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The theory of quantum coherence

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by

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"The arts and the sciences all draw together as the analyst breaks them down into their smallest pieces: at the hypothetical limit, at the very quick of epistemology, there is convergence of speech, picture, song, and instigating force."

Daniel Albright,
Quantum poetics: Yeats, Pound, Eliot, and the science of modernism

Abstract

Quantum coherence, or the property of systems which are in a superposition of states yielding interference patterns in suitable experiments, is the main hallmark of departure of quantum mechanics from classical physics. Besides its fascinating epistemological implications, quantum coherence also turns out to be a valuable resource for quantum information tasks, and has even been used in the description of fundamental biological processes. This calls for the development of a resource theory which rigorously formalizes the notion of coherence, that further allows both to quantify the coherence present in physical systems and to study its manipulation in order to better leverage it. This thesis intends to make a contribution to the recently built resource theory of coherence in a number of ways.

First, we show that coherence, as formalized by its resource theory, is soundly grounded in the physics of interferometers—at least in the context of Strictly Incoherent Operations—and thus embodies its operational foundations.

Second, we note that states can be thought of as constant-output channels, and start to generalize the coherence theory of states to that of channels. In particular, we propose several measures of the coherence content of a channel and further compute them when considering two different classes of free operations: Incoherent Operations and the largest set of Maximally Incoherent Operations.

Finally, we investigate the question whether coherence can witness some other manifestations of non-classicality (we mean, beyond interference effects). In particular, we analyze the connection of coherence to the non-classicality of quantum stochastic processes both in the Markovian and in the non-Markovian regimes.

Resum

La coherència quàntica, o la propietat dels sistemes que es troben en una superposició d'estats capaç de donar lloc a patrons d'interferència en els experiments adequats, és el segell distintiu de la mecànica quàntica. Més enllà de les seves fascinants implicacions epistemològiques, la coherència quàntica resulta també un recurs valuós a l'hora de dur a terme diferents tasques quàntic-informacionals i ha estat fins i tot emprada en la descripció de certs processos biològics. Per aquest motiu s'ha fet necessari el desenvolupament d'una teoria de recursos que formalitzi rigorosament la noció de coherència, i que permeti així quantificar la coherència present en els sistemes físics, així com estudiar la seva manipulació amb vista a un millor aprofitament d'aquest recurs. Aquesta tesi doctoral pretén contribuir a la teoria de la coherència de la següent manera.

En primer lloc, demostrem que la coherència, tal com la teoria la formalitza, està sòlidament ancorada en la física dels interferòmetres —almenys en el context de les Operacions Estrictament Incoherents—, i encarna, per tant, el seu propi principi operacional.

En segon lloc, després de fer notar que els estats poden ser entesos com a canals de "output" constant, emprenem la generalització de la teoria de la coherència dels estats a la teoria dels canals. En concret, proposem diverses maneres de mesurar el contingut en coherència d'un canal quàntic i el calculem considerant dues classes diferents d'operacions del tipus "free": Operacions Incoherents i Operacions Màximament Incoherents.

Finalment, investiguem si la coherència pot ser també testimoni d'alguna manifestació de no classicitat diferent dels propis efectes interferomètrics. En particular, analitzem la connexió de la coherència amb la no classicitat dels processos estocàstics quàntics, tant en el règim markovià com en el no markovià.

Resumen

La coherencia cuántica, o la propiedad de los sistemas que se encuentran en una superposición de estados capaz de dar lugar a patrones de interferencia en los experimentos adecuados, es el sello distintivo de la mecánica cuántica. Más allá de sus fascinantes implicaciones epistemológicas, la coherencia cuántica resulta también un recurso valioso a la hora de llevar a cabo diferentes tareas cuántico-informacionales y ha sido incluso empleada en la descripción de ciertos procesos biológicos. Por este motivo se ha hecho necesario el desarrollo de una teoría de recursos que formalice rigurosamente la noción de coherencia, y que permita así cuantificar la coherencia presente en los sistemas físicos, así como estudiar su manipulación con vistas a un mejor aprovechamiento de este recurso. Esta tesis doctoral pretende contribuir a la teoría de la coherencia del siguiente modo.

En primer lugar, demostramos que la coherencia, tal y como la teoría la formaliza, está sólidamente anclada en la física de los interferómetros —al menos en el contexto de las Operaciones Estrictamente Incoherentes—, con lo que encarna su propio principio operacional.

En segundo lugar, tras hacer notar que los estados pueden ser entendidos como canales de "output" constante, emprendemos la generalización de la teoría de la coherencia de los estados a la teoría de los canales. En concreto, proponemos diversas maneras de medir el contenido en coherencia de un canal cuántico y lo calculamos considerando dos clases diferentes de operaciones de tipo "free": Operaciones Incoherentes y Operaciones Máximamente Incoherentes.

Finalmente, investigamos si la coherencia puede ser también testigo de alguna manifestación de no clasicidad distinta de los propios efectos interferométricos. En particular, analizamos la conexión de la coherencia con la no clasicidad de los procesos estocásticos cuánticos, tanto en el régimen markoviano como en el no markoviano.

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I would also like to express my gratitude to Miguel Ferrero, my Quantum Mechanics professor at university, for having conveyed to us in such an enthusiastic way what I believe is the most important lesson of quantum physics: that the realist worldview of classical physics is not absolute at all—nothing in this world is—, and that this fact allows to bring arts and sciences closer and closer together.

Thanks to Susana Huelga and Martin Plenio, for having generously hosted me in their group in Ulm and for having planted in me the seed of coherence theory which would finally lead to this thesis. Thanks also to Jochen Rau, for his friendship, and for all our deeply motivational conversations. Thanks to Andrea Smirne and Dario Egloff, for having continued to mentor me (both in physics and, most importantly, in life matters), and to the rest of my family in Ulm, for having always turned coldness into warmth.

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Contents

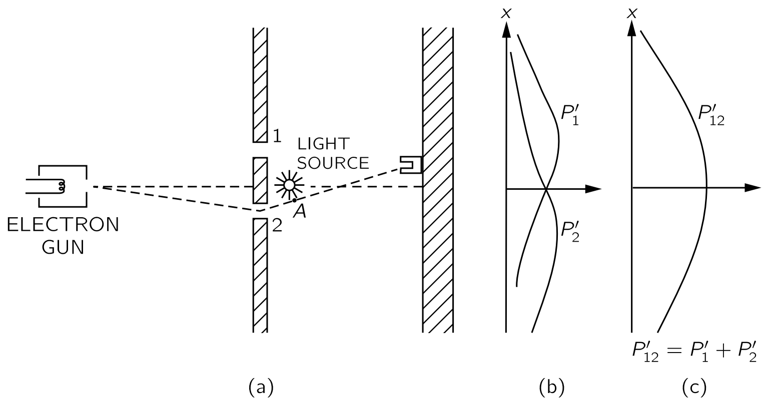
Introduction

"When I gathered flowers, I knew it was myself plucking my own flowering."

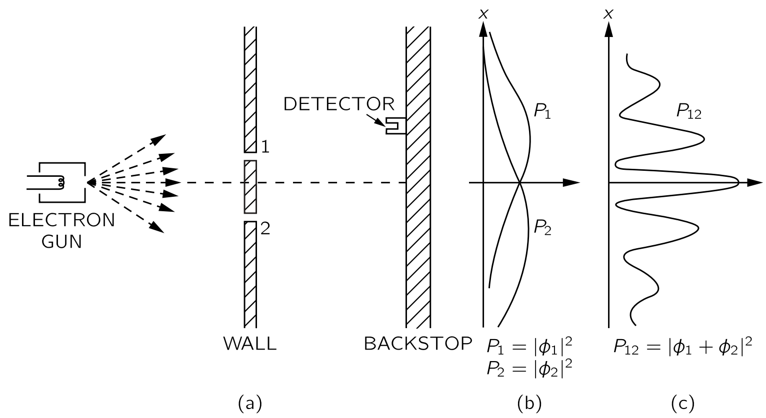
D. H. Lawrence,
"New heaven and earth", Complete Poems

Imagine that an electron gun fires electrons towards a wall which has two holes in it. After going through such a wall, each electron impacts on a screen where a detector has been placed. One could think that, in this scenario, every electron behaves as a proper silver bullet—it is, after all, ourselves (slaves of an imagination nurtured by the macroscopic world we inhabit) who have used the metaphor of a gun in the first place—, in the sense that it goes either through one slit or through the other one. However, we know by now that this is not the only possible outcome of such an experiment. It is true that, when we actually look at the holes to discover which path each electron has decided to take, the detector in our screen reveals that it has behaved as a particle—or a bullet—(see Fig. 1a). Yet if we do not look at the slits, then the detector shows that the electron has performed in the fashion of water and light, that means, as a proper wave—in that both paths have given rise to an interference pattern on the screen—(see Fig. 1b). Feynman was the first to undress such an experiment, that is, to take away all specificity from it, and understand it in its barest form. His conclusions are quoted in the following [1]:

1. The probability of an event in an ideal experiment is given by the square of the absolute value of a complex number ϕ which is called the probability amplitude: P =probability, ϕ =probability amplitude, $P = |\phi|^2$.



(a)



(b)

Figure 1: An experiment with electrons. **a)** Each electron behaves as a particle. **b)** Each electron behaves as a wave. Source: [1].

2. When an event can occur in several alternative ways, the probability amplitude for the event is the sum of the probability amplitudes for each way considered separately. There is interference: $\phi = \phi_1 + \phi_2$, $P = |\phi_1 + \phi_2|^2$ (see Fig. 1b).

3. If an experiment is performed which is capable of determining whether one or another alternative is actually taken, the probability of the event is the sum of the probabilities for each alternative. The interference is lost: $P = P_1 + P_2$ (see Fig. 1a).

We could also think of this in Aristotelian terms. For Aristotle, potentiality is the capacity something has to be in a different and more completed state [2]. For instance, consider a piece of wood which can be carved into a violin or into a flute. According to Aristotle, the wood has at least two different potentialities, for it is potentially a violin and also potentially a flute. Only when the wood is carved can we say it is actually a violin (or a flute). In the same manner, an event may have different potentialities. Feynman tells us that, when it is not possible to know which potentiality has been realized, there is interference. Otherwise, the interference is lost.

This leads us to the following question: how is it possible that knowledge and reality are correlated in this particular way? In other words, why is the potentiality to know—a property inherent to the observing subjects—inextricably linked to the actual future of a physical event? We seem to have come across the empirical manifestation of Wittgenstein's prescription:

> Whereof one cannot speak thereof one must be silent [3].

Indeed, Feynman seems to be telling us that, when we cannot know the path that the electron has taken (since we have not placed a light source by the slits to check it), then we must refrain from posing the question in the first place; nature will only give us a blurry answer in the form of an interference pattern. In Mercè Rodoreda's words:

> [...] allò era una cosa que no es podria saber mai: si a dintre del cargol de mar hi havia onadas quan a l'entrada del forat no hi havia cap orella [4].

At the microscopic level, subject and object appear to coalesce. Heisenberg put it in this way:

> The object of research is no longer nature in itself but rather nature exposed to man's questioning, and to this extent man here also meets himself [5].

When interrogating nature, the scientist encounters herself, as the poet, when gathering flowers, plucks his own flowering [6].

One may have guessed by now that it is the double-slit experiment [1] which we have just been describing, a well-known thought experiment which has already been realized in the laboratory a number of times [7]. Our purpose was to emphasize how quantum mechanics, brimming with counterintuitive features, plunges its roots into it. In fact, quantum coherence, the cornerstone of this thesis, is precisely the property of systems displaying different potentialities—of which the one to be realized cannot be known—, i.e. the property of systems that can be in a superposition of states.

Aristotle gives actuality priority over potentiality. For him, potentialities exist not for the sake of existing, but in order for one of them to be realized. He would be surprised by how things have changed in the contemporary world: quantum states with coherence, that means, raw potentialities by themselves, allow for quantum protocols to achieve an enhanced performance over comparable classical ones, and therefore could be considered prior to incoherent states. One could compare states with coherence with Donna Haraway's cyborg[1], both exhibiting a sort of ubiquity that renders them particularly advantageous:

> The ubiquity and invisibility of cyborgs are precisely why these Sunshine Belt machines are so deadly [8].

In this thesis we will address quantum coherence from a framework that puts its already mentioned value at the center. Such kind of frameworks receive the name of resource theories.

But let us begin by introducing the basic principles of the quantum theory itself.

The quantum theory

Born to parents such as Planck, Bohr, Heisenberg, Born, Einstein, Von Neumann, Dirac, among others, quantum mechanics arrived in the beginning of the 20th century to explain a number of experiments such as the study of blackbody radiation or the photoelectric effect, which could not be understood within the realist paradigm of classical physics. As argued before, such a realist worldview, assuming that sovereign subjects are able to read, in all its truth, a physical world that is separated from them, had to be replaced by a paradigm of *immersion*: systems do not have definite properties that are independent of us observers, but rather these get defined when we measure them. Quantum mechanics demonstrates that subjects are not anymore spectators in the theater which is the world, as Descartes would put it [9], but rather actors and spectators, in Bohr's words [10]. Physicist Paul Davies phrased it as follows [11]:

> The common division of the world into subject and object, inner world and outer world, body and soul, is no longer adequate... Natural science does not simply describe and explain nature; it is part of the interplay between nature and ourselves.

[1] Cyborgs: creatures simultaneously animal and machine, who populate worlds ambiguously natural and crafted [8].

Thus, the new quantum paradigm promised outstanding physical features, such as superposition, non-locality, indeterminism or contextuality, which would remain veiled if the classical worldview had prevailed.

Mathematically, the quantum theory is articulated by the following set of postulates [12]:

1. At each instant the state of a physical system is represented by a ket $|\psi\rangle$ in a Hilbert space $\mathcal{H}$. More generally, a state can be represented by a density operator, which is a trace class, nonnegative Hermitian operator ρ normalized to be of trace 1.

2. The Hilbert space of a composite system is the Hilbert space tensor product of the state spaces associated with the involved systems.

3. Every observable attribute of a physical system is described by a Hermitian matrix on $\mathcal{H}$.

4. The expectation value of the observable A for the system in state $|\psi\rangle$ is given by $\langle\psi| A |\psi\rangle$. In general, the expected value of A in the state ρ is given by $\mathrm{Tr}(A\rho)$. If ρ_ψ is pure (not mixed), i.e. it is the orthogonal projector onto the one-dimensional subspace of $\mathcal{H}$ spanned by $|\psi\rangle$, then $\mathrm{Tr}(A\rho_\psi) = \langle\psi| A |\psi\rangle$.

5. The only possible result of the measurement of an observable A with discrete spectrum is one of the eigenvalues of the corresponding operator A.

6. (Born rule) When a measurement of an observable A is made on a generic state $|\psi\rangle$, the probability of obtaining an eigenvalue a_n is given by the square of the inner product of $|\psi\rangle$ with the eigenstate $|a_n\rangle$, $|\langle a_n|\psi\rangle|^2$. For a state ρ, such probability is given by $\mathrm{Tr}(|a_n\rangle\langle a_n|\rho)$.

7. Immediately after the measurement of an observable A has yielded a value a_n on $|\psi\rangle$, the state of the system is the normalized eigenstate $|a_n\rangle$. If the measurement has been performed on ρ, then the post-measurement state is $\dfrac{|a_n\rangle\langle a_n|\rho|a_n\rangle\langle a_n|}{\mathrm{Tr}(|a_n\rangle\langle a_n|\rho)}$.

8. The time evolution of a quantum system preserves the normalization of the associated ket. The time evolution of the state of a quantum system is described by $|\psi(t)\rangle = U(t, t_0)\,|\psi(t_0)\rangle$, for some unitary operator U. For a mixed state, we have $\rho(t) = U(t, t_0)\rho(t_0)U(t, t_0)^\dagger$.

Having this in mind, let us now introduce what quantum resource theories are all about.

Quantum resource theories

Sticking to Aristotle's worldview, knowledge can be conceived as a garden of two forking paths. On the one hand, there is *theoria* or the activity of contemplation, aiming to answer the question why with no particular practical purpose. *Theoria* is just knowledge-for-the-sake-of-knowledge; things are not regarded in terms of utility, for *theoria*, like art, suffices itself. On the other hand, *poiesis* or creation reaches for a goal that is independent from the action involved in its achievement. Unlike *theoria*, *poiesis* is focused on production, rather than on the creative process itself.

One could then classify scientific theories according to this mindset: depending on the nature of their purposes, they could be separated either into theoretically-inclined or into practically-inclined. For instance, in physics, *theoria* could be linked to the dynamicist tradition, aiming to explain and predict the natural behaviour of systems, whereas *poiesis* could be related to the more pragmatic one, intending to understand phenomena in order to make use of them. Most of the time, though, such a binary classification does not prove accurate enough. Indeed, it is not always trivial to detach purely theoretical intentions from practical consequences, or to remove contemplative joy from a process exclusively conceived towards practicality. Saddled between *theoria* and *poiesis* are, in fact, resource theories.

Every resource theory is born from a constraint, that is, from the fact that some longed-for holy grail exists which cannot be easily achieved. Such a restriction then leads to the definition of two complementary antonyms: the resource (our target) and the free objects (the things that can be easily procured) or free operations (what can be done cheaply). A paradigmatic example of a resource theory—a really early one—is alchemy: gold, the resource, acquires value due to its scarcity; free objects, on the contrary, like iron or zinc, are easy to find; free operations would include, for instance, heating, melting or mixing.

So what is then the goal of a resource theory? To analyze what can still be achieved under the given constraints, how the available resources can be better leveraged and what resources can be interconverted (and how), among other questions. Alchemists, for example, aimed to turn base metals into noble ones by means of simple operations; thermodynamics, another well-known resource theory, was born during the development of steam engines to optimize the manipulation of heat in order to obtain work.

Quantum features, such as superposition and non-locality, can also be cast as proper resources, insofar as they can be exploited to overcome some physical

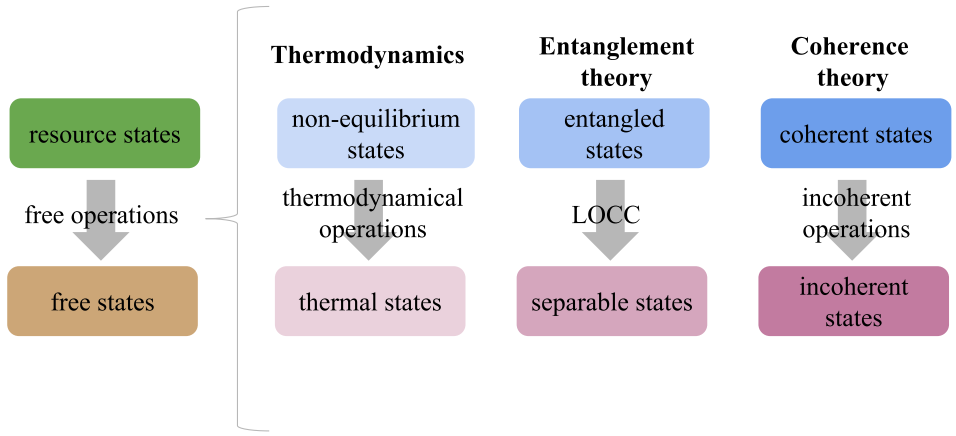


Figure 2: Some examples of quantum resource theories.

constraints that prevent certain operations to be realized. For instance, in a scenario where Alice cannot send a quantum state to Bob's distant lab, shared entanglement between both parties together with classical communication allows Alice to accomplish her task via teleportation [13]. Resource theories based on genuinely quantum resources are called quantum resource theories [14] and are defined by the following elements (see Fig. 2 for some examples):

- *Resource states*: they contain the resource, that is, the property useful to overcome the considered physical constraints.
- *Free operations*: transformations on the considered state space that are cheap or easy to implement.
- *Free states*: they can always be prepared by applying free operations to resource states.

Indeed, the quantitative theory of entanglement [15–18] can be identified as the first example of a theory constructed on the premise that quantum traits are physical resources. This framework assumes that Alice and Bob are allowed to create any local state in their respective labs and to communicate classically with each other, that means, that the free operations of the theory are Local Operations and Classical Communication (LOCC) [19, 20]. The states they are able to produce for free are then separable

$$\rho_{AB} = \sum_i p_i \rho_i^A \otimes \rho_i^B,$$

where $\sum_i p_i = 1$, and any entanglement they may share immediately becomes a valuable resource. Furthermore, thermodynamics can also be formulated at the quantum level in terms of a resource theory [21–29]. There the free operations are energy-preserving unitaries, which, when applied to resourceful non-equilibrium states of a system with Hamiltonian H, may freely yield thermal states of the form

$$\gamma = \frac{e^{-\beta H}}{Z_H},$$

where $\beta = (kT)^{-1}$. Yet another example is the resource theory of asymmetry [30–32], where one is restricted to symmetry-preserving operations, i.e. any channel $\mathcal{E}$ such that for any unitary representation U_G, $g \in G$,

$$\mathcal{E}[U_g(\cdot)U_g^\dagger] = U_g\mathcal{E}(\cdot)U_g^\dagger,$$

as well as to symmetric states, i.e. any state ρ such that $\forall U_g$,

$$U_g \rho U_g^\dagger = \rho.$$

Asymmetry measures, which quantify the resource, are the figure of merit for metrology tasks [32, 33], which pursue highly sensitive measurements of physical parameters—yet another case demonstrating the usefulness of the resource theory approach.

As we have seen, resource theories do not respond to Aristotle's binary classification of scientific theories: their conceptual body both addresses primordial foundations of physical reality and is built up from pragmatic inspiration. It is also worth mentioning that resource theories, because of their intimate relation to problems that are closer to society, are daughters of their time. In fact, only at the historical period when agents are constrained in a particular way it makes sense to develop a framework to study the possibilities accessible to them. For instance, consider a reality where energy was never dissipated: there the resource theory of thermodynamics would not be of particular use, since work would be in fact unlimited.

In this thesis we will address quantum coherence from a resource-theoretic approach. In Chapter 1 we will argue why coherence can be dubbed resourceful and set up the main elements of the coherence theory of states. In Chapter 2 we will prove that coherence, as formulated by its resource theory, is soundly related to the physics of interference, at least under some particular class of free operations. In Chapter 3 we will realize that any state can be thought of as a constant output channel and start to build the coherence theory of channels, from which the theory of states can still be recovered. In particular, we will propose several quantifiers of the coherence content of a map, which we will then compute in the IO-theory (Chapter 4) and in the MIO-theory (Chapter 5). In Chapter 6 we will set the stage for investigating how coherence can be related to other forms of non-classicality (we mean, beyond interference effects), in particular, the non-classicality of quantum stochastic processes. In Chapter 7 we will explore how this can be done when the dynamics that underlies the stochastic process is Markovian and of Lindblad type, and in Chapter 8 we will generalize this framework to also encompass non-Markovian settings.

The various chapters are based on the following papers and preprints:

- **Chapter 2**: Tanmoy Biswas, María García Díaz, and Andreas Winter, *Interferometric visibility and coherence*, 473, Proc. R. Soc. A. (2017). arXiv:1701.05051
- **Chapters 3, 4**: Khaled Ben Dana, María García Díaz, Mohamed Mejatty, and Andreas Winter, *Resource theory of coherence: Beyond states*, Phys. Rev. A 95, 062327 (2017). arXiv:1704.03710
- **Chapters 3, 5**: María García Díaz, Kun Fang, Xin Wang, Matteo Rosati, Michalis Skotiniotis, John Calsamiglia, and Andreas Winter, *Using and reusing coherence to realize quantum processes*, Quantum 2, 100 (2018). arXiv:1805.04045
- **Chapters 6, 7, 8**: Andrea Smirne, Dario Egloff, María García Díaz, Martin B. Plenio, Susana F. Huelga, *Coherence and non-classicality of quantum Markov processes*, Quantum Sci. Technol. 4, 01LT01 (2019). arXiv:1709.05267
- **Chapter 7**: María García Díaz, Benjamin Desef, Matteo Rosati, Dario Egloff, John Calsamiglia, Andrea Smirne, Michalis Skotiniotis, Susana F. Huelga, *Accessible coherence in open quantum system dynamics*, arXiv:1910.05089 (2019).
- **Chapter 8**: Philipp Strasberg and María García Díaz, *Classical quantum stochastic processes*, Phys. Rev. A 100, 022120 (2019). arXiv:1905.03018

Part I

Quantifying quantum coherence

CHAPTER 1

Quantifying the coherence of states

"Bring me the sunset in a cup,
Reckon the morning's flagons up
And say how many Dew"

E. Dickinson,
"Poem 128", Complete Poems

Coherence, i.e. the property of systems which can be in a superposition of states, is perhaps the only mystery of quantum theory [1], causing the departure of the latter from classical ways of thought. As we attempted to show in the Introduction, this intrinsically quantum trait having to do with the potentialities of objects is crucial from an epistemological point of view. In this chapter we will explain how coherence is also of great practical value, that is, why it can be considered a physical resource, and how its corresponding resource theory has been built up.

It should not be difficult to persuade the reader that coherence is indeed resourceful: as a matter of fact, every advantage furnished by quantum technologies over their classical counterparts is underlain by the fact that quantum systems can exist in a superposition state, from condensed matter [34, 35] and thermodynamics [22, 23, 36–38], metrology [39–41], atomic clocks [42] and non-classicality [43], computation [44–48] and communication [49–51] to quantum simulation [52, 53], and has even been used in the description of fundamental biological processes [54–57]. Indeed, the aforementioned advantages make quantum superposition a precious resource: having access to it allows to perform tasks that are otherwise unfeasible; that is the reason why coherence can be cast within the general framework of a quantum resource theory [58].

Building upon the works of Baumgratz *et al.* [59] and Åberg [60] (see also [61]), a full-fledged resource theory of coherence has been under construction in recent years. As one may guess, there agents are restricted to preparing states with no coherence; the resource, therefore, is withheld from them. This comes as no surprise if we take into account that, in many contexts, the noisy evolution of the state will average out superpositions between eigenstates of an observable, making it hard to produce and control the off-diagonal elements with respect to a given set of projectors. Knowing what the constraint to overcome is, we are now ready to present the key elements of the framework (see [62] for a more extensive review).

1.1 Free states: incoherent states

Given a fixed basis $\{|i\rangle\}_{i=0,\ldots,d-1}$ of the d-dimensional Hilbert space $\mathcal{H}$, that comes determined by the problem—for instance, a natural choice of basis in transport phenomena would be the energy basis—, an incoherent state is one that is diagonal in that basis, i.e. it presents no off-diagonal terms or coherences. The set of incoherent states is denoted as $\mathcal{I} \subset \mathcal{S}(\mathcal{H})$. Hence, all density operators $\delta \in \mathcal{I}$ can be written as

$$\delta = \sum_{i=0}^{d-1} \delta_i |i\rangle\langle i|. \tag{1.1}$$

As we can see, coherence is a basis-dependent resource, since the eigenbasis can always be chosen in order to make every state adopt the form of an incoherent one.

1.2 Free operations: a plethora

Free operations are those that do not generate coherence (we will denote them as NCG operations, Φ_{NCG}, for "non-coherence-generating"). Unlike the resource theories of entanglement, thermodynamics and asymmetry, where the free operations are uniquely identified on operational grounds, no such consensus exists within coherence theory, and that is why a plethora of free operations have been proposed in this field (see Fig. 1.1). Each of them defines a different resource theory of coherence.

A comment is in order here: although Fig. 1.1 shows that the set of MIOs is not strictly the largest set of free operations, since there exists TIOs that

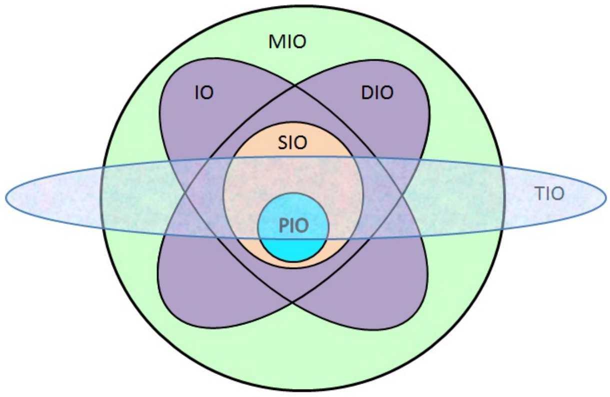


Figure 1.1: Different sets of free operations in the resource theory of coherence. Source: [63].

are not contained in the set of MIOs, in this work we will intentionally leave TIOs out of the theory and consider MIOs as the largest set of free operations. For this reason, we have that every kind of NCG operation that will be considered hereafter is a MIO, and thus necessarily fulfills the MIO properties. In what follows we will explain what these are, and we will enumerate the other sets of NCG operations that we will focus on in this thesis:

- *Maximally incoherent operations (MIOs)* [60]: it is the largest class of free operations, comprising all completely positive and trace-preserving (CPTP) quantum channels $\mathcal{M}$ such that

  $$\mathcal{M}(\mathcal{I}) = \sum_\alpha K_\alpha \mathcal{I} K_\alpha^\dagger \subset \mathcal{I},$$

  where K_α are the Kraus operators of $\mathcal{M}$, fulfilling $\sum_\alpha K_\alpha^\dagger K_\alpha = \mathbb{1}$.

  This is equivalent to the following condition [64]:

  $$\Delta \mathcal{M} \Delta = \mathcal{M} \Delta, \tag{1.2}$$

  where Δ is the complete dephasing map $\Delta = \sum_{i=0}^{d-1} |i\rangle\langle i| \cdot |i\rangle\langle i|$ in the chosen basis $\{|i\rangle\}_{i=0}^{d-1}$.

  In particular, the Choi-Jamiołkowski matrix, $J_\mathcal{M}$, of a MIO operation $\mathcal{M} : A \to B$ is characterized by the following conditions, in addition to the standard ones:

  $$\mathrm{Tr}((|i\rangle\langle i| \otimes |j\rangle\langle j'|) J_\mathcal{M}) = 0, \ \forall i \ \forall \ j \neq j', \tag{1.3}$$

  which is equivalent to requiring that $\mathcal{M}$ does not generate coherence from incoherent states.

- *Incoherent operations (IOs)* [59]: they are the subset of MIOs that admit a Kraus representation with operators K_α such that
$$K_\alpha \mathcal{I} K_\alpha^\dagger \subset \mathcal{I}$$
for all α. Thus, K_α is IO if and only if
$$K_\alpha = \sum_j c_j |k(j)\rangle\langle j|,$$
where $k(j)$ is a function on the index set.

  IOs are defined under the assumption that coherence is never generated in any of the possible outcomes α of a MIO.

- *Strictly incoherent operations (SIOs)* [65]: they are the subset of IOs fulfilling
$$K_\alpha^\dagger \mathcal{I} K_\alpha \subset \mathcal{I}$$
for all α. Thus, K_α is SIO if and only if
$$K_\alpha = \sum_j c_j |k(j)\rangle\langle j|,$$
where $k(j)$ is a one-to-one function on the index set.

  These operations cannot either create coherence or activate the coherence already present in some input (sometimes we will use the word "detect" as a synonym of "activate"), i.e. turn it into the populations which can be measured at a later time [64]:
$$\langle i| K_\alpha \rho K_\alpha^\dagger |i\rangle = \langle i| K_\alpha \Delta[\rho] K_\alpha^\dagger |i\rangle$$
for all α.

Some examples of experimentally relevant NCG operations include the ones responsible for decoherence mechanisms of single qubits such as the phase-damping, the depolarizing and the amplitude-damping channels, as well as permutations of modes of dual-rail qubits in linear optics experiments. In particular, all of the previous operations belong to the set of IOs.

1.3 Maximally resourceful states: cosdits

It remains to characterize the d-dimensional states that contain a maximal amount of resource. In coherence theory these are the d-dimensional maximally coherent states which, for short of notation, we will call *cosdits* (an abbreviation of "coherence of superposition dits"), reserving the term *cosbit* for the unit of quantum coherence—the maximally coherent qubit. Cosdits are the states

$$|\Psi_d\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle \,. \tag{1.4}$$

Note that, for convenience, we will define $\Psi_d := |\Psi_d\rangle\langle\Psi_d|$. A few further remarks are in order:

- Cosdits are maximally resourceful independently of the measure chosen to quantify coherence.
- Any other d-dimensional state may be prepared with certainty by applying NCG operations to a cosdit (see [59] for a proof).

1.4 Coherence measures

Every resource theory requires a functional $F : \mathcal{S}(\mathcal{H}) \to \mathbb{R}_{\geq 0}$ which quantifies the amount of resource present in a state $\rho \in \mathcal{S}(\mathcal{H})$. In the coherence theory specified by a given set of free operations Φ_{NCG} (where NCG could be understood as MIO, IO or SIO, for instance), such a functional is denoted C and must fulfill the following conditions:

1. *Faithfulness*, meaning that coherence must vanish on the set of incoherent states:
$$C(\mathcal{I}) = 0. \tag{1.5}$$
2. *Monotonicity*, which implies that coherence cannot increase under the action of the considered set of NCG operations:
$$C(\rho) \geq C(\Phi_{\mathrm{NCG}}(\rho)). \tag{1.6}$$

These first two conditions are definitely required to speak of a coherence measure. The following ones, which are sometimes demanded axiomatically, are nice if present but not absolutely necessary:

3. *Monotonicity under selective measurements on average (strong monotonicity)*, meaning that the coherence before measuring should be greater than the average coherence of the possible results:
$$C(\rho) \geq \sum_n p_n C(\rho_n), \tag{1.7}$$

where $\rho_n = K_n \rho K_n^\dagger / p_n$ are the possible outcomes of a measurement of $\Phi_{\rm NCG}(\rho) = \sum_n K_n \rho K_n^\dagger$, for $p_n = {\rm Tr}(K_n \rho K_n^\dagger) \geq 0$ with $\sum_n p_n = 1$.

4. *Convexity*, postulated on the basis that coherence must decrease under mixing:
$$\sum_n p_n C(\rho_n) \geq C\left(\sum_n p_n \rho_n\right). \tag{1.8}$$

As a final remark, fulfilment of conditions (3) and (4) implies that condition (2) is also satisfied [59]:

$$C(\Phi_{\rm NCG}(\rho)) = C\left(\sum_n p_n \rho_n\right) \overset{(4)}{\leq} \sum_n p_n C(\rho_n) \overset{(3)}{\leq} C(\rho). \tag{1.9}$$

Moreover, in [66] it has been proven that properties (1)-(4) can be derived from the following three properties: (1), (2) and $C(p_1\rho_1 \oplus p_2\rho_2) = p_1 C(\rho_1) + p_2 C(\rho_2)$.

Having all the previous physically well-motivated definitions in mind, it is now possible to build a *bona fide* coherence quantifier. We are already aware of the formal structure of the functional we are looking for; it is now time to ensure that our monotone will measure coherence and not something else. Let us present some relevant examples of coherence measures.

1.4.1 Distance-based

The coherence of a state is related to its degree of difference with respect to incoherent states. A natural embodiment of such degree of difference is, for instance, the minimum distance existing between the considered state and the set of incoherent states. Thus, distance-based coherence measures are defined as follows [59]:
$$C_D(\rho) = \min_{\delta \in \mathcal{I}} D(\rho, \delta), \tag{1.10}$$
where D is a contractive metric, i.e. $D(\mathcal{E}(\rho), \mathcal{E}(\sigma)) \leq D(\rho, \sigma)$, for any CPTP map $\mathcal{E}$ and any two states $\sigma, \rho \in \mathcal{S}(\mathcal{H})$.

Throughout this thesis we will employ the following distance-based coherence measures; all of them are, at least, faithful and monotonous under some class of NCG operations.

- l_1-*coherence* (IO-monotone) [59]:

$$C_{l_1}(\rho) = \min_{\delta \in \mathcal{I}} ||\rho - \delta||_{l_1} = \sum_{i \neq j} |\rho_{ij}|. \tag{1.11}$$

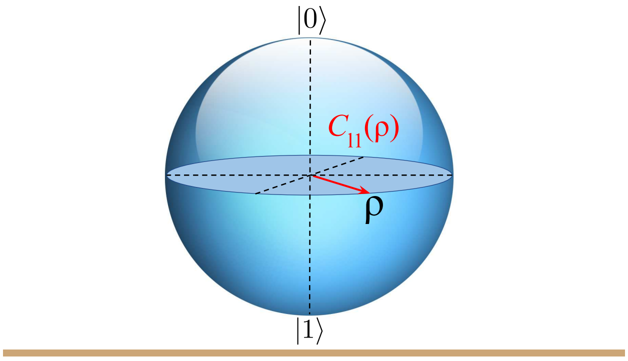


Figure 1.2: The l_1-coherence measure on qubits can be identified with the distance in the Bloch sphere between the state and the z-axis.

The geometrical interpretation of the l_1-coherence measure for qubits is easy to derive. Knowing that the state of a qubit can be expressed as $\rho = \frac{1}{2}(\mathbb{1} + \vec{w}\vec{\sigma})$ where $\vec{w}$ is a real 3-dimensional vector of length $|\vec{w}| \leq 1$ and $\vec{\sigma}$ is the vector of Pauli matrices, we immediately have that:

$$C_{l_1}(\rho) = \frac{1}{2}(|w_1 - iw_2| + |w_1 + iw_2|) = \sqrt{w_1^2 + w_2^2}. \tag{1.12}$$

As we can see, the l_1-coherence of a qubit can be identified with the distance in the Bloch sphere from it to the z-axis (see Fig. 1.2).

Operationally, the l_1-coherence has been proven to quantify the maximum entanglement produced by IOs acting on a system and an incoherent ancilla, as measured by the negativity $\mathcal{N}(\rho) = \operatorname{Tr}|\rho^{T_A}| - 1 = ||\rho^{T_A}||_1 - 1$, where $||\cdot||_1$ is the trace distance and ρ^{T_A} denotes the partial transpose of ρ with respect to subsystem A [67].

As mentioned, the l_1-coherence is monotonic under IO, but not under MIO [68]. We also note that $\log(1 + C_{\ell_1}(\rho))$ is additive under tensor product [69].

- *Relative entropy of coherence* (MIO-monotone) [59]: it is given by the minimum quantum relative entropy between the considered state and the set of incoherent states:

$$C_r(\rho) = \min_{\delta \in \mathcal{I}} S(\rho||\delta) = S(\rho_{\text{diag}}) - S(\rho), \tag{1.13}$$

where ρ_{diag} denotes the diagonal part of the density matrix ρ: $\rho_{\text{diag}} = \sum_i \rho_{ii}|i\rangle\langle i|$ [59] and $S(\rho) = -\operatorname{Tr}(\rho \log \rho)$ is the Von Neumann entropy.

Note that, when computed on pure states $|\varphi\rangle\langle\varphi|$, this measure simply yields

$$C_r(|\varphi\rangle\langle\varphi|) = S(\Delta(|\varphi\rangle\langle\varphi|)), \tag{1.14}$$

and is called *entropy of coherence*.

As we will see in section 5.4, the relative entropy of coherence has been proven to quantify the distillable coherence in the asymptotic limit [65] (under IOs and MIOs) thus taking a similarly central role in coherence theory as the relative entropy of entanglement [16] in the theory of entanglement [70]. In section 1.5.2 we will see that the relative entropy of coherence also quantifies the asymptotic coherence cost under MIOs [71].

- *Trace distance measure of coherence* (IO-monotone only for qubits) [72]:

$$C_{\text{Tr}}(\rho) = \min_{\delta\in\mathcal{I}} \frac{1}{2}||\rho-\delta||_1. \tag{1.15}$$

- *Modified trace distance measure of coherence* (IO-monotone) [66]:

$$C'_{\text{Tr}}(\rho) = \min_{\substack{\delta\in\mathcal{I}\\ \lambda\geq 0}} \frac{1}{2}||\rho-\lambda\delta||_1. \tag{1.16}$$

1.4.2 Robustness of coherence

One can also characterize coherence as the minimal amount of noise one would need to add to a state in order to make it incoherent:

- *Robustness of coherence* (MIO-monotone) [73, 74]:

$$C_R(\rho) = \min_{\tau\in\mathcal{S}(\mathcal{H})} \left\{ s\geq 0 \;\middle|\; \frac{\rho+s\tau}{1+s}\in\mathcal{I}\right\}. \tag{1.17}$$

Moreover, this measure turns out to be a figure of merit in quantum phase discrimination tasks [73]: Suppose a particle goes through an interferometer that applies phases on its state ρ via $U(\vec{\alpha})$. The only thing we know is that d possible interferometers may have acted on the state, each of them with probability $\frac{1}{d}$. Our mission is to guess which interferometer our particle has gone through. Formally, our phase estimation game Θ is defined by the pairs $\Theta = \{\frac{1}{d}, \vec{\alpha}_0 + j\vec{h}_\pi\}_{j=1}^d$, where $\vec{h}_\pi = \frac{2\pi}{d}(\pi(1),\pi(2),...,\pi(d))$ (for a permutation π) is a vector indicating that each interferometer j applies a different vector of equidistributed phases. Then, the robustness of coherence of ρ quantifies the maximum advantage provided by ρ in estimating the correct set of applied phases, with respect to the set of incoherent states $\mathcal{I}$:

$$\max_{\Theta} \frac{P_\Theta^{\text{succ}}(\rho)}{P_\Theta^{\text{succ}}(\mathcal{I})} = 1 + C_R(\rho), \tag{1.18}$$

where $P_\Theta^{\rm succ}(\rho) = \max_{\{M_j\}} \frac{1}{d}\sum_j \mathrm{Tr}(U_j \rho U_j^\dagger M_j)$, for $U_j := U(\vec{\alpha}_0 + j\vec{h}_\pi)$ and a generalized measurement with elements $\{M_j\}$ fulfilling $M_j \geq 0$ and $\sum_j M_j = 1$, denotes the success probability of winning the phase estimation game Θ by using ρ as the input state.

One important result we will make use of throughout the thesis is that the ℓ_1-norm tightly bounds the robustness of coherence [73]

$$\frac{C_{\ell_1}(\rho)}{d-1} \leq C_R(\rho) \leq C_{\ell_1}(\rho), \tag{1.19}$$

where the upper bound becomes an equality for qubits and any pure state (d is the dimension of the state). More generally, it becomes an equality for any state whose density matrix can be made to have non-negative entries under an incoherent unitary [75].

Furthermore, the robustness of coherence is easily computable, since it can be cast as a semidefinite program in primal standard form [74, 75]

$$1 + C_R(\rho) = \min\{\lambda : \ \rho \leq \lambda\sigma,\, \sigma \in \mathcal{I}\}. \tag{1.20}$$

Its equivalent dual form is given by

$$1 + C_R(\rho) = \max\{\mathrm{Tr}\,\rho S : S \geq 0,\, S_{jj} = 1\,\forall j\}, \tag{1.21}$$

which holds because strong duality is fulfilled. The robustness is multiplicative under tensor product of states [69]:

$$1 + C_R(\rho_1 \otimes \rho_2) = (1 + C_R(\rho_1))(1 + C_R(\rho_2)).$$

Finally, it is convenient to define a logged version of the robustness of coherence as:

$$C_{LR} = \log(1 + C_R(\rho)), \tag{1.22}$$

which now becomes an additive quantity under tensor product. In particular, for a cosdit it holds $C_{LR}(\Psi_k) = \log k$.

1.4.3 Coherence rank

- *Coherence rank* (IO-monotone) [43]: for pure states the coherence rank is defined as

$$C_{\rm rank}(\psi) := \min\left\{ r : |\psi\rangle = \sum_{i=1}^{r} c_i\, |c_i\rangle \right\}, \tag{1.23}$$

  where $|c_i\rangle$ is an element of the incoherent basis. That is, the coherence rank of $|\psi\rangle$ is the number of non-zero terms that appear when writing $|\psi\rangle$ in the incoherent basis.

1.4.4 Convex roof quantifiers

A measure C defined for pure states can be extended for mixed states by the convex hull construction to the following coherence monotone:

$$C_{CR}(\rho) = \min_{p_i, |\psi_i\rangle\langle\psi_i|} \sum_i p_i C(|\psi_i\rangle\langle\psi_i|), \tag{1.24}$$

where the minimum is taken over all possible pure state decompositions of $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$.

Some examples of convex roof coherence quantifiers are the coherence of formation and the coherence number:

- *Coherence of formation* (IO-monotone): it extends the entropy of coherence for mixed states:

$$C_f(\rho) = \min_{p_i, |\psi_i\rangle\langle\psi_i|} \sum_i p_i C_r(|\psi_i\rangle\langle\psi_i|) = \min_{p_i, |\psi_i\rangle\langle\psi_i|} \sum_i p_i S(\Delta(|\psi_i\rangle\langle\psi_i|)). \tag{1.25}$$

  As we will explain in section 1.5.2, the coherence of formation has been shown to quantify the asymptotic coherence cost under IOs [65]. Remarkably, this measure violates monotonicity for the class of MIOs [76].

- *Coherence number* (IO-monotone) [77]: the coherence rank can be extended by a convex roof construction to the coherence number:

$$C_{\text{number}}(\rho) := \min_{\rho = \sum_\alpha p_\alpha \psi_\alpha} \max_\alpha C_{\text{rank}}(\psi_\alpha). \tag{1.26}$$

1.5 Some results

Having presented the basic tools of coherence theory, i.e. incoherent states, non-coherence-generating operations, the maximally coherent state and coherence measures, we are now ready to tackle some of the main questions that every resource theory has ever posed, namely how a state can be converted into another one, how much of a resource needs to be consumed in order to produce a given state or how a resource may be distilled from a supply of states at hand.

1.5.1 State interconversion

Alchemists already asked themselves whether base metals could be turned into noble ones by means of feasible operations such as melting or mixing. Analogously, the question whether a given state can be converted into a target one via free operations is investigated by quantum resource theories. In the particular case of coherence theory one wants to know under which conditions the following transformation is possible:

$$\rho \xrightarrow{\text{NCG}} \sigma. \tag{1.27}$$

This problem has been solved completely in the IO-theory, only when pure states are considered:

Theorem 1. *[65, 69] For pure states $|\phi\rangle\langle\phi|$ and $|\psi\rangle\langle\psi|$, the transformation $|\phi\rangle\langle\phi| \xrightarrow{IO} |\psi\rangle\langle\psi|$ is feasible if and only if $\Delta(|\psi\rangle\langle\psi|) \succ \Delta(|\phi\rangle\langle\phi|)$.*

Let us remember that this majorization condition translates as

$$\sum_{i=1}^{k} p_i \downarrow \quad \geq \quad \sum_{i=1}^{k} q_i \downarrow, \tag{1.28}$$

for each k in $1, 2, ..., d$ and where $\downarrow$ implies writing the vector components in descending order, i.e. $\text{spec}(|\psi\rangle\langle\psi|) = (p_1 \geq ... \geq p_d)$ and $\text{spec}(|\phi\rangle\langle\phi|) = (q_1 \geq ... \geq q_d)$.

Explicit protocols for pure state transformations under IOs are given in [78].

Finally, mixed state transformations have been analyzed in the MIO-theory, but only results for qubits have been found:

Theorem 2. *[79] Given two qubits $\rho = \frac{1}{2}(\mathbb{1} + \vec{r}\vec{\sigma})$ and $\sigma = \frac{1}{2}(\mathbb{1} + \vec{s}\vec{\sigma})$, where $\vec{\sigma}$ is the vector of Pauli matrices and $\vec{r}, \vec{s} \in \mathbb{R}^3$, the transformation $\rho \xrightarrow{MIO} \sigma$ is possible if and only if $s_x^2 + s_y^2 \leq r_x^2 + r_y^2$ and $\dfrac{1 - s_z^2}{s_x^2 + s_y^2} \geq \dfrac{1 - r_z^2}{r_x^2 + r_y^2}$.*

As a remark, these conditions can also be cast as monotonicity of two robustness measures, as described by [80]:

$$C_R(\rho) \geq C_R(\sigma) \;\text{ and }\; C_{\Delta,R}(\rho) \geq C_{\Delta,R}(\sigma), \tag{1.29}$$

where

$$C_{\Delta,R}(\rho) = \min_{\tau \in \mathcal{S}(\mathcal{H})} \left\{ s \geq 0 \;\middle|\; \frac{\rho + s\tau}{1 + s} \in \mathcal{I}, \Delta(\tau - \rho) = 0 \right\}. \tag{1.30}$$

1.5.2 Coherence dilution

One special case of state interconversion is coherence dilution:

$$\Psi_d \xrightarrow{\text{NCG}} \sigma. \tag{1.31}$$

In the single-shot scenario, coherence dilution gives rise to a good coherence measure. Indeed, the difficulty to produce a given state by applying NCG operations to a maximally coherent state already gives an intuitive idea of how coherent the state could be. Formally, such coherence measure—called *coherence cost* C_c—is defined as follows [71]:

$$C_c^{\Phi_{\text{NCG}}}(\rho) := \min_{\Lambda \in \Phi_{\text{NCG}}} \{\log_2 M \mid F[\rho, \Lambda(\Psi_M)] = 1\}, \tag{1.32}$$

where $F(\rho, \sigma) = (\text{Tr}[\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}])^2$ is the fidelity between two states ρ and σ.

Several functionals quantifying the coherence cost under MIOs, IOs and SIOs have been provided in [71].

In the asymptotic setting, the coherence cost is also a coherence measure. It is reformulated as the minimal number of coherence units, i.e. cosbits, per copy of ρ required to produce the state ρ asymptotically under NCG operations [65]:

$$C_{c,\infty}^{\Phi_{\text{NCG}}}(\rho) := \min\{R \mid \Psi_2^{\otimes nR} \xrightarrow{\text{NCG}} \overset{1-\epsilon}{\approx} \rho^{\otimes n}, \text{ as } n \to \infty, \epsilon \to 0\}. \tag{1.33}$$

As already pointed out, the asymptotic coherence cost under IOs is given by the coherence of formation: $C_{c,\infty}^{\text{IO}}(\rho) = C_f(\rho)$ [65]. In the MIO-theory, however, the asymptotic coherence cost is proven to be equivalent to the relative entropy of coherence: $C_{c,\infty}^{\text{MIO}}(\rho) = C_r(\rho)$ [71].

1.5.3 Coherence distillation

Another relevant case of state interconversion is coherence distillation:

$$\sigma \xrightarrow{\text{NCG}} \Psi_d. \tag{1.34}$$

In the single-shot setting, coherence distillation also gives rise to a *bona fide* coherence measure. Indeed, one could quantify the coherence of a state by the amount of maximal coherence that can be distilled from it via NCG operations. Formally, such coherence measure—called *distillable coherence* C_d—is defined as follows [81]:

$$C_d^{\Phi_{\text{NCG}}}(\rho) := \log_2 \max_{\Lambda \in \Phi_{\text{NCG}}} \{M \mid F[\Lambda(\rho), \Psi_M] = 1\}. \tag{1.35}$$

The formula for the distillable coherence under MIOs—which is also proven to admit an SDP formulation—has been derived in [81]. The quantifiers of the distillable coherence under IOs and SIOs have been provided by [82].

In the asymptotic scenario, the distillable coherence is a coherence measure as well. It is defined as the maximal number of cosbits per copy of ρ that can be recovered from the state ρ asymptotically under NCG operations [65]:

$$C_{d,\infty}^{\Phi_{\mathrm{NCG}}}(\rho) := \max\{R \mid \rho^{\otimes n} \xrightarrow{\mathrm{NCG}} \overset{1-\epsilon}{\approx} \Psi_2^{\otimes nR}, \text{ as } n \to \infty, \epsilon \to 0\}. \tag{1.36}$$

As already mentioned, the asymptotic distillable coherence under IOs is given by the relative entropy of coherence: $C_{d,\infty}^{\mathrm{IO}}(\rho) = C_r(\rho)$ [65]. The asymptotic distillable coherence for the MIO-theory is given in [81]: $C_{d,\infty}^{\mathrm{MIO}}(\rho) = C_r(\rho)$. Note that, for other sets of NCG operations, $C_{d,\infty}(\rho) \leq C_{c,\infty}(\rho)$. This means that in general coherence theory is irreversible. Furthermore, coherence theory under IOs cannot exhibit "bound coherence", since vanishing distillable coherence $C_r(\rho) = 0$ implies vanishing coherence cost $C_f(\rho) = 0$ [65]. This is in contrast to the SIO-theory of states, which can exhibit bound coherence [83], and to the coherence theory of maps, as we will explain in Chapter 4.

CHAPTER 2

Coherence as interferometric visibility

In the Introduction, we presented the double-slit experiment with quantum particles like electrons in order to highlight how a basic epistemological principle of quantum mechanics is fundamentally grounded on it: in quantum theory, the possibility to know whether an event will occur in one particular way is key to determine whether a particle will interfere with itself or not in a suitable interferometric experiment. Remember that, only when the detector was not placed by the slits with the purpose of finding out which way the electron had chosen, could we see an interference pattern on the target screen. Quantum coherence, the basic hallmark of departure from classical physics, is precisely the property of events leading to such kind of interference.

Furthermore, we compared the behaviour of quantum systems with coherence with that of classical waves like water or light. Indeed, the physics of constructive and destructive interference of waves, along with the concept of coherence, has been well-understood since the development of classical optics in the 19th century. Here, coherence is defined as the correlation of the phases between two or more waves, so that interference may be produced between them. The more correlations there exist, the more coherent the waves are said to be. Fig. 2.1 shows Thomas Young, whose original double-slit experiment revealed the wave nature of light for the first time, and a schematic representation of destructive and constructive interference of waves.

In Chapter 1, we presented the basic elements of the resource theory of coherence. Here, coherence is understood as the degree of difference between a given state and the set of incoherent states, i.e. those states that are

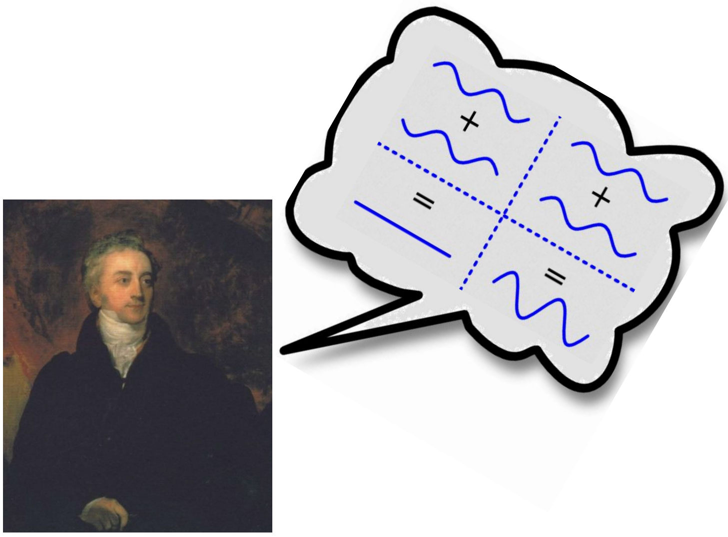


Figure 2.1: Thomas Young and a representation of destructive (left) and constructive (right) interference of waves.

diagonal in the considered basis. However, one could ask: is such a definition of coherence merely axiomatic? What are its operational foundations [84]? More concretely, how can this concept of coherence be connected to its very definition, the observability of an interference pattern in a suitable experiment?

In the present chapter we assume that coherence is the potential of a state to yield visible fringes in a convenient experiment. As we will now demonstrate, it is indeed possible to link the visibility of those fringes with a *bona fide* coherence measure, thus ensuring that the resource theory of coherence is actually rooted in the original physical meaning of coherence. But let us first learn some basic things about interferometers and visibility parameters.

2.1 Interferometers and visibility

Let us consider a multi-path interferometer, in which a single particle can be in one of d possible paths; we will denote the spatial variable by orthogonal vectors $|j\rangle$, $j = 1, \ldots, d$, spanning a d-dimensional Hilbert space $\mathcal{H}$. If we ignore any internal degrees of freedom of the particle, and any other spatial degrees, we have that the entire Hilbert space describing the system is $\mathcal{H}$, and that a pure state inside the interferometer can be written as $|\psi\rangle = \sum_j c_j |j\rangle$,

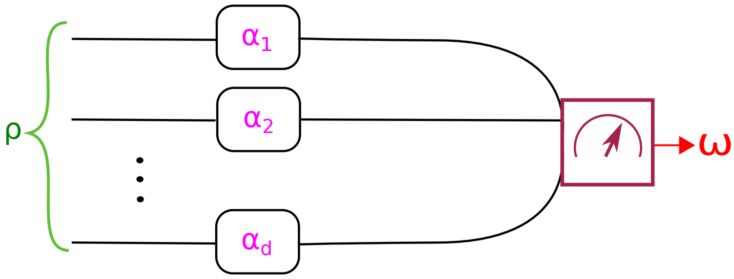


Figure 2.2: Multi-path interferometer. The input state ρ (in green), represents the state of a particle on d paths. Each path can be subjected to a local relative phase α_j (in magenta), after which the particle is detected by means of a suitable measurement (in purple), producing an outcome ω (in red). Source: [85].

and a general mixed state as

$$\rho = \sum_{j,k=1}^{d} \rho_{jk} |j\rangle\langle k|. \tag{2.1}$$

An interferometric experiment (Fig. 2.2) has two main components. First, local phase shifts α_j that can be inserted into the paths, which implement a diagonal phase unitary

$$U(\vec{\alpha}) = \sum_j e^{i\alpha_j} |j\rangle\langle j|, \tag{2.2}$$

so that the state becomes

$$\rho(\vec{\alpha}) = U(\vec{\alpha})\rho U(\vec{\alpha})^\dagger = \sum_{j,k=1}^{d} e^{i(\alpha_j - \alpha_k)} \rho_{jk} |j\rangle\langle k|. \tag{2.3}$$

The second is a detector at the output, that we will characterize as a general POVM $M = (M_\omega)$, with outcomes ω from a suitable space Ω.

If the experimenter has chosen $\vec{\alpha} = (\alpha_1, \ldots, \alpha_d)$, then she will observe outcomes $\omega \in \Omega$ sampled from the Born distribution, i.e. the "interference pattern":

$$P_{M|\rho}(\omega|\vec{\alpha}) = \mathrm{Tr}(U(\vec{\alpha})\rho U(\vec{\alpha})^\dagger M_\omega). \tag{2.4}$$

What is then the signature of interference in such an experiment, where ρ is given? That the distribution $P = P_{M|\rho}$ can vary as a function of the phases α_j. Such degree of variability can be intuitively thought as the visibility of the interference pattern, which we will formalize as a visibility functional $V = V[P]$ on conditional distributions $P(\omega|\vec{\alpha})$. We will assume that such a functional has to capture both the global property of P not being constant, i.e.

it should be 0 for constant $P(\cdot|\vec{\alpha})$ and positive otherwise, and its invariance under permutations and shifts of $\vec{\alpha}$[1] (reflecting the natural symmetries of the experimental setup in Fig. 2.2). A visibility functional $V[P]$ satisfying these two requirements will be called *regular*.

In the literature on interferometers, in particular in the rich discussion on the complementarity between fringe visibility and which-path information [86–90], the topic of visibility has been addressed repeatedly. From it one learns that for $d > 2$ no unique visibility functional seems to exist. Consider, for instance, the simplest case of the well-known Mach-Zehnder interferometer (Fig. 2.3), i.e. $d = 2$. In it we observe interference fringes w.r.t. a relative phase shift, as detailed in the following:

Let us take the density matrix $\rho = \rho_{11}|1\rangle\langle 1| + \rho_{22}|2\rangle\langle 2| + \rho_{12}|1\rangle\langle 2| + \rho_{21}|2\rangle\langle 1|$, the diagonal phase unitary $U(\vec{\alpha}) = e^{i\alpha_1}|1\rangle\langle 1| + e^{i\alpha_2}|2\rangle\langle 2|$, and a measurement with POVM elements $|\mu\rangle\langle\mu|$, $|\mu\rangle = \mu_1|1\rangle + \mu_2|2\rangle$. Now, with $\alpha = \alpha_1 - \alpha_2$, and writing $\rho_{12} = |\rho_{12}|e^{i\beta}$, $\overline{\mu_1}\mu_2 = |\mu_1\mu_2|e^{i\gamma}$, the output probability is

$$\begin{aligned} P_{M|\rho}(\mu|\vec{\alpha}) = \rho_{11}|\mu_1|^2 + \rho_{22}|\mu_2|^2 \\ + 2|\rho_{12}\mu_1\mu_2|\cos(\alpha + \beta + \gamma), \end{aligned} \tag{2.5}$$

whose fluctuation is essentially characterised by the coefficient $|\rho_{12}\mu_1\mu_2|$. Indeed, most analyses conclude this to be the visibility of such an interference pattern.

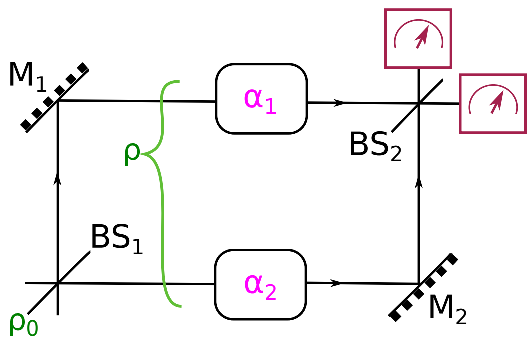


Figure 2.3: Mach-Zehnder two-path interferometer. The initial state is ρ_0, while the state after the interaction with the first beam splitter (BS1) and the first mirror (M1) is ρ (in green), representing the state of a particle on two paths. Each path can be subjected to a phase α_1 and α_2, although only their relative difference $\alpha = \alpha_1 - \alpha_2$ is physically relevant. After the interaction with the second beam splitter (BS2), the interference pattern is observed. Source: [85].

[1]By a shift of $\vec{\alpha}$ we mean applying a common shift to each component α_j of $\vec{\alpha}$.

2.2 Optimal visibility as measure of coherence

Consider the qutrit state $\rho = \frac{1}{3}(|1\rangle + |2\rangle)(\langle 1| + \langle 2|) + \frac{1}{3}|3\rangle\langle 3|$, under a measurement $M^{(0)}$ in the basis $\left\{|1\rangle, \frac{1}{\sqrt{2}}(|2\rangle \pm |3\rangle)\right\}$; the three outcomes all have probability $\frac{1}{3}$, irrespective of the phases in

$$\rho(\alpha) = \frac{1}{3}\left(|1\rangle + e^{i\alpha}|2\rangle\right)\left(\langle 1| + e^{-i\alpha}\langle 2|\right) + \frac{1}{3}|3\rangle\langle 3|.$$

However, if we choose the projective measurement $M^{(1)}$ in the basis

$$\left\{\frac{1}{\sqrt{2}}(|1\rangle \pm |2\rangle), |3\rangle\right\},$$

the detection probabilities are $(\frac{2}{3}\cos^2\frac{\alpha}{2}, \frac{2}{3}\sin^2\frac{\alpha}{2}, \frac{1}{3})$, thus bringing out the coherence in ρ.

This simple example illustrates that an unsuitable choice of measurement may render coherent superposition invisible. Therefore, if we aim to treat the state ρ as a resource, i.e. as a given, of which we are supposed to make the best, it makes sense to optimise the visibility $V[P_{M|\rho}]$ over all possible measurements. This leads to a number that now depends only on the state,

$$C_V(\rho) := \sup_{M=(M_\omega)} V[P_{M|\rho}]. \tag{2.6}$$

Let us now define *weakly affine* visibility functionals as those which, for distributions $P_i(\omega|\vec{\alpha})$, $\omega \in \Omega_i$ (assuming pairwise disjoint Ω_i), and for a probability distribution (q_i), yield $V\left[\overline{P}\right] = \sum_i q_i V[P_i]$, with the averaged distribution $\overline{P} = \sum_i q_i P_i$ on $\Omega = \bigcup_i \Omega_i$. The hypothesis that we will explore in the rest of the paper is that this number, for the large class of regular and weakly affine visibility functionals, is a good coherence measure of ρ, at least in the SIO-theory.

Theorem 3. *For any regular and weakly affine visibility functional $V[P]$, C_V is a coherence measure that is strongly monotonic under SIOs. If V is convex in P, then C_V is convex in ρ.*

Proof. Let a SIO with Kraus operators K_λ be given, acting on a state ρ, so that $q_\lambda \rho_\lambda = K_\lambda \rho K_\lambda^\dagger$ defines the probability of the event λ and the post-measurement state. Observe that, because $K_\lambda = \pi_\lambda D_\lambda$ can be written as a diagonal matrix D_λ followed by a permutation π_λ,

$$\begin{aligned} q_\lambda U(\vec{\alpha})\rho_\lambda U(\vec{\alpha})^\dagger &= U(\vec{\alpha})K_\lambda \rho K_\lambda^\dagger U(\vec{\alpha})^\dagger \\ &= K_\lambda U(\vec{\beta})\rho U(\vec{\beta})^\dagger K_\lambda^\dagger, \end{aligned}$$

with $\beta_j = \alpha_{\pi_\lambda(j)}$. This shows that the probability of seeing outcome λ is q_λ for all $U(\vec{\alpha})$.

Now choose measurements $M^{(\lambda)}$ for each ρ_λ, taking values ω in the disjoint sets Ω_λ, subject to the probability law $P_\lambda = P_{M^{(\lambda)}|\rho_\lambda}$ given by

$$\begin{aligned} P_\lambda(\omega|\vec{\alpha}) &= \operatorname{Tr} U(\vec{\alpha})\rho_\lambda U(\vec{\alpha})^\dagger M_\omega^{(\lambda)} \\ &= \frac{1}{q_\lambda} \operatorname{Tr} U(\vec{\alpha}) K_\lambda \rho K_\lambda^\dagger U(\vec{\alpha})^\dagger M_\omega^{(\lambda)} \\ &= \frac{1}{q_\lambda} \operatorname{Tr} K_\lambda U(\vec{\beta})\rho U(\vec{\beta})^\dagger K_\lambda^\dagger M_\omega^{(\lambda)} \\ &= \frac{1}{q_\lambda} \operatorname{Tr} U(\vec{\beta})\rho U(\vec{\beta})^\dagger K_\lambda^\dagger M_\omega^{(\lambda)} K_\lambda. \end{aligned}$$

Introducing the POVM $\widetilde{M} = \sum_\lambda (K_\lambda^\dagger M_\omega^{(\lambda)} K_\lambda)_{\lambda,\omega}$ with outcomes (λ, ω), we can now invoke weak affinity:

$$\sum_\lambda q_\lambda V[P_\lambda] = V\left[\sum_\lambda q_\lambda P_\lambda\right] = V[P_{\widetilde{M}|\rho}] \leq C_V(\rho),$$

because the measurement $\widetilde{M}$ is eligible for ρ but may be suboptimal. Since the measurements $M^{(\lambda)}$ can be chosen to maximise the left hand side, we obtain

$$\sum_\lambda q_\lambda C_V(\rho_\lambda) \leq C_V(\rho).$$

For the convexity statement, let $\rho = \sum_i p_i \sigma_i$ and choose any measurement M on ρ. Then,

$$V[P_{M|\rho}] = V\left[\sum_i p_i P_{M|\sigma_i}\right] \leq \sum_i p_i V[P_{M|\sigma_i}] \leq \sum_i p_i C_V(\sigma_i),$$

and because M may be chosen to maximise the left hand side, we find $C_V(\sum_i p_i \sigma_i) \leq \sum_i p_i C_V(\sigma_i)$, as claimed. □

As a remark, one might wonder in case we only want to *detect* coherence, whether there is a universal measurement M such that if ρ has coherences, then $V[P_{M|\rho}]$ is positive. The answer is yes, namely any tomographically complete measurement, as long as $V[P]$ has the property that it is non-zero on every non-constant P.

2.3 Examples

In this section we will illustrate how the above theory is not just an abstract construction. In particular, we will consider several visibility parameters, optimize them over all possible POVMs and find out which coherence measure they lead to—or at least considerably simplify the optimization.

2.3.1 Largest difference of intensity

The perhaps simplest and most intuitive parameter of visibility for two-outcome measurements $M = (M_0, M_1 = \mathbb{1} - M_0)$ is the difference between the largest and the smallest value of $P_{M|\rho}(0|\vec{\alpha}) = \operatorname{Tr} U(\vec{\alpha})\rho U(\vec{\alpha})^\dagger M_0$. To make it suitable for measurements with arbitrary outcome sets, we define

$$V_{\max}[P] := \sup_{\vec{\alpha},\vec{\beta}} \frac{1}{2} \left\| P(\cdot|\vec{\alpha}) - P(\cdot|\vec{\beta}) \right\|_1 . \tag{2.7}$$

Note that we do not normalize by the sum of the largest and smallest probability, as is customary in discussions of visibility in classical interferometry, where the basic observable quantities are intensities. There, this appears necessary to obtain a dimensionless visibility; here however, we have the probabilities that are already dimensionless and have an absolute meaning.

Clearly, $V_{\max}$ is regular and weakly affine, so the corresponding coherence measure $C_{\max}$ is a SIO monotone. In fact, it is easy to evaluate it, and the result is

$$\begin{aligned} C_{\max}(\rho) &= \max_{\vec{\alpha}} \frac{1}{2} \left\| U(\vec{\alpha})\rho U(\vec{\alpha})^\dagger - \rho \right\|_1 \\ &= \max_{\vec{\alpha}} \frac{1}{2} \left\| [\rho, U(\vec{\alpha})] \right\|_1 \\ &= \max_{\vec{\alpha},\, M_0} \operatorname{Tr} U(\vec{\alpha})\rho U(\vec{\alpha})^\dagger M_0 - \operatorname{Tr} \rho M_0, \end{aligned} \tag{2.8}$$

because we can always shift $\vec{\beta}$ to $\vec{0}$ by applying $U(-\vec{\beta})$. In particular, the optimal measurement is a two-outcome POVM $(M_0, M_1 = \mathbb{1} - M_0)$, and the value is the largest difference in response probability over POVM elements.

We can compare the result with the trace distance measure of coherence $C_{\mathrm{Tr}}(\rho) = \min_{\delta\in\mathcal{I}} \frac{1}{2}||\rho - \delta||_1$: $C_{\mathrm{Tr}}(\rho) \leq C_{\max}(\rho) \leq 2C_{\mathrm{Tr}}(\rho)$.

Namely, on the one hand, for $\delta \in \mathcal{I}$, we have $\|\rho - \delta\|_1 = \left\| U(\vec{\alpha})\rho U(\vec{\alpha})^\dagger - \delta \right\|_1$, so by the triangle inequality

$$\left\| U(\vec{\alpha})\rho U(\vec{\alpha})^\dagger - \rho \right\|_1 \leq \left\| U(\vec{\alpha})\rho U(\vec{\alpha})^\dagger - \delta \right\|_1 + \|\rho - \delta\|_1,$$

which implies $C_{\max}(\rho) \leq 2C_{\mathrm{Tr}}(\rho)$. On the other hand,

$$\begin{aligned} C_{\mathrm{Tr}}(\rho) &\leq \frac{1}{2}\Big\|\rho - \Delta(\rho)\Big\|_1 \\ &= \frac{1}{2}\left\|\rho - \int \mathrm{d}\vec{\alpha}\, U(\vec{\alpha})\rho U(\vec{\alpha})^\dagger\right\|_1 \\ &\leq \int \mathrm{d}\vec{\alpha}\, \frac{1}{2}\left\|U(\vec{\alpha})\rho U(\vec{\alpha})^\dagger - \rho\right\|_1 \leq C_{\max}(\rho). \end{aligned}$$

In the qubit case, as only the relative phase $\alpha = \alpha_1 - \alpha_2$ matters, it holds

$$\rho - U(\vec{\alpha})\rho U(\vec{\alpha})^\dagger = \begin{bmatrix} 0 & (1 - e^{-i\alpha})\rho_{12} \\ (1 - e^{+i\alpha})\rho_{21} & 0 \end{bmatrix}.$$

Its trace norm clearly is maximized at $\alpha = \pi$, showing $C_{\max}(\rho) = |\rho_{12}| + |\rho_{21}| = C_{\ell_1}(\rho)$, which for qubits is known to equal $2C_{\mathrm{Tr}}(\rho)$. In conclusion, for qubits we have that $C_{\max}(\rho) = 2|\rho_{01}| = C_{\ell_1}(\rho) = 2C_{\mathrm{Tr}}(\rho)$.

2.3.2 Estimating equidistributed phases

Inspired by the previous example, we are motivated to consider guessing problems of a more general kind, where we are trying to estimate the true setting of the phases among several alternatives, based on measurement outcomes. It turns out that a good candidate is the equidistributed set of d phases $\frac{2\pi j}{d}(1, 2, \ldots, d)$, $j = 1, \ldots, d$, and its shifts and permutations:

$$V_{\mathrm{guess}}[P] := -\frac{1}{d} + \max_{\substack{\vec{\alpha}_0,\, \pi \in S_d \\ \Omega_j \cap \Omega_k = \emptyset}} \frac{1}{d} \sum_{j=1}^{d} P(\Omega_j | \vec{\alpha}_0 + j\vec{h}_\pi), \tag{2.9}$$

where $\vec{h}_\pi = \frac{2\pi}{d}\Big(\pi(1), \pi(2), \ldots, \pi(d)\Big)$ is a generating vector of uniformly accelerating phases (w.r.t. the permutation π of coordinates). This quantity is the bias (excess over $\frac{1}{d}$) of the optimal strategy to guess the true value of $j \in \{1, \ldots, d\}$ that defines the phase settings. As defined, this visibility functional is regular and weakly affine, so the corresponding C_{guess} is a coherence monotone under SIO. As a matter of fact, it holds [73]

$$C_{\mathrm{guess}}(\rho) = -\frac{1}{d} + \max_{(M_j)\ \mathrm{POVM}} \frac{1}{d} \sum_{j=1}^{d} \mathrm{Tr}\, \rho(\vec{\alpha}_0 + j\vec{h}_\pi) M_j = \frac{1}{d} C_R(\rho), \tag{2.10}$$

for any $\vec{\alpha}_0$ and any permutation π. Remember that C_R denotes the robustness of coherence measure, whose operational meaning in the context of phase estimation games was already introduced in Section 1.4.2.

In the qubit case, it is well known that the robustness of coherence equals the ℓ_1-measure: $C_R(\rho) = 2|\rho_{01}| = C_{\ell_1}(\rho)$ [73], and so $C_{\text{guess}}(\rho) = |\rho_{01}|$ is just half of that.

2.3.3 Largest sensitivity to phase changes

Looking back at example A, we notice that the points of largest and smallest value of the response probability $I(\vec{\alpha}) = P_{M|\rho}(0|\vec{\alpha}) = \operatorname{Tr} U(\vec{\alpha})\rho U(\vec{\alpha})^\dagger M_0$ to a POVM element M_0 may be quite far apart. In contrast, in many applications of interferometry it is a relatively small phase difference that we want to pick up [91], so we are interested in the largest magnitude of the derivative of $I(\vec{\alpha})$:

$$V_\nabla[P] := \max_{\vec{\alpha},\vec{h}} \left| \frac{\partial I}{\partial \vec{h}}(\vec{\alpha}) \right|, \tag{2.11}$$

where $\vec{\alpha}$ ranges over all phases, and $\vec{h}$ over all direction vectors that are suitably norm bounded. To extend V_∇ to general measurements, we may include a maximisation over all two-outcome coarse grainings. We can easily see that $V_\nabla[P]$ is regular and weakly affine since $I(\vec{\alpha})$ is well-defined probability distribution over $\vec{\alpha}$.

Now, as $I(\vec{\alpha}) = \operatorname{Tr}(\rho U(\vec{\alpha})^\dagger M_0 U(\vec{\alpha}))$, its derivative at (w.l.o.g.) $\vec{0}$ in direction $\vec{h}$ is given by

$$\frac{\partial I}{\partial \vec{h}}(\vec{0}) = -i \operatorname{Tr}[\rho, H] M_0 = -i \operatorname{Tr} \rho [H, M_0], \tag{2.12}$$

where H is the diagonal Hamiltonian with eigenvalues h_j, $H = \operatorname{diag}(\vec{h})$. Note that the derivative at any other point $\vec{\alpha}_0$ is the same, up to conjugating the measurement by $U(\vec{\alpha}_0)$. There are two natural limitations on $\vec{h}$: Geometrically, to obtain the largest gradient of I, we should consider unit vectors $\vec{h}$, meaning $\|H\|_2^2 = \operatorname{Tr} H^2 = 1$; or taking motivation from the Hamiltonian, we should bound its energy range, meaning $\|H\|_\infty \leq 1$. We denote these two scenarios by $p = 2$ and ∞, giving rise to two coherence measures $C_\nabla^{(p)}$. From Eq. (2.12), we directly get

$$C_\nabla^{(p)}(\rho) = \max \frac{1}{2} \big\| [\rho, H] \big\|_1 \text{ s.t. } H \text{ diag.}, \|H\|_p \leq 1. \tag{2.13}$$

Inspecting this formula, we see that the optimisation is convex in H, hence the maximum is attained on an extremal admissible Hamiltonian. For $p = 2$, these have the form $H = \sum_j \epsilon_j \sqrt{t_j} |j\rangle\langle j|$, with $\epsilon_j = \pm 1$ and $\sum_j t_j = 1$. For $p = \infty$, the extremal H have entries ± 1 along the diagonal, and so

$$C_{\nabla}^{(\infty)}(\rho) = \max_{S_+ \dot\cup S_- = [d]} 2 \left\|\Pi_+ \rho \Pi_-\right\|_1, \tag{2.14}$$

where the maximisation over partitions $S_+ \dot\cup S_- = [d]$, with $\Pi_\bullet = \sum_{j \in S_\bullet} |j\rangle\langle j|$, $\bullet = \pm$. In both cases, we obtain a strong SIO monotone, due to the evident weak affinity of V_∇. From Eq. (2.8) we see that $C_\nabla^{(\infty)} \leq C_{\max}$, but equality does not seem to hold in general.

Once again, the qubit case is very simple: For $p = \infty$, the only non-trivial choice is $\Pi_+ = |1\rangle\langle 1|$ and $\Pi_- = |2\rangle\langle 2|$, directly resulting in $C_\nabla^{(\infty)} = 2\left\| |1\rangle\langle 1|\rho|2\rangle\langle 2| \right\|_1 = 2|\rho_{12}|$.

For $p = 2$, we have to consider the Hamiltonian $H = \sqrt{t}|1\rangle\langle 1| \pm \sqrt{1-t}|2\rangle\langle 2|$, yielding

$$[\rho, H] = \begin{bmatrix} 0 & (-\sqrt{t} \pm \sqrt{1-t})\rho_{12} \\ (\sqrt{t} \mp \sqrt{1-t})\rho_{21} & 0 \end{bmatrix}.$$

Its trace norm is maximized for the negative sign choice and at $t = \frac{1}{2}$, and so $C_\nabla^{(2)} = \sqrt{2}|\rho_{12}|$. Therefore, $C_\nabla^{(2)}(\rho) = \sqrt{2}|\rho_{12}|$ and $C_\nabla^{(\infty)}(\rho) = 2|\rho_{12}| = C_{\ell_1}(\rho)$.

2.3.4 Largest Fisher information

Considering further the previous example, we realize that finding the largest derivative of the probability $P(0|\vec{\alpha})$, while strongly motivated by the intuition rooted in intensities, does not necessarily identify the point of strongest statistical sensitivity, which is asking for the largest Fisher information, the natural measure for probability distributions. Looking again at directional estimation of a one-dimensional subfamily $\vec{\alpha} = t\vec{h} + \vec{\alpha}_0$, $t \in \mathbb{R}$, the Fisher information is given by the expected squared logarithmic derivative of the probability distribution:

$$\begin{aligned} \mathcal{F}_{\vec{\alpha}_0}(\vec{h}) &= \sum_{\omega \in \Omega} P(\omega|\vec{\alpha}_0) \left(\frac{\mathrm{d} \ln P(\omega|\vec{\alpha})}{\mathrm{d}t} \Big|_{t=0} \right)^2 \\ &= \sum_{\omega \in \Omega} \frac{1}{P(\omega|\vec{\alpha}_0)} \left(\frac{\mathrm{d}P(\omega|\vec{\alpha})}{\mathrm{d}t} \Big|_{t=0} \right)^2, \end{aligned} \tag{2.15}$$

so we are considering the visibility functional

$$V_{\mathrm{F}}[P] := \max_{\vec{\alpha}_0, \vec{h}} \mathcal{F}_{\vec{\alpha}_0}(\vec{h}), \tag{2.16}$$

where $\vec{\alpha}_0$ varies over the whole space of phases, and $\vec{h}$ over a suitably bounded set of directions. Clearly, $V_{\rm F}$ is regular and weakly affine.

The formula for the Fisher information, optimized over measurements (and $\vec{\alpha}_0$, which w.l.o.g. is $\vec{0}$, by the same reasoning as in previous examples), for estimating $t \approx 0$ in $e^{-itH}\rho e^{itH}$ for a given diagonal Hamiltonian $H = \operatorname{diag}(\vec{h})$ and $\rho = \sum_j \lambda_j |e_j\rangle\langle e_j|$ is known [92, 93] and given by

$$\mathcal{F}_{\rm opt}(\vec{h}) = 2\sum_{jk} \frac{(\lambda_j - \lambda_k)^2}{\lambda_j + \lambda_k} |\langle e_j| H |e_k\rangle|^2. \tag{2.17}$$

Like in the previous example on sensitivity, there are two natural domains of diagonal Hamiltonians H over which to optimize this: Either $\|H\|_2 \leq 1$ or $\|H\|_\infty \leq 1$, leading to two variants $C_{\rm F}^{(2)}(\rho)$ and $C_{\rm F}^{(\infty)}(\rho)$ of the coherence measure.

In either case, the optimal choice of H is extremal subject to the convex constraint, because $\mathcal{F}$ can easily be seen to be convex in H. Namely, each term $|\langle e_j| H |e_k\rangle|$ is convex, hence also its square, and the coefficient in front of it manifestly nonnegative. Thus, we obtain:

$$\begin{aligned} C_{\rm F}^{(2)}(\rho) &= \max_{\substack{\sum_j t_j = 1 \\ \epsilon_j = \pm 1}} \sum_{jk} 2\frac{(\lambda_j - \lambda_k)^2}{\lambda_j + \lambda_k} \\ &\qquad \left| \langle e_j| \left(\sum_j \epsilon_j \sqrt{t_j} |j\rangle\langle j| \right) |e_k\rangle \right|^2, & (2.18) \\ C_{\rm F}^{(\infty)}(\rho) &= \max_{S_+, S_- \subset [d]} \sum_{jk} 2\frac{(\lambda_j - \lambda_k)^2}{\lambda_j + \lambda_k} \left| \langle e_j| (\Pi_+ - \Pi_-) |e_k\rangle \right|^2, & (2.19) \end{aligned}$$

where the first maximization is over diagonal Hamiltonians with Hilbert-Schmidt norm 1; the second over partitions $S_+ \dot\cup S_- = [d]$, with $\Pi_\bullet = \sum_{j \in S_\bullet} |j\rangle\langle j|$, $\bullet = \pm$, so that $H = \Pi_+ - \Pi_-$.

For qubits, the formula for the coherence measure reduces to

$$C_{\rm F}^{(p)}(\rho) = \max 2\frac{(\lambda_1 - \lambda_2)^2}{\lambda_1 + \lambda_2} |\langle e_1| H |e_2\rangle|^2,$$

where the maximization is over $H \in \operatorname{span}\{\mathbb{1}, \sigma_Z\}$ such that $\|H\|_p \leq 1$. Note that $\lambda_1 + \lambda_2 = 1$ and $|\langle e_1| H |e_2\rangle|^2 = \operatorname{Tr} H|e_1\rangle\langle e_1|H|e_2\rangle\langle e_2|$.

This calculation is conveniently done in the Bloch picture, writing $\rho = \frac{1}{2}(\mathbb{1} + \vec{r} \cdot \vec{\sigma})$, with a vector $\vec{r} = r\,\vec{r}^0$ that we decompose as a product of its length $r = |\vec{r}|$ and a unit vector $\vec{r}^0$ (with components r_x^0, r_y^0 and r_z^0). In this way the eigenprojectors of ρ become $|e_{1,2}\rangle\langle e_{1,2}| = \frac{1}{2}(\mathbb{1} \pm \vec{r}^0 \cdot \vec{\sigma})$. In the above maximization, this allows us to identify $\lambda_1 - \lambda_2 = r$ and $r^2 = 2\operatorname{Tr}\rho^2 - 1$.

For $p = \infty$, we already know that $H = \sigma_Z$ is optimal, so

$$\begin{aligned} C_{\mathrm{F}}^{(\infty)}(\rho) &= 2r^2 \operatorname{Tr} \sigma_Z |e_1\rangle\langle e_1|\sigma_Z|e_2\rangle\langle e_2| \\ &= 2r^2 \frac{1}{4} \operatorname{Tr}(\mathbb{1} - r_x^0\sigma_X - r_y^0\sigma_Z + r_z^0\sigma_Z) \\ &\qquad \cdot(\mathbb{1} - r_x^0\sigma_X - r_y^0\sigma_Z - r_z^0\sigma_Z) \\ &= r^2\Big(1 + (r_x^0)^2 + (r_y^0)^2 - (r_z^0)^2\Big) \\ &= 4\Big(\operatorname{Tr}\rho^2 - \operatorname{Tr}\Delta(\rho)^2\Big) \\ &= 8|\rho_{12}|^2 = 2C_{\ell_1}(\rho)^2. \end{aligned}$$

For $p = 2$, the maximization reduces to that of $2r^2 \operatorname{Tr} H|e_1\rangle\langle e_1|H|e_2\rangle\langle e_2|$, with $H = \alpha\mathbb{1} + \beta\sigma_Z$ and $2\alpha^2 + 2\beta^2 \leq 1$. The trace decomposes into four terms, however the three that contain a $\alpha\mathbb{1}$ evaluate to 0, leaving $2\beta^2 r^2 \operatorname{Tr}\sigma_Z|e_1\rangle\langle e_1|\sigma_Z|e_2\rangle\langle e_2|$, which yields (using the optimal choice $2\beta^2 = 1$) $C_{\mathrm{F}}^{(2)}(\rho) = 2(\operatorname{Tr}\rho^2 - \operatorname{Tr}\Delta(\rho)^2) = C_{\ell_1}(\rho)^2$.

For a qubit state ρ, then, it holds that $C_{\mathrm{F}}^{(2)}(\rho) = 4|\rho_{12}|^2 = C_{\ell_1}(\rho)^2$, $C_{\mathrm{F}}^{(\infty)}(\rho) = 2C_{\ell_1}(\rho)^2$.

2.3.5 Largest differential Chernoff bound

We observe that the attainability of the Fisher information presupposes access to many copies of the state and independent measurements, in which setting the Fisher information gives the optimal scaling of the mean squared estimation error with the number of copies. If we allow general collective measurements and at the same time only want to distinguish pairs of nearby states optimally, we are led to the differential Chernoff bound [94]: While the Chernoff bound is defined as $\xi(\rho,\sigma) = \sup_{0\leq s\leq 1} - \ln \operatorname{Tr}\rho^s\sigma^{1-s}$, for states and probability distributions alike [94, 95], it is known that

$$\frac{1}{\mathrm{d}t^2}\mathrm{d}^2\xi\Big(P(\cdot|\vec{\alpha}_0), P(\cdot|\vec{\alpha}_0 + \mathrm{d}t\vec{h})\Big) =: \mathrm{d}_{\vec{h}}\xi^2$$

defines the line element of a Riemannian metric on the parameter space. Thus we let

$$V_{\partial\xi}[P] := \max_{\vec{\alpha}_0,\vec{h}} \mathrm{d}_{\vec{h}}\xi^2. \tag{2.20}$$

As $V_{\partial\xi}$ is regular and weakly affine, we will obtain a strong SIO monotone. Note that this would not work simply fixing a Hamiltonian, as shown in [96, 97].

The differential Chernoff bound, optimized over measurements, for distinguishing $e^{-itH}\rho e^{itH}$ for $t \approx 0$ from ρ in the many-copy regime, with a diagonal Hamiltonian H and $\rho = \sum_j \lambda_j |e_j\rangle\langle e_j|$ is again known [94], and given by $\mathrm{d}_H\xi^2 = \frac{1}{\mathrm{d}t^2}\mathrm{d}^2\xi\big(\rho, e^{-itH}\rho e^{itH}\big)$, which evaluates to

$$\begin{aligned} \mathrm{d}_H\xi^2 &= \frac{1}{2}\sum_{jk}\left(\sqrt{\lambda_j} - \sqrt{\lambda_k}\right)^2 |\langle e_j| H |e_k\rangle|^2 \\ &= \frac{1}{2}\sum_{jk}\left(\lambda_j + \lambda_k - 2\sqrt{\lambda_j\lambda_k}\right) |\langle e_j| H |e_k\rangle|^2 \\ &= \operatorname{Tr}\rho H^2 - \operatorname{Tr}\sqrt{\rho}H\sqrt{\rho}H = -\frac{1}{2}\operatorname{Tr}[\sqrt{\rho}, H]^2, \end{aligned} \tag{2.21}$$

the latter equalling the *Wigner-Yanase skew information*, $I_{\mathrm{WY}}(\rho, H)$ [98].

Like in the previous two examples, there are two natural domains of diagonal Hamiltonians H over which to optimize this: Either $\|H\|_2 \leq 1$ or $\|H\|_\infty \leq 1$, leading to two variants $C^{(2)}_{\partial\xi}(\rho)$ and $C^{(\infty)}_{\partial\xi}(\rho)$ of the coherence measure. Again, $\mathrm{d}_H\xi^2$ is convex in H, thanks to convexity of each term $|\langle e_j| H |e_k\rangle|^2$, and $\left(\sqrt{\lambda_j} - \sqrt{\lambda_k}\right)^2 \geq 0$. Consequently, the optimal H is extremal under the convex norm constraint. For $p = \infty$, this means that the maximum is attained on a difference of two diagonal projectors, $H = \Pi_+ - \Pi_-$. For $p = 2$, however, we can say something even better, using Lieb's concavity theorem [99], which says that for semidefinite H, the Wigner-Yanase skew information is convex in H^2, by writing $H = \sqrt{H^2}$. In general, we split $H = H_+ - H_-$ into positive and negative parts, and find after some straightforward algebra that

$$I_{\mathrm{WY}}(\rho, H) = I_{\mathrm{WY}}(\rho, H_+) + I_{\mathrm{WY}}(\rho, H_-) - 2\operatorname{Tr}\sqrt{\rho}H_+\sqrt{\rho}H_-,$$

which by Lieb's theorem [99] is jointly convex in H_+^2 and H_-^2. Thus we find that the optimal H_+ and H_- must be proportional to rank-one projectors, resulting in the expression claimed for $C^{(2)}_{\partial\xi}(\rho)$.

$$C^{(2)}_{\partial\xi}(\rho) = \max_{j,k,t}\ I_{\mathrm{WY}}\Big(\rho, \sqrt{t}|j\rangle\langle j| - \sqrt{1-t}|k\rangle\langle k|\Big), \tag{2.22}$$

$$\begin{aligned} C^{(\infty)}_{\partial\xi}(\rho) &= \max_{S_+, S_- \subset [d]} I_{\mathrm{WY}}\Big(\rho, \Pi_+ - \Pi_-\Big) \\ &= \max_{S_+ \dot\cup S_- = [d]} 4\operatorname{Tr}\sqrt{\rho}\Pi_+\sqrt{\rho}\Pi_-, \end{aligned} \tag{2.23}$$

where the first maximization is over distinct basis states $j, k \in [d]$ and $0 \leq t \leq 1$; the second over disjoint subsets S_+ and S_- of $[d]$, with $\Pi_\bullet = \sum_{j\in S_\bullet} |j\rangle\langle j|$, $\bullet = \pm$.

Let us analyze the qubit case: For $p = \infty$, the only nontrivial choice is $\Pi_+ = |1\rangle\langle 1|$ and $\Pi_- = |2\rangle\langle 2|$, directly resulting in $C_{\partial\xi}^{(\infty)} = 4\,\mathrm{Tr}\,|1\rangle\langle 1|\sqrt{\rho}|2\rangle\langle 2|\sqrt{\rho} = 4\left|\left(\sqrt{\rho}\right)_{12}\right|^2$.

For $p = 2$, we have to consider the Hamiltonian $H = \sqrt{t}|1\rangle\langle 1| - \sqrt{1-t}|2\rangle\langle 2|$, yielding

$$[\sqrt{\rho}, H] = \begin{bmatrix} 0 & (-\sqrt{t} - \sqrt{1-t})(\sqrt{\rho})_{12} \\ (\sqrt{t} + \sqrt{1-t})(\sqrt{\rho})_{21} & 0 \end{bmatrix}.$$

Thus, $I_{\mathrm{WY}}(\rho, H) = \left(\sqrt{t} + \sqrt{1-t}\right)^2 \left|\left(\sqrt{\rho}\right)_{12}\right|^2$, which is maximized at $t = \frac{1}{2}$, hence $C_{\partial\xi}^{(2)} = 2\left|\left(\sqrt{\rho}\right)_{12}\right|^2$.

That is, for a qubit state ρ, we find that $C_{\partial\xi}^{(2)}(\rho) = 2\left|\left(\sqrt{\rho}\right)_{12}\right|^2$ and $C_{\partial\xi}^{(\infty)}(\rho) = 4\left|\left(\sqrt{\rho}\right)_{12}\right|^2$.

2.3.6 Largest Shannon information

The previous examples should have prepared us for thinking of visibility as an expression of how much information about $\vec{\alpha}$ the output distribution $P(\cdot|\vec{\alpha})$ reveals. So why not take this to the logical conclusion? Noting that P is a channel from multi-phases $\vec{\alpha}$ to outputs ω, in the Shannon theoretic sense, we are motivated to define visibility as the Shannon capacity of P:

$$V_I[P] := C(P) = \sup_{\mu} I(\vec{\alpha} : \omega), \tag{2.24}$$

where μ is a probability measure on the $\vec{\alpha}$, defining a joint distribution $\mu(\vec{\alpha})P(\omega|\vec{\alpha})$ of channel inputs and outputs, and $I(X:Y) = D(\mathbb{P}_{XY}\|\mathbb{P}_X \times \mathbb{P}_Y)$ is the mutual information of two random variables [100]. It can be checked that V_I is regular and weakly affine. Operationally, $V_I[P]$ is the largest communication rate that can be transmitted by a sender, who may encode information into the phase settings $\vec{\alpha}^{(1)}, \ldots, \vec{\alpha}^{(n)}$ of asymptotically many interferometers, to a receiver who decodes the correct message with high probability based on the observations $\omega_1, \ldots, \omega_n$ [100].

To obtain $C_I(\rho)$, we then only need to perform a maximization of the Shannon capacity over all measurements:

$$\begin{aligned} C_I(\rho) = \sup_{(M_\omega)} C(P_{M|\rho}) &= \sup_{\mu} \sup_{(M_\omega)} I(\vec{\alpha} : \omega) \\ &= \sup_{\mu} I_{\mathrm{acc}}\Big(\Big\{\mu(\vec{\alpha}), \rho(\vec{\alpha})\Big\}\Big), \end{aligned} \tag{2.25}$$

where the latter quantity is know as the *accessible information.* These optimizations are by no means easy, and are worked out only in some few cases. In any case, Theorem 3 shows that C_I is a SIO monotone. This might provide some motivation to try to evaluate C_I in certain special cases.

However, due to the Holevo bound [101], and the Holevo-Schumacher-Westmoreland theorem [102, 103] regarding the capacity of the cq-channel $\vec{\alpha} \mapsto \rho(\vec{\alpha})$, we obtain the following:

$$C_I(\rho) \leq S\Big(\Delta(\rho)\Big) - S(\rho) = C_r(\rho) = \sup_n \frac{1}{n} C_I(\rho^{\otimes n}). \tag{2.26}$$

Namely, the Holevo bound [101] upper-bounds the accessible information,

$$\begin{aligned} I_{\text{acc}}\Big(\Big\{\mu(\vec{\alpha}), \rho(\vec{\alpha})\Big\}\Big) &\leq \chi\Big(\Big\{\mu(\vec{\alpha}), \rho(\vec{\alpha})\Big\}\Big) \\ &:= S\left(\int \mu(\mathrm{d}\vec{\alpha})\rho(\vec{\alpha})\right) - \int \mu(\mathrm{d}\vec{\alpha}) S\Big(\rho(\vec{\alpha})\Big). \end{aligned}$$

Here, the second term is always $S(\rho)$ because the $\rho(\vec{\alpha})$ are unitarily rotated versions of ρ, and the first term is maximized by the uniform distribution over all phases:

$$C_I(\rho) \leq S\Big(\Delta(\rho)\Big) - S(\rho) = C_r(\rho). \tag{2.27}$$

Invoking the Holevo-Schumacher-Westmoreland theorem [102, 103] regarding the capacity of the cq-channel $\vec{\alpha} \mapsto \rho(\vec{\alpha})$, we get furthermore

$$\sup_n \frac{1}{n} C_I(\rho^{\otimes n}) = C_r(\rho).$$

In the qubit case, the optimization (2.25) seems to be unknown, but we believe that the maximum is attained on the binary ensemble

$$\Big\{(\frac{1}{2}, \rho_0 = \rho), (\frac{1}{2}, \rho_1 = \sigma_z \rho \sigma_z)\Big\},$$

and the measurement in the eigenbasis of $\rho_0 - \rho_1$, which would yield $C_I(\rho) = 1 - H\Big(\frac{1 \pm 2|\rho_{12}|}{2}\Big) \approx \frac{2}{\ln 2}|\rho_{12}|^2$. On the other hand, $C_r(\rho) = H\Big(\frac{1 \pm \operatorname{Tr} \rho \sigma_Z}{2}\Big) - H\Big(\frac{1 \pm r}{2}\Big)$.

2.4 Chapter summary

- Coherence measures that are SIO-monotones can be derived by optimizing suitable visibility parameteres over detection schemes, thus providing a connection between the resource theory of coherence and the physics of interferometers.
- In this way, one can recover coherence measures that are related to, or identified with, coherence measures already considered by coherence theory.
- For qubits, the single off-diagonal element seems to govern almost all visibility and coherence effects.

2.5 Open questions

- Are visibility based coherence measures monotonic under IOs and MIOs?
- Do our optimized visibility parameters satisfy duality relations with suitable path-information measures?
- Can C_{l_1} and C_f be recovered via visibilities?
- How can our framework be extended to multi-particle interference? This would bring out unique quantum features of interference.

Part II

Exploiting quantum coherence

CHAPTER 3

Quantifying the coherence of maps

"Far-off
at the core of space
at the quick
of time
beats
and goes still
the great swan upon the waters of all endings
the swan within vast chaos, within the electron."

D. H. Lawrence,
"Swan", Complete Poems

So far we have focused on studying the coherence properties of quantum states. However, any state σ can also be conceived as a constant-output channel $\mathcal{N}_\sigma$ such that $\mathcal{N}_\sigma(\cdot) = \sigma$. The question that naturally arises then is, how can the resource theory of coherence be extended to quantum operations, such that the theory of states can still be recovered from it? What kind of framework encompasses the resource properties of an electron's beat, from which those intrinsic to its stillness can still be retrieved? First steps towards building quantum resource theories of channels were given by [61, 64, 104–106].

In the particular case of the coherence theory of maps, free operations are NCG channels, which means that in some sense, any other CPTP map represents a resource. Here, free operations may act on resources by tensor product and composition. The question we would like to address in this chapter is: how to then measure the coherence of a channel? Or better: how to assess its resource character?

As we will see, at least two ways can be proposed to quantify the coherence of an operation. On the one hand, the amount of coherence the considered map is able to generate already gives an intuition on how resourceful the map is. On the other hand, the amount of coherence required to implement the given channel can also give an estimate of its resourcefulness. Both ways suggest how coherence can be exploited in the laboratory: the former, by evaluating how much coherence a quantum process can store in some given state, i.e. how *dynamic* coherence can be converted into *static* coherence; the latter, by quantifying the amount of static coherence required to implement an arbitrary quantum process, which means analyzing how *static* coherence can be turned into *dynamic* coherence.

3.1 Assessing the resourcefulness of a map by the amount of coherence it can generate

In this section we will present several quantifiers of the coherence-generating capabilities of quantum channels:

3.1.1 Coherence power

A first approach to quantify channel coherence can be found in [107, 108], where they define the coherence power $\widehat{P}_C$ of a channel $\mathcal{N} : A \to B$ w.r.t. the coherence measure C as the maximum amount of coherence the channel is able to generate on a state:

$$\widehat{P}_C(\mathcal{N}) := \max_{\rho\in\mathcal{S}(\mathcal{H})} C\Big(\mathcal{N}(\rho)\Big) - C(\rho). \tag{3.1}$$

The same maximization restricted to incoherent input states was introduced by [108, 109]:

$$P_C(\mathcal{N}) := \max_{\rho\in\mathcal{I}} C(\mathcal{N}(\rho)) = \max_{i\in\{0,1,\dots,d\}} C(\mathcal{N}(|i\rangle\langle i|)), \tag{3.2}$$

where $d = |A|$.

3.1.2 Robustness of coherence

Let us define the robustness of coherence C_R of a quantum channel $\mathcal{N}$ as

$$1 + C_R(\mathcal{N}) := \min\left\{\lambda : \ \mathcal{N} \leq \lambda\Phi_{\text{NCG}}\right\}, \tag{3.3}$$

where the inequality $\mathcal{N} \leq \lambda\Phi_{\text{NCG}}$ is understood as completely-positive ordering of super operators, i.e., $\lambda\Phi_{\text{NCG}} - \mathcal{N}$ is a CP map. The smoothed version of this quantity, called *ϵ-robustness of coherence*, is given by

$$C_R^\epsilon(\mathcal{N}) := \min\left\{C_R(\mathcal{L}) : \ \frac{1}{2}\|\mathcal{N} - \mathcal{L}\|_\diamond \leq \epsilon\right\}, \tag{3.4}$$

where $\|\cdot\|_\diamond$ denotes the diamond norm [110, 111].

Recall that the diamond norm is a well-behaved distance which is furthermore endowed with an operational meaning: it quantifies how well one can physically discriminate between two quantum channels [112]. Also note that this definition reduces to the robustness of coherence of states when applied to the constant channel $\mathcal{N}_\sigma(\rho) = \sigma$, i.e., $C_R(\mathcal{N}_\sigma) = C_R(\sigma)$.

If we choose our free operations Φ_{NCG} to be MIOs, it is straightforward to see (Appendix A) that the robustness of coherence of a channel $\mathcal{N}$ i) is a proper coherence measure for channels under MIO, and ii) quantifies the minimum amount of noise, in the form of another channel, that we need to add to $\mathcal{N}$ such that the resulting channel is MIO (just as in the case of states). So one could now ask: why do we then include this measure as a quantifier of the coherence-generating capabilities of a channel? What does adding noise have to do with the ability to generate coherence? Well, in Appendix A we also show that $C_R(\mathcal{N})$ can be easily formulated as a semidefinite program—thus allowing for an efficient numerical computation of this measure—whose dual form permits to link the robustness with the coherence power:

$$C_{LR}(\mathcal{N}) := \log(1 + C_R(\mathcal{N})) = P_{C_{LR}}(\mathcal{N}) = \widehat{P}_{C_{LR}}(\mathcal{N}), \tag{3.5}$$

where C_{LR}, which is additive under tensor product of channels, is the log-robustness of coherence of a channel.

The smoothed version of the log-robustness of a channel will also be of relevance:

$$C_{LR}^\epsilon(\mathcal{N}) := \min\left\{C_{LR}(\mathcal{L}) : \ \frac{1}{2}\|\mathcal{N} - \mathcal{L}\|_\diamond \leq \epsilon\right\}. \tag{3.6}$$

In fact, it can be shown that the (smooth) log-robustness of a channel $\mathcal{N}$ equals the (smooth) maximum relative entropy between $\mathcal{N}$ and a MIO $\mathcal{M}$, minimized over all $\mathcal{M}$, that we define in Appendix A.

3.1.3 Asymptotic coherence generating capacity

Let us now focus on how much pure state coherence can be created asymptotically, using a given operation $\mathcal{N} : A \longrightarrow B$ a large number of times, when

NCG operations are for free.

The most general protocol to generate coherence must use the resource $\mathcal{N}$ and NCG operations according to some predetermined algorithm, in some order. We may assume that the channels $\mathcal{N}$ are invoked one at a time; and we can integrate all NCG operations in between one use of $\mathcal{N}$ and the next into one NCG operation, since the class of NCG opeartions is closed under composition. Thus, a mathematical description of the most general protocol is the following: one starts by preparing an incoherent state ρ_0 on $A \otimes A_0$, then lets act $\mathcal{N}$, followed by a NCG transformation $\Phi_{\text{NCG},1} : B \otimes A_0 \longrightarrow A \otimes A_1$, resulting in the state

$$\rho_1 = \Phi_{\text{NCG},1}\Big((\mathcal{N} \otimes \text{id})\rho_0\Big).$$

Iterating, given the state ρ_t on $A \otimes A_t$ obtained after the action of t realizations of $\mathcal{N}$ and suitable NCG operations, we let $\mathcal{N}$ act and the NCG transformation $\Phi_{\text{NCG},t+1} : B \otimes A_t \longrightarrow A \otimes A_{t+1}$, resulting in the state

$$\rho_{t+1} = \Phi_{\text{NCG},t+1}\Big((\mathcal{N} \otimes \text{id})\rho_t\Big).$$

At the end of n iterations, we have a state ρ_n on $A \otimes A_n$, and we call the above procedure a *coherence generation protocol* of rate R and error ϵ, if $|A_n| = 2^{nR}$ and the reduced state $\rho_n^{A_n} = \text{Tr}_A \, \rho_n$ has high fidelity with the maximally coherent state,

$$\langle \Psi_{2^{nR}} | \, \rho_n^{A_n} \, | \Psi_{2^{nR}} \rangle \geq 1 - \epsilon.$$

The maximum number R such that there exist coherence generating protocols for all n, with error going to zero and rates converging to R, is called the *asymptotic coherence generating capacity* of $\mathcal{N}$, and denoted $C_{\text{gen}}^{\infty}(\mathcal{N})$.

3.2 Assessing the resourcefulness of a map by the amount of coherence required to implement it

As already suggested, another way to assess the resourcefulness of a given transformation is to quantify the amount of resource required to simulate it by means of free operations. In this section we propose several approaches to evaluate the coherence of a channel in this fashion:

3.2.1 Single-shot simulation cost

Our current goal is to quantify the resources required to implement (or simulate) an arbitrary quantum channel via NCG operations by making use of coherent input states, as depicted in Fig. 3.1a.

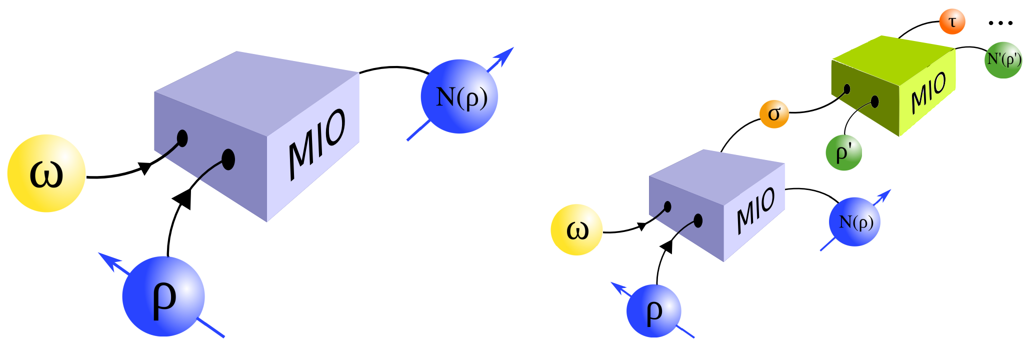


Figure 3.1: Different protocols for implementing a quantum channel $\mathcal{N}$ using an NCG operation. Here, a MIO $\mathcal{M}$ and a coherent resource ω are considered. **a)** The implementation destroys completely the input resource, $\mathcal{M}(\omega \otimes \rho) = \mathcal{N}(\rho)$. **b)** After the implementation, a degraded resource σ is recovered and ready to be recycled: $\mathcal{M}(\omega \otimes \rho) = \sigma \otimes \mathcal{N}(\rho)$. For example it can be used to implement another channel $\mathcal{N}'$: $\mathcal{M}'(\sigma \otimes \rho') = \tau \otimes \mathcal{N}'(\rho')$.

There are many inequivalent ways to quantify the coherence of a given state. Indeed, a state can be more resourceful than another according to a given measure, while the opposite can happen according to a different one. This is due to the existence of *incomparable* resources, i.e., pairs of states that cannot be interconverted in either direction via NCG operations[1]. Nevertheless, in this section we wish to establish general results which hold irrespectively of the particular coherence measure of choice: we will do this at the expense of quantifying the required resource in a somewhat coarse fashion, namely by the smallest size k of a cosdit $|\Psi_k\rangle$ that allows to implement the channel. Indeed, cosdits are maximally coherent states irrespectively of how coherence is quantified, since, as we have seen, any state of the same (or lower) coherence rank can be obtained from them via NCG operations [59]. Restricting our input resources to cosdits might seem a limitation. Nevertheless, we will show that this setting has clear benefits and that it leads to a coherence measure for channels that does take real values.

We can then define an NCG simulation of a quantum channel $\mathcal{N} : A \to B$ up to error ϵ with a cosdit Ψ_k as an NCG transformation $\Phi_{\text{NCG}} : R \otimes A \to B$

[1]For instance, the cosbit cannot be transformed via MIO into the flagpole state $|\varphi_{\frac{2}{3}}\rangle$ (defined below in Eq. (5.20)), since the latter has a greater robustness of coherence $C_R(\varphi) = \frac{5}{3} > 1 = C_R(\Psi_2)$; and at the same time the flagpole $|\varphi_{\frac{2}{3}}\rangle$ cannot be transformed via MIO into the cosbit by Lemma 16.

that satisfies

$$\frac{1}{2}\|\mathcal{N} - \Phi_{\text{NCG}}(\Psi_k \otimes \cdot)\|_\diamond \leq \epsilon. \tag{3.7}$$

The single-shot simulation cost (or simply *simulation cost*) of $\mathcal{N}$, denoted $C^{\epsilon}_{\text{sim}}(\mathcal{N})$, is the smallest $\log k$ satisfying Eq. (3.7), i.e., the minimal coherence rank of the resource allowing for an NCG simulation of $\mathcal{N}$. Henceforth, $C_{\text{sim}}(\mathcal{N})$ implies $\epsilon = 0$, i.e., exact implementation.

3.2.2 Amortized simulation cost

In the previous section, we quantified the coherence of a channel by the minimum rank of the cosdit that allows for an NCG implementation of the channel. However, this provides a somewhat coarse measure of the implementation cost, as cosdits come in discretized units, the smallest being $k = 2$. Moreover, one might speculate that the whole cosdit does not always need to be fully consumed in the implementation, for there exists poorly coherent channels that would certainly do without so much input fuel. In order to quantify the actual resource consumed in the process, we now investigate to which extent some of the coherence of the input resource can be recovered after the channel implementation. For this purpose let us now focus on the setting of Fig. 3.1b, where the resource, initially in a state ω, is recovered in a degraded form σ after fueling the NCG implementation of the target channel $\mathcal{N}(\rho)$. We quantify the minimal resource consumed in this process by the difference between the coherence of ω and σ, when both the input and output states come in standard coherence units, i.e., ω and σ are cosdits. Formally, we define the ϵ-error *amortized cost* of a quantum channel $\mathcal{N}$ as

$$\begin{aligned} C^{\epsilon}_{\text{amo}}(\mathcal{N}) &:= \inf\ (C_{LR}(\Psi_k) - C_{LR}(\Psi_m)) \\ &= \inf(\log \frac{k}{m}) \\ \text{s.t.}\ \ & \Phi_{\text{NCG}}(\Psi_k \otimes \cdot) = \Psi_m \otimes \mathcal{L}(\cdot), \\ & \frac{1}{2}||\mathcal{N} - \mathcal{L}||_\diamond \leq \epsilon, \end{aligned} \tag{3.8}$$

where we recall that $C_{LR}(\Psi_k) = \log k$. In other words the optimization is over all channels $\mathcal{L}$ that are ϵ-close to the target channel $\mathcal{N}$ that can be implemented via an NCG operation $\Phi_{\text{NCG}} : R \otimes A \to S \otimes B$ with cosdit input and output resource states, respectively Ψ_k and Ψ_m, $k \geq m$.

Note that tensor-product structure at the output of the simulation allows complete freedom in reusing σ and $\mathcal{N}(\rho)$ afterwards; an entangled output would unnaturally constrain the recycling operations. For example, it would not allow the implementation of a sequence of channels applied on-the-fly to the same system, i.e., $\mathcal{N}_n \circ \cdots \circ \mathcal{N}_2 \circ \mathcal{N}_1$, whereas this is allowed by Eq. (3.8), see Corollary 13 in Chapter 6.

3.2.3 Asymptotic simulation cost

In the spirit of the previous section, we are interested in the minimum resources required to implement many independent instances of a channel $\mathcal{N}$.

An n-block NCG simulation of a channel $\mathcal{N} : A \longrightarrow B$ with error ϵ and coherent resource Ψ_d (on space D) is an NCG operation $\Phi_{\text{NCG}} : A^n \otimes D \longrightarrow B^n$, such that $\mathcal{N}'(\rho) := \Phi_{\text{NCG}}(\rho \otimes \Psi_d)$ satisfies

$$\begin{aligned}\epsilon &\geq \|\mathcal{N}' - \mathcal{N}^{\otimes n}\|_\diamond \\ &= \sup_{|\phi\rangle \in A^n \otimes C} \|(\mathcal{N}' \otimes \text{id}_C)\phi - (\mathcal{N}^{\otimes n} \otimes \text{id}_C)\phi\|_1.\end{aligned}$$

Here, C is an arbitrary ancilla system.

The rate of the simulation is $\frac{1}{n} \log d$, and the *asymptotic simulation cost* of $\mathcal{N}$, denoted $C_{\text{sim}}^{\infty}(\mathcal{N})$, is the smallest R such that there exist n-block NCG simulations with error going to 0 and rate going to R as $n \to \infty$.

CHAPTER 4

Results in IO-theory

The main purpose of this chapter is to give explicit computations of the coherence of a channel when restricting to IO-theory. As presented in the previous chapter, we will assess the resource character of the considered channel via i) its coherence generating capabilities, and ii) the amount of coherence required to implement it with IOs.

4.1 Asymptotic coherence generating capacity and coherence power

We start by proving the following theorem (for simplicity, we define $\varphi := |\varphi\rangle\langle\varphi|$):

Theorem 4. *For a general CPTP map $\mathcal{N} : A \longrightarrow B$,*

$$C^{\infty}_{gen}(\mathcal{N}) \geq \sup_{|\varphi\rangle \in A\otimes C} C_r\Big((\mathcal{N} \otimes \mathrm{id})\varphi\Big) - C_r(\varphi), \tag{4.1}$$

where the supremum is over all auxiliary systems C and pure states $|\varphi\rangle \in A \otimes C$. Furthermore,

$$C^{\infty}_{gen}(\mathcal{N}) \leq \sup_{\rho \text{ on } A\otimes C} C_r\Big((\mathcal{N} \otimes \mathrm{id})\rho\Big) - C_r(\rho), \tag{4.2}$$

where now the supremum is over mixed states ρ on $A \otimes C$.

If $\mathcal{N}$ is an isometry, i.e. $\mathcal{N}(\rho) = V\rho V^\dagger$ for an isometry $V : A \hookrightarrow B$, the lower bound is an equality, and can be simplified:

$$\begin{aligned} C_{gen}^{\infty}(V \cdot V^\dagger) &= \sup_{|\varphi\rangle \in A \otimes C} C_r\Big((V \otimes \mathbb{1})\varphi(V \otimes \mathbb{1})^\dagger\Big) - C_r(\varphi) \\ &= \max_{|\varphi\rangle \in A} C_r\Big(V\varphi V^\dagger\Big) - C_r(\varphi). \end{aligned} \tag{4.3}$$

Furthermore, the above formulas for the asymptotic coherence generating capacity are related to the *coherence power* (w.r.t. the relative entropy measure) $\widehat{P}_{C_r}(\mathcal{N})$:

$$\widehat{P}_{C_r}(\mathcal{N}) = \max_{\rho \text{ on } A} C_r\Big(\mathcal{N}(\rho)\Big) - C_r(\rho). \tag{4.4}$$

Let us also introduce the same maximization restricted to pure input states,

$$\widetilde{P}_{C_r}(\mathcal{N}) = \max_{|\varphi\rangle \in A} C_r\Big(\mathcal{N}(\varphi)\Big) - C_r(\varphi). \tag{4.5}$$

Note that the only difference to our formulas is that we allow an ancilla system C of arbitrary dimension. If we consider, for a general CPTP map $\mathcal{N}$, the extension $\mathcal{N} \otimes \mathrm{id}_k$ and

$$\widehat{P}_{C_r}^{(k)}(\mathcal{N}) := \widehat{P}_{C_r}(\mathcal{N} \otimes \mathrm{id}_k), \quad \widetilde{P}_{C_r}^{(k)}(\mathcal{N}) := \widetilde{P}_{C_r}(\mathcal{N} \otimes \mathrm{id}_k), \tag{4.6}$$

then we have

$$C_{\mathrm{gen}}^{\infty}(V \cdot V^\dagger) = \sup_k \widetilde{P}_{C_r}^{(k)}(V \cdot V^\dagger) = \widetilde{P}_{C_r}(V \cdot V^\dagger), \tag{4.7}$$

and in general

$$\sup_k \widetilde{P}_{C_r}^{(k)}(\mathcal{N}) \leq C_{\mathrm{gen}}^{\infty}(\mathcal{N}) \leq \sup_k \widehat{P}_{C_r}^{(k)}(\mathcal{N}). \tag{4.8}$$

Proof. We start with the lower bound, Eq. (4.1): For a given ancilla C and $|\varphi\rangle \in A \otimes C$, let $R = C_r\Big((\mathcal{N} \otimes \mathrm{id})\varphi\Big) - C_r(\varphi)$. For any $\epsilon, \delta > 0$, we can choose, by the results of [65], a sufficiently large n such that

$$\begin{aligned} \Psi_2^{\otimes \lfloor nC_r(\varphi) + n\delta \rfloor} &\xmapsto{\mathrm{IO}} \approx \varphi^{\otimes n}, \\ \rho^{\otimes n} &\xmapsto{\mathrm{IO}} \approx \Psi_2^{\otimes \lceil nC_r(\rho) - n\delta \rceil}, \end{aligned}$$

with $\rho = (\mathcal{N} \otimes \mathrm{id})\varphi$, and where $\approx$ refers to approximation of the target state up to ϵ in trace norm. We only have to prove something when $R > 0$, which can only arise if $\mathcal{N}$ is not incoherent, meaning that there exists an initial state $|0\rangle$ mapped to a coherent resource $\sigma = \mathcal{N}(|0\rangle\langle 0|)$, i.e. $C_r(\sigma) > 0$. In the following, assume $R > 2\delta$. Now, we may assume that n is large enough so that with $R_0 = \frac{C_r(\varphi)+\delta}{C_r(\sigma)} + \delta$,

$$\sigma^{\otimes \lfloor nR_0 \rfloor} \xmapsto{\text{IO}} \approx \Psi_2^{\otimes \lfloor nC_r(\varphi)+n\delta \rfloor}.$$

The protocol consists of the following steps:

0: Use $\lfloor nR_0 \rfloor$ instances of $\mathcal{N}$ to create as many copies of σ, and convert them into $\Psi_2^{\otimes \lfloor nC_r(\varphi)+n\delta \rfloor}$ (up to trace norm ϵ).

1-k (repeat): Convert first $\lfloor nC_r(\varphi)+n\delta \rfloor$ of the already created copies of Ψ_2 into n copies of φ; then apply $\mathcal{N}$ to each of them to obtain $\rho = (\mathcal{N} \otimes \mathrm{id})\varphi$; and convert the n copies of ρ to $\Psi_2^{\otimes \lceil nC_r(\rho)-n\delta \rceil}$, incurring an error of 2ϵ in trace norm in each repetition.

At the end, we have $(k-1)n(R-2\delta) + nC_r(\varphi)$ copies of Ψ_2, up to trace distance $O(k^2\epsilon)$, using the channel a total of $kn + nR_0$ times, i.e. the rate is $\geq (R-2\delta)\frac{k-1}{k+R_0}$, which can be made arbitrarily close to R by choosing δ small enough and k large enough (which in turn can be effected by sufficiently small ϵ).

For the upper bound, Eq. (4.2), consider a generic protocol using the channel n times, starting from ρ_0 (incoherent) and generating $\rho_1, \ldots, \rho_n$ step by step along the way, such that ρ_n has fidelity $\geq 1-\epsilon$ with $\Psi_2^{\otimes nR}$. By the asymptotic continuity of C_r [65, Lemma 12], $C_r(\rho_n) \geq nR - 2\delta n - 2$, with $\delta = \sqrt{\epsilon(2-\epsilon)}$, so we can bound

$$\begin{aligned} nR - 2\delta n - 2 &\leq C_r(\rho_n) \\ &= \sum_{t=0}^{n-1} C_r(\rho_{t+1}) - C_r(\rho_t) \\ &\leq \sum_{t=1}^{n-1} C_r\Big((\mathcal{N} \otimes \mathrm{id})\rho_t\Big) - C_r(\rho_t), \end{aligned}$$

where we have used the fact that ρ_0 is incoherent and that $\rho_{t+1} = \mathcal{I}_{t+1}\Big((\mathcal{N} \otimes \mathrm{id})\rho_t\Big)$, with an incoherent operation $\mathcal{I}_{t+1}$, which can only decrease the relative entropy of coherence. However, each term on the right hand sum is of the form $C_r\Big((\mathcal{N} \otimes \mathrm{id})\rho\Big) - C_r(\rho)$ for a suitable ancilla C and a state ρ on $A \otimes C$. Thus, dividing by n and letting $n \to \infty$, $\epsilon \to 0$ shows that $R \leq \sup_{\rho \text{ on } A\otimes C} C_r\Big((\mathcal{N} \otimes \mathrm{id})\rho\Big) - C_r(\rho)$.

For an isometric channel $\mathcal{N}(\rho) = V\rho V^\dagger$, note that the initial state ρ_0 in a general protocol is without loss of generality pure, and that $\mathcal{N}$ maps pure states to pure states. The incoherent operations $\mathcal{I}_t$ map pure states to *ensembles* of pure states, so that following the same converse reasoning as above, we end up upper bounding R by an average of expressions $C_r\Big((\mathcal{N}\otimes\mathrm{id})\rho\Big) - C_r(\rho)$, with pure states ρ, i.e. Eq. (4.3), since we also have $C_{\mathrm{gen}}^{\infty}(\mathcal{N}) \geq C_r\Big((\mathcal{N}\otimes\mathrm{id})\varphi\Big) - C_r(\varphi)$ from the other direction. The fact that no ancilla system is needed, is an elementary calculation. Indeed, for a pure state $|\varphi\rangle \in A\otimes C$,

$$
\begin{aligned}
C_r\Big((\mathcal{N}\otimes\mathrm{id})\varphi\Big) &- C_r(\varphi) \\
&= S\Big((\Delta\circ\mathcal{N}\otimes\Delta)\varphi\Big) - S\Big((\Delta\otimes\Delta)\varphi\Big) \\
&= S\Big((\Delta\circ\mathcal{N}\otimes\mathrm{id})\rho\Big) - S\Big((\Delta\otimes\mathrm{id})\rho\Big), \\
&\qquad\qquad \Big[\text{with } \rho = (\mathrm{id}\otimes\Delta)\varphi \\
&\qquad\qquad\qquad = \sum_i p_i\varphi_i\otimes|i\rangle\langle i|\Big], \\
&= \sum_i p_i\Big(S\Big(\Delta(\mathcal{N}(\varphi_i))\Big) - S\Big(\Delta(\varphi_i)\Big)\Big) \\
&\leq \max_{|\varphi\rangle\in A} S\Big(\Delta(\mathcal{N}(\varphi_i))\Big) - S\Big(\Delta(\varphi_i)\Big),
\end{aligned}
$$

and we are done. □

Remark The same reasoning as in the proof of Theorem 4, replacing C_r with C_f, shows that

$$
\begin{aligned}
C_{\mathrm{gen}}^{\infty}(\mathcal{N}) &\leq \sup_{\rho \text{ on } A\otimes C} C_f\Big((\mathcal{N}\otimes\mathrm{id})\rho\Big) - C_f(\rho) \\
&= \sup_k \widehat{P}_{C_f}(\mathcal{N}\otimes\mathrm{id}_k),
\end{aligned}
\tag{4.9}
$$

with the coherence of formation power, given by $\widehat{P}_{C_f}(\mathcal{N}) := \max_\rho C_f\Big(\mathcal{N}(\rho)\Big) - C_f(\rho)$.

Despite the fact that $C_f(\rho) \geq C_r(\rho)$, since the upper bound is given by a difference of two coherence measures, it might be that for certain channels, the bound (4.9) is better than (4.2), and vice versa for others.

Example: qubit unitaries

We now want to have a closer look at qubit unitaries, for which we would like to find the asymptotic coherence generating capacity.

To start our analysis, we note that a general 2×2-unitary has four real parameters, but we can transform unitaries into each other at no cost by preceding or following them by incoherent unitaries, i.e. combinations of the bit flip σ_x and diagonal (phase) unitaries $\begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{i\beta} \end{pmatrix}$. This implies an equivalence relation among qubit unitaries up to incoherent unitaries. A unique representative of each equivalence class is given by

$$U = U(\theta) = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}, \tag{4.10}$$

where $c = \cos\theta$ and $s = \sin\theta$ and with $0 \leq \theta \leq \frac{\pi}{4}$, so that $c \geq s \geq 0$.

One can calculate $C_{\text{gen}}^{\infty}(U(\theta))$, using the formula from Theorem 4. Clearly,

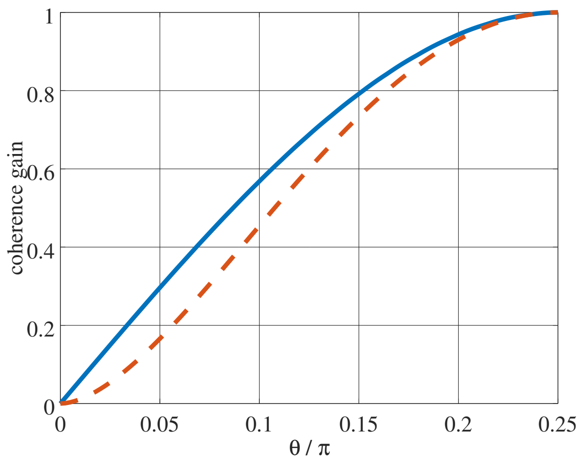


Figure 4.1: Plot of $C_{\text{gen}}^{\infty}(U(\theta)) = \widetilde{P}_{C_r}(U(\theta))$ as a function of $\theta \in [0, \frac{\pi}{4}]$ (solid blue line), and comparison with $h_2(\cos^2\theta)$ (dashed red line), which is the coherence generated by an incoherent input state. In particular, for $\theta \approx 0$, the ratio between the two functions is unbounded. The angle θ is plotted as a fraction of π.

by choosing the test state φ to be pure incoherent,

$$\begin{aligned} C_{\text{gen}}^{\infty}(U(\theta)) &= \widetilde{P}_{C_r}(U(\theta)) \\ &\geq h_2(c^2) = -c^2 \log c^2 - s^2 \log s^2, \end{aligned} \tag{4.11}$$

with $h_2(x) = -x \log x - (1-x)\log(1-x)$ the binary entropy. Perhaps surprisingly, however, this is in general not the optimal state [113, Cor. 5] (see also [114]), meaning that $\widetilde{P}_{C_r}(U(\theta))$ is attained at a coherent test state φ, although no closed form expression seems to be known. In fact, simple manipulations show that we only need to optimise $C_r\big(U(\theta)\varphi U(\theta)^\dagger\big) - C_r(\varphi)$

over states $|\varphi\rangle = U(\alpha)|0\rangle = \cos\alpha\,|0\rangle + \sin\alpha\,|1\rangle$, $0 \le \alpha \le \pi$ (i.e. no phases are necessary). The function to optimise becomes $h_2\Big(\cos^2(\alpha+\theta)\Big) - h_2\Big(\cos^2\alpha\Big)$. Its critical points satisfy the transcendental equation

$$\sin(2\alpha + 2\theta)\ln\tan^2(\alpha+\theta) = \sin(2\alpha)\ln\tan^2\alpha, \tag{4.12}$$

which can be solved numerically. Fig. 4.1 shows that $C_{\text{gen}}^{\infty}(U(\theta)) = \widetilde{P}_{C_r}(U(\theta)) > h_2(\cos^2\theta)$ for across the whole interval, except at the endpoints $\theta = 0, \frac{\pi}{4}$; in Fig. 4.2 we plot the optimal α for $U(\theta)$.

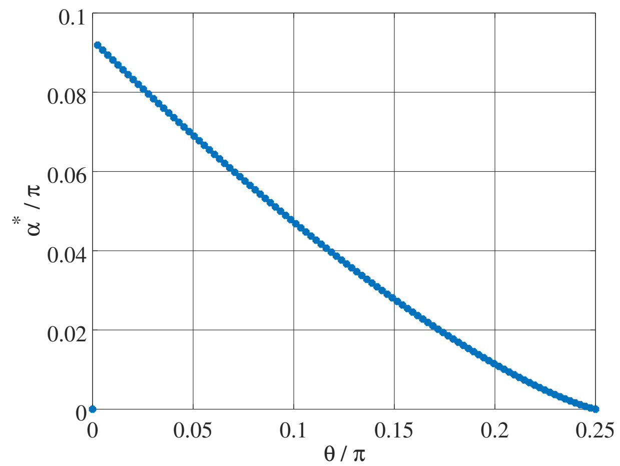


Figure 4.2: The optimal value of α attaining $C_{\text{gen}}^{\infty}(U(\theta)) = \widetilde{P}_{C_r}(U(\theta)) = h_2\Big(\cos^2(\alpha+\theta)\Big) - h_2\Big(\cos^2\alpha\Big)$, as a function of $\theta \in [0, \frac{\pi}{4}]$. It is nonzero except at the endpoints $\theta = 0, \frac{\pi}{4}$. Both angles, θ and α^*, are plotted as fractions of π.

4.2 Single-shot channel simulation

Rather than calculating the single-shot simulation cost of a channel under IOs, in this section we demonstrate that, in fact, any CPTP channel can be implemented by means of IOs and a cosdit. This extends the result in [115, Lemma 2], where the previous statement was first proved for unitary operations.

Theorem 5. *Any CPTP map $\mathcal{N} : A \longrightarrow B$ can be implemented by IOs, using a maximally coherent resource state Ψ_d, where $d = |B|$.*

Proof. Let $\mathcal{N}(\rho) = \sum_\alpha K_\alpha \rho K_\alpha^\dagger$ be a Kraus decomposition of $\mathcal{N}$, with Kraus operators $K_\alpha : A \longrightarrow B$. The idea of the simulation is to use teleportation of the output of $\mathcal{N}$, which involves a maximally entangled state Φ_D on $B' \otimes B''$, a Bell-measurement on system $B \otimes B'$ with outcomes $jk \in \{0, 1, \ldots, d-1\}^2$, and unitaries U_{jk} on B''. The unitaries U_{jk} can be written as $U_{jk} = Z^j X^k$, with the phase and cyclic shift unitaries

$$Z = \begin{pmatrix} 1 & & & \\ & \omega & & \\ & & \ddots & \\ & & & \omega^{d-1} \end{pmatrix}, \quad X = \begin{pmatrix} 0 & & & 1 \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix},$$

where $\omega = e^{\frac{2\pi i}{d}}$ is the d-th root of unity. This scheme can be reduced to a destructive (hence incoherent) POVM on $A \otimes B'$ with outcomes $jk\alpha$, followed by the application of the incoherent U_{jk}. In detail, the probability of getting outcome jk is

$$\Pr\{jk \mid \sigma\} = \operatorname{Tr} \Phi^{(jk)} (K_\alpha \otimes \mathbb{1}) \sigma (K_\alpha \otimes \mathbb{1})^\dagger,$$

where σ is a state on $A \otimes B'$ and the

$$|\Phi^{(jk)}\rangle = (\mathbb{1} \otimes Z^j X^k) |\Phi_d\rangle$$

are the Bell states. We can define the POVM elements $M_{jk\alpha} = (K_\alpha \otimes \mathbb{1})^\dagger \Phi^{(jk)} (K_\alpha \otimes \mathbb{1})$, so that

$$\begin{aligned} \operatorname{Tr}\Big((\mathcal{N} \otimes \mathrm{id})\sigma\Big)\Phi^{(jk)} &= \sum_\alpha \operatorname{Tr}(K_\alpha \otimes \mathbb{1})\sigma(K_\alpha \otimes \mathbb{1})^\dagger \Phi^{(jk)} \\ &= \sum_\alpha \operatorname{Tr} \sigma (K_\alpha^\dagger \otimes \mathbb{1}) \Phi^{(jk)} (K_\alpha \otimes \mathbb{1}) \\ &= \operatorname{Tr}\left[\sigma\left(\sum_\alpha (K_\alpha^\dagger \otimes \mathbb{1}) \Phi^{(jk)} (K_\alpha \otimes \mathbb{1})\right)\right] \\ &= \operatorname{Tr}\left(\sum_\alpha \sigma M_{jk\alpha}\right) = \operatorname{Tr} \sigma M_{jk}, \end{aligned}$$

with $M_{jk} = \sum_\alpha M_{jk\alpha}$. This leads to a new equivalent scheme in which, given a state ρ on A and a maximally entangled state on $B' \otimes B''$, we can apply the measurement M_{jk} on $A \otimes B'$ with outcomes jk, and unitaries U_{jk} acting on B''. Formally, let us define the Kraus operators of the protocol by letting

$$L_{jk\alpha} := [\langle \Phi^{(jk)}| (K_\alpha \otimes \mathbb{1})]^{AB'} \otimes U_{jk}^{B''}. \tag{4.13}$$

It can be checked readily that they satisfy the normalization condition

$$\sum_{jk\alpha} L^{\dagger}_{jk\alpha} L_{jk\alpha} = \sum_{jk\alpha} M_{jk\alpha} \otimes \mathbb{1} = \mathbb{1} \otimes \mathbb{1}.$$

Applying the Kraus operators $L_{jk\alpha}$ on all the system we get

$$\begin{aligned} L_{jk\alpha} \left|\phi\right\rangle^{A} \left|\Psi_d\right\rangle^{B'B''} &= \langle \Phi^{(jk)} | \, (K_\alpha \left|\phi\right\rangle)(\mathbb{1} \otimes U_{jk}) \left|\Psi_d\right\rangle \\ &= \langle \Phi^{(jk)} | \, K_\alpha \left|\phi\right\rangle \left|\Psi^{(jk)}\right\rangle \\ &= \frac{1}{d} K_\alpha \left|\phi\right\rangle . \end{aligned}$$

Hence, $\sum_{jk\alpha} L_{jk\alpha}(\rho \otimes \Phi_d) L^{\dagger}_{jk\alpha} = \sum_\alpha K_\alpha \rho K^{\dagger}_\alpha = \mathcal{N}(\rho)$, and the proof is complete. □

Example: qubit unitaries

So far we have proven that having a cosdit is enough to implement a quantum channel using IOs. However, it is perhaps natural to expect that one could get away with a smaller amount of coherence, given that most target channels are not as coherent as a Hadamard gate. Here we prove that, when we want to implement a qubit unitary with a two-dimensional resource state, this is actually impossible.

Proposition 6. *The only qubit coherent resource state $|r\rangle \in \mathbb{C}^2$ that permits the implementation of $U(\theta)$, $0 < \theta \leq \frac{\pi}{4}$, is the cosbit.*

Proof. We want to know for which state $|r\rangle = c' \left|0\right\rangle + s' \left|1\right\rangle$ the transformation $\left|\psi\right\rangle \left|r\right\rangle \xrightarrow{IO} (U(\theta) \left|\psi\right\rangle) \left|0\right\rangle$ is possible, for a general state $\left|\psi\right\rangle$. Without loss of generality, the incoherent Kraus operators achieving the transformation have the following general form:

$$K = \lambda \Big(U(\theta) \otimes |0\rangle\langle r| + R \otimes |0\rangle\langle r^{\perp}| \Big), \tag{4.14}$$

where $|r^{\perp}\rangle = s' \left|0\right\rangle - c' \left|1\right\rangle$ is the vector orthogonal to $|r\rangle$. We now need to find the form of R such that K is incoherent. For that, we impose incoherence of K when tracing out the ancillary part: $\langle 0|^{A} K \left|0\right\rangle^{A} =: T_0$ and $\langle 1|^{A} K \left|1\right\rangle^{A} =: T_1$, where T_0 and T_1 must be 2-dimensional incoherent operators. We then obtain that $R = s'T_0 - c'T_1$ and $\lambda U = c'T_0 + s'T_1$. The latter condition enforces that either $T_0 \propto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $T_1 \propto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ or viceversa; or $T_0 \propto \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$

and $T_1 \propto \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ or viceversa. From these possibilities, we get 4 possible R matrices, which define 4 different Kraus operators K_i defined, according to Eq. (4.14), by R_i matrices as follows:

$$R_1 = \begin{pmatrix} c\frac{s'}{c'} & -s\frac{c'}{s'} \\ s\frac{c'}{s'} & c\frac{s'}{c'} \end{pmatrix},$$
$$R_2 = \begin{pmatrix} -c\frac{c'}{s'} & s\frac{s'}{c'} \\ -s\frac{s'}{c'} & -c\frac{c'}{s'} \end{pmatrix},$$
$$R_3 = \begin{pmatrix} -c\frac{c'}{s'} & -s\frac{c'}{s'} \\ -s\frac{s'}{c'} & c\frac{s'}{c'} \end{pmatrix},$$
$$R_4 = \begin{pmatrix} c\frac{s'}{c'} & s\frac{s'}{c'} \\ s\frac{c'}{s'} & -c\frac{c'}{s'} \end{pmatrix},$$

and the general incoherent Kraus operator is $K = \lambda_i K_i$ $(i = 1, 2, 3, 4)$. Finally, after imposing $\sum_i |\lambda_i|^2 K_i^\dagger K_i = \mathbb{1}$, we obtain the following conditions on R_i and λ_i:

$$\sum_i |\lambda_i|^2 = 1,$$
$$\sum_i |\lambda_i|^2 R_i^\dagger R_i = \mathbb{1},$$
$$\sum_i |\lambda_i|^2 R_i = 0.$$

It can be verified that these conditions are only fulfilled when $|c'| = |s'| = \frac{1}{\sqrt{2}}$, i.e. $|r\rangle$ is maximally coherent. □

This result might suggest an irreversibility between simulation and coherence generation for qubit unitaries under IOs, but we point out that it does not preclude the possibility of simulations using a higher rank, yet less coherent, resource state (cf. [116], where the analogue is demonstrated for LOCC implementation of bipartite unitaries using entangled resources); or of a simulation of many instances of $U(\theta)$ at a cost lower than 1 per unitary.

4.3 Qubit unitaries: no coherence recycling

When implementing a qubit unitary via IOs and the less coherent resource possible, a cosbit, we find that no coherence is left at the output to be recycled:

Proposition 7. *Any two-qubit incoherent operation $\mathcal{I}$ such that $\mathcal{I}(\rho \otimes |r\rangle\langle r|) = U(\theta)\rho U(\theta)^\dagger \otimes \sigma$ for general ρ, is such that the state σ left behind in the ancilla is necessarily incoherent.*

Proof. If the incoherent implementation of the unitary, instead of mapping two qubits (input and resource state) to one (output), but to two (output plus residual resource), i.e. $\mathcal{I}(\rho \otimes |r\rangle\langle r|) = U(\theta)\rho U(\theta)^\dagger \otimes \sigma$, then first of all σ has to be the same irrespective of the state ρ. Otherwise we would be able, by measuring σ, to learn some information about ρ without disturbing it. Now consider a pure incoherent input state $\rho = |0\rangle\langle 0|$, and note that the desired output state $U(\theta)|\psi\rangle$ has nontrivial coherence. But now observe that $\mathcal{I}$ takes in a state of coherence rank 2 [65], and produces a product of a pure state of coherence rank 2 with another state. Since the coherence rank cannot increase, even under individual Kraus operators [65], it must be the case that σ is incoherent. □

It is still an open question whether this fact generalizes to arbitrary d-dimensional channels, or whether the impossibility of coherence recyling is just intrinsic to qubit unitaries.

4.4 Asymptotic simulation cost

The best general bounds we have on the asymptotic simulation cost under IOs are contained in the following theorem.

Theorem 8. *For any CPTP map $\mathcal{N} : A \longrightarrow B$,*

$$C^\infty_{gen}(\mathcal{N}) \leq C^\infty_{sim}(\mathcal{N}) \leq \log|B|. \tag{4.15}$$

Furthermore,

$$C_{sim}(\mathcal{N}) \geq \max\left\{\sup_k \widehat{P}_{C_r}(\mathcal{N} \otimes \mathrm{id}_k), \sup_k \widehat{P}_{C_f}(\mathcal{N} \otimes \mathrm{id}_k)\right\}. \tag{4.16}$$

Proof. We start with Eq. (4.15): The upper bound is a direct consequence of Theorem 5. The lower bound follows from the fact that $\mathcal{N}$ is implemented using maximally coherent states at rate $R = C^\infty_{\mathrm{sim}}(\mathcal{N})$ and incoherent operations. Generation of entanglement on the other hand uses $\mathcal{N}$ and some more incoherent operations. Since incoherent operations cannot increase the amount of entanglement, the overall process of simulation and generation cannot result in a rate of coherence of more than R.

Regarding Eq. (4.16), the idea is that for $\epsilon > 0$ and n large enough, since the simulation implements a CPTP map $\mathcal{N}'$ that is within diamond norm ϵ from $\mathcal{N}^{\otimes n}$, using incoherent operations and $\Psi_2^{\otimes n(R+\epsilon)}$ as a resource. Applying the simulation to the state $\rho^{\otimes n}$, results in $(\mathcal{N}' \otimes \mathrm{id}_{k^n})\rho^{\otimes n} \approx \Big((\mathcal{N} \otimes \mathrm{id}_k)\rho\Big)^{\otimes n}$, hence we have an overall incoherent operation

$$\Psi_2^{\otimes n(R+\epsilon)} \otimes \rho^{\otimes n} \xmapsto{\mathrm{IO}} (\mathcal{N}' \otimes \mathrm{id}_{k^n})\rho^{\otimes n}.$$

By monotonicity of C_X ($X \in \{r, f\}$) under IO, and $C_X(\Psi_2) = 1$, this means

$$n(R+\epsilon) \geq C_X\Big((\mathcal{N}' \otimes \mathrm{id}_{k^n})\rho^{\otimes n}\Big) - C_X(\rho^{\otimes n}),$$

where we have used additivity of C_r and C_f [65]. Since this holds for all ρ, we obtain

$$\begin{aligned} n(R+\epsilon) &\geq P_X(\mathcal{N}' \otimes \mathrm{id}_{k^n}) \\ &\geq P_X(\mathcal{N}^{\otimes n} \otimes \mathrm{id}_{k^n}) - n\kappa_X\epsilon - 4 \\ &\geq nP_X(\mathcal{N} \otimes \mathrm{id}_k) - n\kappa_X\epsilon - 2, \end{aligned}$$

invoking in the second line Lemma 43 in Appendix B, with $\kappa_r = 4\log|B|$ and $\kappa_f = \log|B| + \log k$, and in the third a tensor power test state. Since ϵ can be made arbitrarily small, and n as well as k arbitrarily large, the claim follows. □

Regarding the existence of bound coherence, let us remember that states do not present it, since for them vanishing distillable coherence implies vanishing coherence cost (see Section 5.4). As we will now explain, this is in contrast to operations, i.e. vanishing asymptotic coherence generating capacity may still yield non-zero asymptotic simulation cost:

Let us consider channel T to be a MIO. We expect the simulation cost of any such T to be positive, $C_{\mathrm{sim}}(T) > 0$. At the same time, $C_{\mathrm{gen}}(T) = 0$ by Theorem 4, because the relative entropy of coherence is a MIO monotone, and the tensor product of MIO transformations is MIO. To obtain an example, we can take any MIO channel for which there exists a state ρ such that $C_f\big(T(\rho)\big) > C_f(\rho)$, since by Theorem 8, $C_{\mathrm{sim}}(T)$ is lower bounded by the difference of the two. (As an aside, we note that this cannot be realised in qubits, because for qubits, any state transformation possible under MIO is already possible under IO, for which C_f is a monotone [75, 76].) Concretely, consider the following states on a $2d$-dimensional system A, which could be called *coherent flower states*, since their corresponding maximally correlated states (cf. [65, 117]) are the well-known flower states [118]. We write them as 2×2-block matrices,

$$\rho_d = \frac{1}{2d}\begin{pmatrix} \mathbb{1} & U \\ U^\dagger & \mathbb{1} \end{pmatrix}, \tag{4.17}$$

where U is the d-dimensional discrete Fourier transform matrix. Via the correspondence between C_r and the relative entropy of entanglement, and between C_f and the entanglement of formation, respectively, of the associated state, we know that $C_r(\rho_d) = 1$ and $C_f(\rho_d) = 1 + \frac{1}{2}\log d$ [118]. By the results of [58], however, for every $\epsilon > 0$ and sufficiently large n, there exists a MIO transformation $T^{(n)} : \binom{2}{\text{cc}}^{\otimes n(1+\epsilon)} \longrightarrow A^n$ with

$$\rho^{(n)} := T^{(n)}\left(\Psi_2^{\otimes n(1+\epsilon)}\right) \overset{\epsilon}{\approx} \rho_d^{\otimes n}. \tag{4.18}$$

By the asymptotic continuity of C_f [65], we have $C_f(\rho^{(n)}) \geq n\left(1 + \frac{1}{2}\log d\right) - n\epsilon\log(2d) - g(\epsilon)$, while of course the preimage $\rho_0 = \Psi_2^{\otimes n(1+\epsilon)}$ has $C_f(\rho_0) \leq n + n\epsilon$, so for ϵ small enough and n large enough, we have a gap:

$$C_{\text{sim}}\left(T^{(n)}\right) \geq \frac{n}{2}\log d - n\epsilon(2 + \log d) - g(\epsilon) > 0, \tag{4.19}$$

invoking Theorem 8. In conclusion, in the IO-theory of maps it is possible to find operations with zero asymptotic generating capacity and non-zero asymptotic simulation cost, i.e. there exists bound coherence.

Example: qubit unitaries

Regarding the implementation of qubit unitary channels, all we can say for the moment is that $C_{\text{sim}}^{\infty}(U(\theta)) \leq 1$, because we can implement each instance of the qubit unitary using a cosbit Ψ_2, as shown in Proposition 6.

4.5 Chapter summary

- We have provided bounds for $C_{\mathrm{gen}}^{\infty}(\mathcal{N})$, and a single-letter formula for the case of unitaries. These bounds are related to the coherence power of the map resulting from appending the identity to $\mathcal{N}$.

- Any channel can be implemented by means of IOs and a cosdit. A cosbit turns out to be the only qubit resource permitting the IO-simulation of a qubit unitary.

- The IO-implementation of a qubit unitary using a cosbit does not allow for coherence recycling.

- Since $C_{\mathrm{gen}}^{\infty}(\mathcal{N}) \leq C_{\mathrm{sim}}^{\infty}(\mathcal{N})$, the IO-theory of channels is not asymptotically reversible (for which one would require $C_{\mathrm{gen}}^{\infty}(\mathcal{N}) = C_{\mathrm{sim}}^{\infty}(\mathcal{N})$). In addition, for qubit unitaries U, $C_{\mathrm{sim}}^{\infty}(U) \leq 1$.

- In contrast to states, operations may exhibit bound coherence, i.e. $C_{\mathrm{gen}}^{\infty}(\mathcal{N}) = 0$ does not imply $C_{\mathrm{sim}}^{\infty}(\mathcal{N}) = 0$.

4.6 Open questions

- Is there any way to compute $\widehat{P}_C^{(k)}(\mathcal{N})$ and $\widetilde{P}_C^{(k)}(\mathcal{N})$, i.e. the bounds to C_{gen}^{∞} and C_{sim}^{∞}, efficiently?
- Can any arbitrary channel be implemented via IOs and a resource state less coherent than a cosdit? For qubit unitaries, at least, this cannot happen. What if we consider higher rank, yet less coherent, resource states?
- Is coherence recycling allowed by IO-implementation of arbitrary channels that are not qubit unitaries?
- What is the single-shot simulation cost of a channel under IOs?

CHAPTER 5

Results in MIO-theory

Analogously to what we presented in the previous chapter, we here provide different computations of the coherence content of a quantum channel, this time considering MIOs as free operations. This way we also benchmark the ultimate performance of coherence generation and channel implementation, since we use the largest set of free operations.

5.1 Single-shot simulation cost

We start by giving an exact expression for the simulation cost of a channel in terms of its smoothed robustness of coherence:

Theorem 9. *For any quantum channel* $\mathcal{N}$ *it holds*

$$C_{\text{sim}}^{\epsilon}(\mathcal{N}) = \log\lceil 1 + C_R^{\epsilon}(\mathcal{N})\rceil, \tag{5.1}$$

where $\lceil\cdot\rceil$ *is the ceiling function.*

Proof. Let $\mathcal{L} := \mathcal{M}(\Psi_k \otimes \cdot) : A \to B$ be the MIO simulation of the induced channel $\mathcal{L} : A \to B$ such that $\frac{1}{2}\|\mathcal{N} - \mathcal{L}\|_\diamond \leq \epsilon$. Noting that $\Psi_k + (k-1)\,\sigma = \mathbb{1}$, with $\sigma = \dfrac{\mathbb{1} - \Psi_k}{k-1}$, define

$$\begin{aligned}\mathcal{M}' := \mathcal{M}\left(\frac{\mathbb{1}}{k} \otimes \cdot\right) &= \frac{1}{k}\mathcal{M}\left((\Psi_k + (k-1)\sigma) \otimes \cdot\right)\\ &= \frac{1}{k}\mathcal{M}(\Psi_k \otimes \cdot) + \left(1 - \frac{1}{k}\right)\mathcal{M}(\sigma \otimes \cdot)\\ &= \frac{1}{k}\mathcal{L} + \left(1 - \frac{1}{k}\right)\mathcal{L}', \end{aligned} \tag{5.2}$$

with $\mathcal{L}' = \mathcal{M}(\sigma \otimes \cdot)$. As $\mathcal{M}$ is MIO, so is $\mathcal{M}' : A \to B$, and the right-hand-side of Eq. (5.2) corresponds to a convex decomposition of $\mathcal{M}'$ in terms of $\mathcal{L}, \mathcal{L}' \in$ CPTP. Hence, from the definition of the maximum-relative entropy, Eq. (A.10) it follows

$$\log k \geq D^{\epsilon}_{\max}(\mathcal{L}\|\mathcal{M}') \geq C^{\epsilon}_{LR}(\mathcal{N}). \tag{5.3}$$

For the converse, let $k \geq 1 + C^{\epsilon}_{R}(\mathcal{N})$ be an integer. By Eq. (A.12), it follows that there exists some CPTP map $\mathcal{L}$ with $\frac{1}{2}||\mathcal{N} - \mathcal{L}||_\diamond \leq \epsilon$ and another CPTP map $\mathcal{L}'$, such that $(\mathcal{L} + (k-1)\,\mathcal{L}')/k$ is MIO. Make the following ansatz for a channel $\mathcal{M}$ that is feasible for the simulation cost:

$$\mathcal{M}(\tau \otimes \rho) := \mathrm{Tr}(\Psi_k\tau)\mathcal{L}(\rho) + \mathrm{Tr}((\mathbb{1} - \Psi_k)\tau)\mathcal{L}'(\rho).$$

The map $\mathcal{M}$ is MIO if and only if $\mathcal{M}(|i\rangle\langle i| \otimes \cdot)$ is MIO for all incoherent basis states $|i\rangle$. This is the case, since $\mathcal{M}(|i\rangle\langle i| \otimes \cdot) = \frac{1}{k}\mathcal{L} + \left(1 - \frac{1}{k}\right)\mathcal{L}'$, which is MIO by construction. Hence $\log k \geq C^{\epsilon}_{\mathrm{sim}}(\mathcal{N})$ and the former can be taken as small as $\lceil 1 + C^{\epsilon}_{R}(\mathcal{N})\rceil$. As the implementation cost is necessarily an integer Eq. (5.1) follows. This completes the proof. □

Theorem 9 can be seen as one-shot coherence dilution in the resource theory of channels: a maximally coherent resource is completely consumed to generate a target one with smaller coherence-generating capability. Indeed, this includes as a special case the one-shot coherence dilution of states studied in [71] and presented in Section 1.5.2, and generalizes the criterion found in [63] for transformations of cosdits Ψ_k to pure states $|\phi\rangle = \sum_i \sqrt{p_i}\,|i\rangle$:

$$\sum_i \sqrt{p_i} \leq \sqrt{k}. \tag{5.4}$$

Thanks to Theorem 9, we can now determine the minimal coherence rank of a cosdit necessary for a MIO implementation of a channel[1].

[1] We also note that, since the the cosdit is the maximally coherent state with a given

Example: qubit unitaries

Let us now focus on implementing the simplest kind of channels, qubit unitary gates. In this case we can compute the quantity defined above:

$$C_{\text{sim}}\left(U_\theta\right) = \begin{cases} 0 & \theta = 0, \\ 1 & \text{otherwise.} \end{cases}, \tag{5.5}$$

as can be readily checked by noticing that the robustness of a qubit state is equal to its ℓ_1-norm of coherence (see Eq. (1.19)): $C_R\Big(U_\theta \left|i\right\rangle\!\left\langle i\right| U_\theta^\dagger\Big) = C_{\ell_1}\Big(U_\theta \left|i\right\rangle\!\left\langle i\right| U_\theta^\dagger\Big) = 2cs = \sin 2\theta$.

5.2 Amortized simulation cost

Interestingly, the amortized cost defined in Eq. (3.8) can be related to the log-robustness of a channel. In order to show it, let us first discuss the exact implementation of a channel via MIOs with coherence recycling, which amounts to taking $\epsilon = 0$ above and $\mathcal{L} \equiv \mathcal{N}$. In this case, thanks to Theorem 9, it is possible to estimate the robustness of coherence left in the resource after the implementation:

Theorem 10. *For a quantum channel $\mathcal{N} : A \to B$, there exists a MIO $\mathcal{M} : R \otimes A \to S \otimes B$ such that $\mathcal{M}(\Psi_k \otimes \cdot) = \sigma \otimes \mathcal{N}(\cdot)$ if and only if*

$$C_{LR}(\mathcal{N}) \leq C_{LR}(\Psi_k) - C_{LR}(\sigma). \tag{5.6}$$

Proof. Let $\mathcal{T} : \mathbb{C} \otimes A \to S \otimes B$ such that $\mathcal{T}(\mathbb{1} \otimes \rho) = \sigma \otimes \mathcal{N}(\rho)$. Here, we have made use of the fact that any state can be identified with a preparation channel from $\mathbb{C}$ to S mapping the unique state $1 \to \sigma$. Now let $\mathcal{M}$ be a MIO such that

$$\mathcal{M}(\Psi_k \otimes \cdot) = \mathcal{T}(\cdot).$$

From Theorem 9 such a simulation is possible if and only if

$$\begin{aligned} k &\geq (1 + C_R(\mathcal{T})) \\ &= (1 + C_R(\sigma))(1 + C_R(\mathcal{N})) \end{aligned}$$

coherence rank, Theorem 9 implies that a generic input resource state ω will not be able to ϵ-simulate a channel if its coherence, as measured by the coherence rank (or by coherence number if the resource is a mixed state), is smaller than the simulation cost: $C_{\text{rank}}(\omega) < \lceil 1 + C_R^\epsilon(N) \rceil$. Indeed, the latter inequality implies that a cosdit ψ'_k of the same rank $k' = C_{\text{rank}}(\omega)$ cannot simulate the channel; moreover, by construction ω is less coherent that ψ'_k, since it can be obtained from the cosdit via $IO \subset MIO$ [59]; hence ω cannot simulate a channel already not simulable with ψ'_k.

where we have used the multiplicativity of the robustness of coherence in the last line. Taking the logarithm on both sides and re-arranging terms gives Eq. (5.6). This completes the proof. □

Note that the bound in Eq. (5.6) can always be attained by some resource state σ_0 that, however, will not be a maximally coherent. If we do impose that the output resource is a cosdit Ψ_m, we obtain the following result:

Corollary 11. *Given a quantum channel $\mathcal{N}$ and an integer $k \geq 1 + C_R(\mathcal{N})$, there always exists a MIO implementation of $\mathcal{N}$ that takes a cosdit resource of coherence rank k and returns a degraded resource in the form of a cosdit of rank $m = \left\lfloor \frac{k}{1+C_R(\mathcal{N})} \right\rfloor$, i.e.*

$$1 + C_R(\mathcal{N}) \leq \frac{k}{m} \leq (1 + C_R(\mathcal{N}))(1 + \frac{1}{m}). \tag{5.7}$$

That is, demanding a cosdit at the output requires an overhead of at most $O(1/m)$ with respect to the optimal ratio that can be attained with a non-maximally coherent output resource; moreover, this overhead can be made arbitrarily small by simply providing a higher-rank cosdit resource at the input. This implies that the amortized cost of a channel is equal to its log-robustness and the same straightforwardly extends to the approximate case, as proved by the following theorem:

Theorem 12. *For any quantum channel $\mathcal{N}$ it holds*

$$C^{\epsilon}_{\text{amo}}(\mathcal{N}) = C^{\epsilon}_{LR}(\mathcal{N}). \tag{5.8}$$

Proof. From Definition 3.2.2 and Corollary 11 it follows immediately that the zero-error amortized cost is equal to the log-robustness. Indeed, if we take the log on each side of Eq. (5.7) and then let $k, m \to \infty$ we obtain $C_{\text{amo}}(\mathcal{N}) = C_{LR}(\mathcal{N})$. this in turn implies that the ϵ-error amortized cost, Eq. (3.8), can be rewritten in terms of the log-robustness of the channel $\mathcal{L}$ as

$$C^{\epsilon}_{\text{amo}}(\mathcal{N}) = \min C_{LR}(\mathcal{L}) \text{ s.t. } \frac{1}{2}||\mathcal{N} - \mathcal{L}||_{\diamond} \leq \epsilon \equiv C_{LR}(\mathcal{N}). \tag{5.9}$$

□

This second key result, together with Theorem 9, establishes the robustness of coherence of a channel as the correct measure to quantify the cosdit resources necessary to implement the channel using a single MIO. Eq. (5.1) determines the minimum coherence rank required at the input, while Eq. (5.8) determines the minimum fraction of input coherence that is actually used in the process. Note however that the latter is a lower bound on the actual coherence consumed when a limited amount of resource can be employed at the input.

Finally, restricting again to the zero-error case, Corollary 11 paves the way for the exact MIO implementation of arbitrary sequences of channels, as depicted in Fig. 3.1b.

Corollary 13. *Any succession of channels $\mathcal{N}_1, \mathcal{N}_2, \cdots, \mathcal{N}_n$ can be implemented on-the-fly by concatenation of MIO implementations at an asymptotically optimal amortized cost*

$$\sum_{i=1}^{n} C_{\text{amo}}\left(\mathcal{N}_i\right) \leq \log \frac{k}{k'} \leq \sum_{i=1}^{n} C_{\text{amo}}\left(\mathcal{N}_i\right) + \frac{n}{k'}, \tag{5.10}$$

where k and $k' \leq k$ are the coherence ranks of the input and output cosdits of the entire protocol.

Note that the additivity of the zero-error log-robustness under tensor product already implies $C_{\text{amo}}\left(\mathcal{N}_1 \otimes \cdots \otimes \mathcal{N}_n\right) = \sum_{i=1}^{n} C_{\text{amo}}\left(\mathcal{N}_i\right)$. However, Corollary 13 is more general and includes this result as a special case. Indeed, it allows the exact implementation of *arbitrary sequences* of channels, and not just their tensor product, at an amortized cost equal to the sum of the single amortized costs of each channel. In particular, this allows the implementation of a concatenation of channels on-the-fly and it justifies the choice of a tensor-product structure at the output of the recycling process. Finally, from Eq. (3.5) we know that the log-robustness of coherence also quantifies the cohering power of a quantum channel. Hence we conclude that the exact amortized cost of a channel coincides with its cohering power.

Example: qubit unitaries

For qubit unitaries it holds:

$$C_{LR}(U_\theta) = \log(1 + \sin 2\theta) = C_{\text{amo}}(U_\theta). \tag{5.11}$$

In contrast with the case of IOs (see Section 4.3), where the implementation of qubit unitaries consumes the entire cosbit resource, Theorem 10 ensures that MIOs do allow for coherence recycling. More precisely, there exists a MIO $\mathcal{M}$ such that $\mathcal{M}(\Psi_2 \otimes \rho) = \sigma \otimes U_\theta \rho U_\theta^\dagger$ for any qubit state ρ if and only if the output resource state σ has coherence

$$C_R(\sigma) \leq \frac{1 - \sin 2\theta}{1 + \sin 2\theta}. \tag{5.12}$$

5.3 Arbitrary resources for MIO implementation of channels

Let us now go beyond the assumptions of the previous sections by considering a scenario where non-maximally coherent states are employed as resources (see [119] for an analogous result in entanglement theory). In particular, we want to study under which conditions the MIO implementation of a quantum channel in the settings of Fig. 3.1a and Fig. 3.1b is still possible.

We begin by introducing a semidefinite program to assess the performance of an arbitrary resource at implementing any target channel; this program yields the best approximate MIO implementation of the target channel with a given resource state (Fig. 3.1a), as measured by the diamond norm.

Proposition 14. *The smallest diamond norm error for the implementation of a quantum channel $\mathcal{N} : A \to B$ via a MIO $\mathcal{M} : R \otimes A \to B$ with a coherent resource $\omega \in R$ is given by the following semidefinite program:*

$$\begin{aligned} &\min \lambda \\ &\text{s.t. } J_{\mathcal{M}} \textit{ is the Choi matrix of } \mathcal{M} \in \textit{MIO} \\ &\quad J_{\mathcal{E}} = \operatorname{Tr}_R((\omega^t \otimes \mathbb{1}_A \otimes \mathbb{1}_B) J_{\mathcal{M}}) \\ &\quad Z \geq 0 \\ &\quad Z \geq J_{\mathcal{N}} - J_{\mathcal{E}} \\ &\quad \lambda \mathbb{1}_A \geq \operatorname{Tr}_B Z, \end{aligned} \tag{5.13}$$

where $J_{\mathcal{N}}$ is the Choi matrix of $\mathcal{N}$, ω^t denotes the transpose of ω, and $J_{\mathcal{E}}$ that of its implementation, $\mathcal{E} = \operatorname{Tr}_B \mathcal{M}(\omega \otimes \cdot)$. Recall that the Choi matrix $J_{\mathcal{M}}$ of a MIO $\mathcal{M}$ as in Eq. (5.13), is fully characterized by $J_{\mathcal{M}} \geq 0$, $\operatorname{Tr}_B J_{\mathcal{M}} = \mathbb{1}_{RA}$ and $\operatorname{Tr}((|i\rangle\langle i|_{RA} \otimes |j\rangle\langle k|_B) J_{\mathcal{M}}) = 0$ $\forall i$ and $\forall j \neq k$.

Proof. The SDP can now be formulated as follows:

$$\begin{aligned} &\min \frac{1}{2} \|\mathcal{E} - \mathcal{N}\|_\diamond \\ &\text{s.t. } X \text{ is the Choi matrix of a MIO } \mathcal{M}, \\ &\quad \mathcal{E} = \operatorname{Tr}_B \mathcal{M}(\omega \otimes \cdot). \end{aligned} \tag{5.14}$$

Using the result of [111], the dual form of the diamond norm distance can be written as

$$\begin{aligned} &\min \lambda \\ &\text{s.t. } Z \geq J_{\mathcal{E}} - J_{\mathcal{N}} \\ &\quad \operatorname{Tr}_B Z \leq \lambda \mathbb{1}_A \\ &\quad Z \geq 0. \end{aligned} \tag{5.15}$$

Using Eq. (C.3) to relate the Choi matrix of $\mathcal{E}$ to that of $\mathcal{M}$ is MIO it is then straightforward to obtain the final formulation. □

In Appendix C.1 we provide a simpler semidefinite program for the case of unitary channels, where the precision of the simulation is assessed in terms of the average gate fidelity $f(U,\mathcal{N})$ [120], rather than by the diamond distance. Specifically, we compute the entanglement fidelity $F(U,\mathcal{N}) = \mathrm{Tr}\, J_U J_{\mathcal{N}}$ (where J_U and $J_{\mathcal{N}}$ are the Choi matrices of U and $\mathcal{N}$, respectively), which fulfills $F(U,\mathcal{N}) = \dfrac{(d+1)f(U,\mathcal{N})-1}{d}$, where d is the dimension of the Hilbert space on which the channels act [120]. From now on we will refer to it as "gate fidelity".

Regarding instead the recycling setting of Fig. 3.1b, thanks to Theorem 12 it is straightforward to write a semidefinite program for the ϵ-error amortized cost of a channel with cosdit input resource, as shown in Fig. 5.4. If, in addition, we ask for the maximum robustness of coherence left in the output resource when a non-maximally coherent input resource is employed, we end up with the following optimization problem:

Proposition 15. *The maximum coherence left in the resource $\sigma \in S$ after the implementation of a quantum channel $\mathcal{N} : A \to B$ via a MIO $\mathcal{M} : R \otimes A \to S \otimes B$ and a coherent resource $\omega \in R$ up to error ϵ in diamond norm is:*

$$\begin{aligned} \max\ & C_R(\sigma) \\ \text{s.t. } & \mathcal{M}(\omega \otimes \cdot) = \sigma \otimes \mathcal{L}(\cdot) \\ & \frac{1}{2}||\mathcal{N} - \mathcal{L}||_\diamond \leq \epsilon. \end{aligned} \tag{5.16}$$

This problem captures exactly the setting of Fig. 3.1b with arbitrary input resource. Note, however, that it cannot be formulated as a semidefinite program, since the tensor-product constraint is non-linear in the optimization variables σ and $\mathcal{L}$. One can devise alternative semidefinite programs by relaxing the constraints in Eq. (5.16), as we discuss in Appendix C.2.

The semidefinite program in Proposition 14 and that for the amortized cost allow for a thorough numerical analysis of our problem, see Figs. 5.2-5.4. Nevertheless, before analysing the numerical results for qubits, let us present some general, analytical results on MIO implementation of channels with non-maximally coherent resources. As a start, according to Theorem 9, any resource state that is MIO-convertible into a cosdit of coherence rank d can be trivially used to implement any quantum channel of log-robustness smaller than $\log d$. The following lemma provides us with a simple necessary condition for the existence of pure state transformation via MIO.

Lemma 16. *Assume that a pure state ψ is transformed to a pure state ϕ via MIO, then*

$$\lambda_1(\psi) \leq \lambda_1(\phi), \tag{5.17}$$

where $\lambda_1(\rho) = \max_i \langle i| \rho |i\rangle$ is the largest diagonal entry of an operator ρ, which coincides with its fidelity of coherence [121, 122] on pure states.

Proof. The geometric measure of coherence, introduced in [121], is defined as

$$C_g(\rho) = 1 - F_C(\rho) \tag{5.18}$$

where

$$F_C(\rho) = \max_{\delta \in \Delta} F(\rho, \delta)^2, \tag{5.19}$$

is the familiar Uhlmann fidelity between two mixed states, $F(\rho, \delta) = \|\sqrt{\rho}\sqrt{\delta}\|_1$. In particular for a pure state $\rho = |\psi\rangle\langle\psi|$, $F(\psi, \delta)^2 = \operatorname{Tr} \psi\delta$, and $F_C(\psi) = \lambda_1(\psi) = \max_i \langle i| \psi |i\rangle$. As the geometric measure of coherence is monotonically decreasing under MIO, it follows that $F_C(\mathcal{M}(\rho)) \geq F_C(\rho)$, for all $\mathcal{M}$ MIO. Setting $\phi = \mathcal{M}(\psi)$ leads to the desired result. □

Condition (5.17) is also sufficient if the target state is maximally coherent, since if $\lambda_1(\psi) \leq \frac{1}{d}$ then there exists a IO⊂MIO that transforms ψ into Ψ_d due to the majorization criterion (see Theorem 1).

Hence, for any channel $\mathcal{N}$ with $C_{LR}(\mathcal{N}) \leq \log d$ there exists a continuous family of resource states of dimension d that allow for the exact MIO simulation of it via conversion to cosdits. We now ask whether there exist any channels that can be implemented using pure resource states $|\omega\rangle$ that are not convertible into cosdits, i.e., $\lambda_1(\omega) > 1/d$. To this end let us define a special class of states that will be useful in this context.

Definition 17. *A d-dimensional* flagpole state *is a pure state of the form*

$$|\varphi_p\rangle = \sqrt{p}\,|0\rangle + \sqrt{\frac{1-p}{d-1}} \sum_{j=1}^{d-1} |j\rangle\,, \;\; \textit{with} \;\; \frac{1}{d} \leq p \leq 1. \tag{5.20}$$

The structure of flagpole states, shown in Fig. 5.1, endows them with several useful properties. Thanks to the majorization criterion, it is easy to show that: i) for all pure states ϕ we can transform $\phi \mapsto \varphi_p$ with a specific value of $p \geq F_C(\phi)$ (see Eq. (5.19)); ii) conversely, we can transform $\varphi_p \mapsto \phi$ for all ϕ such that $p \leq F_C(\phi)$. In other words, φ_p is the most coherent state of fixed coherence rank with fidelity of coherence larger than or equal to p. Moreover, as explained above, a d-dimensional flagpole state with $p \leq \frac{1}{d-1}$ can be converted into a $(d-1)$-dimensional cosdit and thus trivially implements

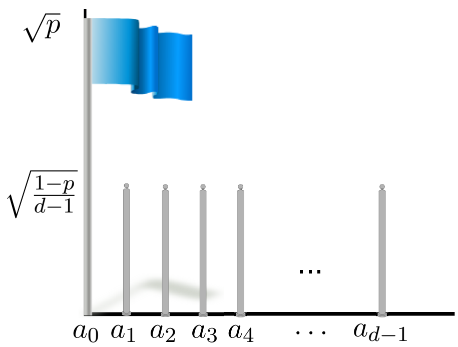


Figure 5.1: Schematic depiction of the coefficients $\{a_j\}_{j=0}^{d-1}$ of a flagpole state $|\varphi_p\rangle$ in the incoherent basis.

any channel of log-robustness smaller or equal than $\log(d-1)$. For all other flagpoles with $p > \frac{1}{d-1}$, which cannot be converted to a cosdit of rank $d-1$, the following theorem holds.

Theorem 18. *For any quantum channel $\mathcal{N} : A \to B$, if*

$$p \leq \frac{1}{1 + C_R(\mathcal{N})}, \tag{5.21}$$

then there exists a MIO $\mathcal{M} : R \otimes A \to B$ such that $\mathcal{M}(\varphi_p \otimes \rho) = \mathcal{N}(\rho)$ for all states ρ, where φ_p is a d-dimensional flagpole state, $d > |B|$.

Proof. Let us decompose the state space R into the following orthogonal subspaces

$$\varphi_p, \ \phi, \ \Pi = \mathbb{1} - \varphi_p - \phi \tag{5.22}$$

where $\varphi_p = |\varphi_p\rangle\langle\varphi_p|$, with $|\varphi_p\rangle$ given by Eq. (5.20), ϕ_1 is the rank-one projector onto the state

$$|\phi\rangle = \sqrt{1-p}\,|0\rangle - \sqrt{\frac{p}{d-1}}\sum_{j=1}^{d-1}|j\rangle\,, \tag{5.23}$$

and Π is a rank-$(d-2)$ orthogonal complement to φ_p and ϕ.

We now make the following ansatz for the MIO channel simulating $\mathcal{N}$ with a flagpole resource φ_p:

$$\begin{aligned}\mathcal{M}(\rho\otimes\sigma) = \mathcal{N}(\rho)\,\mathrm{Tr}(\sigma\varphi_p) + \mathcal{L}_1(\rho)\,\mathrm{Tr}(\sigma\phi_1)\\ + \mathcal{L}_2(\rho)\,\mathrm{Tr}(\sigma\Pi),\end{aligned} \tag{5.24}$$

where $\mathcal{L}_{1,2}$ are CPTP maps. In order for $\mathcal{M}$ to be MIO we require that $\mathcal{M}(\cdot\otimes|j\rangle\langle j|)$ is MIO, $\forall\, j \in (0,\ldots,d-1)$. This leads to the following two conditions

$$\begin{aligned} &p\mathcal{N} + (1-p)\mathcal{L}_1 \text{ is MIO} && \text{for } j = 0 \\ &\frac{1-p}{d-1}\mathcal{N} + \frac{p}{d-1}\mathcal{L}_1 + \left(1 - \frac{1}{d-1}\right)\mathcal{L}_2 \text{ is MIO} && \text{for } j > 0 \end{aligned} \tag{5.25}$$

From Proposition 41 we know that there exists a CPTP map $\mathcal{L}_1$ such that the first condition in Eq. (5.25) is fulfilled if and only if $p \leq (1 + C_R(\mathcal{N}))^{-1}$. To prove that the second condition in Eq. (5.25) is satisfied note that it may be written as

$$\frac{1}{d-1}\mathcal{N}' + \left(1 - \frac{1}{d-1}\right)\mathcal{L}_2 \tag{5.26}$$

where we defined the CPTP map $\mathcal{N}' = (1-p)\mathcal{N} + p\mathcal{L}_1$ whose robustness of coherence is at most $d-1$. By Definition 41 this implies that there exists a CPTP map $\mathcal{L}_2$ such that $(\mathcal{N}' + (d-2)\mathcal{L}_2)/(d-1)$ is MIO. This completes the proof. □

This proves that any pure resource state in dimension larger than 2 is useful for the exact implementation of some coherent unitary channel via MIO. Indeed, any such state can be converted to a flagpole, which in turn can be used for the implementation of a coherent channel, in particular a unitary. Note also that in general the bound on p in Theorem 18 does not single out all flagpoles that can implement a given channel, since it relies on a specific ansatz.

Example: qubit unitaries

Analogously to the previous section, we now address the implementation of qubit unitaries when non-maximally coherent resources are available.

The first question we want to raise is whether it is possible to use a non-maximally coherent pure qubit resource in order to implement a qubit unitary, even one that generates little coherence. We already know that, if the free operations are IOs, this is only possible with a cosbit, no matter how coherent the unitary is. This is still the case under MIOs, even for higher-dimensional unitaries, when restricting to qubit resource states:

Proposition 19. *The only pure qubit resource state $|\omega\rangle \in \mathbb{C}^2$ that permits the MIO implementation of some unitary gate of arbitrary dimension is the cosbit.*

Proof. Let $\{|\omega\rangle, |\omega^\perp\rangle\}$ be an orthonormal basis for $\mathbb{C}^2$, where

$$\begin{aligned} |\omega\rangle &= \alpha|0\rangle + \beta|1\rangle \\ |\omega^\perp\rangle &= \beta|0\rangle - \alpha|1\rangle \end{aligned} \tag{5.27}$$

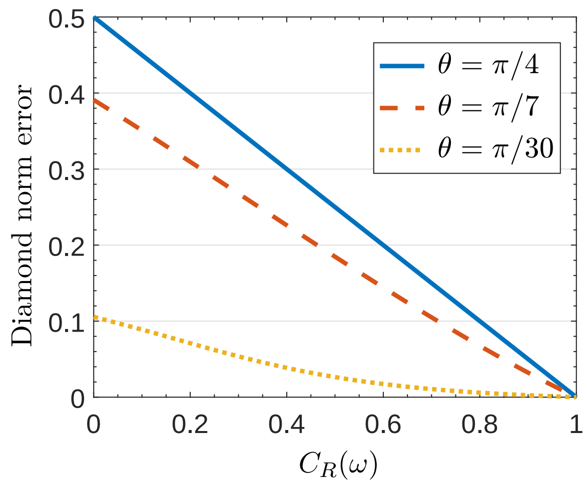


Figure 5.2: Minimum diamond norm distance between the qubit unitary U_θ and a MIO implementation of it with pure qubit resource state $|\omega\rangle$ vs. robustness of coherence of the latter. For $\theta = \pi/30$ (dotted orange line), $\theta = \pi/7$ (dashed red line) and $\theta = \pi/4$ (solid blue line). Note that an exact implementation is possible only with a cosbit resource state.

with $\alpha, \beta \in \mathbb{C}$, $|\alpha|^2 + |\beta|^2 = 1$. The Kraus decomposition of the most general $\mathcal{M}$ is MIO implementing any unitary operator $U : \mathcal{H}_d \to \mathcal{H}_d$, is given by

$$K_i = \lambda_i U \otimes |0\rangle \langle\omega| + R^i \otimes |0\rangle \langle\omega^\perp| , \tag{5.28}$$

where $\{\lambda_i \in \mathbb{C}\}$, and $\{R^i : \mathcal{H}_d \to \mathcal{H}_{d'}, 1 \le d' \le d\}$. As $\mathcal{M}$ is CPTP, $\sum_i K_i^\dagger K_i = \mathbb{1}$ which yields the following conditions on $\{\lambda_i, R^i\}$

$$\begin{aligned} \sum_i |\lambda_i|^2 &= 1 \\ \sum_i |\lambda_i|^2 R^i &= 0 \\ \sum_i |\lambda_i|^2 R^{i\dagger} R^i &= \mathbb{1}. \end{aligned} \tag{5.29}$$

Applying them together with the MIO condition

$$\langle m| \left(\sum_i K_i |jk\rangle \langle jk| K_i^\dagger \right) |n\rangle = 0, \quad \forall\, m \neq n$$

gives rise to the following two equations

$$|\alpha|^2 U_{mj} U_{jn}^\dagger + |\beta|^2 \sum_i |\lambda_i|^2 R_{mj}^i (R^{i\dagger})_{jn} = 0 \tag{5.30}$$

$$|\beta|^2 U_{mj} U_{jn}^\dagger + |\alpha|^2 \sum_i |\lambda_i|^2 R_{mj}^i (R^{i\dagger})_{jn} = 0, \tag{5.31}$$

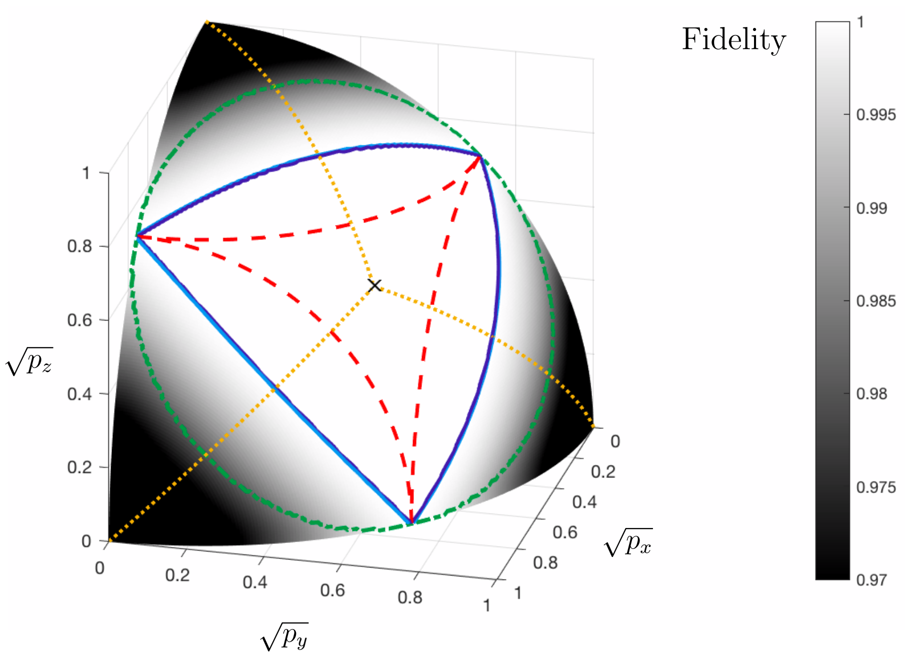


Figure 5.3: Gate fidelity for the MIO implementation of a qubit unitary U_θ with $\theta = \pi/14$ using a generic qutrit ($d = 3$) resource $|\psi\rangle = \sqrt{p_x}\,|0\rangle + \sqrt{p_y}\,|1\rangle + \sqrt{p_z}\,|2\rangle$. The central cross ($\times$) represents the maximally coherent state. The dashed red line around it encloses states that can be transformed with MIO to cosbits, the blue-solid line encloses the states that allow for an exact implementation ($F = 1$), while the dashed-dotted green line encloses all states that cannot be obtained via MIO by diluting a cosbit. The dotted yellow lines represent the family of flagpole states.

corresponding to $k = 0$ and $k = 1$ respectively. Summing both equations above we obtain the additional condition $\sum_i |\lambda_i|^2 R^i_{mj}(R^{i\dagger})_{jn} = -U_{mj}U^\dagger_{jn}$ which, upon substituting into any one of Eqs. (5.30, 5.31) results in

$$(|\alpha|^2 - |\beta|^2)U_{mj}U^\dagger_{jn} = 0, \quad \forall\, j, n, m. \tag{5.32}$$

Now, unless U is the identity, there is at least one pair of values (j, n) such that $0 < |U_{nj}| < 1$. Moreover, as $\sum_m |U_{mj}|^2 = 1$ there also exists at least one $m \neq n$ such that $|U_{mj}| > 0$. Thus for Eq. (5.32) to hold true for all j, n, m, $|\alpha|^2 = |\beta|^2 = \frac{1}{2}$ which implies that $|\omega\rangle = |\Psi_2\rangle$. This completes the proof. □

As an illustration of this fact, Fig. 5.2 shows that only a cosbit resource allows for an exact implementation of the qubit unitary U_θ, for several values of θ. Input coherence can be saved only at the expense of allowing for an approximate implementation of the gate. Moreover, in Fig. 5.3 we give a full characterization of the optimal performance of a MIO simulation, as measured by the gate fidelity, for general qutrit resource states. Without loss of generality we can focus on the upper region of the plot, $p_x, p_y \leq p_z = \lambda_1(\psi)$. Qubit resources are found in the planes defined by $p_x = p_y = 0$, where perfect simulation ($F = 1$) is only reached for cosbits, $p_z = 1/2$. The red dashed line delimits the qutrit states that can be distilled into a cosbit, $p_z \leq 1/2$, and hence also attain $F = 1$. However, perfect simulation can also be attained with other qutrit states: those that fall below the solid blue line in Fig. 5.3. In particular, the qutrit state with the highest value of p_z, i.e. the least coherent qutrit as measured by the fidelity of coherence, that allows for perfect simulation is a flagpole. Indeed, this agrees with the predictions of Theorem 18:

Proposition 20. *A d-dimensional flagpole resource state $|\varphi_p\rangle$ allows for the MIO implementation of a qubit unitary U_θ if*

$$p \leq \frac{1}{1 + \sin 2\theta}, \tag{5.33}$$

where $d \geq 3$ and $0 < \theta \leq \frac{\pi}{4}$. Furthermore, for a qutrit flagpole state $|\varphi_p\rangle \in \mathbb{C}^3$ this is also the highest allowed value of p.

Proof. We shall first prove the statement of the Proposition for the case of qutrit flagpoles. To that end consider the following orthonormal basis for $\mathbb{C}^3$,

$$\begin{aligned} |\varphi_p\rangle \equiv |\phi_0\rangle &= \sqrt{p}\,|0\rangle + \sqrt{\frac{1-p}{2}}(|1\rangle + |2\rangle), \\ |\phi_1\rangle &= \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle), \\ |\phi_2\rangle &= \sqrt{1-p}\,|0\rangle - \sqrt{\frac{p}{2}}(|1\rangle + |2\rangle). \end{aligned}$$

As in the previous case, the Kraus decomposition of the most general MIO $\mathcal{M}$ implementing $U_\theta : \mathbb{C}^2 \to \mathbb{C}^2$ is given by

$$K_i = \lambda_i U_\theta \otimes |0\rangle\langle\phi_0| + R^i \otimes |0\rangle\langle\phi_1| + P^i \otimes |0\rangle\langle\phi_2|\,. \tag{5.34}$$

As $\mathcal{M}$ is CPTP, $\sum_i K_i^\dagger K_i = \mathbb{1}$, which imposes the following conditions on $\{\lambda_i,\, R^i,\, P^i\}$

$$\sum_i |\lambda_i|^2 = 1 \tag{5.35}$$

$$\sum_i |\lambda_i|^2 R^i = \sum_\alpha P_\alpha = 0 \tag{5.36}$$

$$\sum_i |\lambda_i|^2 R^{i\dagger} R^i = \sum_i |\lambda_i|^2 P^{i\dagger} P^i = \mathbb{1} \tag{5.37}$$

$$\sum_i |\lambda_i|^2 R^{i\dagger} P^i = \sum_i |\lambda_i|^2 P^{i\dagger} R^i = 0. \tag{5.38}$$

Applying the above conditions together with the MIO condition

$$\langle m| \left(\sum_\alpha K_\alpha |jk\rangle \langle jk| K_\alpha^\dagger \right) |n\rangle = 0$$

for $m \neq n = 0, 1$, $j = 0, 1$ and $k = 0, 1, 2$ we obtain

$$U_{mj}U^\dagger_{jn}|\phi_0^k|^2 + A_j|\phi_1^k|^2 + B_j|\phi_2^k|^2 + C_j\phi_1^k\phi_2^k = 0, \tag{5.39}$$

where we have defined

$$\begin{aligned} A_j &= \sum_i |\lambda_i|^2 R^i_{mj}(R^{i\dagger})_{jn} \\ B_j &= \sum_i |\lambda_i|^2 P^i_{mj}(P^{i\dagger})_{jn} \\ C_j &= \sum_i |\lambda_i|^2 (R^i_{mj}(P^{i\dagger})_{jn} + P^i_{mj}(R^{i\dagger})_{jn}), \end{aligned}$$

and $\phi_l^k \equiv \langle \phi_l \rangle \, k$. Fixing $m = 0$, $n = 1$, Eq. (5.39) gives rise to six equations which we can solve for A_j, B_j and C_j to obtain

$$\begin{aligned} A_0 &= -A_1 = -\frac{(2p-1)}{p-1}\cos\theta\sin\theta \\ B_0 &= -B_1 = \frac{p}{p-1}\cos\theta\sin\theta \\ C_0 &= C_1 = 0. \end{aligned} \tag{5.40}$$

Now by definition A_j and B_j correspond to the off-diagonal elements of CPTP maps acting on the state $|j\rangle\langle j|$. As the resulting operator must be positive it follows that $|A_j| \leq \frac{1}{2}$, $|B_j| \leq \frac{1}{2}$. Moreover, the conditions on the A_j in Eq. (5.40) are satisfied if B_0, B_1 are given as in Eq. (5.40). Hence

$$|B_j| = \frac{p}{1-p}\cos\theta\sin\theta \leq \frac{1}{2},$$

which implies

$$p \leq \frac{1}{1+\sin 2\theta}.$$

Therefore, there can be no MIO that implements the qubit unitary, U_θ, with a qutrit flagpole state $|\varphi_p\rangle$ such that $p > (1 + \sin 2\theta)^{-1}$.

The proof can be extended to arbitrary d-dimensional flagpole and $(d-1)$-dimensional unitary. We note, however, that in this case we obtain the following upper bound $p \leq \left(1 + \frac{C_R(U)}{d-2}\right)^{-1}$, which coincides with Eq. (5.33) only for $d = 3$. This completes the proof.

We note that it is an interesting open question whether less coherent flagpole states allow for the implementation of qubit unitaries via MIO for the case $d > 3$. □

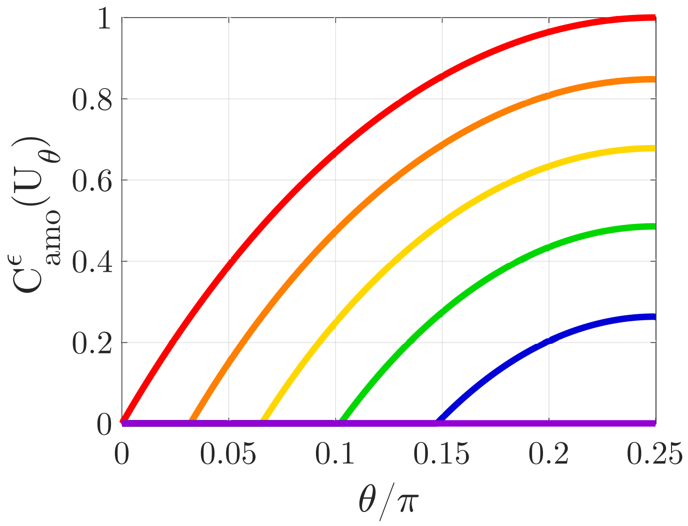


Figure 5.4: Plot of the ϵ-error amortized cost of the qubit unitary U_θ as vs. θ/π, for several error thresholds $\epsilon \in \{0, 0.1, 0.2, 0.3, 0.4, 0.5\}$ (from top to bottom). The amortized cost is higher for more coherent unitaries but it decreases if larger errors are allowed. In particular, when the amortized cost becomes zero it means that the given unitary can be simulated up to the given error with a MIO, which is not necessarily the identity channel, without using any coherence of the input resource state.

In Fig. 5.3 all flagpole states are identified by a yellow dotted line that interpolates between the incoherent state $|0\rangle$ and the cosbit. Finally, the set of states that cannot be obtained via MIO from a cosbit, are enclosed by the dashed-dotted green line which is determined by the intersection of the sphere of qutrit states (in the positive octant) and the plane $\langle\Psi_3\rangle\,\psi = 1$, as shown in Eq. (5.4). Extensive numerical evidence shows that the blue solid line that delimites the region of states that enable an exact MIO implementation is also given by the intersection of the qutrit sphere with a plane of the form $\langle\Phi_\theta\rangle\,\psi \leq f(\theta)$, where the constant $f(\theta)$ and normal vector $|\Phi_\theta\rangle$ can be analytically found by imposing that the plane includes the cosbit and the flagpole saturating Eq. (5.33).

Regarding the case of coherence recycling after the approximate implementation of a qubit unitary, Fig. 5.4 shows the amortized cost of qubit unitaries for several error thresholds. As expected, the amortized cost increases with the coherence of the target unitary, since less coherence can be recycled when implementing more coherent unitaries. Moreover, if we allow a larger error threshold to implement the same unitary, it is possible to obtain more coherence back for the same input resource at the cost of a worse approximation. When the amortized cost becomes zero, Theorem 12 implies that there exists a MIO that implements the target unitary without consuming any coherence. This happens if and only if the simulation cost of the unitary for the same error threshold is zero.

5.4 Asymptotic coherence generating capacity and asymptotic simulation cost

The following theorem can be proven regarding the asymptotic coherence generating capacity of a quantum channel:

Theorem 21. *The asymptotic coherence-generating capacity of a channel* $\mathcal{N}: A \to B$ *is given by*

$$\begin{aligned} C_{gen}^{\infty}(\mathcal{N}) &= \sup_{\rho_{AC}} C_r((\mathcal{N} \otimes \mathrm{id}_C)(\rho_{AC})) \\ &\quad - C_r(\rho_{AC}) && (5.41) \\ &= \sup_{C} \widehat{P}_{C_r}^{(C)}(\mathcal{N}), && (5.42) \end{aligned}$$

where the suprema are over all auxiliary systems C *and the first additionally over states* ρ_{AC} *on* $A \otimes C$.

Proof. The proof is a straightforward extension of Theorem 4, where IOs where considered instead. The key difference here is that the coherence cost of arbitrary states under MIOs is given by the relative entropy of coherence [71], whereas under IOs this is true only for pure states.

We first prove the upper bound: for any coherence-generating protocol, the trace-distance between the final state ρ_n and the cosdit can be upper bounded as $||\rho_n - \Psi_2^{\otimes nR}||_1 \leq 2\sqrt{\epsilon}$. Then the asymptotic continuity of the relative-entropy of coherence [65, Lemma 12] implies that:

$$\left| C_r(\Psi_2^{\otimes nR}) - C_r(\rho_n) \right| \leq 2\sqrt{\epsilon}\, nR + 2h(\sqrt{\epsilon}), \tag{5.43}$$

where $h(x) = -x\log x - (1-x)\log(1-x)$ is the binary-entropy function. Hence we obtain the following chain of inequalities:

$$\begin{aligned} nR(1 - 2\sqrt{\epsilon}) \ - 2h(\sqrt{\epsilon}) &\leq C_r(\rho_n) \\ &= \sum_{j=0}^{n-1} \left(C_r(\rho_{j+1}) - C_r(\rho_j)\right) && (5.44) \\ &\leq n \sup_{\rho \in AC} \left(C_r((\mathcal{N} \otimes \mathrm{id})(\rho)) - C_r(\rho)\right), \end{aligned}$$

where the first inequality follows from Eq. (5.43) and $C_r(\Psi_2^{\otimes nR}) = nR$, while the equality by adding and subtracting the relative-entropy of coherence of each intermediate state of the protocol. The last inequality instead follows from substituting each term of the sum with its sup over all states and the monotonicity of C_r under MIOs. After dividing both sides by n and taking the $n \to \infty$ limit we obtain that the rate of any coherence-generating protocol is upper-bounded by $\sup_C \widehat{P}^{(C)}_{C_r}(\mathcal{N})$ up to an error vanishing with ϵ.

For the lower bound instead we need to exhibit a protocol that asymptotically attains $\sup_C \widehat{P}^{(C)}_{C_r}(\mathcal{N})$. The protocol works as follows: i) we first apply the channel to an incoherent state $|0\rangle\langle 0|$ in order to produce some coherence, i.e., $\sigma = \mathcal{N}(|0\rangle\langle 0|)$; ii) we produce a sufficient number of copies of σ to distill a certain amount of cosbits, which are then used to produce a target state ρ via coherence dilution; iii) we apply the channel to ρ in order to obtain a more coherent state $\rho' = (\mathcal{N} \otimes \mathrm{id})(\rho)$; iv) we distill cosbits from ρ'; v) we use the increased amount of cosbits obtained to iterate the processes (iii-iv) k times. The asymptotic coherence-generating capacity and cost of states under MIOs are both equal to the relative-entropy of coherence [71], so that the conversion rates of processes (ii-iv) described above can be written as

$$\begin{aligned} \sigma^{\otimes m} &\mapsto \Psi_2^{\otimes \lfloor m(C_r(\sigma)-\delta)\rfloor}, \\ \Psi_2^{\otimes \lceil n(C_r(\rho)+\delta)\rceil} &\mapsto \rho^{\otimes n} \\ \rho'^{\otimes n} &\mapsto \Psi_2^{\otimes \lfloor n(C_r(\rho')-\delta)\rfloor}, \end{aligned} \tag{5.45}$$

where $m \gtrsim n(C_r(\rho)+\delta)/(C_r(\sigma)-\delta)$. Note that the coherence production of processes (iii-iv) is not larger than $n(R-2\delta)$, where $R = C_r(\rho') - C_r(\rho)$, and each transformation is accurate up to an error ϵ. Hence, the overall coherence-production rate is bounded by

$$\frac{kn(R-2\delta) + n(C_r(\phi)+\delta)}{kn+m} \xrightarrow[k\to\infty]{} R - 2\delta, \tag{5.46}$$

which can be made arbitrarily close to R by taking δ sufficiently small. We have therefore described a protocol that generates coherence at an asymptotical rate $C_r((N \otimes \mathrm{id})(\phi)) - C_r(\phi)$ for any ϕ_{AC} using MIOs. By taking the sup of this quantity, we obtain the desired lower bound. □

The coherence-generating capacity, despite its nice information theoretic formula, is by no means an easy quantity to compute. Notwithstanding the explicit expression for the relative entropy of coherence, the maximization over the state ρ_{AC} is not necessarily well-behaved. Also, we do not know how large an auxiliary system C is required, or indeed if any at all.

On the other hand, the asymptotic cost of implementing the channel can be expressed using our above results, in particular Theorem 9. Namely, it comes down to

$$C_{\text{sim}}^{\infty}(\mathcal{N}) = \sup_{\epsilon>0} \limsup_{n\to\infty} \frac{1}{n} \log(1 + C_R^{\epsilon}(\mathcal{N}^{\otimes n})). \tag{5.47}$$

Since both quantities—the asymptotic simulation cost and the asymptotic generation capacity—are not easily computable (and since the results of [58] cannot be directly applied to general channel resource theories), it remains to be known whether they coincide, i.e. whether the theory is asymptotically reversible. In general it holds $C_{\text{gen}}^{\infty} \leq C_{\text{sim}}^{\infty}$. There is one special case in which we know what this limit is, namely when for all ρ, $\mathcal{N}(\rho) = \sigma$, i.e. $\mathcal{N}$ is a constant channel. Then, $C_R(\mathcal{N}) = C_R(\sigma)$, and the right hand side of Eq. (5.47) converges [58] to the relative entropy of coherence, $C_r(\sigma)$. More generally, for a cq-channel $\mathcal{N}(|i\rangle\langle j|) = \delta_{ij}\sigma_i$, written in the incoherent basis of the input state, the same reasoning yields

$$\begin{aligned} C_{\text{sim}}^{\infty}(\mathcal{N}) &= C_{\text{gen}}^{\infty}(\mathcal{N}) \\ &= \max_i C_r(\mathcal{N}(|i\rangle\langle i|)), \end{aligned}$$

i.e., the theory is asymptotically reversible when restricted to cq-channels.

Regarding the existence of bound coherence, we cannot ensure whether operations exhibit it in MIO-theory. It is clear that cq-channels $\mathcal{N}$ do not have bound coherence, since for them $C_{\text{gen}}^{\infty}(\mathcal{N}) = 0$ implies $C_{\text{sim}}^{\infty}(\mathcal{N})$. Moreover, for a MIO T there is no bound coherence either, unlike what happened when we considered IOs as free operations (see Section 4.4). It remains an open question whether, in general, the MIO-theory of operations is endowed with bound coherence.

5.5 Chapter summary

- The robustness of coherence of channels determines the single-shot simulation cost of a channel.
- Among qubit resource states, a cosbit is necessary and sufficient to implement any coherent qubit unitary.
- The log-robustness equals the cohering power of the channel and it characterizes the amortized implementation cost with recycling of coherence at the output, as well as the asymptotic cost of realizing exactly many independent instances of the channel.
- MIOs, unlike IOs, allow for coherence recycling.
- A semidefinite program can be used to find the best approximation to any given channel using MIOs and an arbitrary resource.
- Any coherent state in dimension larger than 2, however weakly so, is useful for the exact implementation of some coherent unitary channel.

5.6 Open questions

- Can a closed expression for the asymptotic implementation cost of channels be obtained from the smoothed robustness of coherence?
- Can the coherence left after the approximate implementation of a channel with an arbitrary input resource be formulated as an efficient optimization problem?
- Is the theory reversible, i.e. $C^{\infty}_{\text{gen}}(\mathcal{N}) = C^{\infty}_{\text{sim}}(\mathcal{N})$, for every channel $\mathcal{N}$? We know it is, at least, for cq-channels.
- Is there bound coherence in the MIO-theory of channels? We know that, at least, for cq-channels and MIOs there is no bound coherence.

5.7 Comparison between IO-theory and MIO-theory

	IO	MIO
states	$C_d^\infty(\rho) = C_r(\rho)$	idem
ρ	$C_c^\infty(\rho) = C_f(\rho)$	$C_c^\infty(\rho) = C_r(\rho)$
	$\nexists$ reversibility	$\exists$ reversibility
	$\nexists$ bound coherence	idem
operations	$C_{\text{sim}}^\epsilon(\mathcal{N}) =?$	$C_{\text{sim}}^\epsilon(\mathcal{N}) = \log\lceil 1 + C_R^\epsilon(\mathcal{N})\rceil$
$\mathcal{N}: A \to B$	implementation of $2d$-unitary only w. cosbit	idem
	recycling not always allowed	$C_{\text{amo}}^\epsilon(\mathcal{N}) = C_{LR}^\epsilon(\mathcal{N})$
	$\sup_k \widetilde{P}_{C_r}^{(k)}(\mathcal{N}) \leq C_{\text{gen}}^\infty(\mathcal{N}) \leq \sup_k \widehat{P}_{C_r}^{(k)}(\mathcal{N})$	$C_{\text{gen}}^\infty(\mathcal{N}) = \sup_k \widehat{P}_{C_r}^{(k)}(\mathcal{N})$
	$C_{\text{gen}}^\infty(\mathcal{N}) \leq C_{\text{sim}}^\infty(\mathcal{N}) \leq \log\lvert B\rvert$	$C_{\text{sim}}^\infty(\mathcal{N}) = \sup_{\epsilon>0} \limsup_{n\to\infty} \frac{1}{n} \log(1 + C_R^\epsilon(\mathcal{N}^{\otimes n}))$
	$\exists$ bound coherence	$\exists$ bound coherence ?
	$\nexists$ reversibility	$\exists$ reversibility?

Part III

Quantum coherence and non-classicality

CHAPTER 6

Some preliminary notions

"It troubled me as once I was —
For I was once a Child —
Concluding how an Atom — fell —
And yet the Heavens — held —"

E. Dickinson,
"Poem 600", Complete Poems

So far we have understood that a system is quantum or non-classical when it presents coherence, i.e. it leads to interference phenomena when subjected to an interference experiment in the spirit of Fig. 2.2, where there exists an impossibility to know the path that the system will take. In this respect, it must be pointed out that the system's size should not condition us to think how it will behave: indeed, not only was quantumness demonstrated for electrons [7], but also for composites of up to 430 atoms [123]. Here non-classicality is only associated with superposition or the ability to produce interference, so we should not be misled by the size of the system at hand. Indeed, physicists are still in the quest for the frontier between the classical and quantum realms in terms of a system's proportions.

However, there exist alternative ways in which non-classicality can manifest itself. For instance, in the context of quantum optics, where states of light can be written as $\rho = \int d^2\alpha P(\alpha)|\alpha\rangle\langle\alpha|$ (where $|\alpha\rangle$ are coherent states and $P(\alpha)$ is the Glauber-Sudarshan P distribution), a system is considered non-classical if its corresponding P distribution displays negativities [124]. Yet another kind of non-classicality is present in the statistics obtained when devising proper measurement procedures on entangled systems, typically involving the measurement of non-local or non-commuting observables [125]. Moreover, sequencial measurements of one and the same local observable at different times yield stochastic processes whose potential non-classicality can also be singled out [126–138]. In a few words, although every shade of non-classicality eventually characterizes how our daily world cannot behave, it is important to clarify what we mean by non-classicality before assigning this property to a given phenomenon.

In Chapter 2 we showed how coherence was linked to non-classicality in the sense of interference phenomena. The question we would now like to ask is, how is coherence related to other notions of non-classicality? For example, in [139] they investigated the connection between coherence and non-classicality in the realm of quantum optics, where non-classicality arises from a P distribution presenting negativities.

The aim of the present chapter is to study how coherence is related to the non-classicality of stochastic processes. For that purpose, we will first need to know a bit about open quantum systems, non-coherence-generating-and-detecting (NCGD) dynamics, classical stochastic processes and the Leggett-Garg inequalities.

6.1 Open quantum systems

A realistic description of a quantum system must take into account that every system is open, i.e., it interacts with the surrounding environment [140]. In many circumstances, it is possible to provide such a description by a time-local quantum master equation (QME):

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_{\mathrm{s}}(t) = \mathcal{L}(t)[\rho_{\mathrm{s}}(t)], \tag{6.1}$$

where $\mathcal{L}(t)$ is the *dynamical generator* of the evolution and ρ_{s} the reduced state of the system. Any $\mathcal{L}(t)$ that is both trace- and hermiticity-preserving can be uniquely decomposed as [141]

$$\begin{aligned}\mathcal{L}(t)[\rho_{\mathrm{s}}] = &-\mathrm{i}[H(t), \rho_{\mathrm{s}}] \\ &+ \sum_{i,j=1}^{d^2-1} D_{ij}(t)\Big(F_i\rho_{\mathrm{s}}F_j^\dagger - \frac{1}{2}\{F_j^\dagger F_i, \rho_{\mathrm{s}}\}\Big),\end{aligned} \tag{6.2}$$

where $d < \infty$ is the dimension of the Hilbert space of the system, $H(t)$ a Hermitian operator, $D(t)$ a Hermitian matrix, and $\{F_i\}_{i=1}^{d^2}$ is an orthonormal operator basis with $F_{d^2} = \mathbb{1}/\sqrt{d}$ and $\mathrm{Tr}(F_i^\dagger F_j) = \delta_{ij}$.

Upon integration, the QME leads to a family of trace-preserving (TP) propagators $\mathcal{E}_{t_2,t_1}$ satisfying $\rho_{\rm s}(t_2) = \mathcal{E}_{t_2,t_1}[\rho_{\rm s}(t_1)]\ \forall t_2 \geq t_1 \geq 0$. If the generator $\mathcal{L}(t)$ is of Gorini-Kossakowski-Sudarshan–Lindblad (GKSL) form [141, 142] for all $t \geq 0$, i.e., $D(t) \geq 0$, all propagators are completely positive (CP). In particular, we will first address dynamics for which H and D are time-independent, with $D \geq 0$. This class—which we refer to as *Lindblad dynamics* henceforth—gives rise to CP quantum dynamical semigroups of the form $\mathcal{E}_t = {\rm e}^{t\mathcal{L}}$, where $\mathcal{L}(t) \equiv \mathcal{L}$ is time-independent. Finally we shall refer to *rank-k noise* as those dynamical generators $\mathcal{L}(t)$ in Eq. (6.2) for which D has at most k non-zero eigenvalues.

6.2 Non-coherence-generating-and-detecting (NCGD) dynamics

In Part II we characterized and quantified the coherence generating capabilities of quantum operations. However, the mere ability of a quantum dynamics to generate or detect coherence is of no practical advantage unless this coherence can be harnessed in a beneficial way for some task [104]. In this sense, a prerequisite for a quantum dynamical evolution to generate *resourceful* coherence for a given task is that the coherence it generates can be detected in terms of discriminable statistics of subsequent measurement outcomes associated with this task. Therefore, what we are interested in are the coherence-generating-and-detecting properties of the propagators $\{\mathcal{E}_{t_2,t_1} : t_2 \geq t_1 \geq 0\}$ associated with the dynamics. From this perspective, we define non-coherence-generating-and-detecting dynamics (NCGD) dynamics as follows:

Definition 22. *A dynamics with propagator $\mathcal{E}_{t,s}$, $t \geq s \geq 0$, is NCGD whenever the condition*

$$\Delta \circ \mathcal{E}_{t_3,t_2} \circ \Delta \circ \mathcal{E}_{t_2,t_1} \circ \Delta = \Delta \circ \mathcal{E}_{t_3,t_1} \circ \Delta \tag{6.3}$$

holds for all times $t_3 \geq t_2 \geq t_1 \geq 0$, where $\circ$ denotes composition of maps and $\Delta = \sum_{i=0}^{d-1} |i\rangle\langle i| \cdot |i\rangle\langle i|$ is the complete-dephasing map in the incoherent basis $\{|i\rangle\}_{i=0}^{d-1}$.
Otherwise the dynamics is denoted as CGD.

NCGD dynamics are unable to exploit coherence for delivering a practical advantage and hence can be regarded as free operations. We stress, however, that the class of NCGD dynamical maps is not closed under composition: indeed, it is very easy to obtain CGD dynamics by composing two NCGD dynamical maps; the first coherence-generating but not detecting whilst the second coherence-detecting but not generating.

To compare the notion of NCGD with the literature, we take for a moment $\mathcal{E}_{t_3,t_2} = \mathcal{E}_{t_2,t_1} \equiv \mathcal{E}(t)$, for a fixed $t = \tau$, so that the definition in Eq.(6.3) refers to one map, $\mathcal{E} \equiv \mathcal{E}(\tau)$, which we call NCGD map. There are two interesting subsets of NCGD maps. One is the subset that does not create coherence from incoherent states, which is described by $\Delta \circ \mathcal{E} \circ \Delta = \mathcal{E} \circ \Delta$; this is just the set of MIOs. The other noteworthy subset of NCGD maps is the coherence non-activating set fixed by $\Delta \circ \mathcal{E} \circ \Delta = \Delta \circ \mathcal{E}$; here, since the populations are independent of the initial coherence, the coherence is not a useable resource [64]. Finally, operations that are neither incoherent nor coherence nonactivating may still be NCGD, if the subspaces where coherence is generated are different from the ones detecting it. For instance, consider the completely positive and trace preserving map Λ acting on a basis of linear operators on $\mathbb{C}^2$ as

$$\Lambda : \left\langle \begin{matrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{matrix} \right\rangle \mapsto \begin{matrix} \begin{pmatrix} 0.996 & -0.003i \\ 0.003i & 0.004 \end{pmatrix} & \begin{pmatrix} 0.003 & 0.99 \\ 0 & -0.003 \end{pmatrix} \\ \begin{pmatrix} 0.003 & 0 \\ 0.99 & -0.003 \end{pmatrix} & \begin{pmatrix} 0.004 & 0.003i \\ -0.003i & 0.996 \end{pmatrix} \end{matrix} . \tag{6.4}$$

The map is NCGD, while it both creates coherence and also is able to detect it. Explicitly:

$$\|\Delta \circ \Lambda \circ \Lambda \circ \Delta - \Delta \circ \Lambda \circ \Delta \circ \Lambda \circ \Delta\|_\infty = 0, \tag{6.5}$$

$$\|\Lambda \circ \Delta - \Delta \circ \Lambda \circ \Delta\|_\infty = 0.003, \tag{6.6}$$

$$\|\Delta \circ \Lambda - \Delta \circ \Lambda \circ \Delta\|_\infty = 0.003, \tag{6.7}$$

where $\|\Lambda\|_\infty$ denotes the infinity norm of the 4×4 matrix given by the action of Λ on the basis of operators on $\mathbb{C}^2$; recall that the infinity norm is the maximum among the absolute sums of the columns. Indeed, the same is true for applying the map multiple times: the NCGD condition remains fulfilled, while the above norm increases to over 0.12, as shown in Fig. 6.1.

6.3 Multi-time probabilities and classicality

Suppose that we perform projective measurements of a system's observable $\hat{X}_s = \sum_x x|\psi_x\rangle\langle\psi_x|$ at times $t_n \geq t_{n-1} \geq \ldots \geq t_1$ ($n \in \mathbb{N}$), with discrete outcomes x. Denote the joint probability of obtaining the sequence of outcomes

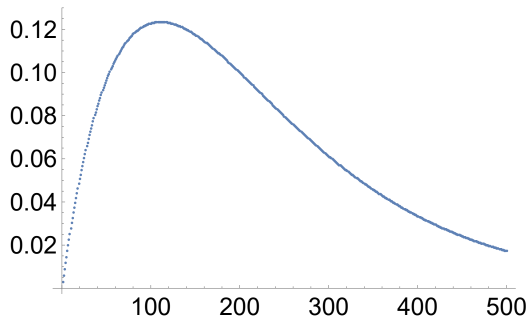


Figure 6.1: Norm of the difference between the coherent map in Eq. (6.4) and the incoherent one (see Eq. (6.6)) as a function of the number of applications. In this example this is the same as the norm of the difference between the coherence non-activating and the actual evolution (see Eq. (6.7)).

x_1 at time t_1, x_2 at time t_2, ..., and x_n at time t_n as

$$Q_n^{\hat{X}_s}\{x_n, t_n; \ldots; x_1, t_1\}. \tag{6.8}$$

Keeping in mind that the whole hierarchy of probabilities cannot be reconstructed practically, as one always deals with a certain finite number of outcomes, we propose the following definition:

Definition 23. *A multi-time statistics is j-Markovian if the corresponding joint probability distribution fulfills the quantum regression theorem (QRT) for any $n \leq j$, $x_1, \ldots x_n$, $t_n \geq \ldots t_1$. The QRT, when associated with projective measurements, reads [143, 144]*

$$\begin{aligned} &Q_n^{\hat{X}_s}\{x_n, t_n; \ldots; x_1, t_1\} \\ &\quad = \mathrm{Tr}_s\Big\{\mathcal{P}_{x_n}\mathcal{E}_{t_n, t_{n-1}} \cdots \mathcal{P}_{x_1}\mathcal{E}_{t_1}\rho_s(0)\Big\}, \end{aligned} \tag{6.9}$$

where $\mathcal{P}_x = |\psi_x\rangle\langle\psi_x| \cdot |\psi_x\rangle\langle\psi_x|$ and $\mathcal{E}_{t,s}$ are the propagators of the open-system dynamics.

A multi-time statistics, as the one in Eq. (6.8), can be traced back to a proper definition of quantum stochastic processes, as introduced in [145, 146] and most recently investigated by means of the so-called comb formalism [147] in [148]. In particular, by the Kolmogorov-Daniell extension theorem, any multi-time statistics can be reproduced by a classical stochastic process if and only if the Kolmogorov consistency conditions are fulfilled [140, 149]:

$$\begin{aligned} &\sum_{x_k} Q_n^{\hat{X}_s}\{x_n, t_n; \ldots; x_1, t_1\} \\ &\quad = Q_{n-1}^{\hat{X}_s}\{x_n, t_n; \ldots; \not{x_k}, \not{t_k}; \ldots; x_1, t_1\}, \end{aligned} \tag{6.10}$$

$\forall k \leq n$, $n > 1$; $\forall t_n \geq \cdots \geq \cdots t_1 \geq 0$; $\forall x_1, \ldots, x_n$.

Again, since practically one can only work with a finite number of outcomes, we propose the following definition:

Definition 24. *A j-classical (jCL) multi-time statistics is a collection of joint probability distributions* $Q_n^{\hat{X}_s}\{x_n, t_n; \ldots ; x_1, t_1\}$ *fulfilling the Kolmogorov conditions in Eq.*(6.10) *for any* $n \leq j$. *We say that it is non-classical if it is not even 2CL.*

6.4 Leggett-Garg (type) inequalitites

Two distinct features of macroscopic reality are remarkable: on the one hand, the properties of the systems have preexisting values, no matter whether they are being measured or not. On the other hand, non-invasive measurability is possible at the macroscopic realm, meaning that measurements are performed on systems without disturbing them. Leggett-Garg inequalities (LGIs) [126], derived from these macroscopic principles, are violated by quantum mechanics and thus reveal the presence of quantum features in a system's evolution. Leggett and Garg initially proposed an rf-SQUID flux (a type of superconducting qubit) as a system on which their inequalities could be tested [126]. Through an experiment that differed from the Leggett-Garg proposal in a number of respects, Palacios-Laloy *et al* announced the first measured violation of LGI twenty-five years later [130]. Let us now explain how the LGIs look like.

Consider an experimental setup similar to the one proposed so far. Here, a dichotomic observable $\hat{X}$ with values in $\{-1, 1\}$ is measured at n different times. The experimental correlations between the obtained outcomes are then computed:

$$C_X(t_j, t_i) = \sum_{x_j, x_i} Q_2^{\hat{X}} \{x_j, t_j; x_i, t_i\}\, x_j x_i.$$

The nth-order Leggett-Garg function K_n is built in the following manner:

$$K_n \equiv C_X(t_2, t_1) + C_X(t_3, t_2) + \ldots + C_X(t_n, t_{n-1}) - C_X(t_n, t_1)$$

It can be then shown that, under the assumptions of MR, the following inequality is fulfilled:

$$K_n \leq n - 2$$

Violation of these Leggett-Garg inequalities, also referred to as temporal Bell inequalities due to the fact that measurements are separated in time but not in space, implies that MR must be abandoned. LGI are therefore useful as an indicator of quantumness.

Taking into account that performing perfectly non-invasive measurements in the lab is a hard job, in [128, 135] they formulated the Leggett-Garg type inequalities (LGtIs), in which the Leggett-Garg non-invasiveness requirement is replaced by an assumption which turns out to be related to Markovianity [135]. Given a dichotomic observable $\hat{X}$ with values in $\{0, 1\}$ and the related correlation function, the LGtI we will consider here reads $|2C_X(t, 0) - C_X(2t, 0)| \leq \langle X(0) \rangle$, with $\langle X(0) \rangle$ the expectation value of $\hat{X}$ at the initial time.

CHAPTER 7

The Lindblad case

As in the previous chapter, here we consider a system that evolves according to a certain evolution, on which we measure an observable projectively at different times. Our purpose is to analyze how the non-classicality of the arising stochastic process can be connected to the coherence of the underlying dynamics, specifically when such a dynamics is of Lindblad type.

Let us start by studying how NCGD Lindblad evolutions look like, as far as their structure is concerned.

7.1 Characterizing NCGD dynamics

In order to check whether or not a given dynamics is NCGD we would, in principle, need to verify Eq. (6.3) for all times $t_3 \geq t_2 \geq t_1$. In this section we provide a finite set of necessary and sufficient conditions certifying the coherence generating and detecting properties of Lindblad dynamical evolutions. More importantly, these conditions pertain directly to the generator of the dynamics in Eq. (6.1) which allows for direct evaluation of the coherence properties of such dynamical evolutions.

Recall that $\mathcal{L}(t)\colon \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ for all times t. Having fixed the complete dephasing superoperator Δ, we can decompose $\mathcal{B}(\mathcal{H})$ into two orthogonal subspaces

$$\mathcal{B}(\mathcal{H}) = \mathcal{B}_{\mathrm{p}}(\mathcal{H}) \oplus \mathcal{B}_{\mathrm{c}}(\mathcal{H}), \tag{7.1a}$$

where

$$\mathcal{B}_{\mathrm{p}}(\mathcal{H}) = \mathrm{Image}(\Delta), \tag{7.1b}$$

$$\mathcal{B}_{\mathrm{c}}(\mathcal{H}) = \mathrm{Kernel}(\Delta), \tag{7.1c}$$

are the subspaces associated with the population and coherence basis elements respectively. In this basis, the matrix representation of the generator $\mathcal{L}(t)$ is given by

$$\mathcal{L}(t) = \begin{pmatrix} \mathcal{L}_{\mathrm{pp}}(t) & \mathcal{L}_{\mathrm{pc}}(t) \\ \mathcal{L}_{\mathrm{cp}}(t) & \mathcal{L}_{\mathrm{cc}}(t) \end{pmatrix}, \tag{7.1d}$$

where, for example, $\mathcal{L}_{\mathrm{pc}} \colon \mathcal{B}_{\mathrm{c}}(\mathcal{H}) \to \mathcal{B}_{\mathrm{p}}(\mathcal{H})$.

We are now ready to formulate our main result, which is a complete characterization of NCGD based solely on the generator of a Lindblad dynamics.

Theorem 25. *For any Lindblad-dynamics generator $\mathcal{L}$ it holds that*

$$\mathit{NCGD} \Leftrightarrow \left(\mathcal{L}_{\mathrm{pc}}\mathcal{L}_{\mathrm{cc}}^{j}\mathcal{L}_{\mathrm{cp}} = 0 \ \forall j \in \{0, \dots, d^2 - d - 1\}\right),$$

where $d = \dim \mathcal{H}$.

Proof. Let us begin by pointing out some general remarks: In the time-independent case, $\mathcal{E}_t = \mathrm{e}^{t\mathcal{L}}$, we can write NCGD as

$$\Delta \mathcal{E}_t \Delta^{\perp} \mathcal{E}_\tau \Delta = 0 \quad \forall t, \tau \geq 0, \tag{7.2}$$

where we defined $\Delta^{\perp} := \mathbb{1} - \Delta$. Expanding the exponential, this is equivalent to

$$\Delta \sum_{n=0}^{\infty} \frac{\mathcal{L}^n t^n}{n!} \Delta^{\perp} \sum_{n'=0}^{\infty} \frac{\mathcal{L}^{n'} \tau^{n'}}{n'!} \Delta = 0 \ \forall t, \tau \geq 0$$
$$\Leftrightarrow \Delta \mathcal{L}^n \Delta^{\perp} \mathcal{L}^{n'} \Delta = 0 \ \forall n, n' \in \mathbb{N}_0. \tag{7.3}$$

Note that for finite-dimensional $\mathcal{H}$ the following equivalence holds:

$$\left(\mathcal{L}_{\mathrm{pc}}\mathcal{L}_{\mathrm{cc}}^{j}\mathcal{L}_{\mathrm{cp}} = 0 \ \forall j \in \left\{0, \dots, \ell - 1\right\}\right) \Leftrightarrow \left(\mathcal{L}_{\mathrm{pc}}\mathcal{L}_{\mathrm{cc}}^{j}\mathcal{L}_{\mathrm{cp}} = 0 \ \forall j \in \mathbb{N}_0\right), \tag{7.4}$$

where $\ell = \dim \mathcal{B}_{\mathrm{c}}(\mathcal{H}) = \dim^2 \mathcal{H} - \dim \mathcal{H} = d^2 - d$. While "$\Leftarrow$" is trivial, by Cayley–Hamilton [150], there are coefficients $\{\alpha_i\}_{i=0}^{\ell} \subset \mathbb{C}$, such that

$$\sum_{i=0}^{\ell-1} \alpha_i \mathcal{L}_{\mathrm{cc}}^{i} = 0. \tag{7.5}$$

Hence any power of $\mathcal{L}_{\mathrm{cc}}$ greater or equal to ℓ can be expressed as a linear combination of powers from 0 to $\ell - 1$. All conditions formulated in the following that rely on infinite matrix powers are therefore already fulfilled if they hold up to the $(\ell-1)^{\text{th}}$ power.

The following structure lemma will be needed henceforth (see proof in Appendix D.1):

Lemma 26. *If* $\mathcal{L}_{\text{pc}}\mathcal{L}_{\text{cc}}^{j}\mathcal{L}_{\text{cp}} = 0$ *for all* $j < n$,

$$\Delta\mathcal{L}^n = \begin{pmatrix} \mathcal{L}_{\text{pp}}^n & \sum_{j=1}^{n} \mathcal{L}_{\text{pp}}^{j-1}\mathcal{L}_{\text{pc}}\mathcal{L}_{\text{cc}}^{n-j} \\ 0 & 0 \end{pmatrix} \quad \textit{and} \quad \mathcal{L}^n\Delta = \begin{pmatrix} \mathcal{L}_{\text{pp}}^n & 0 \\ \sum_{j=1}^{n} \mathcal{L}_{\text{cc}}^{j-1}\mathcal{L}_{\text{cp}}\mathcal{L}_{\text{pp}}^{n-j} & 0 \end{pmatrix}. \tag{7.6}$$

We are now fully equipped to prove Theorem 25. We will first prove

$$\left(\mathcal{L}_{\text{pc}}\mathcal{L}_{\text{cc}}^{j}\mathcal{L}_{\text{cp}} = 0 \;\forall j \in \mathbb{N}_0\right) \Rightarrow \text{NCGD}. \tag{7.7}$$

We directly apply the structure lemma, eq. (7.6), to eq. (7.3). The intermediate $\Delta^{\perp}$ removes the population-to-population entry. Hence, we obtain

$$\begin{aligned} \Delta\mathcal{L}^n\Delta^{\perp}\mathcal{L}^{n'}\Delta &= \begin{pmatrix} 0 & \sum_{j=1}^{n} \mathcal{L}_{\text{pp}}^{j-1}\mathcal{L}_{\text{pc}}\mathcal{L}_{\text{cc}}^{n-j} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \sum_{j'=1}^{n'} \mathcal{L}_{\text{cc}}^{j'-1}\mathcal{L}_{\text{cp}}\mathcal{L}_{\text{pp}}^{n'-j'} & 0 \end{pmatrix} \\ &= \begin{pmatrix} \sum_{j=1}^{n}\sum_{j'=1}^{n'} \mathcal{L}_{\text{pp}}^{j-1}\mathcal{L}_{\text{pc}}\mathcal{L}_{\text{cc}}^{n-j}\mathcal{L}_{\text{cc}}^{j'-1}\mathcal{L}_{\text{cp}}\mathcal{L}_{\text{pp}}^{n'-j'} & 0 \\ 0 & 0 \end{pmatrix}, \end{aligned} \tag{7.8}$$

and we clearly see another $\mathcal{L}_{\text{pc}}\mathcal{L}_{\text{cc}}^{\cdots}\mathcal{L}_{\text{cp}}$ combination, which is zero by the assumption, eq. (7.7).

We will now prove

$$\text{NCGD} \Rightarrow \left(\mathcal{L}_{\text{pc}}\mathcal{L}_{\text{cc}}^{j}\mathcal{L}_{\text{cp}} = 0 \;\forall j \in \mathbb{N}_0\right). \tag{7.9}$$

By assumption,

$$\Delta\mathcal{L}^n\Delta^{\perp}\mathcal{L}\Delta = 0 \;\forall n \in \mathbb{N}_0, \tag{7.10}$$

where we fixed $n' = 1$.

Using induction we will show that if $\mathcal{L}_{\text{pc}}\mathcal{L}_{\text{cc}}^{j}\mathcal{L}_{\text{cp}} = 0$ holds for all $j < n$, then it holds for $j \leq n$. Since this can be done for any $n \in \mathbb{N}_0$ and the case for $j = 0$, $\mathcal{L}_{\text{pc}}\mathcal{L}_{\text{cp}} = 0$, is implied by eq. (7.10) for $n = 1$, the thesis follows.

By hypothesis, $\mathcal{L}_{\text{pc}}\mathcal{L}_{\text{cc}}^{j}\mathcal{L}_{\text{cp}} = 0 \;\forall j < n$. The structure lemma, eq. (7.6), therefore applies and hence [1]

$$\Delta\mathcal{L}^{n+1}\Delta^{\perp} = \begin{pmatrix} 0 & \sum_{j=1}^{n+1} \mathcal{L}_{\text{pp}}^{j-1}\mathcal{L}_{\text{pc}}\mathcal{L}_{\text{cc}}^{n+1-j} \\ 0 & 0 \end{pmatrix},$$

[1] The first identity directly uses the proof of the structure lemma; since we only regard the right column, the lemma also holds for $n+1$.

so that

$$\Delta\mathcal{L}^{n+1}\Delta^{\perp}\mathcal{L}\Delta = \begin{pmatrix} 0 & \sum_{j=1}^{n+1}\mathcal{L}_{\mathrm{pp}}^{j-1}\mathcal{L}_{\mathrm{pc}}\mathcal{L}_{\mathrm{cc}}^{n+1-j} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \mathcal{L}_{\mathrm{cp}} & 0 \end{pmatrix}$$
$$= \begin{pmatrix} \sum_{j=1}^{n+1}\mathcal{L}_{\mathrm{pp}}^{j-1}\mathcal{L}_{\mathrm{pc}}\mathcal{L}_{\mathrm{cc}}^{n+1-j}\mathcal{L}_{\mathrm{cp}} & 0 \\ 0 & 0 \end{pmatrix},$$

and we can again insert the hypothesis to eliminate all terms except $j = 1$.

$$\Delta\mathcal{L}^{n+1}\Delta^{\perp}\mathcal{L}\Delta = \begin{pmatrix} \mathcal{L}_{\mathrm{pc}}\mathcal{L}_{\mathrm{cc}}^{n}\mathcal{L}_{\mathrm{cp}} & 0 \\ 0 & 0 \end{pmatrix};$$

but this, by the assumption of NCGD must be zero, verifying the hypothesis.
□

As stated by this theorem, only $d^2 - d$ conditions need to be verified on the generator of a Lindblad dynamics in order to check whether such an evolution is NCGD. We do not know at this point whether this number of conditions could be reduced even more, but we do know that it must be larger than 2, as shown by the following example:

Consider a 5-level system where coherence is generated, but not detected, between levels 1 and 2 ($\mathcal{L}_{\mathrm{pc}} = 0$, $\mathcal{L}_{\mathrm{cp}} \neq 0$), and where the opposite occurs between levels 4 and 5 ($\mathcal{L}_{\mathrm{cp}} = 0$, $\mathcal{L}_{\mathrm{pc}} \neq 0$). At a first instant, coherence is transferred to levels 1 and 3, and, at the next step, to levels 4 and 5, where it is eventually detected. Such a system is described by a rank-3 noise Lindbladian with Hamiltonian

$$H = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix} \tag{7.11}$$

and jump operators

$$J_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -\mathrm{i} & 0 & 0 & 0 \\ \mathrm{i} & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\mathrm{i} \\ 0 & 0 & 0 & \mathrm{i} & -1 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{7.12}$$

$$\text{and} \quad J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}. \tag{7.13}$$

As expected, such a dynamics is NCGD only up to third order in time, since it can be checked that $\mathcal{L}_{\rm pc}\mathcal{L}_{\rm cp} = 0$ and $\mathcal{L}_{\rm pc}\mathcal{L}_{\rm cc}\mathcal{L}_{\rm cp} = 0$, but $\mathcal{L}_{\rm pc}\mathcal{L}_{\rm cc}^2\mathcal{L}_{\rm cp} \neq 0$. This shows that the fact that the NCGD conditions are fulfilled for $j = 0, 1$ is not enough to ensure that an evolution will be NCGD. One more condition, at least ($j = 2$), has to be verified in order to certify the CGD character of the evolution.

7.1.1 (N)CGD dynamics for qubits

In this section, we will apply Theorem 25 to the special case of qubit dynamics. This will allow us to explicitly give the structure of NCGD dynamics for the case of time-independent qubit generators. Eq. (6.2) in the normalized Pauli operator basis $\{\sigma_i : i = 0, \ldots, 3\}$ can be easily rewritten as

$$\mathcal{L}[\rho_{\rm s}] = \frac{1}{2}\sum_{i,j=0}^{3} \mathsf{L}_{ij}\Big([\sigma_i\rho_{\rm s}, \sigma_j] + [\sigma_i, \rho_{\rm s}\sigma_j]\Big), \tag{7.14}$$

where $\mathsf{L} \in \mathbb{C}^{4\times 4}$ is a Hermitian matrix. We will choose $(\mathbb{1}, \sigma_z)$ as our incoherent basis and (σ_x, σ_y) as the coherent one. With this choice, the matrix representation of Eq. (7.14) in the basis of Eq. (7.1d) is explicitly given by

$$\begin{aligned}
\mathcal{L}_{\rm pp} &= -\begin{pmatrix} 0 & 0 \\ 2\,{\rm Im}\,\mathsf{L}_{12} & \mathsf{L}_{11} + \mathsf{L}_{22} \end{pmatrix} \\
\mathcal{L}_{\rm pc} &= \begin{pmatrix} 0 & 0 \\ {\rm Re}\,\mathsf{L}_{13} - {\rm Im}\,\mathsf{L}_{02} & {\rm Re}\,\mathsf{L}_{23} + {\rm Im}\,\mathsf{L}_{01} \end{pmatrix} \\
\mathcal{L}_{\rm cp} &= \begin{pmatrix} -2\,{\rm Im}\,\mathsf{L}_{23} & {\rm Re}\,\mathsf{L}_{13} + {\rm Im}\,\mathsf{L}_{02} \\ 2\,{\rm Im}\,\mathsf{L}_{13} & {\rm Re}\,\mathsf{L}_{23} - {\rm Im}\,\mathsf{L}_{01} \end{pmatrix} \\
\mathcal{L}_{\rm cc} &= -\begin{pmatrix} \mathsf{L}_{22} + \mathsf{L}_{33} & {\rm Im}\,\mathsf{L}_{03} - {\rm Re}\,\mathsf{L}_{12} \\ -\,{\rm Re}\,\mathsf{L}_{12} - {\rm Im}\,\mathsf{L}_{03} & \mathsf{L}_{11} + \mathsf{L}_{33} \end{pmatrix}.
\end{aligned} \tag{7.15}$$

Theorem 25 states that NCGD is equivalent to

$$\mathcal{L}_{\rm pc}\mathcal{L}_{\rm cp} = \mathcal{L}_{\rm pc}\mathcal{L}_{\rm cc}\mathcal{L}_{\rm cp} = 0. \tag{7.16}$$

In particular, the dynamics is coherence non-activating, i.e., $\mathcal{L}_{\rm pc} = 0$, when

$${\rm Re}\,\mathsf{L}_{13} = {\rm Im}\,\mathsf{L}_{02} \quad \wedge \quad {\rm Re}\,\mathsf{L}_{23} = -\,{\rm Im}\,\mathsf{L}_{01}, \tag{7.17}$$

while it is coherence non-generating, i.e., $\mathcal{L}_{\rm cp} = 0$, when

$$\begin{aligned}
{\rm Re}\,\mathsf{L}_{13} = -\,{\rm Im}\,\mathsf{L}_{02} \quad &\wedge \quad {\rm Re}\,\mathsf{L}_{23} = {\rm Im}\,\mathsf{L}_{01} \\
&\wedge \quad {\rm Im}\,\mathsf{L}_{13} = {\rm Im}\,\mathsf{L}_{23} = 0.
\end{aligned} \tag{7.18}$$

Observe that both coherence non-activating and coherence non-generating dynamics can arise from the simplest open-systems dynamics, namely rank-one Pauli noise. For example, assuming that all contributions L_{0i} arise solely from the Hamiltonian of the system, the following rank-one dissipators $\bar{\mathsf{L}} \in \mathbb{C}^{3\times 3}$,

$$\begin{aligned} \bar{\mathsf{L}}_{\text{non-act.}} &= \boldsymbol{r}\boldsymbol{r}^\top, \quad \boldsymbol{r} = \begin{pmatrix} \operatorname{Im}\mathsf{L}_{02} & -\operatorname{Im}\mathsf{L}_{01} & 1 \end{pmatrix}^\top; \\ \bar{\mathsf{L}}_{\text{non-gen.}} &= \boldsymbol{s}\boldsymbol{s}^\top, \quad \boldsymbol{s} = \begin{pmatrix} -\operatorname{Im}\mathsf{L}_{02} & \operatorname{Im}\mathsf{L}_{01} & 1 \end{pmatrix}^\top, \end{aligned} \tag{7.19}$$

give rise to coherence non-activating and coherence non-generating dynamics respectively.

Note, however, that one can have dynamical evolutions that are capable of both generating and detecting coherence, and yet are still NCGD. This occurs whenever coherence is generated in an orthogonal subspace to the one where it is detected. In the case of qubits this happens precisely when (assuming for simplicity that the denominators involved are different from 0)

$$\frac{\operatorname{Im}\mathsf{L}_{13}}{\operatorname{Im}\mathsf{L}_{23}} = \frac{\operatorname{Re}\mathsf{L}_{13} - \operatorname{Im}\mathsf{L}_{02}}{\operatorname{Re}\mathsf{L}_{23} + \operatorname{Im}\mathsf{L}_{01}} = \frac{\operatorname{Im}\mathsf{L}_{01} - \operatorname{Re}\mathsf{L}_{23}}{\operatorname{Re}\mathsf{L}_{13} + \operatorname{Im}\mathsf{L}_{02}} \tag{7.20}$$

and

$$\begin{aligned} \mathsf{L}_{11} - \mathsf{L}_{22} = &\frac{(\operatorname{Im}\mathsf{L}_{02} - \operatorname{Re}\mathsf{L}_{13})(\operatorname{Im}\mathsf{L}_{03} - \operatorname{Re}\mathsf{L}_{12})}{\operatorname{Im}\mathsf{L}_{01} + \operatorname{Re}\mathsf{L}_{23}} + \\ &\frac{(\operatorname{Im}\mathsf{L}_{01} + \operatorname{Re}\mathsf{L}_{23})(\operatorname{Im}\mathsf{L}_{03} + \operatorname{Re}\mathsf{L}_{12})}{\operatorname{Im}\mathsf{L}_{02} - \operatorname{Re}\mathsf{L}_{13}}. \end{aligned} \tag{7.21}$$

Eq. (7.20) is equivalent to the first condition in Eq. (7.16), $\mathcal{L}_{\text{pc}}\mathcal{L}_{\text{cp}} = 0$; the precise relationship among several coefficients of the dynamical map ensures that coherence is generated in a subspace orthogonal to that of coherence detection. Likewise, Eq. (7.21) rules out the second-order coupling, $\mathcal{L}_{\text{pc}}\mathcal{L}_{\text{cc}}\mathcal{L}_{\text{cp}} = 0$.

Let us illustrate our findings with a concrete, and practically relevant physical example; the Ramsey scheme deployed in interferometry, spectroscopy and atomic clocks. The simplest, non-trivial case of such a scheme is that of rank-one Pauli noise in the same direction as the Hamiltonian evolution—assumed without loss of generality to be $H = \sigma_z$—whose dynamics is given by

$$\mathcal{L}[\rho_{\text{s}}] = -\mathrm{i}\omega[\sigma_z, \rho_{\text{s}}] + \gamma(\sigma_z\rho_{\text{s}}\sigma_z - \rho_{\text{s}}/2), \tag{7.22}$$

where ω is the detuning from the reference field. Note that due to the normalization of the Pauli matrices, $\sigma_z^2 = \frac{\mathbb{1}}{2}$. In the Ramsey scheme, the atoms—approximated as qubits—are first prepared in eigenstates of σ_x, then subjected to the evolution generated by Eq. (7.22), and subsequently measured in the eigenbasis of σ_x. Choosing $\mathcal{B}_\mathrm{p}(\mathcal{H}) = (\mathbb{1}, \sigma_x)$ as our incoherent basis, $\mathcal{B}_\mathrm{c}(\mathcal{H}) = (\sigma_y, \sigma_z)$, and using the matrix representation introduced in Eq. (7.1d), the generator of Eq. (7.22) can be written as

$$\mathcal{L} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\gamma & -\sqrt{2}\omega & 0 \\ 0 & \sqrt{2}\omega & -\gamma & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{7.23}$$

We can assess the CGD properties of such a setup by looking at the distance between the left- and right-hand sides of Eq. (6.3), as measured via the trace distance. Defining

$$\begin{aligned} p_\pm(t_3) &= \mathrm{Tr}\Big(|\pm\rangle\langle\pm|\, \mathcal{E}_{t_3,t_1} \circ \Delta[\rho_\mathrm{s}]\Big) \\ q_\pm(t_3, t_2) &= \mathrm{Tr}\Big(|\pm\rangle\langle\pm|\, \mathcal{E}_{t_3,t_2} \circ \Delta \circ \mathcal{E}_{t_2,t_1} \circ \Delta[\rho_\mathrm{s}]\Big), \end{aligned} \tag{7.24}$$

Figure 7.1 shows the trace distance $\max_{\rho_\mathrm{s}\in\mathcal{I}} \|\boldsymbol{p}(t_3) - \boldsymbol{q}(t_3, t_2)\|$ as a function of the intermediate time t_2 for various values of the ratio γ/ω. The presence of coherence in the dynamics is most prominent half-way through the evolution and, indeed, it is suppressed by a stronger rate γ.

Let us now investigate a complementary scenario in which we also include components orthogonal to the Hamiltonian in the noise. Specifically, consider the open-system dynamics

$$\mathcal{L}[\rho_\mathrm{s}] = -\mathrm{i}\omega[\sigma_z, \rho_\mathrm{s}] + \frac{1}{2}\sum_{i,j=1}^{3} \gamma_{ij}\Big([\sigma_i\rho_\mathrm{s}, \sigma_j] + [\sigma_i, \rho_\mathrm{s}\sigma_j]\Big), \tag{7.25}$$

where $\gamma_{ij} = \gamma_{ji}^*$ are the damping rates; still, our incoherent basis is $(\mathbb{1}, \sigma_x)$.

Coherence non-generating dynamics corresponds to $\gamma_{12} = -\sqrt{2}\omega$ and $\gamma_{13} = 0$, whereas coherence non-activating dynamics is given by $\mathrm{Re}\,\gamma_{12} = \sqrt{2}\omega$ and $\mathrm{Re}\,\gamma_{13} = 0$.

To investigate the more general notion of NCGD dynamics, we look at the matrix representation for the corresponding generator; assuming for the sake of simplicity that all γ_{ij} be real, it reduces to

$$\mathcal{L} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\gamma_{22} - \gamma_{33} & -\sqrt{2}\omega + \gamma_{12} & \gamma_{13} \\ 0 & \sqrt{2}\omega + \gamma_{12} & -\gamma_{11} - \gamma_{33} & \gamma_{23} \\ 0 & \gamma_{13} & \gamma_{23} & -\gamma_{11} - \gamma_{22} \end{pmatrix}. \tag{7.26}$$

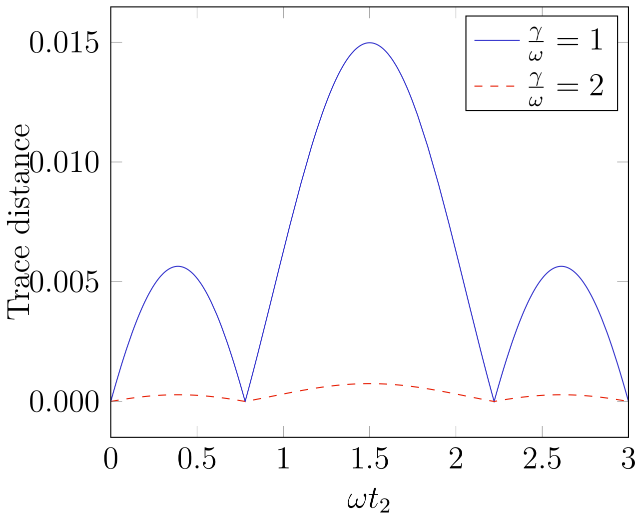


Figure 7.1: Coherence generated and detected, as measured via the trace distance between the probability distributions of Eq. (7.24), maximized over ρ_s, for the open-system evolution described by Eq. (7.22). The total evolution time is fixed to $\omega t_3 = 3$ and the trace distance is plotted as a function of the intermediate time $0 \leq t_2 \leq t_3$.

It can be verified that

$$\begin{aligned} \mathcal{L}_{\rm pc}\mathcal{L}_{\rm cp} &= \mathcal{L}_{\rm pc}\mathcal{L}_{\rm cc}\mathcal{L}_{\rm cp} = 0 \\ \Leftrightarrow 2\omega^2 &= \gamma_{12}^2 + \gamma_{13}^2 \\ \wedge\, \gamma_{13}^2(\gamma_{22} - \gamma_{33}) &= 2\gamma_{12}\gamma_{13}\gamma_{23} \end{aligned} \tag{7.27}$$

so that indeed, the CGD capabilities of the dynamical evolution depend on the damping rates γ_{12} and γ_{13} that mix coherent with incoherent components.

Different behaviors of the CGD capability for various parameter choices are illustrated in Figure 7.2. On the one hand, changing the weights of the noise components can result in even qualitatively different features of the coherences generated and detected along the evolution, characterized, for example, by different locations and number of maxima as a function of the intermediate time t_2.

On the other hand, rather different kinds of noise might exhibit a similar behavior. In fact, compare the case of pure dephasing, see Figure 7.1, and the purely orthogonal noise represented by the solid blue curve in Figure 7.2. The qualitative and even quantitative evolution of the coherences generated and detected is very similar in the two cases. This is particularly relevant since it is well known that, if we want to estimate the value of the frequency ω via the Ramsey scheme, pure dephasing and orthogonal noise will limit the optimal

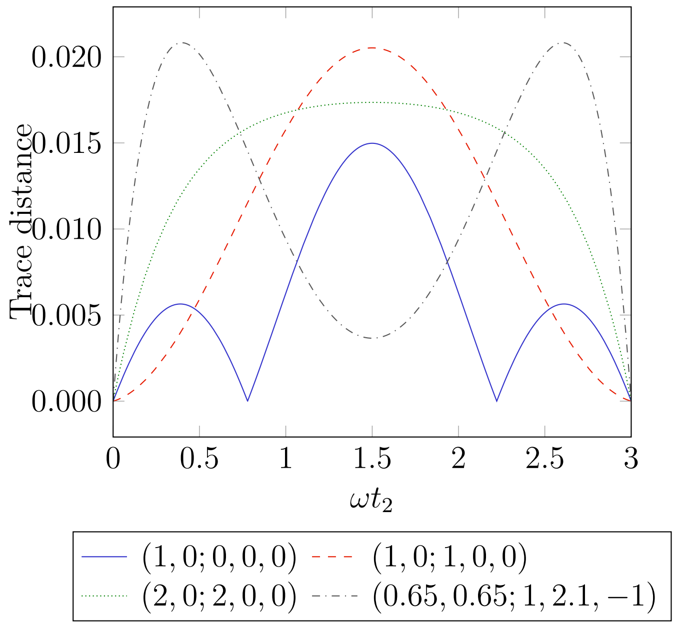


Figure 7.2: Coherence generated and detected, as measured via the trace distance between the two probability distributions of Eq. (7.24), maximized over ρ_s, for an open-system evolution described by Eq. (7.25).
The legend gives, in units of ω, $(\gamma_{11} = \gamma_{22}, \gamma_{33}; \gamma_{12}, \gamma_{13}, \gamma_{23})$; in particular, all but the gray dash-dotted line represent cases of purely orthogonal noise [151], since the non-zero rates are associated in Eq. (7.25) only to Pauli operators in a direction orthogonal to the Hamiltonian $\omega\sigma_3$. The total evolution time is fixed to $\omega t_3 = 3$ and the trace distance is plotted as a function of the intermediate time $0 \leq t_2 \leq t_3$.

achievable precision in a radically different way. Pure dephasing enforces the shot-noise limit [152–155], which is typical of the classical estimation strategies [156]. Note that this is the case even if error-correction techniques are applied [157–159]. Orthogonal noise, instead, allows for super-classical precision [151], which can be even raised to the ultimate Heisenberg limit by means of error correction [157–159]. This provides us with an example of how the capability to generate coherences (in the relevant basis) and later convert them to populations has to be understood as a *prerequisite* to perform tasks which rely on the advantage given by the use of quantum features. However, CGD in itself does not guarantee that such an advantage over any possible classical counterpart is actually achieved.

7.2 Non-classicality and coherence: main result

We now present the main result of this chapter, where we provide the connection between the classicality of a certain statistics and the coherence generating and detecting properties of the underlying dynamics. Note that, since some results that will be presented later are required to prove this theorem, we will postpone its proof to the end of the chapter.

Theorem 27. *Given a non-degenerate reduced observable $\hat{X}_s = \sum_x x|\psi_x\rangle\langle\psi_x|$ and a jM hierarchy of probabilities $Q_n^{\hat{X}_s}\{x_n, t_n; \ldots x_1, t_1\}$, the latter is jCL for any initial diagonal state $\rho(0) = \sum_{x_0} p_{x_0}|\psi_{x_0}\rangle\langle\psi_{x_0}|$ if and only if the dynamics—which must be of Lindblad type—is NCGD.*

Theorem 27 means that if the multi-time statistics is Markovian, the capability of a Lindblad dynamics to generate coherences and turn them into populations is in one-to-one correspondence with non-classicality. In other words, Markovianity guarantees the wanted connection between a property of the coherences, which is fixed by the dynamics, and the classicality of the multi-time probability distributions. This is a direct consequence of the peculiarity of Markovian processes, classical as well as quantum, which allows one to reconstruct the higher order probability distributions from the lowest order one.

7.3 LGtIs and coherence

So far we have focused on investigating the non-classicality of a stochastic process, crystallizing in the violation of the Kolmogorov consistency conditions, in relation to the coherence of the environment. In this section we will analyze the connection of the LGtIs to the coherence properties of the underlying dynamics, which we already expect to be weaker than the one involving the Kolmogorov conditions.

Before relating coherence and the LGtIs we will need to present the following result, which provides yet another operational meaning to (N)CGD.

Proposition 28. *Given a non-degenerate reduced observable $\hat{X}_s = \sum_x x|\psi_x\rangle\langle\psi_x|$ and some Lindblad dynamics, the latter is NCGD if and only if the conditional probabilities $Q_{1|1}^{\hat{X}_s}\{x, t|x_0, 0\}$ satisfy $\forall t \geq s \in \mathbb{R}^+$*

$$Q_{1|1}^{\hat{X}_s}\{x, t|x_0, 0\} = \sum_y Q_{1|1}^{\hat{X}_s}(x, t-s|y, 0) Q_{1|1}^{\hat{X}_s}(y, s|x_0, 0). \tag{7.28}$$

Proof. We first need to introduce the following lemma (see proof in Appendix D.2)

Lemma 29. *The evolution fixed by the Lindblad dynamics* $\left\{\Lambda(t)=e^{\mathcal{L}t}\right\}_{t\in\mathbb{R}^+}$ *is NCGD if and only if*

$$\sum_{y\neq z}\langle\psi_{\tilde{x}}|\,\Lambda(t)[|\psi_y\rangle\langle\psi_z|]\,|\psi_{\tilde{x}}\rangle\,\langle\psi_y|\,\Lambda(\tau)\,[|\psi_x\rangle\langle\psi_x|]\,|\psi_z\rangle=0\;\forall x,\tilde{x};t,\tau\in\mathbb{R}^+. \tag{7.29}$$

We can now prove the Proposition: Using Eq.(3) of the main text and $Q^{\hat{X}_s}_{1|1}\{x,t|y,s\}=\mathrm{Tr}_s\{\mathcal{P}_x e^{\mathcal{L}(t-s)}[|\psi_y\rangle\langle\psi_y|]\}=Q^{\hat{X}_s}_{1|1}\{x,t-s|y,0\}$ we have that

$$\begin{aligned}
&Q^{\hat{X}_s}_{1|1}\{x,t|x_0,0\}-\sum_y Q^{\hat{X}_s}_{1|1}(x,t-s|y,0)Q^{\hat{X}_s}_{1|1}(y,s|x_0,0)\\
&=\langle\psi_x|\,\Lambda(t)\,[|\psi_{x_0}\rangle\langle\psi_{x_0}|]\,|\psi_x\rangle-\sum_y\langle\psi_x|\,\Lambda(t-s)\,[|\psi_y\rangle\langle\psi_y|]\,|\psi_x\rangle\\
&*\langle\psi_y|\,\Lambda(s)\,[|\psi_{x_0}\rangle\langle\psi_{x_0}|]\,|\psi_y\rangle\,,
\end{aligned}$$

so that using the semigroup composition law $\Lambda(t)=\Lambda(t-s)\Lambda(s)$ and the resolution of the identity, the first term in the previous expression can be written as

$$\begin{aligned}
&\langle\psi_x|\,\Lambda(t-s)\Big[\Lambda(s)\,[|\psi_{x_0}\rangle\langle\psi_{x_0}|]\Big]\,|\psi_x\rangle\\
&=\sum_{y,y'}\langle\psi_x|\,\Lambda(t-s)\Big[|\psi_y\rangle\langle\psi_y|\Lambda(s)\,[|\psi_{x_0}\rangle\langle\psi_{x_0}|]\,|\psi_{y'}\rangle\langle\psi_{y'}|\Big]\,|\psi_x\rangle\,,\\
&=\sum_{y,y'}\langle\psi_x|\,\Lambda(t-s)\,[|\psi_y\rangle\,\langle\psi_{y'}|]\,|\psi_x\rangle\,\langle\psi_y|\,\Lambda(s)\,[|\psi_{x_0}\rangle\langle\psi_{x_0}|]\,|\psi_{y'}\rangle\,,
\end{aligned}$$

so that the 'diagonal terms' (with $y=y'$) cancel out with the second contribution and the violation of the homogeneous Chapman-Kolmogorov condition is given by

$$\begin{aligned}
&Q^{\hat{X}_s}_{1|1}\{x,t|x_0,0\}-\sum_y Q^{\hat{X}_s}_{1|1}(x,t-s|y,0)Q^{\hat{X}_s}_{1|1}(y,s|x_0,0)\\
&=\sum_{y\neq y'}\langle\psi_x|\,\Lambda(t-s)\,[|\psi_y\rangle\,\langle\psi_{y'}|]\,|\psi_x\rangle\,\langle\psi_y|\,\Lambda(s)\,[|\psi_{x_0}\rangle\langle\psi_{x_0}|]\,|\psi_{y'}\rangle\,,
\end{aligned} \tag{7.30}$$

which implies that such difference is equal to 0 for any $x_0,x,t\geq s$ if and only if the Lindblad dynamics is NCGD, see Eq.(7.29). □

The condition in Eq.(7.28) is simply the (homogeneous) Chapman - Kolmogorov equation [140, 144, 160], which is always satisfied by a classical Markov (homogeneous) process, but, indeed, not necessarily by a quantum one. Let us also stress that Eq.(7.28) can be in principle easily checked in practice, since the conditional probabilities $Q_{1|1}^{\hat{X}_s}\{x,t|x_0,0\}$ can be reconstructed by preparing the system in one eigenstate of $\hat{X}_s$ and measuring $\hat{X}_s$ itself at a final time t, without the need to access intermediate steps of the evolution.

Now, since the validity of Eq.(7.28) is a sufficient condition for the LGtI to be satisfied, as we will later prove, Proposition 28 directly leads us to the following.

Theorem 30. *Given a Lindblad dynamics, the LGtI is violated only if the dynamics is CGD.*

Proof. The Theorem easily follows from the following Lemma (see proof in Appendix D.3).

Lemma 31. *Consider a dichotomic observable $\hat{X}$ with values in $\{0,1\}$ and such that the conditional probabilities $Q_{1|1}^{\hat{X}}\{x,t;x_0,0\}$ satisfy Eq.(6) of the main text, then the correlation function $C_X(t,0)$ satisfies the LGtI*

$$|2C_X(t,0) - C_X(2t,0)| \leq \langle X(0) \rangle. \tag{7.31}$$

Note that this Lemma holds independently of whether the conditional probabilities are referring to the quantum setting (and hence are defined as in Eq.(1) of the main text) or are directly referring to a classical theory: our proof goes along the same line as that in [129], simply adapting it to (possibly) quantum conditional probabilities. □

This theorem thus clarifies how the LGtI can be used to witness that coherences are generated by the dynamics and subsequently turned into populations.

Finally, Theorem 27 also allows us to clarify to what extent the LGtI is actually related with non-classicality, since it directly implies the following (see proof at the end of the chapter).

Theorem 32. *Given a 2M hierarchy of probabilities, the LGtI is violated only if the hierarchy is non-classical.*

For the sake of clarity, in Fig.7.3 we report a summary of the theorems presented in this chapter, which establish definite connections among the notions of classicality, quantum coherence (in particular NCGD of the dynamics) and LGtI.

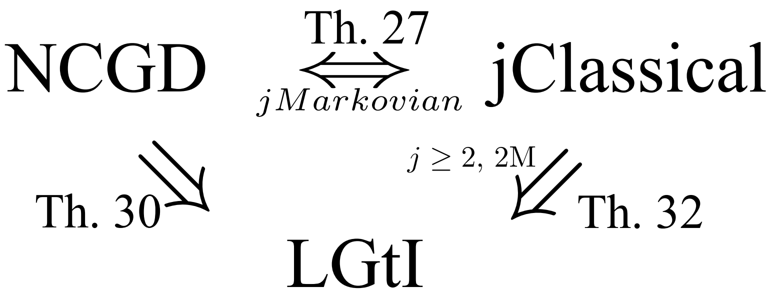


Figure 7.3: Implication structure of the main results of the paper. The notion of j-classicality is given in Definition 24, j-Markovianity in Definition 23 and the property of the evolution of coherence named NCGD, which stands for non-coherence-generating-and-detecting, in Definition 22; finally LGtI denotes the Leggett-Garg type inequality [128] considered here. A Lindblad dynamics is always assumed.

7.4 Proof of Theorem 27 and Theorem 32

Before presenting the proof to Theorem 27, let us give the basic idea behind it. The time-homogeneous Chapman-Kolmogorov equation holds for any classical time-homogeneous Markov process, that is, Markovianity and classicality imply Chapman-Kolmogorov; note that the time-homogeneity of the statistics holds, as a consequence of (2)M and the Lindblad dynamics. For the converse, we can exploit the definition of jM, which provides us with a notion of Markovianity beyond classical processes, i.e., for any quantum statistics. As said, the Markov property (both for classical and nonclassical statistics) connects the multi-time probability distributions to the initial one-time distribution and the conditional probability $Q^{\hat{X}_s}_{1|1}$; as a direct consequence of this, it is then easy to see that, if the Chapman-Komogorov equation holds, jM directly turns into jCL. Theorem 27 thus follows from the equivalence established in Proposition 28.

Explicitly, both Theorems 27 and 32 directly follow from the following Lemma (see proof in Appendix D.4).

Lemma 33. *Given a non-degenerate observable $\hat{X}_s = \sum_x x|\psi_x\rangle\langle\psi_x|$ and a jM hierarchy of probabilities, the latter defines a jCL statistics for any initial diagonal state $\rho(0) = \sum_{x_0} p_{x_0}|\psi_{x_0}\rangle\langle\psi_{x_0}|$ if and only if Eq.(6) of the main text holds for the quantum conditional probability $Q^{\hat{X}_s}_{1|1}\{x, t|x_0, 0\}$.*

Theorem 27 hence directly follows from Lemma 33 and Proposition 28, while Theorem 32 follows, e.g., from Lemmas 31 and 33.

7.5 Chapter summary

- A Lindblad dynamics is non-coherence-generating-and-detecting (NCGD) if and only if its generator $\mathcal{L}$ fulfills the following finite number of conditions: $\mathcal{L}_{\mathrm{pc}}\mathcal{L}_{\mathrm{cc}}^{j}\mathcal{L}_{\mathrm{cp}} = 0 \ \forall j \in \{0, \ldots, d^2 - d - 1\}$, where $d = \dim \mathcal{H}$.
- A j-Markovian multi-time probability distribution is j-classical if and only if the underlying Lindblad dynamics is NCGD.
- The Leggett-Garg type inequalities are fulfilled only if the Lindblad dynamics is NCGD.

7.6 Open questions

- Is it necessary and sufficient to check $d^2 - d$ conditions to verify whether a Lindblad dynamics is NCGD? Can this number be reduced even more? So far we only know that more than 2 conditions are required.

CHAPTER 8

The most general case

Analogously to the previous chapter, here we also want to investigate the role of coherence in finding an answer to the question whether the outcomes of a projectively measured quantum stochastic process are compatible with a classical stochastic process; yet this time we will allow for any kind of underlying dynamics. For this purpose we first need to put forward a more general definition of incoherent dynamics, from which the NCGD definition can be recovered under some particular conditions.

8.1 A problem with the NCGD formulation

When moving away from Markovianity, the coherence generating and detecting properties of the dynamics are not anymore in correspondence with the non-classicality of the statistics. That is, it is in principle possible to encounter a dynamics that is CGD and gives rise to classical statistics, or that is NCGD and yields non-classical statistics. Under the most general kind of dynamics the connection between quantum coherence and the non-classicality of the multi-time statistics is no longer guaranteed. A non-Markovian example of such kind is introduced in the following:

Consider a two-level system, $\mathcal{H}_S = \mathbb{C}^2$, interacting with a continuous degree of freedom, $\mathcal{H}_E = \mathcal{L}(\mathbb{R})$, via the unitary operators defined by $U(t)\,|\ell, p\rangle = e^{i\ell p t}\,|\ell, p\rangle$, where $\{|\ell\rangle\}_{\ell=-1,1}$ is the eigenbasis of the system operator $\hat{\sigma}_z$ and $\{|p\rangle\}_{p\in\mathbb{R}}$ is an improper basis of $\mathcal{H}_E$. Assuming an initial product state and a pure environmental state, $\rho_E(0) = |\varphi_E\rangle\langle\varphi_E|$ with $|\varphi_E\rangle = \int_{-\infty}^{\infty} dp f(p)\,|p\rangle$, the open-system dynamics is pure dephasing, fixed by $\rho_{-11}(t) = \rho_{-11}(0)k(t)$ with $k(t) = \int_{-\infty}^{\infty} dp|f(p)|^2 e^{2ipt}$, where $\rho_{-11} = \langle -1|\, \rho\, |1\rangle$. We consider projective measurements of $\hat{\sigma}_x$, whose eigenbasis is denoted as $\{|+\rangle\,, |-\rangle\}$, and then we assume initial states as $\rho(0) = p_+|+\rangle\langle+| + p_-|-\rangle\langle-|$. For the sake of simplicity, we focus on the two-time statistics and we take as initial condition $\rho(0) = |+\rangle\langle+| \otimes |\varphi_E\rangle\langle\varphi_E|$. Now, let us also fix that both the first and the second outcomes of the measurement of $\hat{\sigma}_x$ yield the same result, +, so that we have

$$\begin{aligned} Q_2^{\hat{\sigma}_x}\{+,t;+,s\} &= \mathrm{tr}_E\{\langle+|\mathcal{U}(t-s)[|+\rangle\langle+|\mathcal{U}(s)[|+\rangle\langle+| \\ &\quad \otimes |\varphi_E\rangle\langle\varphi_E|]\,|+\rangle\langle+|]\,|+\rangle\} \\ &= \frac{1}{4}\mathrm{Re}\left[\frac{1}{2}k(t-2s) + \frac{1}{2}k(t) + k(t-s) + k(s) + 1\right], \end{aligned}$$

where Re denotes the real part. Moreover, since the map and the propagator of the above dynamics are given by

$$\Lambda(t)[\rho] = \begin{pmatrix} \rho_{-1-1} & k(t)\rho_{-11} \\ k^*(t)\rho_{1-1} & \rho_{11} \end{pmatrix}$$

and

$$\Lambda(t,s)[\rho] = \Lambda(t)\circ\Lambda^{-1}(s)[\rho] = \begin{pmatrix} \rho_{-1-1} & \frac{k(t)}{k(s)}\rho_{-11} \\ \frac{k^*(t)}{k^*(s)}\rho_{1-1} & \rho_{11} \end{pmatrix},$$

we get

$$\begin{aligned} Q_{2M}^{\hat{\sigma}_x}\{+,t;+,s\} &= \langle+|\,\Lambda(t,s)\Big[\,|+\rangle\langle+|\,\Lambda(s)[|+\rangle\langle+|]\,|+\rangle\langle+|\,\Big]\,|+\rangle \\ &= \frac{1}{4}\mathrm{Re}\left[\frac{1}{2}k(t)\left(\frac{k^*(s)}{k(s)}+1\right) + \frac{k(t)}{k(s)} + k(s) + 1\right]. \end{aligned}$$

This means that the Markovian description implies

$$Q_2^{\hat{\sigma}_x}\{+,t;+,s\} = Q_{2M}^{\hat{\sigma}_x}\{+,t;+,s\},$$

i.e.,

$$\mathrm{Re}\left[\frac{1}{2}k(t)\frac{k^*(s)}{k(s)} + \frac{k(t)}{k(s)}\right] = \mathrm{Re}\left[\frac{1}{2}k(t-2s) + k(t-s)\right]; \tag{8.1}$$

indeed, a violation of this condition will be enough to prove that the statistics is NM.

In order to check whether the Kolmogorov condition holds for the two-time probabilities, we need also to evaluate the other two-time probability distribution

$$\begin{aligned} Q_2^{\hat{\sigma}_x}\{+,t;-,s\} &= \mathrm{tr}_E\{\langle + | \mathcal{U}(t-s)\Big[\left|-\right\rangle \left\langle -\right| \mathcal{U}(s)[\left|+\right\rangle \left\langle +\right| \\ &\quad \otimes \left|\varphi_E\right\rangle \left\langle \varphi_E\right|] \left|-\right\rangle \left\langle -\right| \Big] \left|+\right\rangle\} \\ &= \frac{1}{4}\,\mathrm{Re}[\frac{1}{2}k(t-2s)+\frac{1}{2}k(t)-k(t-s)-k(s)+1], \end{aligned}$$

as well as the one-time probability

$$\begin{aligned} Q_1^{\hat{\sigma}_x}\{+,t\} &= \mathrm{tr}_E\{\langle + | \mathcal{U}(t)[\left|+\right\rangle \left\langle +\right| \otimes \left|\varphi_E\right\rangle \left\langle \varphi_E\right|] \left|+\right\rangle\} \\ &= \mathrm{Re}[\frac{1}{2}+\frac{1}{2}k(t)]. \end{aligned}$$

Hence, setting $K_{\pm}(t,s) \equiv Q_2^{\hat{\sigma}_x}\{\pm,t;+,s\} + Q_2^{\hat{\sigma}_x}\{\pm,t;-,s\} - Q_1^{\hat{\sigma}_x}\{\pm,t\}$, the statistics is 2-CL if and only if

$$K_+(t,s) = \frac{1}{4}\,\mathrm{Re}[k(t-2s)-k(t)] = 0; \tag{8.2}$$

note that, since $Q_2^{\hat{\sigma}_x}\{-,t;\pm,s\} = Q_1^{\hat{\sigma}_x}\{\pm,s\} - Q_2^{\hat{\sigma}_x}\{+,t;\pm,s\}$, one has $K_-(t,s) = -K_+(t,s)$.

Finally, since we are interested in the connection between classicality and coherences, we want to check whether the dynamics is (N)CGD. With Δ defined with respect to the eigenbasis of $\hat{\sigma}_x$, we have

$$\|(\Delta \circ \Lambda(t,s) - \Delta \circ \Lambda(t,s) \circ \Delta)\Lambda(s)\Delta\|_\infty = \frac{2\,|\mathrm{Im}\,[k(s)]\,\mathrm{Im}\,[k^*(s)k(t)]|}{4|k(s)|^2} \tag{8.3}$$

where Im denotes the imaginary part. For the sake of simplicity, we set the initial time to 0.

NCGD and non-classicality

Let us first consider an initial Lorentzian distribution, $|f(p)|^2 = \Gamma\pi^{-1}/(\Gamma^2+(p-p_0)^2)$, so that the decoherence function is given exactly by an exponential, $k(t) = e^{2ip_0t}e^{-2\Gamma t}$, and the open-system dynamics is fixed by the pure dephasing Lindblad equation [161]:

$$\frac{d}{dt}\rho(t) = -ip_0\,[\hat{\sigma}_z,\rho(t)] + \Gamma\left(\hat{\sigma}_z\rho(t)\hat{\sigma}_z - \rho(t)\right), \tag{8.4}$$

from which we can read the physical meaning of the two parameters, p_0 and Γ defining the Lorentzian distribution in this context.

In particular, for $p_0 = 0$, we get

$$\begin{aligned} Q_2^{\hat{\sigma}_x}\{+,t;+,s\} &= \frac{1}{4}\left(1+e^{-2\Gamma s}+e^{-2\Gamma(t-s)}+\frac{1}{2}e^{-2\Gamma t}\right) \\ &\quad +\frac{1}{8}\left(\cosh\left(2\Gamma(t-2s)\right)-\sinh\left(2\Gamma|t-2s|\right)\right), \end{aligned} \tag{8.5}$$

while the QRT gives us

$$\begin{aligned} Q_{2M}^{\hat{\sigma}_x}\{+,t;+,s\} &= \left\{\langle+|\, e^{\mathcal{L}(t-s)}\left[|+\rangle\langle+|e^{\mathcal{L}s}\left[|+\rangle\langle+|\right]|+\rangle\langle+|\right]|+\rangle\right\} \\ &= \frac{1}{4}\left(1+e^{-2\Gamma s}\right)\left(1+e^{-2\Gamma(t-s)}\right). \end{aligned} \tag{8.6}$$

As shown in Fig. 8.1, these two functions are clearly different, implying that the present statistics is NM, since the QRT is not satisfied. In addition, the statistics is not even classical, as follows from

$$\sum_y Q_2^{\hat{\sigma}_x}\{1,t;y,s\} \neq Q_1^{\hat{\sigma}_x}\{1,t\}, \tag{8.7}$$

which can be easily shown since one has

$$\partial_s \sum_y Q_2^{\hat{\sigma}_x}\{+,t;y,s\} = \Gamma \mathrm{sgn}\{t-2s\}\, e^{-2\Gamma|t-2s|},$$

which is of course different from 0, thus guarantying the inequality in Eq.(8.7). For $p_0 = 0$ the model is furthermore NCGD: as we have here pure dephasing in the z-direction, coherences in the x-direction cannot be even generated. This example clearly shows how the non-classicality of a NM statistics might be fully unrelated even from the presence itself of quantum coherence in the dynamics.

CGD and classicality

In a complementary way, we exemplify how the instants where the multi-time statistics satisfies the Kolmogorov conditions may coincide with instants where coherences are generated and converted into populations. However, we have to leave open the question of whether there is a fully classical statistics (for any sequence of times), while the dynamics of the coherences is non-trivial. Take an initial distribution given by the sum of two Gaussians, $|f(p)|^2 =$

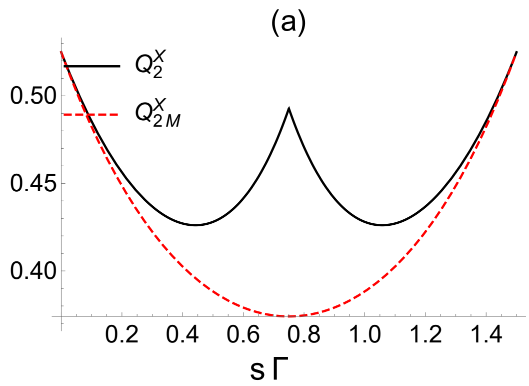


Figure 8.1: Exact two-time probability $Q_2^{\hat{\sigma}_x}\{+,t;+,s\}$ [see Eq.(8.5)] (solid black line) and 2M probability $Q_{2M}^{\hat{\sigma}_x}\{+,t;+,s\}$ [see Eq.(8.6)] (dashed red line) as functions of s, for a Lorentzian $|f(p)|^2$. The parameters are $t = 1.5\Gamma^{-1}, p_0 = 0$.

$\sum_{i=1,2} A_i \exp\left(-(p-p_i)^2/(2\sigma_i^2)\right)$, where $A_1 = \frac{1}{\sqrt{2\pi}\sigma(1+A_\theta)}$, $A_2 = A_\theta A_1$ and $\sigma_1 = \sigma_2 = \sigma$. The decoherence function reduces to

$$k(t) = \frac{e^{-2\sigma^2 t^2}}{A_\theta + 1}\left[e^{2ip_1 t} + A_\theta e^{2ip_2 t}\right]. \tag{8.8}$$

For the specific choice of parameters $A_\theta = \sigma = p_1 = t = 1$, $p_2 = 2p_1$, the functions $Q_2^{\hat{\sigma}_x}\{+,t;+,s\}$ and $Q_{2M}^{\hat{\sigma}_x}\{+,t;+,s\}$ are, in general, different. The present statistics is thus NM, as shown in Fig.8.2 **a)**.

In order for the statistics to be 2-CL, the following condition must hold

$$\begin{aligned} K_+(t,s) &= \frac{1}{4}\mathrm{Re}[k(t-2s) - k(t)] \\ &= \frac{e^{-2\sigma^2(t-2s)^2}}{4(A_\theta+1)}\left[\cos(2p_1(t-2s)) + A_\theta\cos(2p_2(t-2s))\right] \\ &\quad + \frac{e^{-2\sigma^2 t^2}}{4(A_\theta+1)}\left[\cos(2p_1 t) + A_\theta\cos(2p_2 t)\right] = 0. \end{aligned} \tag{8.9}$$

Furthermore, the model is CGD if [see Eq.(8.3)] the quantity

$$N(t,s) := \frac{2\left|\mathrm{Im}\left[k(s)\right]\mathrm{Im}\left[k^*(s)k(t)\right]\right|}{4|k(s)|^2}$$

$$= \beta e^{-2t^2\sigma^2}\left[\sin(2sp_1) + A_\theta \sin(2sp_2)\right]$$

$$\cdot\left[\sin(2(s-t)p_1) + A_\theta(A_\theta\sin(2(s-t)p_2) - \sin(2tp_1 - 2sp_2) + \sin(2sp_1 - 2tp_2))\right],$$

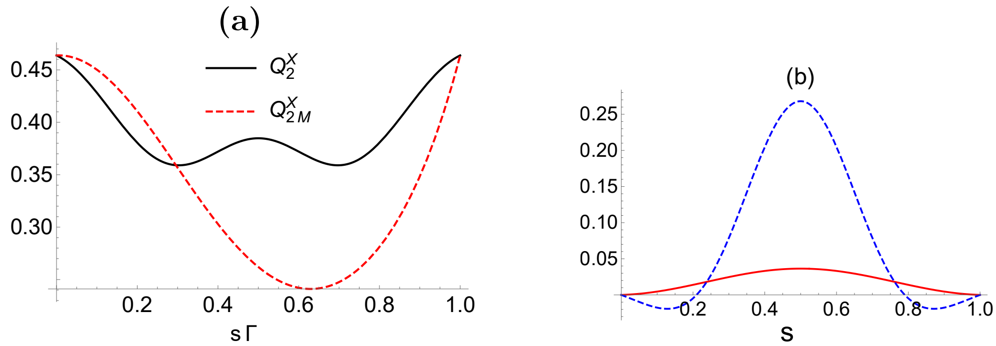


Figure 8.2: **a)** Comparison of the two-time probability distribution as a function of the first time s, for the exact formula (solid, black line) and the 2M statistics (dashed, red line). **b)** Violation of 2CL ($K_+(t,s)$) (dashed, blue line) and violation of NCGD ($N(t,s)$) (solid, red line). For both plots the environmental distribution is given by a sum of two Gaussians, with (in arbitrary units) $A_\theta = \sigma = p_1 = t = 1$, $p_2 = 2p_1$, $s \in [0,t]$.

where $\beta = \frac{1}{2(1+A_\theta)\left|e^{2isp_1}+A_\theta e^{2isp_2}\right|^2}$, is different from 0.

As can be seen in Fig.8.2 **a)**, for the considered choice of parameters the dynamics is NM at instants different from $s = 0.29$. This allows for the existence of scenarios where the possible classicality of the statistics is unrelated to the absence of coherences. As a matter of fact, at the specific instants $s = 0.21$ and $s = 0.79$, where QRT is not satisfied, one finds that $K_+(t,s) = 0$ and $N(t,s) \neq 0$, implying that 2-CL holds together with CGD (Fig.8.2 **b)**).

The previous examples illustrate the essential role of Markovianity to establish a precise link between quantum coherence, as formalized by the CGD condition, and non-classicality. In particular, they imply that the CGD property cannot be used as a witness of non-classicality, without any a-priori information about higher order probabilities. How to then investigate the relation between coherence and non-classicality in a general scenario where Markovianity may not be present? The answer lies in generalizing the definition of NCGD dynamics to that of *incoherent* dynamics.

8.2 Incoherent dynamics (and a revisitation of classical stochastic processes)

We start by reviewing the basic notions of a classical stochastic process which, despite having been presented already in Chapter 6, we will now need to reformulate in a more convenient way. We label the classical distinguishable states of the system of interest by r and we assume that the system gets measured at an arbitrary set of discrete times $\{t_1, \ldots, t_n\}$. We denote the result at time t_i by r_i. Furthermore, for reasons which will become clearer later on, we explicitly denote the initial preparation of the experiment by $\mathcal{A}_0$. We then denote the joint probability distribution to get the sequence of measurement results $\mathbf{r}_n = r_1, \ldots, r_n$ at times $t_1, \ldots, t_n$ given the initial preparation $\mathcal{A}_0$ by

$$p(r_n, t_n; \ldots; r_1, t_1|\mathcal{A}_0) \equiv p(\mathbf{r}_n|\mathcal{A}_0). \tag{8.10}$$

The following definition is standard:

Definition 34. *The probabilities $p(\boldsymbol{r}_n|\mathcal{A}_0)$ are said to be classical with repect to a given preparation procedure $\mathcal{A}_0$ if they fulfill the consistency condition*

$$\sum_{r_k} p(r_\ell, \ldots, r_k, \ldots, r_j|\mathcal{A}_0) = p(r_\ell, \ldots, \not{r_k}, \ldots, r_j|\mathcal{A}_0) \tag{8.11}$$

for all $\ell \geq k \geq j \geq 1$. Here, the probability on the right hand side is constructed by measuring the states r_i of the system only at the set of times $\{t_\ell, \ldots, t_j\} \setminus \{t_k\}$.

Albeit condition (8.11) is in general not fulfilled for quantum dynamics, the joint probability distribution (8.10) is nevertheless a well-defined object in quantum mechanics. For this purpose we assume that the experimentalist measures at time t_k an arbitrary system observable $R_k = \sum_{r_k} r_k P_{r_k}$ with projectors $P_{r_k} = P_{r_k}^2$ and eigenvalues $r_k \in \mathbb{R}$. If all projectors are rank-1, i.e., $P_{r_k} = |r_k\rangle\langle r_k|$, we talk about a non-degenerate system observable, otherwise we call it degenerate. Furthermore, following the conventional picture of open quantum systems [140], we allow the system S to be coupled to an arbitrary environment E. The initial system-environment state at time $t_0 < t_1$ is denoted by $\rho_{SE}(t_0)$. Then, by using superoperator notation, we can express Eq. (8.10) as

$$\begin{aligned} &p(\mathbf{r}_n|\mathcal{A}_0) \\ &= \mathrm{tr}_{SE}\left\{\mathcal{P}_{r_n}\mathcal{U}_{n,n-1}\ldots\mathcal{P}_{r_2}\mathcal{U}_{2,1}\mathcal{P}_{r_1}\mathcal{U}_{1,0}\mathcal{A}_0\rho_{SE}(t_0)\right\} \\ &\equiv \mathrm{tr}_S\left\{\mathfrak{T}[\mathcal{P}_{r_n}, \ldots, \mathcal{P}_{r_2}, \mathcal{P}_{r_1}, \mathcal{A}_0]\right\}. \end{aligned} \tag{8.12}$$

Here, the preparation procedure $\mathcal{A}_0$ is an arbitrary completely positive (CP) and trace-preserving map acting on the system only (we suppress identity operations in the tensor product notation). Notice that the preparation procedure could itself be the identity operation (i.e., 'do nothing') denoted by $\mathcal{A}_0 = \mathcal{I}_0$. Furthermore, $\mathcal{U}_{k,k-1}$ denotes the unitary time-evolution propagating the system-environment state from time t_{k-1} to t_k (we make no assumption about the underlying Hamiltonian here) and $\mathcal{P}_{r_k}$ is the projection superoperator corresponding to result r_k at time t_k. Finally, in the last line of Eq. (8.12) we have introduced the 'process tensor' $\mathfrak{T}$ [162] (also called 'quantum comb' [147, 163] or 'process matrix' [164, 165]). It is a formal but operationally well-defined object: it yields the (subnormalized) state of the system $\tilde{\rho}_S(\mathcal{P}_{r_n}, \dots, \mathcal{P}_{r_2}, \mathcal{P}_{r_1}, \mathcal{A}_0) = \mathfrak{T}[\mathcal{P}_{r_n}, \dots, \mathcal{P}_{r_2}, \mathcal{P}_{r_1}, \mathcal{A}_0]$ conditioned on a certain sequence of interventions $\mathcal{P}_{r_n}, \dots, \mathcal{P}_{r_2}, \mathcal{P}_{r_1}, \mathcal{A}_0$. Its norm, as given by the trace over S, equals the probability to obtain the measurement results $\mathbf{r}_n$. Recently, it was shown that the process tensor allows for a rigorous definition of quantum stochastic processes (or quantum causal models) fulfilling a generalized version of the Kolmogorov-Daniell extension theorem [148].

We can now define what we mean by an 'incoherent' quantum stochastic process:

Definition 35. *For a given set of observables $\{R_k\}$, $k \in \{1, \dots, \ell\}$, we call the dynamics of an open quantum system ℓ-incoherent with respect to the preparation $\mathcal{A}_0$ if all process tensors*

$$\mathfrak{T}\left[\Delta_\ell, \begin{Bmatrix} \Delta_{\ell-1} \\ \mathcal{I}_{\ell-1} \end{Bmatrix}, \dots, \begin{Bmatrix} \Delta_1 \\ \mathcal{I}_1 \end{Bmatrix}, \mathcal{A}_0\right] \tag{8.13}$$

are equal. Here, the angular bracket notation means that at each time step we can freely choose to perform either a dephasing operation (Δ) or nothing ($\mathcal{I}$). If the dynamics are ℓ-incoherent for all $\ell \in \{1, \dots, n\}$, we simply call the dynamics incoherent with respect to the preparation procedure $\mathcal{A}_0$.

This definition is supposed to capture the situation where the experimentalist has no ability to detect the presence of coherence during the course of the evolution. For this purpose we imagine that the experimentalist can manipulate the system in three ways: first, she can prepare the initial system state in some way via $\mathcal{A}_0$ (which could be only the identity operation), she can projectively measure the system observables R_k at times $t_k \in \{t_1, \dots, t_n\}$, and she can detect the final output state by doing state tomography. The question is then: if the final state got dephased with respect to the observable R_ℓ (e.g., by performing a final measurement of R_ℓ), is the experimentalist able to infer whether the system was subjected to additional dephasing operations at earlier times, i.e., can possible coherences at earlier times become manifest in different populations at the final time t_ℓ? If that is not the case, the dynamics are called ℓ-incoherent. We remark that a process that is ℓ-incoherent is not necessarily k-incoherent for $k \neq \ell$. It is therefore important to specify at which (sub)set of times the process is incoherent. In the following we will be only interested in processes which are ℓ-incoherent for all $\ell \in \{1, \ldots, n\}$, henceforth dubbed simply 'incoherent' (with respect to the preparation $\mathcal{A}_0$).

8.2.1 Recovering NCGD from incoherence

Let us now show how the NCGD condition can be recovered from this new definition of incoherence:

Theorem 36. *If the dynamics are Markovian, invertible and incoherent for all possible preparations, then they are also NCGD.*

Proof. Recall that the dynamics of an open quantum system is called NCGD with respect to the set of observables $\{R_k\}$ if

$$\Delta_\ell \Lambda_{\ell,k} \Delta_k \Lambda_{k,j} \Delta_j = \Delta_\ell \Lambda_{\ell,j} \Delta_j \tag{8.14}$$

for all $t_\ell \geq t_k \geq t_j \geq t_1$, where $\Lambda_{\ell,k}$ denotes the 'dynamical map' of the quantum system from time t_k to time t_ℓ.

By assumption of incoherence we have for an arbitrary preparation $\mathcal{A}_0$ and an arbitrary set of times $\{t_\ell, t_k, t_j\}$ with $\ell \geq k \geq j \geq 1$

$$\begin{aligned} &\mathfrak{T}[\Delta_\ell, \ldots, \Delta_k, \ldots, \Delta_j, \ldots, \mathcal{A}_0] \\ &= \mathfrak{T}[\Delta_\ell, \ldots, \mathcal{I}_k, \ldots, \Delta_j, \ldots, \mathcal{A}_0], \end{aligned} \tag{8.15}$$

where the dots denote identity operations. By Markovianity, this means that

$$\Delta_\ell \Lambda_{\ell,k} \Delta_k \Lambda_{k,j} \Delta_j \Lambda_{j,0} \mathcal{A}_0 \rho_0 = \Delta_\ell \Lambda_{\ell,j} \Delta_j \Lambda_{j,0} \mathcal{A}_0 \rho_0. \tag{8.16}$$

Since $\mathcal{A}_0$ is arbitrary and the dynamics are assumed to be invertible, this implies

$$\Delta_\ell \Lambda_{\ell,k} \Delta_k \Lambda_{k,j} \Delta_j = \Delta_\ell \Lambda_{\ell,j} \Delta_j. \tag{8.17}$$

Hence, the dynamics are NCGD. $\square$

The 'converse' of Theorem 36 reads as follows

Theorem 37. *If the dynamics is Markovian and NCGD, the dynamics is incoherent with respect to all preparations that result in a diagonal state (with respect to the observable R_1) at time t_1.*

Proof. Since the dynamics is Markovian and the state at time t_1 is diagonal, we always have

$$\begin{aligned} &\mathfrak{T}\left[\Delta_n, \begin{Bmatrix}\Delta_{n-1}\\ \mathcal{I}_{n-1}\end{Bmatrix}, \ldots, \begin{Bmatrix}\Delta_1\\ \mathcal{I}_1\end{Bmatrix}, \mathcal{A}_0\right] \\ &= \mathfrak{T}\left[\Delta_n, \begin{Bmatrix}\Delta_{n-1}\\ \mathcal{I}_{n-1}\end{Bmatrix}, \ldots, \Delta_1, \mathcal{A}_0\right]. \end{aligned} \tag{8.18}$$

Hence, the dynamics are 'sandwiched' by two dephasing operations at the beginning at time t_1 and at the end at time t_n. By the property of NCGD, we are allowed to introduce arbitrary dephasing/identity operations at any time step t_k, $n > k > 1$. Hence, the dynamics are incoherent. □

8.3 Results for non-degenerate observables

We now have the main tools at hand to rigorously state the question we are posing in this paper: Which conditions does a quantum stochastic process need to fulfill in order to guarantee that the resulting measurement statistics can (or cannot) be explained by a classical stochastic process? That is, when is Eq. (8.11) fulfilled or, in terms of the process tensor, when is

$$\begin{aligned} &\mathrm{tr}_S\{\mathfrak{T}[\mathcal{P}_{r_\ell}, \ldots, \Delta_k, \ldots, \mathcal{P}_{r_j}, \ldots, \mathcal{A}_0]\} \\ &= \mathrm{tr}_S\{\mathfrak{T}[\mathcal{P}_{r_\ell}, \ldots, \mathcal{I}_k, \ldots, \mathcal{P}_{r_j}, \ldots, \mathcal{A}_0]\}? \end{aligned} \tag{8.19}$$

Here, we have introduced the *dephasing operation* at time t_k, $\Delta_k \equiv \sum_{r_k} \mathcal{P}_{r_k}$, which plays an essential role in the following. Furthermore, the dots in Eq. (8.19) denote either projective measurements (if the system gets measured at that time) or identity operations (if the system does not get measured at that time).

Our first main result is the following:

Theorem 38. *If the measurement statistics are classical with respect to $\mathcal{A}_0$, then the dynamics is incoherent with respect to $\mathcal{A}_0$.*

Before we prove it, we remark that this theorem holds for any quantum stochastic process (especially without imposing Markovianity). Furthermore, a classical process for the times $\{t_n, \ldots, t_1\}$ is also classical for all subsets of times and hence, the theorem implies incoherence, i.e., ℓ-incoherence for all $\ell \in \{1, \ldots, n\}$. In the following proof we will only display the case $\ell = n$, as the rest follows immediately.

Proof. We start by noting that

$$\mathfrak{T}[\mathcal{P}_{r_n}, \ldots, \mathcal{P}_{r_1}, \mathcal{A}_0] = p(r_n, \ldots, r_1|\mathcal{A}_0)|r_n\rangle\langle r_n|, \tag{8.20}$$

which is a general identity as we have not made any assumption about the joint probability $p(r_n, \ldots, r_1|\mathcal{A}_0)$. Obviously, if we choose to perform nothing at any time $t_\ell < t_n$, we have

$$\begin{aligned} &\mathfrak{T}[\mathcal{P}_{r_n}, \ldots, \mathcal{I}_\ell, \ldots, \mathcal{P}_{r_1}, \mathcal{A}_0] \\ &= p(r_n, \ldots, r_\ell, \ldots, r_1|\mathcal{A}_0)|r_n\rangle\langle r_n|. \end{aligned} \tag{8.21}$$

But by assumption of classicality, this is equal to

$$\begin{aligned} &\mathfrak{T}[\mathcal{P}_{r_n}, \ldots, \mathcal{I}_\ell, \ldots, \mathcal{P}_{r_1}, \mathcal{A}_0] \\ &= \sum_{r_\ell} p(r_n, \ldots, r_\ell, \ldots, r_1|\mathcal{A}_0)|r_n\rangle\langle r_n| \\ &= \sum_{r_\ell} \mathfrak{T}[\mathcal{P}_{r_n}, \ldots, \mathcal{P}_{r_\ell}, \ldots, \mathcal{P}_{r_1}, \mathcal{A}_0] \\ &= \mathfrak{T}[\mathcal{P}_{r_n}, \ldots, \Delta_\ell, \ldots, \mathcal{P}_{r_1}, \mathcal{A}_0]. \end{aligned} \tag{8.22}$$

Hence, by summing Eq. (8.22) over the remaining $r_k \neq r_\ell$, we confirm

$$\mathfrak{T}[\Delta_n, \ldots, \mathcal{I}_\ell, \ldots, \Delta_1, \mathcal{A}_0] = \mathfrak{T}[\Delta_n, \ldots, \Delta_\ell, \ldots, \Delta_1, \mathcal{A}_0] \tag{8.23}$$

for arbitrary $t_\ell < t_n$ and where the dots denote dephasing operations at the remaining times. We can now pick another arbitrary time $t_k \neq t_\ell$ and repeat essentially the same steps as above to arrive at the conclusion

$$\begin{aligned} &\mathfrak{T}[\Delta_n, \ldots, \mathcal{I}_\ell, \ldots, \mathcal{I}_k, \ldots, \Delta_1, \mathcal{A}_0] \\ &= \mathfrak{T}[\Delta_n, \ldots, \Delta_\ell, \ldots, \Delta_k, \ldots, \Delta_1, \mathcal{A}_0] \end{aligned} \tag{8.24}$$

for any two times $t_k \neq t_\ell$. By repeating this argument further, we finally confirm that the dynamics are incoherent. □

The converse of Theorem 38 holds only in a stricter sense. For this purpose we need the notion of Markovianity as defined in Ref. [162]. In there, it was shown that any Markovian process is operationally CP divisible, i.e., for an arbitrary set of interventions (CP maps) $\mathcal{A}_{r_n}, \ldots, \mathcal{A}_{r_0}$

$$\mathfrak{T}[\mathcal{A}_{r_n}, \ldots, \mathcal{A}_{r_0}] = \mathcal{A}_{r_n}\Lambda_{n,n-1} \ldots \Lambda_{1,0}\mathcal{A}_{r_0}\rho_S(t_0) \tag{8.25}$$

with a family of CP and trace-preserving maps $\{\Lambda_{\ell,k}\}$ fulfilling the composition law $\Lambda_{\ell,j} = \Lambda_{\ell,k}\Lambda_{k,j}$ for any $\ell > k > j$. We remark that a CP divisible process (which is commonly refered to as being 'Markovian') is in general not *operationally* CP divisible (also see the recent discussion in Ref. [166]). In a nutshell, an operationally CP divisible process always fulfills the quantum regression theorem, but a CP divisible process does not.

Furthermore, to establish the converse of Theorem 38 we also need the following definition:

Definition 39. *A Markov process is said to be invertible, if the inverse of any $\Lambda_{k,0}$ exists for all k, i.e., the expression $\Lambda_{\ell,k} = \Lambda_{\ell,0}\Lambda_{k,0}^{-1}$ is well-defined and coincides with the CPTP maps appearing in Eq. (8.25).*

We are now ready to prove the next main theorem:

Theorem 40. *If the dynamics are Markovian, invertible and incoherent for all preparations $\mathcal{A}_0$, then the statistics are classical for any preparation.*

Proof. By using Eq. (8.25) and the property of incoherence, we can conclude that for any two times $t_{\ell+1}, t_\ell \in \{t_1, \dots, t_n\}$ (with $t_{\ell+1} > t_\ell$)

$$\Delta_{\ell+1}\Lambda_{\ell+1,\ell}\Delta_\ell\Lambda_{\ell,0}\mathcal{A}_0\rho_S(t_0) = \Delta_{\ell+1}\Lambda_{\ell+1,\ell}\Lambda_{\ell,0}\mathcal{A}_0\rho_S(t_0). \tag{8.26}$$

Since the dynamics are invertible and incoherent for all preparations $\mathcal{A}_0$, this implies the superoperator identity $\Delta_{\ell+1}\Lambda_{\ell+1,\ell}\Delta_\ell = \Delta_{\ell+1}\Lambda_{\ell+1,\ell}$. By multiplying this equation with $\mathcal{P}_{r_{\ell+1}}$, we arrive at

$$\sum_{r_\ell} \mathcal{P}_{r_{\ell+1}}\Lambda_{\ell+1,\ell}\mathcal{P}_{r_\ell} = \mathcal{P}_{r_{\ell+1}}\Lambda_{\ell+1,\ell}. \tag{8.27}$$

From this general identity we immediately obtain that

$$\begin{aligned}
&\sum_{r_\ell} p(\mathbf{r}_n)\\
&= \mathrm{tr}\left\{\mathcal{P}_{r_n}\Lambda_{n,n-1}\cdots\sum_{r_\ell}\mathcal{P}_{r_{\ell+1}}\Lambda_{\ell+1,\ell}\mathcal{P}_{r_\ell}\dots\mathcal{P}_{r_1}\Lambda_{1,0}\mathcal{A}_0\rho\right\}\\
&= \mathrm{tr}\{\mathcal{P}_{r_n}\Lambda_{n,n-1}\dots\mathcal{P}_{r_{\ell+1}}\Lambda_{\ell+1,\ell}\dots\mathcal{P}_{r_1}\Lambda_{1,0}\mathcal{A}_0\rho\}\\
&= p(r_n, \dots, r_\ell, \dots, r_1).
\end{aligned} \tag{8.28}$$

This concludes the proof as the above argument also holds for all possible subsets of times. □

One remaining open question concerns the assumption of Markovianity. At the moment it is not clear whether relaxing this condition is meaningful as it requires to define the notion of invertibility for a non-Markovian process, which is not unambiguous.

Furthermore, the superoperator identity (8.27) implies that, if we write $\Lambda_{\ell,k}$ as a matrix in an ordered basis where populations precede coherences with respect to the measured observable R_k (input) and R_ℓ (output), it has the form

$$\Lambda_{\ell,k} = \begin{pmatrix} A_{\ell,k} & 0 \\ C_{\ell,k} & D_{\ell,k} \end{pmatrix}, \tag{8.29}$$

where $A_{\ell,k}$ is a stochastic matrix and $C_{\ell,k}$ and $D_{\ell,k}$ are arbitrary matrices, which are only constrained by the requirement of complete positivity.

In addition, the following counterexamples demonstrate that Theorem 40 is also tight in the sense that a process, which is incoherent only for a subset of preparations or which is not invertible, does not imply classical statistics.

A process which is incoherent for one preparation $\mathcal{A}_0$ but not classical for that preparation

Consider an isolated two-level system undergoing purely unitary dynamics. Then, the dynamics are incoherent with respect to any preparation $\mathcal{A}_0$ which maps the system state to a completely mixed state: independent of any dephasing or identity operation, the state will stay at the origin of the Bloch ball for all times.

However, such a dynamics does not necessarily imply classical statistics. Consider, e.g., the measurement basis to be σ_z (with outcomes $\{\uparrow_k, \downarrow_k\}$ at times t_k) and the unitary rotations to be around the y-axis. Furthermore, the time-steps are chosen equidistant in such a way that the rotation is exactly $\pi/2$. It is then easy to confirm that

$$p(\uparrow_3, \uparrow_2, \uparrow_1) = p(\uparrow_3, \downarrow_2, \uparrow_1) = \frac{1}{8}, \tag{8.30}$$

hence, $\sum_{\sigma_2 \in \{\uparrow,\downarrow\}} p(\uparrow_3, \sigma_2, \uparrow_1) = 1/4$. But if we do not perform any measurement at time t_2, we obtain $p(\uparrow_3, \not\sigma_2, \uparrow_1) = 0$. The statistics are therefore non-classical.

A process which is Markovian and incoherent for all preparations but not classical

Consider a Markov process for a two-level system where the map in the first time-step is defined by

$$\Lambda_{2,1} : \rho \mapsto \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{8.31}$$

for any input state ρ. The rest of the dynamics is again unitary as in the previous counterexample. Thus, the dynamics are incoherent for any preparation, but not classical.

8.4 Results for degenerate observables

If the measured observable contains degeneracies, the picture above somewhat reverses. First, Theorem 38 ceases to hold even in the Markovian and invertible regime because it now becomes possible to hide coherences in degenerate subspaces and this can have a detectable effect on the output state (8.13). This is demonstrated with the help of the following example:

Consider two qubits A and B and projective measurements in some fixed basis of qubit A only such that the dephasing operation acts only locally on qubit A: $\Delta = \Delta_A \otimes \mathcal{I}_B$. Thus, the measured observable is degenerate and projects onto two possible subspaces of dimension two. Furthermore, we only consider measurements at two times t_2 and t_1 and assume the dynamics in between these two times to be described by a unitary swap gate, $U_{\text{swap}}|i_A j_B\rangle = |j_B i_A\rangle$. We also assume that the dynamics in between the preparation and the first measurement is trivial, i.e., described by an identity operation.

Now, consider an arbitrary initial state resulting from an arbitrary preparation $\mathcal{A}_0$, denoted as

$$\mathcal{A}_0\rho_0 = \sum_{i_A,i_B,j_A,j_B} \rho_{i_A i_B, j_A j_B} |i_A i_B\rangle\langle j_A j_B|. \tag{8.32}$$

Then, straightforward calculation reveals that

$$p(r_2, r_1) = \text{tr}_{AB}\{\mathcal{P}_{r_2}\mathcal{U}_{\text{swap}}\mathcal{P}_{r_1}\mathcal{A}_0\rho_0\} = \rho_{r_1 r_2, r_1 r_2}, \tag{8.33}$$

$$p(r_2, \not r_1) = \text{tr}_{AB}\{\mathcal{P}_{r_2}\mathcal{U}_{\text{swap}}\mathcal{A}_0\rho_0\} = \sum_j \rho_{j r_2, j r_2}. \tag{8.34}$$

Hence, the process is classical: $\sum_{r_1} p(r_2, r_1) = p(r_2, \not r_1)$.

However, the process is not incoherent. Consider, for instance, the initial state

$$\mathcal{A}_0\rho_0 = |\psi_0\rangle\langle\psi_0|, \quad |\psi_0\rangle = \frac{|0_A\rangle + |1_A\rangle}{\sqrt{2}} \otimes |0_B\rangle. \tag{8.35}$$

Then,

$$\mathfrak{T}[\Delta_2, \mathcal{I}_1, \mathcal{A}_0] = |\psi_1\rangle\langle\psi_1|, \quad |\psi_1\rangle = |0_A\rangle \otimes \frac{|0_B\rangle + |1_B\rangle}{\sqrt{2}}, \tag{8.36}$$

but

$$\mathfrak{T}[\Delta_2, \Delta_1, \mathcal{A}_0] = \frac{|0_A 0_B\rangle\langle 0_A 0_B| + |0_A 1_B\rangle\langle 0_A 1_B|}{2}. \tag{8.37}$$

In contrast, Theorem 40 still holds true for degenerate observables. In fact, in the proof of Theorem 40 we never used that the measured observable is non-degenerate.

8.5 Chapter summary

- The definition of NCGD dynamics is problematic insofar as it does not allow to establish a correspondence between coherence and classicality of non-Markovian processes. *Incoherent* dynamics is put forward to remedy this situation. From it, the NCGD property can be easily recovered.
- When measuring non-degenerate observables, obtaining classical statistics implies that the underlying dynamics is incoherent, no matter of what kind this dynamics is. Conversely, Markovian invertible incoherent dynamics yield classical stochastic processes.
- When measuring degenerate observables, classicality does not imply incoherence anymore, since coherence can be hidden in degenerate subspaces. Yet Markovian invertible incoherent dynamics still give rise to classical processes.

8.6 Open questions

- In which conditions does incoherence imply classicality when considering non-Markovian statistics? For that we would need to propose a definition of invertibility for non-Markovian processes, which is not unambiguous.

Final thoughts

"Those were the days, when we were all at sea. It seems like yesterday to me.
Species, sex, race, class: in those days none of this meant anything at all. No parents, no children, just ourselves, strings of inseparable sisters, warm and wet, indistinguishable one from the other, gloriously indiscriminate, promiscuous and fused. No generations. No future, no past. An endless geographic plane of micromeshing pulsing quanta, limitless webs of interacting blendings, leakings, mergings, weaving through ourselves, running rings round each other, heedless, needless, aimless, careless, thoughtless, amok."

Sadie Plant,
Zeros + Ones

Fog has permeated this thesis. Throughout it we have dealt with the undefined, with the raw potentialities quantum systems have to become something nobody can predict with certainty—a sort of primordial sea where identities have not been written yet. This property of existing blurrily, of being in a *flou*, *brouillé* state—as Schrödinger put it [167]—, which is in turn a superposition of different states, is called quantum coherence.

As explained, coherence is not such an outsider though: indeed, it does manifest itself through intereference phenomena (remember that events which can occur in several alternative ways lead to interference patterns [1]). This means that, at the quantum level, there exists an intertwining between knowledge and physical reality, between subjects and their objects of study. Such an epistemological framework defines what one could call the paradigm of immersion. Unlike classical scenarios, where sovereign subjects are understood as distinct and separate from the systems under investigation, whose properties they can read in all their truth ("As the eye, such the object", in words of William Blake [168]), quantum contexts imply participation, immersion of the subject into the physical problem that is to be solved. The classical dualism subject/object is then replaced by the quantum paradigm, as well as the dualism of Descartes or Plato was confronted by the monism of Spinoza or Aristotle, respectively.

As we have already mentioned, besides its fascinating epistemological implications, quantum coherence turns out to be resourceful in a number of fields ranging from quantum technologies to biology, which calls for the creation of a theoretical framework that rigourosly describes its essence and properties. That is why the resource theory of coherence has been developed, to which this thesis intends to make a contribution.

In the first part of this work we learnt what the basic ingredients of coherence theory are. In particular, after fixing a basis, the free states are those diagonal in that basis and the free operations are non-coherence-generating maps, i.e. maps that leave incoherent states incoherent. Among the great number of non-coherence-generating maps that can be found in the literature, we focused on the largest class, MIOs, as well as in the subclasses of IOs and SIOs. Moreover, we defined the maximally resourceful state of the theory, which we denoted as cosdit, and we explained how a good coherence measure should look like. We also presented some examples of relevant coherence monotones in the MIO-theory, such as the relative entropy of coherence and the robustness of coherence, and in the IO-theory, such as the l_1-norm of coherence, and we remarked that most of such measures are not just axiomatic constructions fulfilling a list of desiderata, but also quantities with an operational meaning.

In particular, we raised the fundamental question of how coherence, as originally conceived by the physics of interferometers, appears in the resource-theoretic framework. The answer we found demonstrates that, at least under SIOs, coherence theory is grounded in the physical and even explains how to measure coherence in the laboratory. Indeed, when fixing SIOs as free operations, coherence measures can be derived by optimizing suitable visibility parameteres over detection schemes. In this way, one can obtain coherence measures that are related to, or even identified with, some of the already known ones.

In the second part we noted that any state can also be thought of as a constant-output channel, meaning that the resource theory of states, which we had been using up to then, could potentially be generalized to that of channels. With that purpose in mind, we proposed several quantifiers of the coherence of a quantum channel, which we assessed in terms of i) how much coherence the channel can produce (some examples of such quantifiers are the coherence power, the robustness of coherence and the asymptotic coherence generating capacity), and ii) how much coherence is required to implement the channel by means of non-coherence-generating operations (some examples of such measures include the simulation cost, both single-shot and asymptotic, and the amortized simulation cost). This broader framework then allows us to exploit coherence in two different ways: i) by turning dynamic coherence into static one, as occurs if we are to store the coherence generated by a certain map in a given system, and ii) by converting static coherence into dynamic coherence, as happens if we have to simulate a given channel using only free operations and a supply of coherent states.

We further computed most of the previously mentioned measures in the IO-theory and in the MIO-theory, and obtained, among others, the following results: first, the IO asymptotic generating capacity of a channel $\mathcal{N}$ is i) bounded by the coherence power of the map resulting from appending the identity to $\mathcal{N}$, and ii) smaller or equal to the asymptotic simulation cost, which makes the theory asymptotically irreversible; second, any channel can be implemented via IOs and a cosdit; third, in the IO-theory operations may exhibit bound coherence, in constrast to states; fourth, the MIO single-shot simulation cost of a channel is given by its robustness of coherence; fifth, under MIOs, the log-robustness of coherence equals the coherence power of the channel and it characterizes the amortized implementation cost with recycling of coherence at the output, as well as the asymptotic cost of realizing exactly many independent instances of the channel; sixth, any coherent state in dimension larger than 2, however weakly so, is useful for the exact implementation of some coherent unitary channel via MIOs; seventh, a semidefinite program can be employed to find the best approximation to any given channel using MIOs and an arbitrary resource. We finished the section by making a comparison between IO-theory and MIO-theory, where we remarked that both the IO/MIO implementation of a qubit unitary are possible if an only if a cosbit is used as fuel, and that, in general, IO-theory does not allow for coherence recycling, in contrast to MIO-theory. We also noted that, whereas the MIO-theory of states is reversible, unlike the IO-theory, we do not know yet whether the same holds for channels.

In the third and last part, we investigated how coherence is related to non-classicality, when the latter is embodied by features beyond the ability to yield interference effects. In particular, our purpose was to determine whether the non-classicality of a quantum stochastic process is connected in some way with the coherence properties of the underlying dynamics. At first we learnt that the coherence-generating capabilities alone of a quantum evolution are not the right property to consider, but rather its coherence-generating-and-detecting (CDG) capabilities. Morever, we found that, when further considering Markovian Lindblad dynamics, a one-to-one correspondence can be established between CGD and non-classicality (*). When moving away from Markovianity, though, this is not true anymore. In fact, to study non-Markovian settings, one has to generalize the definition of NCGD, i.e. not CGD, to that of *incoherence*. This way one can see that incoherence is a necessary condition for classicality, no matter the kind of dynamics one has chosen. Moreover, the converse is only true when restricting to Markovian scenarios with invertible dynamics—which generalizes the result in (*) from time-homogeneous to invertible dynamics—. As desired, the results of the NCGD framework can be easily recovered from those of the more general one.

All in all, this thesis has tried to capture, in a mathematical way, the philosophically beautiful notion of quantum coherence, with the hope of providing a better understanding of how superposition can help us map the fuzzy frontier between the classical and the quantum realms, while also being leverageable as a physical resource.

Theoria and *poiesis* blended in some ineffable tissue.

Finally, we would like to point the reader to some lines of research on which our results may also spur investigation, which go beyond the more specific open questions that were presented at the end of the chapters:

- **Connection with other resource theories.** In [65] it was pointed out that, at least under IOs, coherence theory resembles so closely the theory of *maximally correlated* entangled states, in the sense that the coherence cost and the distillable coherence of a given $\rho = \sum \rho_{ij} |i\rangle\langle j|$ are equal to the entanglement cost and the distillable entanglement, respectively, of $\tilde{\rho} = \sum \rho_{ij} |ii\rangle\langle jj|$. How could the results of this thesis be related to those in entanglement theory? What is more, could there be connections between our results and other resource theories like those of asymmetry and thermodynamics?

- **Experimental applicability of coherence theory.** We have shown that, under SIOs, coherence can be measured in the laboratory using interferometric-based coherence measures. Can this result be extended to the most general setting, the MIO-theory? Moreover, how can MIOs be implemented experimentally? This would allow, for instance, for the design of explicit protocols to simulate quantum channels via MIOs and coherent states.

- **Resource theories of channels.** The resource theory of coherence for channels is still under construction. What one would like to have there is a proper resource theory, where free objects are incoherent maps and free operations are superoperations (or quantum combs [147]) that leave incoherent maps incoherent. First steps towards building a coherence theory of maps have been given by [61, 74, 104, 105]. More interestingly, it would be desirable to extend all these results further to general resource theories of channels, where resources are left completely general [64, 106]. What is more, how do general resource theories look like beyond quantum mechanics, i.e. in the framework of General Probability Theories [169]?

Appendix

APPENDIX A

The robustness of coherence of a channel

Here we prove several properties of the robustness of coherence for quantum channels, introduced in 3.3. We assume $\Phi_{\text{NCG}} = \mathcal{M} \in$ MIOs. Let us start by showing that it is a coherence measure for a coherence theory whose free resources are indeed MIOs.

1. *Faithfulness*: $C_R(\mathcal{N}) = 0$ if and only if $\mathcal{N}$ is MIO. This follows straightforwardly from the definition, since one can take $\lambda = 1$ if and only if $\mathcal{M} \equiv \mathcal{N}$.

2. *Monotonicity*: The robustness of coherence of the channel $\mathcal{N}$ monotonically decreases if we pre- and post-process the input and output using MIO channels $\mathcal{L}'$ and $\mathcal{L}$ respectively, i.e., $C_R(\mathcal{L} \circ \mathcal{N} \circ \mathcal{L}') \leq C_R(\mathcal{N})$, $\forall\, \mathcal{L}$, $\mathcal{L}'$ is MIO. Indeed, take $\lambda \geq 0, \mathcal{M}$ MIO such that $\lambda\mathcal{M} - \mathcal{N} \geq 0$ and concatenate it with the channels $\mathcal{L}, \mathcal{L}'$. Defining $\mathcal{M}' = \mathcal{L} \circ \mathcal{M} \circ \mathcal{L}'$ MIO it follows that $\mathcal{L} \circ \mathcal{N} \circ \mathcal{L}' \leq \lambda\mathcal{M}'$ as well, so that any λ feasible for $\mathcal{N}$ is also feasible for $\mathcal{L} \circ \mathcal{N} \circ \mathcal{L}'$.

3. *Convexity*: $\sum_i p_i C_R(\mathcal{N}_i) \geq C_R(\sum_i p_i \mathcal{N}_i)$ for any probability distribution $\{p_i\}$ and $\{\mathcal{N}_i\}$ a collection of quantum channels. Indeed for each i take the minimum λ_i such that $\mathcal{N}_i \leq \lambda_i \mathcal{M}_i$, $\mathcal{M}_i$ MIO. Then define $\bar{\lambda} = \sum_i p_i \lambda_i$ and note that $\{\widetilde{p}_i = p_i \lambda_i / \bar{\lambda}\}$ is still a probability distribution. By averaging the inequalities over $\{p_i\}$ and rescaling by $\bar{\lambda}$ we get $\sum_i p_i \mathcal{N}_i \leq \bar{\lambda} \sum_i \widetilde{p}_i \mathcal{M}_i$, where the latter is still a MIO. We conclude that $1 + C_R(\sum_i p_i \mathcal{N}_i) \leq \bar{\lambda} = \sum_i \lambda_i p_i$, and thus $\sum_i p_i C_R(\mathcal{N}_i) \geq C_R(\sum_i p_i \mathcal{N}_i)$.

Properties 1-3 above straightforwardly extend to the smooth-robustness of Eq. (3.4) (the faithfulness condition holds up to error ϵ).

Let us now focus on the SDP formulation of the robustness. We want to calculate the dual of the SDP

$$1 + C_R(\mathcal{N}) = \min\{\lambda : \mathcal{M}' \text{ is MIO and } \mathcal{N} \leq \lambda\mathcal{M}' =: \mathcal{M}\},$$

which is equivalent to

$$\begin{aligned} &\min\ \lambda \\ &\text{s.t. } J_{\mathcal{N}} \leq J_{\mathcal{M}} \\ &\quad \operatorname{Tr}_B J_{\mathcal{M}} = \lambda \mathbb{1}_A \\ &\quad \operatorname{Tr} J_{\mathcal{M}}(|i\rangle\langle i| \otimes |j\rangle\langle k|) = 0,\ \forall i,\ \forall j \neq k, \end{aligned} \tag{A.1}$$

where $J_{\mathcal{N}}$, $J_{\mathcal{M}}$ are the Choi matrices of $\mathcal{N}$ and $\mathcal{M}$, respectively. This means that we need to minimize the following Lagrangian:

$$\begin{aligned} L &= \lambda + \operatorname{Tr} X(\operatorname{Tr}_B J_{\mathcal{M}} - \lambda \mathbb{1}_A) - \operatorname{Tr} Y(J_{\mathcal{M}} - J_{\mathcal{N}}) \\ &\quad + \sum_{i,j\neq k} z^i_{jk} \operatorname{Tr} J_{\mathcal{M}}(|i\rangle\langle i| \otimes |j\rangle\langle k|) \\ &= \lambda + \operatorname{Tr}(X \otimes \mathbb{1})J_{\mathcal{M}} - \lambda \operatorname{Tr} X - \operatorname{Tr} Y(J_{\mathcal{M}} - J_{\mathcal{N}}) \\ &\quad + \operatorname{Tr} J_{\mathcal{M}}\Big(\sum_{i,j\neq k} |i\rangle\langle i| \otimes Z_i\Big) \\ &= \lambda + \operatorname{Tr}(X \otimes \mathbb{1})J_{\mathcal{M}} - \lambda \operatorname{Tr} X - \operatorname{Tr} Y(J_{\mathcal{M}} - J_{\mathcal{N}}) \\ &\quad + \operatorname{Tr}(J_{\mathcal{M}} Z) \\ &= \lambda(1 - \operatorname{Tr} X) + \operatorname{Tr} J_{\mathcal{M}}(X \otimes \mathbb{1} - Y + Z) + \operatorname{Tr}(J_{\mathcal{N}} Y), \end{aligned} \tag{A.2}$$

where $Z = \sum_i |i\rangle\langle i| \otimes Z_i$ and Z_i is a zero-diagonal matrix, while $Y \geq 0$. In order to avoid that $\min_{\lambda, J_{\mathcal{M}}} L = -\infty$, we need to impose that $\operatorname{Tr} X = 1$ (so X is a state) and $Y = X \otimes \mathbb{1} + Z$.

To calculate the dual, we maximize the terms of the Lagrangian which do not contain the variables of the primal form, λ and $J_{\mathcal{M}}$:

$$\max\{\operatorname{Tr} J_{\mathcal{N}} Y : \operatorname{Tr} X = 1, Y = X \otimes \mathbb{1} + Z \geq 0\}. \tag{A.3}$$

The objective function is equivalent to

$$\begin{aligned} \operatorname{Tr} J_{\mathcal{N}} Y &= \operatorname{Tr} J_{\mathcal{N}}(X \otimes \mathbb{1}) + \operatorname{Tr}(J_{\mathcal{N}} Z) \\ &= 1 + \operatorname{Tr}(J_{\mathcal{N}} Z) \\ &= 1 + \sum_i \operatorname{Tr} Z_i \mathcal{N}(|i\rangle\langle i|), \end{aligned} \tag{A.4}$$

where we have made use of the identity $\mathrm{Tr}_B((X \otimes \mathbb{1})J_{\mathcal{N}}) = \mathcal{N}(X^T)$, and the fact that $\mathcal{N}$ is CPTP in the second line of Eq. (A.4). As X no longer appears in the objective function, and dephasing it does not affect the constraints, we may assume $X = \sum_i p_i |i\rangle\langle i|$, where p_i are probabilities, without loss of generality. Therefore, we can write

$$\begin{aligned}
&\max \left\{ \sum_i p_i + \sum_i \mathrm{Tr}\, Z_i \mathcal{N}(|i\rangle\langle i|) : p_i \mathbb{1} + Z_i \geq 0\, \forall i \right\} \\
&= \max \sum_i p_i \,\mathrm{Tr}\, S_i \mathcal{N}(|i\rangle\langle i|) \\
&= \max \sum_i p_i (1 + C_R(\mathcal{N}(|i\rangle\langle i|))) \\
&= \max_i \{1 + C_R(\mathcal{N}(|i\rangle\langle i|))\},
\end{aligned} \tag{A.5}$$

where $S_i = \mathbb{1} + p_i^{-1} Z_i \geq 0$ and $\langle j| S_i |j\rangle = 1$ for all j, and we have used the dual form of robustness for states, Eq. (1.21). Moreover, since the latter is multiplicative under tensor product, also the robustness of channels is.

The logged version of the robustness of coherence follows directly from Eq. (A.5), and the equality of the log-robustness of coherence to the cohering power, $P_R(\mathcal{N})$, of Eq. (3.5) is also evident from the definition of the latter. To see that the log-robustness of coherence for a channel is also equal to $\widehat{P}(\mathcal{N})$, observe that for any $\rho \otimes \omega$

$$\begin{aligned}
C_{LR}(\rho \otimes \omega) &\geq C_{LR}(\mathcal{M}(\rho \otimes \omega)) = C_{LR}(\mathcal{N}(\rho) \otimes \sigma) \\
C_{LR}(\rho) + C_{LR}(\omega) &\geq C_{LR}(\mathcal{N}(\rho)) + C_{LR}(\sigma) \\
C_{LR}(\omega) - C_{LR}(\sigma) &\geq C_{LR}(\mathcal{N}(\rho)) - C_{LR}(\rho).
\end{aligned} \tag{A.6}$$

Thanks to Corollary 11, the minimization of the right-hand side of Eq. (A.6) over ω and σ yields the amortized cost (Eq. (3.8)), which is equal to the log-robustness of coherence of $\mathcal{N}$ via Theorem 12. Moreover, the inequality in Eq. (A.6) holds for all $\rho \in \mathcal{H}$. Thus

$$C_{LR}(\mathcal{N}) \geq \max_{\rho \in \mathcal{S}(\mathcal{H})} \left(C_{LR}(\mathcal{N}(\rho)) - C_{LR}(\rho)\right) \geq P_R(\mathcal{N}), \tag{A.7}$$

where the last inequality holds because the maximization in $\widehat{P}_R(\mathcal{N})$ is over a larger convex set than that of $P_R(\mathcal{N})$. As the upper and lower bounds on the coherence power are equal it follows that $C_{LR}(\mathcal{N}) = \widehat{P}_R(\mathcal{N})$.

Similarly, one can write the smoothed robustness of coherence (Definition 3.3) as an SDP in primal and dual form. Using the dual-SDP formulation of the diamond norm [111], the primal SDP of Eq. (3.4) reads

$$
\begin{aligned}
1 + C_R^{\epsilon}(\mathcal{N}) = \min\ & \lambda \\
\text{s.t.}\ & J_{\mathcal{M}} \geq J_{\mathcal{L}}, \\
& \operatorname{Tr}_B J_{\mathcal{M}} = \lambda \mathbb{1}_A, \\
& \operatorname{Tr} J_{\mathcal{M}}(|ij\rangle\langle ik|) = 0,\ \forall i \forall j \neq k, \\
& V \geq J_{\mathcal{L}} - J_{\mathcal{N}}, \\
& \operatorname{Tr}_B V \leq \epsilon \mathbb{1}_A, \\
& \operatorname{Tr}_B J_L = \mathbb{1}_A, \\
& J_L \geq 0,\ V \geq 0.
\end{aligned}
\tag{A.8}
$$

The first three constraints correspond to the simulation of channel $\mathcal{L}$ by MIO and the fourth and fifth constraints capture the diamond norm constraints that $\mathcal{L}$ should be ϵ close to $\mathcal{N}$. The dual form of Eq. (A.8) is given by

$$
\begin{aligned}
\max\ & \operatorname{Tr}(J_{\mathcal{N}}(S - \Delta - \epsilon \Omega \otimes \mathbb{1})) \\
\text{s.t.}\ & S \equiv Y \otimes \mathbb{1} - \sum_i |i\rangle\langle i| \otimes Z_i \geq 0 \\
& S - \Delta \leq W \otimes \mathbb{1} \leq S - \Delta + \Omega \otimes \mathbb{1} \\
& \operatorname{Tr} Z_i = 0,\ \forall i \\
& \operatorname{Tr} W = 0,\ \operatorname{Tr} Y = 1 \\
& \Delta \geq 0,\ \Omega \geq 0,\ Y \geq 0.
\end{aligned}
\tag{A.9}
$$

Finally let us state the connection between the log-robusteness of coherence of a channel (correspondingly its smoothed version) to the maximum relative entropy (correspondingly ϵ-maximum relative entropy) of the latter with respect to a MIO:

Definition 41. *The maximum relative entropy of two channels $\mathcal{N}$ and $\mathcal{M}$ is:*

$$
\begin{aligned}
D_{\max}(\mathcal{N}\|\mathcal{M}) := -\log \max\ & p \\
\text{s.t.,}\ & \mathcal{M} = p\mathcal{N} + (1-p)\mathcal{N}' \\
& p \in [0,1],\ \mathcal{N}' \in CPTP, \\
= \log \min & \left\{ \lambda : \mathcal{N} \leq \lambda \mathcal{M} \right\}.
\end{aligned}
\tag{A.10}
$$

The smoothed version of this quantity is:

$$
D_{\max}^{\epsilon}(\mathcal{N}\|\mathcal{M}) := \min \left\{ D_{\max}(\mathcal{L}\|\mathcal{M}) : \frac{1}{2}\|\mathcal{N} - \mathcal{L}\|_{\diamond} \leq \epsilon \right\}. \tag{A.11}
$$

Proposition 42. *The ϵ-log-robustness of a channel is given by*

$$
C_{LR}^{\epsilon}(\mathcal{N}) = \min \left\{ D_{\max}^{\epsilon}(\mathcal{N}\|\mathcal{M}) : \mathcal{M}\ MIO \right\}. \tag{A.12}
$$

APPENDIX B

Proof of Theorem 8

Here we state the technical lemma required in the proof of Theorem 8.

Lemma 43. *The relative entropy coherence power and the coherence of formation power are asymptotically continuous with respect to the diamond norm metric on channels. To be precise, for* $\mathcal{N}_1, \mathcal{N}_2 : A \longrightarrow B$ *with* $\frac{1}{2}\|\mathcal{N}_1 - \mathcal{N}_2\|_\diamond \leq \epsilon$,

$$\left|\widehat{P}_{C_r}(\mathcal{N}_1 \otimes \mathrm{id}_k) - \widehat{P}_{C_r}(\mathcal{N}_2 \otimes \mathrm{id}_k)\right| \leq 4\epsilon \log |B| + 2g(\epsilon), \tag{B.1}$$

$$\left|\widehat{P}_{C_f}(\mathcal{N}_1 \otimes \mathrm{id}_k) - \widehat{P}_{C_f}(\mathcal{N}_2 \otimes \mathrm{id}_k)\right| \leq \epsilon(\log |B| + \log k) + g(\epsilon), \tag{B.2}$$

where $g(x) = (1+x)h_2\left(\frac{x}{1+x}\right) = (1+x)\log(1+x) - x\log x$.

Proof. For the first bound, observe

$$
\begin{aligned}
&\left|\widehat{P}_{C_r}(\mathcal{N}_1 \otimes \mathrm{id}_k) - \widehat{P}_{C_r}(\mathcal{N}_2 \otimes \mathrm{id}_k)\right| \\
&\quad \leq \max_{\rho^{AC}} \left|C_r\Big((\mathcal{N}_1 \otimes \mathrm{id}_k)\rho\Big) - C_r\Big((\mathcal{N}_2 \otimes \mathrm{id}_k)\rho\Big)\right| \\
&\quad = \max_{\rho^{AC}} \Big|S(BC)_{(\Delta\mathcal{N}_1\otimes\Delta)\rho} - S(BC)_{(\mathcal{N}_1\otimes\mathrm{id}_k)\rho} \\
&\qquad\qquad -S(BC)_{(\Delta\mathcal{N}_2\otimes\Delta)\rho} + S(BC)_{(\mathcal{N}_2\otimes\mathrm{id}_k)\rho}\Big| \\
&\quad = \max_{\rho^{AC}} \Big|S(B|C)_{(\Delta\mathcal{N}_1\otimes\Delta)\rho} - S(B|C)_{(\mathcal{N}_1\otimes\mathrm{id}_k)\rho} \\
&\qquad\qquad -S(B|C)_{(\Delta\mathcal{N}_2\otimes\Delta)\rho} + S(B|C)_{(\mathcal{N}_2\otimes\mathrm{id}_k)\rho}\Big| \\
&\quad \leq \max_{\rho^{AC}} \Big|S(B|C)_{(\Delta\mathcal{N}_1\otimes\Delta)\rho} - S(B|C)_{(\Delta\mathcal{N}_2\otimes\Delta)\rho}\Big| \\
&\qquad\qquad + \Big|S(B|C)_{(\mathcal{N}_1\otimes\mathrm{id}_k)\rho} - S(B|C)_{(\mathcal{N}_2\otimes\mathrm{id}_k)\rho}\Big| \\
&\quad \leq 2\Big(2\epsilon\log|B| + g(\epsilon)\Big),
\end{aligned}
$$

where in the first line we insert the same variable ρ^{AC} to maximise $\widehat{P}_{C_r}(\mathcal{N}_j \otimes \mathrm{id}_k)$ and notice that in that case, the term $C_r(\rho)$ cancels; then in the second line, we use the definition of the relative entropy of coherence and in the third we use chain rule $S(BC) = S(B|C) + S(C)$ for the entropy, allowing us to cancel matching $S(C)$ terms; in the fourth line we invoke the triangle inequality, and finally the Alicki-Fannes bound for the conditional entropy [170] in the form given in [171, Lemma 2].

For the second bound, we start very similarly:

$$
\begin{aligned}
&\left|\widehat{P}_{C_f}(\mathcal{N}_1 \otimes \mathrm{id}_k) - \widehat{P}_{C_f}(\mathcal{N}_2 \otimes \mathrm{id}_k)\right| \\
&\quad \leq \max_{\rho^{AC}} \left|C_f\Big((\mathcal{N}_1 \otimes \mathrm{id}_k)\rho\Big) - C_f\Big((\mathcal{N}_2 \otimes \mathrm{id}_k)\rho\Big)\right| \\
&\quad \leq \epsilon(\log|B| + \log k) + g(\epsilon),
\end{aligned}
$$

where the last line comes directly from the asymptotic continuity of the coherence of formation [65, Lemma 15]. □

APPENDIX C

MIO-simulation with arbitrary resources

C.1 Implementation of unitary gates: an SDP

Here we provide an SDP for the optimal implementation of unitary gates without recycling using the gate fidelity as a figure of merit. We then proceed to prove Proposition 14. For ease of notation, we swap the output subspaces of the MIO implementation map throughout this section, i.e., $\mathcal{M} : R\otimes A \to B\otimes S$. We also consider unnormalized Choi matrices.

Proposition 44. *The optimal gate fidelity of implementation of a unitary $U : A \to B$ by means of a MIO $\mathcal{M} : R \otimes A \to B \otimes S$ and a pure coherent state $\omega \in R$ is given by the following SDP:*

$$\begin{aligned} F = \max\ & \frac{1}{d_A^2} \operatorname{Tr}((\omega^T \otimes J_U \otimes \mathbb{1}_S)X) \\ \text{s.t. }\ & X \text{ is the Choi matrix of a MIO}, \end{aligned} \tag{C.1}$$

where J_U is the Choi matrix of the channel $U \cdot U^\dagger$, $d_A = \dim(A)$, and ω^T denotes the transpose of ω.

Proof. Consider a maximally incoherent operation $\mathcal{M} : R \otimes A \to B \otimes S$. Its Choi matrix is defined as

$$\begin{aligned} J_{\mathcal{M}} &= (\mathrm{id}_{RA} \otimes \mathcal{M})(\Phi_{RR'} \otimes \Phi_{AA'}) \\ &= \sum_{i,k,j,l} |ik\rangle_{RA} \langle jl| \otimes \mathcal{M}\left(|ik\rangle_{R'A'} \langle jl|\right). \end{aligned} \tag{C.2}$$

Suppose that $\mathcal{E} = \mathrm{Tr}_S(\mathcal{M}(\omega \otimes \cdot))$, where $\mathcal{E} : A \to B$. Then the Choi matrix of $\mathcal{E}$, $J_{\mathcal{E}}$, is explicitly given by

$$\begin{aligned} J_{\mathcal{E}} &= \mathrm{Tr}_{RS}((\omega^T \otimes \mathbb{1}_A \otimes \mathbb{1}_B \otimes \mathbb{1}_S) J_{\mathcal{M}}) \\ &= \sum_{ijkl} \langle j| \omega^T |i\rangle |k\rangle_A \langle l| \otimes \mathrm{Tr}_S \left(\mathcal{M}(|ik\rangle_{R'A'} \langle jl|)\right) \\ &= \sum_{k,l} |k\rangle_A \langle l| \otimes \mathrm{Tr}_S \left(\mathcal{M}(\omega \otimes |k\rangle_{A'} \langle l|)\right) \\ &= (\mathrm{id}_A \otimes \mathcal{E})(\Phi_{AA'}) \end{aligned} \tag{C.3}$$

On the other hand, the Choi matrix of the unitary channel is given by $J_U = (\mathbb{1}_A \otimes U)\Phi_{AA'}(\mathbb{1}_A \otimes U^\dagger)$.

Our aim is to compute how well the map $\mathcal{E}$ implements the unitary channel $\mathcal{N}(\rho) = U\rho U^\dagger$. To that end, we use the *gate fidelity*, i.e., the fidelity between the Choi matrices of the corresponding channels:

$$\begin{aligned} F &= \frac{1}{d_A^2} \mathrm{Tr}(J_U J_{\mathcal{E}}) \\ &= \frac{1}{d_A^2} \mathrm{Tr}((\mathbb{1}_R \otimes J_U \otimes \mathbb{1}_S)(\omega^T \otimes \mathbb{1}_A \otimes \mathbb{1}_B \otimes \mathbb{1}_S) J_{\mathcal{M}}) \\ &= \frac{1}{d_A^2} \mathrm{Tr}((\omega^T \otimes J_U \otimes \mathbb{1}_S) J_{\mathcal{M}}). \end{aligned} \tag{C.4}$$

In particular, we want to obtain the optimal gate fidelity of implementation of the given channel. This is an SDP in primal standard form, where $J_{\mathcal{M}} \geq 0$ is the semidefinite variable subject to the constraints $\mathrm{Tr}_{BS} J_{\mathcal{M}} = \mathbb{1}_R \otimes \mathbb{1}_A$ and $\mathrm{Tr}((|ik\rangle_{RA} \langle ik| \otimes |jl\rangle_{SB} \langle j'l'|) J_{\mathcal{M}}) = 0$ for all i, k and all $j \neq j', l \neq l'$. □

C.2 Alternative definitions of coherence left

In this appendix we discuss possible bounds on and alternative definitions of coherence left, when an arbitrary resource is used to simulate a channel with recycling. Let us start by relaxing the tensor-product constraint at the output of the implementation: a simpler problem is that of finding a MIO that simulates the whole tensor-product output up to a given error. This also constitutes an upper bound on the robustness of coherence left.

Proposition 45. *The maximum robustness of coherence left in the resource $\sigma \in S$ after the implementation of a quantum channel $\mathcal{N} : A \to B$ via MIO $\mathcal{M} : R \otimes A \to S \otimes B$ and a coherent resource $\omega \in R$ up to error ϵ in diamond* *norm, i.e., Eq. (5.16), is bounded from above by the following optimization problem:*

$$\begin{aligned} &\max\ C_R(\sigma) \\ &\text{s.t. } \frac{1}{2}||\sigma \otimes \mathcal{N} - \mathcal{M}(\omega \otimes \cdot)||_\diamond \leq \epsilon, \end{aligned} \tag{C.5}$$

which can be expressed as the following semidefinite program:

$$\begin{aligned} \max\ &\sum_{i\neq j} \sigma_{ij} \\ \text{s.t. } &J_{\mathcal{M}} \text{ is the Choi matrix of } \mathcal{M} \text{ MIO} \\ &Z \geq \sigma \otimes J_{\mathcal{N}} - \mathrm{Tr}_R((\omega^T \otimes \mathbb{1}_{SAB})J_{\mathcal{M}}) \\ &\mathrm{Tr}_{SB}\, Z \leq \epsilon \mathbb{1}_A \\ &Z \geq 0. \end{aligned} \tag{C.6}$$

where $J_{\mathcal{N}}$ is the Choi matrix of $\mathcal{N}$.

Proof. It is straightforward to show that Eq. (C.5) is an upper bound on Eq. (5.16). Indeed, we can make the separable-output ansatz $\mathcal{M}(\omega \otimes \cdot) = \sigma \otimes \mathcal{L}(\cdot)$ for the MIO $\mathcal{M}$ simulating the channel in Eq. (C.5) and obtain the problem of Eq. (5.16).

As for the SDP formulation of this problem, the last three conditions in Eq. (C.6) are a simple translation of the diamond norm error one in Eq. (C.5), following [111]. It then remains to show that the robustness of coherence can be re-cast as the sum of off-diagonal elements of $\sigma \in S$. To that end recall the dual formulation, Eq. (1.21), of the robustness of coherence which reads

$$1 + C_R(\sigma) = \max \mathrm{Tr}\, \sigma S = \max \mathrm{Tr}(\sigma \circ S^T)G, \tag{C.7}$$

where S is a non-negative-definite matrix with ones on the diagonal and we have introduced the all-one matrix $G_{ij} = 1$ for all i, j and the Hadamard component-wise product $\sigma \circ S = \sum_{ij} \sigma_{ij} S_{ij} |i\rangle \langle j|$. That this is equal to the sum of all off-diagonal elements of σ follows from

$$\begin{aligned} C_R(\sigma) &= \sum_{i,j} \sigma_{ij}\, S_{ji} - 1 \\ &= \sum_i \sigma_{ii} + \sum_{i\neq j} \sigma_{ij} S_{ji} - 1 \\ &= \sum_{i\neq j} \sigma_{ij} S_{ji}, \end{aligned} \tag{C.8}$$

where we have made use of the fact that $S_{ii} = 1\, \forall\, i$ in the second line above. As the transformation $\sigma \to \sigma \circ S^T$ can be implemented by a MIO CPTP map, given that $(\sigma \circ S^T)_{ij} \leq \sigma_{ij}$ $\forall i \neq j$ (no coherence is created in the transformation), the two objective functions are equivalent under the given constraints, since the diamond norm is contractive under CPTP maps. □

The SDP of Proposition 45 amounts to requiring that the MIO $\mathcal{M}$ implements a good simulation of the overall output $\sigma \otimes \mathcal{N}$ when provided with the input resource ω. Note that this implies that the local reduced output systems of the implementation $\mathcal{M}$ are close to σ and $\mathcal{N}$, but also that their correlations are small, so that the implementation is ϵ-close to a tensor-product one. However, if we increase the allowed dimension of σ it is possible to find states that are ϵ-close to it but have an increasing amount of coherence, e.g., $(1-\epsilon)\sigma + \epsilon\Psi_d$. Hence Eq. (C.5) can give an unbounded amount of coherence left if the output dimension of the resource is allowed to increase.

Two possible ways to remedy this involve changing the objective function such that it either maximizes the ϵ-robustness of coherence of the output resource state, or to find the most coherent state among all output resource states that are ϵ-close to the desired target. Both of these solutions provide an alternative definition of coherence left, physically motivated by the fact that the true output resource is not σ but only a state ϵ-close to it, either in robustness or in trace norm. Unfortunately, the former is not an SDP while the latter is not easily computable. Which of these, or other, definitions of coherence left is better may depend on the recycler's objective and we leave it as an open question for the interested reader.

In Fig. C.1 we plot the robustness of coherence left after the approximate implementation of a qubit unitary U_θ vs. θ/π, for several values of the input coherence and of the error threshold. As expected, for a fixed input coherence, more coherence can be recovered at the output if we accept a worse implementation. Moreover, for sufficiently small error thresholds, there are unitaries that cannot be implemented at all.

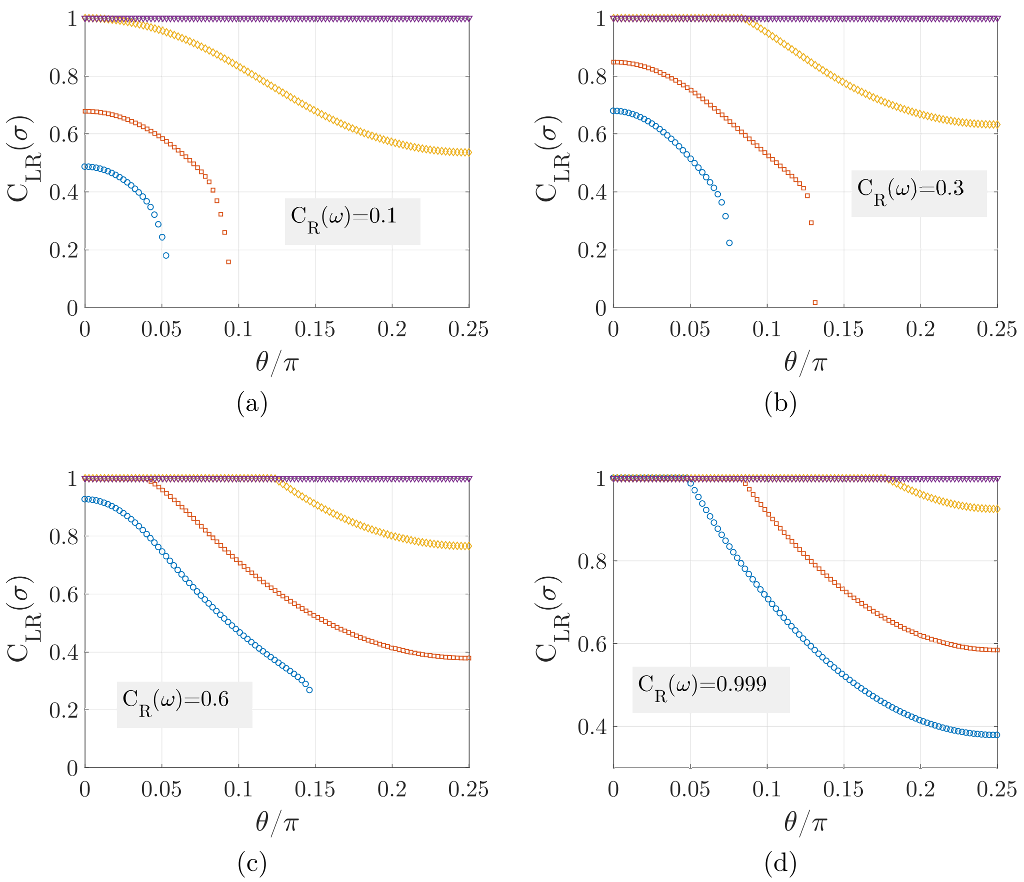


Figure C.1: Maximum robustness of coherence left in the resource after the approximate MIO implementation of the qubit unitary U_θ vs. θ/π for four values of input coherence (a-d). Each plot shows four curves corresponding to different error thresholds $\epsilon \in \{0.15, 0.25, 0.45, 0.75\}$ (from lower to higher curves).

APPENDIX D

Proving some useful lemmas

D.1 Proof of Lemma 26

We will prove the first statement, the second one follows analogously.

For $n = 0$, this is trivially true. Now let $n \mapsto n + 1$.

$$\begin{aligned}
\Delta\mathcal{L}^{n+1} &= \begin{pmatrix} \mathcal{L}_{\mathrm{pp}}^{n} & \sum_{j=1}^{n} \mathcal{L}_{\mathrm{pp}}^{j-1}\mathcal{L}_{\mathrm{pc}}\mathcal{L}_{\mathrm{cc}}^{n-j} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{L}_{\mathrm{pp}} & \mathcal{L}_{\mathrm{pc}} \\ \mathcal{L}_{\mathrm{cp}} & \mathcal{L}_{\mathrm{cc}} \end{pmatrix} \\
&= \begin{pmatrix} \mathcal{L}_{\mathrm{pp}}^{n+1} + \sum_{j=1}^{n} \mathcal{L}_{\mathrm{pp}}^{j-1}\mathcal{L}_{\mathrm{pc}}\mathcal{L}_{\mathrm{cc}}^{n-j}\mathcal{L}_{\mathrm{cp}} & \mathcal{L}_{\mathrm{pp}}^{n}\mathcal{L}_{\mathrm{pc}} + \sum_{j=1}^{n} \mathcal{L}_{\mathrm{pp}}^{j-1}\mathcal{L}_{\mathrm{pc}}\mathcal{L}_{\mathrm{cc}}^{n-j+1} \\ 0 & 0 \end{pmatrix}
\end{aligned}$$

By assumption, the highlighted sum is zero. Note that in the second column, the separate term is precisely the one arising for $j = n + 1$.

$$\Rightarrow \Delta\mathcal{L}^{n+1} = \begin{pmatrix} \mathcal{L}_{\mathrm{pp}}^{n+1} & \sum_{j=1}^{n+1} \mathcal{L}_{\mathrm{pp}}^{j-1}\mathcal{L}_{\mathrm{pc}}\mathcal{L}_{\mathrm{cc}}^{n+1-j} \\ 0 & 0 \end{pmatrix}$$

D.2 Proof of Lemma 29

First note that $\{\Lambda(t)\}_{t\in\mathbb{R}^+}$ satisfies Eq.(7.29) if and only if $\forall x,\tilde{x};t,s\in\mathbb{R}^+$

$$\begin{aligned}
&\langle\psi_{\tilde{x}}|\ \Lambda(t)\circ\Lambda(\tau)\left[|\psi_x\rangle\langle\psi_x|\right]\ |\psi_{\tilde{x}}\rangle \\
&=\sum_{y,z}\langle\psi_{\tilde{x}}|\ \Lambda(t)\Big[|\psi_y\rangle\langle\psi_y|\ \Lambda(\tau)\left[|\psi_x\rangle\langle\psi_x|\right]\ |\psi_z\rangle\langle\psi_z|\Big]\ |\psi_{\tilde{x}}\rangle \\
&=\sum_{y}\langle\psi_{\tilde{x}}|\ \Lambda(t)\Big[|\psi_y\rangle\langle\psi_y|\ \Lambda(\tau)\left[|\psi_x\rangle\langle\psi_x|\right]\ |\psi_y\rangle\langle\psi_y|\Big]\ |\psi_{\tilde{x}}\rangle \\
&=\langle\psi_{\tilde{x}}|\ \Lambda(t)\circ\Delta\circ\Lambda(\tau)\left[|\psi_x\rangle\langle\psi_x|\right]\ |\psi_{\tilde{x}}\rangle\,, \qquad \text{(D.1)}
\end{aligned}$$

since the first and the last equalities are always valid, while the second is Eq.(7.29), adding the diagonal terms on both sides. We first prove that if $\{\Lambda(t)\}_{t\in\mathbb{R}^+}$ is NCGD then the above equality holds. We can in fact rewrite the first line as

$$\begin{aligned}
&\sum_{k,k'}\langle\psi_{\tilde{x}}|\,|\psi_k\rangle\langle\psi_k|\ \Lambda(t)\circ\Lambda(\tau)\Big[|\psi_{k'}\rangle\langle\psi_{k'}||\psi_x\rangle\langle\psi_x||\psi_{k'}\rangle\langle\psi_{k'}|\Big]\ |\psi_k\rangle\langle\psi_k|\,|\psi_{\tilde{x}}\rangle \\
&=\langle\psi_{\tilde{x}}|\ \Delta\circ\Lambda(t)\circ\Lambda(\tau)\circ\Delta\left[|\psi_x\rangle\langle\psi_x|\right]\ |\psi_{\tilde{x}}\rangle \qquad \text{(D.2)}\\
&=\langle\psi_{\tilde{x}}|\ \Delta\circ\Lambda(t)\circ\Delta\circ\Lambda(\tau)\circ\Delta\left[|\psi_x\rangle\langle\psi_x|\right]\ |\psi_{\tilde{x}}\rangle \\
&=\langle\psi_{\tilde{x}}|\ \Lambda(t)\circ\Delta\circ\Lambda(\tau)\left[|\psi_x\rangle\langle\psi_x|\right]\ |\psi_{\tilde{x}}\rangle\,.
\end{aligned}$$

where in the first line we used that only the terms $\tilde{x}=k$,$x=k'$ are non-zero, and in the third line we used that $\{\Lambda(t)\}_{t\in\mathbb{R}^+}$ is NCGD.

For the converse, we start with the assumption that the equality (D.1) holds for any $x,\tilde{x},t,\tau\in\mathbb{R}^+$. The statement then simply follows by the linearity of the propagators since Δ is a projection onto the span of $\{|\psi_x\rangle\langle\psi_x|\}_x$.

D.3 Proof of Lemma 31

First note that

$$C_{X_s}(t,0):=\sum_{x,x_0=0,1}Q_2^{\hat{X}_s}\{x,t;x_0,0\}\,x*x_0=Q_2^{\hat{X}_s}\{1,t;1,0\}\,;$$

since the dichotomic observable has values in $\{0,1\}$; thus

$$\begin{aligned}
&|2C_{X_s}(t,0)-C_{X_s}(2t,0)|=\Big|2Q_2^{\hat{X}_s}\{1,t;1,0\}-Q_2^{\hat{X}_s}\{1,2t;1,0\}\Big| \\
&=\Big|2Q_{1|1}^{\hat{X}_s}\{1,t|1,0\}-Q_{1|1}^{\hat{X}_s}\{1,2t|1,0\}\Big|\,Q_1^{\hat{X}_s}\{1,0\} \\
&=\left|2Q_{1|1}^{\hat{X}_s}\{1,t|1,0\}-\sum_{x_k k=0,1}Q_{1|1}^{\hat{X}_s}\{1,t|x_k,0\}\,Q_{1|1}^{\hat{X}_s}\{x_k,t|1,0\}\right| \\
&Q_1^{\hat{X}_s}\{1,0\}\,, \qquad \text{(D.3)}
\end{aligned}$$

which provides us with Eq.(7.31), since $\langle X_s(0)\rangle = Q_1^{\hat{X}_s}\{1,0\}$, while

$$2Q_{1|1}^{\hat{X}_s}\{1,t|1,0\} - \sum_{x_k k=0,1} Q_{1|1}^{\hat{X}_s}\{1,t|x_k,0\}\, Q_{1|1}^{\hat{X}_s}\{x_k,t|1,0\}$$

is maximized by 1 for $Q_{1|1}^{\hat{X}_s}\{1,t|1,0\} = 1$ or $Q_{1|1}^{\hat{X}_s}\{1,t|1,0\} = 0$ and

$$Q_{1|1}^{\hat{X}_s}\{0,t|1,0\} = 1$$

(as seen using $Q_{1|1}^{\hat{X}_s}\{1,t|0,0\} = 1 - Q_{1|1}^{\hat{X}_s}\{1,t|1,0\}$).

D.4 Proof of Lemma 33

"Only if": the statistics is, in particular, 2CL, so that we have, for any $x, t \geq s \in \mathbb{R}^+$,

$$\sum_y Q_2^{\hat{X}_s}\{x,t;y,s\} = Q_1^{\hat{X}_s}\{x,t\}\,.$$

But then, using the definition of conditional probability $Q_2^{\hat{X}_s}\{x,t;y,s\} = Q_{1|1}^{\hat{X}_s}\{x,t|y,s\} * Q_1^{\hat{X}_s}\{y,s\}$ and, crucially, the time-homogeneity guaranteed by the 2M and the Lindblad equation ($Q_{1|1}^{\hat{X}_s}\{x,t|y,s\} = \mathrm{Tr}_s\{\mathcal{P}_x e^{\mathcal{L}(t-s)}[|\psi_y\rangle\langle\psi_y|]\} = Q_{1|1}^{\hat{X}_s}\{x,t-s|y,0\}$), we can write

$$\sum_y Q_{1|1}^{\hat{X}_s}\{x,t-s|y,0\} * Q_1^{\hat{X}_s}\{y,s\} = Q_1^{\hat{X}_s}\{x,t\}\,.$$

Using the Kolmogorov condition, this time w.r.t. the initial value, and the definition of conditional probability, the previous relation gives

$$\sum_{x_0} Q_1^{\hat{X}_s}\{x_0,0\}\left(\sum_y Q_{1|1}^{\hat{X}_s}\{x,t-s|y,0\} * Q_{1|1}^{\hat{X}_s}\{y,s|x_0,0\} - Q_{1|1}^{\hat{X}_s}\{x,t|x_0,0\}\right) = 0,$$

which directly provides us with the Chapman-Kolmogorov composition law in Eq.(6) of the main text, since, by assumption, we can choose any initial diagonal state $\rho(0) = \sum_{x_0} p_{x_0}|\psi_{x_0}\rangle\langle\psi_{x_0}|$, and then any distribution of $Q_1^{\hat{X}_s}\{x_0,0\} = p_{x_0}$.

"If": Eq.(6) of the main text for the quantum conditional probability $Q_{1|1}^{\hat{X}_s}\{x,t|x_0,0\}$ means that

$$\mathrm{tr}\left\{\mathcal{P}_x e^{\mathcal{L}t}\left[|\psi_{x_0}\rangle\langle\psi_{x_0}|\right]\right\} = \sum_{x_k} \mathrm{tr}\left\{\mathcal{P}_x e^{\mathcal{L}(t-s)}\left[|\psi_{x_k}\rangle\langle\psi_{x_k}|\right]\right\} \mathrm{tr}\left\{\mathcal{P}_{x_k} e^{\mathcal{L}s}\left[|\psi_{x_0}\rangle\langle\psi_{x_0}|\right]\right\}$$

$$\forall x_0, x; t \geq s \in \mathbb{R}^+,$$

which, replaced in Eq.(7.8), implies Eq.(2) of the main text [$s \mapsto t_k - t_{k-1}, x_0 \mapsto x_{k-1}, t \mapsto t_{k+1} - t_{k-1}, x \mapsto x_{k+1}$], so that if the hierarchy is jM it will also be jCL; note that this is the case also for $k = 1$ since we assume the initial state to be diagonal in the selected basis.

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