order both $\{\tilde{I}_j\}$ and $\{I_j\}$ as previously explained, then for each $j \geq 1$ we have that $\tilde{I}_j = I_j - |I|$ and, trivially, for every $j \geq 1$, $|\tilde{I}_j| = |I_j|$. However, for an arbitrary interval I, the sizes of the intervals in $\mathcal{F}(I)$ and $\mathcal{F}(\tilde{I})$ may be completely different. For instance, it could happen that for a given $j \in \mathbb{Z}$, $\mathcal{F}(I)$ had two intervals of size 2^{-j} while $\mathcal{F}(\tilde{I})$ had only one.

Lemma 1.3. *Let I and $\tilde{I}$ be two adjacent intervals of the same length. Fix $\alpha \in \mathbb{R}$. Then there are coverings $\mathcal{G}(I) = \{J_j\}$ and $\mathcal{G}(\tilde{I}) = \{\tilde{J}_j\}$, of I and $\tilde{I}$ respectively, both consisting of intervals of $\mathcal{D}^\alpha$, with $|J_j| = |\tilde{J}_j|$ for any j, and with $|J_{j+2}| \leq |J_j|/2$.*

Proof. Consider the previous coverings $\mathcal{F}(I) = \{I_j\}$ and $\mathcal{F}(\tilde{I}) = \{\tilde{I}_j\}$. If $|I_1| = |\tilde{I}_1|$, then take $J_1 = I_1$ and $\tilde{J}_1 = \tilde{I}_1$. If these sizes are different, assume $|I_1| > |\tilde{I}_1|$ (otherwise the procedure is the same), there exists an integer $k \geq 2$ such that $\sum_{j=1}^k |\tilde{I}_j| = |I_1|$. Note that k exists because all $|I_j|$ (and also $|\tilde{I}_j|$) are dyadic rationals that add up to $|I| = |\tilde{I}|$. Then take $\tilde{J}_j = \tilde{I}_j$ for $1 \leq j \leq k$, and choose pairwise disjoint intervals $J_1, \ldots, J_k \in \mathcal{D}(I_1)$ such that $I_1 = \cup_{j=1}^k J_j$ and that, for each $1 \leq j \leq k$, $|J_j| = |\tilde{J}_j|$. Note that, for $1 \leq j \leq k-2$, we have that $|J_{j+2}| = |\tilde{J}_{j+2}| \leq |\tilde{J}_j|/2$ because of Lemma 1.2. Recursively, consider that we have fixed $\{J_j\}_{j=1}^m \in \mathcal{G}(I)$ and $\{\tilde{J}_j\}_{j=1}^m \in \mathcal{G}(\tilde{I})$, let p, q be the smallest integers such that $I_p \subseteq I \setminus \cup_{j=1}^m J_j$ and $\tilde{I}_q \subseteq \tilde{I} \setminus \cup_{j=1}^m \tilde{J}_j$, and repeat the previous step with I_p and $\tilde{I}_q$. □

Given two finite intervals I_1, I_2, we say that their *minimal common predecessor in* $\mathcal{D}^\alpha$, denoted $P_\alpha(I_1, I_2)$, is the interval $P_\alpha(I_1, I_2) \in \mathcal{D}^\alpha$ such that $I_1 \cup I_2 \subseteq P_\alpha(I_1, I_2)$ and such that for every $J \in \mathcal{D}^\alpha$ that satisfies $I_1 \cup I_2 \subseteq J$, then $P_\alpha(I_1, I_2) \subseteq J$. If $I_1, I_2 \in \mathcal{D}^\alpha$, we define their *distance in the dyadic filtration* $\mathcal{D}^\alpha$, denoted by $\operatorname{dist}_\alpha(I_1, I_2)$, as

$$\operatorname{dist}_\alpha(I_1, I_2) = \log_2 \frac{|P_\alpha(I_1, I_2)|}{|I_1|} + \log_2 \frac{|P_\alpha(I_1, I_2)|}{|I_2|}.$$

Here it is necessary to specify the index α as one could have two intervals I_1, I_2 that were dyadic in two different filtrations $\mathcal{D}^\alpha$ and $\mathcal{D}^\beta$ such that the difference between both distances is as large as desired.

Lemma 1.4. *Consider $f \in \dot{\Lambda}^\alpha_{*d}$ and $I, J \in \mathcal{D}^\alpha$. Then,*

$$\left| \frac{\Delta_1 f(I)}{|I|} - \frac{\Delta_1 f(J)}{|J|} \right| \leq \|f\|_{\dot{\Lambda}_* d} \operatorname{dist}_\alpha(I, J).$$

Proof. Consider the sequences $\{I_j\}_{j=0}^k$ and $\{J_j\}_{j=0}^m$ in $\mathcal{D}^\alpha$ such that $I_0 = I$, $J_0 = J$, $I_k = J_m = P_\alpha(I, J)$, and such that $I_j^* = I_{j+1}$ for $0 \leq j < k$, and such that $J_j^* = J_{j+1}$ for $0 \leq j < m$. One has that

$$\left| \frac{\Delta_1 f(I)}{|I|} - \frac{\Delta_1 f(J)}{|J|} \right| \leq \sum_{j=0}^{k-1} \left| \frac{\Delta_1 f(I_j)}{|I_j|} - \frac{\Delta_1 f(I_{j+1})}{|I_{j+1}|} \right| + \sum_{j=0}^{m-1} \left| \frac{\Delta_1 f(J_j)}{|J_j|} - \frac{\Delta_1 f(J_{j+1})}{|J_{j+1}|} \right|.$$

Each term of these sums is bounded by $\|f\|_{\dot{\Lambda}_* d}$ and, since there are exactly $\operatorname{dist}_\alpha(I, J)$ terms, the result follows. □

For future convenience, given a finite interval I, we will denote its midpoint by $c(I)$. Recall that we denote by I_0 the unit interval, and let us denote by $3I_0$ the interval with centre $c(I_0)$ and length $3|I_0|$.

Lemma 1.5. *Fix $R \geq 1$ and let I and $\tilde{I}$ be two adjacent intervals of length $|I| = |\tilde{I}| < 1/2$. Assume as well that $I \cup \tilde{I} \subset 3I_0$. Let N be the integer such that $2^{-N-1} < |I| \leq 2^{-N}$ and let M be the integer such that $2^{M-1} < R \leq 2^M$. Then, for each $k \geq 1$, one has that*

$$|\{\alpha \in [-R, R] \colon |P_\alpha(I, \tilde{I})| = 2^{k-N}\}| \leq C 2^{M+1} 2^{-k+2},$$

where C does not depend on R nor on the intervals I and $\tilde{I}$.

Proof. Note that for any value of α, one has that $|P_\alpha(I, \tilde{I})| = 2^k 2^{-N}$ for some integer $1 \leq k \leq N+3$. Indeed, the bound $1 \leq k$ is trivial, and we have that $k - N \leq 3$ because of our assumption that $I \cup \tilde{I} \subset 3I_0$ and the way we constructed the dyadic intervals of generation $m < 0$. For $k \geq 2$, the size of the minimal common predecessor in $\mathcal{D}^\alpha$ is exactly 2^{k-N} if and only if there is some $J \in \mathcal{D}^\alpha$, with $|J| = 2^{k-N}$, such that $c(J) \in I \cup \tilde{I}$. For the case $k = 1$, it is only true that if $J \in \mathcal{D}^\alpha$, with $|J| = 2^{1-N}$, is the minimal common predecessor, then $c(J) \in I \cup \tilde{I}$, while the reciprocal does not hold.

Consider $J \in \mathcal{D}_{N-k}$, and consider as well the translated intervals $J + \alpha$, for $\alpha \in [-R, R]$, and their midpoints $c(J + \alpha)$. The set $\{\alpha \in [-R, R] \colon c(J + \alpha) \in I \cup \tilde{I}\}$ has measure bounded by 2^{-N+1}, which is the bound for $|I \cup \tilde{I}|$. Note that it is actually here that we implicitly use that $R \geq 1$ and $|I| < 1/2$, since otherwise it could be that this set had its length bounded by 2^{M+1}. Observe that $[-R, R]$ intersects at most $\max(2^{M+1}2^{N-k+1}, 1)$ intervals of length 2^{k-N}, and in any case at least 1. Hence, since for $k > N+3$ we have that $|\{\alpha \in [-R, R] \colon |P_\alpha(I, \tilde{I})| = 2^{k-N}\}| = 0$, taking $C = 2$ the result holds. □

In the rest of this section, given a locally integrable function f, we will denote its average on I by

$$f(I) := \frac{1}{|I|} \int_I f(x)\, dx.$$

We adopt this convention for clarity and to avoid an excess of subindices later on. Before showing the proof of Theorem 1.3 using the arguments presented by Garnett and Jones in [GJ82], we need a technical lemma for functions in BMO_d. The following result is actually valid for functions in BMO, and can be found in full generality in [Gar07, p. 217], but we only present it here for BMO_d.

Lemma 1.6. *Let f be a function in* BMO_d *and consider two dyadic intervals I and J. Then,*

$$|f(I) - f(J)| \leq 2\, \|f\|_{\mathrm{BMO}d}\, \mathrm{dist}(I, J), \tag{1.14}$$

where $\mathrm{dist}(I, J)$ *denotes the distance between I and J in the standard dyadic filtration.*

Proof. Assume first that J is the predecessor of I, that is, $J = I^*$. Then, we have that

$$|f(I) - f(J)| = \left| \frac{1}{|I|} \int_I (f(x) - f(J))\, dx \right| \leq \frac{2}{|J|} \int_J |f(x) - f(J)|\, dx,$$

and so we have that

$$|f(I) - f(J)| \leq 2\, \|f\|_{\mathrm{BMO}d}\,. \tag{1.15}$$

Now, for general dyadic intervals I and J, consider the sequences $\{I_j\}_{j=0}^{k}$ and $\{J_j\}_{j=0}^{m}$ in $\mathcal{D}$ such that $I_0 = I$, $J_0 = J$, $I_k = J_m = P_0(I, J)$ the minimal common predecessor, and such that $I_j^* = I_{j+1}$ for $0 \leq j < k$, and such that $J_j^* = J_{j+1}$ for $0 \leq j < m$. We can express

$$|f(I) - f(J)| = \sum_{j=0}^{k-1} |f(I_j) - f(I_{j+1})| + \sum_{j=0}^{m-1} |f(J_j) - f(J_{j+1})|.$$

Applying (1.15) to each term and using the definition of distance in the dyadic filtration, equation (1.14) follows. □

Proof of Theorem 1.3. We need to show that, for a fixed $R \geq 1$, there is an absolute constant $C > 0$ such that, for any finite interval $I \subset \mathbb{R}$, we have that

$$\frac{1}{|I|} \int_I |b_R(x) - b_R(I)|\, dx \leq C. \tag{1.16}$$

Because of our assumption on the support of $b^{(\alpha)}$, we can actually restrict to intervals I in $3I_0$ with $|I| < 1$. Using the definition of b_R in terms of the functions $b^{(\alpha)}$ and Fubini's Theorem, we can express

$$b_R(x) - b_R(I) = \frac{1}{2R} \int_{-R}^{R} (b^{(\alpha)}(x+\alpha) - b^{(\alpha)}(I))\, d\alpha.$$

Thus, using again Fubini's Theorem, the left-hand side of (1.16) can be bounded by

$$\frac{1}{2R}\int_{-R}^{R}\frac{1}{|I|}\int_I |b^{(\alpha)}(x+\alpha)-b^{(\alpha)}(I)|\,dx\,d\alpha.$$

Now, for every $\alpha \in [-R,R]$, consider the covering $\mathcal{F}^{\alpha}(I)=\{I_k^{\alpha}\}_{k=0}^{\infty}$ of I by intervals of $\mathcal{D}^{\alpha}$ described in Lemma 1.2. Using these coverings of I, the last quantity is bounded by

$$\begin{aligned}\frac{1}{2R}\int_{-R}^{R}\sum_{k=0}^{\infty}\frac{|I_k^{\alpha}|}{|I|}\frac{1}{|I_k^{\alpha}|}\biggl(\int_{I_k^{\alpha}}\Bigl|b^{(\alpha)}(x+\alpha)-b^{(\alpha)}(I_k^{\alpha})\Bigr|\,dx\\ +\int_{I_k^{\alpha}}\Bigl|b^{(\alpha)}(I_k^{\alpha})-b^{(\alpha)}(I)\Bigr|\,dx\biggr)\,d\alpha.\end{aligned} \tag{1.17}$$

Since for every α we have that $\left\|b^{(\alpha)}\right\|_{\mathrm{BMO}d}\leq 1$, the first term is just bounded by 1. On the other hand, the integrand of the second term is equal to

$$\left|b^{(\alpha)}(I_k^{\alpha})-\sum_{j=0}^{\infty}\frac{|I_j^{\alpha}|}{|I|}b^{(\alpha)}(I_j^{\alpha})\right|\leq\sum_{j=0}^{\infty}\frac{|I_j^{\alpha}|}{|I|}|b^{(\alpha)}(I_k^{\alpha})-b^{(\alpha)}(I_j^{\alpha})|.$$

Applying Lemma 1.6 to each term in this sum, we get that the integrand in the second term of (1.17), after integrating on x, is bounded by

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{|I_k^{\alpha}||I_j^{\alpha}|}{|I|^2}\mathrm{dist}_{\alpha}(I_k^{\alpha},I_j^{\alpha}).$$

Observe that summing over k and j is the same as counting each term twice due to the symmetry of the distance. Consider now the interval $I_*^{\alpha}\in\mathcal{D}^{\alpha}$ such that $I\subsetneq I_*^{\alpha}$ and that it is the smallest one with this property. Then, we can estimate the previous quantity by

$$C\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{|I_k^{\alpha}||I_j^{\alpha}|}{|I|^2}\mathrm{dist}_{\alpha}(I_k^{\alpha},I_*^{\alpha})\leq C\sum_{k=0}^{\infty}\frac{|I_k^{\alpha}|}{|I|}\log\left(\frac{|I_*^{\alpha}|}{|I_k^{\alpha}|}\right),$$

where we used that $\sum_j|I_j^{\alpha}|=|I|$ in the last inequality. Summing over k and applying Lemma 1.2, we get that the previous is bounded by

$$C\left(1+\log\left(\frac{|I_*^{\alpha}|}{|I|}\right)\right).$$

We are just left with integrating with respect to α. To this end, consider the integers M and N such that $2^{M-1}<R\leq 2^M$ and $2^{-N-1}<|I|\leq 2^{-N}$. For $k\geq 1$, consider as well $R_k=\{\alpha\in[-R,R]\colon |I_*^{\alpha}|=2^{k-N}\}$ and note that, by Lemma 1.5, $|R_k|\leq C2^{M+1}2^{-k+2}$, with C independent of R and I. Hence, we can estimate the second term in (1.17) by

$$\frac{C}{R}\sum_{k=1}^{\infty}\int_{R_k}\log\left(\frac{|I_*^{\alpha}|}{|I|}\right)\,d\alpha\leq\frac{C}{2^M}\sum_{k=1}^{\infty}2^{M+1}2^{-k+1}k\leq C,$$

where the last constant does not depend neither on R nor on I, thus finishing the proof. □

Proof of Theorem 1.4. We just need to check that, for a fixed $R \geq 1$, it holds that

$$\sup_I \frac{1}{|I|} |\Delta_1 t_R(I) - \Delta_1 t_R(\tilde{I})| \leq C < \infty,$$

where I ranges over all finite intervals, with $\tilde{I} = I - |I|$, and where C is independent of the value of R. Due to our assumption on the support of $t^{(\alpha)}$, it is enough to check those intervals I with length $|I| < 1/2$ and such that $I \cup \tilde{I} \subset 3I_0$. Fix such an interval I and consider the integer N such that $2^{-N-1} < |I| \leq 2^{-N}$. First, we express

$$\frac{1}{|I|}(\Delta_1 t_R(I) - \Delta_1 t_R(\tilde{I})) = \frac{1}{2R} \int_{-R}^{R} \frac{1}{|I|} \left(\Delta_1 t^{(\alpha)}(\alpha + I) - \Delta_1 t^{(\alpha)}(\alpha + \tilde{I}) \right) d\alpha.$$

Now, for a given α, consider the coverings $\mathcal{G}^\alpha(I) = \{I_j\}_{j=1}^\infty$ and $\mathcal{G}^\alpha(\tilde{I}) = \{\tilde{I}_j\}_{j=1}^\infty$ given in Lemma 1.3, that satisfy $|I_j| = |\tilde{I}_j|$ for $j \geq 1$. We can express

$$\frac{1}{|I|} |\Delta_1 t^{(\alpha)}(\alpha + I) - \Delta_1 t^{(\alpha)}(\alpha + \tilde{I})| \leq \sum_{j \geq 1} \frac{|I_j|}{|I|} \frac{1}{|I_j|} \left| \Delta_1 t^{(\alpha)}(\alpha + I_j) - \Delta_1 t^{(\alpha)}(\alpha + \tilde{I}_j) \right|$$

Observe that $\alpha + I_j \in \mathcal{D}^0$ and, since $t^{(\alpha)} \in \dot{\Lambda}^0_{*d}$, using Lemma 1.2 and Lemma 1.4 we may bound the previous quantity by

$$\sum_{j \geq 1} \frac{|I_j|}{|I|} \left\| t^{(\alpha)} \right\|_{\dot{\Lambda}_{*d}} \operatorname{dist}_\alpha(I_j, \tilde{I}_j) \lesssim \sum_{j \geq 1} 2^{-j/2} \log \left(2^{N+j/2} |P_\alpha(I, \tilde{I})| \right),$$

where we have also used that $\left\| t^{(\alpha)} \right\|_{\dot{\Lambda}_{*d}} \leq 1$ for every α. Summing over j, we get

$$\frac{1}{|I|} |\Delta_1 t^{(\alpha)}(\alpha + I) - \Delta_1 t^{(\alpha)}(\alpha + \tilde{I})| \lesssim 1 + N + \log |P_\alpha(I, \tilde{I})|.$$

Averaging over α, we have

$$\frac{1}{|I|} |\Delta_1 t_R(I) - \Delta_1 t_R(\tilde{I})| \lesssim \frac{1}{R} \int_{-R}^{R} \left(1 + N + \log |P_\alpha(I, \tilde{I})|\right) d\alpha.$$

Set $R_k = \{\alpha \in [-R, R] \colon |P_\alpha(I, \tilde{I})| = 2^{k-N}\}$ and recall that, by Lemma 1.5, $|R_k| \leq C 2^{M+1} 2^{-k+2}$, where M is the integer such that $2^{M-1} < R \leq 2^M$ and C does not depend on R nor on I. Then, we can bound the last quantity by

$$2^{-M+1} \sum_{k \geq 1} \int_{R_k} \left(1 + N + \log |P_\alpha(I, \tilde{I})|\right) d\alpha \lesssim \sum_{k \geq 1} 2^{-k} \left(1 + N + \log 2^{k-N} \right),$$

which is bounded by some positive constant C. Note that the factors depending on N and on M cancel out, which means that this last constant does not depend either on R nor on I. □

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Let $f \in \dot{\Lambda}_*$ and let ε_0 be the infimum in the right-hand side of (1.3). First we show that if $\varepsilon_0 > 0$, whenever $0 < \varepsilon < \varepsilon_0$, there is no function $b \in \mathrm{I}_1(\mathrm{BMO})$ such that $\|f - b\|_{\dot{\Lambda}_*} \leq \varepsilon$. Indeed, assume that there actually is such a function b. Take $\varepsilon < \varepsilon_1 < \varepsilon_0$ and note that, whenever $|\Delta_2 f(x, y)|/y > \varepsilon_1$ we have that $|\Delta_2 b(x,y)|/y > \varepsilon_1 - \varepsilon = \delta > 0$. In particular, this means that $A(f,\varepsilon_1) \subseteq A(b,\delta)$. Thus,

$$\frac{1}{|I|}\int_I\int_0^{|I|} \frac{|\Delta_2 b(x,h)|^2}{y^2}\,\frac{dy\,dx}{y} \geq \frac{\delta^2}{|I|}\int_I\int_0^{|I|} \chi_{A(f,\varepsilon_1)}(x,y)\,\frac{dy\,dx}{y},$$

but the supremum, with I ranging over all finite intervals, of the later quantity is not finite since $\varepsilon_1 < \varepsilon_0$. By Theorem 1.1, this contradicts that $b \in \mathrm{I}_1(\mathrm{BMO})$.

We are left with showing that there exists a universal constant $C > 0$ such that, for any $\varepsilon > \varepsilon_0$, there is $b = b(\varepsilon) \in \mathrm{I}_1(\mathrm{BMO})$ such that $\|f-b\|_{\dot{\Lambda}_*} \leq C\varepsilon$. For any such ε, by assumption we have that

$$M(f,\varepsilon) = \sup_I \frac{1}{|I|}\int_I\int_0^{|I|} \chi_{A(f,\varepsilon)}(x,y)\,\frac{dy\,dx}{y} < \infty. \tag{1.18}$$

Assume now, without loss of generality, that f has support in $I_0 = [0,1]$. We claim that (1.18) implies that $D(f,\varepsilon)$ defined by (1.13) is finite. To see this, take $\varepsilon > \varepsilon_1 > \varepsilon_0$, and let $J \in \mathcal{D}$ be such that $|\Delta S(J)| > \varepsilon/2$, which is equivalent to say that $|\Delta_2 f(c(J^*),|J|)|/|J| > \varepsilon$. By Lemma 1.1, there exists $\delta > 0$ such that if $|x - c(J^*)| < \delta|J|$ and $1-\delta < y/|J| < 1+\delta$, then $|\Delta_2 f(x,y)|/y > \varepsilon_1$. Applying this to every dyadic interval J with $|\Delta S(J)| > \varepsilon/2$, we find the upper bound

$$\frac{1}{|I|}\sum_{\substack{J\in\mathcal{D}(I)\\ |\Delta S(J)|>\varepsilon/2}} |J| \lesssim \frac{1}{|I|}\int_I\int_0^{|I|} \chi_{A(f,\varepsilon_1)}(x,y)\frac{dy\,dx}{y} \leq M(f,\varepsilon_1)$$

for all $I \in \mathcal{D}$. Thus,

$$D(f,\varepsilon) = \sup_{I\in\mathcal{D}} \frac{1}{|I|}\sum_{\substack{J\in\mathcal{D}(I)\\ |\Delta S(J)|>\varepsilon/2}} |J| \lesssim M(f,\varepsilon_1). \tag{1.19}$$

Next, for each $\alpha \in [-1,1]$, define $f^{(\alpha)}(x) = f(x-\alpha) \in \dot{\Lambda}_{*d}$. By (1.19) and Theorem 1.7, $\mathrm{dist}_{\dot{\Lambda}_{*d}}(f^{(\alpha)}, \mathrm{I}_1(\mathrm{BMO})_d) \leq \varepsilon$. Hence, there are $b^{(\alpha)} \in \mathrm{I}_1(\mathrm{BMO})_d$ and $t^{(\alpha)} \in \dot{\Lambda}_{*d}$ such that $f^{(\alpha)} = b^{(\alpha)} + t^{(\alpha)}$, with $\left\|t^{(\alpha)}\right\|_{\dot{\Lambda}_{*d}} \leq \varepsilon$ for all $\alpha \in [-1,1]$. This allows us to express

$$\begin{aligned} f(x) &= \frac{1}{2}\int_{-1}^{1} f^{(\alpha)}(x+\alpha)\,d\alpha \\ &= \frac{1}{2}\int_{-1}^{1} b^{(\alpha)}(x+\alpha)\,d\alpha + \frac{1}{2}\int_{-1}^{1} t^{(\alpha)}(x+\alpha)\,d\alpha. \end{aligned}$$

By Theorem 1.3, taking $R = 1$, the first integral yields a function $b \in \mathrm{I}_1(\mathrm{BMO})$. By Theorem 1.4, with $R = 1$ as well, the second integral yields a function $t \in \dot{\Lambda}_*$ with $\|t\|_{\dot{\Lambda}_*} \leq C\varepsilon$, where the later constant is the same that appears in Theorem 1.4. This completes the proof. □

1.4 The Higher Dimensional Result

For a measurable set $A \subset \mathbb{R}^n$, recall that we denote by $|A|$ its Lebesgue measure. Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $y > 0$ and denote by $Q(x,y)$ the cube with centre at x and with side length $l(Q) = y$ and edges parallel to the axis. For a signed Borel measure μ on $\mathbb{R}^n$, we will treat its densities on cubes as *first divided differences*, and denote them by

$$\Delta_1\mu(x,y) := \frac{\mu(Q(x,y))}{|Q(x,y)|}, \qquad (x,y) \in \mathbb{R}^{n+1}_+,$$

and we also define its *second divided differences on cubes* as

$$\Delta_2\mu(x,y) = \Delta_1\mu(x,y) - \Delta_1\mu(x,2y), \qquad (x,y) \in \mathbb{R}^{n+1}_+.$$

Recall that $\mathbb{R}^{n+1}_+$ denotes the upper half-space $\{(x,y)\colon x \in \mathbb{R}^n,\ y > 0\}$. We say that a signed Borel measure μ on $\mathbb{R}^n$ is a Zygmund measure, $\mu \in \dot{\Lambda}_*$, if it satisfies

$$\|\mu\|_{\dot{\Lambda}_*} := \sup_{(x,y)\in\mathbb{R}^{n+1}_+} |\Delta_2\mu(x,y)| < \infty.$$

Note that there can be Zygmund measures that are singular with respect to the Lebesgue measure (see [Kah69] and [AAN99]). Recall that a real valued function f on $\mathbb{R}^n$ is said to have *bounded mean oscillation in* $\mathbb{R}^n$, $f \in \mathrm{BMO}(\mathbb{R}^n)$, if

$$\|f\|_{\mathrm{BMO}} := \sup_Q \left(\frac{1}{|Q|} \int_Q |f(x) - f_Q|^2\, dx \right)^{1/2} < \infty,$$

where Q ranges over all finite cubes in $\mathbb{R}^n$ with edges parallel to the axis and $f_Q = \frac{1}{|Q|}\int_Q f(x)\, dx$. We will say that a signed Borel measure ν on $\mathbb{R}^n$ is an $\mathrm{I}_1(\mathrm{BMO})$ measure, $\nu \in \mathrm{I}_1(\mathrm{BMO})$, if it is absolutely continuous with respect to the Lebesgue measure and its Radon-Nikodym derivative is

$$d\nu(x) = b(x)dx$$

for some function $b \in \mathrm{BMO}(\mathbb{R}^n)$. Using a characterisation of $\mathrm{BMO}(\mathbb{R}^n)$ functions due to R. Strichartz (see [Str80, pp. 544–547], and in particular Theorem 2.5 and the second example after Theorem 2.6 in the same paper), one can see that such a measure ν satisfies

$$\sup_Q \left(\frac{1}{|Q|} \int_Q \int_0^{l(Q)} |\Delta_2\nu(x,y)|^2 \frac{dy\, dx}{y} \right)^{1/2} < \infty. \tag{1.20}$$

Conversely, whenever ν satisfies equation (1.20), it is an absolutely continuous measure with Radon-Nikodym derivative in $\mathrm{BMO}(\mathbb{R}^n)$ (see [DN02]).

Here we state a version of Theorem 1.2 for Zygmund measures in $\mathbb{R}^n$. For a given Zygmund measure μ and $\varepsilon > 0$, consider the set

$$A(\mu,\varepsilon) = \left\{ (x,y) \in \mathbb{R}^{n+1}_+ \colon |\Delta_2\mu(x,y)| > \varepsilon \right\}.$$

Theorem 1.6. *Let μ be a compactly supported Zygmund measure on $\mathbb{R}^n$. For each $\varepsilon > 0$ consider*

$$M(\mu,\varepsilon) = \sup_Q \frac{1}{|Q|} \int_Q \int_0^{l(Q)} \chi_{A(\mu,\varepsilon)}(x,y) \frac{dy\, dx}{y},$$

where Q ranges over all finite cubes with edges parallel to the axis. Then

$$\mathrm{dist}_{\dot{\Lambda}_*}(\mu, \mathrm{I}_1(\mathrm{BMO})) \simeq \inf\{\varepsilon > 0\colon M(\mu,\varepsilon) < \infty\}.$$

The proof of this result follows the same lines as that of Theorem 1.2. Nonetheless, one has to adapt the auxiliary results used in showing that theorem. First, we state and show the technical estimate in $\mathbb{R}^n$ which is analogous to Lemma 1.1. For convenience, given $t \in \mathbb{R}^{n-1}$ and $y > 0$ we will denote by $q(t, y)$ the cube in $\mathbb{R}^{n-1}$ centred at t, with side length $l(Q) = y$ and edges parallel to the axis.

Lemma 1.7. *Let $\mu \in \dot{\Lambda}_*$ with compact support and assume that $y' > y > y'/2 > 0$ and $|x - x'| < y'/2$. Then*

$$|\Delta_2\mu(x,y) - \Delta_2\mu(x',y')|$$
$$\leq C_n \|\mu\|_{\dot{\Lambda}_*} \left(\frac{y'-y}{y} \left(1 + \log\left(\frac{y}{y'-y} + 1\right)\right) + \frac{|x-x'|}{y} \log\left(\frac{y}{|x-x'|} + 1\right)\right).$$

Here, the constant C_n only depends on the dimension n.

Proof. The proof is split in two steps. First, we find an estimate for the case $y = y'$ and then another one for $x = x'$. We start showing that, for $y > 0$, when $|x - x'| < y/2$

$$|\Delta_2\mu(x,y) - \Delta_2\mu(x',y)| \leq C_n \|\mu\|_{\dot{\Lambda}_*} \frac{|x-x'|}{y} \log\left(\frac{y}{|x-x'|} + 1\right). \tag{1.21}$$

First, if $|x - x'| > y/2$, then

$$|\Delta_1\mu(x,y) - \Delta_1\mu(x',y)| \lesssim \|\mu\|_{\dot{\Lambda}_*} \log\left(\frac{|x-x'|}{y} + 1\right). \tag{1.22}$$

The argument to show this bound is the same as in Lemma 1.1. The only difference is that one has to consider u to be the harmonic extension of μ on the upper half-space $\mathbb{R}^{n+1}_+$, which will be itself a Bloch function, and use the well known fact (see [Ste70, Chapter 5] or [Llo02]) that

$$|\Delta_1\mu(x,y) - u(x,y)| \lesssim \|\mu\|_{\dot{\Lambda}_*}.$$

To show (1.21), assume without loss of generality that $x = (x_1, \dots, x_{n-1}, x_n)$ and $x' = (x_1, \dots, x_{n-1}, x'_n)$ with $x'_n < x_n$, and $|x - x'| < y/2$. If we denote $t = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$, one can see that

$$|Q(x,y)| \left(\Delta_2\mu(x,y) - \Delta_2\mu(x',y)\right) = \mu(l_+) - \mu(l_-) + \frac{\mu(L_-)}{2^n} - \frac{\mu(L_+)}{2^n},$$

where $l_+ = q(t,y) \times [x'_n + y/2, x_n + y/2)$, $l_- = q(t,y) \times [x'_n - y/2, x_n - y/2)$, $L_+ = q(t, 2y) \times [x'_n + y, x_n + y)$ and $L_- = q(t, 2y) \times [x'_n - y, x_n - y)$ are parallelepipeds at opposite sides of the cubes $Q(x,y)$ and $Q(x,2y)$ respectively. We just show how to estimate $|\mu(l_+) - \mu(l_-)|$, as the rest works in the same way. The idea here is to cover l_+ with cubes $\{P_j\}$ and to use a translated cover $\{R_j\}$ for l_-.

In order to cover l_+ with the appropriate cubes, we express first $y/|x - x'| = \sum_{m \geq 0} k_m 2^{-m}$, where $k_0 \geq 2$ and k_m is 0 or 1 for $m \geq 1$ (as in a binary expansion). We construct a generation 0 placing k_0^{n-1} cubes with mutually disjoint interiors of side length $|x - x'|$ at one of the corners of l_-, forming altogether a smaller parallelepiped with one side of length $|x - x'|$ and the rest of length $k_0|x - x'|$. Let us denote by $\{P_0^i\}$ the set of cubes of generation 0. Assume we have constructed cubes up to generation $m - 1$, that is, we have chosen $\{P_j^i\}_{j=0}^{m-1}$. At generation m either we do nothing if $k_m = 0$ or, when $k_m = 1$, we add a layer of cubes $\{P_m^i\}$ of side length $2^{-m}|x - x'|$, such that $\{P_j^i\}_{j=0}^m$ have pairwise disjoint interiors, in order to get a new square based

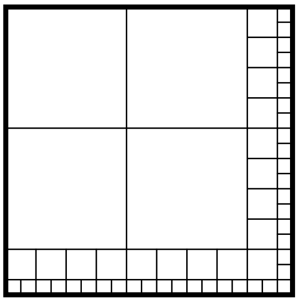

FIGURE 1.1: Parallelepiped l_+ seen from its base (bold square), and the distribution of the cubes P_m^i. Cubes of the same size belong to the same generation.

parallelepiped with one side length $|x-x'|$ and the rest of $(\sum_{j=0}^m k_j 2^{-j})|x-x'|$ (see Figure 1.1). Let $\{P_m^i\}$ be the cubes of generation m and note that their total volume is

$$\sum_i |P_m^i| = |x-x'|^n \left[\left(\sum_{j=0}^{m} k_j 2^{-j} \right)^{n-1} - \left(\sum_{j=0}^{m-1} k_j 2^{-j} \right)^{n-1} \right].$$

If $n \geq 2$ we deduce

$$\sum_i |P_m^i| \lesssim n|x-x'|^n k_m 2^{-m} \left(\sum_{j=0}^{m} k_j 2^{-j} \right)^{n-2}.$$

Since $\sum_{j=0}^m k_j 2^{-j} \leq y/|x-x'|$, we deduce that

$$\sum_i |P_m^i| \lesssim n|x-x'|^2 y^{n-2} k_m 2^{-m} \tag{1.23}$$

Since the distance between the centres of P_m^i and R_m^i is bounded by a fixed multiple of y, applying equation (1.22) we get that

$$|\mu(P_m^i) - \mu(R_m^i)| \lesssim \|\mu\|_{\dot\Lambda_*} |P_m^i| \log\left(\frac{y}{l(P_m^i)} + 1 \right), \tag{1.24}$$

and using (1.24), we have

$$\begin{aligned} |\mu(l_+) - \mu(l_-)| &\leq \sum_m \sum_i |\mu(P_m^i) - \mu(R_m^i)| \\ &\leq C \|\mu\|_{\dot\Lambda_*} \sum_m \sum_i |P_m^i| \log\left(\frac{y}{2^{-m}|x-x'|} + 1 \right). \end{aligned}$$

Summing over i and using (1.23), this is bounded by

$$C \|\mu\|_{\dot\Lambda_*} n|x-x'|^2 y^{n-2} \log\left(\frac{y}{|x-x'|} + 1 \right) \sum_m k_m 2^{-m} m,$$

and we deduce that

$$|\mu(l_+) - \mu(l_-)| \leq C_n \|\mu\|_{\dot\Lambda_*} |x-x'| y^{n-1} \log\left(\frac{y}{|x-x'|} + 1 \right). \tag{1.25}$$

This and the analogue estimate for $|\mu(L_+) - \mu(L_-)|$ yield estimate (1.21).

The second step is to show that, if $y' > y > y'/2 > 0$, then

$$|\Delta_2\mu(x,y') - \Delta_2\mu(x,y)| \leq C_n \|\mu\|_{\dot{\Lambda}_*} \frac{y'-y}{y}\left(1+\log\left(\frac{y}{y'-y}+1\right)\right). \tag{1.26}$$

Let $R(x,y,y') = q(t,y) \times [x_n - y'/2, x_n + y'/2)$, where $t \in \mathbb{R}^{n-1}$ is such that $x = (t, x_n)$. Note that $R(x,y,y')$ is the parallelepiped obtained from dilating the cube $Q(x,y)$ just in one direction. Denote as well

$$\Delta_2\mu(x,y,y') = \frac{\mu(R(x,y,y'))}{|R(x,y,y')|} - \frac{\mu(R(x,2y,2y'))}{|R(x,2y,2y')|}.$$

To show (1.26), it is enough to see that

$$|\Delta_2\mu(x,y,y') - \Delta_2\mu(x,y)| \leq C_n \|\mu\|_{\dot{\Lambda}_*} \frac{y'-y}{y}\left(1+\log\left(\frac{y}{y'-y}+1\right)\right). \tag{1.27}$$

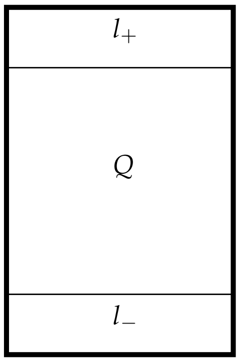

FIGURE 1.2: The parallelepiped R can be decomposed into the cube Q and the square based parallelepipeds l_+ and l_-.

Let us denote $Q = Q(x,y)$, $\tilde{Q} = Q(x,2y)$, $R = R(x,y,y')$ and $\tilde{R} = R(x,2y,2y')$. Note that we can decompose R as the disjoint union $Q \cup l_+ \cup l_-$, where l_+ and l_- are parallelepipeds similar to the ones we used before (see Figure 1.2). In the same way, decompose $\tilde{R} = \tilde{Q} \cup L_+ \cup L_-$, and note that L_+ (and also L_-) can be regarded as the union $\bigcup_{i=1}^{2^n} L_+^i$, where each L_+^i is a translation of l_+. Now, express

$$\Delta_2\mu(x,y) - \Delta_2\mu(x,y,y') = \frac{\mu(Q)}{|Q|} - \frac{\mu(\tilde{Q})}{|\tilde{Q}|} - \frac{\mu(R)}{|R|} + \frac{\mu(\tilde{R})}{|\tilde{R}|} =$$
$$\frac{y'-y}{y'}\left(\frac{\mu(Q)}{|Q|} - \frac{\mu(\tilde{Q})}{|\tilde{Q}|}\right) - \left(\frac{\mu(l_+)}{y^{n-1}y'} - \frac{\mu(L_+)}{2^n y^{n-1}y'}\right) - \left(\frac{\mu(l_-)}{y^{n-1}y'} - \frac{\mu(L_-)}{2^n y^{n-1}y'}\right).$$

The first term is $\Delta_2\mu(x,y)(y'-y)/y'$, which is bounded by $\|\mu\|_{\dot{\Lambda}_*}(y'-y)/y$. We will now show that

$$\left|\frac{\mu(l_+)}{y^{n-1}y'} - \frac{\mu(L_+)}{2^n y^{n-1}y'}\right| \leq C_n \|\mu\|_{\dot{\Lambda}_*} \frac{y'-y}{y}\left(1+\log\left(\frac{y}{y'-y}+1\right)\right) \tag{1.28}$$

The last term is estimated in a similar way. First, we use the decomposition of L_+ to split the difference as follows

$$\left| \frac{\mu(l_+)}{y^{n-1}y'} - \frac{\mu(L_+)}{2^n y^{n-1}y'} \right| \leq \frac{1}{2^n y^{n-1}y'} \sum_{i=1}^{2^n} |\mu(l_+) - \mu(L_+^i)|.$$

For each term in this sum, we can use the estimate in (1.25) for parallelepipeds, just taking into account that now the role of y is taken by $C_n y$ and $y' - y$ plays the role of $|x - x'|$. This gives (1.28), which yields (1.27) and finishes the proof. □

We also need a dyadic version of Theorem 1.6. Here we say that a cube Q in $\mathbb{R}^n$ is a dyadic cube if it is of the form $[k_1 2^{-m}, (k_1+1)2^{-m}) \times \ldots \times [k_n 2^{-m}, (k_n+1)2^{-m})$ where $k_1, \ldots, k_n \in \mathbb{Z}$ and $m \geq 0$, or if it is of the form $[k_1 2^{-m} - t_m, (k_1+1)2^{-m} - t_m) \times \ldots \times [k_n 2^{-m} - t_m, (k_n+1)2^{-m} - t_m)$ where $k_1, \ldots, k_n \in \mathbb{Z}$, $m < 0$ and where t_m is the quantity defined in Section 1.2. We denote the set of dyadic cubes in $\mathbb{R}^n$ by $\mathcal{D}$ and the set of dyadic cubes of side length 2^{-m} by $\mathcal{D}_m$. As we did before, if Q is a given arbitrary cube, we may refer to the set of dyadic cubes contained in Q by $\mathcal{D}(Q)$. For future convenience, given a signed Borel measure μ on $\mathbb{R}^n$, we define the *dyadic second divided difference* as

$$\Delta_2^d \mu(Q) = \Delta_1 \mu(Q) - \Delta_1 \mu(Q^*), \qquad Q \in \mathcal{D},$$

where we used Q^* to denote the unique dyadic cube that contains Q and is such that $l(Q^*) = 2l(Q)$. We will also need the *maximal dyadic second divided difference,* defined by

$$\Delta_2^* \mu(Q) = \max_{Q'} |\Delta_1 \mu(Q') - \Delta_1 \mu(Q)|, \qquad Q \in \mathcal{D},$$

where Q' ranges over all dyadic cubes contained in Q such that $l(Q') = l(Q)/2$. A signed Borel measure μ on $\mathbb{R}^n$ is called a *dyadic Zygmund measure,* $\mu \in \dot{\Lambda}_{*d}$, if

$$\|\mu\|_{\dot{\Lambda}_{*d}} = \sup_{Q \in \mathcal{D}} \Delta_2^* \mu(Q) < \infty.$$

A real valued function f on $\mathbb{R}^n$ is said to have *bounded dyadic mean oscillation*, $f \in \mathrm{BMO}_d(\mathbb{R}^n)$, if

$$\|f\|_{\mathrm{BMO}d} = \sup_{Q \in \mathcal{D}} \left(\frac{1}{|Q|} \int_Q |f(x) - f_Q|^2 \, dx \right)^{1/2} < \infty.$$

We will say that a signed Borel measure ν on $\mathbb{R}^n$ is a *dyadic* $\mathrm{I}_1(\mathrm{BMO})$ *measure,* $\nu \in \mathrm{I}_1(\mathrm{BMO})_d$, if it is absolutely continuous and its derivative is

$$d\nu = b(x) \, dx,$$

where $b \in \mathrm{BMO}_d(\mathbb{R}^n)$. It can be checked that ν is such a measure if and only if it satisfies

$$\sup_{Q \in \mathcal{D}} \left(\frac{1}{|Q|} \sum_{R \in \mathcal{D}(Q)} |\Delta_2^d \nu(R)|^2 |R| \right)^{1/2} < \infty.$$

The analogue of Theorem 1.6 for these dyadic spaces is the following.

Theorem 1.8. *Let μ be a compactly supported measure in $\dot{\Lambda}_{*d}$. For each $\varepsilon > 0$ consider*

$$D(\mu,\varepsilon) = \sup_{Q\in\mathcal{D}} \frac{1}{|Q|} \sum_{\substack{R\in\mathcal{D}(Q)\\ \Delta_2^*\mu(R^*)>\varepsilon}} |R|.$$

Then,

$$\mathrm{dist}_{\dot{\Lambda}_{*d}}(\mu, \mathrm{I}_1(\mathrm{BMO})_d) = \inf\{\varepsilon > 0\colon D(\mu,\varepsilon) < \infty\}. \tag{1.29}$$

Note that, as we did for functions on $\mathbb{R}$, we can rewrite this result in terms of dyadic martingales on $\mathbb{R}^n$. We define a *dyadic martingale on* $\mathbb{R}^n$ as a sequence of functions $S = \{S_m\}_{m=0}^{\infty}$ such that S_m is constant on any cube $Q \in \mathcal{D}_m$ and such that

$$S_m|Q = \frac{1}{2^n} \sum_{\substack{Q'\in\mathcal{D}_{m+1}\\ Q'\subset Q}} S_{m+1}|Q',$$

for all $Q \in \mathcal{D}_m$, $m \geq 0$. Given a measure $\mu \in \dot{\Lambda}_*$, we can define a dyadic martingale by taking

$$S_m(Q) = \Delta_1\mu(Q), \qquad Q \in \mathcal{D}_m,\ m \geq 0, \tag{1.30}$$

and then $\Delta S(Q) = S_m(Q) - S_{m-1}(Q^*) = \Delta_2^d\mu(Q)$, for $Q \in \mathcal{D}_m$, and we can rewrite Theorem 1.8 in terms of martingales. Following this relation between dyadic second divided differences for measures and martingale jumps, we will denote $\Delta^* S(Q) = \Delta_2^*\mu(Q)$.

Proof of Theorem 1.8. Assume that μ is supported on the unit cube $Q_0 = [0,1]^n$. We need to prove that, for a given $\varepsilon > 0$, there is a measure $\nu \in \mathrm{I}_1(\mathrm{BMO})_d$ satisfying $\|\mu - \nu\|_{\dot{\Lambda}_{*d}} \leq \varepsilon$ if and only if $D(\mu,\varepsilon) < \infty$. Denote by ε_0 the infimum in the right-hand side of (1.29).

Given $\varepsilon > \varepsilon_0$, consider the martingale S defined by (1.30). Approximate the martingale S by another dyadic martingale B in the following way. Start taking $B(Q_0) = S(Q_0)$. Then, for $Q \in \mathcal{D}(Q_0)$, set $\Delta B(Q) = \Delta S(Q)$ whenever $\Delta^* S(Q^*) > \varepsilon$, and set $\Delta B(Q) = 0$ otherwise. By construction, it is clear that $|\Delta S(Q) - \Delta B(Q)| \leq \varepsilon$ for any dyadic cube Q. Moreover, for any such cube Q, we have that

$$\frac{1}{|Q|} \sum_{R\in\mathcal{D}(Q)} |\Delta B(R)|^2 |R| = \frac{1}{|Q|} \sum_{\substack{R\in\mathcal{D}(Q)\\ \Delta^* S(R^*)>\varepsilon}} |\Delta S(R)|^2 |R| \lesssim \|\mu\|_{\dot{\Lambda}_{*d}} D(\mu,\varepsilon). \tag{1.31}$$

Define now $b(x) = \lim_n B_n(x) = \sum_{n=1}^{\infty} \Delta B_n(x)$. Using that, for any dyadic martingale, the increments ΔB_j are L^2 orthogonal, we get that

$$\int_{Q_0} b(x)^2\, dx = \int_{Q_0} \sum_{n=1}^{\infty} |\Delta B_n(x)|^2\, dx = \sum_{R\in\mathcal{D}(Q_0)} |\Delta B(R)|^2 |R| < \infty,$$

so that $b \in \mathrm{L}^2$ and it is finite almost everywhere. Hence, the measure ν defined by

$$d\nu = b(x)\, dx,$$

is an absolutely continuous measure that, by (1.31), is an $\mathrm{I}_1(\mathrm{BMO})_d$ measure such that $\|\mu - \nu\|_{\dot{\Lambda}_{*d}} \leq \varepsilon$.

On the other hand, if $\varepsilon_0 > 0$, whenever $0 < \varepsilon < \varepsilon_0$, there exists no measure $\nu \in \mathrm{I}_1(\mathrm{BMO})_d$ satisfying $\|\mu - \nu\|_{\dot{\Lambda}_{*d}} \leq \varepsilon$. Indeed, take $\varepsilon < \varepsilon_1 < \varepsilon_0$ and assume that there is $\nu \in \mathrm{I}_1(\mathrm{BMO})_d$ such that $\|\mu - \nu\|_{\dot{\Lambda}_* d} \leq \varepsilon$. Then, for any $Q \in \mathcal{D}$ such that $\Delta_2^* \mu(Q^*) > \varepsilon_1$, we have that $\Delta_2^* \nu(Q^*) > \varepsilon_1 - \varepsilon = \delta > 0$. Thus

$$\frac{1}{|Q|} \sum_{R \in \mathcal{D}(Q)} |\Delta_2^d \nu(R)|^2 |R| \geq \frac{\delta^2}{|Q|} \sum_{\substack{R \in \mathcal{D}(Q) \\ \Delta_2^* \mu(R^*) > \varepsilon_1}} |R|,$$

but the supremum over $Q \in \mathcal{D}$ of this last quantity is $\delta^2 D(\mu, \varepsilon_1)$, which is infinite since $\varepsilon_1 < \varepsilon_0$. This contradicts that $\nu \in \mathrm{I}_1(\mathrm{BMO})_d$. □

The proof of Theorem 1.6 follows the same lines than the proof of Theorem 1.2. We just mention that the construction used to prove Theorem 1.4 is easily adapted to the setting of $\mathbb{R}^n$, except for the following detail. Let Q be a cube in $\mathbb{R}^n$ and consider the covering $\mathcal{F}(Q) = \{R_j\}$ of Q by maximal dyadic cubes, in the same sense as we did in $\mathbb{R}$. In the case $n = 1$ we could have at most two elements of the same size in $\mathcal{F}(Q)$, but this does not hold for $n \geq 2$. For $n \geq 2$, the amount of cubes R_j in $\mathcal{F}(Q)$ of size $|R_j| = 2^{-kn}|Q|$, for some $k \geq 1$, is of the order of $2^{k(n-1)}$. Using this bound, one sees that the sums appearing in the estimates in the proof of Theorem 1.4 are convergent and bounded by a universal constant.

1.5 An Application to Sobolev Spaces

Fix $1 < p < \infty$. Consider the Sobolev space $\mathrm{W}^{1,p}$ of functions $f \in \mathrm{L}^p$ whose derivative f' in the sense of distributions is also in L^p. Consider as well, in the Zygmund class, the subspace $\dot{\Lambda}_*^p = \mathrm{W}^{1,p} \cap \dot{\Lambda}_*$. For $x \in \mathbb{R}$, consider the truncated cone $\Gamma(x) = \{(t, y) \in \mathbb{R}^2_+ : |x - t| < y < 1\}$. In [Nic18] it is shown that a function $f \in \mathrm{L}^p$ is in the Sobolev space $\mathrm{W}^{1,p}$ if and only if $M(f) \in \mathrm{L}^p$, where

$$M(f)(x) = \left(\int_{\Gamma(x)} \frac{|\Delta_2 f(t, y)|^2}{y^2} \frac{dt\, dy}{y^2} \right)^{1/2}, \qquad x \in \mathbb{R}.$$

The purpose of this section is to prove Theorem 1.5. Following the same scheme as before, we first need a dyadic version of the previous theorem. Let us first recall some more concepts and standard results of Martingale Theory that will be useful later. The *quadratic characteristic* of a dyadic martingale S is the function

$$\langle S \rangle(x) = \left(\sum_{m=1}^{\infty} |\Delta S_m(x)|^2 \right)^{1/2}, \qquad x \in \mathbb{R},$$

and its *maximal function* is

$$S^*(x) = \sup_m |S_m(x) - S_0(x)|, \qquad x \in \mathbb{R}.$$

Given $0 < p < \infty$ and a dyadic martingale S, the Burkholder-Davis-Gundy Inequality (see [BM99]) states that there exists a constant $C = C(p) > 0$ such that

$$C^{-1} \|\langle S \rangle\|_{\mathrm{L}^p} \leq \|S^*\|_{\mathrm{L}^p} \leq C \|\langle S \rangle\|_{\mathrm{L}^p}. \tag{1.32}$$

The *Fatou set* of a dyadic martingale S, denoted by $F(S)$, is defined as

$$F(S) = \{x \in \mathbb{R} : \lim_m S_m(x) \text{ exists and is finite}\}.$$

It is a standard result of Martingale Theory that, for a dyadic martingale S such that $\|S\|_{\dot{\Lambda}_*} < \infty$, its Fatou set is $F(S) = \{x \in \mathbb{R} : \langle S \rangle(x) < \infty\}$, where the equality must be understood up to sets of zero measure (see [Llo02]).

Using the characterisation for the Sobolev space $W^{1,p}$ previously stated, we say that a function b is in the dyadic space $\dot{\Lambda}^p_{*d}$ if its average growth martingale B, as defined in (1.9), has quadratic characteristic $\langle B \rangle \in L^p$ and

$$\|B\|_{\dot{\Lambda}_*} = \sup_{I \in \mathcal{D}} |\Delta B(I)| < \infty.$$

Note that, in fact, $\dot{\Lambda}^p_{*d} = W^{1,p} \cap \dot{\Lambda}_{*d}$. Indeed, if $b \in \dot{\Lambda}^p_{*d}$, by definition $b \in \dot{\Lambda}_{*d}$. Moreover, since its average growth martingale B has quadratic characteristic $\langle B \rangle \in L^p$, $\langle B \rangle(x) < \infty$ for almost every $x \in \mathbb{R}$. Thus, $B(x) = \lim_m B_m(x)$ exists almost everywhere and will satisfy $b'(x) = B(x)$ in the sense of distributions. Using (1.32), $B^* \in L^p$ and, thus, $B \in L^p$ as well, which is the same to say that $b' \in L^p$. We now state the analogous of Theorem 1.5 in this context.

Theorem 1.9. *Let f be a compactly supported function in $\dot{\Lambda}_{*d}$ and fix $1 < p < \infty$. Let S be the average growth martingale of f. For every $\varepsilon > 0$, define the truncated quadratic characteristic*

$$D(f,\varepsilon)(x) = \left(\#\{m : |\Delta S_m(x)| > \varepsilon\}\right)^{1/2}.$$

Then,

$$\operatorname{dist}_{\dot{\Lambda}_{*d}}(f, \dot{\Lambda}^p_{*d}) = \inf\{\varepsilon > 0 : D(f,\varepsilon) \in L^p\}. \tag{1.33}$$

Proof. Let ε_0 be the infimum on (1.33). Assume $0 < \varepsilon < \varepsilon_1 < \varepsilon_0$ and that there is $b \in \dot{\Lambda}^p_{*d}$ such that $\|f - b\|_{\dot{\Lambda}_{*d}} \le \varepsilon$. Let B be the average growth martingale of function b. Whenever $|\Delta S_m(x)| > \varepsilon_1$, we have that $|\Delta B_m(x)| > \varepsilon_1 - \varepsilon = \delta > 0$. Thus,

$$\begin{aligned}\langle B \rangle^2(x) = \sum_{m=1}^{\infty} |\Delta B_m(x)|^2 &\ge \sum_{|\Delta B_m(x)| > \delta} |\Delta B_m(x)|^2 \\ &\ge \frac{\delta^2}{\|f\|^2_{\dot{\Lambda}_{*d}}} D^2(f,\varepsilon_1)\end{aligned}$$

for all $x \in \mathbb{R}$. But, since $\varepsilon_1 < \varepsilon_0$, $D(f,\varepsilon_1) \notin L^p$ and so $\langle B \rangle \notin L^p$, getting in this way a contradiction. Hence, we see that $\operatorname{dist}(f, \dot{\Lambda}^p_{*d}) \ge \varepsilon_0$.

Assume that f is supported on I_0. Consider now $\varepsilon > \varepsilon_0$. Construct a dyadic martingale B with $B(I_0) = S(I_0)$ and such that $\Delta B(I) = \Delta S(I)$ for all $I \in \mathcal{D}(I_0)$ whenever $|\Delta S(I)| > \varepsilon$, but take $\Delta B(I) = 0$ when $|\Delta S(I)| \le \varepsilon$. Note that $\langle B \rangle \in L^p$. Therefore, using (1.32), we see that we can define $b'(x) = \lim_n B_n(x)$ almost everywhere with $b' \in L^p$. Taking now $b(x) = \int_0^x b'(s)\,ds$, we get $b \in \dot{\Lambda}^p_{*d}$ such that $\|f - b\|_{\dot{\Lambda}_{*d}} \le \varepsilon$. This shows that $\operatorname{dist}_{\dot{\Lambda}_{*d}}(f, \dot{\Lambda}^p_{*d}) \le \varepsilon_0$, completing the proof. □

Proof of Theorem 1.5. Let ε_0 be the infimum in (1.4). Assume $0 < \varepsilon < \varepsilon_1 < \varepsilon_0$, take $\delta = \varepsilon_1 - \varepsilon$, and assume that there is $b \in \dot{\Lambda}^p_*$ such that $\|f - b\|_{\dot{\Lambda}_*} \le \varepsilon$. The same argument used in the first part of the proof of Theorem 1.9 allows us to see that

$$M(b)(x) \ge \delta M(b,\delta)(x) \ge \delta M(f,\varepsilon_1)(x)$$

for $x \in \mathbb{R}$. Since $\varepsilon_1 < \varepsilon_0$, we have that $M(f,\varepsilon_1) \notin L^p$ and, thus, $M(b) \notin L^p$, contradicting that $b \in \dot{\Lambda}^p_*$. Hence, $\operatorname{dist}_{\dot{\Lambda}_*}(f, \dot{\Lambda}^p_*) \ge \varepsilon_0$.

Fix $\varepsilon > \varepsilon_0$ so that $M(f,\varepsilon) \in L^p$. For $\alpha \in [-1,1]$, consider $f^{(\alpha)} = f(x+\alpha)$. Note that $M(f^{(\alpha)},\varepsilon) \in L^p$ as well. Using the same argument as in the proof of Theorem

1.2, one can see that this fact implies that $D(f^{(\alpha)}, \varepsilon) \in \mathrm{L}^p$. Thus, for each $\alpha \in [-1, 1]$, the function $f^{(\alpha)}$ satisfies the hypothesis of Theorem 1.9 and may be approximated as $f^{(\alpha)} = b^{(\alpha)} + t^{(\alpha)}$, where $b^{(\alpha)} \in \dot{\Lambda}^p_{*d}$ with $\left\| b^{(\alpha)} \right\|_{\dot{\Lambda}_{*d}} \leq \|f\|_{\dot{\Lambda}_*}$ and $\|t\|_{\dot{\Lambda}_{*d}} \leq \varepsilon$. Apply now Theorem 1.4, with $R = 1$, both with the mapping $\alpha \mapsto b^{(\alpha)}$ and $\alpha \mapsto t^{(\alpha)}$ to obtain respectively functions b and t such that $f = b + t$ and such that $b \in \dot{\Lambda}^p_*$ and $\|t\|_{\dot{\Lambda}_*} \lesssim \varepsilon$. This completes the proof. □

Chapter 2

Approximation in the Zygmund Class using Wavelets

In this chapter, we generalise the results from Chapter 1 to functions defined on $\mathbb{R}^n$. Moreover, we also extend the results not only to the Zygmund class, but also on the spaces of Hölder continuous functions $\dot{\Lambda}_s(\mathbb{R}^n)$. Along this chapter, we will consider $n \geq 1$ to be a fixed integer. For this reason, unless we want to emphasise that we are dealing with functions defined on $\mathbb{R}^n$, we will in general omit this fact.

Recall that we say that a continuous real valued function f on $\mathbb{R}^n$ is in the *homogeneous Hölder class* of order s, with $0 < s < 1$, denoted by $f \in \dot{\Lambda}_s(\mathbb{R}^n)$ or just $f \in \dot{\Lambda}_s$, if

$$\sup_{x,\,|y|>0} \frac{|f(x+y) - f(x)|}{|y|^s} < \infty.$$

Moreover, whenever the previous supremum is finite, it is equivalent to

$$\|f\|_{\dot{\Lambda}_s} := \sup_{x,\,|y|>0} \frac{|f(x+y) - 2f(x) + f(x-y)|}{|y|^s} \tag{2.1}$$

(see [Zyg45] or [Ste70, p. 146]). In this chapter we take the quantity $\|f\|_{\dot{\Lambda}_s}$ as the *Hölder semi-norm* of f. In particular, the space $\dot{\Lambda}_s$ is a quotient space modulo the constant functions, which have zero Hölder semi-norm for any $0 < s < 1$. Given $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, we say that α is a multi-index of length $|\alpha| = \alpha_1 + \ldots + \alpha_n$, and we use the notation $\partial^\alpha = (\partial/\partial x_1)^{\alpha_1} \ldots (\partial/\partial x_n)^{\alpha_n}$. With this notation, if $s = m + t$ with $m \in \mathbb{Z}$ and $0 < t < 1$, we say that a continuous function f is in $\dot{\Lambda}_s$ if it is m times continuously differentiable and $\partial^\alpha f \in \dot{\Lambda}_t$ for every multi-index α with $|\alpha| = m$. In this case, we define the Hölder semi-norm of $f \in \dot{\Lambda}_s$ as $\|f\|_{\dot{\Lambda}_s} := \sup_{|\alpha|=m} \|\partial^\alpha f\|_{\dot{\Lambda}_t}$. Recall as well that we say that a continuous real valued function f on $\mathbb{R}^n$ is in the *homogeneous Zygmund class* $\dot{\Lambda}_*(\mathbb{R}^n)$, or simply $\dot{\Lambda}_*$, if

$$\|f\|_{\dot{\Lambda}_*} := \sup_{x,\,|y|>0} \frac{|f(x+y) - 2f(x) + f(x-y)|}{|y|} < \infty, \tag{2.2}$$

and the quantity $\|f\|_{\dot{\Lambda}_*}$ is called the *Zygmund semi-norm* of f. The space $\dot{\Lambda}_*$ is a quotient space modulo polynomials of degree at most 1. Also, if $s \in \mathbb{Z}$ is such that $s \geq 1$, we say that a continuous function f is in the *homogeneous Zygmund class of regularity s*, denoted by $\dot{\Lambda}^s_*$, if it is $s-1$ times continuously differentiable and $\partial^\alpha f \in \dot{\Lambda}_*$ for every multi-index α with $|\alpha| = s-1$. In this situation, we define the Zygmund semi-norm of $f \in \dot{\Lambda}^s_*$ as $\|f\|_{\dot{\Lambda}^s_*} := \sup_{|\alpha|=s-1} \|\partial^\alpha f\|_{\dot{\Lambda}_*}$. As it was explained in the Introduction, we will restrict ourselves to the spaces $\dot{\Lambda}_s$ for $0 < s < 1$ and to $\dot{\Lambda}_* = \dot{\Lambda}^1_*$, because the higher regularity spaces are studied in the same way. Moreover, in the following results, when we talk about the spaces $\dot{\Lambda}_s$ for $0 < s \leq 1$, we implicitly replace $\dot{\Lambda}_1$ (as it was defined in the Introduction) by $\dot{\Lambda}_*$. Recall that this replacement is justified by the theory of Besov spaces (see for instance [Tri10, Chapter 2]). In particular, the class of Lipschitz functions $\dot{\Lambda}_1$ will play no role in this chapter. For later convenience, because of the definitions (2.1) and (2.2) for the semi-norms of these spaces, given a continuous function f on $\mathbb{R}^n$ we define the *second difference* of f at point x and scale y as

$$\Delta_2 f(x,y) := \sup_{|t|=y} |f(x+t) - 2f(x) + f(x-t)|, \qquad x \in \mathbb{R}^n,\, y > 0.$$

Note that, in the case $n = 1$, this definition does not coincide with that given in Chapter 1, but it is its absolute value. As before, we consider $\Delta_2 f$ as a function defined on the upper half-space $\mathbb{R}^{n+1}_+$.

A locally integrable function f on $\mathbb{R}^n$ is said to have *bounded mean oscillation,* $f \in \mathrm{BMO}$, if

$$\|f\|_{\mathrm{BMO}} := \sup_Q \left(\frac{1}{|Q|} \int_Q |f(x) - f_Q|^2 \, dx \right)^{1/2} < \infty, \tag{2.3}$$

where Q ranges over all finite cubes with sides parallel to the axes in $\mathbb{R}^n$ and where

$$f_Q := \frac{1}{|Q|} \int_Q f(x)\, dx$$

is the average of f on Q. Constant functions have zero BMO norm, so one should consider the set of functions satisfying (2.3) modulo constant functions to study it as a Banach space. Let $\mathcal{F}$ denote the Fourier transform operator. Recall that, given $s \in \mathbb{R}$, we say that a tempered distribution f defined on $\mathbb{R}^n$ belongs to the space $\mathrm{I}_s(\mathrm{BMO})$ if there exists a function $g \in \mathrm{BMO}$ such that $f = \mathcal{F}^{-1}\left[|\xi|^{-s}\mathcal{F}[g]\right]$, where the Fourier transform and its inverse must be understood in the sense of tempered distributions modulo polynomials. This is the same to say that the space $\mathrm{I}_s(\mathrm{BMO})$ is the image of BMO under the Riesz potential I_s, defined by

$$I_s(f) := \mathcal{F}^{-1}\left[|\xi|^{-s}\mathcal{F}[f]\right]$$

for any tempered distribution f modulo polynomials, which is also the same as to say that the fractional laplacian $(-\Delta)^{s/2} f$ is in BMO. These spaces are sometimes called BMO-*Sobolev spaces*. Again, we use the notation $\mathrm{I}_s(\mathrm{BMO})(\mathbb{R}^n)$ whenever it is necessary to avoid ambiguity. For $s \geq 0$, the space $\mathrm{I}_s(\mathrm{BMO})$ is a space of functions, with the particular case $s = 0$ corresponding to the classical BMO. In addition, for $s > 0$ it is actually a space of continuous functions. Moreover, for a given $s \in \mathbb{R}$, if k is a positive integer, then $f \in \mathrm{I}_s(\mathrm{BMO})$ if and only if $\partial^\alpha f \in \mathrm{I}_{s-k}(\mathrm{BMO})$ for every multi-index α of length $|\alpha| = k$. Due to this fact, we will restrict ourselves to the spaces $\mathrm{I}_s(\mathrm{BMO})$ with $0 < s \leq 1$. For a reference on the BMO-Sobolev spaces, including the properties stated here, see [Str80].

R. Strichartz, in [Str80], gave a characterisation of the space $\mathrm{I}_s(\mathrm{BMO})(\mathbb{R}^n)$, for $0 < s < 2$, in terms of its second differences. This was already stated in Theorem 1.1 for compactly supported functions defined on $\mathbb{R}$ and $s = 1$. Here we state it again for compactly supported functions on $\mathbb{R}^n$, with the notation that we are using in this chapter.

Theorem 2.1 (R. Strichartz)**.** *Let $0 < s < 2$. A compactly supported function f is in* $\mathrm{I}_s(\mathrm{BMO})$ *if and only if*

$$\sup_Q \left(\frac{1}{Q} \int_Q \int_0^{l(Q)} \frac{\Delta_2 f(x,y)^2}{y^{2s}} \frac{dy\,dx}{y} \right)^{1/2} < \infty, \tag{2.4}$$

where the supremum ranges over all finite cubes with sides parallel to the axes.

Given a function $f \in \dot{\Lambda}_s$, with $0 < s \leq 1$, we consider its distance to a subspace $X \subseteq \dot{\Lambda}_s$ in terms of the Hölder semi-norm, that is, $\mathrm{dist}_{\dot{\Lambda}_s}(f, X) := \inf_{g \in X} \|f - g\|_{\dot{\Lambda}_s}$. Recall that we say that a cube $Q \subset \mathbb{R}^n$ is a *dyadic cube*, denoted by $Q \in \mathcal{D}$, if it is of the form

$$Q = \{x \in \mathbb{R}^n : 2^j x - k \in [0,1]^n\}$$

for some $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$. Note that this definition is equivalent to that given in Section 1.4 whenever $j \geq 0$. The generalisation of Theorem 1.2 we present in this chapter is the following.

Theorem 2.2. *Let $0 < s \leq 1$ and consider a compactly supported function $f \in \dot{\Lambda}_s$. For each $\varepsilon > 0$ consider the set*

$$S(s,f,\varepsilon) = \left\{ (x,y) \in \mathbb{R}^{n+1}_+ : \Delta_2 f(x,y) > \varepsilon y^s \right\}, \tag{2.5}$$

and the quantity

$$M(S(s,f,\varepsilon)) = \sup_{Q \in \mathcal{D}} \frac{1}{|Q|} \int_Q \int_0^{l(Q)} \chi_{S(s,f,\varepsilon)}(x,y) \frac{dy\,dx}{y}.$$

Then,

$$\mathrm{dist}_{\dot{\Lambda}_s}(f, \mathrm{I}_s(\mathrm{BMO})) \simeq \inf\{\varepsilon > 0 : M(S(s,f,\varepsilon)) < \infty\}. \tag{2.6}$$

The main tool to show this result will be the wavelet characterisations of the function spaces involved. Consider the space $\mathrm{L}^2(\mathbb{R}^n)$ of square integrable functions on $\mathbb{R}^n$. It is known that, for any non-negative $r \in \mathbb{Z}$, there exist real functions ψ_l, for $1 \leq l \leq 2^n - 1$, such that the set

$$\{\psi_{l,j,k}(x) : 1 \leq l \leq 2^n - 1, j \in \mathbb{Z}, k \in \mathbb{Z}^n\},$$

where $\psi_{l,j,k}(x) = 2^{jn/2}\psi_l(2^j x - k)$, forms an orthonormal basis of $\mathrm{L}^2(\mathbb{R}^n)$, and such that for every $N \geq 1$ there is a constant $C_N > 0$ satisfying

$$|\partial^\alpha \psi_l(x)| \leq C_N (1 + |x|)^{-N}, \qquad x \in \mathbb{R}^n,$$

for every $1 \leq l \leq 2^n - 1$ and for every multi-index $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq r$. For a multi-index α and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we denote $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$. Then, these functions also satisfy that

$$\int_{\mathbb{R}^n} x^\alpha \psi_l(x)\, dx = 0,$$

for any multi-index $\alpha \in \mathbb{N}^n$ such that $|\alpha| \leq r$. Such a set of functions is called a wavelet basis of regularity r. For a detailed explanation on how to construct such bases see, for instance, [Mey92]. These functions can be taken to be real valued, and we will assume that this is the case. Moreover, these functions can be chosen to be compactly supported (see [Dau88]), although we will not need to assume this. Recall that $\mathcal{D}$ denotes the set of dyadic cubes in $\mathbb{R}^n$ and, for $j \in \mathbb{Z}$, let us denote the set of those dyadic cubes Q of side length $l(Q) = 2^{-j}$ by $\mathcal{D}_j$. Assuming that $Q \in \mathcal{D}_j$, we let $\tau(Q) := j$ denote the dyadic level of Q. We relate the cubes in $\mathcal{D}$ to the functions in the wavelet basis using the notation $\psi_{(l,Q)}(x) = 2^{nj/2}\psi_l(2^j x - k)$ when $Q = \{x \in \mathbb{R}^n : 2^j x - k \in [0,1]^n\}$. For future convenience, we shall define $\Omega = \{(l, Q) : 1 \le l \le 2^n - 1, Q \in \mathcal{D}\}$ and, for $\omega = (l, Q) \in \Omega$, we denote $|\omega| := \tau(Q)$. Finally, given a cube $Q \subset \mathbb{R}^n$, we denote by $\Omega(Q)$ the set of $(l, P) \in \Omega$ for which $P \subseteq Q$.

Consider a wavelet basis $\{\psi_\omega : \omega \in \Omega\}$ of regularity r. Let f be a function in $\dot{\Lambda}_s$ for some $0 < s \le 1$. The wavelet coefficients of f are

$$c_{(l,Q)}(f) = c_\omega(f) := \int_{\mathbb{R}^n} f(x)\psi_\omega(x)\,dx, \qquad (l, Q) = \omega \in \Omega.$$

Recall that for $f \in \dot{\Lambda}_s$, for $0 < s < 1$, we have that $|f(x)| \le C(1 + |x|)^s$, while for $f \in \dot{\Lambda}_*$ it happens that $|f(x)| \le C(1 + |x| \log |x|)$, with the constant appearing in both bounds depending on the function (see [Mey92, pp. 180–181]). Thus, $c_\omega(f)$ is well-defined due to the fast decay of the wavelet function ψ_ω at infinity. Given a wavelet basis, the wavelet coefficients can actually be defined for a wide class of distributions which depends on the regularity of the basis. However, this level of generality will not be necessary here. In [LM86], P. Lemarié and Y. Meyer characterise when a function f is in the space $\dot{\Lambda}_s$, for $s > 0$, in terms of its wavelet coefficients $\{c_\omega(f)\}$. See also [AB97] and [Mey92, p. 185] for a more detailed explanation.

Theorem 2.3 (P. Lemarié, Y. Meyer). *Let $s > 0$ and consider a wavelet basis $\{\psi_\omega : \omega \in \Omega\}$ of regularity $r > s$. The wavelet series*

$$f(x) = \sum_{\omega \in \Omega} c_\omega(f)\psi_\omega(x), \qquad x \in \mathbb{R}^n,$$

is in $\dot{\Lambda}_s(\mathbb{R}^n)$ if and only if there exists $C > 0$ such that

$$|c_\omega(f)| \le C 2^{-|\omega|(n/2+s)}, \qquad \omega \in \Omega. \tag{2.7}$$

Moreover, if $\|c(f)\|_{\dot{\Lambda}_s}$ is the smallest possible constant in (2.7)*, then $\|c(f)\|_{\dot{\Lambda}_s} \simeq \|f\|_{\dot{\Lambda}_s}$.*

In [LM86], the authors also give a wavelet characterisation for functions in the space BMO (see also [AB97], [Mey92, pp. 154–156] and [Ste93, Section IV.4.5]).

Theorem 2.4 (P. Lemarié, Y. Meyer). *Consider a wavelet basis $\{\psi_\omega : \omega \in \Omega\}$ of regularity $r \ge 1$. If the wavelet series*

$$f(x) = \sum_{\omega \in \Omega} c_\omega(f)\psi_\omega(x), \qquad x \in \mathbb{R}^n,$$

is in BMO, *then*

$$\sup_{Q \in \mathcal{D}} \left(\frac{1}{|Q|} \sum_{\omega \in \Omega(Q)} |c_\omega(f)|^2 \right)^{1/2} \lesssim \|f\|_{\mathrm{BMO}}.$$

Conversely, if the sequence $\{c_\omega : \omega \in \Omega\}$ is such that

$$\sup_{Q \in \mathcal{D}} \left(\frac{1}{|Q|} \sum_{\omega \in \Omega(Q)} |c_\omega|^2 \right)^{1/2} < \infty,$$

then the series $\sum_\omega c_\omega \psi_\omega(x)$ converges in the weak-$*$ topology to a function in BMO.

Similarly, there is an analogous characterisation for $\mathrm{I}_s(\mathrm{BMO})$. This wavelet characterisation, at least for smooth wavelet bases, appears as a particular case of a theorem due to M. Frazier and B. Jawerth (see [FJ90, Sections 2 and 5], and Theorem 2.2 in that article in particular). The more general result for either smooth or compactly supported wavelet bases is stated in [Tri20, p. 16] for the corresponding inhomogeneous spaces, from which one can recover the result for $\mathrm{I}_s(\mathrm{BMO})$. For completeness, we include a proof of this characterisation in Section 2.1. This is a modification of the proof of Theorem 2.4 as it appears in [Mey92, pp. 154–156], following the ideas that lead to the wavelet characterisation of the classical Sobolev spaces (see [Mey92, pp. 168–170] and [MC97, pp. 56–57]). For simplicity, from now on we will assume that $r > s + 1$.

Theorem 2.5. *Let $s > 0$ and consider a wavelet basis $\{\psi_\omega : \omega \in \Omega\}$ of regularity $r > s + 1$. The wavelet series*

$$f(x) = \sum_{\omega \in \Omega} c_\omega(f)\psi_\omega(x), \qquad x \in \mathbb{R}^n,$$

is in $\mathrm{I}_s(\mathrm{BMO})$ if and only if

$$\sup_{Q \in \mathcal{D}} \left(\frac{1}{|Q|} \sum_{\omega \in \Omega(Q)} 4^{s|\omega|} |c_\omega(f)|^2 \right)^{1/2} < \infty. \tag{2.8}$$

Observe that it is an immediate consequence of Theorems 2.3 and 2.5 that $\mathrm{I}_s(\mathrm{BMO}) \subset \dot{\Lambda}_s$ for any $s > 0$, with the inclusion being strict.

Our key tool in proving Theorem 2.2 is the following analogous result in terms of wavelet coefficients.

Theorem 2.6. *Let $0 < s \leq 1$ and consider a wavelet basis $\{\psi_\omega : \omega \in \Omega\}$ of regularity $r > s + 1$. Consider a function $f \in \dot{\Lambda}_s(\mathbb{R}^n)$. For each $\varepsilon > 0$, consider as well the set*

$$W(s, f, \varepsilon) = \bigcup_{j \in \mathbb{Z}} \left\{ Q \in \mathcal{D}_j : \sup_l |c_{(l,Q)}(f)| > \varepsilon 2^{-j(n/2+s)} \right\}$$

and the quantity

$$M_W(s, f, \varepsilon) = \sup_{Q \in \mathcal{D}} \frac{1}{|Q|} \sum_{\substack{P \in W(s,f,\varepsilon), \\ P \subseteq Q}} |P|.$$

Then, we have that

$$\mathrm{dist}_{\dot{\Lambda}_s}(f, \mathrm{I}_s(\mathrm{BMO})) \simeq \inf\{\varepsilon > 0 : M_W(s, f, \varepsilon) < \infty\}. \tag{2.9}$$

Note that the infimum in (2.9) is taken over a non empty set, since for any function $f \in \dot{\Lambda}_s$ with wavelet coefficients $\{c_\omega(f)\}$ one has that $M_W(s, f, \|c(f)\|_{\dot{\Lambda}_s}) = 0$.

Proof of Theorem 2.6. Denote by ε_0 the infimum in (2.9), which we assume to be positive, and assume that $\varepsilon < \varepsilon_0$. Consider a function $g \in \mathrm{I}_s(\mathrm{BMO})$ and assume that $\|c(f) - c(g)\|_{\dot{\Lambda}_s} \leq \varepsilon$, where $\{c_\omega(f)\}$ and $\{c_\omega(g)\}$ are, respectively, the wavelet coefficients of f and g. Take $\varepsilon < \varepsilon' < \varepsilon_0$ and note that, for $\omega \in \Omega$, whenever $|c_\omega(f)| > \varepsilon' 2^{-|\omega|(n/2+s)}$, we have that $|c_\omega(g)| > \delta 2^{-|\omega|(n/2+s)}$, where $\delta = \varepsilon_0 - \varepsilon' > 0$. Thus, for any cube $Q \in \mathcal{D}$, we have that

$$\frac{1}{|Q|} \sum_{\omega \in \Omega(Q)} 4^{|\omega|s} |c_\omega(g)|^2 \gtrsim \frac{\delta^2}{|Q|} \sum_{\substack{P \in W(s,f,\varepsilon') \\ P \subseteq Q}} 2^{-n\tau(P)} = \frac{\delta^2}{|Q|} \sum_{\substack{P \in W(s,f,\varepsilon') \\ P \subseteq Q}} |P|,$$

but the supremum, with Q ranging over all dyadic cubes, of the latter quantity is not finite since $\varepsilon' < \varepsilon_0$. By Theorem 2.5, this contradicts that $g \in \mathrm{I}_s(\mathrm{BMO})$ and, thus, $\mathrm{dist}_{\dot\Lambda_s}(f, \mathrm{I}_s(\mathrm{BMO})) \gtrsim \varepsilon_0$.

If $\varepsilon > \varepsilon_0$, we construct a function $g \in \mathrm{I}_s(\mathrm{BMO})$ such that $\mathrm{dist}_{\dot\Lambda_s}(f,g) \lesssim \varepsilon$. Given the wavelet coefficients $\{c_\omega(f)\}$ of f, we set $c_\omega(g) = c_\omega(f)$ whenever $\omega = (l, P)$ with $P \in W(s, f, \varepsilon)$, and $c_\omega(g) = 0$ otherwise. Clearly $\|c(f) - c(g)\|_{\dot\Lambda_s} \leq \varepsilon$ and, by Theorem 2.3, $\mathrm{dist}_{\dot\Lambda_s}(f,g) \lesssim \varepsilon$. Furthermore, we have that

$$\frac{1}{|Q|} \sum_{\omega \in \Omega(Q)} 4^{|\omega|s} |c_g(\omega)|^2 \lesssim \frac{\|c(f)\|^2_{\dot\Lambda_s}}{|Q|} \sum_{\substack{P \in W(s,f,\varepsilon) \\ P \subseteq Q}} |P|.$$

Since $\varepsilon > \varepsilon_0$, the supremum of the latter quantity, when Q ranges over all dyadic cubes, is finite. Therefore, by Theorem 2.5, $g \in \mathrm{I}_s(\mathrm{BMO})$, as we wanted to show. □

Recall that a compactly supported function $f \in \dot\Lambda_s$ also belongs to the inhomogeneous space Λ_s. A classical way to characterise such spaces is in terms of the hyperbolic derivatives of the Poisson extensions of their functions on the upper half-space. Namely, let us denote by $P_y(x)$ the Poisson kernel on $\mathbb{R}^{n+1}_+$, and by u the harmonic extension $u(x,y) = \mathrm{P}[f](x,y) = (P_y * f)(x)$ of f. Given $0 < s \leq 1$, a function f is in Λ_s if and only if it is uniformly bounded and

$$y^{2-s} \left| \frac{\partial^2 u}{\partial y^2}(x,y) \right| \leq C, \qquad (x,y) \in \mathbb{R}^{n+1}_+. \tag{2.10}$$

Moreover, the smallest constant in (2.10) is comparable to $\|f\|_{\dot\Lambda_s}$. For a reference on this fact, see [Ste70, pp. 141–149]. This motivates us to estimate the distance of a given compactly supported function $f \in \dot\Lambda_s$ to the subspace $\mathrm{I}_s(\mathrm{BMO})$ in terms of these hyperbolic derivatives. Consider the set

$$D(s,f,\varepsilon) = \left\{ (x,y) \in \mathbb{R}^{n+1}_+ : y^2 \left| \frac{\partial^2 \mathrm{P}[f]}{\partial y^2}(x,y) \right| > \varepsilon y^s \right\},$$

that is the set of points in the upper half-space for which the second hyperbolic derivative of f is large with respect to the corresponding scale. We shall show the following result.

Theorem 2.7. *Let $0 < s \leq 1$ and consider a compactly supported function $f \in \dot\Lambda_s$. For each $\varepsilon > 0$ consider the set*

$$D(s,f,\varepsilon) = \left\{ (x,y) \in \mathbb{R}^{n+1}_+ : y^2 \left| \frac{\partial^2 \mathrm{P}[f]}{\partial y^2}(x,y) \right| > \varepsilon y^s \right\} \tag{2.11}$$

and the quantity

$$M(D(s,f,\varepsilon)) = \sup_{Q \in \mathcal{D}} \frac{1}{|Q|} \int_Q \int_0^{l(Q)} \chi_{D(s,f,\varepsilon)}(x,y) \, \frac{dy\,dx}{y}.$$

Then,

$$\operatorname{dist}_{\dot{\Lambda}_s}(f, \mathrm{I}_s(\mathrm{BMO})) \simeq \inf\{\varepsilon > 0\colon M(D(s,f,\varepsilon)) < \infty\}. \tag{2.12}$$

Our proof of Theorems 2.2 and 2.7 will be via Theorem 2.6. However, the reduction is rather non trivial and will be based on careful comparison of the sets $D(s,f,\varepsilon)$, $S(s,f,\varepsilon)$, and the analogous set obtained from the wavelet coefficients:

$$T(s,f,\varepsilon) = \bigcup_{Q \in W(s,f,\varepsilon)} T(Q), \tag{2.13}$$

where for a given dyadic square Q in $\mathbb{R}^n$ we denote by $T(Q)$ its top half cube in $\mathbb{R}^{n+1}_+$. In other words

$$T(Q) = \{(x,y) \in \mathbb{R}^{n+1}_+ \colon x \in Q,\, l(Q)/2 < y < l(Q)\}.$$

Given a set measurable set $A \subseteq \mathbb{R}^{n+1}_+$, we will use the notation

$$M(A) := \sup_{Q \in \mathcal{D}} \frac{1}{|Q|} \int_Q \int_0^{l(Q)} \chi_A(x,y)\, \frac{dy\, dx}{y}.$$

Directly from the definition, it follows that the quantity $M_W(s,f,\varepsilon)$ in Theorem 2.6 is comparable to $M(T(s,f,\varepsilon))$. Our aim is to show that there are inequalities of the type $M(T(s,f,\varepsilon)) \lesssim M(S(s,f,c\varepsilon))$, and similar inequalities between the other pairs, for an absolute constant $c > 0$. The proofs are based on considering inclusions between hyperbolically dilated sets. These then easily yield Theorems 2.2 and 2.7.

The rest of the chapter is structured as follows. First, we show how to prove Theorem 2.5 in Section 2.1. In section 2.2, we study the variability of the second differences, the wavelet coefficients and the hyperbolic derivative with respect to the location in the upper half-space. This is measured in terms of the hyperbolic distance, which we will denote by ρ. Finally, in Section 2.3, we are able to make a rigorous comparison of the sets S, D and T.

2.1 Wavelet Characterisation for the BMO-Sobolev Spaces

Here, for $\omega \in \Omega$, we denote by k_ω the element of the lattice $\mathbb{Z}^n$ such that $\omega = (l, Q)$ for some $1 \le l \le 2^n - 1$ and $Q = \{x \in \mathbb{R}^n \colon 2^{|\omega|}x - k_\omega \in [0,1]^n\}$. Moreover, for each function ψ_l generating our wavelet basis and $s \in \mathbb{R}$, we denote

$$\psi_l^s(x) := \mathcal{F}^{-1}\left[|\xi|^s \mathcal{F}\left[\psi_l\right](\xi)\right](x) = (-\Delta)^{s/2}\psi_l(x)$$

and, for $Q \in \mathcal{D}$, we set

$$\psi^s_{(l,Q)}(x) := 2^{|\omega|n/2}\psi_l^s(2^{|\omega|}x - k_\omega).$$

Observe that, with this convention, we have that $(-\Delta)^{s/2}\psi_\omega(x) = 2^{|\omega|s}\psi^s_\omega(x)$. First, we state without proof two auxiliary lemmas.

Lemma 2.1 ([Mey92, pp. 168–170])**.** *For $|s| < r$, the set of functions $\{\psi^s_\omega\}$ satisfies*

$$|\psi^s_\omega(x)| \lesssim 2^{|\omega|n/2}(1 + |2^{|\omega|}x - k_\omega|)^{-(n+r-s)}, \tag{2.14}$$

$$\int \psi^s_\omega(x)\, dx = 0 \tag{2.15}$$

and

$$\left\| \sum_{\omega\in\Omega} c_\omega \psi^s_\omega(x) \right\|^2_{L^2} \simeq \sum_{\omega\in\Omega} |c_\omega|^2 \tag{2.16}$$

for every sequence $\{c_\omega\} \in l^2(\Omega)$.

Moreover, if $0 < s < r$, *then* ψ^{-s}_ω *is* r *times continuously differentiable and* ψ^s_ω *is in* $\dot{\Lambda}_{r-s}$ *for every* $\omega \in \Omega$.

Lemma 2.2 ([Mey92, p. 155]). *For* $|s| < r$, *consider the set of functions* $\{\psi^s_\omega\}$. *Given a function* $g \in \mathrm{BMO}$ *consider the sequence* $\{c_\omega(g)\}$ *defined by*

$$c_\omega(g) := \int g(x)\psi^s_\omega(x)\, dx.$$

Then, we have that

$$\sup_{Q\in\mathcal{D}} \left(\frac{1}{|Q|} \sum_{\omega\in\Omega(Q)} |c_\omega(g)|^2 \right)^{1/2} \lesssim \|g\|_{\mathrm{BMO}}\,. \tag{2.17}$$

Proof of Theorem 2.5. Assume that (2.8) is satisfied and set

$$g(x) := (-\Delta)^{s/2} f(x) = \sum_{\omega\in\Omega} 2^{|\omega|s} c_\omega \psi^s_\omega(x).$$

To see that $g \in \mathrm{BMO}$, fix a cube $Q \in \mathcal{D}$ of side length 2^{-j_0}. Let us denote by m a positive constant to be determined later. First, split the set Ω into $\Omega_1 = \Omega(mQ)$, $\Omega_2 = \{\omega \in \Omega \setminus \Omega_1 \colon |\omega| \geq j_0\}$ and $\Omega_3 = \Omega \setminus (\Omega_1 \cup \Omega_2)$, and express accordingly $g = g_1 + g_2 + g_3$, where

$$g_i(x) = \sum_{\omega\in\Omega_i} 2^{|\omega|s} c_\omega \psi^s_\omega(x), \qquad \text{for } i = 1, 2, 3.$$

Now, by (2.16), for g_1 we have that

$$\int_Q |g_1(x)|^2\, dx \leq \|g_1\|^2_{L^2} \simeq \sum_{\omega\in\Omega(mQ)} 4^{|\omega|s} |c_\omega|^2 \lesssim m^n |Q|.$$

To get an analogous bound for g_2, we use (2.14) and that for $\omega \in \Omega_2$ and $x \in Q$ we have the lower bound

$$\left| 2^{|\omega|} x - k_\omega \right| \gtrsim m 2^{(|\omega| - j_0)}. \tag{2.18}$$

Thus, for $\omega \in \Omega_2$, if we denote by x_0 the centre of Q, we can choose m large enough so that it holds that

$$\int_Q |\psi^s_\omega(x)|^2\, dx \lesssim |Q| 2^{|\omega| n} \left| 2^{|\omega|} x_0 - k_\omega \right|^{-2(n+r-s)}.$$

Observe that (2.8) implies in particular that $2^{|\omega|s}|c_\omega| \lesssim 2^{-|\omega|n/2}$ for any $\omega \in \Omega$. Thus, by Hölder's inequality we get that

$$\int_Q |g_2(x)|^2\,dx \le \sum_{\omega\in\Omega_2}\sum_{\omega'\in\Omega_2} 2^{|\omega|s}|c_\omega|2^{|\omega'|s}|c_{\omega'}| \int_Q |\psi^s_\omega(x)\psi^s_{\omega'}(x)|\,dx$$
$$\lesssim |Q| \sum_{\omega\in\Omega_2}\sum_{\omega'\in\Omega_2} \left|2^{|\omega|}x_0 - k_\omega\right|^{-(n+r-s)} \left|2^{|\omega'|}x_0 - k_{\omega'}\right|^{-(n+r-s)}.$$

Observe that summing over $\omega \in \Omega_2$ with $|\omega| = j \ge j_0$ is the same as summing over $k \in \mathbb{Z}^n$ such that $\left|2^j x_0 - k\right| \gtrsim m2^{j-j_0}$ because of (2.18). Hence, the previous sum turns out to be bounded by

$$m^{-2(r-s)}|Q| \sum_{j\ge j_0}\sum_{j'\ge j_0} 2^{-(j-j_0)(r-s)}2^{-(j'-j_0)(r-s)} \lesssim m^{-2(r-s)}|Q|.$$

Finally, to obtain the appropriate bound for g_3, note that for each $j < j_0$ there are of the order of m^n values of $\omega = (l, P) \in \Omega_3$ with $|\omega| = j$ such that $P \cap 2^{j_0-j}mQ \ne \emptyset$. For any such ω, we can use that ψ^s_ω are Lipschitz functions if $r > s + 1$ by Lemma 2.1, so that

$$|\psi^s_\omega(x) - \psi^s_\omega(x_0)| \lesssim 2^{|\omega|}2^{|\omega|n/2}|x - x_0|.$$

On the other hand, observe as well that the assumption $r > s + 1$ also implies

$$|\psi^{s+1}_\omega(x)| \lesssim 2^{|\omega|}2^{|\omega|n/2}\left|2^{|\omega|}x - k_\omega\right|^{-(n+r-s-1)}, \qquad \text{as } x \to \infty,$$

which is just (2.14) applied to the functions ψ^{s+1}_l. Thus, m can be chosen large enough so that, if $\omega = (l, P) \in \Omega_3$ is such that $|\omega| = j$ and $P \cap 2^{j_0-j}mQ = \emptyset$, then we have that

$$|\psi^s_\omega(x) - \psi^s_\omega(x_0)| \lesssim 2^{|\omega|}2^{|\omega|n/2}\left|2^{|\omega|}x_0 - k_\omega\right|^{-(n+r-s-1)}|x - x_0|.$$

Using the same argument as for g_2, we get that

$$\sum_{\substack{\omega\in\Omega_3\\|\omega|=j}} 2^{|\omega|s}|c_\omega|\,|\psi^s_\omega(x) - \psi^s_\omega(x_0)| \lesssim m^n 2^j|x - x_0|.$$

Therefore, for $x \in Q$, in which case $|x - x_0| \lesssim 2^{-j_0}$, we get that

$$|g_3(x) - g_3(x_0)| \le \sum_{\omega\in\Omega_3} 2^{|\omega|s}|c_\omega|\,|\psi^s_\omega(x) - \psi^s_\omega(x_0)| \lesssim m^n,$$

and it follows that

$$\int_Q |g_3(x) - g_3(x_0)|^2\,dx \lesssim m^{2n}|Q|,$$

so that $g \in$ BMO as we wanted to see.

Assume now that $g \in \mathrm{BMO}(\mathbb{R}^n)$. Then, if we define the sequence $\{c_\omega(g)\}$ by

$$c_\omega(g) := \int g(x)\psi^{-s}_\omega(x)\,dx, \qquad \omega \in \Omega,$$

by Lemma 2.2 we have that

$$\sup_{Q\in\mathcal{D}}\left(\frac{1}{|Q|}\sum_{\omega\in\Omega(Q)}|c_\omega(g)|^2\right)^{1/2}<\infty.$$

But we have that

$$c_\omega(g)=2^{|\omega|s}\int g(x)(-\Delta)^{-s/2}\psi_\omega(x)\,dx=2^{|\omega|s}\int f(x)\psi_\omega(x)\,dx,=2^{|\omega|s}c_\omega(f),$$

from which (2.8) follows immediately. □

2.2 Properties of the sets S, D and T

Recall that the differential hyperbolic arc length ds at $(x,y)\in\mathbb{R}^{n+1}_+$ is defined by

$$ds^2=\frac{dx^2+dy^2}{y^2}.$$

Geodesics in this metric are circular arcs intersecting $\mathbb{R}^n$ orthogonally and vertical lines. We denote by $\rho(a,b)$ the hyperbolic distance between $a,b\in\mathbb{R}^{n+1}_+$ given by this metric, that is the hyperbolic arc length of the geodesic segment joining a and b. Given a set $A\subseteq\mathbb{R}^{n+1}_+$ and $R>0$, we will denote by A_R the set

$$A_R=\{p\in\mathbb{R}^{n+1}_+\colon\rho(p,A)<R\},$$

which is the R-hyperbolic neighbourhood of A. From now on, fix $0<s\le 1$ and $f\in\dot{\Lambda}_s$, and denote the sets defined in (2.13), (2.5) and (2.11) by $T(\varepsilon)=T(s,f,\varepsilon)$, $S(\varepsilon)=S(s,f,\varepsilon)$ and $D(\varepsilon)=D(s,f,\varepsilon)$ respectively to simplify the notation, whenever there is no ambiguity. We first estimate how the quantities $\Delta_2 f(x,y)$ vary.

Lemma 2.3. *Let $0<s<1$. Consider a function $f\in\dot{\Lambda}_s$. If $1/2<y/y'<2$, then*

$$|\Delta_2 f(x,y)-\Delta_2 f(x',y')|\lesssim\|f\|_{\Lambda_s}\left(|x-x'|^s+|y-y'|^s\right).$$

Proof. Consider $p\in\mathbb{R}^n$ such that $|p|=y$ and note that

$$|(f(x+p)-f(x'+p))-2(f(x)-f(x'))+(f(x-p)-f(x'-p))|\lesssim\|f\|_{\dot{\Lambda}_s}|x-x'|^s$$

since $f\in\dot{\Lambda}_s$. Since this holds uniformly for any such p, we may use the general fact that, for a bounded function H, one has that

$$|\sup_{|p|=y}H(x,p)-\sup_{|p|=y}H(x',p)|\le\sup_{|p|=y}|H(x,p)-H(x',p)|,$$

to deduce that

$$|\Delta_2 f(x,y)-\Delta_2 f(x',y)|\lesssim\|f\|_{\dot{\Lambda}_s}|x-x'|^s. \tag{2.19}$$

On the other hand, if $q=(y'/y)p$, we also have that

$$|(f(x'+p)-f(x'+q))+(f(x'-p)-f(x'-q))|\lesssim\|f\|_{\dot{\Lambda}_s}|p-q|^s=\|f\|_{\dot{\Lambda}_s}|y-y'|^s.$$

This is true uniformly for any such p and, thus, it holds that

$$|\Delta_2 f(x',y)-\Delta_2 f(x',y')|\lesssim\|f\|_{\dot{\Lambda}_s}|y-y'|^s. \tag{2.20}$$

As we join (2.19) and (2.20) the conclusion follows immediately. □

Lemma 2.4. *Consider a function $f \in \dot{\Lambda}_*$. If $|x-x'| < y/2$ and $1/2 < y/y' < 2$, then*

$$|\Delta_2 f(x,y) - \Delta_2 f(x',y')| \lesssim \|f\|_{\dot{\Lambda}_*} \left(|x-x'| \log\left(e + \frac{y}{|x-x'|}\right) + |y-y'| \log\left(e + \frac{y}{|y-y'|}\right) \right).$$

Proof. Consider a smooth function $\widehat{\phi}$ on $\mathbb{R}^n$ such that its support is contained in the annulus $\{\xi \in \mathbb{R}^n : 1/2 < |\xi| < 2\}$, that satisfies $\widehat{\phi}(-\xi) = \widehat{\phi}(\xi)$, and such that if $\widehat{\phi}_j(\xi) = \widehat{\phi}(2^{-j}\xi)$ for $j \in \mathbb{Z}$, then $\sum_{j\in\mathbb{Z}} \widehat{\phi}_j(\xi) = 1$ for $\xi \neq 0$. Now, if $\phi_j = \mathcal{F}^{-1}\widehat{\phi}_j$ and $f_j = \phi_j * f$, the Littlewood-Paley dyadic decomposition of f is $f = \sum_{j\in\mathbb{Z}} f_j$. It is a well known fact (see for instance [Ste93, p. 253]) that, for $s > 0$, a function f is in $\dot{\Lambda}_s$ if and only if $\|f_j\|_{L^\infty} \leq C2^{-js}$ for some positive constant C, and the smallest constant satisfying this bound is equivalent to $\|f\|_{\dot{\Lambda}_s}$. In particular, the case $s = 1$ corresponds to $\dot{\Lambda}_*$, for which we have $\|\partial^\alpha f_j\|_{L^\infty} \leq \|f\|_{\dot{\Lambda}_*} 2^{-j(1-|\alpha|)}$.

Consider $p \in \mathbb{R}^n$ with $|p| = y$. For this particular p we have that

$$\begin{aligned}&|f(x+p) - 2f(x) + f(x-p) - f(x'+p) + 2f(x') - f(x'-p)| \\ &\quad\leq \sum_{j\in\mathbb{Z}} |f_j(x+p) - 2f_j(x) + f_j(x-p) - f_j(x'+p) + 2f_j(x') - f_j(x'-p)|.\end{aligned}$$

We split this sum into those terms for which $2^j < 1/y$, those with $1/y \leq 2^j < 1/|x-x'|$ and those with $2^j \geq 1/|x-x'|$. For the first part, we express

$$f_j(x+p) - 2f_j(x) + f_j(x-p) = \int_{-1}^{1} (1-|u|) \frac{d^2}{du^2} f(x+up)\, du.$$

Using the bound on the third derivatives of f_j we obtain

$$\left| \frac{d^2}{du^2} f_j(x+up) - \frac{d^2}{du^2} f_j(x'+up) \right| \leq |p|^2 2^{2j} |x-x'|.$$

Hence

$$\begin{aligned}&|f_j(x+p) - 2f_j(x) + f_j(x-p) - f_j(x'+p) + 2f_j(x') - f_j(x'-p)| \\ &\qquad\lesssim \|f\|_{\dot{\Lambda}_*} 2^{2j} |x-x'| y^2,\end{aligned}$$

which yields

$$\begin{aligned}\sum_{2^j<1/y} &|f_j(x+p) - 2f_j(x) + f_j(x-p) - f_j(x'+p) + 2f_j(x') - f_j(x'-p)| \\ &\lesssim \|f\|_{\dot{\Lambda}_*} |x-x'| y^2 \sum_{2^j<1/y} 2^{2j} \\ &\lesssim \|f\|_{\dot{\Lambda}_*} |x-x'|.\end{aligned}$$

When $1/y \leq 2^j < 1/|x-x'|$, we use the first derivative bound

$$\begin{aligned}&|f_j(x+p) - 2f_j(x) + f_j(x-p) - f_j(x'+p) + 2f_j(x') - f_j(x'-p)| \\ &\qquad\lesssim \|f\|_{\dot{\Lambda}_*} |x-x'|,\end{aligned}$$

which is deduced in the same way as the previous one. Then we find that

$$\begin{aligned}\sum_{1/y\le 2^j<1/|x-x'|} &|f_j(x+p)-2f_j(x)+f_j(x-p)\\ &\quad -f_j(x'+p)+2f_j(x')-f_j(x'-p)|\\ &\lesssim \|f\|_{\dot{\Lambda}_*}|x-x'|\sum_{1/y\le 2^j<1/|x-x'|}1\\ &\lesssim \|f\|_{\dot{\Lambda}_*}|x-x'|\log\left(e+\frac{y}{|x-x'|}\right).\end{aligned}$$

Finally, using that $\|f_j\|_{L^\infty}\lesssim \|f\|_{\dot{\Lambda}_*}2^{-j}$ we get that the remaining terms are bounded by

$$\begin{aligned}\sum_{2^j\ge 1/|x-x'|} &|f_j(x+p)-2f_j(x)+f_j(x-p)\\ &\quad -f_j(x'+p)+2f_j(x')-f_j(x'-p)|\\ &\lesssim \|f\|_{\dot{\Lambda}_*}\sum_{2^j\ge 1/|x-x'|}2^{-j}\\ &\lesssim \|f\|_{\dot{\Lambda}_*}|x-x'|.\end{aligned}$$

Since all the above bounds are uniform on p with $|p|=y$, we get the estimate

$$|\Delta_2 f(x,y)-\Delta_2 f(x',y)|\lesssim \|f\|_{\dot{\Lambda}_*}|x-x'|\log\left(e+\frac{y}{|x-x'|}\right). \tag{2.21}$$

Now, let p be as before and consider the case $x=x'$ but $y\neq y'$. We take $q=(y'/y)p$, and note that it is enough to estimate the quantity

$$\begin{aligned}&|f(x'+p)-f(x'+q)+f(x'-p)-f(x'-q)|\\ &\qquad\le \sum_{j\in\mathbb{Z}}|f_j(x'+p)-f_j(x'+q)+f_j(x'-p)-f_j(x'-q)|.\end{aligned}$$

We split the previous sum into those terms for which $2^j<1/y$, those with $1/y\le 2^j<1/|y-y'|$ and those with $2^j\ge 1/|y-y'|$, and follow the previous argument with minor changes. Towards estimating the first sum, we observe first the elementary bound

$$\begin{aligned}|g(1,1)-g(1,-1)+g(-1,-1)-g(-1,1)| &= \left|\int_{[-1,1]^2} g_{uv}(u,v)\,du\,dv\right|\\ &\le 4\sup_{(u,v)\in[-1,1]^2}|g_{uv}(u,v)|\end{aligned}$$

As we apply this to function $g(u,v):=f_j\left(x+u\frac{p+q}{2}+v\frac{p-q}{2}\right)$, together with the known bound $\|\partial^\alpha f_j\|_{L^\infty}\le \|f\|_{\dot{\Lambda}_*}2^j$ for multi-indices of length $|\alpha|=2$, it follows that

$$|f_j(x'+p)-f_j(x'+q)+f_j(x'-p)-f_j(x'-q)|\lesssim \|f\|_{\dot{\Lambda}_*}2^j|y-y'|y,$$

and so

$$\sum_{2^j<1/y}|f_j(x'+p)-f_j(x'+q)+f_j(x'-p)-f_j(x'-q)|\lesssim \|f\|_{\dot{\Lambda}_*}|y-y'|.$$

Then, for those terms with $1/y \leq 2^j < 1/|y-y'|$, using the Lipschitz property of f_j we may deduce

$$|f_j(x'+p) - f_j(x'+q) + f_j(x'-p) - f_j(x'-q)| \lesssim \|f\|_{\dot{\Lambda}_*} |y-y'|,$$

which yields

$$\sum_{1/y \leq 2^j < 1/|y-y'|} |f_j(x'+p) - f_j(x'+q) + f_j(x'-p) - f_j(x'-q)| \lesssim \|f\|_{\dot{\Lambda}_*} |y-y'| \log\left(e + \frac{y}{|y-y'|}\right).$$

Finally, use the size estimate $\|f_j\|_{L^\infty} \lesssim \|f\|_{\dot{\Lambda}_*} 2^{-j}$ to get

$$\sum_{2^j \geq 1/|y-y'|} |f_j(x'+p) - f_j(x'+q) + f_j(x'-p) - f_j(x'-q)| \lesssim \|f\|_{\dot{\Lambda}_*} |y-y'|.$$

These bounds are uniform for any p and q such that $|p| = y$ and $q = (y'/y)p$ and, therefore, it is clear that

$$|\Delta_2 f(x',y) - \Delta_2 f(x',y')| \lesssim \|f\|_{\dot{\Lambda}_*} |y-y'| \log\left(e + \frac{y}{|y-y'|}\right). \tag{2.22}$$

The conclusion of the lemma follows from (2.21) and (2.22). □

We study now the variation of $y^2 \partial^2 \mathrm{P}[f] / \partial y^2$. Recall that we denote the hyperbolic metric on the upper half-space by ρ.

Lemma 2.5. *Let $0 < s \leq 1$ and consider a compactly supported function $f \in \dot{\Lambda}_s$. Denote by u the harmonic extension $\mathrm{P}[f]$ of f to $\mathbb{R}^{n+1}_+$. Then, we have that*

$$\left| y^{2-s} \frac{\partial^2 u}{\partial y^2}(x,y) - y'^{2-s} \frac{\partial^2 u}{\partial y^2}(x',y') \right| \lesssim \|f\|_{\dot{\Lambda}_s} \rho\left((x,y),(x',y')\right).$$

Proof. Recall that

$$\left| y^{2-s} \frac{\partial^2 u}{\partial y^2} \right| \lesssim \|f\|_{\dot{\Lambda}_s}.$$

Moreover, this is equivalent to

$$\left| y^{k-s} \frac{\partial^k u}{\partial y^k} \right| \lesssim \|f\|_{\dot{\Lambda}_s}$$

for any integer $k > 2$ (see [Ste70, p. 145]). Define the function

$$g(x,y) := y^{2-s} \frac{\partial^2 u}{\partial y^2}(x,y),$$

which has hyperbolic derivative

$$y \frac{\partial g}{\partial y}(x,y) = (2-s) y^{2-s} \frac{\partial^2 u}{\partial y^2}(x,y) + y^{3-s} \frac{\partial^3 u}{\partial y^3}(x,y).$$

Thus, because of the previous fact, this satisfies

$$\left| y \frac{\partial g}{\partial y}(x,y) \right| \lesssim \|f\|_{\dot{\Lambda}_s}.$$

In other words, g is locally Lipschitz with respect to the hyperbolic metric, which also implies the global Lipschitz property, thus proving the claim. □

Recall that for a given measurable set $A \subseteq \mathbb{R}^{n+1}_+$ we define the quantity

$$M(A) = \sup_{Q \in \mathcal{D}} \frac{1}{|Q|} \int_Q \int_0^{l(Q)} \chi_A(x,y) \, \frac{dy\,dx}{y}.$$

Lemma 2.6. *Consider a collection of dyadic cubes $W \subseteq \mathcal{D}$ and the corresponding top half cubes set*

$$T = \bigcup_{Q \in W} T(Q).$$

Then, if $M(T) < \infty$, it also holds that $M(T_R) < \infty$ for any $R > 0$.

Proof. Consider a dyadic cube Q in $\mathbb{R}^n$. Then, for any cube P such that $Q \subseteq P$, we have that

$$\frac{1}{|P|} \int_P \int_0^{l(P)} \chi_{T(Q)_R}(x,y) \, \frac{dy\,dx}{y} \le \frac{C}{|P|} \int_P \int_0^{l(P)} \chi_{T(Q)}(x,y) \, \frac{dy\,dx}{y}.$$

Indeed, observe first that

$$\int_P \int_0^{l(P)} \chi_{T(Q)}(x,y) \, \frac{dy\,dx}{y} \simeq |Q|.$$

Then, given $R > 0$, there exists $C > 0$ depending on R such that

$$\int_P \int_0^{l(P)} \chi_{T(Q)_R}(x,y) \, \frac{dy\,dx}{y} \le C|Q|.$$

This holds because, for $R > 0$, there are at most C_1 top half cubes $T(Q')$ at hyperbolic distance at most R from $T(Q)$, and the side length of such cubes is at most $C_2 l(Q)$, both constants depending only on R and n. Observe in particular that, if T is a union of top half cubes $T(Q)$, then

$$\int_P \int_0^{l(P)} \chi_T(x,y) \, \frac{dy\,dx}{y} \lesssim \sum_{T(Q) \subseteq T \cap P} |Q|.$$

For an arbitrary cube P in $\mathbb{R}^n$, denote by P' the smallest cube such that $\pi(T(P)_R) \subseteq P'$, where $\pi \colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ stands for the orthogonal projection. Then $|P'| \simeq |P|$.

Using that $T_R \subseteq \bigcup_{T(Q)\subseteq T} T(Q)_R$, we see that

$$\begin{aligned} &\frac{1}{|P|}\int_P \int_0^{l(P)} \chi_{T_R}(x,y)\,\frac{dy\,dx}{y} \\ &\quad\lesssim \frac{1}{|P|}\sum_{\substack{Q\subseteq P' \\ T(Q)\subseteq T}} \int_P \int_0^{l(P)} \chi_{T(Q)_R}(x,y)\,\frac{dy\,dx}{y} \\ &\quad\lesssim \frac{1}{|P'|}\sum_{T(Q)\subseteq T} \int_{P'} \int_0^{l(P')} \chi_{T(Q)}(x,y)\,\frac{dy\,dx}{y}, \end{aligned}$$

and this last quantity is precisely bounded by $M(T)$, which concludes the proof. □

Next, we use Lemma 2.6 to prove the following relations between $T(\varepsilon)$, $S(\varepsilon)$ and $D(\varepsilon)$ and their R-hyperbolic neighbourhoods. The first one being an immediate consequence of Lemma 2.6.

Lemma 2.7. *Let* $0 < s \leq 1$ *and consider a function* $f \in \dot{\Lambda}_s$. *Denote by* $\varepsilon_0 = \inf\{\varepsilon > 0\colon M(T(\varepsilon)) < \infty\}$. *If* $\varepsilon > \varepsilon_0$, *then* $M(T(\varepsilon)_R) < \infty$ *for any* $R > 0$.

Lemma 2.8. *Let* $0 < s \leq 1$ *and consider a function* $f \in \dot{\Lambda}_s$. *Denote by* $\varepsilon_0 = \inf\{\varepsilon > 0\colon M(S(\varepsilon)) < \infty\}$. *If* $\varepsilon > \varepsilon_0$, *then* $M(S(\varepsilon)_R) < \infty$ *for any* $R > 0$.

Proof. Observe that, for $\varepsilon_0 < \varepsilon' < \varepsilon$, the set $S(\varepsilon)$ is contained in $S(\varepsilon')$. Moreover, because of Lemma 2.3 and Lemma 2.4 (depending on if $s = 1$ or not), there exists $\delta > 0$ such that if $|x - x'|/y < \delta$ and $1 - \delta < y/y' < 1 + \delta$, for any $(x,y) \in S(\varepsilon)$ we have that $(x', y') \in S(\varepsilon')$. That is to say that $S(\varepsilon')$ contains an η-neighbourhood, for some $\eta > 0$, of $S(\varepsilon)$, and since $M(S(\varepsilon')) < \infty$, it is also true that $M(S(\varepsilon)_\eta) < \infty$. This shows the claim for any $R < \eta$.

Assume that $R > \eta$, and let us call $U(\varepsilon)$ the union of all top half cubes $T(Q)$ such that Q is a dyadic cube in $\mathbb{R}^n$ and $T(Q) \cap S(\varepsilon)_\eta \neq \emptyset$. Then, for a given cube P we have that

$$\frac{1}{|P|}\int_P \int_0^{l(P)} \chi_{U(\varepsilon)}(x,y)\,\frac{dy\,dx}{y} \leq \frac{1}{|P|}\sum_{T(Q)\subseteq U(\varepsilon)} \int_P \int_0^{l(P)} \chi_{T(Q)_\eta}(x,y)\,\frac{dy\,dx}{y}.$$

Observe now that, since for $T(Q) \subseteq U(\varepsilon)$ the intersection $T(Q) \cap S(\varepsilon)$ is not empty, we have that $T(Q)_\eta \cap S(\varepsilon)_\eta$ contains, at least, a ball of radius comparable to $\eta l(Q)$. Thus, there exists a constant $C > 0$ such that for each $T(Q) \subseteq U(\varepsilon)$, we have that

$$\int_P \int_0^{l(P)} \chi_{T(Q)_\eta}(x,y)\,\frac{dy\,dx}{y} \leq C \int_P \int_0^{l(P)} \chi_{T(Q)_\eta \cap S(\varepsilon)_\eta}(x,y)\,\frac{dy\,dx}{y}.$$

Now, for $T(Q) \subseteq U(\varepsilon)$ there are at most c_1 cubes Q' such that $T(Q')$ intersects $T(Q)_\eta$, where c_1 depends only on η and the dimension n. Thus, we get that

$$\frac{1}{|P|}\int_P \int_0^{l(P)} \chi_{U(\varepsilon)}(x,y)\,\frac{dy\,dx}{y} \lesssim \frac{1}{|P|}\sum_{T(Q)\subseteq U(\varepsilon)} \int_P \int_0^{l(P)} \chi_{T(Q)\cap S(\varepsilon)_\eta}(x,y)\,\frac{dy\,dx}{y},$$

from which it follows that $M(U(\varepsilon)) \lesssim M(S(\varepsilon)_\eta) < \infty$. Next, since $S(\varepsilon)_\eta \subseteq U(\varepsilon)$ by definition, it holds that $S(\varepsilon)_R \subseteq U(\varepsilon)_R$ for any $R > \eta$. In particular, it holds that $M(S(\varepsilon)_R) \leq M(U(\varepsilon)_R)$. But the latter quantity is finite because of Lemma 2.6 applied to $U(\varepsilon)$. Hence, we get that $M(S(\varepsilon)_R) < \infty$ also for $R > \eta$, as we wanted to see. □

Lemma 2.9. *Let $0 < s \leq 1$ and consider a compactly supported function $f \in \dot{\Lambda}_s$. Denote by $\varepsilon_0 = \inf\{\varepsilon > 0\colon M(D(\varepsilon)) < \infty\}$. If $\varepsilon > \varepsilon_0$, then $M(D(\varepsilon)_R) < \infty$ for any $R > 0$.*

Proof. First, if $\varepsilon_0 < \varepsilon' < \varepsilon$, we have that $D(\varepsilon)$ is contained in $D(\varepsilon')$. Now, Lemma 2.5 implies that there exists $\delta > 0$ such that for every $(x, y) \in D(\varepsilon)$, $D(\varepsilon')$ contains a ball of hyperbolic radius δ centred at (x, y). In other words, $D(\varepsilon')$ contains $D(\varepsilon)_\delta$ and, since $\varepsilon' > \varepsilon_0$, this implies that $M(D(\varepsilon)_\delta) < \infty$. This shows the result for $R < \delta$. The result for $R > \delta$ is proved using the same argument as in the proof of Lemma 2.8. □

2.3 Equivalence of Characterisations

The aim of this section is to prove Theorems 2.2 and 2.7. In order to do so, let us fix $0 < s \leq 1$ and $f \in \dot{\Lambda}_s$ compactly supported. We first show some geometric relations between the sets $T(\varepsilon)$, $S(\varepsilon)$ and $D(\varepsilon)$. In this section we denote by $\mathrm{B}_R((x, y))$ the ball of hyperbolic radius R centred at $(x, y) \in \mathbb{R}^{n+1}_+$.

It is important to remark that, a priori, the set $T(\varepsilon)$ depends on the choice of the wavelet basis that we decide to use. Let us consider two different wavelet bases $\{\psi_\omega\}$ and $\{\varphi_\omega\}$, respectively of regularities $r, r' > s$, and denote by $T_\psi(\varepsilon)$ and $T_\varphi(\varepsilon)$ the corresponding sets of top half cubes associated to large wavelet coefficients. Theorem 2.6 holds for any wavelet basis of regularity at least $s + 1$. Hence, either both $T_\psi(\varepsilon)$ and $T_\varphi(\varepsilon)$ are finite, or both are infinite. In the following lemma, we will assume that our wavelet basis has regularity much larger than $s + 1$. However, due to the previous observation, this will have no influence when proving Theorems 2.2 and 2.7.

Lemma 2.10. *Let $0 < s \leq 1$ and consider a function $f \in \dot{\Lambda}_s$. There exists an absolute constant $C > 0$ such that, for any $\varepsilon > 0$, there is $R > 0$ for which $T(\varepsilon) \subseteq S(C\varepsilon)_R$.*

Proof. We want to see that, if

$$y'^{-s} \Delta_2 f(x', y') \lesssim \varepsilon \tag{2.23}$$

for all $(x', y') \in \mathrm{B}_R((x, y))$, then

$$\sup_l |c_{(l,Q)}| \lesssim \varepsilon 2^{-j(n/2+s)}$$

for a fixed dyadic cube Q such that $x \in Q$ and $l(Q)/2 < y < l(Q)$. First of all, we note that without loss of generality, we can assume that $x = 0$ and $y = 1$, and take $Q = Q_0 = [0, 1]$. The general result can be reduced to this case by a translation and a rescaling.

Here we denote our wavelet basis by $\{\psi_{(l,P)}\}$, to emphasise the dependence on the dyadic cubes. Assume that it has large enough regularity r to be determined. The wavelet functions satisfy

$$\int_{\mathbb{R}^n} x^\alpha \psi_{(l,P)}(x)\, dx = 0$$

for multi-indices of length $0 \leq |\alpha| \leq r$, or in other words, their Fourier transforms $\widehat{\psi_{(l,P)}} = \mathcal{F}\left[\psi_{(l,P)}\right]$ satisfy $\partial^\alpha \widehat{\psi_{(l,P)}}(0) = 0$ for such multi-indices. Consider a non negative and radially symmetric smooth function g supported on the annulus $\{x \in \mathbb{R}^n\colon 1/2 \leq |x| \leq 2\}$, and with integral 1. In particular, $\widehat{g} = \mathcal{F}[g]$ is real, smooth and radially symmetric, and it has fast decay in the sense that for each $N \geq 1$ there is a constant C_N such that

$$|\widehat{g}(\xi)| \leq C_N \, (1 + |\xi|)^{-N} .$$

Moreover, $\widehat{g}(\xi) \leq \widehat{g}(0) = 1$ for all $\xi \in \mathbb{R}^n$. More precisely, there is a positive constant $c > 0$ so that

$$1 - \widehat{g}(\xi) \geq c \min(1, |\xi|^2), \qquad \xi \in \mathbb{R}^n. \tag{2.24}$$

Indeed, for ξ at a neighbourhood of the origin we have $1 - \widehat{g}(\xi) \simeq |\xi|^2$ because g has zero first moments due to its symmetry and $\Delta\widehat{g}(0) = -\int_{\mathbb{R}^n} |x|^2 g(x)\, dx < 0$. On the other hand, since g is smooth, we also have $1 - \widehat{g}(\xi) \simeq 1$ as $\xi \to \infty$ due to the fast decay of $\widehat{g}$. Finally, observe that $\widehat{g}(\xi) = 1$ only if $\xi = 0$. This follows from the fact that g is a non negative radially symmetric function and that $e^{i2\pi x\cdot\xi}$ is constant as a function of x on the whole support of g only for $\xi = 0$, so that

$$\left| \int g(x) e^{i2\pi x\cdot\xi}\, dx \right| < \int g(x)\, dx = \widehat{g}(0).$$

Given one of the functions ψ_l generating the wavelet basis, we define the function h via its Fourier transform $\widehat{h} = \mathcal{F}[h]$ by setting

$$\widehat{h}(\xi) = -\frac{\widehat{\psi_l}(\xi)}{\overline{\widehat{g}(0) - \widehat{g}(\xi)}} = -\frac{\widehat{\psi_l}(\xi)}{\overline{1 - \widehat{g}(\xi)}}.$$

Note that, because of (2.24) and the fact that the derivatives of $\widehat{\psi_l}$ vanish up to order at least r, if we take r large enough, say $r > n+1$, then $\widehat{h} \in \mathcal{C}^{n+1}(\mathbb{R}^n)$. Moreover, all the derivatives $\widehat{h}$ are integrable because of the quick decay of the derivatives of $\widehat{\psi_l}$. All this implies that h is smooth and $(1 + |x|^2)^{(n+1)/2} h(x)$ is bounded, so that h is integrable. In particular, h itself is also bounded.

We are now able to estimate the wavelet coefficient in terms of the second differences. Observe that

$$\int_{\mathbb{R}^n} \overline{\psi_l(x)} f(x)\, dx = \int_{\mathbb{R}^n} \overline{\widehat{\psi_l}(\xi)} \widehat{f}(\xi)\, d\xi = \int_{\mathbb{R}^n} \overline{\widehat{h}(\xi)} (\widehat{g}(\xi) - \widehat{g}(0)) \widehat{f}(\xi)\, d\xi.$$

Note as well that, because the properties of g and h, it holds that

$$\begin{aligned}
\iint_{(x,t)\in\mathbb{R}^{2n}} \overline{h(x)} g(t) f(x+t)\, dx\, dt &= \iint_{(x,t)\in\mathbb{R}^{2n}} \overline{h(x)} g(t) f(x-t)\, dx\, dt \\
&= \int_{\mathbb{R}^n} \overline{h(x)} (g * f)(x)\, dx = \int_{\mathbb{R}^n} \overline{\widehat{h}(\xi)} \widehat{g}(\xi) \widehat{f}(\xi)\, d\xi
\end{aligned}$$

and

$$\iint_{(x,t)\in\mathbb{R}^{2n}} g(t) \overline{h(x)} f(x)\, dx\, dt = \widehat{g}(0) \int_{\mathbb{R}^n} \overline{\widehat{h}(\xi)} \widehat{f}(\xi)\, d\xi.$$

Therefore, noting also that g is bounded and that its support lies in the annulus $\{x \in \mathbb{R}^n \colon 1/2 \leq |x| \leq 2\}$, we obtain

$$\begin{aligned}
&\left| \int_{\mathbb{R}^n} \overline{\psi_l(x)} f(x)\, dx \right| \\
&\qquad \leq \frac{1}{2} \iint_{(x,t)\in\mathbb{R}^{2n}} |h(x) g(t)| |f(x+t) - 2f(x) + f(x-t)|\, dx\, dt \\
&\qquad \leq c \iint_{\mathbb{R}^n \times (1/2,2)} |h(x)| \Delta_2 f(x,y)\, dx\, dy.
\end{aligned}$$

We can assume that (2.23) holds on the set $A = \{(x,y) \in \mathbb{R}^{n+1}_+ : |x| < R/2, 1/2 < y < 2\}$. Note that

$$\iint_{(\mathbb{R}^n \times (1/2,2)) \setminus A} |h(x)| \Delta_2 f(x,y)\, dx\, dy \lesssim \|f\|_{\dot{\Lambda}_s} 2^s \int_{|x| \geq R/2} |h(x)|\, dx \lesssim \varepsilon$$

for R large enough. On the other hand, we have by (2.23) that

$$\iint_A |h(x)| \Delta_2 f(x,y)\, dx\, dy \lesssim \varepsilon 2^s \int_{|x| \leq R/2} |h(x)|\, dx \lesssim \varepsilon.$$

And, thus, we get that $|c_{(I,Q)}(f)| \lesssim \varepsilon$, as we wanted to show. □

Recall that for a function $f \in \Lambda_s$ (and in particular for compactly supported functions in $\dot{\Lambda}_s$) with $0 < s \leq 1$, and for an integer $k \geq 2$, one has that the single condition

$$y^{2-s} \left| \frac{\partial^2 \mathrm{P}[f]}{\partial y^2}(x,y) \right| \lesssim \|f\|_{\dot{\Lambda}_s}$$

is equivalent to

$$y^{k-s} \left| \partial^\alpha \mathrm{P}[f](x,y) \right| \lesssim \|f\|_{\dot{\Lambda}_s}, \tag{2.25}$$

for all multi-indices α with $|\alpha| = k$ for any integer $k \geq 2$ (see for example [Ste70, pp. 143–145]). Before we show the corresponding result for $S(\varepsilon)$ in terms of $D(\varepsilon)_R$, we need the following auxiliary lemma.

Lemma 2.11. *Let $0 < s \leq 1$ and consider a function $f \in \Lambda_s$ and an integer $k \geq 2$. There exists $R_0 = R_0(k) > 0$ such that if $R > R_0$ and*

$$y'^{2-s} \left| \frac{\partial^2 \mathrm{P}[f]}{\partial y^2}(x',y') \right| \lesssim \varepsilon, \qquad (x',y') \in \mathrm{B}_R((x,y)), \tag{2.26}$$

then

$$y^{|\alpha|-s} \left| \partial^\alpha \mathrm{P}[f](x,y) \right| \lesssim \varepsilon \tag{2.27}$$

for every multi-index α with $|\alpha| = k$.

Proof. The arguments we use here are the same as those used to prove Lemmas 4 and 5 of [Ste70, pp. 143–145]. We just show that if (2.26) holds for R large enough, then

$$y^{2-s} \left| \frac{\partial^2 \mathrm{P}[f]}{\partial y \partial x_1}(x,y) \right| \lesssim \varepsilon, \tag{2.28}$$

since all the other cases follow the same argument.

Let us denote $u(x,y) = \mathrm{P}[f](x,y)$. Using that for $y > 0$ the Poisson kernel satisfies $P_y(x) = (P_{y/2} * P_{y/2})(x)$, one can express $u(x,y) = (P_{y/2} * u_{y/2})(x)$, where $u_y(t) = u(t,y)$. Thus, one gets

$$\frac{\partial^3 u}{\partial^2 y \partial x_1} = \frac{\partial P_{y/2}}{\partial x_1} * \left. \frac{\partial^2 u}{\partial y^2} \right|_{y/2}.$$

Assume that, for some $R' > 0$, there is $\tilde{R} > 0$ large enough such that it holds that

$$\tilde{y}^{2-s} \left| \frac{\partial^2 \mathrm{P}[f]}{\partial y^2}(\tilde{x},\tilde{y}) \right| \lesssim \varepsilon, \qquad (\tilde{x},\tilde{y}) \in \mathrm{B}_{\tilde{R}}((x',y')), \tag{2.29}$$

for every $(x', y') \in B_{R'}((x,y))$. Note that this can be done requiring (2.26) to hold for $R > R' + \tilde{R}$. Next write

$$\left|\frac{\partial^3 u}{\partial^2 y \partial x_1}(x', y')\right| \leq \int_{|t| > \tilde{R}y'} \left|\frac{\partial P_{y'/2}}{\partial x_1}(t)\right| \left|\frac{\partial^2 u}{\partial y^2}(x' - t, y'/2)\right| dt + \int_{|t| \leq \tilde{R}y'} \left|\frac{\partial P_{y'/2}}{\partial x_1}(t)\right| \left|\frac{\partial^2 u}{\partial y^2}(x' - t, y'/2)\right| dt.$$

Recall that $|\partial P_y / \partial x_1| \lesssim P_y / y$. Thus, we find the bound

$$\int_{|t| > \tilde{R}y'} \left|\frac{\partial P_{y'/2}}{\partial x_1}(t)\right| \left|\frac{\partial^2 u}{\partial y^2}(x' - t, y'/2)\right| dt \lesssim \frac{\|f\|_{\dot{\Lambda}_s}}{\tilde{R}} y'^{s-3}$$

for the first term, while using (2.29) on the second one we get

$$\int_{|t| \leq \tilde{R}y'} \left|\frac{\partial P_{y'/2}}{\partial x_1}(t)\right| \left|\frac{\partial^2 u}{\partial y^2}(x' - t, y'/2)\right| dt \lesssim \varepsilon y'^{s-3}.$$

Summing up, if $\tilde{R}$ is large enough, we have that

$$\left|\frac{\partial^3 u}{\partial^2 y \partial x_1}(x', y')\right| \lesssim \varepsilon y'^{s-3} \tag{2.30}$$

for all $(x', y') \in B_{R'}((x,y))$.

Now, taking into account that f is in the inhomogeneous space Λ_s and that $\left\|(\partial^2/\partial y \partial x_1) P_y\right\|_{L^1} \lesssim y^{-2}$, we have that

$$\left|\frac{\partial^2 u}{\partial y \partial x_1}(x,y)\right| \lesssim y^{-2} \|f\|_{L^\infty},$$

from which it follows that $|(\partial^2/\partial y \partial x_1) u(x,y)|$ tends to zero as $y \to \infty$. Hence, one can express

$$\left|\frac{\partial^2 u}{\partial y \partial x_1}(x,y)\right| \leq \int_y^\infty \left|\frac{\partial^3 u}{\partial y^2 \partial x_1}(x,y')\right| dy' = \int_y^{R'y} \left|\frac{\partial^3 u}{\partial y^2 \partial x_1}(x,y')\right| dy' + \int_{R'y}^\infty \left|\frac{\partial^3 u}{\partial y^2 \partial x_1}(x,y')\right| dy'.$$

Using (2.30) on the first term, we get the bound

$$\int_y^{R'y} \left|\frac{\partial^3 u}{\partial y^2 \partial x_1}(x,y')\right| dy' \lesssim \varepsilon \int_y^{R'y} y'^{s-3}\, dy' \leq \varepsilon y^{s-2}.$$

For the second term, we get

$$\int_{R'y}^\infty \left|\frac{\partial^3 u}{\partial y^2 \partial x_1}(x,y')\right| dy' \lesssim \|f\|_{\dot{\Lambda}_s} \int_{R'y}^\infty y'^{s-3}\, dy' = \frac{\|f\|_{\dot{\Lambda}_s}}{R'^{2-s}} y^{s-2}$$

using the bound $y^{3-s}|(\partial^3/\partial y^2 \partial x_1) u(x,y)| \lesssim \|f\|_{\dot{\Lambda}_s}$. Therefore, adding these two bounds, we get (2.28) if R' is large enough, as we wanted to see. □

Lemma 2.12. *Let $0 < s \leq 1$ and consider a compactly supported function $f \in \dot{\Lambda}_s$. There exists an absolute constant $C > 0$ such that, for any $\varepsilon > 0$, there is $R > 0$ for which $S(\varepsilon) \subseteq D(C\varepsilon)_R$.*

Proof. Fix $(x,y) \in \mathbb{R}^{n+1}_+$, and let us denote $u = \mathrm{P}[f]$. We need to see that, if

$$y'^{2-s}\left|\frac{\partial^2 u}{\partial y^2}(x',y')\right| \leq \varepsilon \tag{2.31}$$

for every $(x',y') \in \mathrm{B}_R((x,y))$, then

$$\frac{\Delta_2 f(x,y)}{y^s} \leq \varepsilon. \tag{2.32}$$

We do so by following the arguments in [Ste70, pp.–147] used to characterise the spaces Λ_s.

Let us fix $p = (p_1, \ldots, p_n)$ with $|p| = 1$. If f was twice continuously differentiable, we would write

$$\begin{aligned}
&|f(x+yp) - 2f(x) + f(x-yp)| \\
&\qquad = \int_0^y \int_{-\sigma}^{\sigma} \frac{d^2}{d\rho^2} f(x+\rho p)\, d\rho\, d\sigma \\
&\qquad = \int_0^y \int_{-\sigma}^{\sigma} \left(\sum_{i,j=1}^n p_i p_j \frac{\partial^2 f}{\partial x_i \partial x_j}\right) d\rho\, d\sigma.
\end{aligned} \tag{2.33}$$

Note that, for $f \in \Lambda_s$ (which is the case since we assume f to be compactly supported) we can express

$$f(x) = \int_0^y y' \frac{\partial^2 u}{\partial y^2}(x,y')\, dy' - y\frac{\partial u}{\partial y}(x,y) + u(x,y)$$

for any $y > 0$. Thus, we can express the second difference of f as

$$\begin{aligned}
&|f(x+yp) - 2f(x) + f(x-yp)| \\
&\quad \leq \int_0^y y' \left|\frac{\partial^2 u}{\partial y^2}(x+yp,y') - 2\frac{\partial^2 u}{\partial y^2}(x,y') + \frac{\partial^2 u}{\partial y^2}(x-yp,y')\right| dy' \\
&\quad + y\left|\frac{\partial u}{\partial y}(x+yp,y) - 2\frac{\partial u}{\partial y}(x,y) + \frac{\partial u}{\partial y}(x-yp,y)\right| \\
&\quad + |u(x+yp,y) - 2u(x,y) + u(x-yp,y)|.
\end{aligned} \tag{2.34}$$

We focus first on the integral term in (2.34). Because of (2.31), we can assume that

$$\left|\frac{\partial^2 u}{\partial y^2}(x',y')\right| \leq \varepsilon y'^{-2+s}$$

for $y/R < y' < y$ and $|x - x'| < Ry$. Therefore, we have that

$$\begin{aligned}
&\int_{y/R}^y y' \left|\frac{\partial^2 u}{\partial y^2}(x+yp,y') - 2\frac{\partial^2 u}{\partial y^2}(x,y') + \frac{\partial^2 u}{\partial y^2}(x-yp,y')\right| dy' \\
&\qquad\qquad \lesssim \int_{y/R}^y \varepsilon y'^{-1+s}\, dy' \lesssim \varepsilon y^s.
\end{aligned}$$

On the other hand, we have that

$$\int_0^{y/R} y' \left| \frac{\partial^2 u}{\partial y^2}(x+yp,y') - 2\frac{\partial^2 u}{\partial y^2}(x,y') + \frac{\partial^2 u}{\partial y^2}(x-yp,y') \right| dy'$$
$$\lesssim \|f\|_{\dot{\Lambda}_s} \int_0^{y/R} y'^{-1+s}\, dy' \lesssim \frac{\|f\|_{\dot{\Lambda}_s}}{R^s} y^s,$$

which will be bounded by εy^s for R large enough.

In order to bound the second and third terms in (2.34), we express them using (2.33). Let $g(t,y) = \frac{\partial u}{\partial y}(t,y)$ and note that, by Lemma 2.11, we have

$$\left| \frac{\partial^2 g}{\partial x_i \partial x_j}(x+\rho p,y) \right| = \left| \frac{\partial^3 u}{\partial x_i \partial x_j \partial y}(x+\rho p,y) \right| \lesssim \varepsilon y^{-3+s}$$

for $|\rho| < y$ if (2.31) holds for R large enough. Thus,

$$y \left| \frac{\partial u}{\partial y}(x+yp,y) - 2\frac{\partial u}{\partial y}(x,y) + \frac{\partial u}{\partial y}(x-yp,y) \right|$$
$$\le y \int_0^y \int_{-\sigma}^{\sigma} \left| \frac{d^2}{d\rho^2} g(x+\rho p,y) \right| d\rho\, d\sigma$$
$$\lesssim \int_0^y \int_{-\sigma}^{\sigma} \varepsilon y^{-2+s}\, d\rho\, d\sigma \lesssim \varepsilon y^s.$$

Similarly, to bound the third term in (2.34), we use that

$$\left| \frac{\partial^2 u}{\partial x_i \partial x_j}(x+\rho p,y) \right| \lesssim \varepsilon y^{-2+s}$$

for $|\rho| < y$ if (2.31) holds for R large enough, again due to Lemma 2.11. The same reasoning as before yields that

$$|u(x+yp,y) - 2u(x,y) + u(x-yp,y)| \lesssim \varepsilon y^s.$$

This shows that

$$|f(x+yp) - 2f(x) + f(x-yp)| \lesssim \varepsilon y^s$$

and, since this bound is uniform on the choice of p, equation (2.32) follows. □

Lemma 2.13. *Let $0 < s \le 1$ and consider a compactly supported function $f \in \dot{\Lambda}_s$. There exists an absolute constant $C > 0$ such that, for any $\varepsilon > 0$, there is $R > 0$ for which $D(\varepsilon) \subseteq T(C\varepsilon)_R$.*

Proof. Fix $(x,y) \in \mathbb{R}^{n+1}_+$. Consider the set G of dyadic cubes of the form $Q = \{x' \in \mathbb{R}^n : 2^j x' - k \in [0,1]^n\}$ such that $y/R < 2^{-j} < yR$ and $|2^j x - k| \lesssim R$, where R is a positive constant to be determined later. We need to verify that by an appropriate choice of R, if

$$\sup_l |c_{(l,Q)}(f)| < \varepsilon 2^{-j(n/2+s)} \tag{2.35}$$

for every $Q \in G$, then

$$y^{2-s} \left| \frac{\partial^2 u}{\partial y^2}(x,y) \right| < \varepsilon. \tag{2.36}$$

Recall that, by Theorem 2.3, we can write

$$f(x) = \sum_{1\le l\le 2^n-1} \sum_{j\in\mathbb{Z}} \sum_{Q\in\mathcal{D}_j} c_{(l,Q)}(f)\psi_{(l,Q)}(x)$$

with $|c_{(l,Q)}(f)| \lesssim 2^{-j(n/2+s)} \|f\|_{\dot{\Lambda}_s}$ and $\{\psi_{(l,Q)}\}$ a wavelet basis of regularity $r > 1$. Moreover, the functions ψ_l can be chosen to be compactly supported. For simplicity, we will assume that this is the case, although the argument would still hold for non compactly supported wavelets (in that case we would use their fast decay). Now, for $j \in \mathbb{Z}$, let us denote

$$f_j(x) = \sum_{1\le l\le 2^n-1} \sum_{Q\in\mathcal{D}_j} c_{(l,Q)}(f)\psi_{(l,Q)}(x),$$

and also consider its harmonic extension $u_j = \mathrm{P}\left[f_j\right]$ on the upper half-space. We estimate first the contribution of u_j to the hyperbolic derivative in (2.36) for j such that $2^{-j} > yR$. Note that, by harmonicity, it is enough to bound $|(\partial^2/\partial x_i^2)u_j(x,y)|$ for $1 \le i \le n$. First observe that

$$\left|\frac{\partial^2 u_j}{\partial x_i^2}(x,y)\right| = \left|\left(P_y * \frac{\partial^2 f_j}{\partial x_i^2}\right)(x,y)\right| \le \|P_y\|_{\mathrm{L}^1} \left\|\frac{\partial^2 f_j}{\partial x_i^2}\right\|_{\mathrm{L}^\infty} = \left\|\frac{\partial^2 f_j}{\partial x_i^2}\right\|_{\mathrm{L}^\infty}.$$

Then, using that for any multi-index α of length $|\alpha| = 2$ we have $|\partial^\alpha \psi_{(l,Q)}| \lesssim 2^{j(n/2+2)}$ for $Q \in \mathcal{D}_j$, the bound on the wavelet coefficients $|c_{(l,Q)}|$ and the bounded overlap of the wavelet functions (due to their compact support), we get that

$$\left|\frac{\partial^2 u_j}{\partial x_i^2}(x,y)\right| \lesssim \|f\|_{\dot{\Lambda}_s} 2^{j(2-s)}.$$

Thus, summing over j, for $2^{-j} > yR$, we get that

$$\sum_{2^{-j}>yR} \left|\frac{\partial^2 u_j}{\partial x_i^2}(x,y)\right| \lesssim \|f\|_{\dot{\Lambda}_s} y^{-2+s} R^{-2+s} \lesssim \varepsilon y^{-2+s}, \tag{2.37}$$

where the last inequality holds for R large enough (not depending on y), since $s \le 1$.

Next, we compute the contribution of u_j to (2.36) for j such that $2^{-j} \le y/R$. In this case, we have that

$$\left|\frac{\partial^2 u_j}{\partial y^2}(x,y)\right| = \left|\left(\frac{\partial^2 P_y}{\partial y^2} * f_j\right)(x,y)\right| \le \left\|\frac{\partial^2 P_y}{\partial y^2}\right\|_{\mathrm{L}^1} \|f_j\|_{\mathrm{L}^\infty}.$$

Here, use that $\left\|(\partial^2/\partial y^2)P_y\right\|_{\mathrm{L}^1} \lesssim 1/y^2$ and that $\left\|f_j\right\|_{\mathrm{L}^\infty} \lesssim \|f\|_{\dot{\Lambda}_s} 2^{-js}$, which holds again because of the bound on the wavelet coefficients and the bounded overlap of the wavelets themselves, to see that

$$\left|\frac{\partial^2 u_j}{\partial y^2}(x,y)\right| \lesssim \|f\|_{\dot{\Lambda}_s} y^{-2} 2^{-js}.$$

Summing now over j, for $2^{-j} \leq y/R$, we get that

$$\sum_{2^{-j} \leq y/R} \left| \frac{\partial^2 u_j}{\partial y^2}(x,y) \right| \lesssim \|f\|_{\dot{\Lambda}_s} y^{-2+s} R^{-s} \lesssim \varepsilon y^{-2+s}, \tag{2.38}$$

where the last inequality holds for R large enough.

For j such that $y/R < 2^{-j} \leq yR$, we express $f_j = g_j + h_j$, where

$$g_j(x) = \sum_{1 \leq l \leq 2^n - 1} \sum_{Q \in \mathcal{D}_j \cap G} c_{(l,Q)}(f) \psi_{(l,Q)}(x).$$

If $y < 2^{-j} \leq yR$, as we did in the case $yR < 2^{-j}$ we have that

$$\begin{aligned} \left| \frac{\partial^2 u_j}{\partial x_i^2}(x,y) \right| &= \left| \left(P_y * \frac{\partial^2 f_j}{\partial x_i^2} \right)(x,y) \right| \\ &\leq \left| \left(P_y * \frac{\partial^2 g_j}{\partial x_i^2} \right)(x,y) \right| + \left| \left(P_y * \frac{\partial^2 h_j}{\partial x_i^2} \right)(x,y) \right|. \end{aligned}$$

Because of (2.35), the first term is bounded by $C\varepsilon 2^{j(2-s)}$. Observe that function h_j only contains wavelets whose support lies on the set $\{t \in \mathbb{R}^n \colon |t - x| \gtrsim yR\}$. Thus, we can bound the second term by

$$C \left\| \frac{\partial^2 h_j}{\partial x_i^2} \right\|_{L^\infty} \int_{|t| \gtrsim yR} P_y(t)\, dt \lesssim \|f\|_{\dot{\Lambda}_s} 2^{j(2-s)} \frac{1}{R}.$$

This yields, by harmonicity, that

$$\sum_{y < 2^{-j} \leq y/R} \left| \frac{\partial^2 u_j}{\partial y^2}(x,y) \right| \lesssim \left(\varepsilon + \frac{\|f\|_{\dot{\Lambda}_s}}{R} \right) y^{-2+s} \lesssim \varepsilon y^{-2+s}, \tag{2.39}$$

where the last inequality holds for R large enough. Similarly, if $y/R < 2^{-j} \leq y$, we write

$$\begin{aligned} \left| \frac{\partial^2 u_j}{\partial y^2}(x,y) \right| &= \left| \left(\frac{\partial^2 P_y}{\partial y^2} * f_j \right)(x,y) \right| \\ &\leq \left| \left(\frac{\partial^2 P_y}{\partial y^2} * g_j \right)(x,y) \right| + \left| \left(\frac{\partial^2 P_y}{\partial y^2} * h_j \right)(x,y) \right|. \end{aligned}$$

Now, the first term is bounded by $C\varepsilon y^{-2} 2^{-js}$ because of condition (2.35). Taking into account that the wavelets appearing in h_j are supported on $\{t \in \mathbb{R}^n \colon |t - x| \gtrsim yR\}$, the second term is bounded by

$$C \|h_j\|_\infty \int_{|t| \gtrsim yR} \left| \frac{\partial^2 P_y}{\partial y^2}(t) \right| dt \lesssim \|f\|_{\dot{\Lambda}_s} 2^{-js} \frac{1}{y^2 R^2}.$$

It follows that

$$\sum_{y/R < 2^{-j} \leq y} \left| \frac{\partial^2 u_j}{\partial y^2}(x,y) \right| \lesssim \left(\varepsilon + \frac{\|f\|_{\dot{\Lambda}_s}}{R^2} \right) y^{-2+s} \lesssim \varepsilon y^{-2+s}, \tag{2.40}$$

where the last inequality holds for R large enough. Since $u = \sum_j u_j$, estimates (2.37), (2.38), (2.39) and (2.40) yield (2.36), as we wanted to see. □

We have now all that we need to prove Theorems 2.2 and 2.7 using Theorem 2.6.

Proof of Theorems 2.2 and 2.7. Let us denote by τ_0 the infimum in (2.9), by σ_0 the infimum in (2.6) and by δ_0 the one in (2.12). Also, let c be the minimum constant appearing in Lemmas 2.10, 2.12 and 2.13. Note that we can assume that $c \leq 1$.

We first show that $\sigma_0 \simeq \tau_0$. Consider $\sigma > 0$ such that $\sigma > c^{-2}\tau_0$. Then, by Lemma 2.7 we have that $M(T(c^2\sigma)_R) < \infty$ for any $R > 0$. This implies, by Lemma 2.13 that $M(D(c\sigma)) < \infty$, so that we can assume in particular that $c\sigma > \delta_0$. Therefore, by Lemma 2.9, it also holds that $M(D(c\sigma)_R) < \infty$ for any $R > 0$ and, by Lemma 2.12, it follows that $M(S(\sigma)) < \infty$. Hence, we get that $\sigma_0 \leq c^{-2}\tau_0$. To get the corresponding lower bound, let $\sigma > 0$ be such that $\sigma < c\tau_0$, assuming that $\tau_0 > 0$ (otherwise, the result is already proved). Then, we have that $M(T(c^{-1}\sigma)) = \infty$, and by Lemma 2.10 there exists some $R > 0$ such that $M(S(\sigma)_R) = \infty$. However, by Lemma 2.8, it has to be that $M(S(\sigma)) = \infty$ as well. Therefore, we get that $\sigma_0 \geq c\tau_0$, as we wanted to see. Following this same reasoning, one can show that $c^2\tau_0 \leq \delta_0 \leq c^{-1}\tau_0$. This concludes the proof of both theorems. □

Chapter 3

Distortion and Distribution of Sets under Inner Functions

In this chapter, we present the details of our research on inner functions. Recall that we denote by $\mathbb{D}$ the open unit disc of the complex plane, and that an analytic mapping $f\colon \mathbb{D} \to \mathbb{D}$ is called an *inner function* if $|\lim_{r\to 1} f(r\xi)| = 1$ for almost every point (a.e.) ξ of the unit circle $\partial\mathbb{D}$. It is of particular interest to us the map induced by an inner function f on $\partial\mathbb{D}$, defined at almost every point $\xi \in \partial\mathbb{D}$ by $f^*(\xi) = \lim_{r\to 1} f(r\xi)$, which we will denote by f as well whenever there is no ambiguity. We are interested in studying certain invariance and distortion properties of measures and Hausdorff contents of sets in the unit circle under the action of inner functions.

Let $f\colon \mathbb{D} \to \mathbb{D}$ be an analytic mapping. We say that a point $p \in \partial\mathbb{D}$ is a *boundary Fatou point* of f if $f(p) = \lim_{r\to 1} f(rp)$ exists and $f(p) \in \partial\mathbb{D}$. Hence, the set of boundary Fatou points of an inner function has full measure. For $0 < \beta < 1$ and $p \in \partial\mathbb{D}$, let $\Gamma_\beta(p) = \{z \in \mathbb{D} : |z - p| < \beta(1 - |z|)\}$ be the *Stolz angle* with opening β and vertex at p. A holomorphic self-map f of the unit disc has *finite angular derivative* at $p \in \partial\mathbb{D}$ if there is a point $\eta \in \partial\mathbb{D}$ and $\beta > 0$ such that the non-tangential limit

$$f'(p) := \lim_{\Gamma_\beta(p)\ni z\to p} \frac{\eta - f(z)}{p - z}$$

exists and is finite. Observe that in this case $\eta = f(p)$. Since this limit will not depend on β, we also use the notation $z \angle p$ to denote $\Gamma_\beta(p) \ni z \to p$ for some $\beta > 0$. We set $|f'(p)| = +\infty$ if the function f does not have a finite angular derivative at the point $p \in \partial\mathbb{D}$. Observe that this is the case if p is not a boundary Fatou point of f. With this convention, for any $p \in \partial\mathbb{D}$, the classical Julia-Carathéodory theorem gives

$$\liminf_{z\to p} \frac{1 - |f(z)|}{1 - |z|} = |f'(p)| > 0, \tag{3.1}$$

in the sense that either the lim inf is finite and equal to $|f'(p)| > 0$ or both quantities are infinite. See for example Chapters IV and V of [Sha93].

We denote by λ the normalized Lebesgue measure on $\partial\mathbb{D}$ and by λ_z the harmonic measure from the point $z \in \mathbb{D}$, given by

$$\lambda_z(E) = \int_E \frac{1 - |z|^2}{|\xi - z|^2}\, d\lambda(\xi),$$

for any measurable set $E \subseteq \partial\mathbb{D}$. A classical result due to Löwner (see for example [Ahl73, p. 12]) says that the Lebesgue measure is invariant under the action of any inner function fixing the origin. Hence, the following conformally invariant version of Löwner's Lemma holds.

Theorem 3.1. *Let $f : \mathbb{D} \to \mathbb{D}$ be an inner function and $z \in \mathbb{D}$. Then*

$$\lambda_z(f^{-1}(E)) = \lambda_{f(z)}(E)$$

for any measurable set $E \subseteq \partial\mathbb{D}$.

Observe that, if $z \in \mathbb{D}$ is a fixed point of f, Theorem 3.1 says that λ_z is invariant under the action of f. However, it may be the case that f has no fixed points in $\mathbb{D}$ but only on $\partial\mathbb{D}$. A point $p \in \partial\mathbb{D}$ is a fixed point for f if $\lim_{r\to 1} f(rp) = p$. Actually, the classical Denjoy-Wolff Theorem (see [Sha93, p. 77]) states that for any analytic self-mapping f on the unit disc which is not an elliptic automorphism, there exists a fixed point $p \in \overline{\mathbb{D}}$ of f, called the *Denjoy-Wolff fixed point* of f, such that the iterates $f^n = f \circ \overset{n)}{\cdots} \circ f$ tend to p uniformly on compacts sets of $\mathbb{D}$. Moreover, p is the unique fixed point of f in $\overline{\mathbb{D}}$ such that $0 < |f'(p)| \leq 1$. See for example [Sha93, Chapter V]. We are interested in analogues of Theorem 3.1 when $z \in \partial\mathbb{D}$. This situation occurs naturally when the Denjoy-Wolff fixed point of f is on the unit circle. In this situation, instead of considering the harmonic measure from a point in the open unit disc, it is natural to measure sets with respect to boundary points. We will consider a measure introduced by Doering and Mañé in [DM91]. Fix a point $p \in \overline{\mathbb{D}}$ and consider the positive measure μ_p on $\partial\mathbb{D}$ defined by

$$\mu_p(E) = \int_E \frac{1}{|\xi - p|^2}\, d\lambda(\xi)$$

for any measurable set $E \subseteq \partial\mathbb{D}$. Observe that for a point $p \in \partial\mathbb{D}$ the measure μ_p is not finite, while for $p \in \mathbb{D}$, it is just a scalar multiple of the harmonic measure given by $\mu_p = (1 - |p|^2)^{-1}\lambda_p$. A very natural interpretation of the measure μ_p when $p \in \partial\mathbb{D}$ is the following. Let $\omega_p\colon \mathbb{D} \to \mathbb{R}^2_+$ be the conformal map from the disc into the upper half-plane $\mathbb{R}^2_+$ such that $\omega_p(p) = \infty$ and $\omega_p(0) = i/2$. Then, for any measurable set $E \subseteq \partial\mathbb{D}$, we have that $\mu_p(E) = |\omega_p(E)|$. Roughly speaking, for a point $p \in \partial\mathbb{D}$, the measure μ_p gives information about the size and the distribution of a set around the point p. Sets having large μ_p measure are those that are highly concentrated around the point p. In particular, if E is an open neighbourhood of p, then $\mu_p(E) = \infty$. Our first result is the following analogue of Theorem 3.1.

Theorem 3.2. *Consider an inner function $f\colon \mathbb{D} \to \mathbb{D}$ and a boundary Fatou point $p \in \partial\mathbb{D}$ of f.*

(a) Assume $|f'(p)| < \infty$. Then

$$\mu_p(f^{-1}(E)) = |f'(p)|\mu_{f(p)}(E)$$

for any measurable set $E \subseteq \partial\mathbb{D}$.

(b) If $|f'(p)| = \infty$ and $E \subseteq \mathbb{D}$ is a measurable set, then $\mu_p(f^{-1}(E)) = \infty$ if $\mu_{f(p)}(E) > 0$ and $\mu_p(f^{-1}(E)) = 0$ if $\mu_{f(p)}(E) = 0$.

As we can see, we still have a general relation between the measure of a set and its preimage under f, independent of the set. Nonetheless, in this case, a distortion factor appears and it is given by the size of the angular derivative at the point p. If $p \in \partial\mathbb{D}$ is the Denjoy-Wolff fixed point of f, this result was previously proved in [DM91].

In [FP92], Fernández and Pestana studied the distortion of Hausdorff contents under inner functions. For any fixed $z \in \mathbb{D}$ and $0 < \alpha < 1$, consider the Hausdorff content defined as

$$M_\alpha(\lambda_z)(E) := \inf \sum_j \lambda_z(I_j)^\alpha,$$

where the infimum is taken over all collections of arcs $\{I_j\}$ of the unit circle such that $E \subseteq \bigcup I_j$. Thus $M_\alpha(\lambda_0)(E)$ is the standard Hausdorff content of E, which is denoted by $M_\alpha(E)$. Observe that if $z \in \mathbb{D}$ and τ is the automorphism of $\mathbb{D}$ which interchanges z and 0, then $M_\alpha(\lambda_z)(E) = M_\alpha(\tau^{-1}(E))$ for any $E \subseteq \partial\mathbb{D}$. Fernández and Pestana proved the following result, analogous to Theorem 3.2 for Hausdorff contents, stated here in a conformally invariant way.

Theorem 3.3. *For any $0 < \alpha < 1$ there exists a constant $C_\alpha > 0$ such that, if $f : \mathbb{D} \to \mathbb{D}$ is an inner function and $z \in \mathbb{D}$, we have*

$$M_\alpha(\lambda_z)(f^{-1}(E)) \geq C_\alpha M_\alpha(\lambda_{f(z)})(E)$$

for any Borel set $E \subseteq \partial\mathbb{D}$.

It is also shown in [FP92] that there exists an inner function f such that the preimage of a single point has Hausdorff dimension 1. Hence, the converse estimate in Theorem 3.3 is false. It is worth mentioning that a related result for sets $E \subseteq \mathbb{D}$ was established by Hamilton in [Ham93]. He also conjectured that the constant C_α appearing in Theorem 3.3 is actually 1.

For $0 < \alpha < 1$ and $p \in \partial\mathbb{D}$, we define the (p, α)-Hausdorff content of a Borel set $E \subseteq \partial\mathbb{D} \setminus \{p\}$ as

$$M_\alpha(\mu_p)(E) := \inf \sum_j \mu_p(I_j)^\alpha,$$

where the infimum is taken over all collections of arcs $\{I_j\}$ of the unit circle such that $E \subseteq \bigcup I_j$. Note that we decided to exclude point p in the definition of $M_\alpha(\mu_p)$. Nonetheless, if we were to assign $M_\alpha(\mu_p)(\{p\}) = +\infty$, the results presented here would hold trivially for any set E containing p. Our second result is the following analogue of Theorem 3.3 when $z \in \partial\mathbb{D}$.

Theorem 3.4. *Consider an inner function $f : \mathbb{D} \to \mathbb{D}$ and a boundary Fatou point $p \in \partial\mathbb{D}$ of f.*

(a) Assume $|f'(p)| < \infty$. Then for any $0 < \alpha < 1$ there exists a constant $C_\alpha > 0$, independent of f, such that

$$M_\alpha(\mu_p)(f^{-1}(E)) \geq C_\alpha |f'(p)|^\alpha M_\alpha(\mu_{f(p)})(E)$$

for any Borel set $E \subseteq \partial\mathbb{D} \setminus \{f(p)\}$.

(b) Assume $|f'(p)| = \infty$. Then we have that $M_\alpha(\mu_p)(f^{-1}(E)) = \infty$ for any Borel set $E \subseteq \partial\mathbb{D} \setminus \{f(p)\}$ such that $M_\alpha(\mu_{f(p)})(E) > 0$.

The proofs of Theorem 3.2 and Theorem 3.4 are given in Section 3.1. In Section 3.2 we give two applications of our results. The first one concerns a smoothness property of inner functions which omit large sets of the unit disc and it is inspired by a nice result in [FP92, Section 4]. In the second application we obtain analogous results on distortion of sets in the real line under inner mappings of the upper half plane.

3.1 Boundary distortion theorems

We start with some elementary properties of the measure μ_p and the content $M_\alpha(\mu_p)$. Recall that a sequence of points $\{p_n\} \subseteq \mathbb{D}$ converges non-tangentially to a point $p \in \partial\mathbb{D}$ if $\lim p_n = p$ and there exists $\beta > 0$ such that $\{p_n\} \subseteq \Gamma_\beta(p)$.

Lemma 3.1. *Let $p \in \partial\mathbb{D}$. For every sequence of points $\{p_n\} \subseteq \mathbb{D}$ converging non-tangentially to p, we have*

$$\mu_{p_n}(E) \longrightarrow \mu_p(E), \qquad \textit{as } n \to \infty,$$

for any measurable set $E \subseteq \partial\mathbb{D}$.

Proof. Let $\{p_n\}_n \subseteq \mathbb{D}$ be any sequence of points approaching p, and write $\mu_n = \mu_{p_n}$ for every $n \geq 1$. By Fatou's Lemma, we have

$$\liminf_n \mu_n(E) \geq \int_E \lim_n \frac{1}{|\xi - p_n|^2}\, d\lambda(\xi) = \mu_p(E),$$

from which it follows that the result is true when $\mu_p(E) = \infty$. So assume $\mu_p(E) < \infty$. Fix $\varepsilon > 0$ and consider an arc I centred at p and such that $\mu_p(E \cap I) < \varepsilon$. Since $p_n \to p$ non-tangentially, there exists a constant $C > 0$ such that $|\xi - p_n| \geq C|\xi - p|$ for every $\xi \in \partial\mathbb{D}$ and every $n \geq 1$. Hence, we have that $\mu_n(E \cap I) \leq C^{-2}\varepsilon$ for every n. On the other hand, by dominated convergence, we have that

$$\mu_n(E \cap (\partial\mathbb{D} \setminus I)) \longrightarrow \mu_p(E \cap (\partial\mathbb{D} \setminus I)), \qquad \text{as } n \to \infty,$$

from which the result follows. □

Observe that the assumption on the non-tangential convergence of the sequence $\{p_n\}$ to p only enters into play if $p \in \overline{E}$. If $p \notin \overline{E}$, the result holds true for any approaching sequence. However, as the following example shows, Lemma 3.1 fails badly if p_n approaches p tangentially. Fix a point $p \in \partial\mathbb{D}$ and consider a sequence of points $\{\xi_n\} \subseteq \partial\mathbb{D}$ such that $|\xi_n - p| = 1/(2n)$ for every $n \geq 1$. Consider as well the sequence of pairwise disjoint arcs $\{I_n\}$ such that I_n is centred at ξ_n and $\lambda(I_n) = 1/(4n^4)$ for every $n \geq 1$. Now, let $E := \bigcup_n I_n$, $p_n = (1 - \lambda(I_n))\,\xi_n$, and $\mu_n = \mu_{p_n}$, for every $n \geq 1$. Since $(1 - |p_n|)/|p - p_n| \leq 1/n^3 \to 0$, the sequence $\{p_n\}$ converges to p tangentially. For $\xi \in I_n$, we have $|p_n - \xi| \leq 2\lambda(I_n)$ and $\mu_n(I_n) \geq (4\lambda(I_n))^{-1} = n^4$. Now, on one hand we have $\mu_n(E) \geq \mu_n(I_n) \to \infty$, as $n \to \infty$. On the other hand since $|p - \xi| \gtrsim 1/n$ for any $\xi \in I_n$, we have $\mu_p(I_n) \leq n^2\lambda(I_n) = 1/4n^2$ and we deduce

$$\mu_p(E) = \sum_n \mu_p(I_n) < \infty.$$

For $0 < \alpha < 1$ and $z \in \mathbb{D}$ consider the (z, α)-Hausdorff content of a Borel set $E \subseteq \partial\mathbb{D}$ defined as

$$M_\alpha(\mu_z)(E) = \inf \sum_j \mu_z(I_j)^\alpha,$$

where the infimum is taken over all collections of arcs $\{I_j\}$ such that $E \subseteq \bigcup I_j$.

Lemma 3.2. *Given $p \in \partial\mathbb{D}$ and $\beta > 0$, let $\Gamma_\beta(p)$ be the Stolz angle of opening β with vertex at p. Then there exists a constant $C = C(\beta) > 0$ such that*

$$\mu_z(A) \leq C\mu_p(A)$$

for any measurable set $A \subseteq \partial\mathbb{D}$ and any $z \in \Gamma_\beta(p)$. Consequently, for any $0 < \alpha < 1$ we also have $M_\alpha(\mu_z)(A) \leq C^\alpha M_\alpha(\mu_p)(A)$ for any set $A \subseteq \partial\mathbb{D}$ and any $z \in \Gamma_\beta(p)$.

Proof. Observe that there exists a constant $C = C(\beta) > 0$ such that $|\xi - z| \geq C|\xi - p|$ for any $z \in \Gamma_\beta(p)$ and any $\xi \in \partial\mathbb{D}$. Hence, $\mu_z(A) \leq C^{-2}\mu_p(A)$ for any measurable set $A \subseteq \partial\mathbb{D}$ and any $z \in \Gamma_\beta(p)$. This last estimate also gives $M_\alpha(\mu_z)(A) \leq C^{-2\alpha}M_\alpha(\mu_p)(A)$. □

The corresponding result to Lemma 3.1 for Hausdorff contents reads as follows.

Lemma 3.3. *Let $0 < \alpha < 1$ and $p \in \partial\mathbb{D}$. For any sequence of points $\{p_n\} \subseteq \mathbb{D}$ converging non-tangentially to p, we have*

$$\lim_{n\to\infty} M_\alpha(\mu_{p_n})(E) = M_\alpha(\mu_p)(E) \tag{3.2}$$

for any Borel set $E \subseteq \partial\mathbb{D} \setminus \{p\}$.

Proof. Write $\mu_n = \mu_{p_n}$ for every $n \geq 1$. Assume that $M_\alpha(\mu_p)(E) < \infty$. In this case, we split the proof of the result into two parts. First we show that

$$\limsup_{n\to\infty} M_\alpha(\mu_n)(E) \leq M_\alpha(\mu_p)(E), \tag{3.3}$$

and then we prove that

$$\liminf_{n\to\infty} M_\alpha(\mu_n)(E) \geq M_\alpha(\mu_p)(E), \tag{3.4}$$

from which (3.2) follows immediately. To prove (3.3), given $\varepsilon > 0$, take a covering of the set E by open arcs $\{I_j\}$ such that

$$\sum_j \mu_p(I_j)^\alpha \leq M_\alpha(\mu_p)(E) + \varepsilon.$$

Now, by Lemma 3.2, for each interval I_j and for every $n \geq 1$ we have that

$$\mu_n(I_j) \leq C\mu_p(I_j).$$

Thus, by Lemma 3.1 and dominated convergence, we get that

$$\sum_j \mu_n(I_j)^\alpha \longrightarrow \sum_j \mu_p(I_j)^\alpha, \qquad \text{as } n \to \infty.$$

By definition, $M_\alpha(\mu_n)(E) \leq \sum_j \mu_n(I_j)^\alpha$ and, thus (3.3) follows immediately.

We prove inequality (3.4) by considering two cases. Assume first that $p \notin \overline{E}$. Pick $\varepsilon > 0$ and a covering of E by open arcs $\{I_j\}$, such that $\operatorname{dist}(I_j, p) \geq \operatorname{dist}(\overline{E}, p)/2$ for every arc I_j. Observe that, in this situation, there exists $n_0 > 0$ such that if $n > n_0$, we have that

$$\mu_n(I_j) \geq (1-\varepsilon)^{1/\alpha}\mu_p(I_j)$$

for every arc I_j in our covering. Thus, for any such covering of E, if $n > n_0$ we have that

$$\sum_j \mu_n(I_j)^\alpha \geq (1-\varepsilon)M_\alpha(\mu_p)(E).$$

Observe that the infimum of $\sum_j \mu_n(I_j)^\alpha$ when ranging over all coverings $\{I_j\}$ of E by open arcs satisfying that $\operatorname{dist}(I_j, p) \geq \operatorname{dist}(\overline{E}, p)/2$ is, precisely, $M_\alpha(\mu_n)(E)$. Hence, equation (3.4) follows in the case that $p \notin \overline{E}$, and therefore equation (3.2) as well in this situation.

In the case that $p \in \overline{E}$, since we assumed that $M_\alpha(\mu_p)(E) < \infty$, given $\varepsilon > 0$ we can choose $\delta = \delta(\varepsilon) > 0$ such that $M_\alpha(\mu_p)(E \cap I(p,\delta)) < \varepsilon$, where $I(p,\delta)$ denotes the arc with centre at p of length δ. Let us denote $E_\delta = E \setminus I(p,\delta)$. Since $p \notin \overline{E_\delta}$, we already know that

$$\lim_{n\to\infty} M_\alpha(\mu_n)(E_\delta) = M_\alpha(\mu_p)(E_\delta) \geq M_\alpha(\mu_p)(E) - \varepsilon.$$

Hence, for any given $\varepsilon > 0$, we have

$$\liminf_{n\to\infty} M_\alpha(\mu_n)(E) \geq \lim_{n\to\infty} M_\alpha(\mu_n)(E_\delta) \geq M_\alpha(\mu_p)(E) - \varepsilon.$$

This concludes the proof whenever $M_\alpha(\mu_p)(E) < \infty$.

Assume now that $M_\alpha(\mu_p)(E) = \infty$. In this case, for any $N > 0$ we can find $\delta = \delta(N) > 0$ such that $M_\alpha(\mu_p)(E_\delta) > N$, where again $E_\delta = E \setminus I(p,\delta)$. Since $p \notin \overline{E_\delta}$, we have that

$$\lim_{n\to\infty} M_\alpha(\mu_n)(E_\delta) = M_\alpha(\mu_p)(E_\delta) > N.$$

Hence, there exists $n_0 > 0$ such that if $n > n_0$, then $M_\alpha(\mu_n)(E_\delta) > N$. Using that $M_\alpha(\mu_n)(E) \geq M_\alpha(\mu_n)(E_\delta)$, we get (3.2) in the case in which $M_\alpha(\mu_p)(E) = \infty$ as well. □

We will use the following auxiliary result which is certainly well known. It is included because we have not found a precise reference.

Lemma 3.4. *Let f be a holomorphic self map of the unit disc. Let $\{p_n\}$ be a sequence of points in $\mathbb{D}$ converging non-tangentially to a point $p \in \partial\mathbb{D}$. If $|f'(p)| < \infty$, then $\{f(p_n)\}$ converges to $f(p) \in \partial\mathbb{D}$ non-tangentially.*

Proof. Since $|f'(p)| < \infty$ we have that $f(p) \in \partial\mathbb{D}$. Write

$$\frac{1-|f(p_n)|}{|f(p)-f(p_n)|} = \frac{1-|f(p_n)|}{1-|p_n|}\frac{1-|p_n|}{|p-p_n|}\frac{|p-p_n|}{|f(p)-f(p_n)|}.$$

Also because $|f'(p)| < \infty$, by Julia-Carathéodory Theorem, the first and third terms converge respectively to $|f'(p)|$ and $|f'(p)|^{-1}$, and therefore

$$\liminf_n \frac{1-|f(p_n)|}{|f(p)-f(p_n)|} = \liminf_n \frac{1-|p_n|}{|p-p_n|} > 0.$$

□

Note that the assumption of finite angular derivative is necessary in the above statement, even if we ask for the function f to be inner. In fact, it can be proved that there exist inner functions mapping a given Stolz angle to a tangential region (see [Don01]).

We are now ready to prove our main results.

Proof of Theorem 3.2. We can choose a sequence of points $\{p_n\}$ in $\mathbb{D}$ approaching p non-tangentially such that

$$\lim_{n\to\infty} \frac{1-|f(p_n)|^2}{1-|p_n|^2} = |f'(p)| > 0. \tag{3.5}$$

By Theorem 3.1, we have that

$$\mu_{p_n}(f^{-1}(E)) = \frac{1-|f(p_n)|^2}{1-|p_n|^2}\mu_{f(p_n)}(E). \tag{3.6}$$

Lemma 3.1 gives that $\mu_{p_n}(f^{-1}(E)) \to \mu_p(f^{-1}(E))$ as $n \to \infty$. If $|f'(p)| < \infty$, applying Lemma 3.4 one deduces that $f(p_n)$ converges to $f(p)$ non-tangentially. Thus, Lemma 3.1 gives that $\mu_{f(p_n)}(E) \to \mu_{f(p)}(E)$ as $n \to \infty$. Therefore, equations (3.5) and (3.6) prove the statement (a). Assume now that $|f'(p)| = \infty$. If $\mu_{f(p)}(E) = 0$, we have $\lambda(E) = 0$. Hence, by Theorem 3.1, we have that $\lambda(f^{-1}(E)) = 0$ and it follows that $\mu_p(f^{-1}(E)) = 0$. Finally assume $\mu_{f(p)}(E) > 0$. Observe that for any $n \geq 1$ we have $\mu_{f(p_n)}(E) > \lambda(E)/4 > 0$. Thus, since $|f'(p)| = \infty$, the right-hand side of equation (3.6) tends to infinity and, by Lemma 3.1, we deduce that $\mu_p(f^{-1}(E)) = \infty$. □

Proof of Theorem 3.4. We will use Theorem 3.3 in the following form. For $z \in \mathbb{D}$ we have that

$$M_\alpha(\mu_z)(f^{-1}(E)) \geq C_\alpha \left(\frac{1-|f(z)|^2}{1-|z|^2}\right)^\alpha M_\alpha(\mu_{f(z)})(E) \tag{3.7}$$

for any Borel set $E \subseteq \partial\mathbb{D}$. We can choose a sequence of points $\{p_n\}$ in $\mathbb{D}$ approaching p non-tangentially such that

$$\lim_{n\to\infty} \frac{1-|f(p_n)|^2}{1-|p_n|^2} = |f'(p)| > 0. \tag{3.8}$$

Assume $|f'(p)| < \infty$. Applying Lemma 3.3 and equation (3.7), we get

$$\begin{aligned} M_\alpha(\mu_p)(f^{-1}(E)) &= \lim_{n\to\infty} M_\alpha(\mu_{p_n})(f^{-1}(E)) \\ &\geq \limsup_{n\to\infty} C_\alpha \left(\frac{1-|f(p_n)|^2}{1-|p_n|^2}\right)^\alpha M_\alpha(\mu_{f(p_n)})(E) \\ &= C_\alpha |f'(p)|^\alpha \limsup_{n\to\infty} M_\alpha(\mu_{f(p_n)})(E). \end{aligned}$$

By Lemma 3.4, $f(p_n)$ tends to $f(p)$ non-tangentially as $n \to \infty$ and hence, Lemma 3.3 gives that

$$\lim_{n\to\infty} M_\alpha(\mu_{f(p_n)})(E) = M_\alpha(\mu_{f(p)})(E),$$

which finishes the proof of part (a).

Assume now $|f'(p)| = \infty$. Since $M_\alpha(\mu_{f(p)})(E) > 0$, there exists an arc I with centre at $f(p)$ such that $M_\alpha(\mu_{f(p)})(E \setminus I) > 0$. Write $E^* = E \setminus I$. Then there exists $n_0 > 0$ such that $M_\alpha(\mu_{f(p_n)})(E^*) > M_\alpha(\mu_{f(p)})(E^*)/2$ if $n > n_0$. Now,

$$\begin{aligned} M_\alpha(\mu_p)(f^{-1}(E^*)) &= \lim_{n\to\infty} M_\alpha(\mu_{p_n})(f^{-1}(E^*)) \\ &\geq C_\alpha \limsup_{n\to\infty} \left(\frac{1-|f(p_n)|^2}{1-|p_n|^2}\right)^\alpha M_\alpha(\mu_{f(p_n)})(E^*) = \infty. \end{aligned}$$

Hence $M_\alpha(\mu_p)(f^{-1}(E)) = \infty$. □

3.2 Applications

3.2.1 Omitted values

A classical result by Frostman says that any inner function f can omit at most a set of logarithmic capacity zero, that is, $\mathbb{D} \setminus f(\mathbb{D})$ has logarithmic capacity zero (see [Gar07, p. 77]). Conversely, given a relatively compact set K of the unit disc having logarithmic capacity zero, the universal covering map $f\colon \mathbb{D} \to \mathbb{D} \setminus K$ is an inner function (see [Tsu75, p. 323]). Given a set $E \subseteq \mathbb{D}$, its *non-tangential closure* on $\partial\mathbb{D}$, denoted by E^{NT}, is the set of points $\xi \in \partial\mathbb{D}$ for which there exists a sequence $\{z_n\} \subseteq E$ such that $z_n \to \xi$ non-tangentially. We first state an auxiliary result which may have independent interest.

Lemma 3.5. *Consider an inner function $f\colon \mathbb{D} \to \mathbb{D}$ and its set of omitted points $E = \mathbb{D} \setminus f(\mathbb{D})$. Then*

$$f^{-1}(E^{NT}) \subseteq \{\xi \in \partial\mathbb{D}\colon |f'(\xi)| = \infty\}.$$

Proof. Consider a point $\xi \in \partial\mathbb{D}$ such that the angular derivative of f at ξ exists and it is finite, and let $\zeta = f(\xi)$. In other words assume that

$$\lim_{z \angle \xi} \frac{\zeta - f(z)}{\xi - z} = A \tag{3.9}$$

is finite. We want to see that, in this situation, for any opening $\gamma > 1$, there is $0 < s = s(\gamma) < 1$ such that the truncated cone

$$\Gamma_{\gamma,s}(\zeta) = \{w \in \mathbb{D}\colon |\zeta - w| < \gamma(1 - |w|),\ |\zeta - w| < s\}$$

does not intersect E, that is, $\Gamma_{\gamma,s}(\zeta) \subseteq f(\mathbb{D})$. So fix $\gamma > 1$ and consider $\Gamma_{\gamma,s}(\zeta)$ with $0 < s < 1$ to be determined. Fix $w_0 \in \Gamma_{\gamma,s}(\zeta)$. We want to see that there is $z_0 \in \mathbb{D}$ such that $f(z_0) = w_0$. By equation (3.9), we can express

$$f(z) = \zeta + A(z - \xi) + o(|z - \xi|),$$

where $o(|z - \xi|)/|z - \xi| \to 0$ as $z \to \xi$ non-tangentially. Consider $\Gamma_{\beta,r}(\xi)$ with $\beta > 2\gamma$ and $0 < r < 1$ to be determined. Observe that there exists $0 < r_0 < 1$ such that, if $r < r_0$ and $0 < s < |A|r/2$, then for any $z \in \partial\Gamma_{\beta,r}(\xi)$ we have that

$$|(f(z) - w_0) - (\zeta + A(z - \xi) - w_0)| < |\zeta + A(z - \xi) - w_0|.$$

Thus, by Rouché's Theorem, the functions $f(z) - w_0$ and $g(z) - w_0 = \zeta + A(z - \xi) - w_0$ have the same number of zeroes in $\Gamma_{\beta,r}(\xi)$. But $g(z)$ is a degree 1 polynomial and $g(\Gamma_{\beta,r}(\xi)) = \Gamma_{\beta,|A|r}(\zeta) \supseteq \Gamma_{\gamma,s}(\zeta)$, and thus $g(z) - w_0$ has a single zero on $\Gamma_{\beta,r}(\xi)$. Therefore, there is $z_0 \in \Gamma_{\beta,r}(\xi)$ such that $f(z_0) = w_0$, which completes the proof. □

Recall that for $p \in \partial\mathbb{D}$, if we assign $M_\alpha(\mu_p)(\{p\}) = +\infty$, then Theorem 3.4 also holds for any Borel set $E \subseteq \partial\mathbb{D}$ with $p \in E$. As an application of Theorem 3.4 and Lemma 3.5, we have the following result.

Corollary 3.1. *Consider an inner function $f\colon \mathbb{D} \to \mathbb{D}$ and its set of omitted points $E = \mathbb{D} \setminus f(\mathbb{D})$. Let p be a boundary Fatou point of f.*

(a) Assume $|f'(p)| < \infty$. Then for any $0 < \alpha < 1$ there exists a constant $C_\alpha > 0$, independent of f, such that

$$M_\alpha(\mu_p)\left(\{\xi \in \partial\mathbb{D}\colon |f'(\xi)| = \infty\}\right) \geq C_\alpha |f'(p)|^\alpha M_\alpha(\mu_{f(p)})(E^{NT}). \tag{3.10}$$

(b) *Assume* $|f'(p)| = \infty$.
Then $M_\alpha(\mu_p)\left(\{\xi \in \partial\mathbb{D} \colon |f'(\xi)| = \infty\}\right) = \infty$ *whenever* $M_\alpha(\mu_{f(p)})(E^{NT}) > 0$.

3.2.2 Inner functions in the upper half plane

Let $\mathbb{R}^2_+ = \{w \in \mathbb{C} \colon \operatorname{Im}[w] > 0\}$ be the upper half plane. A holomorphic mapping $g \colon \mathbb{R}^2_+ \to \mathbb{R}^2_+$ is an *inner function* of the upper half plane if $\lim_{y\to 0} g(x+iy) \in \mathbb{R}$ for a.e. $x \in \mathbb{R}$. This natural definition agrees with conformal changes of coordinates: given $p \in \partial\mathbb{D}$ denote by w_p the Möbius transformation mapping $\mathbb{D}$ onto $\mathbb{R}^2_+$, the point p to ∞ and, say, the origin to $i/2$. Then, g is an inner function of the upper half plane if and only if $f = w_p^{-1} \circ g \circ w_p$ is an inner function of the unit disc $\mathbb{D}$. Observe that $g(\infty) = \lim_{t\to+\infty} g(it) = \infty$ if and only if $f(p) = p$. A holomorphic mapping g from $\mathbb{R}^2_+$ into itself has a finite angular derivative at ∞ if

$$g'(\infty) = \lim_{t\to+\infty} \frac{it}{g(it)}$$

exists and is finite. Otherwise, we write $|g'(\infty)| = \infty$. Observe that g has a finite angular derivative at infinity if and only if $f = w_p^{-1} \circ g \circ w_p$ has a finite angular derivative at p. Let w denote $w_p(z)$. Moreover, the identity $|g'(\infty)| = |f'(p)|$ holds in the sense that both quantities coincide when they are finite, and if one of them is infinite so is the other. This fact easily follows from the identity

$$\frac{w}{g(w)} = \frac{p+z}{p+f(z)}\,\frac{p-f(z)}{p-z}.$$

Recall that we denote by $|A|$ the Lebesgue measure of a measurable set $A \subseteq \mathbb{R}$. Moreover, for $0 < \alpha < 1$, let $M_\alpha(A)$ denote its α-Hausdorff content. We now state the versions of Theorems 3.2 and 3.4 in this setting.

Corollary 3.2. *Consider an inner function* $g \colon \mathbb{R}^2_+ \to \mathbb{R}^2_+$ *such that* $g(\infty) = \infty$.

(a) *Assume* $|g'(\infty)| < \infty$. *Then*

$$|g^{-1}(A)| = |g'(\infty)||A| \tag{3.11}$$

for any measurable set $A \subseteq \mathbb{R}$. *Moreover, for any* $0 < \alpha < 1$ *there exists a constant* $C_\alpha > 0$, *independent of* g, *such that*

$$M_\alpha(g^{-1}(A)) \geq C_\alpha |g'(\infty)|^\alpha M_\alpha(A) \tag{3.12}$$

for any Borel set $A \subseteq \mathbb{R}$.

(b) *If* $|g'(\infty)| = \infty$ *and* $A \subseteq \mathbb{R}$ *is a measurable set, then* $|g^{-1}(A)| = \infty$ *if* $|A| > 0$ *and* $|g^{-1}(A)| = 0$ *if* $|A| = 0$. *Moreover,* $M_\alpha(g^{-1}(A)) = \infty$ *for any Borel set* $A \subseteq \mathbb{R}$ *such that* $M_\alpha(A) > 0$.

Proof. Note that for any measurable set $A \subseteq \mathbb{R}$ and for $p \in \partial\mathbb{D}$ fixed, we have

$$|A| = \mu_p(w_p^{-1}(A)). \tag{3.13}$$

Consider the inner function $f \colon \mathbb{D} \to \mathbb{D}$ defined by $f = w_p^{-1} \circ g \circ w_p$. Hence, we can express $|g^{-1}(A)| = \mu_p(w_p^{-1}(g^{-1}(A))) = \mu_p(f^{-1}(w_p^{-1}(A)))$. Applying Theorem

3.2 and (3.13) we deduce that $|g^{-1}(A)| = |f'(p)|\mu_p(w_p^{-1}(A)) = |g'(\infty)||A|$ which is (3.11). It follows from (3.13) and w_p being a Möbius map that

$$M_\alpha(\mu_p)(E) = M_\alpha(w_p(E)), \qquad E \subseteq \partial\mathbb{D}. \tag{3.14}$$

Thus, the previous argument shows that (3.12) holds. Part (b) follows from similar considerations. □

Additional Remarks

In Chapter 1 we have used dyadic martingales in order to prove Theorem 1.2. In particular, we have first proved an analogous theorem for dyadic spaces (or for spaces of dyadic martingales), from which we recovered the result for the homogeneous spaces $\dot{\Lambda}_s$. Recall as well that our results were inspired by the problem of finding the closure of the space Λ_1 in Λ_* in terms of the Zygmund semi-norm, and also to that of finding the closure of $\mathbb{H}^\infty(\mathbb{D})$ in $\mathcal{B}(\mathbb{D})$.

We say that a dyadic martingale S defined on $I_0 = [0,1]$ is a martingale with uniformly bounded increments, which we denote here by $S \in \Lambda_*$, if it satisfies

$$\|S\|_{\Lambda_*} := \sup_{j\geq 1} \|\Delta_j S\|_{\mathrm{L}^\infty} < \infty. \tag{R.15}$$

Observe that the quantity $\|S\|_{\Lambda_*}$ for $S \in \Lambda_*$ defines a norm in the space of dyadic martingales with uniformly bounded increments modulo constant martingales. On the other hand, we say that a martingale S defined on I_0 is uniformly bounded, denoted here by $S \in \Lambda_1$, if

$$\|S\|_{\Lambda_1} := \sup_{j\geq 0} \|S_j\|_{\mathrm{L}^\infty} < \infty.$$

One can easily check that, for a dyadic martingale S, if $S \in \Lambda_1$, then $S \in \Lambda_*$. Note that if we consider a function $f \in \Lambda_*$ supported on the interval I_0, then its average growth martingale S (as it was defined by (1.9)) is a martingale with uniformly bounded increments. Similarly, if $f \in \Lambda_1$ is a function supported on the interval I_0, its average growth martingale S is uniformly bounded. These observations justify our notation for these martingale spaces. This brings us to the apparently easier following problem.

Problem. *Find the closure of the uniformly bounded martingales in the norm defined by* (R.15).

There are some necessary and and some sufficient conditions for a martingale with uniformly bounded increments to be in the closure of the uniformly bounded martingales, with their counterparts for the corresponding function spaces. We say that a dyadic martingale S has increments tending to zero, denoted by $S \in \lambda_*$, if

$$\lim_{j\to\infty} \|\Delta_j S\|_{\mathrm{L}^\infty} = 0.$$

It is easy to see that if $S \in \lambda_*$, then S is in the closure of the uniformly bounded martingales, but this is far from being necessary. For example, define a martingale S by $S_0 = 0$ and $\Delta_j S = 0$ for every $j \geq 1$ except for those of the form $j = k!$ for some integer $k \geq 1$, in which case take $\Delta_j S(I_*) = 1$, where

$$I_* = [1 - 2^{-j}, 1],$$

and $\Delta_j S(I) = 0$ for any other $I \in \mathcal{D}_j$. Then $S \notin \lambda_*$, but one can easily see that it can be arbitrarily approximated by uniformly bounded martingales. On the other hand, if a martingale lies on the closure of the uniformly bounded martingales, it must be a BMO martingale in the sense of (1.10). Nonetheless, this is also far from being sufficient. For instance, take the martingale S defined by $S_0 = 0$ and, for every $j \geq 1$, take $\Delta_j S(I_*) = 1$ for

$$I_* = [1 - 2^{-j}, 1]$$

and $\Delta_j S(I) = 0$ for any other $I \in \mathcal{D}_j$. Then S is a BMO martingale, but it is easy to see that it cannot be well approximated by uniformly bounded martingales.

In Chapter 3 we have discussed a problem concerning the distortion of measures and Hausdorff contents of sets on $\partial\mathbb{D}$ by inner functions. This problem was inspired by the results of Fernández and Pestana in [FP92]. In that article, the authors also find how inner functions distort capacities. Recall that for our study, we took the measure μ_p that privileged a point $p \in \partial\mathbb{D}$, as well as the (p, α)-Hausdorff content based on this measure. We have explained that these two quantities measure not only the size of sets, but also how concentrated they are around point p. Unfortunately, we were not able to find an analogous quantity to capacities of sets that took into account the distribution of a set around p, in addition to its size, and thus we could not extend the results on capacities to our context. Nonetheless, it is not unreasonable to think that one could somehow adapt this concept to this setting.

Bibliography

[Ahl73] L. V. Ahlfors. *Conformal invariants: topics in geometric function theory*. Series in Higher Mathematics. McGraw-Hill Book Co., New York-Düsseldorf-Johannesburg, 1973.

[AB97] H. Aimar and A. Bernardis. "Wavelet characterization of functions with conditions on the mean oscillation". *Wavelet theory and harmonic analysis in applied sciences (Buenos Aires, 1995)*. Appl. Numer. Harmon. Anal. Birkhäuser Boston, Boston, MA, 1997, pp. 15–32.

[AAN99] A. B. Aleksandrov, J. M. Anderson, and A. Nicolau. "Inner functions, Bloch spaces and symmetric measures". *Proc. London Math. Soc. (3)* 79.2 (1999), pp. 318–352.

[ACP74] J. M. Anderson, J. Clunie, and C. Pommerenke. "On Bloch functions and normal functions". *J. Reine Angew. Math.* 270 (1974), pp. 12–37.

[AP89] J. M. Anderson and L. D. Pitt. "Probabilistic behaviour of functions in the Zygmund spaces Λ^* and λ^*". *Proc. London Math. Soc. (3)* 59.3 (1989), pp. 558–592.

[BM99] R. Bañuelos and C. N. Moore. *Probabilistic behavior of harmonic functions*. Vol. 175. Progress in Mathematics. Birkhäuser Verlag, Basel, 1999.

[Con13] J. M. Conde. "A note on dyadic coverings and nondoubling Calderón-Zygmund theory". *J. Math. Anal. Appl.* 397.2 (2013), pp. 785–790.

[Dau88] I. Daubechies. "Orthonormal bases of compactly supported wavelets". *Comm. Pure Appl. Math.* 41.7 (1988), pp. 909–996.

[DM91] C. I. Doering and R. Mañé. "The dynamics of inner functions". *Ensaios Matemáticos* 3 (1991), pp. 5–79.

[Don01] J. J. Donaire. "Radial behaviour of inner functions in $\mathcal{B}_0$". *J. London Math. Soc. (2)* 63.1 (2001), pp. 141–158.

[DLN14] J. J. Donaire, J. G. Llorente, and A. Nicolau. "Differentiability of functions in the Zygmund class". *Proc. Lond. Math. Soc. (3)* 108.1 (2014), pp. 133–158.

[DN02] E. Doubtsov and A. Nicolau. "Symmetric and Zygmund measures in several variables". *Ann. Inst. Fourier (Grenoble)* 52.1 (2002), pp. 153–177.

[FS72] C. Fefferman and E. M. Stein. "H^p spaces of several variables". *Acta Math.* 129.3-4 (1972), pp. 137–193.

[FP92] J. L. Fernández and D. Pestana. "Distortion of boundary sets under inner functions and applications". *Indiana Univ. Math. J.* 41.2 (1992), pp. 439–448.

[FJ90] M. Frazier and B. Jawerth. "A discrete transform and decompositions of distribution spaces". *J. Funct. Anal.* 93.1 (1990), pp. 34–170.

[GMP15] P. Galanopoulos, N. Monreal Galán, and J. Pau. "Closure of Hardy spaces in the Bloch space". *J. Math. Anal. Appl.* 429.2 (2015), pp. 1214–1221.

[Gar07] J. B. Garnett. *Bounded analytic functions*. first. Vol. 236. Graduate Texts in Mathematics. Springer, New York, 2007.

[GJ78] J. B. Garnett and P. W. Jones. "The distance in BMO to L^∞". *Ann. of Math. (2)* 108.2 (1978), pp. 373–393.

[GJ82] J. B. Garnett and P. W. Jones. "BMO from dyadic BMO". *Pacific J. Math.* 99.2 (1982), pp. 351–371.

[GZ93] P. G. Ghatage and D. C. Zheng. "Analytic functions of bounded mean oscillation and the Bloch space". *Integral Equations Operator Theory* 17.4 (1993), pp. 501–515.

[Ham93] D.-H. Hamilton. "Distortion of sets by inner functions." *Proc. Amer. Math. Soc.* 117 (1993), pp. 771–774.

[JN61] F. John and L. Nirenberg. "On functions of bounded mean oscillation". *Comm. Pure Appl. Math.* 14 (1961), pp. 415–426.

[JW84] A. Jonsson and H. Wallin. "Function spaces on subsets of $\mathbf{R}^n$". *Math. Rep.* 2.1 (1984), pp. xiv+221.

[Kah69] J.-P. Kahane. "Trois notes sur les ensembles parfaits linéaires". *Enseignement Math. (2)* 15 (1969), pp. 185–192.

[LM86] P. G. Lemarié and Y. Meyer. "Ondelettes et bases hilbertiennes". *Rev. Mat. Iberoamericana* 2.1-2 (1986), pp. 1–18.

[LNS19] M. Levi, A. Nicolau, and O. Soler i Gibert. "Distortion and Distribution of Sets Under Inner Functions". *J. Geom. Anal.* (2019 (in press)).

[LM20] A. Limani and B. Malman. "Generalized Cesàro Operators: Geometry of Spectra and Quasi-Nilpotency". *Int. Math. Res. Not. IMRN* (2020 (in press)).

[Llo02] J. G. Llorente. *Discrete martingales and applications to analysis*. Vol. 87. Report. University of Jyväskylä Department of Mathematics and Statistics. University of Jyväskylä, Jyväskylä, 2002.

[Mak89] N. G. Makarov. "Smooth measures and the law of the iterated logarithm". *Izv. Akad. Nauk SSSR Ser. Mat.* 53.2 (1989), pp. 439–446.

[Mei03] T. Mei. "BMO is the intersection of two translates of dyadic BMO". *C. R. Math. Acad. Sci. Paris* 336.12 (2003), pp. 1003–1006.

[Mey92] Y. Meyer. *Wavelets and operators*. Vol. 37. Cambridge Studies in Advanced Mathematics. Translated from the 1990 French original by D. H. Salinger. Cambridge University Press, Cambridge, 1992.

[MC97] Y. Meyer and R. Coifman. *Wavelets*. Vol. 48. Cambridge Studies in Advanced Mathematics. Calderón - Zygmund and multilinear operators, Translated from the 1990 and 1991 French originals by David Salinger. Cambridge University Press, Cambridge, 1997.

[MN11] N. Monreal Galán and A. Nicolau. "The closure of the Hardy space in the Bloch norm". *St. Petersburg Math. J.* 22.1 (2011), pp. 75–81.

[Nic18] A. Nicolau. "Divided differences, square functions and a law of the iterated logarithm". *Real Anal. Exchange* 43.1 (2018), pp. 155–186.

[NS20] A. Nicolau and O. Soler i Gibert. "Approximation in the Zygmund class". *J. London Math. Soc.* 101.1 (2020), pp. 226–246.

[PS06] A. Poltoratski and D. Sarason. “Aleksandrov-Clark measures”. *Recent advances in operator-related function theory*. Vol. 393. Contemp. Math. Amer. Math. Soc., Providence, RI, 2006, pp. 1–14.

[Sak07] E. Saksman. “An elementary introduction to Clark measures”. *Topics in complex analysis and operator theory*. Univ. Málaga, Málaga, 2007, pp. 85–136.

[SS20] E. Saksman and O. Soler i Gibert. “Wavelet Approximation in the Zygmund Class” (2020 (in preparation)).

[Sha93] J. H. Shapiro. *Composition operators and classical function theory*. Universitext: Tracts in Mathematics. Springer-Verlag, New York, 1993.

[Ste70] E. M. Stein. *Singular integrals and differentiability properties of functions*. Princeton Mathematical Series, N. 30. Princeton University Press, Princeton, N.J., 1970.

[Ste93] E. M. Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*. Vol. 43. Princeton Mathematical Series. With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III. Princeton University Press, Princeton, NJ, 1993.

[Str80] R. S. Strichartz. “Bounded mean oscillation and Sobolev spaces”. *Indiana Univ. Math. J.* 29.4 (1980), pp. 539–558.

[Tri10] H. Triebel. *Theory of function spaces*. Modern Birkhäuser Classics. Reprint of 1983 edition, Also published in 1983 by Birkhäuser Verlag. Birkhäuser / Springer Basel AG, Basel, 2010.

[Tri20] H. Triebel. *Theory of function spaces IV*. Monographs in Mathematics. Birkhäuser / Springer Basel AG, Basel, 2020.

[Tsu75] M. Tsuji. *Potential theory in modern function theory*. Reprinting of the 1959 original. Chelsea Publishing Co., New York, 1975.

[YSY10] W. Yuan, W. Sickel, and D. Yang. *Morrey and Campanato meet Besov, Lizorkin and Triebel*. Vol. 2005. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 2010.

[Zyg45] A. Zygmund. “Smooth functions”. *Duke Math. J.* 12 (1945), pp. 47–76.

[Zyg02] A. Zygmund. *Trigonometric series. Vol. I, II*. Third. Cambridge Mathematical Library. With a foreword by Robert A. Fefferman. Cambridge University Press, Cambridge, 2002.

Colophon

This document was typeset with the help of `MastersDoctoralThesis` script. This style, which is a modification due to Sunil Patel, was created by Steve Gunn.

https://www.latextemplates.com/template/masters-doctoral-thesis