BERNSTEIN-SATO POLYNOMIAL OF PLANE CURVES AND YANO'S CONJECTURE

A thesis submitted to the
UNIVERSITAT POLITÈCNICA DE CATALUNYA

by
GUILLEM BLANCO FERNÁNDEZ

in partial fulfillment of the
requirements for the degree of
DOCTOR OF MATHEMATICS

Maria Alberich-Carramiñana, Advisor
Josep Àlvarez Montaner, Advisor

Barcelona, April 2020

Guillem Blanco Fernández: *Bernstein-Sato polynomial of plane curves and Yano's conjecture,* *Als meus pares, al meu germà i a la Clàudia*

ABSTRACT

The main aim of this thesis is the study of the Bernstein-Sato polynomial of plane curve singularities. In this context, we prove a conjecture posed by Yano in 1982 about the generic b-exponents of an irreducible plane curve.

In a part of the thesis, we study the Bernstein-Sato polynomial using the analytic continuation of the complex zeta function of a singularity. We obtain several results on the vanishing and non-vanishing of the residues of the complex zeta function of plane curves. Using these results we obtain a proof of Yano's conjecture under the hypothesis that the eigenvalues of the monodromy are pair-wise different. In another part of the thesis, we study the periods of integrals in the Milnor fiber and their asymptotic expansion. This asymptotic expansion of the periods can be related to the b-exponents and can be constructed in terms of resolution of singularities. Using these techniques, we can present a proof for the general case of Yano's conjecture.

In addition to the Bernstein-Sato polynomial, we also study the minimal Tjurina number of an irreducible plane curve and we answer in the positive a question raised by Dimca and Greuel on the quotient between the Milnor and Tjurina numbers. More precisely, we prove a formula for the minimal Tjurina number in an equisingularity class of an irreducible plane curve in terms of the multiplicities of the strict transform along the minimal resolution. From this formula, we obtain the positive answer to Dimca and Greuel question.

This thesis also contains computational results for the theory of singularities on smooth complex surfaces. First, we describe an algorithm to compute log-resolutions of ideals on a smooth complex surface. Secondly, we provide an algorithm to compute generators for complete ideals on a smooth complex surface. These algorithms have several applications, for instance, in the computation of the multiplier ideals associated to an ideal on a smooth complex surface.

ABSTRACT

El principal objectiu d'aquesta tesi és l'estudi del polinomi de Bernstein-Sato de singularitats de corbes planes. En aquest context, es demostra una conjectura proposada per Yano el 1982 sobre els b-exponents genèrics d'una corba plana irreductible.

En una part d'aquesta tesi, s'estudia el polinomi de Bernstein-Sato utilitzant la continuació analítica de la funció zeta complexa d'una singularitat. S'obtenen diversos resultat sobre l'anul·lació i no anul·lació del residu de la funció zeta complexa d'una corba plana. Utilitzant aquests resultats, s'obté una demostració de la conjectura de Yano sota la hipòtesi de que els valors propis de la monodromia siguin diferents dos a dos. En un altre part de la tesi, s'estudien els períodes d'integrals en la fibra de Milnor i la seva expansió asimptòtica. Aquesta expansió asimptòtica dels períodes pot ser relacionada amb els b-exponents i pot ser construïda en termes de la resolució de singularitats. Utilitzant aquestes tècniques, es presenta una prova del cas general de la conjectura de Yano.

A més a més del polinomi de Bernstein-Sato, també s'estudia el nombre de Tjurina mínim d'una corba plana irreductible i responem positivament a una pregunta formulada per Dimca i Greuel sobre el quocient entre els nombres de Milnor i Tjurina. Concretament, es demostra una fórmula pel nombre de Tjurina mínim en un classe d'equisingularitat de corbes planes irreductibles en termes de la seqüència de multiplicitats de la transformada estricta al llarg de la resolució minimal. A partir d'aquesta fórmula, s'obté la resposta positiva a la pregunta de Dimca i Greuel.

Aquesta tesi també conté resultats computacionals per la teoria de singularitats en superfícies complexes llises. Primer, es descriu un algorisme que calcula la log-resolució d'ideals en un superfície complexa llisa. En segon lloc, es dona un algorisme per calcular generadors per ideals complets en una superfície complexa llisa. Aquests algorismes tenen diverses aplicacions, com per exemple, en el càlcul d'ideals multiplicadors associats a un ideal en una superfície complexa llisa.

ACKNOWLEDGMENTS

First and foremost, I would like to thank my advisors, Maria Alberich and Josep Àlvarez, for the support and encouragement during all these years. Thanks for introducing me to Mathematical research and suggesting interesting and well-suited problems for me to work on.

I would also like to thank Miguel Ángel Barja for introducing me to Maria and Josep when I was doing my master's degree. Probably this thesis would not exist without his early intervention.

Thanks to Victor González for hosting me for a few days at his place in Hanover in the early stages of my Ph.D. studies. Thanks also to Ferran Dachs for the joint work on the computation of multiplier ideals.

I am very grateful to Ben Lichtin for answering my questions and taking the time to review and give feedback on a big part of this thesis.

Thanks to Patricio Almirón for sharing with me some of his thesis problems and for the discussions held in many different places all over the world.

Special thanks go to Xavier Gómez Mont and Manuel González Villa for inviting me to Guanajuato for two weeks and letting me talk about my work on Yano's conjecture with such great level of detail. I would also like to thank Luis Núñez Betancourt for his hospitality during my stay in Guanajuato and for showing us the best places to eat.

I cannot recall how many times I have visited Sevilla during all these years, thanks to Paco Castro and Luis Narváez for their hospitality every single time. I am also grateful to Alejandro Melle and Patricio for inviting me to Madrid and giving me the chance to talk in the seminar about my work on Yano's conjecture.

I would also like to thank Eduard Casas-Alvero for several productive discussions about some of the results of this thesis.

Vull acabar donant les gràcies a tota la meva família, en especial als meus pares i al meu germà, per tot el suport i la paciència durant els estudis de doctorat, però també des que vaig decidir estudiar matemàtiques. Gràcies per no perdre mai l'esperança en entendre el que faig i seguir preguntant sobre el tema de la meva tesi tot i les meves nul·les capacitats d'explicar-vos-ho.

Per últim, però no menys important, gràcies a la Clàudia per estar cada dia al meu costat i recolzar-me incondicionalment durant tot aquest procés.

PUBLICATIONS

The results of this thesis are part of the following research articles:

Chapter II

- M. Alberich-Carramiñana, J. Àlvarez Montaner, and G. Blanco, *Effective computation of base points of ideals in two-dimensional local rings*, Journal of Symbolic Computation **92** (2019), pp. 93–109.
- M. Alberich-Carramiñana, J. Àlvarez Montaner, and G. Blanco, *Monomial generators of complete planar ideals*, Journal of Algebra and its Applications (2020), In press, DOI: 10.1142/S0219498821500328.
- G. Blanco and F. Dachs-Cadefau, *Computing multiplier ideals in smooth surfaces*, Extended abstracts February 2016, Positivity and valuations (Barcelona, 2016), (M. Alberich-Carramiñana, C. Galindo, A. Küronya, and J. Roé, eds.), Trends in Mathematics. Research Perspectives CRM Barcelona, no. 9, Birkhäuser, Basel, 2018, pp. 57–63

Chapter III

- P. Almirón and G. Blanco, *A note on a question of Dimca and Greuel*, Comptes Rendus Mathématique. Académie des Sciences **357** (2019), no. 2, pp. 205–208.
- M. Alberich-Carraminana, P. Almirón, G. Blanco, and A. Melle-Hernández, *The minimal Tjurina number of irreducible germs of plane curve singularities*, To appear in Indiana University Mathematics Journal, 2019, arXiv: 1904.02652 [math.AG].

Chapter IV

- G. Blanco, *Poles of the complex zeta function of a plane curve*, Advances in Mathematics **350** (2019), no. 9, pp. 396–439.

Chapter V

- G. Blanco, *Yano's conjecture*, 2019, arXiv: 1908.05917 [math.AG]
- G. Blanco, *Topological roots of the Bernstein-Sato polynomial of a plane curve*, In preparation

CONTENTS

LIST OF FIGURES

LIST OF TABLES

INTRODUCTION

Let X be a smooth complex variety and let $\mathfrak{a} \subseteq \mathcal{O}_X$ be an ideal defining a singular variety $\mathrm{Var}(\mathfrak{a})$. Since singularities are of local nature, one usually works locally around one of the singular points $p \in \mathrm{Sing}(\mathfrak{a})$ and considers the stalk $\mathfrak{a}_p \subseteq \mathcal{O}_{X,p}$. There are many invariants that, to some extent, measure the complexity or the mildness of a singularity. One may classify the invariants in two big groups, topological and analytical invariants. The invariants in the first group are invariant under local homomorphisms of the ambient space. On the contrary, the second group are invariants under local change of analytical coordinates but not under local homomorphisms.

One of the main problems in Singularity Theory is to determine to what extent the topology of a singularity constrains its analytical invariants. This problem can be approached in two different ways. On one side, one can try to determine the generic behavior of an analytic invariant within singularities having the same topological type. On the other hand, one can study which part of an analytic invariant is purely determined by the topology of the singularity and which part is not.

In this thesis, we will study this problem for two analytical invariants of plane curve singularities: the Bernstein-Sato polynomial and the Tjurina number. The study of the generic behavior of these two invariants, even for irreducible plane curves, have been long-standing problems that are completely solved in this thesis. For the Tjurina number see, for instance, [Zar86; Tei86; Del78; BGM88; LP90; Per97]. For the Bernstein-Sato polynomial, see [Yan78; Kat81; Kat82; Yan82; Cas87; Cas88; Gea91; BGM92; BMT07; Art+17b; Art+17a; Art+18]. Additionally, we will give some results about the topological information contained in these analytical invariants. This study is partially possible in the case of plane curves because the topological classification of plane curves is well-understood. In addition, for plane curves, topological equivalence is the same as equisingular equivalence. Therefore, resolution of singularities becomes a very useful tool for this study and the results obtained will be described in terms of the numerical data of the resolution.

Finally, this thesis also contains some computational aspects of the theory of singularities in smooth complex surfaces. First, we will present an algorithm that computes the log-resolution of an ideal on a smooth complex surface from any given set of generators. We also have developed an algorithm that allows the computation of explicit generators for complete ideals on a smooth complex surface. These algorithms are especially useful to compute effectively the integral closure or the multiplier ideals of a singularity on a smooth surface. All computations are effective in the sense that both the input and the output are ideals generated by polynomials. The algorithms presented in this thesis, as well as other necessary algorithms in the theory of plane curve singularities, have been implemented in Magma [BCP97] and can be found in Appendix A.

THE BERNSTEIN-SATO POLYNOMIAL

The main invariant of a singularity studied in this thesis is the Bernstein-Sato polynomial. Let $f \in \mathbb{C}[x_1, \dots, x_n]$ be a non-constant polynomial. The Bernstein-Sato polynomial $b_f(s)$ of f is defined as the monic polynomial of smallest degree that fulfills the following functional equation

$$P(s) \cdot f^{s+1} = b_f(s)\, f^s, \tag{0.1}$$

where $P(s)$ is a differential operator in $D_{\mathbb{C}^{n+1}} \otimes \mathbb{C}[s]$, with $D_{\mathbb{C}^{n+1}}$ being the ring of $\mathbb{C}$-linear differential operators and s a formal variable. The Bernstein-Sato polynomial $b_f(s)$ was introduced in the polynomial case, independently, by Bernstein [Ber72] using the theory of algebraic D-modules and by Sato [SS90] in the context of prehomogeneous vector spaces. See Section 1.2 for details.

The above construction remains true in the local case. That is, if $f : (\mathbb{C}^n, \mathbf{0}) \longrightarrow (\mathbb{C}, 0)$ is a germ of a holomorphic function, there exists the local Bernstein-Sato polynomial $b_{f,\mathbf{0}}(s)$ fulfilling a functional equation as in Equation (0.1). The existence of the Bernstein-Sato polynomial in the local case is due to Björk [Bjö74]. Since it is known that the global Bernstein-Sato polynomial $b_f(s)$ equals the least common multiple of all the local Bernstein-Sato polynomials and due to the local nature of singularities, we will mainly work with the local Bernstein-Sato polynomial.

It is a classical result that the roots of the Bernstein-Sato polynomial are negative rational numbers. This is established by Malgrange [Mal75] for isolated singularities, using the Gauss-Manin connection, and by Kashiwara [Kas76] in general, using resolution of singularities. However, in general, little more is known about the roots of the Bernstein-Sato polynomial. It is an analytical invariant of the singularity in the sense established above, see for instance the examples in [Kat81; Kat82]. There are algorithms as those developed by Oaku [Oak97] for computing the Bernstein-Sato polynomial of a singularity, however, since they depend on computations with non-commutative Gröbner bases, it is usually very hard to compute examples of the Bernstein-Sato polynomial.

The roots of $b_f(s)$ are related to other invariants of f. By the results of Malgrange, first in the isolated singularity case [Mal75] and later in general [Mal83], for every root α of $b_f(s)$, the value $\exp(2\pi i \alpha)$ is an eigenvalue of the local monodromy at some point of $f^{-1}(0)$ and every eigenvalue is obtained in this way. For an isolated singularity, these results imply that the degree of $b_f(s)$ is at most the Milnor number μ, see Section 1.4 for the definitions.

The log-canonical threshold of the singularity [Kol97; Mus12] is minus the largest root of $b_f(s)$. For isolated singularities, it coincides with the complex singularity index, a concept that dates back to Arnold [AV88]. The spectral numbers of an isolated singularity, introduced by Steenbrink in [Ste89], in the range $(0, 1]$ are always the opposites in sign to roots of the Bernstein-Sato polynomial, for the non-isolated singularity case we refer to [Bud03]. Similarly, the jumping numbers associated with the multiplier ideals, introduced in [Ein+04], which are in $(0, 1]$, are roots of $b_f(-s)$, see also [BS05] and [Lic86].

YANO'S CONJECTURE

Given the analytical nature of the Bernstein-Sato polynomial, one may ask about the generic behavior of the Bernstein-Sato polynomial among all singularities with a fixed topological type. For plane curves, where the topology of singularities is well-understood, one can ask whether there exists a generic Bernstein-Sato polynomial among all plane curves with a given topological type and whether this generic Bernstein-Sato polynomial can be expressed in terms of the topological data of the singularity.

This is, precisely, for the case of irreducible plane curves, the content of a conjecture posed by Yano in 1982 [Yan82]. In order to state Yano's conjecture, one needs the following characterization of the local Bernstein-Sato polynomial of an isolated singularity due to Malgrange [Mal75]. Let $f : (\mathbb{C}^{n+1}, \mathbf{0}) \longrightarrow (\mathbb{C}, 0)$ be a germ of a holomorphic function defining an isolated singularity. First, since -1 is always a root of $b_{f,\mathbf{0}}(s)$, the reduced Bernstein-Sato polynomial is defined as $\tilde{b}_{f,\mathbf{0}}(s) := b_{f,\mathbf{0}}(s)/(s+1)$. Let

$$''H^n = \frac{\Omega^{n+1}_{X,\mathbf{0}}}{\mathrm{d}f \wedge d\Omega^{n-1}_{X,\mathbf{0}}} \tag{0.2}$$

be the Brieskorn lattice and one considers its saturation $''\widetilde{H}^n = \sum_{k=0}(\partial_t t)^k {}''H^n$. Then, Malgrange's result asserts that $\tilde{b}_{f,\mathbf{0}}(s)$ is equal to the minimal polynomial of the complex endomorphism

$$-\overline{\partial_t t} : {}''\widetilde{H}^n / t {}''\widetilde{H}^n \longrightarrow {}''\widetilde{H}^n / t {}''\widetilde{H}^n\,, \tag{0.3}$$

induced by the Gauss-Manin connection. Finally, one defines the *b-exponents* as the roots of the characteristic polynomial of $\overline{\partial_t t}$, see Section 1.7 for the exact results and definitions.

Yano's conjecture deals with the generic behavior of the b-exponents of the singularity instead of the roots of the Bernstein-Sato polynomial. With the notations from Section 2.8 the conjecture reads as follows:

Conjecture (Yano [Yan82])**.** *For generic curves in some μ-constant deformation of an irreducible germ of a plane curve having characteristic sequence $(n, \beta_1, \ldots, \beta_g)$, the b-exponents $\{\alpha_1, \alpha_2, \ldots, \alpha_\mu\}$ are given by the generating function R. Precisely,*

$$R\big((n, \beta_1, \ldots, \beta_g), t\big) := t + \sum_{i=1}^{g} t^{\frac{r_i}{R_i}} \frac{1-t}{1-t^{\frac{1}{R_i}}} - \sum_{i=0}^{g} t^{\frac{r'_i}{R'_i}} \frac{1-t}{1-t^{\frac{1}{R'_i}}} = \sum_{i=1}^{\mu} t^{\alpha_i}. \tag{0.4}$$

The numbers r_i, R_i (resp. r'_i, R'_i) are the numerical data at the rupture divisors (resp. dead-end divisors) of the minimal embedded resolution.

Yano's conjecture is known to be true in the case that f has a single Puiseux pair, see the work of Cassou-Noguès [Cas88]. More recently, Artal-Bartolo, Cassou-Noguès, Luengo, and Melle-Hernández [Art+17b], proved the case of two Puiseux pairs under the hypothesis that the eigenvalues of the monodromy of f are pair-wise different. The assumption on the eigenvalues of the monodromy appears naturally in this context because under this hypothesis the roots of the local Bernstein-Sato polynomial and the b-exponents coincide.

Using the results from Chapter IV, we can give a proof of Yano's conjecture that works under the assumption that the eigenvalues of the monodromy are pair-wise different. This proof uses the poles of the complex zeta function of f and their relation with the Bernstein-Sato functional equation via integration by parts. The hypothesis on the eigenvalues of the monodromy ensures that all the b-exponents can be recovered from the roots of the Bernstein-Sato polynomial. The proof of the general case will be the content of Chapter V and it is based on the asymptotic expansion of the periods of integrals in the Milnor fiber. These periods of integrals are solutions to the Gauss-Manin connection and they give direct information about the b-exponents.

POLES OF THE COMPLEX ZETA FUNCTION OF A PLANE CURVE

Let k be either $\mathbb{R}$ or $\mathbb{C}$ and let $\varphi(x) \in C_c^\infty(k^n)$ be an infinitely many times differentiable function with compact support. Define the archimedean zeta function f^s of a non-constant polynomial $f(x) \in k[x_1, \ldots, x_n]$ as the distribution

$$\langle f^s, \varphi \rangle = \int_{k^n} |f(x)|^{\delta s} \varphi(x)\, dx, \tag{0.5}$$

for $s \in \mathbb{C}$, $\mathrm{Re}(s) > 0$, where $\delta = 1$ if $k = \mathbb{R}$ and $\delta = 2$ if $k = \mathbb{C}$. In the 1954 edition of the International Congress of Mathematicians, I. M. Gel'fand [Gel57] posed the following problem: first, determine whether f^s is a meromorphic function of s with poles forming several arithmetic progressions; second, study the residues at those poles.

The problem is solved for some specific polynomials having simple singularities in the book of Gel'fand and Shilov [GS64], by regularizing the integral in Equation (0.5). It is not after Hironaka's resolution of singularities [Hir64a; Hir64b], that Bernstein and S. I. Gel'fand [BG69], and independently Atiyah [Ati70], give a positive answer to Gel'fand's first question. Both results use resolution of singularities to reduce the problem to the monomial case, already settled in [GS64], and give a sequence of candidates poles for f^s from the resolution data. These constructions are detailed in Section 7.1.

A different approach to the same problem is considered by Bernstein [Ber71; Ber72], who develops the theory of D-modules and proves the existence of the Bernstein-Sato polynomial $b_f(s)$ to analytically continue the archimedean zeta function f^s. One verifies, using the functional equation in Equation (0.1) and integration by parts, see Section 7.2, that the poles of f^s are among the rationals $s = \alpha - \nu$, with $b_f(\alpha) = 0$ and $\nu \in \mathbb{Z}_{\geq 0}$. Loeser [Loe85] shows the equality between both sets for reduced plane curves and isolated quasi-homogeneous singularities.

In Chapter IV we examine the original questions of Gel'fand and we use resolution of singularities to study the possible poles and the residues of the complex zeta function of general plane curves. These results generalize some of the ideas and results of Lichtin [Lic85; Lic89] in the case of irreducible plane curves. Lichtin computed the residue at the first poles of each rupture divisor. The main results of Chapter IV are the following:

- For any candidate pole σ of f^s, we give a formula for its residue expressed as an integral along the exceptional divisor associated to σ, see Proposition 8.4
- In Theorem 8.10, we prove that most non-rupture divisors do not contribute to the poles of f^s, and consequently to the roots of $b_f(s)$.

Since the residues characterize whether some poles of f^s are roots of $b_f(s)$, this last result answers, for reduced plane curves, a question raised by Kollár [Kol97] on which exceptional divisors contribute to roots of the Bernstein-Sato polynomial. It is already well-known that, for plane curves, non-rupture divisors do not contribute to topological invariants such as the eigenvalues of the monodromy [ACa75; Neu83], the jumping numbers [ST07], or the poles of Igusa's local zeta function [Loe88]. For irreducible plane curves, we use Teissier's monomial curve [Tei86], see Section 2.6, associated with the semigroup of f to refine our previous results:

- In Theorem 9.7, we obtain an optimal set of candidates for the poles of f^s in terms of the rupture divisors, the characteristic sequence, and the semigroup of f.

- From this, in Theorem 9.8, we prove that if f_{gen} is generic among all plane branches with fixed characteristic sequence (in the sense that the coefficients of some μ-constant deformation are generic), all the candidates are indeed poles of f_{gen}^s.
- As a consequence, in Corollary 9.9, we prove Yano's conjecture for any number of characteristic exponents under the assumption that the eigenvalues of the monodromy of f are pairwise different.

PERIODS OF INTEGRALS IN THE MILNOR FIBER

Let $f : (\mathbb{C}^{n+1}, \mathbf{0}) \longrightarrow (\mathbb{C}, 0)$ be a germ of a holomorphic function defining an isolated singularity. Instead of dealing with the poles of the complex zeta function as in Chapter IV, the proof of Yano's conjecture presented in Chapter V focuses on the Gauss-Manin connection of an isolated singularity f, which gives direct information about the b-exponents.

The main idea of the proof is to construct certain solutions of the Gauss-Manin connection that are directly associated with the b-exponents of f. These solutions will be periods of differential forms along vanishing cycles $\gamma(t)$ on the Milnor fiber of f,

$$\int_{\gamma(t)} \frac{\omega}{\mathrm{d}f} = \sum_{\lambda \in \Lambda} \sum_{\alpha \in L(\lambda)} \sum_{0 \le k \le n} a_{\alpha-1,k} t^{\alpha-1} (\ln t)^k, \tag{0.6}$$

see Section 10.1 for the exact definitions. These periods of integrals were originally considered by Malgrange in [Mal74a; Mal74b], where he proved that they have a certain asymptotic expansion encoding the structure of the monodromy and the roots of the Bernstein-Sato polynomial. The exact relation between the b-exponents and these periods of integrals is given by a theorem of Varchenko [Var80], see Section 10.2. His result links the so-called *geometric sections* in the cohomology of the Milnor fiber, constructed via these periods of integrals, and Malgrange's characterization of $\tilde{b}_{f,\mathbf{0}}(s)$ in terms of the Brieskorn lattice.

In [Var82], Varchenko uses the first term of each asymptotic expansion series to construct a mixed Hodge structure for the cohomology of the Milnor fiber. He uses resolution of singularities and a process of semi-stable reduction to determine these first terms of the expansions, as we review in Section 10.5. In Chapter V, we will generalize this idea,

- In Section 10.6, we determine all the terms of the asymptotic expansions of the periods of integrals using the divisorial valuations of the resolution.

The main problem arising in arbitrary dimension is how to determine whether a given coefficient of these asymptotic expansions, equivalently a geometric section, is non-zero. However, this is possible in the case of plane curves.

The first terms of the periods of integrals in the case of plane curves were originally determined by Lichtin [Lic89], in the irreducible case, and by Loeser [Loe88], for general curves. In order to show that a given coefficient of an asymptotic expansion is not zero, we will use the same idea as in [Loe88]. Namely, since the exceptional divisors of plane curves are just projective lines, the coefficients of the expansions become integrals of multivalued differential forms on a punctured projective line. Therefore, one can use cohomology with coefficients on local systems and a result of Deligne and Mostow [DM86] on multivalued forms on the projective line.

Determining the whole asymptotic expansions of the periods of integrals is more complicated than just giving the first terms. In contrast with the initial terms, the higher-order terms can change along μ-constant deformations of f. Given an irreducible f, for a Yano's candidate to be a b-exponent, one needs that a certain higher-order term in the asymptotic expansion of some period of integral is non-zero.

- We show in Proposition 11.11 that one can make the corresponding term of the asymptotic expansion associated with a candidate non-zero when f is generic in a certain μ-constant deformation.
- This depends on the existence of a particular μ-constant deformation, see Proposition 2.18, whose existence is proved using Teissier's monomial curve and its deformations [Tei86].

In this way, we can show that a single candidate is indeed a generic b-exponent in some μ-constant deformation of f.

In order to show that all the candidate b-exponents are generic in the same μ-constant deformation of f, as predicted by Yano's conjecture, we will use a semicontinuity argument for the b-exponent. More precisely,

- We generalize a semicontinuity argument of Varchenko [Var80], which is only valid when all the eigenvalues of the monodromy of f are pair-wise different. Our result works under the assumption of the existence of certain dual geometric sections with respect to a basis of generalized monodromy eigenvectors, see Section 10.4.
- For irreducible plane curves, we prove that such dual geometric sections do indeed exist. Consequently, we show that the b-exponents of some μ-constant deformation depend upper-semicontinuously on the parameters of the deformation, see Theorem 11.10.
- In Theorem 11.12, we show how Yano's conjecture follows from these results.

Using the results from Chapter V, we finish this chapter with Theorem 12.3, about the set of topological roots of the Bernstein-Sato polynomial. Precisely, for a irreducible plane curve, we provide a set of topological roots for the Bernstein-Sato polynomial that contain both the opposites in sign of the jumping numbers between $(0, 1]$ and the real parts of the poles of Igusa's zeta function, see Section 12.1 for the exact definitions. For curves with one and two Puiseux pairs, these sets of topological roots already appeared in [Cas87] and [Art+17b].

ON THE TJURINA NUMBER OF PLANE CURVES

The goal of Chapter III is to study another analytic invariant of isolated singularities, the Tjurina number. Let $f : (\mathbb{C}^{n+1}, \mathbf{0}) \longrightarrow (\mathbb{C}, 0)$ be a germ of an analytic function defining an isolated singularity. Taking analytical coordinates, the Tjurina number of f is defined as

$$\tau = \dim_{\mathbb{C}} \frac{\mathbb{C}\{x_0, \dots, x_n\}}{(f, \partial f/\partial x_0, \dots, \partial f/\partial x_n)}. \tag{0.7}$$

The geometric significance of the Tjurina number comes from the fact that τ is the dimension of the miniversal deformation of f. In the case of irreducible plane curve singularities, the generic behavior of the Tjurina number within a fixed topological class has been studied extensively, see [Zar86; Tei86; Del78; BGM88; LP90; Per97]. By semicontinuity, notice that the generic Tjurina number in a topological class coincides with the minimal Tjurina number τ_{min}.

Briançon, Granger and Maisonobe [BGM88] give recursive formulas to compute τ_{min} for the equisingularity classes corresponding to plane branches with a single Puiseux exponent. More generally, Peraire [Per97] gives an algorithm that computes the minimal Tjurina number τ_{min} from the semigroup of a plane branch.

Our interest in the Tjurina number comes from the following question posed by Dimca and Greuel in [DG18] on the quotient of the Tjurina number τ, an analytical invariant, and the Milnor number μ, a topological invariant.

Question. *It is true that $\mu/\tau < 4/3$ for any reduced plane curve singularity?*

Rewriting $\tau > \frac{3}{4}\mu$, this question can be viewed as a topological constraint to the Tjurina number imposed by the Milnor number.

In the first section of Chapter III, we give some evidence to the question of Dimca and Greuel for the case of branches with one Puiseux pair and semi-quasi-homogeneous plane curve singularities, see Proposition 5.2 and Proposition 5.4. For the case of branches with one Puiseux pair, we use a result of Delorme [Del78], where he gives a recursive formula for the generic dimension of the moduli space of a plane branch with one Puiseux exponent. For the case of semi-quasi-homogeneous plane curves, we use the recursive formulas of Briançon, Granger and Maisonobe [BGM88]. This is joint work with Almirón, see [AB19].

In the second part of Chapter III, we present a formula for the minimal Tjurina number of a irreducible plane curve in terms of the multiplicities of the strict transform along an embedded resolution of the curve, see Theorem 6.4. This result uses the work of Genzmer [Gen16], where a formula for the dimension of the generic component of the moduli space of a plane branch is deduced in terms of the embedded resolution. Our formula for the Tjurina number of a plane branch allows us to give a positive answer to Dimca and Greuel question, see Corollary 6.6. This is joint work with Alberich-Carramiñana, Almirón, and Melle-Hernández, see [Alb+19].

A complete positive answer to the question of Dimca and Greuel is given by Almirón in [Alm19] using some known cases of Durfee's conjecture.

EFFECTIVE COMPUTATION OF COMPLETE PLANAR IDEALS

The contents of Chapter II are more computational. Precisely, in Chapter II we give algorithms for the explicit computation of generators for several types of ideals associated with singularities in smooth surfaces, for instance, the multiplier ideals. The results of Chapter II are joint work with Alberich-Carramiñana and Àlvarez Montaner.

Let $(X, \mathbf{0})$ be a germ of smooth complex surface and $\mathcal{O}_{X,\mathbf{0}}$ the ring of germs of holomorphic functions in a neighborhood of $\mathbf{0}$. Let $\mathfrak{a} \subseteq \mathcal{O}_{X,\mathbf{0}}$ be an ideal and let $\pi : (X', E) \longrightarrow (X, \mathbf{0})$ be a log-resolution of $\mathfrak{a}$. In the first sections of Chapter II we present Algorithm 3.13, an algorithm that computes explicitly the minimal log-resolution of $\mathfrak{a}$ from any given set of generators of $\mathfrak{a}$. This is a first step towards the explicit constructions in the second section of this chapter. The results of this chapter generalize the results of Alberich-Carramiñana in [Alb04] in the case of a pencil of curves.

More generally, in the second section of Chapter II, we consider $\pi : (X', E) \longrightarrow (X, \mathbf{0})$ a proper birational morphism that can be achieved as a sequence of blow-ups on a set of points. Given an effective $\mathbb{Z}$-divisor D in X', we may consider its associated (sheaf) ideal $\pi_* \mathcal{O}_{X'}(-D)$ whose stalk at $\mathbf{0}$ we simply denote as H_D. These types of ideals were systematically studied by Zariski in [Zar38]. They are complete ideals of $\mathcal{O}_{X,\mathbf{0}}$ and $\mathfrak{m}$-primary whenever D has exceptional support. Among the class of divisors defining the same complete ideal, we may find a unique maximal representative which happens to have the property of being antinef, see Section 2.3. Actually, Zariski [Zar38] showed that that above correspondence is, in fact, an isomorphism of semigroups between the set of complete $\mathfrak{m}$-primary ideals and the set of antinef divisors with exceptional support.

The aim of the second section is to make the correspondence explicit computationally. Namely, given any antinef divisor D in X', we present Algorithm 4.3, an algorithm that computes a system of generators of the ideal H_D. Moreover, the algorithm produces generators that are monomials in any given set of maximal contact elements of the morphism π, see Section 2.5 for the exact definitions. The algorithm is based on the following results: Zariski's decomposition of complete ideals into simple ones, the unloading procedure to compute antinef closures, and the theory of adjacent ideals on smooth surfaces. These preliminary results will be introduced in Section 2.3.

As a first application of the algorithms developed in Chapter II, we provide a method to compute the integral closure of any ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,\mathbf{0}}$. More precisely, given a log-resolution $\pi : (X', E) \longrightarrow (X, \mathbf{0})$ of the ideal $\mathfrak{a}$, let F_π be the effective Cartier divisor such that $\mathfrak{a} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F_\pi)$. Then, the integral closure $\overline{\mathfrak{a}}$ is the ideal H_{F_π}. Another geometric procedure to compute generators for $\overline{\mathfrak{a}}$ is given by Casas-Alvero in [Cas98].

The usefulness of the algorithms presented in Chapter II becomes still more apparent when dealing with families of complete ideals dominated by the same log-resolution. This is the case of multiplier ideals. Combining the algorithm given in [AÀD16] with Algorithms 3.13 and 4.3, we can give an effective method that, given any set of generators of a planar ideal $\mathfrak{a}$, returns a set of generators of the corresponding multiplier ideals $\mathcal{J}(\mathfrak{a}^\lambda)$. Since multiplier ideals are invariant up to integral closure, we obtain a result that resembles a formula given by Howald [How01] in the sense that the multiplier ideals of a monomial ideal (in the set of maximal contact elements) are monomial as well.

Another interesting family of complete ideals was considered by Teissier in [Tei86]. These ideals are described by valuative conditions given by the intersection multiplicity of the elements of $\mathcal{O}_{X,\mathbf{0}}$ with a fixed germ of a plane curve. With the help of Algorithm 4.3, we can provide an explicit system of generators for these ideals.

I

PRELIMINARIES

In this chapter, we will introduce all the necessary definitions and results that will be required throughout this thesis. This chapter will be divided into two sections. We start reviewing the basics on resolution of singularities, and we introduce the main invariants that we will consider in this thesis, the Bernstein-Sato polynomial, and a related invariant, the multiplier ideals and its associated jumping numbers. For the local study of singularities, we will introduce the Milnor fiber and the monodromy, the Gauss-Manin connection, the Brieskorn lattice, and we will explain the relationship with the Bernstein-Sato polynomial given by Malgrange.

The second section will contain results and definitions for the local study of singularities in complex smooth surfaces. Resolution of singularities is well-understood in this context using the theory of infinitely near points. We will also introduce bases for divisors with exceptional support, antinef divisors, the unloading process to compute antinef closures, and maximal contact elements. For plane curve singularities, we will introduce the semigroup, the minimal embedded resolution via toric morphisms, the monomial curve and its deformations. Finally, we will present Yano's conjecture and some known results about multiplier ideals on a smooth complex surface.

Almost all of the results in this section are well-known results that can be found in the literature. We include the proof of some small lemmas that are probably well-known by the experts in the area, but for which no proof has been found in the literature. In addition, in this section, we include two novel results, Proposition 2.17 and Proposition 2.18, that will be key results for the proof of Yano's conjecture in Chapters IV and V.

1 COMPLEX ALGEBRAIC SINGULARITIES

All the results and objects presented in this section are valid in any dimension.

1.1 *Resolution of singularities*

Let X be a smooth complex algebraic variety of dimension n with structure sheaf $\mathcal{O}_X$. Denote by $\mathrm{Div}(X)$ the free abelian group of prime Weyl divisors. We will write a divisor has $D = \sum n_i D_i$, where D_i are prime divisors and only finitely many integers n_i are non-zero. Similarly, $\mathrm{Div}_{\mathbb{Q}}(X) := \mathrm{Div}_{\mathbb{Q}}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ will denote the set of $\mathbb{Q}$-divisors. Let us start by defining the concept of a *log-resolution* of an ideal sheaf $\mathfrak{a}$ of $\mathcal{O}_X$.

Definition 1.1. A divisor $D = \sum_i D_i$ on a smooth complex algebraic variety is a simple normal crossing (SNC) divisor if each irreducible component D_i is smooth, and D is defined in a neighborhood of any point by an equation in local analytic coordinates of the type

$$z_1 \cdots z_k = 0, \tag{1.1}$$

for some $1 \leq k \leq n$. We say that a divisor $\sum_i a_i D_i$ has simple normal crossings support if $\sum D_i$ is a SNC divisor.

That is, the singularities of D should look no worse than a union of coordinate hyperplanes.

Definition 1.2. Let $\mathfrak{a} \subset \mathcal{O}_X$ be an ideal sheaf. A *log-resolution* of the pair $(X, \mathfrak{a})$ (or of $\mathfrak{a}$, for short), is a proper birational morphism $\pi : X' \longrightarrow X$ such that

1. X' is a smooth complex algebraic variety,

2. $\mathfrak{a} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F_\pi)$ for some effective Cartier divisor F_π,

3. $F_\pi + E$ is a divisor with SNC support, where $E = \mathrm{Exc}(\pi)$ is the exceptional locus.

The existence of a log-resolution, sometimes also called *principalization*, for any sheaf of ideals on any variety over a field of characteristic zero is a result of Hironaka [Hir64a; Hir64b]. Moreover, one can always construct a log-resolution which is a composition of blow-ups along smooth centers. One defines a log-resolution of a divisor $D \in \mathrm{Div}(X)$ as a log-resolution of the associated ideal sheaf $\mathcal{O}_X(-D)$, and hence $F_\pi = \pi^* D$. The notion of log-resolution can be similarly extended to $\mathbb{Q}$-divisors.

Remark 1.1. A log-resolution of an ideal $\mathfrak{a}$ is similar to an *embedded resolution* of the variety defined in $\mathfrak{a}$. In an embedded resolution one requires that the exceptional divisor as SNC support and the strict transform has simple normal crossings with the exceptional divisor, see [Cut04] for the exact definitions. ■

Let U be the maximal Zariski open set on X such that $\pi|_U$ is an isomorphism. Then, the *strict transform* of a divisor $D \in \mathrm{Div}(X)$ is defined as the Zariski closure on X' of the set $\pi^{-1}(D \cap U)$. The divisor F_π will have a decomposition into irreducible components that we will denote as

$$F_\pi = \sum_{i=1}^{r} N_i E_i + \sum_{i=1}^{s} M_i S_i, \tag{1.2}$$

where E_i are the irreducible exceptional divisors. Furthermore, when $\mathfrak{a} = \mathcal{O}_X(-D)$ for some $D \in \mathrm{Div}(X)$ the other irreducible components S_i are, precisely, the irreducible components of the strict transform of D. Notice also that, D is reduced if and only if $M_i = 1$ for $i = 1, \ldots, s$.

Let X be a smooth complex algebraic variety, and let $\omega_X = \Omega_X^n$ be the *canonical bundle* of X, where Ω_X^n is the sheaf of differentials of X. A *canonical divisor* K_X of X is the Cartier divisor satisfying $\omega_X = \mathcal{O}_X(K_X)$.

Definition 1.3. Let $\pi : X' \longrightarrow X$ be a birational morphism between smooth complex varieties. Then, the *relative canonical divisor* is defined as

$$K_\pi = K_{X'} - \pi^* K_X. \tag{1.3}$$

The relative canonical divisor is an effective divisor supported on E, not just an equivalence class. The determinant of the Jacobian matrix, $\det(d\pi)$, defines the local equation of this divisor. Therefore, the divisor K_π will have a decomposition into irreducible components that we will write as

$$K_\pi = \sum_{i=1}^{r} k_i E_i. \tag{1.4}$$

1.2 *The Bernstein-Sato polynomial*

Let $R := \mathbb{C}[x_1, \dots, x_n]$ be the ring of complex polynomials in n variables. Denote by $D := \mathbb{C}[x_1, \dots, x_n]\langle \partial_1, \dots, \partial_n \rangle$ the *Weyl algebra*, where ∂_i is the partial derivative operator with respect to x_i. Therefore, in this section, we will set $X := \mathbb{C}^n$.

The Weyl algebra is a non-commutative ring with the relations $\partial_i x_i - x_i \partial_i = 1$. Any element of D can be written as finite sum $P = \sum_{\underline{\alpha}, \underline{\beta} \geq 0} x^{\underline{\alpha}} \partial^{\underline{\beta}}$, the *normal form* of P, where $\underline{\alpha}, \underline{\beta}$ are multi-indices. If s is another variable that commutes with all the x_i, ∂_i, let $D[s] := D \otimes_{\mathbb{C}} \mathbb{C}[s]$, the polynomial ring in s with coefficients in D. In this case, any $P \in D[s]$ can be written as $P(s) = \sum_{i=0}^{m} s^i P_i$, where the P_i are elements of D.

For any non-constant polynomial $f \in R$, we will associate with the singularity of $f = 0$ the main invariant that we will study in this work, which is derived from the theory of D-modules associated with f.

Definition 1.4. Let s be a new variable, and denote by $R_f[s] \cdot f^s$ the free module generated by f^s over the localized ring $R_f[s] := R[f^{-1}, s]$. The chain rule,

$$\partial_i \cdot \left(\frac{g}{f^k} \cdot f^s \right) = \partial_i \cdot \left(\frac{g}{f^k} \right) \cdot f^s + \frac{sg}{f^{k+1}} \cdot \frac{\partial f}{\partial x_i} \cdot f^s, \tag{1.5}$$

for each $g(x,s) \in R[s]$, induces a structure of left $D[s]$-module on $R_f[s] \cdot f^s$.

Next, define the *parametric annihilator* of f by

$$\operatorname{Ann}_{D[s]}(f^s) = \{P(s) \in D[s] \mid P(s) \cdot f^s = 0\}, \tag{1.6}$$

and by

$$\mathcal{M}_f(s) = D[s] / \operatorname{Ann}_{D[s]}(f^s), \tag{1.7}$$

the cyclic $D[s]$-module generated by $1 \cdot f^s \in R_f[s] \cdot f^s$.

The Bernstein-Sato functional equation introduced by Bernstein in [Ber72] asserts the existence of a differential operator $P(s) \in D[s]$ and a non-zero polynomial $b_{f,P}(s) \in \mathbb{C}[s]$ such that the following relation holds,

$$P(s) \cdot f^{s+1} = b_{f,P}(s) f^s. \tag{1.8}$$

In other words, there exist an element of the form $P(s) \cdot f - b_{f,P}(s) \in \operatorname{Ann}_{D[s]}(f^s)$. Any such differential operator $P(s) \in D[s]$ is only determined up to an element of $\operatorname{Ann}_{D[s]}(f^s)$. Since all the polynomials $b_{f,P}(s)$ satisfying Equation (1.8) for some differential operator $P(s) \in D[s]$ form an ideal in $\mathbb{C}[s]$, one defines

Definition 1.5. The monic generator of the ideal in $\mathbb{C}[s]$ generated by all the $b_{f,P}(s)$ fulfilling Equation (1.8) is the *Bernstein-Sato polynomial* $b_f(s)$ of f.

Denote $\rho_f \subset \mathbb{C}$ the set of roots of $b_f(s)$. Since $s = -1$ is always a root of $b_f(s)$, one usually defines the *reduced Bernstein-Sato polynomial* as $\tilde{b}_f(s) := b_f(s)/(s+1)$.

An equivalent way of defining the Bernstein-Sato polynomial $b_f(s)$ is as the minimal polynomial of the action of s in the $\mathbb{C}[s]$-module $\mathcal{M}(s)/\mathcal{M}(s+1)$. The Bernstein-Sato polynomial $b_f(s)$ of a polynomial f was independently discovered by Sato [SS90] in the context of the theory of prehomogeneous vector spaces.

Remark 1.2. All the above remains essentially true if f is a holomorphic (and even formal) power series in $\mathbb{C}\{x_1,\dots,x_n\}$. The proof of the existence of the Bernstein-Sato for this case is due to Björk [Bjö74]. In this situation, we usually call it the *local Bernstein-Sato polynomial*, in contrast to the *global* $b_f(s)$, and we denote it by $b_{f,\mathbf{0}}(s)$ or by $b_{f,p}$ if $p \in X$. Similarly, denote $\tilde{b}_{f,\mathbf{0}}(s)$ for the local reduced Bernstein-Sato polynomial. ■

The relation between the global $b_f(s)$ and the local $b_{f,p}(s), p \in X$ is the following, see [MN91],

$$b_f(s) = \mathrm{lcm}_{p\in \mathrm{Var}(f)}\, b_{f,p}(s), \tag{1.9}$$

that is, the global Bernstein-Sato is the least common multiple of all the local Bernstein-Sato polynomials. Computing the Bernstein-Sato polynomial for a specific $f \in R$ is, in general, very hard, even computationally, see Remark 1.3 below. If f is smooth at $p \in \mathrm{Var}(f)$, then one can check that $b_{f,p}(s) = s+1$. Therefore, after Equation (1.9), $b_f(s) = s+1$ for all smooth hypersurface defined by $f \in R$. The converse is also true, see [BM96].

Explicit formulas for the Bernstein-Sato polynomial are rare. The normal crossing case is easy to compute,

$$f = \prod_i x_i^{a_i}, \quad P(s) = \prod_i \partial_i^{a_i}, \quad \text{and} \quad b_f(s) = \prod_i \prod_{j=1}^{a_i} (s + j/a_j). \tag{1.10}$$

Notice that from this and Equation (1.9), computing the Bernstein-Sato in the case $n=1$ is trivial. Another easy example,

$$f = \sum_{i=1}^m x_i^2, \quad P(s) = \sum_{i=1}^m \partial_i^2, \quad \text{and} \quad b_f(s) = (s+1)\left(s+\frac{m}{2}\right). \tag{1.11}$$

The case of quasi-homogeneous polynomials with an isolated singularity is also well understood, see [Yan78] and [Mal75]. There are many examples worked out in [Yan78]. See also [Kat81] and [Kat82].

Remark 1.3. The first general algorithm for computing the Bernstein-Sato polynomial is due to Oaku [Oak97], using the theory of non-commutative Gröbner basis in the Weyl algebra. Nowadays, the computer algebra systems Macaulay2 [M2], Singular [Sing], and Risa/Asir [Asir] have packages that allow the computation of Bernstein-Sato polynomials. However, due to the high complexity of the Gröbner basis algorithm, the computation of many examples is not feasible. ■

The roots of the Bernstein-Sato polynomial $b_f(s)$ are negative rational numbers, i.e. $\rho_f \subset \mathbb{Q}_{<0}$, see the works of Malgrange [Mal75], for the case of isolated singularities, and Kashiwara [Kas76], for the general case. Kashiwara [Kas76] uses resolution of singularities to reduce the problem to the computation in the normal crossing case. This method also gives a set of candidates in terms of the multiplicities of f along the resolution. There is a refinement by Lichtin [Lic89] that takes into account also the multiplicities of the relative canonical divisor K_π. Using the notations from Section 1.1, the candidates are the following,

$$\rho_f \subset \left\{ -\frac{k_i + 1 + \nu}{N_i} \mid \nu \in \mathbb{Z}_{\geq 0} \right\} \cup \left\{ -\frac{\nu+1}{M_i} \mid \nu \in \mathbb{Z}_{\geq 0} \right\}. \tag{1.12}$$

Usually, the set of roots of the Bernstein-Sato polynomial is much smaller than the set of candidates in (1.12). Determining which divisors of a resolution contribute to the roots of the Bernstein-Sato is an open problem.

In this context, the smallest root of $b_f(-s)$ is called the *log-canonical threshold* $\operatorname{lct}(f)$ of f. From the point of view of singularity theory, this notion goes back to Arnold [AV85] who called it the *complex singularity exponent*. See Section 1.3 for an analytic approach close to the original motivation in [AV85]. The name of this invariant comes from the context of birational geometry and log-canonical models, see [Sho93] for the original definition in this context. We will denote $\operatorname{lct}_{\mathbf{0}}(f)$ or $\operatorname{lct}_p(f)$, $p \in X$ in the local case. One can show, see for instance [Mus12, Thm. 1.1], that

$$\operatorname{lct}(f) = \min_{i,j} \left\{ \frac{k_i + 1}{N_i}, \frac{1}{M_j} \right\}. \tag{1.13}$$

For more information on the log-canonical threshold and its relation with other invariants of the singularity see [Kol97] and [Mus12]. The only other general result about the roots ρ_f of $b_f(s)$ is the following bound by Saito [Sai94] which uses the log-canonical threshold.

Theorem 1.6 ([Sai94, Thm. 0.4])**.** *For any non-constant $f \in R$,*

$$\rho_f \subset [-n + \operatorname{lct}(f), -\operatorname{lct}(f)]. \tag{1.14}$$

The bounds from Theorem 1.6 are sharp in the case of a quasi-homogeneous singularity with an isolated singularity.

Remark 1.4. The definition of the Bernstein-Sato polynomial $b_f(s)$ has been generalized in two different ways. Sabbah [Sab87a; Sab87b] defined the Bernstein-Sato ideal of a tuple of polynomials $\boldsymbol{f} = (f_1, \dots, f_r)$. Budur, Mustaţă and Saito [BMS06] defined the Bernstein-Sato polynomial $b_{\mathfrak{a}}(s)$ of an arbitrary ideal $\mathfrak{a} \subseteq \mathbb{C}[x_1, \dots, x_n]$. After the results in [Mus19], one has the same kind of relations between the roots of $b_{\mathfrak{a}}(s)$ and the numerical data of a log-resolution of the ideal $\mathfrak{a}$. ■

Before ending this section, some remarks for the local case. If $f : (\mathbb{C}^n, \mathbf{0}) \longrightarrow (\mathbb{C}, 0)$ is a germ of a holomorphic function, then one can define the local Bernstein-Sato $b_{f,\mathbf{0}}(s)$ of f as the local Bernstein-Sato of any representative of f in $\mathbb{C}\{x_1, \dots, x_n\}$. Indeed,

Lemma 1.7. *Let $f : (\mathbb{C}^n, \mathbf{0}) \longrightarrow (\mathbb{C}, 0)$ be a germ of a holomorphic function. Then, the local Bernstein-Sato $b_{f,\mathbf{0}}(s)$ of f is independent of the representative of f in $\mathbb{C}\{x_1, \dots, x_n\}$.*

Proof. Abusing the notation, let $f \in \mathbb{C}\{x_1, \dots, x_n\}$ be a representative of the germ f. Then, consider uf with $u(\mathbf{0}) \neq 0$ a unit, another representative for the germ. Hence, if $P(s, x, \partial) \cdot f^{s+1} = b_{f,\mathbf{0}}(s) f^s$, then

$$u^{-1} P(s, x, \partial - (s+1) \nabla \log u) \cdot (uf)^{s+1} = b_f(s) (uf)^s. \tag{1.15}$$

and vice versa, see [Yan78, pg. 119]. Similarly, let $\varphi : U \longrightarrow \mathbb{C}^n$ be an analytic change of coordinates for $\mathbf{0} \in U \subset \mathbb{C}^n$ sufficiently small. If $\bar{x}_1, \dots, \bar{x}_n$ denote the new coordinates, then we have

$$\bar{\partial}_i = \sum_{j=1}^n \frac{\partial \varphi_j}{\partial \bar{x}_i} \partial_j \qquad \text{and} \qquad \partial_i = \sum_{j=1}^n \frac{\partial \varphi_j^{-1}}{\partial x_i} \bar{\partial}_j. \tag{1.16}$$

Therefore, $P(s, x, \partial) \cdot f^{s+1} = b_{f,\mathbf{0}}(s) f^s$ if and only if

$$P(s, \bar{x}, \bar{\partial}) \cdot (f \circ \varphi)^{s+1} = b_{f \circ \varphi, \mathbf{0}}(s) (f \circ \varphi)^s. \tag{1.17}$$

□

1.3 *Multiplier ideals*

Let X be a smooth complex algebraic variety of dimension n with structure sheaf $\mathcal{O}_X$ and an ideal $\mathfrak{a} \subseteq \mathcal{O}_X$. We use the notions introduced in Section 1.1 and take $\pi : X' \longrightarrow X$ a log-resolution of $\mathfrak{a}$. If $D = \sum_i a_i D_i$ is a $\mathbb{Q}$-divisor D on X, the *round-up* $\lceil D \rceil$ of D is the integral divisor $\lceil D \rceil = \sum_i \lceil a_i \rceil D_i$.

Historically, multiplier ideals first appeared in commutative algebra in the work of Lipman [Lip93]. In an analytical context, they appear in the work of Nadel [Nad90]. However, we will use the following geometric definition. For a complete reference about multiplier ideals, the reader is referred to [Laz04].

Definition 1.8. The *multiplier ideal* sheaf associated to $\mathfrak{a}$ and some rational number $\lambda \in \mathbb{Q}_{>0}$ is defined as

$$\mathcal{J}(X, \mathfrak{a}^\lambda) = \mathcal{J}(\mathfrak{a}^\lambda) := \pi_* \mathcal{O}_{X'}(\lceil K_\pi - \lambda F_\pi \rceil). \tag{1.18}$$

One can define an analogous multiplier ideal associated to a $\mathbb{Q}$-divisor on X by means of the same construction. Before reviewing the more important properties of multiplier ideals, let us motivate the definition of multiplier ideals from an analytic point of view, see [Laz04, §9.3.D].

For simplicity, let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial and $p \in \mathbb{C}^n$ a point of $\mathrm{Var}(f)$, possibly singular. The multiplier ideals arise from an attempt to measure the singularity of f at p by means of integration. Concretely, for sufficiently small $\lambda \in \mathbb{R}_{>0}$ the integral

$$\int_{\overline{B}_\epsilon(p)} \frac{|\mathrm{d}\underline{x}|^2}{|f|^{2\lambda}}, \tag{1.19}$$

where $|\mathrm{d}\underline{x}|^2 := \mathrm{d}\underline{x}\, \overline{\mathrm{d}\underline{x}}$, will converge, where $\overline{B}_\epsilon(p)$ is a small enough closed ball around the point p. In this context we recover one of the original definitions of the *log-canonical threshold*, sometimes called the *complex singularity exponent* of f at p in this context,

$$\mathrm{lct}_p(f) = \sup \left\{ \lambda \in \mathbb{R}_{>0} \;\middle|\; \exists\, \epsilon \ll 1 \text{ such that } \int_{\overline{B}_\epsilon(p)} \frac{1}{|f|^{2\lambda}} < \infty \right\}. \tag{1.20}$$

Resolution of singularities is useful in this context since

$$\int_{\overline{B}_\epsilon(p)} \frac{|\mathrm{d}\underline{x}|^2}{|f|^{2s}} = \int_{\pi^{-1}\left(\overline{B}_\epsilon(p)\right)} \frac{|d\pi|^2}{|\pi^* f|^{2\lambda}}, \tag{1.21}$$

noticing that $\pi^{-1}\big(\overline{B}_\epsilon(p)\big)$ is still a compact set since π is proper, and we can use the local monomial form of F_π and K_π to study the convergence of the integral. Namely, in one of the finitely many local charts U of $\pi^{-1}\big(\overline{B}_\epsilon(p)\big)$ the integral reads as,

$$\int_{\pi^{-1}\left(\overline{B}_\epsilon(p)\right)} \frac{|u(\underline{z})\, z_1^{k_1} z_2^{k_2} \cdots z_r^{k_r}|^2}{|z_1^{a_1} z_2^{a_2} \cdots z_r^{a_r}|^{2\lambda}} |\mathrm{d}\underline{z}|^2. \tag{1.22}$$

Here $u(\underline{z})$ is a unit, $r \leq n$ and we have used $a_i \in \mathbb{Z}$ to denote both the multiplicities of the exceptional divisors and of the strict transform. By Fubini's theorem the integral in (1.22) will be convergent if and only if $k_i - \lambda a_i > -1$, that is, $\lambda < (k_i + 1)/a_i$, for all $i = 1, \ldots, r$. Hence, we recover the formula Equation (1.13) from Section 1.2. The analytic definition of the log-canonical threshold leads naturally to the analytic definition of multiplier ideals.

Definition 1.9. Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ and $p \in \mathrm{Var}(f)$. The multiplier ideal of f at p associated with a rational number $\lambda \in \mathbb{Q}_{>0}$ is

$$\mathcal{J}(f^\lambda)_p = \left\{ g \in \mathbb{C}[x_1, \ldots, x_n] \mid \exists\, \epsilon \ll 1 \text{ such that } \int_{\overline{B}_\epsilon(p)} \frac{|g|^2}{|f|^{2\lambda}} < \infty \right\}. \tag{1.23}$$

Thus, from the analytical definition, the multiplier ideals contain polynomials that can be used as multipliers to make the integral converge. From this same definition, it is easy to see that the multiplier ideals are, in fact, ideals of $\mathbb{C}[x_1, \ldots, x_n]$. The same construction can be carried out for an ideal $\mathfrak{a} \subseteq \mathbb{C}[x_1, \ldots, x_n]$ substituting $|f|^2$ by $\sum_i |f_i|^2$ with f_i a set of generators of $\mathfrak{a}$ giving also a definition for $\mathrm{lct}_p(\mathfrak{a})$, $p \in \mathrm{Var}(\mathfrak{a})$.

We return now to the properties of the multiplier ideals from the algebro-geometric perspective. The first one being that the multiplier ideals are independent of the chosen log-resolution.

Theorem 1.10 ([EV92, §7.3]). *Let $\pi' : X' \longrightarrow X$, $\pi'' : X'' \longrightarrow X$ be two log-resolutions of an ideal $\mathfrak{a} \subseteq \mathcal{O}_X$. Then,*

$$\pi'_* \mathcal{O}_{X'}(\lceil K_{\pi'} - \lambda F_{\pi'} \rceil) = \pi''_* \mathcal{O}_{X''}(\lceil K_{\pi''} - \lambda F_{\pi''} \rceil), \tag{1.24}$$

for any $\lambda \in \mathbb{Q}_{>0}$.

We continue with some basic properties of the multiplier ideals that basically follow from the definition.

Proposition 1.11.

- *Let $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \mathcal{O}_X$ be two ideals, then for any $\lambda \in \mathbb{Q}_{>0}$,*

$$\mathcal{J}(\mathfrak{a}_1^\lambda) \subseteq \mathcal{J}(\mathfrak{a}_2^\lambda). \tag{1.25}$$

- *For any ideal $\mathfrak{a} \subseteq \mathcal{O}_X$,*

$$\mathfrak{a} \subseteq \mathcal{J}(\mathfrak{a}). \tag{1.26}$$

Multiplier ideals are integrally closed ideal sheaves, this follows from the fact that direct images of proper birational morphisms are integrally closed, see for instance [Laz04, Prop. 9.6.11]. Additionally, for any ideal $\mathfrak{a} \subseteq \mathcal{O}_X$, let $\overline{\mathfrak{a}} \subseteq \mathcal{O}_X$ be the integral closure of $\mathfrak{a}$, then $\mathcal{J}(\mathfrak{a}^\lambda) = \mathcal{J}(\overline{\mathfrak{a}}^\lambda)$, since $\overline{\mathfrak{a}} = \pi_* \mathcal{O}_{X'}(-F_\pi)$, see [Laz04, §III.9.6]. Consequently, $\overline{\mathfrak{a}} \subseteq \mathcal{J}(\mathfrak{a})$.

Theorem 1.12 (Skoda's Theorem, [Laz04, Thm. 9.6.21]). *Let $\mathfrak{a} \subseteq \mathcal{O}_X$ be an ideal sheaf of the smooth complex algebraic variety X of dimension n.*

1. *Given any integer $m \geq n$,*

$$\mathcal{J}(\mathfrak{a}^m) = \mathfrak{a} \cdot \mathcal{J}(\mathfrak{a}^{m-1}), \tag{1.27}$$

consequently, iterating the process

$$\mathcal{J}(\mathfrak{a}^m) = \mathfrak{a}^{m-n+1} \cdot \mathcal{J}(\mathfrak{a}^{n-1}), \tag{1.28}$$

for every $m \geq n$. In particular, $\mathcal{J}(\mathfrak{a}^m) \subseteq \mathfrak{a}^{m-n+1}$.

2. *More generally, for every non-zero ideal $\mathfrak{b} \subseteq \mathcal{O}_X$ and $\lambda \in \mathbb{Q}_{>0}$*

$$\mathcal{J}(\mathfrak{a}^m \cdot \mathfrak{b}^\lambda) = \mathfrak{a}^{m-n+1} \cdot \mathcal{J}(\mathfrak{a}^{n-1} \cdot \mathfrak{b}^\lambda), \tag{1.29}$$

for every $m \geq n$.

If X is affine, a different version of Skoda's theorem exists in which the role of the dimension n of the ambient space is changed by the number of generators of the ideal $\mathfrak{a}$, see [Ein+04, Remark 1.16].

Theorem 1.13 (Skoda's Theorem, [Ein+04])**.** *Assume that X is affine and let $\mathfrak{a} \subseteq \mathcal{O}_X$ be an ideal generated by r elements. Then,*

$$\mathcal{J}(\mathfrak{a}^{m+1}) = \mathfrak{a} \cdot \mathcal{J}(\mathfrak{a}^m), \tag{1.30}$$

for $m \geq r$.

Multiplier ideals come together with a set of numerical invariants, the *jumping numbers*, that were studied systematically by Ein, Lazarsfeld, Smith and Varolin [Ein+04]. However, they already appeared implicitly in some works of Libgober [Lib83], Lichtin [Lic86] and Loeser and Vaquié [LV90]. Notice that,

$$\lceil K_\pi - \lambda F_\pi \rceil \geqslant \lceil K_\pi - (\lambda + \varepsilon) F_\pi \rceil \tag{1.31}$$

for any $\varepsilon > 0$, and the equality holds if ε is small enough. Then, for $\mathfrak{a} \subseteq \mathcal{O}_X$ and for a fixed $p \in \operatorname{Var}(\mathfrak{a})$, there is an increasing discrete sequence

$$0 = \lambda_0(\mathfrak{a}, p) < \lambda_1(\mathfrak{a}, p) < \cdots < \lambda_i(\mathfrak{a}, p) < \cdots \tag{1.32}$$

of rational numbers $\lambda_i := \lambda_i(\mathfrak{a}, p)$, if $\mathfrak{a}$ and p are clear from the context, characterized by the properties that

$$\mathcal{J}(\mathfrak{a}^\lambda)_p = \mathcal{J}(\mathfrak{a}^{\lambda_i})_p \qquad \text{if} \qquad \lambda \in [\lambda_i, \lambda_{i+1}), \tag{1.33}$$

and $\mathcal{J}(\mathfrak{a}^{\lambda_{i+1}})_p \subsetneq \mathcal{J}(\mathfrak{a}_i^\lambda)_p$ for every i. In other words, we have the following nested sequence of ideals

$$\mathcal{O}_{X,p} \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda_1})_p \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda_2})_p \supsetneq \cdots \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda_i})_p \supsetneq \cdots. \tag{1.34}$$

Definition 1.14 (Jumping numbers)**.** The rational numbers $\lambda_i := \lambda_i(\mathfrak{a}, p)$ are the *jumping numbers* or *jumping coefficients* of the ideal sheaf $\mathfrak{a} \subseteq \mathcal{O}_X$ at $p \in \operatorname{Var}(\mathfrak{a})$.

From the definition of multiplier ideals, and if $\pi : (X', E) \longrightarrow (X, p)$ is a log-resolution of the ideal $\mathfrak{a}$ at $p \in \operatorname{Var}(\mathfrak{a})$, then the set of jumping numbers must be inside the following set of *candidate jumping numbers*

$$\left\{ \frac{k_i + m}{a_i} \;\middle|\; m \in \mathbb{Z}_{>0} \right\}, \tag{1.35}$$

where the notations are the same as in Equation (1.22). As in the case of the Bernstein-Sato polynomial, Section 1.2, usually, the set of jumping numbers is much smaller than this set. Even for multiplier ideals, determining which divisors contribute to jumping numbers is also an open problem.

The first jumping number $\lambda_1(\mathfrak{a}, p)$ coincides with the log-canonical threshold $\operatorname{lct}_p(\mathfrak{a})$. After Skoda's Theorem Theorem 1.12, the jumping numbers of an ideal $\mathfrak{a}$ exhibit a periodicity after $n-1$. That is, $\lambda > n - 1$ is a jumping number if and only if $\lambda + 1$ is. For principal ideals, $\mathfrak{a} = (f), f \in \Gamma(X, \mathcal{O}_X)$, the variant of Skoda's Theorem 1.13, implies the periodicity of the jumping numbers $\lambda_i(f, p)$ appearing in $(0, 1]$.

We will end this section with some results concerning the relation between the jumping numbers of a principal ideal generated by $f \in \mathbb{C}[x_1, \ldots, x_n]$ and the Bernstein-Sato polynomial of f.

Theorem 1.15 ([Ein+04], [BS05]). *Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be a non-constant polynomial. Then if $\lambda_i = \lambda_i(f, p)$ is a jumping number of f in $(0, 1]$, then $b_f(-\lambda_i) = 0$.*

The same statement holds true when f is replaced by a germ of a holomorphic function at $p \in \mathbb{C}^n$. The proof of Theorem 1.15 in [Ein+04] uses the analytic definition of the multiplier ideals and the Bernstein-Sato functional equation to integrate by parts. In contrast, in [BS05] the results only use the theory of D-modules.

Remark 1.5. There are general algorithms, like those developed by Shibuta [Shi11] and Berkersch and Leykin [BL10], that given a set of generators of $\mathfrak{a}$, compute the list of jumping numbers and a set of generators of the corresponding multiplier ideals. These algorithms use the theory of Bernstein-Sato polynomials and require the use of non-commutative Gröbner bases in the Weyl algebra. They have been implemented in Macaulay2 [M2]. However, it is difficult to compute examples due to the complexity of these algorithms. ■

1.4 *Milnor fiber and monodromy*

Let $f : (\mathbb{C}^{n+1}, \mathbf{0}) \longrightarrow (\mathbb{C}, 0)$ be a germ of a holomorphic function. For $0 < \delta \ll \epsilon \ll 1$, let $B_\epsilon \subset \mathbb{C}^{n+1}$ be the ball of radius ϵ centered at the origin, $T \subset \mathbb{C}$ the disk of radius δ centered at zero, and $T' := T \setminus \{0\}$ the punctured disk. Abusing the notation, we will also denote by f a representative of the germ. Set

$$X := B_\epsilon \cap f^{-1}(T), \quad X' := X \setminus f^{-1}(0), \quad X_t := B_\epsilon \cap f^{-1}(t), \quad t \in T. \tag{1.36}$$

The restriction $f' : X' \longrightarrow T'$ is a smooth fiber bundle such that the diffeomorphism type of the fiber X_t is independent of the choice of δ, ϵ and $t \in T'$, see [Mil68, §4]. Any of the fibers X_t, or rather its diffeomorphism type, is called the *Milnor fiber* of f. The complex homology $H_i(X_t, \mathbb{C})$ (resp. cohomology $H^i(X_t, \mathbb{C})$) groups of the Milnor fiber are finite-dimensional vector spaces that vanish above degree n since X_t has the homotopy type of a finite CW-complex of dimension n [Mil68, Thm. 5.1].

The action of the *fundamental group* $\pi(T', t)$ of the base of the fibration induces a diffeomorphism h on each fiber X_t which is usually called the *geometric monodromy*. Alternatively, this same action induces h_* (resp. h^*) an endomorphism of $H_*(X_t, \mathbb{C})$ (resp. $H^\bullet(X_t, \mathbb{C})$), the *algebraic (complex) monodromy* of f. The algebraic monodromy depends only on the singularity $(f^{-1}(0), \mathbf{0})$. The fundamental result about the structure of the monodromy endomorphism is the so-called *Monodromy Theorem*.

Theorem 1.16 (Monodromy [SGA7-I; Cle69; Bri70]). *The operator h_* is quasi-unipotent, that is, there are integers p and q such that $(h_*^p - \mathrm{id})^q = 0$. In other terms, the eigenvalues of the monodromy are roots of unity. Moreover, one can take $q = n + 1$.*

If f defines an *isolated singularity*, the topology of the Milnor fiber is quite simple. In this case, the Milnor fiber has the homotopy type of a bouquet of μ n-dimensional spheres, where μ is the Milnor number of the singularity, see [Mil68, Thm. 6.5]. Therefore, $\widetilde{H}_i(X_t, \mathbb{C}) = 0$ for $i \neq n$ and $\dim_{\mathbb{C}} H_n(X_t, \mathbb{C}) =: \mu$. Furthermore, the Milnor number coincides with

$$\mu = \dim_{\mathbb{C}} \frac{\Omega^{n+1}_{X,\mathbf{0}}}{\mathrm{d}f \wedge \Omega^n_{X,\mathbf{0}}}. \tag{1.37}$$

By the Oka-Grauert principle [Gra58], the *Milnor fibration* $f' : X' \longrightarrow T'$ of a singularity f defines a holomorphic vector bundle $f^* : H^n \longrightarrow T'$ on T', where

$$H^n := \bigcup_{t \in T'} H^n(X_t, \mathbb{C}) \tag{1.38}$$

and f^* is the natural projection of H^n to T'. Similarly, one has the dual vector bundle $f_* : H_n \longrightarrow T'$. The vector bundle H^n (resp. H_n) is sometimes called the *cohomological* (resp. *homological*) *Milnor fibration*. We will denote by $\mathcal{H}^n$ (resp. $\mathcal{H}_n$) the locally free sheaf of holomorphic sections of the vector bundle H^n (resp. H_n). Since f' is locally trivial,

Lemma 1.17. *The (co)-homological Milnor fibration is a locally constant vector bundle.*

Proof. Fix a trivialization of the Milnor fibration, with $\{U_i\}_{i \in I}$ an open cover of T' and $g_{i,j} : U_i \cap U_j \to \mathrm{Diff}(X_t)$ be the transition functions. These transition functions induce maps $\overline{g}_{i,j} : U_i \cap U_j \to \mathrm{Aut}(H^p(X_t, \mathbb{C}))$ in cohomology determining the holomorphic structure on H^p. Let $t_0, t_1 \in U_i \cap U_j$ be two points in the base joined by an arc γ. Then, $g_{i,j}(t_0)$ and $g_{i,j}(t_0)$ are homotopic maps from X_t to itself via $g_{i,j}(\gamma(\tau)), \tau \in [0,1]$. Hence, the induced maps in cohomology coincide, i.e. $\overline{g}_{i,j}(t_0) = \overline{g}_{i,j}(t_1)$. The proof in homology is the same. $\square$

Obviously, the same holds true in homology. Denote by Λ the set of all eigenvalues of the monodromy operator h^*. Fixed $\lambda \in \Lambda$, denote by H^p_λ the set of vectors of H^p that are annihilated by $(h^* - \lambda\,\mathrm{id})^{n+1}$ and by f^*_λ the natural projection from H^n_λ to T'. Then, H^n_λ is a holomorphic subbundle of H^n such that $H^n = \bigoplus_{\lambda \in \Lambda} H^n_\lambda$.

1.5 *The Gauss-Manin connection*

Let us start with some generalities about holomorphic connections. Let $E \longrightarrow S$ be a holomorphic vector bundle over a complex manifold S. Denote by $\mathcal{F}$ its sheaf of holomorphic sections.

Definition 1.18. A holomorphic *connection* on E is a $\mathbb{C}$-linear map

$$\nabla : \mathcal{F} \longrightarrow \Omega^1_S \otimes_{\mathcal{O}_S} \mathcal{F} =: \Omega^1_S(\mathcal{F}). \tag{1.39}$$

satisfying the *Leibniz identity*,

$$\nabla(g\sigma) = \mathrm{d}g \otimes \sigma + g \otimes \nabla(\sigma), \tag{1.40}$$

for g a section of $\mathcal{O}_S$ and σ a section of $\mathcal{F}$.

Let $\Theta_S := \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_S) = \mathrm{Hom}_{\mathcal{O}_S}(\Omega_S, \mathcal{O}_S)$ be the sheaf of vector fields in the base space S, a connection defines a *covariant derivative* along a section v of Θ_S by $\nabla_v(\sigma) = \langle \nabla\sigma, v \rangle$, where $\langle \cdot, \cdot \rangle$ is induced by the pairing $\Omega^1_S \times \Theta_S \to \mathcal{O}_S$. Then, ∇_v defines a $\mathbb{C}$-linear homomorphism $\nabla_v : \mathcal{F} \to \mathcal{F}$ still satisfying the Leibniz identity. When S is one-dimensional, giving a covariant derivative is the same as giving a connection so, in this case, we will use the term connection indistinctly.

A connection $\nabla =: \nabla_0 : \mathcal{F} \to \Omega^1_S \otimes_{\mathcal{O}_S} \mathcal{F}$ can be extended to a $\mathbb{C}$-linear homomorphism of sheaves

$$\nabla_i : \Omega^i_S \underset{\mathcal{O}_S}{\otimes} \mathcal{F} \longrightarrow \Omega^{i+1}_S \underset{\mathcal{O}_S}{\otimes} \mathcal{F} \tag{1.41}$$

by means of the equality

$$\nabla_i(\omega \otimes \sigma) = \mathrm{d}\omega \otimes \sigma + (-1)^i \omega \wedge \nabla_{i-1}(\sigma), \qquad \omega \in \Omega^i_S, \quad \sigma \in \mathcal{F}. \tag{1.42}$$

with ω a section of Ω^i_S and σ a section of $\mathcal{F}$. Here $\omega \wedge \nabla_{i-1}(\sigma)$ represents the image of the section $\omega \otimes \nabla_{i-1}(\sigma)$ of $\mathcal{F} \otimes (\otimes_1^{i-1}\Omega_S)$ by the natural map

$$\mathcal{F} \otimes (\otimes_1^{i-1}\Omega_S) \longrightarrow \mathcal{F} \otimes (\wedge_1^{i-1}\Omega_S) = \mathcal{F} \otimes \Omega_S^{i-1}. \tag{1.43}$$

The $\mathbb{C}$-linear homomorphism $R = \nabla_1 \circ \nabla_0 : \mathcal{F} \to \Omega^2_S(\mathcal{F})$ is called the *curvature* of the connection. A connection is said to be *integrable* or *flat* if $R = 0$. The integrability of a connection is equivalent to the identity $\nabla_{[X,Y]} = [\nabla_X, \nabla_Y]$, for all X, Y sections of Θ_S, of the covariant derivative. In the sequel, all connections will be assumed to be flat.

A local section σ of $\mathcal{F}$ is said to be *flat* or *horizontal* if $\nabla(\sigma) = 0$. The integrability of a connection ∇ implies the integrability of the differential equation $\nabla(\sigma) = 0$ for any initial condition $\sigma(s) = v_0 \in E_s, s \in S$. Cauchy's Theorem on the existence of solutions of differential equations implies the existence of a basis of flat local sections. Therefore, the kernel of the connection $\ker \nabla$ defines a *locally constant sheaf* of $\mathbb{C}$-vector spaces on S. A locally constant sheaf of $\mathbb{C}$-vector spaces of dimension r is usually called a *local system* of rank r.

Let now $e_1, \ldots, e_n$ be a local basis of a vector bundle $E \longrightarrow T$ of dimension n. One can locally define a flat connection by setting $\nabla(e_i) = 0$ for all $i = 1, \ldots, n$. If the vector bundle is locally constant, this definition extends to a global connection. Indeed, if $\{U_i\}_{i\in I}$ and $g_{i,j} : U_i \cap U_j \to \mathrm{Aut}(E)$ is a trivialization of E, $\nabla(e_k) = \nabla(\sum_l g_{i,j} e_l) = \sum_l \mathrm{d}g_{i,j} e_l = 0$ on $U_i \cap U_j$, since the transition functions are constant. A locally constant vector bundle is usually called a *flat vector bundle*.

Theorem 1.19 ([Del70, Thm. 2.17]). *There is an equivalence of categories between local systems and flat vector bundles.*

We have seen that one direction is given by the kernel of the connection. For the other, if $\mathcal{L}$ is a local system, the connection ∇ on the locally free sheaf $\mathcal{L} \otimes_{\mathbb{C}} \mathcal{O}_T$ is simply given by $\nabla(\sigma \cdot g) = \sigma \cdot \mathrm{d}(g)$. Notice also that if $\mathcal{L} = \underline{\mathbb{C}}_X$ is just the constant sheaf, then $\mathcal{F} = \mathcal{O}_T$ and the connection $\nabla(f) = \mathrm{d}f$, coincides with the exterior derivative.

Definition 1.20. The (co)-homological *Gauss-Manin connection* is the canonical integrable connection

$$\nabla_p : \mathcal{H}_p \longrightarrow \Omega^1_{T'} \otimes_{\mathcal{O}_{T'}} \mathcal{H}_p \qquad\qquad \nabla^p : \mathcal{H}^p \longrightarrow \Omega^1_{T'} \otimes_{\mathcal{O}_{T'}} \mathcal{H}^p, \tag{1.44}$$

on the (co)-homological Milnor fibration.

In the sequel, we will mainly study the cohomological version of the Gauss-Manin connection in the case that $f : (\mathbb{C}^{n+1}, \mathbf{0}) \longrightarrow (\mathbb{C}, 0)$ has an isolated singularity. Therefore, after the discussion in Section 1.4, we will only care about the homology and cohomology groups of degree n.

Taking a basis vector field d/dt on T', consider the associated covariant derivative in cohomology $\partial_t^* := \nabla^n_{d/dt} : \mathcal{H}^n \longrightarrow \mathcal{H}^n$, that we will also call the Gauss-Manin connection. Let $\mathcal{L}^* := \ker \nabla^n$ be the local system associated with the (cohomological) Gauss-Manin connection. Notice that, in this setting, $\mathcal{L}^*$ coincides with $R^n f'_* \underline{\mathbb{C}}_X$. That is, if ϕ is a locally trivial fibration, $R^q\phi_*\underline{\mathbb{C}}$ are always local systems. The subbundles $H^n_\lambda, \lambda \in \Lambda$, defined in Section 1.4 also carry a connection ∇^n_λ which coincides with the restriction of ∇^n. Let $\mathcal{L}^*_\lambda = \ker \nabla^n_\lambda$ be the local system generated by the sections of $\mathcal{L}^*$ that are annihilated by the endomorphism $(h^* - \lambda\,\mathrm{id})^{n+1}$.

We will end this section with the following relation between the differentiation of a section of $\mathcal{O}_T$ and the covariant derivative of the Gauss-Manin connection. First, let $\langle\cdot,\cdot\rangle : \mathcal{H}^n \times \mathcal{H}_n \longrightarrow \mathcal{O}_{T'}$ be the non-degenerate pairing induced by the pairing $H^n(X_t, \mathbb{C}) \times H_n(X_t, \mathbb{C}) \longrightarrow \mathbb{C}$ between homology and cohomology.

Lemma 1.21 ([Kul98, §I.2.6 Prop. 1]). *Let ω be a section of $\mathcal{H}^n$ and γ be a locally constant section of $\mathcal{H}_n$, then*

$$\frac{d}{dt}\langle\omega,\gamma\rangle = \langle\partial^*_t\omega,\gamma\rangle. \tag{1.45}$$

Proof. The equality is true for horizontal sections of $\mathcal{H}^n$. Indeed, in that case, $\langle\omega,\gamma\rangle \in \mathbb{C}$ and both sides are zero. Let us see that it is true for $g\omega$ with g a section of $\mathcal{O}_{T'}$. First,

$$\frac{d}{dt}\langle g\omega,\gamma\rangle = \frac{d}{dt}(g\langle\omega,\gamma\rangle) = \frac{dg}{dt}\langle\omega,\gamma\rangle + g\langle\partial_t\omega,\gamma\rangle. \tag{1.46}$$

On the other side,

$$\langle\partial_t(g\omega),\gamma\rangle = \langle\frac{dg}{dt}\omega + g\partial_t\omega,\gamma\rangle = \frac{dg}{dt}\langle\omega,\gamma\rangle + g\langle\partial_t\omega,\gamma\rangle. \tag{1.47}$$

Finally, the formula follows because horizontal sections generate $\mathcal{H}^n$. □

1.6 *Relative differential forms*

In this subsection, we will review some of the results of Brieskorn [Bri70] about the algebraic description of the Gauss-Manin connections. Let $f : X \longrightarrow T$ be the Milnor fibration, see Section 1.4, of a germ $f : (\mathbb{C}^{n+1}, \mathbf{0}) \longrightarrow (\mathbb{C}, 0)$. With the notations from Section 1.4, $f' : X' \longrightarrow T'$ is a locally trivial fibration. Notice that we do not require yet that f is an isolated singularity at the origin.

The complex of sheaves of *relative differentials* of the morphism $f : X \longrightarrow T$ is

$$\Omega^\bullet_{X/T} : 0 \longrightarrow \mathcal{O}_X \xrightarrow{\mathrm{d}_r} \Omega^1_{X/T} \xrightarrow{\mathrm{d}_r} \cdots \xrightarrow{\mathrm{d}_r} \Omega^n_{X/T} \xrightarrow{\mathrm{d}_r} \Omega^{n+1}_{X/T} \longrightarrow 0, \tag{1.48}$$

where $\Omega^p_{X/T} := \Omega^p_X / f^*\Omega^1_T \wedge \Omega^{p-1}_X = \Omega^p_X / \mathrm{d}f \wedge \Omega^{p-1}_X$ and d_r is the induced differential in the quotient. Some basic properties of the relative differentials are the following.

Lemma 1.22. *The relative differentials $(\Omega^\bullet_{X/T}, \mathrm{d}_r)$ are a chain complex of $\mathcal{O}_X$-modules. Furthermore, $\Omega^p_{X/T}$ is an $\mathcal{O}_T$-module via the action $g \cdot \overline{\omega} \mapsto f^*(g)\overline{\omega}$, for g a section of $\mathcal{O}_T$, and $\overline{\omega}$ a section of $\Omega^p_{X/T}$ and d_r is $\mathcal{O}_T$-linear.*

Proof. We will check first that d_r is well-defined. Let η be a section of Ω^{p-1}_X, then $\mathrm{d}_r(\overline{\omega + \mathrm{d}f \wedge \eta}) = \overline{\mathrm{d}\omega + \mathrm{d}(\mathrm{d}f \wedge \eta)} = \overline{\mathrm{d}\omega + \mathrm{d}f \wedge \mathrm{d}\eta} = \overline{\mathrm{d}\omega} = \mathrm{d}_r(\overline{\omega})$, and d_r is well defined. The relative differentials form a complex since $\mathrm{d}_r \circ \mathrm{d}_r(\overline{\omega}) = \mathrm{d}_r(\overline{\mathrm{d}\omega}) = \overline{\mathrm{d}\mathrm{d}\omega} = 0$. Finally, then $\mathrm{d}_r(g \cdot \overline{\omega}) = \overline{\mathrm{d}(f^*(g)\omega)} = \overline{\mathrm{d}f \wedge f^*(g')\omega + f^*(g)\mathrm{d}\omega} = \overline{f^*(g)\mathrm{d}\omega} = g \cdot \overline{\mathrm{d}\omega} = g \cdot \mathrm{d}_r(\overline{\omega})$. □

Lemma 1.23. *The sheaf $\Omega^{n+1}_{X/T}$ is concentrated at* $\operatorname{Sing}(f)$. *Let $p \in \operatorname{Sing}(f)$ and let $J_p(f)$ be the ideal generated by the partial derivatives of f at p, then $\Omega^{n+1}_{X/T,p} \cong \mathcal{O}_{X,p}/J_p(f)$.*

Proof. The first part is easy to see since, locally around $p \notin \operatorname{Sing}(f)$, X_t can be given by $x_0 = 0$ for some local coordinate x_0 and any $\omega_p \in \Omega^{n+1}_{X,p}$ is equal to $\omega_p = g\mathrm{d}x_0 \wedge \cdots \wedge \mathrm{d}x_n = g\mathrm{d}f \wedge \cdots \wedge \mathrm{d}x_n$. Hence, ω_p is zero in $\Omega^{n+1}_{X/T,p}$. For the second part, take $\eta = \sum x_I \mathrm{d}x_I \in \Omega^n_{X,p}$ with $x_0, \ldots, x_n$ local coordinates at $p \in \operatorname{Sing}(f)$. Consequently, we have a first inclusion

$$\mathrm{d}f \wedge \eta = \sum \partial_i f \alpha_I \mathrm{d}x_i \wedge x_I = \Big(\sum (-1)^i \partial_i f \alpha_I\Big) \mathrm{d}x_0 \wedge \cdots \wedge \mathrm{d}x_n \in J_p(f)\mathrm{d}x. \tag{1.49}$$

For the second inclusion, take $\omega \in J_p(f)\mathrm{d}x$, since then

$$\omega = \Big(\sum \alpha_i \partial_i f\Big)\mathrm{d}x = \mathrm{d}f \wedge \Big(\sum (-1)^i \alpha_i\Big)\mathrm{d}x_0 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_n \in \mathrm{d}f \wedge \Omega^n_{X,p}. \tag{1.50}$$

□

The *relative de Rham cohomology* with respect to the morphism $f : X \longrightarrow T$ is

$$\mathcal{H}^p(X/T) := \mathbb{R}^p f_* \Omega^\bullet_{X/T}. \tag{1.51}$$

If the morphism f is smooth it can be shown that $\mathcal{H}^p(X/T)$ is just $R^p f_* \underline{\mathbb{C}}_X \otimes_{\underline{\mathbb{C}}_T} \mathcal{O}_T$, see [Kul98, p. 3.3.1]. This means that $\mathcal{H}^p(X/T)$ is a natural extension to T of the sheaf $\mathcal{H}^p$ in T' described in Section 1.4, that is, $\mathcal{H}^p(X/T)\big|_{T'} = \mathcal{H}^p(X'/T') \cong \mathcal{H}^p$. Furthermore, in our setting, f is a Stein morphism and, consequently, the first spectral sequence for the hypercohomology in Equation (1.51) degenerates, which implies that

$$\mathcal{H}^p(X/T) = \mathcal{H}^p(f_* \Omega^\bullet_{X/T}). \tag{1.52}$$

Theorem 1.24 ([Bri70, Satz 1.5])**.** *If f is an isolated singularity, the sheafs $\mathcal{H}^p(X/T)$ are coherent sheaves on T.*

The isomorphism between $\mathcal{H}^p(X'/T')$ and $\mathcal{H}^p$ induces a connection on $\mathcal{H}^p(X/T)\big|_{T'}$ which is still called the Gauss-Manin connection. It is one of the fundamental results of Brieskorn in [Bri70] that it is possible to give an algebraic description of the Gauss-Manin connection on $\mathcal{H}^p(X/T)\big|_{T'}$ which extends naturally to the whole T.

The way to obtain the *algebraic Gauss-Manin connection* on $\mathcal{H}^p(X/T)$ is as follows. Since we have a de Rham description of the cohomology bundle $\mathcal{H}^p$, cohomology classes can be integrated. Integration of a relative differential form $[\omega] \in \Gamma(X, \Omega^p_{X/T})$ along a locally constant cycle γ of $\mathcal{H}_p$

$$I(\tilde{t}) := \int_{\gamma(\tilde{t})} \omega, \qquad \tilde{t} \in \widetilde{T}', \tag{1.53}$$

gives a non-degenerate pairing $\mathcal{H}^p(X'/T') \times \mathcal{H}_p \longrightarrow \mathcal{O}_{\widetilde{T}'}$. Here $\mathcal{O}_{\widetilde{T}'}$ denotes the sheaf of holomorphic forms on the universal cover $\widetilde{T}'$ of T'. The monodromy action h_* on $\mathcal{H}_p$ implies that $I(t)$ is, in general, a multivalued holomorphic form on T'. This pairing corresponds, over simply connected open sets of T', to the non-degenerate pairing $\mathcal{H}^p \otimes \mathcal{H}_p \longrightarrow \mathcal{O}_{T'}$.

Proposition 1.25 ([Fer+77, §II.2])**.** *The integral $I(\tilde{t})$ on Equation (1.53) is well-defined.*

Proof. Denote $\Gamma_i : \Delta_p \longrightarrow X_t$ the fundamental p-simplexes of X_t, that is Γ_i differentiable and Δ_p the p–th standard simplex. First, two representatives of $[\omega]$ differ by an element of $\mathrm{d}f \wedge \Omega_X^{p-1} + \mathrm{d}\Omega^{p-1}$, that is, of the form $\mathrm{d}f \wedge \omega' + \mathrm{d}\omega''$ with $\omega', \omega'' \in \Gamma(X, \Omega_X^{p-1})$. Then, if $\Gamma = \sum_i \lambda_i \Gamma_i$ is such that $\gamma(t) = [\Gamma(\tilde{t})]$,

$$\begin{aligned} \int_{\Gamma(\tilde{t})} (\mathrm{d}f \wedge \omega') + \mathrm{d}\omega'' &= \sum_i \lambda_i \int_{\Gamma_i(\tilde{t})} \mathrm{d}f \wedge \omega + \int_{\Gamma(\tilde{t})} \mathrm{d}\omega'' \\ &= \sum_i \lambda_i \int_{\Delta_p} \mathrm{d}(f \circ \Gamma_i) \wedge (\Gamma_i)_* \omega' + \int_{\partial\Gamma(\tilde{t})} \omega'' = 0, \end{aligned} \tag{1.54}$$

since $\Gamma(\tilde{t})$ does not have a boundary and $f \circ \Gamma_i$ is constant. Secondly, two representatives of $\gamma(\tilde{t})$ differ by the boundary of a $(p+1)$-simplex Γ' of X_t, that is, $\partial\Gamma'(\tilde{t}) = \partial(\sum_j \lambda'_j \Gamma'_j(\tilde{t}))$. Then, since $\mathrm{d}\omega = \mathrm{d}f \wedge \eta$ with $\eta \in \Gamma(X, \Omega_X^p)$,

$$\int_{\partial\Gamma'(\tilde{t})} \omega = \int_{\Gamma'(\tilde{t})} \mathrm{d}\omega = \sum_j \lambda'_j \int_{\Gamma'_j(\tilde{t})} \mathrm{d}f \wedge \eta = \sum_j \lambda'_j \int_{\Delta_{p+1}} \mathrm{d}(f \circ \Gamma'_j) \wedge (\Gamma'_j)_* \eta = 0, \tag{1.55}$$

since again $f \circ \Gamma'_j$ is constant. □

In the sequel, the integrals $I(t)$ from Equation (1.53) will be interpreted as holomorphic functions $I(t)$ on T' defined on an arbitrary open sector with center the origin $\mathbf{0} \in T$. Let us show next that $I(t)$ is a holomorphic function of t in any such sector. To that end, we need the following result of Leray.

Theorem 1.26 (Leray's Residue Theorem, [Ler59, p. 28])**.** *Let $\omega \in \Gamma(X, \Omega_X^p)$ and $\gamma(t) \in H(X_t, \mathbb{C})$, then*

$$\int_{\gamma(t)} \omega = \frac{1}{2\pi\imath} \int_{\delta\gamma(t)} \frac{\mathrm{d}f \wedge \omega}{f - t}, \tag{1.56}$$

where $\delta : H_p(X_t, \mathbb{C}) \longrightarrow H_{p+1}(X \setminus X_t, \mathbb{C})$ is the Leray coboundary operator.

The formula given in Equation (1.56) is called Leray's Residue Theorem because $\omega\big|_{X_t}$ is the *Poincaré residue* of $(\mathrm{d}f \wedge \omega)/(f - t)$.

Proposition 1.27 ([Bri70])**.** *Let $[\omega] \in \Gamma(X, \mathcal{H}^p(X/T))$ and let $\gamma(t)$ be a locally constant section of $\mathcal{H}^p$. Then*

$$\frac{d}{dt} \int_{\gamma(t)} \omega = \int_{\gamma(t)} \eta, \tag{1.57}$$

where $\mathrm{d}\omega = \mathrm{d}f \wedge \eta$ since $\mathrm{d}_r([\omega]) = 0$. In particular, the function $I(t)$ from Equation (1.53) is a holomorphic function in any open sector centered at the origin.

Proof. Using Theorem 1.26,

$$\begin{aligned} \frac{d}{dt} I(t) = \frac{d}{dt} \int_{\sigma(t)} \omega &= \frac{d}{dt} \left(\frac{1}{2\pi\imath} \int_{\delta\sigma(t)} \frac{\mathrm{d}f \wedge \omega}{f - t} \right) = \frac{1}{2\pi\imath} \int_{\delta\sigma(t)} \frac{\mathrm{d}f \wedge \omega}{(f - t)^2} \\ &= \frac{1}{2\pi\imath} \int_{\delta\sigma(t)} \left[\frac{\mathrm{d}\omega}{f - t} - \mathrm{d}\left(\frac{\omega}{f - t} \right) \right] = \frac{1}{2\pi\imath} \int_{\delta\sigma(t)} \frac{\mathrm{d}\omega}{f - t} \\ &= \frac{1}{2\pi\imath} \int_{\delta\sigma(t)} \frac{\mathrm{d}f \wedge \eta}{f - t} = \int_{\sigma(t)} \eta. \end{aligned} \tag{1.58}$$

□

After Proposition 1.27 and Lemma 1.21 the expression of the covariant derivative of the Gauss-Manin connection on $\mathcal{H}^p(X/T)$ must be given by $\partial_t([\omega]) = [\eta]$. In other terms, $\nabla([\omega]) = dt \otimes [\eta]$ where η is such that $d\omega = df \wedge \eta$. It remains to show that this expression gives a well-defined connection on $\mathcal{H}^p(X/T)$, that is, that η represents a cohomology class on $\mathcal{H}^p(X/T)$. To that end, consider $(\Omega^\bullet, df)$ the *Koszul complex* of f,

$$(\Omega^\bullet, df) : 0 \longrightarrow \mathcal{O}_X \xrightarrow{\wedge df} \Omega^1_X \xrightarrow{\wedge df} \cdots \xrightarrow{\wedge df} \Omega^n_X \xrightarrow{\wedge df} \Omega^{n+1}_X \longrightarrow 0. \tag{1.59}$$

Denote by $\mathcal{K}^p(f) := \operatorname{Ker}(\Omega^p_X \xrightarrow{\wedge df} \Omega^{p+1}_X)/\operatorname{Im}(\Omega^{p-1}_X \xrightarrow{\wedge df} \Omega^p_X)$ the cohomology sheaves of the complex $(\Omega^\bullet_X, df)$. It is clear that at the points $p \notin \operatorname{Sing}(f)$ the complex is exact, that is $df \wedge \omega = 0$ implies that $\omega = df \wedge \eta$. On the other hand, if we assume that $\operatorname{Sing}(f) = \{\mathbf{0}\}$, i.e. f is an isolated singularity, the following lemma of de Rham implies the exactness of the complex for $p < n$.

Lemma 1.28 (Division lemma, [Rha54]). *Let R be a noetherian commutative ring. If $g = (g_1, \ldots, g_m) \in R^m$ is a regular sequence on R, then the Koszul complex of g,*

$$K^\bullet(g) : 0 \to R \to R^m \xrightarrow{\wedge g} \Lambda^2 R^m \xrightarrow{\wedge g} \cdots \xrightarrow{\wedge g} \Lambda^m R^m \to 0, \tag{1.60}$$

has the cohomology groups

$$H^p(K^\bullet(g)) = \begin{cases} 0, & p < m, \\ R/(g_1, \ldots, g_m), & p = m. \end{cases} \tag{1.61}$$

That is, given $\omega \in \Lambda^p R^m$ there exists $\eta \in \Lambda^{p-1} R^m$ such $\omega = g \wedge \eta$ if and only if $g \wedge \omega = 0$.

Indeed, since f has an isolated singularity at the origin, the partial derivatives of f at $\mathbf{0}$ form a regular sequence on the ring $\mathcal{O}_{X,\mathbf{0}}$. This implies that, for $p < n$, the Gauss-Manin connection on $\mathcal{H}^p(X/T)$ is given by

$$\nabla_f : \mathcal{H}^p(X/T) \longrightarrow \Omega^1_T \otimes_{\mathcal{O}_T} \mathcal{H}^p(X/T), \qquad \text{with} \qquad \nabla_f([\omega]) = [\eta], \tag{1.62}$$

where $d\omega = df \wedge \eta$. Since $0 = d(d\omega) = -df \wedge d\eta$ implies that $d\eta = df \wedge \zeta$, $[\eta]$ is a cocycle in $\mathcal{H}^p(X/T)$. This map is independent of the representative of $[\omega]$ in $\mathcal{H}^p(X/T)$, indeed, the difference of any two representatives ω, ω' lies in $df \wedge \Omega^{p-1}_X + d\Omega^{p-1}_X$. Hence, if $d\omega = df \wedge \eta$, then $d\omega' = df \wedge (\eta - d\alpha)$ for some section α of Ω^{p-1}_X, and $\nabla_f([\omega']) = [\eta - d\alpha] = [\eta] = \nabla_f([\omega])$. Two representatives of $[\eta]$ must fulfill $0 = df \wedge (\eta - \eta')$ and Lemma 1.28 implies that $\eta = \eta' + df \wedge \beta$, β a section of Ω^{p-1}_X. Finally, let us check that it satisfies the Leibniz rule,

$$d(f^*(g)\omega) = df \wedge f^*(g')\omega + f^*(g)d\omega = df \wedge (dg/dt \cdot \omega + g \cdot \eta), \tag{1.63}$$

for $g \in \Gamma(T, \mathcal{O}_T)$.

In general, in the case $p = n$ or if f is not an isolated singularity, one obtains what is sometimes called a *connection of a pair*, the definition will be clear after the following definition. We have the following $\mathbb{C}$-linear morphism satisfying the Leibniz rule that we keep denoting by ∇_f and calling it the Gauss-Manin connection,

$$\nabla_f : \mathcal{H}^p(X/T) \longrightarrow \Omega^1_T \otimes_{\mathcal{O}_T} {}'\mathcal{H}^p, \tag{1.64}$$

where

$$'\mathcal{H}^p := \operatorname{Coker} d_r = f_*\Omega^{p+1}_{X/T}/\operatorname{Im}(\Omega^p_{X/T} \xrightarrow{d_r} \Omega^{p+1}_{X/T}). \tag{1.65}$$

For the case of an isolated singularity and $p = n$, Brieskorn [Bri70] considered a slightly bigger and more natural sheaf. The division lemma Lemma 1.28 implies the existence of the following short exact sequence

$$0 \longrightarrow \Omega^n_{X/T} \xrightarrow{\mathrm{d}f\wedge} \Omega^{n+1}_X \longrightarrow \Omega^{n+1}_{X/T} \longrightarrow 0, \tag{1.66}$$

Taking the quotient by the subsheaves $\mathrm{d}_r\Omega^{n-1}_{X/T}$ and applying f_*, one gets the short exact sequence

$$0 \longrightarrow {}'\mathcal{H}^n \xrightarrow{\mathrm{d}f\wedge} {}''\mathcal{H}^n \longrightarrow f_*\Omega^{n+1}_{X/T} \longrightarrow 0, \tag{1.67}$$

where

$$''\mathcal{H}^n := f_*\Omega^{n+1}_X/\mathrm{d}f \wedge \mathrm{d}(f_*\Omega^{n-1}_X). \tag{1.68}$$

After Equation (1.62), there is also a connection $\nabla_f : \mathrm{d}f \wedge {}'\mathcal{H}^n \longrightarrow {}''\mathcal{H}^n$, which is simply $[\omega] \mapsto [\mathrm{d}\omega]$. The identification of the new sheaf in Equation (1.68) with $\mathcal{H}^n$ over T' is realized in the following way. Let $\omega \in \Gamma(X, \Omega^{n+1}_X)$ representing a section of $''\mathcal{H}^n$. Since $\mathrm{d}f \wedge \omega = 0$, we can write $\omega = \mathrm{d}f \wedge \eta$, for some $\eta \in \Gamma(X, \Omega^n_X)$, locally around the points of $X_t, t \neq 0$. The restriction of any such η to X_t does not depend on the chosen η. In this way, we obtain a form on $\Omega^n_{X'/S'}$ which is usually denoted by $\omega/\mathrm{d}f$ and it is called the *Gel'fand-Leray* form of ω.

1.7 *The Brieskorn lattice*

This section is a continuation of the previous Section 1.6, and we will make use of the same notations. The reason why we had to work with connections between pairs when trying to extend the Gauss-Manin connection to T is because the Gauss-Manin connection is meromorphic at the origin $\mathbf{0} \in T$. Notice that when f has an isolated singularity at the origin, we get the usual notion of a connection in the levels $p < n$ where the cohomology groups of the Milnor fiber are trivial. Let us start this section introducing the theory of meromorphic connections.

Denote by $k := \mathcal{O}_{T,0}[t^{-1}]$ the field of germs of meromorphic functions at the origin of T. Let V be a finite-dimensional k-vector space.

Definition 1.29. A *meromorphic connection* ∇ on V is a $\mathbb{C}$-linear map $\nabla : V \longrightarrow \Omega^1_{T,0} \otimes_{\mathcal{O}_{T,0}} V$ satisfying the Leibniz rule, $\nabla(gv) = dg \otimes v + g \otimes \nabla v$.

If ∇ is a connection on a $\mathcal{O}_{T,\mathbf{0}}$-module E, it extends naturally to a connection $\overline{\nabla}$ on the k-vector space $E \otimes_{\mathcal{O}_{T,0}} k$ via

$$\overline{\nabla}(s \otimes t^{-k}) = \nabla(s) \otimes t^{-k} - s \otimes kt^{-k-1}, \qquad s \in E. \tag{1.69}$$

Similarly, if $\nabla : E \longrightarrow \Omega^1_{T,0} \otimes_{\mathcal{O}_{T,0}} F$ is a connection of a pair of free $\mathcal{O}_{T,0}$-modules $E \subset F$ such that F/E is torsion, i.e. $\dim_{\mathbb{C}} F/E < \infty$, then the connection $\overline{\nabla}$ defines a meromorphic connection on $V = E \otimes_{\mathcal{O}_{T,0}} k = F \otimes_{\mathcal{O}_{T,0}} k$. We will always denote by $\partial_t := \nabla_{d/dt}$ the covariant derivative of the connection.

Definition 1.30. A *lattice* on a k-vector space V is a finitely generated $\mathcal{O}_{T,0}$-submodule L of V such that $V = L \otimes_{\mathcal{O}_{T,0}} k$. The *pole order* of a meromorphic connection ∇ on a lattice L is the minimum $p \in \mathbb{Z}$ such that $t^p\partial_t L \subseteq L$.

Notice that because $V = L \otimes_{\mathcal{O}_{T,0}} k$, L is torsion-free. Then, since L is a torsion-free finitely generated module over a principal ideal domain, it is free of rank $\dim_k V$. Furthermore, any pair of lattices L, L' of V are related by $t^p L \subset L'$ for some $p \in \mathbb{Z}$.

A very relevant class of meromorphic connections are those that have *regular singularities*. There are several definitions for the notion of regularity of a singular point of a connection, in terms of the rate of growth of the solutions or using the matrix of the associated meromorphic differential equation, see for instance [CL55, §4]. We will use the following algebraic definition.

Definition 1.31. A lattice L of the k-vector space V is called *saturated*, if it is stable under the operator $t\partial_t$, that is $t\partial_t L \subseteq L$. A meromorphic connection $\nabla : V \longrightarrow \Omega^1_{T,0} \otimes_{\mathcal{O}_{T,0}} V$ is called *regular* if there exists a saturated lattice in V.

If L is a lattice of V and ∇ is a regular meromorphic connection on V, a stable lattice can be constructed from L by a *saturation* process. That is, $\widetilde{L} := \sum_{p=0}^{\infty} (t\partial_t)^p L$ is a saturated lattice for ∇ on V. Since $\mathcal{O}_{T,0}$ is noetherian, only a finite number of terms are necessary to construct the saturation of L.

The above discussion translates, in the following way, to the case of the Gauss-Manin connection of an isolated singularity.

Proposition 1.32 ([Bri70, Prop. 1.6]). *If f has an isolated singularity at the origin, one has $\mathcal{H}^p(X/T)_0 \cong H^p(\Omega^\bullet_{X/T,\mathbf{0}})$ as $\mathcal{O}_{T,0}$-modules.*

Let us call $H^n := H^n(\Omega^\bullet_{X/T,\mathbf{0}})$. Similarly,

$$\begin{aligned} {}'H^n &:= ({}'\mathcal{H}^n)_0 = \Omega^n_{X,\mathbf{0}} / (\mathrm{d}f \wedge \Omega^{n-1}_{X,\mathbf{0}} + \mathrm{d}\Omega^{n-1}_{X,\mathbf{0}}), \\ {}''H^n &:= ({}''\mathcal{H}^n)_0 ={}'' H^n = \Omega^{n+1}_{X,\mathbf{0}} / (\mathrm{d}f \wedge \mathrm{d}\Omega^{n-1}_{X,\mathbf{0}}). \end{aligned} \tag{1.70}$$

The $\mathcal{O}_{T,0}$-module ${}''H^n$ is usually called the *Brieskorn* lattice, the reason for this name will be justified in a moment. Notice that,

$${}'H^n / H^n = \Omega^{n+1}_X / \mathrm{d}f \wedge \mathrm{d}\Omega^{n-1}_X = {}''H^n / \mathrm{d}f \wedge {}'H^n, \tag{1.71}$$

are finite-dimensional $\mathbb{C}$-vector spaces of dimension μ.

Proposition 1.33 ([Seb70], [Bri70]). *The $\mathcal{O}_{T,0}$-modules $H^n, {}'H^n$ and ${}''H^n$ are free of rank μ, where μ is the Milnor number of f.*

Freeness is due to Sebastiani [Seb70], see also [Mal74a, Thm. 5.1]. The result for H^n follows from the isomorphism in Proposition 1.32, the coherence result in Theorem 1.24, the isomorphism $\mathcal{H}^n(X'/T') \cong \mathcal{H}^n$ and the result of Milnor given in Section 1.4. For the other two modules, it follows similarly from this discussion and Equation (1.71).

All three $\mathcal{O}_{T,0}$-modules from Proposition 1.33 are then lattices on the k-vector space $V = H^n \otimes_{\mathcal{O}_{T,0}} k = {}'H^n \otimes_{\mathcal{O}_{T,0}} k = {}''H^n \otimes_{\mathcal{O}_{T,0}} k$, where the Gauss-Manin connection $\nabla_{f,0}$ between the stalks of both pairs defines a meromorphic connection $\overline{\nabla}_{f,0}$ on V. This connection is usually called the *meromorphic Gauss-Manin connection*. Moreover, since these are lattices on V, we have the reverse inclusions

$$t^{\kappa_f}\, {}''H^n \subset \mathrm{d}f \wedge {}'H^n \qquad \text{and} \qquad t^{\kappa_f}\, {}'H^n \subset H^n, \tag{1.72}$$

where $\kappa_f \in \mathbb{Z}$ is the minimum integer such that $f^{\kappa_f} \in J_0(f)$. Indeed, for such an integer, we have $f^{\kappa_f} \mathrm{d}\underline{x} = \mathrm{d}f \wedge \xi$ for some $\xi \in \Omega^n_{X,\mathbf{0}}$. Then, for any $\omega \in \Omega^n_{X,\mathbf{0}}$,

$$\mathrm{d}(f^{\kappa_f}\omega) = \kappa_f f^{\kappa_f - 1}\mathrm{d}f \wedge \omega + f^{\kappa_f}\mathrm{d}\omega = \mathrm{d}f \wedge \left(\kappa_f f^{\kappa_f-1}\omega + \alpha\xi\right), \tag{1.73}$$

where $\mathrm{d}\omega = \alpha \mathrm{d}\underline{x}, \alpha \in \mathcal{O}_{X,\mathbf{0}}$. Similarly, for $g \in \mathcal{O}_{X,\mathbf{0}}$, $f^{\kappa_f} g \mathrm{d}\underline{x} = \mathrm{d}f \wedge g\xi$. This gives the following well-defined map,

$$t^{\kappa_f}\nabla_{f,0} : {}''H^n \longrightarrow \Omega^1_{T,0} \otimes_{\mathcal{O}_{T,0}} {}''H^n, \tag{1.74}$$

given by

$$t^{\kappa_f}\nabla_{f,0}([g\mathrm{d}\underline{x}]) = \mathrm{d}t \otimes \left[\mathrm{d}(g\xi) - \kappa_f f^{\kappa_f-1} g \mathrm{d}\underline{x}\right]. \tag{1.75}$$

This map shows that the Gauss-Manin connection has a pole of order κ_f in the Brieskorn lattice ${}''H^n$. The above expression for $t^{\kappa_f}\nabla_{f,0}$ enables an effective way of computing the connection matrix of $\nabla_{f,0}$ up to arbitrary high order. In this context, one of the most important results of Brieskorn in [Bri70] is that,

Theorem 1.34 ([Bri70, Satz 2])**.** *The Gauss-Manin connection $\nabla_{f,0}$ of an isolated singularity is a meromorphic connection with regular singularities.*

After Theorem 1.34, one can consider the saturation of the Brieskorn lattice ${}''H^n$ with respect to the Gauss-Manin connection $\nabla_{f,0}$. Namely,

$${}''\widetilde{H}^n = \sum_{p=0}^{\infty} (t\partial_t)^p({}''H^n), \tag{1.76}$$

where ∂_t is the covariant derivative of the Gauss-Manin connection $\nabla_{f,0}$. Once this object has been introduced, we can state the following classical result of Malgrange.

Theorem 1.35 ([Mal75])**.** *If $f : (\mathbb{C}^2, \mathbf{0}) \longrightarrow (\mathbb{C}, 0)$ is an isolated singularity, the reduced Bernstein-Sato polynomial $\tilde{b}_{f,\mathbf{0}}(s)$ of f is equal to the minimal polynomial of the endomorphism*

$$-\overline{\partial_t t} : {}''\widetilde{H}^n / t\,{}''\widetilde{H}^n \longrightarrow {}''\widetilde{H}^n / t\,{}''\widetilde{H}^n, \tag{1.77}$$

of $\mathbb{C}$-vector spaces of dimension μ, the Milnor number of f.

The saturated lattice is stable under $t\partial_t$, however, recall that $\partial_t t - t\partial_t = 1$. The following corollaries are also classical results of Malgrange.

Corollary 1.36 ([Mal75])**.** *If α is a root of $\tilde{b}_{f,\mathbf{0}}(s)$, then $\exp(-2\pi\imath\alpha)$ is an eigenvalue of the monodromy of f at the origin.*

The analogous result for non-isolated singularities is obtained by Malgrange in [Mal83]. The next result was already mentioned in Section 1.2 and is a consequence of the previous corollary and Theorem 1.16.

Corollary 1.37 ([Mal75])**.** *The roots of $b_{f,\mathbf{0}}(s)$ are negative rational numbers.*

As mentioned earlier in Section 1.2, this result also holds for arbitrary singularities, see [Kas76]. We end this section with a definition.

Definition 1.38 (b-exponents)**.** The roots of the characteristic polynomial of the endomorphism $\overline{\partial_t t}$ from (1.77) in Theorem 1.35 are called the *b-exponents* of an isolated singularity.

2 PLANE CURVE SINGULARITIES

The results in this section are framed in the context of the local geometry of smooth surfaces. The first part of the section introduces the theory of infinitely near points, bases for divisors with exceptional support, antinef divisors, the unloading procedure to compute antinef closures, and maximal contact elements. The second part deals with invariants for plane curves singularities such as the semigroup, the minimal embedded resolution or the monomial curve and its deformations. At the end of this section, we present Yano's conjecture and review the state of the art for multiplier ideals in complex smooth surfaces.

2.1 *Proper birational morphisms*

The goal of this section is to review the specifics of log-resolutions of ideals and, in general, the properties of proper birational morphisms on smooth complex surfaces. In this section, we will work locally on a smooth complex surface $(X, \mathbf{0})$, which we can, and we will, think as $(\mathbb{C}^2, \mathbf{0})$ whenever it is convenient.

Let X be a *smooth complex surface* and $\mathcal{O}_{X,\mathbf{0}}$ be the ring of germs of holomorphic functions in a neighborhood of a smooth point $\mathbf{0} \in X$, which after taking local coordinates, we can always identify with $\mathbb{C}\{x, y\}$, the ring of convergent power series in two variables over the complex numbers. In the sequel, we will always denote $\mathfrak{m} := \mathfrak{m}_{X,\mathbf{0}} \subset \mathcal{O}_{X,\mathbf{0}}$ the maximal ideal at $\mathbf{0}$. Given a proper ideal $\mathfrak{a} \subset \mathcal{O}_{X,\mathbf{0}}$, we always have a decomposition $\mathfrak{a} = (f) \cdot \mathfrak{a}'$, where $f \in \mathcal{O}_{X,\mathbf{0}}$ is the greatest common divisor of the elements of $\mathfrak{a}$ and $\mathfrak{a}'$ is $\mathfrak{m}$-primary.

Let $\pi : (X', E) \longrightarrow (X, \mathbf{0})$ be a *proper birational morphism*, with X' a smooth complex surface and $E = \mathrm{Exc}(\pi)$ the exceptional locus. Any such morphism can be achieved by a sequence of point blow-ups

$$\pi : (X', E) := (X_{r+1}, E_{r+1}) \longrightarrow (X_r, E_r) \longrightarrow \cdots \longrightarrow (X_0, \mathbf{0}) := (X, \mathbf{0}), \qquad (2.1)$$

with $X_{i+1} := \mathrm{Bl}_{p_i} X_i$ the surface obtained by blowing-up a point $p_i \in X_i$. Since we are in a local framework the points p_i are always assumed to be on the exceptional divisor of the surface X_i. The set K of points that have been blown-up gives an index for the exceptional components $\{E_p\}_{p \in K}$ of E. These points are sometimes called *infinitely near points* to the origin $\mathbf{0} \in X$ supporting the divisor E and we can establish a *proximity relation* between them.

Namely, we say that a point $q \in K$ is *proximate* to $p \in K$ if and only if q belongs to the exceptional divisor E_p corresponding to p as proper or infinitely near point. We will denote this relation as $q \to p$ and we can collect all these relations by means of the *proximity matrix* $P = (P_{p,q})$ defined as:

$$P_{p,q} := \begin{cases} \;\;1 & \text{if } \; p = q, \\ -1 & \text{if } \; q \to p, \\ \;\;0 & \text{otherwise.} \end{cases} \qquad (2.2)$$

Notice that an infinitely near point q is proximate to just one or two points. In the former case, we say that q is a *free point*, and in the later, it is a *satellite point*. Besides, one can establish a *partial ordering* in K. Namely, $q \leqslant p$ if and only if p is infinitely near to q. Another important matrix associated with a proper birational morphism is the *intersection matrix*, defined as $N = (E_p \cdot E_q)$. The proximity matrix is related to the intersection matrix by the formula $N = -P^t \cdot P$, see [Cas00, §4.5].

Definition 2.1. An irreducible exceptional divisor E_i of a proper birational morphism π will be called a *rupture divisor* if $\chi(E_i^\circ) < 0$, where χ denotes the Euler-Poincaré characteristic and $E_i^\circ := E_i \setminus \cup_{j \neq i} E_j$. On the other hand, the divisor E_i will be called a *dead-end divisor* if $\chi(E_i^\circ) = 1$.

Remark 2.1. When working with the morphism π of the minimal embedded resolution of a plane curve f, the definition of E_i° will also take into account the strict transform of f. That is, in this case, $E_i^\circ := E_i \setminus \cup_{D_j \neq E_i} D_j$ with $D_j \in \mathrm{Supp}(F_\pi)$. This means that the set of rupture divisors of the minimal resolution of f can be bigger than the set of rupture divisors of the morphism π. ■

Since $E_i \cong \mathbb{P}^1_{\mathbb{C}}$, we will have that $\chi(E_i^\circ) = 2 - r_i$, where r_i is the number of irreducible components of F_π intersecting E_i. The canonical examples of proper birational morphisms are the log-resolutions of ideals $\mathfrak{a} \subset \mathcal{O}_{X,\mathbf{0}}$. In this local context, we recover the definition of log-resolution from Section 1.1.

Definition 2.2. A *log-resolution* of a proper ideal $\mathfrak{a} \subset \mathcal{O}_{X,\mathbf{0}}$ is a proper birational morphism $\pi : (X', E) \longrightarrow (X, \mathbf{0})$ with X' smooth and such that there exists a Cartier divisor F_π satisfying $\mathfrak{a} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F_\pi)$ and such that F_π has simple normal crossings support.

We will use here the same notation as in Section 1.1, for the irreducible components and the multiplicities of the *total transform* divisor F_π and the relative canonical divisor K_π. Namely,

$$F_\pi = \sum_{i=1}^{r} N_i E_i + \sum_{j=1}^{s} M_j S_j, \qquad K_\pi = \sum_{i=1}^{r} k_i E_i, \tag{2.3}$$

for an arbitrary total ordering of the exceptional components $E_i := E_{p_i}, p_i \in K$ which we always assume compatible with the partial ordering of the infinitely near points. Given a divisor D on X' we will denote by D_{exc} its exceptional part.

The affine components S_j are always zero if $\mathfrak{a}$ is $\mathfrak{m}$-primary and the multiplicities M_j are one whenever f is reduced. In the two-dimensional case, a log-resolution coincides with an *embedded* resolution of the ideal, this is no longer the case in higher dimensions. In addition, in the two-dimensional case, there is a distinguished log-resolution, the *minimal log-resolution*, in which all the unnecessary (-1)–curves are contracted.

In Section 3, we will present an explicit way of constructing the minimal log-resolution of a two-dimensional ideal in a smooth complex surface from the embedded resolutions of a given set of generators of the ideal.

As a final remark, there are many, sometimes equivalent, ways of encoding the combinatorial information of a proper birational morphism, or equivalently of a log-resolution, in a weighted graph, for instance: the Enriques diagram [EC15, §IV.I] [Cas00, §3.9], the dual graph [Cas00, §4.4] [Wal04, §3.6], the Eisenbud-Neumann diagrams [EN85], or the Eggers-Wall tree [Wal03], to name a few.

2.2 *Divisor basis*

Let $\Lambda_\pi := \bigoplus_{p\in K} \mathbb{Z}E_p$ be the lattice of integral divisors in X' with exceptional support. We have two different bases of this $\mathbb{Z}$-module given by the *strict transforms* and the *total transforms* of the exceptional components. For simplicity, we will also denote the strict transforms by E_p and the total transforms by $\overline{E}_p$. In particular, any divisor $D \in \Lambda_\pi$ can be presented in two different ways

$$D = \sum_{p\in K} v_p(D)E_p = \sum_{p\in K} e_p(D)\overline{E}_p, \tag{2.4}$$

where the integers $v_p(D)$ (resp. $e_p(D)$) are the *values* (resp. *multiplicities*) of D. In Section 2.1 we used the notation, $N_i = v_{p_i}(F_\pi)$ with $p_i \in K$ for the total transform divisor F_π of a log-resolution. For the particular case where $D = \mathrm{Div}(\pi^* g)_{\mathrm{exc}}, g \in \mathcal{O}_{X,\mathbf{0}}$, and π dominates the log-resolution of the ideal (g) one usually denotes $v_p(g) := v_p(D)$ and $e_p(g) := e_p(D)$ for all $p \in K$. In this situation, the integers $e_p(f), p \in K$ are the multiplicities of the strict transform at the exceptional divisors $E_p, p \in K$.

The total transform basis turns out to be very convenient to describe the relative canonical divisor K_π of any proper birational morphism. Indeed, after [Har77, Prop. 3.3], one has that the relative canonical divisor is just

$$K_\pi = \sum_{p\in K} \overline{E}_p. \tag{2.5}$$

Fixed an exceptional irreducible component $E_p, p_i \in K$, the function $v_p : \mathcal{O}_{X,\mathbf{0}} \longrightarrow \mathbb{Z}_{\geq 0} \cup \{\infty\}$ that assigns $g \mapsto v_p(g)$ is a *rank one discrete valuation*. That is, for all $g_1, g_2 \in \mathcal{O}_{\mathbb{C},\mathbf{0}}$ it satisfies the properties: $v_p(g_1g_2) = v_p(g_1) + v_p(g_2), v_p(g_1 + g_2) \geq \min\{v_p(g_1), v_p(g_2)\}$ and $v_p(g_1) = \infty$ if and only if $g_1 = 0$. We will call it the valuation associated with the exceptional divisor E_p. For consistency, if the exceptional divisor is denoted as $E_i := E_{p_i}, p_i \in K$, the valuation will be denoted v_i.

The relation between values and multiplicities is given by the proximity relations

$$v_q(D) = e_q(D) + \sum_{p\to q} v_p(D). \tag{2.6}$$

These relations provide a base change formula $\boldsymbol{e}^t = P \cdot \boldsymbol{v}^t$, where we collected the multiplicities and values in the vectors $\boldsymbol{e} = (e_p(D))_{p\in K}$ and $\boldsymbol{v} = (v_p(D))_{p\in K}$, respectively.

Aside from the total and strict transform basis $\{\overline{E}_p\}_{p\in K}$ and $\{E_p\}_{p\in K}$ of the lattice Λ_π of exceptional divisors, we may also consider the *branch basis* $\{B_p\}_{p\in K}$ defined as the dual of $\{-E_p\}_{p\in K}$ with respect to the intersection form. Any divisor $D \in \Lambda_\pi$ has a presentation

$$D = \sum_{p\in K} \rho_p(D)B_p, \tag{2.7}$$

where $\rho_p(D) = -D \cdot E_p$ is the *excess* at p and the relation between excesses and multiplicities is given by $\boldsymbol{\rho}^t = P^t\boldsymbol{e}^t$, where $\boldsymbol{\rho} = (\rho_p(D))_{p\in K}$ denotes the vector of excesses. The branch basis will be useful in Section 2.3 to announce Zariski's Unique Factorization Theorem for complete ideals.

The *support* $\mathrm{Supp}(D)$ of a divisor $D \in \Lambda_\pi$ is the union of irreducible divisors of D, i.e. if $D = \sum_p v_p(D)E_p$ is the expression of D in the strict transform basis then $\mathrm{Supp}(D) = \{E_p \mid v_p(D) \neq 0\}$. We will also consider the same construction in the total transform basis. Namely, let $D = \sum_p e_p(D)\overline{E}_p$ be the expression of D in the total transform basis, then we define its *support in the total transform basis* as $\mathrm{Supp}_{\overline{E}}(D) = \{\overline{E}_p \mid e_p(D) \neq 0\}$. To avoid confusion with the usual notion of support, we will always be explicit when referring to the support in the total transform basis.

Definition 2.3. Given a point $p \in K$, define $f_p \in \mathcal{O}_{X,\mathbf{0}}$ as *any* irreducible element such that its strict transform by the proper birational morphism π intersects transversally E_p at a smooth point of E. Then, we have that $B_p = \mathrm{Div}(\pi^* f_p)_{\mathrm{exc}}$. Conversely, any $g \in \mathcal{O}_{X,\mathbf{0}}$ with $B_p = \mathrm{Div}(\pi^* g)_{\mathrm{exc}}$ is irreducible and its strict transform intersects transversally E_p at a smooth point of E.

Notice that since the elements $f_p \in \mathcal{O}_{X,\mathbf{0}}$ are irreducible, the points $q \in K$ such that $e_q(B_p) \neq 0$ are totally ordered. Furthermore, the resolution of any $f_p \in \mathcal{O}_{X,\mathbf{0}}, p \in K$ is dominated by π. Moreover, any f whose resolution is dominated by π can be written as a product $f = \prod_p f_p^{\rho_p}$ of suitable elements f_p from Definition 2.3.

We will end this section with a brief discussion on *intersection multiplicity*. Let $C_1 : g_1 = 0$ and $C_2 : g_2 = 0$ be two germs of curves with $g_1, g_2 \in \mathcal{O}_{X,\mathbf{0}}$, then $[C_1, C_2]$ will denote the multiplicity intersection of C_1 and C_2 at the origin $\mathbf{0}$. Recall, this can be defined as

$$[C_1, C_2] = \dim_{\mathbb{C}} \mathcal{O}_{X,\mathbf{0}}/(g_1, g_2), \tag{2.8}$$

and does not depend on the representatives g_1, g_2 of C_1, C_2, see [Cas00, §2.6]. Similarly, for $D_1, D_2 \in \mathrm{Div}(X')$, the usual intersection multiplicity of the divisors D_1 and D_1, see [Har77, §V.1], will be denoted by $D_1 \cdot D_2$. Noether's intersection formula [Cas00, Thm. 3.3.1] gives a relation between these two intersection multiplicities and the total transform basis, namely if $D_i := \mathrm{Div}(\pi^* g_i)_{\mathrm{exc}}$ for $i = 1, 2$, then

$$[C_1, C_2] = \sum_{p \in K} e_p(D_1) e_p(D_2) = -\Big(\sum_{p \in K} e_p(D_1)\overline{E}_p\Big) \cdot \Big(\sum_{p \in K} e_p(D_2)\overline{E}_p\Big) = -D_1 \cdot D_2. \tag{2.9}$$

We will make use of the adjunction formula for surfaces in the forthcoming chapters.

Proposition 2.4 (Adjunction formula, [Har77, Prop. §V.1.5])**.** *If C is a non-singular curve of genus g on a smooth surface X, and if K_X is a canonical divisor on X, then,*

$$2g - 2 = C \cdot (C + K_X). \tag{2.10}$$

Finally, this small lemma will be useful in the sequel.

Lemma 2.5. *For any $p \in K$ and $f \in \mathcal{O}_{X,\mathbf{0}}$, one has $v_p(f) = D \cdot B_p$ with $D = \mathrm{Div}(\pi^* f)_{\mathrm{exc}}$.*

Proof. By construction B_p is the dual to $-E_p$ with respect to the intersection form, hence $D \cdot B_p$ equals the coefficient of the irreducible exceptional divisor E_p in the strict transform basis, that is $v_p(D)$ which we denote by $v_p(f)$. □

2.3 *Complete ideals an antinef divisors*

Given an effective divisor $D = \sum_{p \in K} v_p E_p \in \Lambda_\pi$, we may consider its associated ideal sheaf $\pi_* \mathcal{O}_{X'}(-D)$. Its stalk at $\mathbf{0}$ is

$$H_D = \{f \in \mathcal{O}_{X,\mathbf{0}} \mid v_p(f) \geq v_p(D) \text{ for all } E_p \leq D\}. \tag{2.11}$$

The ideal H_D is *complete*, see [Zar38], and $\mathfrak{m}$-primary since D has only exceptional support. Complete ideals are closed under all standard operations on ideals, except addition: the intersection, product, and quotient of complete ideals are complete.

Recall that an effective divisor $D \in \mathrm{Div}(X')$ is called *antinef* if $\rho_p = -D \cdot E_p \geq 0$, for every exceptional component E_p, $p \in K$. This notion is equivalent, in the total transform basis, to

$$e_p(D) \geq \sum_{q \to p} e_q(D), \quad \text{for all } p \in K. \tag{2.12}$$

These are usually called *proximity inequalities*, see [Cas00, §4.2]. By means of the relation given in Equation (2.11), Zariski [Zar38] establishes an isomorphism of semigroups between the set of ideals H_D and the set of antinef divisors in Λ_π.

Given a non-antinef divisor D, one can compute an equivalent antinef divisor $\widetilde{D}$, called the *antinef closure*, under the equivalence relation that both divisors define the same ideal, i.e. $\pi_* \mathcal{O}_{X'}(-D) = \pi_* \mathcal{O}_{X'}(-\widetilde{D})$, and via the so-called *unloading procedure*. This is an inductive procedure that was already described in the work of Enriques [EC15, §IV.II.17], see also [Cas00, §4.6] for more details. The version that we present here is the one considered in [AÀD16], where it is proved that Algorithm 2.6 finished in a finite number of steps.

Algorithm 2.6 (Unloading procedure)**.**
Input: *A divisor* $D = \sum d_p E_p \in \Lambda_\pi$.
Output: *The antinef closure* $\widetilde{D}$ *of* D.

Repeat:

- *Define* $\Theta := \{E_p \in \Lambda_\pi \mid \rho_p = -D \cdot E_p < 0\}$.
- *Let* $n_p = \lceil \rho_p / E_p^2 \rceil$ *for each* $E_p \in \Theta$. *(Notice that* $(D + n_p E_p) \cdot E_p \leq 0$*).*
- *Define a new divisor as* $\widetilde{D} := D + \sum_{E_p \in \Theta} n_p E_p$.

Until *the resulting divisor* $\widetilde{D}$ *is antinef.*

A fundamental result of Zariski [Zar38] establishes the unique factorization of complete ideals into simple complete ideals, an ideal being simple if it is not the product of ideals different from the unit ideal. A reinterpretation of this result in a more geometrical context is given by Casas-Alvero in [Cas00, §8.4].

Theorem 2.7 ([Zar38], [Cas00, §8.4])**.** *Let* $D \in \Lambda_\pi$ *be an antinef divisor expressed as* $D = \sum_{p \in K} \rho_p B_p$ *in the branch basis. Then,*

$$H_D = \prod_{p \in K} H_{B_p}^{\rho_p}, \tag{2.13}$$

with H_{B_p} *being simple complete ideals for any* $p \in K$.

In the sequel, we will call *simple divisor* the unique antinef divisor defining a simple complete ideal. As a corollary of Theorem 2.7, simple divisors in Λ_π will always be equal to B_p for some $p \in K$.

2.4 *The semigroup of a plane branch*

Let $f : (\mathbb{C}^2, \mathbf{0}) \longrightarrow (\mathbb{C}, 0)$ be an irreducible plane curve. Irreducible plane curves are usually called *plane branches*. To any plane branch, we associate a sequence of positive integers, the *characteristic sequence* $(n, \beta_1, \ldots, \beta_g), n, \beta_i \in \mathbb{Z}_{>0}$ of f, where n is the algebraic multiplicity of f. After an analytic change of variables, we can assume, and we will, that $n < \beta_1 < \cdots < \beta_g$. The characteristic sequence can be obtained from the Puiseux parameterization of f, see [Cas00, §5.1]. See also Section 2.5.

The characteristic sequence of a plane branch is a complete equisingular invariant of the singularity, see [Cas00, Cor. 5.5.4]. That is, any two plane branches with the same characteristic sequence have, combinatorially, the same embedded resolution. The reader is referred to [Cas00, §3.8] for the exact definition. Notice that for f non-irreducible, to describe the equisingularity class one needs both the characteristic sequence and the intersection numbers of the different branches of f [Cas00, Thm. 3.8.6]. Since equisingular plane curves are topologically equivalent, [Bra28] and [Zar32], the characteristic exponents are also a complete topological invariant of plane branches.

Define the integers $e_i := \gcd(e_{i-1}, \beta_i), e_0 := n$, notice that they satisfy $e_0 > e_1 > \cdots > e_g = 1$ and $e_{i-1} \nmid \beta_i$. Set $n_i := e_{i-1}/e_i$ for $i = 1, \ldots, g$ and, by convention, $\beta_0 := 0$ and $n_0 := 0$. The integers $n_1, \ldots, n_g$ are strictly larger than 1 and we have that $e_{i-1} = n_i n_{i+1} \cdots n_g$ for $i = 1, \ldots, g$. In particular, $n = n_1 \cdots n_g$. The fractions $m_i / n_1 \cdots n_i$, with m_i defined as $m_i := \beta_i / e_i$, are the reduced *characteristic exponents* appearing in the Puiseux series of f. The tuples (m_i, n_i) satisfy $\gcd(m_i, n_i) = 1$ and are usually called the *Puiseux pairs*.

Let $\mathcal{O}_f := \mathcal{O}_{\mathbb{C}^2,\mathbf{0}}/(f)$ be the local ring of f. The Puiseux parameterization of f gives an injection $\mathcal{O}_f \hookrightarrow \mathbb{C}\{t\}$. If we denote the t-adic valuation of $\mathcal{O}_f$ by ν_f, then $\Gamma \subseteq \mathbb{Z}_{\geq 0}$ denotes the associated *semigroup* to f, that is

$$\Gamma := \{\nu_f(g) \in \mathbb{Z}_{\geq 0} \mid g \in \mathcal{O}_f \setminus \{0\}\}. \tag{2.14}$$

Since f is irreducible there exists a minimum integer $c \in \mathbb{Z}_{>0}$, the *conductor* of Γ, such that $(t^c) \cdot \mathbb{C}\{t\} \subseteq \mathcal{O}_f$. As a result, any integer in $[c, \infty)$ must belong to Γ, which implies that $\mathbb{Z}_{\geq 0} \setminus \Gamma$ is finite. Since $\mathbb{Z}_{\geq 0} \setminus \Gamma$ is finite, we can find a minimal generating set $\langle \overline{\beta}_0, \overline{\beta}_1, \ldots, \overline{\beta}_g \rangle$ of Γ, i.e. $\overline{\beta}_i$ are the minimal integers such that $\overline{\beta}_i \notin \langle \overline{\beta}_0, \overline{\beta}_1, \ldots, \overline{\beta}_{i-1} \rangle$, with $\overline{\beta}_0 < \overline{\beta}_1 < \cdots < \overline{\beta}_g$ and $\gcd(\overline{\beta}_0, \overline{\beta}_1, \ldots, \overline{\beta}_g) = 1$.

It is well known that the valuation $\nu_f : \mathcal{O}_f \longrightarrow \mathbb{Z}_{\geq 0}$ given by f coincides with the intersection multiplicity with f, see [Cas00, §2.6]. Precisely, this means that if $C : f = 0$, then for any $g \in \mathcal{O}_{X,\mathbf{0}}$ one has that $\nu_f(\bar{g}) = [C, C']$ with $C' : g = 0$. In the sequel, when working with the valuation ν_f no distinction will be made between g and its class $\bar{g}$ inside $\mathcal{O}_f$.

The semigroup generators can be computed from the characteristic sequence in the following way, see [Zar86, §II.3],

$$\overline{\beta}_i = (n_1 - 1) n_2 \cdots n_{i-1} \beta_1 + (n_2 - 1) n_3 \cdots n_{i-1} \beta_2 + \cdots + (n_{i-1} - 1) \beta_{i-1} + \beta_i, \tag{2.15}$$

for $i = 2, \ldots, g$ and with $\overline{\beta}_0 = n, \overline{\beta}_1 = \beta_1$. Recursively this can be expressed as

$$\overline{\beta}_i = n_{i-1} \overline{\beta}_{i-1} - \beta_{i-1} + \beta_i, \qquad i = 2, \ldots, g. \tag{2.16}$$

In a similar way, one can express the characteristic sequence in terms of the generators of the semigroup. Consequently, both the characteristic sequence and the semigroup are complete topological invariants of the singularity defined by f.

By Equation (2.15), $\gcd(e_{i-1}, \overline{\beta}_i) = e_i$ with $e_0 = \overline{\beta}_0 = n$ and $e_{i-1} \nmid \overline{\beta}_i$. In the same way as before, we define the sequence of integers $\overline{m}_i := \overline{\beta}_i / e_i$ which will be useful in the sequel to describe some invariants of the singularity. The integers $\overline{m}_i$ for $i = 1, \ldots, g$ can be obtained recursively using Equation (2.16), namely

$$\overline{m}_i = n_i n_{i-1} \overline{m}_{i-1} - n_i m_{i-1} + m_i, \quad i = 2, \ldots, g, \tag{2.17}$$

with $\overline{m}_0 = 1, \overline{m}_1 = m_1$. Note that Equation (2.17) implies $\gcd(\overline{m}_i, n_i) = 1$ for $i = 1, \ldots, g$. Finally, we define $q_i := (\beta_i - \beta_{i-1})/e_i = m_i - n_i m_{i-1}$ for $i = 1, \ldots, g$. Alternatively, by Equation (2.16), these quantities are $q_i = (\overline{\beta}_i - n_{i-1}\overline{\beta}_{i-1})/e_i = \overline{m}_i - n_i n_{i-1} \overline{m}_{i-1}$ for $i = 2, \ldots, g$ and $q_1 = \overline{m}_1 = m_1$.

The conductor c of Γ can be computed from the data introduced so far in the following way, $c = n_g \overline{\beta}_g - \beta_g - (n-1)$, see [Zar86, §II.3]. Combining this formula with Equation (2.16), one gets the following formula for the conductor in terms of the semigroup data

$$c = \sum_{i=1}^{g} (n_i - 1)\overline{\beta}_i - \overline{\beta}_0 + 1. \tag{2.18}$$

As a consequence of these formulas, one gets an expression for the Milnor number μ of the singularity. Recall that, in local coordinates, $\mu = \dim_{\mathbb{C}} \mathbb{C}\{x, y\} / \langle \partial f / \partial x, \partial f / \partial y \rangle$, for any representative f of a germ of a reduced plane curve, not necessarily irreducible. Then, for plane branches, the Milnor number μ of f coincides with the conductor c, see [Cas00, Prop. 5.8.7, Cor. 6.4.3]. Therefore, μ can be computed from the semigroup $\Gamma = \langle \overline{\beta}_0, \overline{\beta}_1, \ldots, \overline{\beta}_g \rangle$ using Equation (2.18).

We end this section with some properties of semigroups coming from plane branches. The following lemma is a fundamental property of these kinds of semigroups.

Lemma 2.8 ([Tei86, Lemma 2.2.1]). *If $\Gamma = \langle \overline{\beta}_0, \overline{\beta}_1, \ldots, \overline{\beta}_g \rangle$ is the semigroup of a plane branch, then one has*

$$n_i \overline{\beta}_i \in \langle \overline{\beta}_0, \overline{\beta}_1, \ldots, \overline{\beta}_{i-1} \rangle. \tag{2.19}$$

The property in Equation (2.19) from Lemma 2.8 together with the inequality $n_i \overline{\beta}_i < \overline{\beta}_{i+1}$, which follows directly from $\beta_i < \beta_{i+1}$ and Equation (2.16), characterize the semigroups coming from plane branches.

Proposition 2.9 ([Tei86, Prop. 3.2.1]). *Let $\Gamma = \langle \overline{\beta}_0, \overline{\beta}_1, \ldots, \overline{\beta}_g \rangle \subseteq \mathbb{Z}_{\geq 0}$ be a semigroup such that $\mathbb{Z}_{\geq 0} \setminus \Gamma$ is finite. Then Γ is the semigroup of a plane branch if and only if $n_i \overline{\beta}_i \in \langle \overline{\beta}_0, \overline{\beta}_1, \ldots, \overline{\beta}_{i-1} \rangle$ for $i = 1, \ldots, g$, and $n_i \overline{\beta}_i < \overline{\beta}_{i+1}$, for $i = 1, \ldots, g-1$.*

The reverse implication of Proposition 2.9 will be proved at the end of the next section.

2.5 *Maximal contact elements*

Let $f : (\mathbb{C}^2, \mathbf{0}) \longrightarrow (\mathbb{C}, 0)$ be a germ of an irreducible plane curve singularity together with a proper birational morphism π that dominates the log-resolution of (f). Then, f having characteristic sequence $(n; \beta_1, \ldots, \beta_g)$ is equivalent to the fact that a Puiseux series [Cas00, §1.2] of f has the form

$$s(x) = \sum_{\substack{j \in (e_0) \\ \beta_0 \leq j < \beta_1}} a_j x^{j/n} + \sum_{\substack{j \in (e_1) \\ \beta_1 \leq j < \beta_2}} a_j x^{j/n} + \cdots + \sum_{\substack{j \in (e_{g-1}) \\ \beta_{g-1} \leq j < \beta_g}} a_j x^{j/n} + \sum_{\substack{j \in (e_g) \\ j \geq \beta_g}} a_j x^{j/n}, \tag{2.20}$$

with $e_i, i = 0, \dots, g$ being the integers defined in Section 2.4.

Let $\Gamma = \langle \overline{\beta}_0, \cdots, \overline{\beta}_g \rangle$ be the semigroup of f and denote $C : f = 0$ the curve defined by f. We are interested in elements $f_i \in \mathcal{O}_{X,\mathbf{0}}$ such that $\nu_f(f_i) = [C, C_i] = \overline{\beta}_i$, with $C_i : f_i = 0, f_i \in \mathcal{O}_{X,\mathbf{0}}$, for every $i = 0, \dots, g$. These curves will be called *maximal contact elements* of f. Of course, the elements f_i are not unique; for instance, if f is not tangent to the y-axis, then $f_0 = x$ and $f_1 = y + a_1 x + a_2 x^2 + \cdots$, for some $a_1, a_2, \cdots \in \mathbb{C}$.

In general, these elements can be explicitly computed. If f is in Weierstrass form, maximal contact elements correspond to *approximate roots* of f in the sense of Abhyankar [AM73a], [AM73b]. Alternatively, if one has a Puiseux series of f as in Equation (2.20), then the equations of a maximal contact element f_i have Puiseux series:

$$s_i(x) = \sum_{\substack{j \in (e_0) \\ \beta_0 \leq j < \beta_1}} a_j x^{j/n} + \cdots + \sum_{\substack{j \in (e_{i-1}) \\ \beta_{i-1} \leq j < \beta_i}} a_j x^{j/n} + \cdots \quad \text{for} \quad i = 1, \dots g, \tag{2.21}$$

where the non-explicit terms are assumed not to increase the *polydromy order*, [Cas00, §1.2], n/e_{i-1} of s_i, and either $f_0 = x$ or $f_0 = y$ depending on whether f is tangent to the x-axis or the y-axis respectively, see [Cas00, §5.8]. Notice that the multiplicity at the origin of these maximal contact elements is $n/e_{i-1} = n_1 \dots n_i$ for $i = 1, \dots, g$. It is not hard to see that the semigroups Γ_i of the singular maximal contact elements f_i are then

$$\Gamma_i = \langle n_1 n_2 \cdots n_{i-1}, n_2 \cdots n_{i-1} \overline{m}_1, \dots, n_{i-1} \overline{m}_{i-2}, \overline{m}_{i-1} \rangle, \quad \text{for} \quad i = 2, \dots, g+1. \tag{2.22}$$

Similarly, its characteristic sequence is $(n_1 n_2 \cdots n_{i-1}; n_2 \cdots n_{i-1} m_1, \dots, n_{i-1} m_{i-2}, m_{i-1})$. For convenience, and whenever it makes sense, we will assume that the $(g+1)$–th maximal contact element f_{g+1} is f itself.

Let $p_i \in K, i = 1, \dots, g$ be the infinitely near points associated with the rupture divisors E_{p_i} of the minimal log-resolution of (f). Similarly, denote by $q_i \in K, i = 0, \dots, g$ the points for which E_{q_i} is a dead-end divisor of the minimal log-resolution of (f). Observe that these points are totally ordered in the following way: $q_0 \leqslant q_1 \leqslant p_1 \leqslant q_2 \leqslant \cdots \leqslant q_g \leqslant p_g$. Then, one has the following results about the valuation of maximal contact elements of plane branches at these divisors.

Lemma 2.10. *The semigroup Γ_i of the i–th maximal contact element corresponds to the semigroup of the valuation $v_{p_{i-1}}$ of the $(i-1)$–th rupture divisor $E_{p_{i-1}}$.*

Proof. This lemma follows from Lemma 2.5, since for any $g \in \mathcal{O}_{X,\mathbf{0}}$, $v_{p_{i-1}}(g) = \mathrm{Div}(\pi^* g)_{\mathrm{exc}} \cdot B_{p_{i-1}}$, and $\mathrm{Div}(\pi^* f_p)_{\mathrm{exc}} = B_p$. ☐

Lemma 2.11. *For the dead-end divisors E_{q_i} with $i = 0, \dots, g$, we have:*

$$v_{q_i}(f_j) = n_{i+1} \cdots n_{j-1} \overline{m}_i \qquad \text{for} \qquad 0 \leq i \leq j-1, \tag{2.23}$$

and

$$v_{q_i}(f_j) = n_{j-1} \cdots n_{i-1} \overline{m}_{j-1} \qquad \text{for} \quad j \leq i \leq g. \tag{2.24}$$

For the rupture divisors E_{p_i} with $i = 1, \dots, g$, we have:

$$v_{p_i}(f_j) = n_i \cdots n_{j-1} \overline{m}_i \qquad \text{for} \qquad 0 \leq i \leq j-1, \tag{2.25}$$

and

$$v_{p_i}(f_j) = n_{j-1} \cdots n_i \overline{m}_{j-1} \qquad \text{for} \qquad j \leq i \leq g. \tag{2.26}$$

Proof. The first part of the lemma for the dead-end divisors E_{q_i} follows from the semigroup of the j–th maximal contact element in Equation (2.22), Lemma 2.5, and [Cas00, §5.8]. The first part for rupture divisors follows from the fact that $v_{p_i}(g) = n_i v_{q_i}(g)$ for any $g \in \mathcal{O}_{X,\mathbf{0}}$, which follows from Enriques' Theorem [Cas00, Thm. 5.5.1]. For the second part observe that $v_{q_i}(f_j) = v_{p_{i-1}}(f_j)$ if $i \geq j$ since the multiplicities at the points between p_{i-1} and q_j of the strict transform of f_j are zero. Then, $v_{q_j}(f_j) = v_{p_{j-1}}(f_j) = n_{j-1}\overline{m}_{j-1}$ and the rest of the lemma follows from induction and the equality $v_{p_i}(f_j) = n_i v_{q_i}(f_j)$ ☐

As a consequence of this lemma notice that if $j = g+1$, then $v_{q_i}(f_{g+1}) := v_{q_i}(f) = N_{q_i} = \overline{\beta}_i$ and $v_{p_i}(f) = N_{p_i} = n_i\overline{\beta}_i$, since $e_i = n_{i+1}\cdots n_g$ and $\overline{\beta}_i = e_i\overline{m}_i$. Recall that the notation N_{q_i}, N_{p_i} is used when $v_{q_i}(f), v_{p_i}(f)$ are interpreted as the multiplicities of the total transform of f along the embedded resolution. Related to this discussion is the following result on the multiplicities of the relative canonical divisor K_π at these same divisors.

Lemma 2.12 ([Wal04, Thm. 8.5.2]). *For the dead-end divisors E_{q_i} with $i = 0, \ldots, g$, we have*

$$v_{q_i}(K_\pi) = k_{q_i} = \lceil (m_i + n_1 \cdots n_i)/n_i \rceil - 1. \tag{2.27}$$

For the rupture divisors E_{p_i} with $i = 1, \ldots, g$, we have

$$v_{p_i}(K_\pi) = k_{p_i} = m_i + n_1 \cdots n_i - 1. \tag{2.28}$$

We introduce now the notion of *maximal contact elements* for a proper birational morphism $\pi : X' \longrightarrow X$.

Definition 2.13. The maximal contact elements of π with exceptional divisor E are those $f_p \in \mathcal{O}_{X,\mathbf{0}}$ with $p \in K$ considered in Definition 2.3, such that the divisor E_p is a dead-end, i.e. $\chi(E_i^\circ) = 1$.

One can interpret dead-end divisors in terms of the dual graph of π when the vertex associated with E_p is removed the dual graph remains connected. This means that a dead-end divisor E_p will always correspond to a free point of K or the origin $\mathbf{0}$ because satellite points are always proximate to two other points.

A *set of maximal contact elements* $\{f_i\}_{i\in I}$ contains a unique $f_{p_i} =: f_i$ for each dead-end vertex $p_i \in K$. Since these elements are determined by a finite number of valuative conditions, the elements f_i can always be chosen to be polynomials instead of power series. Moreover, since the maximal contact elements are irreducible, one usually constructs them via its Puiseux series in an intrinsic way, i.e. by truncating the series in (2.21) up to the last summation. As one should expect, this definition coincides with the one given above for an irreducible element $f \in \mathcal{O}_{X,\mathbf{0}}$. The elements $f_p \in \mathcal{O}_{X,\mathbf{0}}$ such that E_p is a dead-end divisor of the minimal log-resolution of (f) are exactly those with Puiseux series as in Equation (2.21).

Example 2.1. Let $\mathfrak{a} = ((y^2 - x^3)^2, x^2y^3) \subseteq \mathcal{O}_{X,O}$ be an ideal. Consider the minimal log-resolution $\pi : (X', E) \longrightarrow (X, \mathbf{0})$ of $\mathfrak{a}$ consisting of blowing-up four points with the configuration given by the dual graph in Figure 2.1.

Figure 2.1: Dual graph of the ideal in Example 2.1.

Therefore, $F_\pi = 4E_{p_0} + 6E_{p_1} + 12E_{p_2} + 13E_{p_3} + 26E_{p_4}$. The dead-end points are precisely p_0, p_1, p_3 and a set of maximal contact elements for π is $\{f_0, f_1, f_2\}$ with

$$\begin{aligned} f_0 &:= x + a_{2,0}x^2 + a_{0,2}y^2 + a_{1,1}xy + \cdots, \\ f_1 &:= y + b_2x^2 + b_3x^3 + \cdots, \\ f_2 &:= y^2 - x^3 + \sum_{3i+2j>6} c_{i,j}x^iy^j. \end{aligned}$$

Different choices of $a_{i,j}, b_i, c_{i,j} \in \mathbb{C}$ will give different sets of maximal contact elements. For simplicity, all the coefficients are chosen to be zero, $f_0 = x, f_1 = y, f_2 = y^2 - x^3$, which are polynomials. ■

2.6 *The monomial curve and its deformations*

Let $\Gamma = \langle \overline{\beta}_0, \overline{\beta}_1, \ldots, \overline{\beta}_g \rangle \subseteq \mathbb{Z}_{\geq 0}$ be a semigroup such that $\mathbb{Z}_{\geq 0} \setminus \Gamma$ is finite, that is $\gcd(\overline{\beta}_0, \ldots, \overline{\beta}_g) = 1$, not necessarily the semigroup of a plane branch. We use the notations and definitions from Section 2.4. Following Teissier [Tei86, §I.1.2], let $(C^\Gamma, \mathbf{0}) \subset (X, \mathbf{0})$ be the curve defined via the parameterization

$$C^\Gamma : u_i = t^{\overline{\beta}_i}, \quad 0 \leq i \leq g, \tag{2.29}$$

where $X := \mathbb{C}^{g+1}$. The germ $(C^\Gamma, \mathbf{0})$ is irreducible since $\gcd(\overline{\beta}_0, \ldots, \overline{\beta}_g) = 1$ and its local ring $\mathcal{O}_{C^\Gamma, \mathbf{0}}$ equals

$$\mathbb{C}\{C^\Gamma\} := \mathbb{C}\{t^{\overline{\beta}_0}, \ldots, t^{\overline{\beta}_g}\} \longhookrightarrow \mathbb{C}\{t\}, \tag{2.30}$$

which has a natural structure of graded subalgebra of $\mathbb{C}\{t\}$. The branch $(C^\Gamma, \mathbf{0})$ is usually called the *monomial curve* associated with the semigroup Γ. The first important property of the monomial curve C^Γ is the following

Theorem 2.14 ([Tei86, Thm. 1]). *Every branch $(C, \mathbf{0})$ with semigroup Γ is isomorphic to the generic fiber of a one-parameter complex analytic deformation of $(C^\Gamma, \mathbf{0})$.*

Moreover, with some extra structure on the semigroup Γ, it is possible to obtain more structure on the singularity of $(C^\Gamma, \mathbf{0})$ and even explicit equations.

Proposition 2.15 ([Tei86, Prop. 2.2]). *If Γ satisfies the condition in Lemma 2.8, then the branch $(C^\Gamma, \mathbf{0}) \subset (X, \mathbf{0})$ is a quasi-homogeneous complete intersection with equations*

$$h_i := u_i^{n_i} - u_0^{l_0^{(i)}} u_1^{l_1^{(i)}} \cdots u_{i-1}^{l_{i-1}^{(i)}} = 0, \quad 1 \leq i \leq g, \tag{2.31}$$

and weights $\overline{\beta}_0, \overline{\beta}_1, \ldots, \overline{\beta}_g$, where

$$n_i \overline{\beta}_i = \overline{\beta}_0 l_0^{(i)} + \cdots + \overline{\beta}_{i-1} l_{i-1}^{(i)} \in \langle \overline{\beta}_0, \ldots, \overline{\beta}_{i-1} \rangle. \tag{2.32}$$

In the sequel, we will always assume that the semigroup Γ fulfills (2.19). Since C^Γ is an isolated singularity one has the existence of the miniversal deformation of $(C^\Gamma, \mathbf{0})$, see for instance [Tei86, Add.]. After Theorem 2.14, every branch with semigroup Γ, is analytically isomorphic to one of the fibers of the miniversal deformation of $(C^\Gamma, \mathbf{0})$. Moreover, Teissier [Tei86, §I.2] proves the existence of the miniversal semigroup constant deformation of $(C^\Gamma, \mathbf{0})$, that is a miniversal deformation such that all the fibers have constant semigroup.

Theorem 2.16 ([Tei86, §I.2])**.** *There exists a germ of a flat morphism*

$$G : (X_\Gamma, C^\Gamma) \longrightarrow (\mathbb{C}^{\tau_-}, \mathbf{0}) \tag{2.33}$$

such that it is a miniversal semigroup constant deformation of $(C^\Gamma, \mathbf{0})$. Furthermore, for every representative of the morphism G, and for any branch $(C, \mathbf{0})$ with semigroup Γ, there exists $\boldsymbol{v}_C \in \mathbb{C}^{\tau_-}$ such that $(G^{-1}(\boldsymbol{v}_C), \mathbf{0})$ is analytically isomorphic to $(C, \mathbf{0})$.

Since C^Γ is a complete intersection and we have quasi-homogeneous equations for C^Γ, the miniversal semigroup constant deformation can be made explicit, see [Tei86, §I.2]. This is essentially the proof of Theorem 2.16. Consider the Tjurina module of the complete intersection $(C^\Gamma, \mathbf{0})$,

$$T^1_{C^\Gamma, \mathbf{0}} = \mathcal{O}^g_{X,\mathbf{0}} \Big/ \left(\operatorname{Jac} \boldsymbol{h} \cdot \mathcal{O}^{g+1}_{X,\mathbf{0}} + \langle h_1, \ldots, h_g \rangle \cdot \mathcal{O}^g_{X,\mathbf{0}} \right), \tag{2.34}$$

where $\operatorname{Jac} \boldsymbol{h} \cdot \mathcal{O}^{g+1}_{X,\mathbf{0}}$ is the submodule of $\mathcal{O}^g_{X,\mathbf{0}}$ generated by the columns of the Jacobian matrix of the morphism $\boldsymbol{h} = (h_1, \ldots, h_g)$. Since $(C^\Gamma, \mathbf{0})$ is an isolated singularity, $T^1_{C^\Gamma, \mathbf{0}}$ is a finite-dimensional $\mathbb{C}$–vector space of dimension the Tjurina number of the singularity, usually denoted by τ. Moreover, since $(C^\Gamma, \mathbf{0})$ is a Gorenstein singularity , $\tau = 2 \cdot \#(\mathbb{Z}_{>0} \setminus \Gamma)$, see [Tei86, Prop. 2.7].

We will denote by $\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_\mu \in \mathcal{O}^{g+1}_{X,\mathbf{0}}$ representatives for a basis of the $\mathbb{C}$–vector space $T^1_{C^\Gamma, \mathbf{0}}$. One can check that we can take these representatives as vectors $\boldsymbol{\phi}_r$ in $\mathcal{O}^{g+1}_{X,\mathbf{0}}$ having only one non-zero monomial entry $\phi_{r,i}$ for some index $0 \leq i \leq g$. Since $(C^\Gamma, \mathbf{0})$ is quasi-homogeneous, we can endow $T^1_{C^\Gamma, \mathbf{0}}$ with a structure of graded module.

Denote, after a possible reordering, $\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_{\tau_-}$ the representative of a basis of $T^1_{C^\Gamma, \mathbf{0}}$ with negative weight. Recall from Theorem 2.16 that τ_- was used to denote the dimension of the base of the miniversal semigroup constant deformation of $(C^\Gamma, \mathbf{0})$. Then X_Γ is defined from Equation (2.31) by the equations

$$H_i := h_i + \sum_{r=1}^{\tau_-} v_r \phi_{r,i}(u_0, \ldots, u_g) = 0, \quad 1 \leq i \leq g, \tag{2.35}$$

with the weight of $\phi_{r,i}$ strictly bigger than $n_i \overline{\beta}_i$, see [Tei86, Thm. 3]. Finally, the morphism G is just the projection of the v_r components to $\mathbb{C}^{\tau_-}$.

We focus now on the fibers of the miniversal semigroup constant deformation of $(C^\Gamma, \mathbf{0})$ that are plane, that is that can be embedded in a smooth complex surface. First, notice that the classes of the vectors $u_2 \boldsymbol{e}_1 := (u_2, 0, \ldots, 0), \ldots, u_g \boldsymbol{e}_{g-1} := (0, \ldots, u_g, 0)$ on $T^1_{C^\Gamma, \mathbf{0}}$ are linearly independent over $\mathbb{C}$. If we assume now that Γ is the semigroup of a plane branch, not just that Lemma 2.8 is fulfilled, we also have that $n_i \overline{\beta}_i < \overline{\beta}_{i+1}$ for

$i = 1, \dots, g-1$, and the vectors $u_2\boldsymbol{e}_1, \dots, u_g\boldsymbol{e}_{g-1}$ are part of the miniversal semigroup constant deformation of $(C^\Gamma, \mathbf{0})$.

Then consider, the following semigroup constant deformation of $(C^\Gamma, \mathbf{0})$,

$$C : H_i' := h_i - u_{i+1} = 0, \quad 1 \leq i \leq g. \tag{2.36}$$

Define $f_0 := x, f_1 := y$ and set recursively,

$$f_{i+1} := f_i^{n_i} - f_0^{l_0^{(i)}} f_1^{l_1^{(i)}} \cdots f_{i-1}^{l_{i-1}^{(i)}} \quad \text{with} \quad n_i\overline{\beta}_i = \overline{\beta}_0 l_0^{(i)} + \cdots + \overline{\beta}_{i-1} l_{i-1}^{(i)}, \tag{2.37}$$

for $i = 1, \dots, g$. Then, we can embed $(C, \mathbf{0})$ in the plane in such a way that it has equation $f := f_{g+1}$. Such an equation looks like

$$\left(\cdots \left(\left(\cdots \left(\left(y^{n_1} - x^{l_0^{(1)}} \right)^{n_2} - x^{l_0^{(2)}} y^{l_0^{(2)}} \right)^{n_3} - \cdots \right)^{n_i} - x^{l_0^{(i)}} \cdots f_{i-1}^{l_{i-1}^{(i)}} \right)^{n_{i+1}} - \cdots \right)^{n_{g-1}} - x^{l_0^{(g)}} \cdots f_{g-1}^{l_{g-1}^{(g)}}. \tag{2.38}$$

This particular construction proves the reverse implication of Proposition 2.9.

From the monomial curve and its deformation, we can deduce the following two novel propositions that provide the existence of certain μ-constant deformations, where μ is the Milnor number, of irreducible plane curves that will be used in the next chapters.

First, recall that, since the semigroup is a complete topological invariant, a semigroup constant deformation is equivalent to a topologically trivial deformation. For reduced plane curves, since the Milnor number μ is a topological invariant, topologically trivial implies μ-constant. It is a result of Lê Dũng Tráng and Ramanujam [TR76] that the converse is also true. Hence, in the case of plane branches, the semigroup constant deformations are the same as the μ-constant deformations.

The first result provides a family of μ-constant deformation that looks like f_{g+1} in Equation (2.37), in such a way that any other plane branch with the same topological class is analytically equivalent to a fiber of some deformation in the family.

Proposition 2.17. *Let $\Gamma = \langle \overline{\beta}_0, \overline{\beta}_1, \dots, \overline{\beta}_g \rangle$ be a plane branch semigroup. Consider, with the same notation as above,*

$$f_{i+1} = f_i^{n_i} - \lambda_i f_0^{l_0^{(i)}} f_1^{l_1^{(i)}} \cdots f_{i-1}^{l_{i-1}^{(i)}} + \sum_{\overline{\beta}_0 k_0 + \cdots + \overline{\beta}_i k_i > n_i \overline{\beta}_i} t_{\underline{k}}^{(i)} f_0^{k_0} f_1^{k_1} \cdots f_i^{k_i}, \tag{2.39}$$

with $\lambda_1 := 1, \lambda_i \neq 0$ for $i = 2, \dots, g$ and the sum being finite. Define,

$$f_{t,\lambda}(x, y) := f_{g+1}(x, y; \underline{t}^{(1)}, \dots, \underline{t}^{(g)}; \lambda_2, \dots, \lambda_g). \tag{2.40}$$

Then, $\{f_{t,\lambda}(x, y)\}_{\lambda \in \mathbb{C}^{g-1}}$ is an infinite family of μ-constant deformations, all having semigroup Γ, with the property that any other plane branch with semigroup Γ is analytically equivalent to a fiber of one element of the family.

Proof. Consider the semigroup constant deformation of the monomial curve $(C^\Gamma, \mathbf{0})$ with semigroup Γ from Theorem 2.16. Since the semigroup Γ is of a plane branch, we can assume that the set of vectors $u_2\boldsymbol{e}_1, \ldots, u_g\boldsymbol{e}_{g-1}$ are part of the semigroup constant deformation. Then, X_Γ will have equations

$$C : H_i = h_i - v_{i+1}u_{i+1} + \sum_{r=g+1}^{\tau_-} v_r \phi_{r,i}(u_0, \ldots, u_g) = 0, \quad 1 \leq i \leq g. \tag{2.41}$$

The embedding dimension of $(C, \mathbf{0})$ is equal to $g + 1 - \operatorname{rk} \operatorname{Jac} \boldsymbol{H}(\mathbf{0})$, see for instance [JP00, Thm. 4.3.6]. Since all the monomials in H_i have (non-weighted) degree bigger than 2, except for those in the vectors $u_2\boldsymbol{e}_1, \ldots, u_g\boldsymbol{e}_{g-1}$, the rank of the Jacobian matrix is $g-1$ if and only if $v_2 \cdots v_{g-1}$ is non-zero. Thus, the embedding dimension of $(C, \mathbf{0})$ is 2 if and only if all $v_2, \ldots, v_{g-1}$ are different from zero.

Finally, performing elimination on the variables $u_2, \ldots, u_g$ one obtains a plane branch equation similar to Equation (2.39) with $\lambda_i = v_2^{n_2} \cdots v_i^{n_i} \neq 0$ and a finite number of deformation monomials with coefficients that are polynomials in the variables v_r. Therefore, there is an inclusion of the parameter space of X_Γ into the parameters of the family of deformations $f_{t,\lambda}$. □

We will sometimes drop the dependency on the parameters $\boldsymbol{\lambda} \in \mathbb{C}^{g-1}$ and denote just $f_t(x, y)$. Although we are considering a deformation f_t with a finite number of deformation terms, we can always assume that we have deformation terms of high enough order in the summations of Equation (2.39). Indeed, adding extra terms to the summation does not change the fact that the family contains all plane curves up to analytic isomorphism or that the deformation has constant semigroup.

The second novel result shows the existence of a very particular type of one-parameter μ-constant deformation of irreducible plane curves and it will be key in the proof of Yano's conjecture in Chapter V.

Proposition 2.18. *Let $f : (\mathbb{C}^2, \mathbf{0}) \longrightarrow (\mathbb{C}, 0)$ be a plane branch. Let E_i be a rupture divisor of the minimal resolution of f with divisorial valuation v_i. Then, for any $v > n_i\overline{\beta}_i$ there exists a one-parameter μ-constant deformation of f of the form $f + tg_t$ such that $v_i(g_t) = v$, for all values of the parameter t.*

Proof. By the previous discussion, there exists a fiber of the miniversal semigroup constant deformation of the monomial curve $(C^\Gamma, \mathbf{0})$ that is analytically equivalent to the germ of curve $(C, \mathbf{0})$ defined by f. Precisely, with the notations above,

$$C : h_j + \lambda_j u_{j+1} + l_j = 0 \quad \text{for} \quad 0 \leq j \leq g, \tag{2.42}$$

with $\lambda_j \neq 0$ and where we set $u_{g+1} := 0$ for convenience. Here the elements l_j are linear combinations of the non-zero components of the elements $\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_{\tau_-}$ that are different from $u_2\boldsymbol{e}_1, \ldots, u_g\boldsymbol{e}_{g-1}$, see Equation (2.35). Set now $v' := n_i\overline{m}_i + v - n_i\overline{\beta}_i$, then v' belongs to the semigroup

$$\Gamma_{i+1} = \langle n_1 n_2 \cdots n_i, n_2 \cdots n_i \overline{m}_1, \ldots, n_i \overline{m}_{i-1}, \overline{m}_i \rangle \tag{2.43}$$

introduced in Equation (2.22) which corresponds to the divisorial valuation v_i, see Lemma 2.10. Indeed, notice that v' is strictly larger than the conductor of Γ_{i+1}. Therefore, there exists $(\alpha_0, \ldots, \alpha_i) \in \mathbb{Z}_{\geq 0}^{i+1}$ such that

$$v' = \alpha_0 n_1 n_2 \cdots n_i + \alpha_1 n_2 \cdots n_i \overline{m}_1 + \cdots + \alpha_g \overline{m}_i. \tag{2.44}$$

Consider the one-parameter deformation of C given by deforming the i–th equation of (2.42) in the following way

$$h_i + \lambda_i u_{i+1} + l_i + t u_0^{\alpha_0} u_1^{\alpha_1} \cdots u_i^{\alpha_i} = 0. \tag{2.45}$$

First, we must check that this deformation is semigroup constant, which is equivalent to seeing that $\deg(u_0^{\alpha_0} \cdots u_g^{\alpha_g}) > n_i \overline{\beta}_i$. Indeed,

$$\begin{aligned} \deg(u_0^{\alpha_0} u_1^{\alpha_1} \cdots u_g^{\alpha_g}) &= \alpha_0 \overline{\beta}_0 + \alpha_1 \overline{\beta}_1 + \cdots + \alpha_g \overline{\beta}_g \\ &= e_i(\alpha_0 n_1 n_2 \cdots n_i + \alpha_1 n_2 \cdots n_i \overline{m}_1 + \cdots + \alpha_g \overline{m}_i) \\ &= n_i \overline{\beta}_i + e_i(v - n_i \overline{\beta}_i) > n_i \overline{\beta}_i. \end{aligned} \tag{2.46}$$

Now, eliminating the variables $u_2, u_3, \ldots, u_g$ successively from Equation (2.42) one gets a deformation of f of the form $f + t g_0 + t^2 g_1 + \cdots$, where the dots mean higher-order terms on t. For plane branches, a family with constant semigroup is μ-constant, so this proves the first part of the lemma.

For the second part, one can see that when the first $u_2, u_3, \ldots, u_{i-1}$ variables are eliminated, $u_i = 0$ defines a plane branch f_i that is one of the maximal contact element of f from Section 2.5 with $\nu_f(f_i) = \overline{\beta}_i = \deg u_i$. Finally, after making the corresponding computations,

$$g_0 = f_0^{\alpha_0} \cdots f_i^{\alpha_i} f_{i+1}^{n_{i+1}-1} \cdots f_g^{n_g - 1} + \cdots \tag{2.47}$$

with precisely

$$v_i(f_0^{\alpha_0} \cdots f_i^{\alpha_i} f_{i+1}^{n_{i+1}-1} \cdots f_g^{n_g-1}) = v. \tag{2.48}$$

In order to check Equation (2.48), recall that from Lemma 2.11, we have $v_i(f_j) = n_i \cdots n_{j-1} \overline{m}_i$, for $i \leq j$, and then

$$\begin{aligned} v_i(f_0^{\alpha_0} \cdots f_i^{\alpha_i} f_{i+1}^{n_{i+1}-1} \cdots f_g^{n_g-1}) &= v' + (n_{i+1} - 1) v_i(f_{i+1}) + \cdots + (n_g - 1) v_i(f_g) \\ &= v' + (n_i - 1) n_i \overline{m}_i + \cdots + (n_g - 1) n_{g-1} \cdots n_i \overline{m}_i \\ &= n_g n_{g-1} \cdots n_i \overline{m}_i + v - n_i \overline{\beta}_i = v. \end{aligned} \tag{2.49}$$

A similar computation shows that any other term in g_0, or in higher-order powers of t, has a value strictly larger than v for the valuation v_i of the divisor E_i. □

2.7 *Toric resolutions*

Let $f : (\mathbb{C}^2, \mathbf{0}) \longrightarrow (\mathbb{C}, 0)$ be a germ of an irreducible plane curve with semigroup $\Gamma = \langle \overline{\beta}_0, \overline{\beta}_1, \ldots, \overline{\beta}_g \rangle$. A classical way to obtain the minimal embedded resolution by point blow-ups of an irreducible plane curve f from its characteristic sequence $(n, \beta_1, \ldots, \beta_g)$ is using Enriques' Theorem, see [Casoo, Thm. 5.5.1]. However, here we will describe the approach of Oka given in [Oka96] and describe the minimal resolution of f as a composition of toric morphisms.

There exists a minimal resolution map π of f that decomposes into $g \geq 1$ toric morphisms. For $i = 1, \ldots, g$,

$$\pi^{(i)} := \pi_1 \circ \cdots \circ \pi_i : (X^{(i)}, E^{(i)}) \xrightarrow{\pi_i} (X^{(i-1)}, E^{(i-1)}) \xrightarrow{\pi_{i-1}} \cdots \xrightarrow{\pi_1} (X^{(0)}, \mathbf{0}) := (\mathbb{C}^2, \mathbf{0}), \tag{2.50}$$

where π_i is a toric morphism for a suitable choice of coordinates on $X^{(i-1)}$ and we define $\pi := \pi^{(g)}$. Each morphism π_i resolves one characteristic exponent of the plane branch f in the sense that the strict transform of f on $X^{(i)}$ has one characteristic exponent less than the strict transform on $X^{(i-1)}$. In this way, $X^{(i)}$ always contains one more rupture divisor E_{p_i} than $X^{(i-1)}$. We will denote by U_i, V_i some affine open sets, and by $(x_i, y_i), (z_i, w_i)$ their respective coordinates, such that $U_i \cup V_i$ contains the i–th rupture divisor E_{p_i} on $X^{(i)}$ after the i–th toric modification π_i. In these coordinates, recalling the definitions of the integers n_i, q_i from Section 2.4, the toric morphism is given by

$$\pi_i(x_i, y_i) = \left(x_i^{n_i} y_i^{a_i}, x_i^{q_i} y_i^{b_i}\right) \quad \text{and} \quad \pi_i(z_i, w_i) = \left(z_i^{c_i} w_i^{n_i}, z_i^{d_i} w_i^{q_i}\right), \tag{2.51}$$

with $a_i, b_i, c_i, d_i \in \mathbb{Z}_{\geq 0}$ such that $n_i b_i - q_i a_i = 1, q_i c_i - n_i d_i = 1$ and $a_i q_i + d_i n_i = n_i q_i - 1$. These toric morphisms can be thought as a composition of point blow-ups. In the sequel, we will associate with each plane branch singularity these series of four integers a_i, b_i, c_i, d_i for every $i = 1, \ldots, g$. They are determined, although not explicitly, by the semigroup Γ of f since they depend on the continued fraction expansion of q_i/n_i.

If the singularity is still not resolved at $X^{(i)}$, one needs to perform an analytic change of coordinates around the unique singular point of the strict transform of f on $X^{(i)}$ in order for π_{i+1} to be toric. These new coordinates, let us say $(\bar{x}_i, \bar{y}_i)$ and $(\bar{z}_i, \bar{w}_i)$, are such that $\pi_*^{(i)} \bar{y}_i = \pi_*^{(i)} \bar{z}_i$ defines a germ $f_i : (\mathbb{C}^2, \mathbf{0}) \longrightarrow (\mathbb{C}, 0)$ that is a maximal contact element of f in the sense of Section 2.5. By construction, each of these maximal contact elements f_i are resolved by the corresponding $\pi^{(i)}$. In the case of the plane curves constructed in Section 2.6, the maximal contact element f_i coincide with the elements f_i defined in Equation (2.37) and Equation (2.39).

We will continue with a novel result about the resolution of the elements of the semigroup constant deformations $\{f_{t,\lambda}\}_{\lambda \in \mathbb{C}^{g-1}}$ from Proposition 2.17 that will be used in Chapter IV. Since having constant semigroup means that all the fibers of all the elements of the family are equisingular, the toric resolution of the plane branches in the family f_t is the same modulo the coordinates needed at each $X^{(i)}$. The following proposition describes locally the equations of f_t around the rupture divisors after pulling back by $\pi^{(i)}$.

Proposition 2.19. *Let E_{p_i} be the i–th rupture divisor on the surface $X^{(i)}$ and let U_i, V_i be the corresponding charts containing E_{p_i} with local coordinates (x_i, y_i) and (z_i, w_i), respectively. Then,*

- *The equations of the total transform of f_t are given by*

$$x_i^{n_i \overline{\beta}_i} y_i^{a_i \overline{\beta}_i} u_1(x_i, y_i) \tilde{f}_t(x_i, y_i), \qquad z_i^{(c_i n_{i-1} \overline{m}_{i-1} + d_i) e_{i-1}} w_i^{n_i \overline{\beta}_i} u_2(x_i, y_i) \tilde{f}_t(z_i, w_i), \tag{2.52}$$

where u_1, u_2 are units at any point of E_{p_i}.

- *The equations $\tilde{f}_t$ of the strict transform of f are*

$$\tilde{f}_t(x_i, y_i) = \tilde{f}_{g+1}(x_i, y_i; \underline{t}^{(1)}, \ldots, \underline{t}^{(g)}), \quad \tilde{f}_t(z_i, w_i) = \tilde{f}_{g+1}(z_i, w_i; \underline{t}^{(1)}, \ldots, \underline{t}^{(g)}). \tag{2.53}$$

- *The $(i+1)$–th maximal contact element has equations*

$$\begin{aligned}\tilde{f}_{i+1}(x_i,y_i) &= y_i - \lambda_i + \sum_{\overline{\beta}_0 k_0+\cdots+\overline{\beta}_i k_i > n_i\overline{\beta}_i} t^{(i)}_{\underline{k}} x_i^{\rho^{(i)}_{i+1}(\underline{k})} y_i^{A^{(i)}_{i+1}(\underline{k})} u^{(i)}_{\underline{k}}(x_i,y_i), \qquad (U_i \text{ chart}),\\ \tilde{f}_{i+1}(z_i,w_i) &= 1 - \lambda_i z_i + \sum_{\overline{\beta}_0 k_0+\cdots+\overline{\beta}_i k_i > n_i\overline{\beta}_i} t^{(i)}_{\underline{k}} z_i^{C^{(i)}_{i+1}(\underline{k})} w_i^{\rho^{(i)}_{i+1}(\underline{k})} u^{(i)}_{\underline{k}}(z_i,w_i), \qquad (V_i \text{ chart}),\end{aligned} \tag{2.54}$$

  where $u^{(i)}_{\underline{k}}$ are units at any point of E_{p_i}.

- *The remaining maximal contact elements $f_{j+1}, j > i$ have strict transforms given by*

$$\tilde{f}_{j+1} = \tilde{f}_j^{n_j} - \lambda_j x_i^{\rho^{(i)}_{j+1}(\underline{l}_j)} y_i^{A^{(i)}_{j+1}(\underline{l}_j)} u^{(j)}_{\underline{0}} \tilde{f}_{i+1}^{l^{(j)}_{i+1}} \cdots \tilde{f}_{j-1}^{l^{(j)}_{j-1}} + \sum_{\overline{\beta}_0 k_0+\cdots+\overline{\beta}_j k_j > n_j\overline{\beta}_j} t^{(j)}_{\underline{k}} x_i^{\rho^{(i)}_{j+1}(\underline{k})} y_i^{A^{(i)}_{j+1}(\underline{k})} u^{(j)}_{\underline{k}} \tilde{f}_{i+1}^{k_{i+1}} \cdots \tilde{f}_j^{k_j} \tag{2.55}$$

  in the U_i chart, and similarly in V_i. As before, $u^{(j)}_{\underline{0}}, u^{(j)}_{\underline{k}}$ are units everywhere on E_{p_i} and we denote $\underline{l}_j := (l^{(j)}_0, l^{(j)}_1, \ldots, l^{(j)}_{j-1}, 0)$ the integers from Proposition 2.15.

Finally, $\rho^{(i)}_{j+1}, A^{(i)}_{j+1}, C^{(i)}_{j+1}, j \geq i$, are the following linear forms:

$$\begin{aligned}\rho^{(i)}_{j+1}(\underline{k}) &= \sum_{l=0}^{i} n_{l+1}\cdots n_i \overline{m}_l k_l + n_i\overline{m}_i \sum_{l=i+1}^{j} n_{i+1}\cdots n_{l-1}k_l - n_i\cdots n_j\overline{m}_i,\\ A^{(i)}_{j+1}(\underline{k}) &= a_i \sum_{l=0}^{i-1} n_{l+1}\cdots n_{i-1}\overline{m}_l k_l + (a_i n_{i-1}\overline{m}_{i-1} + b_i)k_i\\ &\quad + a_i\overline{m}_i \sum_{l=i+1}^{j} n_{i+1}\cdots n_{l-1}k_l - a_i\overline{m}_i n_{i+1}\cdots n_j,\\ C^{(i)}_{j+1}(\underline{k}) &= c_i \sum_{l=0}^{i-1} n_{l+1}\cdots n_{i-1}\overline{m}_l k_l + (c_i n_{i-1}\overline{m}_{i-1} + d_i)k_i\\ &\quad + n_i(c_i n_{i-1}\overline{m}_{i-1} + d_i) \sum_{l=i+1}^{j} n_{i+1}\cdots n_{l-1}k_l - (c_i n_{i-1}\overline{m}_{i-1} + d_i)n_i\cdots n_j.\end{aligned} \tag{2.56}$$

Proof. The results follow from the inductive procedure of applying the toric transformations from Equation (2.51) to the equations of f_t in Proposition 2.17. At each $X^{(i)}$ the analytic coordinates which make the morphism π_i toric are described by $\bar{y}_i = \widetilde{f}_{i+1}, \bar{x}_i = x_i u_i$ in the U_i chart, for some unit u_i. The expressions for the linear forms $\rho^{(i)}_{j+1}, A^{(i)}_{j+1}, C^{(i)}_{j+1}$ follow, recursively, from the relations

$$\begin{aligned}\rho^{(i)}_{j+1}(\underline{k}) &= n_i\rho^{(i-1)}_{j+1}(\underline{k}) + q_i k_i + n_i q_i \sum_{l=i+1}^{j} n_{i+1}\cdots n_{l-1}k_l - n_i\cdots n_j q_i,\\ A^{(i)}_{j+1}(\underline{k}) &= a_i\rho^{(i-1)}_{j+1}(\underline{k}) + b_i k_i + a_i q_i \sum_{l=i+1}^{j} n_{i+1}\cdots n_{l-1}k_l - a_i q_i n_{i+1}\cdots n_j,\\ C^{(i)}_{j+1}(\underline{k}) &= c_i\rho^{(i-1)}_{j+1}(\underline{k}) + d_i k_i + n_i d_i \sum_{l=i+1}^{j} n_{i+1}\cdots n_{l-1}k_l - d_i n_i\cdots n_j.\end{aligned} \tag{2.57}$$

□

Following the notations from Proposition 2.19, we will usually fix the index i to denote that we have resolved the singularity up to the i–th rupture divisor on $X^{(i)}$. On the other hand, the index j will make reference to the j–th maximal contact element. At any step of the resolution process, one has that $1 \leq i \leq j \leq g$.

Corollary 2.20. *Let $\rho_{j+1}^{(i)}(\underline{k}), A_{j+1}^{(i)}(\underline{k}), C_{j+1}^{(i)}(\underline{k})$ be the linear forms in Proposition 2.19. Then,*

$$A_{j+1}^{(i)}(\underline{k}) + C_{j+1}^{(i)}(\underline{k}) + \sum_{l=i+1}^{j} n_{i+1}\cdots n_{l-1}k_l = \rho_{j+1}^{(i)}(\underline{k}) + n_{i+1}\cdots n_j. \tag{2.58}$$

Proof. From the relations between a_i, b_i, c_i, d_i, one can deduce that $a_i + c_i = n_i, b_i + d_i = q_i$. The result follows from adding $A_{j+1}^{(i)}$ and $C_{j+1}^{(i)}$ and using these relations. □

The following corollary will be useful in the next chapters. In the case of plane branches, it describes the multiplicity of the total transform along the two exceptional divisors preceding a certain rupture divisor in terms of the toric morphism. For the sake of simplifying the notation, and since no confusion may arise since we will always work with a fixed rupture divisor, $E_i := E_{p_i}, p_i \in K$, we will denote by D_1, D_2, D_3 the irreducible components in $\mathrm{Supp}(F_\pi)$ such that $E_i \cap D_i \neq \emptyset$. Furthermore, we will assume that D_1 and D_2 precede E_i in the resolution process.

Corollary 2.21. *Let E_i be a rupture divisor of the minimal embedded resolution of f. Then, the multiplicities of D_1 and D_2 in F_π are*

$$N_1 = a_i\overline{\beta}_i \qquad \text{and} \qquad N_2 = (c_i n_{i-1}\overline{m}_{i-1} + d_i)e_{i-1}. \tag{2.59}$$

2.8 *Yano's conjecture*

We will next present the conjecture posed in 1982 by Yano [Yan82] about the generic b-exponents, see Definition 1.38, of irreducible germs of plane curve singularities. The conjecture predicts that for generic curves in some μ-constant deformation of f, the whole set of μ b-exponents can be completely determined from the characteristic sequence of the topological class.

Accordingly, let $f : (\mathbb{C}^2, \mathbf{0}) \longrightarrow (\mathbb{C}, 0)$ be a germ of a holomorphic function defining an irreducible plane curve singularity with characteristic sequence $(n, \beta_1, \ldots, \beta_g)$, where n is the algebraic multiplicity of f at the origin and $g \geq 1$ is the number of characteristic pairs. With the same notations as [Yan82, §2], define

$$\begin{aligned} &r_i := \frac{\beta_i + n}{e_i}, \quad R_i := \frac{\beta_i e_{i-1} + \beta_{i-1}(e_{i-2} - e_{i-1}) + \cdots + \beta_1(e_0 - e_1)}{e_i}, \\ &r'_0 := 2, \quad r'_i := r_{i-1} + \left\lfloor \frac{\beta_i - \beta_{i-1}}{e_{i-1}} \right\rfloor + 1 = \left\lfloor \frac{r_i e_i}{e_{i-1}} \right\rfloor + 1, \\ &R'_0 := n, \quad R'_i := R_{i-1} + \beta_i - \beta_{i-1} = \frac{R_i e_i}{e_{i-1}}. \end{aligned} \tag{2.60}$$

where the integers $e_i, i = 0, \ldots, g$ were defined in Section 2.4. Inspired by A'Campo formula [ACa75] for the eigenvalues of the monodromy of an isolated singularity, Yano defines the following polynomial with fractional powers in t

$$R\big((n, \beta_1, \ldots, \beta_g), t\big) := \sum_{i=1}^{g} t^{\frac{r_i}{R_i}} \frac{1-t}{1-t^{\frac{1}{R_i}}} - \sum_{i=0}^{g} t^{\frac{r'_i}{R'_i}} \frac{1-t}{1-t^{\frac{1}{R'_i}}} + t, \tag{2.61}$$

and proves that $R\big((n, \beta_1, \ldots, \beta_n), t\big)$ has non-negative coefficients. Finally,

Conjecture (Yano [Yan82]). *For generic curves in some μ-constant deformation of an irreducible germ of plane curve having characteristic sequence $(n, \beta_1, \ldots, \beta_g)$, the b-exponents $\{\alpha_1, \alpha_2, \ldots, \alpha_\mu\}$ are given by the generating function R. That is,*

$$\sum_{i=1}^{\mu} t^{\alpha_i} = R\big((n, \beta_1, \ldots, \beta_g), t\big). \tag{2.62}$$

Yano's conjecture was proved in the case of one Puiseux pair, $g = 1$, by Cassou-Noguès [Cas88]. More recently, Artal-Bartolo, Cassou-Noguès, Luengo, and Melle-Hernández [Art+17b] proved the case of two Puiseux pairs under the hypothesis that the eigenvalues of the monodromy of f are pair-wise different. This hypothesis on the eigenvalues of the monodromy ensures that the minimal and the characteristic polynomial of the endomorphism (1.77) from Theorem 1.35 coincide. This means that the roots of the Bernstein-Sato polynomial and the b-exponents are the same.

Remark 2.2. With the notations from Section 2.5 and Lemma 2.12, the integers defined in Equation (2.60) are equal to

$$r_i = k_{p_i}, \quad R_i = N_{p_i}, \quad r'_i = k_{q_i}, \quad R'_i = N_{q_i}. \tag{2.63}$$

where E_{p_i}, E_{q_i} are, respectively, the rupture and dead-end exceptional divisors of the minimal resolution of f. ■

2.9 *Multiplier ideals and jumping numbers*

The multiplier ideals, introduced in Section 1.3, and the associated jumping numbers are quite well-understood in the case of a smooth surface $(X, \mathbf{0})$. The first results about the jumping numbers of ideals in smooth complex surfaces are due to Järvilehto [Jär11], for simple complete ideals, and Naie [Nai09] for irreducible plane curve singularities. We will use the notation introduced in Section 2.4 for the semigroup of a plane branch to describe these results.

Theorem 2.22 ([Jär11, Thm. 6.2]). *Let $\mathfrak{a} \subseteq \mathcal{O}_{X,\mathbf{0}}$ be a simple complete $\mathfrak{m}$-primary ideal and assume that the generic curve of the ideal has semigroup $\Gamma = \langle \overline{\beta}_0, \overline{\beta}_1, \ldots, \overline{\beta}_g \rangle$. Then, the jumping numbers of $\mathfrak{a}$ are*

$$\bigcup_{\nu=1}^{g-1} \left\{ \frac{\overline{m}_\nu i + n_{\nu-1} j + n_\nu \overline{m}_\nu k}{n_\nu \overline{\beta}_\nu} \;\middle|\; i, j \in \mathbb{Z}_{>0}, k \in \mathbb{Z}_{\geq 0}, \frac{i}{n_\nu} + \frac{j}{\overline{m}_\nu} \leq 1 \right\} \tag{2.64}$$

and

$$\left\{ \frac{\overline{\beta}_g i + n_g j}{n_g \overline{\beta}_g} \;\middle|\; i, j \in \mathbb{Z}_{\geq 0}, \frac{i}{n_g} + \frac{j}{\overline{\beta}_g} \leq 1 \right\}, \tag{2.65}$$

and all the jumping numbers are contributed by the rupture divisor $E_{p_i}, i = 1, \ldots, g$.

For irreducible plane curve singularities, Naie [Nai09] also presents a formula in terms of the semigroup of the singularity. Due to the relation of the jumping numbers of an isolated singularity with the spectral numbers of the singularity, see [Ein+04, Remark 3.10], Naie's result already appeared in [Sai00], where Saito computed the spectral numbers of an irreducible curve singularity in terms of the characteristic exponents.

Theorem 2.23 ([Sai00, Thm. 1.5], [Nai09, Thm. 3.1]). *The jumping numbers in* $(0,1]$ *of an irreducible plane curve singularity with semigroup* $\Gamma = \langle \overline{\beta}_0, \overline{\beta}_1, \ldots, \overline{\beta}_g \rangle$ *are*

$$\bigcup_{\nu=1}^{g} \left\{ \frac{\overline{m}_\nu i + n_\nu j + n_\nu \overline{m}_\nu k}{n_\nu \overline{\beta}_\nu} \;\middle|\; i,j \in \mathbb{Z}_{>0}, 0 \leq r < e_\nu, \frac{i}{n_\nu} + \frac{j}{\overline{m}_\nu} < 1 \right\}, \tag{2.66}$$

and all the jumping numbers are contributed by the rupture divisors $E_{p_i}, i = 1, \ldots, g$.

For arbitrary plane curve singularity, it is also true, see the work of Smith and Thompson [ST07, Thm. 3.1], that the only exceptional divisors contributing to jumping numbers are the rupture divisors of the minimal embedded resolution.

More generally, there is an algorithm by Tucker [Tuc10] that computes all the jumping numbers of an ideal $\mathfrak{a}$ from a log-resolution of $\mathfrak{a}$. However, we will present the algorithm by Alberich-Carramiñana, Àlvarez Montaner and Dachs-Cadefau [AÀD16] that computes both the jumping numbers $\lambda_i(\mathfrak{a}) := \lambda_i(\mathfrak{a}, \mathbf{0})$ of any ideal of $\mathcal{O}_{X,\mathbf{0}}$ and the integral divisor on the resolution surface X' defining the i–th multiplier ideal. The algorithm in [AÀD16] is based on the unloading procedure, see Algorithm 2.6, and on the following result.

Theorem 2.24 ([AÀD16, Thm 3.5]). *Let* $\mathfrak{a} \subseteq \mathcal{O}_{X,\mathbf{0}}$ *be an ideal and let* $D_{\lambda'} = \sum a_i^{\lambda'} E_i$ *be the antinef closure of* $\lfloor \lambda' F_\pi - K_\pi \rfloor$ *for a given* $\lambda' \in \mathbb{Q}_{>0}$. *Then,*

$$\lambda = \min_i \left\{ \frac{k_i + 1 + a_i^{\lambda'}}{a_i} \right\} \tag{2.67}$$

is the jumping number consecutive to λ'. *Here* $F_\pi = \sum_i a_i D_i$.

Algorithm 2.25 (Jumping Numbers and Multiplier Ideals, [AÀD16]).
Input: *A log-resolution of the ideal* $\mathfrak{a} \subseteq \mathcal{O}_{X,\mathbf{0}}$, *a positive integer* $N \in \mathbb{Z}_{>0}$.
Output: *The first* N *jumping numbers of* $\mathfrak{a}$ *and the corresponding multiplier ideals.*

- *Set* $\lambda_0 := 0, D_{\lambda_0} := 0$. ***For*** $j = 1, \ldots, N$:
  - Jumping number: *Compute*

$$\lambda_j = \min_i \left\{ \frac{k_i + 1 + a_i^{\lambda_{j-1}}}{a_i} \right\}.$$

  - Multiplier ideal: *Compute the antinef closure* $D_{\lambda_j} = \sum e_i^{\lambda_j} E_i$ *of* $\lfloor \lambda_j F_\pi - K_\pi \rfloor$ *using Algorithm 2.6.*

Both the results in [Tuc10] and the algorithm in [AÀD16] are more general than the ones presented above since both results also work if $(X, \mathbf{0})$ has at most, a rational singularity.

In Chapter II, we will present some results that combined with Algorithm 2.25 will allow the computation of generators for all the multiplier ideals of $\mathfrak{a}$ starting from any set of generators of $\mathfrak{a}$.

II EFFECTIVE COMPUTATION OF COMPLETE PLANAR IDEALS

This chapter is divided in two sections. The first section contains an algorithm that computes the log-resolution of an ideal in a smooth complex surface from any set of generators. The second section develops an algorithm that computes generators for the complete ideal given by the push-forward by a proper birational morphism of the ideal sheaf defined by a divisor with exceptional support. This chapter contains the more algorithmic results of this thesis.

At the end of the chapter, we also include some of the applications of these algorithms. For instance, combining these two algorithms with Algorithm 2.25, one has an effective method to compute the multiplier ideals of an ideal in a smooth complex surface. The combination of the three algorithms is effective in the sense that both the input and the output are given by polynomials. The code for the algorithms presented in this chapter, as well as for other required algorithms from the theory of plane curve singularities, can be found in Appendix A.

3 COMPUTING LOG-RESOLUTIONS OF PLANAR IDEALS

In this section, we present an algorithm to compute the log-resolution of a planar ideal in a smooth surface from any given set of generators of the ideal. Although explicit computation of log-resolutions of general ideals by blowing-up smooth centers is a well-studied and solved problem, see [Vil89; BEV05; Frü07], the result in this section can be seen as the analog for log-resolutions of ideals to the classical resolution of plane curve singularities via the Newton-Puiseux expansion and Enriques' Theorem [Cas00, §1, §5.5]. The results of this section are joint work with Alberich-Carramiñana and Àlvarez Montaner and can be found in [AÀB19].

3.1 *A characterization of the log-resolution of an ideal*

Let $(X, \mathbf{0})$ be a germ of a smooth complex surface and let $\mathfrak{a} \subseteq \mathcal{O}_{X,\mathbf{0}}$ be an ideal and $\mathfrak{m} = \mathfrak{m}_{X,\mathbf{0}} \subseteq \mathcal{O}_{X,\mathbf{0}}$ be the maximal ideal. The aim of this section is to give a characterization of the minimal log-resolution divisor F_π, see Definition 2.2, associated to $\mathfrak{a} = (f_1, \ldots, f_r)$. This description will be more suitable for the computational purposes of the rest of the section. Then, we will describe a divisor $G = \sum_{p \in K'} v_p E_p$ with multiplicities v_p depending on the values of the curves $C_i : f_i = 0, i = 1, \ldots, r$ and we will then prove that it is equal to F_π.

Remark 3.1. For the sake of simplicity, we will assume that $\mathfrak{a}$ is $\mathfrak{m}$-primary. Otherwise, $\mathfrak{a} = (f) \cdot \mathfrak{a}'$ with $\mathfrak{a}'$ being $\mathfrak{m}$-primary and $f \in \mathcal{O}_{X,\mathbf{0}}$. Then, the minimal log-resolution divisor of $\mathfrak{a}$ will be obtained by combining the minimal log-resolution divisors of (f) and $\mathfrak{a}'$, plus some extra blow-ups. ■

Definition 3.1. Let $\mathfrak{a} = (f_1, \dots, f_r) \subseteq \mathcal{O}_{X,\mathbf{0}}$ be an $\mathfrak{m}$-primary ideal. For any point p equal or infinitely near to $\mathbf{0}$ we define the value $v_p := \min\{v_p(f_1), \dots, v_p(f_r)\}$, and recursively on the proximate points, we define $h_{\mathbf{0}} = 0$ and

$$h_p := \sum_{p \to q} v_q. \tag{3.1}$$

Define also the divisor $G = \sum_{p \in K'} v_p E_p$, where K' is the set of points p such that $h_p < v_p$.

Notice that, after Equation (2.6), the multiplicities e_p in the total transform basis are $e_p = v_p - h_p$, for each $p \in K'$, i.e. $G = \sum_{p \in K'} e_p \overline{E}_p$.

For G to be a minimal log-resolution divisor, we need to check that K' is finite and that for any $p \in K'$, all the preceding points also belong to K'. To do so, we will start with a technical lemma.

Lemma 3.2. *Let $\mathfrak{a} = (f_1, \dots, f_r) \subseteq \mathcal{O}_{X,\mathbf{0}}$ be an $\mathfrak{m}$-primary ideal. Then, for any $f \in \mathfrak{a}$ and any p proper or infinitely near to the origin, we have $v_p(f) \geq \min\{v_p(f_1), \dots, v_p(f_r)\}$.*

Proof. Assume that $f = g_1 f_1 + \cdots + g_r f_r$, for $g_1, \dots, g_r \in \mathcal{O}_{X,\mathbf{0}}$. Using the fact that $v_p(\cdot)$ is a discrete valuation in $\mathcal{O}_{X,\mathbf{0}}$, see [Cas00, §4.5], we have:

$$\begin{aligned} v_p(f) = v_p(g_1 f_1 + \cdots + g_r f_r) &\geq \min\{v_p(g_1 f_1), \dots, v_p(g_r f_r)\} \\ = \min_i \{v_p(g_i) + v_p(f_i)\} &\geq \min\{v_p(f_1), \dots, v_p(f_r)\}, \end{aligned} \tag{3.2}$$

where in the last inequality we used that $v_p(g) \geq 0, \forall g \in \mathcal{O}_{X,\mathbf{0}}$. □

Next, we prove that the definition of the divisor G does not depend on the generators of the ideal $\mathfrak{a}$.

Lemma 3.3. *Let $\mathfrak{a} = (f_1, \dots, f_r) \subseteq \mathcal{O}_{X,\mathbf{0}}$ be an $\mathfrak{m}$-primary ideal. The multiplicities $v_p, p \in K'$ of the divisor G associated to $\mathfrak{a}$ do not depend on the generators of the ideal. In other words, $v_p = \min_{f \in \mathfrak{a}}\{v_p(f)\}$.*

Proof. It is clear that $v_p = \min_i\{v_p(f_i)\} \geq \min_{f \in \mathfrak{a}}\{v_p(f)\}$ since $\{f_1, \dots, f_r\} \subset \mathfrak{a}$. On the other hand, by Lemma 3.2, $v_p(f) \geq \min_i\{v_p(f_i)\}$, for all $f \in \mathfrak{a}$, hence $\min_{f \in \mathfrak{a}}\{v_p(f)\} \geq v_p$ and the result follows. □

Lemma 3.4. *Under the assumptions of Definition 3.1, the inequality $h_p \leq v_p$ holds for any point p equal or infinitely near to $\mathbf{0}$.*

Proof. The inequality is clear when $p = \mathbf{0}$. Now, assume that p is free, so it is proximate to one point $p \to q$. Then, we have:

$$h_p = v_q = \min_i\{v_q(f_i)\} \leq \min_i\{v_p(f_i)\} = v_p. \tag{3.3}$$

If p is satellite, it is proximate to two points $p \to q$ and $p \to q'$. Then, we have:

$$\begin{aligned} h_p = v_q + v_{q'} &= \min_i\{v_q(f_i)\} + \min_i\{v_{q'}(f_i)\} \\ &\leq \min_i\{v_q(f_i) + v_{q'}(f_i)\} = \min_i\{v_p(f_i)\} = v_p. \end{aligned} \tag{3.4}$$

□

Lemma 3.5. *Let $\mathfrak{a} = (f_1, \ldots, f_r) \subseteq \mathcal{O}_{X,0}$ be an $\mathfrak{m}$-primary ideal. If there exists a generator f_i such that $h_p = v_p = v_p(f_i)$, then we have $e_p(f_i) = 0$ and $v_q = v_q(f_i)$ for any point q proper or infinitely near to the origin such that $p \to q$.*

Proof. If p is a free point, we take the unique point q such that $p \to q$. Notice that

$$v_p(f_i) = v_p = h_p = v_q = \min_j \{v_q(f_j)\}, \tag{3.5}$$

so we have $v_p(f_i) \leq v_q(f_i)$. It follows from Equation (2.6) that $v_p(f_i) = v_q(f_i)$ and $e_p(f_i) = 0$, hence, $v_q = v_q(f_i)$.

If p is satellite, we take the points q and q' such that $p \to q$, $p \to q'$. We have

$$v_p(f_i) = v_p = h_p = v_q + v_{q'}, \tag{3.6}$$

thus $v_p(f_i) \leq v_q(f_i) + v_{q'}(f_i)$. Using Equation (2.6), we obtain $v_p(f_i) = v_q(f_i) + v_{q'}(f_i)$ and $e_p(f_i) = 0$. Finally, if $v_q < v_q(f_i)$ or $v_{q'} < v_{q'}(f_i)$, then $h_p = v_q + v_{q'} < v_p(f_i)$, so we get a contradiction. □

Proposition 3.6. *Under the assumptions of Definition 3.1, if $p \in K'$, then any point q preceding p also belongs to K'.*

Proof. We will prove the converse statement: assume that $q \notin K'$, i.e. $h_q = v_q$. We will prove $h_p = v_p$ for any p in the first neighborhood of q, and it will follow inductively $h_p = v_p$, i.e. $p \notin K'$, for any point p infinitely near to q.

Assume that $q \notin K'$ and let p be a point in the first neighborhood of q, in particular $p \to q$. Consider a generator f_i such that $v_q = \min_j\{v_q(f_j)\} = v_q(f_i)$, hence $h_q = v_q = v_q(f_i)$. If p is satellite, we take the second point q' such that $p \to q'$. Then, by Lemma 3.5

$$h_p = v_q + v_{q'} = v_q(f_i) + v_{q'}(f_i) = v_p(f_i), \tag{3.7}$$

and by Lemma 3.4, $h_p = v_p(f_i) = v_p$. If p is free, the same reasoning is valid by taking $v_{q'} = v_{q'}(f_i) = 0$. □

Since we are assuming that $\mathfrak{a}$ is $\mathfrak{m}$-primary, we will assume that the minimal log-resolution divisor has the expression $F_\pi = \sum_{p \in K} N_p E_p$ in the strict transform basis. Notice that since the log-resolution of $\mathfrak{a}$ can be constructed by successive blow-ups, we can assume that F_π is antinef. We will show next that the divisor G equals the minimal log-resolution divisor F_π of $\mathfrak{a}$ and we will conclude that K' is finite.

Proposition 3.7. *Let $F_\pi = \sum_{p \in K} N_p E_p$ be the minimal log-resolution divisor of the $\mathfrak{m}$-primary ideal $\mathfrak{a} = (f_1, \ldots, f_r) \subseteq \mathcal{O}_{X,0}$ and $G = \sum_{p \in K'} v_p E_p$ as given in Definition 3.1. Let $p \in K$, then $p \in K'$ and the equality of multiplicities $N_p = v_p$ is satisfied.*

Proof. Let $f = g_1 f_1 + \cdots + g_r f_r$ be an element of $\mathfrak{a}$ such that the minimal log-resolution divisor of (f) equals F_π, such elements always exist [Cas00, Thm. 7.2.13]. From Lemma 3.2 we obtain

$$N_p = v_p(f) \geq \min_i \{v_p(f_i)\} = v_p. \tag{3.8}$$

Now, by [Cas00, Cor. 7.2.16], we may find a system of generators $\mathfrak{a} = (h_1, \ldots, h_s)$ such that the minimal log-resolution divisor of each (h_i) equals F_π. Then,

$$v_p = \min_i \{v_p(f_i)\} \geq \min_i \{v_p(h_i)\} = N_p, \tag{3.9}$$

after applying Lemma 3.2, once again, to the elements f_i expressed in terms of $h_1, \ldots, h_s$. Therefore, the equality $v_p = N_p$ follows. Since the same equality holds for all the points preceding p, we infer

$$v_p - h_p = N_p - \sum_{p \to q} N_q = e_p(F_\pi) > 0, \tag{3.10}$$

that is, $v_p > h_p$, since $e_p(F_\pi)$ is the multiplicity of $\overline{E}_p$ in the minimal log-resolution divisor F_π which is antinef, Equation (2.6). □

Theorem 3.8. *The minimal log-resolution divisor F_π and the divisor G are equal. In particular, K' is finite.*

Proof. From Proposition 3.7 we already have $K \subseteq K'$. We will prove the other inclusion using induction on the order of neighborhood which a point $p \in K'$ belongs to.

For $p = \mathbf{0}$, it is clear that p belongs to both K' and K. Now, assume that the assertion is true for all the points preceding p, which are in K' by Proposition 3.6. Let $q \in K$ be the antecessor of p. By [Cas00, Lemma 7.2.6], $p \in K$ if and only if $0 < \min_{f \in \mathfrak{a}}\{e_p(\check{f}_p)\}$, where $e_p(\check{f}_p)$ is the multiplicity of the strict transform of a f at p relative to the proper birational morphism where the points after p have not been blown-up. This is equivalent, by Equation (2.6), to

$$\min_{f \in \mathfrak{a}}\{v_p(f)\} > \sum_{p \to s} N_s. \tag{3.11}$$

By Lemma 3.3, $v_p = \min_{f \in \mathfrak{a}}\{v_p(f)\}$. Thus, applying Proposition 3.7 to the points preceding p, we have that p belongs to K if and only if,

$$v_p > \sum_{p \to s} N_s = \sum_{p \to s} v_s = h_p. \tag{3.12}$$

□

Remark 3.2. Theorem 3.8 is a generalization for $\mathfrak{m}$-primary ideals of [Alb04, Thm. 2.5] which describes the log-resolution of the general element of a pencil $\lambda_1 f_1 + \lambda_2 f_2$ of curves $f_1, f_2 \in \mathcal{O}_{X,\mathbf{0}}$, $\lambda_1, \lambda_2 \in \mathbb{C}$. ■

Corollary 3.9. *Let $F_\pi = \sum_{p \in K} N_p E_p$ be the minimal log-resolution divisor of a $\mathfrak{m}$-primary ideal $\mathfrak{a} = (f_1, \ldots, f_r) \subseteq \mathcal{O}_{X,\mathbf{0}}$. Let $D = \sum_{p \in K''} v_p(D) E_p = \sum_{p \in K''} e_p(D) \overline{E}_p$ be any divisor with*

$$v_p(D) = \min_i\{v_p(f_i)\}, \quad \text{for all } p \in K'', \tag{3.13}$$

or, alternatively,

$$e_p(D) = v_p(D) - \sum_{p \to q} v_q(D), \quad \text{for all } p \in K'', \tag{3.14}$$

and such that, for any $p \in K''$, $e_p(D) \neq 0$. Then $K'' \subseteq K$.

Proof. Since, by definition, $p \in K$ if and only if $e_p > 0$, clearly $K'' \subseteq K$. Then, the result follows using Theorem 3.8. □

3.2 *An algorithm to compute the log-resolution of an ideal*

In this section we will describe an algorithm that allows to compute the minimal log-resolution divisor of any ideal $\mathfrak{a} = (a_1, \ldots, a_r) \subseteq \mathcal{O}_{X,\mathbf{0}}$. First we recall that we have a decomposition $\mathfrak{a} = (f) \cdot \mathfrak{a}'$, where $f \in \mathcal{O}_{X,\mathbf{0}}$ is the greatest common divisor of the generators of $\mathfrak{a}$ and $\mathfrak{a}' = (f_1, \ldots, f_r)$ is $\mathfrak{m}$-primary. Moreover, the minimal log-resolution divisor of $\mathfrak{a}$ is described in terms of the minimal log-resolution divisors of $\mathfrak{a}'$ and (f), see Remark 3.1.

The log-resolution of f, which coincides with its minimal embedded resolution, is easy to describe using, for instance, the Newton-Puiseux algorithm, see [Cas00]. The bulk of the process is then in the computation of the minimal log-resolution F_π of $\mathfrak{a}'$. Before proceeding to describe the algorithm we need to introduce several technical results that will allow us to compute F_π in terms of the minimal log-resolutions of the given set of generators.

Blowing-up free and satellite points

Again, for the sake of simplicity, we will assume throughout this subsection that our ideal $\mathfrak{a} = (f_1, \ldots, f_r) \subseteq \mathcal{O}_{X,\mathbf{0}}$ is $\mathfrak{m}$-primary. In order to compute the minimal log-resolution divisor F_π, we will start with the minimal log-resolution divisor of the product of the generators $f_1 \cdots f_r$, which gives a first approximation. Then, using the following results, we will blow-up and blow-down the necessary free and satellite points to obtain the divisor F_π.

In the sequel, we assume that $G' = \sum_{p \in K'} v_p E_p$ is a divisor with $v_p = \min_i \{v_p(f_i)\}$ and such that any point $p \in K$ that also belongs to the minimal log-resolution of $(f_1 \cdots f_r)$ is already in K'.

The following set of technical results will allow us to decide which points we need to blow-up or blow-down from the minimal log-resolution surface of $f_1 \cdots f_r$ in order to obtain the minimal log-resolution surface (X', E) of $\mathfrak{a}$ where the minimal log-resolution divisor F_π lives. The first result states that all the free points supporting the exceptional part of F_π lie on the generators.

Lemma 3.10. *Let K_i be the infinitely points supporting the exceptional part of the divisors of the minimal log-resolution of the generators $f_i, i = 1, \ldots, r$. Let q be a free point such that $q \notin K_i$ for $i = 1, \ldots, r$, then $q \notin K$*

Proof. Let $q \to p$. If $q \notin K_i$ for all i, then $v_q(f_i) = v_p(f_i)$ for all $i = 1, \ldots, r$ and $v_q = \min_i \{v_q(f_i)\} = \min_i \{v_p(f_i)\} = v_p$, hence $e_q = v_q - v_p = 0$ and $q \notin K$. $\square$

The next result characterizes the free points supporting the exceptional part of F_π that are not part of the minimal log-resolution of $(f_1 \cdots f_r)$.

Proposition 3.11. *Let G' be a divisor as above. Let $q \notin K'$ be a free point proximate to $p \in K'$. Then, q is in K if and only if any generator f_i with $v_p(f_i) = v_p$ satisfies $e_q(f_i) > 0$.*

Proof. We shall apply Theorem 3.8 to characterize whether q belongs to K. By definition, $v_q = \min_j \{v_q(f_j)\}$ and $v_q(f_j) = e_q(f_j) + v_p(f_j)$, for $j = 1, \ldots, r$. Set $\Lambda_q := \{j \mid e_q(f_j) > 0\}$. Comparing

$$v_q = \min_{j \in \Lambda_q, k \notin \Lambda_q} \{v_p(f_k), v_p(f_j) + e_q(f_j)\} \quad \text{and} \quad v_p = \min_i \{v_p(f_i)\}, \tag{3.15}$$

we infer that $e_q = v_q - v_p > 0$ if and only if $v_p(f_k) > v_p$, for all $k \notin \Lambda_q$, which is equivalent to $\{i \mid v_p(f_i) = v_p\} \subseteq \Lambda_q$. $\square$

Remark 3.3. Under the hypothesis of Proposition 3.11 we observe that there might be two generators, say f_i, f_j, such that $e_q(f_i) > 0$, $e_q(f_j) > 0$ although the point q is not singular for the reduced germ of $f_1 \cdots f_r = 0$. This may happen when f_i and f_j have a common factor which is not a common factor of the rest of generators. This is a subtle difference with respect to the case of pencils treated in [Alb04]. ■

Our next result deals with the satellite points not already in K'. Notice that these missing satellite points will not lie on any generator, otherwise they would belong to the points supporting the exceptional part of the minimal log-resolution divisor of $(f_1 \cdots f_r)$.

Proposition 3.12. *Let G' be a divisor as above. Let $q \notin K'$ be a satellite point proximate to $p, p' \in K'$. Then, q is in K if and only if for each generator f_i either $v_p(f_i) > v_p$ or $v_{p'}(f_i) > v_{p'}$.*

Proof. Let us start by proving the converse implication. We know that $q \notin f_j$ for any $j = 1, \ldots, r$, otherwise q would be in K'. We want to see that $e_q = v_q - v_p - v_{p'} > 0$. Then, $v_q = \min_j\{v_q(f_j)\} = \min_j\{v_p(f_j) + v_{p'}(f_j)\}$ and the last equality is true because $e_q(f_j) = 0$. By hypothesis, $v_p(f_j) + v_{p'}(f_j) > v_p + v_{p'}$, for any j, hence $v_q > v_p + v_{p'}$ as we wanted.

For the other implication, let us assume the contrary, that is, there exists a generator f_i such that $v_p(f_i) = v_p$ and $v_{p'}(f_i) = v_{p'}$. We know that $q \notin f_i$, otherwise it would be in K'. By definition, $v_q = \min_j\{v_q(f_j)\} = \min_j\{v_p(f_j) + v_{p'}(f_j)\} = v_p(f_i) + v_{p'}(f_i) = v_p + v_{p'}$, implying that $e_q = 0$, which is a contradiction with the fact that $q \in K$ $\square$

The algorithm

In this subsection we go back to our original setup, so let $\mathfrak{a} = (a_1, \ldots, a_r) \subseteq \mathcal{O}_{X,\mathbf{0}}$ be an ideal that admits a decomposition $\mathfrak{a} = (f) \cdot \mathfrak{a}'$, where $f \in \mathcal{O}_{X,\mathbf{0}}$ is the greatest common divisor of the generators of $\mathfrak{a}$ and $\mathfrak{a}' = (f_1, \ldots, f_r)$ is $\mathfrak{m}$-primary. With all the technical results stated above and the relation between the minimal log-resolutions of $\mathfrak{a}, \mathfrak{a}'$ and (f), see Remark 3.1, we present the algorithm.

Algorithm 3.13. *(Minimal log-resolution of an ideal)*
Input: *An ideal $\mathfrak{a} = (a_1, \ldots, a_r) \subseteq \mathcal{O}_{X,\mathbf{0}}$.*
Output: *The log-resolution divisor $F_\pi = \sum_{p \in K} N_p E_p + \sum_{i=1}^{s} M_i S_i$ of $\mathfrak{a}$.*

1. *Find $f = \gcd(a_1, \ldots, a_r)$ and set $a_i = f f_i$.*

2. *Find the minimal log-resolution $\overline{\pi} : (\overline{X}, E) \longrightarrow (X, \mathbf{0})$ of $(f \cdot f_1 \cdots f_r)$. Let $\overline{D}_i := \operatorname{Div}(\overline{\pi}^* f_i)_{\text{exc}} = \sum_{p \in \overline{K}} v_p(f_i) E_p$. Compute $v_p = \min_i\{v_p(f_i)\}$ for $p \in \overline{K}$ and define $\overline{G} = \sum_{p \in \overline{K}} v_p E_p$. Set also $\overline{D} := \operatorname{Div}(\overline{\pi}^* f)$.*

3. *Define $G' = \sum_{p \in K'} v_p E_p$ from $\overline{G}$ by blowing-up, if necessary, the missing free points using Proposition 3.11. Define D' by pulling-back $\overline{D}$.*

4. *Define $G'' = \sum_{p \in K''} v_p E_p$ from G' by blowing-up, if necessary, the missing satellite points using Proposition 3.12. Define D'' by pulling-back D'.*

5. *Compute, recursively on the order of neighborhood p belongs to, the multiplicities $e_p = v_p - \sum_{p\to q} v_q$. Define $G = \sum_{p\in K} v_p E_p$ from G'' by blowing-down the points such that $e_p = 0$ and $e_p(D'') = 0$. Define D by pulling-back D''.*

6. *Return $F_\pi := G + D$.*

The next result proves the correctness of Algorithm 3.13.

Theorem 3.14. *Algorithm 3.13 computes $F_\pi = \sum_{p\in K} N_p E_p + \sum_{i=1}^{s} M_i S_i$, the minimal log-resolution divisor of the ideal $\mathfrak{a}$.*

Proof. Since the divisor $\overline{G}$ from step 2 fulfills the hypothesis of Proposition 3.11 and Proposition 3.12, we can use them to blow-up the remaining points.

After step 4 all remaining points to get F_π have been blown-up. Indeed, if we had to blow-up a missing point in the first neighborhood of a point already in K', it would have to be free as we have blown-up all the missing satellites in the last step. This free point would have to be on a generator, by Lemma Lemma 3.10, and it would have to be after one of the new satellite points, otherwise it would have been blown-up in the fourth step. But that is impossible because the new satellite points cannot be on a generator, and hence, neither can do any of its successors.

By Corollary 3.9, after blowing-down the points p in K'' such that $e_p = 0$ and $e_p(D) = 0$, the set of points parameterizing the resulting divisor is inside K and, since no point remains to be blown-up, it must be equal to K. □

As a corollary Theorem 3.14 we obtain the following generalization of the classical result that the equisingularity class of a plane curve is determined by the equisingularity class and the intersection multiplicity of its branches.

Corollary 3.15. *Given an ideal $\mathfrak{a} = (a_1, \ldots, a_r) \subseteq \mathcal{O}_{X,0}$, the equisingular class of a general element of $\mathfrak{a}$ is determined by the equisingularity class of each generator a_i, and the intersection multiplicities of every pair of branches from different generators.*

Proof. General elements in $\mathfrak{a}$ have the same minimal log-resolution divisor as $\mathfrak{a}$. By Theorem 3.14, the relative position of the infinitely near points in F_π are completely determined by the equisingularity class of each a_i and the intersection multiplicities between any pair of branches of different generators. □

Example 3.1. Consider the ideal

$$\mathfrak{a} = (a_1, a_2, a_3) = \big((y^5 + x^7)^2 + y^{10}x, x^8(y^3 + x^5), y^8(y^2 - x^3)\big) \subseteq \mathbb{C}\{x, y\}. \tag{3.16}$$

The steps of Algorithm 3.13 are performed as follows:

1. We have that $g = \gcd(a_1, a_2, a_3) = 1$, so the ideal is $\mathfrak{m}$-primary. Then, $f_i := a_i$.

2. The minimal log-resolution of $(a_1 a_2 a_3)$ is described by means of the proximity matrix

$$P_{\overline{K}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}. \tag{3.17}$$

The multiplicities $\{v_p(a_i)\}_{p\in\overline{K}}, i = 1,2,3$ are the following:

$$\begin{aligned} v(a_1) &= [10\ 10\ 14\ 14\ 28\ 40\ 70\ 72\ 74\ 75\ 150\ 151\ 28\ 42\ 42]^t, \\ v(a_2) &= [11\ 19\ 13\ 13\ 25\ 36\ 61\ 61\ 61\ 61\ 122\ 122\ 25\ 39\ 40]^t, \\ v(a_3) &= [10\ 10\ 19\ 27\ 30\ 40\ 70\ 70\ 70\ 70\ 140\ 140\ 31\ 49\ 49]^t. \end{aligned} \tag{3.18}$$

Therefore, $v_p = \min_i\{v_p(f_i)\}$ for $p \in \overline{K}$ is:

$$v = [10\ 10\ 13\ 13\ 25\ 36\ 61\ 61\ 61\ 61\ 122\ 122\ 25\ 39\ 40]^t. \tag{3.19}$$

The corresponding divisor $\overline{G} = \sum_{p\in\overline{K}} v_p E_p$ is represented using the dual graph in Figure 3.1.

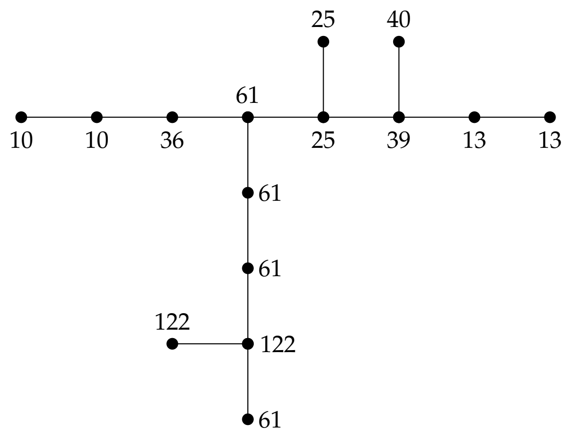

Figure 3.1: Dual graph of $\overline{G}$ from Algorithm 3.13 in Example 3.1.

3. There are two missing free points that we have to blow-up. The infinitely near points K' parameterizing the divisor G' are given by the proximity matrix

$$P_{K'} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}. \tag{3.20}$$

The updated multiplicities $v(a_i), i = 1, 2, 3$ are

$$\begin{aligned} v(a_1) &= [10\ 10\ 14\ 14\ 28\ 40\ 70\ 72\ 74\ 75\ 150\ 151\ 28\ 42\ 42\ 42\ 42]^t, \\ v(a_2) &= [11\ 19\ 13\ 13\ 25\ 36\ 61\ 61\ 61\ 61\ 122\ 122\ 25\ 39\ 40\ 41\ 42]^t, \\ v(a_3) &= [10\ 10\ 19\ 27\ 30\ 40\ 70\ 70\ 70\ 70\ 140\ 140\ 31\ 49\ 49\ 49\ 49]^t. \end{aligned} \tag{3.21}$$

Thus, we have

$$v = [10\ 10\ 13\ 13\ 25\ 36\ 61\ 61\ 61\ 61\ 122\ 122\ 25\ 39\ 40\ 41\ 42]^t, \tag{3.22}$$

and the divisor G' is represented by the dual graph in Figure 3.2.

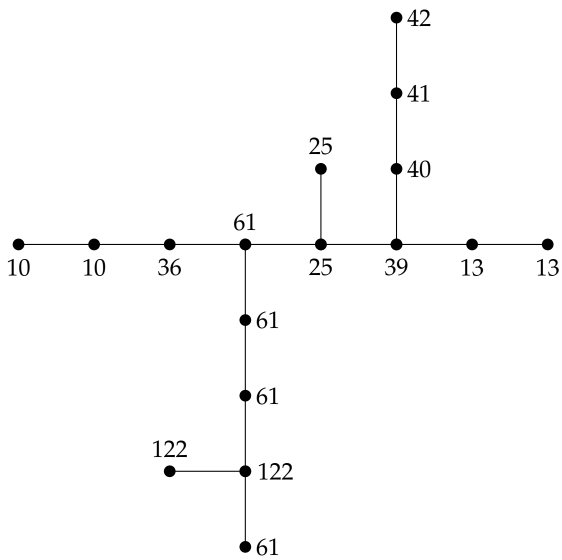

Figure 3.2: Dual graph of G' from Algorithm 3.13 in Example 3.1.

4. There are four missing satellite base points. The points in K'' are given by the proximity matrix

$$P_{K''} = \begin{bmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
-1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
-1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&-1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
-1&0&-1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
-1&0&0&0&-1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&-1&-1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&-1&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&-1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&-1&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&-1&-1&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&-1&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&-1&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&-1&0&-1&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&-1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&1&0&0&0&0\\
-1&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&1&0&0\\
-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&1&0\\
-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&1
\end{bmatrix}. \tag{3.23}$$

The updated multiplicities are:

$$\begin{aligned}
v(a_1) &= [10\ 10\ 14\ 14\ 28\ 40\ 70\ 72\ 74\ 75\ 150\ 151\ 28\ 42\ 42\ 42\ 42\ 50\ 60\ 70\ 80]^t,\\
v(a_2) &= [11\ 19\ 13\ 13\ 25\ 36\ 61\ 61\ 61\ 61\ 122\ 122\ 25\ 39\ 40\ 41\ 42\ 47\ 58\ 69\ 80]^t,\\
v(a_3) &= [10\ 10\ 19\ 27\ 30\ 40\ 70\ 70\ 70\ 70\ 140\ 140\ 31\ 49\ 49\ 49\ 49\ 50\ 60\ 70\ 80]^t,\\
v &= [10\ 10\ 13\ 13\ 25\ 36\ 61\ 61\ 61\ 61\ 122\ 122\ 25\ 39\ 40\ 41\ 42\ 47\ 58\ 69\ 80]^t.
\end{aligned} \tag{3.24}$$

and the corresponding divisor G'' is represented by the dual graph in Figure 3.3.

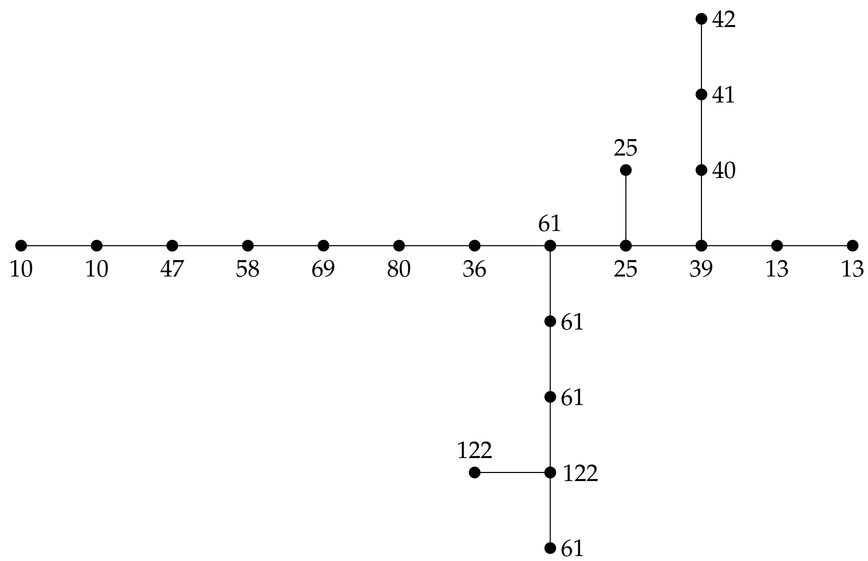

Figure 3.3: Dual graph of G'' from Algorithm 3.13 in Example 3.1.

5. Using the base change formula $e^t = P_{K''} v^t$ we get

$$e = [10\ 0\ 3\ 0\ 2\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1]^t. \tag{3.25}$$

Thus, blowing-down the points with multiplicity zero, we finally obtain the minimal log-resolution divisor $F_\pi = \sum_{p \in K} N_p E_p$. The points in K are represented by the proximity matrix

$$P_K = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}, \tag{3.26}$$

and

$$N = [10\ 13\ 25\ 36\ 39\ 40\ 41\ 42\ 47\ 58\ 69\ 80]^t. \tag{3.27}$$

Equivalently, the dual graph of F_π is in Figure 3.4.

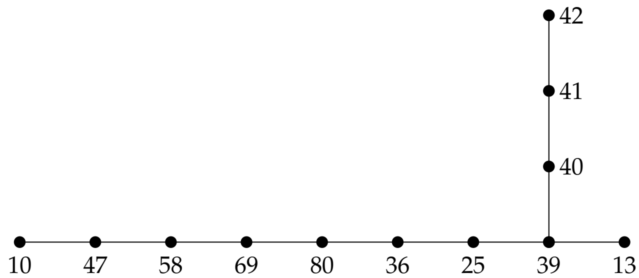

Figure 3.4: Dual graph of F_π from Algorithm 3.13 in Example 3.1.

■

3.3 *Newton-Puiseux revisited*

Taking a closer look at Algorithm 3.13, we see that all the steps can be effectively computed once we have a precise description of the minimal log-resolution divisor from step 2. The aim of this section is to provide an algorithm that solves the following problem:

Given a set of elements $f_1, \ldots, f_r \in \mathbb{C}\{x,y\}$, provide a method to compute the minimal log-resolution $\pi : (X', E) \longrightarrow (\mathbb{C}^2, \mathbf{0})$ of $(f = f_1 \cdots f_r)$ with log-resolution divisor $F_\pi = \sum_{p \in K} N_p E_p + \sum_{i=1}^{s} M_i S_i$ together with the divisors $D_i := \mathrm{Div}(\pi^* f_i) = \sum_{p \in K} v_p(f_i) E_p + \sum_{j=1}^{s_i} \alpha_{i,j} S_{i,j}$.

We point out that in the case that f is reduced, we can compute the minimal log-resolution using the Newton-Puiseux algorithm and Enriques' Theorem [Cas00, §1, §5.5]. However, we are in a more general situation that requires some extra work. The Puiseux Factorization Theorem [Cas00, §1.5] states that any $g \in \mathbb{C}\{x,y\}$ can be decomposed as

$$g(x,y) = u x^{\alpha_0} g_1^{\alpha_1} \cdots g_\ell^{\alpha_\ell} = u x^{\alpha_0} \prod_{i=1}^{\ell} \prod_{j=1}^{\nu_i} (y - \sigma_i^j(s_i))^{\alpha_i}, \quad \alpha_1, \ldots, \alpha_l \in \mathbb{N} \tag{3.28}$$

where $u \in \mathbb{C}\{x,y\}$ is a unit, $g_1,\ldots,g_\ell \in \mathbb{C}\{x,y\}$ are irreducible, $s_i \in \mathbb{C}\langle\langle x\rangle\rangle$ are Puiseux series such that $g_i(x,s_i(x)) = 0$, $\nu_i = \mathrm{ord}_y(g_i(0,y))$, and σ_i^j is the automorphism of $\mathbb{C}((x^{1/\nu_i}))$ generated by $x^{1/\nu_i} \mapsto e^{2\pi\sqrt{-1}j/\nu_i}x^{1/\nu_i}$.

From the above factorization one can compute all the minimal log-resolution data. It is a classical result, see [Cas00, §5.5], that the Puiseux series $s_i, i = 1,\ldots,\ell$ completely determine the minimal log-resolution of an element $g \in \mathbb{C}\{x,y\}$. In order to compute the values $v_p(g)$ for any point p, we can use the fact that v_p are valuations, thus

$$v_p(g) = \alpha_0 v_p(x) + \alpha_1 v_p(g_1) + \cdots + \alpha_\ell v_p(g_\ell). \tag{3.29}$$

In addition, the values $v_p(x), v_p(g_i), i = 1,\ldots,\ell$, can also be deduced from their associated Puiseux series s_i of (g_i). However, notice that the algebraic multiplicities α_i play their role in Equation (3.29).

The Newton-Puiseux algorithm, that traditionally has been used to obtain the Puiseux decompositions, only works for reduced elements. This means that you cannot recover the algebraic multiplicities of the Puiseux series in Equation (3.28). Another problem that arises when applying the Newton-Puiseux algorithm to a product $f = f_1\cdots f_r$ is that you cannot find which factor f_i contains each resulting Puiseux series.

To overcome such inconvenients, we will present a modified version of the Newton-Puiseux algorithm that, given a set of elements $f_1,\ldots,f_r \in \mathbb{C}\{x,y\}$ not necessarily reduced or irreducible, will compute the Puiseux decomposition of the product $f = f_1\cdots f_r$, that is, the Puiseux series of f together with their algebraic multiplicities in each of the factors $f_1,\ldots,f_r$.

The Newton-Puiseux algorithm is obviously restricted to compute a partial sum of each Puiseux series in the decomposition (3.28) as the series are potentially infinite. Thus, the algorithm computes enough terms of each series so they do not share terms from a certain degree onward. In this situation we will say that the series have been pair-wise *separated*. In particular, this means that a partial sum of Puiseux series s might be enough to separate s inside a factor, but not inside the whole product $f = f_1\cdots f_r$. Hence, applying the Newton-Puiseux algorithm to the factors $f_1,\ldots,f_r$ does not provide as much information as applying the Newton-Puiseux algorithm to the product.

Similarly, if one obtains just the Puiseux series of the product it is not possible to recover the Puiseux decomposition of each factor. The modification of the Newton-Puiseux algorithm that we will present provides all the information needed to recover both the decomposition of each factors and the decomposition of the whole product at the same time. One of the key ingredients is the *square-free factorization*.

Definition 3.16. Let R be a unique factorization domain. The *square-free factorization* of an element $h \in R[[x]]$ is

$$h = h_1 h_2^2 \cdots h_n^n, \tag{3.30}$$

such that $h_i \in R[[x]], i = 1,\ldots,n$ are reduced, pair-wise coprime elements, and h_n is a non-unit.

Notice that some of the $h_i, i = 1,\ldots,n-1$ can be units. The non-unit factors in Equation (3.30) will be called *square-free factors* and are unique up to multiplication by a unit. The square-free factorization can be computed efficiently, see for instance [Yun76].

We will not explain all the details for the traditional Newton-Puiseux algorithm, for that we refer the reader to [Casoo, §1.5]. We will just recall that it is an iterative algorithm that at the i–th step computes the i–th term of one of the Puiseux series s. The first term of $s(x) =: s^{(0)}(x_0)$ is computed from $f(x,y) =: f^{(0)}(x_0, y_0)$, and the i–th term of s is computed as the first term of $s^{(i)}(x_i) \in \mathbb{C}\langle\langle x_i \rangle\rangle$ from $f^{(i)}(x_i, y_i) \in \mathbb{C}\{x_i, y_i\}$ which are defined recursively from $s^{(i-1)}$ and $f^{(i-1)}(x_{i-1}, y_{i-1})$ by means of a change of variables.

The basic idea behind our new algorithm is to apply the traditional Newton-Puiseux algorithm to the reduced part of f, $\bar{f}$, while the square-free factors of each $f_i, i = 1, \ldots, r$ are transformed using the changes of variables given by $\bar{f}$. The Newton-Puiseux algorithm applied on $\bar{f}$ will tell when all the branches have been separated, i.e. the stopping condition. The square-free factors will encode, at the end, the algebraic multiplicities of the resulting Puiseux series in each factor.

The modified Newton-Puiseux algorithm works as follows:

- Compute the element $f = f_1 \cdots f_r$ and $\bar{f} = f / \gcd(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$. Define $x_0 := x, y_0 := y, f^{(0)} := \bar{f}$, and
$$S^{(0)} := \{h_{j,k} \in \mathbb{C}\{x_0, y_0\} \mid h_{j,k} \text{ square-free factor of } f_k, k = 1, \ldots, r \text{ with multiplicity } j \in \mathbb{N}\}$$
- **Step (i)**: The i–th iteration runs as in the traditional algorithm and we compute x_{i+1}, y_{i+1} and $f^{(i+1)}$. In addition, we compute $S^{(i+1)}$ from $S^{(i)}$ in the following way:
$$S^{(i+1)} = \{x_{i+1}^{-\beta_{i,j,k}} h_{j,k}^{(i)}(x_{i+1}, y_{i+1}) \in \mathbb{C}\{x_{i+1}, y_{i+1}\} \mid x_{i+1}^{\beta_{i,j,k}+1} \nmid h_{j,k}^{(i)}(x_{i+1}, y_{i+1}), \; h_{j,k}^{(i+1)} \text{ non-unit}, h_{j,k}^{(i)} \in S^{(i)}\}$$
- The algorithm ends at the same step the traditional Newton-Puiseux algorithm ends for the reduced part $\bar{f}$.

In order to prove the correctness of this modification we will need the following results.

Lemma 3.17 ([Casoo, p. 1.6.3]). *For any $j > i \geq 0$, the multiplicity of $s^{(i)}$ as Puiseux series of $f^{(i)}$ equals the multiplicity of $s^{(j)}$ as Puiseux series of $f^{(j)}$.*

In the current context the following lemma follows from the definitions.

Lemma 3.18. *Two elements of $\mathbb{C}\{x, y\}$ are coprime if and only if they share no Puiseux series and no x factor.*

Proposition 3.19. *The set $S^{(i)}$ contains the square-free factors of $f_k^{(i)}$ for any $i \geq 0$ and any $k = 1, \ldots, r$.*

Proof. By induction on $i \geq 0$. By construction, $S^{(0)}$ contains the square-free factors of $f_k^{(0)} := f^k$, for $k = 1, \ldots, r$. Assume now that $S^{(i)}$ contains the square-free factors of $f_k^{(i)}$.

If two elements of $h_{n,k}^{(i+1)}, h_{m,k}^{(i+1)}$ are not coprime, they would share a Puiseux series or an x factor, by Lemma 3.18. The x factor is not possible by definition of $S^{(i+1)}$. If they share a Puiseux series $s^{(i+1)}$, $s^{(i)}$ would be a series of $h^{(i)}_{n,k}$ and $h^{(i)}_{m,k}$, contradicting the induction hypothesis. Since $h^{(i)}_{j,k}$ is reduced so is $h^{(i+1)}_{j,k}$, by Lemma 3.17. Since Equation (3.30) still holds after applying the change of variables two both sides, the result follows. □

Proposition 3.20. *Assume $s \in \mathbb{C}\langle\langle x\rangle\rangle$ has been separated from the rest of the series of $\bar{f}$ at the i–th step of the algorithm. Then, s is a Puiseux series of $f_k \in \mathbb{C}\{x,y\}$ with algebraic multiplicity $j \in \mathbb{N}$ if and only if $h^{(i)}_{j,k} \in S^{(i)}$.*

Proof. For the direct implication, assume that s is a Puiseux series of f_k with multiplicity $j \in \mathbb{N}$. Then, s is a Puiseux series of $h^{(0)}_{j,k} \in S^{(0)}$ and no other square-free factor, by Lemma 3.18. Now, by Lemma 3.17, $s^{(i)}$ is a root of $h^{(i)}_{j,k}$ and it belongs to $S^{(i)}$ because it is a non-unit. For the converse, since s has been separated, $f^{(i)}$ has no other Puiseux series other than $s^{(i)}$ and its conjugates. By Proposition 3.19, there must be a unique $h^{(i)}_{j,k}$ square-free factor of $f^{(i)}_k$ in $S^{(i)}$. Finally, by Lemma 3.17, if the algebraic multiplicity of $s^{(i)}$ is $j > 0$ in $f^{(i)}_k$, so is the algebraic multiplicity of s in f_k. □

It follows from Proposition 3.20 that, when the algorithm stops at the i–th step after s has been separated, the set $S^{(i)}$ contains the information about the factors and the algebraic multiplicities of the Puiseux series s.

4 MONOMIAL GENERATORS OF COMPLETE PLANAR IDEALS

This section contains the second algorithms presented in this thesis together with the proof of its correctness. Given a proper birational morphism $\pi : (X', E) \longrightarrow (X, \mathbf{0})$ between smooth complex surfaces, this algorithm computes a set of generators for the push-forward of the ideal sheaf $\mathcal{O}_{X'}(-D)$ for D a divisor in X' with support on E. Furthermore, the generators of these ideals are monomials in a set of maximal contact elements of the morphism π. At the end of this sections we give some applications of this algorithm to three different problems: the computation of the integral closure, multiplier ideals, and ideals defined by curve valuations. The results of this sections are joint work with Alberich-Carramiñana and Àlvarez Montaner and can be found in [AMB20].

4.1 *An algorithm to compute H_D*

Let $(X, \mathbf{0})$ be a germ of smooth complex surface and let $\pi : (X', E) \longrightarrow (X, \mathbf{0})$ be a proper birational morphism. In this section we present an algorithm which computes a set of generators for the $\mathfrak{m}$-primary ideal H_D, see Section 2.3, for any divisor D with exceptional support in X', i.e. $D \in \Lambda_\pi$.

We start by briefly describing the main ideas behind Algorithm 4.3. We start with a divisor $D \in \Lambda_\pi$ which can be assumed to be antinef. After Theorem 2.7, D decomposes into simple divisors $D = \rho_{q_1} B_{q_1} + \cdots + \rho_{q_r} B_{q_r}$ with all $\rho_{q_i} > 0$. For each simple divisor B_{q_i}, $q_i \neq \mathbf{0}$, we compute the antinef closure of $B_{q_i} + E_\mathbf{0}$ which we denote $\widehat{D}_i$. This new divisor describes a particular *adjacent ideal* $H_{\widehat{D}_i}$ *below* $H_{B_{q_i}}$, i.e. an ideal $H_{\widehat{D}_i} \subsetneq H_{B_{q_i}}$ such that $\dim_\mathbb{C} H_{B_{q_i}} / H_{\widehat{D}_i} = 1$. Next, we find, among the set of maximal contact elements of π, see Definition 2.13, an element $f \in \mathcal{O}_{X,\mathbf{0}}$ belonging to $H_{B_{q_i}}$ but not to $H_{\widehat{D}_i}$. Now, $\widehat{D}_i$ is no longer simple but has smaller support than B_{q_i} in the total transform basis. Therefore we may repeat the same procedure with $D := B_{q_j}, j < i$ until $D = B_{\mathbf{0}} := \mathrm{Div}(\pi^*\mathfrak{m})$.

The first part of the algorithm can be represented by a tree where each vertex is an antinef divisor. The leaves of the tree are all $B_{\mathbf{0}} = \mathrm{Div}(\pi^*\mathfrak{m})$ and the root is the initial divisor D. The second part of the algorithm traverses the tree bottom-up computing, in each node, the ideal associated to the divisor of that node. Using the notations from the paragraph above, given any node in the tree with divisor D, the ideal H_D is computed multiplying the ideals in child nodes and adding the element f to the resulting generators.

Before giving a more explicit description of the algorithm, let us first state two technical results. The first one presents some properties of adjacent ideals based on results obtained by Fernández-Sánchez, see [Fer03; Fer05; Fer06], in the study of sandwiched singularities and the Nash conjecture of arcs on these singularities.

Proposition 4.1. *Let H_D be the complete ideal defined by an antinef divisor $D \in \Lambda_\pi$. Consider $\widehat{D}$ the antinef closure of $D + E_{\mathbf{0}}$, obtained from $D + E_{\mathbf{0}}$ by unloading on a given[1] subset of points $T \subseteq K$. Then, $H_{\widehat{D}} \subsetneq H_D$ are adjacent ideals if and only if $\rho_{\mathbf{0}}(D) = 0$.*

Furthermore, if $H_{\widehat{D}} \subsetneq H_D$ are adjacent then,

1. *$\sum_{p\in T} E_p$ is the connected component of $\sum_{p\in K, \rho_p(D)=0} E_p$ containing $E_{\mathbf{0}}$;*
2. *$e_{\mathbf{0}}(\widehat{D}) = e_{\mathbf{0}}(D) + 1$, and $e_p(D) - 1 \le e_p(\widehat{D}) \le e_p(D)$ for any $p \in K$, $p \neq \mathbf{0}$. Moreover $\rho_{\mathbf{0}}(\widehat{D}) > 0$.*
3. *If $p \in K \setminus T$ and p is proximate to some point in T then, $e_p(\widehat{D}) = e_p(D) - 1$.*

Proof. Consider $r := \rho_{\mathbf{0}}(D) + 1$ new free points $p_1, \dots, p_r$ lying on $E_{\mathbf{0}}$. Let $\pi' : (Y', E') \longrightarrow (X, \mathbf{0})$ be the composition of π with the sequence of blow-ups of the points $p_1, \dots, p_r$. Denote by $\overline{G} \in \mathrm{Div}(Y')$ the pullback of any $G \in \mathrm{Div}(X')$. For simplicity, denote the strict and the total transform basis by $\{E_p\}_{p\in K'}$ and $\{\overline{E}_p\}_{p\in K'}$ respectively in the lattice $\Lambda_{\pi'}$.

Clearly, both $\overline{D} + E_{\mathbf{0}}$ and $\overline{D} + E_{p_1} + \cdots + E_{p_r}$ are not antinef, whereas $\overline{D} + E_{p_1} + \cdots + E_{p_i}$ are antinef for all $1 \le i < r$. Moreover, when applying the unloading procedure described in Algorithm 2.6, we find that the antinef closures of $\overline{D} + E_{\mathbf{0}}$ and $\overline{D} + E_{p_1} + \cdots + E_{p_r}$ are the same, say $\widehat{D'}$, and $e_{p_i}(\widehat{D'}) = 0$ for all $1 \le i \le r$. Indeed, the first step of the unloading procedure applied to $\overline{D} + E_{\mathbf{0}}$ or $\overline{D} + E_{p_1} + \cdots + E_{p_r}$ gives the same divisor $\overline{D} + E_{\mathbf{0}} + E_{p_1} + \cdots + E_{p_r}$. Furthermore, $\widehat{D'}$ is the pullback of the antinef closure $\widehat{D}$ of $D + E_{\mathbf{0}}$ in $\mathrm{Div}(X')$ and hence they define the same complete ideal $H_{\widehat{D'}} = H_{\widehat{D}}$.

Now, from [Cas00, §4.7], the codimension of a complete ideal H_G defined by a divisor $G \in \Lambda_\pi$ satisfies

$$\dim \mathcal{O}_{X,\mathbf{0}}/H_G = \sum_{p\in K} \frac{e_p(\widetilde{G})(e_p(\widetilde{G})+1)}{2} \le \sum_{p\in K} \frac{e_p(G)(e_p(G)+1)}{2}, \tag{4.1}$$

where $\widetilde{G}$ is the antinef closure of G. Hence,

$$H_{\widehat{D}} = H_{\overline{D}+E_{p_1}+\cdots+E_{p_r}} \subsetneq H_{\overline{D}+E_{p_1}+\cdots+E_{p_{r-1}}} \subsetneq \dots \subsetneq H_{\overline{D}+E_{p_1}} \subsetneq H_D \tag{4.2}$$

1 T is the set of points $p \in K$ that parameterize the support of $\widehat{D} - (D + E_{\mathbf{0}})$. Notice that T may be empty.

is a chain of adjacent complete ideals, giving $\dim_{\mathbb{C}} H_{B_{q_i}}/H_{\widehat{D}_i} = r = \rho_{\mathbf{0}}(D)+1$. Therefore, $H_{\widehat{D}} \subsetneq H_D$ are adjacent if and only if $\rho_{\mathbf{0}}(D) = 0$.

Finally, from [AF07, Prop. 2.1] and [Fer03, Cor. 4.6] claim $i)$ follows. Claim $ii)$ and $iii)$ are consequences of [Fer05, Lemma 4.2] and [Fer06, Lemma 2.2], see also [AF10, Prop. 3.1]. □

Remark 4.1. Although there may be multiple adjacent ideals $H_{\widehat{D}}$ to a fixed ideal H_D, the adjacent ideal considered in Proposition 4.1 is unique with the property that $e_{\mathbf{0}}(\widehat{D}) = e_{\mathbf{0}}(D)+1$, so we will refer to it as *the adjacent ideal* to H_D. This property turns out to be crucial for the finiteness of the algorithm. ■

Remark 4.2. Notice that if D is a simple divisor such that $H_D \neq \mathfrak{m}$, then $\rho_{\mathbf{0}}(D) = 0$ and $H_{\widehat{D}}$ is always adjacent. Furthermore, the unloading step is always required, i.e. T is always non empty in this case. ■

Lemma 4.2. *Let H_D be the complete ideal defined by an antinef divisor $D \in \Lambda_\pi$. The divisor $D+\overline{E}_{\mathbf{0}}$ is antinef and $H_{D+\overline{E}_{\mathbf{0}}} = \mathfrak{m}H_D$.*

Proof. Clearly $\mathfrak{m} = \{f \in \mathcal{O}_{X,\mathbf{0}} \mid e_{\mathbf{0}}(f) \geq 1\}$. Thus, $\mathfrak{m} = H_{\overline{E}_{\mathbf{0}}}$, and since $\overline{E}_{\mathbf{0}} = B_{\mathbf{0}}$ it is antinef and the result follows by the correspondence between antinef divisor and complete ideals in section Section 2.3. □

The following algorithm computes generators for the ideals $H_D, D \in \Lambda_\pi$ that are monomial in any set of maximal contact elements of the morphism π. This algorithms is inspired in the work [Cas98] of Casas-Alvero.

Algorithm 4.3. *(Generators for H_D)*
Input: *A proper birational morphism $\pi : (X', E) \to (X, \mathbf{0})$ and an antinef divisor $D \in \Lambda_\pi$.*
Output: *Generators for the ideal H_D.*

1. *Compute and fix a set of maximal contact elements $\{f_i\}$ with $i \in I$ of π.*
2. *Set $D^{(0)} := D$ and proceed from step (0.1).*

Step (i):

- *i.1 Decompose $D^{(i)}$ into $d_i := \#\{p \in K \mid \rho_p(D^{(i)}) > 0\}$ simple divisors.*
- *i.2 For each $j = 1, \ldots, d_i$, consider $q_j \in \{p \in K \mid \rho_p(D^{(i)}) > 0\}$ and assume $B_{q_j} = \sum_{p \in K} e_p \overline{E}_p$.*
  - *i.j.1* ***Stop at the maximal ideal:*** *If $B_{q_j} = B_{\mathbf{0}} := Div(\pi^*\mathfrak{m})$, then set $H_{B_{q_j}} = (f_{i_0}, f_{i_1})$ for $i_0, i_1 \in I$ such that they are smooth and transverse at O, then stop. Otherwise, proceed from i.j.2.*
  - *i.j.2* ***Compute the adjacent ideal to $H_{B_{q_j}}$:*** *Perform unloading on the divisor $B_{q_j} + E_{\mathbf{0}}$ to get its antinef closure $\widehat{D}_j$.*
  - *i.j.3* ***Select a maximal contact element in $H_{B_{q_j}} \setminus H_{\widehat{D}_j}$*** *: Let $p \in K$ be the last free point such that $e_p \neq 0$. Take $\tau_j \in I$ such that $e_p(f_{\tau_j}) = 1$ and $e_{\mathbf{0}}(f_{\tau_j}) \leq e_{\mathbf{0}}$. Define the integer $e_j := e_{\mathbf{0}}(B_{q_j})/e_{\mathbf{0}}(f_{\tau_j})$.*
  - *i.j.4* ***Recursive step****: Assume that $H_{\widehat{D}_j}$ has been computed after performing step $(i+1)$ with $D^{(i+1)} := \widehat{D}_j$.*

i.j.5 ***Set:***

$$H_{B_{q_j}} = \left(f_{\tau_j}^{e_j}\right) + H_{\widehat{D}_j}.$$

i.3 ***Apply Zariski's factorization theorem:*** *Compute the product*

$$H_{D^{(i)}} = \prod_{j=1}^{d_i} H_{B_{q_j}}^{\rho_{q_j}},$$

giving generators $h_1, \dots, h_{s_i}$.

i.4 ***Set, using Nakayama's Lemma:***

$$H_{D^{(i)}} = \Big(h_k \mid \pi^* h_k \notin \mathcal{O}_{X'}\big(-D^{(i)} - \overline{E}_{\mathbf{0}}\big),\ k = 1, \dots, s_i\Big)\mathcal{O}_{X,\mathbf{0}}.$$

3. ***Return:*** $H_D = H_{D^{(0)}}$.

Remark 4.3. In order to clarify some steps of the algorithm we point out the following:

· At step 1 of Algorithm 4.3 a set of maximal contact elements $\{f_i\}_{i \in I}$ of π is fixed. The specific choice of the germs $f_i = 0$ nor of the equations f_i do not affect the output of the algorithm: the monomial expression remains the same for whatever choice, since the algorithm only uses the information of the equisingularity types of the maximal contact elements.

· At steps ($i.j.$1) and ($i.j.$3) of Algorithm 4.3 we have to choose maximal contact elements. These choices are not necessary unique as several maximal contact elements may fulfill the required conditions.

· Since the sheaf ideals $\mathcal{O}_{X'}(-D)$, with $D \in \mathrm{Div}(X')$, are defined by valuations, testing whether the pullback of an element f belongs to $\mathcal{O}_{X'}(-D)$ or not is only a matter of comparing the values $v_p(\mathrm{Div}(\pi^* f))$ and $v_p(D)$ for all $p \in K$.

· It is clear from Nakayama's Lemma and Lemma 4.2 that a set of elements of $\mathcal{O}_{X,\mathbf{0}}$ is a system of generators of H_D if and only if its classes modulo $H_{D+\overline{E}_{\mathbf{0}}}$ are a system of generators of $H_D / H_{D+\overline{E}_{\mathbf{0}}}$ as $\mathbb{C}$-vector space. Equivalently, any element of $H_{D+\overline{E}_{\mathbf{0}}}$ is redundant in a system of generators of H_D.

■

Example 4.1. We will compute H_D for the divisor D and the morphism π from Example 2.1. Let us fix the set of maximal contact elements $f_0 = x, f_1 = y, f_2 = y^2 - x^3$. The steps of Algorithm 4.3 applied to $D = 4E_{\mathbf{0}} + 6E_{p_1} + 12E_{p_2} + 13E_{p_3} + 26E_{p_4}$ will be illustrated in Figure 4.1 by means of the tree-shaped graph in Figure 2.1.

Each vertex of the tree contains an antinef divisor. In this example, we use dual graphs to represent them. The root node contains the initial divisor D. Dashed arrows connect simple divisors B_{q_j} with its corresponding adjacent $\widehat{D}_j$ from step ($i.j.$2) of Algorithm 4.3. The maximal contact elements from step ($i.j.$3) that belong to $H_{B_{q_j}}$ but not to $H_{\widehat{D}_j}$ are represented next to dashed arrows. Solid arrows connect $\widehat{D}_j =: D^{(i+1)}$ with each of its irreducible components B_p, with $p \in K$. Finally, the weight $\rho_p^{(i)}$ of each divisor $B_p, p \in K$, in $\widehat{D}^{(i)}$ is written next to the solid arrows.

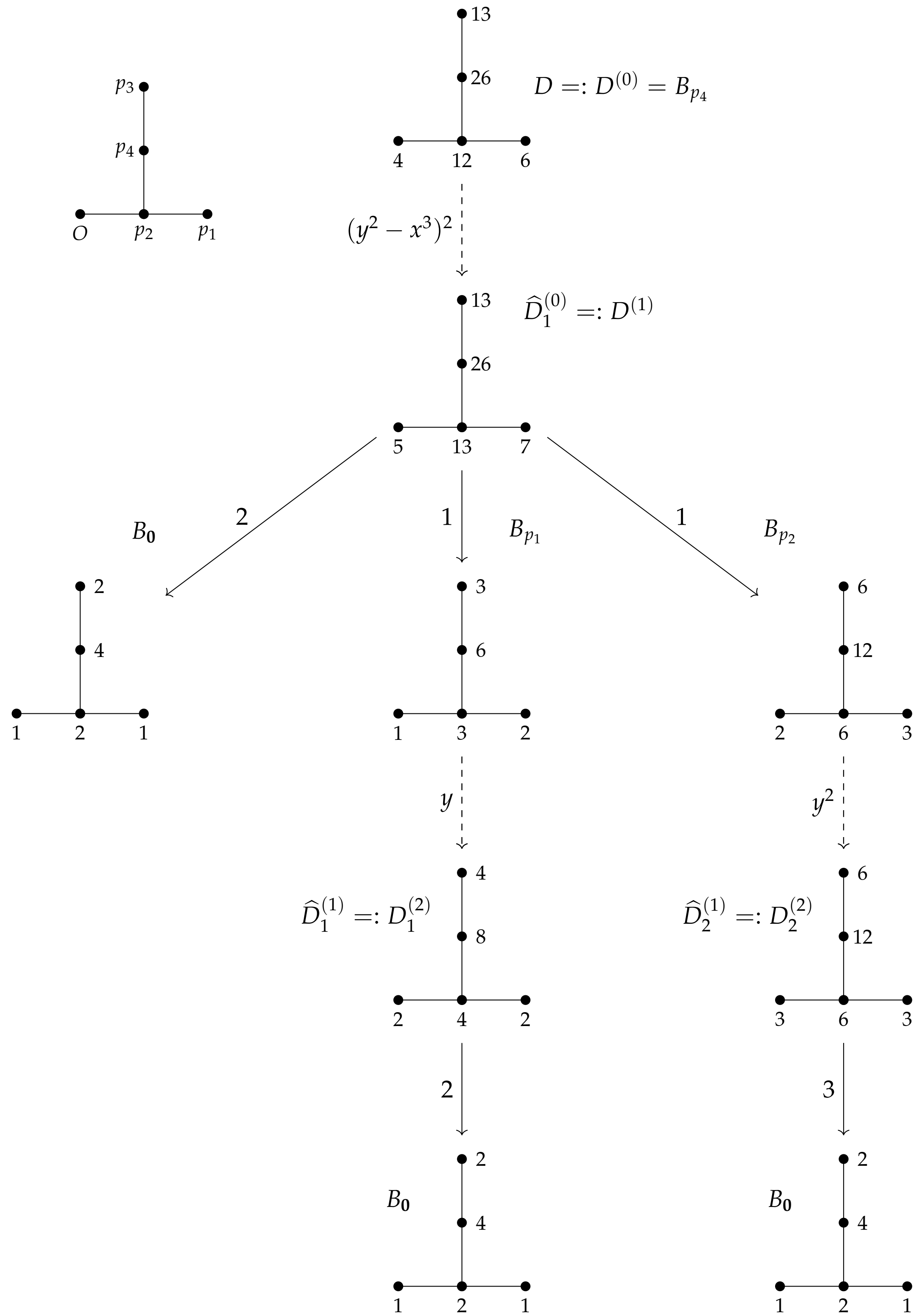

Figure 4.1: Tree of divisors from Algorithm 4.3 in Example 4.1.

We have added some extra indices to the divisors appearing in the algorithm to highlight at which step we encounter them. Hopefully it does not create any confusion since its meaning should be clear from the context. The generators of the ideals associated to the divisors in each intermediate step are then:

- $H_{B_0} = \mathfrak{m} = (x, y)$.
- $H_{B_{p_1}} = (y) + H_{D_1^{(2)}} = (y) + \mathfrak{m}^2 = (y, x^2, xy, y^2)$.
- $H_{B_{p_2}} = (y^2) + H_{D_2^{(2)}} = (y^2) + \mathfrak{m}^3 = (y^2, x^3, x^2y, xy^2, y^3)$.
- $H_{D^{(1)}} = B_{\mathbf{0}}^2 \cdot B_{p_1} \cdot B_{p_2} = (x,y)^2 \cdot (y, x^2, xy, y^2) \cdot (y^2, x^3, x^2y, xy^2, y^3)$
$= (x^7, \cancel{x^6y}, \cancel{x^5y^2}, \cancel{x^4y^3}, \ldots, x^5y, x^4y^2, \ldots, \cancel{x^3y^3}, \cancel{x^2y^4}, x^2y^3, xy^4, y^5)$
$= (x^7, x^5y, x^4y^2, x^2y^3, xy^4, y^5)$.
- $B_{p_4} = ((y^2-x^3)^2) + H_{D^{(1)}} = ((y^2-x^3)^2, x^7, x^5y, x^4y^2, x^2y^3, xy^4, y^5)$.
- $H_D := H_{D^{(0)}} = B_{p_4} = ((y^2-x^3)^2, x^7, x^5y, x^4y^2, x^2y^3, xy^4, \cancel{y^5})$.

The crossed out elements are those that are redundant by step (i.4) of Algorithm 4.3 and Lemma 4.2. Observe that, although many crossed out elements are actually multiple of other elements, step (i.4) and Lemma 4.2 allows us to remove y^5 which is not multiple of any other element.

Remark 4.4. As an outcome of the algorithm, we see that H_D admits the following monomial expression

$$H_D = (f_2^2, f_0^7, f_0^5 f_1, f_0^4 f_1^2, f_0^2 f_1^3, f_0 f_1^4), \tag{4.3}$$

in the set of maximal contact elements $f_0 = x, f_1 = y, f_2 = y^2 - x^3$ associated to π that we fixed in the beginning. It is worth remarking that we would get the same monomial expression for any other set of maximal contact elements chosen in the beginning.

However, we might get a different monomial expression depending on the maximal contact elements (or powers of) that we choose in step ($i.j$.3) of Algorithm 4.3. In this example, when choosing an element in $H_{B_{p_3}}$ that does not belong to $H_{\widehat{D}_2^{(1)}}$ we took $f_1^2 = y^2$, but we could also have chosen $f_2 = y^2 - x^3$. In the later case the final system of generators is

$$H_D = \left((y^2-x^3)^2, x^2y(y^2-x^3), xy^2(y^2-x^3), x^7, x^5y, x^4(y^2-x^3), x^4y^2\right), \tag{4.4}$$

so we get the monomial expression

$$H_D = (f_2^2, f_0^2 f_1 f_2, f_0 f_1^2 f_2, f_0^7, f_0^5 f_1, f_0^4 f_2, f_0^4 f_1^2). \tag{4.5}$$

■

■

4.2 *Correctness of the algorithm*

In this section we will prove that Algorithm 4.3 described in Section 4.1 is correct. First, we need to check that it terminates after a finite number of steps. The key point is to prove that the divisor $\widehat{D}_j$ defining the adjacent ideal to the simple ideal $H_{B_{q_j}}$ has smaller support in the total transform basis than B_{q_j}.

Lemma 4.4. *Using the notations in Algorithm 4.3, assume that B_{q_j} is a simple divisor different from $B_{\mathbf{0}} = \mathrm{Div}(\pi^*\mathfrak{m})$. Let $\widehat{D}_j$ be the antinef closure of $B_{q_j} + E_{\mathbf{0}}$ computed in step (i.j.2). Then, $\widehat{D}_j$ has smaller support than B_{q_j} in the total transform basis. That is,*

$$|\mathrm{Supp}_{\overline{E}}(B_{q_j})| > |\mathrm{Supp}_{\overline{E}}(\widehat{D}_j)|. \tag{4.6}$$

Proof. Since $B_{q_j} \neq B_{\mathbf{0}}$, the excess of B_{q_j} at O is $\rho_{\mathbf{0}}(B_{q_j}) = 0$. Then, according to Proposition 4.1, the ideal defined by $B_{q_j} + E_{\mathbf{0}}$ is an adjacent ideal below $H_{B_{q_j}}$. Let $T \subseteq K$ be the points on which unloading is performed to obtain the antinef closure $\widehat{D}_j$ from $B_{q_j} + E_{\mathbf{0}}$. By Proposition 4.1, T are the points $p \in K$ whose associated exceptional divisor E_p belongs to the same connected component as $E_{\mathbf{0}}$ in $\sum_{O \leq p < q_j} E_p$. Observe that $\sum_{0 \leq p < q_j} E_p$ has either one or two components, according if q_j is either free or satellite. In both cases, q_j is proximate to the point $p \in T$ whose exceptional divisor cuts E_{q_j}, i.e. $E_p \cdot E_{q_j} = 1$. Hence, using Proposition 4.1 again, the multiplicity at q_j of $\widehat{D}_j$, after performing unloading on $B_{q_j} + E_{\mathbf{0}}$, decreases by one. Since B_{q_j} is simple, the multiplicity of B_{q_j} at q_j is one. Hence, the multiplicity of $\widehat{D}_j$ at q_j is zero, giving the desired result. □

In the next proposition we prove that Algorithm 4.3 terminates after a finite number of steps. To emphasize the dependence of the divisors on a specific step (i) of the algorithm we will use the notation $B_{q_j}^{(i)}$ and $\widehat{D}_j^{(i)}$.

Proposition 4.5. *Algorithm 4.3 terminates after a finite number of steps.*

Proof. As noted in Section 2.2, the points $q \in K$ such that $e_q(B_p) \neq 0$ are totally ordered. Then, using Equation (2.12), the sequence of multiplicities of B_p decrease along those points. Hence, we have $|B_p|_{\overline{E}} = 1$ for some $p \in K$ if and only if $p = \mathbf{0}$ and then, $B_p = \mathrm{Div}(\pi^*\mathfrak{m})$.

Using the notations in Algorithm 4.3, assume that we are in step (i) and we have a simple divisor B_{q_j} in step $(i.j.1)$. If $q_j = \mathbf{0}$, then $B_{\mathbf{0}} = \mathrm{Div}(\pi^*\mathfrak{m})$ and we are done. Otherwise, since $q_j \neq \mathbf{0}$, we have that $|\mathrm{Supp}_{\overline{E}}(B_{q_j})| > |\mathrm{Supp}_{\overline{E}}(\widehat{D}_j^{(i)})|$ by Lemma Lemma 4.4. Since $D^{(i+1)} := \widehat{D}_j^{(i)}$ admits a decomposition $D^{(i+1)} = \sum_{p \in K} \rho_p^{(i+1)} B_p$, we have $|\mathrm{Supp}_{\overline{E}}(D^{(i+1)})| \geq |\mathrm{Supp}_{\overline{E}}(B_p)|$ for all B_p with $\rho_p^{(i+1)} > 0$. Hence, $|\mathrm{Supp}_{\overline{E}}(B_{q_j})| > |\mathrm{Supp}_{\overline{E}}(B_p)|$, for all p with $\rho_p^{(i+1)} > 0$, and the result follows by induction. □

Lemma 4.6. *Let B_q be a branch basis divisor associated to a satellite point $q \in K$. Let $\Gamma_q = \langle \overline{\beta}_0, \overline{\beta}_1, \ldots, \overline{\beta}_r \rangle$ be the semigroup of a $C_q : f_q = 0, f_q \in \mathcal{O}_{X,\mathbf{0}}$ associated to B_q and take $C : f_r = 0, f_r \in \mathcal{O}_{X,\mathbf{0}}$ such that $[C_q, C] = \overline{\beta}_r$. Then, $B_q^2 = [C_q, e_{r-1}C]$. Furthermore, $v_p(f_r^{e_{r-1}}) \geq v_p(B_q)$, for all $p \leq q$.*

Proof. Assume that $B_q = \sum_{p \in K} e_p \overline{E}_p$. The first claim follows from the following computation:

$$\begin{aligned} B_q^2 &= \sum_{p \in K} e_p^2 = \sum_{i=1}^{r} (\beta_i - \beta_{i-1}) e_{i-1} = \sum_{i=1}^{r-1} (e_{i-1} - e_i)\beta_i + e_{r-1}\beta_r \\ &= e_{r-1}\overline{\beta}_r = e_{r-1}[C_q, C] = [C_q, e_{r-1}C]. \end{aligned} \tag{4.7}$$

where the second equality is true since $q \in K$ is a satellite point, see [Cas00, §5.10, Ex. 5.6], and the fourth equality comes from Equation (2.16)

To prove the second claim, after Lemma 2.5, it suffices to check the inequalities $[e_{r-1}C, C_p] \geq [C_q, C_p]$ for any $p \leq q$ with $C_p : f_p = 0$. We will use well-known properties of the ultrametric $d_{\mathcal{C}}$ distance, introduced in [Pło86], defined over the space $\mathcal{C}$ of plane branches as

$$\frac{1}{d_{\mathcal{C}}(C, D)} := \frac{[C, D]}{e_{\mathbf{0}}(C) e_{\mathbf{0}}(D)}, \tag{4.8}$$

for any $C, D \in \mathcal{C}$. Hence, the inequalities above are equivalent to $d_{\mathcal{C}}(e_{r-1}C, C_p) \geq d_{\mathcal{C}}(C_q, C_p)$ for any $p \leq q$, since in our case

$$\frac{[e_{r-1}C, C_p]}{e_{\mathbf{0}}(e_{r-1}C) e_{\mathbf{0}}(C_p)} = \frac{e_{g-1}[C, C_p]}{e_{g-1} e_{\mathbf{0}}(C) e_{\mathbf{0}}(C_p)} = \frac{1}{d_{\mathcal{C}}(C, C_p)}. \tag{4.9}$$

Notice that $f_q = f_{q_g}$ for some point q_g, $\mathbf{0} \leq q_g \leq q$, which corresponds to a dead-end in the dual graph of B_q. Now we summarize the results on the ultrametric space of plane branches from [AAG11, Thm. 3.1, Thm. 3.2, Prop. 3.4] adapted to our setting:

- $d_{\mathcal{C}}(C_q, C_p) = d_{\mathcal{C}}(C, C_p)$, if $\mathbf{0} \leq p < q_g$.
- $d_{\mathcal{C}}(C_q, C_p) = d_{\mathcal{C}}(C, C_p)$, if $q_g < p \leq q$ and in the dual graph of B_q the vertex of p lies on the segment joining the vertexes q and q_g.
- $d_{\mathcal{C}}(C_q, C_p) > d_{\mathcal{C}}(C, C_p)$, otherwise.

Hence, the second claim follows. □

Proposition 4.7. *Using the notations in Algorithm 2.6, at any step (i) of the algorithm, there exists a power of a maximal contact element $f_{\tau_j}^{e_j} \in \mathcal{O}_{X,\mathbf{0}}$ as required at step $(i.j.3)$ and such element belongs to $H_{B_{q_j}}$ but not to $H_{\widehat{D}_j}$.*

Proof. We are going to break the proof of the first statement in two cases depending on whether the point $q_j \in K$ is free or satellite. With the notations from step $(i.j.3)$, $p \in K$ will be the last free point such that $e_p(B_{q_j}) \neq 0$.

Assume first that q_j is free, i.e. $p = q_j$. If, in addition, the vertex of q_j is a dead-end of the dual graph of π we are done, since $f_{\tau_j} = f_{q_j}$ and $f_{q_j} \in H_{B_{q_j}}$. If q_j is not a dead-end of the dual graph, there is a dead-end $q \in K$ and a totally ordered sequence $q_j \leq p_1 \leq \cdots \leq p_r \leq q$ of free points such that $e_{q_j}(B_q) = e_{p_1}(B_q) = \cdots = e_{p_r}(B_q) = e_q(B_q) = 1$, by Equation (2.12). Therefore, $f_{\tau_j} = f_q$ with $e_{q_j}(f_q) = 1$ and $B_q = B_p + \overline{E}_{p_1} + \cdots + \overline{E}_{p_r} + \overline{E}_q$, which implies, by Equation (2.6), that $f_q \in H_{B_q} \subsetneq H_{B_{q_j}}$. In both cases, we have that $e_j = e_{\mathbf{0}}(B_{q_j}) / e_{\mathbf{0}}(f_{\tau_j}) = 1$.

Now, assume that q_j is satellite and hence $p < q_j$. Let $\Gamma_{q_j} = \langle \overline{\beta}_0, \overline{\beta}_1, \ldots, \overline{\beta}_g \rangle$ be the semigroup of $C_{q_j} : f_{q_j} = 0$. By [Cas00, §5.8], p has the property that any $C : f_p =$

$0, f_p \in \mathcal{O}_{X,\mathbf{0}}$ satisfies $[C_{q_j}, C] = B_{q_j} \cdot B_p = \overline{\beta}_g$. However, it may happen that $p \in K$ is not a dead-end. In this case, using the same argument as before, there is a dead-end $q \in K$ and a totally ordered sequence $p \leq p_1 \leq \cdots \leq p_r \leq q$ of free points and $B_q = B_p + \overline{E}_{p_1} + \cdots + \overline{E}_{p_r} + \overline{E}_q$. Since p is the last free point of B_{q_j}, $B_{q_j} \cdot \overline{E}_{p_i} = 0$ for $i = 1, \ldots, r$ and also $B_{q_j} \cdot \overline{E}_q = 0$. Hence, $[C', C_{q_j}] = B_q \cdot B_{q_j} = \overline{\beta}_g$ with $C' : f_q = 0$, i.e. we can take $f_{\tau_j} = f_q$. We can then apply Lemma 4.6 to B_{q_j} with $f_g = f_{\tau_j}$ yielding that $f_{\tau_j}^{e_j} \in H_{B_{q_j}}$ with $e_j = e_{\mathbf{0}}(B_{q_j})/e_{\mathbf{0}}(f_{\tau_j})$.

Finally, if $f_{\tau_j}^{e_j} \in \mathcal{O}_{X,\mathbf{0}}$ fulfills the requirements of step $(i.j.3)$, then $e_{\mathbf{0}}(f_{\tau_j}^{e_j}) = e_{\mathbf{0}}(B_{q_j})$, but $e_{\mathbf{0}}(\widehat{D}_j) > e_{\mathbf{0}}(B_{q_j})$ by Proposition 4.1, therefore we have that $f_{\tau}^{e_j} \notin H_{\widehat{D}_j}$. □

Theorem 4.8. *Let $\pi : (X', E) \longrightarrow (X, \mathbf{0})$ be a proper birational morphism and let $D \in \Lambda_\pi$. Then, Algorithm 4.3 computes a set of generators for H_D that are monomial in any given set of maximal contact elements of π.*

Proof. Let us prove that the i–th step of the algorithm returns a system of generators of $H_{D^{(i)}}$ which has the desired properties. By Zariski's Factorization Theorem 2.7, it is enough to focus on computing generators for each simple ideal $H_{B_{q_j}}, j = 1, \ldots, d_i$, in the decomposition of $D^{(i)}$. Fixing B_{q_j} at step $(i.2)$, we will make induction on the order of the neighborhood that $q_j \in K$ belongs to, and we will show that Algorithm 4.3 computes generators for $H_{B_{q_j}}$ which are monomials in the set of maximal contact elements.

If $q_j = \mathbf{0}$, then $B_{\mathbf{0}} = \mathrm{Div}(\pi^*\mathfrak{m})$ and step $(i.j.1)$ returns $H_{D^{(i)}} = \mathfrak{m}$, since a pair of smooth transverse elements generate $\mathfrak{m}$. By construction, any set of maximal contact elements contain such a pair of elements.

Assume now that $q_j \neq O$ and that the algorithm computes the generators of the ideals associated to B_p for $p < q_j$. By Proposition 4.1, $H_{\widehat{D}_j} \subsetneq H_{B_{q_j}}$ are adjacent ideals. Since $\widehat{D}_j = \sum_{p<q_j} \rho_p^{(i)} B_p$, we can apply the induction hypothesis to the simple divisors $B_p, p < q_j$ such that $\rho_p^{(i)} \neq 0$ and apply Theorem 2.7 to get

$$H_{\widehat{D}_j} = \prod_{p<q_j} H_{B_p}^{\rho_p^{(i)}} \subsetneq H_{D^{(i)}}. \tag{4.10}$$

At this point is it enough to add any element that belongs to $H_{B_{q_j}}$ but not to $H_{\widehat{D}_j}$ to get a system of generators of $H_{B_{q_j}}$. By Proposition 4.7, the element chosen at step $(i.j.3)$ has the desired properties, namely, it is a power of a maximal contact element. Finally, we can remove unnecessary elements from the system of generators of $H_{D^{(i)}}$ using Lemma 4.2.

The dependency on the set of maximal contact elements $\{f_i\}_{i\in I}$ is only used in step $(i.j.3)$. The conditions required to $\{f_p\}_{i\in I}$ depend only on a finite number of valuations associated to the exceptional divisors of π. These conditions are fulfilled by an infinite number of elements which can be part of a set of maximal contact elements. □

Remark 4.5. We would like to stress the generality of the monomial generators in Theorem 4.8. Consider the monomials $\boldsymbol{z}^{\boldsymbol{\alpha}} = \prod_{i\in I} z_i^{\alpha_i}, \boldsymbol{\alpha} = (\alpha_i)_{i\in I}$ in the variables $z_i, i \in I$. Each variable z_i formally represents all possible elements f_p for a fixed dead-end of the dual graph of π. They all have the same value for the valuations associated to the exceptional divisors. Take now any set of maximal contact elements $\boldsymbol{f} = \{f_{p_i}\}_{i \in I}$ and denote $\boldsymbol{z}_{\boldsymbol{f}}^{\boldsymbol{\alpha}} = \prod_{i \in I} f_{p_i}^{\alpha_i}$ the specialization $z_i \mapsto f_{p_i}$.

The result of Theorem 4.8 is that Algorithm 4.3 returns formally $(\boldsymbol{z}^{\boldsymbol{\alpha}_1}, \dots, \boldsymbol{z}^{\boldsymbol{\alpha}_r})$ and that for any two sets of maximal contact elements $\boldsymbol{f} = \{f_{p_i}\}_{i \in I}$ and $\boldsymbol{g} = \{g_{p_i}\}_{i \in I}$, both specializations $H_{D,\boldsymbol{f}} = (\boldsymbol{z}_{\boldsymbol{f}}^{\boldsymbol{\alpha}_1}, \dots, \boldsymbol{z}_{\boldsymbol{f}}^{\boldsymbol{\alpha}_r})$ and $H_{D,\boldsymbol{g}} = (\boldsymbol{z}_{\boldsymbol{g}}^{\boldsymbol{\alpha}_1}, \dots, \boldsymbol{z}_{\boldsymbol{g}}^{\boldsymbol{\alpha}_r})$ are equal. ■

Corollary 4.9. *The set of monomial expressions returned by Algorithm 4.3 is a topological invariant of D, i.e. it is an invariant of the weighted dual graph of D.*

Proof. From the proof of Theorem 4.8 it follows that the algorithm only uses the information of the equisingularity types of the maximal contact elements and of the dual graph of D weighted by the natural numbers $v_p(D)$ for $p \in K$. □

4.3 *Applications to some families of complete ideals*

Let $(X, \mathbf{0})$ be a germ of smooth complex surface and let $\pi : (X', E) \longrightarrow (X, \mathbf{0})$ be a proper birational transform. The ideal sheaves $\mathcal{O}_{X'}(D)$ and its pushforward, for some $D \in \Lambda_\pi$, arise in many different contexts. The goal of this sections is to show how Algorithm 4.3 and Theorem 4.8 applies to different problems. Our approach is specially useful when studying families of divisors $\{D_i\}_{i \in I}$ in Λ_π since all the generators for all the ideals H_{D_i} will be given as monomials in any set of maximal contact elements.

Integral closure

Let $\mathfrak{a} \subseteq \mathcal{O}_{X,0}$ be an ideal which can be assumed to be $\mathfrak{m}$-primary after considering the decomposition $\mathfrak{a} = (a) \cdot \mathfrak{a}'$ with $a = \gcd(\mathfrak{a})$. Let $\pi : (X', E) \longrightarrow (X, \mathbf{0})$ be a log-resolution of the ideal $\mathfrak{a}$, i.e., a proper birational morphism such that there exists F_π an effective Cartier divisor such that $\mathfrak{a} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F_\pi)$. Then, the integral closure $\bar{\mathfrak{a}}$ of $\mathfrak{a}$ is just the ideal H_{F_π}.

Therefore, we have a very simple method to compute the integral closure of any planar ideal that boils down to the following steps:

- *Compute the divisor F_π of the minimal log-resolution of $\mathfrak{a}$ by using Algorithm 3.13.*
- *Compute a set of generators for the ideal H_{F_π} using Algorithm 4.3.*

Let us illustrate this situation with an small example.

Example 4.2. Let $\mathfrak{a} = ((y^2 - x^3)^2, x^2y^3) \subseteq \mathcal{O}_{X,0}$ be an ideal. One can compute the minimal log-resolution divisor of $\mathfrak{a}$ using Algorithm 3.13. The minimal log-resolution and its associated divisor F_π of $\mathfrak{a}$ are precisely the proper birational morphism π and the divisor D from Example 2.1.

Namely, $F_\pi = 4E_0 + 6E_{p_1} + 12E_{p_2} + 13E_{p_3} + 26E_{p_4}$ and we can take $f_0 = x, f_1 = y, f_2 = y^2 - x^3$ as a set of maximal contact elements of the minimal log-resolution π. Thus, from the computation in Example 4.1, one deduces that

$$\bar{\mathfrak{a}} = H_{F_\pi} = \pi_* \mathcal{O}_{X'}(-F_\pi) = ((y^2 - x^3)^2, x^7, x^5y, x^4y^2, x^2y^3, xy^4). \qquad (4.11)$$

■

The following results follows directly from Theorem 4.8 and its corollaries.

Theorem 4.10. *Let $\mathfrak{a} \subseteq \mathcal{O}_{X,\mathbf{0}}$ be an ideal. There exists a set of generators of the integral closure $\mathfrak{a}$ that are monomial in any given set of maximal contact elements of the minimal log-resolution of $\mathfrak{a}$.*

Corollary 4.11. *Let $\pi : (X', E) \longrightarrow (X, \mathbf{0})$ be a proper birational morphism. Any complete ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,\mathbf{0}}$ whose log-resolution is dominated by π admits a system of generators given by monomials in any set of maximal contact elements associated to π.*

Corollary 4.12. *The set of monomial expressions returned by Algorithm 4.3 for the integral closure of an ideal $\mathfrak{a}$ is an equisingular invariant of $\mathfrak{a}$.*

Multiplier ideals

Let $\pi : (X', E) \longrightarrow (X, \mathbf{0})$ be a log-resolution of an ideal $\mathfrak{a} \subseteq \mathcal{O}_X$ and let F_π be the divisor such that $\mathfrak{a} \cdot \mathcal{O}'_X = \mathcal{O}_{X'}(-F_\pi)$. Recall the definition of the *multiplier ideal* associated to $\mathfrak{a}$ and some rational number $\lambda \in \mathbb{Q}_{>0}$ introduced in Section 1.3,

$$\mathcal{J}(\mathfrak{a}^\lambda) = \pi_* \mathcal{O}_{X'}(\lceil K_\pi - \lambda F_\pi \rceil)_{\mathbf{0}}, \tag{4.12}$$

Combining the algorithms from Section 3 and Section 2.9 with Algorithm 4.3 we may provide a method that, given a set of generators of a planar ideal $\mathfrak{a}$, returns the set of jumping numbers and a set of generators of the corresponding multiplier ideals, see [BD18]. Namely, we have to perform the following steps:

- *Compute the divisor F_π of the minimal log-resolution of $\mathfrak{a}$ by using Algorithm 3.13.*
- *Compute the sequence of jumping numbers $\{\lambda_j\}_{j \in \mathbb{Z}_{\geq 0}}$ and the divisor corresponding to the associated multiplier ideals $\{\mathcal{J}(\mathfrak{a}^{\lambda_j})\}_{j \in \mathbb{Z}_{\geq 0}}$, i.e. the antinef closures D_{λ_j} of $\lfloor \lambda_j F_\pi - K_\pi \rfloor$, using Algorithm 2.25.*
- *Compute a set of generators for the ideals $H_{D_{\lambda_j}}$ using Algorithm 4.3.*

This method is illustrated with the following

Example 4.3. Consider the ideal $\mathfrak{a} = ((y^2 - x^3)^3, x^3(y^2 - x^3)^2, x^6 y^3) \subseteq \mathcal{O}_{X,\mathbf{0}}$. The minimal log-resolution of $\mathfrak{a}$ can be computed using the algorithm from Algorithm 3.13 and it is represented by means of the following dual graph:

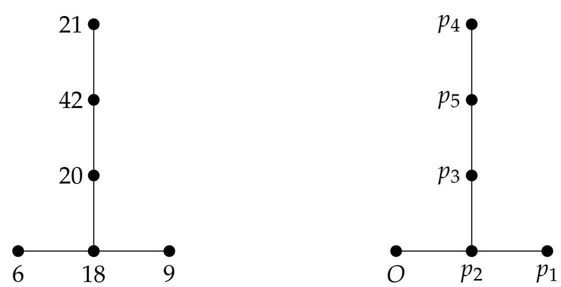

Figure 4.2: Dual graph of the ideal in Example 4.3.

The divisor F_π such that $\mathfrak{a} \cdot \mathcal{O}'_X = \mathcal{O}_{X'}(-F_\pi)$ is $F_\pi = 6E_0 + 9E_1 + 18E_2 + 20E_3 + 21E_4 + 42E_5$. A set of maximal contact elements for the minimal log-resolution of $\mathfrak{a}$ is, for instance, $f_0 = x, f_1 = y, f_2 = y^2 - x^3$.

The jumping numbers smaller than 1 computed using the Algorithm 2.25 and generators for the associated multiplier ideals computed using Algorithm 4.3 can be found in Table 4.1.

λ_i	$\mathcal{J}(\mathfrak{a}^{\lambda_i})$
$\frac{5}{18}$	x, y
$\frac{7}{18}$	y, x^2
$\frac{4}{9}$	x^2, xy, y^2
$\frac{1}{2}$	xy, y^2, x^3
$\frac{23}{42}$	y^2, x^3, x^2y
$\frac{25}{42}$	$y^2 - x^3, x^2y, xy^2, x^4$
$\frac{11}{18}$	x^2y, xy^2, y^3, x^4
$\frac{9}{14}$	xy^2, y^3, x^4, x^3y
$\frac{29}{42}$	$x(y^2 - x^3), y(y^2 - x^3), x^3y, x^2y^2, x^5$
$\frac{13}{18}$	y^3, x^3y, x^2y^2, x^5
$\frac{31}{42}$	$y(y^2 - x^3), x^2y^2, xy^3, x^2(y^2 - x^3), x^4y$
$\frac{7}{9}$	$x^2y^2, xy^3, y^4, x^5, x^4y$
$\frac{11}{14}$	$x^2(y^2 - x^3), xy(y^2 - x^3), y^2(y^2 - x^3), x^4y, x^3y^2, x^6$
$\frac{5}{6}$	$xy(y^2 - x^3), y^2(y^2 - x^3), x^3(y^2 - x^3), x^3y^2, x^2y^3, x^5y$
$\frac{37}{42}$	$xy(y^2 - x^3), y^2(y^2 - x^3), x^3(y^2 - x^3), x^2y^3, xy^4, x^5y, x^4y^2, x^7$
$\frac{8}{9}$	$y^2(y^2 - x^3), x^5y, x^3(y^2 - x^3), x^2y^3, x^2y(y^2 - x^3), xy^4, x^4y^2, x^7$
$\frac{13}{14}$	$y^2(y^2 - x^3), x^3(y^2 - x^3), x^2y(y^2 - x^3), xy^4, y^5, x^4y^2, x^3y^3, x^7, x^6y$
$\frac{17}{18}$	$x^2y(y^2 - x^3), xy^2(y^2 - x^3), y^3(y^2 - x^3), x^4y^2, x^4(y^2 - x^3), x^3y^3, x^6y$
$\frac{41}{42}$	$x^2y(y^2 - x^3), xy^2(y^2 - x^3), y^3(y^2 - x^3), x^4(y^2 - x^3), x^3y^3, x^2y^4, x^6y, x^5y^2, x^8$

Table 4.1: The jumping numbers smaller than 1 and generators of the associated multiplier ideal for $\mathfrak{a} = ((y^2 - x^3)^3, x^3(y^2 - x^3)^2, x^6y^3)$.

■

It is a known result that the multiplier ideals $\mathcal{J}(\mathfrak{a}^\lambda)$ associated to $\mathfrak{a}$ are the same when taking the completion $\bar{\mathfrak{a}}$ of $\mathfrak{a}$, i.e. $\mathcal{J}(\mathfrak{a}^\lambda) = \mathcal{J}(\bar{\mathfrak{a}}^\lambda)$. As a corollary of Theorem 4.8 and Corollary 4.11 we obtain the following result which resembles Howald's Theorem [How01] on the fact that multiplier ideals of monomial ideals are also monomial.

Theorem 4.13. *Let $\mathfrak{a} \subseteq \mathcal{O}_{X,0}$ be an ideal and consider its completion $\bar{\mathfrak{a}}$, that can be generated by monomials in any given set of maximal contact elements. Then, the multiplier ideals $\mathcal{J}(\mathfrak{a}^\lambda)$*

are also generated by monomials in the same set of maximal contact elements of the minimal log-resolution of $\mathfrak{a}$.

Valuation filtration

Let $f : (\mathbb{C}^2, \mathbf{0}) \longrightarrow (\mathbb{C}, 0)$ be a plane branch and let ν_f be the valuation induced by the intersection multiplicity. Let $\mathfrak{V}_i$ denote the ideal of all the elements in $\mathcal{O}_{X,\mathbf{0}}$ with valuation greater or equal to i:

$$\mathfrak{V}_i := \{g \in \mathcal{O}_{X,\mathbf{0}} \mid \nu_f(g) \geq i\}. \tag{4.13}$$

These ideals form a filtration in $\mathcal{O}_{X,\mathbf{0}}$:

$$\mathcal{O}_{X,\mathbf{0}} = \mathfrak{V}_0 \supseteq \mathfrak{V}_1 \supseteq \cdots \supseteq \mathfrak{V}_i \supseteq \mathfrak{V}_{i+1} \supseteq \cdots \tag{4.14}$$

such that $\mathfrak{V}_i \cdot \mathfrak{V}_j \subset \mathfrak{V}_{i+j}$ and $\cap_{i \in \mathbb{Z}_{\geq 0}} \mathfrak{V}_i = (f)$. For instance, this type of filtration was considered by Teissier in [Tei86]. Since, the ideals $\mathfrak{V}_i$ are defined by valuations, they are complete, and hence, have the form $\pi_* \mathcal{O}_{X'_i}(D_i)_\mathbf{0}$ for some effective divisor D_i in some surface X'_i.

Example 4.4. Consider the plane branch, $f = (y^2 - x^3)^2 - x^5 y \in \mathcal{O}_{X,\mathbf{0}}$. A minimal embedded resolution $\pi : (X', E) \longrightarrow (X, \mathbf{0})$ of f is the same as the one given for the ideal in Example 2.1. Using Algorithm 4.3 we obtain the generators of the filtration in (4.14) until the last $\mathfrak{V}_i$ dominated by the minimal log-resolution of (f), which we collect in Table 4.2

■

i	$\mathfrak{V}_i$
$1,2,3,4$	x, y
$5,6$	y, x^2
$7,8$	xy, x^2, y^2
$9,10$	xy, y^2, x^3
$11,12$	x^2y, y^2, x^3
13	$y^2 - x^3, x^2y, xy^2, x^4$
14	x^2y, xy^2, y^3, x^4
$15,16$	xy^2, y^3, x^4, x^3y
17	$x(y^2 - x^3), y(y^2 - x^3), x^3y, x^2y^2, x^5$
18	y^3, x^3y, x^2y^2, x^5
19	$y(y^2 - x^3), x^2(y^2 - x^3), x^4y, x^2y^2, xy^3$
20	$x^4y, x^2y^2, xy^3, y^4, x^5$
21	$x^2(y^2 - x^3), xy(y^2 - x^3), y^2(y^2 - x^3), x^4y, x^3y^2, x^6$
22	$xy^3, y^4, x^4y, x^3y^2, x^6$
23	$xy(y^2 - x^3), y^2(y^2 - x^3), x^3y^2, x^3(y^2 - x^3), x^2y^3, x^5y$
24	$y^4, x^3y^2, x^2y^3, x^6, x^5y$
25	$(y^2 - x^3)^2, x^3(y^2 - x^3), x^2y(y^2 - x^3), x^5y, x^4y^2, x^7$
26	$(y^2 - x^3)^2, x^2y^3, xy^4, x^5y, x^4y^2, x^7$

Table 4.2: The ideals $\mathfrak{V}_i$ of the filtration associated to the plane branch $f = (y^2 - x^3)^2 - x^5y$ for $i = 1, \ldots, 26$.

III

ON THE TJURINA NUMBER OF PLANE CURVES

Let $f : (\mathbb{C}^{n+1}, \mathbf{0}) \longrightarrow (\mathbb{C}, 0)$ be a germ of a holomorphic function defining an isolated singularity. Taking local coordinates at the origin, the Tjurina number τ of f is defined as the codimension of the Jacobian ideal, that is

$$\tau := \dim_{\mathbb{C}} \frac{\mathbb{C}\{x_0, \ldots, x_n\}}{(f, \partial f/\partial x_0, \ldots, \partial f/\partial x_n)}. \tag{4.1}$$

Its significance comes from the fact that the Tjurina number of f is the dimension of the miniversal deformation of f. We are interested in the following question posed by Dimca and Greuel in [DG18] between the quotient of the Milnor and Tjurina number.

Question. *It is true that $\mu/\tau < 4/3$ for any reduced plane curve singularity?*

By the semicontinuity of the dimension, the Tjurina number achieves a minimal value τ_{min}, *the minimal Tjurina number*, within a fixed topological class. In this chapter, we will study the question of Dimca and Greuel using the minimal Tjurina number of a topological class. In the first section, we present some results towards a positive answer to the question for the case of semi-quasi-homogeneous plane curve singularities. In the second section, we give a formula for the minimal Tjurina number of an equisingularity class in terms of its minimal resolution and we deduce a positive answer to the question in this case.

A complete positive answer to the question of Dimca and Greuel is given by Almirón in [Alm19] using some known cases of Durfee's conjecture.

5 QUASI-HOMOGENEOUS PLANE CURVES

In the first part of this section, we will provide a positive answer to the question of Dimca and Greuel for irreducible plane curves with one Puiseux exponent using results of Delorme [Del78]. Then, in the second part, we give a different proof for semi-quasi-homogeneous singularities using results of Briançon, Granger, and Maisonobe [BGM88]. By a well-known result of Zariski [Zar86, §VI.2], the latter case contains the former. However, we include both proofs as the approaches are fundamentally different and, at the time of its publication, they might have led to different more general cases of the question. The results of this section give the first evidence towards a possible positive answer to the Dimca and Gruel question. The results of this section are joint work with Almirón and are published in [AB19].

5.1 *The case of one Puiseux pair*

In this section, we assume that $f : (\mathbb{C}^2, \mathbf{0}) \longrightarrow (\mathbb{C}, 0)$ defines a curve with a single Puiseux pair (n, m). We will denote by $\Gamma = \langle n, m \rangle, n < m$ with $\gcd(n, m) = 1$ the semigroup of f. Ebey proves in [Ebe65] that the moduli space of curves having a given semigroup is in bijection with a constructible algebraic subset of some affine space. For this, he shows that the moduli space is a quotient of an affine space by an algebraic group. Consequently, Zariski [Zar86, §VI] defines the generic component of the moduli space as the variety representing the generic orbits of this group action.

Following the ideas of Zariski in [Zar86], Delorme [Del78] computed the dimension $q_{n,m}$ of the generic component of the moduli space of plane branches with a single Puiseux pair (n, m).

Theorem 5.1 ([Del78, Thm. 32]). *Consider the continued fraction representation $m/n = [h_1, h_2, \ldots, h_k]$, with $k \geq 2, h_1 > 0$ and $h_2 > 0$. Define, inductively, the following numbers*

$$r_k := 0, \quad t_k := 1, \quad r_{i-1} := r_i + t_i h_i, \quad t_{i-1} := \begin{cases} 0, & \text{if } t_i = 1 \text{ and } r_{i-1} \text{ even}, \\ 1, & \text{otherwise}. \end{cases} \tag{5.1}$$

Then, the dimension $q_{n,m}$ of the generic component of the moduli space is given by

$$q_{n,m} = \frac{(n-4)(m-4)}{4} + \frac{r_0}{4} + \frac{(2-t_1)(h_1-2)}{2} - \frac{t_1 t_2}{2}. \tag{5.2}$$

In particular, except for the case $(n, m) = (2, 3)$,

$$\frac{(n-4)(m-4)}{4} \leq q_{n,m} \leq \frac{(n-3)(m-3)}{2}. \tag{5.3}$$

The bound on the left-hand side of Equation (5.3) is sharp, consider, for instance, the characteristic pair $n = 8, m = 11$. In the Appendix [Tei86] of [Zar86], Teissier, using the monomial curve C^Γ, proves that, in general, the dimension q of the generic component of the moduli space of plane branch with semigroup Γ is given by

$$q = \tau_- - (\mu - \tau_{min}), \tag{5.4}$$

where τ_- is the dimension of the miniversal semigroup constant deformation of the monomial curve C^Γ. For one characteristic exponent we have that τ_- is the number of points of the standard lattice of $\mathbb{R}^2$ that are in the interior of the triangle defined by the lines $\alpha = m - 1, \ \beta = n - 1, \ \alpha n + \beta m = nm$, see [Zar86, §VI.2]. Therefore, it is easy to see that

$$\tau_- = \frac{(n-3)(m-3)}{2} + \left[\frac{m}{n}\right] - 1, \tag{5.5}$$

where $[\,\cdot\,]$ denotes the integer part. In this case, the Milnor number is $\mu = (n-1)(m-1)$. Combining the lower bound in Equation (5.3) and Equation (5.4) one obtains the following lower bound for τ_{min}

$$\frac{(n-4)(m-4)}{4} + (n-1)(m-1) - \frac{(n-3)(m-3)}{2} - \frac{m}{n} + 1 \leq \tau_{min}. \tag{5.6}$$

except for the case $(n, m) = (2, 3)$.

Proposition 5.2. *For any plane branch with one characteristic exponent, $\mu/\tau < 4/3$.*

Proof. It is sufficient to prove the inequality for the τ_{min} of each characteristic pair (n, m). Dividing μ by the expression in Equation (5.6) and rewriting

$$\frac{\mu}{\tau} \leq \frac{\mu}{\tau_{min}} \leq \frac{4n(n-1)(m-1)}{3n^2 m - 2n^2 - 2nm + 6n - 4m}, \tag{5.7}$$

assuming always that $(n, m) \neq (2, 3), n < m$. The upper bound in Equation (5.7) is strictly smaller than $4/3$ if and only if $0 < m(n-4) + n(n+3)$. Therefore, the result holds if $n \geq 4$. The cases $n = 2$ and $n = 3$ follow from computing the τ_{min} using Theorem 5.1

Indeed, let $n = 2$ and $m = 2h_1 + 1, h_1 > 1$ so the continued fraction representation is $m/n = [h_1, 2]$. Then, $r_0 = 2, t_1 = 0, t_2 = 1$ and $q_{2,m} = h_1 - m/2 - 1/2 = 0$. Analogously, if $n = 3$, then $m = 3h_1 + 1$ or $m = 3h_1 + 2$; the continued fractions are either $m/n = [h_1, 3]$ or $m/n = [h_1, 1, 2]$. Then, $r_0 = 3 + h$ or $r_0 = 2 + h$, $t_2 = 1$ or $t_2 = 0$, respectively, and $t_1 = 1$ in either case. Consequently, in both cases, $q_{3,3h_1+1} = -m/4 + 3h_1/4 + 1/4 = 0$ and $q_{3,3h_1+1} = -m/4 + 3h_1/4 + 1/2 = 0$. Finally, since $\tau_- = 0$ if $n = 2$ and $\tau_- = h_1 - 1$ if $n = 3$,

$$\frac{\mu}{\tau_{min}} = 1 < \frac{4}{3}, \qquad \frac{\mu}{\tau_{min}} < \frac{6m-6}{5m-3} < \frac{6}{5} < \frac{4}{3}, \tag{5.8}$$

for $n = 2, m \geq 3$ and $n = 3, m \geq 4$, respectively. □

5.2 *Semi-quasi-homogeneous singularities*

We assume now that f is a semi-quasi-homogeneous singularity with weights $w = (n, m)$ such that $\gcd(n, m) \geq 1$ and $n, m \geq 2$. This means that $f = f_0 + g$ is a deformation of the initial term $f_0 = y^n - x^m$ such that $\deg_w(f_0) < \deg_w(g)$. In [BGM88], Briançon, Granger and Maisonobe, using the technique of *escaliers*, give recursive formulas to compute the τ_{min} of this type of singularities. Their main result is the following:

Theorem 5.3 ([BGM88, §I.6]). *For semi-quasi-homogeneous singularities with initial term $y^n - x^m$,*

$$\tau_{min} = (m-1)(n-1) - \sigma(m, n). \tag{5.9}$$

The number $\sigma(a, b)$ is defined recursively for any non-negative integers a, b as follows. If $a, b \leq 2$ then $\sigma(a, b) := 0$. Otherwise, we can express $a = bq + r, 0 \leq r < b, q \geq 1$. For the cases $r = 0, 1, b-1, b/2$ there are closed formulas for $\sigma(a, b)$ denoted by $\Sigma_0, \Sigma_1, \Sigma_{b-1}, \Sigma_{b/2}$, see Table 1 in [BGM88]. If none of the above cases hold, define recursively, see Tables 2 and 3 in [BGM88], a finite sequence $(a_0, b_0), (a_1, b_1), \ldots, (a_k, b_k)$ with $(a_0, b_0) = (m, n)$, $\sigma(a_k, b_k)$ is in one of the previous cases, and for $i = 0, \ldots, k-1$:

(A) If $\gcd(a_i, b_i) = 1$, we can find $ub_i - va_i = 1$ with $2 \leq u < a_i$. Letting $\gamma := [\frac{a_i - 1}{u}]$, we have two subcases:

(AE) If γ is even, define $a_{i+1} = a_i - \gamma u, b_{i+1} = b_i - \gamma v$, then

$$\sigma(a_i, b_i) := \frac{(a_i-2)(b_i-2)}{4} - \frac{(a_{i+1}-2)(b_{i+1}-2)}{4} - \frac{\gamma}{4} + \sigma(a_{i+1}, b_{i+1}). \tag{5.10}$$

(AO) If γ is odd, define $a_{i+1} = (\gamma+1)u - a_i, b_{i+1} = (\gamma+1)v - b_i$, and

$$\sigma(a_i, b_i) := \frac{(a_i-2)(b_i-2)}{4} - \frac{(a_{i+1}-2)(b_{i+1}-2)}{4} - \frac{\gamma+1}{4} + \sigma(a_{i+1}, b_{i+1}). \tag{5.11}$$

(B) Otherwise, $a_i = \alpha a', b_i = \alpha b'$ with $\alpha \geq 2, \gcd(a', b') = 1$, and we can find a Bezout's identity $ub' - va' = 1$ with $1 \leq u < a'$. We have again two subcases:

(BP) If α is even,

$$\sigma(a_i, b_i) := \frac{(a_i-2)(b_i-2)}{4} - \frac{\alpha}{2}. \tag{5.12}$$

(BO) If α is odd, define $a_{i+1} = |a' - 2u|$ and $b_{i+1} = |b' - 2v|$, and

$$\sigma(a_i, b_i) := \frac{(a_i-2)(b_i-2)}{4} - \frac{\alpha}{2} - \frac{(a_{i+1}-2)(b_{i+1}-2)}{4} + \sigma(a_{i+1}, b_{i+1}). \tag{5.13}$$

Proposition 5.4. *For any semi-quasi-homogeneous singularities with initial term $y^n - x^m$,*

$$\mu/\tau < 4/3. \tag{5.14}$$

Proof. Observe that in the recursive cases (A) and (BO),

$$\sigma(a, b) \leq \frac{(a-2)(b-2)}{4} - \frac{(a_k-2)(b_k-2)}{4} + \sigma(a_k, b_k), \tag{5.15}$$

where $\sigma(a_k, b_k)$ is either zero or has a closed-form. Notice also that $a_i b_{i+1} > b_i a_{i+1}$ for all $i = 0, \ldots, k-1$. From these observations, one can deduce that, in general,

$$(n-1)(m-1) - \frac{(m-2)(n-2)}{4} - \kappa(n, m) \leq \tau_{min}, \tag{5.16}$$

where $\kappa(n, m) = m/4n$ if $\sigma(a_k, b_k)$ is $\Sigma_0, \Sigma_1, \Sigma_{b-1}$ with b odd, $\kappa(n, m) = 5/4$ if $\sigma(a_k, b_k)$ is $\Sigma_0, \Sigma_1, \Sigma_{b-1}$ with b even or $\Sigma_{b/2}$ with $b/2$ odd, and $\kappa(n, m) = 0$ if $\sigma(a_k, b_k)$ is $\Sigma_{b/2}$ with $b/2$ even or in the case (BP). In any case,

$$\frac{\mu}{\tau} \leq \frac{\mu}{\tau_{min}} \leq \frac{4(n-1)(m-1)}{3nm - 2n - 2m - 4\kappa(n, m)}, \tag{5.17}$$

which is bounded by $4/3$ if and only if $n + m + \kappa(n, m) > 3$, which is true for $n, m \geq 2$. □

5.3 *A family with two Puiseux pairs*

In [LP90], Luengo and Pfister study the family of irreducible plane curve singularities with semigroup $\langle 2p, 2q, 2pq + d \rangle$ such that $\gcd(p, q) = 1, p < q$ and d odd. The Milnor number of this family equals

$$\mu = (2p-1)(2q-1) + d. \tag{5.18}$$

Studying the kernel of the Kodaira-Spencer map, they prove, see [LP90, pg. 259], that τ is constant in each equisingularity class and equals,

$$\tau = \mu - (p-1)(q-1). \tag{5.19}$$

One can easily check that $\mu/\tau < 4/3$ for all the semigroups of the family.

6 THE MINIMAL TJURINA NUMBER OF A PLANE BRANCH

In this section, we will prove a formula for the minimal Tjurina number in an equisingularity class of plane branches. The formula is expressed in terms of the multiplicities of the strict transform along the resolution. As a consequence, we can give a positive answer to the question of Dimca and Greuel in the case of irreducible plane curves. The results of this section are joint work with Almirón, Alberich-Carramiñana, and Melle-Hernández and will appear in [Alb+19]. Similar results appeared simultaneous in [GH20].

6.1 *Semigroup constant deformations of C^Γ*

Recall the definition of the monomial curve C^Γ associated with a numerical semigroup Γ from Section 2.6. After Theorem 2.14, every branch $(C,\mathbf{0})$ with semigroup Γ is analytically isomorphic to one of the fibers of the miniversal deformation $G:(X,\mathbf{0})\longrightarrow(D,\mathbf{0})$ of C^Γ. The dimension of the base D of the miniversal deformation of $(C^\Gamma,\mathbf{0})$ equals μ, the Milnor number of the equisingularity class of plane branches with semigroup Γ, see [Tei86, Prop. 2.7].

After [Tei86, Thm. 3], we will denote by τ_- the dimension of the base $(D_\Gamma,\mathbf{0})$ of the miniversal semigroup constant deformation of $(C^\Gamma,\mathbf{0})$. Let us denote by $(C_{\boldsymbol{v}},\mathbf{0}),\boldsymbol{v}\in D_\Gamma$ any fiber of the miniversal semigroup constant deformation of $(C^\Gamma,\mathbf{0})$. We will denote by $\tau(C_{\boldsymbol{v}})$ the dimension of the base of the miniversal deformation of $(C_{\boldsymbol{v}},\mathbf{0})$. Similarly, we denote by $q(C_{\boldsymbol{v}})$ the dimension of the base of the miniversal semigroup constant deformation of the fiber $(C_{\boldsymbol{v}},\mathbf{0})$.

By the Product Decomposition Theorem [Tei86, Addendum 2.1], the germ of D_Γ at any $\boldsymbol{v}$ is a product

$$(D_\Gamma,\boldsymbol{v})\cong(\mathbb{C}^{\mu-\tau(C_{\boldsymbol{v}})}\times D_{\Gamma,\boldsymbol{v}},\mathbf{0}), \tag{6.1}$$

where $D_{\Gamma,\boldsymbol{v}}$ is the base of the miniversal semigroup constant deformation of $(C_{\boldsymbol{v}},\mathbf{0})$. Thus, one has the following relation, see [Tei86, §II.3.4],

$$\tau(C_{\boldsymbol{v}})-q(C_{\boldsymbol{v}})=\mu-\tau_-. \tag{6.2}$$

Let $\pi:(X,E)\longrightarrow(\mathbb{C}^2,\mathbf{0})$ be a resolution of any plane branch with semigroup Γ. Since all such plane branches are equisingular, $F_{\pi,\mathrm{exc}}=\sum_{p\in K}e_p\overline{E}_p+S$ is the expression of the exceptional part of the total transform divisor, regardless of the chosen branch. In the same way that the Milnor number μ can be expressed in terms of the sequence $\{e_p\}_{p\in K}$, see for instance [Cas00, p. 6.4],

$$\mu=\sum_{p\in K}e_p(e_p-1), \tag{6.3}$$

the same is possible for τ_- as we are going to show next. Assume now that $(C_{\boldsymbol{v}},\mathbf{0}),\boldsymbol{v}\in D_\Gamma$ is a plane branch and take $f\in\mathbb{C}\{x,y\}$ any equation of $(C_{\boldsymbol{v}},\mathbf{0})$.

Proposition 6.1. *The dimension of the miniversal μ-constant unfolding of f equals τ_-.*

Proof. The miniversal unfolding of the equation f has a base of dimension μ. Let us denote by ϑ the dimension of the base of the miniversal μ-constant unfolding of f. The codimension of the μ-constant stratum is then $\mu-\vartheta$. Now, since the miniversal unfolding of f is a versal deformation of $(C_{\boldsymbol{v}},\mathbf{0})$, the codimension of the μ-constant strata of both deformations coincide. The curve $(C_{\boldsymbol{v}},\mathbf{0})$ being plane implies that μ-constant is equivalent to constant semigroup, [TR76], and hence, $\tau(C_{\boldsymbol{v}})-q(C_{\boldsymbol{v}})=\mu-\vartheta$. Finally, by Equation (6.2), ϑ equals τ_-. $\square$

Finally, the dimension of the μ-constant stratum of the miniversal unfolding of any reduced $f\in\mathbb{C}\{x,y\}$, which we will also denote by τ_- after Proposition 6.1, is computed by Mattei [Mat91] and Wall [Wal84] in terms of the sequence of multiplicities of the strict transform along an embedded resolution of the germ $(C,\mathbf{0})$.

Theorem 6.2 ([Mat91, Thm. 4.2.1], [Wal84, Thm. 8.1])**.** *The dimension of the μ-constant stratum of the miniversal unfolding of f equals*

$$\tau_- = \sum_{p\in K} \frac{(e'_p-2)(e'_p-3)}{2}, \tag{6.4}$$

where

(a) $e'_p := e_p$ *if p is the origin,*

(b) $e'_p := e_p + 1$ *if p is free and $e_p > 0$,*

(c) $e'_p := e_p + 2$ *if p is satellite and $e_p > 0$,*

(d) $e'_p := 2$ *otherwise.*

Remark 6.1. It might be worth noticing that the quantity on the left (or right) hand-side of Equation (6.2) coincides with the codimension τ^{es} of the equisingularity ideal I^{es} of $(C_v, \mathbf{0})$, see for instance [GLS07] and the references therein. Similarly, it can be seen that τ_- also equals the modality of f in the sense of Wall [Wal84]. ■

6.2 *The dimension of the generic component of the moduli space*

Following the notations from the last section, let us begin this section by defining the moduli space of plane branches with semigroup Γ. Analytically equivalence of germs induces an equivalence relation $\sim$ in D_Γ. The topological space $D_\Gamma/\sim$, with the quotient topology, will be denoted by $\widetilde{M}_\Gamma$ and it is called the *moduli space* associated to the semigroup Γ. Let $m : D_\Gamma \longrightarrow \widetilde{M}_\Gamma$ be the natural projection and let $D_\Gamma^{(2)}$ be the following subset of D_Γ

$$D_\Gamma^{(2)} := \{v \in D_\Gamma \mid (G^{-1}(v), \mathbf{0}) \text{ is a plane branch}\}. \tag{6.5}$$

Then, Teissier proves in [Tei86] that $D_\Gamma^{(2)}$ is an analytic open dense subset of D_Γ and that $m(D_\Gamma^{(2)})$ is the moduli space M_Γ of plane branches with semigroup Γ in the sense of Zariski [Zar86]. Moreover, $\widetilde{M}_\Gamma = M_\Gamma$ if and only if Γ is generated by two elements.

Following [Zar86], we define the generic curve C_v of the moduli space $\widetilde{M}_\Gamma$ as the fiber corresponding to the generic point v in the base space D_Γ of the miniversal semigroup constant deformation. By the discussion in the previous paragraph, C_v is a plane branch and coincides with the generic fiber defined in M_Γ. Since $\tau(C_v)$ coincides with the dimension of the Tjurina algebra of $(C_v, \mathbf{0})$ and it is upper-semicontinuous, we can define $\tau_{min} := \tau(C_v)$. Similarly, after Equation (6.2), it makes sense to define $q_{min} := q(C_v)$.

The quantity q_{min} is the dimension of the generic component of the moduli space $\widetilde{M}_\Gamma$ of branches with semigroup Γ, see [Tei86, Thm. 6]. Therefore, after applying Equation (6.2) to the generic branch $(C_v, \mathbf{0})$, see [Tei86, §II.3.6],

$$\tau_{min} - q_{min} = \mu - \tau_-, \tag{6.6}$$

and, after Theorem 6.2, computing q_{min} is equivalent to computing τ_{min}.

Recently, Genzmer in [Gen16] computed the dimension of the generic component of the moduli space q_{min} for any plane branch $(C, \mathbf{0})$ in terms of the sequence of multiplicities of the strict transform along the minimal embedded resolution of the germ $(C, \mathbf{0})$. With the same notations from Theorem 6.2,

Theorem 6.3 ([Gen16]). *The dimension of the generic component q_{min} of the moduli space of plane branches with semigroup Γ equals*

$$q_{min} = \sum_{p \in K} \sigma(e'_p), \tag{6.7}$$

and

$$\sigma(k) := \begin{cases} \dfrac{(k-2)(k-4)}{4}, \text{ if } k \text{ is even}, \\ \dfrac{(k-3)^2}{4}, \text{ if } k \text{ is odd}. \end{cases} \tag{6.8}$$

6.3 *The minimal Tjurina number of an equisingularity class*

In this section, we present a closed formula for the minimal Tjurina number τ_{min} of an equisingularity class of a plane branch $(C, \mathbf{0})$ in terms of the sequence of multiplicities $\{e_p\}_p$. From the formula for the τ_{min}, some consequences will be inferred, including the main result of this paper which is a positive answer to Dimca and Greuel question on the quotient μ/τ of any plane branch.

Theorem 6.4. *For any equisingular class of germs of irreducible plane curve singularity,*

$$\tau_{min} = \sigma(n) + \frac{n^2+3n-6}{2} + \sum_{\substack{p \in K, \\ p \text{ free}}} \frac{(e_p-1)(e_p+2)+2\sigma(e_p+1)}{2} + \sum_{\substack{p \in K, \\ p \text{ sat.}}} \frac{e_p(e_p-1)+2\sigma(e_p+2)}{2}.$$

Proof. Using Equation (6.6) we have $\tau_{min} = q_{min} + \mu - \tau_-$, from Theorems 6.2 and 6.3 the claimed formula follows. □

A fortiori, we can see from Theorem 6.4 that the τ_{min} of an equisingularity class depends only on the minimal resolution of $(C, \mathbf{0})$ and not on the minimal embedded resolution. Furthermore, the formula works for any resolution of $(C, \mathbf{0})$, minimal or not.

This formula for τ_{min} enables us to give a positive answer to Dimca and Greuel question, in the case of any plane branch. Before proving this, we need the following property of the sequence of multiplicities.

Lemma 6.5. *For any plane branch singularity of multiplicity n,*

$$\sum_{\substack{p \in K, \\ p \text{ sat.}}} e_p = n - 1. \tag{6.9}$$

Proof. Consider the finite sequence of positive multiplicities $\{e_p\}_p$ along the minimal embedded resolution of the plane branch. From Enriques' Theorem [Casoo, Thm. 5.5.1] one has that $n + \sum_{p\,\text{free}} e_p = \beta_g$, where β_g/n is the last characteristic exponent. On the other hand, from [Casoo, Ex. 5.6], $\sum_p e_p = \beta_g + n - 1$, where this summation runs on all points p equal or infinitely near to the origin. Since all the satellite points for which e_p is positive are included in the sequence of points blown-up in the minimal embedded resolution, the result follows. □

Finally, we get the announced positive answer to Dimca and Greuel question as a corollary of Theorem 6.4.

Corollary 6.6. *For any plane branch singularity,*

$$\frac{\mu}{\tau} < \frac{4}{3}. \tag{6.10}$$

Proof. It is enough to prove the inequality for the τ_{min} of each equisingularity class of plane branches. We will show that $4\tau_{min} - 3\mu > 0$. From Theorem 6.4 and Equation (6.3) we have that

$$4\tau_{min} - 3\mu = 4\sigma(n) - n^2 + 9n - 12 + \sum_{\substack{p\in K,\\ p\,\text{free}}} \big(4\sigma(e_p+1) - (e_p-1)(e_p-4)\big) + \sum_{\substack{p\in K\\ ,p\,\text{sat.}}} \big(4\sigma(e_p+2) - e_p(e_p-1)\big)\,.$$

Now, since $\sigma(k) \geq (k-2)(k-4)/4$,

$$4\tau_{min} - 3\mu \geq 3n - 4 + \sum_{\substack{p\in K,\\ p\,\text{free}}} (e_p - 1) - \sum_{p\,\text{sat.}} e_p. \tag{6.11}$$

Finally, using Lemma 6.5

$$4\tau_{min} - 3\mu \geq 2n - 3 + \sum_{\substack{p\in K,\\ p\,\text{free}}} (e_p - 1) > 0, \tag{6.12}$$

and the result follows, since $n \geq 2$. □

As a direct consequence of Theorem 6.4 we also obtain the following lower bound for τ:

Corollary 6.7. *For any plane branch,*

$$\begin{aligned} \tau &\geq \frac{3n^2}{4} - 1 \quad \text{if} \quad n \text{ is even},\\ \tau &\geq \frac{3}{4}(n^2-1) \quad \text{if} \quad n \text{ is odd}. \end{aligned} \tag{6.13}$$

The bound in Corollary 6.7 is sharp, as one can easily check for generic curves in the equisingularity class of the singularities $y^n - x^{n+1} = 0$, i.e. the minimal Tjurina number in the equisingularity class of the singularities $y^n - x^{n+1} = 0$ coincides with the bound of Corollary 6.7. In fact, from Theorem 6.4 one can see that these are the only topological types of singular plane branches for which the bound is reached.

IV

THE COMPLEX ZETA FUNCTION

In this chapter, we study the complex zeta function of a singularity with a special emphasis in the case of plane curve singularities. The first part of the chapter will introduce the complex zeta function of a singularity, its analytic continuation as a distribution using resolution of singularities, and its connection with the Bernstein-Sato polynomial.

In the second part, we will present some results about the vanishing and non-vanishing of the residues at the poles in the case of plane curves, generalizing some results of Lichtin [Lic85; Lic89]. Using these results we can ask, for the case of reduced plane curves, a question raised by Kollár [Kol97] on which exceptional divisors contribute to roots of the Bernstein-Sato polynomial Finally, using the study of the residues in the case of plane branches, we will give a proof of Yano's conjecture for irreducible plane curves with any number of Puiseux pairs under the topological restriction that the eigenvalues of the monodromy are pair-wise different. The results of this chapter are published in [Bla19a].

7 ANALYTIC CONTINUATION OF COMPLEX POWERS

In this section, we will review the basic results on regularization of complex powers appearing in the book of Gel'fand and Shilov [GS64]. We will see how resolution of singularities is used to construct the analytic continuation of the complex zeta function of an arbitrary polynomial $f \in \mathbb{C}[z_1, \ldots, z_n]$. On the other hand, the Bernstein-Sato polynomial, see Section 1.2, is used to construct the analytic continuation of f^s in a different way.

7.1 *Regularization of complex powers*

We will take the set of *test functions of complex variable* as the set of smooth, compactly supported functions $\varphi : \mathbb{C}^n \longrightarrow \mathbb{C}$. The space of such functions is denoted by $C_c^\infty(\mathbb{C}^n)$. Alternatively, we can consider the larger space of test functions consisting of Schwartz functions. From the analytic continuation principle, one deduces that there are no holomorphic compactly supported functions. Therefore, any $\varphi \in C_c^\infty(\mathbb{C}^n)$ has a holomorphic and an antiholomorphic part, i.e. $\varphi = \varphi(z, \bar{z})$.

Let $f(z) \in \mathbb{C}[z_1, \ldots, z_n]$ be a non-constant polynomial. We define a parametric family of *distributions* of complex variable $f^s : C_c^\infty(\mathbb{C}^n) \longrightarrow \mathbb{C}$ given by

$$\langle f^s, \varphi \rangle := \int_{\mathbb{C}^n} \varphi(z, \bar{z}) |f(z)|^{2s} dz d\bar{z}, \tag{7.1}$$

which is well-defined for any $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$. The dependence of f^s on the parameter s is holomorphic as we can differentiate under the integral symbol to obtain another well-defined distribution, namely

$$\frac{d}{ds}\langle f^s, \varphi \rangle = \int_{\mathbb{C}^n} \varphi(z, \bar{z}) |f(z)|^{2s} \log |f(z)|^2 dz d\bar{z} = \left\langle \frac{df^s}{ds}, \varphi \right\rangle, \quad \mathrm{Re}(s) > 0. \tag{7.2}$$

The distribution f^s or the function $\langle f^s, \varphi\rangle$ are usually called the *complex zeta function* of f. This name goes back to Gel'fand [Gel57]. In [GS64], it is shown how one can obtain the analytic continuation of $\langle f^s, \varphi\rangle$ by *regularizing* the integral in Equation (7.1), for some classes of polynomials. The concept of *regularization* is better understood after the following example.

If one takes the function $z^{-3/2}$, in general, its integral $\langle z^{-3/2}, \varphi\rangle$ against a test function $\varphi(z,\bar{z})$ will diverge. However, if $\varphi(z,\bar{z})$ vanishes at zero, the integral converges. Any distribution whose action on the elements of $C_c^\infty(\mathbb{C})$ vanishing at zero coincides with the action of $z^{-3/2}$ is a regularization of $z^{-3/2}$. The regularization of a function with algebraic singularities is unique up to functionals concentrated in the zero locus, see [GS64, p. I.1.7]. For a fixed $s \in \mathbb{C}$, the *canonical regularization* of the function z^s, in the sense that it is the most natural, is presented in the following proposition.

Proposition 7.1 ([GS64, B1.2], Gel'fand-Shilov regularization). *For any $m \in \mathbb{Z}_{\geq 0}$, the regularization of the distribution $z^s : C_c^\infty(\mathbb{C}) \longrightarrow \mathbb{C}$ is given by*

$$\begin{aligned}\langle z^s, \varphi\rangle = &\int_{|z|\leq 1}\left[\varphi(z,\bar{z}) - \sum_{k+l=0}^{m-1}\varphi^{(k,l)}(0,0)\frac{z^k\bar{z}^l}{k!l!}\right]|z|^{2s}dz d\bar{z} \\ &+ \int_{|z|>1}\varphi(z,\bar{z})|z|^{2s}dzd\bar{z} - 2\pi i\sum_{k=0}^{m-1}\frac{\varphi^{(k,k)}(0,0)}{(k!)^2(s+k+1)}, \quad \operatorname{Re}(s) > -m-1,\end{aligned} \tag{7.3}$$

where $\varphi^{(i,j)} := \partial^{i+j}\varphi/\partial z^i \partial\bar{z}^j$. Hence, z^s has poles at $s=-k-1$ for $k \in \mathbb{Z}_{\geq 0}$ with residues

$$\operatorname*{Res}_{s=-k-1} z^s = -\frac{2\pi i}{(k!)^2}\delta_0^{(k,k)}, \tag{7.4}$$

where $\delta_0^{(i,j)}$ are the distributional derivatives of the Dirac's delta function defined by $\langle \delta_0^{(i,j)}, \varphi\rangle := (-1)^{i+j}\varphi^{(i,j)}(0,0)$. Furthermore, in the strip $-m-1 < \operatorname{Re}(s) < -m$, Equation (7.3) reduces to

$$\langle z^s, \varphi\rangle = \int_{\mathbb{C}}\left[\varphi(z,\bar{z}) - \sum_{k+l=0}^{m-1}\varphi^{(k,l)}(0,0)\frac{z^k\bar{z}^l}{k!l!}\right]|z|^{2s}dzd\bar{z}. \tag{7.5}$$

For a fixed $\varphi \in C_c^\infty(\mathbb{C})$, Proposition 7.1 gives the meromorphic continuation to the whole complex plane of the holomorphic function of s defined by the integral $\langle z^s, \varphi\rangle$. For any polynomial f, we will talk indistinguishably about the meromorphic continuation or the (canonical) regularization of its complex zeta function f^s.

Remark 7.1. Although in Proposition 7.1 the test function φ is assumed to be in $C_c^\infty(\mathbb{C})$, the proof of the result only uses the fact that φ is infinitely differentiable near 0 and compactly supported. This means that the same result works for a meromorphic φ with poles away from 0 and compact support. In particular, if $\varphi(z,\bar{z};s) \in C^\infty(U)$, where U is a neighborhood of 0, and compactly supported, the poles of $\langle z^s, \varphi(z,\bar{z};s)\rangle$ will be the negative integers $\mathbb{Z}_{<0}$ together with the poles of $\varphi(z,\bar{z};s)$ in s away from U. ∎

7.2 *Resolution of singularities & Bernstein-Sato polynomial*

Resolution of the singularities of $f \in \mathbb{C}[z_1,\ldots,z_n]$ is used by Bernstein and Gel'fand [BG69] and Atiyah [Ati70] to reduce the problem of finding the analytic continuation of f^s to the monomial case considered in Proposition 7.1. Using the notations from

Section 1.1, let $\pi : X' \longrightarrow \mathbb{C}^n$ be a resolution of f with $F_\pi := \sum_i N_i E_i + \sum_j M_i S_i$ the total transform divisor and $K_\pi := \sum_i k_i E_i$ the relative canonical divisor, and suppose that $E := \mathrm{Exc}(\pi) = \sum_i E_i$ is the exceptional locus. Let $\{U_\alpha\}_{\alpha\in\Lambda}$ be an affine open cover of X'. Take $\{\eta_\alpha\}$ a partition of unity subordinated to the cover $\{U_\alpha\}_{\alpha\in\Lambda}$. That is, $\eta_\alpha \in C^\infty(\mathbb{C}^n)$ (not necessarily with compact support), $\sum_\alpha \eta_\alpha \equiv 1$, with only finitely many η_α being non-zero at a point of X' and $\mathrm{Supp}(\eta_\alpha) \subseteq U_\alpha$. Then, with a small abuse of notation,

$$\begin{aligned}\langle f^s, \varphi\rangle &= \int_{X'} |\pi^* f|^{2s} (\pi^*\varphi) |d\pi|^2 \\ &= \sum_{\alpha\in\Lambda} \int_{U_\alpha} |z_1|^{2(N_{1,\alpha}s + k_{1,\alpha})} \cdots |z_n|^{2(N_{n,\alpha}s+k_{n,\alpha})} |u_\alpha(z)|^{2s} |v_\alpha(z)|^2 \varphi_\alpha(z,\bar{z}) dz d\bar{z},\end{aligned} \tag{7.6}$$

where $\varphi_\alpha := \eta_\alpha \pi^* \varphi$ for each $\alpha \in \Lambda$ and $u_\alpha(\mathbf{0}), v_\alpha(\mathbf{0}) \neq 0$. The resolution morphism π being proper implies that both E and $\pi^{-1}(\mathrm{Supp}(\varphi))$ are compact sets. Since the singularities of the integral $\langle f^s, \varphi\rangle$ are produced by the zero set of f, in order to study the poles of f^s, it is enough to consider a finite affine open cover $\{U_\alpha\}_{\alpha\in\Lambda}$ of E consisting of neighborhoods of points $p_\alpha \in E$ and such that $\mathrm{Supp}(\varphi) \subseteq \pi(\cup_\alpha U_\alpha)$.

From Equation (7.6) and Proposition 7.1, we see that each divisor D_i in the support of F_π generates a set of candidate poles of f^s, namely

$$-\frac{k_i + 1 + \nu}{N_i}, -\frac{\nu}{M_i}, \quad \nu \in \mathbb{Z}_{\geq 0}. \tag{7.7}$$

The opposite in sign to the largest pole coincides with the log-canonical threshold $\mathrm{lct}(f)$ of f, see Section 1.2 and Section 1.3, and sets the maximal region of holomorphy of $\langle f^s, \varphi\rangle$ for a general $\varphi \in C_c^\infty(\mathbb{C}^n)$. Resolution of singularities then solves Gel'fand's first question in [Gel57]. However, nothing is said about the residues of f^s at those poles. Moreover, the set of candidates is usually much large than the actual poles of f^s.

On the other hand, the Bernstein-Sato polynomial and its functional equation, introduced in Section 1.2, together with integration by parts can be used to obtain the analytic continuation of f^s in a different way.

Proposition 7.2. *The complex zeta function f^s admits a meromorphic continuation to $\mathbb{C}$ with poles among the rational numbers $\alpha - k$ with $b_f(\alpha) = 0$ and $k \in \mathbb{Z}_{\geq 0}$.*

Proof. We can use the functional Equation (1.8) and integration by parts to analytically continue Equation (7.1) in the following way

$$\begin{aligned}\langle f^s, \varphi\rangle = \int_{\mathbb{C}^n} \varphi(z,\bar{z}) |f(z)|^{2s} dz d\bar{z} &= \frac{1}{b_f^2(s)} \int_{\mathbb{C}^n} \varphi(z,\bar{z}) \left[P(s)\cdot f^{s+1}(z)\right] \left[\bar{P}(s)\cdot f^{s+1}(\bar{z})\right] dz d\bar{z} \\ &= \frac{1}{b_f^2(s)} \int_{\mathbb{C}^n} \bar{P}^* P^*(s) \big(\varphi(z,\bar{z})\big) |f(z)|^{2(s+1)} dz d\bar{z}.\end{aligned} \tag{7.8}$$

The last term of Equation (7.8) defines an analytic function whenever $\mathrm{Re}(s) > -1$, except for possible poles at $b_f^{-1}(0)$, and it is equal to $\langle f^s, \varphi\rangle$ in $\mathrm{Re}(s) > 0$. If a differential operator has the form $P(s) = \sum_{\underline{\beta}} a_{\underline{\beta}}(s,z) (\frac{\partial}{\partial z})^{\underline{\beta}}$, we have considered

$$\bar{P}(s) := \sum_{\underline{\beta}} a_{\underline{\beta}}(\bar{s},\bar{z}) \left(\frac{\partial}{\partial \bar{z}}\right)^{\underline{\beta}}, \quad P^*(s) := \sum_{\underline{\beta}} (-1)^{|\underline{\beta}|} \left(\frac{\partial}{\partial z}\right)^{\underline{\beta}} a_{\underline{\beta}}(z,s), \tag{7.9}$$

the *conjugate* and *adjoint operator* of $P(s)$, respectively. Iterating the process we get

$$\langle f^s, \varphi \rangle = \frac{\langle f^{s+k+1}, \bar{P}^* P^*(s+k) \cdots \bar{P}^* P^*(s)(\varphi) \rangle}{b_f^2(s) \cdots b_f^2(s+k)}, \quad \mathrm{Re}(s) > -k-1, \tag{7.10}$$

and the result follows. □

The set of poles of the complex zeta function f^s is known to be exactly the set $\alpha - k$ with $b_f(\alpha) = 0$ and $k \in \mathbb{Z}_{\geq 0}$ for reduced plane curve singularities and isolated quasi-homogeneous singularities, see [Loe85, Thm. 1.9]. Therefore, at least in these cases, the divisors contributing to poles of the complex zeta function f^s are the same divisors that contribute to roots of the Bernstein-Sato polynomial $b_f(s)$. However, even in these cases, it is not straightforward to relate poles of f^s with roots of $b_f(s)$. In general, from Theorem 1.6 and Proposition 7.2, one has that,

Corollary 7.3. *Every pole $\sigma \in [-n + \mathrm{lct}(f), -\mathrm{lct}(f)]$ of f^s such that $\sigma + k$ is not a root of $b_f(s)$ for all $k \in \mathbb{Z}_{>0}$ is a root of $b_f(s)$.*

8 POLES AND RESIDUES FOR PLANE CURVES

Let $f : (\mathbb{C}^2, \mathbf{0}) \longrightarrow (\mathbb{C}, 0)$ be a plane curve not necessarily reduced or irreducible. After fixing local coordinates x, y, and with a small abuse of notation, let $f \in \mathbb{C}\{x, y\}$ be an equation of the germ, assuming it is defined in a neighborhood $U \subseteq \mathbb{C}^2$ of the origin. Following Section 7.1, define the complex zeta function f^s of a local singularity as

$$\langle f^s, \varphi \rangle := \int_U |f(x,y)|^{2s} \varphi(z) dz, \quad \text{for} \quad \mathrm{Re}(s) > 0, \tag{8.1}$$

with $\varphi \in C_c^\infty(U)$ and $z := (x, y, \bar{x}, \bar{y})$. The poles of $\langle f^s, \varphi \rangle$ do not depend on the equation of the germ or the local coordinates x, y. As discussed in Section 7.1, f^s must be understood in the distributional sense. In this section, we will use the minimal embedded resolution of singularities of f to study the structure of the residues of f^s at any candidate pole σ. The residue will be expressed as an improper integral along the exceptional divisor associated with σ. In order to do so, we first present the straightforward generalization of Proposition 7.1 to the two-dimensional case and see how poles of order two might arise. Finally, we will use the residue formula to prove that most non-rupture divisors do not contribute to poles of the complex zeta function f^s.

8.1 *Regularization of monomials in two variables*

The result from Proposition 7.1 can be easily generalized to the two-dimensional case, mimicking the proof in [GS64], to see how poles of order two arise. Let $\varphi(z_1, z_2) \in C_c^\infty(\mathbb{C}^2)$ which is, in fact, a function of $z = (z_1, z_2, \bar{z}_1, \bar{z}_2)$, and consider

$$\langle z_1^{s_1} z_2^{s_2}, \varphi \rangle = \int_{\mathbb{C}^2} |z_1|^{2s_1} |z_2|^{2s_2} \varphi(z) dz, \tag{8.2}$$

which is absolutely convergent for $\mathrm{Re}(s_1) > -1$, $\mathrm{Re}(s_2) > -1$ since φ has compact support.

Let $\Delta_0 = D_1 \times D_2$ be the polydisc formed by the discs of radius one centered at the origin, i.e. $D_1 = \{|z_1| \leq 1\}$ and $D_2 = \{|z_2| \leq 1\}$. We can decompose $\mathbb{C}^2$ as the disjoint union

$$\Delta_0 \cup (D_1 \times \mathbb{C} \setminus D_2) \cup (\mathbb{C} \setminus D_1 \times D_2) \cup (\mathbb{C} \setminus D_1 \times \mathbb{C} \setminus D_2). \tag{8.3}$$

Using the notation $z^k = z_1^{k_1} z_2^{k_2} \bar{z}_1^{k_3} \bar{z}_2^{k_4}$, the integral in Equation (8.2) on the region Δ_0 can be written as

$$\int_{\Delta_0} |z_1|^{2s_1} |z_2|^{2s_2} \left(\varphi(z) - \sum_{|k| \leq m} \frac{\partial^k \varphi(\mathbf{0})}{\partial z^k} \frac{z^k}{k!} \right) dz - \sum_{|k| \leq m} \frac{\partial^k \varphi}{\partial z^k}(\mathbf{0}) \frac{4\pi^2}{k!(s_1 + k_1 + 1)(s_2 + k_2 + 1)}, \tag{8.4}$$

where, in the second summation, we have $k_1 = k_3, k_2 = k_4$. The integral in the left-hand side is holomorphic on the regions $\mathrm{Re}(s_1) > -m - 1, \mathrm{Re}(s_2) > -m - 1$.

With a small abuse of the notation, let $z_1 = (z_1, \bar{z}_1), z_2 = (z_2, \bar{z}_2)$ and $z_1^{k_1} = z_1^{k_{1,1}} \bar{z}_1^{k_{1,2}}$. On the region $D_1 \times \mathbb{C} \setminus D_2$, the integral in Equation (8.2) is

$$\begin{aligned} &\int_{D_1 \times \mathbb{C} \setminus D_2} |z_1|^{2s_1} |z_2|^{2s_2} \left(\varphi(z) - \sum_{|k_1| \leq m} \frac{\partial^{k_1} \varphi}{\partial z_1^{k_1}}(0, z_2) \frac{z_1^{k_1}}{k_1!} \right) dz_1 dz_2 \\ &\qquad -2\pi i \sum_{|k_1| \leq m} \frac{\displaystyle\int_{|z_2| > 1} |z_2|^{2s_2} \frac{\partial^{k_1} \varphi}{\partial z_1^{k_1}}(0, z_2) dz_2}{(k_{1,1}!)^2 (s_1 + k_{1,1} + 1)}, \end{aligned} \tag{8.5}$$

where, in the second sum, $k_{1,1} = k_{1,2}$. The left-hand integral is holomorphic in $\mathrm{Re}(s_1) > -m - 1$. By symmetry, a similar expression holds in the other region, $\mathbb{C} \setminus D_1 \times D_2$. On the last region, $\mathbb{C} \setminus D_1 \times \mathbb{C} \setminus D_2$, the integral in Equation (8.2) is absolutely convergent for all $s_1, s_2 \in \mathbb{C}$.

From the regularization of $z_1^{s_1} z_2^{s_2}$ constructed above, we can see that the residue of $z_1^{s_1} z_2^{s_2}$ at a simple pole $s_1 = -k_1 - 1, k_1 \in \mathbb{Z}_{>0}$, i.e. the coefficient of $(s_1 + k_1 + 1)^{-1}$, is given by the following function of s_2

$$\operatorname*{Res}_{s_1 = -k_1 - 1} \langle z_1^{s_1} z_2^{s_2}, \varphi \rangle = -\frac{2\pi i}{(k_1!)^2} \int_{\mathbb{C}} |z_2|^{2s_2} \frac{\partial^{2k_1} \varphi}{\partial z_1^{k_1} \partial \bar{z}_1^{k_1}}(0, s_2) dz_2 d\bar{z}_2. \tag{8.6}$$

The residue in Equation (8.6) is a meromorphic function of s_2 which will have simple poles at the points $s_2 = -k_2 - 1, k_2 \in \mathbb{Z}_{>0}$. A pole of $\mathrm{Res}_{s_1 = -k_1 - 1} z_1^{s_1} z_2^{s_2}$ as a function of s_2 means that $z_1^{s_1} z_2^{s_2}$ has a pole of order one in both s_1 and s_2.

8.2 *The residue at the poles*

Let $E_p, p \in K$ be an irreducible exceptional divisor of the minimal embedded resolution of $f : (\mathbb{C}^2, \mathbf{0}) \longrightarrow (\mathbb{C}, 0)$. We will denote by $D_1, D_2, \ldots, D_r \in \mathrm{Div}(X')$ the other prime components (exceptional or not) of F_π crossing E_p. By definition, dead-end divisors have only one divisor crossing them, which will be denoted D_1. On the other hand, rupture divisors have at least three divisors crossing them, i.e. $r \geq 3$. In any other case, $r = 2$. We will denote by $N_1, N_2, \ldots, N_r$ (resp. $k_1, k_2, \ldots, k_r$) the coefficients of $D_1, D_2, \ldots, D_r$ in F_π (resp. in K_π). Since no confusion arises, we drop the explicit dependence on $p \in K$.

For each $E_p, p \in K$ we consider two affine charts U_p, V_p containing E_p that arise after the blow-up of a neighborhood of p in any chart containing p. The origin of the charts U_p, V_p are neighborhoods of opposite points in the projective line E_p. Usually, these points are the intersection points of E_p with two other components of F_π which we will assume to be D_1 and D_2. For simplicity, if this is not the case, we will set D_1, N_1, k_1 (or D_2, N_2, k_2) to be zero.

In order to define the complex zeta function f^s on X', we need to work locally with coordinates. Accordingly, let $(x_p, y_p), (z_p, w_p)$ be the natural holomorphic coordinates of U_p, V_p centered at the origins of both charts which, by construction, are the origin and the infinity point on a $\mathbb{P}^1_{\mathbb{C}}$, or vice versa. The coordinates $(x_p, y_p), (z_p, w_p)$ are related at the intersection $U_p \cap V_p$ by $x_p = z_p^{\kappa_p} w_p, y_p z_p = 1$, and where the integer value $\kappa_p \in \mathbb{Z}_{>0}$ has a very precise geometric meaning, namely, $\kappa_p = -E_p \cdot E_p$.

Following the discussion in Section 7.1, each exceptional component E_p contributes with a sequence of candidate poles to the meromorphic continuation of f^s. Indeed, with the notations above,

$$\left\{ \sigma_{p,\nu} = -\frac{k_p + 1 + \nu}{N_p} \;\middle|\; \nu \in \mathbb{Z}_{\geq 0} \right\}, \quad p \in K. \tag{8.7}$$

The set $\mathcal{W}_K := \{U_p, V_p\}_{p \in K}$ forms a finite affine open cover X', which applied to the construction presented in Section 7.1, Equation (7.6), results in the following proposition. First, denote by $\{\eta_{1,q}, \eta_{2,q}\}_{q \in K}$ a partition of unity subordinated to the open cover $\mathcal{W}_K$.

Proposition 8.1. *Using the affine open cover $\mathcal{W}_K$, the part of the complex zeta function f^s on Equation (7.6) involving the exceptional divisor E_p can be written as the sum of the integrals over just two affine charts $U_p, V_p \in \mathcal{W}_K$ containing E_p, namely*

$$\begin{aligned} &\int_{U_p} |x_p|^{2(N_p s + k_p)} |y_p|^{2(N_1 s + k_1)} \Phi_1(x_p, y_p; s) \eta_1 dx_p dy_p d\bar{x}_p d\bar{y}_p \quad + \\ &\int_{V_p} |z_p|^{2(N_2 s + k_2)} |w_p|^{2(N_p s + k_p)} \Phi_2(z_p, w_p; s) \eta_2 dz_p dw_p d\bar{z}_p d\bar{w}_p, \end{aligned} \tag{8.8}$$

where $\Phi_1(x_i, y_i; s), \Phi_2(z_i, w_i; s)$ are infinitely many times differentiable at neighborhoods of the points $p_1 = E_p \cap D_1$ and $p_2 = E_p \cap D_2$. More precisely,

$$\Phi_1 := |u_1|^{2s} |v_1|^2 (\pi^* \varphi)\big|_{U_p}, \quad \Phi_2 := |u_2|^{2s} |v_2|^2 (\pi^* \varphi)\big|_{V_p}, \tag{8.9}$$

with the elements u_1, v_1 (resp. u_2, v_2) being units in the local ring at the points p_1 (resp. p_2). Finally, η_1 and η_2 have compact support and $\eta_1 E_p\big|_+ \eta_2 E_p\big|_\equiv 1$.

Proof. Let us denote by $\mathcal{W}_{p_1}, \mathcal{W}_{p_2}$ all the elements in $\mathcal{W}_K$ that contain p_1 and p_2. By construction, $\mathcal{W}_{p_1}$ is disjoint from $\mathcal{W}_{p_2}$, since there can be no affine open set in our collection $\mathcal{W}$ containing both p_1 and p_2, and the union of $\mathcal{W}_{p_1}$ and $\mathcal{W}_{p_2}$ contains all the charts from $\mathcal{W}_K$ containing E_p. Applying Equation (7.6) from Section 7.1, the part of f^s on X' where the divisor E_p appears is a sum of the integrals over the affine open sets from $\mathcal{W}_{p_1}$ and $\mathcal{W}_{p_2}$,

$$\sum_{U_q \in \mathcal{W}_{p_1}} \int_{U_q} |\pi^* f|^{2s}\big|_{U_q} (\pi^* \varphi)\big|_{U_q} |d\pi|^2 \eta_{1,q} + \sum_{V_q \in \mathcal{W}_{p_2}} \int_{V_q} |\pi^* f|^{2s}\big|_{V_q} (\pi^* \varphi)\big|_{V_q} |d\pi|^2 \eta_{2,q}. \tag{8.10}$$

Let us see that we can reduce (8.10) to Proposition 8.1. The proof is the same for both summations in (8.10). Since the elements of $\mathcal{W}_{p_1}$ are blow-up charts, the difference between the union and the intersection of all the elements in $\mathcal{W}_{p_1}$ is contained in a finite number of lines. Given that a finite number of lines have measure zero, they do not affect the integral, and we can replace the left-hand summation of (8.10) by

$$\sum_{U_q \in \mathcal{W}_{p_1}} \int_{U_q} |\pi^* f|^{2s}\big|_{U_q} (\pi^* \varphi)\big|_{U_q} |d\pi|^2 \eta_{1,q} = \int_{\cap U_q} |\pi^* f|^{2s}\big|_{\cap U_q} (\pi^* \varphi)\big|_{\cap U_q} |d\pi|^2 \eta_1. \quad (8.11)$$

This equality is true since $|\pi^* f|^{2s} \pi^* \varphi |d\pi|^2$ is a global section on X' and its restriction on the U_q agrees on the overlap of all $U_q \in \mathcal{W}_{p_1}$. Concerning the partitions of unity, we just set $\eta_1 := \sum_{U_q \in \mathcal{W}_{p_1}} \eta_{1,q}$. Finally, by the same argument as before, we can replace $\cap_{U_q \in \mathcal{W}_{p_1}} U_q$ by any $U_p \in \mathcal{W}_{p_1}$ yielding Proposition 8.1. Notice that, by definition of $\mathcal{W}_{p_1}$, no other $\eta \in \{\eta_{1,q}, \eta_{2,q}\}_{q \in K}$, except for those in η_1, has p_1 in its support. □

Before presenting the formula for the residue, let us introduce the following rational numbers associated with a candidate pole $\sigma_{p,\nu}$ of an irreducible exceptional divisor $E_p, p \in K$. They will play an important role in the analysis of the residues.

Definition 8.2 (Residue numbers)**.** Let $\sigma_{p,\nu}, \nu \in \mathbb{Z}_{\geq 0}$ be a candidate pole of f^s associated with an exceptional divisor $E_p, p \in K$ intersecting the divisors $D_1, D_2, \ldots, D_r \in \operatorname{Div}(X')$. Define the *residue numbers* as

$$\epsilon_{i,\nu} := N_i \sigma_{p,\nu} + k_i \in \mathbb{Q} \quad \text{for} \quad i = 1, \ldots, r. \quad (8.12)$$

For the ease of notation, we will omit the dependence of $\epsilon_{1,\nu}, \epsilon_{2,\nu}$ on $p \in K$. The following relations between $\epsilon_{1,\nu}, \epsilon_{2,\nu}, \ldots, \epsilon_{r,\nu}$ holds.

Lemma 8.3. *For any $\nu \in \mathbb{Z}_{\geq 0}$, we have*

$$\epsilon_{1,\nu} + \epsilon_{2,\nu} + \cdots + \epsilon_{r,\nu} + \kappa_p \nu + 2 = 0. \quad (8.13)$$

Proof. Consider the $\mathbb{Q}$–divisor $\sigma_{p,\nu} F_\pi + K_\pi$. Applying the adjunction formula for surfaces Proposition 2.4, $(\sigma_{p,\nu} F_\pi + K_\pi) \cdot E_p = \kappa_p - 2$, recall $\kappa_p = -E_p \cdot E_p$. On the other hand,

$$(\sigma_{p,\nu} F_\pi + K_\pi) \cdot E_p = \sum_{i=0}^{r} \epsilon_{i,\nu} - \kappa_p (N_p \sigma_{p,v} + k_p). \quad (8.14)$$

Since $N_p \sigma_{p,\nu} + k_p = -\nu - 1$, the result follows. □

A first instance of the numbers $\epsilon_{i,\nu}$ and of Equation (8.13), in the case of rupture divisors of irreducible plane curves and $\nu = 0$, already appeared in an article of Lichtin [Lic85].

The formula for the residue at a candidate pole $\sigma_{p,\nu}$ is presented next. The residue is expressed as an improper integral, see Remark 8.1 below, along the divisor E_p having singularities of orders $\epsilon_{1,\nu}, \epsilon_{2,\nu}, \ldots, \epsilon_{1,r}$ at the intersection points of E_p with $D_1, D_2, \ldots, D_r$.

Proposition 8.4. *The residue of the complex zeta function f^s at a candidate pole $s = \sigma_{p,\nu}$ is given by*

$$\begin{aligned} \operatorname*{Res}_{s=\sigma_{p,\nu}} \langle f^s, \varphi \rangle &= \frac{-2\pi i}{(\nu!)^2} \int_{\mathbb{C}} |y_p|^{2\epsilon_{1,\nu}} \frac{\partial^{2\nu} \Phi_1}{\partial x_p^\nu \partial \bar{x}_p^\nu} (0, y_p; \sigma_{p,\nu}) dy_p d\bar{y}_p & & (U_p \text{ chart}) \\ &= \frac{-2\pi i}{(\nu!)^2} \int_{\mathbb{C}} |z_p|^{2\epsilon_{2,\nu}} \frac{\partial^{2\nu} \Phi_2}{\partial w_p^\nu \partial \bar{w}_p^\nu} (z_p, 0; \sigma_{p,\nu}) dz_p d\bar{z}_p, & & (V_p \text{ chart}). \end{aligned} \quad (8.15)$$

Proof. Applying Equation (8.6) to Proposition 8.1 with $s_2 = N_1\sigma_{p,\nu} + k_1$ and $N_2\sigma_{p,\nu} + k_2$, respectively, we obtain that the residue of f^s at $s = \sigma_{p,\nu}$ is

$$\begin{aligned} \operatorname*{Res}_{s=\sigma_{p,\nu}} \langle f^s, \varphi\rangle = \frac{-2\pi i}{(\nu!)^2}\bigg(\int_{\mathbb{C}} |y_p|^{2(N_1\sigma_{p,\nu}+k_1)} \frac{\partial^{2\nu}\Phi_1\eta_1}{\partial x_p^\nu \partial \bar{x}_p^\nu}(0, y_p; \sigma_{p,\nu}) dy_p d\bar{y}_p \\ + \int_{\mathbb{C}} |z_p|^{2(N_2\sigma_{p,\nu}+k_2)} \frac{\partial^{2\nu}\Phi_2\eta_2}{\partial w_p^\nu \partial \bar{w}_p^\nu}(z_p, 0; \sigma_{p,\nu}) dz_p d\bar{z}_p \bigg). \end{aligned} \tag{8.16}$$

The differential form $|\pi^* f|^{2s}(\pi^*\varphi)|d\pi|^2$ is a global section on X' such that its restriction to U_p, resp. V_p, has Φ_1, resp. Φ_2, as factor. The global section differs only from Φ_1, Φ_2 by the exceptional part of D_1, D_2 in the total transform $|\pi^* f|^{2s}$. Thus, at the intersection $U_p \cap V_p$, having that $y_p z_p = 1, w_p = x_p^{\kappa_p} y_p$, one checks that

$$\begin{aligned} \Phi_2(z_p, w_p; \sigma_{p,\nu}) &= \Phi_2(y_p^{-1}, x_p^{\kappa_p} y_p; \sigma_{p,\nu}) \\ &= |y_p|^{-2(\epsilon_{3,\nu}+\cdots+\epsilon_{r,\nu})}\Phi_2(y_p, x_p^{\kappa_p} y_p; \sigma_{p,\nu}) \\ &= |y_p|^{-2(\epsilon_{3,\nu}+\cdots+\epsilon_{r,\nu})}\Phi_1(x_p, y_p; \sigma_{p,\nu}). \end{aligned} \tag{8.17}$$

Now, taking derivatives on both sides with respect to w_p and $\bar{w}_p$ and setting $w_p, \bar{w}_p = 0$ yields,

$$\frac{\partial^{2\nu}\Phi_2}{\partial w_p^\nu \partial \bar{w}_p^\nu}(z_p, 0; \sigma_{p,\nu}) = |y_p|^{-2(\epsilon_{3,\nu}+\cdots+\epsilon_{r,\nu}+\kappa_p\nu)} \frac{\partial^{2\nu}\Phi_1}{\partial x_p^\nu \partial \bar{x}_p^\nu}(0, y_p; \sigma_{p,\nu}). \tag{8.18}$$

This, together with Lemma 8.3, shows that the differential forms

$$|y_p|^{2\epsilon_{1,\nu}} \frac{\partial^{2\nu}\Phi_1}{\partial^\nu x_p \partial^\nu \bar{x}_p}(0, y_p; \sigma_{p,\nu}) \mathrm{d}y_p \wedge \mathrm{d}\bar{y}_p, \quad |z_p|^{2\epsilon_{2,\nu}} \frac{\partial^{2\nu}\Phi_2}{\partial w_p^\nu \partial \bar{w}_p}(z_p, 0; \sigma_{p,\nu}) \mathrm{d}w_p \wedge \mathrm{d}\bar{w}_p, \tag{8.19}$$

define a global section on E_p. As a consequence, it suffices to use $z_p y_p = 1$ in either of the integrals in Equation (8.16), together with the fact that $\eta_1|_{E_p} + \eta_2|_{E_p} \equiv 1$. □

Corollary 8.5. *The residue of the complex zeta function f^s at $s = \sigma_{p,\nu}$ is given, in the U_p chart, by*

$$\begin{aligned} \operatorname*{Res}_{s=\sigma_{p,\nu}} \langle f^s, \varphi\rangle =& \frac{-2\pi i}{(\nu!)^2} \int_{|y_p|\le R} |y_p|^{2\epsilon_{1,\nu}} \frac{\partial^{2\nu}\Phi_1}{\partial x_p^\nu \partial \bar{x}_p^\nu}(0, y_p; \sigma_{p,\nu}) dy_p d\bar{y}_p \\ &+ \frac{-2\pi i}{(\nu!)^2} \int_{|y_p|>R} |y_p|^{2\epsilon_{1,\nu}} \frac{\partial^{2\nu}\Phi_1}{\partial x_p^\nu \partial \bar{x}_p^\nu}(0, y_p; \sigma_{p,\nu}) dy_p d\bar{y}_p, \quad R > 0, \end{aligned} \tag{8.20}$$

and analogously for the other chart V_p.

Proof. The function η_1 can be chosen to be continuous and such that its restriction to E_p is $\eta_1|_{E_p} \equiv 1$ in $|y_p| \le R$ and 0 in $|y_p| > R$. Because $z_p y_p = 1$ on the overlap of any two charts of E_p, $\eta_2|_{E_p}$ must be identically 1 in $|z_p| < 1/R$, i.e. $|y_p| > R$, and zero in the complement. The results follow now from the proof of the previous proposition. Indeed, substitute such η_1 and η_2 in Equation (8.16) and use the fact that the integrand is a global section on E_p. □

Remark 8.1. The value of the residue $\mathrm{Res}_{s=\sigma_{p,\nu}}\langle f^s, \varphi\rangle$ must be understood as the analytic continuation of the functions,

$$\begin{aligned} I_1(\alpha_1, \beta_3, \ldots, \beta_r) &= \int_{\mathbb{C}} |y_p|^{2\alpha_1} \frac{\partial^{2\nu}\Phi_1}{\partial x_p^\nu \partial \bar{x}_p^\nu}(0, y_p; \beta_3, \ldots, \beta_r) dy_p d\bar{y}_p, \\ I_2(\alpha_2, \beta_3, \ldots, \beta_r) &= \int_{\mathbb{C}} |z_p|^{2\alpha_2} \frac{\partial^{2\nu}\Phi_2}{\partial w_p^\nu \partial \bar{w}_p^\nu}(z_p, 0; \beta_3, \ldots, \beta_r) dz_p d\bar{z}_p, \end{aligned} \tag{8.21}$$

at the rational points $(\epsilon_{1,\nu}, \epsilon_{3,\nu}, \ldots, \epsilon_{r,\nu})$ and $(\epsilon_{2,\nu}, \epsilon_{3,\nu}, \ldots, \epsilon_{r,\nu})$, respectively, as these points will usually be outside the region of convergence of the integrals defining I_1, I_2. For simplicity of the notation, we always present Φ_1, Φ_2 depending only on a single variable $\sigma_{p,\nu}$, as $\epsilon_{3,r}, \ldots, \epsilon_{r,\nu}$ are in fact $N_3\sigma_{p,\nu} + k_3 = \epsilon_{3,\nu}, \ldots, N_r\sigma_{p,\nu} + k_r = \epsilon_{r,\nu}$. ■

Using the residue formula from Proposition 8.4, we can detect whether the complex zeta function f^s has poles of order one or two. Since the residue of $z_1^{s_1} z_2^{s_2}$ as a function of s_2 has poles, the residue of f^s at $s = \sigma_{p,\nu}$ can be infinity, in the sense that the analytic continuation of the functions in Equation (8.21) at the points $\epsilon_{i,\nu}$ has a pole. Therefore,

Lemma 8.6. *If the residue $\mathrm{Res}_{s=\sigma_{p,\nu}} f^s$ is infinite, then $\sigma_{p,\nu}$ is a pole of order two. Conversely, if $\mathrm{Res}_{s=\sigma_{p,\nu}} f^s$ is zero, $\sigma_{p,\nu}$ is neither a pole of order one or two. In particular, a necessary condition for $\sigma_{p,\nu}$ to be a pole of order two is that $\epsilon_{i,\nu} \in \mathbb{Z}_{<0}$ for some $i = 1, \ldots, r$.*

Proof. In the proof of Proposition 8.4, we have used the expression for the residue of $z_1^{s_1} z_2^{s_2}$ at $s_1 = N_p\sigma_{p,\nu} + k_p = -\nu - 1$ with $s_2 = N_i\sigma_{p,\nu} + k_i = \epsilon_{i,\nu}$, to deduce the expression for the residue of f^s at $s = \sigma_{p,\nu}$. Thus, in terms of the Laurent expansions, a term $(s_1 + \nu_1 + 1)^{-1}(s_2 + \nu_2 + 1)^{-1}$ from $z_1^{s_1} z_2^{s_2}$ becomes a term $(N_p s + k_p + \nu_1 + 1)^{-1}(N_i s + k_i + \nu_2 + 1)^{-1}$, which may generate a pole if $\epsilon_{i,\nu} \in \mathbb{Z}_{<0}$. □

Finally, we end this section with the following important observation. As in the monomial case $\langle z^s, \varphi\rangle$ considered in Proposition 7.1, where the residue is interpreted in terms of the derivatives of the test function φ, and consequently, in terms of the Dirac's delta function, the same holds for any f^s. The derivatives of Φ_1, Φ_2 involve taking derivatives on $\pi^*\varphi$ which, by the product rule of differentiation and the fact that $(\pi^*\varphi)\big|_{E_p} = \varphi(\mathbf{0})$, imply that

$$\operatorname*{Res}_{s=\sigma_{p,\nu}} f^s \in \big\langle \delta_{\mathbf{0}}^{(0,0,0,0)}, \delta_{\mathbf{0}}^{(1,0,0,0)}, \delta_{\mathbf{0}}^{(0,1,0,0)}, \ldots, \delta_{\mathbf{0}}^{(\nu,\nu,\nu,\nu)} \big\rangle_{\mathbb{C}}. \tag{8.22}$$

Therefore, the residue of f^s at any candidate pole must be also understood as a distribution in this precise sense. As a consequence, the residue of f^s at a candidate pole will be zero when all the coefficients of the linear combination in Equation (8.22) are zero. In a similar way, the residue will be non-zero when just one of the coefficients is non-zero.

8.3 *Residues at non-rupture divisors*

The exact expression of the residue is quite involved due to the presence of the ν–th derivative of Φ_1 or Φ_2. However, from the study of the derivatives of the factors of Φ_1 or Φ_2, we will show when the residues at a candidate $\sigma_{p,\nu}$ is zero for a non-rupture exceptional divisor E_p. The proof uses the following technical results.

Lemma 8.7 (Faà di Bruno's formula, [Com74, p. III.3.4]). *Let g, h be two infinitely many times differentiable functions. Then,*

$$\frac{d^\nu}{dx^\nu} g(h(x)) = \sum_{k=1}^{\nu} \frac{d^k g}{dx^k}(h(x)) B_{\nu,k}\left(\frac{dh}{dx}(x), \frac{d^2h}{dx^2}(x), \dots, \frac{d^{\nu-k+1}h}{dx^{\nu-k+1}}(x)\right), \tag{8.23}$$

where $B_{\nu,k}$ are the partial exponential Bell polynomials

$$B_{\nu,k}(x_1, x_2, \dots, x_{\nu-k+1}) := \sum \frac{\nu!}{j_1! j_2! \cdots j_{\nu-k+1}!}\left(\frac{x_1}{1!}\right)^{j_1}\left(\frac{x_2}{2!}\right)^{j_2} \cdots \left(\frac{x_{\nu-k+1}}{(\nu-k+1)!}\right)^{j_{\nu-k+1}}, \tag{8.24}$$

and where the summation takes places over all integers $j_1, j_2, j_3, \dots, j_{\nu-k+1}$, such that

$$\begin{aligned} j_1 + j_2 + j_3 + \cdots + j_{\nu-k+1} &= k, \\ j_1 + 2j_2 + 3j_3 + \cdots + (\nu-k+1)j_{\nu-k+1} &= \nu. \end{aligned} \tag{8.25}$$

For instance, in the chart U_p around E_p, we are interested in the situation where $g(x) = x^s$ and $h(x_p)$ is equal to $u_1(x_p, y_p)$ from Proposition 8.1, and we set $x_p = 0$ after deriving. In this case, Lemma 8.7 reads as

$$\frac{\partial^\nu u_1^s}{\partial x_p^\nu}(0, y_p) = \sum_{k=1}^{\nu} (s)_k \big(u_1(0, y_p)\big)^{s-k} B_{\nu,k}\left(\frac{du_1}{dx_p}(0, y_p), \frac{d^2u_1}{dx_p^2}(0, y_p), \dots, \frac{d^{\nu-k+1}u_1}{dx_p^{\nu-k+1}}(0, y_p)\right), \tag{8.26}$$

where $(s)_k := s(s-1)\cdots(s-k+1)$. And similarly in the other chart V_p.

Proposition 8.8 ([GS64, p. I.3.8]). *For any $\alpha, \alpha' \in \mathbb{C}$ such that $\alpha' - \alpha = n \in \mathbb{Z}$, the analytic continuation of the sum*

$$I_n(\alpha) := \int_{|z| \le R} z^{\alpha'} \bar{z}^\alpha dz d\bar{z} + \int_{|z| > R} z^{\alpha'} \bar{z}^\alpha dz d\bar{z} \quad \text{for any} \quad R > 0. \tag{8.27}$$

is zero everywhere, i.e. $I_n(\alpha) \equiv 0$.

Proof. Using polar coordinates[1]

$$-2i \int_0^R \int_0^{2\pi} r^{2\alpha+n+1} e^{2\pi i n\theta} d\theta dr - 2i \int_R^\infty \int_0^{2\pi} r^{2\alpha+n+1} e^{2\pi i n\theta} d\theta dr. \tag{8.28}$$

However,

$$\int_0^{2\pi} e^{2\pi i n\theta} d\theta = \begin{cases} 0, & n \neq 0, \\ 2\pi, & n = 0. \end{cases} \tag{8.29}$$

Hence, the result follows if $n \neq 0$. In the case that $n = 0$, the first integral defines a holomorphic function in $\mathrm{Re}(\alpha) > -1$. It can be analytically continued by means of

$$-4\pi i \int_0^R r^{2\alpha+1} dr = -2\pi i \frac{R^{2(\alpha+1)}}{\alpha+1} \quad \text{for} \quad \alpha \neq -1. \tag{8.30}$$

Similarly, the other integral defines a holomorphic function in $\mathrm{Re}(\alpha) < -1$, and the analytic continuation to the whole complex plane is

$$-4\pi i \int_R^\infty r^{2\alpha+1} dr = 2\pi i \frac{R^{2(\alpha+1)}}{\alpha+1} \quad \text{for} \quad \alpha \neq -1. \tag{8.31}$$

Finally, the sum of the analytic continuation of both integrals is identically zero. □

1 By definition, $z^{\alpha'} \bar{z}^\alpha := |z|^{\alpha'+\alpha} e^{i(\alpha'-\alpha)\arg z}$, which, for integral $\alpha' - \alpha$, is a single-valued function of z.

In the following proposition, we generalize a calculation attributed to Cohen appearing in an article of Barlet [Bar86] and used by Lichtin in [Lic89]. The original result gives a closed form for the integral in Equation (8.32) below in the case $R_{0,0}(\alpha,\beta)$. We provide a formula for the general case $R_{n,m}(\alpha,\beta)$.

Proposition 8.9. *For $\alpha,\alpha',\beta,\beta' \in \mathbb{C}$, such that $\alpha'-\alpha = n \in \mathbb{Z}$ and $\beta'-\beta = m \in \mathbb{Z}$, the integral*

$$R_{n,m}(\alpha,\beta) := \int_{\mathbb{C}} z^{\alpha'}\bar{z}^{\alpha}(1-\lambda z)^{\beta'}(1-\bar{\lambda}\bar{z})^{\beta} dz d\bar{z} = R_{-n,-m}(\alpha',\beta'), \quad \lambda \in \mathbb{C}^*, \tag{8.32}$$

is absolutely convergent for $\mathrm{Re}(\alpha'+\alpha) > -2$, $\mathrm{Re}(\beta'+\beta) > -2$ *and* $\mathrm{Re}(\alpha'+\alpha+\beta'+\beta) < -2$. *It defines a meromorphic function on $\mathbb{C}^2$ equal to*

$$R_{n,m}(\alpha,\beta) = -2\pi i \lambda^{-\alpha'-1}\bar{\lambda}^{-\alpha-1}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma+1)}{\Gamma(-\alpha-n)\Gamma(-\beta-m)\Gamma(-\gamma-n-m)}, \tag{8.33}$$

where $\gamma := -\alpha-\beta-n-m-2$.

Proof. Let us prove first the case $m=0$. Since $R_{n,0}(\alpha,\beta) = R_{-n,0}(\alpha',\beta')$, we can assume $n \in \mathbb{Z}_{\geq 0}$. Using polar coordinates $\lambda z = re^{i\theta}$, we have that $(1-\lambda z)^{\beta'}(1-\bar{\lambda}\bar{z})^{\beta} = |1-\lambda z|^{2\beta} = (1-2r\cos\theta + r^2)^{\beta}$ and

$$R_{n,0}(\alpha,\beta) = -2i\lambda^{-\alpha'-1}\bar{\lambda}^{-\alpha-1}\int_0^{2\pi}\int_0^{\infty} r^{2\alpha+n+1}e^{in\theta}(1-2r\cos\theta+r^2)^{\beta} dr d\theta. \tag{8.34}$$

For simplicity, we can set $\lambda = 1$. We have,

$$(1-2r\cos\theta+r^2)^{\beta} = (1+r^2)^{\beta}\left(1-\frac{2r\cos\theta}{1+r^2}\right)^{\beta} \tag{8.35}$$

and since $|2r/(1+r^2)| \leq 1$, we may expand the binomial,

$$\left(1-\frac{2r\cos\theta}{1+r^2}\right)^{\beta} = \sum_{k=0}^{\infty}\binom{\beta}{k}\frac{(-2)^k r^k}{(1+r^2)^k}\cos^k\theta. \tag{8.36}$$

The angular part of the integral is just

$$\int_0^{2\pi} e^{in\theta}\cos^k\theta \, d\theta = \begin{cases} \dfrac{2\pi}{4^l}\dbinom{2l}{l-s}, & \text{if} \quad k = 2l \geq n = 2s, \quad l \in \mathbb{Z}_{\geq 0}, s \in \mathbb{Z}_{\geq 0}, \\ \dfrac{\pi}{4^l}\dbinom{2l+1}{l-s}, & \text{if} \quad k = 2l+1 \geq n = 2s+1, \quad l \in \mathbb{Z}_{\geq 0}, s \in \mathbb{Z}_{\geq 0}, \\ \quad 0, & \quad \text{otherwise.} \end{cases} \tag{8.37}$$

The integral then reads as

$$R_{2s,0}(\alpha,\beta) = -4\pi i\int_0^{\infty} r^{2\alpha+2s+1}(1+r^2)^{\beta}\sum_{l=0}^{\infty}\binom{2l}{l-s}\binom{\beta}{2l}\frac{r^{2l}}{(1+r^2)^{2l}}dr, \tag{8.38}$$

for n even, and

$$R_{2s+1,0}(\alpha,\beta) = 4\pi i\int_0^{\infty} r^{2\alpha+2s+2}(1+r^2)^{\beta}\sum_{l=0}^{\infty}\binom{2l+1}{l-s}\binom{\beta}{2l+1}\frac{r^{2l+1}}{(1+r^2)^{2l+1}}dr, \tag{8.39}$$

for n odd. Using that $\binom{\beta}{k} = (-1)^k \frac{\Gamma(k-\beta)}{\Gamma(-\beta)k!}$ and interchanging the summation and integral signs,

$$\begin{aligned} R_{2s,0}(\alpha,\beta) &= \frac{-4\pi i}{\Gamma(-\beta)} \sum_{l=0}^{\infty} \frac{\Gamma(2l-\beta)}{(l-s)!(l+s)!} \int_0^\infty r^{2\alpha+2s+2l+1}(1+r^2)^{\beta-2l} dr, \\ R_{2s+1,0}(\alpha,\beta) &= \frac{-4\pi i}{\Gamma(-\beta)} \sum_{l=0}^{\infty} \frac{\Gamma(2l+1-\beta)}{(l-s)!(l+s+1)!} \int_0^\infty r^{2\alpha+2s+2l+3}(1+r^2)^{\beta-2l-1} dr. \end{aligned} \tag{8.40}$$

Now, for $\mathrm{Re}(\mu) > 0$ and $\mathrm{Re}(2\nu+\mu) < 0$,

$$\int_0^\infty x^{\mu-1}(1+x^2)^\nu dx = \frac{1}{2}\mathrm{B}\left(\frac{\mu}{2}, -\nu-\frac{\mu}{2}\right) = \frac{1}{2}\Gamma\left(\frac{\mu}{2}\right)\Gamma\left(-\nu-\frac{\mu}{2}\right)\Gamma(-\nu)^{-1}. \tag{8.41}$$

See, for instance, [GR15, pp. 3.251–2]. For $\mu_l = 2(\alpha+s+l+1)$ and $\nu_l = \beta - 2l$,

$$R_{2s}(\alpha,\beta) = \frac{-2\pi i}{\Gamma(-\beta)} \sum_{l=0}^{\infty} \frac{\Gamma(\alpha+s+l+1)\Gamma(l-\alpha-s-\beta-1)}{(l-s)!(l+s)!}, \tag{8.42}$$

since $\mathrm{Re}(\mu_l) > 0$ and $\mathrm{Re}(2\nu_l + \mu_l) < 0$ for all $l \in \mathbb{Z}_{\geq 0}$. Analogously, for $n \geq 0$ odd, $\mu_l = 2(\alpha+s+l+2)$ and $\nu_l = \beta - 2l - 1$,

$$R_{2s+1,0}(\alpha,\beta) = \frac{-2\pi i}{\Gamma(-\beta)} \sum_{l=0}^{\infty} \frac{\Gamma(\alpha+s+l+2)\Gamma(l-\alpha-s-\beta-1)}{(l-s)!(l+s+1)!}. \tag{8.43}$$

Since $\Gamma(-k)^{-1} = 0$ for $k \in \mathbb{Z}_{\geq 0}$, we can write

$$\begin{aligned} R_{2s,0}(\alpha,\beta) &= \frac{-2\pi i}{\Gamma(-\beta)} \sum_{l=0}^{\infty} \frac{\Gamma(\alpha+l+1)\Gamma(l-\alpha-2s-\beta-1)}{\Gamma(l-2s+1)\, l!}, \\ R_{2s+1,0}(\alpha,\beta) &= \frac{-2\pi i}{\Gamma(-\beta)} \sum_{l=0}^{\infty} \frac{\Gamma(\alpha+l+1)\Gamma(l-\alpha-2s-\beta-2)}{\Gamma(l-2s)\, l!}. \end{aligned} \tag{8.44}$$

Finally, for $\mathrm{Re}(c) > \mathrm{Re}(a+b)$,

$$\sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)k!} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} {}_2F_1(a,b;c;1) = \frac{\Gamma(a)\Gamma(b)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \tag{8.45}$$

where ${}_2F_1(a,b;c;z)$ is the hypergeometric function. For the last equality see, for instance, [GR15, pp. 9.122–1]. For $n = 2s$, set $a = \alpha+1, b = -\alpha-\beta-2s-1$ and $c = -2s+1$. Similarly, for $n = 2s+1$, $a = \alpha+1, b = -\alpha-\beta-2s-2$ and $c = -2s$. Then,

$$R_{2s,0}(\alpha,\beta) = R_{2s+1,0}(\alpha,\beta) = -2\pi i \frac{\Gamma(\alpha+1)\Gamma(-\alpha-\beta-n-1)\Gamma(\beta+1)}{\Gamma(-\beta)\Gamma(-n-\alpha)\Gamma(\alpha+\beta+2)}. \tag{8.46}$$

For the case where $m \neq 0$, having proved the result for $n \in \mathbb{Z}, m = 0$, we can assume without loss of generality that $m \in \mathbb{Z}_{\geq 0}$. Keeping $\lambda = 1$, notice that

$$R_{n,m}(\alpha,\beta) = \int_{\mathbb{C}} z^{\alpha'} \bar{z}^{\alpha} |1-z|^{2\beta}(1-z)^m dz d\bar{z} = \sum_{j=0}^{m} \binom{m}{j} (-1)^j R_{n+j,0}(\alpha,\beta). \tag{8.47}$$

Hence,

$$R_{n,m}(\alpha,\beta) = -2\pi i \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(-\beta)\Gamma(\alpha+\beta+2)} \sum_{j=0}^{m} \binom{m}{j} (-1)^j \frac{\Gamma(-\alpha-\beta-n-j-1)}{\Gamma(-n-j-\alpha)}. \tag{8.48}$$

Since the binomial coefficient is zero if $j > m$, we can consider the infinite sum. Expanding the binomial coefficient in terms of the Gamma function

$$R_{n,m}(\alpha,\beta) = -2\pi i \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(-\beta)\Gamma(\alpha+\beta+2)\Gamma(-m)} \sum_{j=0}^{\infty} \frac{\Gamma(j-m)\Gamma(-\alpha-\beta-n-j-1)}{\Gamma(-n-j-\alpha)\, j!}. \tag{8.49}$$

Using the functional equation $\Gamma(z+1) = z\Gamma(z)$ at each term

$$\begin{aligned} R_{n,m}(\alpha,\beta) = &-2\pi i \frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(-\alpha-\beta-n-1)\Gamma(\alpha+\beta+n+2)}{\Gamma(-\beta)\Gamma(\alpha+\beta+2)\Gamma(-m)\Gamma(-\alpha-n)\Gamma(\alpha+n+1)} \\ &\cdot \sum_{j=0}^{\infty} \frac{\Gamma(j-m)\Gamma(\alpha+n+j+1)}{\Gamma(\alpha+\beta+n+j+2)\, j!}. \end{aligned} \tag{8.50}$$

Applying Equation (8.45) once again, since $\mathrm{Re}(\beta'+\beta) > -2$ implies $\mathrm{Re}(\beta+m) > -1$,

$$R_{n,m}(\alpha,\beta) = -2\pi i \frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(-\alpha-\beta-n-1)\Gamma(\alpha+\beta+n+2)\Gamma(\beta+m+1)}{\Gamma(-\beta)\Gamma(\alpha+\beta+2)\Gamma(-\alpha-n)\Gamma(\alpha+\beta+n+m+2)\Gamma(\beta+1)}. \tag{8.51}$$

And we get the desired result using the functional equation $\Gamma(z+1) = z\Gamma(z)$ once again. □

It is now possible to prove the following result regarding non-rupture divisors. Recall that, with the notations set in the first paragraph of Section 8.2, the divisors crossing a non-rupture exceptional divisor E_p can only be D_1, D_2, D_3, with at least one being non-zero. Recall also the residue numbers $\epsilon_{1,\nu}, \epsilon_{2,\nu}, \epsilon_{3,\nu}$ from Definition 8.2.

Theorem 8.10. *Let $f : (\mathbb{C}^2, \mathbf{0}) \longrightarrow (\mathbb{C}, 0)$ be any plane branch. Let $E_p, p \in K$ be a non-rupture exceptional divisor with sequence of candidate poles $\sigma_{p,\nu}, \nu \in \mathbb{Z}_{\geq 0}$. Then,*

- *If $D_3 = 0$,*
$$\operatorname*{Res}_{s=\sigma_{p,\nu}} f^s = 0, \quad \text{for all} \quad \nu \in \mathbb{Z}_{\geq 0}. \tag{8.52}$$

- *If $D_3 \neq 0$,*
$$\operatorname*{Res}_{s=\sigma_{p,\nu}} f^s = 0, \quad \text{if} \quad \epsilon_{3,\nu} \notin \mathbb{Z}. \tag{8.53}$$

Proof. Let us first begin with D_3 non-zero. In this case, we must also have D_1 or D_2 non-zero. We can assume, for instance, D_1 non-zero.

By the definition of $D_1, D_2, \ldots, D_r$, if D_1 and D_3 are non-zero, D_3 is the only divisor crossing E_p in the V_p chart. In the (z_p, w_p) coordinates, this means that $u_2^s(z_p, 0)v_2(z_p, 0)$ has the form $(1-\lambda y_p)^{N_3 s + k_3}$ for some $\lambda \in \mathbb{C}^*$. By Faà di Bruno's formula in Lemma 8.7 and Equation (8.26), the ν–th holomorphic and antiholomorphic derivatives of Φ_2 at $z_p, \bar{z}_p = 0, s = \sigma_{p,\nu}$ is an algebraic function involving the terms $z_p^{k'} \bar{z}_p^{k} (1-\lambda z_p)^{\epsilon_{3,\nu}-l'} (1 - \bar{\lambda}\bar{y}_p)^{\epsilon_{3,\nu}-l}$ with $k, k', l, l' \in \mathbb{Z}_{\geq 0}$. Hence, the residue reduces to a finite sum involving the integrals $R_{n,m}(\alpha,\beta)$ from Equation (8.32) which, by Proposition 8.9, are proportional to

$$\frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma+1)}{\Gamma(-\alpha-n)\Gamma(-\beta-m)\Gamma(-\gamma-n-m)}, \quad n, m \in \mathbb{Z}. \tag{8.54}$$

Applying Lemma 8.3 to this situation implies that, $\epsilon_{1,\nu} + \epsilon_{3,\nu} + \kappa_p \nu + 2 = 0$. Notice that $\epsilon_{3,\nu} \notin \mathbb{Z}$ implies $\epsilon_{1,\nu} \notin \mathbb{Z}$. Since the poles of $\Gamma(z)$ are located at the negative integers, and $\Gamma(z)$ has no zeros, $\Gamma(\beta+1), \Gamma(\gamma+1), \Gamma(-\beta-m)^{-1}, \Gamma(-\gamma-n-m)^{-1} \in \mathbb{C}^*$. Finally, consider the non-negative integers k, k' in $z_p^{k'} \bar{z}_p^k$ above. The quotient $\Gamma(\alpha+1)/\Gamma(-\alpha-n)$ is zero since $\epsilon_{2,\nu} = 0$ and $\alpha + 1 = k + 1 > 0, -\alpha - n = -\alpha' = -k' \leq 0$.

If D_3 is zero and E_p is a non-rupture divisor, it may happen that D_2 is zero or not. If D_2 is not zero, it must cross E_p in the only point of V_p not in U_p. In both cases, in the chart U_p, none of the components u_1^s, v_1 of Φ_1 will cross E_p. In the (x_p, y_p) coordinates this means that $u_1^s(0, y_p) v_1(0, y_p) \in \mathbb{C}^*$ for all $y_p \in \mathbb{C}$. Applying Faà di Bruno's formula again, the holomorphic and antiholomorphic derivatives of Φ_1 restricted to $x_p, \bar{x}_p = 0$ are just polynomials in y_p and $\bar{y}_p$. By Corollary 8.5 and Proposition 8.8, the residue is zero. □

Example 8.1. There are examples where $D_3 \neq 0$ and $\epsilon_{3,\nu} \in \mathbb{Z}$ and the corresponding $\sigma_{p,\nu}$ has non-zero residue. For instance, $f = (y^2 - x^3)(y - x^2)^3$. Its minimal embedded resolution is given by

$$F_\pi = 5E_{p_0} + 9E_{p_1} + 15E_{p_2} + C_1 + 3C_2, \quad K_\pi = E_{p_0} + 2E_{p_1} + 4E_{p_2}. \tag{8.55}$$

In this case, E_{p_1} is a non-rupture exceptional divisor with E_{p_2} and C_2 crossing E_{p_1}. It can be checked that $\sigma_{p_1,0} = -1/3$ is a pole of order two of f^s. Here, $D_1 = E_{p_2}, D_3 = C_2$ and $\epsilon_{1,0} = \epsilon_{3,0} = -1$. ■

9 THE SET OF POLES OF THE COMPLEX ZETA FUNCTION OF A PLANE BRANCH

In this section, we restrict our study of the poles of the complex zeta function to the case of plane branch singularities. Throughout the rest of this work, we will fix a plane branch semigroup $\Gamma = \langle \overline{\beta}_0, \overline{\beta}_1, \ldots, \overline{\beta}_g \rangle$ and we stick to the notations from Section 2.4. Instead of taking any germ $f : (\mathbb{C}^2, \mathbf{0}) \longrightarrow (\mathbb{C}, 0)$ with semigroup Γ, we will work with the family $\{f_{t,\lambda}\}_{\lambda \in \mathbb{C}^{g-1}}$ from Proposition 2.17. Since this family contains at least one representative for each analytic type in the equisingularity class of the semigroup Γ, we can give an optimal set of candidates for the poles of the complex zeta function f^s of all plane branches with semigroup Γ. Furthermore, we will prove that if f_{gen} is generic among all branches with semigroup Γ (in the sense that the coefficients of a μ-constant deformation are generic), all the candidates are indeed poles of f_{gen}^s. As a corollary, we prove Yano's conjecture under the assumption that eigenvalues of the monodromy are pairwise distinct.

9.1 *Residues at rupture divisors*

After Theorem 8.10 in the preceding section, the only divisors that will contribute to poles of the complex zeta function of a plane branch will be rupture divisors and the divisor of the strict transform. Following the discussion in Section 2.7, a singular plane branch will have exactly $g \geq 1$ rupture divisors, denoted $E_{p_1}, \ldots, E_{p_g}$, where g is the number of characteristic exponents of f_t. This first observation reduces the list of candidate poles to $\sigma_{i,\nu}, \nu \in \mathbb{Z}$ for $i = 1, \ldots, g$ contributed by E_{p_i}, in addition to the negative integers corresponding to the strict transform $\widetilde{C}$. With the notations from Lemma 2.11

$$\sigma_{i,\nu} = -\frac{m_i + n_1 \cdots n_i + \nu}{n_i \overline{\beta}_i}, \quad \nu \in \mathbb{Z}_{\geq 0}. \tag{9.1}$$

We will basically use the same results about the residue presented in Section 8.2. However, in this case, a better understanding of the total transform around the rupture divisors is needed. To that end, we will make use of Proposition 2.19. The only thing that will differ from Section 8.2 is that, instead of computing the residue on the minimal resolution surface X', we proceed inductively and resolve up to the i–th rupture divisor, compute the residue on $X^{(i)}$ and blow-up up to the $(i+1)$–th rupture divisor. This process simplifies the notation and fits more naturally with the toric resolutions from Section 2.7.

First of all, we have to analyze the residue numbers from Equation (8.12) in the case of a rupture divisor of a plane branch. In this case, Lemma 8.3 is $\epsilon_{1,\nu} + \epsilon_{2,\nu} + \epsilon_{3,\nu} + \kappa_{p_i}\nu + 2 = 0$, since rupture divisors of plane branches only have three divisors, D_1, D_2, D_3, crossing them. Since we will be working on $X^{(i)}$, the divisor previously written as D_3 is the strict transform $\widetilde{C}$ of f on $X^{(i)}$ and thus, $\epsilon_{3,\nu} = e_i\sigma_{i,\nu}$, because $E_{p_i} \cdot \widetilde{C} = e_i$. Similarly, on the surface $X^{(i)}$, $\kappa_{p_i} = -E_{p_i} \cdot E_{p_i} = 1$. Hence,

$$\epsilon_{1,\nu} + \epsilon_{2,\nu} + e_i\sigma_{i,\nu} + \nu + 2 = 0. \tag{9.2}$$

It is then enough to study the relation of $\epsilon_{1,\nu}, \epsilon_{2,\nu}, \nu \in \mathbb{Z}_{\geq 0}$ with the semigroup Γ. We point out that Lichtin [Lic85] studied the residue numbers $\epsilon_{1,\nu}, \epsilon_{2,\nu}$ for the case $\nu = 0$. Using the notations and definitions from Section 2.4.

Proposition 9.1 ([Lic85, Prop. 2.12]). *The residue numbers $\epsilon_{1,0}, \epsilon_{2,0}$ associated with a rupture divisor E_{p_i} of a plane branch are given by*

$$\epsilon_{1,0} + 1 = \frac{1}{n_i}, \quad \epsilon_{2,0} + 1 = \frac{m_{i-1} - n_{i-1}\overline{m}_{i-1} + n_1 \cdots n_{i-1}}{\overline{m}_i}. \tag{9.3}$$

Corollary 9.2. *For any $\nu \in \mathbb{Z}_{\geq 0}$, the residue numbers are*

$$\epsilon_{1,\nu} + 1 = -\frac{a_i}{n_i}\nu + \frac{1}{n_i}, \quad \epsilon_{2,\nu} + 1 = -\frac{c_i n_{i-1}\overline{m}_{i-1} + d_i}{\overline{m}_i}\nu + \frac{m_{i-1} - n_{i-1}\overline{m}_{i-1} + n_1 \cdots n_{i-1}}{\overline{m}_i}. \tag{9.4}$$

Proof. The result follows from the definition of $\epsilon_{1,\nu}, \epsilon_{2,\nu}$ in Equation (8.12) and using Proposition 9.1 and Corollary 2.21. □

Secondly, we focus our attention on the functions Φ_1, Φ_2 from Proposition 8.1 but now in the case of the family f_t. Around the rupture divisor E_{p_i} on $X^{(i)}$, we have that $\Phi_{1,t} = |u_1|^{2s}|v_1|^2|\tilde{f}_t|^{2s}(\pi^{(i)})^*\varphi|_{U_i}$ and $\Phi_{2,t} = |u_2|^{2s}|v_2|^s|\tilde{f}_t|^{2s}(\pi^{(i)})^*\varphi|_{V_i}$. Since the only factor of $\Phi_{1,t}, \Phi_{2,t}$ crossing E_{p_i} is the strict transform $\tilde{f}_t$, by Faà di Bruno's formula in Lemma 8.7 applied to the equations in Proposition 2.19, the ν–th derivatives of $\Phi_{1,t}, \Phi_{2,t}$ at $x_p, \bar{x}_p = 0, w_p, \bar{w}_p = 0$ with $s = \sigma_{i,\nu}$ are a finite sum with summands that look like

$$y_p^{k'}\bar{y}_p^{k}(y_i - \lambda_i)^{e_i(\sigma_{i,\nu} - l')}(\bar{y}_i - \bar{\lambda}_i)^{e_i(\sigma_{i,\nu} - l)}, \quad z_p^{k'}\bar{z}_p^{k}(1 - \lambda_i z_p)^{e_i(\sigma_{i,\nu} - l')}(1 - \bar{\lambda}_i\bar{z}_p)^{e_i(\sigma_{i,\nu} - l)}, \tag{9.5}$$

with $k', k, l', l \in \mathbb{Z}_{\geq 0}$. Therefore, it makes sense to consider the order and the degree of y_i, z_i in $\Phi_{1,t}, \Phi_{2,t}$. On the U_i chart, they will be denoted by

$$0 \leq \operatorname{ord}_{y_i} \frac{\partial^\nu \Phi_{1,t}}{\partial x^\nu \partial \bar{x}^\nu}(0, y_i; \sigma_{i,\nu}) \leq \deg_{y_i} \frac{\partial^\nu \Phi_{1,t}}{\partial x^\nu \partial \bar{x}^\nu}(0, y_i; \sigma_{i,\nu}), \tag{9.6}$$

and respectively on the V_i chart. By the symmetry of the holomorphic and antiholomorphic parts, the orders and degrees are exactly the same if considered with respect to the conjugated variables $\bar{y}_i, \bar{z}_i$. By convention, the order and degree of zero are $+\infty$ and 0, respectively. Let us first present some technical lemmas.

Lemma 9.3. *Let $f_1, \dots, f_n \in C^\infty(\mathbb{C}^2)$ be functions such that*

$$\alpha\nu \leq \operatorname{ord}_y \frac{\partial^\nu f_i}{\partial x^\nu}(0,y), \quad \deg_y \frac{\partial^\nu f_i}{\partial x^\nu}(0,y) \leq \beta\nu, \quad \text{for} \quad i = 1, \dots, n, \tag{9.7}$$

and $\alpha, \beta \in \mathbb{Q}_{>0}$. Then,

$$\alpha\nu \leq \operatorname{ord}_y \frac{\partial^\nu (f_1 \cdots f_n)}{\partial x^\nu}(0,y), \qquad \deg_y \frac{\partial^\nu (f_1 \cdots f_n)}{\partial x^\nu}(0,y) \leq \beta\nu, \tag{9.8}$$

and,

$$\alpha\nu \leq \operatorname{ord}_y \frac{\partial^\nu f_1^s}{\partial x^\nu}(0,y), \quad \deg_y \frac{\partial^\nu f_1^s}{\partial x^\nu}(0,y) \leq \beta\nu \quad \text{for all} \quad s \in \mathbb{C}. \tag{9.9}$$

Proof. The first inequalities follow from the general Leibniz rule,

$$\frac{\partial^\nu (f_1 \cdots f_n)}{\partial x^\nu}(0,y) = \sum_{k_1 + \dots + k_n = \nu} \binom{n}{k_1, \dots, k_n} \frac{\partial^{k_1} f_1}{\partial x^{k_1}}(0,y) \cdots \frac{\partial^{k_n} f_n}{\partial x^{k_n}}(0,y). \tag{9.10}$$

Similarly, the second follows from Faà di Bruno's formula and the definition of the partial exponential Bell polynomials, see Lemma 8.7 and Lemma 8.7. Specifically, they follow from the second part of Equation (8.25) □

Lemma 9.4. *Let $f(x,y) \in C^\infty(\mathbb{C}^2)$ and let $\pi(x,y) = (x^n y^a, x^m y^b)$ with $n, m, a, b \in \mathbb{Z}_{\geq 0}$. Furthermore, assume that $nb - ma \geq 0$. Then,*

$$\frac{a}{n}\nu \leq \operatorname{ord}_y \frac{\partial^\nu (f \circ \pi)}{\partial x^\nu}(0,y), \quad \deg_y \frac{\partial^\nu (f \circ \pi)}{\partial x^\nu}(0,y) \leq \frac{b}{m}\nu \quad \text{for all} \quad \nu \in \mathbb{Z}_{\geq 0}. \tag{9.11}$$

Proof. Consider the Taylor expansion of f at the origin of order $\tau > \nu$,

$$f(x,y) = \sum_{i,j=0}^{\tau-1} \frac{\partial^{i+j} f}{\partial x^i \partial y^j}(0,0) \frac{x^i}{i!} \frac{y^j}{j!} + R_\tau(x,y) x^\tau y^\tau, \tag{9.12}$$

where R_τ is the residual. Composing with π, we get

$$f(\pi(x,y)) = \sum_{i,j=0}^{\tau-1} \frac{\partial^{i+j} f}{\partial x^i \partial y^j}(0,0) \frac{x^{ni+mj}}{i!} \frac{y^{ai+bj}}{j!} + R_\tau(x^n y^a, x^m y^b) x^{(n+m)\tau} y^{(a+b)\tau}. \tag{9.13}$$

At the ν–th derivative of this Taylor polynomial with respect to x and restricted to $x = 0$, we must have that $ni + mj = \nu$ for some integers $i, j \geq 0$. Notice that if there are no such $i, j \geq 0$ the derivative is zero, and the bounds are trivially fulfilled. Finally,

$$\begin{aligned} \operatorname{ord}_y \frac{\partial^\nu (f \circ \pi)}{\partial x^\nu}(0,y) &= \min_{ni+mj=\nu} \{ai + bj\} \geq \min_{j \geq 0} \left\{ \frac{a}{n}\nu + \frac{(nb - ma)j}{n} \right\} \geq \frac{a}{n}\nu, \\ \deg_y \frac{\partial^\nu (f \circ \pi)}{\partial x^\nu}(0,y) &= \max_{ni+mj=\nu} \{ai + bj\} \leq \max_{i \geq 0} \left\{ \frac{b}{m}\nu - \frac{(nb - ma)i}{m} \right\} \leq \frac{b}{m}\nu, \end{aligned} \tag{9.14}$$

where in the first inequality of each equation used that $i = (\nu - mj)/n$ and $j = (\nu - ni)/m$, respectively. □

Recall the linear forms $\rho_{j+1}^{(i)}(\underline{k}), A_{j+1}^{(i)}(\underline{k}), C_{j+1}^{(i)}(\underline{k})$ from Proposition 2.19.

Lemma 9.5. *For any $\nu \in \mathbb{Z}_{\geq 0}$, we have*

$$\frac{a_i}{n_i}\nu \leq \min_{\rho^{(i)}_{j+1}(\underline{k})=\nu} A^{(i)}_{j+1}(\underline{k}), \qquad \frac{c_i n_{i-1}\overline{m}_{i-1}+d_i}{\overline{m}_i}\nu \leq \min_{\rho^{(i)}_{j+1}(\underline{k})=\nu} C^{(i)}_{j+1}(\underline{k}), \tag{9.15}$$

$$\max_{\rho^{(i)}_{j+1}(\underline{k})=\nu} A^{(i)}_{j+1}(\underline{k}) \leq \frac{a_i n_{i-1}\overline{m}_{i-1}+b_i}{\overline{m}_i}\nu + n_{i+1}\cdots n_j, \quad \max_{\rho^{(i)}_{j+1}(\underline{k})=\nu} C^{(i)}_{j+1}(\underline{k}) \leq \frac{c_i}{n_i}\nu + n_{i+1}\cdots n_j. \tag{9.16}$$

Proof. For the first inequality, solve for k_0 in the constrain $\rho^{(i)}_{j+1}(k_0,\ldots,k_j) = \nu$ and substitute in $A^{(i)}_{j+1}(\underline{k})$. After some cancellations, we get

$$\frac{a_i}{n_i}\nu + a_i n_{i-1}\overline{m}_{i-1}k_i + b_i k_i - \frac{a_i}{n_i}\overline{m}_i k_i = \frac{a_i}{n_i}\nu + \frac{k_i}{n_i}, \tag{9.17}$$

where in the equality we applied $\overline{m}_i = n_i n_{i-1}\overline{m}_{i-1} + q_i$, from Equation (2.17), and that $n_i b_i - a_i q_i = 1$. Since $k_i \geq 0$, we obtain the lower bound for the minimum of $A^{(i)}_{j+1}(\underline{k})$ restricted to $\rho^{(i)}_{j+1}(\underline{k}) = \nu$. The argument for the lower bound for the minimum of $C^{(i)}_{j+1}(\underline{k})$ works similarly. Instead, solve for k_i in the constrain $\rho^{(i)}_{j+1}(k_0,\ldots,k_i,\ldots,k_j) = \nu$ and substitute in $C^{(i)}_{j+1}(\underline{k})$. Applying Equation (2.17) when necessary and that $q_i c_i - n_i d_i = 1$, we obtain

$$\frac{c_i n_{i-1}\overline{m}_{i-1}+d_i}{\overline{m}_i}\nu + \frac{1}{\overline{m}_i}\sum_{l=0}^{i-1} n_{l+1}\cdots n_i \overline{m}_l k_l, \tag{9.18}$$

which gives again the lower bound since $k_l \geq 0$. Having obtained the lower bounds for the minimums of $A^{(i)}_{j+1}$ and $C^{(i)}_{j+1}$ we can use Corollary 2.20 to obtain the upper bounds for the maximums. Indeed,

$$A^{(i)}_{j+1}(\underline{k}) + C^{(i)}_{j+1}(\underline{k}) \leq \rho^{(i)}_{j+1}(\underline{k}) + n_{i+1}\cdots n_j, \tag{9.19}$$

since $k_l \geq 0$. Hence,

$$\max_{\rho^{(i)}_{j+1}(\underline{k})=\nu} A^{(i)}_{j+1}(\underline{k}) \leq \nu + n_{i+1}\cdots n_j - \min_{\rho^{(i)}_{j+1}(\underline{k})=\nu} C^{(i)}_{j+1}(\underline{k}) = \frac{a_i n_{i-1}\overline{m}_{i-1}+b_i}{\overline{m}_i}\nu + n_{i+1}\cdots n_j, \tag{9.20}$$

since $a_i + c_i = n_i$ and $b_i + d_i = q_i$. We can argue similarly for the remaining bound. □

All these technical lemmas are used in the proof of the following proposition.

Proposition 9.6. *With the notations above, we have that*

$$\frac{a_i}{n_i}\nu \leq \mathrm{ord}_{y_i} \frac{\partial^{2\nu}\Phi_{1,t}}{\partial x_i^\nu \partial \bar{x}_i^\nu}(0, y_i; \sigma_{i,\nu}), \quad \frac{d_i}{q_i}\nu \leq \mathrm{ord}_{z_i} \frac{\partial^{2\nu}\Phi_{2,t}}{\partial w_i^\nu \partial \bar{w}_i^\nu}(z_i, 0; \sigma_{i,\nu}), \tag{9.21}$$

$$\deg_{y_i} \frac{\partial^{2\nu}\Phi_{1,t}}{\partial x_i^\nu \partial \bar{x}_i^\nu}(0, y_i; \sigma_{i,\nu}) \leq \frac{b_i}{q_i}\nu + e_i, \quad \deg_{z_i} \frac{\partial^{2\nu}\Phi_{2,t}}{\partial w_i^\nu \partial \bar{w}_i^\nu}(z_i, 0; \sigma_{i,\nu}) \leq \frac{c_i}{n_i}\nu + e_i, \tag{9.22}$$

for all $\nu \in \mathbb{Z}_{\geq 0}$, and the same bounds hold for the conjugated variables $\bar{y}_i, \bar{z}_i$.

Proof. We will do the proof for $\Phi_{1,t}$ since the proof for $\Phi_{2,t}$ works similarly. Recall that $\Phi_{1,t} = |u_1|^{2s}|v_1|^2|\tilde{f}_t|^{2s}(\pi^{(i)})^*\varphi\big|_{U_i}$. By Lemma 9.3, it is enough to prove the bounds for each holomorphic or antiholomorphic factor. The factors $u_1, v_1, (\pi^{(i)})^*\varphi\big|_{U_i}$ are the pull-back by the toric morphism π_i of some invertible elements in the unique point of the total transform of f_t that is not a simple normal crossing on $X^{(i-1)}$. Therefore, by Section 2.7 and Lemma 9.4, their orders (resp. degrees) with respect to y_i are bounded by $a_i\nu/n_i$ (resp. $b_i\nu/q_i$). It remains to prove the bounds for the strict transform $\tilde{f}_t$.

For the strict transform, consider Proposition 2.19 and proceed by induction from $\tilde{f}_{i+1}$. Analyzing Equation (2.54), the only part depending on x_i is the summation. By Lemma 9.3, it is enough to show that each factor of each summand fulfills the bounds. By the same argument as before, the units $u_{\underline{k}}^{(i)}$ satisfy the bounds. On the other hand, Lemma 9.5 assures the bounds for the monomials in x_i, y_i. The lower-bound for the order is clear. For the upper-bound on the degree just notice that

$$\frac{a_i n_{i-1}\overline{m}_{i-1} + b_i}{\overline{m}_i} = \frac{a_i n_{i-1}\overline{m}_{i-1} + b_i}{n_i n_{i-1}\overline{m}_{i-1} + q_i} < \frac{b_i}{q_i}, \tag{9.23}$$

since $\overline{m}_i = n_i n_{i-1}\overline{m}_{i-1} + q_i$ and $n_i b_i - q_i a_i = 1$. Therefore, we are done for $\tilde{f}_{i+1}$. By induction, if all $\tilde{f}_{k+1}, i \leq k < j$ satisfy the bounds, so does $\tilde{f}_{j+1}$. To see this, it is just a matter of applying Lemma 9.3, Lemma 9.4, Lemma 9.5, and the induction hypothesis to Equation (2.55). □

We are ready to present the main result of this section.

Theorem 9.7. *For any plane branch singularity $f : (\mathbb{C}^2, \mathbf{0}) \longrightarrow (\mathbb{C}, 0)$, the poles of the complex zeta function f^s are simple and contained in the sets*

$$\left\{ \sigma_{i,\nu} = -\frac{m_i + n_1 \cdots n_i + \nu}{n_i \overline{\beta}_i} \;\middle|\; \nu \in \mathbb{Z}_{\geq 0},\ \overline{\beta}_i\sigma_{i,\nu}, e_{i-1}\sigma_{i,\nu} \notin \mathbb{Z} \right\}, \quad i = 1, \ldots, g, \tag{9.24}$$

contributed by the rupture divisors E_{p_i}, together with the negative integers $\mathbb{Z}_{<0}$, contributed by the strict transform $\widetilde{C}$.

Proof. In order to prove this result for any plane branch, it is enough to restrict the study to the family of f_t from Proposition 2.17. By the previous discussion, we only have to show that the candidates $\sigma_{i,\nu}$ such that $\overline{\beta}_i\sigma_{i,\nu}, e_{i-1}\sigma_{i,\nu} \in \mathbb{Z}$ have always residue zero. The first important observation is that $\overline{\beta}_i\sigma_{i,\nu}, e_{i-1}\sigma_{i,\nu} \in \mathbb{Z}$, if and only if, $\epsilon_{1,\nu}, \epsilon_{2,\nu} \in \mathbb{Z}$, respectively. To see this, consider the definitions $\epsilon_{1,\nu} = N_1\sigma_{1,\nu} + k_1, \epsilon_{2,\nu} = N_2\sigma_{2,\nu} + k_2$ from Equation (8.12). Hence, $\epsilon_{1,\nu}, \epsilon_{2,\nu} \in \mathbb{Z}$, if and only if, $N_1\sigma_{i,\nu}, N_2\sigma_{i,\nu} \in \mathbb{Z}$, respectively. By Proposition 2.19, $N_1 = a_i\overline{\beta}_i$ and $N_2 = (c_i n_{i-1}\overline{m}_{i-1} + d_i)e_{i-1}$, and the remark follows because

$$\gcd(n_i\overline{\beta}_i, a_i\overline{\beta}_i) = \overline{\beta}_i \gcd(n_i, a_i) = \overline{\beta}_i, \tag{9.25}$$

since $n_i b_i - q_i a_i = 1$, and

$$\gcd(n_i\overline{\beta}_i, (c_i n_{i-1}\overline{m}_{i-1} + d_i)e_{i-1}) = e_{i-1}\gcd(\overline{m}_i, c_i n_{i-1}\overline{m}_{i-1} + d_i) = e_{i-1}, \tag{9.26}$$

since $\overline{m}_i = n_i n_{i-1}\overline{m}_{i-1} + q_i$ and $q_i c_i - n_i d_i = 1$. The argument to show that the residue is zero if $\epsilon_{1,\nu} \in \mathbb{Z}$ or $\epsilon_{2,\nu} \in \mathbb{Z}$ is fundamentally different for each case. Let us begin by the case where $\overline{\beta}_i\sigma_{i,\nu} \in \mathbb{Z}$ and $e_{i-1}\sigma_{i,\nu} \notin \mathbb{Z}$.

In order to study the residues of the candidates $\sigma_{i,\nu}$ of the i–th rupture divisor E_{p_i} such that $\overline{\beta}_i\sigma_{i,\nu} \in \mathbb{Z}$, we place ourselves in the chart U_i of $X^{(i)}$ with local coordinates (x_i, y_i). The origin of this chart is the intersection point $E_{p_1} \cap D_1$. The only point of the total transform on $X^{(i)}$ that is not a simple normal crossing is the intersection of the strict transform of f_t with E_{p_i}, with E_{p_i} being the only exceptional divisor at that point. Therefore, we can apply the formula for the residue in Section 1.1 for the U_i chart.

From the preceding discussion, the derivatives of $\Phi_{1,t}$ appearing in the residue formula are a finite sum of terms that look like $y_p^{k'}\bar{y}_p^{k}(y_i-\lambda_i)^{e_i(\sigma_{i,\nu}-l')}(\bar{y}_i-\bar{\lambda}_i)^{e_i(\sigma_{i,\nu}-l)}$. Consequently, we can reduce the residue to a finite sum of the integrals from Equation (8.32), which, by Proposition 8.9, are equal to

$$-2\pi i\lambda^{-\alpha-n-1}\bar{\lambda}^{-\alpha-1}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma+1)}{\Gamma(-\alpha-n)\Gamma(-\beta-m)\Gamma(-\gamma-n-m)}, \quad n, m \in \mathbb{Z}. \tag{9.27}$$

As noted earlier, for E_{p_i} on $X^{(i)}$ we have that $\epsilon_{1,\nu}+\epsilon_{2,\nu}+e_i\sigma_{i,\nu}+\nu+2=0$. If we are assuming that $\epsilon_{1,\nu} \in \mathbb{Z}$ but $\epsilon_{2,\nu} \notin \mathbb{Z}$, i.e. $e_{i-1}\sigma_{i,\nu} \notin \mathbb{Z}$, it must happen that $e_i\sigma_{i,\nu} \notin \mathbb{Z}$. This implies that $\Gamma(\beta+1), \Gamma(-\beta-n), \Gamma(\gamma+1), \Gamma(-\gamma-n-m) \in \mathbb{C}^*$. However, the remaining factor, $\Gamma(\alpha+1)/\Gamma(-\alpha-n)$, is always zero, since $\epsilon_{1,\nu} \in \mathbb{Z}$ implies that $\alpha, \alpha' \in \mathbb{Z}$,

$$\alpha+1=\epsilon_{1,\nu}+1+k \geq -\frac{a_i}{n_i}\nu+\frac{1}{n_i}+\frac{a_i}{n_i}\nu=\frac{1}{n_i}>0, \tag{9.28}$$

and $-\alpha-n=-\alpha'<1$, by Corollary 9.2 and Proposition 9.6. This proves that the residue at the candidate poles $\sigma_{i,\nu}$ such that $\overline{\beta}_i\sigma_{i,\nu} \in \mathbb{Z}, e_{i-1}\sigma_{i,\nu} \notin \mathbb{Z}$ is zero.

We move now to the case where $e_{i-1}\sigma_{i,\nu} \in \mathbb{Z}, \overline{\beta}_i\sigma_{i,\nu} \notin \mathbb{Z}$, i.e. $\epsilon_{2,\nu} \in \mathbb{Z}$ and $e_i\sigma_{i,\nu} \notin \mathbb{Z}$. For this case observe that the previous argument, applied to the residue in the V_i chart, only works for the first rupture divisor. For $i=1$, $\epsilon_{2,\nu}+1=-d_1\nu/q_1+1/q_1$, since $n_0, m_0=0$ and $\overline{m}_1=q_1$. Otherwise, the formula for $\epsilon_{2,\nu}$ from Corollary 9.2 and the bound for the order of $\Phi_{2,t}$ in Proposition 9.6 do not match. Thus, from now on, we will assume $i \geq 2$.

To study the residue at these poles, we consider $\mathrm{Res}_{s=\sigma_{i,\nu}} f_t^s$ as a function of $\lambda_i \in \mathbb{C}$ and we place ourselves in the V_i chart. Since λ_i is the intersection coordinate of the strict transform with E_{p_i}, if we let $\lambda_i \to 0$ or $\lambda_i \to \infty$, we are in the situation of a non-rupture divisor and the residue is zero. By Proposition 8.9 and since $\epsilon_2 \in \mathbb{Z}$, the residue is a Laurent series on $\lambda_i, \bar{\lambda}_i \neq 0$. Deriving under the integral sign in the formula for the residue from Corollary 8.5 we increase the order in $z_i, \bar{z}_i$ of $\Phi_{2,t}$ by one unit. Therefore, after deriving enough times we can assume that in Equation (9.27), $\alpha+1>0, -\alpha' \leq 0, \alpha, \alpha' \in \mathbb{Z}$. This implies that the principal part in $\lambda_i, \bar{\lambda}_i$ of $\mathrm{Res}_{s=\sigma_{i,\nu}} f_t^s(\lambda_i)$ must be zero. However, the same argument is true if we consider the residue as a function of $\lambda_i^{-1}, \bar{\lambda}_i^{-1}$. Hence, the residue is independent of $\lambda_i, \bar{\lambda}_i$. This implies that the residue is zero overall.

It remains to show that the residue is zero in the case that $\overline{\beta}_i\sigma_{i,\nu} \in \mathbb{Z}$ and $e_{i-1}\sigma_{i,\nu} \in \mathbb{Z}$. Both conditions imply that, $e_i\sigma_{i,\nu} \in \mathbb{Z}$. To see that the residue is zero in this situation, it is just a matter of combining the previous arguments and recalling that the Gamma function has only simple poles. After deriving with respect to λ_i in the V_i chart, we can get Equation (9.27) with $\alpha+1>0$ and $-\alpha-n=-\alpha' \leq 0$, i.e., the factor $\Gamma(\alpha+1)/\Gamma(-\alpha-n)$ is zero. Assume we have derived d' times with respect to λ_i, then

$$\alpha'=\epsilon_{2,\nu}+d'+k' \qquad \beta'=e_i\sigma_{i,\nu}-e_il'-e_id'. \tag{9.29}$$

Consequently, since $e_i \geq 1$,

$$\begin{aligned} \gamma + 1 = -\alpha' - \beta' - 1 = -\epsilon_{2,\nu} + 1 - k' - e_i\sigma_{i,\nu} + e_i l' - 2 \geq \epsilon_{1,\nu} + \nu + 1 - k' + e_i \\ \geq -\frac{a_i}{n_i}\nu + \frac{1}{n_i} + \nu - \frac{c_i}{n_i}\nu - e_i + e_i = \frac{1}{n_i} > 0, \end{aligned} \tag{9.30}$$

by Corollary 9.2 and Proposition 9.6. Similarly, $-\gamma' - n - m < 1$. Therefore, the factor $\Gamma(\gamma+1)/\Gamma(-\gamma-n-m)$ is also zero. Since $e_{i-1}\sigma_{i,\nu} \in \mathbb{Z}$, the piece, $\Gamma(\beta+1)/\Gamma(-\beta-m)$, has a pole. However, Equation (9.27) is zero because the poles of $\Gamma(z)$ are simple.

Finally, we need to see that the negative integers, the candidates coming from the strict transform $\widetilde{C}$, are poles. We can argue directly from the definition of f^s given in Equation (8.1). Take $\mathbf{0} \neq p \in f_t^{-1}(0) \cap U$ at which the equation f_t can be taken as one of the holomorphic coordinates. Thus, we reduce the problem to the monomial case and, by Proposition 7.1, the negative integers are simple poles. The poles contributed by rupture divisors are also simple because they must have $\epsilon_{1,\nu}, \epsilon_{2,\nu}, e_i\sigma_{i,\nu} \notin \mathbb{Z}$ for all $\nu \in \mathbb{Z}_{\geq 0}$. These conditions imply that Equation (9.27) cannot have a pole and hence, the residue does not have a pole. By Section 8.1, all the poles of f^s are simple. □

We point out that the candidate poles $\sigma_{1,0} > \sigma_{2,0} > \cdots > \sigma_{g,0}$ are always poles of f^s for any plane branch f as shown by Lichtin in [Lic85; Lic89].

Example 9.1. There are examples where the candidate poles of f^s that are in the sets from Equation (9.24) vary in a μ–constant deformations of f. For instance, consider $f = y^4 - x^9$ and the μ–constant deformation $f_t = y^4 - x^9 + tx^5y^2$. For the unique rupture divisor, the sequence of candidate poles is $\sigma_{1,\nu} = -(13+\nu)/36, \nu \in \mathbb{Z}_{\geq 0}$. Taking $\nu = 2$, $\sigma_{1,2} = -5/12, \epsilon_{1,2} = -9/4, \epsilon_{2,2} = -4/3$, and

$$\operatorname*{Res}_{s=\sigma_{1,2}} f_t = -16\pi^2\sigma_{1,2}^2 \frac{\Gamma(\epsilon_{1,2}+3)\Gamma(\sigma_{1,2})\Gamma(\epsilon_{2,2}+2)}{\Gamma(-\epsilon_{1,2}-2)\Gamma(-\sigma_{1,2}+1)\Gamma(-\epsilon_{2,2}-1)} |t|^2 \delta_{\mathbf{0}}^{(0,0,0,0)}. \tag{9.31}$$

Therefore, $\sigma_{1,2} = -5/12$ is a pole, if and only if, $t \neq 0$. ■

9.2 *Generic poles*

Studying the residues in terms of the deformation parameters of f_t, we can get open conditions in which a certain candidate pole is indeed a pole, as seen in Example 9.1. Recalling Equation (8.22), the first observation is that, in terms of $\boldsymbol{t}$,

$$\operatorname*{Res}_{s=\sigma_{i,\nu}} f_{\boldsymbol{t}}^s = \sum_{k',l',k,l=0}^{\nu} p_{k',l'}(\boldsymbol{t}) p_{k,l}(\bar{\boldsymbol{t}})\, \delta_{\mathbf{0}}^{(k',l',k,l)}, \tag{9.32}$$

with $p_{k',l'}(\boldsymbol{t}) = \overline{p_{k,l}(\bar{\boldsymbol{t}})}$ if $k' = k$ and $l' = l$. The following theorem shows that, actually, any candidate is a pole in a certain Zariski open set in the deformation space of f_t.

Theorem 9.8. *For any $M_1, \ldots, M_g \in \mathbb{Z}_{\geq 0}$, generic plane branches f_{gen} in the equisingularity class corresponding to the semigroup $\Gamma = \langle \overline{\beta}_0, \overline{\beta}_1, \ldots, \overline{\beta}_g \rangle$ satisfy that*

$$\left\{ \sigma_{i,\nu} = -\frac{m_i + n_1 \cdots n_i + \nu}{n_i \overline{\beta}_i} \;\middle|\; 0 \leq \nu < M_i,\ \overline{\beta}_i\sigma_{i,\nu}, e_{i-1}\sigma_{i,\nu} \notin \mathbb{Z} \right\}, \quad i = 1, \ldots, g, \tag{9.33}$$

are simple poles of the complex zeta function f_{gen}^s.

Proof. By Theorem 9.7, we only have to check the candidates in the sets given in Equation (9.24). In the case that $\nu = 0$, the residue for the candidates $\sigma_{1,0} > \sigma_{2,0} > \cdots > \sigma_{g,0}$ does not depend on $\boldsymbol{t}$ and consists only of one of the integrals from Proposition 8.9 which is non-zero by Proposition 9.1.

Therefore, fix a candidate pole $\sigma_{i,\nu}$ with $0 < \nu < M_i$. It is enough to show that one of the polynomials $p_{k',l'}(\boldsymbol{t})$ on the parameters $\boldsymbol{t}$ in the expression from Equation (9.32) is not identically zero. Consider the residue formula from Proposition 8.4 in, for instance, the U_i chart and recall that $\Phi_{1,\boldsymbol{t}} = |u_1|^{2s}|v_1|^2|\widetilde{f}_{\boldsymbol{t}}|^{2s}(\pi^{(i)})^*\varphi|_{U_i}$. Now, considering Equation (2.54), we claim that the strict transform has a deformation term

$$t_{\underline{k}}^{(i)} x_i^{\rho_{i+1}^{(i)}(\underline{k})} y_i^{A_{i+1}^{(i)}(\underline{k})} u_{\underline{k}}^{(i)}(x_i, y_i) \tag{9.34}$$

for a certain $\underline{k}$ such that $\rho_{i+1}^{(i)}(\underline{k}) = \nu$. Indeed, this happens since $\rho_{i+1}^{(i)}(\underline{k}) = \nu$ is equivalent, by Equation (2.56), to

$$n_1 \cdots n_i k_0 + n_2 \cdots n_i \overline{m}_1 k_1 + \cdots + \overline{m}_i k_i = n_i \overline{m}_i + \nu. \tag{9.35}$$

But this is an identity in the semigroup Γ_{i+1} of the maximal contact element f_{i+1}, see Equation (2.22), and such a $\underline{k}$ always exists because $n_i\overline{m}_i + \nu$ is bigger than the conductor of Γ_{i+1}. The deformation parameter $t_{\underline{k}}^{(i)}$ for such a $\underline{k}$ can only appear when $f_{\boldsymbol{t}}$ is derived ν times. By Faà di Bruno's formula, this implies that the polynomial $p_{0,0}(\boldsymbol{t})$ in Equation (9.32) is equal to $\zeta t_{\underline{k}}^{(i)} + \cdots$, where the dots represent other terms on $\boldsymbol{t}$ not containing $t_{\underline{k}}^{(i)}$. The coefficient $\zeta \in \mathbb{C}$ is different from zero since it has the form of Equation (8.33) and we are assuming that $\overline{\beta}_i\sigma_{i,\nu}, e_{i-1}\sigma_{i,\nu} \notin \mathbb{Z}$. Hence, the condition $p_{0,0}(\boldsymbol{t}) \neq 0$ gives a non-empty Zariski open subset on the deformation space of $f_{\boldsymbol{t}}$ in which $\sigma_{i,\nu}$ is a pole.

In the case that there is a resonance between two poles, i.e. $\sigma_{i,\nu} = \sigma_{i',\nu'}$ for $i \neq i'$, the condition that the residues of $\sigma_{i,\nu}$ and $\sigma_{i',\nu'}$ do not cancel out gives another Zariski open set. The intersection of all the open sets defines generic plane branches f_{gen}. □

Consider the sets from Equation (9.33) with $M_i = n_i\overline{\beta}_i$, namely

$$\Pi_i := \left\{ \sigma_{i,\nu} = -\frac{m_i + n_1 \cdots n_i + \nu}{n_i \overline{\beta}_i} \;\middle|\; 0 \leq \nu < n_i\overline{\beta}_i,\ \overline{\beta}_i\sigma_{i,\nu}, e_{i-1}\sigma_{i,\nu} \notin \mathbb{Z} \right\}, \tag{9.36}$$

for $i = 1, \ldots, g$ and define $\Pi := \bigcup_{i=1}^{g} \Pi_i$. An easy computation shows that there are exactly μ elements in Π, counted with possible multiplicities,

$$|\Pi| = \sum_{i=1}^{g} n_i\overline{\beta}_i - \overline{\beta}_i - n_i e_i + e_i = \sum_{i=1}^{g}(n_i - 1)\overline{\beta}_i + \sum_{i=1}^{g} e_i - e_{i-1} = \sum_{i=1}^{g}(n_i - 1)\overline{\beta}_i - n + 1 = \mu, \tag{9.37}$$

using Equation (2.18) in the last equality. The sets Π_i are precisely the b-exponents in Yano's conjecture from Section 2.8. Indeed, the relation between the notations in Equation (2.60) and the resolution data in Section 2.5 is clear. Namely, $R_i = N_{p_i} = n_i\overline{\beta}_i, R_i' = N_{q_i} = \overline{\beta}_i, r_i = k_{p_i} + 1 = m_i + n_1 \cdots n_i$ and $r_i' = k_{q_i} + 1 = \lceil (m_i + n_1 \cdots n_i)/n_i \rceil$, see Remark 2.2. To see the equality between the exponents in Equation (2.61) and the set Π is enough to notice that $R_i = N_{p_i} = n_i N_{q_i} = n_i R_i'$ and $r_i = k_{p_i} + 1 = n_i(k_{q_i} + 1) = n_i r_i'$.

The results of Malgrange [Mal75] and Barlet [Bar84] imply that the elements of Π generate all the eigenvalues of the monodromy. The characteristic polynomial of the monodromy is a topological invariant of the singularity, see A'Campo formula [ACa75]. Consequently, it can be computed from the semigroup Γ of f, see [Neu83]. The hypothesis that the eigenvalues of the monodromy are pairwise different is a condition on the equisingularity class, i.e. on the semigroup Γ, implying that there are exactly μ different elements in Π.

As a corollary of Theorem 9.8, we can deduce Yano's conjecture for any number of characteristic exponents if we assume that the eigenvalues of the monodromy are different.

Corollary 9.9. *Let $\Gamma = \langle \overline{\beta}_0, \overline{\beta}_1, \ldots, \overline{\beta}_g \rangle$ be a semigroup defining an equisingularity class of plane branches. If the eigenvalues of the monodromy in the equisingularity class associated with the semigroup Γ are pairwise different, then Yano's conjecture holds.*

Proof. We must check that all the μ different elements of Π are roots of the Bernstein-Sato polynomial $b_{gen,\mathbf{0}}(s)$ of f_{gen}. If $\sigma_{i,\nu} \in \Pi_i$ is in between $-\mathrm{lct}(f) - 1$ and $-\mathrm{lct}(f)$, then $\sigma_{i,\nu}$ is automatically a root of $b_{gen,\mathbf{0}}(s)$ by Corollary 7.3. Otherwise, by Corollary 7.3, we must check that $\sigma_{i,\nu} + 1$ is not a root of $b_{gen,\mathbf{0}}(s)$. By contradiction, assume $\sigma_{i,\nu} + 1$ is a root of $b_{gen,\mathbf{0}}(s)$. By [Loe85, Thm. 1.9], the roots of the Bernstein-Sato polynomial of a reduced plane curve can only be of the form $\sigma - k$, with $k \in \mathbb{Z}_{\geq 0}$ and σ a pole of f^s. Hence, by Theorem 9.7, $\sigma_{i,\nu} + 1 = \sigma_{i',\nu'} \in \Pi_{i'}, i \neq i'$. But this is impossible since, as eigenvalues of the monodromy, they are both equal, contradicting the hypothesis. Finally, by the definitions in Section 2.8, if the Bernstein-Sato polynomial has exactly μ different roots they must coincide with the opposites in sign to the b-exponents. $\square$

V

PERIODS OF INTEGRALS IN THE MILNOR FIBER

In this chapter, we will study periods of integrals in the Milnor fiber. These periods are multivalued functions in the base of the Milnor fiber that have an asymptotic expansion that encodes invariants of the singularity as the monodromy or the roots of the Bernstein-Sato polynomial. Furthermore, the asymptotic expansion of the periods can be constructed using resolution of singularities. Therefore, these periods of integrals can be used to study these invariants in terms of resolution of singularities.

The asymptotic expansion of the periods has been studied by Malgrange [Mal74a; Mal74b], Varchenko [Var80; Var82], Lichtin [Lic89], and Loeser [Loe88; Loe90]. We will present some new results in the case of plane curves, generalizing previous results of Varchenko [Var80], Lichtin [Lic89], and Loeser [Loe88]. Using these new results we will give a proof for the general case of Yano's conjecture. Finally, in the last section, and using these ideas, we will deduce some results about topological roots of the Bernstein-Sato polynomial of a plane branch.

10 ASYMPTOTICS OF INTEGRALS AND COHOMOLOGY OF THE MILNOR FIBER

Let $f : (\mathbb{C}^{n+1}, \mathbf{0}) \longrightarrow (\mathbb{C}, 0)$ be a germ of a holomorphic map with an isolated singularity at the origin. In this section, we will introduce the main object of study of this chapter: the periods of integrals in the Milnor fiber and their asymptotic expansion. We will also introduce the notion of geometric and elementary sections that are due to Varchenko, the relation between these sections and the b-exponents, and a semicontinuity result for the b-exponents. Finally, we will show how one can construct the full asymptotic expansion of the periods using resolution of singularities.

Throughout this section, we will use the notations and results introduced in Chapter I for the Milnor fiber, see Section 1.4, the Gauss-Manin connection, see Section 1.5, and the Brieskorn lattice, see Section 1.6.

10.1 *Periods of integrals*

Let $\eta \in \Gamma(X, \Omega^n_X)$ be a holomorphic n-form on X and let $\gamma(t), t \in T'$ be a locally constant section of the fibration $H_n = \cup_{t \in T'} H_n(X_t, \mathbb{C})$. Since the restriction of η to X_t is a holomorphic form of maximal degree, and thus it is closed, the integral

$$I(t) := \int_{\gamma(t)} \eta, \tag{10.1}$$

already considered in Section 1.6, of η along the cycle $\gamma(t)$ is well-defined. Furthermore, by the monodromy action, $\gamma(t)$ is a multivalued function of t with values in $H_n(X_t, \mathbb{C})$ and $I(t)$ defines a multivalued function on T'. After Leray's Residue Theorem Theorem 1.26, it is a (multivalued) holomorphic function since it holds that

$$I'(t) = \frac{d}{dt} \int_{\gamma(t)} \eta = \int_{\gamma(t)} \frac{\mathrm{d}\eta}{\mathrm{d}f}, \tag{10.2}$$

where $\mathrm{d}\eta/\mathrm{d}f$ is the Gel'fand-Leray form of $\mathrm{d}\eta$, introduced in Section 1.6, which denotes the restriction to X_t of any holomorphic form ξ such that $\xi \wedge \mathrm{d}f = \mathrm{d}\eta$. In local coordinates, if $\mathrm{d}\eta = g\,\mathrm{d}x_0 \wedge \cdots \wedge \mathrm{d}x_n$, then, on the set $\{x \in X \mid \partial f/\partial x_i \neq 0\}$, $\mathrm{d}\eta/\mathrm{d}f$ is defined by the restriction to X_t of the form

$$(-1)^i \left(\frac{\partial f}{\partial x_i}\right)^{-1} g\,\mathrm{d}x_0 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_n. \tag{10.3}$$

Now, let $\gamma(t)$ be a *vanishing cycle*, that is, a locally constant section of H_n such that $\gamma(t) \to 0$ as $t \to 0$. Malgrange [Mal74a, Lemma 4.5] proves that if $\gamma(t)$ is such a cycle, $I(t) \to 0$ as $t \to 0$ in any sector $|\arg t| \leq C, C \in \mathbb{R}_+$. In addition, assume that $\gamma(t)$ is a generalized eigenvector of the monodromy automorphism h_n of $H_n(X_t, \mathbb{C})$, i.e. $(h_n - \lambda\,\mathrm{id})^p \gamma(t) = 0$ with p minimal and $\lambda \in \mathbb{C}^*$ a root of unity. Then, since the Gauss-Manin connection is regular, one has that $I(t)$ has the following expansion, see [Mal74a, Eq. 4.7.1],

$$I(t) = \int_{\gamma(t)} \eta = \sum_{\substack{\alpha \in L(\lambda) \\ 0 \leq q < p}} a_{\alpha,q} t^{\alpha} (\ln t)^q, \quad a_{\alpha,q} \in \mathbb{C}, \tag{10.4}$$

where $L(\lambda)$ is the set of $\alpha > 0$ such that $\lambda = \exp(-2\pi\imath\alpha)$. Notice from this that if $\gamma(t)$ is not a vanishing cycle, then necessarily $\lambda = 1$, since $I(t)$ has the same expression as Equation (10.4) plus a non-zero constant term.

Since for any top form $\omega \in \Gamma(X, \Omega_X^{n+1})$ there exists $\eta \in \Gamma(X, \Omega_X^n)$ such that $\omega = \mathrm{d}\eta$, we will consider only the integrals of $\mathrm{d}\eta/\mathrm{d}f$ along any vanishing cycle $\gamma(t) \in H_n(X_t, \mathbb{C})$. Then, by the Monodromy Theorem and Equation (10.4)

$$\int_{\gamma(t)} \frac{\omega}{\mathrm{d}f} = \sum_{\lambda \in \Lambda} \sum_{\alpha \in L(\lambda)} \sum_{0 \leq k \leq n} a_{\alpha-1,k} t^{\alpha-1} (\ln t)^k. \tag{10.5}$$

10.2 *Geometric sections*

For every top form $\omega \in \Gamma(X, \Omega_X^{n+1})$ and every $t \in T'$, the form $\omega/\mathrm{d}f$ in X_t defines an element of $H^n(X_t, \mathbb{C})$. Hence, every such ω defines a section $s[\omega]$ of the bundle H^n. Indeed, by the previous section, if $\gamma(t)$ is a locally constant section of the fibration H_n, the pairing $\langle s[\omega], \gamma \rangle$ given by Equation (10.5) is holomorphic and hence $s[\omega]$ is a holomorphic section of the bundle H^n, i.e. an element of $\mathcal{H}^n$. Following the terminology of Varchenko [Var80, §4], these sections will be called *geometric sections*.

Given w a local section of $\mathcal{H}^n$ and γ a locally constant section of H_n, one has that $\frac{d}{dt}\langle w, \gamma \rangle = \langle \partial_t^* w, \gamma \rangle$, since horizontals sections generate $\mathcal{H}^n$ over $\mathcal{O}_{T'}$. Consequently, by Equation (10.2), the Gauss-Manin connection ∂_t^* on $\mathcal{H}^n$ applied to the geometrical sections can be computed as $\partial_t^* s[\omega] = s[\mathrm{d}(\omega/\mathrm{d}f)]$.

The complex numbers $a_{\alpha,k}$ appearing in Equation (10.5) depend on ω and $\gamma(t)$. For a fixed ω, $a_{\alpha,k}$ is a linear function on the space of locally constant sections of H_n. As a consequence, the number $a_{\alpha,k}$ defines a locally constant section $A_{\alpha,k}^{\omega}(t)$ of the fibration H^n by the rule $\langle A_{\alpha,k}^{\omega}(t), \gamma(t) \rangle := a_{\alpha,k}(\omega, \gamma)$. By construction, the geometrical sections $s[\omega]$ are defined by

$$s[\omega] := \sum_{\lambda \in \Lambda} \sum_{\alpha \in L(\lambda)} \sum_{0 \leq k \leq n} t^{\alpha-1} (\ln t)^k A_{\alpha-1,k}^{\omega}(t). \tag{10.6}$$

The sections $A^{\omega}_{\alpha,k}(t)$ will be called *locally constant geometric sections.*

Lemma 10.1 ([Var80, Lemma 4]). *The local system $\mathcal{L}^* := \ker \nabla^n$ is generated by the locally constant sections $A^{\omega}_{\alpha,k}$, where $\omega \in \Gamma(X, \Omega^{n+1}_X)$ and the α, k are the same as in Equation (10.5).*

Let S_α be the sheaf of all locally constant sections of the bundle H^n generated by the sections $A^{\omega}_{\alpha,k}$ with fixed $\alpha \in \mathbb{Q}$ and where $\omega \in \Gamma(X, \Omega^{n+1}_X)$, $k = 0, \ldots, n$. Since $A^{\omega}_{\alpha,k} = A^{f\omega}_{\alpha+1,k}$, one has that $S_\alpha \subseteq S_{\alpha+1}$. After Lemma 10.1, $\mathcal{L}^* = \bigoplus_\alpha S_\alpha$.

Following [Var80, §4], for every $\alpha \in \cup_{\lambda \in \Lambda} L(\lambda)$ one defines the holomorphic subbundle $f^*_{\lambda,\alpha} : H^n_{\lambda,\alpha} \longrightarrow T'$ of H^n_λ generated over $\mathcal{O}_{T'}$ by the sections of S_α. According to [Var80, §2], these subbundles have the following properties. If $\alpha, \alpha' \in L(\lambda)$ and $\alpha > \alpha'$, then $H^n_{\lambda,\alpha'}$ is a subbundle of $H^n_{\lambda,\alpha}$ and for $\alpha \in L(\lambda)$ sufficiently large then $H^n_{\lambda,\alpha} = H^n_\lambda$. Moreover, these subbundles are invariant under the covariant derivative of the connection ∇^*_λ and the monodromy endomorphism h^*.

10.3 *Elementary sections*

We will proceed now to define the *elementary sections* of $\mathcal{H}^n$ in the sense of Varchenko [Var80, §6]. In order to do that, one needs to understand first the natural action of the monodromy on the local system $\mathcal{L}^*$ in terms of the sections $A^{\omega}_{\alpha,k}$. Let $\omega \in \Gamma(X, \Omega^{n+1}_X), \lambda \in \Lambda, \alpha \in L(\lambda)$ and assume that, at least, one of the sections $A^{\omega}_{\alpha,0}, \ldots, A^{\omega}_{\alpha,n}$ is not equal to zero. Let $p = \max\{k \in \mathbb{Z} \mid A^{\omega}_{\alpha,k} \neq 0\}$.

The natural action of the monodromy h_* on the homology bundle H_n translates into a natural action of $(h^*)^{-1}$ on the sections $A^{\omega}_{\alpha,k}$ via the integral in Equation (10.5). Namely, Varchenko shows in [Var80, Lemma 5] that

$$(h^*)^{-1} A^{\omega}_{\alpha,k} = \lambda^{-1} \sum_{j=k}^{p} \frac{(2\pi\imath)^{j-k}}{(j-k)!} A^{\omega}_{\alpha,j}. \tag{10.7}$$

Therefore, the sections $A^{\omega}_{\alpha,0}, \ldots, A^{\omega}_{\alpha,p}$ are in the locally constant subsheaf $\mathcal{L}^*_\lambda$ of $\mathcal{L}^*$ that is invariant under h^* and $A^{\omega}_{\alpha,0}$ is a cyclic section of this subsheaf. After Equation (10.7), instead of the operator $h^* - \lambda\,\mathrm{id}$, we will consider the operator $(h^*)^{-1}\lambda - \mathrm{id}$ on $\mathcal{L}^*_\lambda$. Notice that, $((h^*)^{-1}\lambda - \mathrm{id})^{n+1} = 0$ on $\mathcal{L}^*_\lambda$. Now, define the operator $\ln((h^*)^{-1}\lambda)$ on $\mathcal{L}^*_\lambda$ by the formula

$$\ln((h^*)^{-1}\lambda) := \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} ((h^*)^{-1}\lambda - \mathrm{id})^j \tag{10.8}$$

as in [Var80, Lemma 5]. Hence, from Equation (10.7),

$$A^{\omega}_{\alpha,k} = \left(\frac{\ln((h^*)^{-1}\lambda)}{2\pi\imath} \right)^k A^{\omega}_{\alpha,0}, \tag{10.9}$$

and, therefore,

$$t^\alpha \sum_{k=0}^{p} (\ln t)^k A^{\omega}_{\alpha,k}(t) = \exp\left[\ln t \left(\alpha\,\mathrm{id} + \frac{\ln((h^*)^{-1}\lambda)}{2\pi\imath} \right) \right] A^{\omega}_{\alpha,0}(t). \tag{10.10}$$

Then, Varchenko [Var80] defines the *elementary section* associated with a locally constant section $A \in \mathcal{L}^*_\lambda$ and $\alpha \in L(\lambda)$ as the section $s_\alpha[A]$ of $\mathcal{H}_\lambda := \mathcal{L}^*_\lambda \otimes_{\mathbb{C}_{T'}} \mathcal{O}_{T'}$ defined by

$$s_\alpha[A](t) := \exp\left[\ln t \left(\alpha\,\mathrm{id} + \frac{\ln((h^*)^{-1}\lambda)}{2\pi\imath} \right) \right] A(t). \tag{10.11}$$

We end this section with the following properties of elementary sections.

Lemma 10.2 ([Var80, Lemma 9]). *Let $\lambda \in \Lambda, \alpha \in L(\lambda)$ and A a section of $\mathcal{L}^*_\lambda$, then:*

1. *The sections $s_\alpha[A]$ are holomorphic univalued sections of the vector bundle H^n_λ.*
2. *If the sections $A_0, \ldots, A_p \in \mathcal{L}^*_\lambda$ are linearly independent at every fiber, then the sections $s_\alpha[A_0], \ldots, s_\alpha[A_p]$ are linearly independent at every point $t \in T'$.*
3. *The action of the covariant derivative on an elementary section is:*

$$t\partial_t^* s_\alpha[A] = \alpha s_\alpha[A] + (2\pi\imath)^{-1} s_\alpha[\ln((h^*)^{-1}\lambda)A]. \tag{10.12}$$

Next, we will present the relation, given by Varchenko in [Var80, §8], between the elementary sections and the saturation of the Brieskorn lattice appearing in Malgrange's result in Theorem 1.35.

Consider the quotient vector bundle $H^n_{\lambda,\alpha}/H^n_{\lambda,\alpha-1}$ over T' which will be denoted by F_α. Let $\mathcal{F}_\alpha$ be the locally free sheaf of sections of the vector bundle F_α. We will denote by $\mathcal{G}_\alpha$ the subsheaf of $\mathcal{F}_\alpha$ generated by the image of the elementary sections $s_\alpha[A]$, with A a section of S_α, under the projection map

$$\pi_\alpha : H^n_{\lambda,\alpha} \longrightarrow H^n_{\lambda,\alpha}/H^n_{\lambda,\alpha-1} = F_\alpha. \tag{10.13}$$

After Lemma 10.2, for every value $t \in T'$, the sections $\mathcal{G}_\alpha$ generate the whole fiber of F_α. The restriction of the connection ∇^*_λ of H^n_λ to $H^n_{\lambda,\alpha}$ induces a connection $\nabla^*_{\lambda,\alpha}$ in the quotient bundle F_α. Therefore, since the sheaf $\mathcal{G}_\alpha$ is annihilated by this connection $\nabla^*_{\lambda,\alpha}$, $\mathcal{G}_\alpha$ is a local system equal to $\ker \nabla^*_{\lambda,\alpha}$. Furthermore, by Lemma 10.2, the operator $\partial_t^* t$ maps elementary sections to elementary sections and thus, it induces an endomorphism D_α on $\mathcal{G}_\alpha$ which has eigenvalues equal to α at every fiber. For more details see [Var80, Lemma 10].

If $j : T' \hookrightarrow T$ denotes again the open inclusion and $j_!$ is the extension by zero, then we have that $j_!\mathcal{G}_\alpha \neq j_*\mathcal{G}_\alpha$, meaning that the stalk $(j_*\mathcal{G}_\alpha)_0$ is not zero. Indeed, by Lemma 10.2, the elementary sections whose image under π_α generate $\mathcal{G}_\alpha$ are univalued. We will continue to denote D_α the extension to $j_*\mathcal{G}_\alpha$ of the endomorphism D_α of $\mathcal{G}_\alpha$.

Theorem 10.3. *[Var80, Thm. 13] Let $\mathcal{G}_\alpha, \alpha \in L(\lambda), \lambda \in \Lambda$ be the locally constant sheaves defined above and consider the locally constant sheaf $\mathcal{G} := \bigoplus_{\lambda\in\Lambda}\bigoplus_{\alpha\in L(\lambda)} \mathcal{G}_\alpha$ of complex vector spaces with the endomorphism $D := \bigoplus_{\lambda\in\Lambda}\bigoplus_{\alpha\in L(\lambda)} D_\alpha$. Then, there exists a natural isomorphism of complex vectors spaces between $(j_*\mathcal{G})_0$ and $\widetilde{H}''_{f,\mathbf{0}}/t\widetilde{H}''_{f,\mathbf{0}}$ and under this isomorphism, the endomorphism $\partial_t t$ in $\widetilde{H}''_{f,\mathbf{0}}/t\widetilde{H}''_{f,\mathbf{0}}$ corresponds to D_0.*

The set of b-exponents of an isolated singularity is, therefore, after Theorem 10.3, contained in the set of positive rational numbers of the form $\alpha \in L(\lambda), \lambda \in \Lambda$.

10.4 *Semicontinuity of the b-exponents*

In [Var80, §11], Varchenko proves the semicontinuity of the b-exponents under μ-constant deformations of the singularity in the case that the eigenvalues of the monodromy endomorphism are pair-wise different. In this section, we will generalize his result to any isolated singularity under the extra assumption of the existence of certain *dual locally constant geometric sections*. First, let us review the results from [Var80, §11].

Fix $\lambda \in \Lambda$ an eigenvalue of the monodromy. We have seen in Section 10.2 that the vector bundles $H^n_{\lambda,\alpha}, \alpha \in L(\lambda)$ form an increasing filtration in H^n_λ. Denote by d_α the dimension of the bundle $H^n_{\lambda,\alpha}$. Then, $d_\alpha \leq d_{\alpha+1}$ and $d_\alpha = \dim_{\mathbb{C}} H^n_\lambda$, for $\alpha \gg 0$, which is exactly the number of eigenvalues of the monodromy that are equal to λ. Then, for the quotient bundles F_α defined in the previous section, we have that

$$\sum_{\lambda \in \Lambda} \sum_{\alpha \in L(\lambda)} \dim_{\mathbb{C}} F_\alpha = \mu, \tag{10.14}$$

since $\dim_{\mathbb{C}} F_\alpha = d_\alpha - d_{\alpha-1}$. If one assumes that the monodromy has pair-wise different eigenvalues, then $\dim_{\mathbb{C}} H^n_\lambda = 1$ for all $\lambda \in \Lambda$. Therefore, there is only a single $\alpha \in L(\lambda)$ that can be a b-exponent, and such α is characterized by the fact that $\dim_{\mathbb{C}} F_\alpha = 1$.

Let $f_y : (\mathbb{C}^{n+1}, \mathbf{0}) \longrightarrow (\mathbb{C}, 0)$, with $y \in I_\eta := \{z \in \mathbb{C} \mid |z| < \eta\}, 0 < \eta \ll 1$, be a one-parameter μ-constant deformation of the isolated singularity $f =: f_0$. Recall that under μ-constant deformations neither the eigenvalues nor the Jordan form of the monodromy endomorphism change, [TR76]. Then, if one denotes by $d_\alpha(y)$ the dimension of the corresponding bundle $H^n_{\lambda,\alpha}(y)$ of the isolated singularity f_y, we have

Proposition 10.4 ([Var80, Cor. 19])**.** *The dimension $d_\alpha(y)$ of $F_\alpha(y)$ depends lower-semicontinuously on the parameter y.*

From this, and under the assumption that the eigenvalues of the monodromy are pair-wise different, Varchenko [Var80, Cor. 21] deduces a lower-semicontinuity for the roots of the Bernstein-Sato polynomial of f. Since the b-exponents are the opposites in sign to the roots of $b_{f,\mathbf{0}}(s)$, one has an analogous upper-semicontinuity for the b-exponents of f. Next, we will construct suitable subbundles of $H^n_{\lambda,\alpha}$ such that its dimension completely characterizes the existence of b-exponents even if the eigenvalues of the monodromy are not pair-wise different.

Let us fix $\gamma_1(t), \dots, \gamma_\mu(t)$ a basis of generalized eigenvectors of the monodromy endomorphism in the homology of each fiber $X_t, t \in T'$. Similarly to the sheaf of locally constant sections $S_\alpha, \alpha \in L(\lambda)$ introduced in Section 10.2, define $S_{\gamma_i}, i = 1, \dots, \mu$ the sheaf of locally constant sections of the bundle H^n generated by the locally constant geometric sections $A^\omega_{\alpha,k}$, $k = 0, \dots, n$ with $\omega \in \Gamma(X, \Omega^{n+1}_X)$ such that

$$\langle A^\omega_{\alpha,k}(t), \gamma_i(t) \rangle \neq 0 \qquad \text{and} \qquad \langle A^\omega_{\alpha,k}(t), \gamma_j(t) \rangle = 0, \tag{10.15}$$

with $\gamma_j(t)$ any other eigenvector of the basis different from $\gamma_i(t)$. The locally constant sections in S_{γ_i} will be called *dual locally constant geometric sections* to $\gamma_i(t)$. Notice that, after Equation (10.4), if $\gamma(t)$ is a generalized eigenvector of eigenvalue λ, then it is necessary that $\lambda = \exp(-2\pi\imath\alpha)$ for $\langle A^\omega_{\alpha,k}(t), \gamma(t) \rangle$ to be non-zero.

It is not a priori clear that dual locally constant geometric sections should exist. In Sections 11.3 and 11.5, we will show that, for irreducible plane curve singularities, dual locally constant geometric sections exist with respect to a certain basis of eigenvectors of the monodromy. As we will see in Sections 11.3 and 11.5, the geometric picture behind the duality (10.15) is that two locally constant sections of H^n with the same $\alpha \in L(\lambda)$ will be linearly independent because they will be dual to two linearly independent eigenvectors of eigenvalue λ. At the same time, these eigenvectors will be linearly independent because they will vanish to different rupture divisors of the minimal embedded resolution of f.

Let $\gamma_\lambda(t)$ be a generalized eigenvector of the basis $\gamma_1(t), \dots, \gamma_\mu(t)$ with eigenvalue $\lambda \in \Lambda$. We define the holomorphic subbundle $f^*_{\gamma_\lambda} : H^n_{\gamma_\lambda} \longrightarrow T'$ of the bundle H^n_λ as the bundle generated by the locally constant sections in S_{γ_λ}. These subbundles are also invariant under the covariant derivative associated with the connection ∇^*_λ in H^n_λ. Consider the vector bundle $H^n_{\gamma_\lambda} \cap H^n_{\lambda,\alpha}$, which will be denoted by $H^n_{\gamma_\lambda,\alpha}$, and which is a subbundle of $H^n_{\lambda,\alpha} \subset H^n_\lambda$. The bundle $H^n_{\gamma_\lambda,\alpha}$ is also invariant by the covariant derivative of the connection ∇^*_λ of H^n_λ. Notice that $\dim_{\mathbb{C}} H^n_{\gamma_\lambda} = 1$ and if the monodromy has only one eigenvalue equal to λ, then $H^n_\lambda = H^n_{\gamma_\lambda}$.

Following Section 10.3, define the quotient bundles $H^n_{\gamma_\lambda,\alpha}/H^n_{\gamma_\lambda,\alpha-1}$ which will be denoted by $F_{\gamma_\lambda,\alpha}$. Let $\mathcal{F}_{\gamma_\lambda,\alpha}$ denote the locally free sheaf of sections of $F_{\gamma_\lambda,\alpha}$. Then, $\mathcal{G}_{\gamma_\lambda,\alpha}$ is the subsheaf of $\mathcal{F}_{\gamma_\lambda,\alpha}$ generated by the image of the elementary sections $s_\alpha[A]$, with A a section in $S_\alpha \cap S_{\gamma_\lambda}$, under the projection map

$$\pi_{\gamma_\lambda,\alpha} : H^n_{\gamma_\lambda,\alpha} \longrightarrow H^n_{\gamma_\lambda,\alpha}/H^n_{\gamma_\lambda,\alpha-1} =: F_{\gamma_\lambda,\alpha}. \tag{10.16}$$

All the quotient bundles and subbundles of H^n presented so far are related by the following diagram,

$$\begin{tikzcd}
H^n_{\gamma_\lambda,\alpha} \arrow[r, hook] \arrow[d, two heads] & H^n_{\lambda,\alpha} \arrow[r, hook] \arrow[d, two heads] & H^n_\lambda \arrow[r, hook] & H^n \\
F_{\gamma_\lambda,\alpha} \arrow[r, hook] & F_\alpha & & \\
G_{\gamma_\lambda,\alpha} \arrow[r, hook] \arrow[u, hook] & G_\alpha. \arrow[u, hook] & &
\end{tikzcd} \tag{10.17}$$

One can check that the subsheaf $\mathcal{G}_{\gamma_\lambda,\alpha}$ has the same properties as the subsheaf $\mathcal{G}_\alpha$ described in the previous section. The important point is that the dimensions of the vector bundles $H^n_{\gamma_\lambda,\alpha} \subseteq H^n_{\gamma_\lambda}, \alpha \in L(\lambda)$, denoted $d_{\gamma_\lambda,\alpha}$, are either zero or one. In addition, $d_{\gamma_\lambda,\alpha} = \dim_{\mathbb{C}} H^n_{\gamma_\lambda} = 1$, for $\alpha \gg 0$. Therefore, this construction allows us to characterize the existence of a certain b-exponent in terms of the dimensions of $F_{\gamma_\lambda,\alpha}$. A candidate b-exponent α for $\alpha \in L(\lambda)$, associated with the generalized eigenvector γ_λ of the monodromy, is a b-exponent, if and only if, $\dim_{\mathbb{C}} F_{\gamma_\lambda,\alpha} = d_{\gamma_\lambda,\alpha} - d_{\gamma_\lambda,\alpha-1} = 1$.

As before, let $f_y : (\mathbb{C}^{n+1}, \mathbf{0}) \longrightarrow (\mathbb{C}, 0), y \in I_\eta$ be a one-parameter μ-constant deformation of an isolated singularity. Following [Var80, §11] and the notations from Section 1.4, let $\mathcal{X} := \{(x,y) \in B_\epsilon \times I_\eta \mid f_y(x) = t \in T_\delta\}$. Denote by $\Phi : \mathcal{X} \longrightarrow T_\delta$ the application given by $(x,y) \mapsto (f_y(x), y)$. Let

$$X_{t,y} := \mathcal{F} \cap \Phi^{-1}(t,y) \qquad \text{and} \qquad \mathcal{X}' := \mathcal{X} \setminus \Phi^{-1}(\{0\} \times I_\eta), \tag{10.18}$$

then $\Phi' : \mathcal{X}' \longrightarrow T'_\delta \times I_\eta$ is a locally trivial fibration. As in Section 1.5, this means that the associated homological and cohomological μ-dimensional bundles carry an integrable connection. Furthermore, the (co)homological bundle of the restriction of Φ' over $T'_\delta \times \{y\}$ is canonically isomorphic to the (co)homological Milnor fibration of the singular point of the fiber f_y, see [Var80, Cor. 17].

In particular, this means that we can fix a basis $\gamma_1(t,y), \gamma_2(t,y), \dots, \gamma_\mu(t,y)$ of the homological Milnor fibration of f_y for $y \in I_\eta$, given by eigenvectors of the monodromy endomorphism, and they can be extended by parallel transport, to a basis of the homological bundle associated with the locally trivial fibration $\Phi : \mathcal{X}' \longrightarrow T'_\delta \times I_\eta$.

After the above discussion, we are in the same situation as in [Var80, Cor. 21], replacing the distinct eigenvalues of the monodromy by the distinct generalized eigenvectors of the monodromy. Therefore, assuming the existence of dual locally constant geometric sections, there is a single b-exponent $\alpha \in L(\lambda)$ associated with each generalized eigenvector of the monodromy. The following proposition then follows from the same argument as in [Var80, Cor. 21].

Proposition 10.5. *Let $\gamma_\lambda(t,y)$ be a generalized eigenvector of the monodromy of eigenvalue $\lambda \in \Lambda$. Assume that there exist dual locally constant geometric sections to $\gamma_\lambda(t,y)$ for all fibers of the deformation, that is, $\dim_{\mathbb{C}} H^n_{\gamma_\lambda}(y) = 1$ for all $y \in I_\eta$. Then, the b-exponent associated with $\gamma_\lambda(t,y)$ depends upper-semicontinuously on the parameter y of the μ-constant deformation.*

Proof. For the semicontinuity of the dimension $d_{\gamma_\lambda,\alpha}(y)$, one argues as in the proof of Proposition 10.4 given in [Var80] since the bundle $H^n_{\gamma_{\lambda,\alpha}}$ is a subbundle of $H^n_{\lambda,\alpha}$. Then, after Theorem 10.3 and since $\dim_{\mathbb{C}} H^n_{\gamma_\lambda}(y) = 1$ for all $y \in I_\eta$, one has that $\alpha \in L(\lambda)$ is a b-exponent of f_y, if and only if, $d_{\gamma_\lambda,\alpha}(y) - d_{\gamma_\lambda,\alpha-1}(y) = 1$. □

10.5 *Resolution of singularities and semi-stable reduction*

In this section, we show how to use a resolution of singularities of f to study the integrals of relative differential forms along vanishing cycles following the ideas of Varchenko in [Var82, §4].

With the notations and definition from Section 1.1, let $\pi : \overline{X} \longrightarrow X$ be a resolution of singularities of f. For reasons that will become clear later, it is convenient to pass from the resolution manifold $\overline{X}$ to another space where the normal crossings of the exceptional divisor E are preserved but F_π is *reduced*. This is indeed possible if we relax the smoothness conditions of $\overline{X}$. This process is called *semi-stable reduction* and the reader is referred to [Ste77, §2] for the details.

Let e be a positive integer such that the e–th power of the monodromy is unipotent. By A'Campo's description of the monodromy in terms of a resolution of f given in [ACa75], we can take $e = \operatorname{lcm}(N_1, N_2, \ldots, N_r)$. Define

$$\widetilde{T} := \{\tilde{t} \in \mathbb{C} \mid |\tilde{t}| < \delta^{1/e}\} \tag{10.19}$$

and let $\sigma : \widetilde{T} \longrightarrow T$ be given by $\sigma(\tilde{t}) = \tilde{t}^e$. Denote by $\widetilde{X}$ the normalization of the fiber product $\overline{X} \times_T \widetilde{T}$ and by $n : \widetilde{X} \longrightarrow \overline{X} \times_T \widetilde{T}$ the normalization morphism. Let $\rho : \widetilde{X} \longrightarrow \overline{X}$ and $\tilde{f} : \widetilde{X} \longrightarrow \widetilde{T}$ be the natural maps. Finally, denote $\widetilde{X}_t := \tilde{f}^{-1}(t)$ and $\widetilde{D} := \rho^*(D)$ for any divisor D on $\overline{X}$. We have the following commutative diagram

$$\begin{array}{ccccc} \widetilde{X} & \xrightarrow{\rho} & \overline{X} & \xrightarrow{\pi} & X \\ \downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle \pi^* f} & & \downarrow{\scriptstyle f} \\ \widetilde{T} & \xrightarrow{\sigma} & T & = & T. \end{array} \tag{10.20}$$

An orbifold of dimension $n+1$ is a complex analytic space which admits an open covering $\{U_i\}$ such that each U_i is analytically isomorphic to Z_i/G_i where $Z_i \subset \mathbb{C}^{n+1}$ is an open ball and G_i is finite subgroup of $GL(n+1, \mathbb{C})$. Similarly, a divisor D on an orbifold $\widetilde{X}$ is an orbifold normal crossing divisor if locally $(\widetilde{X}, \widetilde{D}) = (Z, F)/G$ with

$Z \subset \mathbb{C}^{n+1}$ an open domain, $G \subset GL(n+1,\mathbb{C})$ a small subgroup acting on Z and $F \subset Z$ a G–invariant divisor with normal crossings. The singularities of an orbifold are concentrated in codimension at least two.

Lemma 10.6 ([Ste77, Lemma 2.2]). *$\widetilde{X}$ is an orbifold and the divisor of $(\pi\rho)^*f$ is a reduced divisor with orbifold normal crossings.*

In terms of the local coordinates of $\overline{X}$, the orbifold $\widetilde{X}$ is presented as follows. Fix U an affine coordinate chart of $\overline{X}$ with coordinates $x_0,\ldots,x_n$, for which there are non-negative integers k and $N_0,\ldots,N_k$ such that $(\pi^*f)(x_0,\ldots,x_n) = x_0^{N_0}\cdots x_k^{N_k}$. Then, on the open neighborhood $U \times \widetilde{T}$ in $\overline{X} \times \widetilde{T}$, we have $x_0^{N_0}\cdots x_k^{N_k} = \tilde{t}^e$. Set $d := \gcd(N_0,\ldots,N_k)$, the preimage of $U \times \widetilde{T}$ in $\widetilde{X}$ consists of d disjoint open sets which we denote by $U_1, U_2, \ldots, U_d$. On one of these subsets U_j, there are coordinates $y_0,\ldots,y_k,\tau$ related by $\tau = y_0\cdots y_k$. The map $\rho|_{U_j} : U_j \longrightarrow U \times_T \widetilde{T}$ is given by $\tilde{t} = \tau\exp(2\pi\imath j/d)$ and $x_i = y_i^{e/N_i}$ if $0 \le i \le k$ and $x_i = y_i$ if $i > k$.

Let $G = \mathbb{Z}/(e/N_0) \times \cdots \times \mathbb{Z}/(e/N_k)$ be the group that acts on $\mathbb{C}\{y_0,\ldots,y_k\}$ according to the following rules

$$(a_0,\ldots,a_k)\cdot y_i = \begin{cases} \exp(2\pi\imath a_j N_j/e)\cdot y_j & \text{if } 0 \le j \le k, \\ y_j & \text{if } j > k. \end{cases} \tag{10.21}$$

Let $G' := \{g \in G \mid g\tau = \tau\} \subset GL(n+1,\mathbb{C})$. Then, the holomorphic functions and differential forms on U_j are the usual ones in terms of the coordinates $y_0,\ldots,y_n$ subject to the condition that they must be invariant under G', i.e. $g\cdot(y_0\cdots y_k) = y_0\cdots y_k$. In this context, differential calculus is completely analogous to the usual one on manifolds.

Assume that $\omega \in \Gamma(X,\Omega_X^{n+1})$ is a top holomorphic form. Let $v_i(\omega)$ be the order of vanishing of $\pi^*\omega$ along the exceptional component E_i, then the order of vanishing $\tilde{v}_i(\omega)$ of $(\pi\rho)^*\omega$ along $\widetilde{E}_i$ is $e(v_i(\omega)+1)/N_i - 1$, see [Var82, Lemma 4.4]. Clearly, $v_i(\omega) = k_i$ if $\omega = \mathrm{d}x_0 \wedge \cdots \wedge \mathrm{d}x_n$, and thus $\tilde{v}_i(\omega) = e(k_i+1)/N_i - 1$. Now, take $\widetilde{\omega}$ a section of $\Omega_{\widetilde{X}}^{n+1}$, since locally $\tilde{f}(y_0,\ldots,y_n) = y_0\cdots y_k$, the relative form $\widetilde{\omega}/\mathrm{d}\tilde{f}$ is well-defined on

$$\widetilde{E}^\circ := \bigcup_{i=1}^{r} \widetilde{E}_i^\circ \quad \text{where} \quad \widetilde{E}_i^\circ := \widetilde{E}_i \setminus \bigcup_{j\neq i}(\widetilde{E}_i \cap \widetilde{E}_j). \tag{10.22}$$

The following lemma is easy to establish.

Lemma 10.7 ([Var82, Lemma 4.3]). *If $\pi\rho$ also denotes the restriction to $\widetilde{X}_t, \tilde{t} \in \widetilde{T}^*$, of the map $\pi\rho : \widetilde{X} \longrightarrow X$, then, for all $\tilde{t} \in \widetilde{T}^*$,*

$$\left.\frac{(\pi\rho)^*(\omega)}{\mathrm{d}\tilde{f}}\right|_{\widetilde{X}_{\tilde{t}}} = e\tilde{f}^{e-1}(\pi\rho)^*\left(\left.\frac{\omega}{\mathrm{d}f}\right|_{X_{\tilde{t}^e}}\right). \tag{10.23}$$

10.6 *Full asymptotic expansion*

In this section, we will construct the full expansion of the integrals in Equation (10.5) from a resolution of singularities of f and the powers of the exceptional divisors. We will use the same notations from the previous section. In the sequel, we will always fix an exceptional component E_i of a resolution of the germ $f : (\mathbb{C}^{n+1},\mathbf{0}) \longrightarrow (\mathbb{C},0)$.

Let $\omega \in \Gamma(X, \Omega_X^{n+1})$ be a holomorphic differential form of maximal degree on X. Consider on $E_i^\circ \times T \subset \overline{X}$ the analytic coordinates such that $\bar{f} = x_0^{N_i} = t$. Expressing $\overline{\omega} := \pi^*\omega$ in these fixed analytic coordinates, we can consider the following decomposition of $\overline{\omega}$

$$\overline{\omega} = \overline{\omega}_0 + \overline{\omega}_1 + \cdots + \overline{\omega}_\nu + \cdots, \tag{10.24}$$

where $\overline{\omega}_\nu$ is a holomorphic form with degree $v_i(\omega) + \nu$ in x_0. Since E_i° is quasi-projective each form $\overline{\omega}_\nu, \nu \in \mathbb{Z}_+$ extends to a section of $\Omega_{\overline{X}}(-\nu E_i) := \Omega_{\overline{X}} \cdot \mathcal{O}_{\overline{X}}(-\nu E_i)$, where $\mathcal{O}_{\overline{X}}(-E_i)$ is the ideal sheaf of the effective divisor E_i. We will call $\overline{\omega}_\nu$ the ν–th piece of ω associated with the divisor E_i. For the ease of notation, we will omit the dependence of the pieces $\overline{\omega}_\nu$ on the index i as we will always work with a fixed divisor E_i.

Set $\widetilde{X}_i^\circ := n^{-1}(E_i^\circ \times \widetilde{T}) \subset \widetilde{X}$, where n is the normalization morphism from the previous section. Recall that on $\widetilde{X}_i^\circ$ there are local coordinates such that $\tilde{f}(y_0, \ldots, y_n) = y_0 = \tilde{t}$ with $\widetilde{E}_i^\circ : y_0 = 0$. Recall that if $\widetilde{\omega}_\nu = \rho^*\overline{\omega}_\nu$, then the orders of vanishing have the following relation: since $v_i(\overline{\omega}_\nu) = v_i(\omega) + \nu$, then $\tilde{v}_i(\widetilde{\omega}_\nu) = e(v_i(\omega) + 1 + \nu)/N_i - 1$. With all these considerations, the following lemma follows from a local computation.

Lemma 10.8. *Let $\tilde{\gamma}(\tilde{t})$ be any n-cycle on $\widetilde{X}_i^\circ$. Then,*

$$\int_{\tilde{\gamma}(\tilde{t})} \frac{\widetilde{\omega}}{\mathrm{d}\tilde{f}} = \sum_{\nu \geq 0} \tilde{t}^{\,\tilde{v}_i(\widetilde{\omega}_\nu)} \int_{\tilde{\gamma}(\tilde{t})} R_i(\widetilde{\omega}_\nu), \tag{10.25}$$

where $R_i(\widetilde{\omega}_\nu) := \tilde{f}^{-\tilde{v}_i(\widetilde{\omega}_\nu)} \widetilde{\omega}_\nu / \mathrm{d}\tilde{f}$ is a well-defined holomorphic n-form on $\widetilde{E}_i^\circ$.

Notice that since $\overline{\omega}_0$ is always different from zero, $R_i(\widetilde{\omega}_0)$ is always a non-zero n-form on $\widetilde{E}_i^\circ$. However, this may not be the case for the other terms $R_i(\widetilde{\omega}_\nu), \nu > 0$.

Now we can obtain the expression of Equation (10.5) in terms of the resolution data by pushing down to X the expressions from Equation (10.25). Namely, after Lemma 10.7, the left-hand side of Equation (10.25) reads as

$$\int_{\tilde{\gamma}(\tilde{t})} \frac{(\pi\rho)^*\omega}{\mathrm{d}\tilde{f}} = e\tilde{t}^{e-1} \int_{\tilde{\gamma}(\tilde{t}^e)} (\pi\rho)^* \Big(\frac{\omega}{\mathrm{d}f}\Big). \tag{10.26}$$

Define the numbers

$$\sigma_{i,\nu}(\omega) := \frac{v_i(\omega) + 1 + \nu}{N_i}, \qquad \nu \in \mathbb{Z}_+. \tag{10.27}$$

In particular, $\sigma_{i,\nu}(\mathrm{d}x_0 \wedge \cdots \wedge \mathrm{d}x_n) = (k_i + 1 + \nu)/N_i$. Then

$$\int_{\tilde{\gamma}(\tilde{t}^e)} (\pi\rho)^* \Big(\frac{\omega}{\mathrm{d}f}\Big) = e^{-1} \sum_{\nu \geq 0} \tilde{t}^{\,e(\sigma_{i,\nu}(\omega) - 1)} \int_{\tilde{\gamma}(\tilde{t})} R_i(\widetilde{\omega}_\nu), \tag{10.28}$$

since $\tilde{v}_i(\widetilde{\omega}_\nu) - e + 1 = e(\sigma_{i,\nu}(\omega) - 1)$. Finally, since $\tilde{t}^e = t$, the following lemma follows.

Lemma 10.9. *For any n-cycle $\tilde{\gamma}(\tilde{t})$ on $\widetilde{X}_i^\circ$, let $\gamma(t) := \rho_* \tilde{\gamma}(\tilde{t}^e)$, then*

$$\int_{\pi_*\gamma(t)} \frac{\omega}{\mathrm{d}f} = \sum_{\nu \geq 0} t^{\sigma_{i,\nu}(\omega) - 1} \int_{\gamma(\tilde{t})} R_{i,\nu}(\omega), \tag{10.29}$$

where $R_{i,\nu}(\omega) := e^{-1}(\rho^{-1})^* R_i(\widetilde{\omega}_\nu)$ is a multivalued n-form on E_i° that does not depend on the integer e.

The missing logarithmic terms in Equation (10.29), when compared to Equation (10.5), are in the integrals on the right-hand side of Equation (10.29). Indeed, some of these integrals may blow-up to infinity as $\tilde{t}$ tends to zero. This would mean that, after Equation (10.5), there is a logarithmic term associated with the exponent $\sigma_{i,\nu}(\omega) - 1$.

Remark 10.1. Since $v_i(\omega) \geq k_i$, after Lemma 10.9 the set of candidates for the b-exponents of an isolated singularity f are the set of rational numbers

$$\sigma_{i,\nu}(\mathrm{d}x_0 \wedge \cdots \wedge \mathrm{d}x_n) = (k_i + 1 + \nu)/N_i, \nu \in \mathbb{Z}_+ \tag{10.30}$$

associated with each exceptional divisor E_i of a resolution of f. Obviously, this set of candidates coincides with the well-known set of candidates for the roots of the Bernstein-Sato polynomial from Equation (1.12) in Section 1.2. ■

Consequently, at least for those terms such that the integral as t tends to zero of $R_{i,\nu}(\omega)$ along some cycle $\gamma(\tilde{t})$ is well-defined, the multivalued n-form $R_{i,\nu}(\omega)$ on E_i defines a locally constant cohomology class $A^{\omega}_{\sigma_{i,\nu}-1,0}(t)$ of the bundle that is dual to the vector bundle generated at each fiber by $\pi_*\gamma(t)$, $t \in T'$. Indeed, define the pairing

$$\langle A^{\omega}_{\sigma_{i,\nu}-1,0}(t), \pi_*\gamma(t)\rangle := \lim_{\tilde{t}\to 0} \int_{\gamma(\tilde{t})} R_{i,\nu}(\omega) \in \mathbb{C}. \tag{10.31}$$

Here, and in the sequel, we denote $A^{\omega}_{\sigma_{i,\nu}-1,0}$ instead of $A^{\omega}_{\sigma_{i,\nu}(\omega)-1,0}$ for the ease of notation.

Let $D_{i,j}$ be the intersection of E_i with another irreducible component $D_j \in \operatorname{Supp}(F_\pi)$. Then, the $D_{i,j}$ are divisors on E_i since F_π is a simple normal crossing divisor. By definition, $E_i^\circ = E_i \setminus \cup_j D_{i,j}$.

Lemma 10.10. *The n-forms $R_{i,\nu}(\omega)$ on E_i° are multivalued along the divisors $D_{i,j}$ with order of vanishing*

$$\varepsilon_{j,\nu}(\omega) := -N_j\sigma_{i,\nu}(\omega) + v_j(\overline{\omega}_\nu) = -N_j\frac{v_i(\omega) + 1 + \nu}{N_i} + v_j(\overline{\omega}_\nu). \tag{10.32}$$

Proof. The proof follows from a local computation. Let x_0, x_1 be coordinates around a general point of $D_{i,j}$, with $x_0 = x_1 = 0$ a local equation for $D_{i,j}$ and $f = x_0^{N_i}x_1^{N_j}$. Then, for any $\omega \in \Gamma(X, \Omega_X^{n+1})$, $\overline{\omega}_\nu = x_0^{v_i(\omega)+\nu}x_1^{v_j(\overline{\omega}_\nu)} v \mathrm{d}x_0 \wedge \ldots \mathrm{d}x_n$, with v a unit. Following Section 10.5, we set $x_0 = y_0^{e/N_i}, x_1 = y_1^{e/N_j}$ and $x_k = y_k$ otherwise. Hence,

$$\widetilde{\omega}_\nu = \frac{e}{N_i}\frac{e}{N_j} y_0^{e(v_i(\omega)+1+\nu)/N_i - 1} y_1^{e(v_j(\overline{\omega}_\nu)+1)/N_j - 1} v \mathrm{d}y_0 \wedge \cdots \wedge y_n, \tag{10.33}$$

and $f = z_0 z_1$. Now, on $\widetilde{E}_i^\circ$, $f = \bar{y}_0$ with $y_0 = \bar{y}_0/y_1$. Making the substitution on $\widetilde{\omega}_\nu$,

$$\widetilde{\omega}_\nu = \frac{e}{N_i}\frac{e}{N_j} \bar{y}_0^{e\sigma_{i,\nu}(\omega)-1} y_1^{e(v_j(\overline{\omega}_\nu)+1)/N_j - e\sigma_{i,\nu}(\omega) - 1} v \mathrm{d}\bar{y}_0 \wedge \cdots \wedge y_n. \tag{10.34}$$

Finally, with the notations from Lemma 10.8, $R_i(\widetilde{\omega}_\nu)$ is given locally around $y_1 = 0$ on $\widetilde{E}_i^\circ$ by the expression

$$\frac{e}{N_j} y_1^{e(v_j(\overline{\omega}_\nu)+1)/N_j - e\sigma_{i,\nu}(\omega) - 1} v \mathrm{d}y_1 \wedge \cdots \wedge y_n. \tag{10.35}$$

Finally, to get the local expression for the $R_{i,\nu}(\omega)$ from Lemma 10.9, one simply undoes the first change of variables, i.e. $y_1 = x_1^{N_j/e}$ and $y_k = x_k$ otherwise, obtaining

$$y_1^{v_j(\overline{\omega}_\nu)-N_j\sigma_{i,\nu}(\omega)} v \mathrm{d}x_1 \wedge \cdots \wedge x_n, \tag{10.36}$$

as we wanted to show. □

Since no confusion may arise we drop the dependency on the index i of the divisor E_i when referring to the numbers $\varepsilon_{j,\nu}(\omega)$. These numbers are similar, although not exactly the same, to the residue numbers considered in Definition 8.2.

Given $\omega \in \Gamma(X, \Omega_X^{n+1})$, in order to prove that one candidate $\sigma_{i,\nu}(\omega)$ is a b-exponent of f, several things must be checked. First, the ν–th piece $\overline{\omega}_\nu$ of ω associated with the exceptional divisor E_i must be non-zero. Then, one must show that there exists a cycle $\gamma(\tilde{t})$ such that the integral of $R_{i,\nu}(\omega)$ along $\gamma(\tilde{t})$ is non-zero when $\tilde{t}$ tends to zero. After Section 10.2, this would give a locally constant section $A^{\omega}_{\sigma_{i,\nu}-1,0}$. Finally, after Theorem 10.3, for $\sigma_{i,\nu}(\omega)$ to be a b-exponent it is enough that $A^{\omega}_{\sigma_{i,\nu}-1,0} \notin S_{\sigma_{i,\nu}-2}$, since then the image of $s_{\sigma_{i,\nu}-1}[A^{\omega}_{\sigma_{i,\nu}-1,0}]$ in $\mathcal{G}_{\sigma_{i,\nu}-1} \subseteq \mathcal{F}_{\sigma_{i,\nu}-1}$ will be non-zero.

We will devote the rest of this chapter to check these conditions, under some hypothesis of genericity, for the candidate b-exponents of irreducible plane curve singularities.

11 THE CASE OF PLANE CURVE SINGULARITIES

In this section, we will study the object present in the previous section in the case of plane curve singularities. In this context, we will be able to prove the existence of non-zero locally constant sections. This will be possible because resolution of singularities of plane curves is well-understood and because of a result of Deligne and Mostow that is presented in the sequel. Finally, using all the techniques developed in this chapter, we will present a proof of Yano's conjecture in its full generality. The results of this section can be found in [Bla19b].

11.1 *Multivalued forms on the punctured projective line*

In this section, we will review the basic facts from Section 2 of [DM86] about multivalued holomorphic forms on the punctured projective line defining cohomology classes in the cohomology groups with coefficients on a local system.

Let $\mathbb{P} := \mathbb{P}^1_{\mathbb{C}}$ be the complex projective line, and $S := \{s_1, s_2, \ldots, s_r\}$ be a set of $r \geq 1$ distinct points on $\mathbb{P}$ and $(\alpha_s)_{s\in S}$ be a family of complex numbers satisfying $\prod_{s\in S} \alpha_s = 1$. With these data there is, up to non-unique isomorphism, a unique local system L of rang one in $\mathbb{P} \setminus S$ such that the monodromy of L around each $s \in S$ is the multiplication by α_s. We will denote by $L^\vee$ the dual local system with monodromies $\alpha_s^{-1}, s \in S$.

In order to work with the locally constant sections of L, we fix complex numbers $(\mu_s)_{s\in S}$ such that $\alpha_s = e^{2\pi\imath\mu_s}, s \in S$. Let z be a local coordinate near $s \in S$. Any local section u of $\mathcal{O}(L)$ (resp. $\Omega^1(L)$) in a neighborhood of s can be written as $u = z^{-\mu_s} e f$ (resp. $u = z^{-\mu_s} e f \mathrm{d}z$) with e a non-zero multivalued section of L and f holomorphic in a punctured neighborhood of s. We define u to be meromorphic in $s \in S$ if f is, and we define the valuation of u at s as $v_s(u) := v_s(f) - \mu_s$.

The holomorphic L-valued de Rham complex $\Omega^\bullet(L)$: $\mathcal{O}(L) \longrightarrow \Omega^1(L)$ with the natural connecting morphism $\mathrm{d}(ef) = e\mathrm{d}f$ is a resolution of L by coherent sheaves. Therefore, one can interpret $H^*(\mathbb{P} \setminus S, L)$ as the hypercohomology on $\mathbb{P} \setminus S$ of $\Omega^\bullet(L)$. Since $\mathbb{P} \setminus S$ is Stein, $H^q(\mathbb{P} \setminus S, \Omega^p(L)) = 0$ for $q > 0$ and the hypercohomology $\mathbb{H}^*(\mathbb{P} \setminus S, \Omega^\bullet(L))$ gives

$$H^*(\mathbb{P} \setminus S, L) = H^*\Gamma(\mathbb{P} \setminus S, \Omega^\bullet(L)). \tag{11.1}$$

Let $j : \mathbb{P} \setminus S \longrightarrow \mathbb{P}$ be the inclusion. Similarly, since j is a Stein morphism, the higher direct images $R^q j_* \Omega^p(L)$ vanish for $q > 0$ and

$$H^*(\mathbb{P} \setminus S, L) = H^*\Gamma(\mathbb{P} \setminus S, \Omega^\bullet(L)) = H^*\Gamma(\mathbb{P}, j_*\Omega^\bullet(L)) = \mathbb{H}^*(\mathbb{P}, j_*\Omega^\bullet(L)). \tag{11.2}$$

It is convenient to replace the complex of sheaves $j_*\Omega^\bullet(L)$ by the subcomplex $j_*^m\Omega^p(L)$ of meromorphic forms. The analytic Atiyah-Hodge Lemma [HA55, Lemma 17] implies that

$$\mathbb{H}^*(\mathbb{P}, j_*^m\Omega^\bullet(L)) \cong \mathbb{H}^*(\mathbb{P}, j_*\Omega^\bullet(L)) = H^*(\mathbb{P} \setminus S, L). \tag{11.3}$$

Since $j_*^m\Omega^p(L)$ is an inductive limit of line bundles with degrees tending to infinity, one can show that $H^q(\mathbb{P}, j_*^m\Omega^p(L))$ vanishes for $q > 0$. Indeed, if $D = \sum_{s\in S} s$ is the divisor on $\mathbb{P}$ associated with S, then

$$H^q(\mathbb{P}, j_*^m\Omega^q(L)) = H^q(\mathbb{P}, \varinjlim_n \Omega^p(L) \otimes \mathcal{O}(nD)) = \varinjlim_n H^q(\mathbb{P}, \Omega^p(L) \otimes \mathcal{O}(nD)) = 0, \tag{11.4}$$

since $H^q(\mathbb{P}, \Omega^p(L) \otimes \mathcal{O}(nD))$ vanishes for $n \gg 0$. Finally, this means that the cohomology groups $H^q(\mathbb{P} \setminus S, L)$ can be computed as the cohomology of the complex of L–valued forms on $\mathbb{P}$ meromorphic along D,

$$H^*(\mathbb{P} \setminus S, L) \cong H^*\Gamma(\mathbb{P}, j_*^m\Omega^\bullet(L)). \tag{11.5}$$

Define the line bundle $\mathcal{O}(\sum \mu_s s)(L)$ as the subsheaf of $j_*^m\mathcal{O}(L)$ whose local holomorphic sections are the local sections u of $j_*^m\mathcal{O}(L)$ such that the integer $v_s(u) + \mu_s$ is greater or equal than zero, i.e. $v_s(u) \geq -\mu_s$. If u is a meromorphic section of $\mathcal{O}(\sum_s \mu_s s)(L)$ and if $s \in S$, one has that $\deg_s(u) = v_s(u) + \mu_s$. The same holds true for $x \in \mathbb{P} \setminus S$ if one defines $\mu_x = 0$ for $x \in \mathbb{P} \setminus S$. By [DM86, Prop. 2.11.1], the degree of the line bundle $\mathcal{O}(\sum \mu_s s)(L)$ is equal to $\sum_{s\in S} \mu_s$. Similarly, $\deg \Omega^1(\sum \mu_s s)(L) = \sum_{s\in S} \mu_s - 2$.

The following proposition is a slight generalization of [DM86, Prop. 2.14], where the differential form is assumed to be invertible in $\mathbb{P} \setminus S$. We will allow ω to have zeros in $\mathbb{P} \setminus S$. Denote by $\delta_x \in \mathbb{N}$ the order of vanishing of ω in the points $x \in \mathbb{P} \setminus S$.

Proposition 11.1. *Let $\omega \in \Gamma(\mathbb{P}, \Omega^1(\sum \mu_s s - \sum \delta_x x)(L))$. Assume that $\sum_{s\in S} \mu_s \leq r - 1$ and that $\alpha_s \neq 1$ for all $s \in S$. Then, ω defines a non-zero cohomology class in $H^1(\mathbb{P} \setminus S, L)$.*

Proof. After Equation (11.5), we want to show that equation $\mathrm{d}u = \omega$ is impossible for $u \in \Gamma(\mathbb{P}, j_*^m\mathcal{O}(L))$. For any section of $j_*^m\mathcal{O}(L)$ verifying the relation, one has that $v_x(\omega) \geq v_x(u) - 1$, for $x \in \mathbb{P}$. The equality may fail if $v_x(u) = 0$ or if $v_x(\omega)$ is a non-negative integer. In any case, $v_x(u)$ is always a non-negative integer if $x \in \mathbb{P} \setminus S$. This implies that u must be a global section of the line bundle $\mathcal{O}(\sum(\mu_s - 1)s - \sum \delta'_x x)(L)$ with $\delta'_x > 0, x \in \mathbb{P} \setminus S$. But the degree of this line bundle is

$$\sum_{s\in S}(\mu_s - 1) - \sum_{x\in\mathbb{P}\setminus S} \delta'_x \leq \sum_{s\in S} \mu_s - r \leq -1, \tag{11.6}$$

which is impossible. $\square$

11.2 *Periods of plane curve singularities*

In the sequel, let $f : (\mathbb{C}^2, \mathbf{0}) \longrightarrow (\mathbb{C}, 0)$ be a germ of a holomorphic function defining a reduced plane curve singularity, not necessarily irreducible. Using the notations from Section 10.5 we will fix any embedded resolution $\pi : \overline{X} \longrightarrow X$ of f. Notice that, the exceptional divisor of any such resolution is composed exclusively of rational curves, i.e. $E_i \cong \mathbb{P}^1_{\mathbb{C}}$.

Using the same notation from Sections 10.5 and 10.6, let us fix an exceptional divisor E_i from a resolution of f. Assume that one has a holomorphic 2-form $\omega \in \Gamma(X, \Omega^2_X)$ such that the piece $\overline{\omega}_\nu$ of degree ν associated with the i–th rupture divisor E_i is non-zero. The following argument to find cycles C on rupture exceptional divisors such that for certain candidate exponents $\sigma_{i,\nu} - 1$ on has that

$$\lim_{\tilde{t} \to 0} \int_{\gamma(\tilde{t})} R_{i,\nu}(\omega) = m \int_C R_{i,\nu}(\omega) \neq 0, \quad m \in \mathbb{Z}, \tag{11.7}$$

is basically due to Loeser [Loe88, §III.3] using the results of Deligne and Mostow reviewed in Section 11.1. Thus, as noticed earlier in Section 10.6, this implies that the multivalued form $R_{i,\nu}(\omega)$ on E_i defines a non-zero locally constant section $A^{\omega}_{\sigma_{i,\nu}-1,0}$.

In the case of plane curves, the divisors $D_{i,j}$ on E_i are just points which we will denote by p_j, dropping its dependence on E_i since no confusion may arise. Let

$$S_{i,\nu}(\omega) := \{p_j \in E_i \mid p_j = E_i \cap D_j \text{ with } D_j \in \operatorname{Supp}(F_\pi) \text{ and } \varepsilon_{j,\nu}(\omega) \neq 0\}, \tag{11.8}$$

and let L be the local system on $E_i \setminus S_{i,\nu}(\omega)$ with monodromies $e^{-2\pi\imath\varepsilon_{j,\nu}(\omega)}$ at the points $p_j \in S_{i,\nu}(\omega)$. The forms $R_{i,\nu}(\omega)$, for $\nu > 0$, might not be invertible in E_i°. Hence, denote by $q_k \in E_i^\circ, k = 1, \dots, r$ the points where $R_{i,\nu}(\omega)$ has zeros of order $\delta_{k,\nu}(\omega) > 0$. Then, the multivalued form $R_{i,\nu}(\omega)$ defines an element of $\Gamma(E_i, \Omega^1(-\sum \varepsilon_{j,\nu}(\omega)p_j - \sum \delta_{k,\nu}(\omega)q_k)(L))$ in the sense of Section 11.1.

The following lemma is key to apply the results of Deligne and Mostow from Section 11.1. Other versions of this result in the case $\nu = 0$ can be found in the works of Lichtin [Lic85] and Loeser [Loe88].

Proposition 11.2. *For any holomorphic form $\omega \in \Gamma(X, \Omega^2_X)$,*

$$\sum_{j=1}^{r} \varepsilon_{j,\nu}(\omega) + \sum_{k=1}^{s} \delta_{k,\nu}(\omega) = -2 - \nu E_i^2. \tag{11.9}$$

Proof. Consider the $\mathbb{Q}$-divisor $-\sigma_{i,\nu}(\omega)F_\pi + \operatorname{Div}(\overline{\omega}_\nu)$ on $\overline{X}$. Let us compute the intersection number of this divisor with E_i in two different ways. First, notice that the intersection number $(-\sigma_{i,\nu}(\omega)F_\pi + \operatorname{Div}(\overline{\omega}_\nu)) \cdot E_i$ equals

$$\sum_{j=1}^{r} \varepsilon_{j,\nu}(\omega) + \sum_{k=1}^{s} \delta_{k,\nu}(\omega) - E_i^2. \tag{11.10}$$

On the other hand, recall that the ideal sheaf of E_i in $\overline{X}$ is $\mathcal{O}_{\overline{X}}(-E_i)$ and that $\overline{\omega}_\nu$ is a section of $\Omega^2_{\overline{X}} \otimes \mathcal{O}_{\overline{X}}(-\nu E_i)$. Hence, $\operatorname{Div}(\overline{\omega}_\nu) \cdot E_i = (K_\pi - \nu E_i) \cdot E_i = -2 - E_i^2 - \nu E_i^2$, by the adjunction formula for surfaces Proposition 2.4. $\square$

Corollary 11.3. *Assume $\omega \in \Gamma(X, \Omega_X^2)$ is such that the divisors in* $\operatorname{Supp}(\operatorname{Div}(\pi^*\omega))$ *and in* $\operatorname{Supp}(F_\pi)$ *intersecting E_i are the same, then*

$$\sum_{j=1}^{r} \varepsilon_{j,\nu}(\omega) \geq -2. \tag{11.11}$$

Proof. This follows from the assumption and the fact that the line bundle $\mathcal{O}_{\overline{X}}(-\nu E_i) \otimes \mathcal{O}_{E_i}$ on E_i has a degree equal to $-\nu E_i^2$. $\square$

Remark 11.1. Notice that this result is not true without the hypothesis on the support of the divisor of $\pi^*\omega$. For instance, let $f = (y^2 - x^3)^2 - x^5 y$ and $\omega = (y^2 + x^3)\mathrm{d}x \wedge \mathrm{d}y$. On the first rupture divisor E_1, one has that $N_1 = 12, v_1(\omega) = 10$. Then, for $\nu = 0$, $\sigma_{1,0}(\omega) = -11/12$ and $\varepsilon_{1,0}(\omega) = -2/3, \varepsilon_{2,0}(\omega) = -1/2, \varepsilon_{3,0}(\omega) = -11/6$. In addition, $R_{i,0}(\omega)$ has an extra zero of multiplicity 1 in E_1° given by the strict transform of $y^2 + x^3$. Then, $\varepsilon_{1,0}(\omega) + \varepsilon_{2,0}(\omega) + \varepsilon_{3,0}(\omega) = -3 \leq -2$. ■

Assuming that ω satisfies Corollary 11.3, if E_i is a rupture divisor, i.e. $\chi(E_i^\circ) = 2 - r < 0$, and we suppose that the candidate $\sigma_{i,\nu}(\omega) - 1$ is such that $\varepsilon_{j,\nu}(\omega) \notin \mathbb{Z}$, then Proposition 11.1 holds and the multivalued form $R_{i,\nu}(\omega)$ defines a non-zero cohomology class on $H^1(E_i^\circ, L)$. The consequence of this is that, since the pairing between homology and cohomology is non-degenerate, there exists a twisted cycle $C \in H_1(E_i^\circ, L^\vee)$ such that

$$\int_C R_{i,\nu}(\omega) \neq 0. \tag{11.12}$$

Following [Loe88], let $p : F \longrightarrow E_i^\circ$ be the finite cover associated with the local system L on E_i°. This finite cover is characterized by the fact that $p_* \underline{\mathbb{C}}_F = L$. By definition, the twisted cycle $C \in H_1(E_i^\circ, L^\vee)$ is identified with a cycle in $H_1(F, \mathbb{C})$. Recall now the morphism $\rho : \widetilde{X} \longrightarrow \overline{X}$ from Section 10.5. The restriction of ρ to $\widetilde{E}_i$ is a ramified covering of degree N_i, the multiplicity of the divisor E_i, ramified at the points $E_i \cap D_j$ with monodromies $\exp(2\pi\imath N_j/N_i)$. Therefore, since the monodromies of F are

$$\exp(2\pi\imath\varepsilon_{j,\nu}(\omega)) = \exp\left(-2\pi\imath(k_i + 1 + \nu)\frac{N_j}{N_i}\right) = \exp\left(2\pi\imath\frac{N_j}{N_i}\right)^{-(k_i+1+\nu)}, \tag{11.13}$$

the restriction ρ_i of ρ to $\widetilde{E}_i^\circ$ factorizes as

$$\rho_i : \widetilde{E}_i^\circ \xrightarrow{\;q\;} F_0 \xrightarrow{\;p|_{F_0}\;} E_i^\circ, \tag{11.14}$$

where F_0 is a given connected component of F. Now, since q is also a finite covering, there exists an integer m and a cycle $\tilde{\gamma}$ in $H_1(\widetilde{E}_i^\circ, \mathbb{C})$ such that $q_*\tilde{\gamma} = mC$. Finally, since $\tilde{f}$ is a locally trivial fibration in a neighborhood of $\widetilde{E}_i^\circ$, using tubular neighborhoods, we can extend $\tilde{\gamma}$ to a family of locally constant cycles $\tilde{\gamma}(\tilde{t})$ in $H_1(\widetilde{X}_t, \mathbb{C})$ with $\tilde{t} \in \widetilde{T}'$ such that they vanish to $\tilde{\gamma}(0) := \tilde{\gamma}$, see [Var82, §4.3].

Setting $\gamma(t) := \rho_*\tilde{\gamma}(\tilde{t}^e)$ for every point $t \in T'$ in the base we have obtained, under some assumptions on the candidate exponent $\sigma_{i,\nu}(\omega) - 1$ and the exceptional divisor E_i, a family of locally constant cycles in $H_1(\overline{X}_t, \mathbb{C})$ such that they satisfy Equation (11.7). The precise result is stated in the proposition below.

Definition 11.4. A candidate b-exponent $\sigma_{i,\nu}(\omega)$ will be called *non-resonant* if the numbers $\varepsilon_{j,\nu}(\omega)$, defined in Equation (10.32), belong to $\mathbb{Q} \setminus \mathbb{Z}$.

Notice that the non-resonance condition is independent of the form $\omega \in \Gamma(X, \Omega^2_X)$ chosen to define $\sigma_{i,\nu}(\omega)$. That is, if $\omega' \in \Gamma(X, \Omega^2_X)$ is another differential form such that $\sigma_{i,\nu}(\omega') = \sigma_{i,\nu}(\omega)$ and $\sigma_{i,\nu}(\omega)$ is non-resonant, then $\sigma_{i,\nu}(\omega')$ is also non-resonant.

Proposition 11.5. *Let $\omega \in \Gamma(X, \Omega^2_X)$ be a differential form satisfying Corollary 11.3 and such that the ν–th piece $\overline{\omega}_\nu$ of ω for the rupture divisor E_i is non-zero. Assume also that $\sigma_{i,\nu}(\omega)$ is non-resonant. Then, the multivalued differential form $R_{i,\nu}(\omega)$ on E_i defines a non-zero locally constant geometric section $A^{\omega}_{\sigma_{i,\nu}-1,0}(t)$ of the vector bundle H^1.*

Proof. We have seen that under the hypothesis of the proposition there exists a vanishing cycle $\gamma(t)$ such that $\langle A^{\omega}_{\sigma_{i,\nu}-1,0}(t), \gamma(t)\rangle$ is non-zero. For any other vanishing cycle $\gamma'(t)$ of the bundle H_1, if its limit cycle γ' defines a cycle C' of $H_1(E_i^\circ, L^\vee)$, the pairing $\langle A^{\omega}_{\sigma_{i,\nu}-1,0}(t), \gamma'(t)\rangle = \langle R_{i,\nu}(\omega), C'\rangle$ is well-defined since $R_{i,\nu}(\omega)$ defines a cohomology class of $H^1(E_i^\circ, L)$. It may happen that γ' does not define a cycle in E_i°. However, in this case, γ' defines a locally finite homology class C' of $H_1^{lf}(E_i^\circ, L^\vee)$. Now, since $\sigma_{i,\nu}(\omega)$ is non-resonant, one has that $H_1^{lf}(E_i^\circ, L^\vee) \cong H_1(E_i^\circ, L^\vee)$, see [DM86, Prop. 2.6.1], and C' can be replaced by a cycle C'' in $H_1(E_i^\circ, L^\vee)$ for which $\langle A^{\omega}_{\sigma_{i,\nu}-1,0}(t), \gamma'(t)\rangle = \langle R_{i,\nu}(\omega), C''\rangle$ is well-defined. □

If in Proposition 11.5 one sets $\nu = 0$ and takes, for instance, $\omega = \mathrm{d}x \wedge \mathrm{d}y$ one obtains the results of Lichtin [Lic89, Prop. 1], for the case of irreducible plane curves, and Loeser [Loe88, Prop. III.3.2], for general plane curves. In this situation the first piece $\overline{\omega}_0$ for any exceptional divisor E_i is always non-zero. For irreducible plane curves, Lichtin [Lic85, Prop. 2.12] proves that the exponents $\sigma_{i,0}(\mathrm{d}x \wedge \mathrm{d}y)$ are always non-resonant. This result is related to the fact that for irreducible plane curve singularities the monodromy endomorphism is semi-simple, see [Trá72, Thm. 3.3.1].

11.3 *Dual locally constant geometric sections*

In this section, we will continue to work on a fixed exceptional divisor E_i of the minimal resolution. We will show that, under some assumptions on the combinatorics of the exceptional divisors, the locally constant geometric sections $A^{\omega}_{\sigma_{i,\nu}-1,0}$ from Proposition 11.5 are dual with respect to the exceptional divisor E_i. This concept of duality with respect to E_i will be clear at the end of the section, but it essentially means that the locally constant section $A^{\omega}_{\sigma_{i,\nu}-1,0}$ will be dual to some eigenvector of the monodromy with respect to a basis of cycles vanishing to E_i°. This is a first step towards constructing a basis of locally constant geometric sections of H^1 dual to a certain basis of H_1.

Fixing a locally constant geometric section $A^{\omega}_{\sigma_{i,\nu}-1,0}$ from Proposition 11.5, we first construct the cycle which will be dual to $A^{\omega}_{\sigma_{i,\nu}-1,0}$. In order to do that, take the non-zero cycle $\gamma(t) \in H_1(\overline{X}_t, \mathbb{C})$, given by Proposition 11.5, such that $\langle A^{\omega}_{\sigma_{i,\nu}-1,0}(t), \gamma(t)\rangle \neq 0$, and consider the projection $\gamma_\lambda(t)$ of $\gamma(t)$ to the subbundle of H_1 annihilated by $(h_* - \lambda\,\mathrm{id})^2$, where $\lambda := \exp(-2\pi\imath\sigma_{i,\nu}(\omega))$. The first important observation is that Equations (10.4) and (10.5), imply that $\langle A^{\omega}_{\sigma_{i,\nu}-1,0}(t), \gamma(t)\rangle = \langle A^{\omega}_{\sigma_{i,\nu}-1,0}(t), \gamma_\lambda(t)\rangle$, which implies $\gamma_\lambda(t) \neq 0$.

Denote by $\overline{X}_{i,t}$ the subset of the Milnor fiber $\overline{X}_t$ over the subset E_i° of the exceptional fiber $\overline{X}_0$. These are the points where, locally, the Milnor fiber can be written as $x_0^{N_i} = t$ with $x_0^{N_i} = 0$ being a local equation of E_i°. Locally around these points $\overline{X}_{i,t}$ is an N_i-fold covering of E_i°. Let $\phi_t : \overline{X}_{i,t} \longrightarrow E_i^\circ$ be the projection map. In this situation, the *geometric* *monodromy* acting on $\overline{X}_{i,t}$ is just a deck transformation permuting the elements of each fiber.

Denoting by h' the restriction of the monodromy map to $\overline{X}_{i,t}$ and by $j : \overline{X}_{i,t} \hookrightarrow \overline{X}_t$ the open inclusion, we obtain the following commutative diagram,

$$\begin{tikzcd} \overline{X}_{i,t} \arrow[r, "h'"] \arrow[d, hook, "j"] & \overline{X}_{i,t} \arrow[r, "\phi_t"] \arrow[d, hook, "j"] & E_i^\circ \arrow[d, hook] \\ \overline{X}_t \arrow[r, "h"] & \overline{X}_t \arrow[r] & \overline{X}_0. \end{tikzcd} \tag{11.15}$$

This commutative diagram then induces a commutative diagram on homology. Namely,

$$\begin{tikzcd} H_1(\overline{X}_{i,t}, \mathbb{C}) \arrow[r, "h'_*"] \arrow[d, "j_*"] & H_1(\overline{X}_{i,t}, \mathbb{C}) \arrow[r, "(\phi_t)_*"] \arrow[d, "j_*"] & H_1(E_i^\circ, \mathbb{C}) \arrow[d] \\ H_1(\overline{X}_t, \mathbb{C}) \arrow[r, "h_*"] & H_1(\overline{X}_t, \mathbb{C}) \arrow[r] & H_1(\overline{X}_0, \mathbb{C}). \end{tikzcd} \tag{11.16}$$

The vanishing cycles in $j_* H_1(\overline{X}_{i,t}, \mathbb{C})$ are precisely those vanishing cycles from $H_1(\overline{X}_t, \mathbb{C})$ that vanish to a cycle in E_i° associated with the exceptional divisor E_i. By construction, the vanishing cycle $\gamma(t)$ from above belongs to $j_* H_1(\overline{X}_{i,t}, \mathbb{C}) \subset H_1(\overline{X}_t, \mathbb{C})$, and this means that there exists $\zeta_i(t) \in H_1(\overline{X}_{i,t}, \mathbb{C})$ such that $j_* \zeta_i(t) = \gamma(t)$. We will show in Proposition 11.7 that $\gamma_\lambda(t)$ is also an element of $j_* H_1(\overline{X}_{i,t}, \mathbb{C})$ and that it is also an eigenvector with eigenvalue λ of h'_*.

First, recall that the characteristic polynomial $\Delta(t)$ of the monodromy endomorphism $h_* : H_n(\overline{X}_t, \mathbb{C}) \longrightarrow H_n(\overline{X}_t, \mathbb{C})$ is determined by A'Campo [ACa75] in terms of a resolution of the singularity of $f : (\mathbb{C}^{n+1}, \mathbf{0}) \longrightarrow (\mathbb{C}, 0)$. Precisely, A'Campo proved the following formula for $\Delta(t)$:

$$\Delta(t) = \left[\frac{1}{t-1} \prod_{i=1}^{r} \left(t^{N_i} - 1 \right)^{\chi(E_i^\circ)} \right]^{(-1)^{n+1}}. \tag{11.17}$$

For the two-dimensional case, i.e. $n = 1$, one has a similar result to Equation (11.17) for the action of the monodromy h'_* on $H_n(\overline{X}_{i,t}, \mathbb{C})$.

Proposition 11.6. *Let E_i be a rupture divisor with multiplicity N_i. Then, the characteristic polynomial $\Delta_i(t)$ of the monodromy endomorphism $h_* : H_1(\overline{X}_{i,t}, \mathbb{C}) \longrightarrow H_1(\overline{X}_{i,t}, \mathbb{C})$ is equal to*

$$\Delta_i(t) = (t^{N_i} - 1)^{-\chi(E_i^\circ)} (t^{c_i} - 1), \tag{11.18}$$

where $c_i = \gcd(N_j \mid D_j \cap E_i \neq \emptyset, D_j \in \operatorname{Supp}(F_\pi))$.

The proof of Proposition 11.6, which uses basically the same ideas as the proof of Equation (11.17) given in [ACa75], will be the content of the next section. Since $\sigma_{i,\nu}(\omega) = (v_i(\omega) + 1 + \nu)/N_i$ and $\lambda = \exp(-2\pi\imath\sigma_{i,\nu}(\omega))$, the subspace of generalized eigenvectors with eigenvalue λ is different from zero and we consider the projection $\zeta_{i,\lambda}(t)$ of $\zeta_i(t)$. Since the diagram (11.16) is commutative, one also has the following commutative diagram

$$\begin{tikzcd} H_1(\overline{X}_{i,t}, \mathbb{C}) \arrow[r, "j_*"] \arrow[d] & H_1(\overline{X}_t, \mathbb{C}) \arrow[d] \\ \ker(h'_* - \lambda\,\mathrm{id})^2 \arrow[r] & \ker(h_* - \lambda\,\mathrm{id})^2, \end{tikzcd} \tag{11.19}$$

the vertical arrows being the natural projections. Hence, as we wanted to show, one has that $\gamma_\lambda(t) = j_*\zeta_{i,\lambda}(t)$, since $\gamma(t) = j_*\zeta_i(t)$.

Proposition 11.7. *Under the assumptions of Proposition 11.5, let $\sigma_{i,\nu}(\omega)$ be non-resonant associated with a rupture divisor E_i such that $\chi(E_i^\circ) = -1$. Then, $\gamma_\lambda(t)$ is an eigenvector of the monodromy of eigenvalue $\lambda = \exp(-2\pi\imath\sigma_{i,\nu}(\omega))$ belonging to the subspace $j_*H_1(\overline{X}_{i,t}, \mathbb{C})$. Furthermore, it is dual to $A^{\omega}_{\sigma_{i,\nu}-1,0}(t)$ with respect to any basis of the monodromy restricted to the subspace $j_*H_1(\overline{X}_{i,t}, \mathbb{C})$.*

Proof. If we assume that $\chi(E_i^\circ) = -1$, then by Proposition 11.6 there is only one eigenvalue equal to λ in $\Delta_i(t)$. Indeed, notice that since $\sigma_{i,\nu}(\omega)$ is non-resonant, the eigenvalue λ cannot be a root of the factor $t^{c_i} - 1$ of $\Delta_i(t)$. First, this implies that $\gamma_\lambda(t)$ is an eigenvector of the monodromy,

$$h_*\gamma_\lambda(t) = h_*j_*\zeta_{i,\lambda}(t) = j_*h'_*\zeta_{i,\lambda}(t) = \lambda j_*\zeta_{i,\lambda}(t) = \lambda\gamma_\lambda(t). \tag{11.20}$$

Secondly, this implies that $A^{\omega}_{\sigma_{i,\nu}-1,0}(t)$ is dual to $\gamma_\lambda(t)$ with respect to any basis of the vector space $j_*H_1(\overline{X}_{i,t}, \mathbb{C})$ since there is only one eigenvalue equal to $\exp(-2\pi\imath\sigma_{i,\nu}(\omega))$. That is, $\langle A^{\omega}_{\sigma_{i,\nu}-1,0}(t), \gamma_j(t)\rangle = 0$ for any $\gamma_j(t) \neq \gamma_\lambda(t)$ on a basis of the monodromy restricted to $j_*H_1(\overline{X}_{i,t}, \mathbb{C})$. ☐

In Section 11.5, for irreducible plane curves, we will merge the cycles and the locally constant geometric sections with respect to each exceptional divisor E_i to construct a basis of locally constant geometric sections of the bundle H^1 dual to a basis of H_1. This will be possible because the monodromy of an irreducible plane curve is semi-simple [Trá72, Thm. 3.3.1].

11.4 *Partial characteristic polynomial of the monodromy*

In the celebrated work [ACa75] of A'Campo, the characteristic polynomial of the monodromy $h_* : H_n(X_t, \mathbb{C}) \longrightarrow H_n(X_t, \mathbb{C})$ is computed in terms of a resolution of the singularity. A'Campo constructs a homotopic model F_t of the Milnor fiber $\overline{X}_t$ so that there is a continuous retraction $c_t : F_t \longrightarrow X_0$ from the general to the exceptional fiber, which is compatible with the geometric monodromy. Then, he uses Leray's spectral sequence associated with this map to compute the Lefschetz number of the monodromy which determines the zeta function of the monodromy.

We will show next how the same argument works to prove Proposition 11.6. Here, we will use the map $\phi_t : \overline{X}_{i,t} \longrightarrow E_i^\circ$ of the unramified covering. Over the sets E_i°, the homotopic model F_t and the fiber $\overline{X}_t$ are homeomorphic. To simplify the notation, in the sequel, we will denote also by h the restriction of the monodromy h to the sets $\overline{X}_{i,t} \subset \overline{X}_t$. First, recall the zeta function of the monodromy

$$Z_h(t) := \prod_{q\geq 0} \det\left(\mathrm{id} - th^*; H^q(\overline{X}_{i,t}, \mathbb{C})\right)^{(-1)^{q+1}}. \tag{11.21}$$

If we denote the Lefschetz numbers of the monodromy in the following way

$$\Lambda(h^k) = \sum_{q\geq 0}(-1)^q \,\mathrm{tr}\left((h^*)^k; H^q(\overline{X}_{i,t}, \mathbb{C})\right), \tag{11.22}$$

the zeta function $Z_h(t)$ depends on the integers $\Lambda(h^k)$ through the following well-known inversion formula. Let $s_1, s_2, \ldots$ be the integers defined by $\Lambda(h^k) = \sum_{i|k} s_i$ for $k \geq 0$, then

$$Z_h(t) = \prod_{i \geq 0} (1 - t^i)^{-s_i/i}. \tag{11.23}$$

Now, one can use Leray's spectral sequence of the map ϕ_t to compute the Lefschetz numbers $\Lambda(h^k)$. Define the *sheaf of cycles vanishing to the divisor* E_i as $\Psi_i^q := R^q {\phi_t}_* \mathbb{C}_{\overline{X}_{i,t}}$. If one replaces ϕ_t by the retraction c_t, one gets the usual sheaf of vanishing cycles, see for instance [SGA7-II]. The second page of Leray's spectral sequence of the map ϕ_t is then equal to

$$E_2^{p,q} = H^p(E_i^\circ, \Psi_i^q), \tag{11.24}$$

and the spectral sequence converges to $E_\infty^{p+q} = H^{p+q}(\overline{X}_{i,t}, \mathbb{C})$. Since ϕ_t is compatible with the geometric monodromy, the monodromy endomorphism h^* induces a monodromy action T on the sheaf of vanishing cycles of the divisor E_i, namely

$$T(U) := \left(h|_{\phi_t^{-1}(U)}\right)^* : H^*(\phi_t^{-1}(U), \mathbb{C}) \longrightarrow H^*(\phi_t^{-1}(U), \mathbb{C}), \tag{11.25}$$

with the actions $T_2^{p,q}$ on the terms $E_2^{p,q}$ converging to $T_\infty^{p+q} = h^* : H^{p+q}(\overline{X}_{i,t}, \mathbb{C}) \longrightarrow H^{p+q}(\overline{X}_{i,t}, \mathbb{C})$. Therefore, one has that

$$\Lambda(h^k) = \Lambda(T^k, E_\infty^{\bullet\bullet}) = \cdots = \Lambda(T^k, E_3^{\bullet\bullet}) = \Lambda(T^k, E_2^{\bullet\bullet}), \tag{11.26}$$

see for instance [Spa81, Thm. 4.3.14]. It is then enough to compute $\Lambda(T^k, E_2^{\bullet\bullet})$. Again, one argues similarly to [ACa75]. Since ϕ_t is a locally trivial fibration, the sheaves of vanishing cycles of the divisor E_i, $\Psi_i^\bullet$, are complex local systems. Notice that when one considers all the exceptional fiber $\overline{X}_0$, the usual sheaves of vanishing cycles are not local systems but constructible sheaves. Hence,

$$\Lambda(T^k, E_2^{\bullet\bullet}) = \sum_{p,q \geq 0} \operatorname{tr}\left(T^k; H^p(E_i^\circ, \Psi_i^q)\right) = \chi(E_i^\circ) \Lambda(T^k, (\Psi_i^\bullet)_s), \tag{11.27}$$

with $s \in E_i^\circ$. The stalk $(\Psi^\bullet)_s$ is identified with the cohomology of the Milnor fibration of the equation of E_i at $s \in E_i^\circ$, and this identification is compatible with both monodromies, see [ACa73]. Since locally at $s \in E_i^\circ$, E_i is $x_0^{N_i} = 0$ for some local coordinate x_0, then counting fixed points

$$\Lambda(T^k, (\Psi_i^\bullet)_s) = \begin{cases} 0, & \text{if} \quad N_i \nmid k, \\ N_i, & \text{if} \quad N_i \mid k. \end{cases} \tag{11.28}$$

Finally, the zeta function of the monodromy $h_* : H_n(\overline{X}_{i,t}, \mathbb{C}) \longrightarrow H_n(\overline{X}_{i,t}, \mathbb{C})$ equals to

$$Z_h(t) = \left(1 - t^{N_i}\right)^{-\chi(E_i^\circ)}. \tag{11.29}$$

It remains to compute the characteristic polynomial $\Delta_1(t)$ of the monodromy action from the zeta function.

Proof of Proposition 11.6. If we now restrict to the case $n = 1$, then the only homology groups are $H_i(\overline{X}_{i,t}, \mathbb{C})$ for $i = 0, 1$. Hence, in terms of the zeta functions the characteristic polynomial $\Delta_1(t)$ reads as

$$\Delta_1(t) = t^{b_1} \left[\frac{t^{b_0} - 1}{t} Z_h(1/t) \right] \tag{11.30}$$

where b_0 and b_1 are, respectively, the dimensions of $H_0(\overline{X}_{i,t}, \mathbb{C})$ and $H_1(\overline{X}_{i,t}, \mathbb{C})$.

Let us now compute the dimension of these homology groups. First, one has that $\chi(\overline{X}_{i,t}) = N_i\chi(E_i^\circ)$ since $\overline{X}_{i,t}$ is a N_i-fold unramified covering of E_i°. Second, the number of connected components c_i of a covering equals the index of the fundamental group of the base in the covering group.

Fix a point $x \in E_i^\circ$. The fundamental group $\pi_1(E_i^\circ, x)$ has rank $r_i - 1$, where r_i is the number of missing points from E_i°, and it is generated by loops $\gamma_1, \ldots, \gamma_{r_i}$ around the missing points with the relation $\gamma_{r_i}\gamma_{r_{i-1}} \cdots \gamma_1 = 1$. On the other hand, the covering group is cyclic and is generated by the monodromy action h. Hence, the action of a loop γ_j around the intersection of E_i with E_j in the covering group is $h^{N_i/\gcd(N_i,N_j)}$. Therefore, the index of $\pi_1(E_i^\circ, x)$ in the covering group is

$$c_i = \gcd(N_j \mid E_j \cap E_i \neq \emptyset, E_j \in \operatorname{Supp}(F_\pi)) = b_0, \tag{11.31}$$

and $b_1 = c_i - N_i\chi(E_i^\circ)$. Finally,

$$\Delta_i(t) = (t^{N_i} - 1)^{-\chi(E_i^\circ)}(t^{c_i} - 1). \tag{11.32}$$

□

11.5 *Generic b-exponents*

Let $f : (\mathbb{C}^2, \mathbf{0}) \longrightarrow (\mathbb{C}, 0)$ be a germ of a holomorphic function defining an irreducible plane curve with semigroup $\Gamma = \langle \overline{\beta}_0, \ldots, \overline{\beta}_g \rangle$. Given E_i a rupture divisor of the minimal embedded resolution of f, take $\sigma_{i,\nu}(\omega)$ a non-resonant candidate b-exponent associated with E_i, see Definition 11.4.

Lemma 11.8. *A candidate b-exponent $\sigma_{i,\nu}(\omega)$ is non-resonant, if and only if, $\overline{\beta}_i\sigma_{i,\nu}(\omega) \notin \mathbb{Z}$ and $e_{i-1}\sigma_{i,\nu}(\omega) \notin \mathbb{Z}$.*

Proof. The candidate $\sigma_{i,\nu}(\omega)$ is non-resonant if $\varepsilon_{j,\nu}(\omega) \notin \mathbb{Z}$, for all $D_j \cap E_i \neq \emptyset, D_j \in \operatorname{Supp}(F_\pi)$. By the definition of $\varepsilon_{j,\nu}(\omega)$, this is the same as $N_j\sigma_{i,\nu}(\omega) \notin \mathbb{Z}$. Since for plane branches there are only three D_1, D_2, D_3 divisors crossing E_i in the support of F_π, by Proposition 11.2, and since the $\delta_{k,\nu}(\omega)$ are integers, it is enough to check the non-resonance condition for two of the crossing divisors. Therefore, assume D_1, D_2 are the divisors preceding E_i in the minimal resolution. Hence, $N_1, N_2 < N_i = n_i\overline{\beta}_i$ and $N_j\sigma_{i,\nu} \notin \mathbb{Z}$ is equivalent to $\gcd(N_i, N_j)\sigma_{i,\nu} \notin \mathbb{Z}$, $j = 1, 2$. However, after a possible reordering, $\gcd(N_i, N_1) = \overline{\beta}_i$ and $\gcd(N_i, N_2) = e_{i-1}$, see [Wal04, Prop. 8.5.3]. □

For the rest of the section, $\omega \in \Gamma(X, \Omega_X^2)$ will be a fixed top differential form such that $\omega = g dx \wedge dy$ with $g(\mathbf{0}) \neq 0$. For simplicity, we could take $\omega = dx \wedge dy$. After Lemmas 2.11 and 2.12, we can write the candidates associated with such ω along each rupture divisor E_i in terms of the semigroup Γ in the following way,

$$\sigma_{i,\nu}(\omega) = \frac{m_i + n_1 \cdots n_i + \nu}{n_i\overline{\beta}_i}, \qquad \nu \in \mathbb{Z}_+. \tag{11.33}$$

Notice now that the set of candidates from Yano's conjecture, see Equation (2.61), are exactly

$$\bigcup_{i=1}^{g} \left\{ \sigma_{i,\nu}(\omega) = \frac{m_i + n_1 \cdots n_i + \nu}{n_i\overline{\beta}_i} \;\middle|\; 0 \leq \nu < n_i\overline{\beta}_i,\ \overline{\beta}_i\sigma_{i,\nu}(\omega), e_{i-1}\sigma_{i,\nu}(\omega) \notin \mathbb{Z} \right\}. \tag{11.34}$$

To see the equality between the exponents of Equation (2.61) and the set in (11.34), it is enough to notice that $R_i' = \overline{\beta}_i$ and $r_i' = \lceil (m_i + n_1 \cdots n_i)/n_i \rceil$, see Remark 2.2. Hence, $R_i = N_i = n_i\overline{\beta}_i = n_i R_i'$ and $r_i = k_i + 1 = m_i + n_1 \cdots n_i = n_i r_i'$.

If we consider A'Campo formula in the case of plane branches, it is easy to see that there are exactly μ elements in the sets from (11.34), counted with possible multiplicities. Therefore, $\lambda = \exp\left(2\pi\imath\sigma_{i,\nu}(\omega)\right)$ with $i = 1, \dots, g$ and $0 \leq \nu < n_i\overline{\beta}_i$ is the set of all the eigenvalues of the monodromy of a plane branch.

Proposition 11.9. *Let* $\lambda = \exp(-2\pi\imath\sigma_{i,\nu}(\omega)), 0 \leq \nu < n_i\overline{\beta}_i$, *be an eigenvalue of the monodromy. For any* $f_y : (\mathbb{C}^2, \mathbf{0}) \longrightarrow (\mathbb{C}, 0), y \in I_\delta$, *$\mu$-constant deformation of f, there exists a differential form* $\eta_y \in \Gamma(X, \Omega^2_X)$ *such that* $A^{\eta_y}_{\sigma_{i,0}-1,0}(t, y)$ *is non-zero for all fibers of the deformation and* $\exp(-2\pi\imath\sigma_{i,0}(\eta_y)) = \exp\left(-2\pi\imath\sigma_{i,\nu}(\omega)\right)$.

Proof. First, recall that a μ-constant deformation is topologically trivial, see [TR76], and hence equisingular. Recall also that the semigroup Γ_i of the divisorial valuation v_i associated with the rupture divisor E_i with candidate exponent $\sigma_{i,\nu}(\omega)$ is finitely generated, see Equation (2.22). Take $k \gg 0$, such that $\nu' = \nu + kN_i$ is larger than the conductor of the semigroup Γ_i. Now, let $h_y \in \Gamma(X, \mathcal{O}_X)$ with $v_i(h_y) = \nu'$ and define $\eta_y = h_y \mathrm{d}x \wedge \mathrm{d}y$. Notice that h_y can always be chosen such that η_y satisfies Corollary 11.3. Then, because

$$\sigma_{i,0}(\eta_y) = \frac{k_i + \nu' + 1}{N_i} = \frac{k_i + \nu + kN_i + 1}{N_i} = \sigma_{i,\nu}(\omega) + k, \tag{11.35}$$

the eigenvalues of the monodromy are the same, and $\sigma_{i,0}(\eta_y)$ is non-resonant since $\sigma_{i,\nu}(\omega)$ is non-resonant by Lemma 11.8. Finally, the first piece $\overline{\eta}_{y,0}$ of η_y associated with E_i is non-zero. Therefore, by Proposition 11.5, the locally constant section $A^{\eta_y}_{\sigma_{i,0}-1,0}(t, y)$ defined by $R_{i,0}(\eta_y)$ is non-zero. □

We can now use the previous proposition together with Proposition 11.7 to construct dual bases of locally constant sections of the bundles H_1 and H^1 for all fibers of a one-parameter μ-constant deformation of a plane branch.

Theorem 11.10 (Semicontinuity). *If* $f : (\mathbb{C}^2, \mathbf{0}) \longrightarrow (\mathbb{C}, 0)$ *is a plane branch, the b-exponents of a one-parameter μ-constant deformation of f depend upper-semicontinuously on the parameter.*

Proof. For a fixed $1 \leq 1 \leq g$, let $\lambda := \exp(-2\pi\imath\sigma_{i,\nu}(\omega)), 0 \leq \nu < n_i\overline{\beta}_i$ be an eigenvalue of the monodromy with $\sigma_{i,\nu}(\omega)$ from (11.34). After Proposition 11.9, there is a differential form η_y with $\lambda = \exp\left(-2\pi\imath\sigma_{i,0}(\eta_y)\right)$ such that there exists a non-zero locally constant section $A^{\eta_y}_{\sigma_{i,0}-1,0}(t, y)$ for all values of the parameter y. Since for plane branches $\chi(E_i^\circ) = -1$, we can apply Proposition 11.7 to this section, and for $t \neq 0$, we obtain the existence of $\gamma_\lambda(t, y)$ a representative 1-cycle of an eigenvector of the monodromy of the subspace $j_* H_1(\overline{X}_{i,t}, \mathbb{C})$ with eigenvalue λ.

The set of all homology classes of all such cycles for all $\sigma_{i,\nu}(\omega), 0 \leq \nu < n_i\overline{\beta}_i, i = 1, \dots, g$, in (11.34) gives a basis of eigenvectors $\gamma_\lambda(t, y)$ of the monodromy endomorphism which are dual to the corresponding $A^{\eta_y}_{\sigma_{i,0}-1,0}(t, y)$, for all fibers of the μ-constant deformation. Indeed, since we have exactly μ cycles and all these subspaces $j_* H_1(\overline{X}_{j,t}, \mathbb{C})$ are direct summands in $H_1(\overline{X}_t, \mathbb{C})$, one has that

$$H_1(\overline{X}_t, \mathbb{C}) = j_* H_1(\overline{X}_{1,t}, \mathbb{C}) \oplus j_* H_1(\overline{X}_{2,t}, \mathbb{C}) \oplus \cdots \oplus j_* H_1(\overline{X}_{g,t}, \mathbb{C}).$$

After Proposition 11.7, any such $A^{\eta_y}_{\sigma_{i,0}-1,0}$ is dual to the eigenvectors forming a basis of $j_*H_1(\overline{X}_{i,t},\mathbb{C})$. Finally, since any other cycle not in $j_*H_1(\overline{X}_{i,t},\mathbb{C})$ must vanish to a different rupture divisor, i.e. other than E_i, we obtain the desired duality.

For this precise basis of eigenvectors of the monodromy we have constructed, we have shown the existence of dual locally constant geometric sections for all fibers of the deformation. That is, using the notations from Section 10.4, for all $\gamma_\lambda(t,y)$ in the basis, $\dim_{\mathbb{C}} H^1_{\gamma_\lambda}(y) = 1$, for all values of the parameter y. Therefore, we can apply Proposition 10.5, and all the b-exponents of any one-parameter μ-constant deformation depend upper-semicontinuously on the parameter. □

Finally, Yano's conjecture will follow from Theorem 11.10 and the following proposition. For any $\nu \in \mathbb{Z}_+$, we can show that, generically in a one-parameter μ-constant deformation of f, the piece $\overline{\omega}_\nu$ of degree ν of ω associated with a rupture divisor E_i is non-zero, and hence $R_{i,\nu}(\omega)$ is non-zero.

Proposition 11.11. *For any $\sigma_{i,\nu}(\omega), \nu \in \mathbb{Z}_+$ non-resonant, there exists $f_y : (\mathbb{C}^2, \mathbf{0}) \longrightarrow (\mathbb{C}, 0), y \in I_\delta$, a one-parameter μ-constant deformation of f such the locally constant section $A^{\omega}_{\sigma_{i,\nu}-1,0}(t,y)$ is non-zero for generic fibers of the deformation.*

Proof. Assume that $\nu > 0$. Let $f_y = f + yg_y$ be the one-parameter μ-constant deformation of f from Proposition 2.18 with $v_i(g_y) = N_i + \nu = n_i\overline{\beta}_i + \nu$. Recall that, since the deformation is μ-constant, all the fibers are equisingular. Thus, locally at a point p in E_i°, let x denote a local defining function for E_i° and z the other coordinate. Then, near p we can write

$$f_y = x^{N_i} + yx^{N_i+\nu}u_y(x,z), \tag{11.36}$$

since $v_i(g_y) = N_i + \nu$ and where $u_y(0,z)$ is not identically zero. Then, this is equal to $x^{N_i}(1 + yx^\nu u_y)$ and the curves f_y can be written, locally around the same point of E_i°, as $\bar{x}^{N_i}$ for a new coordinate $\bar{x}$.

Focusing on the differential form, we have that $\omega = x^{k_i}z^{b_i}v(x,z), b_i \geq 0$, with $v(x,z)$ a local unit. Now, expand ω in series and apply the change of coordinates $x = \bar{x}(1 - y\bar{x}^\nu \bar{u}_y(\bar{x},z))$ which comes from inverting $\bar{x} = x(1 + yx^\nu u_y(x,z))^{1/N_i}$ with respect to x, and $\bar{u}_y(0,z)$ is not identically zero. That is,

$$\omega_y = \sum_{\alpha,\beta\geq 0} a_{\alpha,\beta}\bar{x}^{k_i+\alpha}(1 - y\bar{x}^\nu\bar{u}_y)^{k_i+\alpha}z^{b_i+\beta}\mathrm{d}\left(\bar{x}(1 - y\bar{x}^\nu\bar{u}_y)\right) \wedge \mathrm{d}z, \tag{11.37}$$

where the differential form now depends on the deformation parameter y. We have to check that, generically on y, the ν–th piece $\omega_{y,\nu}$ of ω_y is non-zero. In order to study $\omega_{\nu,y}$, we look at the terms of ω_y with degree $k_i + \nu$ in $\bar{x}$. Since $\mathrm{d}x \wedge \mathrm{d}z = [1 - (\nu+1)y\bar{x}^\nu\bar{u}_y - yx^{\nu+1}\partial\bar{u}_y/\partial x]\mathrm{d}\bar{x} \wedge \mathrm{d}z$, the only relevant terms from Equation (11.37) are,

$$a_{\nu,\beta}\bar{x}^{k_i+\nu}z^\beta, \quad -a_{0,\beta}k_iy\bar{x}^{k_i+\nu}\bar{u}_yz^\beta, \quad -a_{0,\beta}(\nu+1)y\bar{x}^{k_i+\nu}\bar{u}_yz^\beta. \tag{11.38}$$

Since $a_{0,0} \neq 0$, $\omega_{\nu,y}$ is non-zero for $y \neq 0, 0 < |y| \ll 1$, as we wanted to show. Since $\sigma_{i,\nu}(\omega_y)$ is non-resonant, and since the pull-back of ω has exceptional support, we can apply Proposition 11.5. This implies that $R_{i,\nu}(\omega_y)$, and hence $A^{\omega}_{\sigma_{i,\nu}-1,0}(t,y)$, is non-zero as required.

Finally, notice that for the case $\nu = 0$, it is enough to consider the trivial deformation since $\overline{\omega}_0$ is always different from zero. □

Theorem 11.12 (Yano's conjecture)**.** *Let $f : (\mathbb{C}^2, \mathbf{0}) \longrightarrow (\mathbb{C}, 0)$ be a germ of a holomorphic function defining an irreducible plane curve with semigroup $\Gamma = \langle \overline{\beta}_0, \overline{\beta}_1, \dots, \overline{\beta}_g \rangle$. Then, for generic curves in some μ-constant deformation of f, the b-exponents are*

$$\bigcup_{i=1}^{g} \left\{ \sigma_{i,\nu} = \frac{m_i + n_1 \cdots n_i + \nu}{n_i \overline{\beta}_i} \;\middle|\; 0 \leq \nu < n_i \overline{\beta}_i,\ \overline{\beta}_i \sigma_{i,\nu}, e_{i-1} \sigma_{i,\nu} \notin \mathbb{Z} \right\}. \tag{11.39}$$

Proof. Let $\sigma_{i,\nu} = \sigma_{i,\nu}(\omega)$ be a candidate b-exponent from the set (11.39). By Lemma 11.8, $\sigma_{i,\nu}(\omega)$ is non-resonant and, as a consequence of Proposition 11.11, we have the existence, generically in a μ-constant deformation of f, of non-zero locally constant geometric section $A^{\omega}_{\sigma_{i,\nu}-1,0}$ given by the exponent $\sigma_{i,\nu}(\omega) - 1$ associated with the rupture divisor E_i. After Theorem 10.3, we have that since $0 \leq \nu < N_i$, then $\sigma_{i,\nu}(\omega)$ is a b-exponent of these generic curves, because the projection of $s_{\sigma_{i,\nu}(\omega)-1}[A^{\omega}_{\sigma_{i,\nu}-1,0}]$ in the quotient bundle $F_{\sigma_{i,\nu}(\omega)-1}$, see Section 10.3, is non-zero. Indeed, $A^{\omega}_{\sigma_{i,\nu}-1,0}$ is not in the subbundle $H^1_{\lambda,\sigma_{i,\nu}(\omega)-2}$ because $\sigma_{i,\nu}(\omega) - 1, 0 \leq \nu < n_i \overline{\beta}_i$, is strictly smaller than $\sigma_{i,0}(\omega)$.

Finally, we can use the upper-semicontinuity result from Theorem 11.10, in order to apply this argument to all the candidate b-exponents from (11.39). In this case, since $\sigma_{i,\nu}(\omega) - 1, 0 \leq \nu < n_i \overline{\beta}_i$, is smaller than $\sigma_{i,0}(\omega)$ for all $i = 1, \dots, g$, the upper-semicontinuity implies that when a single candidate b-exponent has been set generically, further deformations do move that b-exponent. This way, we obtain a μ-constant deformation of the original curve f such that all the candidates from (11.39) are the b-exponents of generic fibers of this μ-constant deformation. □

12 TOPOLOGICAL ROOTS OF THE BERNSTEIN-SATO POLYNOMIAL OF A PLANE CURVE

In this section, we present a result about topological roots of the Bernstein-Sato polynomial of a plane branch. The set of roots of the Bernstein-Sato polynomial that we are going to describe are called topological because they are constant among plane branches that are topologically equivalent. That is, these roots do not exhibit a jumping behavior as some other roots of the Bernstein-Sato polynomial, see [Kat81; Kat82] or [Var80, §11]. Furthermore, this set of topological roots of the Bernstein-Sato polynomial contains both the opposites in sign to the jumping numbers in $[0, 1)$ and the real part of the poles of Igusa's zeta function. The results in this section will appear in [Bla].

12.1 *Igusa's zeta function*

Let $f \in \mathbb{Z}[x_0, \dots, x_n]$ be a non-constant polynomial and fix a prime numbers p. *Igusa's zeta function* of f is defined by the following p-adic integral

$$Z_p(s; f) := \int_{\mathbb{Z}_p^n} |f(x)|_p^s |dx|, \tag{12.1}$$

for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$. Here $\mathbb{Z}_p$ denotes the ring of p-adic integers inside the field of p-adic numbers $\mathbb{Q}_p$. The p-adic numbers $\mathbb{Q}_p$ carry an absolute value defined by

$$|a|_p = p^{-ord_p(a)}. \tag{12.2}$$

Then, $\mathbb{Z}_p$ is the closed ball of radius one under this absolute value. The measure $|dx|$ used to define the integral in Equation (12.1) is the Haar measure on $\mathbb{Q}_p$ normalized in such a way that $\mathbb{Z}_p$ has measure 1. The function $Z_p(s; f)$ is a holomorphic function on the half-plane $\{s \in \mathbb{C} \mid \operatorname{Re}(s) > 0\}$ and it is a result of Igusa [Igu74; Igu75] that it admits a meromorphic extension to the whole complex plane. Moreover, Igusa proves that $Z_p(s; f)$ is a meromorphic function of p^{-s}.

Remark 12.1. Igusa's zeta function of f is the non-archimedean version of the complex zeta function of a complex polynomial studied in Chapter IV. One can define similar Igusa zeta functions depending on a p-adic test function. However, p-adic functions are locally constant and one can always reduce the analysis to the unit ball as in Equation (12.1). ■

The rationality of $Z_p(s; f)$ is obtained using a resolution of singularities of f. Assuming $\pi : X \longrightarrow \mathbb{A}^n_{\mathbb{Q}}$ is an embedded resolution of f and denoting by $E_i, i \in I$ the irreducible components of $\pi^{-1}(\operatorname{Var}(f))$, let a_i be the multiplicity of E_i in the divisor of $\pi^* f$ and k_i the multiplicity of E_i in the relative canonical divisor. After Denef's formula [Den87, Thm 2.4], the real parts of the poles of $Z_p(s; f)$ are contained in the set $\{-(k_i + 1)/a_i \mid i \in I\}$. Comparing this with Equation (1.12), these results lead naturally to the following conjecture.

Conjecture (Strong Monodromy). *For almost all prime numbers p, if σ is a pole of Igusa's zeta function $Z_p(s; f)$, then* $\operatorname{Re}(s)$ *is a pole of the Bernstein-Sato polynomial $b_f(s)$.*

The adjective strong in the name of the above conjecture is due to the existence of a related conjecture, the Monodromy conjecture, linking the real parts of the roots of $Z_p(s; f)$ with the eigenvalues of the monodromy of f at some point of $\operatorname{Var}(f)$. Notice that, after the results of Malgrange [Mal75; Mal83], the Strong Monodromy conjecture implies the Monodromy conjecture, hence the name.

Loeser proved in [Loe88] that the Strong Monodromy conjecture is true for plane curves, i.e. the case $n = 1$. The proof involves the study of the periods of integrals in the Milnor fiber in terms of resolutions of singularities as in Section 10. The important point in this case is that, except for the case where there is a double root of $b_{f,\mathbf{0}}(s)$, the real parts of the poles of $Z_p(s; f)$ are non-resonant in the sense of Definition 11.4. That is, using the notations from Section 10.6,

Proposition 12.1 ([Loe88, Prop. II.3.1])**.** *Let E_i be an exceptional divisor of the minimal resolution of f. Consider the numbers $\varepsilon_{j,0}(\mathrm{d}x \wedge \mathrm{d}y)$ from Equation (10.32) associated with the divisors D_j from the resolution that cross E_i. If $\chi(E_i^\circ) < 1$, then $-2 < \varepsilon_{j,0}(\mathrm{d}x \wedge \mathrm{d}y) < 0$.*

In [Loe90], Loeser proved the Strong Monodromy conjecture for singularities with non-degenerate Newton polygon assuming that the real parts of the poles of $Z_p(s; f)$ are non-resonant. The Strong Monodromy conjecture has also been proved for some special types of hyperplane arrangements, see [BSY11; Wal17; BW17].

Remark 12.2. The Strong Monodromy conjecture can be generalized to arbitrary varieties by considering the Bernstein-Sato polynomial of a variety, see Remark 1.4. In this context, it has been proved in the case of monomial ideals [JY07], and some determinantal varieties [Lőr+17]. After Mustaţă [Mus19, Thm. 1.4], the Strong Monodromy conjecture for hypersurface implies the conjecture for arbitrary varieties. ■

12.2 *Topological roots*

Let $f : (\mathbb{C}^2, \mathbf{0}) \longrightarrow (\mathbb{C}, 0)$ be a holomorphic map with a singularity at the origin defining a plane curve singularity.

Definition 12.2. A root σ of the local Bernstein-Sato polynomial $b_{f,\mathbf{0}}(s)$ of f is called a *topological root* if, for any equisingular deformation of f, σ is a root of the local Bernstein-Sato polynomial of each fiber of the deformation.

Recall that, in the case $n = 1$, equisingular deformations are the same as the topologically trivial ones. By the classic result of Lê Dũng Tráng and Ramanujam [TR76], if f has an isolated singularity, topologically trivial deformations are equivalent to μ-constant deformations.

The jumping numbers in $(0, 1]$ of f at the origin, see Section 1.3, are, by definition and Theorem 1.15, topological roots of the Bernstein-Sato polynomial $b_{f,\mathbf{0}}(-s)$ of f. Similarly, since the Strong Monodromy conjecture is true, the real parts of the poles of Igusa's zeta function are also topological roots of the Bernstein-Sato polynomial. It is also well-known that, generally, not all the roots of the Bernstein-Sato are topological roots, see for instance the examples in [Yan78] or [Kat81; Kat82].

However, the intersection of the opposites in sign to the jumping numbers in $(0, 1]$ and the real parts of the poles of Igusa's zeta function is just the log-canonical threshold of f at the origin. For the case of irreducible plane curves, the next result shows that there is a bigger set of topological roots of the Bernstein-Sato polynomial containing both the opposites in sign to the jumping numbers in $(0, 1]$ and the real parts of the poles of Igusa's zeta function $Z_p(s; f)$.

Theorem 12.3. *Let $f : (\mathbb{C}^2, \mathbf{0}) \longrightarrow (\mathbb{C}, 0)$ be an irreducible plane curve. Let $E_i, i = 1, \dots, g$ be the rupture divisors of the minimal resolution of f with resolution data (N_i, k_i) and let Γ_i be the value semigroup of the divisorial valuation of E_i. Denote $N_j^{(i)}, j = 1, 2, 3$, the multiplicities of the other divisors of the resolution crossing E_i. Then,*

$$\bigcup_{i=1}^{g} \left\{ \sigma_{i,\nu} = -\frac{k_i + 1 + \nu}{N_i} \;\middle|\; \nu \in \Gamma_i,\ 0 \leq \nu < N_i,\ N_j^{(i)} \sigma_{i,\nu} \notin \mathbb{Z} \text{ for } j = 1, 2, 3 \right\} \cup \{-1\} \tag{12.3}$$

is a set of topological roots of the Bernstein-Sato of f that contains the opposite in sign to the jumping numbers of f in $(0, 1]$ and the real parts of the poles of Igusa's zeta function of f.

Proof. We will begin by showing that all the rational numbers in (12.3) are topological roots of $b_{f,\mathbf{0}}(s)$. In order to do that, we will use the notations from Chapter V and fix a Milnor representative $f : X \longrightarrow T$. Since the statement is trivially true for -1 and it is the only root contributed by the strict transform, for the sake of simplicity, we will omit this case in the discussion below.

First, since $\nu \in \Gamma_i$, there exists $g \in \Gamma(X, \mathcal{O}_X)$ such that $v_i(g) = \nu$. Therefore, after Section 10.6, considering $\omega = g \mathrm{d}x \wedge \mathrm{d}y$, one has that with $\sigma_{i,0}(\omega) = (k_i + 1 + \nu)/N_i = -\sigma_{i,\nu}$. By the third condition in (12.3), all the rational numbers $\sigma_{i,0}(\omega) = -\sigma_{i,\nu}$ in (12.3) are non-resonant in the sense of Definition 11.4. Since E_i are rupture divisors, after the discussion in Section 11.2,

$$\lim_{t \to 0} t^{1 - \sigma_{i,0}(\omega)} \int_{\gamma(t)} \frac{\omega}{\mathrm{d}f} = a_{\sigma_{i,0} - 1, 0}(\omega) \neq 0, \tag{12.4}$$

for some cycle $\gamma(t)$. Then $A^{\omega}_{\sigma_{i,0}-1,0}(t) \neq 0$ and, since $0 \leq \nu < N_i$, by Theorem 10.3, the numbers $\sigma_{i,\nu}$ are roots of the local Bernstein-Sato $b_{f,\mathbf{0}}(s)$. Notice that $\sigma_{i,\nu}$ are topological roots since for any fiber of an equisingular deformation of f, the element $g \in \Gamma(X, \mathcal{O}_X)$ with $v_i(g) = \nu$ will always exist. Furthermore, the $g \in \Gamma(X, \mathcal{O}_X)$ can always be chosen such that ω satisfies the hypothesis of Corollary 11.3.

Let us now check that the real parts of the poles of Igusa's zeta function are in (12.3). The only exceptional divisors of a resolution contributing to such poles are the rupture divisors [Loe88, Lemme IV.2.3] of the minimal resolution. That is, they are $\sigma_{i,0} = -(k_i+1)/N_i, i = 1, \ldots, g$. It remains to check that these numbers are indeed non-resonant in the sense of Definition 11.4. In the irreducible case, this follows from Proposition 9.1 and the definitions in Section 2.4.

Since the only contributing exceptional divisors to the jumping numbers of a plane curve are the rupture divisors of the minimal resolution [ST07, Thm. 3.1], it remains to check that the jumping numbers in $(0,1]$ are non-resonant. In order to do that we will use the formula given in Theorem 2.23. Furthermore, in Corollary 2.21 we have an explicit expression for $N_1^{(i)}, N_2^{(i)}$ in terms of the semigroup. Therefore, if $\lambda_{i,\nu}$ is an element of the i–th set in Theorem 2.23,

$$N_1^{(i)}\lambda_{i,\nu} = N_1^{(i)}\frac{k_i+1+\nu}{N_i} = a_i\overline{\beta}_i\frac{\overline{m}_i\iota + n_ij + n_i\overline{m}_ik}{n_i\overline{\beta}_i} = \frac{a_i}{n_i}(\overline{m}_i\iota + n_ij + n_i\overline{m}_ik). \tag{12.5}$$

Then, $a_i(\overline{m}_i\iota + n_ij + n_i\overline{m}_ik) \equiv a_i\overline{m}_i\iota \pmod{n_i}$. Since $\gcd(\overline{m}_i, n_i) = \gcd(a_i, n_i) = 1$, see Sections 2.4 and 2.7, $N_1^{(i)}\lambda_{i,\nu} \in \mathbb{Z}$ if and only if $\iota \equiv 0 \pmod{n_i}$. But the later is impossible under the conditions in Theorem 2.23. The other case works similarly,

$$N_2^{(i)}\lambda_{i,\nu} = (c_in_{i-1}\overline{m}_{i-1} + d_i)e_{i-1}\frac{\overline{m}_i\iota + n_ij + n_i\overline{m}_ik}{n_i\overline{\beta}_i} = \frac{c_in_{i-1}\overline{m}_{i-1} + d_i}{\overline{m}_i}(\overline{m}_i\iota + n_ij + n_i\overline{m}_ik). \tag{12.6}$$

Then, $\gcd(c_in_{i-1}\overline{m}_{i-1} + d_i, \overline{m}_i) = 1$, since $q_i = n_in_{i-1}\overline{m}_{i-1} + q_i$ and $q_ic_i - n_id_i = 1$, see again Sections 2.4 and 2.7. Thus, $N_1^{(i)}\lambda_{i,\nu} \in \mathbb{Z}$ if and only if $n_ij \equiv 0 \pmod{\overline{m}_i}$, which is impossible under the conditions in Theorem 2.23. □

Remark 12.3. Notice that the hypothesis of irreducibility is only used to apply Proposition 9.1 and Theorem 2.23. For the poles of $Z_p(s; f)$, the general case can be worked out by using Proposition 12.1 and distinguishing the case of a double root of $b_{f,\mathbf{0}}(s)$. However, how to prove that the jumping numbers of a reduced plane curve are non-resonant is not a priori clear. At the time of finishing this thesis, the generalization of Theorem 12.3 to reduced curves is work in progress. ■

One can find examples where the elements in (12.3) are all the topological roots of $b_{f,\mathbf{0}}(s)$. See, for instance, Example 12.1 below. How to prove that the set from (12.3) is the set of *all* topological roots of a plane curve is also not a priori clear.

Example 12.1. Consider the irreducible plane curve $f = (y^2 - x^3)^2 - x^5y$ with semigroup $\Gamma = \langle 4, 6, 13\rangle$. The minimal log-resolution divisor is $F_\pi = 4E_0 + 6E_1 + 12E_2 + 13E_3 + 26E_4$ and $K_\pi = E_0 + 2E_1 + 4E_2 + 5E_3 + 10E_4$. The rupture divisors are E_2 and

E_4 and the associated valuations have semigroups $\Gamma_2 = \langle 2, 3 \rangle$ and $\Gamma_4 = \Gamma$. Then, the topological roots of $b_{f,\mathbf{0}}(s)$ from Theorem 12.3 are

$$\begin{aligned} &E_2 : -\frac{5}{12}, -\frac{7}{12}, -\frac{11}{12}, -\frac{13}{12}. \\ &E_4 : -\frac{11}{26}, -\frac{15}{26}, -\frac{17}{26}, -\frac{19}{26}, -\frac{21}{26}, -\frac{23}{26}, -\frac{25}{26}, -\frac{27}{26}, -\frac{29}{26}, -\frac{31}{26}, -\frac{33}{26}, -\frac{35}{26}. \end{aligned} \tag{12.7}$$

and -1. One can check that these are, in fact, all the roots of $b_{f,\mathbf{0}}(s)$. This is due to the fact that f has no non-trivial μ-constant deformations, see [Tei86, §II.1]. In contrast, the real parts of the poles of Igusa's zeta function are $-\frac{5}{12}, -\frac{11}{26}$, and the jumping numbers in $(0, 1]$ are $\frac{5}{12}, \frac{11}{12}, \frac{15}{26}, \frac{17}{26}, \frac{19}{26}, \frac{21}{26}, \frac{23}{26}, \frac{25}{26}$. ■

A

APPENDIX

This appendix contains the implementation in Magma [BCP97] of the Algorithms 3.13, 2.6, 2.25, and 4.3, as well as the applications from Section 4.3. We also include the implementation of other classical algorithms in the theory of plane curves, some of them being prerequisites for the algorithms present in this thesis. For instance: the Newton-Puiseux algorithm, resolution of plane curves via Enriques' Theorem, algorithms for numerical semigroups of curves, the computation of the monomial curve and its deformations, etc. The code has been tested in Magma v2.24–4.

List of functions:

- `ASufficiencyBound`: Computes a lower-bound for the A-sufficiency degree of a plane curve, see [Cas00, §7.7].
- `CharExponents`: Returns the characteristic exponents associated with a Puiseux series, a plane branch equation or a semigroup.
- `Conductor`: Returns the conductor of a plane curve semigroup from the minimal set of generators, the characteristic exponents, a plane branch equation or a Puiseux series.
- `ContactMatrix`: The contact matrix between the branches of a plane curve.
- `DeformationCurve`: The miniversal semigroup constant deformation of the monomial curve associated to a plane curve semigroup.
- `ESufficiencyDegree`: Computes a lower-bound for the E-sufficiency degree of a plane curve, see [Cas00, §7.5].
- `Filtration`: Returns the filtration of the local ring by the complete ideals defined by the valuation of a plane branch.
- `FiltrationRupture`: Returns the filtration of the local ring by the complete ideals defined by the valuation of the i–th rupture divisor of a plane branch.
- `GenericBExponents`: Returns the generating sequence for the generic b-exponents from the characteristic sequence or the semigroup using Yano's formula.
- `IsCharSequence`: Whether the input is a valid characteristic sequence or not.
- `IsPlaneCurveSemiGroup`: Whether the input is a plane curve semigroup or not.
- `JumpingNumbers`: The jumping numbers in $(0, 1]$ of a plane branch from its semigroup or characteristic sequence.
- `LogResolution`: Computes the proximity matrix and total transform multiplicities of the minimal log-resolution of a bivariate polynomial ideal.
- `MaxContactElements`: Computes a set of maximal contact elements of a plane curve singularity.

- `MilnorAlgebraAdapted`: Constructs an adapted basis of the Milnor algebra of an isolated singularity f from a given basis of the Tjurina algebra by successive multiplication by f.
- `MilnorAlgebra`: Computes a monomial basis for the Milnor algebra of an isolated singularity.
- `MilnorNumber`: The Milnor number of an isolated singularity. Computes the Milnor number from the semigroup or characteristic sequence of a plane branch.
- `MonomialCurve`: Equations for the monomial curve associated to a plane curve semigroup.
- `MultiplierIdeals`: Returns generators for the multiplier ideals of a singularity in a smooth complex surface.
- `NewtonPolygon`: The Newton polygon of a plane curve.
- `PolarInvariants`: The polar invariants of a plane curve, see [Cas00, §6].
- `ProximityMatrix`: The proximity matrix and the multiplicities of the strict transform of a plane curve singularity.
- `PuiseuxExpansion`: The Puiseux expansions of a plane curve singularity.
- `SemiGroup`: Computes the minimal set of generators for the semigroup of a plane branch from the characteristic sequence, a Puiseux series, an equation or the proximity matrix.
- `SemiGroupMembership`: Whether a number belongs to a given semigroup or not.
- `Spectrum`: Computes the singularity spectrum of a plane branch from the semigroup, the characteristic sequence or an equation.
- `TjurinaAlgebra`: Computes a monomial basis for the Tjurina algebra of an isolated singularity.
- `TjurinaAlgebraAdapted`: Computes a basis for the Tjurina algebra of a plane branch using maximal contact elements.
- `TjurinaFiltration`: Returns a filtration of the Tjurina ideal, see [Per97].
- `TjurinaGaps`: Computes the gaps of the Tjurina algebra, see [Per97].
- `TjurinaNumber`: The Tjurina number of an isolated singularity.
- `TopologicalRootsBS`: Returns the topological roots of the Bernstein-Sato polynomial of a plane branch.
- `WeierstrassEquation`: Computes the Weierstrass equation of a plane branch from a Puiseux series.

LogResolution.m

```
import "ProximityMatrix.m": ProximityMatrixImpl, CoefficientsVectorBranch;

ExpandWeightedCluster := procedure(~P, ~EE, ~CC, ~S, b)
 // Expand the proximity matrix.
 P := InsertBlock(ScalarMatrix(Ncols(P) + 1, 1), P, 1, 1); N := Ncols(P);
 // Expand branches multiplicities.
 EE := [InsertBlock(ZeroMatrix(IntegerRing(), 1, N), EEi, 1, 1)
   : EEi in EE];
 // If a free points, it has mult. 1 in the branch & expand the Puiseux series.
 if b ne -1 then
  EE[b][1, N] := 1;
  // Number of points of branch b appearing in BP(I)
  m := #[e : e in Eltseq(EE[b]) | e ne 0];
  // If we do not have enough terms of Puiseux already computed...
  if #CC[b] lt m + 1 then
   SS := PuiseuxExpansionExpandReduced(S[b][1], S[b][3]:
    Terms := m + 1 - #CC[b], Polynomial := true)[1];
   S[b][1] := SS[1]; S[b][3] := SS[2];
   CC[b] := CoefficientsVectorBranch(S[b][1], m + 1);
  end if;
 end if;
end procedure;

ComputeLogResolutionData := procedure(~P, ~EE, ~CC, ~S, N, ~E, ~C, ~V, ~v)
 // Compute the multiplicities of each generator in G.
 E := [ZeroMatrix(IntegerRing(), 1, Ncols(P)) : i in [1..N]];
 // Hold information about the branches in each generator.
 for i in [1..#S] do for m in S[i][2] do
  E[m[2]] := E[m[2]] + m[1] * EE[i];
 end for; end for;
 // Merge the coefficients of each branch.
 C := [#CC gt 0 select Parent(CC[1][2]) else RationalField()
  | <0, 1> : i in [1..Ncols(P)]];
 for i in [1..#EE] do
  I := [j : j in [1..Ncols(P)] | EE[i][1][j] ne 0];
  for j in [1..#I] do C[I[j]] := CC[i][j]; end for;
 end for;
 // Values for each generator in G & each (initial) base point.
 Pt_inv := Transpose(P^-1); V := [e * Pt_inv : e in E];
 v := ZeroMatrix(IntegerRing(), 1, Ncols(P));
 for i in [1..Ncols(P)] do v[1][i] := Min([vj[1][i] : vj in V]); end for;
end procedure;

intrinsic LogResolution(I::RngMPolLoc : Coefficients := false) -> []
{ Computes the proximity matrix matrix and total transform multiplicities of
 the minimal log-resolution of a bivariate polynomial ideal I }
 // Generators in G & fixed part F.
 G := Basis(I); F := Gcd(G); G := [ExactQuotient(g, F) : g in G];

 ////////////// Compute all information ////////////////
 S := PuiseuxExpansion(G: Polynomial := true);
 P, EE, CC := ProximityMatrixImpl([*<s[1], 1> : s in S*]: ExtraPoint := true);

 E := []; // Multiplicities of each generator.
 C := []; // Coefficients of BP(I).
 V := []; // Vector a values for each generator.
 v := []; // Virtual values of BP(I).
 ComputeLogResolutionData(~P, ~EE, ~CC, ~S, #G, ~E, ~C, ~V, ~v);
```

```

  /////////////// Add new free points /////////////////////
  lastFree := [i : i in [1..Ncols(P)] | (&+P[1..Ncols(P)])[i] eq 1];
  points2test := #lastFree; idx := 1;
  // For each last free point on a branch...
  while points2test gt 0 do
    // Values for each gen. at p.
    p := lastFree[idx]; Vp := [vi[1][p] : vi in V];
    // Generators achieving the minimum.
    GG := [i : i in [1..#Vp] | Vp[i] eq Min(Vp)];
    // If the multiplicities of all the generators achieving the minimum
    // at p is > 0 add new point.
    if &and[E[g][1][p] ne 0 : g in GG] then
      // The (unique) branch of the generator 'g' where 'p' belongs.
      assert(#[i : i in [1..#EE] | EE[i][1, p] ne 0] eq 1);
      b := [i : i in [1..#EE] | EE[i][1, p] ne 0][1];
      ExpandWeightedCluster(~P, ~EE, ~CC, ~S, b); P[Ncols(P)][p] := -1;
      ComputeLogResolutionData(~P, ~EE, ~CC, ~S, #G, ~E, ~C, ~V, ~v);
      // We may need to add more free points after the points we added.
      lastFree cat:= [Ncols(P)]; points2test := points2test + 1;
    end if;
  points2test := points2test - 1; idx := idx + 1;
  end while;

  /////////////// Add new satellite points /////////////////////
  points2test := Ncols(P) - 1; p := 2; // Do not start at the origin.
  while points2test gt 0 do
    // Values for the generators at point p.
    Vp := [vi[1][p] - v[1][p] : vi in V];
    // Points p is proximate to && Points proximate to p.
    p_prox := [i : i in [1..Ncols(P)] | P[p][i] eq -1];
    prox_p := [i : i in [1..Ncols(P)] | P[i][p] eq -1];
    Q := [q : q in p_prox | &+Eltseq(Submatrix(P, prox_p, [q])) eq 0];
    for q in Q do
      // Values for the generators at point q.
      Vq := [vi[1][q] - v[1][q] : vi in V];
      if &*[Vp[i] + Vq[i] : i in [1..#Vp]] ne 0 then
        ExpandWeightedCluster(~P, ~EE, ~CC, ~S, -1);
        P[Ncols(P)][p] := -1; P[Ncols(P)][q] := -1;
        ComputeLogResolutionData(~P, ~EE, ~CC, ~S, #G, ~E, ~C, ~V, ~v);
        // We may need to add more satellite points after the points we added.
        points2test := points2test + 1;
      end if;
    end for;
  points2test := points2test - 1; p := p + 1;
  end while;

  /////////////// Remove non base points ////////////////
  // Multiplicities for the cluster of base points.
  e := v * Transpose(P); I := [i : i in [1..Ncols(P)] | e[1][i] ne 0];
  // Remove points not in the cluster of base points.
  P := Submatrix(P, I, I); v := Submatrix(v, [1], I); C := C[I];

  // Select 1 as affine part iff F is a unit.
  F := Evaluate(F, <0, 0>) ne 0 select Parent(F)!1 else F;
  if Coefficients then return P, v, F, C;
  else return P, v, F; end if;
end intrinsic;
```

Filtration.m

```
import "ProximityMatrix.m": ProximityMatrixImpl,
                     ProximityMatrixBranch,
                     MultiplicityVectorBranch,
                     CoefficientsVectorBranch;
import "IntegralClosure.m": IntegralClosureIrreducible,
                     Unloading, ProductIdeals,
                     ClusterFactorization, MaxContactElements;
import "LogResolution.m": ExpandWeightedCluster;
import "SemiGroup.m": TailExponentSeries;

// Helper funcition
ConvertToIdeal := func<I, Q | [&*[g[1]^g[2] : g in f] : f in I]>;

FiltrationRuptureImpl := function(P, e, c, i, niBi)
  // Compute a set of max. cont. elem. of the curve.
  Q<x, y> := LocalPolynomialRing(Parent(c[1][2]), 2, "lglex"); ZZ := Integers();
  Pt := Transpose(P); Pt_inv := Pt^-1;
  Cv := MaxContactElements(P, e*Pt_inv, c, Q); N := Ncols(P);
  // Compute the maximal ideal values.
  max := ZeroMatrix(IntegerRing(), 1, N); max[1][1] := 1; max := max*Pt_inv;

  // Compute the position of the i-th rupture divisor.
  VS := RSpace(ZZ, N); isSat := &+[VS | Pt[i] : i in [1..N]]; R := [];
  for p in [2..N] do // Construct the set of rupture points.
    // Points proximate to 'p' that are free.
    prox_p_free := [i : i in [p + 1..N] | Pt[p][i] eq -1 and isSat[i] ne -1];
    if (isSat[p] eq -1 and (#prox_p_free ge 1 or p eq N)) or
       (isSat[p] ne -1 and #prox_p_free ge 2) then R cat:= [p]; end if;
  end for; Ri := R[i];

  // Construct the i-th cluster.
  vi := ZeroMatrix(IntegerRing(), 1, N); H := [];
  while vi[1][Ri] lt niBi do
    // Unload K_i to get a strictly consistent cluster.
    vi[1][Ri] +:= 1; vi := Unloading(P, vi);

    // Compute generators for the complete ideal H_i.
    Hi := [IntegralClosureIrreducible(P, P*Transpose(v_j), v_j, Cv, max, Q) :
      v_j in ClusterFactorization(P, vi)];
    Hi := [g[1] : g in ProductIdeals(Hi) |
      &or[g[2][1][i] lt (vi + max)[1][i] : i in [1..N]]];
    H cat:= [<Hi, vi[1][Ri]>];
  end while; return H;
end function;

intrinsic FiltrationRupture(f::RngMPolLocElt, i::RngIntElt : N := -1, Ideal := true) -> []
{ Returns the filtration of the local ring by the complete ideals defined
  by the valuation of the \( i \)--th rupture divisor of a plane branch }
require i gt 0: "Second␣argument␣must␣be␣a␣positive␣integer";

  Q := Parent(f); S := PuiseuxExpansion(f: Polynomial := true); ZZ := Integers();
  if #S gt 1 or S[1][2] gt 1 then error "The␣curve␣must␣be␣irreducible"; end if;
  s := S[1][1]; P, e, c := ProximityMatrixImpl([<s, 1>]); G := SemiGroup(P);
  if i gt #G - 1 then error "Rupture␣divisor␣index␣too␣large"; end if;
  Ei := [i gt 1 select Gcd(Self(i - 1), G[i]) else G[1] : i in [1..#G]];
  n := G[1]; Ni := [0] cat [ZZ!(Ei[i] div Ei[i + 1]) : i in [1..#G - 1]];
  if N lt 0 then N := Ni[i + 1] * G[i + 1]; end if;

```

```
  F := FiltrationRuptureImpl(P, e[1], c[1], i, N);
  if Ideal eq true then return [<ConvertToIdeal(I[1], Q), I[2]> : I in F];
  else return F; end if;
end intrinsic;

FiltrationImpl := function(s, f, e, M)
  // Compute an upper bound for the necessary points.
  KK := (e*Transpose(e))[1][1]; N := Max(Ncols(e) + M - KK, Ncols(e));
  // Get the proximity matrix with all the necessary points.
  s := PuiseuxExpansionExpandReduced(s, f: Terms := M - KK - 1)[1];
  P := ProximityMatrixBranch(s, N); Pt := Transpose(P); Pt_inv := Pt^-1;
  e := MultiplicityVectorBranch(s, N); c := CoefficientsVectorBranch(s, N);

  // Compute a set of max. cont. elem. of the curve.
  Q<x, y> := LocalPolynomialRing(Parent(c[1][2]), 2, "lglex");
  Cv := MaxContactElements(P, e*Pt_inv, c, Q);
  // Compute the maximal ideal values.
  max := ZeroMatrix(IntegerRing(), 1, N); max[1][1] := 1; max := max*Pt_inv;

  // Construct the i-th cluster.
  ei := ZeroMatrix(IntegerRing(), 1, N); m_i := 0; H := [];
  while m_i lt M do
    // Get the last points with multiplicity zero.
    I := [i : i in [1..N] | ei[1][i] eq 0][1];
    ei[1][I] := 1; vi := ei*Pt_inv;
    // Unload K_i to get a strictly consistent cluster.
    vi := Unloading(P, vi); ei := vi*Pt;

    // Compute generators for the complete ideal H_i.
    Hi := [IntegralClosureIrreducible(P, P*Transpose(v_j), v_j, Cv, max, Q) :
      v_j in ClusterFactorization(P, vi)];
    Hi := [g[1] : g in ProductIdeals(Hi) |
      &or[g[2][1][i] lt (vi + max)[1][i] : i in [1..N]]];

    // Fill the gaps in the filtration.
    KK_i := &+[e[i] * ei[1][i] : i in [1..N]]; // Intersection [K, K_i]
    H cat:= [<Hi, KK_i>]; m_i := KK_i;
  end while; return H;
end function;

intrinsic Filtration(f::RngMPolLocElt : N := -1, Ideal := true) -> []
{ Returns the filtration of the local ring by the complete ideals
  defined by the valuation of a plane branch }

  Q := Parent(f); S := PuiseuxExpansion(f: Polynomial := true);
  if #S gt 1 or S[1][2] gt 1 then error "the curve must be irreducible"; end if;
  s := S[1][1]; f := S[1][3]; _, e, _ := ProximityMatrixImpl([<s, 1>]);
  KK := e[1]*Transpose(e[1]); // Curve auto-intersection.

  F := FiltrationImpl(s, f, e[1], N lt 0 select KK[1][1] else N);
  if Ideal eq true then return [<ConvertToIdeal(I[1], Q), I[2]> : I in F];
  else return F; end if;
end intrinsic;

TjurinaFiltrationImpl := function(S, f)
  // Get the proximity matrix with all the necessary points.
  P, E, C := ProximityMatrixImpl(S); N := NumberOfColumns(P);
  Pt := Transpose(P); Pt_inv := Pt^-1; R := Parent(f);
  // The Tjurina ideal & its standard basis.
```

```
  J := JacobianIdeal(f) + ideal<R | f>; J := ideal<R | StandardBasis(J)>;

  // Compute a set of max. cont. elem. of the curve.
  A := Parent(C[1][1][2]); Q := LocalPolynomialRing(A, 2, "lglex");
  vi := E[1]*Pt_inv; Cv := MaxContactElements(P, vi, C[1], Q); ZZ := IntegerRing();
  // Add the curve itself as a max. contact element.
  Cvf := <[<Q!f, 1>], vi, E[1]>; Cv cat:= [Cvf];
  // Compute the maximal ideal values.
  max := ZeroMatrix(ZZ, 1, N); max[1][1] := 1; max := max*Pt_inv;

  // Construct the i-th cluster.
  ei := ZeroMatrix(ZZ, 1, N); JJ := []; Hi := ideal<R | 1>; Ji := Hi meet J;
  while Hi ne Ji do
    // Enlarge, if necessary, the cluster with one point on the curve.
    I := [i : i in [1..N] | ei[1][i] eq 0];
    if #I eq 0 then
      ExpandWeightedCluster(~P, ~E, ~C, ~S, -1); N := N + 1; P[N][N - 1] := -1;
      E[1][1][N] := 1; ei := E[1]; Pt := Transpose(P); Pt_inv := Pt^-1;
      // Expand (i.e. blow-up an extra point) the maximal ideal.
      max := ZeroMatrix(ZZ, 1, N); max[1][1] := 1; max := max*Pt_inv;

      newCv := []; // Expand (i.e. blow-up an extra points) the max. cont. elem.
      for i in [1..#Cv] do
        Ei := InsertBlock(ZeroMatrix(ZZ, 1, N), Cv[i][3], 1, 1);
        if i eq #Cv then Ei[1][N] := 1; end if; // The last max. cont. elem. is f.
        newCv cat:= [<Cv[i][1], Ei*Pt_inv, Ei>];
      end for; Cv := newCv;
    else ei[1][I[1]] := 1; end if;

    // Unload K_i to get a strictly consistent cluster.
    vi := ei*Pt^-1; vi := Unloading(P, vi); ei := vi*Pt;

    // Compute generators for the complete ideal H_i.
    Hi := [IntegralClosureIrreducible(P, P*Transpose(v_j), v_j, Cv, max, Q)
      : v_j in ClusterFactorization(P, vi)];
    Hi := [g[1] : g in ProductIdeals(Hi) | &or[g[2][1][i] lt
      (vi + max)[1][i] : i in [1..N]]];
    Hi := ideal<R | ConvertToIdeal(Hi, R)>; Ji := Hi meet J;
    // Ignore the begining of the filtration.
    if Ji eq J then continue;
    else JJ cat:= [<Ji, vi[1][Ncols(vi)]>];
    end if;
  end while; return JJ;
end function;

intrinsic TjurinaFiltration(f::RngMPolLocElt) -> []
{ Returns an adapted filtration of the Tjurina ideal of an irreducible
  plane curve }

  Q := Parent(f); S := PuiseuxExpansion(f: Polynomial := true);
  if #S gt 1 or S[1][2] gt 1 then error "the␣curve␣must␣be␣irreducible"; end if;
  return TjurinaFiltrationImpl(S, f);
end intrinsic;
```

IntegralClosure.m

```
import "SemiGroup.m": Euclides, TailExponentMatrix, InversionFormula;

// Given a Puiseux series s, returns its associated Weierstrass equation.
```

```
intrinsic WeierstrassEquation(s::RngSerPuisElt, Q::RngMPolLoc) ->
 RngMPolLocElt
{ Computes the Weierstrass equation associated to a Puiseux series }

 x := Q.1; y := Q.2; C, m, n := Coefficients(s); G := CyclicGroup(n);
 g := n gt 1 select G.1^(n - 1) else G.0;

 V := [&+[Q | C[i + 1] * x^((m + i - s) div n) : i in [0..#C - 1] |
   (i + m) mod n eq s] : s in [0..n - 1]];
 M := Matrix([V[Eltseq(g^i)] : i in [0..n - 1]]);
 for i in [1..n] do
   InsertBlock(~M, x * Submatrix(M, [i], [1..i - 1]), i, 1);
 end for;
 return Determinant(ScalarMatrix(n, y) - M);
end intrinsic;

// Factorices the weighted cluster (P, v) as a unique sum of
// irreducible weighted clusters.
ClusterFactorization := function(P, v)
 N := Transpose(P) * P; Ninv := N^-1; exc := v * N; n := Ncols(P);
 B := []; // For each point with strictly positive excess.
 for i in [i : i in [1..n] | exc[1][i] gt 0] do
   p := i; I := [p];
   while p ne 1 do // Traverse the cluster back to the origin
     p := [j : j in Reverse([1..n]) | P[p][j] eq -1][1]; I := [p] cat I;
   end while;
   v := ZeroMatrix(IntegerRing(), 1, n); v[1][i] := exc[1][i];
   v := v * Ninv; B cat:= [v];
 end for; return B;
end function;

// Returns the curve going sharply through (P, v).
// Prerequisite: The cluster (P, v, c) must be irreducible.
// && the last point of (P, v) must be free.
SharplyCurve := function(P, v, c, Q)
 m := Gcd(Eltseq(v)); v := v div m; G := SemiGroup(P);
 M := CharExponents(G) cat [TailExponentMatrix(P)];
 // If the curve is the y-axis.
 if #G eq 1 and #c gt 1 and &and[c[i][1] eq 0: i in [2..#c]] then
 return Q.1; end if;
 // If the curve is inverted.
 if c[2][1] eq 0 then M := InversionFormula(M, P, c); end if;
 P<t> := PuiseuxSeriesRing(Parent(c[1][2])); s := P!0; k := 1; n := M[1][2];
 for i in [2..#M] do
   mj := M[i - 1][1]; nj := M[i - 1][2]; mi := M[i][1]; h0 := (mi - mj) div nj;
   s +:= &+[P | c[k + l][2] * t^((mj + l * nj) / n) : l in [0..h0]];
   k +:= &+Euclides(mi - mj, nj)[1];
 end for; return WeierstrassEquation(s, Q)^m;
end function;

// Computes the maximal contact elements of a weighted cluster.
MaxContactElements := function(P, v, c, Q)
 P_inv := P^-1; Pt := Transpose(P); Pt_inv := Transpose(P_inv);
 n := Ncols(P); isSat := &+[Pt[i] : i in [1..n]];
 // Dead-end points are always free.
 freePoints := [p : p in [1..n] | isSat[p] eq 0]; curvPoints := [];
 for p in freePoints do
   // Points proximate to 'p'.
   prox_p := [i : i in [p + 1..n] | Pt[p][i] eq -1];
```

```
  // Points proximate to 'p' that are satellites.
  prox_p_sat := [q : q in prox_p | isSat[q] eq -1];
  // Select 'p' is if it has no proximate points in K (dead end) or
  // all its proximate points in K are satellite (rupture point).
  if #prox_p eq 0 or #prox_p eq #prox_p_sat then
   curvPoints cat:= [p]; end if;
 end for; C := []; // Now, construct equations for each max. cont. elem.
 for p in curvPoints do
  i_p := ZeroMatrix(IntegerRing(), 1, n); i_p[1][p] := 1;
  e_p := i_p*P_inv; Ip := [i : i in [1..n] | e_p[1][i] ne 0];
  v_p := e_p*Pt_inv; f_p := SharplyCurve(Submatrix(P, Ip, Ip),
   Submatrix(v_p, [1], Ip), c[Ip], Q);
  C cat:= [<[<f_p, 1>], v_p, e_p>];
 end for;
 // Let's check if we need to add x or y as a max. cont. elem. This will always
 // happen in the irreducible case. Otherwise, we might have smooth max. cont.
 // elem. playing the role of x and/or y.
 e_O := ZeroMatrix(IntegerRing(), 1, n); e_O[1][1] := 1; v_O := e_O*Pt_inv;
 if #[f : f in C | LeadingMonomial(f[1][1][1]) eq Q.1] eq 0 then
  C := [<[<Q.1, 1>], v_O, e_O>] cat C; end if;
 if #[f : f in C | LeadingMonomial(f[1][1][1]) eq Q.2] eq 0 then
  C := [<[<Q.2, 1>], v_O, e_O>] cat C; end if;
 return C;
end function;

// Unloads the weighted cluster represented by (P, v) where v are virtual values.
Unloading := function(P, v)
 N := Transpose(P) * P; n := Ncols(P); R := CoefficientRing(P);
 while #[r : r in Eltseq(v * N) | r lt 0] gt 0 do
  p := [i : i in [1..n] | (v * N)[1][i] lt 0][1];
  lp := ZeroMatrix(R, 1, n); lp[1][p] := 1;
  rp := (lp * N * Transpose(lp))[1][1];
  np := Ceiling(-(v * N * Transpose(lp))[1][1] / rp);
  v +:= np * lp;
 end while; return v;
end function;

// Returns the datum corresponding to multipling the datum of the
// max. contact elements of f and g.
ProductMaxContElem := function(f, g)
 fg := f[1] cat g[1]; S := {h[1] : h in fg};
 return <[<s, &+[h[2] : h in fg | h[1] eq s]> : s in S],
  f[2] + g[2], f[3] + g[3]>;
end function;

// Returns the datum corresponding to raising the datum representing
// the max. cont. elem. f to alpha \in \mathbb{N}.
PowerMaxContElem := function(f, alpha)
 return <[<f_i[1], alpha * f_i[2]> : f_i in f[1]], alpha * f[2], alpha * f[3]>;
end function;

// Multiplies together all the 'ideals' of max. cont. elem. in the sequence S.
ProductIdeals := function(S)
if #S eq 0 then return []; end if;
return Reverse([i gt 1 select SetToSequence({ProductMaxContElem(f, g) :
  f in S[i], g in Self(i - 1)}) else S[1] : i in [1..#S]])[1];
end function;

// Raises the 'ideal' of max. cont. elem. to the alpha-th power.
```

```
PowerIdeal := function(I, alpha)
  return Reverse([i gt 1 select ProductIdeals([I] cat [Self(i - 1)])
    else I : i in [1..alpha]])[1];
end function;

forward IntegralClosureIrreducible;

IntegralClosureIrreducible := function(P, e, v_i, Cv, max, Q)
  // If the cluster is a power of the maximal ideal, select all the
  // possible generators for a maximal ideal from the max. cont. elem.
  alpha := Gcd(Eltseq(v_i)); v_i := v_i div alpha;
  if v_i eq max then
    X := [f : f in Cv | LeadingMonomial(f[1][1][1]) eq Q.1];
    Y := [f : f in Cv | LeadingMonomial(f[1][1][1]) eq Q.2];
    _, i1 := Max([(f[3] * e)[1][1] : f in X]);
    _, i2 := Max([(f[3] * e)[1][1] : f in Y]);
    return PowerIdeal([X[i1], Y[i2]], alpha);
  end if;

  // Find the max. cont. elem. going through the current cluster.
  Pt := Transpose(P); n := Ncols(P); isFree := &+[Pt[i] : i in [1..n]];
  e_i := v_i*Pt; p := [j : j in Reverse([1..n]) | isFree[j] eq 0 and
    e_i[1][j] ne 0][1]; // Last free point.
  Fs := [f : f in Cv | f[3][1][p] gt 0 and f[3][1][1] le v_i[1][1]][1];
  beta := v_i[1][1] div Fs[3][1][1];

  // Increase the value at the origin & unload.
  v := v_i; v[1][1] +:= 1; v := Unloading(P, v);
  // Apply Zariski theorem on complete ideals and recurse.
  Is := [IntegralClosureIrreducible(P, e, v_j, Cv, max, Q) :
    v_j in ClusterFactorization(P, v)];

  // Compute (f^\beta) + \prod_{i=1}^{#II} I_i and clean gens.
  return PowerIdeal([PowerMaxContElem(Fs, beta)] cat [g : g in ProductIdeals(Is)
    | &or[g[2][1][i] lt (v_i + max)[1][i] : i in [1..n]]], alpha);
end function;

intrinsic IntegralClosure(I::RngMPolLoc : Ideal := true) -> []
{ Computes the integral closure of a bivariate polynomial ideal }
  // Compute the log-resolution of I
  P, v, f, c := LogResolution(I : Coefficients := true);
  Pt := Transpose(P); n := Ncols(P); R := Parent(f);
  // If the ideal is principal its integral closure is itself.
  if n eq 0 then
    if Ideal then return ideal<R | f>; else return [[<R!f, 1>]]; end if;
  end if;

  // Compute the maximal contact elements of the log-resolution morphism.
  Q<x, y> := LocalPolynomialRing(Parent(c[1][2]), 2, "lglex");
  Cv := MaxContactElements(P, v, c, Q); e := P*Transpose(v);
  // Sort max. contact elements by increasing intersection mult. with F_\pi.
  Sort(~Cv, func<x, y | (y[3]*e - x[3]*e)[1][1]>);
  // Compute the maximal ideal values.
  max := ZeroMatrix(IntegerRing(), 1, n); max[1][1] := 1; max := max*Pt^-1;

  // Compute systems of generators of irreducible cluster.
  Is := [IntegralClosureIrreducible(P, P*Transpose(v_i), v_i, Cv, max, Q) :
    v_i in ClusterFactorization(P, v)];
  // Multiply all the ideals together, add the affine part and clean gens.
```

```
  Ibar := [g[1] cat (f eq 1 select [] else [<Q!f, 1>]) : g in ProductIdeals(Is)
    | &or[g[2][1][i] lt (v + max)[1][i] : i in [1..n]]];

  // Return the internal representation of the monomials in the max. contact
  // elements or return a sequence of ideals instead.
  ConvertToIdeal := func<I, Q | [Q!(&*[g[1]^g[2] : g in f]) : f in I]>;
  if Ideal then return ConvertToIdeal(Ibar, R); else return Ibar; end if;
end intrinsic;

// Computes a set of maximal contact elements of an element f.
intrinsic MaxContactElements(f::RngMPolLocElt) -> []
{ Computes a set of maximal contact elements of a plane curve f }

  P, e, c := ProximityMatrix(f : Coefficients := true); R := Parent(f);
  Q<x, y> := LocalPolynomialRing(Parent(c[1][2]), 2); Pt := Transpose(P);
  return [R!fi[1][1][1] : fi in MaxContactElements(P, e*Pt^-1, c, Q)];
end intrinsic;

// Computes generators for \pi^* O_{X'}(-D)_O
intrinsic GeneratorsOXD(P::Mtrx, v::Mtrx, c::SeqEnum[Tup], R::RngMPolLoc) -> []
{ Computes monomial generators for the stalk at zero of the ideal
  associated to the divisor D given by the values in v}
require Ncols(P) eq Ncols(v) and Ncols(P) eq #c: "Dimensions␣do␣not␣agree";

  // Compute the maximal contact elements of the morphism.
  Q<x, y> := LocalPolynomialRing(Parent(c[1][2]), 2, "lglex");
  Cv := MaxContactElements(P, v, c, Q); e := P*Transpose(v);
  // Sort max. contact elements by increasing intersection multiplicity with D.
  Sort(~Cv, func<x, y | (y[3]*e - x[3]*e)[1][1]>); n := Ncols(P);
  // Compute the maximal ideal values.
  max := ZeroMatrix(IntegerRing(), 1, n); max[1][1] := 1;
  max := max*Transpose(P)^-1;

  // Compute systems of generators of irreducible cluster.
  Is := [IntegralClosureIrreducible(P, P*Transpose(v_i), v_i, Cv, max, Q) :
    v_i in ClusterFactorization(P, v)];
  // Multiply all the ideals together, add the affine part and clean gens.
  Ibar := [g[1] : g in ProductIdeals(Is) | &or[g[2][1][i] lt
    (v + max)[1][i] : i in [1..n]]];

  ConvertToIdeal := func<I, Q | [Q!(&*[g[1]^g[2] : g in f]) : f in I]>;
  return ConvertToIdeal(Ibar, R);
end intrinsic;
```

Jacobian.m

```

// MilnorNumber not available for local polynomial rings.
intrinsic MilnorNumber(f::RngMPolLocElt) -> RngIntElt
{ The Milnor number of f }
  R := Parent(f); J := JacobianIdeal(f); RJ := R/J;
  if HasFiniteDimension(RJ) then return Dimension(RJ);
  else return Infinity(); end if;
end intrinsic;

// The Milnor number of the singularity defined by a plane curve semigroup.
intrinsic MilnorNumber(G::[RngIntElt]) -> RngIntElt
{ The Milnor number of a semigroup }
require IsPlaneCurveSemiGroup(G): "Argument␣must␣be␣a␣plane␣curve␣semigroup";
```

```

 E := [i gt 1 select Gcd(Self(i - 1), G[i]) else G[1] : i in [1..#G]];
 N := [E[i - 1] div E[i] : i in [2..#G]]; g := #G - 1; n := G[1];
 return &+[(N[i] - 1) * G[i + 1] : i in [1..g]] - n + 1;
end intrinsic;

// The Milnor number of the singularity defined by a plane curve char. seq.
intrinsic MilnorNumber(n::RngIntElt, M::[RngIntElt]) -> RngIntElt
{ The Milnor number of a characteristic sequence }
 G := SemiGroup(n, M); return MilnorNumber(G);
end intrinsic;

// TjurinaNumber not available for local polynomial rings.
intrinsic TjurinaNumber(f::RngMPolLocElt) -> RngIntElt
{ The Tjurina number of f }
 R := Parent(f); Jf := JacobianIdeal(f) + ideal<R | f>; RJf := R/Jf;
 if HasFiniteDimension(RJf) then return Dimension(RJf);
 else return Infinity(); end if;
end intrinsic;

// A monomial basis for the finite dimension algebra R/(J(f) + f)
intrinsic TjurinaAlgebra(f::RngMPolLocElt) -> []
{ Monomial basis for the Tjurina algebra }
 tau := TjurinaNumber(f); R := Parent(f); g := Rank(R);
require tau ne Infinity(): "Argument␣must␣be␣an␣isolated␣singularity.";

 Jf := JacobianIdeal(f) + ideal<R | f>; RJf := R/Jf;
 F := hom<RJf -> R | [R.i : i in [1..g]]>;
 return Reverse(Setseq(F(MonomialBasis(RJf))));
end intrinsic;

// A monomial basis for the finite dimension algebra R/J(f)
intrinsic MilnorAlgebra(f::RngMPolLocElt) -> []
{ Monomial basis for the Milnor algebra of an isolated singularity }
 mu := MilnorNumber(f); R := Parent(f); g := Rank(R);
require mu ne Infinity(): "Argument␣must␣be␣an␣isolated␣singularity.";

 J := JacobianIdeal(f); RJ := R/J; F := hom<RJ -> R | [R.i : i in [1..g]]>;
 return Reverse(Setseq(F(MonomialBasis(RJ))));
end intrinsic;

// By the Briancon-Skoda theorem, for every every hypersurface defining
// an isolated singularity there exists a minimal kappa such that
// f^kappa belong to the Jacobian ideal.
intrinsic JacobianPower(f::RngMPolLocElt) -> RngIntElt
{ Computes the minimal kappa s.t. f^kappa \in J(f), f an isolated singularity }
 mu := MilnorNumber(f);
require mu ne Infinity(): "Argument␣must␣be␣an␣isolated␣singularity.";

 J := JacobianIdeal(f); kappa := 1;
 while NormalForm(f^kappa, J) ne 0 do kappa +:= 1; end while;
 return kappa;
end intrinsic;

// The gaps in the Tjurina filtration of the curve.
intrinsic TjurinaGaps(f::RngMPolLocElt) -> []
{ The gaps of the Tjurina ideal of an irreducible plane curve }
 R := Parent(f); g := Rank(R); tau := TjurinaNumber(f);
require tau ne Infinity(): "Argument␣must␣be␣an␣isolated␣singularity.";
```

```
  M := CharExponents(f); n := M[1][2]; M := [m[1] : m in M[2..#M]];
  G := SemiGroup(n, M); c := Conductor(G);

  // Elements nu < c + min(n, m1) - 1 s.t nu \in Gamma.
  Nu1 := [i : i in [0..(c + Min(n, M[1]) - 1) - 1] | SemiGroupMembership(i, G)];
  FJf := TjurinaFiltration(f); // i is a gap iff FJf[i] eq FJf[i+1].
  Nu2 := [i - 1 + (c + Min(n, M[1]) - 1) : i in [1..#FJf-1]
    | FJf[i][1] eq FJf[i+1][1]]; return Nu1, Nu2;
end intrinsic;

// An special adapted basis for the finite dimension algebra R/(J(f) + f)
intrinsic TjurinaAlgebraAdapted(f::RngMPolLocElt) -> []
{ An adapted basis for the Tjurina algebra }
  R := Parent(f); g := Rank(R); G := SemiGroup(f);
  Nu1, Nu2 := TjurinaGaps(f); Cv := MaxContactElements(f); B := [];
  for alpha in Nu1 cat Nu2 do
    _, b := SemiGroupMembership(alpha, G); B cat:= [b];
  end for; assert(#B eq TjurinaNumber(f));
  S := [&*[Cv[i]^beta[i] : i in [1..#G]] : beta in B];
  return S;
end intrinsic;

// An 'adapted' basis of the Milnor algebra constructed from any basis
// of the Tjurina algebra.
intrinsic MilnorAlgebraAdapted(f::RngMPolLocElt, RJf::[RngMPolLocElt]) -> []
{ Constructs an adapted basis of the Milnor algebra of an isolated singularity f
  from a given basis of the Tjurina algebra by successive multiplication by f }
  R := Parent(f); J := JacobianIdeal(f); kappa := JacobianPower(f);
  for i in [2..kappa] do
    Ji := J + ideal<R | f^i>; tau_i := Dimension(R/Ji);
    Ii := [R | f^(i-1) * gi : gi in RJf | not (f^(i-1) * gi) in Ji];
    RJf cat:= Ii[1..(tau_i - #RJf)];
  end for; return RJf;
end intrinsic;
```

LocalPolynomialRing.m

```
// SquarefreePart not available for local polynomial rings.
intrinsic SquarefreePart(f::RngMPolLocElt) -> RngMPolLocElt
{ Return the squarefree part of f, which is the largest (normalized)
  divisor g of f which is squarefree. }

  P := Parent(f); Q := PolynomialRing(CoefficientRing(P), Rank(P));
  return P!SquarefreePart(Q!f);
end intrinsic;

// SquarefreeFactorization not available for local polynomial rings.
intrinsic SquarefreeFactorization(f::RngMPolLocElt) -> SeqEnum
{ Factorize into squarefree polynomials the polynomial f. }

  P := Parent(f); Q := PolynomialRing(CoefficientRing(P), Rank(P));
  return [<P!g[1], g[2]> : g in SquarefreeFactorization(Q!f)];
end intrinsic;

// JacobianMatrix not available for local polynomial rings.
intrinsic JacobianMatrix(poly_list::[RngMPolLocElt]) -> GrpMat
{ Returns the matrix with (i,j)'th entry the partial derivative of the i'th
  polynomial in the list with the j'th indeterminate of its parent ring. }
```

```

  P := Parent(poly_list[1]); Q := PolynomialRing(CoefficientRing(P), Rank(P));
  return ChangeRing(JacobianMatrix([Q | f : f in poly_list]), P);
end intrinsic;

// Jacobian ideal not available for local polynomial rings.
intrinsic JacobianIdeal(p::RngMPolLocElt) -> RngMPolLoc
{ Returns the ideal generated by all first partial derivatives
  of the polynomial. }

  R := Parent(p); n := Rank(R);
  return ideal<R | [Derivative(p, i) : i in [1..n]]>;
end intrinsic;

// Polynomial not available for local polynomial rings.
intrinsic Polynomial(C::SeqEnum[RngElt], M::SeqEnum[RngMPolLocElt]) -> RngMPolLocElt
{ The multivariate polynomial whose coefficients are C and monomials are M
  (so that Polynomial(Coefficients(f), Monomials(f))) equals f. }

  P := Parent(M[1]); Q := PolynomialRing(CoefficientRing(P), Rank(P));
  return P!Polynomial(C, ChangeUniverse(M, Q));
end intrinsic;
```

Misc.m

```
import "ProximityMatrix.m": ProximityMatrixImpl;
import "SemiGroup.m": Euclides;

intrinsic MonomialCurve(G::[RngIntElt]) -> []
{ Computes the monomial curve assocaited to a semigroup of a
  plane curve }
require IsPlaneCurveSemiGroup(G): "G␣is␣not␣the␣semigroup␣of␣a␣plane␣curve";

  E := [i gt 1 select Gcd(Self(i - 1), G[i]) else G[1] : i in [1..#G]];
  N := [E[i - 1] div E[i] : i in [2..#G]];

  R := PolynomialRing(RationalField(), G); I := [R | ];
  AssignNames(~R, ["u" cat IntegerToString(i) : i in [0..#G - 1]]);
  for i in [1..#G - 1] do
    _, L_i := SemiGroupMembership(N[i] * G[i + 1], G[[1..i]]);
    I cat:= [R.(i + 1)^N[i] - &*[R.j^L_i[j] : j in [1..i]]];
  end for; return I;
end intrinsic;

intrinsic MonomialCurve(n::RngIntElt, M::[RngIntElt]) -> []
{ Computes the monomial curve associated to a characteristic sequence }

  G := SemiGroup(n, M);
  return MonomialCurve(G);
end intrinsic;

intrinsic DeformationCurve(G::[RngIntElt]) -> []
{ Computes the deformations of the monomial curve associated to the
  semigroup G }

  I := MonomialCurve(G); g := #I; R := Universe(I); ZZ := Integers();
  Ei := [i gt 1 select Gcd(Self(i - 1), G[i]) else G[1] : i in [1..#G]];
  Ni := [0] cat [ZZ!(Ei[i] div Ei[i + 1]) : i in [1..g]];
  nB := [-Ni[i+1] * G[i+1] : i in [1..g]];
```

```

  M := EModule(R, nB); N := ideal<R | I> * M;
  J := Transpose(JacobianMatrix(I));
  T_1 := N + sub<M | [M ! m : m in RowSequence(J)]>;

  Groebner(T_1); LT := [LeadingTerm(m) : m in Basis(T_1)]; D_mu := [];
  for i in [1..g] do
    LT_i := ideal<R | [m[i] : m in LT | m[i] ne 0]>;
    M_i := [M.i * m : m in MonomialBasis(quo<R | LT_i>)];
    D_mu cat:= [m : m in M_i | WeightedDegree(m) gt 0];
  end for;

  RR := LocalPolynomialRing(RationalField(), Rank(R) + #D_mu, "lglex");
  AssignNames(~RR, ["t" cat IntegerToString(i) : i in [0..#D_mu - 1]] cat
                ["u" cat IntegerToString(i) : i in [0..g]]);
  phi := hom<R -> RR | [RR.i : i in [#D_mu + 1..Rank(RR)]]>;
  II := [RR | phi(f) : f in I];
  for i in [1..#D_mu] do
    e_i := Column(D_mu[i]);
    II[e_i] +:= RR.i * phi(D_mu[i][e_i]);
  end for; return II;
end intrinsic;

intrinsic ESufficiencyDegree(f::RngMPolLocElt) -> RngIntElt
{ Computes the E-sufficiency degree of a plane curve }
require Evaluate(f, <0, 0>) eq 0: "Curve␣must␣be␣non-empty";

  branches := PuiseuxExpansion(f); P, E, _ := ProximityMatrixImpl(branches);
  ZZ := IntegerRing(); VS := RSpace(ZZ, Ncols(P));

require &+[ZZ | Gcd(Eltseq(e)) : e in E] eq #E: "Curve␣must␣be␣reduced";

  Pt := Transpose(P); N := Ncols(P); isSat := &+[VS | Pt[i] : i in [1..N]];
  // Construct subset T of free points of K.
  freePoints := [p : p in [1..N] | isSat[p] eq 0]; T := []; exc := &+E*P;
  for p in freePoints do
    // Points proximate to 'p'.
    prox_p := [i : i in [p + 1..N] | Pt[p][i] eq -1];
    // Points proximate to 'p' that are satellites.
    prox_p_sat := [q : q in prox_p | isSat[q] eq -1];
    // Select 'p' if all its proximate points in K are
    // satellite and its excess is equal to 1.
    if #prox_p eq #prox_p_sat and exc[1][p] eq 1 then T cat:= [p]; end if;
  end for;
  // Apply theorem 7.5.1 (Casas-Alvero)
  QQ<n> := PolynomialRing(RationalField()); Pt := ChangeRing(Pt, QQ);
  e := ChangeRing(&+E, QQ); i_O := ZeroMatrix(QQ, 1, N); i_O[1][1] := 1;
  u := (i_O*n - e)*Pt^-1; ns := [ZZ | ];
  for p in [1..N] do
    a := Roots(u[1][p])[1][1]; b := Ceiling(a);
    ns cat:= [p in T select b else (a eq b select a + 1 else b)];
  end for; E := Max(ns); return E;
end intrinsic

intrinsic PolarInvariants(f::RngMPolLocElt) -> []
{ Computes the polar invariants of a plane curve }
require Evaluate(f, <0, 0>) eq 0: "Curve␣must␣be␣non-empty";

  branches := PuiseuxExpansion(f); P, E, _ := ProximityMatrixImpl(branches);
```

```
  ZZ := IntegerRing(); VS := RSpace(ZZ, Ncols(P));

require &+[ZZ | Gcd(Eltseq(e)) : e in E] eq #E: "Curve␣must␣be␣reduced";

  Pt := Transpose(P); N := Ncols(P); isSat := &+[VS | Pt[i] : i in [1..N]];
  Pinv := P^-1; e := Transpose(&+E); exc := Pt*e; R := [1];
  for p in [2..N] do // Construct the set of rupture points.
    // Points proximate to 'p' that are free.
    prox_p_free := [i : i in [p + 1..N] | Pt[p][i] eq -1 and isSat[i] ne -1];
    if (isSat[p] eq -1 and (#prox_p_free ge 1 or exc[p][1] gt 0)) or
      (isSat[p] ne -1 and #prox_p_free ge 2) then R cat:= [p]; end if;
  end for; I := [];
  // For each rupture point compute the polar invariant.
  for p in R do
    i_p := ZeroMatrix(ZZ, 1, N); i_p[1][p] := 1;
    e_p := i_p*Pinv; I cat:= [(e_p*e)[1][1] / e_p[1][1]];
  end for; return I;
end intrinsic;

intrinsic ASufficiencyBound(f::RngMPolLocElt) -> RngIntElt
{ Computes a lower-bound for the A-sufficiency degree of a plane curve }

  I := PolarInvariants(f);
  a := 2*Max(I); b := Ceiling(a);
  return a eq b select a + 1 else b;
end intrinsic;

intrinsic Spectrum(G::[RngIntElt]) -> []
{ The singularity spectrum of an irreducible plane curve singularity }
require IsPlaneCurveSemiGroup(G): "G␣is␣not␣the␣semigroup␣of␣a␣plane␣curve";

  E := [i gt 1 select Gcd(Self(i - 1), G[i]) else G[1] : i in [1..#G]];
  N := [E[i - 1] div E[i] : i in [2..#G]]; g := #G - 1; n := G[1]; S := [];

  for i in [1..#G - 1] do
    Mi := G[i+1] div E[i+1];
    S cat:= [(Mi*j + N[i]*k + r*N[i]*Mi)/(N[i]*G[i+1]) :
      j in [1..N[i] - 1], k in [1..Mi - 1], r in [0..E[i+1] - 1] |
        Mi*j + N[i]*k lt N[i]*Mi];
  end for; return S cat [2 - s : s in S];
end intrinsic;

intrinsic Spectrum(n::RngIntElt, M::[RngIntElt]) -> []
{ The singularity spectrum of an irreducible plane curve singularity }

  return Spectrum(SemiGroup(n, M));
end intrinsic;

intrinsic Spectrum(f::RngMPolLocElt) -> []
{ The singularity spectrum of an irreducible plane curve singularity }

  return Spectrum(SemiGroup(f));
end intrinsic;

intrinsic GenericBExponents(n::RngIntElt, M::[RngIntElt]) -> RngSerPuisElt
{ Computes the generating sequence for the generic b-exponents
  from the characteristic sequence using Yano's formula }

  G := SemiGroup(n, M); M := [n] cat M;
```

```
  E := [i gt 1 select Gcd(Self(i - 1), G[i]) else G[1] : i in [1..#G]];

  Rk := [i gt 1 select (E[i - 1] div E[i]) * (Self(i - 1) + M[i] - M[i - 1])
    else n : i in [1..#G]];
  rk := [(M[i] + n) div E[i] : i in [1..#G]];
  Rk_ := [n] cat [(Rk[i] * E[i]) div E[i - 1] : i in [2..#G]];
  rk_ := [2] cat [(rk[i] * E[i]) div E[i - 1] + 1 : i in [2..#G]];

  P<t> := PuiseuxSeriesRing(RationalField());
  s := &+[P | t^(rk[i]/Rk[i]) * (1 - t)/(1 - t^(1/Rk[i])) : i in [2..#Rk]] -
       &+[P | t^(rk_[i]/Rk_[i]) * (1 - t)/(1 - t^(1/Rk_[i])) :
         i in [1..#Rk_]] + t;
  return ChangePrecision(s, Infinity());
end intrinsic;

intrinsic GenericBExponents(G::[RngIntElt]) -> RngSetPuisElt
{ Computes the generating sequence for the generic b-exponents
  from the semigroup using Yano's formula }

    C := CharExponents(G); n := C[1][2]; C := [C[i][1] : i in [2..#C]];
    return GenericBExponents(n, C);
end intrinsic;
```

MultiplierIdeals.m

```
import "ProximityMatrix.m": ProximityMatrixImpl;
import "IntegralClosure.m": IntegralClosureIrreducible, Unloading, ProductIdeals,
                            ClusterFactorization;

// Reference: Naie - "Jumping numbers of a unibranch curve on a smooth surface"
intrinsic JumpingNumbers(G::[RngIntElt]) -> []
{ Compute the Jumping Numbers < 1 of an irreducible plane curve
  from its semigroup }
require IsPlaneCurveSemiGroup(G): "G␣must␣be␣the␣semigroup␣of␣a␣plane␣curve";

  E := [i gt 1 select Gcd(Self(i - 1), G[i]) else G[1] : i in [1..#G]];
  RSet := func<p, q | [a*p+b*q : a in [1..q], b in [1..p] | a*p+b*q lt p*q]>;

  g := #G - 1; JN := [];
  for i in [1..g] do
    p := E[i] / E[i + 1]; q := G[i + 1] / E[i + 1]; R := RSet(p, q);
    Rmj := [k*p*q + alpha : k in [0..E[i+1] - 1], alpha in R];
    JN cat:= [[beta / Lcm(E[i], G[i + 1]) : beta in Sort(Rmj)]];
  end for; return JN;
end intrinsic;

// Reference: Naie - "Jumping numbers of a unibranch curve on a smooth surface"
intrinsic JumpingNumbers(n::RngIntElt, M::[RngIntElt]) -> []
{ Compute the Jumping Numbers < 1 of an irreducible plane curve from its
  char. exponents }

  G := SemiGroup(n, M); return JumpingNumbers(G);
end intrinsic;

intrinsic MultiplierIdeals(f::RngMPolLocElt : MaxJN := 1) -> []
{ Computes the Multiplier Ideals and its associated Jumping Number for an
  plane curve in a smooth complex surface using the algorithm
  of Alberich-Alvarez-Dachs }

```

```
  P, E, C := ProximityMatrix(f: Coefficients := true); QQ := Rationals();
  EQ := ChangeRing(E, QQ); PQ := ChangeRing(P, QQ); PQTinv := Transpose(PQ)^-1;
  N := Ncols(P); F := EQ*PQTinv; K := Matrix([[QQ | 1 : i in [1..N]]]);
  K := K * PQTinv; ZZ := Integers(); k := Parent(f);

  JN := 0; S := [];
  while JN lt MaxJN do
    D := Unloading(PQ, Matrix([[QQ | Floor(ei) : ei in Eltseq(JN*F - K)]]));
    lastJN := JN;
    JN, i := Min([(K[1][i] + 1 + D[1][i])/F[1][i] : i in [1..N]]);
    S cat:= [<GeneratorsOXD(P, ChangeRing(D, ZZ), C, k), lastJN>];
  end while; return S;
end intrinsic;

intrinsic MultiplierIdeals(I::RngMPolLoc : MaxJN := 1) -> []
{ Computes the Multiplier Ideals and its associated Jumping Number for an
  m-primary ideal in a smooth complex surface using the algorithm
  of Alberich-Alvarez-Dachs }
require Gcd(Basis(I)) eq 1: "Ideal␣must␣be␣m-primary";

  P, F, _, C := LogResolution(I: Coefficients := true); QQ := Rationals();
  F := ChangeRing(Matrix(F), QQ); PQ := ChangeRing(P, QQ); ZZ := Integers();
  PQTinv := Transpose(PQ)^-1; k := Universe(Basis(I)); N := Ncols(P);
  // Compute relative canonical divisor
  K := Matrix([[QQ | 1 : i in [1..N]]]); K := K * PQTinv;

  JN := 0; S := [];
  while JN lt MaxJN do
    D := Unloading(PQ, Matrix([[QQ | Floor(ei) : ei in Eltseq(JN*F - K)]]));
    lastJN := JN;
    JN, i := Min([(K[1][i] + 1 + D[1][i])/F[1][i] : i in [1..N]]);
    S cat:= [<GeneratorsOXD(P, ChangeRing(D, ZZ), C, k), lastJN>];
  end while; return S;
end intrinsic;

intrinsic TopologicalRootsBS(G::[RngIntElt]) -> []
{ Compute the topological roots of the Bernstein-Sato polynomial of a
  topological class given by the semigroup G }

  P, E := ProximityMatrix(G); QQ := Rationals(); ZZ := Integers();
  N := Ncols(P); P := ChangeRing(P, QQ); Pt := Transpose(P);
  E := ChangeRing(E, QQ); PTinv := Pt^-1; F := E*PTinv;
  K := Matrix([[QQ | 1 : i in [1..N]]]); K := K * PTinv;

  VS := RSpace(ZZ, N); R := []; isSat := &+[VS | Pt[i] : i in [1..N]];
  for p in [2..N] do // Construct the set of rupture points.
    // Points proximate to 'p' that are free.
    prox_p_free := [i : i in [p + 1..N] | Pt[p][i] eq -1 and isSat[i] ne -1];
    if (isSat[p] eq -1 and (#prox_p_free ge 1)) then R cat:= [p]; end if;
  end for; R cat:= [N];

  JN := [];
  Ei := [i gt 1 select Gcd(Self(i - 1), G[i]) else G[1] : i in [1..#G]];
  for i in [1..# G - 1] do
    JNi := 0; Si := []; r := R[i]; lct := (K[1][r] + 1)/F[1][r];
    Fi := Matrix([[QQ | 0 : j in [1..N]]]); Fi[1][r] := F[1][r];
    while JNi lt 1 + lct do
      D := Unloading(P, Matrix([[QQ | Floor(ei) : ei in Eltseq(JNi*Fi - K)]]));
      JNi := (K[1][r] + 1 + D[1][r])/F[1][r];
```

```
    if Denominator(Ei[i]*JNi) ne 1 and Denominator(G[i + 1]*JNi) ne 1 then
      Si cat:= [JNi];
    end if;
   end while; JN cat:= [Si[1..#Si - 1]];
  end for; return JN;
end intrinsic;
```

NewtonPolygon.m

```
// Hack to return 0 when 0 * Infinity() is computed.
MyProd := function(a, b)
  if a eq 0 or b eq 0 then return 0;
  else return a * b; end if;
end function;

// Counter clockwise turn.
CcwTurn := function(p1, p2, p3)
  return (MyProd(p2[1] - p1[1], p3[2] - p1[2]) -
       MyProd(p2[2] - p1[2], p3[1] - p1[1])) le 0;
end function;

// Magma's NewtonPolygon function does not work in our case.
intrinsic NewtonPolygon(f::RngMPolLocElt) -> SeqEnum
{ The newton polygon for the bivariate polynomial f. }
require Rank(Parent(f)) eq 2: "Argument␣must␣be␣a␣bivariate␣polynomial";

  NP := [];
  for p in Sort([Exponents(m) : m in Monomials(f)]) cat [[Infinity(), 0]] do
   while #NP ge 2 and CcwTurn(NP[#NP - 1], NP[#NP], p) do
     Prune(~NP);
   end while; NP cat:= [p];
  end for; Prune(~NP); return NP;
end intrinsic;

NewtonSide := procedure(p, q, ~f, ~S)
  g := Gcd(p[2] - q[2], q[1] - p[1]);
  n := (p[2] - q[2]) div g;
  m := (q[1] - p[1]) div g;
  k := (p[2] * q[1] - p[1] * q[2]) div g;

  // Select which exponents are on the side generated by p & q.
  Q<X, Y> := PolynomialRing(IntegerRing(), 2);
  side := n * X + m * Y - k;
  C, M := CoefficientsAndMonomials(f);
  onSide := [C[i] * M[i] : i in [1..#M] |
   Evaluate(side, <Exponents(M[i])[1], Exponents(M[i])[2]>) eq 0];

  // Construct the equation associated with the pq side.
  P<Z> := PolynomialRing(CoefficientRing(Parent(f)));
  h := Evaluate(&+onSide, <1, Z>);
  E, _ := Support(h); C := Coefficients(h);
  beta0 := Reverse(Sort([Exponents(m) : m in onSide]))[1][2];
  S cat:= [<n, m, &+[C[e + 1] * Z^((e - beta0) div n) : e in E]>];
end procedure;

intrinsic NewtonSides(f::RngMPolLocElt, NP::SeqEnum) -> SeqEnum
{ Returns the sides of the Newton polygon. }
require Rank(Parent(f)) eq 2: "Argument␣must␣be␣a␣bivariate␣polynomial";

```

```
  S := [];
  for i in [1..#NP-1] do
    NewtonSide(NP[i], NP[i+1], ~f, ~S);
  end for; return S;
end intrinsic;
```

ProximityMatrix.m

```
import "SemiGroup.m": Euclides, TailExponentSeries;

PuiseuxInfo := function(s)
  if Type(s) eq RngMPolLocElt then return [<[<0,0>], [0, Infinity()]>]; end if;
  E := CharExponents(s); T := TailExponentSeries(s);
  I := []; E cat:= [T]; n := E[1][2];
  // For each characteristic exponent...
  for i in [2..#E] do
    mj := E[i - 1][1]; nj := E[i - 1][2]; mi := E[i][1]; ni := E[i][2];
    h0 := (mi - mj) div nj; sat := Euclides(mi - mj, nj)[1];
    free := [<e, Coefficient(s, e)> : e in [(mj + k*nj)/n : k in [0..h0]]];
    Append(~I, <free, sat>);
  end for; return I;
end function;

ContactNumberExp := function(expInfoA, expInfoB)
  ContactNum := 0;
  // Free points associated with the char. exponent.
  freeA := expInfoA[1]; freeB := expInfoB[1];
  // Satellite points associated with the char. exponent.
  satA := expInfoA[2]; satB := expInfoB[2];
  // Compare free points.
  for i in [1..Min(#freeA, #freeB)] do
    if freeA[i] eq freeB[i] then ContactNum := ContactNum + 1;
    else return ContactNum, false; end if;
  end for;
  // If the num. of free points is not the same, no more points can be shared.
  if #freeA ne #freeB then return ContactNum, false; end if;
  // Compare satellite points.
  satA[#satA] := satA[#satA] - 1; satB[#satB] := satB[#satB] - 1;
  for i in [2..Min(#satA, #satB)] do
    ContactNum := ContactNum + Min(satA[i], satB[i]);
    if satA[i] ne satB[i] then return ContactNum, false; end if;
  end for;
  // If the number of stairs is not the same, no more points can be shared.
  if #satA ne #satB then return ContactNum, false; end if;
  // Otherwise, all the points are shared.
  return ContactNum, true;
end function;

ContactNumber := function(branchInfoA, branchInfoB)
  ContactNum := 0;
  // For each characteristic exponent...
  for r in [1..Min(#branchInfoA,#branchInfoB)] do
    // Get the contact num. of this char. exponent and wheter
    // or not we should compare more points.
    C, cont := ContactNumberExp(branchInfoA[r], branchInfoB[r]);
    ContactNum := ContactNum + C;
    if not cont then break; end if;
  end for; return ContactNum;
end function;
```

```

ContactMatrix := function(branches)
 //Add a dummy term so compare exact branches is easier.
 max := Max([0] cat [s[1] eq 0 select 0 else
   Ceiling(Degree(s[1])) : s in branches]) + 1;
 branches := [* <s[1] + (Parent(s[1]).1)^max, s[2]> : s in branches *];
 info := [PuiseuxInfo(s[1]) : s in branches];
 contact := ScalarMatrix(#branches, 1);
 // For each pair of branches compute their contact number.
 for i in [1..#branches] do
   for j in [i+1..#branches] do
     contactNum := ContactNumber(info[i], info[j]);
     contact[i][j] := contactNum; contact[j][i] := contactNum;
   end for;
 end for; return contact;
end function;

ProximityMatrixSemiGroup := function(H, maxContact : ExtraPoint := false)
 // Smooth inverted branches could have 2 char exps.
 if #H eq 2 and #H[1] eq 2 and H[1][1] eq 0 then Prune(~H); end if;
 // Dimension of the proximity matrix.
 N := Max(&+[IntegerRing() | &+h : h in Prune(H)], maxContact);
 if ExtraPoint then N := N + 1; end if;
 // Construct a proximity matrix with free points only.
 P := ScalarMatrix(N, 1);
 for i in [2..N] do P[i][i - 1] := -1; end for;
 // Fill in satellite points proximities.
 for i in [1..#H] do
   // Inverted axis case.
   if i eq 1 and H[1][1] eq 0 then j0 := 3; else j0 := 2; end if;
   Hi := H[i]; Hi[#Hi] := Hi[#Hi] - 1;
   for j in [j0..#Hi] do
     l := &+[IntegerRing() | &+H[k] : k in [1..i - 1]] + &+Hi[1..j - 1];
     for k in [1..Hi[j]] do P[l + k + 1, l] := -1; end for;
   end for;
 end for; return P;
end function;

ProximityMatrixBranch := function(s, maxContact : ExtraPoint := false)
 // If the branch is the y-axis.
 if Type(s) eq RngMPolLocElt then
   if ExtraPoint then maxContact := maxContact + 1; end if;
   // Construct a proximity matrix with free points only.
   P := ScalarMatrix(maxContact, 1);
   for i in [2..maxContact] do P[i][i - 1] := -1; end for;
   return P;
 end if; // Otherwise, the branch is represented by a Puiseux series.
 H := [charExps[2] : charExps in PuiseuxInfo(s)];
 return ProximityMatrixSemiGroup(H, maxContact : ExtraPoint := ExtraPoint);
end function;

MultiplicityVectorBranch := function(s, maxContact: ExtraPoint := false)
 // If the branch is the y-axis.
 if Type(s) eq RngMPolLocElt then
   if ExtraPoint then maxContact := maxContact + 1; end if;
   return Vector([1 : i in [1..maxContact]]);
 end if; // Otherwise, the branch is represented by a Puiseux series.
 M := []; E := CharExponents(s);
 for i in [2..#E] do
```

```
  mj := E[i-1][1]; nj := E[i-1][2]; mi := E[i][1]; ni := E[i][2];
  Hs := Euclides(mi - mj, nj)[1]; Ns := Euclides(mi - mj, nj)[2];
  for j in [1..#Hs] do M cat:= [Ns[j] : k in [1..Hs[j]]]; end for;
 end for;
 M cat:= [1 : i in [1..(maxContact - #M)]];
 if ExtraPoint then M cat:= [1]; end if;
 return Vector(M);
end function;

CoefficientsVectorBranch := function(s, maxContact)
 // If the branch is the y-axis
 R := CoefficientRing(Parent(s));
 if Type(s) eq RngMPolLocElt then
  return [<1, R!0>] cat [<0, R!1> : i in [1..maxContact]];
 end if; // Otherwise, the branch is represented by a Puiseux series.
 I := PuiseuxInfo(s); C := [];
 for i in [1..#Prune(I)] do
  C cat:= [<1, freePoint[2]> : freePoint in I[i][1]];
  Hi := I[i][2]; Hi[#Hi] := Hi[#Hi] - 1;
  C cat:= [<0, R!1> : j in [1..&+Hi[2..#Hi]]];
 end for;
 C cat:= [<1, freePoint[2]> : freePoint in I[#I][1]];
 if #C lt maxContact then
  C cat:= [<1, R!0> : i in [1..(maxContact - #C)]];
 end if;
 // The 0 Puiseux series must be treated separately.
 if s eq 0 then C cat:= [<1, R!0>]; end if;
 return C;
end function;

function ProximityMatrixImpl2(contactMat, branchesProx)
 if #branchesProx eq 0 then return <ScalarMatrix(0, 0), []>; end if;
 ////////////////////////// Base case //////////////////////////////
 // If there is only branch, return its prox. matrix.
 if #branchesProx eq 1 then
  return <branchesProx[1], [[i : i in [1..Ncols(branchesProx[1])]]]>;
 end if;
 ///////////////////// Compute the splits ////////////////////////////
 // Substract one to all the contact numbers except the diagonal ones.
 N := Nrows(contactMat); ZZ := IntegerRing();
 contactMat := contactMat - Matrix(N, [ZZ | 1: i in [1..N^2]])
  + ScalarMatrix(N, 1);
 // Identify each current branch with an ID from 1 to #branches.
 C := contactMat; remainingBranches := [i : i in [1..N]]; S := [];
 // Splits will contain lists of branches ID, where two branches will
 // be in the same list iff they do not separate in the current point.
 while #remainingBranches ne 0 do
  // Get the contact number of the first remaining branch.
  branchCont := ElementToSequence(C[1]);
  // Get the positions of the branches with contact > 1 & contact = 1.
  sameBranchIdx := [i : i in [1..#branchCont] | branchCont[i] ne 0];
  otherBranchIdx := [i : i in [1..#branchCont] | branchCont[i] eq 0];
  // Save the branches with contact > 1 together and remove them since they
  // have been splitted from the rest of branches.
  Append(~S, remainingBranches[sameBranchIdx]);
  remainingBranches := remainingBranches[otherBranchIdx];
  // Compute the contact matrix of the remaining brances.
  C := Submatrix(C, otherBranchIdx, otherBranchIdx);
 end while;
```

```
  ///////////// Compute the prox. matrix of each subdiagram /////////////
  // Substract one to all the contact numbers and erase the
  // first point of the proximity matricies of the current
  // branches since we are moving down the Enriques diagram.
  newBranchProx := [*RemoveRowColumn(branchP, 1, 1) : branchP in branchesProx*];
  // Traverse each sub-diagram recursivaly.
  splitResult := [* ProximityMatrixImpl2(Submatrix(contactMat, split, split),
   newBranchProx[split]) : split in S *];
  ///////////////// Merge the prox. matrix of each split ////////////////
  // Create the matrix that will hold the proximity branch of this subdiagram.
  numPoints := &+[ZZ | Ncols(X[1]) : X in splitResult] + 1;
  P := ScalarMatrix(numPoints, 1); rowPoint := []; k := 1;
  // For each set of branches that splits in this node...
  for s in [1..#S] do
   // Get the proximity matrix & the position of the points
   // (relative to that prox. matrix) of the s-th subdiagram.
   X := splitResult[s]; M := X[1]; splitRowPoint := X[2];
   // Copy the submatrix M inside P with the top left entry in (k+1, k+1)
   InsertBlock(~P, M, k + 1, k + 1);
   // Sum k+1 and add the new point ({0}) to the position of the
   // points relative to the prox. matrix of the subdiagram.
   splitRowPoint := [[1] cat [p + k : p in pp] : pp in splitRowPoint];
   rowPoint cat:= splitRowPoint;
   // Use the information in splitRowPoint to set the proximities of
   // the current point into the new prox. matrix (P):
   // For each branch in this subdiagram...
   for i in [1..#S[s]] do
    Q := branchesProx[S[s][i]];
    // For each element int the first column...
    for j in [1..Ncols(Q)] do P[splitRowPoint[i][j]][1] := Q[j][1]; end for;
   end for;
   k := k + Ncols(M);
  end for;
  // Make sure rowPoint is returned in the original order.
  SS := []; for split in S do SS cat:= split; end for;
  SS := [Position(SS, i) : i in [1..#SS]];
  return <P, rowPoint[SS]>;
end function;

function ProximityMatrixImpl(branches: ExtraPoint := false)
  // Compute the proximity matrix, the contact matrix, the mult.
  // vector of each branch and its coefficients.
  contactMat := ContactMatrix(branches);
  branchProx := [* ProximityMatrixBranch(branches[i][1],
   Max(ElementToSequence(contactMat[i])) :
   ExtraPoint := ExtraPoint) : i in [1..#branches] *];
  branchMult := [* branches[i][2] * MultiplicityVectorBranch(branches[i][1],
   Max(ElementToSequence(contactMat[i])) :
   ExtraPoint := ExtraPoint) : i in [1..#branches] *];
  branchCoeff := [ CoefficientsVectorBranch(branches[i][1],
   Max(ElementToSequence(contactMat[i])) + 1) : i in [1..#branches] ];
  // Get the proximity matrix of f and the position of each infinitely
  // near point inside the prox. matrix.
  X := ProximityMatrixImpl2(contactMat, branchProx);
  // Finally, rearrange each point's multiplicity so its position is coherent
  // coherent with the prox. matrix P.
  P := X[1]; R := X[2]; E := [RMatrixSpace(IntegerRing(), 1, Ncols(P)) | ];
  for i in [1..#branches] do
   Append(~E, ZeroMatrix(IntegerRing(), 1, Nrows(P)));
```

```
    for j in [1..#R[i]] do E[i][1, R[i][j]] := branchMult[i][j]; end for;
  end for; return P, E, branchCoeff;
end function;

intrinsic ProximityMatrix(f::RngMPolLocElt: ExtraPoint := false,
                    Coefficients := false, Branches := false) -> []
{ Computes the proximity matrix of the resolution of a plane curve }
  // Get the general Puiseux expansion of f.
  branches := PuiseuxExpansion(f);
  P, E, C := ProximityMatrixImpl(branches: ExtraPoint := ExtraPoint);
  if not Coefficients then
    if not Branches then return P, &+E; else return P, E; end if;
  end if;
  CC := [Parent(C[1][1]) | <1, 0> : i in [1..Nrows(P)]];
  for i in [1..#E] do
    I := [j : j in [1..Ncols(P)] | E[i][1][j] ne 0];
    for j in [1..#I] do CC[I[j]] := C[i][j]; end for;
  end for;
  if not Branches then return P, &+E, CC;
  else return P, E, CC; end if;
end intrinsic;

intrinsic ProximityMatrix(G::[RngIntElt] : ExtraPoint := false) -> []
{ Computes the proximity matrix of the resolution of a plane curve
  with semigroup G }

  ZZ := Integers(); N := Gcd(G); G := [ZZ!(g/N) : g in G];
  require IsPlaneCurveSemiGroup(G): "Argument␣must␣be␣a␣plane␣curve␣semigroup";
  C := CharExponents(G) cat []; n := C[1][2]; I := [];
  // For each characteristic exponent...
  for i in [2..#C] do
    mj := C[i - 1][1]; nj := C[i - 1][2]; mi := C[i][1]; ni := C[i][2];
    h0 := (mi - mj) div nj; sat := Euclides(mi - mj, nj)[1];
    Append(~I, sat);
  end for; I cat:= [[0]];
  P := ProximityMatrixSemiGroup(I, 1 : ExtraPoint := ExtraPoint);
  e := ZeroMatrix(ZZ, 1, Ncols(P)); e[1, Ncols(P)] := N;
  return P, e*P^-1;
end intrinsic;

intrinsic ContactMatrix(f::RngMPolLocElt) -> []
{ Computes de contact numbers of the branches of f }

  S := PuiseuxExpansion(f);
  P, E := ProximityMatrixImpl(S); N := Ncols(P);
  C := ScalarMatrix(#S, 0);
  for i in [1..#S] do for j in [i + 1..#S] do
      C[i][j] := &+[E[i][1][k] * E[j][1][k] : k in [1..N]];
  end for; end for;
  return C + Transpose(C);
end intrinsic;
```

PuiseuxExpansion.m

```
xFactor := function(f)
  return Min([IntegerRing() | Exponents(t)[1] : t in Terms(f)]);
end function;

yFactor := function(f)
```

```
  return Min([IntegerRing() | Exponents(t)[2] : t in Terms(f)]);
end function;

forward PuiseuxExpansionLoop;

intrinsic PuiseuxExpansion(f::RngMPolLocElt : Terms := -1,
                                   Polynomial := false) -> [ ]
{ Computes the Puiseux expansion of any bivariate polynomial }
require Rank(Parent(f)) eq 2: "Argument␣must␣be␣a␣bivariate␣polynomial";
  // If Nf start on the right of the x-axis, we have an x-factor.
  yBranch := (xFactor(f) gt 0) select [* <Parent(f).1,
    [<xFactor(f), 1>], Parent(f).1> *] else [* *];

  P<x, y> := LocalPolynomialRing(AlgebraicClosure(
    CoefficientRing(Parent(f))), 2, "lglex");
  S := yBranch cat SequenceToList(PuiseuxExpansionLoop(P!SquarefreePart(f),
    [<P!g[1], g[2], 1> : g in SquarefreeFactorization(f)], Terms - 1));
  if not Polynomial then return [* <s[1], s[2][1][1]> : s in S *];
  else return [* <s[1], s[2][1][1], s[3]> : s in S *]; end if;
end intrinsic;

intrinsic PuiseuxExpansion(L::[RngMPolLocElt] : Terms := -1,
                                     Polynomial := false) -> [ ]
{ Computes the Puiseux expansion for the product of all the elements of L }
require #L gt 0: "Argument␣must␣be␣a␣non-empty␣list";
require &and[Rank(Parent(f)) eq 2 : f in L]:
  "Elements␣of␣L␣must␣be␣bivariate␣polynomials";

  f := &*L;
  P<x, y> := LocalPolynomialRing(AlgebraicClosure(
    CoefficientRing(Parent(f))), 2, "lglex");
  // If Nf start on the right of the x-axis, we have an x-factor.
  yBranch := (xFactor(f) gt 0) select [* <Parent(f).1,
    [<xFactor(L[i]), i> : i in [1..#L] | xFactor(L[i]) ne 0], x> *] else [* *];

  sqFreePart := P!SquarefreePart(f); sqFreeFact := [];
  for i in [1..#L] do
    sqFreeFact cat:= [<P!g[1], g[2], i>: g in SquarefreeFactorization(L[i])
      | Evaluate(L[i], <0, 0>) eq 0];
  end for;
  S := yBranch cat SequenceToList(PuiseuxExpansionLoop(sqFreePart,
    sqFreeFact, Terms - 1));

  // Return the polynomial residue if requested.
  if not Polynomial then return [* <s[1], s[2]> : s in S *];
  else return S; end if;
end intrinsic;

PuiseuxExpansionLoop := function(f, L, terms)
  Q<t> := PuiseuxSeriesRing(CoefficientRing(Parent(f)));
  x := Parent(f).1; y := Parent(f).2;
  // Step (i.a): Select only those factors containing the 0 branch.
  S := yFactor(f) gt 0 select [<Q!0, [<g[2], g[3]> : g in L
    | yFactor(g[1]) ne 0], y>] else [];
  // Step (i.b): For each side...
  for F in NewtonSides(f, NewtonPolygon(f)) do
    n := F[1]; m := F[2]; P := F[3];
    // Apply the change of variables (1).
    C := Reverse(Coefficients(n eq 1 select f else Evaluate(f, 1, x^n), 2));
```

```
    CL := [<Reverse(Coefficients(n eq 1 select g[1] else
      Evaluate(g[1], 1, x^n), 2)), g[2], g[3]> : g in L];
    // For each root...
    for a in [<Root(a[1], n), a[2]> : a in Roots(P)] do
      // Apply the change of variables (2) & get the sub-solution recursively.
      ff := [i gt 1 select C[i] + Self(i - 1) * x^m * (a[1] + y) else C[1]
         : i in [1..#C]][#C];
      LL := [<[i gt 1 select Cj[1][i] + Self(i - 1) * x^m * (a[1] + y) else
        Cj[1][1] : i in [1..#Cj[1]]][#Cj[1]], Cj[2], Cj[3]> : Cj in CL];
      // Select only those factors that contain the current branch.
      LL := [g : g in LL | NewtonPolygon(g[1])[1][2] ne 0];
      // If the mult. of a is greater than 1 continue.
      R := (a[2] ne 1 and terms lt -1) or terms gt 0 select
        PuiseuxExpansionLoop(ff, LL, terms - 1) else
          [<Q!0, [<g[2], g[3]> : g in LL], ff>];
      // Undo the change of variables.
      S cat:= [<t^(m/n) * (a[1] + Composition(s[1], t^(1/n))), s[2], s[3]>
           : s in R];
    end for;
  end for; return S;
end function;

forward PuiseuxExpansionReducedLoop;

intrinsic PuiseuxExpansionReduced(f::RngMPolLocElt : Terms := -1,
                                           Polynomial := false) -> [ ]
{ Computes the Puiseux expansion of a reduced bivariate polynomial }
require Rank(Parent(f)) eq 2: "Argument␣must␣be␣a␣bivariate␣polynomial";

  P := LocalPolynomialRing(AlgebraicClosure(
    CoefficientRing(Parent(f))), 2, "lglex");
  // If Nf start on the right of the x-axis, we have an x-factor.
  yBranch := (xFactor(f) gt 0) select [<Parent(f).1, P.1>] else [];

  S := yBranch cat PuiseuxExpansionReducedLoop(
    P!SquarefreePart(f), Terms - 1);
  if Polynomial then return S; else return [s[1] : s in S]; end if;
end intrinsic;

intrinsic PuiseuxExpansionExpandReduced(s::RngSerPuisElt, f::RngMPolLocElt
                              : Terms := 1, Polynomial := false) -> [ ]
{ Expands the Puiseux expansion s of a reduced bivariate polynomial }
require Rank(Parent(f)) eq 2: "Argument␣f␣must␣be␣a␣bivariate␣polynomial";

  n := ExponentDenominator(s); x := Parent(s).1;
  m := s eq 0 select 0 else Degree(s);

  S := Terms gt 0 select PuiseuxExpansionReducedLoop(f, Terms - 1)
      else [<PuiseuxSeriesRing(CoefficientRing(Parent(f)))!0, f>];
  P<t> := PuiseuxSeriesRing(CoefficientRing(Parent(s)));
  if Polynomial then return [<s + t^m * Composition(si[1], t^(1/n)), si[2]>
    : si in S];
  else return [s + t^m * Composition(si[1], t^(1/n)): si in S]; end if;
end intrinsic;

intrinsic PuiseuxExpansionExpandReduced(x::RngMPolLocElt, f::RngMPolLocElt
                              : Terms := 1, Polynomial := false) -> [ ]
{ Expands the Puiseux expansion s of a reduced bivariate polynomial }
require Rank(Parent(f)) eq 2: "Argument␣f␣must␣be␣a␣bivariate␣polynomial";
```

```

  if Polynomial then return [<x, x>]; else return [x]; end if;
end intrinsic;

PuiseuxExpansionReducedLoop := function(f, terms)
  Q<t> := PuiseuxSeriesRing(CoefficientRing(Parent(f)));
  x := Parent(f).1; y := Parent(f).2;
  // Step (i.a): Select only those factors containing the 0 branch.
  S := yFactor(f) gt 0 select [<Q!0, y>] else [];
  // Step (i.b): For each side...
  for F in NewtonSides(f, NewtonPolygon(f)) do
    n := F[1]; m := F[2]; P := F[3];
    // Apply the change of variables (1).
    C := Reverse(Coefficients(n eq 1 select f else Evaluate(f, 1, x^n), 2));
    // For each root...
    for a in [<Root(a[1], n), a[2]> : a in Roots(P)] do
      // Apply the change of variables (2) & get the sub-solution recursively.
      g := [i gt 1 select C[i] + Self(i - 1) * x^m * (a[1] + y) else C[1]
         : i in [1..#C]][#C];
      R := (a[2] ne 1 and terms lt -1) or terms gt 0 select
        PuiseuxExpansionReducedLoop(g, terms - 1) else [<Q!0, g>];
      // Undo the change of variables.
      S cat:= [<t^(m/n) * (a[1] + Composition(s[1], t^(1/n))), s[2]> : s in R];
    end for;
  end for; return S;
end function;
```

SemiGroup.m

```
Euclides := function(m, n)
  hs := []; ns := [];
  while n ne 0 do
    Append(~hs, m div n); Append(~ns, n);
    r := m mod n; m := n; n := r;
  end while; return <hs, ns>;
end function;

intrinsic CharExponents(s::RngSerPuisElt) -> []
{ Returns the characteristic exponents of a Puiseux series }
  C, m, n := ElementToSequence(s);
  // Exponents appearing in the series s
  E := [m + i - 1 : i in [1 .. #C] | C[i] ne 0];
  charExps := [<0, n>]; ni := n;
  while ni ne 1 do
    // m_i = min{ j | a_j != 0 and j \not\in (n_{i-1}) }
    mi := [e : e in E | e mod ni ne 0][1];
    // n_i = gcd(n, m_1, ..., m_k)
    ni := Gcd(ni, mi); Append(~charExps, <mi, ni>);
  end while; return charExps;
end intrinsic;

intrinsic CharExponents(f::RngMPolLocElt) -> []
{ Returns the characteristic exponents of an irreducible bivariate polynomials }
  S := PuiseuxExpansion(f);
  if #S ne 1 then error "Argument␣must␣be␣a␣irreducible␣series"; end if;
  if S[1][2] ne 1 then error "Argument␣must␣be␣a␣reduced␣series"; end if;
  return CharExponents(S[1][1]);
end intrinsic;

```

```
intrinsic CharExponents(G::[RngIntElt]) -> []
{ Computes the characteristic exponents from the generators of the semigroup }
  require IsPlaneCurveSemiGroup(G): "G␣is␣not␣the␣semigroup␣of␣a␣plane␣curve";

  M := [G[1]]; N := [G[1]];
  for i in [2..#G] do
    M cat:= [ &+[j ne i select -(N[j - 1] - N[j]) div N[i - 1] * M[j]
      else G[j] : j in [2..i]] ]; N cat:= [Gcd(M)];
  end for;
  return [<0, N[1]>] cat [<M[i], N[i]> : i in [2..#M]];
end intrinsic;

TailExponentSeries := function(s)
  C, m, n := ElementToSequence(s);
  // Exponents appearing in the series s
  E := [m + i - 1 : i in [1..#C] | C[i] ne 0];
  charExps := CharExponents(s); g := #charExps;
  if s eq 0 then return <0, 1>; end if;
  return [<e, 1> : e in Reverse(E) | e ge charExps[g][1]][1];
end function;

TailExponentMatrix := function(P)
  E := CharExponents(SemiGroup(P)); N := Ncols(P);
  // If last point is satellite there is no tail exponent
  if &+Eltseq(P[N]) eq -1 then error "no␣free␣point"; end if;
  Pt := Transpose(P); isSat := &+[Pt[j]: j in [1..N]];
  p := ([i : i in Reverse([1..N]) | isSat[i] eq -1] cat [0])[1] + 1;
  return <E[#E][1] + (N - p), 1>;
end function;

intrinsic SemiGroup(n::RngIntElt, M::[RngIntElt]) -> []
{ Computes a minimal set of generators for the semigroup associated
  to a set of charactetistic exponents }
require IsCharSequence(n, M) : "Argument␣must␣be␣a␣valid␣char.␣sequence";

  E := [i gt 1 select Gcd(Self(i - 1), M[i - 1]) else n : i in [1..#M + 1]];
  G := [i gt 2 select ( (Self(i - 1) - M[i - 2]) * E[i - 2] div E[i - 1] ) +
    M[i - 1] + ( (E[i - 2] - E[i - 1]) div E[i - 1] ) * M[i - 2]
       else [n, M[1]][i] : i in [1..#M + 1]];
  return Sort(G);
end intrinsic;

intrinsic SemiGroup(s::RngSerPuisElt) -> []
{ Computes a minimal set of generators for the semigroup of the
  Puiseux series of an irreducible plane curve }

  M := CharExponents(s); // (G)amma starts with <n, m, ...>
  return SemiGroup(M[1][2], [M[i][1] : i in [2..#M]]);
end intrinsic;

intrinsic SemiGroup(f::RngMPolLocElt) -> []
{ Computes a minimal set of generators for the semigroup of
  and irreducible plane curve }

  S := PuiseuxExpansion(f);
  if #S ne 1 then error "Argument␣must␣be␣an␣irreducible␣series"; end if;
  if S[1][2] ne 1 then error "Argument␣must␣be␣a␣reduced␣series"; end if;

  return SemiGroup(S[1][1]);
```

```
end intrinsic;

intrinsic SemiGroup(P::Mtrx : UseExtraPoints := false) -> []
{ Returns the minimal set of generators for the semigroup of and irreducible
  plane curve from its weighted cluster of singular points }
require Ncols(P) eq Nrows(P):
  "Proximity␣matrix␣does␣not␣have␣the␣required␣dimensions";
require CoefficientRing(P) eq Integers():
  "Proximity␣matrix␣must␣be␣defined␣over␣the␣integers";
require IsInvertible(P):
  "Proximity␣matrix␣must␣be␣invertible";
require IsUnipotent(P):
  "Proximity␣matrix␣must␣be␣unipotent";

  if Ncols(P) eq 0 then return [1, 0]; end if;
  N := Ncols(P); e := ZeroMatrix(Integers(), 1, N); e[1, N] := 1; e := e*P^-1;
  Pt := Transpose(P); v := e*Pt^-1; isSat := &+[Pt[i] : i in [1..N]];
  G := [v[1][1]]; G cat:= [v[1][i] : i in [1..N - 1] |
    isSat[i] ne -1 and isSat[i + 1] eq -1];
  if UseExtraPoints and &+Eltseq(P[N]) ne -1 then return G cat [v[1, N]];
  else return G; end if;
end intrinsic;

forward SemiGroupMemberImpl;

SemiGroupMemberImpl := procedure(v, i, ~G, ~B, ~b, ~V)
  if v lt 0 or i gt #G then b := 0; return; end if;
  if B[v + 1, i] ne -1 then b := B[v + 1, i]; return; end if;
  V[i] := V[i] + 1;
  SemiGroupMemberImpl(v - G[i], i, ~G, ~B, ~b, ~V);
  B[v + 1, i] := b; if b eq 1 then return; end if;
  V[i] := V[i] - 1;
  SemiGroupMemberImpl(v, i + 1, ~G, ~B, ~b, ~V);
  B[v + 1, i] := b; if b eq 1 then return; end if;
end procedure;

intrinsic SemiGroupMembership(v::RngIntElt, G::[RngIntElt]) -> BoolElt
{ Returns whether or not an integer v belongs to a numerical semigroup G and
  the coordinates v in the semigroup }

  V := [0 : i in [1..#G]];
  if v lt 0 then return false, V; end if;
  // Any semigroup is valid.
  B := Matrix(v + 1, #G, [IntegerRing() | -1 : i in [1..(v + 1) * #G]]);
  for i in [1..#G] do B[0 + 1][i] := 1; end for;
  b := 0; SemiGroupMemberImpl(v, 1, ~G, ~B, ~b, ~V);
  return b eq 1, V;
end intrinsic;

InversionFormula := function(M0, P, c)
  // Compute the exp. of the last free pt. depending of the first char. exp.
  N := Ncols(P); Pt := Transpose(P); isSat := &+[Pt[j]: j in [1..N]];
  p := ([i : i in [1..N] | isSat[i] eq -1] cat [N + 1])[1] - 1;
  m := ([i : i in [2..p] | c[i][1] ne 0] cat [0])[1] - 1;
  // Depending on whether 'm' is 0 or not, we have case (a) or case (b).
  M1 := [<0, m eq -1 select M0[2][1] else Lcm(M0[1][2], m)>]; k := #M0 - 1;
  M1 cat:= [<M0[1][2], Gcd(M1[1][2], M0[1][2])>]; i0 := m eq -1 select 2 else 1;
  ni := M1[2][2]; // ni := gcd(n, m_1, ..., m_i)
  for i in [i0..k] do // \bar{mi} = mi + n - \bar{n}
```

```
    mi := M0[i + 1][1] + M0[1][2] - M1[1][2];
    ni := Gcd(ni, mi); M1 cat:= [<mi, ni>];
  end for; return M1;
end function;

intrinsic IsPlaneCurveSemiGroup(G::[RngIntElt]) -> BoolElt
{ Whether the semigroup is a plane curve semigroup or not }

  if Gcd(G) ne 1 then return false; end if;
  // e_i := gcd(\bar{m}_{i-1}, \bar{m}_i)
  E := [i gt 1 select Gcd(Self(i - 1), G[i]) else G[1] : i in [1..#G]];
  // n_i := e_i / e_{i + 1}
  N := [1] cat [E[i] div E[i + 1] : i in [1..#G - 1]];
  // n_i != 1 for all i (iff e_i > e_{i+1})
  if Position(N[2..#N], 1) ne 0 then return false; end if;
  // n_i \bar{m}_{i} \in < m_{0}, ..., m_{i-1} > &&
  // n_i \bar{m}_i < \bar{m}_{i + 1} &&
  if not &and[SemiGroupMembership(N[i] * G[i], G[[1..i - 1]]) : i in [2..#G]] or
     not &and[N[i] * G[i] lt G[i + 1] : i in [1..#G - 1]] then return false;
  end if; return true;
end intrinsic;

intrinsic IsCharSequence(n::RngIntElt, M::[RngIntElt]) -> BoolElt
{ Whether the inputs is a valid characteristic sequence or not }
  // e_i := gcd(e_{i-1}, m_i)
  E := [i gt 1 select Gcd(Self(i - 1), M[i - 1]) else n : i in [1..#M + 1]];
  if Sort(E) eq Reverse(E) and E[#E] eq 1 and E[#E - 1] ne 1 then
  return true; else return false; end if;
end intrinsic;

intrinsic Conductor(G::[RngIntElt]) -> RngIntElt
{ Returns the conductor of the semigroup G }
require IsPlaneCurveSemiGroup(G): "Argument␣must␣be␣a␣plane␣curve␣semigroup";

  E := [i gt 1 select Gcd(Self(i - 1), G[i]) else G[1] : i in [1..#G]];
  N := [E[i - 1] div E[i] : i in [2..#G]]; g := #G - 1; n := G[1];
  return &+[(N[i] - 1) * G[i + 1] : i in [1..g]] - n + 1;
end intrinsic;

intrinsic Conductor(n::RngIntElt, M::[RngIntElt]) -> RngIntElt
{ Returns the conductor of the char. exponents (n, M) }

  return Conductor(SemiGroup(n, M));
end intrinsic;

intrinsic Conductor(s::RngSerPuisElt) -> RngIntElt
{ Returns the conductor of the Puiseux series s }

  return Conductor(SemiGroup(s));
end intrinsic;

intrinsic Conductor(f::RngMPolLocElt) -> RngIntElt
{ Returns the conductor of the irreducible plane curve f }

  return Conductor(SemiGroup(f));
end intrinsic;

forward SemiGroupCoordImpl;

```

```
SemiGroupCoordImpl := function(v, i, G);
  if v eq 0 then return [[0 : i in [1..#G]]]; end if;
  if v lt 0 or i gt #G then return []; end if; CC := [];

  for k in Reverse([0..(v div G[i])]) do
    T := SemiGroupCoordImpl(v - k * G[i], i + 1, G);
    for j in [1..#T] do T[j][i] := k; CC cat:= [T[j]]; end for;
  end for; return CC;
end function;

intrinsic SemiGroupCoord(v::RngIntElt, G::[RngIntElt]) -> []
{ Return the coordinates of an integer v in the numerical semigroup G }

  return SemiGroupCoordImpl(v, 1, G);
end intrinsic;

intrinsic SemiGroup(L::[SeqEnum]) -> []
{ Constructs a semigroup from the semigroup of each characteristic exponent }
require #L ne 0: "List␣must␣be␣non-empty";
require &and[#S eq 2 : S in L]: "Input␣semigroup␣must␣have␣two␣elements";

  P := ProximityMatrix(L[1]); ZZ := Integers();
  for i in [2..#L] do
    Qi := ProximityMatrix(L[i]); N := Ncols(P); Ni := Ncols(Qi);
    P := DiagonalJoin(P, ZeroMatrix(ZZ, Ni - 1, Ni - 1));
    InsertBlock(~P, Qi, N, N);
  end for; return SemiGroup(P);
end intrinsic;
```

BIBLIOGRAPHY

[ACa73] N. A'Campo, *Le nombre de Lefschetz d'une monodromie*, Indagationes Mathematicae **35** (1973), pp. 113–118.

[ACa75] N. A'Campo, *La fonction zêta d'une monodromie*, Commentarii Mathematici Helvetici **50** (1975), pp. 233–248.

[AM73a] S. S. Abhyankar and T. T. Moh, *Newton-Puiseux expansion and generalized Tschirnhausen transformation. I*, Journal für die Reine und Angewandte Mathematik **260** (1973), pp. 47–83.

[AM73b] S. S. Abhyankar and T. T. Moh, *Newton-Puiseux expansion and generalized Tschirnhausen transformation. II*, Journal für die Reine und Angewandte Mathematik **261** (1973), pp. 29–54.

[AAG11] I. Abío, M. Alberich-Carramiñana, and V. González-Alonso, *The ultrametric space of plane branches*, Communications in Algebra **39** (2011), no. 11, pp. 4206–4220.

[Alb04] M. Alberich-Carramiñana, *An algorithm for computing the singularity of the generic germ of a pencil of plane curves*, Communications in Algebra **32** (2004), no. 4, pp. 1637–1646.

[Alb+19] M. Alberich-Carraminana, P. Almirón, G. Blanco, and A. Melle-Hernández, *The minimal Tjurina number of irreducible germs of plane curve singularities*, To appear in Indiana University Mathematics Journal, 2019, arXiv: 1904.02652 [math.AG].

[AÀB19] M. Alberich-Carramiñana, J. Àlvarez Montaner, and G. Blanco, *Effective computation of base points of ideals in two-dimensional local rings*, Journal of Symbolic Computation **92** (2019), pp. 93–109.

[AÀD16] M. Alberich-Carramiñana, J. Àlvarez Montaner, and F. Dachs-Cadefau, *Multiplier ideals in two-dimensional local rings with rational singularities*, Michigan Mathematical Journal **65** (2016), no. 2, pp. 287–320.

[AF07] M. Alberich-Carramiñana and J. Fernández-Sánchez, *Equisingularity classes of birational projections of normal singularities to a plane*, Advances in Mathematics **216** (2007), no. 2, pp. 753–770.

[AF10] M. Alberich-Carramiñana and J. Fernández-Sánchez, *On adjacent complete ideals above a given complete ideal*, Proceedings of the Royal Society of Edinburgh. Section A **140** (2010), no. 2, pp. 225–239.

[AMB20] M. Alberich-Carramiñana, J. Àlvarez Montaner, and G. Blanco, *Monomial generators of complete planar ideals*, Journal of Algebra and its Applications (2020), In press, DOI: 10.1142/S0219498821500328.

[Alm19] P. Almirón, *On the quotient of Milnor and Tjurina numbers for two-dimensional isolated hypersurface singularities*, 2019, arXiv: 1910.12843 [math.AG].

[AB19] P. Almirón and G. Blanco, *A note on a question of Dimca and Greuel*, Comptes Rendus Mathématique. Académie des Sciences **357** (2019), no. 2, pp. 205–208.

[AV85] V. I. Arnold and S. M. Gusein-Zade A. Varchenko, *Singularities of Differentiable Maps*, vol. 1, Monographs in Mathematics, no. 82, Birkhäuser, Boston, MA, 1985.

[AV88] V. I. Arnold and S. M. Gusein-Zade A. Varchenko, *Singularities of Differentiable Maps*, vol. 2, Monographs in Mathematics, no. 83, Birkhäuser, Boston, MA, 1988.

[Art+17a] E. Artal Bartolo, Pi. Cassou-Noguès, I. Luengo, and A. Melle Hernández, *Bernstein polynomial of 2-Puiseux pairs irreducible plane curve singularities*, Methods and Applications of Analysis. **24** (2017), no. 2, pp. 185–214.

[Art+17b] E. Artal Bartolo, Pi. Cassou-Noguès, I. Luengo, and A. Melle Hernández, *Yano's conjecture for two-Puiseux-pair irreducible plane curve singularities*, Publications of the Research Institute for Mathematical Sciences **53** (2017), no. 1, pp. 211–239.

[Art+18] E. Artal Bartolo, Pi. Cassou-Noguès, I. Luengo, and A. Melle Hernández, *On the b-exponents of generic isolated plane curve singularities*, Journal of Singularities **18** (2018), pp. 36–49.

[Ati70] M. F. Atiyah, *Resolution of singularities and division of distributions*, Communications on Pure and Applied Mathematics **23** (1970), no. 2, pp. 145–150.

[BW17] A. Bapat and R. Walters, *The strong topological monodromy conjecture for Weyl hyperplane arrangements*, Mathematical Research Letters **24** (2017), no. 4, pp. 947–954.

[Bar84] D. Barlet, *Contribution effective de la monodromie aux développements asymptotiques*, Annales Scientifiques de l'École Normale Supérieure Quatrième Série **17** (1984), no. 2, pp. 293–315.

[Bar86] D. Barlet, *Le calcul de la forme hermitienne canonique pour* $X^a + Y^b + Z^c = 0$, Séminaire d'analyse P. Lelong – P. Dolbeault – H. Skoda, années 1983/1984, (P. Lelong, P. Dolbeault, and H. Skoda, eds.), Lecture Notes in Mathematics, no. 1198, Springer, Berlin, 1986, pp. 35–46.

[BL10] C. Berkesch and A. Leykin, *Algorithms for Bernstein-Sato polynomials and multiplier ideals*, Proceedings of the 2010 International Symposium on Symbolic and Algebraic Computation (Munich, 2010), Association for Computing Machinery, New York, NY, 2010, pp. 99–106.

[Ber71] J. Bernstein, *Modules over a ring of differential operators. Study of the fundamental solutions of equations with constant coefficients*, Functional Analysis and its Applications **5** (1971), no. 2, pp. 1–16.

[Ber72] J. Bernstein, *The analytic continuation of generalized functions with respect to a parameter*, Functional Analysis and its Applications **6** (1972), no. 4, pp. 26–40.

[BG69] J. Bernstein and S. I. Gel'fand, *Meromorphic continuation of the function* P^λ, Functional Analysis and its Applications **3** (1969), no. 1, pp. 84–85.

[Bjö74] J.-E. Björk, *Dimensions of modules over algebras of differential operators*, Fonctions analytiques de plusieurs variables et analyse complexe, Colloque International du C.N.R.S. no. 208 (Paris, 1972), Agora Mathematica, no. 1, Gauthier-Villars, Paris, 1974, pp. 6–11.

[Bla19a] G. Blanco, *Poles of the complex zeta function of a plane curve*, Advances in Mathematics **350** (2019), no. 9, pp. 396–439.

[Bla19b] G. Blanco, *Yano's conjecture*, 2019, arXiv: 1908.05917 [math.AG].

[Bla] G. Blanco, *Topological roots of the Bernstein-Sato polynomial of a plane curve*, In preparation.

[BD18] G. Blanco and F. Dachs-Cadefau, *Computing multiplier ideals in smooth surfaces*, Extended abstracts February 2016, Positivity and valuations (Barcelona, 2016), (M. Alberich-Carramiñana, C. Galindo, A. Küronya, and J. Roé, eds.), Trends in Mathematics. Research Perspectives CRM Barcelona, no. 9, Birkhäuser, Basel, 2018, pp. 57–63.

[BCP97] W. Bosma, J. Cannon, and C. Playoust, *The Magma algebra system. I. The user language*, Journal of Symbolic Computation **24** (1997), no. 3–4, pp. 235–265.

[Bra28] K. Brauner, *Zur Geometrie der Funktionen zweier komplexer Veräderlicher. II. Das Verhalten der Funktionen in der Umgebung ihrer Verzweigungsstellen*, Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg **6** (1928), no. 1, pp. 1–55.

[BEV05] A. Bravo, S. Encinas, and O. Villamayor, *A simplified proof of desingularization and applications*, Revista Matemática Iberoamericana **21** (2005), no. 2, pp. 349–458.

[BGM92] J. Briançon, F. Geandier, and Ph. Maisonobe, *Déformation d'une singularité isolée d'hypersurface et polynômes de Bernstein*, Bulletin de la Société Mathématique de France **120** (1992), no. 1, pp. 15–49.

[BGM88] J. Briançon, M. Granger, and Ph. Maisonobe, *Le nombre de modules du germe de courbe plane $x^a + y^b = 0$*, Mathematische Annalen **279** (1988), no. 3, pp. 535–551.

[BM96] J. Briançon and P. Maisonobe, *Caractérisation géométrique de l'existence du polynôme de Bernstein relatif*, Algebraic Geometry and Singularities, (A. Campillo and L. Narváez-Macarro, eds.), Progress in Mathematics, no. 134, Birkhäuser, Basel, 1996, pp. 215–236.

[BMT07] J. Briançon, Ph. Maisonobe, and T. Torrelli, *Matrice magique associée à un germe de courbe plane et division par l'idéal jacobien*, Université de Grenoble. Annales de l'Institut Fourier **57** (2007), no. 3, pp. 919–953.

[Bri70] E. Brieskorn, *Die Monodromie der isolierten Singularitäten von Hyperflächen*, Manuscripta Mathematica **2** (1970), pp. 103–161.

[Bud03] N. Budur, *On Hodge spectrum and multiplier ideals*, Mathematische Annalen **327** (2003), no. 2, pp. 257–270.

[BMS06] N. Budur, M. Mustaţă, and M. Saito, *Bernstein-Sato polynomials of arbitrary varieties*, Compositio Mathematica **142** (2006), no. 3, pp. 779–797.

[BS05] N. Budur and M. Saito, *Multiplier ideals, V-filtration, and spectrum*, Journal of Algebraic Geometry **14** (2005), no. 2, pp. 269–282.

[BSY11] N. Budur, M. Saito, and S. Yuzvinsky, *On the local zeta functions and the b-functions of certain hyperplane arrangements.* Journal of the London Mathematical Society. Second Series **84** (2011), no. 3, pp. 631–648.

[Cas98] E. Casas-Alvero, *Filtrations by complete ideals and applications*, Collectanea Mathematica **49** (1998), no. 2–3, pp. 265–272.

[Cas00] E. Casas-Alvero, *Singularities of plane curves*, London Mathematical Society Lecture Note Series, no. 276, Cambridge University Press, Cambridge, 2000.

[Cas87] Pi. Cassou-Noguès, *Étude du comportement du pôlynome de Bernstein lors d'une déformation à μ constant de $X^a + Y^b$ avec $(a, b) = 1$*, Compositio Mathematica **63** (1987), no. 3, pp. 291–313.

[Cas88] Pi. Cassou-Noguès, *Polynôme de Bernstein générique*, Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg **58** (1988), no. 1, pp. 103–124.

[Cle69] C. H. Clemens, *Picard-Lefschetz theorem for families of nonsingular algebraic varieties acquiring ordinary singularities*, Transactions of the American Mathematical Society **136** (1969), pp. 93–108.

[CL55] E. Coddington and N. Levinson, *Theory of ordinary differential equations.* McGraw-Hill, Inc., New York-Toronto-London, 1955.

[Com74] L. Comtet, *Advanced combinatorics. The art of finite and infinite expansions*, D. Reidel Publishing Company, Dordrecht, 1974.

[Cut04] S. D. Cutkosky, *Resolution of singularities*, Graduate Studies in Mathematics, no. 63, American Mathematical Society, Providence, RI, 2004.

[Sing] W. Decker, G.-M. Greuel, G. Pfister, and H. Schönemann, Singular 4.1.1 — *A computer algebra system for polynomial computations*, Available at http://www.singular.uni-kl.de, 2018.

[Del70] P. Deligne, *Equations Différentielles à Points Singuliers Réguliers*, Lecture Notes in Mathematics, no. 163, Springer, Berlin, 1970.

[SGA7-II] P. Deligne and N. Katz, *Séminaire de Géométrie Algébrique du Bois-Marie (1967/69); SGA 7 II*, Lecture Notes in Mathematics, no. 340, Springer, Berlin, 1973.

[DM86] P. Deligne and G. D. Mostow, *Monodromy of hypergeometric functions and nonlattice integral monodromy*, Institut des Hautes Études Scientifiques. Publications Mathématiques **63** (1986), pp. 5–89.

[Del78] C. Delorme, *Sur les modules des singularités des courbes planes*, Bulletin de la Société Mathématique de France **106** (1978), no. 4, pp. 417–446.

[Den87] J. Denef, *On the degree of Igusa's local zeta function.* American Journal of Mathematics **109** (1987), no. 6, pp. 991–1008.

[DG18] A. Dimca and G.-M. Greuel, *On 1-forms on isolated complete intersection curve singularities*, Journal of Singularities **18** (2018), pp. 114–118.

[Ebe65] S. Ebey, *The classification of singular points of of algebraic curves*, Transactions of the American Mathematical Society **118** (1965), pp. 454–471.

[Ein+04] L. Ein, R. Lazarsfeld, K. E. Smith, and D. Varolin, *Jumping coefficients of multiplier ideals*, Duke Mathematical Journal **123** (2004), no. 3, pp. 469–506.

[EN85] D. Eisenbud and W. Neumann, *Three-dimensional link theory and invariants of plane curve singularities*, Annals of Mathematics Studies, no. 110, Princeton University Press, Princeton, N.J., 1985.

[EC15] F. Enriques and O. Chisini, *Lezioni sulla teoria geometrica delle equazioni e delle funzioni algebriche*, N. Zanichelli, Bologna, 1915.

[EV92] H. Esnault and E. Viehweg, *Lectures on Vanishing Theorems*, DMV Seminar, no. 20, Birkhäuser Verlag, Basel, 1992.

[Fer03] J. Fernández-Sánchez, *On sandwiched singularities and complete ideals*, Journal of Pure and Applied Algebra **185** (2003), no. 1–3, pp. 165–175.

[Fer05] J. Fernández-Sánchez, *Nash families of smooth arcs on a sandwiched singularity*, Mathematical Proceedings of the Cambridge Philosophical Society **138** (2005), no. 1, pp. 117–128.

[Fer06] J. Fernández-Sánchez, *Some results relating to adjacent ideals in dimension two*, Journal of Pure and Applied Algebra **207** (2006), no. 2, pp. 387–395.

[Fer+77] J. Ferrer, V. Navarro, F. Panyella, and F. Puerta, *El polinomio de Bernstein y la monodromia, un curso de L. Puig*, Seminario de Matemáticas I, no. 10, Edited by Université Paris VII, 1977.

[Frü07] A. Frühbis-Krüger, *Computational aspects of singularities*, Singularities in geometry and topology (Trieste, 2005), (J.-P. Brasselet, J. Damon, Lê Dũng Tráng, and M. Oka, eds.), World Scientific Publising, Hackensack, NJ, 2007, pp. 253–327.

[Gea91] F. Geandier, *Déformations à nombre de Milnor constant: quelques résultats sur les polynômes de Bernstein*, Compositio Mathematica **77** (1991), no. 2, pp. 131–163.

[Gel57] I. M. Gel'fand, *Some aspects of functional analysis and algebra*, Proceedings of the International Congress of Mathematicians (Amsterdam, 1954), (J. C. H. Gerretsen and J. de Groot, eds.), vol. 1, Erven P. Noordhoff N. V., Groningen and North-Holland Pub. Co., Amsterdam, 1957, pp. 253–276.

[GS64] I. M. Gel'fand and G. E. Shilov, *Generalized functions*, vol. 1, Academic Press, New York, NY, 1964.

[Gen16] Y. Genzmer, *Dimension of the moduli space of a curve in the complex plane*, 2016, arXiv: `1610.05998 [math.DS]`.

[GH20] Y. Genzmer and M. E. Hernandes, *On the Saito basis and the Tjurina number for plane branches*, Transactions of the American Mathematical Society (2020), In press, DOI: `10.1090/tran/8019`.

[GR15] I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products*, (D. Zwillinger and V. Moll, eds.), 8th ed., Elsevier/Academic Press, Amsterdam, 2015.

[Gra58] H. Grauert, *Analytische Faserungen über holomorph-vollständigen Räumen*, Mathematische Annalen **135** (1958), pp. 263–273.

[M2] D. R. Grayson and M. E. Stillman, Macaulay2 `1.14` — *A software system for research in algebraic geometry*, Available at `http://www.math.uiuc.edu/Macaulay2/`, 2019.

[GLS07] G.-M. Greuel, C. Lossen, and E. Shustin, *Introduction to Singularities and Deformations*, Springer Monographs in Mathematics, Springer, Berlin, 2007.

[SGA7-I] A. Grothendieck, *Séminaire de Géométrie Algébrique du Bois-Marie (1967/69); SGA 7 I*, Lecture Notes in Mathematics, no. 288, Springer, Berlin, 1972.

[Har77] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, no. 52, Springer, Berlin, 1977.

[Hir64a] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero: I*, Annals of Mathematics **79** (1964), no. 1, pp. 109–203.

[Hir64b] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero: II*, Annals of Mathematics **79** (1964), no. 2, pp. 205–326.

[HA55] W. V. D. Hodge and M. F. Atiyah, *Integrals of the second kind on an algebraic variety*, Annals of Mathematics **62** (1955), no. 1, pp. 56–91.

[How01] J. A. Howald, *Multiplier ideals of monomial ideals*, Transactions of the American Mathematical Society **353** (2001), no. 7, pp. 2665–2671.

[Igu74] J.-i. Igusa, *Complex powers and asymptotic expansions. I. Functions of certain types*, Journal für die Reine und Angewandte Mathematik **268-269** (1974), pp. 110–130.

[Igu75] J.-i. Igusa, *Complex powers and asymptotic expansions. II*, Journal für die Reine und Angewandte Mathematik **278-279** (1975), pp. 307–321.

[JY07] M. Mustaţă J. Howald and C. Yuen, *On Igusa zeta functions of monomial ideals*, Proceedings of the American Mathematical Society **135** (2007), no. 11, pp. 3425–3433.

[Jär11] T. Järvilehto, *Jumping numbers of a simple complete ideal in a two-dimensional regular local ring*, Memoirs of the American Mathematical Society **214** (2011), no. 1009.

[JP00] T. de Jong and G. Pfister, *Local analytic geometry. Basic theory and applications.* Advanced Lectures in Mathematics, Friedr. Vieweg & Sohn, Braunschweig, 2000.

[Kas76] M. Kashiwara, *B-functions and holonomic systems*, Inventiones Mathematicae **38** (1976), no. 1, pp. 33–53.

[Kat81] M. Kato, *The b-function of a μ-constant deformation of $x^7 + y^5$*, Bulletin of the College of Science. University of the Ryukyus **32** (1981), pp. 5–10.

[Kat82] M. Kato, *The b-function of a μ-constant deformation of $x^9 + y^4$*, Bulletin of the College of Science. University of the Ryukyus **33** (1982), pp. 5–8.

[Kol97] J. Kollár, *Singularities of pairs*, Algebraic Geometry (Santa Cruz, 1995), (J. Kollár, R. Lazarsfeld, and D. R. Morrison, eds.), part 2, Proceedings of Symposia in Pure Mathematics, no. 62, American Mathematical Society, Providence, RI, 1997, pp. 221–287.

[Kul98] V. S. Kulikov, *Mixed Hodge Structures and Singularities*, Cambridge Tracts in Mathematics, no. 132, Cambridge University Press, Cambridge, 1998.

[Laz04] R. Lazarsfeld, *Positivity in algebraic geometry. II. Positivity for vector bundles, and multiplier ideals*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, no. 49, Springer, Berlin, 2004.

[Ler59] J. Leray, *Le calcul différentiel et intégral sur une variété analytique complexe*, Bulletin de la Société Mathématique de France **87** (1959), pp. 81–180.

[Lib83] A. Libgober, *Alexander invariants of plane algebraic curves*, Singularities (Arcata, California, 1981), (P. Orlik, ed.), part 1, Proceedings of Symposia in Pure Mathematics, no. 40, American Mathematical Society, Providence, RI, 1983, pp. 135–143.

[Lic85] B. Lichtin, *Some algebro-geometric formulae for poles of* $|f(x,y)|^s$, American Journal of Mathematics **107** (1985), no. 1, pp. 139–162.

[Lic86] B. Lichtin, *An upper semicontinuity theorem for some leading poles of* $|f|^{2s}$, Complex analytic singularities, Advanced Studies in Pure Mathematics, no. 8, North-Holland, Amsterdam, 1986, pp. 241–272.

[Lic89] B. Lichtin, *Poles of* $|f(z,w)|^{2s}$ *and roots of the B-function*, Arkiv för Matematik **27** (1989), no. 1-2, pp. 283–304.

[Lip93] J. Lipman, *Adjoints and polars of simple complete ideals in two-dimensional regular local rings*, Bulletin de la Société Mathématique de Belgique. Série A **45** (1993), no. 1-2, pp. 223–244.

[Loe85] F. Loeser, *Quelques conséquences locales de la théorie de Hodge*, Université de Grenoble. Annales de l'Institut Fourier **35** (1985), no. 1, pp. 75–92.

[Loe88] F. Loeser, *Fonctions d'Igusa p-adiques et polynômes de Bernstein*, American Journal of Mathematics **110** (1988), no. 1, pp. 1–21.

[Loe90] F. Loeser, *Fonctions d'Igusa p-adiques, polynômes de Bernstein, et polyèdres de Newton*, Journal für die Reine und Angewandte Mathematik **412** (1990), pp. 75–96.

[LV90] F. Loeser and M. Vaquié, *Le polynôme d'Alexander d'une courbe plane projective*, Topology. An International Journal of Mathematics **2** (1990), pp. 163–173.

[Lőr+17] A. C. Lőrincz, C. Raicu, U. Walther, and J. Weyman, *Bernstein-Sato polynomials for maximal minors and sub-maximal Pfaffians*, Advances in Mathematics **307** (2017), pp. 224–252.

[LP90] I. Luengo and G. Pfister, *Normal forms and moduli spaces of curve singularities with semigroup* $(2p, 2q, 2pq + d)$, Compositio Mathematica **76** (1990), no. 1-2, pp. 247–264.

[Mal74a] B. Malgrange, *Intégrales asymptotiques et monodromie*, Annales Scientifiques de l'École Normale Supérieure Quatrième Série **7** (1974), no. 3, pp. 405–430.

[Mal74b] B. Malgrange, *Sur les polynômes de I. N. Bernstein*, Russian Mathematical Surveys **29** (1974), no. 4, pp. 81–88.

[Mal75] B. Malgrange, *Le polynôme de Bernstein d'une singularité isolée*, Lecture Notes in Mathematics **4** (1975), pp. 98–119.

[Mal83] B. Malgrange, *Polynôme de Bernstein-Sato et cohomologie évanescente*, Analysis and topology on singular spaces, II, III (Luminy, 1981), vol. 101-102, Astérisque, Société Mathématique de France, Paris, 1983, pp. 243–267.

[Mat91] J. F. Mattei, *Modules de feuilletages holomorphes singuliers: I équisingularité*, Inventiones Mathematicae **103** (1991), no. 2, pp. 297–325.

[MN91] Z. Mebkhout and L. Narváez-Macarro, *La théorie du polynôme de Bernstein-Sato pour les algèbres de Tate et de Dwork-Monsky-Washnitzer*, Annales Scientifiques de l'École Normale Supérieure Quatrième Série **24** (1991), no. 2, pp. 227–256.

[Mil68] J. Milnor, *Singular Points of Complex Hypersurfaces*, Annals of Mathematics Studies, no. 61, Princeton University Press, Princeton, N.J., 1968.

[Mus12] M. Mustaţă, *IMPANGA lecture notes on log canonical thresholds*, Contributions to algebraic geometry, Impanga Lecture Notes (Będlewo, 2010), (P. Pragacz, ed.), EMS Series of Congress Reports, Notes by T. Szemberg, European Mathematical Society, Zürich, 2012, pp. 407–442.

[Mus19] M. Mustaţă, *Bernstein-Sato polynomials for general ideals vs. principal ideals*, 2019, arXiv: 1906.03086 [math.AG].

[Nad90] A. M. Nadel, *Multiplier ideal sheaves and Kähler-Einstein metrics of positive scalar curvature*, Annals of Mathematics **132** (1990), no. 3, pp. 549–596.

[Nai09] D. Naie, *Jumping numbers of a unibranch curve on a smooth surface*, Manuscripta Mathematica **128** (2009), no. 1, pp. 33–49.

[Neu83] W. D. Neumann, *Invariants of plane curve singularities*, Knots, braids and singularities (Plans-sur-Bex, 1982), Monographies de L'Enseignement Mathématique, no. 31, Enseignement Mathématique, Geneva, 1983, pp. 223–232.

[Asir] M. Noro and T. Takeshima, *Risa/Asir – a computer algebra system*, ISSAC'92 – Papers from the International Symposium on Symbolic and Algebraic Computation (Berkeley, 1992), (P. S. Wang, ed.), Association for Computing Machinery, New York, 1992, pp. 387–396.

[Oak97] T. Oaku, *An algorithm of computing b-functions*, Duke Mathematical Journal **87** (1997), no. 1, pp. 115–132.

[Oka96] M. Oka, *Geometry of plane curves via toroidal resolution*, Algebraic Geometry and Singularities, (A. Campillo and L. Narváez-Macarro, eds.), Progress in Mathematics, no. 134, Birkhäuser, Basel, 1996, pp. 215–236.

[Per97] R. Peraire, *Tjurina number of a generic irreducible curve singularity*, Journal of Algebra **196** (1997), no. 1, pp. 114–157.

[Pło86] A. Płoski, *Remarque sur la multiplicité d'intersection des branches planes*, Bulletin of the Polish Academy of Sciences **33** (1986), no. 11–12, pp. 601–605.

[Rha54] G. de Rham, *Sur la division de formes et de courants par une forme par une forme linéaire*, Commentarii Mathematici Helvetici **28** (1954), pp. 346–352.

[Sab87a] C. Sabbah, *Proximité évanescente. I. La structure polaire d'un $\mathcal{D}$-module.* Compositio Mathematica **62** (1987), no. 3, pp. 283–328.

[Sab87b] C. Sabbah, *Proximité évanescente. II. Équations fonctionnelles pour plusieurs fonctions analytiques.* Compositio Mathematica **64** (1987), no. 2, pp. 213–241.

[Sai94] M. Saito, *On microlocal b-function*, Bulletin de la Société Mathématique de France **122** (1994), pp. 163–184.

[Sai00] M. Saito, *Exponents of an irreducible plane curve singularity*, 2000, arXiv: math/0009133 [math.AG].

[SS90] M. Sato and T. Shintani, *Theory of prehomogeneous vector spaces (algebraic part) – the English translation of Sato's lecture from Shintani's note*, Nagoya Mathematical Journal **120** (1990), pp. 1–34.

[Seb70] M. Sebastiani, *Preuve d'une conjecture de Brieskorn*, Manuscripta Mathematica **2** (1970), pp. 301–308.

[Shi11] T. Shibuta, *Algorithms for computing multiplier ideals*, Journal of Pure and Applied Algebra **215** (2011), no. 12, pp. 2829–2842.

[Sho93] V. V. Shokurov, *Three-dimensional log perestroikas. With an appendix in English by Yujiro Kawamata*, Rossiĭskaya Akademiya Nauk. Izvestiya. Seriya Matematicheskaya **56** (1993), no. 1, pp. 95–202.

[ST07] K. E. Smith and H. M. Thompson, *Irrelevant exceptional divisors for curves on a smooth surface*, Algebra, geometry and their interactions, (A. Corso, J. Migliore, and C. Polini, eds.), Contemporary Mathematics, no. 448, American Mathematical Society, Providence, RI, 2007, pp. 245–254.

[Spa81] E. H. Spanier, *Algebraic topology*, Springer, Berlin, 1981.

[Ste77] J. H. M. Steenbrink, *Mixed Hodge structure on the vanishing cohomology*, Real and complex singularities (Oslo, 1976), Proc. Ninth Nordic Summer School/NAVF Sympos. Math. Sijthoff and Noordhoff, Alphen aan den Rijn, 1977, pp. 525–563.

[Ste89] J. H. M. Steenbrink, *The spectrum of hypersurface singularities*, Actes du Colloque de Théorie de Hodge (Luminy, 1987), Astérisque, no. 179-180, Société Mathématique de France, Paris, 1989, pp. 163–184.

[Tei86] B. Teissier, *Appendix*, in [Zar86], 1986.

[Trá72] Lê Dũng Tráng, *Sur les noeuds algébriques*, Compositio Mathematica **25** (1972), no. 3, pp. 281–321.

[TR76] Lê Dũng Tráng and C. P. Ramanujam, *The invariance of Milnor's number implies the invariance of the topological type*, American Journal of Mathematics **98** (1976), no. 1, pp. 67–78.

[Tuc10] K. Tucker, *Jumping numbers on algebraic surfaces with rational singularities*, Transactions of the American Mathematical Society **362** (2010), no. 6, pp. 3223–3241.

[Var80] A. N. Varchenko, *Gauss-Manin connection of isolated singular point and Bernstein polynomial*, Bulletin des Sciences Mathématiques. 2e Série **104** (1980), pp. 205–223.

[Var82] A. N. Varchenko, *Asymptotic Hodge structure in the vanishing cohomology*, Mathematics of the USSR-Izvestiya **18** (1982), no. 3, pp. 469–512.

[Vil89] O. Villamayor, *Constructiveness of Hironaka's resolution*, Annales Scientifiques de l'École Normale Supérieure Quatrième Série **22** (1989), no. 1, pp. 1–32.

[Wal84] C. T. C. Wall, *Notes on the classification of singularities*, Proceedings of the London Mathematical Society **48** (1984), no. 3, pp. 461–513.

[Wal03] C. T. C. Wall, *Chains on the Eggers tree and polar curves*, Revista Matemática Iberoamericana **19** (2003), no. 2, pp. 745–754.

[Wal04] C. T. C. Wall, *Singular points of plane curves*, London Mathematical Society Student Texts, no. 63, Cambridge University Press, Cambridge, 2004.

[Wal17] U. Walther, *The Jacobian module, the Milnor fiber, and the D-module generated by f^s*, Inventiones Mathematicae **207** (2017), no. 3, pp. 1239–1287.

[Yan78] T. Yano, *On the theory of b-functions*, Publications of the Research Institute for Mathematical Sciences **14** (1978), no. 1, pp. 111–202.

[Yan82] T. Yano, *Exponents of singularities of plane irreducible curves*, Sci. Rep. Saitama Univ. Ser. **10** (1982), no. 2, pp. 21–28.

[Yun76] D. Y. Y. Yun, *On square-free decomposition algorithms*, Proceedings of the Third ACM Symposium on Symbolic and Algebraic Computation (Yorktown Heights, 1976), (R. D. Jenks, ed.), Association for Computing Machinery, New York, NY, 1976, pp. 26–35.

[Zar32] O. Zariski, *On the topology of algebroid singularities*, American Journal of Mathematics **54** (1932), no. 3, pp. 453–465.

[Zar38] O. Zariski, *Polynomial ideals defined by infinitely near points*, American Journal of Mathematics **60** (1938), no. 1, pp. 151–204.

[Zar86] O. Zariski, *Le problème des modules pour les branches planes*, Hermann, Publishers in Arts and Science, Paris, 1986.

INDEX

Colophon

This thesis was typeset in LaTeX using the typographical look-and-feel `classicthesis`. All the graphics in this thesis are generated using `PGF/TikZ`. The bibliography is typeset using `biblatex`.