Network creation games: structure vs anarchy

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Abstract

In an attempt to understand how Internet-like networks and social networks behave, different models have been proposed and studied throughout history to capture their most essential aspects and properties. Network Creation Games are a class of strategic games widely studied in Algorithmic Game Theory that model these networks as the outcome of decentralised and uncoordinated interactions. In these games the different players model selfish agents that buy links towards the other agents trying to minimise an individual function. This cost is modelled as a function that usually decomposes into the creation cost (cost of buying links) and the usage cost (measuring the quality of the connection to the network). Due to the agents' selfish behaviour, stable configurations in which all players are happy with the current situation, the so-called *Nash equilibria*, do not have to coincide with any socially optimal configuration that could be established if a centralised authority could decide by all players. In this way, the *price of anarchy* is the measure that quantifies precisely the ratio between the most costly Nash equilibrium versus any optimal network from a social point of view.

In this work, we study the price of anarchy and Nash equilibria in different scenarios and situations, in order to better understand how the selfish behaviour of agents in these networks affects the quality of the resulting networks. We propose this study from two different perspectives.

In the first one, we study one of the most emblematic models of Network Creation Games called *sum classical network creation game* [20]. This is a model that is based on two different parameters: n, the number of nodes, and α, a function of n that models the price per link. Throughout history it has been shown that the price of anarchy is constant for $\alpha = O(n^{1-\delta})$ with $\delta \geq 1/\log n$ and for $\alpha > 4n - 13$. It has been conjectured that the price of anarchy is constant regardless of the parameter α. In this first part we show, first of all, that the price of anarchy is constant even when $\alpha \geq n(1+\epsilon)$ with $\epsilon > 0$ any positive constant, thus enlarging the range of values α for which the price of anarchy is constant. Secondly, regarding the range $\alpha < n/C$ with $C > 4$ any positive constant, we know that equilibria induce a class of graphs called *distance-uniform*. Then, we study the diameter of the distance-uniform graphs in an attempt to obtain information about the topology of equilibria for the range $\alpha < n/C$ with $C > 4$ any positive constant.

In the second perspective we propose and study two new models that we call *celebrity games*. These two models are based on the analysis of decentralized networks with heterogeneous players, that is, players with different degrees of relevance within the corresponding network, a feature that has not been studied in much detail in the literature. To capture this natural property, we introduce a *weight* for each player in the network. Furthermore, these models take into account a critical distance β, a threshold value. Each player aim to be not farther than β from the other players and decides whether to buy links to other players depending on the price per link and their corresponding weights. Moreover, the larger is the weight of a player farther than β, larger is the corresponding penalty. Thus, in these new models players strive to have the minimum possible number of links and at the same time they want to minimise as much as possible the penalty for having players farther from β. They differ in how the penalty corresponding to the players further than β is computed. For both models we obtain upper and lower bounds of the price of anarchy as well as the main topological properties and characteristics of their equilibria.

Resum

En un intent per entendre com xarxes de naturalesa similar a les de l'Internet i les xarxes socials es comporten, al llarg de la història s'han proposat i estudiat diferents models que tracten de capturar-ne els aspectes i les propietats més essencials. Els jocs de formació de xarxes són una classe de jocs estratègics molt estudiats en teoria de jocs algorísmica que modelitzen aquestes xarxes com el resultat de la interacció descentralitzada dels agents que la integren. En aquests jocs els diferents jugadors modelitzen agents egoistes que compren enllaços cap als altres jugadors intentant minimitzar una funció individual. Aquest cost es modela com una funció que es descomposa en dues components: el cost de creació (cost relatiu a la compra dels enllaços del mateix jugador) i, en segon lloc, el cost d'utilització (mesura de la qualitat de connexió a la xarxa resultant). Degut al comportament egoista dels agents, les situacions estables que s'assoleixen, els anomenats *equilibris de Nash*, no tenen per què coincidir amb les configuracions òptimes des del punt de vista social que es podrien establir si existís una entitat centralitzadora que decidís per tots els jugadors. Justament, el *preu de l'anarquia* és la mesura que quantifica la diferència que hi ha entre l'equilibri de Nash més costós versus l'òptim des del punt de vista social.

En aquesta tesis, estudiarem aquests dos conceptes claus, el preu de l'anarquia i els equilibris de Nash, en escenaris i situacions diferents, per tal d'entendre millor com el comportament egoista dels agents d'aquestes xarxes n'afecta la seva qualitat. Proposem duess perspectives diferents a aquest estudi.

En primer lloc, estudiem un dels models més emblemàtics dels jocs de formació de xarxes que anomenem *sum classical network creation game* [20]. Aquest és un model de xarxes que es basa en dos paràmetres diferents: n, nombre de nodes, i α, una funció de n que modelitza el preu per enllaç. Al llarg de la història s'ha demostrat que el preu de l'anarquia és constant per a $\alpha = O(n^{1-\delta})$ i $\delta \geq 1/\log n$, així com per a $\alpha > 4n - 13$. A més s'ha conjecturat que el preu de l'anarquia és constant independentment del paràmetre α. Pel que fa al rang $\alpha < n/C$ amb $C > 4$ constant, sabem que els equilibris indueixen una classe de grafs que s'anomenen *distance-uniform*. En aquesta primera part es demostra, en primer lloc, que el preu de l'anarquia és constant inclús quan $\alpha > n(1+\epsilon)$ amb $\epsilon > 0$ qualsevol constant positiva allargant, doncs, el rang de valors del paràmetre α pels quals el preu d'anarquia és constant. En segon lloc, s'estudia el diàmetre dels grafs distance-uniform en un intent d'obtenir informació sobre la topologia dels equilibris per al rang $\alpha < n/C$ amb $C > 4$ constant.

El segon punt de vista que considerem consisteix en proposar i estudiar dos nous models de creació de xarxes que anomenem els *celebrity games*. Aquests dos models parteixen de l'anàlisi de les xarxes decentralitzades amb agents heterogenis, com és el cas d'agents que tenen diferents graus de rellevància dins de la xarxa corresponent, un tret fins ara poc estudiat en els models de la literatura. Per capturar aquesta característica natural s'introdueixen *pesos*, un per a cada agent de la xarxa. D'altra banda, una altra característica que es considera en la proposta d'aquests dos models nous és el concepte de *distància crítica* que captura el llindar β a partir del qual, nodes que estiguin més llunyans que el valor β del jugador en consideració penalitzen a tal jugador. Així, el que es persegueix en aquests dos nous models és tenir el mínim nombre d'enllaços possibles i al mateix temps, minimitzar el màxim possible la penalització dels jugadors més llunyans de β d'acord amb els seus pesos. Els dos models que estudiem es diferencien en com es calcula l'afectació o penalització dels jugadors més llunyans de β. Pels dos models obtenim fites superiors i inferiors del preu de l'anarquia així com les propietats i característiques topològiques principals dels equilibris.

Resumen

En un intento para entender como redes de naturaleza similar a las del Internet y la redes sociales se comportan, al largo de la historia se han propuesto y estudiado modelos que tratan de capturar los aspectos y las propiedades más esenciales. Los juegos de formación de redes son una clase de juegos estratégicos muy estudiados en la teoría de juegos algorísmica que modelizan estas redes como el resultado de la interacción descentralizada de los agentes que la integran. En estos juegos los distintos jugadores modelizan agentes egoístas que compran enlaces hacia los otros jugadores intentando minimizar una función individual. Este coste se modeliza como una función que se descompone en el coste de creación (coste relativo a la compra de los enlaces del mismo jugador) y, en segundo lugar, el coste de utilitzación (mesura de la cualidad de conexión a la red resultante). Debido al comportamiento egoísta de los agentes, las situaciones estables que se consiguen, los llamados *equilibrios de Nash*, no coinciden necesariamente con las configuraciones óptimas des del punto de vista social que se podrían establecer si existiera una entidad centralizadora que tomara una decisión por todos los jugadores. Justamente, el *precio de la anarquía* es la medida que cuantifica la diferencia que hay entre los equilibrios de Nash más costosos versus el óptimo des del punto de vista social.

En esta tesis, estudiaremos estos dos conceptos claves, el precio de la anarquía y los equilibrios de Nash en escenarios y situaciones diferentes, con la intención de entender mejor como el comportamiento egoísta de los agentes de estas redes afecta su cualidad. Proponemos dos perspectivas distintas para este estudio.

En primer lugar, estudiamos uno de los modelos más emblemáticos de los juegos de formación de redes que llamamos *sum classical network creation game* [20]. Este es un modelo de redes que se basa en dos parámetros distintos: n, el número de nodos de la red, y α, una función de n que modeliza el precio por enlace. Al largo del tiempo se ha demostrado que el precio de la anarquía es constante por $\alpha = O(n^{1-\delta})$ con $\delta \geq 1/\log n$, así como para $\alpha > 4n - 13$. Además se ha conjeturado que el precio de la anarquía es constante independientemente del parámetro α. Respecto al rango $\alpha < n/C$ con $C > 4$ constante, sabemos que los equilibrios inducen una clase de grafos que se llama *distancia-uniforme*. En esta primera parte se demuestra, primero, que el precio de la anarquía es constante incluso cuando $\alpha > n(1+\epsilon)$ con $\epsilon > 0$ cualquier constante positiva engrandeciendo, pues, el rango de valores del parámetro α por los cuales el precio de la anarquía es constante. En segundo lugar, se estudia el diámetro de los grafos distancia-uniforme en un intento de obtener información sobre la topología de los equilibrios para el rango $\alpha > n/C$ con $C > 4$ constante.

El segundo punto de vista que consideremos consiste en proponer y estudiar dos modelos de creación de redes nuevos que llamamos los *celebrity games*. Estos dos modelos parten del análisis de la redes descentralizadas con agentes heterogéneos, como puede ser el caso de agentes que tienen distintas grados de relevancia dentro de la red correspondiente, una característica hasta ahora poco estudiada en los modelos de la literatura. Para capturar esta característica natural se introducen *pesos* para cada agente de la red. Por otro lado, otra característica que se considera en la propuesta de estos dos models nuevos es el concepto de *distancia crítica* que captura el nivel β a partir del cual, nodos que estén más lejanos que el valor β del jugador en consideración penalizan a tal jugador. Así, el que se persigue en estos dos nuevos modelos es tener el mínimo número de enlaces posibles y al mismo tiempo minimizar la máxima de la penalizaciones de los jugadores más lejanos de β de acuerdo con sus pesos. Los dos modelos que estudiamos se diferencian en como se calcula la afectación o penalización de los jugadores más lejanos de β. Para los dos modelos obtenemos cotas superiores e inferiores del precio de la anarquía así como la propiedades y características topológicas principales de los equilibrios.

Acknowledgments [1]

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Contents

Chapter 1

Introduction

1.1 Motivation and Context

This work studies some aspects and properties of communication networks having a nature quite similar to the one of Internet-like networks or social networks. Mostly, these networks are formed by agents following some selfish interests lacking coordination among them. The study of communication networks is a classical subject inside Theoretical Computer Science that can be analysed following distinct perspectives. In its beginnings, Network Design was a line of research that assumed that a central authority creates the network trying to fulfill some optimization criteria. Whilst this perspective has lead to interesting results which now have become well-known classical aspects about networks (maximum edge disjoint paths, maximum node disjoint paths, k-median, k-center, as well as many others), it turns to be that most of the known Internet-like networks and social networks are not obtained following the orders of a central authority, but rather, following the selfish interest of the multiple agents that integrate such networks. Once these fundamental key ingredients defining the nature of such networks are identified, Algorithmic Game Theory provides us a right point of view to better understand this combination.

In Algorithmic Game Theory, Network Creation Games is a subfamily of strategic games that models the collective attempt of creating a network. Each player or agent buys links to other agents and selfishly tries to minimise some objective function that considers both the creation cost (the cost of buying links) and some usage cost (the quality of the connection to the network).

Now, let us describe the most relevant concepts to better understand the paradigm of the Network Creation Games. First of all, any Network Creation Game is specified by a finite set of players or agents. A *strategy* for any player consists of a subset of the other players to whom the player at consideration wants to buy links. Then, a *strategy profile* is any configuration consisting of the choices of all the players which naturally defines a network or an *outcome graph*. Each player has associated an *individual cost* which value depends not only on its own strategy but also on the strategy profile. The *best response* of a player to a strategy profile is the strategy (or strategies) that minimises his cost. A *Nash equilibrium* (or abbreviated NE) is any strategy profile in which each player cannot strictly decrease its cost function when the strategies of the other players are fixed. In other words, a NE is a configuration such that the strategy of every player is a best response for that player. We are interested in measuring somehow the quality of a specific network. The *social cost* quantifies the overall cost of any strategy profile and it is usually defined as the sum of the individual costs of the players from the configuration. In this way, any best (or worst) strategy profile from a certain collection with respect the social cost is any configuration from the collection minimising (or maximising) the social cost. In particular, an *optimal network* is any best configuration and the *social optimum* is the social cost of any such network. Now, notice that a NE network might not be an optimal network. The *price of anarchy* (or abbreviated PoA) is the key concept that quantifies the distance (with respect the social cost) between the worst equilibrium and any optimal network and is defined as the ratio between the maximum social cost of any equilibrium and the social optimum. Similarly, the *price of stability* (or abbreviated PoS) quantifies the distance between the best equilibrium and any optimal network and is defined as the ratio between the minimum social cost of any equilibrium and the social optimum.

1.2 This Thesis

The main goal of this thesis is to get a better comprehension of how the selfish behaviour of the agents participating in the creation of communication networks, affects the quality of such networks from a social point of view. This is the main question of the thesis and we address it by considering two different approaches. The first one is the study of the most emblematic and classical network creation game and the second one is the proposal and study of two new original models capturing some new features of communication and social networks not present in any other previous model of the literature. For both approaches the analysis that we perform consists in studying two fundamental and crucial concepts. First, the PoA, which gives a kind of quantitative answer to the main question. Secondly, when the exact value of the PoA is hard to calculate or even approximate via upper and lower bounds, we study the topology of equilibria, which might give us a kind of qualitative answer to the main question.

Now, let us describe in a little more detail these two distinct approaches.

Our starting point is, as we have explained previously, the study of the very famous and classical model of communication networks, the *Sum Classical Network Creation Game* introduced by Fabrikant, Luthra, Maneva, Papadimitriou, and Shenker in [20], which we abbreviate as SUM NCG. The authors analyse a very simple yet tractable model in which there are mainly two parameters: n the number of nodes that form the network and α, a function of n, representing the cost of buying exactly one link. Interestingly, it has been proved that the price of anarchy for this model is constant (asymptotically speaking) for almost every function α of n and, in fact, it has been conjectured that the PoA is constant independently of the value α [17]. This means that although, in some sense, anarchy is present when creating Internet-like networks, the resulting networks might not be far from any optimal network built by a centralised authority. In the first part of this thesis we focus our attention on this open question: what is the PoA in the SUM NCG? We study this question providing two distinct results. First, we are able to enlarge the range of the parameter α for which the PoA is known to be constant. And second, for the remaining range of the parameter α in which it is not known whether the PoA is constant, we shed light on the topology of equilibria studying a general related class of graphs known as distance-uniform graphs. As you can see in the historical overview (Section 2.4), there is a large number of related papers that have been published in the last years since the appearance of the SUM NCG in 2003 improving subsequently the upper bounds for the PoA for distinct values of α [27, 1, 17, 29, 28, 11]. This constitutes a strong evidence that the problem we study is relevant for the community.

In the second part of this thesis, we focus our attention on the proposal and study of two new original models that help us to get a better understanding of some features and properties that most Internet-like networks as well as social networks satisfy. To this end, after analysing the current state-of-the-art to identify the diverse variety of Network Creation Games existing in the literature, we propose two new models for which we discuss and analyse bounds on the PoA and properties regarding the topology of equilibria. The introduction of these models, called *celebrity games*, allows us to study how heterogeneous players (every player has some weight or relevance that might differ from one node to the other) and a critical distance (defining the radius for which the players at distance greater than this threshold penalise the player at consideration), affect the quality of equilibria with respect any optimal network.

1.3 The Articles Included in this Thesis

Recall that this work is an article-based thesis. The following list contains the four papers in which this thesis is based on:

1. Àlvarez, C., Messegué, A. "On the Price of Anarchy for High-Price Links". *Web and Internet Economics.* 316–329, 2019. **(Best paper award)** [Chapter 3].

2. M. Lavrov, P.S. Loh, A. Messegué. "Distance-Uniform Graphs with Large Diameter". *SIAM Journal on Discrete Mathematics* 33(2) 994–1005, 2019 [Chapter 4].

3. C. Àlvarez, M. J. Blesa, A. Duch, A. Messegué and M. J. Serna. "Celebrity games." *Theor. Comput. Sci.* , 648 56–71, 2016 [Chapter 5].

4. Àlvarez, C., Messegué, A. "Max Celebrity Games". *Algorithms and Models for the Web Graph - 13th International Workshop, WAW 2016, Montreal, QC, Canada, December 14-15, 2016, Proceedings*, 10088 88–99, 2016 [Chapter 6].

1.4 The Structure and Outline of this Thesis

In Chapter 2 we provide the reader the preliminaries needed to understand the analysis of Part I and Part II of the thesis as well as an extensive discussion of the state-of-the-art related to the most relevant Network Creation Games. Then the thesis is divided in two parts. The Part I is dedicated to the study of the classical model SUM NCG. In Chapter 3, we show that the PoA for the SUM NCG is constant when $\alpha > n(1+\epsilon)$, where ϵ is any positive small constant. In Chapter 4, we investigate the topology and diameter of a certain class of graphs called distance-uniform, which are related to equilibria for the SUM NCG when $\alpha < n/C$ with $C > 4$. This result provides us new information about the topology of the SUM NCG equilib ria for the same range of α.

In Part II of this thesis, we propose and study the celebrity models. These models consider the combination of two key features. The first is the assumption that the network is heterogeneous, which is modelled assuming that players have associated different relevances or weights. The second is the existence of a critical distance β, indicating that all the players that are further than β from a player u penalise to the cost function of u. Depending on how the penalties are defined with respect the weights of the players further than β, we obtain the *Sum Celebrity model* and the *Max Celebrity model*. In Chapter 5 we motivate and introduce the Sum Celebrity model and make a deep study of its main features. The most important results that we obtain are non-trivial upper bounds on the PoA in terms of some of the parameters of the game as well as some topological properties of equilibria, mainly, upper and lower bounds on the diameter of equilibria. In Chapter 6 we continue the analysis of the celebrity models introducing and studying the max celebrity model. The most important results that we obtain are analogous to the ones in Chapter 5 with some exceptions.

In order to close the work, in Chapter 7 we summarise the main results and we discuss some open questions that can be of further interest. Finally, the Bibliography contains the information that correspond to the citations appearing in the text from each chapter excluding the references made in the original articles included in Chapter 3, 4, 5 and 6.

Chapter 2

Preliminaries

In this section, we basically provide the reader the main concepts and frameworks to get a correct understanding of the forthcoming chapters. As we pointed out in the introduction, to understand the main question of this thesis we model Internet-like networks and social networks as decentralised network creation where selfish agents interact with each other pursuing their egoistic interests in such a way that a resulting network is formed. Therefore, we need to clearly understand the concept of *graph*, which is the main mathematical object that we use to work with any network (Subsection 2.1); we also review *strategic games* and related concepts, which are the abstract tool we use to model the interaction of selfish agents pursuing their egoistic interests (Subsection 2.2); we give definition of the classical Network Creation Game introduced by Fabrikant et al. in 2003 which is a particular strategic game (Subsection 2.3); and finally, we also summarise the main results of the literature related to the classical model and we examine other network creation models (Subsection 2.4).

2.1 Graphs

A *graph* is a pair $G = (V, E)$ where V is the set of vertices or nodes of G, a finite set, and E is the set of edges of G, a subset of pairs (u, v) with $u, v \in V$ and $u \neq v$. When only G is specified we will use the notation $V(G), E(G)$ to refer to the corresponding sets V, E, respectively, of G. We distinguish between *directed graph* or *undirected graph.* A graph is directed when the pairs (u, v) and (v, u) represent distinct edges. For the contrary, in an undirected graph the pairs (u, v) and (v, u) represent the same edge and we will also use the notation uv to refer to such an edge.

Let $G = (V, E)$ be a graph. If G is undirected, the *degree* of a node $u \in V$ is the cardinality of the set of edges $uv \in E$. Similarly, if G is directed, then we distinguish between the *out-degree*, the cardinality of the set of edges $(u, v) \in E$, noted as $deg^+(u)$, and the *in-degree*, the cardinality of the set of edges $(v, u) \in E$, noted as $deg^-(u)$.

A *path* between two nodes $u, v \in V(G)$ is a sequence of nodes $w_1, w_2, ..., w_k$ with $w_1 = u$, $w_k = v$ and $(w_i, w_{i+1}) \in E(G)$ for each $i < k$. We will use the notation $w_1 - ... - w_k$ to refer to such a path. Then, the *length* of the path is $k - 1$. A *minimal length path* between $u \in V(G)$ and $v \in V(G)$ is a path between u, v having the minimum length. Two nodes $u, v \in V(G)$ are said to be *connected* in G iff there exist a path between u, v. We say that an undirected graph G is *connected* iff there exists a path between any two nodes $u, v \in V(G)$.

The *undirected distance* or just the *distance* to simplify, between two connected nodes $u, v \in V(G)$ is the length of any minimal length path between u, v. If u, v are not connected then we consider that the distance between u, v is ∞. For every r and $u \in V(G)$ we consider $A_r(u)$ the set of nodes at distance r from u, that is, $A_r(u) = \{v \mid d_G(u, v) = r\}$. Similarly, for every r and $u \in V(G)$, we consider $B_r(u)$ the set of nodes at distance at most r from u, that is, $B_r(u) = \{v \mid d_G(u, v) \leq r\}$.

The *local diameter* of a node u from a graph G is the maximum distance from u to any other node from G. Then, the *diameter* of G, noted as $diam(G)$, is the maximum possible local diameter of any node from G whereas the *eccentricity* of G refers to the minimum possible local diameter of any node from G. Also, for every strictly positive integer k, the $k-$th power of an undirected graph G, noted by G^k, is the graph with same vertices as G but with edges between two nodes u, v iff $d_G(u, v) \leq k$.

A *subgraph* H from a graph G, noted as $H \subseteq G$ is a pair $(V(H), E(H))$ such that $V(H) \subseteq V(G), E(H) \subseteq E(G)$. Given a subgraph $H \subseteq G$, we extend in a natural way the notion of degree restricted to H. If G is undirected then $deg_H(u)$ for $u \in V(H)$ is the number of edges uv with $v \in V(H)$. Similarly, if G is directed, then $deg_H^+(u)$ and $deg_H^-(u)$ for $u \in V(H)$ are the number of edges (u, v) and (v, u) with $v \in V(H)$, respectively. Also, we write $deg(H)$ the *average degree* of H, whenever G is undirected, that is, the sum of all the degrees $deg_H(u)$ for $u \in V(H)$ divided over the number of nodes from H.

A *tree* is a connected graph with no cycles. In a connected graph $G = (V, E)$ a vertex is a *cut vertex* if its removal increases the number of connected components of G. A graph is biconnected if it has no cut vertices. We say that $H \subseteq G$ is a *biconnected component* of G if H is a maximal biconnected subgraph of G. More specifically, H is such that there is no other distinct biconnected subgraph of G containing H as a subgraph.

2.2 Strategic Games

A *strategic game* is defined by a tuple $\Gamma = (N, (\mathcal{S}_i)_{i \in N}, (c_i)_{i \in N})$ where:

- $N = \{1, 2, ..., n\}$ is a finite set of n players.
- Each player $i \in N$ selects one *strategy* s_i from a set of strategies S_i.
- c_i is a cost function for every agent $i \in N$.

The distinct combinations of all the possible strategies for every player define all the possible outcomes. A *strategy profile* is any such possible configuration represented as $s = (s_1, ..., s_n)$ with $s_i \in \mathcal{S}_i$ for each $i \in N$. The *strategy space*, noted with $\mathcal{S}$, is the collection of all such $n-$dimensional vector of strategies:

$$\mathcal{S} = \mathcal{S}_1 \times ... \times \mathcal{S}_n$$

Once defined the strategy space, the *cost function* c_i for player i models the interest of such player in terms of not only the current strategy of the player, but also in terms of the strategies of the other players. This is achieved assigning a real value to each possible configuration:

$$c_i : \mathcal{S} \rightarrow \mathbb{R}$$

For a strategy profile $s = (s_1, ..., s_n)$ let s_{-i} be the strategies of all the players except for player i, so that we use the convention $s = (s_{-i}, s_i)$ for every $i \in N$. This notation is useful to introduce the following definitions.

Definition 1. *Given an strategy profile s and a player $i \in N$, $s'_i \in \mathcal{S}_i$ is a* best response *relative to player i associated to s if*

$$\forall s''_i \in \mathcal{S}_i : c_i(s_{-i}, s'_i) \leq c_i(s_{-i}, s''_i)$$

That is, the best response is any choice that player would pick when playing optimally.

Now, we are interested in studying *Nash equilibria*, which are configurations in which every player or agent is happy with his current strategy and has no interest in deviating unilaterally. Such configurations are defined as follows:

Definition 2. *A strategy profile $s \in \mathcal{S}$ is a Nash equilibrium (or abbreviated* NE*) iff:*

$$\forall i \in N, \forall s'_i \in \mathcal{S}_i : c_i(s_{-i}, s'_i) \geq c_i(s)$$

Therefore, in a NE, each player has a best response as a strategy.

Now, when evaluating the quality of the communication networks we need some measure of how good a created network is for the whole society integrated by all the players or agents. The *social cost* is the concept we can use to make this measure and is defined for a strategy profile $s = (s_1, ..., s_n)$.

Definition 3. *Given a strategy profile $s = (s_1, ..., s_n)$, the social cost of s, $c(s)$, is defined as the sum of the individual costs of all the players. That is*

$$c(s) = \sum_i c_i(s)$$

Once we have this objective function measuring the quality of the created network, we refer to the *social optimum* (or abbreviated OPT) as the value of the minimum possible social cost.

Definition 4. *The social optimum of Γ or abbreviated,* $\text{OPT}(\Gamma)$*, is:*

$$\text{OPT}(\Gamma) = \min_{s \in \mathcal{S}} c(s)$$

We can think of a NE as a stable configuration obtained following the selfish interests of the players. Obviously, the social optimum value is not necessarily attained by a NE. Therefore, it is natural to consider what it is the loss, in terms of efficiency, of the worst NE in comparison with the social optimum. The so called *price of anarchy* captures precisely this concept. That is, if $\mathcal{E}$ is the set of NE strategy profiles for Γ then:

Definition 5. *The price of anarchy (or abbreviated PoA) is*

$$PoA(\Gamma) = \max_{s \in \mathcal{E}} c(s) / \min_{s \in \mathcal{S}} c(s)$$

Finally, a related concept to the price of anarchy is the *price of stability* (or abbreviated PoS).

Definition 6. *The price of stability (or abbreviated PoS) is*

$$PoS(\Gamma) = \min_{s \in \mathcal{E}} c(s) / \min_{s \in \mathcal{S}} c(s)$$

2.3 The Sum NCG

In the first part of this thessis we study the classical model called *Sum Network Creation Game* model, or SUM NCG for short. A game in this model is a particular strategic game defined by a tuple $\langle N, \alpha \rangle$ where $N = \{1, ..., n\}$ is the set of players and the parameter α models the cost of buying any individual link [20]. In this model the players interact buying links to the other players so that a network is created. Therefore, the available actions or choices for every player $i \in N$ is a subset $s_i \in \mathcal{S}_i = \mathcal{P}(N \setminus \{i\})$ representing the subset of players to which i establishes links.

The natural network of all the players is called the *communication network* or *outcome graph.* Given a strategy profile $s = (s_1, ..., s_n)$ the outcome graph $G[s]$ is defined as follows:

$$G[s] = (N, \{ij \mid i \in s_j \vee j \in s_i\})$$

Then, the cost function modelling the selfish interests of the players in the game for a strategy profile $s = (s_1, ..., s_n)$ has two components: the *creation cost* and the *usage cost*. The creation cost for a player $i \in N$ is $\alpha|s_i|$ and it quantifies the cost of buying $|s_i|$ links. In contrast, the usage cost for a player i is $\sum_{j \neq i} d_{G[s]}(i, j)$ if $G[s]$ is connected or ∞ otherwise. Therefore, the total cost incurred for player i is

$$c_i(s) = \begin{cases} \infty & \text{if } G[s] \text{ is not connected} \\ \alpha|s_i| + \sum_{j \neq i} d_{G[s]}(i, j) & \text{otherwise} \end{cases}$$

2.4 Historical Overview

Network Creation Games is an extensive area in constant growth since the appearance in 2003 of the model SUM NCG from Fabrikant et al. [20]. In this paper the authors introduce the well-known classical model to study the quality of the stable Internet-like networks obtained following the selfish interests of its agents. After this work, several authors have been improving previous results and studying variations and extensions of this classical model. Here we address, first, the progress around the SUM NCG classical model, which provides the context for the Part I of this thesis, and after this, we dive into some related models to the SUM NCG to motivate the models that we introduce and study in the Part II of this thesis.

2.4.1 The Sum NCG model

In the seminal paper [20] the authors define the NCG model, show that the problem of calculating the Best Response is NP-hard for $\alpha = 2$ and they characterise the social optima for the distinct ranges of α. They also show that PoA $= O(1)$ for $\alpha = O(1)$ and PoA ≤ 5 for tree equilibria. Finally, they state the *Tree conjecture*, saying that there exists a constant A such that for every $\alpha > A$ every NE graph G is a tree. In 2006, the conjecture was refuted by Albers et al. in [1]. In 2010 the tree conjecture is reformulated as follows: for any $\alpha > n$ every NE is a tree [29].

In the subsequent years, several authors continue to further study the model enlarging the range of the parameter α for which the PoA is constant. In the case of high-price links the property that the PoA for trees is constant has been very useful to prove that PoA is constant. In Table 1 we can see how the interval of values α for which all NE are trees has been progressively amplified and then, the PoA is constant in such intervals.

	$\alpha > 4n - 13$ – $17n$	$17n$ – $65n$	$65n$ – $273n$	$273n$ – $12n \log n$	$12n \log n$ – $10n^{3/2}$	$10n^{3/2}$ – ∞
All NE *are trees*	[11] (2018)	[7] (2017)	[28] (2015)	[29] (2013)	[1] (2006)	[27] (2003)

Table 1. Summary of the progress of showing constant PoA for high price per link (citation and year).

In the case of low-price links the relation between the PoA and the diameter is also crucial. In Table 2 we can see the distinct improvements showing constant PoA for such ranges of α.

	$\alpha = 0$ – $O(1)$	$O(1)$ – $O(\sqrt{n})$	$O(\sqrt{n})$ – $O(n^{1-\delta})$
Constant PoA	[20] (2003)	[1] (2006)	[17] (2007)

Table 2. Summary of the progress of showing constant PoA for low price per link (citation and year).

For the remaining range of the parameter α the best upper bound known on the PoA is $2^{O(\sqrt{\log n})}$ [17]. In fact, in the same paper, the authors conjecture that the PoA is constant for any α, a conjecture that we call the *Constant PoA conjecture*. This is undoubtedly one of the major open conjectures for the SUM NCG, together with the reformulated Tree conjecture, into which so many people have contributed. In Table 3 we have collected the best upper bounds known for the PoA for distinct ranges of α.

α	0	1	2	$\sqrt[3]{n/2}$	$\sqrt{n/2}$	$O(n^{1-\delta})$	$4n-13$	$12n\log n$	∞
PoA	1	$\leq \frac{4}{3}$ ([20])	≤ 4 ([17])	≤ 6 ([17])	$\Theta(1)$ ([17])	$2^{O(\sqrt{\log n})}$ ([17])	< 5 ([11])	1.5 ([1])	

Table 3. Summary of the best known bounds for the *PoA* for the SUM NCG.

This long list of incremental improvements in the history provide the evidence that the problem of showing constant PoA for a wider range of α is really tough. Furthermore, notice that the relationship between the Tree conjecture and the Constant PoA conjecture for $\alpha > n$ shows that in some situations is very important to understand the topology of equilibria to get a better understanding of the PoA.

Precisely, in Chapter 3, we keep studying topological properties of equilibria for $\alpha > n$ that allow us to show constant PoA for the range $\alpha > n(1+\epsilon)$ with $\epsilon > 0$ any constant. This is achieved by showing that any biconnected component H of any equilibrium graph G has constant size when $\alpha > n(1+\epsilon)$ and that $diam(G) < diam(H) + 250$ for $\alpha > n$. Furthermore, in Chapter 4, we provide new insights regarding the topology of equilibria for the range $\alpha < n/C$ with $C > 4$ a constant, by studying some topological properties of distance-uniform graphs.

2.4.2 Variations of the Sum NCG

In this section we review a selected set of variations and extensions of the SUM NCG that motivate the definition of the games studied in part II. Other interesting good examples can be found in [25, 2, 30, 9, 14, 16, 13]. Also, in order to get a broader overview of this subject, we address the reader to [21] and [22].

In the classical model, the general assumption that the selfish agents want to be well-connected in the resulting network buying as few links as possible, is captured by the individual cost function. Recall that the individual cost of the players can be decomposed into the creation cost, which is the cost of buying links, plus the usage cost, that corresponds to the quality of the connection to the network. This main idea is present also in most of the other models that we are going to mention, but as we shall see, distinct assumptions can lead to distinct features and behaviours.

The SUM NCG considers that link creation is unilateral, that is, that any player can buy any link to any other player without asking for permission to that player. In the *bilateral game* [15] a link is created iff both endpoints agree and then, the cost of buying that link is equally split for the two players.

Also, in the classical model all the links have the same price α so that the creation cost for a player is the number of bought links times α. A variant of having uniform priced links can be found here [18], in which it is assumed that there exists a host graph that indicates which links can be bought or not. This models the natural property that in the reality certain links cannot exist due to physical limitations.

Regarding the usage cost of a node, in the classical model it is specified by the sum of the distances to all the other nodes. This goes with the intuition that nodes that are far away from the current player penalise the player accordingly to their distance. The larger the distance is, the larger the penalisation gets for the player. Obviously, this is not the only way in which we can compute a penalisation for the players that are far from us. A good example of this is the *Max Network Creation Game* [17], the model that we obtain from the SUM NCG when we change the summation of the distances to all the nodes for the maximum of such distances.

Another variation is the *disconnected equilibria* model [12] that contemplates the existence of disconnected equilibria by introducing an extra parameter $\beta > 1$. The authors keep the same definition from the SUM NCG except for the following little modification in the usage cost. Given a player i, if player $j \neq i$ is in the same connected component we add the distance between i and j to the usage cost, like it is done in the SUM NCG. Otherwise, we add a term β to the usage cost of i. Notice that this model could be thought as if there was a distance (which in this case could be n), such that the penalisation that a player i gets for every player further than this distance, contributes in β units.

This idea of a distance or threshold that defines which players affect in a negative way the corresponding agent is considered in other models. In the *MaxBD* (and *SumBD*) games from [10] the authors introduce the concept of bounded distance R_i for every player i and do not consider any link cost α. They define the cost function for player i as the number of bought links if the maximum of all the distances in the MaxBD (or the sum of all the distances in the SumBD) to the other players is smaller than R_i, otherwise it is infinite. In this way, this model forces that in every NE, every player i has the other players close in the sense of closeness given by the parameter R_i and at the same time, player i cannot strictly reduce the number of bought links still satisfying this bounded distance condition.

Inspired by all the previous models we introduce the *Sum Celebrity Games* and the *Max Celebrity Games*. We preserve the setting in which the individual cost of a player is given by the creation cost plus the usage cost. Regarding the creation cost, in both celebrity models we consider the unilateral version with α a parameter that stands for the price per link without no further restrictions. Regarding the usage cost, in the celebrity models there is a critical distance β that defines the threshold for which the players further than this distance affect the player at consideration in a negative way. However, in all the models we have considered above the players or agents are thought to have the same relevance with respect the other players. In our celebrity games, we assume that players may have distinct relevance. The different degrees of relevance are expressed by associating different positive weights to the players. The idea is that having a large weight means that the player is more relevant in the network so the majority of the agents would like to be closer to this node. In the Sum Celebrity model we calculate the affectation of having nodes at distance further than β as the sum of their weights whereas in the max celebrity model as the maximum of their weights. Therefore, celebrity games are the first example of a model considering heterogeneous players in which the requirement of being close to a global critical distance has to be balanced against the node weight of the players. In Chapters 5 and 6 we are going to introduce in more detail these two models and study their PoA as well as non-trivial topological properties of equilibria.

Part I

On the PoA for the Classical NCG

Chapter 3

The Price of Anarchy for High-Price Links

3.1 Summary

The seminal model of Fabrikant et al. introduced in [20] initiated the study of non-centralised Internet-like networks and the efficiency of such networks with respect the social optimum via the PoA. For this model it has been conjectured that the PoA is constant and up to now this conjecture has been proved to be true for the range $\alpha = O(n^{1-\delta})$ with δ any constant with $\delta \geq 1/\log n$ [17] and for $\alpha > 4n - 13$ [11] (you can see Table 3 from Section 2.4.1 for a better description of the best known upper bounds on the PoA for the distinct ranges of α). Regarding the topology of equilibria, it has been conjectured that every NE graph is a tree for $\alpha > n$ (the tree conjecture) and this conjecture has been proved to be true for the range $\alpha > 4n - 13$ [11].

The main result of this chapter is that for any positive constant $\epsilon > 0$, the PoA is constant even for $\alpha > n(1 + \epsilon)$.

We have been able to prove this strong result in a simple and elegant way after two incremental attacks to the hard problem of bounding the PoA by a constant which can be found in the following papers:

1. *Network Creation Games: Structure vs Anarchy* [7]. We prove that the tree conjecture is true for $\alpha > 17n$ and that the PoA is constant for $\alpha > 9n$.

2. *On the Constant Price of Anarchy Conjecture* [5]. We prove that for $\alpha > n(1 + \epsilon)$ the PoA is constant, but in a more complicated way.

This result is relevant in multiple perspectives. Firstly, our contribution reinforces the conjecture that the PoA is constant for any α. Secondly, our results show new features and properties about the topology of equilibria for the range $\alpha > n(1 + \epsilon)$ that were previously unknown. In particular, we show that for $\alpha > n(1+\epsilon)$ the size of any biconnected component, if it exists, is at most a constant. Hence, this could be understood as an intermediate result towards settling the tree conjecture. Therefore, we do not only enlarge the range of the parameter α for which the PoA is known to be constant, we also provide some insights indicating that maybe we are close to validate the tree conjecture for the same range $\alpha > n(1 + \epsilon)$.

3.2 Article: On the Price of Anarchy for High-Price Links

Àlvarez, C., Messegué, A. "On the Price of Anarchy for High-Price Links". *Web and Internet Economics.* 316–329, 2019. **(Best paper award)**

On the Price of Anarchy for High-Price Links

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Abstract. We study Nash equilibria and the price of anarchy in the classic model of Network Creation Games introduced by Fabrikant, Luthra, Maneva, Papadimitriou and Shenker in 2003. This is a selfish network creation model where players correspond to nodes in a network and each of them can create links to the other $n-1$ players at a prefixed price $\alpha > 0$. The player's goal is to minimise the sum of her cost buying edges and her cost for using the resulting network. One of the main conjectures for this model states that the price of anarchy, i.e. the relative cost of the lack of coordination, is constant for all α. This conjecture has been confirmed for $\alpha = O(n^{1-\delta})$ with $\delta \geq 1/\log n$ and for $\alpha > 4n-13$. The best known upper bound on the price of anarchy for the remaining range is $2^{O(\sqrt{\log n})}$.
We give new insights into the structure of the Nash equilibria for $\alpha > n$ and we enlarge the range of the parameter α for which the price of anarchy is constant. Specifically, we prove that for any small $\epsilon > 0$, the price of anarchy is constant for $\alpha > n(1+\epsilon)$ by showing that any biconnected component of any non-trivial Nash equilibrium, if it exists, has at most a constant number of nodes.

1 Introduction

Many distinct network creation models trying to capture properties of Internet-like networks or social networks have been extensively studied in Computer Science, Economics, and Social Sciences. In these models, the players (also called nodes or agents) buy some links to other players creating in this way a network formed by their choices. Each player has a cost function that captures the need of buying few links and, at the same time, being well-connected to all the remaining nodes of the resulting network. The aim of each player is to minimise her cost following her selfish interests. A stable configuration in which every player or agent has no incentive in deviating unilaterally from her current strategy is called a *Nash equilibrium* (NE). In order to evaluate the social impact of the resulting network, the *social cost* is introduced. In this setting the social cost is defined as the sum of the individual costs of all the players. Since there is no coordination among the different players, one can expect that stable networks do not minimise the social cost. The *price of anarchy* (PoA) is a measure that quantifies how far is the worst NE (in the sense of social cost) with respect to any optimal configuration that minimises the social cost. Specifically, the PoA is defined as the ratio between the maximum social cost of NE and the social cost of the optimal configuration. If we were able to prove formally that the PoA is constant, then we could conclude that the equilibrium configurations in the selfish network creation games are so good in terms of social cost.

Since the introduction of the classical network creation game by Fabrikant et al. in [12], many efforts have been done in order to analyse the quality of the resulting equilibrium networks. The *constant PoA conjecture* is a well-known conjecture that states that the PoA is constant independently of the price of the links. In this work we provide a new understanding of the structure of the equilibrium networks for the classical network creation game [12]. We focus on the equilibria for high-price links and show that in the case that an equilibrium is not a tree, then the size of any of its biconnected components is upper bounded by a constant. This is the key ingredient to prove later that, for any small $\epsilon > 0$, the PoA is constant for $\alpha > n(1+\epsilon)$ where α is the price per link and n is the number of nodes.

Let us first define formally the model and related concepts.

1.1 Model and definitions

The *sum classic network creation game* Γ is defined by a pair $\Gamma = (V, \alpha)$ where $V = \{1, 2, ..., n\}$ denotes the set of players and $\alpha > 0$ a positive parameter. Each player $u \in V$ represents a node of an undirected graph and α represents the cost of establishing a link.

A *strategy* of a player u of Γ is a subset $s_u \subseteq V \setminus \{u\}$, the set of nodes for which player u pays for establishing a link. A strategy profile for Γ is a tuple $s = (s_1, \ldots, s_n)$ where s_u is the strategy of player u, for each player $u \in V$. Let $\mathcal{S}$ be the set of all strategy profiles of Γ. Every strategy profile s has associated a *communication network* that is defined as the undirected graph $G[s] = (V, \{uv \mid v \in s_u \vee u \in s_v\})$. Notice that uv denotes the undirected edge between u and v.

Let $d_G(u, v)$ be the distance in G between u and v. The cost associated to a player $u \in V$ in a strategy profile s is defined by $c_u(s) = \alpha|s_u| + D_{G[s]}(u)$ where $D_G(u) = \sum_{v \in V, v \neq u} d_G(u, v)$ is the sum of the distances from the player u to all the other players in G. As usual, the social cost of a strategy profile s is defined by $C(s) = \sum_{u \in V} c_u(s)$.

A Nash Equilibrium (NE) is a strategy vector s such that for every player u and every strategy vector s' differing from s in only the u component, $s_u \neq s'_u$, satisfies $c_u(s) \leq c_u(s')$. In a NE s no player has incentive to deviate individually her strategy since the cost difference $c_u(s') - c_u(s) \geq 0$. Finally, let us denote by $\mathcal{E}$ the set of all NE strategy profiles. The price of anarchy (PoA) of Γ is defined as $PoA = \max_{s \in \mathcal{E}} C(s) / \min_{s \in \mathcal{S}} C(s)$.

It is worth observing that in a NE $s = (s_1, ..., s_n)$ it never happens that $u \in s_v$ and $v \in s_u$, for any $u, v \in V$. Thus, if s is a NE, s can be seen as an orientation of the edges of $G[s]$ where an arc from u to v is placed whenever $v \in s_u$. It is clear that a NE s induces a graph $G[s]$ that we call *NE graph* and we mostly omit the reference to such strategy profile s when it is clear from context. However, notice that a graph G can have different orientations. Hence, when we say that G is a NE graph we mean that G is the outcome of a NE strategy profile s, that is, $G = G[s]$.

Given a graph G we denote by $X \subseteq G$ the subgraph of G induced by $V(X)$. In this way, given a graph $G = G[s] = (V, E)$, a node $v \in V$, and $X \subseteq G$, the *outdegree of* v in X is defined as $deg^+_X(v) = |\{u \in V(X) \mid u \in s_v\}|$, the *indegree of* v in X as $deg^-_X(v) = |\{u \in V(X) \mid v \in s_u\}|$, and, finally, the *degree of* v in X as $deg_X(v) = deg^+_X(v) + deg^-_X(v)$. Notice that $deg_X(v) = |\{u \in V(X) \mid uv \in E\}|$. Furthermore, the average degree of X is defined as $deg(X) = \sum_{v \in V(X)} deg_X(v)/|V(X)|$.

Furthermore, remind that in a connected graph $G = (V, E)$ a vertex is a *cut vertex* if its removal increases the number of connected components of G. A graph is biconnected if it has no cut vertices. We say that $H \subseteq G$ is a *biconnected component* of G if H is a maximal biconnected subgraph of G. More specifically, H is such that there is no other distinct biconnected subgraph of G containing H as a subgraph. Given a biconnected component H of G and a node $u \in V(H)$, we define $S(u)$ as the connected component containing u in the subgraph induced by the vertices $(V(G) \setminus V(H)) \cup \{u\}$. The *weight* of a node $u \in V(H)$, denoted by $|S(u)|$ is then defined as the number of nodes of $S(u)$. Notice that $S(u)$ denotes the set of all nodes v in the connected component containing u induced by $(V(G) \setminus V(H)) \cup \{u\}$ and then, every shortest path in G from v to any node $w \in V(H)$ goes through u.

In the following sections we consider G to be a NE for $\alpha > n$ and $H \subseteq G$, if it exists, a non-trivial biconnected component of G, that is, a biconnected component of G of at least three distinct nodes. Then we use the abbreviations d_G, d_H to refer to the diameter of G and the diameter of H, respectively, (although $d_G(u, v)$ denotes the distance between u, v in G), and n_H the size of H.

1.2 Historical overview

We now describe the progress around the central question of giving improved upper bounds on the PoA of the network creation games introduced by Fabrikant et al. in [12].

First of all, let us explain briefly two key results that are used to obtain better upper bounds on the PoA. The first is that the PoA for trees is at most 5 ([12]). The second one is that the PoA of any NE graph is upper bounded by its diameter plus one unit ([9]). Using these two results it can be shown that the PoA is constant for almost all values of the parameter α. Demaine et al. in [9] showed constant PoA for $\alpha = O(n^{1-\delta})$ with $\delta \geq \frac{1}{\log n}$ by proving that the diameter of equilibria is constant for the same range of α. In the view that the PoA is constant for a such a wide range of values of α, Demaine et al. in [9] conjectured that the PoA is constant for any α. This is what we call the *constant PoA conjecture*. More recently, Bilò and Lenzner in [7] demonstrated constant PoA for $\alpha > 4n - 13$ by showing that every NE is a tree for the same range of α. For the remaining range Demaine et al. in [9] determined that the PoA is upper bounded by $2^{O(\sqrt{\log n})}$.

The other important conjecture, the *tree conjecture*, stated by Fabrikant et al. in [12], still remains to be solved. The first version of the tree conjecture said that there exists a positive constant A such that every NE is a tree for $\alpha > A$. This was later refuted by Albers et al. in [3]. The reformulated tree conjecture that is believed to be true is for the range $\alpha > n$. In [20] the authors show an example of a non-tree NE for the range $\alpha = n - 3$ and then, we can deduce that the generalisation of the tree conjecture for $\alpha > n$ cannot be extended to the range $\alpha > n(1 - \delta)$ with $\delta > 0$ any small enough positive constant. Notice that the constant PoA conjecture and the tree conjecture are related in the sense that if the tree conjecture was true, then we would obtain that the PoA is constant for the range $\alpha > n$ as well.

Let us describe the progress around these two big conjectures considering first the case of large values of α and after the case of small values of α.

For *large values* of α it has been shown constant PoA for the intervals $\alpha > n^{3/2}$ [17], $\alpha > 12n \log n$ [3], $\alpha > 273n$ [19], $\alpha > 65n$ [20], $\alpha > 17n$ [1] and $\alpha > 4n - 13$ [7], by proving that every NE for each of these ranges is a tree, that is, proving that the tree conjecture holds for the corresponding range of α.

The main approach to prove the result in [19, 20, 1] is to consider a biconnected (or 2-edge-connected in [1]) component H from the NE network, and then to establish non-trivial upper and lower bounds for the average degree of H, noted as $deg(H)$. More specifically, it is shown that $deg(H) \leq f_1(n, \alpha)$ for every $\alpha \geq c_1 n$ and $deg(H) \geq f_2(n, \alpha)$ for every $\alpha \geq c_2 n$, with c_1, c_2 constants and $f_1(n, \alpha), f_2(n, \alpha)$ functions of n, α. From this it can be concluded that there cannot exist any biconnected component H for any α in the set $\{\alpha \mid f_1(n, \alpha) < f_2(n, \alpha) \wedge \alpha \geq \max(c_1, c_2)n\}$, and thus every NE is a tree for this range of α.

In [19, 20], to prove the upper bound on the term $deg(H)$ the authors basically consider a BFS tree T rooted at a node u minimising the sum of distances in H and define a *shopping vertex* as a vertex from H that has bought at least one edge of H but not of T. The authors show that every shopping vertex has bought at most one extra edge and that the distance between two distinct shopping vertices is lower bounded by a non-trivial quantity that depends on α and n. By combining these two properties the authors can give an improved upper bound on $deg(H)$ which is close to 2 from above when α is large enough in comparison to n. On the other hand, to prove a lower bound on $deg(H)$ the authors show that in H there cannot exist too many nodes of degree 2 close together.

In [1], the authors use the same upper bound as the one in [20] for the term $deg(H)$ but give an improved lower bound better than the one from [20]. To show this lower bound we introduce the concept of *coordinates* and *2-paths*. For $\alpha > 4n$, the authors prove that every minimal cycle is directed and then use this result to show that there cannot exist long $2-$paths.

In contrast, Bilò and Lenzner in [7] consider a different approach. Instead of using the technique of bounding the average degree, they introduce, for any non-trivial biconnected component H of a graph G, the concepts of *critical pair*, *strong critical pair*, and then, show that every minimal cycle for the corresponding range of α is directed. The authors play with these concepts in a clever way in order to reach the conclusion.

In a very preliminary draft [2], we take another perspective and conclude that given $\epsilon > 0$ any positive constant, the PoA is constant for $\alpha > n(1+\epsilon)$. Specifically, in [2], we prove that if the diameter of a NE graph is larger than a given positive constant, then the graph must be a tree. Such proposal represents an interesting approach to the same problem but the calculations and the proofs are very involved and hard to follow. In this work we present in a clear and elegant way the stronger result that, for the same range of α, the size of any biconnected component of any non-tree NE is upper bounded by a constant.

For *small values* of α, among the most relevant results, it has been proven that the PoA is constant for the intervals $\alpha = O(1)$ [12], $\alpha = O(\sqrt{n})$ [3, 17] and $\alpha = O(n^{1-\delta})$ with $\delta \geq 1/\log n$ [9].

The most powerful technique used in these papers is the one from Demaine et al. in [9]. They show that the PoA is constant for $\alpha = O(n^{1-\delta})$ with $\delta > 1/\log n$, by studying a specific setting where some disjoint balls of fixed radius are included inside a ball of bigger radius. Considering the deviation that consists in buying the links to the centers of the smaller balls, the player performing such deviation gets closer to a majority of the nodes by using these extra bought edges (if these balls are chosen adequately). With this approach it can be shown that the size of the balls grows in a very specific way, from which then it can be derived the upper bounds for the diameter of equilibria and thus for the PoA.

1.3 Our contribution

Let us consider a weaker version of the tree conjecture that considers the existence of biconnected components in a NE having some specific properties regarding their size.

Conjecture 1 (The biconnected component conjecture). For $\alpha > n$, any biconnected component of a non-tree NE graph has size at most a prefixed constant.

Let $\epsilon > 0$ be any positive constant. We show that the restricted version of this conjecture where $\alpha > n(1+\epsilon)$ is true (Section 5, Theorem 3). This result jointly with $d_G \leq d_H + 250$ (Theorem 1, Section 4) for $\alpha > n$, whenever H exists, imply that d_G is upper bounded by a prefixed constant, too. Recall that, the diameter of any graph plus one unit is an upper bound on the PoA and the price of anarchy for trees is constant. Hence, we can conclude that the PoA is constant for $\alpha > n(1+\epsilon)$.

In order to show these results, we introduce a new kind of sets, the A sets, satisfying some interesting properties and we adapt some well-known techniques and then, combine them together in a very original way. Let us describe the main ideas of our approach:

– Inspired by the technique considered in [9] which is used to relate the diameter of G with the size of G, we obtain an analogous relation between the diameter of H and the size of H (Section 3, Proposition 4), that can be expressed as $d_H = 2^{O(\sqrt{\log n_H})}$.
– We improve the best upper bound known on $deg(H)$ (Section 5, Theorem 2). We show this crucial result by using a different approach than the one used in the literature. We consider a node $u \in V(H)$ minimising the sum of distances and, instead of lower bounding the distance between two shopping vertices, we introduce and study a natural kind of subsets, the A sets (Section 2). Each A set corresponds to a node $v \in V(H)$ and a pair of edges e_1, e_2 where $v \in V(H)$ and $e_1, e_2 \in E(H)$ are two links bought by v. The A sets play an important role when upper bounding the cost difference of player v associated to the deviation of the same player that consists in selling e_1, e_2 and buying a link to u (Section 2, Proposition 1 and Proposition 2). By counting the cardinality of these A sets we show that the term $deg(H)$ can be upper bounded by an expression in which the terms n, α, n_H, and d_H appear (Section 2, Proposition 3). By using the relation $d_H = 2^{O(\sqrt{\log n_H})}$ we can refine the upper bound for the $deg(H)$ even more. Subsequently, we consider the technique used in [19, 20, 1], in which lower and upper bounds on the average degree of H are combined to reach a contradiction whenever H exists, i.e. whenever G is a non-tree NE graph.

2 An upper bound for $deg(H)$ in terms of the size and the diameter of H

Remind that in all the sections we consider that G is a NE of a network creation game $\Gamma = (V, \alpha)$ where $\alpha > n$. If G is not a tree then we denote by H a maximal biconnected component of G.

In this section we give an intermediate upper bound for the term $deg(H)$ that will be useful later to derive the main conclusion of this paper.

Let $u \in V(H)$ be a prefixed node and suppose that we are given $v \in V(H)$ and $e_1 = (v, v_1), e_2 = (v, v_2)$ two links bought by v. The *A set of* $v, e_1 = (v, v_1), e_2 = (v, v_2)$, noted as $A_{e_1,e_2}(v)$, is the subset of nodes $z \in V(G)$ such that every shortest path (in G) starting from z and reaching u goes through v and the predecessor of v in any such path is either v_1 or v_2.

Therefore, notice that $v \notin A_{e_1,e_2}(v)$ and the following remark always hold:

Remark 1. Let e_1, e_2, e_1', e_2' be four distinct edges such that e_1, e_2 are bought by v and e_1', e_2' are bought by v'. If $d_G(u, v) = d_G(u, v')$ then the A set of v, e_1, e_2 and the A set of v', e_1', e_2' are disjoint even if $v = v'$.

Notice that the definition of the A sets depends on $u \in V(H)$, a prefixed node. For the sake of simplicity we do not include u in the notation of the A sets. Proposition 1 and Proposition 2 are stated for any general $u \in V(H)$ but in Corollary 1 we impose that u minimises the function $D_G(\cdot)$ in H.

For any $i = 1, 2$, we define the A^i *set of* $v, e_1 = (v, v_1), e_2 = (v, v_2)$, noted as $A^i_{e_1,e_2}(v)$, the subset of nodes z from $A_{e_1,e_2}(v)$ for which there exists a shortest path (in G) starting from z and reaching u such that goes through v and the predecessor of v in such path is v_i.

With these definitions, $A_{e_1,e_2}(v) = A^1_{e_1,e_2}(v) \cup A^2_{e_1,e_2}(v)$ and $A^i_{e_1,e_2}(v) = \emptyset$ iff $d_G(u, v_i) = d_G(u, v) - 1$ or $d_G(u, v_i) = d_G(u, v)$. Furthermore, the subgraph induced by $A^i_{e_1,e_2}(v)$ is connected whenever $A^i_{e_1,e_2}(v) \neq \emptyset$.

Now, suppose that $e_1, e_2 \in E(H)$ and think about the deviation of v that consists in deleting e_i for $i = 1, 2$ and buying a link to u. Let ΔC be the corresponding cost difference and define *crossings*(X, Y) for subsets of nodes $X, Y \subseteq V(G)$ to be the set of edges xy with $x \in X$, $y \in Y$. Then we derive formulae to upper bound ΔC in the two only possible complementary cases: (i) *crossings*$(A^1_{e_1,e_2}(v), A^2_{e_1,e_2}(v)) \neq \emptyset$ and (ii) *crossings*$(A^1_{e_1,e_2}(v), A^2_{e_1,e_2}(v)) = \emptyset$.

In case (i), $A^1_{e_1,e_2}(v), A^2_{e_1,e_2}(v) \neq \emptyset$ so that the subgraphs induced by $A^1_{e_1,e_2}(v), A^2_{e_1,e_2}(v)$ are both connected. This trivially implies that the graph induced by $A_{e_1,e_2}(v) = A^1_{e_1,e_2}(v) \cup A^2_{e_1,e_2}(v)$ is connected as well. Therefore, since H is biconnected and $e_1, e_2 \in E(H)$ by hypothesis, there must exist at least one connection distinct from e_1, e_2 joining $A_{e_1,e_2}(v)$ with its complement. Taking this fact into the account we obtain the following result:

Proposition 1. *Let us assume that crossings*$(A^1_{e_1,e_2}(v), A^2_{e_1,e_2}(v)) \neq \emptyset$ *and* xy *is any connection distinct from* e_1, e_2 *between* $A_{e_1,e_2}(v)$ *and its complement, with* $x \in A_{e_1,e_2}(v)$*. Furthermore, let* l *be the distance between* v_1, v_2 *in the subgraph induced by* $A_{e_1,e_2}(v)$*. Then* ΔC*, the cost difference for player* v *associated to the deviation of the same player that consists in deleting* e_1, e_2 *and buying a link to* u*, satisfies the following inequality:*

$$\Delta C \leq -\alpha + n + D_G(u) - D_G(v) + (2d_G(v, x) + l)|A_{e_1,e_2}(v)|$$

Proof. The term $-\alpha$ is clear because we are deleting the two edges e_1, e_2 and buying a link to u. Now let us analyse the difference of the sum of distances in the deviated graph G' vs the original graph. For this purpose, suppose wlog that $x \in A^1_{e_1,e_2}(v)$ and let z be any node from G. We distinguish two cases:

(A) If $z \notin A_{e_1,e_2}(v)$ then:

(1) Starting at v, follow the connection vu.

(2) Follow a shortest path from u to z in the original graph.

In this case we have that:

$$d_{G'}(v, z) \leq 1 + d_G(u, z)$$

(B) If $z \in A_{e_1,e_2}(v)$ then there exists some i such that $z \in A^i_{e_1,e_2}(v)$. Consider the following path (see the figure below for clarifications):

(1) Starting at v, follow the connection vu, which corresponds to one unit distance.

(2) Follow a path from u to y contained in the complementary of $A_{e_1,e_2}(v)$. Since $y \notin A_{e_1,e_2}(v)$ we have that $d_G(u, y) \leq d_G(u, v) + d_G(v, x) + 1$. Therefore, in this case we count at most $d_G(u, v) + d_G(v, x) + 1$ unit distances.

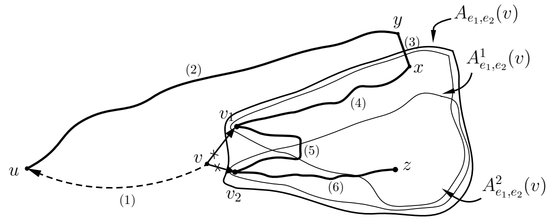

Fig. 1. The new path from z to v in the deviated graph G'

(3) Cross the connection yx, which corresponds to one unit distance.
(4) Go from x to v_1 inside $A_{e_1,e_2}(v)$ giving exactly $d_G(x,v)-1$ unit distances.
(5) Go from v_1 to v_i inside $A_{e_1,e_2}(v)$ giving at most l unit distances.
(6) Go from v_i to z inside $A_{e_1,e_2}(v)$ giving exactly $d_G(v,z)-1$ unit distances.
In this case we have that:

$$\begin{aligned} d_{G'}(v,z) \leq & \overbrace{1}^{(1)} + \overbrace{d_G(u,v)+d_G(v,x)+1}^{(2)} + \overbrace{1}^{(3)} + \overbrace{d_G(x,v)-1}^{(4)} + \overbrace{l}^{(5)} + \overbrace{d_G(v,z)-1}^{(6)} \\ =& 1+d_G(u,z)+(2d_G(v,x)+l) \end{aligned}$$

Combining the two inequalities we reach the conclusion:

$$\Delta C \leq -\alpha + \sum_{z\in V(G)} (d_{G'}(v,z)-d_G(v,z)) \leq -\alpha + n + D_G(u) - D_G(v) + (2d_G(v,x)+l)|A_{e_1,e_2}(v)|$$

□

In case (ii), we assume that $crossings(A^1_{e_1,e_2}(v), A^2_{e_1,e_2}(v)) = \emptyset$. Since H is biconnected and $e_1, e_2 \in E(H)$ by hypothesis, for each i such that $A^i_{e_1,e_2}(v) \neq \emptyset$ there must exist at least one connection distinct from e_i joining $A^i_{e_1,e_2}(v)$ with its complement. Taking this fact into the account we obtain the following result:

Proposition 2. *Let us assume that* $crossings(A^1_{e_1,e_2}(v), A^2_{e_1,e_2}(v)) = \emptyset$ *and let* $I \subseteq \{1,2\}$ *be the subset of indices* i *for which* $A^i_{e_1,e_2}(v) \neq \emptyset$*. Furthermore, suppose that for each* $i \in I$*,* x_iy_i *is any connection distinct from* e_i *between* $A^i_{e_1,e_2}(v)$ *and its complement, with* $x_i \in A^i_{e_1,e_2}(v)$*. Then* ΔC*, the cost difference of player* v *associated to the deviation of the same player that consists in deleting* e_1, e_2 *and buying a link to* u*, satisfies the following inequality:*

$$\Delta C \leq -\alpha + n + D_G(u) - D_G(v) + \max(0, 2\max_{i\in I} d_G(v,x_i))|A_{e_1,e_2}(v)|$$

Proof. The term $-\alpha$ is clear because we are deleting e_1, e_2 and buying a link to u. Now let us analyse the difference of the sum of distances in the deviated graph G' vs the original graph. To this purpose, let z be any node from G. We distinguish two cases:
(A) If $z \notin A_{e_1,e_2}(v)$ then:
(1) Starting at v, follow the connection vu.
(2) Follow a shortest path from u to z in the original graph.
In this case we have that:

$$d_{G'}(v,z) \leq 1 + d_G(u,z)$$

(B) If $z \in A_{e_1,e_2}(v)$ then there exists some i such that $z \in A^i_{e_1,e_2}(v)$. Consider the following path:
(1) Starting at v, follow the connection vu, which corresponds to one unit distance.

(2) Follow a path from u to y contained in the complementary of $A_{e_1,e_2}(v)$. Since $y \notin A_{e_1,e_2}(v)$ we have that $d_G(u,y) \leq d_G(u,v) + d_G(v,x_i) + 1$. Therefore, in this case we count at most $d_G(u,v) + d_G(v,x_i) + 1$ unit distances.

(3) Cross the connection $y_i x_i$, which corresponds to one unit distance.

(4) Go from x_i to v_i giving exactly $d_G(x_i,v) - 1$ unit distances.

(5) Go from v_i to z giving exactly $d_G(v,z) - 1$ unit distances.

In this case we have that:

$$\begin{aligned} d_{G'}(v,z) \leq & \overbrace{1}^{(1)} + \overbrace{d_G(u,v) + d_G(v,x) + 1}^{(2)} + \overbrace{1}^{(3)} + \overbrace{d_G(x_i,v) - 1}^{(4)} + \overbrace{d_G(v,z) - 1}^{(5)} \\ =& 1 + d_G(u,z) + 2d_G(v,x_i) \end{aligned}$$

Combining the two inequalities we reach the conclusion:

$$\Delta C \leq -\alpha + \sum_{z \in V(G)} (d_{G'}(v,z) - d_G(v,z)) \leq -\alpha + n + D_G(u) - D_G(v) + \max(0, 2\max_{i \in I} d_G(v,x_i))|A_{e_1,e_2}(v)|$$

□

Now, notice the following simple fact:

Remark 2. If $z_1, z_2 \in V(H)$ then any shortest path from z_1 to z_2 is contained in H. This is because otherwise, using the definition of cut vertex, any such path would visit two times the same cut vertex thus contradicting the definition of shortest path. Therefore, if $z_1, z_2 \in V(H)$ then $d_G(z_1,z_2) = d_H(z_1,z_2) \leq d_H$.

Combining the formulae from Proposition 1 and Proposition 2 together with this last remark, we can obtain a lower bound for the cardinality of any A set of v, e_1, e_2 when u satisfies a very specific constraint:

Corollary 1. *If $u \in V(H)$ is such that $D_G(u) = \min_{z \in V(H)} \{D_G(z)\}$, then $|A_{e_1,e_2}(v)| \geq \frac{\alpha - n}{4d_H}$*

Proof. Let us analyse the properties that are fulfilled for the distinct elements in this setting:

First, u minimises the sum of distances on $V(H)$. Therefore, $D_G(u) - D_G(v) \leq 0$ for any $v \in V(H)$.

Now, let xy be any crossing between $A_{e_1,e_2}(v)$ and its complement with x in $A_{e_1,e_2}(v)$ and y in the complementary of $A_{e_1,e_2}(v)$. Consider also x', y' be the nodes from $V(H)$ such that $x \in S(x')$ and $y \in S(y')$. If $z \in V(H)$, then either $S(z)$ is a subset of $A_{e_1,e_2}(v)$, if $z \in A_{e_1,e_2}(v)$, or $S(z)$ is a subset of the complementary of $A_{e_1,e_2}(v)$ otherwise, by the definition of the A sets and by the definition of cut vertex. Therefore, $S(x')$ is a subset of $A_{e_1,e_2}(v)$ and $S(y')$ is a subset of the complementary of $A_{e_1,e_2}(v)$. Furthermore, by the definition of biconnected component, any crossing or connection between $S(z_1)$ and $S(z_2)$ with $z_1, z_2 \in V(H)$ and $z_1 \neq z_2$, if it exists, must definitely be $z_1 z_2$. Therefore, $x = x', y = y'$ and as a result $x, y \in V(H)$. Then by Remark 2, the distance from x to v is at most d_H. In a similar way, it can be deduced that if $x_i y_i$ is any crossing between $A^i_{e_1,e_2}(v)$ and its complement, then $x_i, y_i \in V(H)$ and therefore, the distance from x_i to v is at most d_H. As a conclusion, both expressions $d_G(v,x)$ and $d_G(v,x_i)$, appearing in the formulae from Proposition 1 and Proposition 2, respectively, are at most d_H.

Moreover, whenever $e_1, e_2 \in E(H)$ and $crossings(A^1_{e_1,e_2}(v), A^2_{e_1,e_2}(v)) \neq \emptyset$, any shortest path connecting v_1 and v_2 inside $A_{e_1,e_2}(v)$ is contained in H and has length at most $2d_H$. This implies that the expression l appearing in the formula from Proposition 1 is at most $2d_H$.

With all these results, we deduce that, the expressions multiplying $|A_{e_1,e_2}(v)|$ in the rightmost term of the two inequalities from Proposition 1 ($2d_G(v,x) + l$) and Proposition 2 ($\max(0, 2\max_{i \in I} d_G(v,x_i))$) can be upper bounded by $4d_H$.

Imposing that G is a NE then we obtain the conclusion.

□

Now we use this last formula to give an upper bound for the average degree of H. Recall that we are working in the range $\alpha > n$:

Proposition 3.

$$deg(H) \leq 2 + \frac{16 d_H (d_H + 1) n}{n_H (\alpha - n)}$$

Proof. For any node $v \in V(H)$ let $Z(v)$ be any maximal set of distinct and mutually disjoint pairs of edges from H bought by v. Let X be defined as the set of tuples $(\{e_1, e_2\}, v)$ with $v \in V(H)$ and $\{e_1, e_2\}$ a pair of edges from $Z(v)$. Now define $S = \sum_{(\{e_1,e_2\},v)\in X} |A_{e_1,e_2}(v)|$. On the one hand, using Corollary 1:

$$S \geq \frac{\alpha - n}{4 d_H} |X|$$

On the other hand, for each distance index i, let S_i be the sum of the cardinalities of the A sets for all the tuples $(\{e_1, e_2\}, v) \in X$ with $d_G(u, v) = i$. By Remark 1, $S_i \leq n$. Therefore:

$$|X| \frac{\alpha - n}{4 d_H} \leq S = S_0 + ... + S_{d_H} \leq n(d_H + 1)$$

Next, notice that there are exactly $\lfloor \frac{deg_H^+(v)}{2} \rfloor$ pairs in $Z(v)$ for each v considered. Furthermore, $\lfloor \frac{deg_H^+(v)}{2} \rfloor = deg_H^+(v)/2$ if $deg_H^+(v)$ is even and $\lfloor \frac{deg_H^+(v)}{2} \rfloor = (deg_H^+(v) - 1)/2$ otherwise. Hence:

$$|X| \geq \sum_{v \in V(H)} \frac{deg_H^+(v) - 1}{2} = \frac{|E(H)| - |V(H)|}{2}$$

Finally:

$$deg(H) = \frac{2|E(H)|}{|V(H)|} \leq 2 + \frac{4|X|}{|V(H)|} \leq 2 + \frac{16(d_H + 1) n d_H}{n_H (\alpha - n)}$$

□

3 The diameter of H vs the number of nodes of H

In this section we establish a relationship between the diameter and the number of the vertices of H which allows us to refine the upper bound for the term $deg(H)$ using the main result of the previous subsection.

We start extending the technique introduced by Demaine et al in [9]. Instead of reasoning in a general G, we focus our attention to the nodes from H reaching an analogous result. Since for $\alpha > 4n - 13$ every NE is a tree it is enough if we study the case $\alpha < 4n$.

For any integer value k and $u \in V(H)$ we define $N_{k,H}(u) = \{v \in V(H) \mid d_G(u, v) \leq k\}$, the set of nodes from $V(H)$ at distance at most k from u. With this definition in mind then $S_k(u) = \cup_{v \in N_{k,H}(u)} S(v)$ is the set of all nodes inside $S(v)$ for all $v \in V(H)$ at distance at most k from u. In other words, $S_k(u)$ is the set of all nodes z such that the first cut vertex that one finds when following any shortest path from z to u is at distance at most k from u.

Furthermore, for any integer k we define $m_k = \min_{u \in V(H)} |N_{k,H}(u)|$. That is, m_k is the minimum cardinality that any k-neighbourhood in H can have.

Lemma 1. *Let H be a biconnected component of G. For any integer $k \geq 0$, either there exists a node $u \in V(H)$ such that $|S_{4k+1}(u)| > n/2$ or, otherwise, $m_{5k+1} \geq m_k k/4$.*

Proof. If there is a vertex $u \in V(H)$ with $|S_{4k+1}(u)| > n/2$, then the claim is obvious. Otherwise, for every vertex $u \in V(H)$, $|S_{4k+1}(u)| \leq n/2$. Let u be any node from $V(H)$ minimising the cardinality of the balls of radius $5k+1$ intersected with $V(H)$. That is, u is any node from $V(H)$ with $|N_{5k+1,H}(u)| = m_{5k+1}$. Let $Z = \{v_1, ..., v_l\}$ be any maximal set of nodes from $V(H)$ at distance $4k+1$ from u (in H) with the property that every two distinct nodes $v_i, v_j \in Z$, we have that $d_G(v_i, v_j) \geq 2k+1$ (see the left picture from Figure 2 for a visual clarification).

Now, consider the deviation of u that consists in buying the links to every node from Z and let G' be the new graph resulting from such deviation. Let $z \in S(w)$ with $w \in V(H)$ and $d_G(w, u) \geq 4k+1$ and consider any shortest path (in H) from w to u. Let w_π be the node from any such shortest path at distance $4k+1$ from u. By the maximality of Z there exists at least one node $v_w \in Z$ for which $d_G(v_w, w_\pi) \leq 2k$. The original distance between z and u is $d_G(z, u) = d_G(z, w) + d_G(w, u)$. In contrast, the distance between z and u in G' satisfies the following inequality (see the right picture from Figure 2 for a visual clarification):

$$d_{G'}(z, u) \leq 1 + d_G(v_w, w_\pi) + d_G(w_\pi, w) + d_G(w, z)$$

$$\leq 1 + 2k + (d_G(u, w) - (4k+1)) + d_G(w, z) = -2k + d_G(u, w) + d_G(w, z)$$

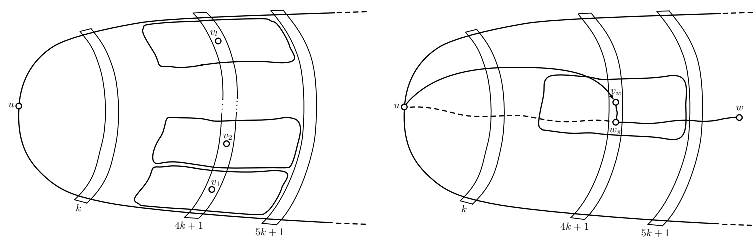

Fig. 2. The setting of nodes from the proof (left) and the alternative path from w to u in the deviated graph (right)

Therefore, $d_G(z, u) - d_{G'}(z, u) \geq 2k$. Since we are assuming that $|S_{4k+1}(u)| \leq n/2$ then this means that $\sum_{\{v \in V(H) | d_G(v,u) > 4k+1\}} |S(v)| \geq n/2$, that is, the sum of the weights of the nodes from H at distance strictly greater than $4k+1$ from u is greater than or equal $n/2$. Then ΔC, the cost difference for u associated to such deviation, satisfies:

$$\Delta C \leq l\alpha - 2k\left(\frac{n}{2}\right) \leq 4nl - kn$$

Since G is a NE then from this we conclude that $l \geq k/4$.

Finally, notice that the distance between two nodes in Z is at least $2k+1$ implying that the set of all the balls of radius k with centers at the nodes from Z are mutually disjoint. Therefore, $m_{5k+1} = |N_{5k+1,H}(u)| \geq l m_k \geq m_k k/4$.

□

Lemma 2. *If $r < d_H/4 - 4$ then $|S_r(u)| \leq n/2$ for every node $u \in V(H)$.*

Proof. Suppose the contrary and we reach a contradiction, that is, suppose that there exists some $u \in V(H)$ with $|S_r(u)| > n/2$ and $r < d_H/4 - 4$. Let $t \in V(H)$ be any node at distance $d_H/2$ from u, which always exists. We consider the deviation of t that consists in buying a link to u and we define G' to be the new graph resulting from such deviation. Let $z \in S_r(u)$ with $w \in V(H)$ such that $z \in S(w)$. The distance between t and w in G is at least $d_H/2 - r$ so the distance between t and z in G is at least $d_H/2 - r + d_G(w,z)$. In contrast, the distance between t and w in G' is at most $1+r$, so the distance between t and z in G' is at most $1 + r + d_G(w,z)$. Therefore:

$$d_G(z,t) - d_{G'}(z,t) \geq d_H/2 - 2r - 1 > d_H/2 - 2(d_H/4 - 4) - 1 = 7$$

Then $d_G(z,t) - d_{G'}(z,t) \geq 8$ and thus ΔC, the cost difference of t associated to such deviation, satisfies:

$$\Delta C \leq \alpha - 8|S_r(u)| \leq 4n - 8|S_r(u)| < 4n - \frac{8}{2}n = 0$$

A contradiction with the fact that G is a NE.

□

Combining these results we are able to give an extension of the result from Demaine et al in [9]:

Proposition 4. $d_H < 5^{\sqrt{2\log_5 n_H}+5}$.

Proof. Consider the following sequence of numbers $(a_i)_{i\geq 0}$ defined in the following way:
(i) $a_0 = 21$.
(ii) $a_{i+1} = 5a_i + 1$ for $i \geq 0$.

It is easy to check that $a_k = 21 \cdot 5^k + \frac{5^k - 1}{5-1}$ for any $k \geq 0$ so that $22 \cdot 5^k > a_k \geq 21 \cdot 5^k$. With this definition and using the two previous results we reach the conclusion that whenever $4a_i + 1 < d_H/4 - 4$, then $|S_{4a_i+1}(u)| \leq n/2$ for all $u \in V(H)$, by Lemma 2, implying $m_{a_{i+1}} \geq m_{a_i}\frac{a_i}{4}$, by Lemma 1. Iterating the recurrence relation we can see that whenever $i \geq 0$ and $4a_i + 1 < d_H/4 - 4$, then:

$$m_{a_{i+1}} \geq \frac{a_i a_{i-1} ... a_1 a_0}{4^{i+1}} m_{a_0}$$

Since $a_0 = 21$ then $m_{a_0} \geq 21$. Therefore:

$$m_{a_{i+1}} \geq 21 \left(\frac{21}{4}\right)^{i+1} 5^{i+(i-1)+...+1+0} > 5^{i^2/2}$$

Now, consider the value k such that $4a_k + 1 < d_H/4 - 4 \leq 4a_{k+1} + 1$. On the one hand, $n_H \geq m_{a_{k+1}} > 5^{k^2/2}$ so this implies that $k \leq \sqrt{2\log_5 n_H}$. On the other hand, $d_H/4 \leq 4a_{k+1} + 5 < 22 \cdot 4 \cdot 5^{k+1}$. Therefore, $d_H < 5^{k+5} \leq 5^{\sqrt{2\log_5 n_H}+5}$, as we wanted to see.

□

4 The diameter of G vs the diameter of H.

In this section we establish a relationship between the diameter of G and the diameter of H when $\alpha > n$. Since for $\alpha > 4n - 13$ every NE is a tree it is enough if we study the case $n < \alpha < 4n$.

We show that in this case, the distance between any pair $w, z \in V(G)$ where $z \in S(w)$, is upper bounded by 125 from where we can conclude that $d_G < d_H + 250$. To obtain these results we basically exploit the fact that G is a NE graph together with key topological properties of biconnected components:

Proposition 5. *Let $w \in V(H)$ and $z \in S(w)$ maximising the distance to w. Then $d_G(z,w) < 125$.*

Proof. Let Z be the subgraph of G induced by $S(w)$ and W the subgraph of G induced by w together with the set of nodes $V(G) \setminus S(w)$. Then, define $r = d_G(z,w) = \max_{t\in V(Z)} d_G(w,t)$, $s = \max_{t\in V(W)} d_G(w,t)$ (see the figure below for clarifications). With these definitions it is enough to show that $r < 125$. Notice that, for instance, if $S(w) = \{w\}$ then the result trivially holds.

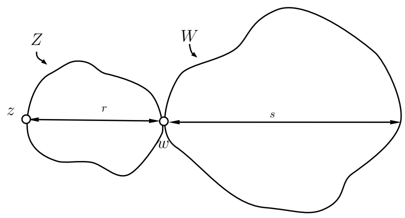

Fig. 3. The most important subsets, nodes and distances from the setting.

First, let us see that $\min(r, s) \leq 8$.

Let v any node maximising the distance to w in W and ΔC_1 and ΔC_2 the corresponding cost differences of players z and v, respectively, associated to the deviations of the same players that consist in buying a link to w. Then:

$$\Delta C_1 \leq \alpha - |V(W)|(r-1)$$

$$\Delta C_2 \leq \alpha - |V(Z)|(s-1)$$

Adding up the two inequalities and using that $\alpha < 4n$:

$$\Delta C_1 + \Delta C_2 \leq 2\alpha - (\min(r,s)-1)(|V(Z)| + |V(W)|) < 8n - (\min(r,s)-1)n$$

Since G is a NE graph then $\Delta C_1 + \Delta C_2 \geq 0$ and from here we deduce that $\min(r, s) \leq 8$, as we wanted to see.

If $r \leq 8$ then we are done. Therefore we must address the case $s \leq 8$.

Next, since H is a non-trivial biconnected component, there exist nodes $t, t' \in V(H)$ such that they are adjacent in H, t has bought the link $e = (t, t')$ and one of the two following cases happen: either (i) t is at distance 1 from w, t' is at distance 1 or 2 from w or (ii) t' is at distance 1 from w and t at distance 2 from w (see the figure below for a clarification).

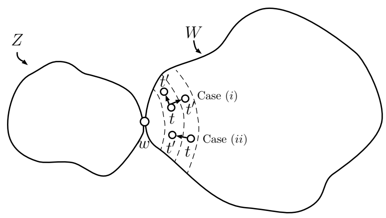

Fig. 4. An image depicting the setting for case (i) and case (ii).

In case (i) we deduce that $|S(w)| = |V(Z)| \leq n\frac{4s-2}{4s-1} \leq n\frac{30}{31}$. This is because of the following reasoning. Let ΔC_{delete} be the corresponding cost difference of player t associated to the deviation of the same player that consists in deleting the edge e. Since H is biconnected then there exists a loop going through e and contained in H of length at most $4s+1$. Notice that when deleting e, t only increases the distances maybe to the nodes from $V(W) \setminus \{w\}$ but not to the nodes from $V(Z)$ by at most $4s-1$ distance units. Therefore:

$$\Delta C_{delete} \leq -\alpha + (4s-1)(n - |V(Z)|) < -n + (4s-1)(n - |V(Z)|)$$

Since G is a NE graph then $\Delta C_{delete} \geq 0$ and from here, using the hypothesis $s \leq 8$, we deduce the conclusion:

$$|V(Z)| < \frac{-n + n(4s-1)}{4s-1} = n\frac{4s-2}{4s-1} \leq \frac{30}{31}n$$

In case (ii) we deduce that $|S(w)| = |V(Z)| \leq n/2$. This is because of the following reasoning. Let ΔC_{swap} be the corresponding cost difference of player t associated to the deviation of the same player that consists in swapping the edge e for the link (t, w). Notice that when performing such swap, t only increases the distances maybe to the nodes from $V(W) \setminus \{w\}$ but strictly decreases for sure, one unit distance to all the nodes from $V(Z)$. Therefore:

$$\Delta C_{swap} \leq -|V(Z)| + (n - |V(Z)|) \leq n - 2|V(Z)|$$

Since G is a NE graph then $\Delta C_{swap} \geq 0$ and from here we deduce the conclusion $|V(Z)| \leq n/2$. Hence, we have obtained that either $|S(w)| \leq \frac{30}{31}n$, in case (i), or $|S(w)| \leq \frac{n}{2}$, in case (ii).

Finally, consider the deviation of z that consists in buying the link to w. Then the corresponding cost difference ΔC_{buy} satisfies the following inequality:

$$\Delta C_{buy} \leq \alpha - (r-1)(n - |S(w)|) < 4n - (r-1)(n - |S(w)|)$$

Since G is a NE graph, then $\Delta C_{buy} \geq 0$ so that we conclude that $r < \frac{4n}{n - |S(w)|} + 1$. Using this property we conclude that $r < 125$ in case (i) and $r \leq 8$ in case (ii), so we are done.

□

As a consequence:

Theorem 1. $d_G < d_H + 250$.

5 Combining the results

Finally, in this section we combine the distinct results obtained so far to prove the main conclusion.

On the one hand, combining Proposition 3 with Proposition 4 we reach the following result for the average degree of H:

Theorem 2.

$$deg(H) < 2 + \frac{16n}{\alpha - n}\frac{5^{2\sqrt{2\log_5 n_H}+10}}{n_H}$$

On the other hand, recall that from Lemma 4 and Lemma 2 from [19] and [20], respectively, the general lower bound $deg(H) \geq 2 + \frac{1}{16}$ that works for any α can be obtained.

With these results in mind we are now ready to prove the following strong result:

Theorem 3. *Let $\epsilon > 0$ be any positive constant and $\alpha > n(1+\epsilon)$. There exists a constant K_ϵ such that every biconnected component H from any non-tree Nash equilibrium G has size at most K_ϵ.*

Proof. Let G be any non-tree NE graph. Then there exists at least one biconnected component H. By Theorem 2 when $\alpha > n(1+\epsilon)$ we have that $deg(H) < 2 + \frac{16}{\epsilon}\frac{5^{2\sqrt{2\log_5 n_H}+10}}{n_H}$. On the other hand, we know that for any α, $deg(H) \geq 2 + \frac{1}{16}$. Then this implies that there exists a constant K_ϵ upper bounding the size of H, otherwise we would obtain a contradiction comparing the asymptotic behaviour of the upper and lower bounds obtained for $deg(H)$ in terms of n_H.

□

In other words, the biconnected component conjecture holds for $\alpha > n(1+\epsilon)$.

Furthermore, recall that it is well-known that the diameter of any graph plus one unit is an upper bound for the PoA and the PoA for trees is constant. Therefore, we conclude that:

Theorem 4. *Let $\epsilon > 0$ be any positive constant. The price of anarchy is constant for $\alpha > n(1+\epsilon)$.*

Proof. Let G be a NE. If G is a tree we are done, because the PoA for trees is at most 5. Therefore to prove the result consider that G is a non-tree configuration. Then, G has at least one non-trivial biconnected component H. On the one hand, by Theorem 3, there exists a constant K_ϵ that upper bounds the size of H. This implies that $d_H \leq n_H \leq K_\epsilon$. On the other hand, by Theorem 1, $d_G \leq d_H + 250$. In this way, $d_G \leq K_\epsilon + 250$ and since $K_\epsilon + 250$ is a constant, then the conclusion follows because the PoA is upper bounded by the diameter plus one unit.

□

6 The conclusions

The most relevant contribution we have made in this article is to show that the price of anarchy is constant for $\alpha > n(1+\epsilon)$. We have not been able to prove the tree conjecture for $\alpha > n$ by showing that there cannot exist any non-trivial biconnected component H for the same range of α. Instead, we have proved that for $\alpha > n(1+\epsilon)$, if H exists, then it has a constant number of nodes. This property implies constant PoA for the same range of α. The technique we have used relies mostly on the improved upper bound on the term $deg(H)$ for $\alpha > n$. However, as in [19, 20], our refined upper bound still depends on the term $n/(\alpha - n)$, that tends to infinity when α approaches n from above. This makes us think that either our technique can be improved even more to obtain the conclusion that the tree conjecture claims or it might be that there exist some non-tree equilibria when α approaches n from above.

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Chapter 4

The Diameter of Distance Uniform Graphs

4.1 Summary

In the previous chapter we have shown that in the SUM NCG, for any positive constant $\epsilon > 0$ the PoA is constant for $\alpha > n(1+\epsilon)$. In order to investigate what happens in the range of α in which we do not still know whether the PoA is constant, we study the diameter of distance-uniform graphs which are closely related to equilibria in the SUM NCG for the range $\alpha < n/C$ with $C > 4$ a constant. In the following we explain briefly how this class of graphs were introduced and why they are relevant to the network creation games.

In [2] Alon et al. introduce a new network creation game, the *sum basic network creation game*, with no parameter α and in which every node wishes to minimise its average distance to all the other nodes. An equilibrium for this model is called a *sum basic equilibrium graph* and it is defined as an undirected graph G for which every edge uv and every node w, swapping the edge uv with the edge uw does not strictly decrease the total sum of distances from u to all other nodes.

In the same work, the authors study the diameter of such sum basic equilibria. To this end, they introduce the so called *distance-uniform graphs*. Specifically, given any ϵ we say that a graph G is ϵ*-distance-uniform* (and $\epsilon-$*distance-almost-uniform*) iff there exists a distance index d, which is called the *critical distance*, such that for every node $u \in V(G)$ the size of any subset of nodes at distance d (and d or $d+1$) is at least $n(1-\epsilon)$, where n is the size of G.

An interesting result from [2], Theorem 13, is that high-diameter sum basic equilibrium graphs are related to distance-uniform graphs. More precisely, every sum basic equilibrium graph G with n nodes, $n \geq 24$ and diameter $d > 2\log n$ *induces* an $\epsilon-$distance-almost-uniform graph G' with n vertices and diameter $\Theta(\epsilon d/\log n)$ and an $\epsilon-$distance-uniform graph G'' with n vertices and diameter $\Theta(\epsilon d/\log^2 n)$. When the authors say that G *induces* a graph G' they mean that G' is the xth power of G, $G' = G^x$ for some $x \geq 1$.

Using the simple relationship that exists between the diameter of the kth power of G ($k \geq 1$) with respect the diameter of G, $diam(G^k) = \lceil diam(G)/k \rceil$, then, upper bounding the diameter of distance-uniform graphs allows us to upper bound the diameter of sum basic equilibrium graphs. With this in mind, the authors propose the following conjecture for distance-uniform graphs that would imply poly-logarithmic diameter for equilibria for the sum basic network creation game:

Conjecture 14 [2]. *Distance-almost-uniform graphs have diameter* $O(\log n)$

In [7], we show that for any constant $C > 4$ and $\alpha < n/C$, the 4th power of any NE in the SUM NCG is an $\epsilon-$distance-almost-uniform graph. If Conjecture 14 was true then, using the relationship between the diameter and the PoA in the SUM NCG, we would have that, for any constant $C > 4$ and $\alpha < n/C$, PoA $= O(\log n)$, improving in this way the best upper bound known of $2^{O(\sqrt{\log n})}$.

Motivated by all these considerations we study the diameter of distance-uniform graphs. It is not hard to see that the diameter and critical distance of distance-almost-uniform graphs are closely related. Let G be an $\epsilon-$distance-almost-uniform graph and let d be its critical distance. Let us see that if $d < diam(G)/2-1$ then $\epsilon \geq 1/2$. If $u, v \in V(G)$ such that $d_G(u,v) = diam(G)$ then, $(A_d(u) \cup A_{d+1}(u)) \cap (A_d(v) \cup A_{d+1}(v)) = \emptyset$ and therefore, $n > |A_d(u)| + |A_{d+1}(u)| + |A_d(v)| + |A_{d+1}(v)|$. Since G is $\epsilon-$distance-almost-uniform we have that $|A_d(u)| + |A_{d+1}(u)| + |A_d(v)| + |A_{d+1}(v)| \geq 2n(1-\epsilon)$. Hence, we can conclude that if $\epsilon < 1/2$ then the critical distance d satisfies the relation $d \geq diam(G)/2 - 1$.

Therefore, since we are mostly interested in the asymptotics, we study the critical distance as a function of n, ϵ, instead of studying directly the diameter. Specifically, my main contribution in the paper presented in this chapter shows that in any ϵ-distance-almost-uniform graph with n nodes, the critical distance d satisfies $d = 2^{O\left(\log n/\log(\epsilon^{-1})\right)}$. Furthermore, in the same article, it is also proved that for every ϵ with $\frac{1}{n} \leq \epsilon \leq \frac{1}{\log n}$ there exist ϵ-distance-uniform graphs with n nodes, such that the critical distance d satisfies $d = 2^{\Omega\left(\log n/\log(\epsilon^{-1})\right)}$.

Therefore, Conjecture 14 is false and thus, we cannot conclude that PoA $= O(\log n)$ for the SUM NCG when $\alpha < n/C$ with $C > 4$, at least, going through this direction. However, we have discovered interesting topological properties satisfied by distance-uniform graphs that can help us to better understand the topology of equilibria for our SUM NCG when $\alpha < n/C$ with $C > 4$.

4.2 Article: Distance-Uniform Graphs with Large Diameter

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Distance-Uniform Graphs with Large Diameter

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August 5, 2018

Abstract

An ϵ-distance-uniform graph is one with a critical distance d such that from every vertex, all but at most an ϵ-fraction of the remaining vertices are at distance either d or $d+1$. Motivated by the theory of network creation games, Alon, Demaine, Hajiaghayi, and Leighton made the following conjecture of independent interest: that every ϵ-distance-uniform graph (and, in fact, a broader class of ϵ-distance-almost-uniform graphs) has critical distance at most logarithmic in the number of vertices n. We disprove this conjecture, and characterize the asymptotics of this extremal problem. Specifically, for $\frac{1}{n} \leq \epsilon \leq \frac{1}{\log n}$, we construct ϵ-distance-uniform graphs with critical distance $2^{\Omega(\frac{\log n}{\log \epsilon^{-1}})}$. We also prove an upper bound on the critical distance of the form $2^{O(\frac{\log n}{\log \epsilon^{-1}})}$ for all ϵ and n. Our lower bound construction introduces a novel method inspired by the Tower of Hanoi puzzle, and may itself be of independent interest.

1 Introduction

Much research has focused on combinatorial objects featuring some form of regularity or uniformity. These have produced vast bodies of literature ranging across error-correcting codes, strongly regular graphs, graphs from algebraic constructions, and probabilistic constructions. The constructions themselves often challenge human creativity, as the search space of discrete systems is exponential.

This paper studies a more generous notion of uniformity, which was introduced by Alon, Demaine, Hajiaghayi, and Leighton in [2]. There, they were motivated by the analysis of network creation games, but raised this as a deeper, purely graph-theoretic problem of independent interest. We say that an n-vertex graph is *ϵ-distance-uniform* for some parameter $\epsilon > 0$ if there is a value d, called the *critical distance*, such that, for every vertex v, all but at most ϵn of the other vertices are at distance exactly d from v. Distance-uniform graphs exist for some, but not all, possible triplets (n, ϵ, d); a trivial example is the complete graph K_n, which is distance-uniform with $\epsilon = \frac{1}{n}$ and $d = 1$. So it is natural to try to characterize which triplets (n, ϵ, d) are realizable as distance-uniform graphs.

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More generally, we consider ϵ-distance-almost-uniform graphs. Here, the definition is relaxed to only require that all but ϵn of the other vertices are at distance exactly d or exactly $d+1$ from v. All distance-uniform graphs are distance-almost-uniform, but distance-almost-uniform graphs are occasionally easier to construct. We prove results for distance-almost-uniform graphs in cases where this is more general.

1.1 From network creation games to distance uniformity

The concept of ϵ-distance-uniformity is interesting from multiple perspectives, both theoretical and applied. In the applied domain, it turns out that equilibria in a certain network creation game can be used to construct distance-uniform graphs. As a result, understanding distance-uniform graphs tells us which equilibria are possible.

The use of the Internet has been growing significantly in the last few decades. This has motivated interest in *network creation games*: models that try to simulate the creation of Internet-like networks. These models can be thought as games in which players (located at different nodes in a network) interact selfishly with each other trying to connect the network. These players have a cost function that they wish to minimize: they want to be well-connected to other nodes, but pay a price for adding edges to the network. Depending on the specific cost functions for the players and the set of possible actions (usually every player can alter the network only by deleting some edges and/or buying new edges incident to him/her), we obtain distinct models; see [4, 5, 8, 9, 10, 11] for some examples.

The central elements of interest studied in network creation games are the concepts of *Nash equilibrium* (NE) and the *Price of anarchy* (PoA). A NE is a configuration from which no player is willing to deviate unilaterally, provided that the other players keep their own positions without any change. As one can expect, equilibria may not necessarily coincide with *optimal networks* (configurations having the minimal possible sum of individual costs), because equilibria are stable networks obtained by following the selfish interests of the players. The PoA is a ratio that quantifies the lack of coordination from equilibria built following the selfish interests of the players versus any optimal network that could be built in a centralised way looking for maximal social welfare. A high PoA implies that the worst NE in terms of the sum of the individual costs of the players is far more costly than any optimal network, whereas a low PoA implies that all equilibria are not that far from optimal networks.

One of these models, the *sum classic network creation game*, is the model introduced by Fabrikant et al. in [11] in which every player pays a constant price α for every edge he/she buys in order to be connected to the resulting network. The cost function for this game is α times the number of edges bought by each player (creation cost) plus the sum of distances to the other players (usage cost). Although it can be seen as a simple model, this is in fact the seminal model of the network creation games and has been studied largely along time. One of the main still-unsettled conjectures in the field of network creation games is that the price of anarchy for the sum classic network creation game is bounded by a constant independent of α. As it can be seen in the literature [1, 9, 6, 13, 14], this is a hard problem, and there still exists an interval where it is not known if the conjecture is true or not. In [9] it was shown that the largest diameter of any NE plus one unit serves as an upper bound for the PoA. This result transforms the problem of upper bounding the PoA to the problem of upper bounding the (largest) diameter of equilibria. This is why the study of the diameter of equilibria for the sum classic network creation game is so important.

On the other hand, in [2], Alon et al. propose a simpler model, the *sum basic network creation game* that avoids the use of the parameter α but still gets at the heart of the sum classic network creation game. In the sum basic network creation game, players do not create edges, but can swap one incident edge for another. The notion of *swap equilibrium* is the analogous concept in this game to the NE defined above. Specifically, a swap equilibrium is a network for which no player can strictly decrease the average distance to the other players when swapping any of its incident edges for another incident edge. As a hope to better understand the PoA for the classic network creation game, in view of the relationship previously explained between the diameter of equilibria and the PoA, Alon et al. study the diameter of the sum basic network creation game and shows an upper bound of $2^{O(\sqrt{\log n})}$ (Theorem 9 from [2]). In the same article, they conjecture that ϵ-distance-almost-uniform graphs have logarithmic diameter (Conjecture 14 from [2]). This last conjecture together with an interesting relationship between swap equilibria and ϵ-distance-almost-uniform graphs (Theorem 13 from [2]) would allow to improve the bound for the diameter of swap equilibria to be at most polylogarithmic in n.

However, as we refute this conjecture, a direct improvement on the diameter of equilibria for the sum basic network creation game cannot be obtained, at least going in this direction. Therefore, we cannot improve the PoA for the sum classic network creation game, either. On the other hand, a recent result from Alvarez et al. [3] shows that every fourth power of a equilibrium for the sum classic network creation game for an appropriate range of the parameter α is an ϵ-distance-almost-uniform graph. These relationships make us believe that ϵ-distance-almost-uniform graphs play a very important role in network creation games. As it seems, they are at the core of the difficulty of the classic problem of upper bounding the PoA, and this is why we think that this family of graphs deserve a detailed study.

1.2 Preliminary notions

Before we go on to describe previously known results about distance-uniform graphs and the progress we have made, we define several useful notions in graph theory.

For a vertex v of a graph G, let $\Gamma_i(v)$ denote the set $\{w \in V(G) \mid d(v,w) = i\}$: the vertices at distance exactly i from v. In particular, $\Gamma_0(v) = \{v\}$ and $\Gamma_1(v)$ is the set of all vertices adjacent to v. Let

$$N_j(v) = \bigcup_{i=0}^{j} \Gamma_i(v)$$

denote the set of vertices within distance at most j from v.

For a vertex v of a graph G, a *breadth-first search tree* rooted at v is a spanning tree T constructed in order of increasing distance from v. We begin by setting T to the subgraph consisting only of v itself. Once all the vertices of $\Gamma_i(v)$ have been added to t for some i, we "explore" them in arbitrary order. When we explore $w \in \Gamma_i(v)$, we add all of w's neighbors in $\Gamma_{i+1}(v)$ to T if they were not in T already, as well as the edges joining them to w. When this process is complete, the resulting tree T has the property that every vertex in $\Gamma_i(v)$ is also at distance exactly i from v in T, not just in G. We refer to the sets $\Gamma_r(v)$ as *layers* of the breadth-first search tree T.

1.3 Previous results on distance uniformity

The application to network creation games motivates the already natural question: in an ϵ-distance-uniform graph with n vertices and critical distance d, what is the relationship between the parameters ϵ, n, and d? Specifically, can we derive an upper bound on d in terms of ϵ and n? Up to a constant factor, this is equivalent to finding an upper bound on the diameter of the graph, which must be between d and $2d$ as long as $\epsilon < \frac{1}{2}$.

Random graphs provide one example of distance-uniform graphs. In [7], Bollobás shows that for sufficiently large $p = p(n)$, the diameter of the random Erdős–Rényi random graph $\mathcal{G}_{n,p}$ is asymptotically almost surely concentrated on one of two values. In fact, from every vertex v in $\mathcal{G}_{n,p}$, a breadth-first search tree expands by a factor of $O(np)$ at every layer, reaching all or almost all vertices after about $\log_{np} n$ steps. Such a graph is also expected to be distance-uniform: the biggest layer of the breadth-first search tree will be much bigger than all previous layers.

More precisely, suppose that we choose $p(n)$ so that the average degree $r = (n-1)p$ satisfies two criteria: that $r \gg (\log n)^3$, and that for some d, $r^d/n - 2\log n$ approaches a constant C as $n \to \infty$. Then it follows from Lemma 3 in [7] that (with probability $1 - o(1)$) for every vertex v in $\mathcal{G}_{n,p}$, the number of vertices at each distance $k < d$ from v is $O(r^k)$. It follows from Theorem 6 in [7] that the number of vertex pairs in $\mathcal{G}_{n,p}$ at distance $d+1$ from each other follows a Poisson distribution with mean $\frac{1}{2}e^{-C}$, so there are only $O(1)$ such pairs with probability $1-o(1)$. As a result, such a random graph is ϵ-distance-uniform with $\epsilon = O(\frac{\log n}{r})$, and critical distance $d = \log_r n + O(1)$.

This example provides a compelling image of what distance-uniform graphs look like: if the breadth-first search tree from each vertex grows at the same constant rate, then most other vertices will be reached in the same step. In any graph that is distance-uniform for a similar reason, the critical distance d will be at most logarithmic in n. In fact, Alon et al. [2] conjecture that all distance-almost-uniform graphs have diameter $O(\log n)$.

Alon et al. prove an upper bound of $O(\frac{\log n}{\log \epsilon^{-1}})$ in a special case: for ϵ-distance-uniform graphs with $\epsilon < \frac{1}{4}$ that are Cayley graphs of Abelian groups. In this case, if G is the Cayley graph of an Abelian group A with respect to a generating set S, one form of Plünnecke's inequality (see, e.g., [15]) says that the sequence

$$|\underbrace{S + S + \cdots + S}_{k}|^{1/k}$$

is decreasing in k. Since $S, S+S, S+S+S, \dots$ are precisely the sets of vertices which can be reached by $1, 2, 3, \dots$ steps from 0, this inequality quantifies the idea of constant-rate growth in the breadth-first search tree; Theorem 15 in [2] makes this argument formal.

1.4 Our results

In this paper, we disprove Alon et al.'s conjecture by constructing distance-uniform graphs that do not share this behavior, and whose diameter is exponentially larger than these examples. We also prove an upper bound on the critical distance (and diameter) showing our construction to be best possible in one asymptotic sense. Specifically, we show the following two results:

Theorem 1.1. *In any ϵ-distance-almost-uniform graph with n vertices, the critical distance d satisfies*

$$d = 2^{O\left(\frac{\log n}{\log \epsilon^{-1}}\right)}.$$

Theorem 1.2. *For any ϵ and n with $\frac{1}{n} \le \epsilon \le \frac{1}{\log n}$, there exists an ϵ-distance-uniform graph on n vertices with critical distance*

$$d = 2^{\Omega\left(\frac{\log n}{\log \epsilon^{-1}}\right)}.$$

Note that, since a $\frac{1}{\log n}$-distance-uniform graph is also $\frac{1}{2}$-distance-uniform, Theorem 1.2 also provides a lower bound of $d = 2^{\Omega(\frac{\log n}{\log \log n})}$ for any $\epsilon > \frac{1}{\log n}$.

Combined, these results prove that the maximum critical distance is $2^{\Theta(\frac{\log n}{\log \epsilon^{-1}})}$ whenever they both apply. A small gap remains for sufficiently large ϵ: for example when ϵ is constant as $n \to \infty$. In this case, Theorem 1.1 gives an upper bound on d which is polynomial in n, while the lower bound of Theorem 1.2 grows slower than any polynomial.

The family of graphs used to prove Theorem 1.2 is interesting in its own right. We give two different interpretations of the underlying structure of these graphs. First, we describe a combinatorial game, generalizing the well-known Tower of Hanoi puzzle, whose transition graph is ϵ-distance-uniform and has large diameter. Second, we give a geometric interpretation, under which each graph in the family is the 1-skeleton of the convex hull of an arrangement of points on a high-dimensional sphere.

2 Upper bound

Before proceeding to the proof of Theorem 1.1, we begin with a simple argument that is effective for an ϵ which is very small:

Lemma 2.1. *The minimum degree $\delta(G)$ of an ϵ-distance-uniform graph G satisfies $\delta(G) \ge \epsilon^{-1} - 1$.*

Proof. Suppose that G is ϵ-distance-uniform, n is the number of vertices of G, and d is the critical distance: for any vertex v, at least $(1-\epsilon)n$ vertices of G are at distance exactly d from v.

Let v be an arbitrary vertex of G, and fix an arbitrary breadth-first search tree T, rooted at v. We define the *score* of a vertex w (relative to T) to be the number of vertices at distance d from v which are descendants of w in the tree T.

There are at least $(1-\epsilon)n$ vertices at distance d from v, and all of them are descendants of some vertex in the neighborhood $\Gamma_1(v)$. Therefore the total score of all vertices in $\Gamma_1(v)$ is at least $(1-\epsilon)n$.

On the other hand, if $w \in \Gamma_1(v)$, each vertex counted in the score of w is at distance $d-1$ from w. Since at least $(1-\epsilon)n$ vertices are at distance d from w, at most ϵn vertices are at distance $d-1$, and therefore the score of w is at most ϵn.

In order for $|\Gamma_1(v)|$ scores of at most ϵn to sum to at least $(1-\epsilon)n$, $|\Gamma_1(v)|$ must be at least $\frac{(1-\epsilon)n}{\epsilon n} = \epsilon^{-1} - 1$. □

This lemma is enough to show that in a $\frac{1}{\sqrt{n}}$-distance-uniform graph, the critical distance is at most 2. Choose a vertex v: all but $\sqrt{n}$ of the vertices of G are at the critical distance d from v, and $\sqrt{n}-1$ of the vertices are at distance 1 from v by Lemma 2.1. The remaining uncounted vertex is v itself. It is impossible to have $d \geq 3$, as that would leave no vertices at distance 2 from v.

We can prove a similar result for ϵ-distance-almost-uniform graphs. For the remainder of this section, we will deal with ϵ-distance-almost-uniform graphs, in order to prove a more general result.

Lemma 2.2. *In an ϵ-distance-almost-uniform graph G with critical distance $d \geq 2$, for any vertex v, $|\Gamma_2(v)| \geq \epsilon^{-1} - 1$.*

Proof. Suppose that G is ϵ-distance-almost-uniform, n is the number of vertices of G, and d is the critical distance: for any vertex v, at least $(1-\epsilon)n$ vertices of G are at distance d or $d+1$ from v.

As before, let v be an arbitrary vertex of G, and fix an arbitrary breadth-first search tree T, rooted at v. This time, we define the *score* of a vertex w (relative to T) to be the number of vertices at distance d or $d+1$ from v which are descendants of w in the tree T.

There are at least $(1-\epsilon)n$ vertices at distance d or $d+1$ from v, and all of them are descendants of some vertex in the neighborhood $\Gamma_2(v)$. Therefore the total score of all vertices in $\Gamma_2(v)$ is at least $(1-\epsilon)n$.

On the other hand, if $w \in \Gamma_2(v)$, each vertex counted in the score of w is at distance $d-1$ or $d-2$ from w. Since at least $(1-\epsilon)n$ vertices are at distance d or $d+1$ from w, at most ϵn vertices are at distance $d-1$ or $d-2$, and therefore the score of w is at most ϵn.

In order for $|\Gamma_2(v)|$ scores of at most ϵn to sum to at least $(1-\epsilon)n$, $|\Gamma_2(v)|$ must be at least $\frac{(1-\epsilon)n}{\epsilon n} = \epsilon^{-1} - 1$. □

For larger ϵ, the bound of Lemma 2.2 becomes ineffective, but we can improve it by a more general argument of which Lemma 2.2 is just a special case.

Lemma 2.3. *Let G be an ϵ-distance-almost-uniform graph with critical distance d. Suppose that for some integers N and r with $2r+2 \leq d$, we have $|N_r(v)| \geq N$ for each $v \in V(G)$. Then we have $|N_{3r+2}(v)| \geq N\epsilon^{-1}$ for each $v \in V(G)$.*

Proof. Let v be any vertex of G, and let $\{w_1, w_2, \ldots, w_t\}$ be a maximal collection of vertices in $\Gamma_{2r+2}(v)$ such that $d(w_i, w_j) \geq 2r+1$ for each $i \neq j$ with $1 \leq i, j \leq t$.

We claim that for each vertex $u \in \Gamma_d(v) \cup \Gamma_{d+1}(v)$—for each vertex u at distance d or $d+1$ from v—there is some i with $1 \leq i \leq t$ such that $u \in N_{d-1}(w_i)$. To see this, consider any shortest path from v to u, and let $u_\pi \in \Gamma_{2r+2}(v)$ be the $(2r+2)^{\text{th}}$ vertex along this path. (Here we use the assumption that $2r+2 \leq d$.) From the maximality of $\{w_1, w_2, \ldots, w_t\}$, it follows that $d(w_i, u_\pi) \leq 2r$ for some i with $1 \leq i \leq t$. But then,

$$d(w_i, u) \leq d(w_i, u_\pi) + d(u_\pi, u) \leq 2r + (d+1-(2r+2)) = d-1.$$

So $u \in N_{d-1}(w_i)$.

To state this claim differently, the sets $N_{d-1}(w_1), \ldots, N_{d-1}(w_t)$ together cover $\Gamma_d(v) \cup \Gamma_{d+1}(v)$. These sets are all small while the set they cover is large, so there must be many of them:

$$(1-\epsilon)n \leq |\Gamma_d(v)| + |\Gamma_{d+1}(v)| \leq \sum_{i=1}^{t} |N_{d-1}(w_i)| \leq \sum_{i=1}^{t} \epsilon n = t\epsilon n,$$

which implies that $t \geq \frac{(1-\epsilon)n}{\epsilon n} = \epsilon^{-1} - 1$.

The vertices $v, w_1, w_2, \ldots, w_t$ are each at distance at least $2r+1$ from each other, so the sets $N_r(v), N_r(w_1), \ldots, N_r(w_t)$ are disjoint.

By the hypothesis of this lemma, each of these sets has size at least N, and we have shown that there are at least ϵ^{-1} sets. So their union has size at least $N\epsilon^{-1}$. Their union is contained in $N_{3r+2}(v)$, so we have $|N_{3r+2}(v)| \geq N\epsilon^{-1}$, as desired. □

We are now ready to prove Theorem 1.1. The strategy is to realize that the lower bounds on $|N_r(v)|$, which we get from Lemma 2.3, are also lower bounds on n, the number of vertices in the graph. By applying Lemma 2.3 iteratively for as long as we can, we can get a lower bound on n in terms of ϵ and d, which translates into an upper bound on d in terms of ϵ and n.

More precisely, set $r_1 = 2$ and $r_k = 3r_{k-1} + 2$, a recurrence which has closed-form solution $r_k = 3^k - 1$. Lemma 2.2 tells us that in an ϵ-distance-almost-uniform graph G with critical distance d, $N_{r_1}(v) \geq \epsilon^{-1}$. Lemma 2.3 is the inductive step: if, for all v, $N_{r_k}(v) \geq \epsilon^{-k}$, then $N_{r_{k+1}}(v) \geq \epsilon^{-(k+1)}$, as long as $2r_k + 2 \leq d$.

The largest k for which $2r_k + 2 = 2 \cdot 3^k \leq d$ is $k = \left\lfloor \log_3 \frac{d}{2} \right\rfloor$. So we can inductively prove that

$$n \geq N_{r_{k+1}}(v) \geq \epsilon^{-(\lfloor \log_3 \frac{d}{2} \rfloor + 1)}$$

which can be rearranged to get

$$\frac{\log n}{\log \epsilon^{-1}} - 1 \geq \left\lfloor \log_3 \frac{d}{2} \right\rfloor.$$

This implies that

$$d \leq 2 \cdot 3^{\frac{\log n}{\log \epsilon^{-1}}} = 2^{O\left(\frac{\log n}{\log \epsilon^{-1}}\right)},$$

proving Theorem 1.1.

3 Lower bound

To show that this bound on d is tight, we need to construct an ϵ-distance-uniform graph with a large critical distance d. We do this by defining a puzzle game whose state graph has this property.

3.1 The Hanoi game

We define a *Hanoi state* to be a finite sequence of nonnegative integers $\vec{x} = (x_1, x_2, \dots, x_k)$ such that, for all $i > 1$, $x_i \neq x_{i-1}$. Let

$$\mathcal{H}_{r,k} = \left\{\vec{x} \in \{0, 1, \dots, r\}^k : \vec{x} \text{ is a Hanoi state}\right\}.$$

For convenience, we also define a *proper Hanoi state* to be a Hanoi state $\vec{x}$ with $x_1 \neq 0$, and $\mathcal{H}^*_{r,k} \subset \mathcal{H}_{r,k}$ to be the set of all proper Hanoi states. While everything we prove will be equally true for Hanoi states and proper Hanoi states, it is more convenient to work with $\mathcal{H}^*_{r,k}$, because $|\mathcal{H}^*_{r,k}| = r^k$: there are r possibilities for x_1, since $x_1 \neq 0$, and r possibilities for each x_i with $i > 1$, since $x_i \neq x_{i-1}$.

In the *Hanoi game on* $\mathcal{H}_{r,k}$, an initial state $\vec{a} \in \mathcal{H}_{r,k}$ and a final state $\vec{b} \in \mathcal{H}_{r,k}$ are chosen. The state $\vec{a}$ must be transformed into $\vec{b}$ via a sequence of moves of two types:

1. An *adjustment* of $\vec{x} \in \mathcal{H}_{r,k}$ changes x_k to any value in $\{0, 1, \dots, r\}$ other than x_{k-1}. For example, $(1, 2, 3, 4)$ can be changed to $(1, 2, 3, 0)$ or $(1, 2, 3, 5)$, but not $(1, 2, 3, 3)$.
2. An *involution* of $\vec{x} \in \mathcal{H}_{r,k}$ finds the longest tail segment of $\vec{x}$ on which the values x_k and x_{k-1} alternate, and swaps x_k with x_{k-1} in that segment. For example, $(1, 2, 3, 4)$ can be changed to $(1, 2, 4, 3)$, or $(1, 2, 1, 2)$ to $(2, 1, 2, 1)$.

We define the Hanoi game on $\mathcal{H}^*_{r,k}$ in the same way, but with the added requirement that all states involved should be proper Hanoi states. This means that involutions (or, in the case of $k = 1$, adjustments) that would change x_1 to 0 are forbidden.

The name "Hanoi game" is justified because its structure is similar to the structure of the classical Tower of Hanoi puzzle. In fact, though we have no need to prove this, the Hanoi game on $\mathcal{H}^*_{3,k}$ is isomorphic to a Tower of Hanoi puzzle with k disks.

It is well-known that the k-disk Tower of Hanoi puzzle can be solved in $2^k - 1$ moves, moving a stack of k disks from one peg to another. In [12], a stronger statement is shown: only $2^k - 1$ moves are required to go from any initial state to any final state. A similar result holds for the Hanoi game on $\mathcal{H}_{r,k}$:

Lemma 3.1. *The Hanoi game on $\mathcal{H}_{r,k}$ (or $\mathcal{H}^*_{r,k}$) can be solved in at most $2^k - 1$ moves for any initial state $\vec{a}$ and final state $\vec{b}$.*

Proof. We induct on k to show the following stronger statement: for any initial state $\vec{a}$ and final state $\vec{b}$, a solution of length at most $2^k - 1$ exists for which any intermediate state $\vec{x}$ has $x_1 = a_1$ or $x_1 = b_1$. This auxiliary condition also means that if $\vec{a}, \vec{b} \in \mathcal{H}^*_{r,k}$, all intermediate states will also stay in $\mathcal{H}^*_{r,k}$.

When $k = 1$, a single adjustment suffices to change $\vec{a}$ to $\vec{b}$, which satisfies the auxiliary condition.

For $k > 1$, there are two possibilities when changing $\vec{a}$ to $\vec{b}$:

- If $a_1 = b_1$, then consider the Hanoi game on $\mathcal{H}_{r,k-1}$ with initial state $(a_2, a_3, \dots, a_k)$ and final state $(b_2, b_3, \dots, b_k)$. By the inductive hypothesis, a solution using at most $2^{k-1} - 1$ moves exists.

Apply the same sequence of adjustments and involutions in $\mathcal{H}_{r,k}$ to the initial state $\vec{a}$. This has the effect of changing the last $k-1$ entries of $\vec{a}$ to $(b_2, b_3, \ldots, b_k)$. To check that we have obtained $\vec{b}$, we need to verify that the first entry is left unchanged.

The auxiliary condition of the inductive hypothesis tells us that all intermediate states have $x_2 = a_2$ or $x_2 = b_2$. Any move that leaves x_2 unchanged also leaves x_1 unchanged. A move that changes x_2 must be an involution swapping the values a_2 and b_2; however, $x_1 = a_1 \neq a_2$, and $x_1 = b_1 \neq b_2$, so such an involution also leaves x_1 unchanged.

Finally, the new auxiliary condition is satisfied, since we have $x_1 = a_1 = b_1$ for all intermediate states.

- If $a_1 \neq b_1$, begin by taking $2^{k-1} - 1$ moves to change $\vec{a}$ to $(a_1, b_1, a_1, b_1, \ldots)$ while satisfying the auxiliary condition, as in the first case.

  An involution takes this state to $(b_1, a_1, b_1, a_1, \ldots)$; this continues to satisfy the auxiliary condition.

  Finally, $2^{k-1} - 1$ more moves change this state to $\vec{b}$, as in the first case, for a total of $2^k - 1$ moves. □

If we obtain the same results as in the standard Tower of Hanoi puzzle, why use the more complicated game in the first place? The reason is that in the classical problem, we cannot guarantee that any starting state would have a final state $2^k - 1$ moves away. With the rules we define, as long as the parameters are chosen judiciously, each state $\vec{a} \in \mathcal{H}_{r,k}$ is part of many pairs $(\vec{a}, \vec{b})$ for which the Hanoi game requires $2^k - 1$ moves to solve.

The following lemma almost certainly does not characterize such pairs, but provides a simple sufficient condition that is strong enough for our purposes.

Lemma 3.2. *The Hanoi game on $\mathcal{H}_{r,k}$ (or $\mathcal{H}^*_{r,k}$) requires exactly $2^k - 1$ moves to solve if $\vec{a}$ and $\vec{b}$ are chosen with disjoint support: that is, $a_i \neq b_j$ for all i and j.*

Proof. Since Lemma 3.1 proved an upper bound of $2^k - 1$ for all pairs $(\vec{a}, \vec{b})$, we only need to prove a lower bound in this case.

Once again, we induct on k. When $k = 1$, a single move is necessary to change $\vec{a}$ to $\vec{b}$ if $\vec{a} \neq \vec{b}$, verifying the base case.

Consider a pair $\vec{a}, \vec{b} \in \mathcal{H}_{r,k}$ with disjoint support, for $k > 1$. Moreover, assume that $\vec{a}$ and $\vec{b}$ are chosen so that, of all pairs with disjoint support, $\vec{a}$ and $\vec{b}$ require the least number of moves to solve the Hanoi game. Since we are proving a lower bound on the number of moves necessary, this assumption is made without loss of generality: if even $\vec{a}$ and $\vec{b}$ require at least $2^k - 1$ moves, so too must any other pair of states in $\mathcal{H}_{r,k}$ with disjoint support.

In a shortest path from $\vec{a}$ to $\vec{b}$, every other move is an adjustment: if there were two consecutive adjustments, the first adjustment could be skipped, and if there were two consecutive involutions, they would cancel out and both could be omitted. Moreover, the first move is an adjustment: if we began with an involution, then the involution of $\vec{a}$ would be a state closer to $\vec{b}$ yet still with disjoint support to $\vec{b}$, contrary to our initial assumption. By the same argument, the last move must be an adjustment.

Given a state $\vec{x} \in \mathcal{H}_{r,k}$, let its *abbreviation* be $\vec{x}' = (x_1, x_2, \ldots, x_{k-1}) \in \mathcal{H}_{r,k-1}$. An adjustment of $\vec{x}$ has no effect on $\vec{x}'$, since only x_k is changed. If $x_k \neq x_{k-2}$, then an involution of $\vec{x}$ is an adjustment of $\vec{x}'$, changing its last entry x_{k-1} to x_k. Finally, if $x_k = x_{k-2}$, then an involution of $\vec{x}$ is also an involution of $\vec{x}'$.

Therefore, if we take a shortest path from $\vec{a}$ to $\vec{b}$, omit all adjustments, and then abbreviate all states, we obtain a solution to the Hanoi game on $\mathcal{H}_{r,k-1}$ that takes $\vec{a}'$ to $\vec{b}'$. By the inductive hypothesis, this solution contains at least $2^{k-1} - 1$ moves, since $\vec{a}'$ and $\vec{b}'$ have disjoint support. Therefore the shortest path from $\vec{a}$ to $\vec{b}$ contains at least $2^{k-1} - 1$ involutions. Since the first, last, and every other move is an adjustment, there must be 2^{k-1} adjustments as well, for a total of $2^k - 1$ moves. □

Now let the *Hanoi graph* $G^*_{r,k}$ be the graph with vertex set $\mathcal{H}^*_{r,k}$ and edges joining each state to all the states that can be obtained from it by a single move. Since an adjustment can be reversed by another adjustment, and an involution is its own inverse, $G^*_{r,k}$ is an undirected graph.

For any state $\vec{a} \in \mathcal{H}^*_{r,k}$, there are at least $(r-k)^k$ other states with disjoint support to $\vec{a}$, out of $|\mathcal{H}^*_{r,k}| = r^k$ other states, forming a $\left(1 - \frac{k}{r}\right)^k > 1 - \frac{k^2}{r}$ fraction of all the states. By Lemma 3.2, each such state $\vec{b}$ is at distance $2^k - 1$ from $\vec{a}$ in the graph $G^*_{r,k}$, so $G^*_{r,k}$ is ϵ-distance uniform with $\epsilon = \frac{k^2}{r}$, $n = r^k$ vertices, and critical distance $d = 2^k - 1$.

Having established the graph-theoretic properties of $G^*_{r,k}$, we now prove Theorem 1.2 by analyzing the asymptotic relationship between these parameters.

Proof of Theorem 1.2. Begin by assuming that $n = 2^{2^m}$ for some m. In the monotonically decreasing sequence

$$\frac{2^{2m}}{2^{2^0}}, \frac{2^{2(m-1)}}{2^{2^1}}, \ldots, \frac{2^{2(m-i)}}{2^{2^i}}, \ldots, \frac{2^0}{2^{2^m}},$$

the first ($i = 0$) term is $2^{2m} > 1 \geq \epsilon$, while the last ($i = m$) term is exactly $\frac{1}{n} \leq \epsilon$. So let a be the least value of i for which $\frac{2^{2(m-i)}}{2^{2^i}} \leq \epsilon$, and let $b = m - i$; then we have $a + b = m$ and

$$\frac{2^{2b}}{2^{2^a}} \leq \epsilon < \frac{2^{2(b+1)}}{2^{2^{a-1}}}.$$

Setting $r = 2^{2^a}$ and $k = 2^b$, the Hanoi graph $G^*_{r,k}$ has n vertices and is ϵ-distance uniform, since $\frac{k^2}{r} \leq \epsilon$. Moreover, our choice of a and b guarantees that $\epsilon < \frac{4k^2}{\sqrt{r}}$, or $\log \epsilon^{-1} \geq \frac{1}{2}\log r - 2\log 2k$. Since $n = r^k$, $\log n = k \log r$, so

$$\log \epsilon^{-1} \geq \frac{1}{2k}\log n - 2\log 2k.$$

We show that $k \geq \frac{\log n}{6 \log \epsilon^{-1}}$. Since $\epsilon \leq \frac{1}{\log n}$, this is automatically true if $k \geq \frac{\log n}{6 \log\log n}$, so assume that $k < \frac{\log n}{6\log\log n}$. Then

$$\frac{1}{3k}\log n > 2\log\log n > 2\log 2k,$$

so

$$\log \epsilon^{-1} \geq \frac{1}{2k} \log n - 2 \log 2k > \frac{1}{2k} \log n - \frac{1}{3k} \log n = \frac{1}{6k} \log n,$$

which gives us the desired inequality $k \geq \frac{\log n}{6 \log \epsilon^{-1}}$. The Hanoi graph $G^*_{r,k}$ has critical distance $d = 2^k - 1 = 2^{\Omega(\frac{\log n}{\log \epsilon^{-1}})}$, so the proof is finished in the case that n has the form 2^{2^m} for some m.

For a general n, we can choose m such that $2^{2^m} \leq n < 2^{2^{m+1}} = \left(2^{2^m}\right)^2$, which means in particular that $2^{2^m} \geq \sqrt{n}$. If $\epsilon < \frac{2}{\sqrt{n}}$, then the requirement of a critical distance of $2^{\Omega(\frac{\log n}{\log \epsilon^{-1}})}$ is only a constant lower bound, and we may take the graph K_n. Otherwise, by the preceding argument, there is a $\frac{\epsilon}{2}$-distance-uniform Hanoi graph with 2^{2^m} vertices; its critical distance d satisfies

$$d \geq 2^{\Omega\left(\frac{\log \sqrt{n}}{\log(\epsilon/2)^{-1}}\right)} = 2^{\Omega\left(\frac{\log n}{\log \epsilon^{-1}}\right)}.$$

To extend this to an n-vertex graph, take the blow-up of the 2^{2^m}-vertex Hanoi graph, replacing every vertex by either $\lfloor n/2^{2^m} \rfloor$ or $\lceil n/2^{2^m} \rceil$ copies.

Whenever v and w were at distance d in the original graph, the copies of v and w will be at distance d in the blow-up. The difference between floor and ceiling may slightly ruin distance uniformity, but the graph started out $\frac{\epsilon}{2}$-distance-uniform, and $\lceil n/2^{2^m} \rceil$ differs from $\lfloor n/2^{2^m} \rfloor$ at most by a factor of 2. Even in the worst case, where for some vertex v the $\frac{\epsilon}{2}$-fraction of vertices not at distance d from v all receive the larger number of copies, the resulting n-vertex graph will be ϵ-distance-uniform. □

3.2 Points on a sphere

In this section, we identify $G_{r,k}$, the graph of the Hanoi game on $\mathcal{H}_{r,k}$, with a graph that arises from a geometric construction.

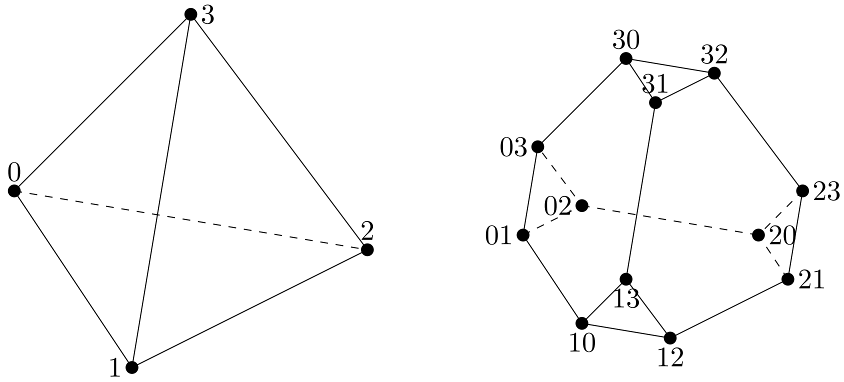
Fix a dimension r. We begin by placing $r+1$ points on the r-dimensional unit sphere arbitrarily in general position (though, for the sake of symmetry, we may place them at the vertices of an equilateral r-simplex). We identify these points with a graph by taking the 1-skeleton of their convex hull. In this starting configuration, we simply get K_{r+1}.

Next, we define a truncation operation on a set of points on the r-sphere. Let $\delta > 0$ be sufficiently small that a sphere of radius $1-\delta$, concentric with the unit sphere, intersects each edge of the 1-skeleton in two points. The set of these intersection points is the new arrangement of points obtained by the truncation; they all lie on the smaller sphere, and for convenience, we may scale them so that they are once again on the unit sphere. An example of this is shown in Figure 1.

Proposition 3.1. *Starting with a set of $r+1$ points on the r-dimensional sphere and applying k truncations produces a set of points such that the 1-skeleton of their convex hull is isomorphic to the graph $G_{r,k}$.*

Proof. We induct on k. When $k = 1$, the graph we get is K_{r+1}, which is isomorphic to $G_{r,1}$.

From the geometric side, we add an auxiliary statement to the induction hypothesis: given points p, q_1, q_2 such that, in the associated graph, p is adjacent to both q_1 and q_2, there is a 2-dimensional



(a) A tetrahedron (b) A truncated tetrahedron

Figure 1: An example of truncation

face of the convex hull containing all three points. This is easily verified for $k = 1$: in the r-simplex, any three vertices are contained in a two dimensional face.

Assuming that the induction hypotheses are true for $k-1$, fix an isomorphism of $G_{r,k-1}$ with the set of points after $k-1$ truncations, and label the points with the corresponding vertices of $G_{r,k-1}$. We claim that the graph produced after one more truncation has the following structure:

1. A vertex that we may label $(\vec{x}, \vec{y})$ for every ordered pair of adjacent vertices of $G_{r,k-1}$.
2. An edge between $(\vec{x}, \vec{y})$ and $(\vec{y}, \vec{x})$.
3. An edge between $(\vec{x}, \vec{y})$ and $(\vec{x}, \vec{z})$ whenever both are vertices of the new graph.
4. No other edges: the graph is r-regular.

The first claim is immediate from the definition of truncation: we obtain two vertices from the edge between $\vec{x}$ and $\vec{y}$. We choose to give the name $(\vec{x}, \vec{y})$ to the vertex closer to $\vec{x}$. The edge between $\vec{x}$ and $\vec{y}$ remains an edge, and now joins the vertices $(\vec{x}, \vec{y})$ and $(\vec{y}, \vec{x})$, verifying the second claim.

By the auxiliary condition of the induction hypothesis, the vertices labeled $\vec{x}$, $\vec{y}$, and $\vec{z}$ lie on a common 2-face whenever $\vec{x}$ is adjacent to both $\vec{y}$ and $\vec{z}$. After truncation, $(\vec{x}, \vec{y})$ and $(\vec{x}, \vec{z})$ will also be on this 2-face; since they are adjacent along the boundary of that face, and are both extreme points of the convex hull, they are joined by an edge, verifying the third claim.

To verify the fourth claim, we check that no other edges can exist. There is no edge between $(\vec{x}, \vec{y})$ and $(\vec{z}, \vec{x})$, where $\vec{x}, \vec{y}, \vec{z}$ are distinct, because these are non-adjacent vertices all on the 2-face containing $\vec{x}$, $\vec{y}$, and $\vec{z}$. The remaining case is an edge between $(\vec{x}, \vec{y})$ and $(\vec{z}, \vec{w})$, where $\vec{x}, \vec{y}, \vec{z}, \vec{w}$ are all distinct. In this case, the convex hull of $\vec{x}, \vec{y}, \vec{z}, \vec{w}$ is an affine transformation of the tetrahedron in Figure 1a. After truncation, points corresponding to the convex hull shown in Figure 1b will remain (though they might not be vertices of the truncation, if not all four of the vertices $\vec{x}, \vec{y}, \vec{z}, \vec{w}$ were adjacent). The segment joining $(\vec{x}, \vec{y})$ to $(\vec{z}, \vec{w})$ is in the interior of that convex hull, so it cannot be an edge of the truncation.

To finish the geometric part of the proof, we verify that the auxiliary condition remains true. There are two cases to check. For a vertex labeled $(\vec{x}, \vec{y})$, if we choose the neighbors $(\vec{x}, \vec{z})$ and $(\vec{x}, \vec{w})$, then any two of them are joined by an edge, and therefore they must lie on a common 2-dimensional face (which contains only these three vertices). If we choose the neighbors $(\vec{x}, \vec{z})$ and $(\vec{y}, \vec{x})$, then the points continue to lie on the 2-dimensional face inherited from the face through $\vec{x}$, $\vec{y}$, and $\vec{z}$ of the previous convex hull.

Now it remains to construct an isomorphism between the 1-skeleton graph of the truncation, which we will call T, and $G_{r,k}$. We identify the vertex $(\vec{x}, \vec{y})$ of T with the vertex $(x_1, x_2, \ldots, x_{k-1}, y_{k-1})$ of $G_{r,k}$. Since $x_{k-1} \neq y_{k-1}$ after any move in the Hanoi game, this k-tuple really is a Hanoi state. Conversely, any Hanoi state $\vec{z} \in \mathcal{H}_{r,k}$ corresponds to a vertex of T: let $\vec{x} = (z_1, z_2, \ldots, z_{k-1})$, and let $\vec{y}$ be the state obtained from $\vec{x}$ by either an adjustment of z_{k-1} to z_k, if $z_k \neq z_{k-2}$, or else an involution, if $z_k = z_{k-2}$. Therefore the map we define is a bijection between the vertex sets.

Both T and $G_{r,k}$ are r-regular graphs, therefore it suffices to show that each edge of T corresponds to an edge in $G_{r,k}$. Consider an edge joining $(\vec{x}, \vec{y})$ with $(\vec{x}, \vec{z})$ in T. This corresponds to vertices $(x_1, x_2, \ldots, x_{k-1}, y_{k-1})$ and $(x_1, x_2, \ldots, x_{k-1}, z_{k-1})$ in $G_{r,k}$; these are adjacent, since we can obtain one from the other by an adjustment.

Next, consider an edge joining $(\vec{x}, \vec{y})$ to $(\vec{y}, \vec{x})$. If $\vec{x}$ and $\vec{y}$ are related by an adjustment in $G_{r,k-1}$, then they have the form $(x_1, \ldots, x_{k-2}, x_{k-1})$ and $(x_1, \ldots, x_{k-2}, y_{k-1})$. The vertices corresponding to $(\vec{x}, \vec{y})$ and $(\vec{y}, \vec{x})$ in $G_{r,k}$ are $(x_1, \ldots, x_{k-2}, x_{k-1}, y_{k-1})$ and $(x_1, \ldots, x_{k-2}, y_{k-1}, x_{k-1})$, and one can be obtained from the other by an involution.

Finally, if $\vec{x}$ and $\vec{y}$ are related by an involution in $G_{r,k-1}$, then that involution swaps x_{k-1} and y_{k-1}. Therefore such an involution in $G_{r,k}$ will take $(x_1, \ldots, x_{k-1}, y_{k-1})$ to $(y_1, \ldots, y_{k-1}, x_{k-1})$, and the vertices corresponding to $(\vec{x}, \vec{y})$ and $(\vec{y}, \vec{x})$ are adjacent in $G_{r,k}$. □

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Part II

Celebrity Games

Chapter 5

Sum Celebrity Games

5.1 Summary

In the previous chapters we have studied the most emblematic model of the Network Creation Games. The techniques used to improve constant bounds for the PoA in the subsequent papers proposed by many authors are interesting and diverse, and this confirms that there is a very nice and rich combinatorial structure behind the model. However, one criticism to the same model is that it is far too simple and it does not capture the complexity of real Internet-like networks as well as social networks. For this reason many models have appeared either trying to extend the SUM NCG or proposing variations that are in some sense inspired by the same model. In the historical overview we have seen some of the examples that motivate the definition of two new models.

In the following we introduce the *Sum Celebrity model*, or abbreviated SUM CG, as an attempt to capture some characteristics of such networks that are not present in this classical model.

Like in the SUM NCG, we model the SUM CG as a strategic game specified by a finite set of players $V = \{1, ..., n\}$ and a parameter $\alpha > 0$ indicating the price that each player has to pay to create an incident link. The players wish to buy links in order to be connected in the resulting network. Hence, the strategy profile s_i of any player $i \in V$ is any subset of $V \setminus \{i\}$ indicating to which other players i has bought a link. In this way, the resulting network formed by the strategy profile $s = (s_1, ..., s_n)$ is defined as the communication network $G[s] = (V, \{ij \mid i \in s_j \vee j \in s_i\})$, exactly as it was defined in the SUM NCG.

In the classical model, all the players have the same relevance. However, this does not happen in real life scenarios where the players may be different in many senses. In the Internet, and specially in the social networks, there are players that have more relevance than others. Trying to capture this fact we introduce the *weight* $w_i > 0$ for every player $i \in V$, representing the amount of relevance of player i.

Therefore it seems natural to assume that every player wishes to have the more relevant nodes not far from a certain distance. Hence we introduce the critical distance $\beta \in \{1, ..., n-1\}$ indicating that the weight of nodes further than β of a player $i \in V$ will be taken into account in the cost function of i.

Finally, we define how these weights affect the player under consideration. In our SUM CG model we assume that the affectation is the summation of the weights. Given a tuple $\langle V, \alpha, (w_i)_{i \in V}, \beta \rangle$, with $V, \alpha, (w_i)_{i \in V}, \beta$ satisfying the previous restrictions, the SUM CG is the strategic game in which $\mathcal{S}_i = 2^{V \setminus \{i\}}$ for every $i \in V$ and where the individual cost of any player is

$$c_{G[s]}(i) = \alpha |s_i| + \sum_{\{j \mid d_{G[s]}(i,j) > \beta\}} w_j$$

Once we have introduced the model, in the article presented in this chapter we investigate how the combination of the distinct elements that define the SUM CG affect the quality of equilibria as well as their topology. We address the cases $\beta > 1$ and $\beta = 1$ separately.

The first thing we study for $\beta > 1$, is the problem of computing the Best Response which we show that it is NP-hard when $\alpha = 2$.

Then, to study the PoA and the topology of equilibria, it is useful to distinguish between two distinct classes of games that depend on whether there exist a certain kind of players with large weight or relevance, nodes which we call *celebrities.* Specifically, a celebrity is a node $i \in V$ such that $w_i > \alpha$. Then, *star celebrity games* are games containing at least one celebrity and *non star celebrity games* are games not containing any celebrity. A feature that star celebrity games have is that every NE is connected, a property that we need, for instance, when studying the diameter of equilibria.

For both star celebrity games and non star celebrity games we study upper and lower bounds for the PoA. Regarding the upper bounds we compare the PoA vs the critical distance β and the PoA vs the total amount of weight. On the positive side, although such upper bounds are not tight, they imply low PoA when β is close to n and when W is close to α as one can expect. On the negative side, although in very specific cases, we can show games that have a very large PoA.

Regarding the topology, we see that the diameter of equilibria is related to the critical distance β. This result goes with the intuition that in any connected equilibrium, players do not wish to have many other players further than β, otherwise their penalisation might get large. Moreover, recall the MaxBD model as a network creation game introduced in section 2.4.2. We also see that there exists a connection between equilibria for the MaxBD where the bounded distance has the same value β for every player, with equilibria for our Sum Celebrity games where every player is a celebrity.

Finally, we also study the particular case of $\beta = 1$. Equilibria for such games can be analysed as configurations in which every player i pays w_j for every non adjacent node j, α for every adjacent node j for which i has bought the link (i, j), and 0 for every adjacent node j that has bought the link (j, i). These particular restrictions allow us to show that the problem of computing the Best Response can be done in polynomial time and, furthermore, that the PoA is at most 2.

In the forthcoming paper we define and analyse in detail all these properties of Sum Celebrity Games.

5.2 Article: Celebrity Games

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Celebrity Games

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Abstract

We introduce *Celebrity games*, a new model of network creation games. In this model players have weights (being W the sum of all the player's weights) and there is a critical distance β as well as a link cost α. The cost incurred by a player depends on the cost of establishing links to other players and on the sum of the weights of those players that remain farther than the critical distance. Intuitively, the aim of any player is to be relatively close (at a distance less than β) from the rest of players, mainly of those having high weights. The main features of celebrity games are that: computing the best response of a player is NP-hard if $\beta > 1$ and polynomial time solvable otherwise; they always have a pure Nash equilibrium; the family of celebrity games having a connected Nash equilibrium is characterized (the so called *star celebrity games*) and bounds on the diameter of the resulting equilibrium graphs are given; a special case of star celebrity games share its set of Nash equilibrium profiles with the MaxBD games with uniform bounded distance β intoduced in (Bilò et al., 2012). Moreover, we analyze the Price of Anarchy (PoA) and of Stability (PoS) of celebrity games and give several bounds. These are that: for non-star celebrity games PoA = PoS = $max\{1, W/\alpha\}$; for star celebrity games PoS = 1 and PoA = $O(min\{n/\beta, W\alpha\})$ but if the Nash Equilibrium is a tree then the PoA is $O(1)$; finally, when $\beta = 1$ the PoA is at most 2. The upper bounds on the PoA are complemented with some lower bounds for $\beta = 2$.

Keywords: Network creation games, Nash equilibrium, Price of Anarchy, diameter

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1. Introduction

The global growth of Internet and social networks usage has been accompanied by an increasing interest to model theoretically their creation as well as their behavior. In particular, network creation games (NCG) aim to model Social Networks and Internet by simulating the creation of a decentralized and non-cooperative communication network among n players (the network nodes).

From the seminal paper (Fabrikant et al., 2003) several proposals have been made in the area of NCG. In the original model, the goal of each player is to have, in the resulting network, all the other nodes as close as possible while buying as few links as possible (Fabrikant et al., 2003). Several assumptions are made: all the players have the same interest (all-to-all communication pattern with identical weights); the cost of being disconnected is infinite; and the edges paid by one node can be used by others. Formally, a game Γ in this model is defined as a tuple $\Gamma = \langle V, \alpha \rangle$, where V is the set of n nodes and α the cost of establishing a link. A strategy for player $u \in V$ is a subset $S_u \subseteq V - \{u\}$, the set of players for which player u pays for establishing a link. The n players and their joint strategy choices $S = (S_u)_{u \in V}$ create an undirected graph $G[S]$. The cost function for each node u under strategy S is defined by $c_u(S) = \alpha|S_u| + \sum_{v \in V} d_{G[s]}(u, v)$ where $d_{G[s]}(u, v)$ is the distance between nodes u and v in graph $G[S]$. Because of the summation in the cost function this model is informally known as the *Sum game* model. By changing the cost function to $c_u(S) = \alpha|s_u| + \max\{d_{G[S]}(u, v)|v \in V\}$ as proposed in Demaine et al. (2012) one obtains the *Max game* model.

From here on several versions and variants have been considered. Instead of buying links unilaterally, Corbo and Parkes (2005) proposed the possibility of having links formed by bilateral contracting: both endpoints must agree before creating a link between them and the two players share (half-half) the cost of establishing the link. NCG models can be cooperative –a possibility introduced by Albers et al. (2006)– and therefore any node can purchase any amount of any link in the resulting graph, and a link can be created when its cost is covered by a set of players. The model studied in Bilò et al. (2015b) (see also Bilò et al. (2012)) considers the notion of bounded distance per player and propose two variants: the MaxBD game and the SumBD game, corresponding to the original Max and Sum cost models respectively. The cost in those games depends on whether the player's eccentricity is smaller or equal than the associated bounded distance. In that case a player pays the number of established links, otherwise its cost is infinite. For further variants we refer the interested reader to (Demaine et al., 2012; Leonardi and Sankowski, 2007; Brandes et al., 2008; Demaine et al., 2009; Lenzner, 2011; Alon et al., 2013, 2014; Bilò et al., 2015a; Nikoletseas et al., 2015; Cord-Landwehr and Lenzner, 2015; Ehsani et al. , 2015) among others.

We introduce *celebrity games* a NCG where players have different weights and share a common distance bound. As far as we understand, not all the nodes in Internet based networks have the same importance. It is though natural to consider players with different relevance weights. In such a setting, the cost of being far (even if connected) from important nodes (the ones with high weight) should be higher than the cost of having them close. Intuitively, the goal of each player in celebrity games is to buy as few links as possible in order to have the high-weighted nodes (or groups of nodes) closer to the given critical distance. Observe that if the cost of establishing links is higher than the benefit of having close a node (or set of nodes), players might rather prefer to stay either far or even disconnected from it.

Our aim is to study the combined effect of having players with different weights that share a common bounded distance. Although heterogeneous players have been considered recently in the context of NCG under bilateral contracting (Meirom et al., 2014; Àlvarez et al., 2015), and Bilò et al. (2015b) consider the notion of bounded distance, to the best of our knowledge this is the first model that studies how a common critical distance, different weights, and a link cost, altogether affect the individual preferences of the players.

In our model the cost of a player has two components. The first one is the cost of the links established by the node. The second one is the sum of the weights of those nodes that are farther away than the critical distance. More specifically, the parameters of a celebrity game are: a weight to each player; a cost for establishing a link; and a critical distance. Formally, a celebrity game is defined by $\Gamma = \langle V, (w_u)_{u\in V}, \alpha, \beta\rangle$, where V is a set of nodes with weights $(w_u)_{u\in V}$, α is the cost of establishing a link and, β establishes the desirable distance bound. Celebrity games include the MaxBD games introduced in (Bilò et al., 2015b) (see Section 5 for the details). They capture not only the cases in which players are indistinguishable but those cases where the players may have different weights affecting differently the costs of the other players.

We analyze the structural properties of the Nash equilibrium (NE) graphs of celebrity games and their quality with respect to the optimal strategies under the usual social cost. To do so we address the cases $\beta = 1$ and $\beta > 1$ separately. Notice that, for $\beta = 1$, each player u has to decide for every non-edge (u, v) of the graph to pay either α for the link or w_v (the weight of the non-adjacent node v) while, for $\beta > 1$, every player u has to choose for each non-edge (u, v) between buying the link (u, v) and paying α minus the sum of the weights of those nodes whose distance to u will become less or equal than the critical distance β or paying the sum of the weights of the nodes with distance to u greater than β.

For the general case $\beta > 1$ our results can be summarized as follows:

- Computing a best response for a player is NP-hard
- The optimal social cost of a celebrity game Γ depends on the relation between the total sum of the weights W and the cost α of buying a link: $\text{OPT}(\Gamma) = \min\{\alpha, W\}(n - 1)$. Nevertheless, pure NE always exist and NE graphs are either connected or a set of isolated nodes. Again, the relationship between the cost of establishing a link and the weight of the nodes leads to different types of NE.
- We use the term *celebrity* for a node whose weight is strictly greater than the cost of establishing a link. Having at least one celebrity guarantees that all NE graphs are connected, although there are celebrity games without

celebrities that still have connected NE graphs. In those games having a connected NE graph, a star tree is always a NE graph. We called this subfamily of celebrity games *star celebrity games*.

- For star celebrity games, we obtain a general upper bound of $2\beta + 1$ for the diameter of NE graphs. In particular, if G is a NE tree we show that $diam(G) \leq \beta + 1$, otherwise $\beta/2 < diam(G) \leq 2\beta + 1$. The upper bound can be improved by considering the relationship between α and the maximum and minimum weights, w_{max} and w_{min}, respectively. So, if $w_{min} \leq \alpha < w_{max}$, then $diam(G) \leq 2\beta$. On the contrary, if $\alpha < w_{min}$, then $diam(G) \leq \beta$.
- For star celebrity games with $\alpha < w_{min}$, we show that the set of NE strategy profiles coincides with the set of NE strategy profiles of a MaxBD game with uniform bounded distance β.
- We find several bounds on the Price of Anarchy (PoA) and of stability (PoS). For non-star celebrity games PoS = PoA = $\max\{1, W/\alpha\}$. For star celebrity games the PoS is 1 and we obtain a general upper bound of $O(\min\{n/\beta, W/\alpha\})$ for the PoA. We also show particular games on n players having PoA = $\Omega(n)$, for $\beta = 2$. To complement those results we prove that the PoA on NE trees is constant (special cases like trees are also considered in the literature, see for instance (Alon et al., 2013, 2014; Ehsani et al. , 2015).

Finally, for the particular case $\beta = 1$, we show that computing a best response for a player is polynomial time solvable and that the PoA is at most 2.

The paper is organized as follows. In Section 2 we introduce the basic definitions and the celebrity games model. We also show that computing a best response is NP-hard. In Section 3 we set the fundamental properties of NE and optimal graphs. We characterize star celebrity games and we provide the first bounds for the PoA and the PoS. Section 4 is devoted to the study of the diameter of NE graphs. Section 5 studies the relation between the MaxBD game model and the celebrity game model. In Section 6 we derive the bounds for the PoA. In Section 7 we give the upper bound of the PoA over NE trees and in Section 8 we study the case $\beta = 1$. Finally, we state some conclusions and open problems in Section 9.

2. The Model

In this section we introduce celebrity games and we analyze the complexity of computing a best response. Let us start with some definitions. We use standard notation for graphs and strategic games. All the graphs in the paper are undirected unless explicitly said otherwise. Given a graph $G = (V, E)$ and $u, v \in V$, $d_G(u, v)$ denotes the *distance* between u and v in G, i.e., the length of the shortest path from u to v. The *diameter* (or *eccentricity*) of a vertex $u \in V$ is $diam(u) = \max_{v \in V} d_G(u, v)$ and the *diameter* of G is $diam(G) =$

$\max_{v \in V} diam(v)$. An *orientation* of an undirected graph is an assignment of a direction to every edge of the graph, turning it into a directed graph. A *bridge* is an edge whose deletion increases the number of connected components of the graph. For a weighted set $(V, (w_u)_{u \in V})$ we extend the weight function to subsets in the usual way. For $U \subseteq V$, $w(U) = \sum_{u \in U} w_u$. Furthermore, we set $W = w(V)$, $w_{max} = \max_{u \in V} w_u$ and $w_{min} = \min_{u \in V} w_u$.

Definition 1. *A* celebrity game Γ *is a tuple* $\langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ *where:* $V = \{1, \ldots, n\}$ *is the set of players, for each player* $u \in V$, $w_u > 0$ *is the* weight *of player* u, $\alpha > 0$ *is the cost of establishing a link, and* β, $1 \leq \beta \leq n-1$, *is the* critical distance.

A strategy *for player* u *in* Γ *is a subset* $S_u \subseteq V - \{u\}$, *the set of players for which player* u *pays for establishing a direct link. A* strategy profile *for* Γ *is a tuple* $S = (S_1, \ldots, S_n)$ *that assigns a strategy to each player. Every strategy profile* S *has associated an* outcome graph, *the undirected graph defined by* $G[S] = (V, \{\{u,v\} | u \in S_v \vee v \in S_u\})$.

We denote by $c_u(S) = \alpha |S_u| + \sum_{\{v | d_{G[S]}(u,v) > \beta\}} w_v$ *the* cost *of player* u *in the strategy profile* S. *And, as usual, the* social cost *of a strategy profile* S *in* Γ *is defined as* $C(S) = \sum_{u \in V} c_u(S)$.

Observe that, even though a link might be established by only one of the two players, we assume that once a link is established it can be used in both directions. Note also that players may have different weights. The player's cost function has two components: the cost of establishing links and the sum of the weights of those players who are farther away than the critical distance β. In our model links have uniform length therefore w.l.o.g β is an integer. In what follows we assume that, for a celebrity game $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$, the parameters verify the required conditions. Furthermore, unless specifically stated, we assume $\beta > 1$, the case $\beta = 1$ will be analyzed in Section 8. We use the following notation $n = |V|$, $\mathcal{S}(u)$ is the set of strategies for player u and $\mathcal{S}(\Gamma)$ is the set of strategy profiles of Γ. For a strategy profile $S \in \mathcal{S}(\Gamma)$ and a strategy $S'_u \in \mathcal{S}(u)$, for player u, (S_{-u}, S'_u) represents the strategy profile in which S_u is replaced by S'_u while the strategies of the other players remain unchanged. The *cost difference* $\Delta(S_{-u}, S'_u)$ is defined as $\Delta(S_{-u}, S'_u) = c_u(S_{-u}, S'_u) - c_u(S)$. Observe that, if $\Delta(S_{-u}, S'_u) < 0$, then player u has an incentive to deviate from S_u and select S'_u. A *best response* to $S \in \mathcal{S}(\Gamma)$ for player u is a strategy $S'_u \in \mathcal{S}(u)$ minimizing $\Delta(S_{-u}, S'_u)$.

Let us recall the definition of Nash equilibrium.

Definition 2. *Let* $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ *be a celebrity game. A strategy profile* $S \in \mathcal{S}(\Gamma)$ *is a* Nash equilibrium *of* Γ *if no player has an incentive to deviate from his strategy. Formally, for each player* u *and each strategy* $S'_u \in \mathcal{S}(u)$, $\Delta(S_{-u}, S'_u) \geq 0$.

We denote by $\mathrm{NE}(\Gamma)$ the set of Nash equilibria of a game Γ and we use the term NE to refer to a strategy profile $S \in \mathrm{NE}(\Gamma)$. We say that a graph G is a NE *graph* of Γ if there is $S \in \mathrm{NE}(\Gamma)$ so that $G = G[S]$. We will drop the explicit reference to Γ whenever Γ is clear from the context. It is worth observing that, for $S \in \mathrm{NE}(\Gamma)$, it never happens that $v \in S_u$ and $u \in S_v$, for any $u, v \in V$. Thus, if G is the outcome of a NE S, S corresponds to an orientation of the edges in G. Furthermore, a NE graph G can be the outcome of several strategy profiles but not all the orientations of a NE graph G are NE.

Let $\mathsf{opt}(\Gamma) = \min_{S \in \mathcal{S}(\Gamma)} C(S)$ be the minimum value of the social cost. We use the term OPT strategy profile to refer to one strategy profile with optimal social cost.

Observe that, when in a strategy profile S, two players u and v are such that $u \in S_v$ and $v \in S_u$, the social cost is higher than when only one of them is paying for the connection $\{u, v\}$ and therefore, as for NE, this does not happen in an OPT strategy profile. In the following, as we are interested in NE and OPT strategies, among all the possible strategy profiles having the same outcome graph, we only consider those corresponding to orientations of the outcome graph. In this sense the social cost depends only on the outcome graph, the weights and the parameters. Thus, we can express the social cost of a strategy profile as a function of the outcome graph G as follows

$$C(G) = \alpha|E(G)| + \sum_{u \in V} \sum_{\{v \mid d_G(u,v) > \beta\}} w_v = \alpha|E(G)| + \sum_{\{(u,v) \mid u<v \text{ and } d_G(u,v) > \beta\}} (w_u + w_v).$$

We make use of three particular outcome graphs on n vertexes: K_n, the complete graph; I_n, the independent set; and ST_n the star graph, i.e., a tree in which one of the vertexes, the *central* one, has a direct link to all the other $n-1$ vertexes. For those graphs, we have the following values of the social cost. For $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$, with $|V| = n$, $C(K_n) = \alpha n(n-1)/2$, $C(I_n) = W(n-1)$, for $\beta \geq 1$. Furthermore, $C(ST_n) = \alpha(n-1)$, for $1 < \beta \leq n-1$, and $C(ST_n) = \alpha(n-1) + (n-2)(W - w_c)$ where c is the central vertex, for $\beta = 1$.

We define the PoA and the PoS as usual.

Definition 3. *Let* Γ *be a celebrity game. The* Price of Anarchy *of* Γ *is defined as* $PoA(\Gamma) = \max_{S \in NE(\Gamma)} C(S)/\mathsf{opt}(\Gamma)$ *and the* Price of Stability *of* Γ *as* $PoS(\Gamma) = \min_{S \in NE(\Gamma)} C(S)/\mathsf{opt}(\Gamma)$.

Our first result shows that computing a best response in celebrity games is NP-hard by a reduction from the minimum dominating set problem. The problem becomes tractable for $\beta = 1$ as we show in Section 8.

Proposition 1. *Computing a best response for a player to a strategy profile in a celebrity game is NP-hard, even when* $\beta = 2$ *and restricted to the cases in which all players except possibly one have weights bigger than* α.

Proof. We provide a reduction from the problem of computing a dominating set of minimum size which is a classical NP-hard problem. Recall that a dominating set of a graph $G = (V, E)$ is a set $U \subset V$ such that any vertex $u \in V$ is in U or has a neighbor in U.

Let $G = (V, E)$ be a graph, we associate to G and u a celebrity game $\Gamma = \langle V', (w_v)_{v \in V'}, \alpha, \beta \rangle$, and a strategy profile S as follows:

- The set of players is $V' = V \cup \{u\}$, where u is a new player (i.e. $u \notin V$).
- $\beta = 2$, $\alpha = 1.5$,
- for every $v \in V$, $w_v = 2$.
- The strategy profile S is obtained from an orientation of the edges in G setting $S_u = \emptyset$. Observe that by construction $G[S]$ is the disjoint union of G with the isolated vertex u.

Finally, set u to be the player for which we want to compute the best response to S. Observe that the weight of u has not been defined yet.

Let $D \subseteq V$ be a strategy for player u. Notice that, if D is a dominating set of G, then $c_u(S_{-u}, D) = \alpha|D| + \sum_{x \in V, d(u,x)>2} 2 = \alpha|D|$. If D is not a dominating set of G, $c_u(S_{-u}, D) = \alpha|D| + \sum_{x \in V, d(u,x)>2} 2 > \alpha(|D| + |\{x \in V | d(u,x) > 2\}|$. Then, $D \cup \{x \in V | d(u,x) > 2\}$ is a better response than D and furthermore it is a dominating set. Hence, the best response of player u is a dominating set D of G of minimum size. To conclude the proof just notice that the described reduction is polynomial time computable and that we did not make any assumption on the weight of the node u. □

3. Social Optimum and Nash equilibrium

We analyze here the main properties of OPT and NE strategy profiles in celebrity games. We start analyzing the cost of optimal graphs for the social cost. Then we characterize the family of star celebrity games having a connected NE graph. Finally, we provide exact bounds on the PoA and the PoS in some particular cases.

Proposition 2. *Let $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ be a celebrity game. We have that $opt(\Gamma) = \min\{\alpha, W\}(n-1)$.*

Proof. Let $S \in \text{OPT}(\Gamma)$, and let $G = G[S]$ with connected components $G_1, ..., G_r$, $V_i = V(G_i)$, $k_i = |V_i|$, and $W_i = w(V_i)$, for $1 \leq i \leq r$. Observe that the social cost of a disconnected graph can be expressed as the sum of the social cost of its connected components. Each connected component must be a tree of diameter at most β, otherwise a strategy profile with smaller social cost could be obtained by replacing the connections on V_i by such a tree. We can assume w.l.o.g. that, for $1 \leq i \leq r$, the i-th connected component is a star graph ST_{k_i} of k_i vertexes. Since $C(ST_k) = \alpha(k-1)$ we have that

$$C(G) = \sum_{i=1}^{r} \alpha(k_i - 1) + \sum_{i=1}^{r} k_i(W - W_i) = \alpha(n-r) + nW - \sum_{i=1}^{r} k_i W_i.$$

As $1 \leq k_i \leq n - (r-1)$, we have $W \leq \sum_{i=1}^{r} k_i W_i \leq (n-r+1)W$. Therefore, $\alpha(n-r) + (r-1)W \leq C(G)$. We consider two cases.

Case 1: $\alpha \geq W$. We have $W(n-1) \leq C(G)$. Since $C(I_n) = W(n-1) \leq C(G)$ and G is an optimal graph, then $C(G) = W(n-1)$.

Case 2: $\alpha < W$. Now $\alpha(n-1) \leq C(G)$. As $C(ST_n) = \alpha(n-1) \leq C(G)$, the optimal graph G has a social cost $C(G) = \alpha(n-1)$. We conclude that $\text{OPT} = \min\{\alpha, W\}(n-1)$. □

Now we turn our attention to the study of the NE graph topologies showing that any NE graph is either an independent set or a connected graph.

Proposition 3. *Let* $\Gamma = \langle V, (w_u)_{u\in V}, \alpha, \beta\rangle$ *be a celebrity game. Every* NE *graph of* Γ *is either connected or the graph* I_n*, where* $n = |V|$*.*

Proof. If $n \leq 2$ the proposition follows immediately. When $n > 2$, let us suppose that there is a NE S such that the graph $G = G[S]$ is not connected and different from I_n. In this case G is composed of at least two different connected components G_1 and G_2. Furthermore, as $G \neq I_n$, we can assume that $|V(G_1)| > 1$ as at least one of the connected components contains two vertexes connected by an edge. Let $u \in V(G_1)$ be such that $S_u \neq \emptyset$. Let $x \in S_u$ and $v \in V(G_2)$. Let us consider the strategies $S'_u = S_u \setminus \{x\}$ and $S'_v = S_v \cup \{x\}$. As S is a NE we know that $\Delta(S_{-u}, S'_u) \geq 0$. Let $G' = G[S_{-v}, S'_v]$, observe that $d_{G'}(v, u) = 2 \leq \beta$, therefore $\Delta(S_{-v}, S'_v) \leq -\Delta(S_{-u}, S'_u) - w_u < 0$. This contradicts the hypothesis that S is a NE. □

Next we study the conditions under which particular topologies are NE graphs. Those results prove that celebrity games always have a NE.

Proposition 4. *Every celebrity game* $\Gamma = \langle V, (w_u)_{u\in V}, \alpha, \beta\rangle$ *has a* NE*. Furthermore, when* $\alpha \geq w_{max}$*,* I_n *is a* NE *graph, otherwise* ST_n *is a* NE *graph but* I_n *is not, where* $n = |V|$*.*

Proof. When $\alpha \geq w_{max}$ let us show that I_n is a NE graph. Observe that $G = I_n$ is the outcome of a unique strategy profile S in which $S_u = \emptyset$, for any $u \in V$. Let us consider a player u and a strategy $S'_u \neq \emptyset$. The cost difference of player u is then $\Delta(S_{-u}, S'_u) = \alpha|S'_u| - \sum_{v\in S'_u} w_v = \sum_{v\in S'_u}(\alpha - w_v) \geq 0$. Therefore player u has no incentive to deviate from S_u and I_n is a NE graph.

When $\alpha < w_{max}$, let u be a vertex with $w_u = w_{max}$ and let ST_n be a star graph with n vertexes in which the center is u, let us show that ST_n is a NE graph. Consider the strategy profile S in which $S_u = \emptyset$ and $S_v = \{u\}$, for any $v \in V$ different from u. Observe that the center u is a vertex with maximum weight. As $\beta > 1$ no player will get a cost decrease by connecting to more players. Furthermore, for $u \neq v$, $w_v + \alpha < w_v + w_{max} < W$. Thus $\alpha < W - w_v$ and v will not get any benefit by deleting the actual connection. The only remaining possibility is to reconnect to another vertex, but in such a case the cost cannot decrease. Therefore, ST_n is a NE graph. Notice that in this case I_n can not be a NE, as every player u has incentive to connect with any other player v such that $w_v = w_{max}$. □

To conclude the study of NE we characterize the celebrity games where I_n is the unique NE graph. The negated condition characterizes those games in which ST_n is a NE.

Proposition 5. *Let* $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ *be a celebrity game on* n *players with* $\alpha \geq w_{max}$. *If there is more than one vertex* $u \in V$ *with* $\alpha > W - w_u$, *then* I_n *is the unique* NE *graph of* Γ, *otherwise* ST_n *is a* NE *graph of* Γ.

Proof. Assume that, for two vertexes $u \neq v$, $\alpha > W - w_u$ and $\alpha > W - w_v$, and that there exists a NE graph $G = G[S]$ different from I_n. By Proposition 3, G is connected. Therefore, it has at least $n-1$ edges. Since, $\alpha > W - w_u$ and $\alpha > W - w_v$, we have that $S_u = S_v = \emptyset$, otherwise S would not be a NE. Therefore, there must be a vertex, $z \neq u, v$ such that $|S_z| \geq 2$. Let $x, y \in S_z$ and consider the strategy $S'_z = S_z \setminus \{x, y\}$. Then, $\Delta(S_{-z}, S'_z) \leq -2\alpha + W - w_z$. As G is a NE graph and we have that $2\alpha > W - w_u + W - w_v$, we conclude that $W - w_z \geq 2\alpha > W - w_u + W - w_v$. Hence, $W < w_u + w_v - w_z < w_u + w_v$, which is impossible. In the case that there is at most one vertex u with $\alpha > W - w_u$, the strategy profile S, where $S_u = \emptyset$, and $S_v = \{u\}$, for all $v \neq u$, is a NE. Furthermore $G[S] = ST_n$. □

Corollary 1. *Let* $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ *be a celebrity game on* n *players.* I_n *is the unique* NE *graph of* Γ *if and only if* $\alpha \geq w_{max}$ *and there is more than one vertex* $u \in V$ *such that* $\alpha > W - w_u$.

Observe that in our model it is preferable to be an isolated node than to pay a huge amount for establishing a link. In fact, in a NE graph either all nodes are isolated, or the graph is connected. Hence, selecting an appropriate price per link is a key fact to guarantee the connectivity of the equilibrium graphs.

Finally, using this characterization, we can formally define the subfamily of celebrity games that have always a connected NE graph. Those games have ST_n as a NE graph.

Definition 4. $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ *is a star celebrity game if* Γ *has a* NE *graph that is connected.*

Corollary 2. *For a celebrity game* $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$, *the following statements are equivalent.*

- Γ *is a star celebrity game.*
- *Either* $\alpha < w_{max}$ *or* $\alpha \geq w_{max}$ *and there is at most one* $u \in V$ *for which* $\alpha > W - w_u$.
- ST_n *is a* NE *graph of* Γ.

Putting all together we can compute the PoS and, in some cases, the PoA.

Theorem 1. *Let* Γ *be a celebrity game. Then we have.*

- *If* Γ *is a star celebrity game,* $PoS(\Gamma) = 1$.
- *If* Γ *is not a star celebrity game and* $\alpha \geq W$, *then* $PoS(\Gamma) = PoA(\Gamma) = 1$.
- *If* Γ *is not a star celebrity game and* $\alpha < W$, *then* $PoS(\Gamma) = PoA(\Gamma) = W/\alpha > 1$.

Proof. From Proposition 2, we have that $\mathsf{opt}(\Gamma) = W(n-1)$ if $\alpha \geq W$ and $\mathsf{opt}(\Gamma) = \alpha(n-1)$, otherwise. When Γ is a star celebrity game, by Corollary 2 we know that ST_n is a NE graph. Let us see that in star celebrity games it can only occur that $\alpha < W$. If $\alpha < w_{max}$, clearly $\alpha < W$. If $\alpha \geq w_{max}$, by Corollary 2 there is at most one $u \in V$ for which $\alpha > W - w_u$. Assuming that $w_{u_1} \leq \ldots \leq w_{u_{n-1}} \leq w_{u_n}$, we have that $W > W - w_{u_1} \geq \ldots \geq W - w_{u_{n-1}} \geq W - w_{u_n}$, and then $W - w_{u_{n-1}} \geq \alpha$. Hence, $\text{PoS}(\Gamma) = 1$.

When Γ is not a star celebrity game, I_n is the unique NE graph. Thus, when $\alpha \geq W$ we have, $\text{PoS}(\Gamma) = \text{PoA}(\Gamma) = 1$ and, when $\alpha < W$ we have, $\text{PoS}(\Gamma) = \text{PoA}(\Gamma) = W/\alpha > 1$. □

4. Critical distance and diameter in Nash equilibrium graphs

In this section we analyze the diameter of NE graphs and its relationship with the parameters defining the game. We are interested only in games in which NE graphs with finite diameter exist, thus we only consider star celebrity games. In stating the characterization, nodes with a high weight with respect to the link cost play a fundamental role and it is worth to give them a name.

Definition 5. *Let $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ be a celebrity game. We say that a vertex $u \in V$ is a* celebrity *if $\alpha < w_u$.*

Given a celebrity u, any other node v with $d(u, v) > \beta$ has an incentive to pay for connecting to u. Thus, in any NE graph G, every celebrity node u satisfies that $diam(u) \leq \beta$.

In some of the proofs of the following results we refer to a set of *critical nodes* $z \in V$ of a graph $G = (V, E)$ with respect to a node u and an edge $\{x, y\}$. Critical is used in the sense that as all the shortest paths from u to z use $\{x, y\}$, removing the edge $\{x, y\}$ results in an increase of the distance from u to z. We use the notation

$$A^G_{\{x,y\}}(u) = \{z \in V | \text{ all the shortest paths in } G \text{ from } u \text{ to } z \text{ use the edge } \{x, y\}\}$$

We drop the explicit reference to G whenever G is clear from the context.

Proposition 6. *Let $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ be a star celebrity game. If G is a* NE *graph of Γ, then $diam(G) \leq 2\beta + 1$.*

Proof. Let S be a NE of Γ such that $G = G[S]$. Assume that $diam(G) \geq 2\beta + 2$. Then, there are two nodes $u, v \in V$ such that $d(u, v) = 2\beta + 2$. Consider a shortest path from u to v, $u = u_0, u_1, \ldots, u_{2\beta+1}, u_{2\beta+2} = v$.

Let $A_u = \{x \in V | d(u, x) \leq \beta\}$ and let $A_{u_1} = \{x \in V | d(u_1, x) \leq \beta\}$. Let us show that if a node $x \in A_u \cup A_{u_1}$, then $d(x, v) > \beta$. If $x \in A_u$ then $d(x, v) > \beta$, otherwise $d(u, v) \leq d(u, x) + d(x, v) \leq 2\beta$ contradicting the fact that $d(u, v) = 2\beta + 2$. Moreover, if $x \in A_{u_1}$ then $d(x, v) > \beta$, otherwise $d(u, v) \leq 1 + d(u_1, x) + d(x, v) \leq 2\beta + 1$ which also contradicts the fact that $d(u, v) = 2\beta + 2$.

Consider the edge $\{u, u_1\}$. Then, either $u_1 \in S_u$ or $u \in S_{u_1}$. In the case that $u_1 \in S_u$, let $S'_u = S_u \setminus \{u_1\}$ and $S'_v = S_v \cup \{u_1\}$. Observe that,

$$\Delta(S_{-u}, S'_u) \leq -\alpha + w(A_{\{u,u_1\}}(u) \cap A_u)$$

By the previous remark about distances, we know that all the vertexes $x \in A_{\{u,u_1\}}(u) \cap A_u$ verify $d(x, v) > \beta$, but after adding $\{v, u_1\}$ all of them and u become at distance $\leq \beta$ from v, therefore

$$\Delta(S_{-v}, S'_v) \leq \alpha - w_u - w(A_{\{u,u_1\}}(u) \cap A_u).$$

Hence, $\Delta(S_{-u}, S'_u) + \Delta(S_{-v}, S'_v) \leq -w_u < 0$. Therefore, either $\Delta(S_{-u}, S'_u) < 0$ or $\Delta(S_{-v}, S'_v) < 0$ and then S can not be a NE.

The case $u \in S_{u_1}$, follows in a similar way by interchanging the roles of u and u_1. □

The previous result can be refined to get better bounds on the diameter when all the nodes are celebrities or when at least one of the nodes is a celebrity.

Property 1. *Let $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ be a star celebrity game and let G be a* NE *graph of Γ, then*

- *if $w_{min} \leq \alpha < w_{max}$, $diam(G) \leq 2\beta$ and,*
- *if $\alpha < w_{min}$, $diam(G) \leq \beta$.*

Proof. When $w_{min} \leq \alpha < w_{max}$, there is a celebrity $u \in V$ with $w_u > \alpha$. We know that $diam(u) \leq \beta$. Let x and z be any two different nodes of G, then $d(x, u) \leq \beta$ and $d(z, u) \leq \beta$. Therefore, $d(x, z) \leq d(x, u) + d(z, u) \leq 2\beta$ and the claim follows. When $\alpha < w_{min}$, each $u \in V$ is a celebrity, thus $diam(u) \leq \beta$. Therefore $diam(G) \leq \beta$. □

For NE trees we have a trivial lower bound of 2 on the diameter as a star is a NE graph. For non-tree NE graphs we provide a lower bound on the diameter. We first prove a technical result.

Lemma 1. *Let $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ be a star celebrity game. In a* NE *graph of Γ containing at least one cycle, if u is a node of a cycle and $diam(u) \leq \beta - k$, for some $k \geq 1$, then the length of any cycle containing u is bigger than $2k + 2$.*

Proof. Let us suppose that S is a NE and that $G = G[S]$ contains a cycle C through a node u such that $diam(u) \leq \beta - k$, for some $k \geq 1$. Assume that C is the shortest cycle containing u and that the length ℓ of C verifies $\ell \leq 2k + 2$. We split the proof in two cases, depending on the parity of ℓ.

Case 1: C has odd length, $\ell = 2i + 1$. Let v_1, v_2 be the two vertexes in C that are at distance i of u in C, as C is of minimal length $d(u, v_1) = d(u, v_2) = i$. By our hypothesis, $2i + 1 \leq 2k + 2$ and thus $i \leq k$. Assume w.l.o.g. that $v_2 \in S_{v_1}$ and consider the strategy $S'_{v_1} = S_{v_1} \setminus \{v_2\}$. Let $G' = G[S_{-v_1}, S'_{v_1}]$. Notice that $d_{G'}(v_2, u) = i$. Therefore, $diam_{G'}(v_1) \leq k + \beta - k = \beta$, by selecting a path going through u, so $\Delta(S_{-v_1}, S'_{v_1}) \leq -\alpha < 0$ and G would not be a NE graph.

Case 2: C has even length, $\ell = 2i$. Let v be the antipodal vertex to u, at distance i from u in C and let v_1, v_2 be the two vertexes in C that are at distance $i-1$ of u in C. By our hypothesis, $2i \leq 2k+2$ and thus $i-1 \leq k$. If $v \in S_{v_1}$, consider the strategy $S'_{v_1} = S_{v_1} \setminus \{v\}$. Using the same arguments as in Case 1 and the fact that the distance from v_1 to u in C is $\leq k$, we conclude that S is not a NE. The same happens when $v \in S_{v_2}$. It remains to consider the case in which $v_1, v_2 \in S_v$. Consider the strategy $S'_v = (S_v \cup \{u\}) \setminus \{v_1, v_2\}$. Now all shortest paths in G from v passing through v_1 or v_2 can be rerouted trough u with and increment in length of at most $i-1 \leq k$. Therefore, $diam_{G'}(v) \leq 1+\beta-k \leq \beta$. Thus $\Delta(S_{-v}, S'_v) \leq -\alpha < 0$ and G would not be a NE graph. □

Proposition 7. *Let $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ be a star celebrity game and let G be a* NE *graph of Γ. If G is not a tree, then $diam(G) > \beta/2$.*

Proof. Let G be a NE graph containing at least one cycle. If $diam(G) \geq \beta$, the claim holds. Assume that $diam(G) \leq \beta - 1$. We know that the length of the shortest cycle C is $\leq 2\,diam(G)+1$. Let u be any node of C. Then, we have $diam(u) \leq diam(G) = \beta - (\beta - diam(G))$. By Lemma 1, $2\,diam(G)+1 > 2(\beta - diam(G)) + 2$. The last inequality implies $diam(G) > \beta/2$. □

5. MaxBD network creation games versus celebrity games

In this section we show that MaxBD games are equivalent to celebrity games where all players are celebrities. Let us formalize the definition of MaxBD game taken from (Bilò et al., 2015b).

A *MaxBD game* Γ is defined by a tuple $\langle V, D \rangle$ where $V = \{1, \dots, n\}$ is the set of players and D, $1 \leq D \leq n-1$, is an integer representing the bound on the diameter of each node $v \in V$. Concepts like strategy of a player, strategy profile, and outcome graph are defined as in the celebrity game model. The *cost* of player u in the strategy profile S is $c_u^{\mathsf{MaxBD}}(S) = |S_u|$, if $diam_{G[S]}(u) \leq D$; $c_u^{\mathsf{MaxBD}}(S) = +\infty$, otherwise. The social cost of S is $C^{\mathsf{MaxBD}}(S) = \sum_{u \in V} c_u^{\mathsf{MaxBD}}(S)$. Notice that by the definition of MaxBD game, any strategy profile S that is either a NE or a social OPT satisfies $diam_{G[S]}(u) \leq D$ and, therefore $C(S) = \alpha\, C^{\mathsf{MaxBD}}(S)$.

In the following we show how a MaxBD game can be translated, preserving NE, to different instances of celebrity games. A MaxBD game can be seen as a celebrity game in which the weights of each one of the players are large enough so that buying a link is more suitable than having an eccentricity greater than the given distance bound. On the other hand, we show that every celebrity game with $\alpha < w_{min}$ corresponds to a MaxBD game, again preserving NE.

Proposition 8. *Let V be a set of players and $\beta > 1$. Let $\Gamma = \langle V, \beta \rangle$ be a MaxBD game and $\Gamma' = \langle V, (w_v)_{v \in V}, \alpha, \beta \rangle$ be a celebrity game where $\alpha < w_{min}$. Then,* NE$(\Gamma) =$ NE(Γ').

Proof. Let us prove first that NE$(\Gamma) \supseteq$ NE(Γ'). Assume that $S \in$ NE(Γ'). By Property 1, $diam_{G[S]}(u) \leq \beta$ and this implies that $c_u(S) = \alpha |S_u|$. Let us suppose that S is not a NE for Γ. Then there exists a player $u \in V$ and a strategy S'_u such that $c_u^{\mathsf{MaxBD}}(S') < c_u^{\mathsf{MaxBD}}(S) = |S_u|$, where $S' = (S_{-u}, S'_u)$. Hence, the only possibility is that $diam_{G[S']}(u) \leq \beta$ and $|S'_u| < |S_u|$. Therefore $c_u(S') < c_u(S)$ contradicting the fact that $S \in \text{NE}(\Gamma')$.

It remains to show that $\text{NE}(\Gamma) \subseteq \text{NE}(\Gamma')$. Let $S \in \text{NE}(\Gamma)$. We know that $diam(G[S]) \leq \beta$ and $c_u^{\mathsf{MaxBD}}(S) = |S_u|$, for $u \in V$. For Γ', we have that $c_u(S) = \alpha|S_u|$. Now let us assume that S is not a NE of Γ'. Then, there exists $u \in V$ and a strategy S'_u such that $c_u(S) > c_u(S')$, where $S' = (S_{-u}, S'_u)$. Since $w_v > \alpha$, then we have that $c_u(S') = \alpha|S'_u| + \sum_{\{v | d_{G[(S')]}(u,v) > \beta\}} w_v \geq \alpha(|S'_u| + |\{v \mid d_{G[S']}(u,v) > \beta\}|)$. Consider the strategy profile $S'' = (S_{-u}, S''_u)$, where $S''_u = S'_u \cup \{v \mid d_{G[S']}(u,v) > \beta\}$. We have $diam_{G[(S'')]}(u) \leq \beta$. Thus, $c_u(S'') = \alpha|S''_u|$. Combining the inequalities $c_u(S) = \alpha|S_u| > c_u(S'') = \alpha|S''_u|$. Then, $|S_u| > |S''_u|$ contradicting the fact that $S \in \text{NE}(\Gamma)$. □

The previous correspondences allow us to get a relationship on the PoA and the PoS,

Corollary 3. *Let V be a set of players and $\beta > 1$. Let $\Gamma = \langle V, \beta \rangle$ be a MaxBD game and let $\Gamma' = \langle V, (w_v)_{v \in V}, \alpha, \beta \rangle$ be a celebrity game where $\alpha < w_{min}$. Then,*

- *$PoS(\Gamma) = PoS(\Gamma') = 1$,*
- *$PoA(\Gamma) = PoA(\Gamma')$.*

Proof. We know by Proposition 4 that the star tree is a social optimum as well as a NE for celebrity games when $\alpha < w_{min}$. The same occurs for MaxBD games as it was shown in Theorem 3.3 of (Bilò et al., 2012). Hence, $\text{PoS}(\Gamma) = \text{PoS}(\Gamma') = 1$.

For the celebrity game Γ', we have that

$$\text{PoA}(\Gamma) = \frac{\alpha \max_{S \in \text{NE}(\Gamma)} \{|E(G[S])|\}}{\alpha(n-1)} = \frac{\max_{S \in \text{NE}(\Gamma)} \{|E(G[S])|\}}{(n-1)}.$$

By Proposition 8, $\text{NE}(\Gamma) = \text{NE}(\Gamma')$. Thus NE of Γ' have diameter $\leq \beta$ and then we can conclude that $\text{PoA}(\Gamma) = \text{PoA}(\Gamma')$. □

Hence, the upper bound on the PoA of MaxBD games shown in (Bilò et al., 2015b) is also an upper bound for celebrity games. In the subsequent sections we consider the general case where the assumption $\alpha < w_{min}$ is not required.

We have considered here only the uniform version of the MaxBD games in which the eccentricity bound is equal for all the nodes. Bilò et al. (2015b) considers also a non uniform version in which each node has a different eccentricity requirement. It is easy to extend Proposition 8 to show that the set of NE is preserved provided that the eccentricity bounds are the same in both games and $\alpha < w_{min}$. Therefore, non-uniform celebrity games have unbounded PoA, as it was shown for the non-uniform MaxBD games in (Bilò et al., 2015b).

6. Bounding the price of anarchy

We provide here bounds on the contribution of the edges and the weights to the social cost of NE graphs. Those bounds allow us to provide a bound on the PoA. Our next result establishes an upper bound on the PoA in terms of W and α.

Lemma 2. *For a star celebrity game $\Gamma = \langle V, (w_u)_{u\in V}, \alpha, \beta\rangle$, $PoA(\Gamma) \leq W/\alpha$.*

Proof. Let S be a NE of Γ and let $G = G[S] = (V, E)$. As S is a NE, no player has an incentive to deviate from S. Thus, for any $u \in V$,

$$0 \leq \Delta(S_{-u}, \emptyset) \leq -\alpha|S_u| + w(\{v \mid d(u,v) \leq \beta\}) - w_u.$$

Summing up, for all $u \in V$, we have

$$0 \leq \sum_{u\in V}(-\alpha|S_u| + \sum_{\{v|d(u,v)\leq\beta\}} w_v - w_u) = -\alpha|E| + \sum_{u\in V}\sum_{\{v|d(u,v)\leq\beta\}} w_v - W.$$

Therefore,

$$\begin{aligned} C(G) &= \alpha|E| + \sum_{u\in V}\sum_{\{v|d(u,v)>\beta\}} w_v \\ &\leq \sum_{u\in V}\left(\sum_{\{v|d(u,v)\leq\beta\}} w_v + \sum_{\{v|d(u,v)>\beta\}} w_v\right) - W = (n-1)W. \end{aligned}$$

Hence, $\text{PoA}(\Gamma) \leq \frac{(n-1)W}{\alpha(n-1)} = \frac{W}{\alpha}$. □

Using the previous lemma we can get an $O(n)$ upper bound on the PoA of star celebrity games. Let us see that this upper bound can be improved by bounding the weight component and the link component of the social cost, separately.

Define the *weight component* of the social cost, for a critical distance β, $W(G,\beta)$, as

$$W(G,\beta) = \sum_{u\in V(G)}\sum_{\{v|d(u,v)>\beta\}} w_v = \sum_{\{\{u,v\}|d(u,v)>\beta\}} (w_u + w_v).$$

Lemma 3. *Let $\Gamma = \langle V, (w_u)_{u\in V}, \alpha, \beta\rangle$ be a star celebrity game. In a NE graph G, $W(G,\beta) = O(\alpha n^2/\beta)$.*

Proof. Let S be a NE and $G = G[S]$ be a NE graph. Let $u \in V$ and let $b = diam(u)$. Recall that, by Proposition 6, $b \leq 2\beta + 1$. We have three cases.

Case 1: $b < \beta$. For any node $v \in V \setminus \{u\}$ consider the strategy $S'_v = S_v \cup \{u\}$, and let $G' = G[S_{-v}, S'_v]$. By connecting to u we have $diam_{G'}(v) \leq \beta$ and, as S is a NE, we have

$$\Delta(S_{-v}, S'_v) = \alpha - \sum_{\{x|d_G(x,v)>\beta\}} w_x \geq 0.$$

Therefore we have

$$\sum_{\{x \mid d_G(x,v)>\beta\}} w_x \leq \alpha.$$

As $b < \beta$ we conclude that

$$W(G,\beta) \leq n\alpha.$$

Since $1 < \beta \leq n-1$, we get $n/\beta \leq \alpha n^2/\beta$.

Case 2: $b \geq \beta$ and $b \geq 6$. For $1 \leq i \leq b$, consider the set $A_i(u) = \{v \mid d(u,v) = i\}$ and the sets

$$\begin{aligned} C_1 &= \{v \in V \mid 1 \leq d(u,v) \leq b/3\} = \cup_{1\leq i\leq b/3} A_i(u), \\ C_2 &= \{v \in V \mid b/3 < d(u,v) \leq 2b/3\} = \cup_{b/3<j\leq 2b/3} A_j(u), \\ C_3 &= \{v \in V \mid 2b/3 < d(u,v) \leq b\} = \cup_{2b/3<k\leq b} A_k(u). \end{aligned}$$

As $b = diam(u)$, $A_\ell(u) \neq \emptyset$, $1 \leq \ell \leq b$, and all those sets constitute a partition of $V \setminus \{u\}$. As $b \geq 6$, for each ℓ, $1 \leq \ell \leq 3$, C_ℓ contains vertexes at a $b/3 \geq 2$ different distances. Therefore, for $1 \leq \ell \leq 3$, it must exist i_ℓ such that $A_{i_\ell}(u) \subseteq C_\ell$ and $|A_{i_\ell}(u)| \leq 3n/b$, otherwise the total number of elements in C_ℓ would be bigger than n.

For any $v \in V$, let $S'_v = (S_v \cup A_{i_1}(u) \cup A_{i_2}(u) \cup A_{i_3}(u)) \setminus \{v\}$ and let $G' = G[S_{-v}, S'_v]$. Since $b \leq 2\beta + 1$, we have that $b/3 < \beta$. Hence, by construction, $diam_{G'}(v) \leq \beta$. Therefore, as S is a NE, we have

$$0 \leq \Delta(S_{-v}, S'_v) \leq \frac{9n\alpha}{\beta} - \sum_{\{x \mid d_G(x,v)>\beta\}} w_x.$$

Thus,

$$\sum_{\{x \mid d_G(x,v)>\beta\}} w_x \leq \frac{9n\alpha}{\beta} \text{ and } W(G,\beta) \leq \frac{9n^2\alpha}{\beta}.$$

Case 3: $b \geq \beta$ and $b \leq 6$. Consider the sets $A_i(u) = \{v \mid d(u,v) = i\}$, $0 \leq i \leq b$, and the sets $C_0 = \{v \in V \mid d(u,v) \text{ is even}\}$ and $C_1 = V \setminus C_0$. Both sets are non-empty and one of them must have $\leq n/2$ vertexes. By connecting to all the vertexes in the smaller of those sets the diameter of the resulting graph is 2. Therefore, using a similar argument as in case 2, we get

$$W(G,\beta) \leq \frac{n^2\alpha}{2},$$

which is $O(n^2/\beta)$ as $\beta < 6$. Which concludes the proof. □

Our next result provides a bound for the number of edges in a NE graph.

Lemma 4. *Let $\Gamma = \langle V, (w_u)_{u\in V}, \alpha, \beta\rangle$ be a star celebrity game. In a NE graph G, $|E(G)| \leq n - 1 + \frac{3n^2}{\beta}$.*

Proof. Let S be a NE of Γ and let $G = G[S] = (V, E)$. Let u be a node in V. For any $v \in S_u$, recall that $A_{\{u,v\}}(u)$ denotes the set of nodes z such that all shortest paths from u to z use the edge $\{u, v\}$. Observe that $v \in A_{\{u,v\}}(u)$ and that, for $v, v' \in S_u$ with $v \neq v'$, $A_{\{u,v\}}(u) \cap A_{\{u,v'\}}(u) = \emptyset$.

Let $B(G)$ be the set of bridges of G, recall that $|B(G)| \leq n-1$. For $u \in V$, let $\overline{B}(u) = \{x \in S_u \mid \{u, x\} \notin B(G)\}$. Observe that $|E| = |B(G)| + \sum_{u \in V} |\overline{B}(u)|$.

Let us show that for any $v \in S_u$ such that $\{u, v\}$ is not a bridge, there exists $z \in A_{\{u,v\}}(u)$ such that $d(u, z) > \beta/3$.

Let us suppose that $\{u, v\}$ is not a bridge and that, for every $z \in A_{\{u,v\}}(u)$ $d(u, z) \leq \beta/3$. In such a case there must be some edge $\{x, y\}$ with $x \notin A_{\{u,v\}}(u)$ and $y \in A_{\{u,v\}}(u)$. Furthermore, we can select x so that $x \neq u$ and such that there is a shortest path P from u to x using only vertexes in $V \setminus A_{\{u,v\}}(u)$. Observe that $d(u, x) \leq d(u, y) + 1$. Furthermore, for $z \in A_{\{u,v\}}(u)$, there exists a path from u to z that follows P from u to x, the edge $\{x, y\}$, a shortest path from y to v (part of a shortest path to u through $A_{\{u,v\}}(u)$), and a shortest path from v to z (through $A_{\{u,v\}}(u)$). Notice that $d(u, x) \leq \beta/3 + 1$, $d(y, v) \leq \beta/3 - 1$, and $d(v, z) \leq \beta/3 - 1$. Hence, there is a path from u to z of distance $\leq (\beta/3+1)+1+(\beta/3-1)+(\beta/3-1) = \beta$ which does not use $\{u, v\}$. Thus, u has incentive to remove $\{u, v\}$ since $\Delta(S_{-u}, S_u \setminus \{v\}) = -\alpha < 0$, which contradicts the fact that S is a NE.

Therefore, for $v \in \overline{B}(u)$, there exists $z \in A_{\{u,v\}}(u)$ such that $d(u, z) > \beta/3$ and as all the predecessors of z in a shortest path from u belong to $A_{\{u,v\}}(u)$, we have $|A_{\{u,v\}}(u)| > \beta/3$. Observe that $n \geq \sum_{\{v \in S_u \mid v \in \overline{B}(u)\}} |A_{u,v}(u)| \geq |\overline{B}(u)|(\beta/3)$, thus $|\overline{B}(u)| \leq \frac{3n}{\beta}$. Finally, combining the two bounds, we have $|E| = |B(G)| + \sum_{u \in V} |\overline{B}(u)| \leq (n-1) + \frac{3n^2}{\beta}$. □

Observe that, the previous results jointly with $\mathsf{opt}(\Gamma) = \alpha(n-1)$, leads us to the following upper bound of the PoA.

Theorem 2. *For a star celebrity game* Γ*,* $PoA(\Gamma) = O(\min\{n/\beta, W/\alpha\})$*.*

We finalize this section showing a family of star celebrity games having PoA $= \Omega(n)$, for $\beta = 2$.

Lemma 5. *Let* $k > 2$*,* $\alpha > 0$ *and let* $w = (w_1, \ldots w_k)$ *be a positive weight assignment. There is a star celebrity game* $\Gamma = \Gamma(k, \alpha, w)$ *with* $n = 2k$ *players and* $\beta = 2$ *having* $PoA(\Gamma) > \frac{3n}{8}$*.*

Proof. Consider the game $\Gamma_k = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$, where

- $V = \{u_1, \ldots, u_k\} \cup \{v_1 \ldots, v_k\}$,
- $w(u_i) = \alpha$ and $w(v_i) = w_i$, for $1 \leq i \leq n$,
- $\beta = 2$.

Consider any strategy profile S where, for $1 \leq i \leq k$, $\{u_1, \ldots, u_k\} \cap S_{v_i} = \{u_i\}$ and $S_{u_i} = \emptyset$, and such that in $G[S]$ the subgraph induced by $\{v_1 \ldots, v_k\}$ is a clique. An example of such a strategy, for $k = 4$, is given in Figure 1.

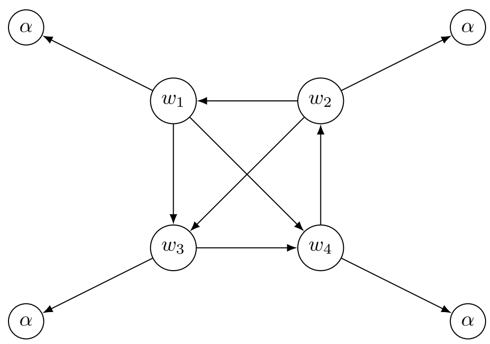

Figure 1: A NE for the game $\Gamma(4, \alpha, w)$.

Observe that there is no vertex in $G[S]$ that is at distance 1 of more than one vertex in $\{u_1, \ldots, u_k\}$. Furthermore, any edge (u, v) lies in the unique shortest path from u to a vertex in $\{u_1, \ldots, u_k\}$. Therefore S is a NE.

We have $C(G[S]) = \alpha\left(\frac{k(k-1)}{2} + k\right) + \alpha k(k-1) = \alpha(3k(k-1) + 2k)/2$. As a star tree is an OPT graph and $n = 2k$, we conclude that

$$\mathrm{PoA}(\Gamma) = \frac{\alpha \frac{\frac{3n}{2}\left(\frac{n}{2}-1\right)+2\frac{n}{2}}{2}}{\alpha(n-1)} = \frac{3n}{8}\frac{(n-1)+\frac{1}{3}}{(n-1)} = \frac{3n}{8}\left(1 + \frac{1}{3(n-1)}\right).$$

□

7. Price of anarchy on Nash equilibrium trees

Now we complement the results of the previous sections by providing a constant upper bound on the PoA when we restrict the NE graphs to be trees. We can find in the literature different models for which the diameter or the PoA can not be proved to be constant on general NE graphs, but they are shown to be constant in the case of NE trees (see for example Alon et al. (2013, 2014); Ehsani et al. (2015)).

In order to get a tighter upper bound for the PoA on NE trees, we first improve the bound on the diameter of NE trees to $\beta + 1$.

Proposition 9. *Let $\Gamma = \langle V, (w_u)_{u\in V}, \alpha, \beta\rangle$ be a star celebrity game. If T is a* NE *tree of Γ, $diam(T) \leq \beta + 1$.*

Proof. Let T be a tree such that $T = G[S]$ where S is a NE of Γ. Let $d = diam(T)$ and let $P : u = u_0, u_1, \ldots, u_d$ be a diametral path of T. Assume that $d > \beta + 1$. For $1 \leq i < d$, let T_i be the connected subtree containing u_i after removing edges $\{u_{i-1}, u_i\}$ and $\{u_i, u_{i+1}\}$. As P is a diametral path, both u and u_d are leaves in T. Furthermore, T_1 and T_{d-1} are star trees. In general, the distance from the leaves of any T_i to both u and u_d is at most d.

We consider two cases depending on who is paying for the connections to the end points of P.

Case 1: $u \in S_{u_1}$ *or* $u_d \in S_{u_{d-1}}$. W.l.o.g. assume that $u_d \in S_{u_{d-1}}$. As S is a NE we have $w_{u_d} \geq \alpha$. Consider the strategy $S'_{u_1} = S_{u_1} \cup \{u_{d-1}\}$, then $\Delta(S_{-u_1}, S'_{u_1}) \leq \alpha - w_{u_d} - w_{u_{d-1}} < 0$ and T can not be a NE graph.

Case 2: $u_1 \in S_u$ *and* $u_{d-1} \in S_{u_d}$. When $\beta \geq 3$. Set $S'_u = S_u - \{u_1\} \cup \{u_2\}$ and $T' = G[(S_{-u}, S'_u)]$. Observe that, for $x \in T_1$, $d_{T'}(u,x) \leq 3 \leq \beta$ and, for $x \notin T_1 \cup \{u\}$, $d_{T'}(u,x) = d_T(u,x) - 1$. Therefore, $\Delta(S_{-u}, S'_u) \leq -w_{u_{\beta+1}} < 0$. Therefore, T is not a NE graph.

The previous argument fails when $\beta = 2$ as there might be $x \in T_1$ with $d_{T'}(u,x) = 3$. From Proposition 6, we know that $d \leq 2\beta + 1 \leq 5$. Let us see that it can not be the case that $d = 4$ or $d = 5$. Let $S'_u = S_u - \{u_1\} \cup \{u_{d-1}\}$ and $S'_{u_d} = S_{u_d} - \{u_{d-1}\} \cup \{u_1\}$. Let $T^1 = G[(S_{-u}, S'_u)]$ and $T^2 = G[(S_{-u_d}, S'_{u_d})]$.

When $d = 4$, for any $x \in T_2$, $d_{T^1}(u,x) = d_T(u,x)$ and $d_{T^2}(u_4,x) = d_T(u_4,x)$. Therefore, we have

$$\Delta(S_{-u}, S'_u) = w(T_1) - w(T_3) - w_{u_4} \text{ and } \Delta(S_{-u_4}, S'_{u_4}) = w(T_3) - w(T_1) - w_u.$$

Thus $\Delta(S_{-u}, S'_u) + \Delta(S_{-u_4}, S'_{u_4}) = -w_u - w_{u_4} < 0$ and one of the two players has an incentive to deviate.

When $d = 5$, we have $\Delta(S_{-u}, S'_u) = w(T_1) + w_{u_2} - w_{u_3} - w(T_4) - w_{u_5}$ and $\Delta(S_{-u_5}, S'_{u_5}) = w(T_4) + w_{u_3} - w_u - w(T_1) - w_{u_2}$. Therefore we have that $\Delta(S_{-u}, S'_u) + \Delta(S_{-u_5}, S'_{u_5}) = -w_u - w_{u_5} < 0$ and one of the two players has an incentive to deviate. □

We need to prove first an auxiliary result.

Lemma 6. *Let* $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ *be a star celebrity game and let* G *be a* NE *graph of* Γ. *If there is* $v \in V$ *with* $diam_G(v) \leq \beta - 1$, *then* $W(G, \beta) \leq \alpha(n-1)$.

Proof. Let $S \in$ NE(Γ) and let $G = G[S]$. Let $u \in V$, $u \neq v$. If $v \notin S_u$, $\Delta(S_{-u}, S_u \cup \{v\}) \geq \alpha - \sum_{\{x | d_G(u,x) > \beta\}} w_x \geq 0$. But, if $v \in S_u$, $diam(u) \leq \beta$.

Hence, $\alpha \geq \sum_{\{x | d_G(u,x) > \beta\}} w_x$ and summing over all $u \neq v$ we have that $\alpha(n-1) \geq W(G, \beta)$. □

The proof of the upper bound for the PoA on NE trees uses the previous statements and examines the particular cases $\beta = 2, 3$.

Theorem 3. *The PoA on* NE *trees of a star celebrity game is at most* 2.

Proof. Let T be a NE tree of Γ. From Proposition 9 we have a bound on the diameter, so we know that $diam(T) \leq \beta + 1$. Since T is a tree, we have that there exists $u \in V$ such that $diam(u) \leq (diam(T) + 1)/2 \leq \beta/2 + 1$. If $\beta \geq 4$, then $diam(u) \leq \beta - 1$. By Lemma 6, $C(T) \leq 2\alpha(n-1)$. Hence, the PoA of NE trees of Γ is at most 2 for $\beta \geq 4$.

In the case of $\beta = 3$, either $diam(T) \leq 3$ or $diam(T) = 4$. In the first case $C(T) = \alpha(n-1)$ and in the second there is u with $diam_T(u) = 2 = \beta - 1$ and we can use Lemma 6.

Finally, we consider the case $\beta = 2$. Notice that the unique tree T with diameter 3 is a double star, a graph that is formed by connecting the centers of two stars. Assume that a NE tree T is formed by T_u, a star with center u, and T_v, a star graph with center v, joined by the edge (u, v). Let L_u (L_v) be the set of leaves in T_u (T_v). As T is a NE graph we have that $w(L_u), w(L_v) \leq \alpha$. Furthermore

$$\begin{aligned} C(T) =& \alpha(n-1) + \sum_{w\in L_u} w(L_v) + \sum_{w\in L_v} w(L_u) \leq \alpha(n-1) + \sum_{w\in L_u} \alpha + \sum_{w\in L_v} \alpha \\ & \leq \alpha(n-1) + \alpha(n-2) \leq 2\alpha(n-1). \end{aligned}$$

□

Note that in a NE tree T, if $\alpha > w_{max}$, for an edge $\{u, v\}$ connecting a leaf u, it must be the case that $v \in S_u$. Then, in the proof of Proposition 9, we only have the case $u_1 \in S_u$. In such case $diam(T) \leq \beta$. Hence, if $\alpha > w_{max}$, the PoA on NE trees is 1.

Corollary 4. *Let $\Gamma = \langle V, (w_u)_{u\in V}, \alpha, \beta\rangle$ be a star celebrity game such that $\alpha > w_{max}$. For any* NE *tree of Γ, $diam(T) \leq \beta$ and therefore the PoA on* NE *trees is* 1.

To tighten the upper bound let us analyze the properties of the NE trees with diameter $\beta + 1$

Lemma 7. *Let T be a* NE *tree of $\Gamma = \langle V, (w_u)_{u\in V}, \alpha, \beta\rangle$ having $diam(T) = \beta+1$, for some $\beta \geq 3$. Let $P = u, u_1, \ldots, u_\beta, v$ be a diametral path in T and let S be a* NE *such that $T = G[S]$. We have that*

1. *u and v are leaves of T.*
2. *$S_u = S_v = \emptyset$.*
3. *$w(u) = w(v) = \alpha$.*
4. *P is the unique diametral path in T.*

Proof. Statement 1 follows from the fact that T is a tree with diameter $\beta + 1$.

To prove the second statement, assume that $S_u \neq \emptyset$. As u is a leaf it must be the case that $S_u = \{u_1\}$. Consider the strategy $S'_u = \{u_2\}$. Taking into account that $d_T(u_2, v) = \beta - 1$ and that the tree rooted at u_2 after deleting (u, u_1) and (u_2, u_3) has depth at most 2, we have that $\Delta(S_{-u}, S'_u) \leq -w(v) < 0$. Contradicting the fact that T is a NE tree. A symmetric argument shows that $S_v = \emptyset$.

To prove the third statement we consider two cases.

Case 1: $w(u) > \alpha$. Let $S'_v = \{u_1\}$, then $\Delta(S_{-v}, S'_v) \leq \alpha - w(u) < 0$. Thus T could not be a NE.

Case 2: $w(u) < \alpha$. By 2 we know that $S_u = S_v = \emptyset$, therefore $u \in S_{u_1}$. Taking $S'_{u_1} = S_{u_1} \setminus \{u_1\}$ we have $\Delta(S_{-u_1}, S'_{u_1}) \leq w(u) - \alpha < 0$. Again S could not be a NE.

We conclude that $w(u) = \alpha$. A symmetric argument shows that $w(v) = \alpha$.

To prove the last statement assume that T has two diametral paths with length $\beta+1$. Let u, v, u', v' be four vertexes such that $d(u,v) = d(u',v') = \beta+1$. We consider two cases.

Case 1: the four vertexes are different. Let P be the shortest path from u to v and P' the shortest path from u' to v'. Let us first show that P and P' must share at least one point. Otherwise let y be the vertex in P that is closest to P' and let x be the vertex in P' that is closest to y. By construction P' lies in the subtree rooted at y after removing the edges in P, thus $d(y,x) > 0$. Therefore, $\max\{d(u,y), d(y,v)\} + d(x,y) + \max\{d(u',x), d(x,v')\} > \beta+1$. Contradicting the fact that T has diameter $\beta+1$.

Thus P and P' share at least one point. Let $x(y)$ be the vertex common to P and P' that is closer to $u(v)$. If there is only one common point $x = y$. Observe that when $x = y$ it must happen that x is the central point of both paths, that is $\beta+1$ must be even and $d(u,x) = d(v,x) = d(u'x) = d(v'x) = (\beta+1)/2$. When $x \neq y$ assume without loss of generality that u' is the vertex in the subtree rooted at x after removing P. In such a case, $d(u',x) = d(u,x) \leq (\beta+1)/2$ and $d(v,y) = d(v',y) \leq (\beta+1)/2$ as otherwise the tree will not have diameter $\beta+1$. Thus $d(u,v') = \beta+1$. By 2 we know that $S_u = \emptyset$ and by 3 that $w(v) = w(v') = \alpha$. Consider the strategy profile, $S'_u = \{y\}$. We have that $\delta(S_{-u}, S'_u) \leq \alpha - w(v) - w(v') < 0$. Therefore T cannot be a NE.

Case 2: two vertexes are the same. Without loss of generality assume that $u' = u$. Let y be the branching point of the paths from u to v and u to v'. As in the previous case, we have that $d(y,v) = d(y,v') \leq (\beta+1)/2$. Considering $S'_u = \{y\}$ we have again that $\delta(S_{-u}, S'_u) \leq \alpha - w(v) - w(v') < 0$. Therefore T cannot be a NE.

We conclude that there are only two vertexes at distance $\beta+1$ in T. □

Putting all together we get an upper bound on the PoA on NE trees when $\beta \neq 2$.

Theorem 4. *The PoA on* NE *trees of a star celebrity game with* $\beta \geq 3$ *and* n *players is at most* $1 + \frac{2}{n-1}$.

Proof. For NE trees with diameter $\leq \beta$ the social cost is $\alpha(n-1)$ but for NE with diameter $\beta+1$, by Lemma 7, the social cost is $\alpha(n-1) + 2\alpha$. As a star is an optimal graph with social cost $\alpha(n-1)$ the claim follows. □

For the case $\beta = 2$ it remains to analyze whether a double star can be a NE for a star celebrity game.

Lemma 8. *Let* T *be a* NE *tree of a star celebrity game* $\Gamma = \langle V, (w_u)_{u\in V}, \alpha, \beta\rangle$ *let* $\beta = 2$. *There is no* NE *tree for* Γ *with diameter 3 except when* $|V| = 4$ *and at least two players have weight* α.

Proof. Assume that a double star T is formed by two starts T_u and T_v with centers u and v respectively and the edge $\{u,v\}$. Let L_u (L_v) be the set of leaves in T_u (T_v). Let S be a NE so that $T = G[S]$. As T is a NE we know that

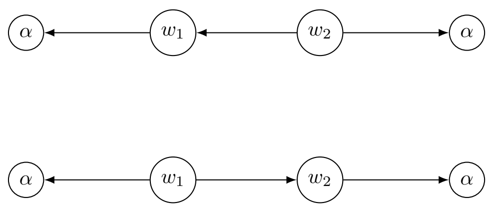

Figure 2: The NE trees with diameter 3

$w(L_u), w(L_v) \leq \alpha$, otherwise by connecting a leaf to the other center their cost will decrease.

Assume that $|L_u|, |L_v| \geq 2$. We have that, for a leaf x, $w(x) < \alpha$. So, $u \in S_x$, for $x \in L_u$, and $v \in S_y$, for $y \in L_v$. Otherwise, u (or v) would benefit by disconnecting to their leaves. For any leaf $x \in L_u$ ($y \in L_v$), consider the strategy $S'_x = \{v\}$ ($S'_y = \{u\}$). For $x \in L_u$, we have $\Delta(S_{-x}, S'_x) = w(L_u) - w(x) - w(L_v) \geq 0$, that is $w(x) \leq w(L_u) - w(L_v)$. For $y \in L_v$, we have $\Delta(S_{-y}, S'_y) = w(L_v) - w(y) - w(L_u) \geq 0$, thus $w(y) \leq w(L_v) - w(L_u)$. Which is impossible as the node weights are positive. Therefore $|L_u| = 1$ or $|L_v| = 1$.

Let us assume w.l.o.g that $L_u = \{x\}$. If $w(x) < \alpha$ and $x \in S_u$, $\Delta(S_u, \emptyset) = w(x) - \alpha < 0$, which is not possible. Therefore, $u \in S_x$. But in such a case $\Delta(S_{-x}, \{v\}) = -w(L_v) < 0$. So, $w(x) = \alpha$.

If $|L_v| > 1$, let $y \in L_v$. As $w(L_v) \leq \alpha$ and $w(y) > 0$, we have $w(y) < \alpha$. Therefore, $v \in S_y$, but then

$$\Delta(S_{-y}, \{u\}) = -w(x) + w(L_v) - w(y) = -\alpha + w(L_v) - w(y) < 0.$$

Contradicting that S is a NE. Thus, $L_v = \{y\}$ and, as for the case $L_u = \{x\}$, we can conclude that $w(y) = \alpha$.

The unique graph satisfying all conditions is a path on 4 vertexes. Furthermore the leaf nodes must have weight α and there are no restrictions for the weights of the internal vertexes. It is easy to see that the unique orientations producing a NE in this particular case are the ones depicted in Figure 2. □

Theorem 5. *The PoA on* NE *trees of star celebrity games is* $\leq 5/3$ *and there are games for which a* NE *tree has cost* $5\,\mathit{opt}/3$.

Proof. For $\beta \geq 3$ and $n \geq 4$, the PoA on NE trees is at most $1 + \frac{2}{n-1} \leq 5/3$, by Theorem 4. For $\beta \geq 3$ and $n < 4$, all trees have diameter at most β, so the PoA on NE trees is 1. For $\beta = 2$ according to Lemma 8 all NE trees have diameter at most β except for P_3 in some cases. When P_3 is a NE we have that $C(P_3) = 3\alpha + 2\alpha = 5\alpha$, giving the upper bound. As there are games for which P_3 is a NE (see Figure 2), the claim follows. □

8. Celebrity games for $\beta = 1$

Let us now analyze the case $\beta = 1$. Observe that every player u for each non-adjacent node v pays w_v, and for each adjacent node pays either α if he has bought the link, or 0, otherwise. Notice that if u establishes the link (u, v), only the node v will take profit of this decision. Contrasting with this, when $\beta > 1$, if player u pays a new link, then all the nodes that get closer to u but not farther than β, will take advantage of this new link.

This particular behavior allows us to show that computing a best response becomes a tractable problem. Furthermore, the structure of NE and OPT graphs is quite different from the case of $\beta > 1$ and we can obtain a tight bound for the PoA.

Proposition 10. *The problem of computing a best response of a player to a strategy profile in celebrity games is polynomial time solvable when $\beta = 1$.*

Proof. Let S be a strategy profile of $\Gamma = \langle V, (w_u)_{u\in V}, \alpha, 1\rangle$ and let $u \in V$. Consider another strategy profile $S' = (S_{-u}, S'_u)$, for some $S'_u \subseteq V \setminus \{u\}$. As $\beta = 1$ we have

$$c_u(S') = \alpha|S'_u| + \sum_{v\notin S'_u} w_v.$$

Note that, when $|S'_u| = k$, the first component of the cost is the same and thus a best response on strategies with k players can be obtained by taking from S'_u the players with the k-th highest weights. Let $S'_u(k)$ be the set of those players and let $W_k = W - w(S'_u(k))$. Thus $c_u(S_{-u}, S'_u(k)) = \alpha k + W_k$. To obtain a best response it is enough to compute the value k for which $c_u((S_{-u}, S'_u(k)))$ is minimum and output $S'_u(k)$. Observe that the overall computation can be performed in polynomial time. □

In order to show a bound for the PoA we prove first some auxiliary results. When $\beta = 1$ pairs of vertexes at distance bigger than one correspond to pairs of vertexes that are not connected by an edge and such a property does not hold for higher values of β.

Proposition 11. *Let $\Gamma = \langle V, (w_u)_{u\in V}, \alpha, 1\rangle$ be a celebrity game. If $G = (V, E)$ is a* NE *graph of Γ, for each $u, v \in V$,*

- *if either $w_u > \alpha$ or $w_v > \alpha$ then $\{u, v\} \in E$,*
- *if both $w_u < \alpha$ and $w_v < \alpha$ then $\{u, v\} \notin E$,*
- *otherwise the edge $\{u, v\}$ might or might not belong to E.*

Proof. Let S be a NE and let $G = G[S] = (V, E)$. Observe that due to the fact that $\beta = 1$, for any player u,

$$c_u(S) = \alpha|S_u| + \sum_{\{v|v\neq u,\{u,v\}\notin E\}} w_v.$$

The cost is thus expressed in terms of the existence or non existence of a connection between pairs of nodes and thus the strategy can be analyzed considering only deviations in which a single edge is added or removed. We analyze the different cases for players u and v.

Case 1: $w_u > \alpha$. For any player $v \neq u$, if the edge $\{u, v\}$ is not present in G the graph cannot be a NE graph as v improves its cost by connecting to u. For the same reason, if the edge is present either $u \in S_v$ or $v \in S_v$. The latter case$v \in S_v$, can happen only when $w_v > \alpha$. Therefore, the player that is paying for the connection will not obtain any benefit by deviating.

Case 2: $w_u, w_v < \alpha$. If the edge $\{u, v\}$ is present in G the graph cannot be a NE graph as the player establishing the connection improves its cost by removing the connection to the other player. For the same reason, if the edge is not present none of the players will obtain any benefit by deviating and paying for the connection.

Case 3: $w_u, w_v = \alpha$. The cost, for any of the players, of establishing the connection or not is the same. In consequence the edge can or cannot be in a NE graph.

Case 4: $w_u = \alpha$ *and* $w_v < \alpha$. Player v is indifferent to be or not to be connected to u, but player u in a NE will never include v in its strategy. Observe that again the edge can or cannot exists in a NE graph but, if it exists, it can only be the case that $u \in S_v$. □

Let us analyze now the structure of the OPT graphs.

Proposition 12. *Let* $G = (V, E)$ *be a* OPT *graph of a celebrity game* $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, 1\rangle$. *For any* $u, v \in V$, *we have*

- *if* $w_u + w_v < \alpha$ *then* $\{u, v\} \notin E$,
- *if* $w_u + w_v > \alpha$ *then* $\{u, v\} \in E$,
- *if* $w_u + w_v = \alpha$ *then* $\{u, v\}$ *might or not be an edge in* G.

Proof. Let S be a strategy profile and let $G = G[S] = (V, E)$ be an OPT graph. As we have seen before as $\beta = 1$, for any player u,

$$c_u(S) = \alpha|S_u| + \sum_{\{v|v \neq u, \{u,v\} \notin E\}} w_v,$$

and we get et the following expression for the social cost

$$C(G) = \alpha|E| + \sum_{\{u,v|u<v, \{u,v\} \notin E\}} (w_u + w_v).$$

The above expression shows that to minimize the contribution to the cost, an edge $\{u, v\}$ can be present in the graph only if $w_u + w_v \geq \alpha$ and will appear for sure only when $w_u + w_v > \alpha$. Thus the claim follows. □

From the previous characterizations we can derive a constant upper bound for the price of anarchy when $\beta = 1$.

Theorem 6. *Let* $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, 1\rangle$ *be a celebrity game.* $PoA(\Gamma) \leq 2$. *Furthermore the, ratio among the social cost of the best and the worst* NE *graphs of* Γ *is bounded by 2.*

Proof. $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, 1\rangle$. Observe that due to the conditions given in Propositions 11 and 12 the social cost of an OPT graph is

$$\sum_{\{\{u,v\} \mid w_u + w_v \geq \alpha\}} \alpha \quad + \sum_{\{\{u,v\} \mid w_u + w_v < \alpha\}} (w_u + w_v),$$

and the social cost of a NE graph with minimum number of edges, i.e., one in which all the optional are not present, is at most

$$\begin{aligned}\sum_{\{\{u,v\} \mid w_u > \alpha \text{ or } w_v > \alpha\}} \alpha + \sum_{\{\{u,v\} \mid w_u, w_v \leq \alpha\}} (w_u + w_v) = \\ = \sum_{\{\{u,v\} \mid w_u > \alpha \text{ or } w_v > \alpha\}} \alpha \\ + \sum_{\{\{u,v\} \mid w_u, w_v \leq \alpha \text{ and } w_u + w_v = \alpha\}} \alpha \\ + \sum_{\{\{u,v\} \mid w_u, w_v \leq \alpha \text{ and } w_u + w_v < \alpha\}} (w_u + w_v) \\ + \sum_{\{\{u,v\} \mid w_u, w_v \leq \alpha \text{ and } w_u + w_v > \alpha\}} (w_u + w_v)\end{aligned}$$

Observe that the difference with the cost of an OPT graph is in the last term

$$D = \{\{u, v\} \mid w_u, w_v \leq \alpha \text{ and } w_u + w_v > \alpha\}.$$

Notice that $\{u, v\} \in D$ contributes to the cost of an OPT graph with α and to the cost of a NE graph with $w_u + w_v$. By taking Γ with $w_u = \alpha$, for any $u \in V$, we can maximize the size of D and this leads to the worst possible NE graph. For such a Γ, I_n is a NE graph and we have that $C(I_n) = \sum_{u,v \in V, u<v}(w_u + w_v) = \alpha n(n-1)$. Furthermore, in any OPT graph of Γ, all the edges will be present, thus we have $\text{OPT} = \alpha n(n-1)/2$. Thus

$$\text{PoA}(\Gamma) \leq \frac{n(n-1)\alpha}{\alpha n(n-1)/2} = 2$$

Observe that when $w_u = \alpha$, for any u, the complete graph is also a NE graph and thus we have that the ratio between the social cost of the worst and the best NE graph is bounded by 2. □

Observe that when $\alpha < w_{min}$ the unique NE is a complete graph which is also an OPT graph. Taking into account that the relationship among celebrity games and MaxBD games provided in Proposition 8 also holds for $\beta = 1$ we can conclude.

Corollary 5. *For $\beta = 1$, the PoA and the PoS of MaxBD games and celebrity games with $\alpha < w_{min}$ is 1.*

9. Conclusions and Open Problems

We have introduced the celebrity games model aiming to address the creation of networks in a scenario where the nodes or players may have different weights and where the requirement of being close to a global critical distance has to be balanced against the node weights. Our results provide further understanding of the structural properties of stable networks. We have shown that the critical distance affects directly the diameter of the stable networks. For star celebrity games the diameter is $\leq 2\beta + 1$ and, in the case that the NE graph is not a tree, the diameter is $> \beta/2$. Furthermore, this critical distance, jointly with player weights and link establishment cost, have implications on the quality of the NE. We have shown that the PoA of star celebrity games is $O(\min\{n/\beta, W/\alpha\})$ and, for $\beta = 2$, we have found games whose PoA is $\Omega(n)$. In contra-position restricting the NE to be trees the PoA is constant.

We can observe that, as one can expect, enlarging the value of the critical distance improves the quality of equilibria. Furthermore, if the total game weight $W = O(\alpha)$, the PoA is $O(1)$. Corresponding to the intuition that when player's weights are negligible players prefer to be isolated. In contrast, when all the players are celebrities, even though their weights could be very different, players prefer to be closer, and the NE graphs have diameter $\leq \beta$. In this latter case, the upper bound on the PoA obtained in Bilò et al. (2015b) for MaxBD games ameliorates the upper bound of celebrity games.

It still remains open to shorten the gap between the lower and upper bounds on the PoA. Our results are only tight for $\beta = 1$ and $\beta = 2$. The cases where β is constant are of particular interest. In the family of graphs providing the lower bound on the PoA not all the nodes are celebrities, so our result has no implication for MaxBD games.

Further questions of interest are to study natural variations of our framework. Among the many possibilities, we propose to analyze celebrity games under (i) the Max cost model (work in progress), (ii) other definitions of the social cost.

Finally, we have not considered the non uniform version where each player u can have its own critical distance β_u. Bilò et al. (2015b) showed that the PoA of MaxBD game is $\Omega(n)$ even for the non uniform model with only two distance-bound values. As we have mentioned before such a negative result for MaxBD games translates to the celebrity games when all the players are celebrities.

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Chapter 6

Max celebrity games

6.1 Summary

In the previous chapter we have introduced and studied a new original model, the Sum Celebrity model or abbreviated, SUM CG. Changing the way in which the weights of the players that are at a distance greater than β affect such player we can obtain very natural variations of the SUM CG. We propose to consider the *max celebrities model*, or abbreviated MAX CG, the model that we obtain keeping all the definitions the same except for the individual cost function of the players, where we change the summation by the maximum. In this way, as in the SUM CG, every MAX CG game is specified by a tuple $\langle V, (w_i)_{i \in V}, \alpha, \beta \rangle$ where, V is the set of players, α is the cost per link, β the critical distance and $(w_i)_{i \in V}$ the weights or relevance degrees of each player, satisfying all of them the same restrictions as in the SUM CG.

Now we move on analysing the MAX CG model paying attention to the analogous elements of interest that we studied in the SUM CG. We address the cases $\beta > 1$ and $\beta = 1$ separately.

The first thing we study for $\beta > 1$, is the problem of computing the Best Response which is NP-hard, too.

Then, we study upper and lower bounds for the PoA. Regarding the upper bounds we compare the PoA vs the critical distance β and the PoA vs the maximum weight of the players $w_{max} = \max_{i \in V} \{w_i\}$. On the positive side, such upper bounds imply low PoA when β is close to n and when w_{max} is close to α, results that are clearly analogous to the SUM CG. On the negative side, contrasting with the SUM CG results, in the MAX CG we find several situations in which games have a very large PoA.

Regarding the topology, we see that the diameter of equilibria is related to the critical distance β, too. Moreover, we also study the connectivity properties of equilibria. Due to the fact that there exist various situations in which equilibria are disconnected, we consider a little variation in the definition of the cost function for the players of any MAX CG game by introducing infinite individual cost exactly for every configuration that has associated a non-connected communication network. When doing so, we assure that every equilibrium graph is connected but then there appear equilibrium graphs that have diameter $n - 1$.

Finally, for the case $\beta = 1$ we investigate the problem of computing the best response as well as the PoA. Although the problem of computing the best response is polynomial time solvable, as in the SUM CG, we are not able to prove an upper bound on the PoA having low value in general. This could be explained because the maximum is a slightly more involved function than the summation so that for $\beta = 1$, although both SUM CG and MAX CG become simple models, the MAX CG is not as tractable as the SUM CG.

6.2 Article: Max Celebrity Games

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Max Celebrity Games

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Abstract. We introduce *Max celebrity games* a new variant of Celebrity games defined in [4]. In both models players have weights and there is a critical distance β as well as a link cost α. In the max celebrity model the cost of a player depends on the cost of establishing links to other players and on the maximum of the weights of those nodes that are farther away than β (instead of the sum of weights as in celebrity games). The main results for $\beta > 1$ are that: computing a best response for a player is NP-hard; the optimal social cost of a celebrity game depends on the relation between α and w_{max}; NE always exist and NE graphs are either connected or a set of $r \geq 1$ connected components where at most one of them is not an isolated node; for the class of connected NE graphs we obtain a general upper bound of $2\beta+2$ for the diameter. We also analyze the price of anarchy (PoA) of connected NE graphs and we show that there exist games Γ such that $\mathrm{PoA}(\Gamma) = \Theta(n/\beta)$; modifying the cost of a player we guarantee that all NE graphs are connected, but the diameter might be $n-1$. Finally, when $\beta = 1$, computing a best response for a player is polynomial time solvable and the $\mathrm{PoA} = O(w_{max}/w_{min})$.

1 Introduction

The increasing use of Internet and social networks, has motivated a great interest to model theoretically their behavior. Fabrikant et al. [15] proposed a game-theoretic model of network creation (NCG) as a simple tool to analyze the creation of Internet as a decentralized and non-cooperative communication network among players (the network nodes).

In this model the goal of each player is to have, in the resulting network, all the other nodes as close as possible paying a minimum cost. It is assumed that: all the players have the same interest (all-to-all communication pattern with identical weights); the cost of being disconnected is infinite; and the links to other nodes paid by one node can be used by others. Formally, a game Γ in this seminal model is defined as a tuple $\Gamma = \langle V, \alpha \rangle$, where V is the set of n nodes and α the cost of establishing a link. A strategy for player $u \in V$ is a subset $S_u \subseteq V - \{u\}$, the set of players for which player u pays for establishing a link. The n players and their joint strategy choices $S = (S_u)_{u \in V}$ create an undirected graph $G[S]$. The cost function for each node u under strategy S is defined by $c_u(S) = \alpha|S_u| + \sum_{v \in V} d_{G[S]}(u, v)$ where $d_{G[S]}(u, v)$ is the distance between nodes u and v in graph $G[S]$. By changing the cost function to $c_u(S) =$

$\alpha|s_u| + \max\{d_{G[S]}(u,v)|v \in V\}$ as proposed in [13] one obtains the max game model.

From here on several versions have been considered to make the model a little more realistic. For different variants we refer the interested reader to [1–3, 6, 9–14, 16, 17, 19] among others.

In Internet as well as in social networks not all the nodes have the same importance. It seems natural to consider nodes with different relevance weights. In such a setting, the cost of being far (even if connected) from high-weight nodes should be greater than the cost of being far from low-weight nodes. In [4] we introduce *celebrity games* with the aim to study the combined effect of having players with different weights that share a common distance bound.

In celebrity games the cost of a player has two components. The first one is the cost of the links established by the node. The second one is the sum of the weights of those nodes that are farther away than the critical distance. Formally, a celebrity game is defined by $\Gamma = \langle V, (w_u)_{u\in V}, \alpha, \beta\rangle$, where V is a set of nodes with weights $(w_u)_{u\in V}$, α is the cost of establishing a link and β establishes the desirable distance bound. The cost function for each node is defined by $c_u(S) = \alpha|S_u| + \sum_{\{v|d_{G[S]}(u,v)>\beta\}} w_v$.

In this paper we extend the study initiated in [4]: we define a max version of the celebrity games that we name *max celebrity games* and we analyze the structure and quality of their Nash equilibria. From now on, let us refer to celebrity games as *sum celebrity games.* In the max celebrity model the cost of a player takes into account the maximum of the weights (worst-case) of those nodes that are farther away than the critical distance, instead of the sum of weights (average-case). The cost function is formally defined by $c_u(S) = \alpha|S_u| + \max_{\{v|d_{G[S]}(u,v)>\beta\}} w_v$. Intuitively, the goal of each player in max celebrity games is to buy as few links as possible in order to have the high-weighted nodes closer to the given critical distance. Observe that if the cost of establishing links is higher than the benefit of having close a node (or set of nodes), players might rather prefer to stay either far or even disconnected from it.

Observe that the main feature of both, sum and max celebrity games, is the combination of bounded distance with players having different weights. Even though heterogeneous players have been considered in NCG under bilateral contracting [5, 18], and the notion of bounded distance has been studied in [8], to the best of our knowledge sum celebrity games is the first model that studies how a common critical distance, different weights, and a link cost, altogether affect the individual preferences of the players. Furthermore, max celebrity games is the first model that focuses on how the maximum weight of those nodes that are farther than β affects the creation of graphs.

In this paper we analyze the structure of Nash equilibrium (NE) graphs of max celebrity games and their quality with respect to the optimal strategies. To do so we address the cases $\beta > 1$ and $\beta = 1$, separately. For $\beta > 1$, every player u has to choose for each non-edge (u, v) between paying the maximum of the weights of the nodes with distance to u greater than β, or buying the link (u, v) and paying α for the link minus the maximum of the weights of those nodes whose distance to u will become less or equal than the critical distance β. While for $\beta = 1$, each player u has to decide for every non-edge (u, v) of the graph to pay either α for the link or at least w_v (the weight of the non-adjacent node v).

For the general case $\beta > 1$ our results can be summarized as follows: computing a best response for a player is NP-hard; the optimal social cost of a celebrity game Γ depends on the relation between α and the maximum weight w_{max}; NE always exist and NE graphs are either connected or a set of $r \geq 1$ connected components where at most one of them is not an isolated node; for the class of connected NE graphs we obtain a general upper bound of $2\beta+2$ for the diameter; we also analyze the quality of connected NE graphs and we show that there exist max celebrity games such that $\mathrm{PoA}(\Gamma) = \Theta(n/\beta)$; we consider a variation of the cost of the player in order to avoid non-connected NE graphs.

Finally, for the particular case $\beta = 1$, we show that computing a best response for a player is polynomial time solvable and that the $\mathrm{PoA} = O(w_{max}/w_{min})$.

The paper is organized as follows. In Section 2 we introduce the basic definitions and the max celebrity model. In Section 3 we study the fundamental properties of optimal graphs and NE graphs. Section 4 studies for $\beta > 1$ the quality of connected NE graphs and considers a modification of the cost of a player in order to guarantee connected NE graphs. In Section 5 we study the complexity of the best response problem and the PoA for the case $\beta = 1$. Finally, in Section 6 we give an outline of the main differences between max and sum celebrity models.

2 The Model

We use standard notation for graphs and strategic games. All the graphs in the paper are undirected unless explicitly said otherwise. For a graph $G = (V, E)$ and $u, v \in E$, $d_G(u, v)$ denotes the distance, i.e. the length of a shortest path, from u to v in G. The *diameter* of a vertex $u \in V$, $diam_G(u)$, is defined as $diam_G(u) = \max_{v \in V}\{d_G(u, v)\}$ and the *diameter* of G, $diam(G)$, is defined as usual as $diam(G) = \max_{v \in V}\{diam_G(v)\}$. An *orientation* of an undirected graph is an assignment of a direction to each edge, turning the initial graph into a directed graph.

For a weighted set $(V, (w_u)_{u \in V})$ we extend the weight function to subsets in the following way. For $U \subseteq V$, $w(U) = \max_{u \in U}\{w_u\}$. Furthermore, we set $w_{max} = \max_{u \in V}\{w_u\}$ and $w_{min} = \min_{u \in V}\{w_u\}$.

Definition 1. *A* max celebrity game *Γ is defined by a tuple $\langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ where: $V = \{1, \ldots, n\}$ is the set of players, for each player $u \in V$; $w_u > 0$ is the* weight *of player u; $\alpha > 0$ is the cost of establishing a link and β, $1 \leq \beta \leq n - 1$, is the* critical distance.

A strategy *for player u is a subset $S_u \subseteq V - \{u\}$ denoting the set of players for which player u pays for establishing a direct link. A* strategy profile *for Γ is a tuple $S = (S_1, S_2, \ldots, S_n)$ defining a strategy for each player. For a strategy profile S, the associated* outcome graph *is the undirected graph $G[S]$ which is defined by $G[S] = (V, \{\{u, v\} | u \in S_v \vee v \in S_u\})$.*

For a strategy profile $S = (S_1, S_2, \ldots, S_n)$, the cost function of player u, denoted by c_u, is defined as $c_u(S) = \alpha|S_u| + W_u$ where $W_u = \max_{\{v \mid d_{G[S]}(u,v) > \beta\}}\{w_v\}$. And as usual, the social cost *of a strategy profile S in Γ is defined as $C(S) = \sum_{u \in V} c_u(S)$. The* social cost *of a graph G in Γ is defined analogously as $C(G) = \alpha|E(G)| + \sum_{u \in V(G)} W_u$.*

Observe that, even though a link might be established by only one player, we consider the outcome graph as an undirected graph, assuming that once a link is bought the link can be used in both directions. In our definition we have considered a general case in which players may have different weights and defined the cost function through properties of the undirected graph created by the strategy profile. The player's cost function takes into account two components: the cost of establishing links and the maximum of the weights of the players that are at a distance greater than the critical distance β.

In the remaining of the paper, we assume that, for $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$, the parameters verify the required conditions. Furthermore, unless specifically stated, we consider $\beta > 1$, the case $\beta = 1$ will be studied in Section 5. We use the following notation, for a game $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$, $n = |V|$. We denote by $\mathcal{S}(u)$ the set of strategies for player u and by $\mathcal{S}(\Gamma)$ the set of strategy profiles of Γ.

As usual, for a strategy profile S and a strategy S'_u for player u, (S_{-u}, S'_u) represents the strategy profile in which S_u is replaced by S'_u while the strategies of the other players remain unchanged. The *cost difference* $\Delta(S_{-u}, S'_u)$ is defined as $\Delta(S_{-u}, S'_u) = c_u(S_{-u}, S'_u) - c_u(S)$. Observe that, if $\Delta(S_{-u}, S'_u) < 0$, player u has an incentive to deviate from S_u. A best response to $S \in \mathcal{S}(\Gamma)$ for player u is a strategy $S'_u \in \mathcal{S}(u)$ minimizing $\Delta(S_{-u}, S'_u)$. Let us remind the definition of Nash equilibrium.

Definition 2. *Let $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ be a max celebrity game. A strategy profile $S = (S_1, S_2, \ldots, S_n)$ is a* Nash equilibria *of Γ if no player has an incentive to deviate from his strategy. Formally, for a player u and each strategy $S'_u \in \mathcal{S}(u)$, $\Delta(S_{-u}, S'_u) \geq 0$.*

We denote by $\mathrm{NE}(\Gamma)$ the set of Nash equilibria of a game Γ. We use the term NE to refer to a strategy profile $S \in \mathrm{NE}(\Gamma)$. We say that a graph G is a NE *graph* if there is $S \in \mathrm{NE}(\Gamma)$ so that $G = G[S]$.

We denote by $\mathsf{opt}(\Gamma)$ the minimum value of the social cost, i.e. $\mathsf{opt}(\Gamma) = \min_{S \in \mathcal{S}(\Gamma)} C(S)$. We denote by $\mathrm{OPT}(\Gamma)$ the set of optimum strategy profiles of Γ w.r.t. the social cost, that is, for $S \in \mathrm{OPT}(\Gamma)$, $C(S) = \mathsf{opt}(\Gamma)$. We use the term OPT *strategy profile* to refer to a $S \in \mathrm{OPT}(\Gamma)$.

It is worth observing that: for $S \in \mathrm{NE}(\Gamma)$, it never happens that $v \in S_u$ and $u \in S_v$, for any $u, v \in V$; a NE graph G can be the outcome of several strategy profiles and not all the orientations of a NE graph G are NE.

In the following we make use of some particular outcome graphs on n vertices: I_n, the independent set; and ST_n a star graph, i.e. a tree in which one of the vertices, the *central* vertex, is connected to all the other $n-1$ vertices.

We define the Price of Anarchy and the Price of Stability as usual.

Definition 3. *Let Γ be a max celebrity game. The* Price of Anarchy *of Γ is defined as $PoA(\Gamma) = \max_{S \in NE(\Gamma)} C(S)/\mathsf{opt}(\Gamma)$ and the* Price of Stability *of Γ is defined as $PoS(\Gamma) = \min_{S \in NE(\Gamma)} C(S)/\mathsf{opt}(\Gamma)$*

The explicit reference to Γ will be dropped whenever Γ is clear from the context. We will refer to $\textsc{ne}(\Gamma)$, $\mathsf{opt}(\Gamma)$, $PoA(\Gamma)$, and $PoS(\Gamma)$ by $\textsc{ne}$, opt, PoA and PoS respectively.

Our first result shows that the computation of a best response in max celebrity games is a NP-hard problem for $\beta \geq 2$. The proof consists in a reduction from the Dominating Set problem. The problem becomes polynomial time computable for $\beta = 1$ as we show in Section 5.

Proposition 1. *Computing a best response for a player to a strategy profile in a max celebrity game is NP-hard even when $\beta = 2$.*

3 Social Optimum and Nash Equilibrium

In this section we analyze some properties of the opt and $\textsc{ne}$ strategy profiles in max celebrity games. We start by giving bounds for opt that depend on the existence of one or more than one connected components.

Proposition 2. *Let $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ be a max celebrity game. We have that $2\alpha(n-1) \geq \mathsf{opt}(\Gamma) \geq \min\{\alpha(n-1), w_{max}(n-1) + w_{min}\}$.*

Proof. Let $S \in \mathrm{OPT}(\Gamma)$ and let $G = G[S]$. Let $G_1, ..., G_r$ be the connected components of G and let $V_i = V(G_i), k_i = |V_i|$, and $W_i = w(V_i)$, for $1 \leq i \leq r$. Assume w.l.o.g that $W_1 \geq W_2 \geq ... \geq W_r$. Observe that the social cost of a disconnected graph can be expressed as the sum of the social cost of the connected components plus the additional contribution of the pairs of vertices that lie in different components. Each connected component must be a tree of diameter at most β, otherwise a strategy profile with smaller social cost could be obtained by replacing the connections on V_i by such a tree. W.l.o.g we can assume that, for $1 \leq i \leq r$, the $i-$th connected component is a star graph ST_{k_i} on k_i vertices. Recall that $C(ST_{k_i}) = \alpha(k_i - 1)$, thus $C(G) = \sum_{i=1}^{r} \alpha(k_i - 1) + \sum_{i=1}^{r} k_i \left(\max_{j \neq i}\{W_j\}\right) = \alpha(n-r) + nW_1 - k_1(W_1 - W_2)$.

Notice that if for some $i > 1$, the i-th connected component is not an isolated node, then the node with maximum weight in this connected component can be moved to G_1. By preserving the connectivity and structure (a star) of G_1, the social cost of the resulting graph is strictly smaller than the cost of the original G. This implies that for every $i > 1$, $k_i = 1$. Hence, $C(G) = \alpha(n-r) + (r-1)W_1 + (n-r+1)W_2$.

If $r = 1$ then $C(G) = \alpha(n-1)$ and we are done. Otherwise, if $r > 1$, then we have the inequality $C(ST_n) \geq C(G)$. This implies that $\alpha \geq \frac{1}{r-1}(W_1(r-1) + W_2(n-r+1)) \geq W_1$.

Then we get the following results. First: $C(G) \geq W_1(n-r) + (r-1)W_1 + (n-r+1)W_2 \geq (n-1)W_1 + W_2 \geq (n-1)w_{max} + w_{min}$.

Secondly, using that $r > 1$: $C(G) = \alpha(n-r) + (r-1)W_1 + (n-r+1)W_2 \leq \alpha(n-r) + (r-1)\alpha + (n-r+1)\alpha \leq 2\alpha(n-1)$.

The relationship between α and w_{max} determines partially the topology of the NE graphs. As one can expect, if $\alpha > w_{max}$, no player has incentive to establish a link then the independent set is the unique NE graph. Otherwise, any NE graph can be connected or disconnected.

Proposition 3. *Every max celebrity game $\Gamma = \langle V, (w_u)_{u\in V}, \alpha, \beta\rangle$ has a* NE. *Furthermore, when $\alpha \leq w_{max}$, ST_n is a* NE *graph, and when $\alpha \geq w_{max}$, I_n is a* NE *graph. If $\alpha > w_{max}$, then I_n is the unique* NE *graph.*

Proof. When $\alpha \leq w_{max}$ let us show that ST_n is a NE graph. Let u_{max} a node with maximum weight and we suppose that it is the central node of the star. If $S_{u_{max}} = \emptyset$ and for every node $v \neq u_{max}$, $S_v = \{u_{max}\}$, then u_{max} cannot improve its actual cost since it is exactly 0. Moreover, the other nodes can only delete the edge to u_{max}. Since such deviation has a cost increment of $-\alpha + w_{max} \geq 0$, then we are done.

When $\alpha \geq w_{max}$, let us show that I_n is a NE graph. Let S be the empty strategy profile, $I_n = G[S]$. Notice that for any player u, if $S'_u \neq \emptyset$, then $\Delta(S_{-u}, S'_u) \geq |S'_u|\alpha - w_{max} \geq (|S'_u| - 1)w_{max} \geq 0$. Finally, if $\alpha > w_{max}$ it is easy to see that the unique NE graph is I_n. Let us suppose that there exist $u, v \in V$ such that $v \in S_u$. If $S'_u = S_u - \{v\}$, then $\Delta(S_{-u}, S'_u) \leq -\alpha + w_{max} < 0$. Hence, if $G \neq I_n$, then G is not a NE graph.

Corollary 1. *Let $\Gamma = \langle V, (w_u)_{u\in V}, \alpha, \beta\rangle$ be a max celebrity game. Then, $PoS(\Gamma) = 1$ for $\alpha \leq w_{max}$ and $PoS(\Gamma) < 2$ for $\alpha > w_{max}$.*

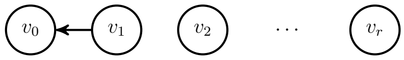
In particular, even in the case that $\alpha < w_{max}$, it can be shown that there exist max celebrity games where I_n is a NE graph. Indeed consider a game with $n \geq 2$ and weights defined as $w_i = (i-1)\alpha$ for $i > 1$ and $w_1 = \alpha$. Then, clearly $\alpha < w_{max}$ and I_n is a NE graph.

Furthermore, for every integer $1 < r \leq n$, there exists non-connected max celebrity games with exactly r different connected components. Moreover, the only connected component that can have more than one node is the one that contains a node with weight w_{max}.

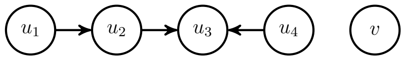
Proposition 4. *Any* NE *graph distinct from I_n has at most one non-trivial connected component. Moreover, for every integer $r \geq 2$ there exists a max celebrity game having a* NE *graph with exactly r connected components.*

Proof. (Sketch) For the first part, let $G_1, ..., G_r$ be the connected components of a NE graph. Assume that a node with the maximum weight is in G_1. If for some $i > 1$, $|G_i| > 1$, then there exist $u, v \in V(G_i)$ such that $u \in S_v$. In this case, v can strictly decrease its cost deleting this edge because the node with maximum weight is still at distance greater than β, contradicting the fact that G is a NE graph.

For the second part we distinguish two cases: $r \geq 3$ and $r = 2$. For the first case, let $n = r + 1$, $V = \{v_0, v_1, ..., v_r\}$, $E = \{\{v_0, v_1\}\}$ with $S_{v_1} = \{v_0\}$, $S_{v_i} = \emptyset$ for $i \neq 1$, as depicted in the figure below. For the weights consider $w_{v_0} = w_1$ and $w_{v_i} = w_2$ for all $i \geq 1$, with $w_1 > w_2$, $w_1 - w_2 = \alpha$ and $\alpha \geq w_1/(n-1), w_2/(n-2)$. We have that this configuration is a NE.



For the case $r = 2$ see the figure below. It is not hard to see that this configuration is also a NE.



4 The price of anarchy

Observe that there exist max celebrity games Γ with $\alpha \leq w_{max}$ having disconnected NE graphs with high social cost in comparison with the optimum. Indeed, consider the example given in Proposition 4 with $w_2 = \frac{w_1(n-2)}{(n-1)}$. The cost of this NE graph is $w_1(n-1) + \frac{w_1(n-2)}{n-1}$ and combining it with $\mathsf{opt} \leq 2\alpha(n-1)$, we get the bound $\mathrm{PoA}(\Gamma) \geq (n-1)/2$. Hence, we focus on the study of the PoA for connected NE graphs. Since the restriction $\alpha \leq w_{max}$ by itself does not exclude the possibility of having non-connected NE graphs, we study the PoA of connected equilibria from two different perspectives: first, we analyze the worst case among all connected NE graphs; second, we introduce a slight modification of the player's cost function in order to guarantee connectivity in the class of NE graphs. Whenever we consider the class of connected NE graphs we compare the social cost of such equilibria with the optimum value among the connected graphs, $\mathsf{opt}(\Gamma) = \alpha(n-1)$.

4.1 PoA and diameter of connected NE graphs

In this subsection we analyze the quality and structure of equilibria in terms of the parameters that define the max celebrity games. Our next result indicates that the price to pay for the anarchy is low when α is close to w_{max}.

Proposition 5. *For every max celebrity game $\Gamma = \langle V, (w_u)_{u\in V}, \alpha, \beta, \rangle$, $PoA(\Gamma) \leq 2(w_{max}/\alpha)$.*

Proof. Let S be a NE of Γ and let $G = G[S] = (V, E)$. Then, no player has an incentive to deviate from S. In particular, for each $u \in V$ we have that $0 \leq \Delta(S_{-u}, \emptyset) = -\alpha|S_u| + W'_u - W_u$ where $W_u = \max_{\{x | d_G(u,x) > \beta\}} w_x$ and $W'_u = \max_{\{x | d_{G[(S_{-u},\emptyset)]}(u,x) > \beta\}} w_x$. By adding for each $u \in V$ the corresponding inequalities, we have that $0 \leq \sum_{u\in V}(-\alpha|S_u| + W'_u - W_u) = -\alpha|E| + \sum_{u\in V} W'_u - \sum_{u\in V} W_u$.

Therefore, $C(G) = \alpha|E| + \sum_{u\in V} W_u \leq \sum_{u\in V} W'_u \leq n w_{max}$ and we can conclude that $\mathrm{PoA}(\Gamma) \leq \frac{n w_{max}}{\alpha(n-1)} \leq 2\frac{w_{max}}{\alpha}$.

The diameter of NE graphs depends directly on the critical distance β.

Proposition 6. *Let $\Gamma = \langle V, (w_u)_{u\in V}, \alpha, \beta \rangle$ be a max celebrity game. In a NE graph G for Γ, $diam(G) \leq 2\beta + 2$.*

Proof. Let $S \in \text{NE}(\Gamma)$ such that $G = G[S]$ is connected. Assume that the node u satisfies that $d_G(u, u_{max}) > \beta$ and $|S_u| > 0$. Then u has incentive to break any of its bought links because after doing so, u_{max} will still remain inside the complementary of the ball of radius β centered at u. Next, assume that $diam(u_{max}) \geq \beta + 2$. Let $u_{max}, u_1, u_2, ..., u_{\beta+2}$ be a path. Then, either $u_{\beta+1} \in S_{u_{\beta+2}}$ or $u_{\beta+2} \in S_{u_{\beta+1}}$. Therefore, since both $u_{\beta+1}, u_{\beta+2}$ are at distances greater than β from u_{max}, G cannot be a NE. This proves that $diam(u_{max}) \leq \beta + 1$ in any connected NE and, as a consequence, that $diam(G) \leq 2\beta + 2$.

Let us provide for a NE graph G, a bound on the contribution of the *weight component* of the social cost of G, $W(G, \beta) = \sum_{\{u \in V(G)\}} W_u$. The following lemma is a reformulation of a similar result that can be found in [4] using a cleaner and simpler argument.

Lemma 1. *Let $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ be a max celebrity game. In a NE graph G for Γ, $W(G, \beta) = O(\alpha n^2/\beta)$.*

Proof. Let S be a NE and $G = G[S]$ be a connected NE graph. Let $u \in V$ be any node in V. Consider the sets $A_i(u) = \{v \mid d_G(u, v) = i\}$. Define for $i = 1, ..., k$, $C_i = \{v \in V \mid (i-1)(\beta-1) \leq d_G(u, v) < i(\beta-1)\} = \cup_{(i-1)(\beta-1) \leq j < i(\beta-1)} A_j(u)$ with k such that $\cup_{i=1}^k C_i = V(G)$. By the pigeonhole principle, for each $i = 1, ..., k$ there exists at least one subindex, call it $j(i)$, for which $(i-1)(\beta-1) \leq j(i) < i(\beta-1)$ and $|A_{j(i)}(u)| \leq |C_i|/(\beta-1)$. In this way, for any node $v \in V(G)$, let $S'_v = (S_v \cup_{i=1}^k A_{j(i)}(u)) - \{v\}$ and let $G' = G[S_{-v}, S'_v]$. By construction, $diam_{G'}(v) \leq \beta$. Therefore, as S is a NE, we have $0 \leq \Delta(S_{-v}, S'_v) \leq \alpha \sum_{i=1}^k \frac{|C_i|}{\beta-1} - W_v = \alpha\left(\frac{n-1}{\beta-1}\right) - W_v$.

Thus, $W(G, \beta) \leq \frac{n(n-1)\alpha}{\beta-1} = O\left(\frac{\alpha n^2}{\beta}\right)$.

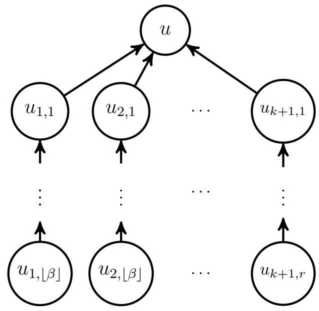
Using the same technique to provide a bound for the number of edges in NE graphs for a sum celebrity games (Proof of Lemma 4 of [4]), we obtain the following result.

Lemma 2. *Let $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ be a max celebrity game. In a NE graph G for Γ, $|E(G)| \leq n - 1 + \frac{3n^2}{\beta}$.*

Corollary 2. *For every max celebrity game $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta, \rangle$, $PoA(\Gamma) = O(n/\beta)$*

Proposition 7. *For every $n > \beta > 1$, there exists a max celebrity game $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta, \rangle$ such that $PoA(\Gamma) = \Omega(n/\beta)$.*

Proof. Given n, let k and r be such that $n-1 = k\lfloor\beta\rfloor + r$, $k \geq 3$ and $0 \leq r < \lfloor\beta\rfloor$. Let $V = \{u\} \cup \left(\cup_{i=1}^k \{u_{i,j} \mid 1 \leq j \leq \lfloor\beta\rfloor\}\right) \cup \{u_{k+1,1}, u_{k+1,2}, ..., u_{k+1,r}\}$. We then define $w_u = W$ and $w_{u_{i,j}} = w$ with W, w such that $w = (k-2)\alpha$ and $W > n\alpha$. In this way we consider the configuration S defined by the relations $S_u = \emptyset$, $S_{u_{i,j}} = \{u_{i,j-1}\}$ for $j \geq 2$ and $S_{u_{i,1}} = \{u\}$ for $i = 1, ..., k+1$, as depicted in the figure below. To prove that $S \in \text{NE}(\Gamma)$ we see that the cost difference associated to any deviation is not negative.



Clearly, u has no incentive of deviating his strategy because his cost is zero. Let us prove that any other node $u_{i,j}$ has no incentive in deviating from its current strategy. We say that a node v is covered with respect a node v' if v is at a distance at most β from v'. We have three cases:

1. The deviation is such that all nodes are covered with respect $u_{i,j}$. In this situation the cost difference is $l\alpha - w$. Notice that every node $u_{h,\lfloor\beta\rfloor}$ with $h \neq i$ can be reached only when a link from $u_{i,j}$ to the path formed by $u_{h,1}, u_{h,2}, ..., u_{h,\lfloor\beta\rfloor}$ is bought. Since initially $u_{i,j}$ has bought one link this leads to the inequality $l \geq k-2$. Therefore $l\alpha - w \geq (k-2)\alpha - (k-2)\alpha = 0$.

2. The deviation is such that u is uncovered with respect $u_{i,j}$. In this situation, since $W > n\alpha$, the cost difference is $l\alpha - w + W \geq 0$, for $-1 \leq l \leq n-1$.

3. The deviation is such that u is covered with respect $u_{i,j}$ but there is at least one node node of weight w uncovered with respect $u_{i,j}$. Then the cost difference is $l\alpha$ for some integer l. The only negative value that l can take is -1, but in such case the configuration leaves u uncovered with respect $u_{i,j}$, a contradiction. Therefore, $l\alpha \geq 0$.

Hence, $S \in \text{NE}(\Gamma)$ and $C(S) > (n-1)w = (n-1)(k-2)\alpha$. Using the bound for the social optimum $\mathsf{opt}(\Gamma) \leq 2\alpha(n-1)$ we have that $\text{PoA}(\Gamma) \geq (k-2)/2$.

Theorem 1. *For every $n > \beta > 1$, there exists a max celebrity game $\Gamma = \langle V, (w_u)_{u\in V}, \alpha, \beta, \rangle$ such that $PoA(\Gamma) = \Theta(n/\beta)$.*

4.2 The PoA when the connectivity of the NE graphs is guaranteed

Let us consider a new cost function that excludes non-connected NE graphs. We define a *connected max celebrity game* Γ^{con} as a max celebrity game $\Gamma^{con} = \langle V, (w_u)_{u\in V}, \alpha, \beta, \rangle$, but now, the cost for each player $u \in V$ in strategy profile S is denoted by $c_u^{con}(S)$ and it is defined as follows: $c_u^{con}(S) = c_u(S)$, if $diam_{G[S]}(u) \leq n-1$; otherwise, $c_u^{con}(S) = \infty$. As usual, the social cost of a strategy profile S in Γ^{con} is defined as $C^{con}(S) = \Sigma_{u\in V} c_u^{con}(S)$. Since for any connected graph G, $C^{con}(G) = C(G) \geq \alpha(n-1)$, then we have that $\mathsf{opt}(\Gamma^{con}) = \alpha(n-1)$. Notice that the same tuple $\langle V, (w_u)_{u\in V}, \alpha, \beta, \rangle$ can define a max celebrity game as well as a connected max celebrity game. In order to distinguish one from the other, we denote by $\Gamma = \Gamma(\langle V, (w_u)_{u\in V}, \alpha, \beta, \rangle)$ the corresponding max celebrity game and by $\Gamma^{con} = \Gamma^{con}(\langle V, (w_u)_{u\in V}, \alpha, \beta, \rangle)$, the corresponding connected max celebrity game.

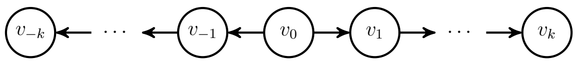
Proposition 8. *Let $\langle V, (w_u)_{u\in V}, \alpha, \beta\rangle$ be a tuple defining $\Gamma = \Gamma(\langle V, (w_u)_{u\in V}, \alpha, \beta\rangle)$ and $\Gamma^{con} = \Gamma^{con}(\langle V, (w_u)_{u\in V}, \alpha, \beta\rangle)$. Then,* $\textsc{ne}(\Gamma) \subsetneq \textsc{ne}(\Gamma^{con})$ *when we consider* $\textsc{ne}(\Gamma)$ *restricted to connected graphs.*

Proof. Let $S \in \textsc{ne}(\Gamma)$ be such that $G = G[S]$ is connected. Let u be a player, let S'_u be a deviation, and let $G' = G[(S_{-u}, S'_u)]$. Let $\Delta(S_{-u}, S'_u)$ and $\Delta^{con}(S_{-u}, S'_u)$ be the corresponding increments in the games Γ and Γ^{con}, respectively. We have that $\Delta^{con}(S_{-u}, S'_u) = \Delta(S_{-u}, S'_u)$, if G' is connected. Otherwise, $\Delta^{con}(S_{-u}, S'_u) = \infty$, $\Delta(S_{-u}, S'_u) < \infty$. Therefore, $\Delta^{con}(S_{-u}, S'_u) \geq \Delta(S_{-u}, S'_u)$ and then, $\textsc{ne}(\Gamma) \subseteq \textsc{ne}(\Gamma^{con})$.

To see that the inclusion might be strict, let us consider that $V = \{u, v\}$, $v \in S_u$, and $S_v = \emptyset$. If $w_v > \alpha$, S is not a NE for Γ. On the other hand, independently of the weights of u, v, S is a NE for Γ^{con}.

Proposition 9. *There are connected max celebrity games that have* NE *graphs with diameter equal to $n - 1$.*

Proof. Let $n = 2k+1$ be a positive integer and let $V = \{v, v_1, v_{-1}, v_2, v_{-2}, \ldots, v_k, v_{-k}\}$. Let S be the strategy profile defined by $v_1, v_{-1} \in S_v$ and $v_{i+1} \in S_{v_i}, v_{-(i+1)} \in S_{v_{-i}}$ for $i \leq k-1$ (see the figure below). Setting the weights $w_x \leq \alpha$ for all $x \in V$ and for any $\beta < (n-1)/4$ it is easy to see that the corresponding graph is indeed a NE.



The bounds on the PoA obtained for the class of connected NE graphs for max celebrity games also hold for connected max celebrity games. The proofs also work for this case.

Theorem 2. *The PoA for the connected max celebrity games satisfies:*
1. For every connected max celebrity game $\Gamma^{con} = \Gamma^{con}(\langle V, (w_u)_{u\in V}, \alpha, \beta\rangle)$, $PoA(\Gamma^{con}) = O(n/\beta)$
2. For every $n > \beta > 1$, there exists a connected max celebrity game $\Gamma^{con} = \Gamma^{con}(\langle V, (w_u)_{u\in V}, \alpha, \beta\rangle)$ such that $PoA(\Gamma^{con}) = \Theta(n/\beta)$.

5 Max celebrity games for $\beta = 1$

When $\beta = 1$, each player u has to decide for every non-edge (u, v) of the graph to pay either α for the link, or at least w_v. It is not difficult to show that the best response of a player can be computed by sorting the weights of the non-adjacent nodes and then, selecting the number of links to be added to the most weighted non-adjacent nodes.

Proposition 10. *The problem of computing a best response of a player to a strategy profile in max celebrity games is polynomial time solvable when $\beta = 1$.*

In the next result we show that the price to pay for the anarchy is low when w_{min} is close to w_{max}.

Theorem 3. *Let $\Gamma = \langle V, (w_u)_{u\in V}, \alpha, 1\rangle$ be a max celebrity game. Then, $PoA(\Gamma) = O(w_{max}/w_{min})$.*

Proof. Let $S \in \text{OPT}(\Gamma)$ and $G = G[S] = (V, A)$. Let $X = \{v \in V \mid deg(v) = n-1\}$ where $deg(v)$ means the degree of v in the undirected graph G. We have that $C(G) \geq \frac{1}{2}\alpha(n-1)|X| + (n-|X|)w_{min}$. Hence, $C(G) \geq nw_{min}$, if $w_{min} \leq \frac{(n-1)}{2}\alpha$ and $C(G) \geq \binom{n}{2}\alpha$, otherwise. To prove the result we distinguish three cases:

First we see that if $w_{min} \leq \alpha(n-1)/2$, then $\text{PoA}(\Gamma) \leq w_{max}/w_{min}$. Indeed, let S be a NE of Γ and let $G = G[S] = (V, E)$. Using the same reasoning as in proposition 5 we have that $C(G) = \sum_{u\in V}(|S_u|\alpha + \max_{\{x|d(u,x)>1\}}\{w_x\}) \leq nw_{max}$. Therefore, if $w_{min} \leq \alpha(n-1)/2$, then $\text{PoA}(\Gamma) \leq w_{max}/w_{min}$, as we wanted to see.

Now, let us see that $PoA(\Gamma) = 1$ for $w_{min} > (n-1)\alpha$. This is because if $G \neq K_n$ then there exists some $v \in V$ with $diam_G(v) > 1$. Then considering the deviation for player v that consists in adding links to all the remaining nodes from the graph we get a cost increment of $k\alpha - w$ for some $k > 0$ and $w \geq w_{min}$. Since $k \leq (n-1)$ then $k\alpha - w \leq (n-1)\alpha - w_{min} < 0$, a contradiction for G being a NE. Thus $G = K_n$ and hence the result.

Finally, we see that for $\frac{n-1}{2}\alpha < w_{min} \leq (n-1)\alpha$ then $\text{PoA}(\Gamma) \leq 3$. Indeed, let S be a NE and $G = G[S] = (V, A)$. For a given $u \in V$ such that $diam_G(u) > 1$, let v be such that $w_v = W_u$. If $w_v > (n-1)\alpha$ then buying from u all the links to the remaining nodes from $V - \{x \mid d_G(u,x) \leq 1\}$ yields a cost increment of at most $(n-1)\alpha - w_v < 0$, a contradiction with G being a NE. Therefore $PoA(\Gamma) \leq (\binom{n}{2}\alpha + n(n-1)\alpha)/\binom{n}{2}\alpha = 3$.

6 Max celebrity games vs Sum celebrity games

The main differences between max and sum celebrity games are that: for $\beta > 1$, in max model there exist other disconnected NE graphs than I_n; in connected NE graphs, PoA $= O(n/\beta)$ in both models, but this is tight for some max games; for $\beta = 1$, PoA $=$O(w_{max}/w_{min}) in max, while in sum PoA ≤ 2. Finally, max celebrity games are equivalent to the MaxBD games (see [8] [7]) when $\alpha < w_{min}/(n-1)$ as they are sum celebrity games when $\alpha < w_{min}$. (See the proof of Proposition 8 in [4] and replace $\alpha < w_{min}$ by $\alpha < w_{min}/(n-1)$).

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Chapter 7

Conclusions and Open Problems

In this thesis, we have tackled the problem of understanding how the selfish behaviour of the agents creating a communication network affects the quality of the outcome with respect the social cost in two distinct attacks. In the first one, we have dealt with the most emblematic model of the network creation games, the SUM NCG [20]. The SUM NCG models in a very simple but elegant way the creation of Internet-like networks by selfish nodes without central coordination by considering that the n players buy links of price α to the other players in order to be well-connected. Up until now, several contributions have proved that the relative cost of the lack of coordination is modest for mostly any function α. If this result was true for every α then it would suggest that the selfish behaviour of the agents from Internet-like networks do not affect severely the quality of such networks with respect the social cost. In the second attack, after analysing natural properties that neither the SUM NCG nor any network creation game capture, at least combined in the way we do, we have proposed and studied two new original models, the SUM CG and MAX CG. These models are inspired by these properties and thus they provide a distinct point of view on the same problem. The main idea for these two models is to consider that the players or nodes from the network might have distinct popularities or relevances. In this way, we then assume that each player wants to have popular nodes closer than a critical distance β by buying the fewest amount of links of price α.

In both attacks we have analysed the price of anarchy, which helps us understand the loss of efficiency of the system due to the selfish behaviour of the agents, and the structure of equilibria, which helps us understand the properties of such networks. Our conclusions are described in the next subsection.

7.1 Conclusions

Firstly, in the Part I we have studied the SUM NCG model and we have shown that the price of anarchy is constant even when $\alpha > n(1+\epsilon)$, where $\epsilon > 0$ any small positive constant, which means that equilibrium networks are not far from the optimal ones. Therefore, our contribution reinforces the conjecture stating that the price of anarchy is constant [17] (table 7.1 shows our contribution in red color).

$\alpha = 0$	1	2	$\sqrt[3]{n/2}$	$\sqrt{n/2}$	$O(n^{1-\delta})$	$n(1+\epsilon)$	$4n-13$	$12n\log n$	∞
PoA	1	$\leq \frac{4}{3}$ ([20])	≤ 4 ([17])	≤ 6 ([17])	$\Theta(1)$ ([17])	$2^{O(\sqrt{\log n})}$ ([17])	$O(1)$	< 5 ([11])	1.5 ([1])

Table 7.1: Summary (updated) of the best known bounds for the PoA for the SUM NCG.

Regarding the structure of equilibria, we have obtained distinct results depending on the range of the parameter α that we consider.

On the one hand, for the same range $\alpha > n(1+\epsilon)$, we have also seen in Chapter 3, that if an equilibrium graph is not a tree, then the size of any biconnected component of the network is at most a constant. Therefore, this tells us that non-tree equilibria for this range, if it is the case that they exist, are very close to a tree. This is a result that connects with the reformulated tree conjecture, stating that equilibria for the range $\alpha > n$ can only be trees [29]. As we could expect, this property follows the general intuition that when α is more expensive the number of bought links in equilibrium networks should decrease in some way and therefore, since equilibria are connected, their topology is close to the one given by a tree. To prove this result we have been inspired by the average degree technique which was first introduced by Mihalák et al. in [29]. This technique bounds the average degree of any biconnected component H, noted as $deg(H)$, in two different ways and deduces that H cannot exist if the two bounds contradict each other when α belongs to a certain range and, thus, G must be a tree for this range of α. Our main contribution consists in giving an improved upper bound for the term $deg(H)$, that in this case leads to a contradiction if the size of H is larger than some constant. To reach this improved upper bound for $deg(H)$, we use a combination of non-trivial relationships between the size and the diameter of G and H. Some of these relationships are obtained introducing new concepts in an original way, like the A sets in Chapter 3, as well as extending well-known techniques, like the growing ball technique from Demaine et al. [17], that allows us to get an improved upper bound for the diameter of H in terms of the size of H (see Proposition 4 from Chapter 3).

Furthermore, in Chapter 4, we have investigated the topology of distance-uniform graphs by studying their diameter. Distance-uniform graphs are introduced by Alon et al. in [2] and they are connected with equilibria for the range $\alpha < n/C$ with $C > 4$ as shown in [7]. Using this connection together with the relationship between the PoA and the diameter of equilibria [17] we deduce that, for the same range of α, the diameter of distance-uniform graphs can upper bound the PoA. The results from Chapter 4 refute the conjecture of Alon et al. in [2], which states that the diameter of distance-uniform graphs is logarithmic. As a consequence, we cannot use, at least alone, this previous connection between the PoA and the diameter of distance-uniform graphs in order to improve directly the upper bound for the PoA for the same range of α.

In Part II, we have proposed two new original models, the SUM CG and MAX CG, which try to capture distinct aspects of communication networks based on some assumptions. One of these assumptions is the possibility of having players with different degree of popularity. Another important assumption is the existence of a critical distance that induces a penalisation for each player according to the relevances or popularities of the nodes that are further than this critical distance. For each of them we have studied both the PoA and the topology of equilibria. Tables 7.2 and 7.3 illustrate our most relevant contributions regarding these models for the distinct scenarios $\beta = 1$ and $\beta > 1$ distinguishing also when we look at certain classes of topologies for the equilibria at consideration. Apart from α, β, n, recall that W corresponds to the sum of all the weights of the players and w_{max}, w_{min} are the maximum and minimum weights, respectively.

	Sum Celebrities		Max Celebrities	
	$\beta = 1$	$\beta > 1$	$\beta = 1$	$\beta > 1$
BR	P	NP-Hard	P	NP-Hard
PoA	≤ 2	$O(\min(n/\beta, W/\alpha))$	$O(w_{max}/w_{min})$	$\Theta(n/\beta), O(w_{max}/\alpha)$

Table 7.2: Summary of the results for computing the best response and bounds for the PoA

As we can see, the results we obtain can differ depending on whether the critical distance β satisfies $\beta = 1$ or $\beta > 1$.

		Sum Celebrities	Max Celebrities
OPT		$\min(\alpha, W)(n-1)$	$\in (\min(\alpha(n-1), w_{min} + (n-1)w_{max}), 2\alpha(n-1))$
Diameter	Connected	$\leq 2\beta + 1$	$\leq 2\beta + 2$
	Trees	$\leq \beta + 1$	
	$w_{min} \leq \alpha$	$\leq \beta$	
Connectivity			r connected components with at most one of which is not an isolated node

Table 7.3: Summary of the bounds on the OPT and the topology of equilibria for the case $\beta > 1$

On the computational side, we have seen that computing the best response is NP-hard when $\beta > 1$ for both SUM CG and MAX CG models whereas when $\beta = 1$ the best response is polynomial time solvable.

Then, to obtain the results regarding the PoA we investigated the social optimum in the first place. Whereas in the SUM CG we have the very specific equality OPT $= \min(\alpha, W)(n-1)$, in the MAX CG we have the two bounds OPT $\leq 2\alpha(n-1)$ and OPT $\geq \min(\alpha(n-1), w_{min}+(n-1)w_{max})$, which are not as precise as the direct equality for the SUM CG but, in some sense, they provide an interval narrow enough.

The most important results correspond to the bounds obtained to study the PoA. For the case $\beta > 1$, we have two kinds of parallel bounds $O(n/\beta)$ and $\Theta(n/\beta)$ for the SUM CG and MAX CG, respectively, which seem to illustrate the dependency of the PoA with respect n and β. In particular, the greater is β in relation with n, the lower is the bound on the PoA. On the other hand, we also have the bounds $O(W/\alpha)$ and $O(w_{max}/\alpha)$ for the SUM CG and MAX CG, respectively, which seem to illustrate a similar dependency with respect the price per link α and the weights or the relevances of the players. In contrast, for $\beta = 1$ we obtain the upper bounds PoA ≤ 2 and PoA $= O(w_{max}/w_{min})$ for the SUM CG and MAX CG, respectively. If instead of looking at a general topology we restrict to the specific tree topology, the PoA for the SUM CG is reduced to be asymptotically constant, a result that resembles that the PoA for trees is at most 5 for the SUM NCG.

Another further aspect of equilibria that we have studied in both models is the diameter of connected equilibria obtaining upper bounds of $2\beta + 1$ and $2\beta + 2$ for the SUM CG and MAX CG models, respectively. Whereas in the MAX CG we have not obtained refinements of this upper bound for specific topologies, in the SUM CG we have improved the upper bound on the diameter when we restrict to more specific configurations. Specifically, we have proved that $diam(G) \leq \beta + 1$ for tree equilibria G and $diam(G) \leq \beta$ for equilibria G in games with $w_{min} \leq \alpha$.

Finally, another curious aspect which becomes of interest in the MAX CG model is what happens with disconnected equilibria. In the same model we see that such equilibria consists of $r \geq 2$ connected components with exactly $r - 1$ connected components that are isolated nodes.

7.2 Open problems and future work

As we can see from the main results of the previous summary there are questions still unresolved that deserve further study. In the following we analyse different possibilities that can be explored in order to dive deeper into the main questions of the thesis.

The Sum NCG model

Two of the most interesting open problems for the SUM NCG are to prove or refute the constant PoA conjecture and the tree conjecture.

The tree conjecture has been proved to be true for $\alpha > 4n-13$, which is a range really close to the range $\alpha > n$. Furthermore, in Chapter 3, we show that for any $\epsilon > 0$ positive constant and $\alpha > n(1+\epsilon)$, the size of any biconnected component from any NE graph G is at most a constant. This last result is weaker than the corresponding validation of the tree conjecture for the same range, but indeed very close to it. This could be one possible line of attack.

On the other hand, one of the most challenging but exciting open problems in the field is to investigate what happens in the remaining range of α between $n^{1-\delta}$ ($\delta \geq 1/\log n$) and $n(1+\epsilon)$. Any new finding regarding the PoA would constitute a very nice contribution: either the PoA is constant in this range, and then the constant PoA would be settled in an affirmative way, or the PoA is non-constant and then it would be somehow surprising.

Let us discuss the two possibilities regarding the PoA for the range of α between $n^{1-\delta}$ and $n(1+\epsilon)$.

Could the conjecture be false?

We know that the diameter plus one unit is an upper bound for the PoA of a graph [17]. We now make the observation that an analogous relationship is satisfied in the reverse direction. Let G be any NE of diameter d and let $B_r(v)$ be the subset of nodes from G at distance at most r from v. For any node $u \in V(G)$ let z be any node at maximum distance with respect u. If z buys a link to u, then z gets closer at least $d/4$ units to any node from $B_{d/8-1}(u)$. Therefore, $0 \leq \alpha - \frac{d}{4}|B_{d/8-1}(u)|$ implying that $|B_{d/8-1}(u)| \leq \frac{4\alpha}{d}$. From here we deduce that $D_G(u) \geq (n - \frac{4\alpha}{d})\frac{d}{8}$ so that $\sum_{u \in V(G)} D_G(u) \geq n\left(n - \frac{4\alpha}{d}\right)\frac{d}{8}$. Furthermore, for $\alpha < n(1+\epsilon)$, the social optimum is $\Theta(n^2)$ and then:

$$\text{PoA} > \frac{\sum_{u \in V(G)} D_G(u)}{\text{OPT}} = \Omega\left(d/8 - \frac{\alpha}{2n}\right) = \Omega\left(d/8 - \frac{1+\epsilon}{2}\right)$$

Then this simple result shows that if we find a NE of non-constant diameter, then the PoA would be non-constant, too, thus the constant PoA conjecture would be false. In this line of research, we have made some progress.

Let a *buying deviation* be any deviation that consists in buying links and, specifically, let a buying deviation of cardinality k be a deviation that consists in buying k links. A *buying* NE is any configuration in equilibrium when restricting only to buying deviations. With this terminology, it can be shown that for any buying deviation of cardinality $k > 1$ strictly decreasing the cost of the corresponding player, there also exists a buying deviation of cardinality 1 strictly decreasing the cost of the same player [19]. This property allows us to dramatically reduce the number of buying deviations that one must consider in order to verify that the network is a buying NE. This together with the previous result regarding the relationship with the PoA and the diameter of equilibria, lead us to see that there exist buying equilibria having non-constant diameter (this was one of our talks at WINE 2018, you can find the poster that was accepted in the Appendix A). Of course, the examples that we have found having non-constant diameter are not equilibria in the general sense. However, we believe that looking for other non-trivial examples could be a nice line of research.

Could the conjecture be true?

If the conjecture was true, then it would be wonderful closing an open problem from so many years. However, the raw result seems really tough and, therefore, an incremental approach seems more reasonable. Hence, it is natural to start studying topological properties of equilibria so that if several results are gathered, then maybe we would be able to validate the conjecture. In this direction we are working in the following results (ongoing research):

1. Regarding the metric properties of equilibrium networks, we have been able to prove that for the range $\alpha < 4n$ and for any constant $\epsilon > 0$, there exists a constant K such that every Kth power of any NE G is an $\epsilon-$distance-almost-uniform graph, thus generalising the result from [7],

which states that for $\alpha < n/C$ with $C > 4$ every fourth power of any equilibrium graph is an $4\alpha/n-$distance-almost-uniform graph.

2. Another important topological parameter from any graph is the degree (both undirected and directed) of its nodes. We have been able to prove that for any biconnected component H of any equilibrium G it holds that $deg_H^+(u) \leq 2n/\alpha + 6$, thus giving a non-trivial upper bound on the number of links from H bought by any specific node $u \in V(H)$ and generalising the result in [5], where we show that the directed degree in any biconnected component when $\alpha > n$ is at most a constant. This confirms the intuition that for high-price links the number of non-essential bought links should be small.

Other future lines of research

Another interesting line of research would be to explore the max network creation game, the analogous model to the SUM NCG in which we change the summation from the cost component for the maximum over all the corresponding distances. As we mentioned in the historical overview, some interesting results have been obtained for this model. One of the main interests in this model is the study of the PoA, as it happens in the SUM NCG game. It has been proved that the PoA for the MAX NCG is constant for $\alpha > 129$ and for $\alpha = O(n^{-1/2})$. These results appear in [29], where the authors improve the upper bounds on the PoA for the SUM NCG as well using a quite similar technique.

To prove our results in the SUM NCG we used new techniques such as the formulae bounding the size of the A sets and extending the growing ball technique to the subgraph given by any biconnected component. We think that it is quite natural to try to use these techniques for the MAX NCG model as well. The aim of this study would be to enlarge the range of the values α for which the PoA is constant for the MAX NCG model.

The SUM CG and MAX CG models

The major topic of interest in the field, the study of the PoA, seems almost resolved for the MAX CG model since the estimation we obtain is asymptotically tight. However, in the SUM CG model we have shown that the PoA is $O(n/\beta)$. It is then an open problem to see if the bound is tight or can be improved. One possibility is to see whether we can apply the growing ball technique considered in [17, 2] to obtain an improved upper bound on the diameter, since this technique can be adapted to distinct scenarios, as it is shown in the literature and as we have seen in our work.

Dynamics

In the models we have studied in this thesis, we have studied properties for a given instance of the game but another point of view could be to consider the so-called *dynamics*. A dynamics for a strategic game studies the behaviour of sequences consisting of instances of the game starting from a given configuration and in which every new instance in the sequence is obtained from the previous one by selecting a player and performing a specific deviation on that player. The central point of study of a dynamics is which starting configurations, which players and which deviations, can be selected in each step in order to guarantee that the whole process converges to an equilibrium or, at least, to a configuration close to an equilibrium. If it is the case that convergence is guaranteed, then it is also natural to study how fast or slow is such process, but it can also be the case that in some situations cycles appear, a matter of interest, too. For instance, a *Best Response dynamics* corresponds to the dynamics process that updates the current instance to the one obtained when the selected node or player chooses any of its best responses strategies. Similarly, a *Better Response dynamics* corresponds to the dynamics process that updates the current instance to the one obtained when the selected node or player chooses any strategy strictly reducing the value of its current cost function. Finally, a *Greedy Best Response dynamics* corresponds to the dynamics process in which the selected player chooses among the deviations that consist in either buying, swapping or deleting exactly one link, any one having the minimum possible cost value among all such deviations.

The following results are ongoing work related to the SUM CG and MAX CG models for distinct dynamics processes, see the draft in [4] for more details:

On the one hand, for $\beta = 1$ in the SUM CG, independently of the initial configuration we choose, every Best Response dynamics has no cycle and, more specifically, we easily obtain a NE in at most $2n$ rounds if we consider the Best Response dynamics given when we select the players $1, 2, ..., n, 1, 2, ..., n$ in this order. In contrast, for $\beta = 1$ in the MAX CG, there exists an initial configuration for which the Best Response dynamics can cycle. However, on the positive side, there are configurations from where certain Best Response dynamics converge to a NE in a reasonable number of steps.

On the other hand, if we focus our attention to the case $\beta > 1$, in some situations we have obtained cycles in the Better Response dynamics when $\beta = 3$ for the SUM CG and when $\beta = 5$ for the MAX CG. However, when considering the Greedy Best Response dynamics, we have seen that the majority of our experiments converge to a NE.

As a conclusion, it seems that exploring this topic could lead to an another interesting line of research.

Extensions of the model

It can be really tough to propose a tractable model of a phenomenon that has such a variety of aspects and that depends on so many variables. This is one of the reasons why there are so many distinct models in the Network Creation Games. Some of them try to capture previously unconsidered properties and others are extensions of already well-known models.

Recently, Yen-Yu et al. in [31] have proposed a new strategic game that precisely, generalises in a very simple way our SUM CG and MAX CG models. More specifically, their model is specified by a finite set of players V and α the cost per link, as in the celebrity models. However, instead of having the same critical distance for each player in the game, in the extension of Yen-Yu et al. each player $i \in V$ has its own critical distance β_i with $2 \leq \beta_i \leq n-1$ and a function $f : V \times 2^V \rightarrow \mathbb{R}$ defines the penalty that each player receives. This function f is defined in such a way that $f(i, V_i)$ is the penalty that player i receives for having exactly the set of players V_i further away from its critical distance β_i and with $f(i, \emptyset) = 0$. Therefore, each player can have different penalty functions and they do not require f to be monotone. These are two very natural assumptions because in the reality and specially in social networks, every agent can have different preferences and having more friends does not imply that the corresponding player will be happier. The strategic game is then defined in terms of the strategies as in the SUM CG and MAX CG where each player i selects its strategy as a subset from $V \setminus \{i\}$. In this way, given a strategy profile S and G the corresponding communication network, the cost of player u is defined as $c_u(S) = \alpha|S_u| + f(u, V_u)$ if G is connected and where V_u is the set of nodes at distance strictly greater than β_u in G. Otherwise, if G is not connected the cost is set to be ∞. In this model the star is a social optimum and the main results the authors show is that the PoA is upper bounded by $O\left(\sum_{i=1}^{n} \frac{1}{\beta_i}\right)$ matching our upper bound $O(n/\beta)$ if β_i are the same for each player.

In conclusion, this is an interesting extension of our celebrity models in which the parameters of the game seem to satisfy very general conditions. Therefore, rather than trying to extend this model, a possible future line of research would be to think about natural constraints that the function f must satisfy in the context of Internet-like networks as well as social networks. Maybe, when restricting to these special cases, we can improve the bounds on the PoA and deduce properties regarding the structure of equilibria as well.

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Appendices

Distance Uniform Graphs in the Network Creation Game

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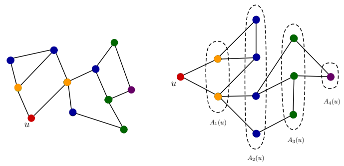
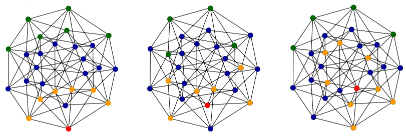
Abstract
We present new connections between distance uniform graphs and the classic model of the Network Creation Games that provide insight into the question of whether the Price of Anarchy (PoA) is constant.

Definitions

Distance Set For a given graph G and any node u from G we denote by $A_r(u)$ the set of nodes at distance r from u.



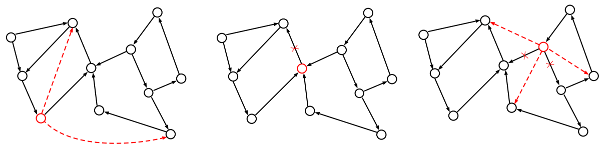
Distance Uniform Graphs ([2]) A given graph G is $\epsilon-$distance-uniform (or $\epsilon-$distance-almost-uniform) iff there exists a distance index r (the critical distance) such that $|A_r(u)| \geq n(1-\epsilon)$ (or $\max(|A_r(u)|, |A_{r+1}(u)|) \geq n(1-\epsilon)$) for every $u \in V(G)$.



Sum Classic Network Creation Game It is defined as a tuple $\langle V, \alpha \rangle$ where:
(i) V is the set of players with $n = |V|$.
(ii) α is the cost of buying a single link.
Every player u chooses a subset $s_u \subseteq V \setminus \{u\}$ which corresponds to the set of players to which u buys a link. An *strategy profile* is a tuple $s = (s_u)_{u \in V}$. For any strategy profile s we define the *communication network* as the undirected graph:

$$G[s] = (V, \{(i,j) \mid i \in s_j \vee j \in s_i\})$$

When the context is clear we will omit the reference to s and we will write G instead of $G[s]$.
The players try to minimise their *cost function* which is, for player u and a strategy profile s, $c_u(s) = |s_u|\alpha + \sum_{v \neq u} d_{G[s]}(u,v)$.
A *Nash equilibrium* (NE) for this model is a strategy profile s such that, for every player $u \in V$, $c_u(s') \geq c_u(s)$ for any other strategy profile s' differing from s in exactly the component for u. A buying NE is a strategy profile s such that, for any player $u \in V$, $c_u(s') \geq c_u(s)$ for any other strategy profile s' differing from s in exactly the component for u which we impose that contains s_u.
We say that G is a (buying) NE graph iff there exists a (buying) NE s such that $G = G[s]$.



Buying deviation | Deliting deviation | General deviation

The *social cost* is $c(s) = \sum_{u \in V} c_u(s)$. With this notation then OPT $= \min_s c(s)$
The *price of anarchy* PoA is a ratio that quantifies the distance in terms of efficiency between the worst NE and any optimal configuration. More precisely, if $\mathcal{E}$ is the set of equilibria:

$$\max_{s \in \mathcal{E}} c(s)/\text{OPT}$$

We set d_{max} to be the maximum diameter of the communication network corresponding to any NE for the corresponding range of the parameter α.

Open Question

Is the PoA constant? Up until now:

α	0 – 1	1 – 2	2 – $\sqrt[3]{n/2}$	$\sqrt[3]{n/2}$ – $\sqrt{n/2}$	$\sqrt{n/2}$ – $O(n^{1-\delta})$	$O(n^{1-\delta})$ – $n(1+\epsilon)$	$n(1+\epsilon)$ – $4n-13$	$4n-13$ – $12n\log n$	$12n\log n$ – ∞
PoA	1	$\leq \frac{4}{3}$ ([5])	≤ 4 ([4])	≤ 6 ([4])	$\Theta(1)$ ([4])	$2^{O(\sqrt{\log n})}$ ([4])	$\Theta(1)$	< 5 ([3])	1.5 ([1])

The lower range | The middle range | The upper range

Distance Uniformity and the Diameter of NE Graphs

Theorem 1 Let G be a NE graph for $\alpha = n/C$ with $C > 4$. Then, there exists an r such that, for every $u \in V$:

$$|A_r(u)| + |A_{r+1}(u)| + |A_{r+2}(u)| + |A_{r+3}(u)| + |A_{r+4}(u)| > n - 4\alpha.$$

Corollary 1 Let G be a NE graph for $\alpha = n/C$ with $C > 4$. Then, G^4 is a $4\alpha/n-$distance-almost-uniform graph.

Corollary 2 For the interval $\alpha \leq n/C$ with $C > 8$, PoA $= \Omega(d_{max})$.

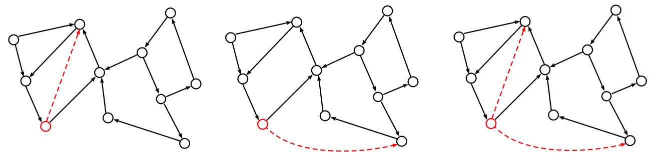
Lemma 1 ([4]) $PoA = O(d_{max})$.

Theorem 2 For the interval $\alpha \leq n/C$ with $C > 8$,

$$\text{PoA} = \Theta(d_{max}).$$

Buying NE Graphs of Non-Constant Diameter

Lemma 2 Let $u, u_1, ..., u_k$ be nodes in G. Let ΔC_i the deviation in u that consists in buying the link uu_i for i with $1 \leq i \leq k$. Also, let ΔC be the deviation that consists in buying in u all the links $uu_1, ..., uu_k$. Then, $\Delta C_1 + ... + \Delta C_k \leq \Delta C$.



$\Delta C_1 \quad + \quad \Delta C_2 \quad \leq \quad \Delta C$

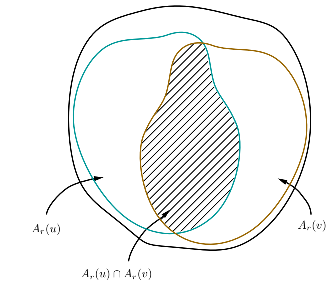
Remark 1 In order to evaluate if G is a buying NE we only need to check $n(n-1)$ deviations, the ones of cardinality one.

Proposition 1 Any $\epsilon-$distance-uniform graph G with $diam(G) \leq (2\epsilon C)^{-1} + 1$ is a buying NE for $\alpha = n/C$.

r the critical distance

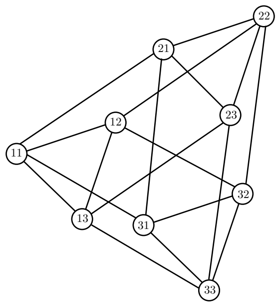
u buys a link to v

$$|A_r(u) \cap A_r(v)| \geq |A_r(u)| + |A_r(v)| - n \geq$$
$$\geq n(1-2\epsilon)$$
$$\Delta C_{buy} \geq \alpha - (diam(G) - 1)(n - |A_r(u) \cap A_r(v)|) \geq$$
$$\geq n/C - ((2\epsilon C)^{-1})(n - n(1-2\epsilon)) \geq 0$$



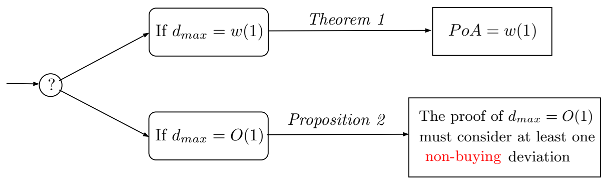
Proposition 2 There exist buying NE graphs of non-constant diameter for $\alpha = n/C$ and $C = O(1)$.

$$V(G) = \{x_1...x_r \mid x_i \in \{1,...,k\}\}$$
$$E(G) = \{(x_1....x...x_r, x_1...y...x_r) \mid x \neq y\}$$
$$|A_i(u)| = \binom{r}{i}(k-1)^i$$
$$diam(G) = r$$
$$|A_r(u)| = (k-1)^r = k^r(1-\tfrac{1}{k})^r \geq$$
$$\geq k^r(1-\tfrac{r}{k}) \geq n(1-\tfrac{r}{k})$$
↓
G is $\frac{r}{k}-$distance-uniform
↓
G is a buying NE for $\alpha = n/C$
if $r \leq 1 + \frac{k}{2rC}$
↓
In particular, for $k = r^3$ and $C = O(1)$
$diam(G) = r = \Omega(\sqrt{\log n})$



Conclusions

Constant or non-constant PoA? For the interval $\alpha \leq n/C$ with $C > 8$:



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