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Substructural Logics and Pragmatic Enrichment

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SUBSTRUCTURAL LOGICS AND PRAGMATIC ENRICHMENT

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Abstract

In this dissertation, we argue for a Pragmatic Logical Pluralism, a pluralist thesis about logic which endorses Classical, Relevant, Linear, and Ordered logic. We justify that the formal languages of these four logics are legitimate codifications of the logical vocabulary and capture legitimate senses of logical consequence. This will be justified given a particular interpretation of the four formal languages: logical consequence and conditional, disjunction, and conjunction of the four different logics codify different and legitimate senses of 'follows from', 'if...then', 'or' and 'and' which diverge in their different pragmatic enrichments. The dissertation is twofold. First, we will explore the effect that the lack of structural rules has on logical connectives, in four substructural logics, and its connection with certain pragmatic enrichments. Second, we will defend a pluralist thesis according to which pragmatics has an important role for capturing the inferential role of logical vocabulary, both of the notions of 'follows from' and the logical constants, although classical logic preserves truth and captures their literal meaning. In sum, we defend a version of logical pluralism based on the plurality of legitimate translations from natural language to formal languages, arguing that more than one translation is legitimate for logical vocabulary, which makes it possible to adopt more than one logic.

Resum

En aquesta tesi presentem el Pluralisme Lògic Pragmàtic, una tesi pluralista sobre la lògica que accepta les lògiques Clàssica, Rellevant, Lineal i Ordenada. Justifiquem que els llenguatges formals d'aquestes quatre lògiques són codificacions legítimes del vocabulari lògic i capturen sentits legítims de la conseqüència lògica. Això es justificarà donant una interpretació particular dels quatre llenguatges formals: la conseqüència lògica i el condicional, la disjunció i la conjunció de les quatre lògiques acceptadas codifiquen diferents i legítims sentits de 'si...llavors', 'o' i 'i', que es distingeixen pels diferents enriquiments pragmàtics que codifiquen. La tesi té dues vessants. Primer, explorem l'efecte que la falta de regles estructurals té en les connectives lògiques de les quatre lògiques presentades, i la seva connexió amb certs enriquiments pragmàtics. Segon, defensem una visió pluralista segons la qual la pragmàtica juga un rol important a l'hora de capturar el rol inferencial del vocabulari lògic, tant per la noció de conseqüència lògica com per les connectives, tot i que la lògica clàssica preserva la veritat i captura el seu significat literal. En resum, defensem una versió del pluralisme lògic basat en la pluralitat de traduccions legítimes del llenguatge natural al llenguatge formal, argumentant que més d'una traducció és legítima pel vocabulari lògic, la qual cosa ens permet adoptar més d'una lògica.

Publications

Some ideas presented here have been published in the following papers:

- Terrés Villalonga, Pilar (2019). From Natural to Formal Language: A Case for Logical Pluralism. *Topoi* 38 (2):333-345.
- Terrés Villalonga, Pilar (forthcoming). Substructural logics, pragmatic enrichment, and the inferential role of logical constants. *Inquiry: An Interdisciplinary Journal of Philosophy*:1-27.

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I have a junior member of Logos since 2012, coordinated first by Genoveva Martí and later by Sven Rosenkranz. In this group I have found the perfect environment for elaborating this thesis, and for this reason I am indebted to all the members of the group that I have had the chance to meet. Particularly important to me have been the people participating in the different reading groups which I have attended during these years, and in which I could present different parts of the dissertation: the Philosophy of Logic reading group, coordinated by José Martínez, and the Graduate Reading Group. In both regular meetings I could share my ideas with actual or formers members of Logos, visiting students and professors, and other researchers, including Mar Alloza, Carlos Benito, Joan Bertran, Bartłomiej Czajka, Laura Delgado, Johannes Findl, Johan Gebo, John Horden, Carlos Jaén, Nasim Mahoozi, Sergi Oms, Claudia Picazo, Lucía Prieto, David Rey, Gonçalo Santos, Sven Rosenkranz, Javier Suárez, Lisa Vogt, Romina Zuppone, and Elia Zardini. I could also benefit from fruitful discussions about different parts of this dissertation with Genoveva Martí and Josep Macià, whose expertise on philosophy of language improved the core of this thesis. With Sergi Oms and José Martínez I read some technical papers and books on substructural logics which clearly had an impact on the pluralist thesis that is defended in this dissertation. I also met in Logos Elia Zardini with whom I had extremely fruitful conversations about both technical and philosophical consequences of my views, and in different stages of development of the research, which allowed me to improve the work. And I had innumerable conversations during the first years of this research with Claudia Picazo, whose insightful perspective on language had a great impact on my own views.

In 2014 and 2015 I participated in an inter-university Reading Group on Logical Pluralism coordinated by Concha Martínez Vidal, and in which I had the chance to meet and discuss a great amount of bibliography with Giuseppe Spolaore, Matteo Plebani, Concha Martínez, José Martínez and Pierdaniele Giaretta. Thanks to all of them too for all the fruitful discussion and shared ideas on the logical pluralism debate.

During my PhD, I completed two research stays. First, in 2015 I spend one month at the University of Cagliari, working under the supervision of Francesco Paoli. I want to thank him for his dedication in reading and discussing important parts of this dissertation, when it was underdeveloped, and which helped me to find better arguments for my own intuitions. Some of the main claims in this dissertation are the result of answering some insightful observations which he made about my initial ideas on logical pluralism. In Cagliari I also had the chance to present my work, and I received interesting comments from Francesca Ervas.

The second stay abroad was one year later, in 2016, this time visiting the University of Bergen during four months, and working under the supervision of Ole T. Hjortland. I want to thank him for his support and dedication in supervising my work during this research stay, and for his valuable comments, pieces of advice and encouragement that he gave me during that period and in other occasions in which I had the chance to meet him. His guidance gave me a necessary perspective on the whole project when it was yet underdeveloped, by suggesting different directions in which my ideas could be expanded, which is reflected on the current form of the thesis.

I had the chance to present parts of this work in several universities and other institutions, where I could benefit from discussions and different perspectives on the topic of my research: in 2014 at the summer School on Mathematical Philosophy for Female Students (MCMP); in 2015 at the VIII Conference of the Spanish Society for Logic, Methodology and Philosophy of Science (University of Barcelona), at the Congrés Català de Filosofia (University of the Balearic Islands), at the 15th Congress of Logic, Methodology and Philosophy of Science (University of Helsinki), and at the Postgraduate Workshop on Knowledge, Reasoning, and Discourse (University of the Basque Country); in 2016 at the Pluralism Workshop 2: Pluralism and Normativity (University of Bologna), at the Pluralism

Week, organised by the Pluralism Global Research Network (Yonsei International Campus, Incheon), at the Third Pluralism workshop (University of Bologna); in 2017 at the Pluralism Workshop: Methodology, Motivations, and Interconnections (University of Bonn); in 2018 at the LanCog Seminar (University of Lisbon), at the Substructural and Pluralism Workshop (Universitat de Barcelona); in 2019 at Pavia Workshop (IUSS Pavia), and at the Context-Sensitivity and Logical consequence workshop (University of Bonn), and I also had a dialogue with the Buenos Aires Logic Group (University of Buenos Aires).

The environment in each one of these events benefited the research, and the insights received at the different conferences are reflected in the final form of this thesis. I want to thank Filippo Ferrari, Sebastiano Moruzzi, Nikolaj Jang Lee Linding Pedersen, Andrea Sereni, Erik Stei, and Elia Zardini for inviting me to participate in some of the mentioned workshops. My experience of this research period has been extremely rewarding thanks to these opportunities, and the development of this dissertation has improved in each one of these events. This is because the audiences at the different workshops provided insightful ideas which are reflected in the final form of this thesis. For this reason I want to thank also Eduardo Barrio, Elke Brendel, Colin Caret, Pablo Cobreros, Bogdan Dicher, Matti Eklund, William Gamaster, Patrick Greenough, Nathan Kellen, Teresa Kouri-Kissel, Jiwon Kim, Ben Martin, Kris McDowell, Francesco Orilia, Ole T. Hjortland, Graham Priest, Lucas Rosenblatt, Greg Restall, Gillian Russell, Gil Sagi, Maria Paola Sforza Fogliani, Paul Smith, Sindre Søderstrøm, Damian Szmuc , Elena Tassoni, Jessica Wilson, Cory Wright, Crispin Wright, Jeremy Wyatt, Andy Yu, Luca Zanetti, Luca Zanetti and Thomas Ede Zimmermann.

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Finally, I want to thank my mother Agustina for understanding how important this project was to me, and for her immense support and illusion for seeing this thesis finished, providing me with the necessary help whenever I needed it; thanks also my brother Miguel, for always encouraging me to find a work-life balance, which has not always been easy. Thanks to mi grandmother and to all my family, for their support, affection, and interest on this project. I dedicate this thesis to my father Gabriel, who has been deeply missed during all this time. I cannot imagine how much more enjoyable this process would have been with his encouragement and support, but I know that his love and inspiration have guided me during all these years.

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A la memoria de mi padre, Gabriel Terrés

Contents

PART I

Logical Pluralism and Pragmatic Enrichment

CHAPTER 1

Introduction

Whenever one enters into the debate on logical pluralism, one feels the need to answer a multitude of questions about the nature of logic before entering into the debate about pluralism itself. Haack [1978] captures many of them:

> What does it mean to say that an argument is valid? that one statement follows from another? that a statement is logically true? Is validity to be explained as relative to some formal system? Or is there an extra-systematic idea that formal system aims to represent? What has being valid got to do with being a good argument? How do formal logical systems help one to assess informal arguments? How like is 'and' and '$\wedge$', for instance, and what should one think of 'p' and 'q' as standing for? Is there one correct formal logic? and what might 'correct' mean here? How does one recognise a valid argument or a logical truth? Which formal systems count as logics, and why? [Haack, 1978, p. 1]

After some reflection, one might try to narrow down the debate, focus on some particular disagreement between two or more logics, and try to figure out how, if possible, it can be solved. In other words, there are two ways of entering into the debate on logical pluralism: it can be a top-down investigation (from the abstract questions about the nature of logic to the particular reconciliation of particular disagreements) or a bottom-up investigation (from the diagnostic of the dispute between two or more specific logics to general facts about logic). For the development of the present dissertation, we found the latter method much more fruitful than the former.

The Problems that Logical Pluralism solves

'Logical Pluralism' is the thesis that there is more than one correct logic, and although the topic is not new (Carnap [1937] defended a pluralist view about logic), it has recently generated a lot of interest, mainly after the work of Beall and Restall in [2000], [2001] [2006]. Logical pluralism responds to the increasing need to deal with the multitude of logics that one finds in the literature. Hence, pluralists have tried to solve the rivalry between different logics that one may adopt and endorse at the same time. That is, the pluralist tries to make compatible some divergences of the following kind:

- ◦ Δ follows from Γ,
- ◦ Δ does not follow from Γ.

One reason for making compatible these two claims for some Γ and Δ is that both derivations, in a loose sense, seem correct, or that both have a reasonable justification for being right. Let us discuss some of these cases as an illustration and a motivation for the version of logical pluralism that will be defended in this dissertation.

Being 'too logical' and ignoring pragmatic enrichment

Logic, in its normative dimension, should guide our reasoning[1]. We consider that someone is irrational if she fails to follow basic inference rules. Consider someone who believes 1:

1. If it starts raining then the streets will get wet, and it will rain today. However, the streets will not get wet.

Given that the following inference is valid, we consider that the speaker is irrational, as she *should* believe the conclusion because the premises entail it *logically*:

1c. $p \supset q, p \Rightarrow q$ [2]

However, sometimes we blame people for being *too logical* in their arguments, that is, ignoring certain subtleties of the language that make some arguments not eligible to be analysed with classical logic. Consider the following examples:

[1]There is a debate about this claim. For a defence of the claim see MacFarlane [2004a] and Steinberger [2017a]. For a rejection see Harman [1986]. It is outside the scope of this thesis to defend the claim so we will assume a normative dimension of logic.

[2]We will use $\Rightarrow$ to refer to logical entailment whenever it is not necessary to specify whether it is proof-theoretic ($\vdash$) or model-theoretic ($\vDash$).

2. The sun will come up tomorrow, hence if it doesn't, it won't matter. [Jackson, 1979]
3. If you pay five euros, and you pay five euros, you get a pack of Camels, and you get a pack of Marlboros. Hence, if you pay five euros, you get a pack of Camels, and you get a pack of Marlboros. [Girard, 1995]
4. If the old king dies of a heart attack and a Republic is declared, Tom will be content. A Republic will be declared tomorrow. Hence, if the old king dies tomorrow, Tom will be content. [Carston, 2002]

The arguments are classically valid:

2c. $p \Rightarrow \neg p \supset q$

3c. $(p \wedge p) \supset (q \wedge r) \Rightarrow p \supset (q \wedge r)$

4c. $(p \wedge q) \supset r, q \Rightarrow p \supset r$

However, in the three cases, there seems to be a mismatch between natural language and the classical formalisation, which makes the antecedent true but the conclusion unacceptable. In 2, the conditional in the consequent expresses a connection which seems false, although the premise is acceptable. In 3, there is a waste of the 5 euros when one buys either of the two items; one can have either of the two options, but one cannot have both things together. To understand why 4 has an unacceptable conclusion, consider the following situation: Tom wants a Republic to be declared; however, he also appreciates the old king. Now that the king is old, he wishes that a Republic is announced the day that the king dies. So Tom will be content if the two facts happen in a specific order, and in no case, he does wish that the declaration of a Republic happen before the natural death of the king.

What these examples illustrate is that classical logic is not always a correct guide for reasoning. In the three cases, one would blame the speaker for using logic too rigidly, without considering the nuances in natural language that cannot be captured with classical vocabulary. That is, logical connectives in classical logic not only diverge from our actual use of the connectives in natural language but also their inferential role seems to be different from the inferential role that we attribute to them. Hence, if logic has a normative dimension, this second feature should be addressed. In other words, classical logic does not always seem to coincide with what Frege argued that logic should be:

> [T]he most general laws, which prescribe universally the way in which one ought to think if one is to think at all. [Frege, 1968]

From natural to formal language: a pluralist perspective

Given the counterexamples above, one might have mixed intuitions. On the one hand, there is a formal language which can easily explain why the arguments are valid; on the other hand, given that they are, at the very least, misleading, we might also endorse a deviant logic with a language that is capable of invalidating them. We will present a pluralist thesis which aims to resolve such mixed intuitions.

A first approximation to logical pluralism is the claim that there is more than one correct logic, in the following loose sense (which will be specified later) described by Haack [1978],

> [A] logical system is correct if the formal arguments which are valid in that system correspond to informal arguments which are valid in the extra-systematic sense, and the wffs which are logically true in the system correspond to statements which are logically true in the extra-systematic sense. The monist holds that there is a unique logical system which is correct in this sense, the pluralist that there are several. [Haack, 1978, p. 222]

If there is some logical system that can explain the invalidity of 2-4, such a logical system can be considered eligible to be correct, and it can coexist with other systems, such as classical logic, which can also explain why the arguments are valid.

Substructural logics, as we will see, invalidate many of the counterintuitive uses of logical constants, and in particular, some of they refute the previous inferences 2-4. Paoli [2002] illustrates with natural language examples the reasons for rejecting structural rules and endorsing substructural logics. He also suggests that one can explain the mismatch between classical logic and natural language pragmatically [Paoli, 2002, p. 17]. However, we seem to be in front of a dichotomy: either we accept that some (or all) the counterexamples to classical logic show that this logic does not correctly capture the semantics of logical constants, nor their inferential role (so we must endorse a logic that does so, as a substructural logic), or we are capable of explaining the counterexamples pragmatically, which reestablishes the validity of classical logic.

We suggest a pluralist thesis, to which we refer to as *Pragmatic Logical Pluralism*, which avoids this dichotomy and endorses classical and some substructural logics at the same time. In particular, we will endorse four propositional logics: classical logic (as presented in [Gentzen, 1964]), relevant logic (as presented in [Paoli, 2002] and [Read, 1988]), linear logic (as presented in [Paoli, 2002] and [Girard, 1987]), and ordered logic (as presented in [Abrusci, 1991]). This pluralism is possible by following a third option regarding the dichotomy above: we agree with the thesis that substructural logics can be explained pragmatically, and that classical logic captures the literal meaning of the logical constants. However, we also argue that certain pragmatic enrichments of logical vocabulary should be considered for encoding legitimate notions of 'follows from' and legitimate formalisations of logical connectives.

In more detail, the pluralist thesis presented here is based on a plurality of formalisations of logical connectives based on the pragmatic enrichment[3] of the logical vocabulary that specific logics codify: we will argue that the semantics do not entirely determine the inferential role and the truth conditions of logical vocabulary, but some pragmatic contributions also affect its truth conditions and its inferential role. Hence, there are two or more readings for the premises and conclusion of the above inferences, and this plurality corresponds to the legitimacy of the plurality of formal languages that capture them. In other words, the previously presented intuition of 'being too logical' is explained by the mismatch between semantics and pragmatics.

In sum, the dissertation is twofold. First, we will explore the effect that the lack of structural rules has on logical connectives, in four substructural logics, and its connection with certain pragmatic enrichments. Second, we will defend a pluralist thesis according to which pragmatics have an important role in capturing the inferential role of logical vocabulary, both of the notions of 'follows from' and the logical constants, although classical logic preserves truth and captures their literal meaning.

Overview of the thesis

The work is divided into three main parts. The first part is devoted to situate Pragmatic Logical Pluralism in the contemporary debate on logical pluralism and to present the main ingredients for this version of pluralism. In Chapter 2, we will introduce the debate on logical pluralism. Following [Priest, 2006] and [Cook, 2010] we will distinguish levels from versions of logical pluralism: there are different levels where pluralism can emerge, but the interesting versions need to display a plurality of genuine consequence relations. Second, we will offer a new presentation of the main versions of logical pluralism in the literature and argue for the need of Pragmatic Logical Pluralism as a response to the problems that other proposals have. In Chapter 3, we will introduce the basic phenomenon which underlies the present version of logical pluralism: the theory of pragmatic enrichment. We will present the view on the distinction between semantics and pragmatics that can explain how the different logics presented can be endorsed, following the work of

[3]We borrow the term from Recanati [2004], although the present explanation will diverge from his theory. However, we think that it captures a general phenomenon independently of the particular theoretical framework in which it is embedded. We will see the details in Chapter 2.

Recanati [2004], Carston [2002], and Bach [1994]. In particular, we will see that logical vocabulary can be pragmatically enriched by observing or violating certain Gricean maxims (note that although we use Gricean maxims to explain the pragmatic enrichment, this pragmatic content will be distinguished from the Gricean implicatures). This implies that it is possible to identify the literal meaning of the logical connectives with their classical counterparts, and at the same time defend that logical connectives can contribute to what is said by diverging from this classical codification.

The second part is the central one, in which the legitimacy of different substructural logics will be defended, using the theory presented in Chapter 3. We will explain in two separate chapters, with a similar structure, how classical, relevant, linear, and ordered logic capture legitimate senses of logical vocabulary to encode logical consequence. Chapter 4 argues for the legitimacy of both Classical Logic and Relevant Logic. First, we will present the Gentzen sequent calculus for classical logic [Gentzen, 1964], and defend that the structural rules in the calculus (which will be rejected by the different substructural logics) preserve truth. Moreover, we will argue that the classical conditional, disjunction, and conjunction codify the literal meaning of 'if...then', 'or', and 'and'. Second, we will defend a view about the relevant vocabulary that resolves the apparent rivalry with classical logic. The effect that the lack of weakening in *LR* has on the conditional, disjunction, and conjunction, (following [Paoli, 2002]) will be explained pragmatically. In particular, we will argue for the legitimacy of relevant logic, relying on a relationship between structural rules and the Gricean maxims: we will show that the rejection of weakening in LR allows us to formalise a notion of 'follows from' which avoids the violation of the maxim of Quantity or Relation. As a result, both notions of logical consequence (both the classical full-structural classical relation and the substructural relevant relation) encode legitimate senses of 'follows from': the presence of weakening makes the classical relation truth-preservational, while its rejection in relevant logic encodes an alternative notion of 'follows from', avoiding truth-preserving arguments which violate certain normative rules for reasoning. Moreover, we will argue that the logical connectives in *LR* are pragmatically enriched by observing the Quantity and Relation maxims, which make them suitable for the relevant consequence relation.

Chapter 5 focuses on the legitimacy of Linear and Ordered Logic (as presented in [Girard, 1987] and [Abrusci, 1991] respectively). To resolve the apparent rivalry between these two logics and classical or relevant logic, we will rely again on an interpretation of their vocabulary as pragmatically enriched (and again using the Gricean maxims). Linear logic rejects weakening and contraction and Ordered Logic also rejects exchange, and this is interpreted given the particular reading that sequents have in both logics: in linear and ordered logic 'Γ entails Δ' is usually interpreted by formalising a *use* of the premises to get the conclusion (contrary to *LK* and *LR*). This different reading makes them capable of capturing senses of the connectives, which are pragmatically enriched following the manner maxims 'be ordered', 'avoid repetition', or 'avoid ambiguity'. Hence linear and ordered linear logic require the conditional, disjunction, and conjunction to connect uses of the premises that are sensitive to order or to repetition, given the special reading of sequents and their pragmatic enrichment.

Finally, in part 3, we will see some of the virtues that this version of pluralism has, and some of the objections that can be raised against pluralism in general and Pragmatic Logical Pluralism in particular and how to solve them. First, in Chapter 6, we will present three applications that this version of pluralism has in resolving certain paradoxes: the paradoxes of material conditional, the Free Choice Permission paradox (following the work of [**?**], although solving some problems that such approach has), and also how it can give a natural framework for the Lottery Paradox (reviewing the work of [Paoli, 2005] and [Zardini, 2015a] but giving a novel pragmatic diagnostic). We consider the explanation for these paradoxes a virtue of the pluralist thesis defended here given that it can offer a natural solution that is overall more natural than those solutions from monist positions. And in chapter 7 we will discuss some general and particular objections to logical pluralism. First, we will respond to those general criticisms that are launched on other versions of pluralism (presented in Chapter 2), such as the plurality of inferential roles, or the plurality of meanings of logical constants given the plurality of logics presented in the two previous chapters. Second, we will turn our attention to the possible criticisms that this pragmatic pluralist thesis may receive.

Chapter 2

Logical Pluralism: Levels and Versions

Logical pluralism is the thesis that there is more than one correct logic. In other words, logical pluralism aims to resolve the apparent rival claims 'Δ follows from Γ' and 'Δ does not follow from Γ'. But there are many ways of resolving this disagreement, which depend on the underlying assumptions about what logic is. As Field [2009] argues, there are trivial versions of pluralism which are true but uninteresting, and there are other much more interesting but false versions. So logical pluralism can be seen as a project or field of study rather than a thesis: the study of finding out how two logics can be endorsed, or how far our tolerance towards rival logics can go.

This chapter is divided into three main parts: first, we will distinguish levels and versions of logical pluralism, and we will devote the first part to clearly state what the different levels in which a pluralist thesis about logic can be situated are. Second, we will present the main versions of logical pluralism in the literature. Third, we will briefly introduce Pragmatic Logical Pluralism, emphasizing its connection with the different versions of pluralism introduced.

2.1 Levels of Pluralism

The debate on logical pluralism has received numerous contributions, not only proposals for and against some pluralist thesis, but also classifications of versions of logical pluralism (i.e., Priest [2006], Field [2009], Cook [2010], Eklund [2012b]). And together with the classifications of logical pluralism, some authors have also argued that there are different *levels* of logical pluralism.

Following this path, and in order to better classify the different proposals, we can distinguish *levels* from *versions* of logical pluralism. The levels of logical pluralism correspond to the different definitions of logic or the different stages about which one can be a pluralist: from pure logic as a mathematical structure to the science of correct reasoning. The versions of pluralism are the different theories of how to be a pluralist at a given level, considering that a version of logical pluralism is that one which situates the pluralist thesis at the higher level that one considers as defining what logic is.

More specifically, we follow the work of Priest [2006] for classifying the different levels of logical pluralism. Priest's classification distinguishes three main stages in which the pluralist thesis can be situated, which will be named as follows:

I. Pure Logical Pluralism (PLP): there is a plurality of pure logics.

II. Applied Logical Pluralism (ALP): there is a plurality of logics that can be fruitfully applied to interesting phenomena.

III. Reasoning Logical Pluralism (RLP): there is a plurality of logic for analysing correct reasoning.

It is important to note that each version of pluralism implies a particular assumption about the nature of logic: before presenting the pluralist thesis, one needs to specify what it is pluralist about. What one defines as *logic* determines what kind of pluralism, if any, one can endorse. Hence, given one determinate view about what logic is, there are two possibilities regarding the pluralist debate: either arguing that there is more than one such logic or arguing that there can be just one. These considerations mean that a pluralist and a monist can agree about a pluralist thesis (i.e. that there is more than one 'logic', in the sense of the pluralist), but disagree that this is logical pluralism because the monist considers that logic is something else and that it is unique. They would be talking past each other, with different senses of *logic*.

2.1.1 Pure Logical Pluralism

There are many logics in the literature: one can find logic manuals that present classical, relevant, and linear logic (just to mention the ones that we will endorse in this project). So a trivial version of pluralism is the claim that these logics are equally legitimate as pure systems. We will call the pluralist thesis at this level *Pure Logical Pluralism* (PLP). Priest [2006] considers this version of pluralism trivial,

> [T]here are many pure logics (...) Each is a well-defined mathematical structure with a proof-theory, model theory, etc. There is no question of rivalry between them at this level. This can occur only when one requires a logic for application to some end. Then the question of which logic is right arises. If one is asking

> about pure logics, then, pluralism is uncontentiously correct. [Priest, 2006, p. 195]

This level of pluralism is what Cook calls *Mathematical Logical Pluralism*,

> [T]here is more than one logic. [Cook, 2010, p. 494]

Which he also considers trivially true,

> Whether or not MLP [Mathematical Logical Pluralism] is true is a purely mathematical issue, as its truth-value hinges solely on the existence of certain mathematical structures. Further, MLP is trivial (...) Thus, this sort of pluralism is not what philosophers have in mind when engaging in the logical pluralism/monism debate. Nevertheless, were MLP false, then more substantial versions of pluralism would be nonstarters. [Cook, 2010, p. 494]

We can also situate at this level what Eklund [2017] calls *mapping pluralism*,

> [T]here are the different possible languages mentioned, perhaps adding that they are all legitimate objects of theoretical investigation. [Eklund, 2017, p. 1]

We do not want to consider this plurality claim a version of substantial pluralism, as we do not consider logic to be *just* a pure mathematical structure. We consider that *logic* is something different.

2.1.2 Applied Logical Pluralism

We assume that the logical vocabulary is interpreted in order to discriminate those inferences that are valid from those that are invalid. Hence, if logic is not just a pure mathematical structure, it might be the application of that structure to some domain, interpreting the logical vocabulary accordingly. And this can be the second level in our scale of what *logic* is, and where an interesting version of logical pluralism can be situated. In Priest's words,

> Fix, then, on some one application. A pure logic is applied by interpreting it in some way or other. ... [I]t then becomes a theory of how the domain in which it is interpreted behaves. [Priest, 2006, p. 195]

We will call the pluralist thesis at this level *Applied Logical Pluralism* (ALP). Cook distinguishes two levels which we subsume into ALP (as we assume that mathematics is already a philosophically interesting phenomenon): first, *Mathematical Application Pluralism*,

> There is more than one logic that can be fruitfully applied (in the general sense of applied mathematics). [Cook, 2010, p. 494]

Second, *Philosophical Logical Pluralism*,

> There is more than one logic that can be fruitfully applied to philosophically interesting phenomena. [Cook, 2010, p. 494]

And finally Eklund's [2017] *purpose pluralism* can also be situated here,

> [T]here is the view that different languages (truth predicates, etc.) are best for different purposes. Call this *purpose pluralism*. [Eklund, 2017, p. 5]

One possible interpretation of ALP is that once a domain or purpose is specified, there is one correct logic that can be applied to it. That is, pluralism is an external phenomenon for each specific application. However, at this level, we can also find a version of pluralism within one specific domain, as an internal phenomenon. We can then distinguish the following:

- External ALP: different logics can be applied to different phenomena, and once the phenomenon is determined, there is one logic which can be fruitfully applied to it,
- Internal ALP: different logics can be applied to different phenomena, and there is some phenomenon to which more than one logic can be fruitfully applied.

First, as an illustration of external ALP, one can argue that quantum mechanics can be fruitfully applied to codify certain physical phenomena at the quantum level and that Boolean algebra can be fruitfully applied to computer circuits. The logical vocabulary in both cases follow different rules, and hence, more than one logic is embraced for different purposes. Second, for an illustration of internal ALP, one can fruitfully apply both Quantum and Boolean algebras in the same computer domain, giving fruitful and different results in each case.

Although pluralism at this level is not as straightforward as PLP, it can be defended as true without yet endorsing logical pluralism. One can agree with the fact that different formal languages can be *theories* about different phenomena, both for the external and for the internal version of ALP, but disagree that this is evidence of a plurality of logics in a stronger sense. This is so because one or more of the following facts: (i) the logical vocabulary is interpreted technically and does not correspond to our intended meaning of the connectives when we reason with them, (ii) the connection between premises and conclusions in the different domains is not determined by logical consequence but some other technical connection determined by the formalised phenomena.

If one agrees with ALP but still defends a monism view on logic, this is because one considers that *logic* is to be identified with one particular application, which is correct reasoning, which can be analysed with just one formal language.

2.1.3 Reasoning Logical Pluralism

The last level in which we can situate the pluralist thesis, and which we will call *Reasoning Logical Pluralism* (RLP), is the level where we focus on the formalization of *genuine* logical consequence. We are interested in the application of logic in logical consequence, as Priest [2006] argues,

> [T]he canonical application [of logic]: the application of a logic in the analysis of reasoning - which was also traditionally called 'logic', of course (...) The central purpose of an analysis of reasoning is to determine what follows from what - what premises support what conclusions - and why. [Priest, 2006, p. 196]

At this level we can situate Cook's *Logical Consequence Pluralism*,

> There is more than one logic such that CP holds for that logic.

The CP or *Correctness Principle* for a logic $\langle L, \vdash \rangle$ being the following,

> CP: Given any recursive mapping I from L to statements in our natural language (i.e. an interpretation) which agrees with T on LV, and given any statement ϕ and set of statements Δ from L:
>
> $I(\phi)$ is a logical consequence of $I(\Delta)$ if and only if $\Delta \vdash \phi$. [Cook, 2010, p. 495]

where LV is a subset of the primitive symbols of L (the logical vocabulary of L), and T is a (partial) translation function from LV to appropriate bits of natural language.

Although we are introducing the notion of consequence, it can still be uncontroversial if we do not establish a division between logical and non-logical vocabulary. The divergence in the languages of propositional and first-order logics are a model for this version of pluralism [Cook, 2010, p. 496] which becomes uncontroversial.

We need a stronger presentation of the pluralist thesis, which is what Cook calls *Substantial Logical Pluralism* (and which we also subsume under RLP),

> Given a formal language L and an identification of the logical vocabulary LV of L (and a projection T of LV onto natural language) there exists distinct $\vdash_a$ and $\vdash_b$ such that CP holds for both $\langle L, \vdash_a \rangle$ and $\langle L, \vdash_b \rangle$.

And finally, Eklund's [2017] *goodness pluralism* is another presentation of RLP,

> Even given some particular purpose, perhaps some canonical purpose, different languages serve that purpose equally well. [Eklund, 2017, p. 5]

Localism and Globalism We can draw an interesting distinction within Reasoning Logical Pluralism. On the one hand, one can argue that different domains require different logics in order to *reason* about them, that is, *logical consequence* changes from domain to domain[1]. This position is Haack's [1978][2] notion of Local Pluralism:

> Local Pluralism: different logical systems are applicable to (i.e. correct with respect to) different areas of discourse. [Haack, 1978, p. 223]

On the other hand, one might argue that according to Localism one is inferring a conclusion about some specific phenomenon from some premises *plus* some extra information about that specific area. One might argue that logic abstracts from any particular content of the discourse about which we might be reasoning, in order to be fruitfully applied to *any* subject matter. Given this generality, one can defend Haack's [1978] Global Pluralism:

> Global Pluralism: logical principles should apply irrespective of subject-matter (...) the global pluralist denies either that the classical and deviant logician are really using 'valid'/'logically true' in the same sense, or else that they are really disagreeing about one and the same argument/statement. [Haack, 1978, p. 223]

Now we are in the position of talking about versions of logical pluralism: interesting versions of pluralism need to either (a) endorse more than one logic at this last level, or (b) argue that this last level does not correspond to *logic*, and endorse pluralism at a lower level.

[1] Note that this is similar but not equivalent to external ALP: fruitful application is different to genuine logical consequence. Consider, for instance, a relevant logician who uses the material conditional in computer circuits. The relevant logician, in this case, endorses ALP but not RLP. The relevant logician would be a partisan of RLP if she defended that in order to reason about circuits, classical logic is required.

[2] See also [Priest, 2006, p. 174] for the distinction between Localism and Globalism.

2.2 Versions of Logical Pluralism

Once we have agreed that the interest of logical pluralism is to make compatible two apparently incompatible claims about logical consequence, that is, about whether Δ follows from Γ for some Γ and Δ, we can turn our attention to this possibility, and how it should be defended.

One possible demarcation of versions of logical pluralism is based on the two possible origins of the rivalry: divergence about what Γ and Δ express (usually defended by a divergence over the logical constants in Γ and Δ, but not necessarily) or divergence about validity[3]. Given two logics L and L', we have the following two possibilities:

- Language divergence: $\Gamma_l \Rightarrow \Delta_l$ but $\Gamma_{l'} \not\Rightarrow \Delta_{l'}$
- Consequence divergence: $\Gamma \Rightarrow_l \Delta$ but $\Gamma \not\Rightarrow_{l'} \Delta$

Given that there are syntactic and semantic presentations of a logic, there are syntactic and semantic presentations of the rivalry among logics, which entails semantic and proof-theoretic presentations of the pluralist thesis for each origin of the divergence.

- Language divergence: $\Gamma_l \Rightarrow \Delta_l$ but $\Gamma_{l'} \not\Rightarrow \Delta_{l'}$
  - $\Gamma_l \vDash \Delta_l$ but $\Gamma_{l'} \nvDash \Delta_{l'}$
  - $\Gamma_l \vdash \Delta_l$ but $\Gamma_{l'} \nvdash \Delta_{l'}$
- Consequence divergence: $\Gamma \Rightarrow_l \Delta$ but $\Gamma \not\Rightarrow_{l'} \Delta$
  - $\Gamma \vdash_L \Delta$ but $\Gamma \nvDash_{l'} \Delta$
  - $\Gamma \vdash_L \Delta$ but $\Gamma \nvdash_{l'} \Delta$

For each kind of divergence, there can be several versions of both logical monism and logical pluralism. Logical monism claims that the divergence is genuine, while logical pluralism claims that the disagreement is just apparent. This classification is a map of the possible divergences depending on the presentations of a logic. However, it is not possible to neatly distinguish each one of them, as inferential roles are connected to truth conditions, and validity is connected to logical constants (see the phenomenon of the internalisation of validity in [Hjortland, 2014b]). Hence, although we speak of versions of logical pluralism, they can also be seen as *presentations* of logical pluralism, which have an impact on the rest of dimensions of logic.

[3]About this division, see Haack [1978], discussed in [Hjortland, 2013]

2.2.1 Language Pluralism

Under this classification, the most natural candidates to demonstrate a divergence about languages are the logical connectives. And their divergence can be motivated by two different sources, their truth conditions or their inferential roles. The divergence in the truth conditions of the logical constants as a source of logical pluralism is not widespread, and at first glance, this claim seems to go against a very basic intuition about the logical connectives, if one shares the following view:

> If we give different truth conditions for the connectives, we are giving the formal connectives different meanings. When we apply the logics to vernacular reasoning we are, therefore, giving different theories of the meanings of the vernacular connectives. We have a case of theoretical pluralism; and the theories cannot both be right - or if they are, we simply have a case of ambiguity. [Priest, 2006, p. 204]

We will find a divergence about the truth conditions of the logical constants as a consequence, rather than as a cause, of other presentations of logical pluralism. That is, the divergences about the truth conditions of the logical constants are, in some cases, a consequence of a pluralist thesis that is motivated for other reasons. And the intuition expressed in the previous quote has been raised as an objection to such versions of pluralism.

It is more natural to endorse a divergence about the inferential role of the connectives than a divergence about their truth conditions. This view has been traditionally attributed to Carnap [1937], and recently Shapiro [2006] and Cook [2010], Eklund [2012b], Kouri Kissel [2018] have defended pluralism with a similar approach; we will see them in what follows. Moreover, there are two more versions of pluralism that should be included inside this section about language pluralism which are motivated by different phenomena than the variation of the inferential role of the logical constants. They vary more general facts about Γ and Δ: Varzi's logical relativism and Russell's pluralism of bearers of truth; these versions will be presented at the end of the section.

Tolerance

Carnap is considered the first philosopher to lead the debate on logical pluralism, and he has been vindicated as such by Beall and Restall [2006] and Restall [2002]. He shares with other proposals of logical pluralism the motivation of endorsing or making compatible two or more (apparently) rival logics. And the spirit behind this motivation is the Principle of Tolerance, according to which one can be tolerant and endorse more than one set of syntactic rules for the logical vocabulary.

> [L]et any postulates and rules of inference be chosen arbitrarily; then this choice, whatever it may be, will determine what meaning is to be assigned to the fundamental logical symbols. By this method, also, the conflict between the divergent points of view on the problem of the foundations of mathematics disappears (...) The standpoint we have suggested - we will call it the Principle of Tolerance - (...) relates not only to mathematics, but to all questions of logic. [Carnap, 1937, p.xv]

Following Kouri Kissel [2019], we can identify two main principles as the origin of Carnap's pluralism: first, Carnap assumes that a change in logic is a change of language, by an identification between a formal language and a logic:

> In logic there are no morals. Everyone is at liberty to build his own logic, i.e. his own language, as he wishes. All that is required of him is that, if he wishes to discuss it, he must state his methods clearly, and give syntactical rules instead of philosophical arguments. [Carnap, 1937, p. 52]

Second, a Principle of Tolerance towards more than one language, which generates a plurality of languages, and given the identification between language and logic, a pluralism of logics.

> Principle of Tolerance: It is not our business to set up prohibitions but to arrive at conventions. [Carnap, 1937, p. 51]

Moreover, this tolerance principle is, in turn, sustained in two other principles. First, we can distinguish between internal and external questions about a given language:

> For Carnap, a theoretical question is one asked relative to a linguistic framework. It is internal to a linguistic framework, and asked assuming the rules of that framework. A non-theoretical question is one asked about reality itself, without a linguistic framework in mind. It is external to any given framework. [Kouri Kissel, 2019, p. 2]

Second, the external question about *the* correct language is either pragmatic or a pseudo-question:

> There are two types of external questions: pragmatic questions and pseudo-questions. A pragmatic question is a question about which framework is best for a given purpose, and can be answered. Pseudo questions (non-pragmatic external questions) cannot be answered, on Carnap's view, and this is the sense in which they are illegitimate. [Kouri Kissel, 2019, p. 2]

According to this view it impossible or nonsensical to determine which is the correct formalization or inferential role of 'and', 'or', or 'if...then'. Only after one has specified the application about which one is theorizing, one can determine the best formalization of conjunction, disjunction or conditional; otherwise, it is an illegitimate question. And given that there can be more than one formalization for the logical connectives legitimately applied to different contexts, logical pluralism is correct.

Reaction: ALP but not RLP An initial reaction to this version of logical pluralism is that we are dealing with a version of Applied Logical Pluralism but not Reasoning Logical Pluralism.[4] Although one might agree with the kind of pluralism that Carnap endorses, there is still room for monism if one rejects the claim that there cannot be an external question about *the* correct language. This external question, if possible, might restore monism, as its answer can give a monist an answer to RLP, despite the pluralism at the ALP level.

We have seen, however, that if it is justified to state that the level in which the pluralist thesis is defended is also the last level in which one situates what logic is, this is a legitimate version of logical pluralism. Hence, the fact that Carnap explicitly rejects the possibility of the external question about the best language to encode logical consequence makes his an interesting version of logical pluralism.

Modelling

The work of Shapiro [2006] and Cook [2002] is a kind of pluralism since they claim that 'the job of formal logic is to model a natural language' [Russell, 2016], where models are structures that intend to characterize some feature of a phenomenon; language, in this case, and there can be rival and correct models of the same language.

[4] Our diagnostic diverges from Cook's, who argues that Carnap's pluralism falls within RLP, although it is not controversial as it is a version of what Cook calls Logical Consequence Pluralism, rather than a version of Substantial Logical Pluralism. The divergence is originated in the claim, which Cook seems to endorse and that we do not, that Carnap is codifying correct reasoning or logical consequence, as we have previously stated. The impossibility of formulating an external question about the correct logic, if any, makes it impossible to classify Carnap's pluralism as claiming that there is more than one correct logic *for* logical consequence.

First, Shapiro argues that a formal language is a mathematical model of natural language,

> The present claim is that a formal language is a mathematical model of a natural language, in roughly the same sense as, say, a Turing machine is a model of calculation, a collection of point masses is a model of a system of physical objects, and the Bohr construction is a model of an atom. In other words, a formal language displays certain features of natural languages, or idealizations thereof, while ignoring or simplifying other features. [Shapiro, 2006, p. 49]

And logical pluralism is a natural consequence of this view,

> [W]ith mathematical models generally, there is typically no question of 'getting it exactly right'. For a given purpose, there may be bad models - models that are clearly incorrect - and there may be good models, but it is unlikely that one can speak of the one and only correct model (...) The models may be serving different purposes. [Shapiro, 2006, p. 50]

Cook [2002] has a similar view of the relation between natural and formal language,

> [T]he formalism is not a description of what is really occurring but is instead a fruitful way to represent the phenomenon, that is, it is merely one tool among many that can further our understanding of the discourse in question. In particular, not every aspect of the model need correspond to actual aspects of the phenomenon being modelled. [Cook, 2002, p. 234]

This form of pluralism seems to revive the spirit of Carnap's pluralism, in the sense that it is not concerned about the demarcation of a language to give a pluralism of consequence.

> The crucial insight of the logic-as-modeling view for our purposes is that there can be multiple, incompatible, competing models of the same phenomenon. [Cook, 2010]

However, Cook and Shapiro argue that the phenomena that these rival formal languages capture can be logical consequence itself. Hence, contrary to Carnap's, we can classify their version of pluralism as RLP.

Reaction Modelling solves the previous problem in Carnap, so we agree that this version of pluralism can be classified as RLP. Our view, as will be developed, is sympathetic to Cook and Shapiro motivation for pluralism: different formal languages might correctly capture legitimate codifications of logical consequence. However, one might worry that we lack a proper justification of why this pluralism emerges, that is, what are the mechanisms that explain that more than one formal language might be suitable for reasoning (and not just for capturing certain features of natural language, while just one of the possible formalization captures genuine logical consequence). In other words, we lack a proper justification why there is more than one normative constraint for the logical vocabulary in natural language.

Polysemy

Kouri Kissel [2018] has recently defended a polysemous view on the meaning of logical constants. According to her view, connectives switch their meaning from context to context but are connected with a pre-theoretic connective (which can neither be identified with any of the connectives, nor with the natural language counterpart) in a polysemous way.

Polysemy is defended as follows,

> Two words are polysemous when they are ambiguous but related. So, while 'bat' is ambiguous between the mammal and the baseball equipment, it is not polysemous. On the other hand, 'wood' is ambiguous between the thing trees are made of and the thing a lot of trees make up, they are related to each other, and so are polysemous. I will make no claims here about whether all polysemous words are always related by a 'pre-theoretic' notion. [Kouri Kissel, 2018, p. 7]

Consider, for instance, the natural connective 'not' and its formal counterparts in rival logics as classical or intuitionistic logic. According to Kouri Kissel, there is a pre-theoretic negation [2018, p. 7], which is related to each negation in each rival logic endorsed by the pluralist as polysemous words are connected. This pre-theoretic negation can neither be identified with the natural language 'not' nor with the classical formal $\neg$ or any other formal negation. Negation can only be precisified relative to a given context of discourse (e.g. classical or intuitionistic reasoning about mathematics). Moreover, similarly to Carnap, Kouri Kissel argues that the external question about *the* meaning of the logical connective is a pseudo-question if made outside a specific context of discourse.

Reaction Polysemy is an interesting term to refer to the connection between the different connectives in different logics, as there is something that connects them, that is, they do not have the same name just by chance. Similarly to the theory of modeling, we are sympathetic to such a view, the intuition that there is a plurality of formal languages that correctly codify logical vocabulary is defended in the present thesis. Our worry is whether polysemy is the exact phenomenon that connects the different senses of the logical connectives across logics, and also their natural language counterpart.

Although this might be the case for some connectives and some pairs of logics, we are sure that a more precise diagnostic can be given to the connection between logical constants in the logics discussed here. Kouri Kissel agrees that the view is vague [2018, p. 11], so this vagueness can be precissified in some cases, as we will show.

In sum, we show that logical connectives are 'connected', as Kouri suggests, but not because they are polysemous but because they have the same meaning and different pragmatic enrichments.

Contextual pluralism

Caret [2017] defends a contextualist version of pluralism, according to which each context of reasoning picks a suitable notion of validity.

> [C]ontextualism about validity is the view that 'valid' and its cognates include a parameter for a deductive standard. In simple terms, a deductive standard is an admissible class of cases that function as logically salient alternatives. Each context selects for a deductive standard and this, in turn, gives content to validity attributions in that context. Validity always requires a connection between premises and conclusion that rules out counterexamples, but the salient range of cases may vary from one context to another. [Caret, 2017, p. 14]

Caret's view assumes the range of logics defended by Beall and Restall (presented below) and suggests that the context picks the correct logic to be applied, depending on certain standards imposed by the context to the notion of validity.

Reaction Contextualism about logic, as it is presented by Caret, depends on a previous specification of which logics are legitimate (and Caret seems to endorse at least the logics embraced by Beall and Restall). Although we are sympathetic to Caret's view and it can be a natural reasoning towards a complete version of RLP, we argue that in order to be a justified version of RLP we lack an explanation of *why* different contexts require different logics, and why the particular logics considered as valid by Caret (and Beall and Restall) are the logics embraced in the pluralist thesis.

Multitude

Eklund [2012b] also refers to a version of pluralism within a Carnapian perspective, which he calls *multitude* to distinguish it from Beall and Restall's [2006] logical pluralism: Eklund uses 'logical pluralism' to refer uniquely to Beall and Restall's theory, contrary to multitude, which is a more general term. Before presenting this version, we need to clarify two points: first, for our purposes, and given the generality with which we are treating the notion of logical pluralism, Eklund's notion of multitude is a version of logical pluralism and Beall and Restall's version of pluralism is just one version among many. Second, it is important to point out that Eklund himself does not endorse this version of pluralism, as he remains sceptical about whether it is valid or not.

The multitude version of pluralism is Carnapian in spirit, given his tolerance towards a plurality of languages,

> (MULTITUDE): there are different logics of different languages, and the choice between these languages is merely one of expediency. [Eklund, 2012a, p. 9 833]

> [F]or some different logics, the sorts of utterances and inferences those employing those logics accept and reject are such that there is a possible language such that under the hypothesis that those employing those logics speak that language, their use is correct. [Eklund, 2012b, p. 218]

But Eklund distinguishes his view from the traditional Carnapian pluralism given that multitude does not imply conventionalism about the meaning of logical connectives. While conventionalism relies on convention for making logical truths true, multitude has a more general view and it does not rely on one specific view about what makes the logical truth true: multitude is true just in case there are two or more languages that validate different logical principles, whichever are the principles that make them true.

And to the claim that multitude might mean that logical disputes are merely verbal disputes, Eklund claims that, although logicians may disagree over which is the best language to use, they can do so using the same language. Arguing that different logics should follow different rules does not entail the actual use of different languages [Eklund, 2012b, p. 221]. Eklund's response to this possible objection shows that multitude does not depend, as Carnapian pluralism does, on the impossibility of an external question about *the* correct logic. In Eklund's view, the object language and the metalanguage might diverge.

Reaction We agree with the pluralist thesis, and with the possibility of the external question which was not possible in Carnap's view (recall that we argued that Carnap's pluralism was a version of ALP rather than RLP because of this particular point). Hence, multitude is compatible with a version of RLP. However, without the justification of why the different languages codify genuine logical consequence, we are in front of an incomplete version of RLP.

Logic of Ambiguity

Allo [2013] argues that logical connectives in classical logic are defective for codifying disjunction, which is an ambiguous connective, a claim also made by Paoli [2007]. We will see later how substructural languages split each logical connective into two, but given that Allo focuses on the disjunction to illustrate his claim, we will briefly introduce such ambiguity: in classical logic, 'or' is formalised with $\vee$, while in relevant and linear logic there are two connectives: $\sqcup$ and $+$ which validate different inferences:

$$\frac{A}{A \sqcup B}\sqcup\mathrm{I} \qquad\qquad \frac{A + B \qquad \neg A}{B}+\mathrm{E}$$

Now certain paradoxes of classical logic, such as explosion, $(A, \neg A \vdash B)$, can be shown to be invalid, given this ambiguity of 'or'. The argument of explosion can be shown to be a fallacy of equivocation, as the different steps in the derivation correspond to the two different disjunctions:

$$\frac{\dfrac{A}{A \vee B}\vee\mathrm{I} \qquad \neg A}{B}\vee\mathrm{E}$$

In general, substructural logics exhibit two senses of the logical connectives that classical logic ignores. And regarding this ambiguity, Allo adopts the following three assumptions [2013, p. 59]:

- Some substructural logic is a correct logic, preferable above classical logic.
- At least one natural language connective is ambiguous.
- The connectives of a substructural logic provide all the resources we need for disambiguating all natural language connectives.

These three principles can easily be assumed by monists who endorse some substructural language as the correct formalization of the logical constants, as Paoli [2007] does with linear languages and Anderson and Belnap [1977] or Read [1988] with relevant logic. However, contrary to Paoli and other monists, Allo defends that this ambiguity opens up the possibility for logical pluralism.

> The thesis I would like to defend is that the thorny issue of logical deviance and the disagreement between logics can to some extent be elucidated through the use of ambiguous connectives. That is, by using an ambiguous language as the locus to record explicit disagreements, we can understand the rivalry between classical and substructural logic as a genuine disagreement about what follows from what. [Allo, 2013, p. 77]

The pluralism thesis emerges from the acceptance of at least two logics: on the one hand, some substructural language (for instance, linear) correctly disambiguates the connectives in natural language, and it is called 'the logic of disambiguation'. On the other hand, some other logic has ambiguous connectives, as natural language does. The interest of such a logic is that it is capable of capturing the natural language, and shows at the same time that it is a defective language. That is, a logic with ambiguous connectives will be capable of explaining how to reason with 'or', given that this connective is also ambiguous. Hence, rather than only endorsing a logic that avoids this ambiguity, one might also endorse an ambiguous logic.

> A system which allows one to reason from ambiguous premises is, both by monist and pluralist standards, a better or at least an equally good candidate for being (a) a prima facie candidate for being the correct logic, or (b) a logic properly called so. [Allo, 2013, p. 62]

Allo argues that this ambiguous logic cannot be Classical logic, as those inferences that are invalid in substructural logics but valid in classical logic are 'fallacies of equivocation' [Allo, 2013, p. 58], such as the law of explosion, mentioned earlier. Classical logic is a noisy logic,

> Classical logic allows one to reason from ambiguous premises, but in doing so, it commits a fallacy of equivocation. A classical retreat is, therefore (but also in view of our first assumption) out of question. Hence, a logic which does not conflate different connectives even when confronted with premises containing ambiguous connectives remains to be given. [Allo, 2013, p. 62]

We need another logic, with ambiguous connectives but capable of avoiding these fallacies. This logic will be an equivocal but not a noisy logic. What Allo develops in his paper [2013] is such an equivocal logic. The basic idea for it is the following:

> Given a set of ambiguous premises Γ, one can (without making any further assumption about that set) deduce an ambiguous formula A from this set iff

> all possible disambiguations of Γ entail some disambiguation of A. [Allo, 2013, p. 67]

Leaving aside the details of such a logic, it is clear how a pluralist thesis emerges from Allo's proposal: while linear, or some other substructural logic captures the genuine and disambiguated meaning of the logical constants, we need another logic which is capable of reasoning with ambiguous connectives, given that our use of them is also ambiguous. This logic, however, has to avoid the fallacies of equivocation that classical logic commits when reasoning with these connectives. Hence, more than one formal language is legitimate to capture logical consequence.

Problem: the relation between natural and formal language Allo's thesis is motivated by the same kind of phenomenon that motivates the present dissertation. The ambiguity suggested by substructural logics regarding the logical connectives in natural language seems a legitimate phenomenon about which one should be a pluralist. The expressive power of substructural logics is compelling, and logical pluralism seems the correct attitude to have towards them, given that in natural language we do not always distinguish the intensional from the extensional version of 'if...then', 'or' and 'and'. It seems reasonable to try to make them compatible with some stronger logic.

However, Allo's resolution of the phenomenon diverges from the present one, and we suggest that the ambiguity thesis is less compelling than the enrichment thesis defended here. The phenomenon that Allo presents regarding disjunction does not seem to be a case of ambiguity: the word 'bank' is ambiguous, and it is so because it expresses two different and completely disconnected things. They share the same name by chance. The intensional and extensional 'or' in linear logic share some common features which we are interested in capturing (and Allo's thesis is the evidence for this): a logic that is capable of reasoning with genuine ambiguous terms does not seem an interesting logic, contrary to Allo's system. Whether something follows from all the disambiguations of an ambiguous term seems a hostage to fortune, rather than an interesting property for reasoning.

Hence, rather than ambiguity, we could (and will) argue that 'or' can be specified and determined in more than one way, but in all its specifications there is a basic meaning, which is captured by the strongest logic (in our case, classical). These specifications are not part of the meaning of 'or', but they express an extra content, which substructural logics are capable of codifying. The connection between $\sqcup$ and $+$ seems closer to the connection between scarlet and crimson rather than between bank_1 and bank_2. And one would not say that the word 'red' is ambiguous between 'scarlet' and 'crimson', but that 'scarlet' and 'crimson' are senses of red.

Logical relativity

Varzi distinguishes what he calls *Tarskian* relativism from *Carnapian* relativism. First, and following Tarski, Varzi defends a version of logical pluralism based on a plurality of valid models, which is in turn based on the plurality of demarcations of logical vocabulary. He states his view as follows,

> One logic or many? I say - many. Or rather, I say there is one logic for each way of specifying the class of all possible circumstances, or models, i.e., all ways of interpreting a given language. But because there is no unique way of doing this, I say there is no unique logic except in a relative sense. [Varzi, 2002, p. 1]

So Varzi argues for the plurality of demarcations, from which a pluralist thesis emerges,

> One can draw the line between the logical and the extra-logical vocabulary in many ways, and depending on how one draws the line one can think of the models that fix the meaning of the logical terms as constituting the class of all models. [Varzi, 2002, p. 1]

Second, following Carnap, Varzi claims that once a specific demarcation of the logical vocabulary has been established, then one can be a pluralist on the Carnapian sense seen above.

Reaction Let us put aside Carnapian relativism since it was discussed above. In the previous discussion about levels of pluralism, we have considered, following Cook [2010], that there are two different stages of pluralism within RLP (page 15). Tarskian pluralism falls under Logical Consequence Pluralism but not under Substantial Logical Pluralism, and hence, it is uncontroversial (note that this is not to say that Varzi's version of pluralism is uncontroversial, note that his Carnapian view on logic makes his version a substantial one).

Plurality of bearers of truth

There is another version in the literature that can be classified here: a version of pluralism that emerges from a plurality of interpretations of truth-bearers in Γ and Δ, as defended by Gillian Russell [2008].

Russell's version of pluralism [2008] does not depend on a variation of the truth conditions of the logical constants but on the variation of the truth-bearers in Γ and Δ in an inference $\Gamma \vDash \Delta$. Russell discusses Beall and Restall's versions of logical pluralism [2006], which is a kind of pluralism that emerges from a plurality (or what Russell considers an ambiguity) on the notion of consequence, as we will see later in this section. Inspired by this fact, she claims that we can find other ambiguities that give rise to other versions of pluralism:

> Having discovered one ambiguity in the definition of validity, one might wonder whether there are others, and whether they might also generate different logics. If there were, these logics might turn out to be logics with which we are already familiar, or they might turn out to be new things entirely. One place where we can find an ambiguity is in the notion of an *argument*. [Russell, 2008, p. 596]

The premises and conclusion of an argument are truth-bearers, but this characterization opens a plurality of possibilities, as truth-bearers can be 'sentences, propositions, characters, statements, utterances, occurrences of sentences, beliefs and judgements' [Russell, 2008, p. 596]. Logical Pluralism emerges from the ambiguity of the arguments depending on what truth-bearers we chose. The author concedes that propositions have a privileged status as the primary truth bearers, while the rest depend on the truth conditions of the proposition they express. Let B_i be some truth bearer that expresses the proposition P_i: the biconditional $B_1 ... B_n \vDash B_m$ iff $P_1 ... P_n \vDash P_m$ fails in both directions.

For the right-to-left direction of the biconditional, Russell presents the following counterexample:

$$\vDash \text{I am here now}$$

Although this expression, as a non-propositional truth bearer will always express something true, the proposition that it expresses in each case is not a logical truth (I am here now, but this fact does not express a logical truth). Hence, there are valid derivations of a logic which pick a non-propositional truth bearer which are invalid for a logic which ranges on propositional content.

And for the other direction of the biconditional, Russell considers the following counterexample:

$$\text{Hesperus is Hesperus} \vDash \text{Hesperus is Phosphorus}$$

Both sentences express the same proposition, but a non-propositional truth bearer seems to fail to establish a strong logical connection between the premise and the conclusion. Hence, there are valid derivations considering the truth-bearers as propositions which are invalid if the truth-bearers are non-propositional.

In these examples, there is more than one answer to the question about the validity of the arguments, as there is more than one truth bearer of logical arguments. Hence, logical pluralism emerges as a consequence of the ambiguity that the author attributes to the notion of *argument*.

Reaction This seems a legitimate form of pluralism, and it can be a form of RLP. In our previous classification of versions of pluralism we have assumed that there is no equivocation about the bearers of truth, and we assume that more interesting versions of RLP emerge when the truth-bearers are fixed; that is, we wish to show that there might be more substantial forms of pluralism which emerge from different phenomena than the equivocation over the truth-bearers.

2.2.2 Consequence Pluralism

Recent versions of logical pluralism have focused on the plurality of valid consequence relations within a single language. Hence, in contraposition to language pluralisms (both about truth conditions and inferential role), we can define a logical consequence pluralism; the main proposal of this project is a version of Consequence Pluralism presented model-theoretically by Beall and Restall [2000],[2001],[2006]. We will see them in what follows.

Moreover, there is also a version of Consequence Pluralism with a proof-theoretic presentation. It is presented by Russell as follows:

> Another issue is the fact that one can generate different logics, not by varying the rules governing any particular expression, but rather by varying the more general structural rules of the logic, which govern things like whether or not one is allowed multiple conclusions, and whether or not a premise can be used more than once in a proof. (Restall 2000; Paoli 2003) This suggests that even if the meanings of the logical expressions are governed by the rules that tell you how they can be used in proofs (as Carnap suggests) two logics can agree on those rules, whilst disagreeing on the relation of logical consequence. Hence even if you have successfully chosen a language, it seems that you might not yet have determined a logic. [Russell, 2016]

We can find in the literature three presentations of logical pluralism as a variation of the notion of consequence: Restall [2014], Hjortland [2013], and Barrio et al. [2015], the three of them varying the structural rules in a Gentzen calculus of classical and substructural logics.

Beall and Restall's pluralism

Let us first see the contrast between language pluralism (identified with Carnap in this case) and consequence pluralism:

> For Carnap, diverging analyses of the validity of an argument result from diverging languages. The disputants in a dispute over the validity of an argument literally disagree in their reading of the argument. Given the disambiguation of the choice of a language, there can be no real disagreement, provided that the disputants know what language they speak. This is especially bad for a pluralist, who has appeared to have disagreed with herself. She is simply confused, equivocating from one language to another. Instead of saying that the argument is valid on one sense and invalid in another, she must be saying 'I can read the argument like this or I can read it like that'. I take it that this conclusion is unacceptable to my pluralism. If accepting different logics commits one to accept different languages for those logics, then our pluralism is primarily one of languages (which come with their logics in tow) instead of logics. To put it graphically, as a pluralist, I wish to say that
>
> - $A, \neg A \vdash_k B$ but $A, \neg A \nvdash_r B$
>
> A and $\neg A$ together, *classically* entail B, but A and $\neg A$ together do not *relevantly* entail B. On the other hand, Carnap wishes to say that
>
> - $A, \neg_k A \vdash B$ but $A, \neg_r A \nvdash B$
>
> A together with its *classical* negation entails B, but A together with its *relevant* negation need not entail B. [Restall, 2002, pp. 4-5]

In effect, Carnap and Beall and Restall's pluralism diverge: as we have seen, the pluralism in Carnap arises as a consequence of the impossibility of an external question, which also makes it impossible to identify the meaning of the connectives across logics; but this is possible in Beall and Restall. Hence, Beall and Restall's pluralism might be an extension of Carnap's, in which the external question about the correct formalization of a language to capture logical consequence is allowed, which makes it possible to argue for the identity of languages across logics.

Leaving aside the comparison with Carnap's thesis, let us turn our attention to the defence of pluralism in Beall and Restall [2000], [2001] and [2006]: they define their pluralism as 'the view that there is more than one genuine deductive consequence relation' [Beall and Restall, 2006] and this is so because 'the pre-theoretic notion of logical consequence is not formally defined, and it does not have sharp edges' [2006].

Their logical pluralism is the result of five principles. First, a reformulation of the Tarskian account of validity, GTT (which results in a model-theoretic pluralism):

> Generalised Tarskian Thesis (GTT): An argument is valid$_x$ if and only if, in every case$_x$ in which the premises are true, so is the conclusion. [Beall and Restall, 2006, p. 29]

Second, the observation that the index x on *valid* and *cases* shows that validity is relative to the specification of cases. Therefore, each specification of cases will determine a relation of logical consequence.

In order to focus on a divergence that this dissertation will resolve, consider the divergence between classical and relevant logic on the validity of $A \wedge \neg A \vdash B$[5]. On the one hand, cases can be complete and consistent, as possible words are, which leads to the 'necessary truth-preservation (NTP) account of validity' [Beall and Restall, 2006]. Or they can be Tarskian models, with a non-empty domain D and an interpretation I of the language over D. In both cases, the logic that results is classical, and there will not be a counterexample to $A \wedge \neg A \vdash B$. On the other hand, relevant logic restricts classical logic considering cases to be situations, which are restricted parts of the world, which may be incomplete and inconsistent. If cases can be incomplete and inconsistent, there can be counterexamples to explosion in which A is both true and false and B is false, which makes the inference from $A \wedge \neg A$ to B invalid.

The third ingredient for Logical Pluralism is a restriction on the instances of GTT to those that generate necessary, normative, and formal instances of consequence. In order to see the importance of this point, observe that a possible precisification of cases in GTT could be the actual world. In this case, there would be inferences $\Gamma \vdash \Delta$, which would be valid just because some formula in Γ is false or some formula in Δ is true in the actual world. However, that precisification of cases would not give a normative, formal and necessary relation of consequence.

The fourth principle is the fact that a *logic* is determined by any admissible instance of GTT. This point guarantees that this version of logical pluralism is a version of RLP, rather than a version of ALP.

Finally, the fifth necessary point for Beall and Restall's version of pluralism is the claim that there is more than one precisification of the notion of case, and Beall and Restall argue

[5]Beall and Restall's pluralism endorses at least classical, relevant and intuitionistic logic. We focus here on classical and relevant logic given that these two logics are endorsed by my version of pluralism.

that there are at least four (that determine three different logics): possible worlds, Tarskian models, situations, and constructions.

In sum, these are the five principles (stated in [Beall and Restall, 2006, p. 35]) that lead to Beall and Restall's version of Logical Pluralism:

1. The settled core of *consequence* is given in GTT,
2. An instance of GTT is obtained by a *specification of the cases*$_x$ in GTT, and a specification of the relation *is true in a case*$_x$. Such a specification can be seen as a way of spelling out *truth conditions*,
3. An instance of GTT is *admissible* if it satisfies the settled role of consequence, and if its judgements about consequence are necessary, normative, and formal,
4. A *logic* is given by an admissible instance of GTT,
5. There are *at least two* different admissible instances of GTT.

Reactions Beall and Restall's pluralism has been the target of many objections, and some of them are specific to their version of pluralism, but some others can be extended to other proposals. The following two objections, the collapse challenge and the meaning-variance objection, can be generalised, as Stei [2017] does with the collapse problem, and as Williamson [2014] and Priest [2006] do with the meaning-variance argument. Both challenges will be discussed in Chapter 7.

Collapse Challenge The Collapse Challenge is stated by Priest [2006] as follows:

> Suppose that one is a pluralist of the kind in question. Let s be some situation about which we are reasoning; suppose that s is in different classes of situations, say, K_1 and K_2. Should one use the notion of validity appropriate for K_1 or for K_2? We cannot give the answer 'both' here. Take some inference that is valid in K_1 but not K_2, $\alpha \vdash \beta$, and suppose that we know (or assume) α holds in s; are we, or are we not entitled to accept that β does? Either we are or we are not; there can be no pluralism about this. In fact, the answer is that we are. Since s is in K_1, and the inference is truth-preserving in all situations in K_1. In other words, if we know that a situation about which we are reasoning is in class K, we are justified in reasoning with validity defined over the restricted class of situations K. [Priest, 2006, p. 203]

What this challenge shows is the apparent contradiction in endorsing two rival logics, and the claim that logic is defined as truth preservation (unless one is a pluralist about truth, which is not the case for Beall and Restall). The consequence is that one of the two logics, K_1 or K_2, fails to diagnose as valid an argument which preserves truth.

In fact, Beall and Restall's answer to the challenge is that β is true in s [2006], which shows that one needs to reject the claim that both logics are *just* truth-preserving. Hence, as Read [2006] argues, Beall and Restall's pluralism implies the supremacy of classical logic over non-classical logics as relevant logic, as all relevantly valid inferences are also classically valid, but not vice-versa, so classical logic preserves truth while relevant logic preserves truth *plus relevance*. In sum, relevant logic fails to diagnose as valid some inferences that are truth-preserving, and hence, valid, according to the pluralist.

Meaning-variance Another criticism to Beall and Restall's pluralism is the claim that their theory implies a version of language pluralism. Beall and Restall want to avoid a plurality of languages, arguing that genuine logical pluralism needs to be a pluralism within a fixed language, in which two or more notions of consequence coexist. If they do not give a criterion of the sameness of meaning across the logics that they endorse, the following is an argument against their pluralism:

> Classical and intuitionist connectives have different meanings (...) Thus, take negation as an example. Classical and intuitionist negations have different truth conditions. But difference in truth conditions entails difference in meaning. Hence, the two connectives have different meanings. Now, either vernacular negation is ambiguous or it is not. If it is not, then, since the different theories attribute it different meanings, they cannot both be right. We have a simple theoretical pluralism. The other possibility is that vernacular negation is ambiguous. Thus, it may be argued that vernacular negation sometimes means classical negation and sometimes means intuitionist negation. But if this were right, we would have two legitimate meanings of negation, and the correct way to treat this formally would be to have two corresponding negation signs in the formal language, the translation manual telling us how to disambiguate when translating into formalese. In exactly this way, it is often argued that the English conditional is ambiguous, between the subjunctive and indicative. This does not cause us to change logics, we simply have a formal language with two conditional symbols, say $\supset$ and $\rightarrow$, and use both. This is pluralism in a sense, but the sense is just one of ambiguity. [Priest, 2006, p. 199]

Priest's argument goes as follows: a connective is either ambiguous or unambiguous. If it is ambiguous, we should disambiguate it in a formal language using two different symbols, and this is not pluralism; if it is not ambiguous, there is one theory that correctly captures the meaning of the connective, and one that fails in doing so.

Also Hjortland [2013] notes that pluralism in the consequence relation implies meaning-variance:

> There is, however, an argument to the effect that GTT pluralism entails a corresponding [meaning-variance]-type pluralism. The idea is as follows: different (admissible) precisifications have different classes of cases. But if the meaning of a logical connective is specified truth-conditionally, it is natural to assume that the corresponding truth conditions are given by the class of cases in question. Thus, for the signature classical precisification, i.e., where the cases are classical models, the truth-conditions are *truth-in-a-model*-conditions. Similarly, for Beall and Restall's preferred relevant system, since truth-preservation is *truth-in-a-situation*-preservation, truth-conditions are *truth-in-a-situation*-conditions. [Hjortland, 2013, p. 361]

Priest and Hjortland are clear in their criticism of Beall and Restall. The version of logical pluralism that Beall and Restall defend entail a kind of meaning-variance pluralism. Moreover, as Priest argues, this is an unwelcome consequence. It is necessary to give an explanation or criterion for the sameness of meaning of the logical constants across logics. And they do:

> The clauses can both be equally accurate in exactly the same way as different claims about a thing can be equally true: they can be equally true of one and the same object simply in virtue of being *incomplete* claims about the object. What is required is that such incomplete claims do not conflict, but the clauses governing negation do not *conflict*. [Beall and Restall, 2006, p. 98]

However, there is a further objection regarding this response about the sameness of meaning, raised by Hjortland [2013], according to which Beall and Restall's response alienate the semantics of a connective (that would be the combination of all the partial semantics in each logic) from its inferential role in each logic. There is no logic that underlies the meaning of a connective, but only logics for partial features, and this draws an inconvenient distance between the two dimensions, which otherwise should be much more connected,

> This leads to an uncomfortable gap between the semantics of a connective and its underlying logic. One motivation for the model-theoretic semantics for a connective is that it induces the logic of a connective by determining a class of arguments sound with respect to it. [Hjortland, 2013, p. 364]

In other words, the partial view about the meaning of the connectives for Beall and Restall implies that one cannot explain the logic underlying a connective *because* of its meaning, but just because of a partial dimension of it, leaving the justification for the logic worse justified than for monist theories.

Nihilism or Universalism A related problem is presented by Bueno and Shalkowski [2009], who argue that Beall and Restall's version of pluralism is reduced to either nihilism or to universalism. The source of the criticism is in the necessity constraint that Beall and Restall require for the different logics embraced in their view, together with the universal quantification over cases,

> For the premisses to necessitate a conclusion is a matter of them doing the right thing in all cases. Having recognized that cases may or may not be complete, and that they may or may not be consistent, to do the right thing over all cases is to do the right thing regardless of whether a case is complete or consistent. Consequence relations suited to reasoning about complete situations fail to satisfy the necessity constraint, since by following them exclusively we do not manage our inferences correctly over incomplete cases. [Bueno and Shalkowski, 2009, p. 299]

This observation can easily drive us to nihilism. First, the set of valid inferences across the three logics in Beall and Restall is extremely reduced. Moreover, nothing prevents us from expanding Beall and Restall's pluralism to other logics, and making the common core of valid inferences empty.

In order to avoid nihilism, Beall and Restall could consider the following strategy:

> A natural manoeuvre is to shift the domain of quantification when considering each logic. [Bueno and Shalkowski, 2009, p. 300]

However, that strategy would lead to universalism, as one could ignore counterexamples to any invalid inference, and hence being able to make any inference valid:

> By ignoring the cases that invalidate classical inferences, we can declare that classical logics satisfy the necessity constraint. Likewise for the other well-known logical systems. If we are even more selective in our attention to invalidating cases, however, any set of inferences can be declared to satisfy the

> necessity constraint. Pick any pattern of inference you like. Ignore all invalidating cases. Voila! A genuine consequence relation is born! [Bueno and Shalkowski, 2009, p. 300]

Modalism

A solution suggested by Bueno and Shalkowski for their own objections against Beall and Restall's pluralism leads to another version of the view: modalism. We have seen how Beall and Restall's pluralism emerges from the quantification in the definition of logical consequence over different kinds of cases, which makes necessity depend on truth preservation across different kinds of cases. However, Bueno and Shalkowski argue that logical consequence is genuine only if it ranges over all *possible* cases. This consideration does not prevent us from being pluralists, rather, it opens the possibility of a version of pluralism as a result of varying the 'domain of possible' [Bueno and Shalkowski, 2009, p. 312].

To be more specific, Bueno and Shalkowski define logical consequence avoiding a quantification over cases, but which relies on the notion of 'impossibility':

> An argument is valid if, and only if, the conjunction of its premises with the negation of its conclusion is impossible. [Bueno and Shalkowski, 2009, p. 295]

With this definition of consequence in hand, a plurality of logics emerges if it is legitimate to expand the 'domain of possible', by focusing on different reasoning subjects. It is the case, according to Bueno and Shalkowski, that contradictions are possible in certain domains of discourse [p. 309], but not in others, which gives rise to a version of pluralism.

> There are different ways of expanding the domain of the possible: by entertaining inconsistent situations, by considering incomplete situations such as all and only the contents of one's office, by allowing for scenarios in which the notion of identity cannot be applied to certain things (such as some cases considered in certain interpretations of quantum mechanics). By expanding and exploring the possible in this way, the modalist can identify contexts in which the validity of certain classical inferences is violated. [Bueno and Shalkowski, 2009, 321]

Consider, for instance, that we are reasoning about the world. In this case, we will assume that our domain of discourse is complete and consistent, and classical logic will be appropriate. However, in another situation we might be interested in reasoning about the *claims* that have been made about the world. There will clearly be inconsistencies in all the data that we should included in the domain of the discourse, and these inconsistencies are completely *possible*. Hence, in such a case a different than classical logic is needed, probably a paraconsistent logic.

Reaction This version of pluralism is certainly a version of RLP and, in particular, a version of localism. It is necessary to specify about what we are reasoning in order to endorse or reject an inference such as $A \wedge \neg A \vdash B$. In other words, it is not the interpretation of the logical vocabulary what determines what follows from what, but the particular context in which one is reasoning.

Sequent Calculus

The following three versions of logical pluralism are presented proof-theoretically, admitting more than one legitimate behaviour for the logical consequence operator on a sequent calculus. Before presenting them, let us first introduce the Gentzen sequent calculus. The proofs are formed by sequents, with the following form:

$$A, \Gamma \vdash \Delta, B$$

In general, we can read the sequent as 'Δ or B follows from A and Γ' or 'A and Γ logically imply Δ or B', where Γ and Δ are sequences of formulas, separated by commas, read conjunctively on the left and disjunctively on the right. And we can introduce rules that allow us to go from some sequents to others: they can be rules that affect a logical connective, which are called operational rules, or they can affect the comma, which are called structural rules (See Appendix A for complete calculus systems). As an illustration, let us see the derivation of Disjunctive Syllogism in classical logic LK:

$$\begin{prooftree}
\AxiomC{$A \vdash A$}
\RightLabel{WR}
\UnaryInfC{$A \vdash A, B$}
\RightLabel{ER}
\UnaryInfC{$A \vdash B, A$}
\AxiomC{$B \vdash B$}
\RightLabel{WR}
\UnaryInfC{$B \vdash B, A$}
\RightLabel{$\vee$L}
\BinaryInfC{$A \vee B \vdash B, A$}
\RightLabel{$\neg$L}
\UnaryInfC{$\neg A, A \vee B \vdash B$}
\RightLabel{EL}
\UnaryInfC{$A \vee B, \neg A \vdash B$}
\end{prooftree}$$

The derivation starts with axioms, $A \vdash A$ and $B \vdash B$, and then applies rules that affect both logical connectives ($\vee$L and $\neg$L), or the commas (WR, ER, EL). The result is the sequent $A \vee B, \neg A \vdash B$, that is, B follows from $A \vee B$ and $\neg A$ assuming that the rules and the axioms are valid too.

These rules can be modified: one might present alternative rules for the behaviour of some logical connective, or one can reject some structural rule that affects the commas, and as a consequence affect $\vdash$. This second choice, a plurality of norms for $\vdash$, is taken by the following versions of pluralism.

Intratheoretic pluralism: Hjortland

Hjortland [2013] presents a version of logical pluralism which emerges from a plurality of consequence relations within a single language, which the author denominates intra-theoretic pluralism,

> [A] pluralism not of logical theories but of logical consequence relations within one and the same theory. Let us call it *intra-theoretic pluralism*. [Hjortland, 2013]

As an illustration of the kind of pluralism that Hjortland defends, he considers two logics: Kleene 3-valued logic (K3), and Logic of Paradox (LP). They have the same truth values, $V = \{0, i, 1\}$ and the same interpretation for logical connectives, but in K3 only 1 is a designated value while in LP both 1 and i are. This creates divergences in the consequence relation: the Law of Excluded Middle is invalid in K3 but valid in LP[6], while Ex Falso Quodlibet is valid in L3 but invalid in LP[7]. That is, K3 is paracomplete, and LP is paraconsistent.

As Hjortland shows, we can create a single proof system for both logics, a 3-sided sequent calculus with values $0, i, 1$. Let us see first how a 3-sided sequent system would work. One natural reading for the typical 2-sided sequent $\Gamma \vdash \Delta$ would be 'either Γ is false or Δ is true', that is, 'either Γ has truth value 0 or Δ has truth value 1'. Now, we can change the $\vdash$ symbol for $|$, and instead of having just 0 and 1 as possible truth values, we can have three truth values, 0, i and 1, and express a 3-sided sequent as:

$$\Gamma_0|\Gamma_i|\Gamma_1$$

which can be read as 'either Γ_0 has truth value 0, or Γ_i has truth value i, or Γ_1 has truth value 1'.

Now each structural rule or connective has three rules, which affect each side in the sequent, instead of the typical Right and Left rules in the 2-sided sequent calculus. Consider the following as illustrations:

$$\frac{\Gamma_0|\Gamma_i|\Gamma_1}{\Gamma_0, A|\Gamma_i|\Gamma_1}\,K_0 \qquad \frac{\Gamma_0|\Gamma_i|\Gamma_1}{\Gamma_0|\Gamma_i, A|\Gamma_1}\,K_i \qquad \frac{\Gamma_0|\Gamma_i|\Gamma_1}{\Gamma_0|\Gamma_i|\Gamma_1, A}\,K_1$$

[6]Note that if p has value i, then $\vdash p \vee \neg p$ also has value i and is truth-preserving in LP but not in K3.
[7]Note that if p has value i and q has value 0, $(p \wedge \neg p) \vdash q$ is truth-preserving in K3 but not in LP.

$$\frac{\Gamma_0, A|\Gamma_i|\Gamma_1 \qquad \Gamma_0, B|\Gamma_i|\Gamma_1}{\Gamma_0, A \vee B|\Gamma_i|\Gamma_1} \vee_0 \qquad\qquad \frac{\Gamma_0|\Gamma_i|\Gamma_1, A, B}{\Gamma_0|\Gamma_i|\Gamma_1, A \vee B} \vee_1$$

$$\frac{\Gamma_0|\Gamma_i, A, B|\Gamma_1 \qquad \Gamma_0, A|\Gamma_i, A|\Gamma_1 \qquad \Gamma_0, B|\Gamma_i, B|\Gamma_1}{\Gamma_0|\Gamma_i, A \vee B|\Gamma_1} \vee_i$$

Let us turn now to LP and K3. They share the same rules for connectives and structural rules. But:

a. $\Gamma \vdash_{K3} \Delta$ just in case the sequent $\Gamma|\Gamma|\Delta$ is derivable.

b. $\Gamma \vdash_{LP} \Delta$ just in case the sequent $\Gamma|\Delta|\Delta$ is derivable.

Hjortland [2013] shows the divergence in the extension of the notion of consequence with the following example:

$$\frac{\dfrac{\dfrac{A|A,\neg A|A}{|A,\neg A|A,\neg A}}{|A,\neg A|A \vee \neg A} \qquad \dfrac{A|A|A,\neg A}{A|A|A \vee \neg A} \qquad \dfrac{\neg A|\neg A|A,\neg A}{\neg A|\neg A|A \vee \neg A}}{|A \vee \neg A|A \vee \neg A}$$

While there is a proof for LEM in LP, it cannot be derived in K3. Hence, changes in the derivability properties of $\vdash$ change the extension of logical consequence, without changes in the meaning of the logical constants.

Reaction: Scope of this version of pluralism One possible reaction to this version of logical pluralism is that it lacks motivation for $\vdash_{K3}$ and $\vdash_{LP}$ as legitimate codings of logical consequence. Without such justification, this version of pluralism is closer to being a version of ALP rather than RLP, as it is compatible with the philosophical justification that one of the two, or some other formal consequence $\vdash$, captures genuine logical consequence. But if such justification was provided, it opens the possibility of a genuine and interesting version of logical pluralism, which can be orthogonal to other versions: while it is possible to adopt more than one logic rejecting and endorsing different structural rules, one might open the possibility of a pluralism which emerges with the same structural rules but with different evaluations, as Hjortland suggests.

Meta-inferential pluralism: Barrio et al.

Barrio et al. [2018] open the possibility of a pluralism which embraces classical and the substructural logic ST, a non-transitive logic which is presented in [Ripley, 2012] [Cobreros et al., 2012] [Cobreros et al., 2013].

The pluralism emerges given that these two different logics coincide in all the inferences but not in their meta-inferences. That is, although LK and ST coincide in the sense presented in the introduction ('Δ follows from Γ', page 4) and which we have been defined as the source of disagreement that might lead to logical pluralism, they disagree about the following:

- $\Gamma \vdash \Delta$ follows from $\Gamma' \vdash \Delta'$
- $\Gamma \vdash \Delta$ does not follow from $\Gamma' \vdash \Delta'$

In order to put things more formal, consider the following definitions entirely extracted from [Barrio et al., 2018]. Let $FOR(\mathcal{L})$ be the formulas of $\mathcal{L}$. An inference $\Gamma \vdash \Delta$ on $\mathcal{L}$ is an ordered pair $\langle \Gamma, \Delta \rangle$ where $\Gamma, \Delta \subseteq FOR(\mathcal{L})$, and $SEQ^0(\mathcal{L})$ is the set of inferences in $\mathcal{L}$.

Inferences are evaluated as follows in LK and ST. On the one hand, LK has a Boolean valuation: $v \vDash_C \Gamma \vdash \Delta$ iff it is not the case that $v(A) = 1$ for all A in Γ and $v(B) = 0$ for all B in Δ; and an inference $\Gamma \vdash \Delta$ is valid in LK ($\vDash_C \Gamma \vdash \Delta$) iff it is satisfied by all valuations. On the other hand, ST has a Strong-Kleene-valuation: $v \vDash_{ST} \Gamma \vdash \Delta$ iff it is not the case that $v(A) = 1$ for all A in Γ and $v(B) = 0$ for all B in Δ; an inference $\Gamma \vdash \Delta$ is valid in ST ($\vDash_{ST} \Gamma \vdash \Delta$) iff it is satisfied by all valuations.

With these definitions in hand, Barrio et al. argue for the following fact:

i. For all $\Gamma, \Delta \subseteq FOR(\mathcal{L})$:

$$\vDash_{ST} \Gamma \vdash \Delta \qquad \text{iff} \qquad \vDash_{CL} \Gamma \vdash \Delta$$

That is, LK and ST coincide with respect to the validity of all inferences in the logic. That would make them equivalent, but this is not the case when we focus on the validity of their meta-inferences. In order to see this, we need to know what a meta-inference is and how is it evaluated.

A metainference $\Theta \vdash_1 B$ on $\mathcal{L}$ is an ordered pair $\langle \Theta, B \rangle$ where $\Theta \subseteq SEQ^0(\mathcal{L})$ and $B \in SEQ^0(\mathcal{L})$. Moreover, they are evaluated as follows: on the one hand, a valuation of a meta-inference $v \vdash_K \Theta \vdash_1 B$ iff $v \nvDash \theta$ for some $\theta \in \Theta$, or $v \vDash_C B$; a meta-inference is valid in C ($\vDash_C \Theta \vdash_1 B$) iff it is satisfied by all boolean valuations. On the other hand, a meta-inference is valid in ST ($\vDash_C \Theta \vdash_1 B$) iff it is satisfied by all SK-valuations.

With these definitions in hand, we can present the second fact (let $\Rightarrow$ expresses the entailment connection between some group of sequents and another sequent):

ii. There are some $\Theta \subseteq SEQ^0(\mathcal{L})$ and $B \in SEQ^0(\mathcal{L})$ such that

$$\nvDash_{ST} \Theta \Rightarrow B \qquad \text{and} \qquad \vDash_{LK} \Theta \Rightarrow B$$

That is, LK and ST do not coincide with respect to their meta-inferences. For instance,

$$\frac{\vdash A \qquad \vdash \neg A}{\vdash B}$$

is valid in LK but not in ST, given that it requires to use transitivity (or cut) in order to be derived. And this is why ST can manage paradoxes like the liar, but LK trivializes with them.

Reaction Again, we lack a proper justification of the philosophical reasons for endorsing each logic of the view. However, with proper justification that both LK and ST codify legitimate rules for *reasoning about logics*, this could be a version of RLP. While many interesting versions of pluralism disagree at the first level of validity, this version of pluralism is orthogonal to the position one might have about pluralism at this lower level, and it opens the possibility of a further pluralism once a particular logic at the lower level is determined. In other words, the view shows that there is more than one way in which a logic can be one among many: either because its valid inferences disagree with those of other logics, or because its meta-inferences do, or both.

Structural pluralism: Restall

Restall suggests a version or presentation of pluralism different from (but compatible with) his own (and Beall's) theory presented above. In this alternative presentation, he considers the Gentzen sequent calculus as a presentation of different logics, and endorses different admissible logical consequence formalizations varying the admissible structural rules, while keeping the rules of the logical constants fixed. Consider, as Restall does, negation. In classical and intuitionistic logic negation has the following right and left rules:

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\neg L \qquad\qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\neg R$$

However, its behaviour diverges from classical to intuitionistic logic due to restrictions on structural rules which affect $\vdash$. In classical logic, there can be more than one formula on the right side of the sequent, while in intuitionistic logic the number of formulas that can appear on the right side of the formula is limited to one. Hence, ¬R in intuitionistic logic can only be applied when Δ is empty. This difference in the structural rules for ⊢ invalidates the following inference in intuitionistic logic:

$$\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{\vdash \neg A, A}\neg L}{\vdash A \vee \neg A, A}\vee R}{\vdash A \vee \neg A, A \vee \neg A}\vee R}{\vdash A \vee \neg A}CR$$

If we agree with Restall that the right and left rules are meaning constitutive for ¬ while structural rules, despite affecting its behaviour, do not affect its meaning, then we must agree with him that Bivalence is not meaning-constitutive for negation (its derivation requires that two formulas can appear on the right side of a sequent). Moreover, if we agree that both the classical and intuitionistic formalizations of ⊢ are admissible (that is, we have reasons both to accept and reject the possibility that there can be more than one formula on the right side of a sequent), then we must endorse both classical and intuitionistic logic as two logics which share the same language. That is, we endorse a version of consequence pluralism.

Reaction: the meaning of the logical constants A first reaction to Restall's version of pluralism is that the meaning of the logical constants is extremely reduced. Some simple examples will clarify the point: Restall concludes that Bivalence is not meaning-constitutive for negation, which might be easily endorsed by a philosopher with pluralist intuitions: we still grasp the meaning of 'not' even if bivalence is not part of its meaning. However, consider the conjunction: there are simple inferences that are not meaning constitutive for it (if the theory is correct), but without which we seem to be unable to grasp the meaning of 'and', for instance:

- $A \wedge B \vdash B \wedge A$,
- $A \vdash A \wedge A$

None of these two inferences can be derived without structural rules. This implies that these very simple derivations are not meaning-constitutive for the conjunction, but depend on the nature of ⊢. The counterintuitive result of this fact is that the meaning of the connective alone does not determine most of our intuitions about the meaning of 'and', and it is not clear even for logicians (let alone competent speakers of English) what is the meaning of 'and'.

Reaction: justification for the plurality of $\vdash$ A second reaction to Restall's presentation of consequence inferential role pluralism is that we are not presented with convincing reasons to endorse or reject the different structural rules for $\vdash$. For instance, we do not have a justification for the fact that one or more formulas are admissible on the right side of a sequent. And this criticism can be extended to the rest of the structural rules: without a justification for why weakening, contraction or exchange should be endorsed or rejected for formalizing 'follows from' we do not have a justification for a version of RLP, and we only have a version of ALP, given that their rejection might imply that we are failing to capture logical consequence.

2.3 The route to Logical Pluralism: Pragmatic Logical Pluralism

In this last section, we introduce the present version of logical pluralism, *Pragmatic Logical Pluralism*, motivating the reasons for endorsing it; and we situate this version with respect to all the previous defences of logical pluralism, pointing out their similarities and differences.

2.3.1 Pragmatic Logical Pluralism

Logical consequence

It is evident from our classification of levels of pluralism that we are seeking a version of pluralism that makes compatible apparently rival claims about the validity of certain inferences. That is, a version of logical pluralism that makes compatible different notions of consequence. Following the proof-theoretical view on logical consequence of Gentzen [1964], Prawitz [1971], [1985], and more recently Read [2015] and Shapiro [2005], we propose a version of logical pluralism based on the existence of a proof from the premises to the conclusions[8]:

> Δ is a logical consequence$_x$ of Γ iff there is a deduction of Δ from Γ by a finite number of legitimate$_x$ rules of inference.

The pluralist thesis emerges from the plurality of legitimate deduction rules for one language. The second part of this dissertation will argue for such plural legitimacy. We will defend that there are at least three different notions of 'logical consequence' by varying the notion of legitimacy in the previous definition. First, logical consequence can be understood as truth preservation,

[8]We leave aside semantics characterizations of logical consequence.

LK: a rule of inference from Γ to Δ is legitimate if it preserves truth.

Second, logical consequence can be seen as a relation between premises and conclusions in which the conclusions are the result of performing certain operations with the premises,

LR: a rule of inference from Γ to Δ is legitimate if operating with Γ we get Δ.

Third, logical consequence can be seen as a relation between premises and conclusions in which the conclusions are the result of the use of some premises (in which the *use* might possibly change the initial conditions expressed by the premises, as will be detailed in Chapter 5),

LL/*LO*: a rule of inference from Γ to Δ is legitimate if it uses Γ to get Δ.

Note that we have criticised the lack of justification of this plurality of consequence relations in the majority of proposals presented, arguing that without a proper justification of the reasons for endorsing the different formalizations of logical consequence we only have a version of ALP rather than RLP. We aim to avoid this unwelcome consequence, and hence, the plurality of rules of inference will be explained with a proper justification of the reason why this plurality of rules capture genuine consequence.

Logical Connectives

Pragmatic Logical Pluralism also embraces a variety of formal languages which codify different behaviours of the logical connectives. It is well known that each logical consequence relation presented above determine different behaviours for the logical connectives. Without further explanation, this would be enough to criticise pragmatic logical pluralism with a similar criticism to Beall and Restall's pluralism: it is not a pluralism about consequence but about languages, and hence, motivated by equivocation rather than by a genuine plurality. However, this dissertation aims to justify that each divergent behaviour of the logical connectives in the different logics respond to different dimensions of the *same* logical connectives. In particular, we argue that each logic embraced captures various pragmatic enrichments that 'if...then', 'or' and 'and' exhibit in natural language.

Finally, our logical pluralism will consider that the truth bearers that play the role of premises and conclusion in derivations are *utterances*, rather than *propositions*. One and the same utterance might express different propositions depending on the particular pragmatic enrichments that one might derive from it. Hence, restricting our discourse to utterances, we are capable of considering different interpretations, and thus different inferential roles, for a unique set of premises.

2.3.2 The situation of Pragmatic Logical Pluralism in the debate

The version of logical pluralism we defend is indebted to many of the versions seen above. It shares with all of them the aim of making compatible more than one logic and coincides with some of them in certain diagnostics about the reconciliation of rival logics. Moreover, it will be argued that Pragmatic Logical Pluralism is a version of Reasoning Logical Pluralism.

Pragmatic logical pluralism can be seen as another presentation of Logical Pluralism as a Consequence Pluralism, in the spirit of Restall's [2014], and Hjortland's [2013] (and not so much in line with Barrio et al.'s [2015] given the focus on inferences rather than meta-inferences) but which solves the problems already mentioned for these proposals. Moreover, it is orthogonal to Hjortland's version, given that it endorses different logics, and it ranges over two-sided sequent calculi. What Pragmatic Logical Pluralism shares with both Hjortland and Restall's proposal is that the origin of the pluralism is on the proof-theoretical notion of consequence. In particular, it accepts that there is more than one proof-theoretical symbol $\vdash$ that legitimately encodes 'follows from', even though there are important differences between the proposals: our treatment of the logics focuses on the differences among the languages in the different logics, explaining these differences as a requirement for the plurality of consequence relations.

Pragmatic Logical Pluralism shares with Allo [2013] the range of logics it can endorse, but disagrees on the explanation of the origin of this plurality. We agree with Allo on the intuitions about the need of making compatible substructural languages with a stronger logic (classical logic in our case), but the present proposal disagrees with the ambiguity proposal and interprets substructural languages as pragmatically enriched versions of the classical language. Hence, there is no *defective* language, in the sense that none of the languages is either noisy or ambiguous. That is, we are capable of explaining why logical constants have different inferential roles without needing to consider that some of the languages embraced do not legitimately capture the behavior of logical connectives.

Moreover, it has similarities and differences with the spirit of Carnap's Tolerance and the influence that Carnap has on Shapiro [2013], Cook [2010], Eklund [2012b], and Kouri Kissel [2019]. The common idea with Carnapian tolerance is the acceptance of different inferential roles for the logical connectives. However, there are significant differences between them: not any inferential role, even if it is justified within a language framework, is accepted for the logical connectives. We suggest that the different inferential roles for the logical connectives will respond to a pragmatic enrichment. Hence, we have a justification for why 'if...then', 'or' and 'and' diverge in different frameworks. In sum, those reasons that make Carnap's proposal a version of Applied Logical Pluralism rather than Reasoning Logical Pluralism are rejected, and it can be justified that the logics embraced here capture genuine logical consequence. We will argue that there is a linguistic connection between natural language and a plurality of formal languages, similarly to Kouri Kissel [2018], but contrary to her view, this phenomena will not be a polysemous relation, but a relation between the literal meaning and its possible pragmatic enrichments. This fact implies that there is no meaning variance across logics, and can better articulate the connection between the behaviour of logical connectives in the different logics.

In sum, Pragmatic Logical Pluralism will naturally explain how more than one formal language captures legitimate senses of logical vocabulary[9], focusing on (i) the different legitimate notions of logical consequence and (ii) the inferential roles and truth conditions of the logical constants. Both (i) and (ii) will be interpreted as different pragmatic enrichments of one language, whose literal meaning is captured by classical logic. Moreover, contrary to other versions of logical pluralism, the plurality of formal languages emerge as a natural and desirable consequence of this pragmatic interpretation.

[9]We will use the term *logical vocabulary* to refer to both logical connectives and logical consequence, distinguishing it from *logical connectives* or *logical constants*.

CHAPTER 3

Semantics and Pragmatics of Logical Vocabulary

The present dissertation aims to endorse more than one logic, that is, we argue for the legitimacy of more than one language, both for the codification of 'follows from' and for the logical constants 'and', 'if...then' and 'or'. To defend a version of Reasoning Logical Pluralism, as opposed to a version of Application Logical Pluralism we need to demonstrate that the different formal languages that codify the logical vocabulary can be legitimately used to encode logical consequence and the correct inferential role of the logical constants.

In this chapter, we will turn our attention to the mechanisms that explain how there can be more than one legitimate codification of the logical vocabulary. And we will do this focusing on the philosophy of language, and on the different theories about the meaning of logical particles. The philosophy of logic and the philosophy of language have a common interest in determining the semantics of the logical particles. And although the philosophy of language is primarily interested in the use of the logical constants in natural language, we can learn about their inferential role and truth conditions in this tradition. Moreover, certain dimensions of the logical connectives which have caught the attention of philosophers of language have been ignored by logicians, but with the recent growth of substructural logics, these dimensions can be captured in a formal language. Hence, there can be a fruitful dialogue between the two traditions.

3.1 The logical vocabulary

3.1.1 Ordinary vs Ideal language

We have seen in Chapter 1 that there seems to be a mismatch between the natural language connectives 'if...then', 'or', and 'and' and the formal language symbols $\supset$, $\vee$, $\wedge$. This mismatch is not only about our actual use of the connectives, which can be biased and defective (see for instance the experiments on the actual use of the conditional by competent speakers of English, in what is known as *the Wason selection task* in [Stenning and Van Lambalgen, 2008]), but also about a pre-theoretic normative dimension of the vocabulary. Not only do we understand different things by 'and' and $\wedge$, but in certain contexts, it seems that 'and' *should* follow different rules from the rules for $\wedge$. It is this second mismatch that is the important one for this dissertation and for the discussion of logical pluralism in general.

Consider again the illustrations in the Introduction, in which this mismatch is motivated:

1. The sun will come up tomorrow, hence if it doesn't, it won't matter. [Jackson, 1979]

2. If you pay 5 euros and you pay 5 euros, you get a packet of Camels and you get a packet of Marlboros. Hence, if you pay 5 euros, you get a packet of Camels, and you get a packet of Marlboros. [Girard, 1995]

3. If the old king dies of a heart attack and a Republic is declared, Tom will be content. A Republic will be declared tomorrow. Hence, if the old king dies tomorrow, Tom will be content. [Cohen, 1971]

Consider, in particular, the simple sentences that are contained in them and which are the origin of the mismatch between natural language and the formal counterpart:

4. If the sun doesn't come up tomorrow, it won't matter.

5. If you pay 5 euros and you pay 5 euros, you get a packet of Camels, and you get a packet of Marlboros.

6. The old king will die of a heart attack and a Republic will be declared.

These are the particular uses of 'if...then', 'or' and 'and' that depart from their classical counterpart, and somehow seem to express something different from their classical truth conditions: the 'if...then' in 4 expresses a connection between the antecedent and consequent; the 'and' in 5 expresses an accumulative reading of paying, in which you need to pay twice in order to get the two packets of cigarettes; the 'and' in 6 expresses an order relation between the two conjuncts. A question to be formulated is whether such information that distinguishes them from $\supset$, $\vee$ and $\wedge$ can or should be encoded. That is, the question is whether there can be a formal language (classical or otherwise) that captures the meaning of the uses of the connectives in natural language, and which could encode their inferential role in order to invalidate the inferences 1-3.

There are two main theories about the connection between natural and formal language: the *ideal language philosophy* and the *natural language philosophy* (see the distinction in [Recanati, 2004]). A useful simplification of the divergence between the two traditions is to say that according to the ideal language philosophers, a formal language *illuminates* the genuine structure of natural language, while according to natural language philosophers, a formal language *simplifies* what natural language expresses. But both aim to find the connection between 'and' and $\wedge$.

On the one hand, Strawson [2011] was on the natural language philosopher side when he argued that it is wrong to identify formal language with their natural counterparts:

> We have to ask, with regard to each of those constants, whether there is any expression of ordinary speech which has at least a standard use identifiable with the meaning of that constant. It is quite common to suggest, with certain reservations, the following identifications: '$\neg$' with 'not', or 'it is not the case that'; '$\wedge$' with 'and'; '...$\vee$...' with 'either ...or ...'; '...$\supset$...' with 'if...then...'; '...$\leftrightarrow$...' with 'if and only if'. Of these identifications the first two are the least misleading. We shall find that the remainder are not only misleading, but definitely wrong. [Strawson, 2011, p. 78]

This thesis, applied to the previous examples, would motivate us to look for the normative role of 'and', 'if...then', and 'or' in 1-3 outside logic: the normative role of $\wedge$ does not necessarily apply to the normative role of 'and'.

On the other hand, the intuition behind the ideal language philosophy can be summarised in what Geach calls the 'Frege point': a proposition has the same content independently of whether it is asserted or not, as there is a content which is immutable and independent of the conditions under which the utterance that expressed the proposition is used. This is the content that can be expressed in a formal language.

> The aim of the formalists, I think, was to achieve the precise expression of scientific truths in a representation system which would wear its logic on its face, its logical implications following transparently from its form, as is the case with the predicate calculus, for instance. [Carston, 2002, p. 48]

The divergence with the ordinary language philosophers is the view that there can be a formal language that captures the structure of the imperfect natural language. The defective (according to Strawson) identification between 'and' and $\wedge$ can be solved, if not for the classical $\wedge$, at least for one conjunction in some alternative formal language.

> I believe that I can best make the relation of my ideography to ordinary language clear if I compare it to that which the microscope has to the eye. Because of the range of its possible uses and the versatility with which it can adapt to the most diverse circumstances, the eye is far superior to the microscope. Considered as an optical instrument, to be sure, it exhibits many imperfections, which ordinarily remain unnoticed only on account of its intimate connection with our mental life. But, as soon as scientific goals demand great sharpness of resolution, the eye proves to be insufficient. The microscope, on the other hand, is perfectly suited to precisely such goals, but that is just why it is useless for all others. [Frege, 1879, p. 6]

Hence, contrary to the natural language conclusion about 1-3, according to this last tradition we should either explain how the normative role of the logical connectives in natural language corresponds to their classical counterparts, or abandon classical logic in favour of some divergent logic.

Finally, there is a third position based on a connection between formal and natural language, which can be seen as a mid-point between the two traditions, and which will be discussed in this chapter. Although Strawson was in the natural language tradition, he already recognised some parallelism between natural and formal language,

> But we shall also find that even the most mistaken of these identifications [between formal and natural language] has a point: we shall find not only some degree of formal parallelism (which could be noted independently of interpretation) but some degree of interpenetration of meanings of the interpreted expressions of the system and of ordinary speech respectively. We could not, of course, find the latter without the former. [Strawson, 2011, p. 78]

The intuition is the following: we should not fully identify 'and' with $\wedge$, but we should find some connection between the two. And it will be in this connection, together with the divergence we find between the natural and formal language of logic that we can find the ingredients for the pluralist thesis defended here.

3.2 The Gricean reconciliation

3.2.1 Conversational implicatures

Let us begin with the most important proposal of a connection and reconciliation between the two traditions: the Gricean theory. Grice agreed with Strawson that there is a mismatch between formal and natural language.

> It is a commonplace of philosophical logic that there are, or appear to be, divergences in meaning between, on the one hand, at least some of what I shall call the FORMAL devices- $\neg$, $\wedge$, $\vee$, $\supset$, $(\forall x)$, $(\exists x)$, (ιx) (when these are given a standard two-valued interpretation)- and, on the other, what are taken to be their analogs or counterparts in natural language- such expressions as *not*, *and*, *or*, *if*, *all*, *some*, (or *at least one*), *the*. [Grice, 1989c, p. 41]

However, Grice was capable of finding an explanation for the mismatch, identifying the connectives in natural language with their formal counterparts, captured in the following two conclusions,

> (i) My main initial effort has been to develop the idea that the conventional (lexical) meaning of 'if' is that which is provided by a truth-table for material implication. (ii) Thought a stronger conditions than that provided by the truth-table is often implied, it is a mistake to regard this implication as lexical in origin, rather than as a conversational implicature. [Grice, 1989b, p. 83]

Before entering into the particular mismatch between natural and formal language, we will see what an implicature is in the Gricean theory. Consider the following dialogue between two spectators of a Jazz concert:

7. a. Are you enjoying the concert?
   b. I am not a fan of Jazz music.

The interlocutor is not answering the question (which requires a positive or negative answer), but she has *implicated* a negative response, without saying it. This content, which is not stated but is conveyed in the dialogue, is an implicature.

With more detail, an implicature is a content of an utterance p which is not part of its literal meaning, but which can be derived from the act of saying p. We can distinguish different kinds of implicatures. First, an implicature is generalised whenever it does not depend on the particular context of the utterance of p, but is only associated with the content expressed; otherwise, it is particularised. Second, an implicature is conventional whenever it is part of the linguistic meaning of p but does not affect the truth conditions of p (for instance, the contrast between conjuncts in 'but'), and it is conversational whenever it can be derived from the assumption that the speaker is observing the following Cooperative Principle,

> Make your conversational contribution such as is required, at the stage at which it occurs, by the accepted purpose or direction of the talk exchange in which you are engaged. [Grice, 1989c, p. 45]

Which is specified by the following four maxims (and submaxims):

- Quality: try to make your contribution one that is true: ((i) do not say what you believe to be false; (ii) do not say that for which you lack adequate information).

- Quantity ((i) make your contribution as informative as is required; (ii) do not make your contribution more informative than is required).

- Relation: be relevant.

- Manner: be perspicuous ((i) avoid obscurity of expression; (ii) avoid ambiguity, (iii) be brief; (iv) be orderly).

And now we have the mechanism to explain how the negative answer in the previous dialogue emerged: assuming that the interlocutor is being cooperative, and with her contribution she is trying to be relevant to the question, one can infer that her answer implicates (although does not actually say) that she is not enjoying the concert.

There are two characteristics that can help us see that this is not said, but implicated[1]: First, cancellation [Grice, 1989c]. One can cancel the implicature carried by some utterance without contradicting what has already been said. For instance, the interlocutor in the previous dialogue might add the following (b′), without contradicting herself:

7′. a Are you enjoying the concert?

b′ I am not a fan of Jazz music. *But this band is wonderful.*

Second, reinforcement [Levinson, 2000]: one can explicitly say what has been merely implicated, without repeating or making a redundant contribution. For instance, the interlocutor might add the following (b″):

7″. a Are you enjoying the concert?

b″ I am not a fan of Jazz music. *And this is no exception.*

[1] Grice considers up to five characteristics, however, for our purposes the following two (which are usually considered as the main ones) are sufficient.

3.2.2 A minimal sense of what is said

In the Gricean picture, we can reconstruct the process by virtue of which a given utterance expresses more information than what it actually encodes. In Recanati's words:

> Interpretation is construed as a two-step procedure: (i) The interpreter accesses the literal interpretations of all constituents in the sentence and uses them to compute the proposition literally expressed, with respect to the context at hand; (ii) on the basis of this proposition and general conversational principles he or she infers what the speaker means (which may be distinct from what is said, that is, from the proposition literally expressed). [Recanati, 2004, p. 27]

Consider the previous dialogue in the Jazz concert. The interpreter has access to the literal meaning of the utterance, according to which the interlocutor is not a fan of Jazz music. With this information in hand, and the fact that the interlocutor has said this in answer to the question about whether she is enjoying the concert, the interpreter can derive the implicature that the interlocutor is not enjoying the concert: if she was, she would have said so.

The process, in more detail, is composed of two steps. The first step is saturation, which connects the sentence meaning with what is said,

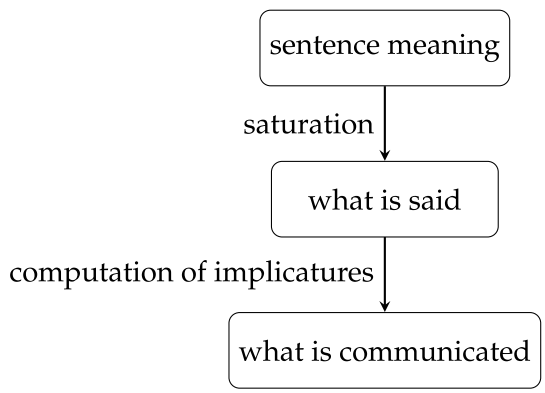
> Saturation is the process whereby the meaning of the sentence is completed and made propositional through the contextual assignment of semantic values to the constituents of the sentence whose interpretation is context-dependent (and, possibly, through the contextual provision of 'unarticulated' propositional constituents, if one assumes, as some philosophers do, that such constituents are sometimes needed to make the sentence fully propositional). This process takes place whenever the meaning of the sentence includes something like a 'slot' requiring completion or a 'free variable' requiring contextual instantiation. [Recanati, 2004, p. 7]

In this process of saturation, reference fixing of the indexicals and disambiguation are the only two phenomena in which the context of the utterance, and hence some pragmatic information, contributes to the meaning of the proposition expressed by an utterance.

> The small gap between the conventional linguistic meaning that contributes to truth-conditional content and the complete truth-conditional content or 'what is said', is, on Grice's conception ..., bridged by reference assignment and by

> sense selection in the case of ambiguous words or structures. [Carston, 2002, p. 105]

Secondly, after the process of saturation, we have a complete proposition, identified with *what is said*. This is identified with the meaning of the utterance and is its semantic dimension. This stage gives rise to the Gricean optional process of the derivation of implicatures, with which we reach the stage of what is communicated. In sum, the Gricean process is summarised as follows by Recanati [2004]:



The connectives in the Gricean analysis

This general explanation of implicatures is also the theory that explains the mismatch between formal and natural language. With the Cooperative Principle and the Maxims in hand we distinguish between what is said and what is implicated by the particles 'and', 'or', 'if...then', identifying their meaning with *LK*'s formalization and explaining pragmatically the apparent mismatch, under the assumption that our discourse is governed by the CP and the maxims.

In the particular case of logical vocabulary, the first pragmatic contribution of saturation does not affect logical vocabulary, according to the Gricean view. Hence, its contribution is fixed at the level of sentence meaning, remains unaltered by saturation and contributes with its literal meaning to what is said, and can communicate something different to its literal meaning by the optional process of the derivation of implicatures.

Consider the examples 4 - 6. The Gricean can explain, first, that 4 expresses a connection between the antecedent and the consequent of the conditional because we assume that the speaker is observing the maxim of quantity. Second, that 5 expresses an accumulation of actions (rather than an iteration of the same information) given that we assume that the speaker is observing the maxim of manner, and in particular the submaxim 'be brief'. Third, that 6 expresses an ordered sequence of conjuncts by observing the maxim of manner, and in particular, the submaxim 'be ordered'.

In conclusion, following Grice, we have a 'pragmatic account', rather than a 'semantic account' of the divergences between natural and formal language for the logical constants. Given that Grice can explain the apparent mismatch between the natural 'if...then', 'and', 'or' and the classical $\rightarrow$, $\wedge$, $\vee$, we can say that classical formal language captures the semantic dimension of the logical constants and ignores their pragmatic enrichment.

3.3 Pragmatic contributions to *what is said*

3.3.1 Implicatures and Enrichments

Philosophers of language such as Recanati [2004], Carston [2002], or Bach [1994], have argued that the pragmatic contribution to what is said is not limited to the indexical fixing and disambiguation of ambiguous words. There are other pragmatic contributions which determine the content of an utterance. One simple way of illustrating this fact is with the following phenomenon of embedded implicatures [Recanati, 2003], or intrusive constructions [Cohen, 1971], which affect all the Gricean theory, and in particular their explanation of the meaning of the logical connectives.

Embedded implicatures We have seen how Grice is capable of explaining the mismatch between classical formal language and natural language. And we have seen how a speaker, by saying p, means q, although the literal meaning of p does not change and q is the result of p plus its implicatures. In the Gricean picture, this pragmatic contribution is calculated once a complete utterance p has been uttered.

However, the embedding of certain expressions under the scope of a logical operator sometimes bears its conversational implicature, which contributes to the meaning of the whole expression. The is known as 'intrusive construction' [Cohen, 1971] or 'embedded implicatures' [Recanati, 2003], and the Gricean schema cannot be extended to them. This limitation was illustrated by Cohen with an example we have already seen, 3, or by Carston with the illustration 8. Moreover, we will argue that certain paradoxes of the conditional are explained by the same phenomena, as the example 9 by Cooper,

3. If the old king dies of a heart attack and a Republic is declared, then Tom will be quite content. [Cohen, 1971]

8. Driving home and drinking several beers is better than drinking several beers and driving home. [Carston, 2002]

9. It is not the case that if there is a good god, the prayers of evil people will be answered. [Cooper, 1968] (as cited in [Priest, 2008, p. 15])[2]

The natural way to understand 3, 8-9 is the following, in which the pragmatics associated to 'and', 'three' and 'some' contribute to the meaning of the whole expression:

3′. If the old king dies of a heart attack and *then* a Republic is declared, then Tom will be quite content.

8′ Driving home *and then* drinking several beers is better than drinking several beers *and then* driving home.

9′ It is not the case that *there is a connection* between the existence of a good god and the prayers of evil people being answered.

This phenomenon of embedded implicatures escapes the Gricean analysis, given that they do not emerge as a consequence of the semantic meaning of an utterance. This poses two problems regarding the Gricean view: first, it changes the natural explanatory order of the implicatures. The Gricean explanation requires a proposition to be expressed as a ground for deriving the implicature, but in these embedded cases the implicature is not the result of a proposition expressed. Second, if the implicatures are required to compute the meaning of the whole utterance, the division between semantics and pragmatics is blurred, and their connections should be reformulated, as the clear cut that the Gricean theory suggested is not so clear.

The phenomenon is widely recognised (although the explanations it has received diverge), and even Grice was aware that pragmatic enrichments were a problem for his theory:

> It certainly does not seem reasonable to subscribe to an absolute ban on the possibility that an embedding locution may govern the standard non-conventional implicatum rather than the conventional import of the embedded sentence [Grice, 1989d, p. 375]

Having reached at this point, we can see that the counterexamples to classical logic in 1-3 seen above (p.50) are counterintuitive because the implicatures that arise in 4-6 are intuitively embedded, and classical logic fails to capture this. In other words, the Gricean theory can explain the implicature of the simple sentences when they are isolated (4-6), but not the whole utterance in which they are embedded (1-3).

[2] The case of conditionals is especially interesting given the amount of bibliography that it has generated. In Chapter 4 we will better illustrate the explanation outlined here, and in Chapter 6 we will apply the Pragmatic Pluralist Thesis to solve the paradoxes of the material conditional.

From embedded implicatures to enrichments Given that embedded implicatures are a different phenomenon from Gricean implicatures, we should abandon the term 'implicature' for them. Hence, following Recanati [2004][3] let us call these embedded contents *pragmatic enrichments*, or simply *enrichments*, to contrast them with implicatures, which emerge in a Gricean manner.

It is important to notice that implicatures and enrichments are two different things. There is no change from an implicature to an enrichment if an implicature is embedded. That is, it is not the case that the order of a conjunction, for instance, is a simple implicature when the conjunct is just asserted, and becomes something different when it is embedded.

> These observations are at one with the view that pragmatic inference plays a fundamental role in determining the proposition expressed; however, they do not have to be taken as entailing that what is an implicature (a propositional form distinct from the proposition expressed) of a simple sentence/utterance changes its status when that simple sentence is embedded, becoming then part of the proposition expressed (the truth-conditional content). Rather, we have a pragmatic contribution to the proposition expressed in both cases (unembedded and embedded) and an implicature in neither. [Carston, 2002, p. 193]

The connection between embedding and enrichment is that a certain content is an enrichment rather than an implicature *if* it is embeddable. In order to distinguish the first from the second pragmatic contribution to a certain utterance, Recanati suggests the following Scope Principle:

> Scope Principle: A pragmatically determined aspect of meaning is part of what is said (and, therefore, not a conversational implicature) if - and, perhaps, only if - it falls within the scope of logical operators such as negation and conditionals. [Recanati, 1993, p. 271]

Carston also has a similar view,

> The interesting fact is that some pragmatically derived meaning does fall in the scope of logical operators and some does not, so that we have a test for distinguishing pragmatic contributions to the proposition expressed from conversational implicatures. [Carston, 2002, p. 193]

One can embed the enrichment of 'if the sun goes out of existence, it won't matter', but one cannot embed the implicature of 'I am not a fan of Jazz'. Consider the following modification of an example by Carston [2002]:

[3]Recanati in [2003] refers to them as 'embedded implicatures'; however, he abandons this term in [2004].

10. a. Are you entitled to a compensation?
    b. Well, there was a hole and I fell.

10b. implicates an affirmative answer to the question 10a., so the subject is entitled to a compensation (this is derived assuming the maxim of Relation). But in order to compute this affirmative answer we need to interpret a conjunction that departs from the classical $\wedge$, as the two conjuncts alone do not entail the implicature, but only their connection. This example illustrates the difference between the two kinds of pragmatic content: enrichments and implicatures. And only the first can be embedded, although they do not need to be embedded in order to emerge.

3.3.2 An alternative notion of what is said...

In the Gricean picture, implicatures and enrichments are mixed up. Hence, now that we are aware of the phenomenon of enrichment, which affects what is said, we should abandon the previous two-step schema. We need a modification of the very notion of what is said, in which disambiguation and reference-fixing of the indexicals are not the only pragmatic contributions, but in which we include pragmatic enrichments.

We have two alternatives: either we substitute the Gricean minimal sense of what is said presented above for an enriched one, and allow the enrichments at the same level as disambiguation and reference fixing, or we add a new sense of what is said, which coexists with the minimal one.

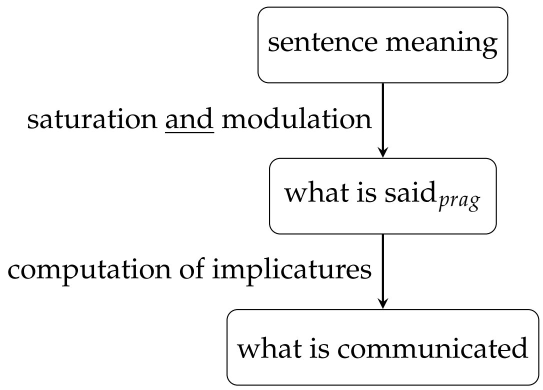
The first option, to substitute the minimal Gricean notion of what is said for a richer one, is a suggestion from Recanati [2003], explained by the process of free enrichment,

> Various pragmatic processes come into play in the very determination of what is said; not merely saturation - the contextual assignment of values to indexicals and free variables in the logical form of the utterance - but also free enrichment and other processes which are not linguistically triggered but are pragmatic through and through. [Recanati, 2004, p. 21]

According to Recanati, we need to distinguish two moments in the pragmatic contribution to an utterance: first, the *primary pragmatic processes*, which are those in which certain pragmatics are required in order to compute the complete meaning of the utterance. Recanati includes the phenomenon of saturation but also that of *modulation*, which refers to those pragmatic contributions required to understand what is said: 'pragmatic processes at work in the very determination of what is said' [Recanati, 2003, p. 316]. The primary pragmatic process is then composed of saturation and modulation.

Second, the *secondary pragmatic processes*, which are those that the traditional Gricean view considers as the pragmatic computation. They require a complete utterance p in order to derive the information that q. These ‘are post-propositional inferences à la Grice: in interpreting an utterance, what is implied, in the intuitive sense, is inferentially derived from the speaker’s saying what s/he says‘ [Recanati, 2003, p. 316].

Compare the previous minimal schema with the following, to which Recanati refers to as the *availability based approach* [Recanati, 2004, p. 16],



About modulation,

> This family of primary pragmatic processes I call ‘modulation‘, as opposed to saturation (...). Modulation takes as input the meaning of some expression (whether simple or complex) and returns as output a pragmatically derived meaning serving as compositional value. [Recanati, 2003, p. 319]

What is distinctive about Recanati’s [2004] approach, is that both saturation and modulation contribute to determining one and the same level: there is no sense of what is said that is prior to the different pragmatic enrichments:

> This puts pragmatic enrichment in the same ballpark as the assignment of contextual values to indexicals and free pronouns. [Recanati, 2013, p. 2]

Carston in [2002] and [2013] has a similar diagnostic for what is said, according to which there is no minimal proposition expressed without some pragmatic contribution.

> [W]e have to distinguish two notions which (...) have been run together: there is linguistic meaning, the information encoded in the particular lexical-syntactic form employed, and there is the thought or proposition which it is being used to express, that is, what is said. [Carston, 2002, p. 17]

The thesis depends on her underdeterminacy thesis.

> Linguistic meaning underdetermines what is said (...) the linguistic semantics of the utterance, that is, the meaning encoded in the linguistic expressions used, the relatively stable meanings in a linguistic system, meanings which are widely shared across a community of users of the system, underdetermines the proposition expressed (what is said). [Carston, 2002, p. 19]

Consider the following two utterances (from [Carston, 2002]):

11. I haven't eaten lunch

12. I haven't eaten frogs' legs

Although they are both complete propositions, according to Carston the contribution to *what is said* of 'I haven't eaten' is different in each case: in normal circumstances, in the first case, it expresses that the speaker has not eaten lunch today, and in the second, the speaker has not eaten frogs' legs in her life. In sum: sentence meaning is not equivalent to *what is said* [Carston, 2002, p. 49],

> [W]hat examples [above] demonstrate is that disambiguation, indexical resolution and *enrichment* are pragmatic processes that contribute to the recovery of the proposition expressed by an utterance. [Carston, 2002, p. 98][4]

We can, therefore, identify Carston's [2002] view on what is said with Recanati's what is said$_{prag}$, in contrast with the Gricean what is said$_{min}$.

3.3.3 ...or an additional sense of what is said?

We will not endorse Recanati's or Carston's model in this project. One reason for this is that, as Bach [1994] argues, in Recanati's model one cannot distinguish 13s. from 13f., given that there is no level in which one considers 13s. without the pragmatic enrichment, or modulation:

13. If they married and had a child, their parents must be happy,

13s. **Saturation**: If they [Ali and Bel] married and had a child, their parents must be happy

[4]Emphasis added.

13f. **Free enrichment**: If they [Ali and Bel] married and had a child [in this particular order] their parents must be happy.

According to Recanati, there is no level of *what is said* before the free enrichment. However, we would not say that 13s. says the same as 13f. Moreover, we are able to grasp a minimal sense of meaning before free enrichment, independent of the intention of the speaker. This is especially important for the meaning of logical constants: there is a literality in the conjunction 'and' in 13 which is captured in 13s. but not in 13f., which we would like to equate to the proposition formalised with $\wedge$.

The approach to pragmatic enrichment that we will endorse reestablishes a minimal sense of what is said, in which the contribution of 'and' can be formalised with $\wedge$, independently of the fact that a speaker who utters 13 wants to express a temporal dimension of the conjunction. This is possible with the distinction between *completions* and *expansion* by Bach [1994] (which are parallel to the two processes of saturation and modulation by Recanati, but that according to the author were part of the same step).

> [There are] two ways in which a speaker can, even without using any ambiguous or indexical expressions and without speaking figuratively or indirectly, mean something without making it fully explicit. The first way arises whenever an utterance, even after disambiguation and reference fixing, does not by virtue of linguistic meaning express a complete proposition. When a sentence is in this way semantically underdeterminate, understanding its utterance requires a process of *completion* to produce a full proposition. The second way occurs when the utterance does express a complete proposition (possibly as the result of completion) but some other proposition, yielded by what I call the process of *expansion*, is being communicated by the speaker. [Bach, 1994, p. 125][5]

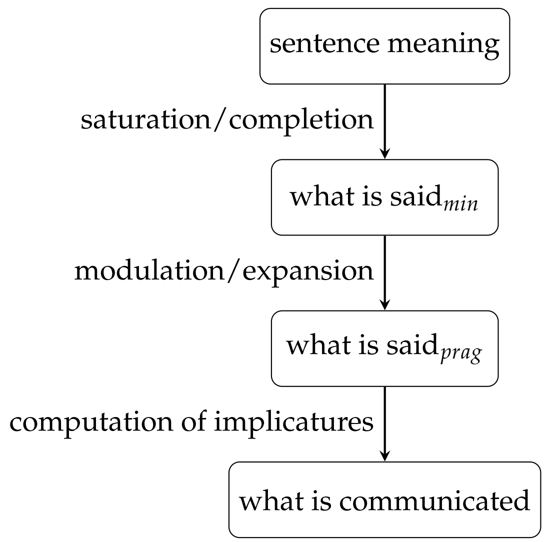
The two steps belong to two different processes,

> Because the utterance of a semantically underdeterminate sentence leaves out a conceptual element (or a relation between conceptual elements), the process of completion is required before a proposition is yielded. The process of expansion is not required in this sense - it is not mandated conceptually but merely pragmatically. [Bach, 1994, p. 127]

And they both need to be distinguished from the Gricean *implicatures*. This is why Bach refers to the result of both expansion and completion as *impliciture* [Bach, 1994, p.

[5]Emphasis added.

126]. Implicitures contribute to determining what is said, in the same manner as the first primary pragmatic process did for Recanati. But contrary to Recanati, completion and expansion belong to two different steps in the process. Hence, in contrast to the two-step model above, we are seeking a schema similar to this (also presented, although not defended, in [Recanati, 2004]):



The disanalogy with Recanati is clear:

> Now Recanati has argued that what I call the expansion of what is said *is* what is said. [Bach, 1994, p. 137]

In sum, Bach's [1994] proposal is closer to our intuitions about the different levels of information carried by an utterance, and it is also closer to our pluralist intuitions about the logical vocabulary, as we will see below. [6]

Excursus. In the next two chapters, we will see that substructural logics encode what is said$_{prag}$ while classical logic encodes what is said$_{min}$. We are assuming in this dissertation a three-step schema in which two notions of what is said coexist. In this manner, we can make room for a minimal sense of what is said which is captured with classical logic, as we will see. However, the analysis we are making of the pragmatic contribution captured by substructural languages is independent of the decision of considering a three-steps

[6] Although our view is closer to Bach's than to Recanati, we will keep referring to the pragmatic content which departs from the literal meaning of the vocabulary as *enrichment* or *pragmatic enrichment* rather than *implicitures*.

process or a two-step process in which what is said is pragmatically enriched (as Recanati or Carston suggests). The classical reading of the connectives, in such a case, would be limited to the sentence meaning, but the first computation of the meaning of an utterance would already be formalised in a substructural language. *End of Excursus.*

3.3.4 Mechanisms of Enrichment

We have explained above the mechanisms to derive implicatures, which is the last step in our schema. And we have said nothing about the mechanism to derive enrichments. All we know is that the mechanisms are not Gricean: contrary to implicatures, one does not compute a pragmatic enrichment q from a proposition p and the fact that p has been uttered in a given context, observing the Cooperative Principle and maxims.

However, although the mechanisms are clearly not Gricean, some authors claim that maxims can also operate at this sub-sentential level. That is, we can detach maxims from their role of deriving implicatures, and use them to explain pragmatic enrichments. After all, maxims are general norms which govern any discourse. In particular, Simons [2010] has argued that maxims can be used to fix the reference of indexicals. Hence, the role of maxims is not limited to generating implicatures, but may also help to enrich the meaning of the utterance:

> The basis for the general case is that subordinate clauses do not serve merely to contribute to the propositional content expressed in an utterance. Typically, these clauses themselves serve identifiable discourse functions. *Cooperativity requires these functions to be fulfilled as well as possible*. To put this a different way: interpreters can pay attention to parts of sentences independently of the containing sentence, and can reason about why the speaker produced just that sentence-part in attempting to convey her communicative intention. This reasoning, I suggest, is what gives rise to 'local' conversational inferences. [Simons, 2010, p. 8][7]

However, we need further articulation to understand how exactly maxims play a role in the derivation of pragmatic content at a sub-sentential level, given that it is not straightforward in all the examples.

Consider, for instance, the negation of a conditional, as 14,

14. It is false that if the Sun explodes, it won't matter.

[7]Emphasis added.

What we understand here is the negation of the enrichment of the conditional, that is, that there is a connection between these two events happening. It might be the case, or not, that the Sun explodes, but if it does, it will matter. However, the negation of the material conditional is equivalent to the affirmation of the antecedent and the negation of the consequent. It is not clear how the maxim of Quantity plays a role in deriving the intended meaning rather than the literal meaning in this case.

The importance of reasons We need further development to understand the above and other examples of embedded connectives and the embedding of their pragmatic enrichments. In sum, what we need is to know what the Primary Pragmatic Processes are in the above three-step schema; or to know what the mechanisms that allow us to derive what is said$_{prag}$ from what is said$_{min}$ are.

We argue that the enrichment (at least for the logical vocabulary) is produced by assuming certain *reasons* to assert the literal meaning of an utterance: the process of enrichment of an utterance p is to choose one among many reasons to assert p, and implicitly assume that those reasons are also communicated. Consider the following two utterances:

15. Charles Darwin joined the HMS Beagle for a trip around the world and wrote *The Voyage of the Beagle*.

16. Charles Darwin wrote *The Voyage of the Beagle* and joined the HMS Beagle for a trip around the world.

Both 15 and 16 say$_{min}$ exactly the same: Darwin joined the HMS Beagle for a trip around the world and ($\wedge$) Darwin wrote *The Voyage of the Beagle*. However, they do not say$_{prag}$ the same:

15e. Charles Darwin joined the HMS Beagle for a trip around the world and (then) wrote *The Voyage of the Beagle*.

16e. Charles Darwin wrote *The Voyage of the Beagle* and (then) joined the HMS Beagle for a trip around the world.

One mechanism to derive what is said$_{prag}$ from what is said$_{min}$ is by a simple assumption, which is usually automatic and sub-personal, along the following lines:

15r. There are reasons for asserting 15 which observe the submaxim of orderliness.

16r. There are reasons for asserting 16 which observe the submaxim of orderliness.

Moreover, these assumptions can be embedded under the scope of another operator. Whenever one wants to negate what is said$_{prag}$ by 16, that is, one wants to negate 16e, one is also negating 16r.:

17. It is not the case that there are reasons for asserting 16 which observe the submaxim of orderliness.

Consider now the embedding of 15 under the scope of a conditional:

18. If Charles Darwin joined the HMS Beagle for a trip around the world and wrote *The Voyage of the Beagle*, it must be the case that the trip was inspiring.

If the speaker intends to convey that the trip inspired the book, the utterance is, strictly speaking, false. The utterance only makes sense if the antecedent is embedded with its order implicature:

19. If there are reasons to assert that Charles Darwin joined the HMS Beagle for a trip around the world and wrote *The Voyage of the Beagle*, observing the submaxim of orderliness, it must be the case that the trip was inspiring.

What generates, then, what is said$_{prag}$ from what is said$_{min}$ are assumptions about the reasons for asserting the utterance. We can generalize the previous examples and say, for any maxim M, and any connective $\bullet$, that we can distinguish the following reasons to assert $p \bullet q$:

- There are reasons for asserting $p \bullet q$ which observe the maxim M.
- There are reasons for asserting $p \bullet q$ which flout the maxim M.

These reasons enrich the meaning of $\bullet$ and can be embedded under the scope of other operators, helping to capture the enriched sense of utterances. We will see below, and in the next two chapters, how substructural languages distinguish different reasons to assert utterances with a connective, and hence, capture enriched senses of those connectives.

Given that maxims might play a role both in the codification of implicatures and in the codification of enrichments, we will use the expression *relevant/manner implicatures* to refer to the particular implicatures that can be derived using relevant/manner maxims; and we will use the expression *relevant/manner enrichments* to refer to the particular enrichments that can be computed with the aid of relevant/manner maxims.

Cancellation and reinforcements of enrichments The nature of pragmatic enrichments is a matter of dispute: a semanticist might still argue that what we have labelled as pragmatic enrichments is semantic content rather than pragmatic content. That would multiply the meaning of the logical connectives, and turn the logical pluralist proposal of this dissertation a thesis about languages grounded on an equivocation between them.

However, the two mechanisms presented above that allowed to identify conversational implicatures (i.e. cancellation and reinforcement), also affect pragmatic enrichments,

> Implicatures are, as Grice observed, cancellable and can be vague or indeterminate, but the same is true of implicitures' [Bach(1994: 140)]

Consider the enrichment of order in the previous example, 15. One can easily cancel or reinforce the enrichment without contradiction:

20. Charles Darwin joined the HMS Beagle for a trip around the world and wrote *The Voyage of the Beagle*, not necessarily in this order,

21. Charles Darwin joined the HMS Beagle for a trip around the world and wrote *The Voyage of the Beagle*, in this particular order.

Hence, a similar argument against the semantization of conversational implicatures will help us identifying the enrichments along with the dissertation.

3.4 Substructural logics as pragmatically enriched

We conclude this chapter with a brief introduction to what will be discussed in the following two chapters, in which we will see how we can apply all the information about pragmatic enrichments to substructural logics. In particular, we will see that what is said$_{prag}$ is encoded by substructural logics, while what is said$_{min}$ is formalised by classical logic. Hence, this phenomenon sustains our pluralist view of logic.

So far we have presented the different views on the pragmatic enrichment of different expressions, assuming that logical constants are capable of being enriched. We have also presented many examples using logical constants. Although the paradigmatical cases of underdeterminacy are certain properties which are much more context-sensitive than logical connectives[8] we are arguing that the contribution of a logical connective to an

[8]See for instance the discussion about the utterance 'the leaf is green', and the meaning of green, by Travis [2008].

utterance is not fully determined by its literal meaning. We will see in the following two chapters the details for such enrichment of 'if...then', 'or', and 'and'.

It is important to clarify the direction of the explanation of this thesis: we are not claiming that substructural logics are capable of formalizing any pragmatic enrichment that a logical connective might suffer. The explanation goes in the opposite direction: whenever two of the different logics endorsed in this dissertation diverge about the validity of some inference, this divergence can be explained with a variation in the interpretation of the logical vocabulary: one of the logics is capturing and encoding certain information that the other logic ignores (and explains pragmatically).

3.4.1 'Follows from' and its different senses

This thesis endorses four logics: classical logic *LK*, relevant logic *LR*, linear logic *LL*, and ordered logic *LO*. Recall that we have defined logical consequence as follows:

> Δ is a logical consequence$_x$ of Γ iff there is a deduction of Δ from Γ by a finite number of legitimate$_x$ rules of inference.

Each substructural logic diverges from the rest in the acceptance or rejection of some structural rule. The presence or absence of each structural rule affects the behaviour of $\vdash$ and of logical connectives, and we can explain the difference in the behaviour of $\vdash$ observing that the rejection of each structural is related to certain Gricean maxim. Hence, we will give pragmatic justifications for arguing that it is legitimate to endorse and to reject structural rules, giving rise to different legitimate notions of consequence.

***LK* and the preservation of Quality** Classical logic *LK* precisifies the above definition of logical consequence by considering legitimacy as truth-preservation:

> *LK*: a rule of inference from Γ to Δ is legitimate$_K$ if it preserves truth.

A logic that preserves truth is a logic which observes the maxim of Quality. The maxim of Quality is considered as the most important maxim, which always needs to be observed in order to participate in any discourse [Grice, 1989c]. Moreover, its relation with logical formalization is different from the rest of the maxims: any logic should preserve, from the premises to the conclusion, this maxim, as any logic should preserve truth.[9] That is, if the premises are true, the conclusion must be true. In other words, if the premises observe the maxim of Quality, the conclusion should also observe the maxim of Quality.

[9] I want to thank Genoveva Martí for a conversation that motivates this observation.

LR **and the preservation of Relation** Relevant logic *LR* links the legitimacy of an inference with the derivation of the conclusion from the premises,

> *LR*: a rule of inference from Γ to Δ is legitimate$_R$ if it derives Δ operating with Γ.

This connection between premises and conclusion requires the observation of the maxims of Quantity and Relation. Following Geurts [2010, p. 13] we subsume the maxim of Quantity under the maxim of Relation. Both maxims are related to *what* information is given, to the amount of information and the fact that certain information is given. As we will see in more detail in the following section, whenever $\vdash$ is understood as 'follows from' the presence of weakening allows for the violation of both the maxim of Relation and of Quantity, given that $B, A \vdash A$ and $A \vdash A, B$. This fact implies that this encoding of logical consequence captures relation enrichments of the connectives.

*LL***,** *LO***, and Manner** Linear Logic *LL* and Ordered Logic *LO* capture the legitimacy in a derivation with the use of the premises to get the conclusion,

> *LL*/*LO*: a rule of inference from Γ to Δ is legitimate$_{L/O}$ iff it uses Γ to get Δ.

This is linked to Manner maxims, which are related to *how* the information is presented. Note that when $\vdash$ is understood as the natural language expression 'follows from', Exchange allows for the violation of the fourth maxim of Manner, since $A, B \dashv\vdash B, A$ and Contraction allows for the violation of the third maxim of Manner, given that $A, A \dashv\vdash A$. This fact implies that such encodings of logical consequence are enriched with the implicatures that the Manner maxim allows us to derive, that is, the manner implicatures.

3.4.2 Pragmatically enriched connectives

The justification of substructural connectives runs as follows: The lack of a structural rule for each particular $\vdash$, and its corresponding enrichment entails that those instances of logical constants that violate the maxims associated with that structural rule and for which the derived enrichments are false can be distinguished from those that do not violate the maxims and for which the derived enrichments are true.

First, $\vdash_R$ requires that logical constants be enriched with relation maxims - and *LR*'s language distinguishes enriched instances ($+,\times$) from non-enriched ones ($\sqcup$,$\sqcap$). Substructural logics that reject weakening allow a distinction to be made between those instances of logical constants that violate the Relation maxims and for which the relation enrichments are false, and those that do not and for which the relation enrichments are true.

Second, $\vdash_L$ and $\vdash_O$ require that logical constants are enriched with the relation and manner maxims - and LL's and LO's languages distinguish enriched instances ($\oplus/\oplus$,$\otimes/\otimes$) from non-enriched ones ($\sqcup_L/\sqcup_O$,$\sqcap_L/\sqcup_O$). Specifically, those substructural logics without exchange and/or contraction distinguish between the instances of logical constants that violate Manner maxims and for which the manner enrichments are false, and those that do not and for which the manner enrichments are true.

The legitimacy of the substructural formalization of logical constants requires the observation that there is a dependency between the particular enrichment of the notion of consequence and the logical vocabulary in each case, as we will see with more detail in the two following chapters.

PART II

Legitimacy of Substructural Languages

CHAPTER 4

Classical and Relevant Logic

In this chapter, we will introduce classical and relevant logic as valid codifications of logical consequence and logical vocabulary. Regarding classical logic we will argue that it encodes a truth-preserving relation, justifying its structural rules in the Gentzen sequent calculus as truth-preserving rules. And regarding relevant logic we will see three things: first, it formalizes 'follows from' in such a way that it avoids the violation of Relation and Quantity maxims; second, *LR*'s formalizations of conjunction, disjunction and conditional distinguish those instances that violate the Relation and Quantity maxims (and for which the derived enrichments are false) from those that do not violate such maxims (and for which the derived enrichments are true); third, the inferential role of pragmatically enriched vocabulary in *LR* (which obviously diverges from the classical) captures some of our intuitions about how to reason with logical vocabulary in natural language.

4.1 Classical logic

We will see in the second part of this chapter and in the following one that substructural logics capture enriched senses of the logical connectives. And that this enrichment is explained because they encode the different reasons one might have to assert the connectives. Hence, the explanation presupposes a minimal meaning of the logical constants (one which is used to distinguish reasons to assert *that* connective). Hence, we need to identify first the connective that is enriched, that is, the literal meaning of 'and', 'or', and 'if...then'. And we argue that this underlying meaning is the classical one.

In sum, we will see two things about *LK*. First, $\vdash_K$ is a truth-preserving relation; second, $\supset$, $\vee$, and $\wedge$ capture the literal meaning of 'if...then', 'or', and 'and', required by $\vdash_K$.

4.1.1 The logic

The presentation of classical logic in this dissertation will follow the propositional fragment of Gentzen's [1964] classical logic *LK*. In such logic multiple conclusions are allowed (in contrast to Gentzen's calculus for Intuitionistic logic).[1]

Notation The notation we will use coincides with [Gentzen, 1964] except for the symbol of logical consequence,

	Gentzen [1964]	Paoli [2002]	Our notation
Logical consequence	$\rightarrow$	$\Rightarrow$	$\vdash$
Negation	$\neg$	$\neg$	$\neg$
Conditional	$\supset$	$\rightarrow$	$\supset$
Disjunction	$\vee$	$\vee$	$\vee$
Conjunction	$\wedge$	$\wedge$	$\wedge$

4.1.2 Consequence: justification of structural rules

To justify the truth-preserving nature of $\vdash_K$, let us first present the nature of sequent calculus. We have already seen in Chapter 2 that sequent calculus establishes connections between sequents, in the following way:

$$A, \Gamma \vdash \Delta, B$$

Which is read as 'Δ or B follows from A and Γ' or 'A and Γ logically entail Δ or B', where Γ and Δ are sequences of formulas, separated by commas, read conjunctively on the left and disjunctively on the right. In sequent calculus, we relate valid sequents, rather than formulas. That is, we can establish metarules that allow us to derive from the validity of a sequent $\Gamma \vdash \Delta$ the validity of another sequent $\Gamma' \vdash \Delta'$:

$$\frac{\Gamma \vdash \Delta}{\Gamma' \vdash \Delta'}$$

The metarules in a sequent calculus are therefore about the normative force that some valid inferences transmit to other valid inferences: if the sequent $\Gamma \vdash \Delta$ is valid, then some other sequent $\Gamma' \vdash \Delta'$ is also valid. Hence, $\vdash$ can have different normative forces, which will be determined by the accepted structural rules for each logic.

And one possible interpretation of $\vdash$ is truth preservation. Let us observe how if this is the case, weakening, contraction, and exchange are acceptable rules for $\vdash$.

[1] For further reference about multiple conclusions see [Restall, 2005] and the discussion in [Beall and Restall, 2006].

Weakening Let us see that weakening preserves truth. To prove this, let $\Gamma \vdash \Delta$ be a truth-preserving relation: whenever all the formulas in Γ are true, one of the formulas in Δ is true. Hence, adding an extra formula A in the premises will not change this fact, as if it did, $\Gamma \vdash \Delta$ would not be truth-preserving, to begin with. Equivalently, given that one of the formulas in Δ is true if all the formulas in Γ are true, the addition of an extra formula in the conclusion cannot change this fact.

$$\frac{\Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}\,\mathrm{WL} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A}\,\mathrm{WR}$$

Contraction Let $A, A, \Gamma \vdash \Delta$ be a truth-preserving relation. This means that if A, A and the formulas in Γ are all true, some formula in Δ is also true, or that there is a valid derivation with A, A and Γ as premises and Δ as a consequence. The repetition of A in the premises might be the symptom that it is necessary to *use* A twice to derive Δ. But given that the truth of some formula is not affected by the derivations we might perform with them, one can contract A, and use this single premise as many times as one needs. Similarly, if from the truth of the formulas in Γ one derives the truth of one formula in Δ, A, A, this will not be affected by the contraction of A.

$$\frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}\,\mathrm{CL} \qquad \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}\,\mathrm{CR}$$

Exchange The validity of exchange for a truth-preserving relation is straightforward with observations about sequents similar to those made above. Let $\Gamma, A, B, \Gamma' \vdash \Delta$ be a truth-preserving relation. Whether some pairs of formulas are in a certain order will not affect their truth value, nor the truth value of Δ. And for the same is true of the conclusion.

$$\frac{\Gamma, A, B, \Gamma' \vdash \Delta}{\Gamma, B, A, \Gamma' \vdash \Delta}\,\mathrm{EL} \qquad \frac{\Gamma \vdash \Delta, A, B, \Delta'}{\Gamma \vdash \Delta, B, A, \Delta'}\,\mathrm{ER}$$

In order to understand the divergence between *LK* and substructural logics we should keep in mind that sequents are valid inferences. The invalidity of a sequent $\Gamma \vdash \Delta$ in a substructural logic means that there is some interpretation of the logical vocabulary for which there is no inferential force to go from Γ to Δ. There are two possibilities for this invalidity: the sequent can be classically valid or invalid. If the sequent is classically invalid, it does not preserve truth, and substructural invalidity is a natural consequence. However, if the inference is classically valid, its substructural invalidity is due to the presence of some structural rule in its derivation, that is, it is invalid because there is some irrelevance (weakening), or because there is some repetition of the premises (contraction), or because there is some change in the order of the premises or the conclusion (exchange). The three cases can be explained by a stronger than a truth-preserving normative constraint to the sequents, which can be explained by a pragmatic enrichment of the interpretation of the logical vocabulary, as will be demonstrated.

4.1.3 Connectives

Internalization of validity

Logical connectives internalize structural rules [Hjortland, 2014b], that is, one cannot fully split operational from structural rules in a calculus. Consider, for instance, the following left rule for conjunction (which is equivalent to $\wedge$L in a calculus with weakening and contraction):

$$\frac{A, B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}\ \wedge\mathrm{L}'$$

Given this rule, the conjunction internalizes the properties that the comma has in the sequent. If $\vdash$ lacks contraction, for instance, $A \wedge A \nvdash A$, or if $\vdash$ lacks exchange, $A \wedge B \nvdash B \wedge A$, while these derivations would be valid in a logic with these structural rules.

The phenomenon of internalization of validity is related to the *antecedent context of deducibility* [Belnap, 1962], as it means that we do not define connectives *ab initio*, but that the already valid inferences in the connective-free fragment for $\vdash$ determine which of these connectives will be admissible for a logic with $\vdash$.

Hacking [1979] follows the same line, arguing that the notion of consequence presented above already determines the properties of the logical connectives:

> [G]iven a certain pure notion of truth and consequence, all the desirable semantic properties of the constants are determined. [Hacking, 1979, p. 299]

> I claim ... that the operational rules 'fix the meanings of the logical connectives' in the sense of giving a semantics, only if classical notions of truth and logical consequence are already assumed. [Hacking, 1979, p. 300]

Finally, Došen [1989] goes one step further, identifying logical connectives with punctuation marks in a consequence relation,

> Logical form is primarily exhibited by structural deductions, and when logical constants are introduced they serve, so to speak, as punctuation marks of the object language, for some structural features of deductions. [Došen, 1989, p. 366]

Hence, once $\vdash_K$ is justified as a truth-preserving relation between formulas, the justification of their classical left and right rules in the Gentzen sequent calculus is straightforward if, following Došen [1989], we interpret them as punctuation marks: in the object language, constants mimic some already accepted properties in the metalanguage: we have interpreted $\vdash$ as logical consequence, the comma at the left as conjunction and the comma at the right as disjunction. Logical connectives then reflect those implicit assumptions in $\vdash_K$. In other words, we want logical connectives to be conservative. Došen illustrates this claim with the conditional $\supset$,

> The connective of implication in $A \supset B$ says that A and B are connected in this formula of the object language like a premise and a conclusion in a deduction. [Došen, 1989, p. 366]

And in a similar way, we can introduce the conjunction $A \wedge B$ as the connective which says that A and B are connected in this formula of the object language like premises in a deduction. And finally, the connective of disjunction $A \vee B$ says that A and B are connected in this formula of the object language like conclusions in a deduction.

The rules that satisfy such connections with the properties of the classical consequence relation $\vdash$ are the following[2],

$$\frac{A, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}\wedge\mathrm{L}_1 \qquad \frac{B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}\wedge\mathrm{L}_2 \qquad \frac{\Gamma \vdash \Delta, A \qquad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}\wedge\mathrm{R}$$

$$\frac{\Gamma \vdash \Delta, A \qquad B, \Pi \vdash \Sigma}{A \supset B, \Gamma, \Pi \vdash \Delta, \Sigma}\supset L \qquad \frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \supset B}\supset R$$

$$\frac{A, \Gamma \vdash \Delta \qquad B, \Gamma \vdash \Delta}{A \vee B, \Gamma \vdash \Delta}\vee L \qquad \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \vee B}\vee R_1 \qquad \frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \vee B}\vee R_2$$

See Appendix A for a complete presentation of the Gentzen calculus for classical logic LK, and for further references.

[2]Our aim in this dissertation is to study the effect that structural rules have on logical constants, and how to explain their behaviour in certain substructural rules. Our focus will be on the conditional, the disjunction and the conjunction, while the behaviour of negation will not play a significant role in the research. The reason for this is that, as we will see, the logics endorsed here split the three connectives we will study, while negation is maintained as unique, except in Ordered Logic, where we will introduce two presentations of negation.

4.2 Relevant Logic

Relevant Logic is the result of rejecting the rule of weakening in *LK*. This rejection is motivated by the behaviour of the material conditional $\supset$ in *LK*, and also the behaviour of $\vdash$. In this section, we will see the motivations for rejecting weakening, the effect that this rejection has on the logical connectives, and the fact that we can explain the codification of logical vocabulary as a pragmatic enrichment of classical vocabulary, which affects the inferential role of the connectives.

4.2.1 The logic

Among the different presentations of relevant logic (see [Anderson et al., 1992],[1977], [Read, 1988], [Mares, 2004]) we are interested in the effect that the lack of weakening has on logical connectives. Hence, we will consider what Paoli calls LR^{ND}[3], which is the result of rejecting weakening in *LK*. We will refer to this logic simply as *LR*[4]. This rejection has an effect on the conditional, and also on the rest of the connectives: each classical binary connective is split into two relevant connectives, an extensional and an intensional one, which can be proven to be equivalent in *LK* due to the effect of weakening. In the rest of this section, I will focus on the Gentzen-style presentation of relevant logic *LR*, which rejects weakening as a structural rule and duplicates logical connectives.

Notation Our notation for *LR* coincides with Paoli [2007][5]:

[3] The result of rejecting weakening from *LK* is a non-distributive logic in which $A \wedge (B \vee C) \nvdash (A \wedge B) \vee (A \wedge C)$. Most relevant logics include this derivation as valid (see [Anderson and Belnap, 1977], [Read, 1988], [Mares, 2004]), however this invalidity is not problematic in our case. We are not defending, as monist relevant logicians do, that *LR* is the calculus for codifying a unique notion of logical consequence. For us, it is the calculus for codifying certain pragmatic phenomena which affect logical vocabulary, and the origin of such pragmatic phenomena is explained by the lack of weakening and all the effects that it has on the logical vocabulary. This view on *LR* is not in tension with the validity of distribution for preserving truth, as we endorse that this is codified in *LK*.

[4] It is important to note that, although we are using Paoli's presentation of *LR*, this is not the logic that Paoli considers as the correct logic. Paoli defends linear logic, *LL*, a logic without weakening and contraction, which differs from *LR* in that the extensional and intensional connectives are completely independent while in *LR* the intensional imply the extensional ones.

[5] The symbol * indicates that the connective is not present.

	Paoli [2002]	Paoli [2007]	Read [1988]	Our notation
Negation	$\neg$	$\neg$	$\sim$	$\neg$
Extensional conditional	$*$	$\rightsquigarrow$	$*$	$\rightsquigarrow$
Intensional conditional	$\rightarrow$	$\rightarrow$	$\supset$	$\rightarrow$
Extensional disjunction	$\vee$	$\sqcup$	$\vee$	$\sqcup$
Intensional disjunction	$+$	$+$	$+$	$+$
Extensional conjunction	$\wedge$	$\sqcap$	&	$\sqcap$
Intensional conjunction	$\times$	$\times$	$\times$	$\times$

4.2.2 Relation Enrichment of Consequence

Paradoxes of material conditional and consequence

Relevant logic aims to avoid certain paradoxes classical logic has about the conditional and the notion of consequence, such as the following derivations in classical logic LK:

$$\dfrac{\dfrac{A \vdash A}{B, A \vdash A}\,WL}{A \vdash B \supset A}\supset R \qquad\qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B \vdash B}{B \vdash B, A}\,\mathrm{WR}}{B \vdash A, B}\,\mathrm{ER}}{\neg B, B \vdash A}\,\neg\mathrm{L}}{B, \neg B \vdash A}\,\mathrm{EL}}{\neg B \vdash B \supset A}\supset\!\mathrm{R}$$

One of the main criticisms that relevant logic makes about LK is that the indicative conditional does not behave like $\supset$. One would not say, in natural language, the following:

22. The theory of evolution is correct. Hence, if God created all the animal species, then the theory of evolution is correct.

23. Santa Claus does not exist. Hence, if Santa Claus exists, he does not have a beard.

But these should be formalised as $A \vdash B \supset A$ and $\neg B \vdash B \supset A$, which we have seen to be valid.

The relevantist strategy to solve these counterexamples to the classical conditional is to reject weakening, and there are two main reasons for this choice: first, this rule is present in similar paradoxes which do not involve the conditional but do involve negation, disjunction, and conjunction, as we see:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B \vdash B}{B \vdash B \vee \neg B}\,\vee\mathrm{R}_1}{\vdash B \vee \neg B, \neg B}\,\neg\mathrm{R}}{\vdash B \vee \neg B, B \vee \neg B}\,\vee\mathrm{R}_2}{\vdash B \vee \neg B}\,\mathrm{CR}}{A \vdash B \vee \neg B}\,\mathrm{WR}$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{A \vdash A}{A \wedge \neg A \vdash A}\wedge L_1}{\neg A, A \wedge \neg A \vdash}\neg L}{A \wedge \neg A, A \wedge \neg A \vdash}\wedge L_2}{A \wedge \neg A \vdash}CL}{A \wedge \neg A \vdash B}WL$$

Second, weakening has a crucial role in the derivation of paradoxes, as it is responsible for introducing irrelevant information in a derivation:

> [U]pholding weakening amounts to failing to take at face value the expression 'assertable on the basis of': if I am in a position to assert B on the basis of the information provided by A, I need not be in a position to assert B on the basis of both A and C - where C is just an idle assumption, irrelevant to my conclusion. [Paoli, 2007, p. 559]

However, from the classical point of view, paradoxes do not pose any problem: they are an immediate consequence of logic being truth-preserving. Hence, there are two possible formalizations of 'follows from': one with and one without weakening.

In the rest of this section, we will focus on the Gentzen-style presentation of relevant logic LR, which rejects weakening as a structural rule and duplicates the conditional, disjunction, and conjunction into two versions, an intensional and an extensional connective, following Paoli [2002, 2007].

Weakening and Relation Enrichment

The presence of weakening makes it possible to add irrelevant information to a derivation, either in the premises or in the conclusion. And this irrelevant information invalidates those inferences in which 'follows from' (and the logical connectives, as will be demonstrated), observe and are enriched with the relation maxims. First, whenever $\vdash$ in a sequent $\Gamma \vdash \Delta$ is relevantly enriched, one cannot derive $A, \Gamma \vdash \Delta$ (without losing the pragmatic enrichment of $\vdash$) given the irrelevance of A for Δ and the conjunctive reading of the premises in a sequent.

$$\frac{\Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}\, WL$$

Second, one can neither derive from $\Gamma \vdash \Delta$ (in which $\vdash$ is relevantly enriched) that $\Gamma \vdash \Delta, B$, given that B weakens what follows from Γ and hence one violates the maxim of quantity, given the disjunctive reading of the commas in the conclusion,

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A}\, WR$$

Given this, *LK* and *LR* encode two natural views about what logic is. We have seen the reasons for rejecting weakening: *LR* rejects the paradoxes of material implication and formalizes a sense of 'follows from' in which the conclusion is the result of operating with the premises. In addition, the rejection of weakening prevents the violation of two Gricean maxims, making $\vdash_R$ a divergent normative relation for reasoning. However, we have reasons to accept weakening as well: $\vdash_K$ reflects a notion of 'follows from' understood as a truth-preserving relation, although the conclusion might not require all the information contained in the premises for its derivation, or the information in the premises might entail something stronger than the conclusion.

4.2.3 The effect of structural rules on logical constants

The pluralism presented here departs from variations on the inferential role of $\vdash$. In particular, these variations are the result of restricting the structural rules of weakening, contraction, and exchange for the classical $\vdash$. Although the obvious target of these restrictions is the notion of consequence, the rejection of structural rules has an important effect on logical constants too. Contraction, exchange and weakening also affect the behaviour of logical connectives. In particular, as is well known, in substructural logics the conditional, conjunction and disjunction, which are unique in *LK*, split into two different connectives: an intensional and an extensional version, with different expressive powers in these logics.

> [I]f we relinquish some or all of the structural rules in Gentzen's *LK* (or for that matter, in *LJ*), a plethora of new connectives emerges in place of the original four. Structural rules flatten this expressive wealth by reducing the defining rules of some connectives to the defining rules of other ones. [Paoli, 2002, p. 15]

Conditional The conditional splits into two connectives when certain structural rules (and in particular, weakening) are not present: an intensional conditional $\rightarrow$ and an extensional conditional $\rightsquigarrow$, introduced by Paoli [2007]:

$$\frac{\Gamma \vdash \Delta, A \qquad B, \Pi \vdash \Sigma}{A \rightarrow B, \Gamma, \Pi \vdash \Delta, \Sigma} \rightarrow L \qquad\qquad \frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \rightarrow B} \rightarrow R$$

$$\frac{\Gamma \vdash \Delta, A \qquad B, \Gamma \vdash \Delta}{A \rightsquigarrow B, \Gamma \vdash \Delta} \rightsquigarrow L \qquad \frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, A \rightsquigarrow B} \rightsquigarrow R_1$$

$$\frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \rightsquigarrow B} \rightsquigarrow R_2$$

In the presence of all structural rules, as in *LK*, these two connectives are equivalent, but the unavoidable use of structural rules to prove their equivalence makes them different connectives in substructural logics.

Disjunction As in the case of the conditional, the absence of weakening splits the classical disjunction $\vee$ into two different connectives: an extensional disjunction, $\sqcup$ and an intensional one, $+$:

$$\frac{A, \Gamma \vdash \Delta \qquad B, \Gamma \vdash \Delta}{A \sqcup B, \Gamma \vdash \Delta} \sqcup L \qquad \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \sqcup B} \sqcup R_1$$

$$\frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \sqcup B} \sqcup R_2$$

$$\frac{A, \Gamma \vdash \Delta \qquad B, \Pi \vdash \Sigma}{A + B, \Gamma, \Pi \vdash \Delta, \Sigma} +L \qquad \frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A + B} +R$$

Conjunction Finally, conjunction also splits into two different connectives in *LR*, an intensional $\times$ and extensional $\sqcap$ one.

$$\frac{A, \Gamma \vdash \Delta}{A \sqcap B, \Gamma \vdash \Delta} \sqcap L_1 \qquad \frac{\Gamma \vdash \Delta, A \qquad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \sqcap B} \sqcap R$$

$$\frac{B, \Gamma \vdash \Delta}{A \sqcap B, \Gamma \vdash \Delta} \sqcap L_2$$

$$\frac{A, B, \Gamma \vdash \Delta}{A \times B, \Gamma \vdash \Delta} \times L \qquad \frac{\Gamma \vdash \Delta, A \qquad \Pi \vdash \Sigma, B}{\Gamma, \Pi \vdash \Delta, \Sigma, A \times B} \times R$$

This effect of the structural rules will be a fundamental phenomenon to demonstrate the validity of the different substructural logics that emerge from the rejection of weakening, contraction, and exchange. See Appendix A for a presentation of the Gentzen sequent calculus for Relevant Logic *LR*.

4.2.4 Relation Enrichment of Connectives

We need to find criteria for the translation of logical constants. For each connective, we should distinguish two dimensions: its literal meaning (what is said$_{min}$) and its enriched meaning (what is said$_{prag}$). On the one hand, we have assumed that the classical formalization capture the literal truth-conditions of each logical constant, which is precisely what the classical notion of logical consequence requires. On the other hand, one logical constant can have two different inferential roles, depending on different enrichments, which in turn depend on the grounds or reasons one has for asserting it. In effect, the lack of weakening affects the behaviour of logical constants and allows us to identify two inferential roles for each connective: the distinction between extensional and intensional constants corresponds to two different uses of the connectives, those that violate the Gricean maxim of Quantity or Relation (and for which the derived enrichment is false) and those that do not (and for which the derived enrichment is true).

It might be useful for this section to highlight the interderivability connections between connectives (see table 2 [Read, 1988, p. 38]). We will see also the connections between intensional and extensional connectives in each section.

	Extensional connectives	Intensional connectives
Conjunction/disjunction	$\neg(A \sqcap B) \equiv \neg A \sqcup \neg B$	$\neg(A \times B) \equiv \neg A + \neg B$
Conjunction/conditional	$\neg(A \sqcap B) \equiv A \rightsquigarrow \neg B$	$\neg(A \times B) \equiv A \rightarrow \neg B$
Disjunction/conditional	$\neg A \sqcup B \equiv A \rightsquigarrow B$	$\neg A + B \equiv A \rightarrow B$

Conditional

The absence of weakening, as we have seen, avoids the paradoxes of the material conditional, $\supset$. In LR we can define an intensional conditional $\rightarrow$ with the same Left and Right rule as $\supset$ but for which the paradoxes are invalid. Moreover, as Paoli [2007, p. 561] shows, we can also define an extensional conditional, which validates the paradoxes of the material conditional, but for which Modus Ponens is invalid. We, therefore, have two conditionals: an intensional ($\rightarrow$) and an extensional one ($\rightsquigarrow$). In LK, $\supset$ is the connective for which these two derivations are valid: $\neg A \vdash A \supset B$, $B \vdash A \supset B$ (considered as the paradoxes of the material conditional according to relevant logicians), and also Modus Ponens, $A \supset B, A \vdash B$. However, in LR, these inferences are valid for different connectives: the paradoxes are valid only for $\rightsquigarrow$, $\neg A \vdash A \rightsquigarrow B$, $B \vdash A \rightsquigarrow B$ [6] but Modus Ponens

[6]Note that $\vdash$ differs for each logic (so instead of $\vdash$ we might formalize consequence as $\vdash_K$, $\vdash_R$), but we will distinguish them with subindices just in case the formulas used in the sequent are not sufficient for

is valid only for $\rightarrow$, $A \rightarrow B, A \vdash B$. Moreover, the intensional conditional entails the extensional, but not the other way around: $A \rightarrow B \vdash A \rightsquigarrow B$, but $A \rightsquigarrow B \nvdash A \rightarrow B$.

The enrichment schema Pérez-Otero [2001, p. 251] introduces a schema to derive the conversational implicature usually carried by a conditional, which we will call *the Enrichment schema*. A similar derivation is in [Rieger, 2006]. We have expanded the schema to other connectives, and we use it to distinguish the two versions of each constant in *LR*. We will outline two kinds of reasons for introducing each logical constant, distinguishing those that violate the Gricean maxim of Quantity from those that do not. We will then see how the first group correspond to the extensional versions, while the second corresponds to the intensional ones, and finally, we will see how these differences also require different inferential roles, which are also captured by *LR*.

We have already seen in Chapter 2 that for each connective and each maxim, we can distinguish two different reasons to assert it, in particular for the conditional. Given this and the fact that according to *LK*, $A \supset B \equiv \neg A \vee B$, we deduce the following Enrichment schema for the conditional distinguishing the following reasons to assert 'if A then B':

- E-reasons: reasons that allow us to introduce a connective but violating a Gricean maxim. In this particular case, one asserts a conditional for E-reasons if one has reasons to assert $\neg A$ or reasons to assert B.

- I-reasons: reasons that allow us to assert a connective without violating a Gricean maxim. In this particular case, this is a connection between the antecedent and consequent, such that the antecedent cannot be true without the consequent being true.

LK does not distinguish between the two cases, but in *LR* we can: in effect, E-reasons correspond to the extensional conditional and I-reasons to the intensional one,[7]

- E-conditional: $\Gamma \vdash \neg A \Rightarrow \Gamma \vdash A \rightsquigarrow B$ / $\Gamma \vdash B \Rightarrow \Gamma \vdash A \rightsquigarrow B$

- I-conditional: $A, \Gamma \vdash B \Rightarrow \Gamma \vdash A \rightarrow B$

disambiguating them. Otherwise we assume that each $\vdash$ is disambiguated by the vocabulary in the sequent in which it appears.

[7] We formalize the enrichment schema as a meta-inference: we derive that a sequent $\Gamma \vdash \Delta$ is valid given that another sequent $\Gamma' \vdash \Delta'$ is also valid, which represent the initial reasons that give rise to the reasons for asserting the connective. For example, having reasons to assert $\neg A$ is represented as $\Gamma \vdash \neg A$. The connection between the two sequents is represented as $\Rightarrow$.

We can distinguish the two conditionals in natural language. Consider the following instance of a 'Dutchman conditional'[8], introduced for E-reasons:

1. If Gödel was not a logician, then pigs can fly.

Or consider those sentences, in the context of a card game, in which a player asserts the following while looking at her hand:

2. If I have a black card, then I have a red card.[9]

Clearly, these are cases in which 'if A then B' is asserted on the grounds of not-A or B (that is, for E-reasons); and in which the relation enrichments are false and should be formalised with the extensional conditional $\rightsquigarrow$ in LR.

Consider now the following sentences, which are introduced for I-reasons (and hence formalised with $\rightarrow$):

3. If you walk every day, you will feel better.

4. If it rains, the match will be cancelled.

3 and 4 are examples of uses of the conditional that express a connection between the antecedent and the consequent, that is, uses of the conditional for which the relation implicature is true. These would be formalised in LR with the intensional conditional $\rightarrow$.

The Relevant Conditional and the Indirectness condition The relevant conditional $\rightarrow$ captures the connection between antecedent and consequent on the different examples of natural language that we have seen above. The I-reasons to assert 'if A then B' have been argued by relevant monists as the only legitimate reasons to assert the conditional, that is, 'if A then B' can only be asserted if the following is the case:

- There is a connection between A and B such that A cannot be the case without B being the case.

In [1989b] Grice discusses what he calls the *the Indirectness Condition*, a condition for asserting 'if A then B' which would negate his analysis of the material conditional. It expresses a stronger connection between antecedent and consequent of a conditional that is part of the meaning of 'if...then' and defended by those that reject the classical material analysis of the indicative conditionals. It is expressed as follows by Grice:

[8]'Dutchman conditional' refers to those expressions in which the consequent is clearly false ('pigs can fly', 'I am a Dutchman') to imply that the antecedent is false too, but in which there is no connection between them.

[9]Similar examples on [Grice, 1989b].

> [T]here are no truth-conditional reasons for saying 'if A then B'.

Grice argues, in order to reject that the indirectness condition is part of the meaning of 'if...then' that it does not seem to be the case that a conditional is always asserted on the grounds of some non-truth-conditional reason that connects the antecedent and the consequent. And he argues for this with similar examples to the ones that we have used to illustrate $\rightsquigarrow$. In effect, Grice argues that there are some cases in which one seems to be able to assert 'if A then B' without the indirectness condition being true, as in the following example:

5. I know just where Smith is and what he is doing, but all I will tell you is that if he is in the library he is working [Grice, 1989b]

The same observations can be made with respect to the I-reasons as the only reasons for asserting a conditional: the relevantist rejection of E-reasons for asserting a conditional finds counterexamples in natural language, in which the enrichment is cancelled.

Now we can see that $\vdash_R$ requires the enriched versions of the connectives in certain inferences. As we have seen above, these two kinds of conditional diverge in their inferential roles: $A \rightsquigarrow B, A \nvdash B$ but $A \rightarrow B, A \vdash B$.

First, the invalidity of $A \rightsquigarrow B, A \vdash B$ in LR is straightforward if we observe that if we add the sequent to our calculus, weakening is derivable:

$$\dfrac{A \rightsquigarrow B, A \vdash B \qquad \dfrac{\dfrac{A \vdash A}{\neg A, A \vdash}\,\neg\mathrm{L}}{\neg A \vdash A \rightsquigarrow B}\,{\rightsquigarrow}\mathrm{R}}{A, \neg A \vdash B}\,\mathrm{Cut}$$

It is illegitimate to use Modus Ponens with those conditionals that are asserted for E-reasons. The relevantist notion of 'follows from' requires the relation enrichment of the conditional to be true. Let us see why in detail: first, if the conditional is asserted on the grounds of the falsity of the antecedent, $\neg A$, the addition of A as a premise, rather than allowing one to derive B, forces one to retract from 'if A then B'. If one, after asserting 1, discovered that Gödel was not a logician, one would not be able to derive that pigs can fly legitimately. Second, if a conditional is asserted on the grounds of the truth of the consequent B, it is not Modus Ponens that is used to derive B, but B itself. If the player that asserts 2 on the grounds of having a red card then realizes that she also has a black card, it would be illegitimate to derive B *because of* A and $A \rightsquigarrow B$. The maxim of Relation would be flouted.[10]

On the other hand, Modus Ponens can be used with the intensional conditional, as weakening is not required:

$$\frac{A \vdash A \qquad B \vdash B}{A \rightarrow B, A \vdash B} \rightarrow\text{L}$$

To use Modus Ponens with 3 and 4 is completely legitimate. It is trivial given that there is a connection between A and B such that B is derived from A: whenever one asserts 3, one can derive from the information that she walks every day that her interlocutor feels better.

Disjunction

In the case of the disjunction, in *LK*, $\vee$ is both the connective for which Addition and Disjunctive Syllogism are valid: $A \vdash A \vee B$ and $A \vee B, \neg A \vdash B$. In *LR* these properties are valid for the two different disjunctions: Addition is valid only for $\sqcup$, $A \vdash A \sqcup B$; and Disjunctive Syllogism is valid only for +, $A + B, \neg A \vdash B$. Moreover, the intensional disjunction entails the extensional, but not the other way around: $A + B \vdash A \sqcup B$, but $A \sqcup B \nvdash A + B$.

As we did for the conditional, we can distinguish extensional and intensional reasons to assert 'A or B':

- E-reasons: there are reasons for asserting A / there are reasons for asserting B,
- I-reasons: there is a connection between A and B such that the negation of one disjunct implies the other.

To assert a disjunction for E-reasons violates the Gricean maxim of Quantity, while a disjunction asserted for I-reasons does not. Notice that the E-reasons correspond to the rules of introduction for $\sqcup$, while the I-reasons correspond to the reason for introducing $+$:

- E-conjunction: $\Gamma \vdash A \Rightarrow \Gamma \vdash A \sqcup B$ / $\Gamma \vdash B \Rightarrow \Gamma \vdash A \sqcup B$
- I-conjunction: $\Gamma \vdash \neg A \rightarrow B \Rightarrow \Gamma \vdash A + B$

[10] One might think of cases in which one seems to be legitimised to use MP with a conditional asserted for E-reasons. For instance, if someone asserts 2 and *another player* discovers that the speaker has a black card, she will correctly derive that she also has a red card - and there seems to be nothing wrong with it. We will come back to this in Chapter 7, page 190.

There are many examples of both kinds of disjunction in natural language. Consider the following sentences, introduced for E-reasons:

5. You will enjoy the book, or I am a Dutchman.

6. The cake is either in the kitchen or in the garden (but I won’t tell you where).

7. Socrates was a man or he was a stone. [Read, 1988, p. 142]

In these cases, the disjunction is asserted on the grounds of the truth of one of the disjuncts, independently of any connection between the two, and the relation enrichment is false for them. Hence, they should be formalised in *LR* with the extensional disjunction $\sqcup$.

Now consider the following sentences, introduced for I-reasons:

8. You should either work or study.

9. I know *Blade Runner*’s plot: either I read the book or I watched the movie.

Clearly, there seems to be a connection between the disjuncts, which indicates that the relation enrichment is true. As a result, they have to be formalised in *LR* using the intensional disjunction $+$.

Again, these two versions of disjunction have two different inferential roles as the relevant expression ‘follows from‘ requires the relation enrichment to be true in certain inferences. It is illegitimate to reason with the Disjunctive Syllogism with a disjunction that is asserted for E-reasons. Note that if the sequent were valid, weakening would be valid too:

$$\dfrac{\dfrac{\dfrac{A \vdash A}{A \vdash A \sqcup B}\,\sqcup\mathrm{R} \qquad A \sqcup B, \neg A \vdash B}{A, \neg A \vdash B}\,\mathrm{Cut} \qquad \dfrac{A \vdash A}{\vdash A, \neg A}\,\neg\mathrm{R}}{A \vdash A, B}\,\mathrm{Cut}$$

In order to see how this fits with our intuitions, consider first the following reasoning about 7 by Read:

> Let A be ‘Socrates was a man‘ and B ‘Socrates was a stone‘. It follows from the fact that Socrates was a man that Socrates was a man or a stone. So ‘$A \vee B$‘ is true. But it does not follow that if Socrates was not a man he was a stone. (...) Hence the reasoning is blocked, and B does not follow from ‘$A \vee B$‘, and ‘$\neg A$‘, nor from A and its negation. [Read, 1988, p. 142]

In effect, from a relevant perspective, B does not follow from $\neg A$ and $A \sqcup B$, as the addition of $\neg A$ spoils the reasoning (we would probably conclude that Socrates was a woman if one day we discover that he was not a man).

Second, as Paoli [2007, p. 566] notes, we can find relevantly invalid instances of Disjunctive Syllogism without needing to have inconsistent premises: as an illustration, consider the following modification of Read's counterexample to Disjunctive Syllogism:

> Let A be 'Socrates was a man' and B 'Socrates was a stone'. It follows from the fact that Socrates was a man that Socrates was a man or a stone. So '$A \vee B$' is true. We also know that Socrates was not a stone, and hence $\neg B$. But A does not follow *from* $A \vee B$ and $\neg B$, but from A alone.

Or consider 6. Imagine that one asserts it in the context of a game, and on the grounds of one's knowledge that the cake is in the kitchen. Imagine that one forgets the exact place where one left it, and starts looking for it. One asserted 6, and cannot find it in the kitchen. Should one look for the cake in the garden? Of course not. It would be illegitimate to use Disjunctive Syllogism in this case. Although one knows that the disjunction is true, one knows it on the grounds of one of the disjuncts. Under no circumstances would the evidence that one disjunct is false entail that the other is the case. It would just spoil the derivation.

On the other hand, it is completely natural to reason with the intensional Disjunctive Syllogism, given that $A + B \equiv \neg A \to B$ as weakening is not necessary for the derivation:

$$\dfrac{\dfrac{A \vdash A \qquad B \vdash B}{A + B \vdash A, B}\ {+}\mathrm{L}}{A + B, \neg A \vdash B}\ \neg\mathrm{L}$$

In effect, given 9, one is safe to conclude (or at least can legitimately conclude) that if one has not seen the movie, then one has read the book. There is a certain connection between the disjuncts that is not present in 6, which makes it legitimate to use Disjunctive Syllogism.

Or contrast the previous examples about Socrates with this instance of disjunctive syllogism:

10. Socrates was a man or a woman, and he was not a man. Hence, Socrates was a woman.

We know that Socrates was a human being, so he was probably a man or a woman. Imagine that we do not know which, so we are asserting the disjunction without violating the maxim of quantity. Now, the discovery of the negation of any of the two possibilities legitimately makes us assert the other disjunct.

Grice rejected a stronger than the truth-conditional sense of 'or'. While we agree that this is the case for the minimal or literal sense of disjunction, we reject this for a pragmatic sense of meaning. Grice's reasons for this rejection is that a negated disjunction can only have one sense, i.e. 'It's not the case that A or B',

> [I]f *'or'* is to be supposed to possess a strong sense, then it should be possible to suppose it (or) to bear this sense in a reasonable wide range of linguistic settings; it ought to be possible, for example, to say *it is not the case that A or B*... But this, in the examples mentioned, does not seem to be possible; in anything but perhaps a very special case to say *it is not the case that A or B* seems to amount to saying that neither A nor B. [Grice, 1989a]

However, as we have seen in Chapter 3, pragmatic enrichments can be embedded, and in particular, they can be embedded under the scope of a negation. The following illustrations, in which we can clearly distinguish two senses of 'not (A or B)', are an example of this,

11. It is not the case that the murderer entered through the back door or through the front door. He entered through the window.
12. It is not the case that the murder entered either through the front door or through the back door. He *might* have entered through the window.

While 11 expresses the negation of each disjunct, 12 expresses the negation of the impossibility of a third option, independently of the truth value of A or B.[11]

In *LR*, we are capable of formalizing these two senses:

[11]These two different senses of the negation are studied by the phenomenon labelled as 'metalinguistic negation', a non-truth-functional negation. In Horn's words, it is defined as:

> [A] device for objecting to a previous utterance on any grounds whatever, including the conventional or conversational implicata it potentially induces, its morphology, its style or register, or its phonetic realization.' [as cited in Carston [1996]]

The literature on this topic has focused on the possible senses of 'not' while we are considering the different senses of 'or'. That is, we are interpreting the metalinguistic negation as the negation of a pragmatic enrichment rather than the negation of a literal disjunction. Hence, we need to argue that the variation is on the 'or' and not on the 'not' in certain cases. We give two observations towards this conclusion.

First, the phenomenon also affects other connectives. If, for instance, the disjunction is not negated but embedded in the antecedent of a conditional, there are two ways in which 'or' is interpreted, which correspond to the two senses in which it can also be negated:

- Negation of E-disjunction: $\neg(A \sqcup B)$
- Negation of I-disjunction: $\neg(A + B) \equiv \neg(\neg A \to B)$

Grice has a response to the counterexample just given: he calls 11 a *contradictory disagreement*, and 12 a *substitutive disagreement*, which is not properly a negation of the disjunction. It is not that one disagrees with the disjunction, as one does not exclude that A or B could be the case. But with this response, Grice is begging the question, as it presupposes that $\neg(\neg A \to B)$ is not the negation of a disjunction. However, $\neg A \to B$ is the definition of the intensional disjunction +, and the negation of the intensional conditional derived from it is the only requirement for the negation of the intensional disjunction.

Conjunction

The case for conjunction is different from the previous two cases: an affirmed conjunction does not violate the Gricean maxim of Quantity or of Relation, because given the truth conditions for 'A and B', there is only one reason why someone would assert it, which is that both A and B are the case. The enrichments of the affirmation of a conjunction are usually of order, causality or addition and not related to the effect of Relation maxims but Manner maxims:

13. She jumped and broke her leg.
14. They enjoyed the movie and watched it twice.
15. If you spend \$1 and you spend \$1 you get two candies.[12]

These kinds of implicatures can be captured by a distinction of two uses of the conjunctions in a system without the structural rules of exchange, contraction or both, which will be studied in the next chapter.

Turning our attention for the moment into the two conjunctions in *LR*, it is pertinent to note that the intensional conjunction in LR has been usually considered as problematic,

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13. If the thief entered through the front door or through the back door then he was inside the house,
14. If the thief entered through the front door or through the back door then he did not consider entering through the window.

Second, the disagreement between the two views depends on a presupposition: that the sense of 'or' is unique, and hence, there is just one way of negating it. However, once we agree that the connectives can be pragmatically enriched it is natural that such enrichment can be negated too.

[12] Similar examples on [Girard, 1995].

> Stephen Read, among others, has developed a binary connective, called 'fusion', which is an 'intensional conjunction.' The usual conjunction-introduction and conjunction-elimination rules do not hold for it, in full generality (...) he does argue that this connective is the right way to interpret (or regiment) some uses of the word 'and' in ordinary English, and perhaps also in some of the languages of mathematics. He claims that there is an ambiguity in the natural language term 'and'. I just said that it is 'natural' to hold that if a connective does not obey the indicated introduction and elimination rules, then it is not conjunction. It seems that this is not held by every logician. [Shapiro, 2014, p. 97-98]

One of the reasons for this controversy with the relevant conjunctions might be the fact that the intensional and extensional conjunctions are not distinguished when are asserted but only whenever are embedded. In effect, the two conjunctions in *LR* capture two uses of the conjunction when they are embedded under the scope of a negation or in the antecedent of a conditional. In those cases the Enrichment schema is applicable.

The difference between the conjunction in *LK* and *LR* is that those inferences that are valid for the same connective, $\wedge$, in *LK*, are valid for different connectives in *LR*. For instance, $\wedge$ is a conjunction for which $\neg A \vdash \neg(A \wedge B)$ and $\neg(A \wedge B) \vdash A \supset \neg B$ holds. But in *LR*, these inferences are valid for the two different conjunctions: $\neg A \vdash \neg(A \sqcap B)$ is only valid for $\sqcap$ and $\neg(A \times B) \vdash A \to \neg B$ is only valid for $\times$. Also, in *LK* the following inferences are valid for $\supset$: $A \supset C \vdash (A \wedge B) \supset C$ and $(A \wedge B) \supset C \vdash A \supset (B \supset C)$. However, in *LR*, they are valid for different conjunctions: $A \to C \vdash (A \sqcap B) \to C$ is only valid for $\sqcap$ and $(A \times B) \to C \vdash A \to (B \to C)$ is only valid for $\times$. Moreover, the extensional conjunction entails the intensional one, but not the other way around: $A \sqcap B \vdash A \times B$, but $A \times B \nvdash A \sqcap B$.

The Enrichment schema for a negated conjunction distinguishes E and I reasons to assert 'not (A and B)':

- E-reasons: not A/not B
- I-reasons: there is some connection between A and B that makes them incompatible: i.e. one conjunct excludes the other, A entails $\neg B$.

In uses of the negation of a conjunction, the assertion 'not (A and B)' due to E-reasons violates the Quantity maxim, and hence they imply that conjunctions are asserted for I reasons. This distinguishes the extensional and the intensional conjunction in *LR*,

- E-conjunction: $\Gamma \vdash \neg A \Rightarrow \Gamma \vdash \neg(A \sqcap B)$ / $\Gamma \vdash \neg B \Rightarrow \Gamma \vdash \neg(A \sqcap B)$

- I-conjunction: $\Gamma \vdash A \to \neg B \Rightarrow \Gamma \vdash \neg(A \times B)$

As an illustration of the distinction, consider the following examples from natural language, 16 asserted for E-reasons and 17 asserted for I-reasons:

16. It is not the case that she is blonde and has blue eyes (because she is not blonde).

17. We won't go to Paris and to London this summer (we cannot afford it).

Clearly, the grounds for asserting 17 and 16 are of different kinds. One asserts 17 because of a certain incompatibility between the conjuncts, that is, the truth of one of the conjuncts excludes the other, that is, for I-reasons. The enrichment is true in this case, and it should be formalised in *LR* with $\times$. In the case of 16, one negates the conjunction because of the falsity of one of the conjuncts, independently of its relationship with the other (that is, for E-reasons) and the enrichment is false. As a result, it should be formalised in *LR* with $\sqcap$.

Furthermore, they have different inferential roles. Note that $\neg(A \sqcap B), A \vdash \neg B$ is invalid in *LR*, as if the sequent was valid, weakening would be valid too:

$$\dfrac{\dfrac{\neg(A \sqcap B), A \vdash \neg B \qquad \dfrac{\dfrac{B \vdash B}{A \sqcap B \vdash B}\,\sqcap\mathrm{L}}{\neg B \vdash \neg(A \sqcap B)}\,\neg\mathrm{L}/\neg\mathrm{R}}{A, \neg B \vdash \neg B}\,\mathrm{Cut}}{A, B \vdash B}\,\mathrm{E}/\neg R/\neg L$$

Now consider 16. If the grounds one has for asserting it are the knowledge that she is not blonde, it would be illegitimate to infer that she does not have blue eyes upon discovering that she *is* in fact blonde. The discovery would just spoil the derivation. It would be equally illegitimate to infer that she is not blonde with the following reasoning, it is only $\neg A$ that *relevantly* entails $\neg A$, as B is arbitrary, and the argument violates the maxim of Relation.

A similar inference with $\times$ does not require weakening so it is perfectly correct in *LR*:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A \qquad B \vdash B}{A, B \vdash A \times B}\,\times\mathrm{R}}{B, A \vdash A \times B}\,\mathrm{EL}}{A \vdash A \times B, \neg B}\,\neg\mathrm{R}}{A \vdash \neg B, A \times B}\,\mathrm{ER}}{\neg(A \times B), A \vdash \neg B}\,\neg\mathrm{L}$$

Consider 17: given the connection between the conjuncts it seems perfectly right to derive that the speaker will not go to London from the fact that she will go to Paris given their incompatibility.

The enrichment schema for the conjunction embedded in the antecedent of a conditional distinguishes extensional and intensional reasons for asserting 'if A and B, then C':

- E-reasons: A implies C / B implies C,
- I-reasons: both A and B are required to derive C.

The E-reasons violate the Gricean maxim of Relation and this is captured in LR:

- E-conjunction: $\Gamma \vdash A \rightarrow C \Rightarrow \Gamma \vdash (A \sqcap B) \rightarrow C$ / $\Gamma \vdash B \rightarrow C \Rightarrow \Gamma \vdash (A \sqcap B) \rightarrow C$
- I-conjunction: $\Gamma \vdash A \rightarrow (B \rightarrow C) \Rightarrow \Gamma \vdash (A \times B) \rightarrow C$

The I-reasons to embed the conjunction in the antecedent of a conditional are read as 'with' in natural language. 'A with B' is usually formalised as $A \wedge B$ but pragmatically implies that both are used to obtain the conclusion. LR is capable of capturing this. Consider the following examples, 18 asserted for E-reasons, 19 asserted for I-reasons:

18. If I have a coffee and eat a croissant, then I have my dose of caffeine.
19. If I have a coffee and eat a croissant, then I have a full breakfast.

The consequent of 19 is asserted on grounds of both conjuncts in the antecedent, so 19 is asserted for I-reasons, while the consequent of 18 is asserted on grounds of just one of the conjuncts in the antecedent and 18 is asserted for E-reasons.

In this case, the difference in the inferential role of both conjuncts lies in the property of importation. It is natural to import the conditional with the intensional conjunction, but not with the extensional conjunction.

First, note that $(A \sqcap B) \rightarrow C, A \vdash B \rightarrow C$ is invalid in LR, as if the sequent was valid, weakening would be valid (note that we can derive $A, B \vdash C$ from $B \vdash C$):

$$\dfrac{\dfrac{(A \sqcap B) \to C, A \vdash B \to C \qquad \dfrac{\dfrac{\dfrac{B \vdash B}{A \sqcap B \vdash B}\,\sqcap\mathrm{L} \qquad C \vdash C}{A \sqcap B, B \to C \vdash C}\,\to\mathrm{L}}{B \to C \vdash (A \sqcap B) \to C}\,\to\mathrm{R}}{A, B \to C \vdash B \to C}\,\mathrm{Cut} \qquad B \to C, B \vdash C}{A, B, B \to C \vdash C}\,\mathrm{Cut} \qquad \dfrac{B \vdash C}{\vdash B \to C}\,\to\mathrm{R}$$
$$\overline{\qquad\qquad A, B \vdash C \qquad\qquad}\,\mathrm{Cut}$$

Now consider 18: from LR's perspective it is illegitimate to say that, if I have a coffee, then, if I have a croissant then I have my dose of caffeine. The croissant has nothing to do with the caffeine, and it seems illegitimate to derive the conditional.

In contrast, consider 19: given that both conjuncts are necessary to infer the consequent, it is completely natural to say that, if I have a coffee, then, if I have a croissant then I have a full breakfast. The sequent is derivable in LR:

$$\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A \qquad B \vdash B}{A, B \vdash A \times B}\,\times\mathrm{R} \qquad C \vdash C}{(A \times B) \to C, A, B \vdash C}\,\to\mathrm{L}}{B, (A \times B) \to C, A \vdash C}\,\mathrm{EL}}{(A \times B) \to C, A \vdash B \to C}\,\to\mathrm{R}$$

Intuition for LR's conjunctions A classical criticism to LR is its lack of a connective like $\wedge$ in LK, which has the following two properties:

$A, B \vdash A \wedge B$

$A \wedge B \vdash A$

In effect, one of the least controversial claims in logic is that the truth conditions for the conjunction should be those of $\wedge$. Hence, the relevant monist has problems explaining the behaviour of the conjunction.

The pluralist perspective presented here can shed light on this. On the one hand, we maintain $\wedge$ as capturing the literal truth conditions of the conjunction, which makes the above properties for the conjunction truth-preserving. On the other hand, in the context of logical consequence understood as LR (that is, understood as a normative relation which avoids the violation of Relation maxims) it is desirable to split the conjunction as LR does. Apart from the fact that a connective such as $\wedge$ would reestablish weakening, there is a non-*ad hoc* explanation: whenever $\vdash$ is enriched with relation enrichments there is a sense of conjunction for which A does not follow from A and B, and this is the sense that $\times$ captures:

> Naturally, in view of the fact that a conjunction must function as a unity, it cannot be asserted that the conjunction of p and q *entails* p, for q may be totally irrelevant to and independent of p, in which case, p *and* q do not entail p, but it is only p that entails p. (Nelson, as cited in [Humberstone, 2011, p. 658]).

In effect, whenever one infers p from p and q one is violating the maxim of Relation, as q is irrelevant for the derivation, with just p entailing p. Recall the Enrichment schema for a conjunction as the antecedent of a conditional: there is a violation of the Gricean maxim of Quantity or Relation whenever the conclusion C is derived from a conjunction (A and B) if the conclusion C is known to be derivable from one of the conjuncts, A.

In sum, we have seen how each logical connective is split into two, and that they diverge about the reasons one might have to assert it. Moreover, the reasons one has to assert a connective are related to the Relation maxims: these reasons can be encoded. See the following table for a summary.

Natural Language	Reasons to assert	*LK*	*LR*
if A then B	not A / B	$\supset$	$\rightsquigarrow$
	given A, B		$\rightarrow$
A or B	A / B	$\vee$	$\sqcup$
	the rejection of A entails B, or viceversa		$+$
not (A and B)	not A / not B	$\wedge$	$\sqcap$
	incompatibility between A and B		$\times$
if (A and B) then C	if A then C / if B then C	$\wedge$	$\sqcap$
	both A and B are required for C		$\times$

CHAPTER 5

Linear and Ordered Logic

The two following logics, Linear (LL) and Ordered Logic (OL), are the result of rejecting contraction and exchange, in addition to the rejection of weakening. The logics are therefore weaker than classical and relevant logic, and codify a stronger notion of consequence in which the *use* of the premises is codified, and therefore related to the manner maxim.

5.1 Linear Logic

5.1.1 The logic

The linear logic considered here is the propositional fragment without exponentials of the linear logic presented in [Girard, 1987] and [Girard, 1995], that is, MALL (for Multiplicative Additive Linear Logic). It is extracted from the presentation in [Paoli, 2002]. Linear Logic is usually presented with two operators, the exponentials, which can re-establish the expressive power of classical logic. However, given the pluralist perspective adopted here (which endorses classical logic) we will not consider them: the aim of this project focuses on the effect of the structural rules on the behaviour of conditional, disjunction, and disjunction, and the presence of exponentials does not have a direct relationship with this research.

Notation The notation that we will use for Linear Logic is different from the usual notation used in the linear tradition. This decision is not arbitrary, but motivated by the coherence with the notation for LR[1] [2].

[1] The symbol * indicates that the connective is not present.

[2] We will use subindices to distinguish the extensional connectives in the different substructural logics only when the rest of the symbols in the sequent are not sufficient for disambiguiating them.

	Girard [1987]	Paoli [2002]	Paoli [2007]	our notation
Negation	$-^{\perp}$	$\neg$	$\neg$	$-^{\perp}$
Extensional conditional	$*$	$*$	$\rightsquigarrow$	$\rightsquigarrow_L$
Intensional conditional	$\multimap$	$\rightarrow$	$\rightarrow$	$\multimap$
Extensional disjunction	$\oplus$	$\vee$	$\sqcup$	$\sqcup_L$
Intensional disjunction	$\parr$	$+$	$+$	$\oplus$
Extensional conjunction	&	$\wedge$	$\sqcup$	$\sqcap_L$
Intensional conjunction	$\otimes$	$\times$	$\times$	$\otimes$

5.1.2 Linear sequents

The tradition of linear logic has its origin in the work of Girard [1995] and [1987], and since then an extensive bibliography has been generated. There are *uses* of linear logic which find the linear language a good formalization for a given context or domain. This is the case of **?**, who models deontic logic with linear logic, interpreting the sequents as ranging over actions permitted (and solving certain paradoxes that classical logic faces, such as the Free Choice Permission paradox, which will be discussed in chapter 7). And another example is Dixon et al. [2009] who use linear logic to model planning, a domain of reasoning in which we need to track the effect that certain actions have on the given background. Finally, there is an extensive bibliography on the application of linear logic to computer science. Even Girard interprets this logic as applicable to different domains or contexts without contradicting classical logic,

> Linear logic is not an alternative logic; it should rather be seen as an extension of usual logic. [Girard, 1995, p.1]

Hence, a version of logical pluralism which endorses classical and linear logic would be either a version of Applied Logical Pluralism, or a version of Localism, depending on whether Linear Logic applied to these different discourses is considered genuine logical consequence or not.

We will show that Linear Logic has to be considered as a consequence relation whenever there are certain pragmatic enrichments of the logical vocabulary, without endorsing Localism: we will argue that Linear Logic captures genuine logical consequence, and the fact that certain domains have been defended as exhibiting a Linear consequence will be defended arguing that there are domains which usually exhibit such enrichments.

We aim to show that Linear Logic should receive attention from philosophers, and in particular from the debate on logical pluralism, as relevant logic does. The current pluralist framework can explain how classical, relevant and linear logics can coexist. And one of the first issues that we should address is the treatment of the notion of consequence of linear logic as a genuine formalization of 'follows from'.

A general reading of $\vdash_L$

In the literature on Linear Logic there are different attempts to interpret $\vdash_L$. Our aim is to find an interpretation which embraces and explains all of them as as a pragmatically enriched sense of 'follows from'.

Actions Our starting point is Girard's interpretation of $\vdash_L$. In [1995] he interprets $\vdash_L$ as a *causal* relation, in which the left side of a sequent is seen as the causal conditions for the right side. Moreover, Girard introduces the notion of 'reaction', a notion which expresses the effect that the sequent itself, or the process of arriving to the conclusion from the premises, has on those premises,

> [R]eal implication is causal. A causal implication cannot be iterated since the conditions are modified after its use; this process of modification of the premises (conditions) is known in physics as reaction. [Girard, 1995]

Although Girard considers that the linear sequent formalizes causality, we will argue that it formalizes a notion of logical consequence. However, we will make use of the notions of *use* and of *reaction* as both notions are fundamental to understand $\vdash_L$: both terms are related and are illustrative of the difference between the linear consequence and the classical or relevant. The fact that premises are used to get the conclusion is related to the fact that they can be wasted or modified given the conclusion. And this use leads to the reaction, which needs to be distinguished from *implication*: in a linear sequent $A \vdash_L B$, the conclusion B is the linear implication of A, while the reaction refers to the modification of the premises (A) in the process (for instance, it might be the case that A is no longer available once we got B). As an illustration, consider Girard's interpretation of $A \vdash_L B$, as expressing something along the following lines:

1. If I *pay* one Euro I *get* one coffee.

While the implication of paying one Euro is getting a coffee, the reaction is loosing one Euro, so one might not be able to pay another euro, that is, of using the premise again.

In this interpretation Girard treats the premises and conclusions of linear sequents as *actions*. Following this path, Bellot et al. define an action as 'anything that can be done' [1999]. In the same vein, Restall [1994] interprets a linear sequent with the following simple expression,

> As a result of A, B. [Restall, 1994, p. 159]

Moreover, without mentioning the term *reaction*, Restall adds its core idea:

> As I perform actions of different types, I change the world around me. [Restall, 1994, p. 159]

Resources Actions are not the only interpretation of the linear sequents that we find in the literature. Danos and Di Cosmo [1992] argue that saying that 'Γ implies Δ' is understood in *LL* as a trading relation between the resources in Γ and the products in Δ,

> [I]f you give me the goods in Γ, I will give you one of the goods in Δ. [Danos and Di Cosmo, 1992]

Premises are interpreted as resources for the conclusion, and the repetition of a premise is interpreted as the need for a repetition of that resource. Contrast 1 with 2:

2. If I *have* one Euro, I *can get* one pencil.

Under this interpretation there is no action codified in Γ nor in Δ, but there is a reaction on Γ given Δ: the premises are *used* to get the conclusion, and *wasted* in the process.

Abilities In this second interpretation of linear sequents one usually introduces a modal operator in the actions that one *can* perform given the situation in the premises, that is, we add a modal operator 'can' for the things that allow us to do the 'good' in the premises. In this vein, Blass [1997] introduces the notion of *ability*, which is not mentioned in Girard [1995] (despite the spirit of Blass' reading, which follows Girard's),

> Linear logic comes even closer to intuition if, following Girard, we think of it as being about actions or *abilities*. [Blass, 1997, p. 4][3]

Having one Euro gives its owner the capacity or ability to buy one coffee. This interpretation as abilities will help us understand more complex examples capable of being formalised with linear logic such as the following examples by Zardini [2016],

> 'Every piece of wood that makes 4 chairs makes 1 bed' seems true, but 'Every piece of wood that makes 4 chairs makes 1 bed and makes 4 chairs' does not seem so; 'Every lion that needs 4 steaks per day needs 1 joint of roast beef

[3]Emphasis added.

> per day' seems true, but 'Every lion that needs 4 steaks per day needs 1 joint of roast beef per day and needs 4 steaks per day' does not seem so; 'Every quantity of energy that heats 4 houses moves 1 train' seems true, but 'Every quantity of energy that heats 4 houses moves 1 train and heats 4 houses' does not seem so. [Zardini, 2016, p. 306]

Consider the first case:

3. Every piece of wood that makes 4 chairs makes one bed $\not\Rightarrow$ Every piece of wood that makes 4 chairs makes one bed and makes 4 chairs.

What the example expresses is the capacity of some objects to be transformed into more than one thing, but the incapacity to be transformed into more than one of those things *together*. 3 can be reformulated as follows:

3′. Every piece of wood that has the capacity to make 4 chairs has the capacity to make 1 bed $\not\Rightarrow$ Every piece of wood that has the capacity to make 4 chairs has the capacity to make 1 bed and to make 4 chairs.

Given that we are focusing on the propositional fragment of linear logic, we should avoid the 'every' of these examples, focusing on one particular piece of wood,

3′. If a piece of wood has the capacity to make 4 chairs then it has the capacity to make 1 bed $\not\Rightarrow$ If a piece of wood has the capacity to make 4 chairs then it has the capacity to make 1 bed and to make 4 chairs.[4]

Again, the common characteristic of these examples with the previously seen is that the relation between premises and conclusion is such that there is a *reaction* on the premises given the conclusion: the reaction of the relation between the piece of wood and the one bed is that whenever it is performed, the piece of wood loses its capacity of making four chairs.

[4]In order to be completely fair with the linear formalization the conditional form of these examples is a case of mixed inference, and this will be mentioned in Chapter 7, although it is outside the scope of this project: The transformation of the piece of wood into 4 chairs and into a bed can be formalised with a linear conditional, but the connection between these two capacities seems a relevant relation to be formalised with the relevant $\rightarrow$. The closest example that can be formalised purely in the propositional fragment of linear logic is:

3″. This piece of wood has the capacity of making 4 chairs and it has the capacity of making 1 bed $\not\Rightarrow$ This piece of wood has the capacity of making 4 chairs and making 1 bed.

Finally, in order to see that these examples have a similar interpretation to the simpler cases 1 and 2, note that those previous simple examples can be reformulated in a similar form as in Zardini's examples:

4. Any amount of money that buys one pencil buys one pen, but the amount of money that buys one pencil does not buy one pencil and buy one pen.

Information interpretation Mares and Paoli [2014] argue for an interpretation of LL according to which $\vdash$ represents information transmission. Formulas in sequents are treated as tokens of pieces of information, from which other tokens are extracted. This reading contrasts with classical logic in which the truth of the premises gives grounds for the truth of the conclusion.

This interpretation does not fit well with our previous readings of $\vdash_L$ given that, although it shares with them the presence of 'use' of the premises to get to the conclusion, there is no reaction in information transmission.[5]

A general reading of $\vdash_L$ All these interpretations of linear logic (except transmission of information, i.e. actions, resources, and abilities) share a common property: they express a connection between the premises and conclusions in which there is a use and a reaction. This is common in causal relations, but causal relations can be included as implication relations (see [Zardini, 2019] for an analysis of the connection), and hence, in natural language, we can express implication relation which are grounded in causality. Note that this is a stronger relation than the classical (in which we just establish a truth-preserving relation between statements), and also a stronger relation than relevant (in which we establish a relation between the truth of utterances, but in which the premises are relevant for the truth of the conclusion). In sum, in the linear notion of consequence we determine that the premises are responsible for the conclusion and somehow *used* for arriving to that conclusion. Hence, we suggest a reading of a sequent that goes in line with and expands

[5]The reaction on the premises given the conclusion gives us the resources to interpret the following two facts about LL, which are difficult to interpret under the informational reading:

- $A \multimap B, A \nvdash_L B \otimes A$
- $A \vdash_L A$

In the first case, given that A is used to get B, it cannot appear again conjuncted to be in the conclusion. This is clear whenever there is a reaction. And an advocate of the information transmission reading may respond that the A in the conclusion is a different token from the A in the premises, which would explain the invalidity of the sequent. However, if this is the case, it is not clear how a token of A entails a different token of A in the second case.

the different interpretations that we find in the literature: a relation that explains its traditional reading as ranging over events or facts, but also over resources and their outcomes, or certain circumstances and the abilities that they offer. What is fundamental in a linear relation is the presence of a reaction, and in this sense, it might be useful to understand this relation in the following manner,

$\Gamma \vdash_L \Delta$: using Γ we get Δ

Assuming that the term *use* is accompanied by a possible change of Γ given Δ, when Δ has been used and, hence, possibly transformed.

5.1.3 Manner Enrichment of Consequence

We have seen how relevant logic is related to the relation and quantity Gricean maxims, and we have suggested in previous chapters that linear logics are related to manner maxims. However, the connection is not as straightforward as in the case of relevance, and it has to do with certain particularities of the manner maxim.

Manner maxim does not affect all the utterances' connectives, and it is strongly linked to the content of the proposition expressed and our knowledge about the world. For instance, the following two conjunctions are enriched with the manner submaxim of order, but its contribution is different in each case:

5. She jumped and broke her leg,

6. She shut down the computer and went for a walk.

While both conjunctions are enriched with an order enrichment, the first is also enriched with a cause/effect relation, while the second is not. And the reason why this is so is our knowledge about the world, and about how these kinds of events usually happen. In sum: our knowledge about the world and about the content of the utterance affects the particular enrichments, especially the manner enrichments, that the utterance may have.

In a similar way, our knowledge about the world (about the connection between the Euros and the world, our knowledge of the things that can be accumulated and the things that cannot, etc.) affects the linear connection of consequence. The notion of consequence is enriched with a reaction given the particular information of the premises and conclusion. In other words, the reaction expressed in some inferences is a pragmatic enrichment but not part of the literal meaning of *follows from*. Let us see in what follows why Weakening and Contraction should be rejected given this kind of pragmatic enrichment.

Weakening

Weakening is rejected in Linear Logics, and the reasons for rejecting the right and left side are equivalent to the reasons already seen,

$$\frac{\Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}\,\mathrm{WL} \qquad\qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A}\,\mathrm{WR}$$

We have seen in Chapter 4 the relevant reasons to reject this structural rule. These reasons are still in force in linear logics: the relevant connection between premises and consequences is present and reinforced in the linear connection: our reading 'using A we get B' requires that both A and B are relevant to each other.

Moreover, there is a further reason for rejecting weakening. Irrelevant information in a relation in which we understand that there is a reaction on the premises would entail the false idea that some irrelevant premise is wasted, or that the premises are wasted due to some irrelevant conclusion, which would not only be misleading but false.

Finally, there is another reason to reject weakening on the left side: the result of one A might be different to the result of twice A. Paying one Euro and paying one Euro can be understood as paying one Euros *twice*, which implies different things from paying one Euro *once*. Hence, in linear logic we *should be brief*, so we should not repeat an utterance because that would entail that two copies of the premises are required for the conclusion.

Contraction

Linear logic rejects contraction. Contraction has been rejected by some theorists ([Zardini, 2015b],[2011],[Restall, 1994],[1993]), although their motivation is not interpreted as a pragmatic enrichment of the vocabulary[6]. So we need to justify this claim, and see how this rejection is explained for an enrichment of the logical vocabulary.

Instability We find a reason for rejecting contraction in [Zardini, 2015b], which can be adopted with some modifications here. Zardini rejects contraction for those formulas which formalize states of affairs which are *unstable*, such as the liar, $l = \neg T(l)$. Unstable state of affairs are those which do not co-obtain with their consequences[7]:

> Stable states-of-affairs, if they obtained, would co-obtain with all of their consequences; unstable states-of-affairs would not. [Zardini, 2011]

[6] See [Rosenblatt, 2019] for a review of the reasons for rejecting contraction, related to the semantic paradoxes.

[7] Notice the similarity of this property to the notion of reaction seen above

Although a response to the liar paradox is outside the scope of this project, instability can apply to other expressions different from the liar. In fact, Zardini suggests that there is a similarity between the instability of the liar and the instability of physical states:

> [T]he behavior of an unstable state-of-affairs as the one expressed by $\neg Tl$ resembles the behavior of physical states: both kinds of states, if they obtained, would lead to other states with which they would not coobtain - although it is precisely the obtaining of the former states that would lead to the obtaining of the latter states. For example, in the case of physical states, the state in which one moving object is about to enter into collision with another moving object, if it obtained, would typically lead to a state where the direction and velocity of the two objects are different from and hence incompatible with the original ones. Hence, the former state would lead to a state with which it would not coobtain - although it is precisely the obtaining of the former that would lead to the obtaining of the latter. [Zardini, 2011, p.404]

Hence, while A implies (in a linear sense) B, it is perfectly natural to imagine that they are not true together, in a similar way in which in a natural causation between C and D in which C is the case at time t_0 and D is the case at time t_1, C ceases to be the case at time t_1.

Contrary to Zardini, for whom instability is an intrinsic property of certain states of affairs, we will characterize instability as the result of the reaction that using the premises to get a conclusion has on those same premises. In other words, whenever a connection between some premises and some conclusions is accompanied by a reaction, the premises of the inference are unstable with respect to their linear consequences.

For instance, having one euro is unstable with respect to buying one coffee, but having one euro is not unstable with respect to having some money in ones wallet. In the first case the connection is linear, in the second, classical or relevant.

The reason why contraction fails for formulas which formalize unstable states of affairs (for Zardini) is that it might be the case that for some A and B, B follows from twice A but not once A, if A expresses an unstable state of affairs.

$$\frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}\,\mathrm{CL}$$

> If φ expresses the state-of-affairs s_0, $\varphi, \varphi \vdash \psi$ may hold, let's suppose, only because s_0 and some state-of-affairs s_1 consequence of s_0 *together* directly lead to the state-of-affairs s_2 expressed by ψ. If s_0 is however unstable, it does not follow that s_0 *by itself* leads to s_2, and so it does not follow that $\varphi \vdash \psi$ holds.

> For, although s_0 does of course by itself lead to its consequence s_1, by its instability s_0 need not co-obtain with s_1, while we're supposing that s_0 can lead to s_2 only together with s_1. Failure of contraction is thus the logical symptom of an underlying unstable metaphysical reality. [Zardini, 2015b, p.404]

As an illustration, consider the instability of A with respect to the linear connection ($\vdash$) with Δ in this example is explained thus: paying half a Euro (A) does not get one a coffee (Δ), unless one has already payed half a Euro (A). Paying half a Euro, combined with paying half a Euro, gets one what costs one Euro. But having already payed half a Euro may not co-obtain with having half a Euro to pay. This is why we need twice A and hence A cannot be contracted: one A creates the situation such as 'one has already payed half a coffee', and another A is the event that in such situation results in Δ.[8]

The right part of the structural Contraction, CR, is harder to connect (at least, directly) with the Manner Maxim,

$$\frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}\ \mathrm{CR}$$

Given the rules of negation, one can easily transform twice A as conclusions into twice $A^{\perp}$[9] as the premises, and the rejection of the left rule of contraction justifies why CR should be invalid. In other words, contraction on the right side of a sequent would validate inferences such as:

$$\frac{A^{\perp}, A^{\perp}, \Gamma \vdash \Delta}{A^{\perp}, \Gamma \vdash \Delta}\ \mathrm{CR}$$

Which cannot be valid with current interpretation of sequents.

5.1.4 Manner Enrichment of Connectives

In parallel with the case of relevant logic, linear connectives split into two different ones (see the sequent calculus for Linear Logic LL in Appendix A). And similarly to relevant logic, their divergence from classical behaviour is naturally explained given the new sense of consequence: the linear behaviour of $\vdash$ requires a linear behaviour of the connectives.

[8] In [2019] Zardini gives a different line of argument about why contraction fails for unstable states of affairs. However, for the purpose of this dissertation, this earlier justification can better explain the rejection of this structural rule.

[9] Recall that the negation in LL is formalised as $-^{\perp}$.

> Once we decide to be careful about resources, we start to be suspicious about the traditional $\wedge$ and $\vee$ connectives too, and we discover that each of them is now split in two. [Danos and Di Cosmo, 1992]

A first grasp of the connectives in Linear Logic and their expressive power is introduced by Lafont, as cited in [Danos and Di Cosmo, 1992]. Consider the following illustration of a menu at a restaurant:

Menu **à** 75 FF	
Entree:	Quiche lorraine **ou** Saumon fume
et	
Plat:	Pot-au-feu **ou** Filet de canard
et	
Fruit:	selon saison (Banane **ou** Raisin **ou** Oranges **ou** Ananas)
	ou
	dessert au choix (Mistere**,** Glace**,** Tarte aux pommes)

Let us briefly see the intuition behind each connective in natural language that appears in the menu. First, there is a connection between enjoying the menu and paying 75FF ($\multimap$). This connection can also be seen as a disjunction: one can choose between not paying 75FF or getting the menu at the restaurant ($\oplus$). There is a further disjunction that connects each option of the 'selon saison', which expresses that one of the options is available, but its availability depends on the season of the year ($\sqcup$). There are also two kinds of conjunctions: the conjunction that connects each part of the menu (Entree, Plat, Fruit) which expresses that the customer will get one of each ($\otimes$) and the conjunction, expressed by a comma, in the 'dessert au choice' that expresses the availability of three alternatives among which the customer has to choose ($\sqcap$). The menu can thus be formalised in *LL*:

$$75FF \multimap ((Q \sqcap S) \otimes (P \sqcap F) \otimes ((B \sqcup R \sqcup O \sqcup A) \sqcap (M \sqcap G \sqcap T)))$$

Or put equivalently:

$$75FF^{\perp} \oplus ((Q \sqcap S) \otimes (P \sqcap F) \otimes ((B \sqcup R \sqcup O \sqcup A) \sqcap (M \sqcap G \sqcap T)))$$

Conditional

In linear logic the division between extensional and intensional connectives expresses the implication, conjunction or disjunction within a consequence relation with a reaction. As will be shown below, intensional connectives enrich their expressive power with respect to the extensional connectives in linear logic and with respect to the intensional connectives in the stronger logic *LR*. The distinctions can be grasped by the different reasons one might have to introduce a given connective, as will be shown below.

The linear conditional expresses a relevant connection between two utterances with a reaction between them. For instance, following the interpretation of linear logic as ranging over actions by Girard, the conditional expresses something along the following lines,

> An action type $A \multimap B$ is a way of replacing any specific dollar by a specific pack of camels. [Girard, 1995, p. 2]

Or considering the resource interpretation, it is read as follows,

> $A \multimap B$ (...) indicates that the resource A is consumed and the resource B is produced as a result. [Dixon et al., 2009, p. 254]

In any case, the linear interpretation of 'if...then' is accompanied by a reaction which affects the antecedent (i.e. losing a dollar or consuming a resource).

Intensional vs Extensional conditional Contrary to *LR*, in which there was an entailment connection between the intensional and extensional connectives, in *LL* these two are independent. In particular, it is neither the case that $A \multimap B \vdash A \rightsquigarrow B$ nor that $A \rightsquigarrow B \vdash A \multimap B$.

We can add two new reasons to assert a conditional 'if A then B' from Γ:

- LE-reasons: using Γ one gets $A^{\perp}$ / using Γ one gets B.
- LI-reasons[10]: There is a connection between A and B such that using Γ and A we get B as a result.

This is captured in Linear logic:

- LE-conditional: $\Gamma \vdash_L A^{\perp} \Rightarrow \Gamma_L \vdash A \rightsquigarrow B$ / $\Gamma \vdash_L B \Rightarrow \Gamma \vdash A \rightsquigarrow B$

[10]We have referred to extensional and intensional reasons in *LR* as E/I-reasons, given that there was no ambiguity with other E/I-reasons yet. From now on, we will refer as RE/RI, LE/LI (or OE/OI) to the different reasons in each logic.

◦ LI-conditional: $A, \Gamma \vdash_L B \Rightarrow \Gamma \vdash A \multimap B$

The manner enrichment, which emerges from our knowledge about the world, and the kind of connection between two utterances, according to which the antecedent is modified in producing the consequent, is true for $\multimap$. We can grasp these differences in natural language, being 7 asserted for LE-reasons and 8 for LI-reasons:

7. If I have 10 Euro in my pocket I will get a menu at the restaurant (on the grounds of being invited to have lunch at restaurant by its chef).

8. If you have 10 Euro you can get a menu at the restaurant (giving that 10 Euro is the price of the menu).

Extensional connectives Note that while the difference between $\rightsquigarrow$ and $\rightarrow$, and between $\rightarrow$ and $\multimap$ is clear (as explained below), the difference between extensional connectives in the different substructural logics, $\rightsquigarrow_R$ and $\rightsquigarrow_L$, is not so clear, given that the reasons to assert the extensional connectives in each logic seem to remain equivalent, but not their inferential role.

However, extensional connectives across logics are not exactly equivalent, although closely related. Note that an extensional connective $A \bullet B$ is asserted from some background Γ, that is, we are presenting connectives as reasons for asserting $A \bullet B$ given Γ as $\Gamma \vdash A \bullet B$. However, $\vdash$ is different in each logic and expresses different things: for instance, the grounds for deriving $A \rightsquigarrow B$ from Γ in LR is that $\neg A$ or B follow *relevantly* from Γ, while in LL the grounds are that *using* Γ one gets $\neg A$ or B.

Intensional$_l$ vs Intensional$_r$ conditionals An interesting distinction, given the present pluralist perspective, is between the intensional conditionals in LR and LL: that is, the connection between A and B in $A \rightarrow B$ and $A \multimap B$: while $\multimap$ expresses the use of A to get B, $\rightarrow$ expresses a relevant connection between A and B. Hence, we can distinguish two reasons for asserting an enriched conditional 'if A then B':

◦ RI-reasons: There is a connection between A and B such that the antecedent cannot be true without the consequent being also true.

◦ LI-reasons: There is a connection between A and B such that using A we get B as a result.

In many contexts (as in the menu example) there is a reaction in the derivation of B from A (for instance, the unavailability of A, given B). This idea of reaction is not present in *LR*, and this is how we can distinguish the intensional conditionals in the two logics, given their connection with the consequence relation:

- RI-conditional: $A, \Gamma \vdash_R B \Rightarrow \Gamma \vdash A \to B$
- LI-conditional: $A, \Gamma \vdash_L B \Rightarrow \Gamma \vdash A \multimap B$

And they can be distinguished in natural language, 9 introduced for RI-reasons and 10 introduced for LI-reasons:

9. If you have 1,000,000 Euro you can appear in the Forbes' list (not because you have to pay)[11],

10. If you have 75FF you can get a menu at the restaurant (giving the 75FF in exchange).

In order to see that $\multimap$ can express this transaction expressed in 10, let us see that in *LL* one cannot derive from A and $A \multimap B$ both $A \otimes B$. Note that if $A, A \multimap B \vdash A \otimes B$ is added to the calculus, contraction can be derived. In order to proof the invalidity of this sequent we add a sequent in which B is substituted by $A \multimap B$, and we show that supposing $A, A \vdash B$, we can derive $A \vdash B$, that is, we can derive the right rule of contraction:

$$\dfrac{\dfrac{A \multimap (A \multimap B), A \vdash A \otimes (A \multimap B) \qquad \dfrac{\dfrac{A, A \vdash B}{A \vdash A \multimap B}\,{\multimap}\mathrm{R}}{\vdash A \multimap (A \multimap B)}\,{\multimap}\mathrm{R}}{A \vdash A \otimes (A \multimap B)}\,\mathrm{Cut} \qquad \dfrac{\dfrac{A \vdash A \qquad B \vdash B}{A \multimap B, A \vdash B}\,{\multimap}\mathrm{L}}{(A \multimap B) \otimes A \vdash B}\,{\otimes}\mathrm{L}}{A \vdash B}\,\mathrm{Cut}$$

However, the sequent can be derived in *LR* using contraction, that is, in *LR*, $A, A \to B \vdash A \times B$ is valid.

In sum, the two conditionals in the two different logics express two different things, given the absence of reaction in *LR* and its presence in *LL*.

[11]Notice also the difference with $\rightsquigarrow$: in $A \to B$ there is a connection between having 1,000,000 Euro and being able to appear on the list, contrary to the case of $A \rightsquigarrow B$, which can be asserted on grounds of one not having the 1,000,000 Euro or one being able to appear on the list independently of the amount of money one has.

Understanding negation

In order to understand the disjunction in *LL* we first need to understand the role of negation, which is usually represented as $A^{\perp}$. There have been different attempts in the literature to give a natural reading of negation in Linear Logic, which we will briefly review.

First, Girard, given the change of positions of negation in the Gentzen calculus, and given that $\vdash$ is interpreted as a causal relation, interprets negation in the following way:

> [A]ction of type A = reaction of type $A^{\perp}$ [Girard, 1995, p. 3]

However, this interpretation changes the sense of reaction above. In fact, this characterization in Girard seems to misleadingly imply that from A one can derive $A^{\perp}$, which is false.

Another interpretation is in Bellot et al. [1999], who interpret a negative action $A^{\perp}$ as '*undo* action A' [Bellot et al., 1999]. However, the term 'undo' seems to imply that the action A has been performed, and then one reverses it. Moreover, it only fits the interpretation of formulas as formalizing actions, but not our more general reading as utterances in a relation with a reaction.

Given that we have extended the range of expressions susceptible to being formalised in linear logic, a reading of $A^{\perp}$ needs to be one which comprehends the retraction of some action, but also the lack of some resource, the incapability of certain transformations, or even the negation of some situation as in *LK* and *LR*. But this interpretation is not straightforward and needs to avoid one important challenge originated by the following valid inference in *LL*:

$$A \multimap B \vdash_L B^{\perp} \multimap A^{\perp}$$

This fact, which might seem obvious in *LK* and *LR*, is hard to explain from the interpretations of negation in *LL* found in the literature: consider A as 'I have one Euro' and B 'I can get one pencil'. Now, it is not clear how to interpret the *use* of *not* being able to get a pencil in order to *not* having one euro. Or even worse, there might be cases in which the enrichment of $\multimap$ seems to vanish. Let A be 'this piece of wood makes one bed' and B be 'this piece of wood makes four chairs'. Assuming that $A \multimap B$ is the case, it should also be the case that $B^{\perp} \multimap A^{\perp}$. However, the enrichment according to which $B^{\perp}$ is affected by a reaction given $A^{\perp}$, and might be no longer the case, seems to vanish, and the inference $B^{\perp} \multimap A^{\perp} \vdash B^{\perp} \multimap (A^{\perp} \otimes B^{\perp})$ seems valid (it is not clear how the incapability of a piece of wood of making four chairs might no longer be the case given its use in the incapability of that piece of wood making one bed). In other words, the enrichment of $\multimap$ vanishes and it is transformed into $\supset$ or $\rightarrow$.

In order to avoid these unwelcome consequences, our interpretation of $-^{\perp}$ follows another part of Bellot's work, in which the author refers to $A^{\perp}$ as 'erase the situation in which A holds' [Bellot et al., 1999]. In brief, we will interpret the negation as follows:

$A^{\perp}$: Avoid A

It is clear that whatever avoids the possibility of buying one pencil has to avoid also the possibility of having one euro given their connection, or that whatever avoids the possibility of a piece of wood of making four chairs also avoids the possibility of it making one bed. Moreover, this interpretation can explain why $B^{\perp} \multimap A^{\perp} \vdash B^{\perp} \multimap (A^{\perp} \otimes B^{\perp})$ is invalid: once B is avoided, it might be the case that one cannot avoid it *again* given the avoidance of A.

Disjunction

Extensional vs Intensional The two disjunctions in LL are introduced for different reasons. Again, contrary to LR, it is neither the case that $A \oplus B \vdash A \sqcup B$ nor that $A \sqcup B \vdash A \oplus B$.

The difference between the extensional $\sqcup$ and the intensional $\oplus$ is related to the presence or lack of connection between disjuncts. While the extensional disjunction $\sqcup$ can be asserted for similar reasons as asserting $\sqcup$ in LR, the intensional disjunction expresses a resource connection between disjuncts, given that $A \oplus B \vdash \neg A \multimap B$. Hence, in LL we can distinguish the following two kinds of reasons to assert 'A or B' given Γ,

- LE-reasons: using Γ one gets A / using Γ one gets B,
- LI-reasons: there is a connection between A and B such that using Γ and $A^{\perp}$ (or avoiding A) we get B as a result / using Γ and $B^{\perp}$ (or avoiding B) we get A as a result

These reasons are captured by the disjunctions in LL:

- LE-disjunction: $\Gamma \vdash_L A \Rightarrow \Gamma \vdash A \sqcup B$ / $\Gamma \vdash B \Rightarrow \Gamma \vdash A \sqcup B$
- LI-disjunction $\Gamma \vdash A^{\perp} \multimap B \Rightarrow \Gamma \vdash A \oplus B$ / $\Gamma \vdash B^{\perp} \multimap A \Rightarrow \Gamma \vdash A \oplus B$

And for the distinction in natural language, consider 11 asserted for LE-reasons and 12 asserted for LI-reasons:

11. Either I kept 75 FF. in your pocket, or I had lunch at the restaurant yesterday (asserted because the speaker kept 75 FF. in her pocket),

12. Either you keep 75 FF. in your pocket, or you get a menu at the restaurant (asserted on grounds of a connection between both actions).

In Danos & Di Cosmo's words,

> The $\oplus$ connective tells us that if we give in 75 FF. (that sadly disappear immediately afterward: the fact that we give in the francs is expressed by the little $\perp$) for one 'entree' and one 'plat' and either a 'dessert' or some 'fruit'. So a $\oplus$ connective somewhere in a formula tells us that we are faced with a trading situation: $A^{\perp} \oplus B$ means we can get A or B in the very particular sense that if we got an A somewhere (we are not too poor) either we keep this A or we get a B by exchanging it with our A. We cannot get both, because we live in a capitalistic world. [Danos and Di Cosmo, 1992][12]

Relevant Intensional vs Linear Intensional disjunction

Again, there is an interesting distinction between $\oplus$ and $+$. Given the connection between the conditional and the disjunction, the distinction between the relevant and linear intensional disjunction is similar to the first case. We can distinguish two reasons to assert an intensional disjunction:

- RI-reasons: There is a connection between A and B such that the negation of one entails the other,
- LI-reasons: There is a connection between A and B such that using $A^{\perp}$ we get B as a result.

The two reasons are distinguished in LR and LL:

- RI-disjunction: $\Gamma \vdash \neg A \rightarrow B \Rightarrow \Gamma \vdash B + A$
- LI-disjunction: $\Gamma \vdash A^{\perp} \multimap B \Rightarrow \Gamma \vdash A \oplus B$

And in natural language, 13 asserted for RI-reasons and 14 asserted for LI-reasons:

13. Either you don't have 1,000,000 Euro or you can appear in Forbes' list.
14. Either you don't pay 75 FF or you get a menu in the restaurant (75FF is the price of the menu).

[12]Notation modified.

From 13 one can deduce that if one appears in Forbes' list, one still has 1,000,000 Euro, while this is impossible to derive in *LL*. In effect, $\neg A + B, A \vdash B \times A$ is a valid sequent in *LR* but $\neg A \oplus B, A \vdash B \otimes A$ is invalid in *LL*. Note that if $\neg A \oplus B, A \vdash B \otimes A$ is added to *LL*, contraction can be derived.[13]

Moreover, the sequent can be derived in *LR* using contraction:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg A \vdash \neg A \qquad B \vdash B}{\neg A + B \vdash \neg A, B}\,+\mathrm{L}}{\neg A + B \vdash B, \neg A}\,\mathrm{ER}}{A, \neg A + B \vdash B}\,\neg\mathrm{L}}{\neg A + B, A \vdash B}\,\mathrm{EL} \qquad A \vdash A}{\neg A + B, A, A \vdash B \times A}\,\times\mathrm{R}}{\neg A + B, A \vdash B \times A}\,\mathrm{CL}$$

Conjunction

Conjunction is a key connective in linear logic. It has received much attention due to the expressive power of both extensional and intensional conjunction. Let us see their differences in detail.

Extensional vs Intensional conjunction

Girard [1995] presents a clear and intuitive difference between $\sqcap$ and $\otimes$,

> In linear logic, two conjunctions $\otimes$ (...) and $\sqcap$ (...) coexist. They correspond to two radically different uses of the word 'and'. Both conjunctions express the availability of two actions; but in the case of $\otimes$, both will be done, whereas in the case of $\sqcap$, only one of them will be performed (but we shall decide which one) [Girard, 1995, p.2].[14]

Recall the Menu example again. There were two conjunctions, $\otimes$ and $\sqcap$: one that connected each 'part' of the menu (those parts of the menu that one gets, together), $\otimes$, and another one expressing the availability of different options, but which are not available together (one has to choose one), $\sqcap$. Two senses of 'and' capture these differences: 'with' ($\otimes$), and 'any' ($\sqcap$).

[13]Given the equivalence between $\neg A \oplus B$ and $A \multimap B$, it can be easily proven given the derivation in page 112.

[14]Notation modified.

> Now, given an action of type $A \multimap B$ and an action of type $A \multimap C$, there will be no way of forming an action of type $A \multimap (B \otimes C)$, since for 1 you will never get what costs 2 (there will be an action of type $A \otimes A \multimap B \otimes C$, namely getting two packs for 2). However, there will be an action of type $A \multimap B \sqcap C$, namely the superimposition of both actions. [Girard, 1995, p. 2]

Let us distinguish with more detail the extensional from the intensional conjunction in LL. Again, it is interesting to note that it is neither the case that $A \otimes B \vdash A \sqcap B$ nor that $A \sqcap B \vdash A \otimes B$. A first difference between the two is parallel to the differences between extensional and intensional conjunction in LR, that is, the different reasons to assert 'if (A and B) then C':

- LE-reasons: A is used to get C / B is used to get C,
- LI-reasons: A and B together are used to get C.

Which is captured in LL:

- LE-conjunction: $\Gamma \vdash A \multimap C \Rightarrow \Gamma \vdash (A \sqcap B) \multimap C$ / $\Gamma \vdash B \multimap C \Rightarrow \Gamma \vdash A \sqcap B \multimap C$
- LI-conjunction: $\Gamma \vdash A \multimap (B \multimap C) \Rightarrow \Gamma \vdash (A \otimes B) \multimap C$

There are similar examples to the distinction between the extensional and intensional conditional in LR, except that the connection between antecedent and consequent in LL is linear. While the conjunction in 15 should be formalised with $\sqcap$, the conjunction in 16 should be formalised with $\otimes$:

15. If I have a coffee and a croissant, then I have my dose of caffeine,[15]
16. If I pay two Euro and order a coffee, I get a coffee.

Another difference is that $\sqcap$ contracts while $\otimes$ does not. Hence, we can distinguish two reasons for saying 'If A and A, then B:

- LE-reasons: one A is used for getting B
- LI-reasons: twice A are used for getting B

This is distinguished in LL:

[15] Note that one can understand this example linearly given that there is a reaction on the coffee whenever one gets ones' dose of caffeine - that is, that the coffee is no longer available.

- LE-conjunction: $\Gamma \vdash (A \sqcap A) \multimap B \Leftrightarrow \Gamma \vdash A \multimap B$
- LI-conjunction: $\Gamma \vdash (A \otimes A) \multimap B \not\Leftrightarrow A \multimap B$

And the examples in natural language are simple, being the conjunction in 17 extensional and in 18 intensional:

17. If one walks one kilometer and walks one kilometer one can reach the town which is one kilometer away from here,
18. If one walks one kilometer and walks one kilometer one can reach the town which is two kilometers away from here.

Another difference, which motivates the intuitive reading of 'and' in the literature (and in the menu example seen above) is a connection with the linear conditional, which determines the following reasons to assert 'if C then A and B':

- LE-reasons: one can get *any* among A and B,
- LI-reasons: one can get *both* A and B.

Fomalised in LL as follows:

- LE-conjunction: $\Gamma \vdash A$ and $\Gamma \vdash B \Rightarrow \Gamma \vdash (A \sqcap B)$
- LI-conjunction: $\Gamma \vdash A$ and $\Gamma \vdash B \not\Rightarrow \Gamma \vdash (A \otimes B)$

Notice that twice Γ should be required to derive $A \otimes B$, which means that both A and B are available at the same time, given this required repetition. In natural language, 19 is introduced for LE-reasons while 20 is introduced for LI-reasons:

19. If you have 5 Euro you get a packet of Camels and a packet of Marlboros (alternative reading of 'and'),
20. If you have 5 and 5 Euro you get a packet of Camels and a packet of Marlboros (cumulative reading of 'and').

Note that the validity of $C \multimap A, C \multimap B \vdash C \multimap (A \otimes B)$ would validate contraction. This is proven substituting A by $C \multimap A$, and B by C. We suppose that $C, C \vdash A$ and derive $C \vdash A$:

$$
\begin{prooftree}
\AxiomC{$C \multimap (C \multimap A), C \multimap C \vdash C \multimap ((C \multimap A) \otimes C)$}
\AxiomC{$C, C \vdash A$}
\RightLabel{$\multimap$R}
\UnaryInfC{$C \vdash C \multimap A$}
\RightLabel{$\multimap$R}
\UnaryInfC{$\vdash C \multimap (C \multimap A)$}
\RightLabel{Cut}
\BinaryInfC{$C \multimap C \vdash C \multimap ((C \multimap A) \otimes C)$}
\AxiomC{$C \vdash C$}
\AxiomC{$(C \multimap A) \otimes C \vdash (C \multimap A) \otimes C$}
\RightLabel{$\multimap$L}
\BinaryInfC{$C \multimap ((C \multimap A) \otimes C), C \vdash (C \multimap A) \otimes C$}
\RightLabel{Cut}
\BinaryInfC{$C \multimap C, C \vdash (C \multimap A) \otimes C^*$}
\end{prooftree}
$$

...

$$
\begin{prooftree}
\AxiomC{$C \multimap C, C \vdash (C \multimap A) \otimes C^*$}
\AxiomC{$C \vdash C$}
\AxiomC{$A \vdash A$}
\RightLabel{$\multimap$L}
\BinaryInfC{$C \multimap A, C \vdash A$}
\RightLabel{$\otimes$L}
\UnaryInfC{$(C \multimap A) \otimes C \vdash A$}
\RightLabel{Cut}
\BinaryInfC{$C \multimap C, C \vdash A$}
\AxiomC{$C \vdash C$}
\RightLabel{$\rightarrow$R}
\UnaryInfC{$\vdash C \multimap C$}
\RightLabel{Cut}
\BinaryInfC{$C \vdash A$}
\end{prooftree}
$$

Is $\sqcap$ a kind of disjunction? One concern about the extensional conjunction in *LL* is that it can be seen as a kind of disjunction rather than a conjunction. Consider the instances of $\sqcap$ in the menu, as in the dessert part, $(M \sqcap G \sqcap T)$. Given that one has to choose among M, G and T, in classical logic one might opt to formalize this with the classical $\vee$. It would not be strange for a waiter to announce the desserts by saying 'for dessert, we have Mistere or Glace or Tarte aux pommes'; but it would not be strange either for the waiter to connect the three options with 'and'.

However, recall that $\vee$ is compatible with just one of the possibilities being possible, while in a restaurant, when such a choice is offered, all the possibilities are available (although choosing one of them excludes the other options). So in natural language $\sqcap$ expresses a kind of conjunction with an expressive power that is impossible to grasp in classical logic.

Moreover, from its introduction rule we can better grasp its conjunctive character:

$$
\frac{D \vdash M \qquad D \vdash G}{D \vdash M \sqcap G}\ \sqcap\text{R}
$$

The particularity of $\sqcap$ is that the premise from which each conjunct is wasted given the linear reaction, which cannot be captured in *LK* or *LR*. We can consider $\sqcap$ as combining alternatives,

> It follows from the resource consciousness of linear logic that there are two possible meanings for conjunction according to whether one use of $A \sqcap B$ means one use of each conjunct or one use of only one conjunct with the user allowed to choose the conjunct - that's why it is a conjunction rather than a disjunction. [Blass, 1997, p. 3]

Relevant Intensional vs linear intensional conjunction

Again, there are interesting differences between the intensional connectives in LL and in LR. One first observation about their connection is that the second and third previously presented differences between $\sqcap$ and $\otimes$ coincide with a distinction between $\otimes$ and $\times$. First, $\times$ contracts as $\sqcap$ does:

- RI-conjunction: $\Gamma \vdash A \times A \rightarrow B \Rightarrow \Gamma \vdash A \rightarrow B$
- LI-conjunction: $\Gamma \vdash A \sqcap A \multimap B \Rightarrow \Gamma \vdash A \multimap B$

Second, the connection between conjunction and conditional that motivated the distinction in the menu example is also a distinction between $\times$ and $\otimes$:

- $\Gamma \vdash A$ and $\Gamma \vdash B \Rightarrow \Gamma \vdash A \times B$
- $\Gamma \vdash A$ and $\Gamma \vdash B \not\Rightarrow \Gamma \vdash A \otimes B$

We have already seen how introducing the sequent leads to Contraction (p. 119), and now we can see how one can derive the sequent in LR using contraction:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{C \vdash C \qquad A \vdash A}{C \rightarrow A, C \vdash A}\,{\rightarrow}\mathrm{L} \qquad \dfrac{C \vdash C \qquad B \vdash B}{C \rightarrow B, C \vdash B}\,{\rightarrow}\mathrm{L}}{C \rightarrow A, C \rightarrow B, C, C \vdash A \times B}\,{\times}\mathrm{R}}{C \rightarrow A, C \rightarrow B, C \vdash A \times B}\,\mathrm{CL}}{C, C \rightarrow A, C \rightarrow B \vdash A \times B}\,\mathrm{EL}}{C \rightarrow A, C \rightarrow B \vdash C \rightarrow (A \times B)}\,{\rightarrow}\mathrm{R}$$

Given that $\vdash_L$ entails a reaction, the fact that C is sufficient both for A and for B does not imply that this C guarantees A and B together. But this is valid in LR, and this is given the effect that contraction has on the conjunction $\times$.

However it is straightforward that $\sqcap$ and $\times$ do not coincide in general. And we can see this with the following derivation which is valid only for $\times$ but not for $\sqcap$, and neither for $\otimes$. We have seen above that $\sqcap$ and $\otimes$ are distinguished whenever they are embedded in the antecedent of a conditional. This embedding also distinguishes $\times$ from $\otimes$, because in LL we introduce the term 'use' which is not present in LR. Hence, we can distinguish the following reasons for saying 'if A and B then C,

- RI-reasons: A together with B entail C,
- LI-reasons: A together with B are used for C.

And see that the reaction on A and B prevents the validity in LL of a valid sequent in LR:

- RI-conjunction: $(A \times B) \rightarrow C, A \rightarrow B \vdash A \rightarrow C$
- LI- conjunction: $(A \otimes B) \multimap C, A \multimap B \nvdash A \multimap C$

And in natural language, 21 is introduced for RI reasons while 22 for LI reasons:

21. If I have a coffee and a croissant, then I have a full breakfast, and whenever I have a croissant, I also have a coffee. Hence, if I have a croissant, I have a full breakfast,

22. If I have two Euro and a croissant, I can have a full breakfast (because I can get a coffee with the two Euro). If I have two Euro I can have a croissant. Hence, If I have two Euro I can have a full breakfast.

For a summary of the reasons to assert each connective in Linear Logic see the following table (and for further references see Appendix C).

Natural Language	Reasons to assert	LK	LL
if A then B	not A / B using A one gets B	$\supset$	$\rightsquigarrow_L$ $\multimap$
A or B	A / B avoiding A one gets B, or viceversa	$\vee$	$\sqcup_L$ $\oplus$
if (A and B), then and C	if A then C / if B then C if both A and B, then C	$\wedge$	$\sqcap_L$ $\otimes$
if A then (B and C)	if A then (B and C) (any) if A then (B and C) (both)	$\wedge$	$\sqcap_L$ $\otimes$

5.2 Ordered Logic

5.2.1 The logic

Ordered logic, or a logic without Exchange, has not received as much attention as linear or relevant logic have. Girard, for instance, refers to this logic as the *Far West logic* [1995, p.6], while Paoli just mentions its existence in [2002, p.xi,1 28-30], without developing it. Its formal treatment is extremely complex, and the philosophical interest of a logic without a structural rule as basic as exchange has not generated much interest.

However, in order to complete our research, we need to pay attention to the effect that the lack of exchange has on logical connectives and on logical consequence. And we will turn our attention to those authors that have developed such a logic: The main results are owed to Michele Abrusci, who has investigated first the intuitionistic non-commutative linear logic in [Abrusci, 1990] and [Abrusci and Ruet, 1999], and the non-commutative linear classical logic in [Abrusci, 1991] (what distinguishes what Abrusci calls 'intuitionistic' from 'classical' is the multiple conclusion property for sequents, which is not permitted in the former, but is permitted in the latter). We will focus on this second one.

Notation The notation adopted here has significant differences with the notation used in the tradition of linear logic, especially for the conjunction and disjunction [16] [17].

	Abrusci	Our notation
Post negation	$-^{\perp}$	$-^{\perp}$
Retro negation	${}^{\perp}-$	${}^{\perp}-$
Extensional conditional	*	$\rightsquigarrow_O$
Post conditional	$\multimap$	$-\!\bullet$
Retro conditional	$\circ\!-$	$\bullet\!-$
Extensional disjunction	$\oplus$	$\sqcup_O$
Intensional disjunction	$\parr$	$\oplus$
Extensional conjunction	$\&$	$\sqcap_O$
Intensional conjunction	$\otimes$	$\otimes$

A note of caution Our interest is to study the effect that removing a particular structural rule has on logical connectives, and we illustrate the motivation behind the connectives in substructural logics with examples from natural language. This was quite natural in relevant logic, and a bit more complicated in linear logic. Now, in ordered logic, the examples will be purely indicative. And there is a good reason for this: the kind of enrichment of relevant logic is general, in the sense that, given a suitable context, any utterance can be seen as relevantly enriched (whether or not we are interested in that enrichment for a particular inference). However, linear logic and ordered logic express pragmatic enrichments related to the Manner maxims, which, as has been illustrated above (page 105) depend on

[16] The symbol * indicates that the connective is not present.

[17] We will use subindices to distinguish the extensional connectives in the different substructural logics only when the rest of the symbols in the sequent are not sufficient for disambiguiating them.

the content of the utterances and hence are not general. And here the problem emerges: most of the examples in natural language are mixed, in the sense that they are not enriched uniformly with the enrichments of a given linear logic, and with them alone. This will be discussed further in chapter 7.

5.2.2 Manner Enrichment of Consequence II

In the presentation of LL, we have assumed that the utterances are commutative. But it is not surprising that we many times establish connections between utterances in which the order of those expressions matter. One could be interested, for instance, in capturing the order of events after one pays for a menu in a restaurant. Hence, the following two descriptions would not be equivalent:

23. If one pays 75FF one gets a Fruit, a Plat, and an Entree,

24. If one pays 75FF one gets an Entree, a Plat, and a Fruit.

This is even more clear for those interpretations of linear logic (which are embraced in our more general interpretation) as ranging over actions,

> Application of actions is not commutative. Having an argument and then making up is not the same sort of action as making up and then having an argument.[18] [Restall, 2013]

In a logic without exchange we can express new nuances of the connectives. Let us explain the reasons for rejecting this structural rule.

Exchange The presence of Exchange allows one to ignore the order of the premises or conclusions, which can be translated into priority or order of the premises or conclusions. First, whenever a conclusion follows from a sequence of premises which are enriched observing the submaxim of orderliness, one cannot derive the conclusion from a different order of the premises:

$$\frac{\Gamma, A, B, \Gamma' \vdash \Delta}{\Gamma, B, A, \Gamma' \vdash \Delta}\ \mathrm{EL}$$

[18] In this quote Restall is already formalizing the order with 'and then.' But this can be seen as a reinforcement of the enrichment, as the example works as well with a simple 'and'.

If A and B, in this particular order, have Δ as a result, it might be the case that Δ is only the result of A and B in this particular order. Hence, the enrichment of the sequent explains why the structural rule EL should be rejected.

Second, the presence of exchange in the conclusion, and given the disjunctive reading of the comma, allows one to ignore the priority of the disjuncts, that is, the fact that given the premises we get some result, and only if this first result does not happen, the second will,

$$\frac{\Gamma \vdash \Delta, A, B, \Delta'}{\Gamma \vdash \Delta, B, A, \Delta'}\ \mathrm{ER}$$

The examples below about the disjunction in *LO* will clarify this priority, and show how the structural rules ER should be rejected.

Cut The only structural rule in *LO* is cut, presented as follows in [Abrusci, 1991]:

$$\frac{\Gamma \vdash \Delta', A, \Delta'' \qquad \Gamma', A, \Gamma'' \vdash \Delta}{\Gamma', \Gamma, \Gamma'' \vdash \Delta', \Delta, \Delta''}\ \mathrm{Cut}$$

if $\Delta' = \Gamma'' = \emptyset$ or $\Delta'' = \Gamma' = \emptyset$ or $\Gamma' = \Gamma'' = \emptyset$, or $\Delta' = \Delta'' = \emptyset$[19]

Note first that the rules for the connectives and Cut in *LO* add a complication with regard to the previous systems. Given that exchange is not present, we cannot isolate those formulas with which we operate in a derivation in order to introduce them at the right or left side of another sequent. Hence, we need to duplicate the side sequents Γ and Δ in order to reflect the possible positions a formula may have, and how to operate in each case.

Second, some rules in *LO* are presented with the condition of the emptiness of some of these side formulas, as in this case. This guarantees that there will be no conflict cases in which the order of the side formulas is not clear after applying the rule (see footnote 19 for further references).

[19] This entails that the rule can be applied in the following four cases:

$$\frac{\Gamma \vdash A, \Delta'' \qquad \Gamma', A \vdash \Delta}{\Gamma', \Gamma \vdash \Delta, \Delta''} \quad \frac{\Gamma \vdash \Delta', A \qquad A, \Gamma'' \vdash \Delta}{\Gamma, \Gamma'' \vdash \Delta', \Delta} \quad \frac{\Gamma \vdash \Delta', A, \Delta'' \qquad A \vdash \Delta}{\Gamma \vdash \Delta', \Delta, \Delta''} \quad \frac{\Gamma \vdash A \qquad \Gamma', A, \Gamma'' \vdash \Delta}{\Gamma', \Gamma, \Gamma'' \vdash \Delta}$$

These restrictions to the cut rule, and similar ones to $\oplus$L and $\otimes$R, are the result of the lack of exchange in *LO*. The reason for this is that these restrictions, together with some necessary movements of the side formulas using the ordered negations $-^{\perp}$, ${}^{\perp}-$ presented below, make possible to isolate the formulas which are combined by the rule (either eliminated by cut, conjuncted or disjuncted), and avoid conflict cases in which the order of the side formulas is undetermined (see [Abrusci, 1991, p.1406,1447,1449-50]).

5.2.3 Manner Enrichment of Connectives II

The rules are entirely from the work of [Abrusci, 1991], except the conditionals, which are developed by us guided by Abrusci's definition of the conditionals with the aid of the rules of disjunction and negation. The complete calculus for *LO* is shown in Appendix A.

Conditional

Abrusci distinguishes two kinds of conditionals ([1991],[1999]), the linear post implication ($\multimap$) and the linear retro implication ($\multimapinv$). They are distinguished by the position of the antecedent and the auxiliary formulas (in a logic with exchange, these two can be proven to be equivalent - hence, the classical $\supset$ is split into three rather than into two in *LO*). Restall [2013] also distinguishes two conditionals ($\rightarrow$ and $\leftarrow$), in such a way that Restall's $\rightarrow$ is closer to Abrusci's $\multimapinv$ and vice versa (with respect to the order of the premises).

We do not find the rules for $\multimapdot$ and $\multimapdotinv$ in Abrusci's work, but only rules for $\oplus$ and their equivalence with the conditionals. We deduce that the following are the rules for such connectives:

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma', B, \Gamma'' \vdash \Delta'}{\Gamma', B \multimapdotinv A, \Gamma, \Gamma'' \vdash \Delta', \Delta}\ {\multimapdotinv}\mathrm{L} \qquad\qquad \frac{\Gamma, A \vdash \Delta, B}{\Gamma \vdash \Delta, B \multimapdotinv A}\ {\multimapdotinv}\mathrm{R}$$

$$\frac{\Gamma \vdash \Delta, A \qquad \Gamma', B, \Gamma'' \vdash \Delta'}{\Gamma', \Gamma, A \multimapdot B, \Gamma'' \vdash \Delta, \Delta'}\ {\multimapdot}\mathrm{L} \qquad\qquad \frac{A, \Gamma \vdash B, \Delta}{\Gamma \vdash A \multimapdot B, \Delta}\ {\multimapdot}\mathrm{R}$$

What these two conditionals distinguish is different perspectives on the connection between antecedent and consequent, and the auxiliary conditions that are required to reach the conclusion. On the one hand, consider $\multimapinv$R. Whenever B is the result of a sequence of utterances Γ, $B \multimapinv A$ expresses the connection between the last of these events (A) and the conclusion (given that the middle events happen too). On the other hand, consider $\multimap$R. $A \multimap B$ expresses the connection between the first event (A) which is sufficient for the conclusion (B), assuming that all the previous events have also happened. Given that the two connectives are indistinguishable in the absence of side sequents (that is, $\vdash A \multimap B \Leftrightarrow \vdash B \multimapinv A$), what distinguishes these two conditionals is the position of the antecedent with respect to a series of other facts to reach the conclusion. Both connections are conditional-like in natural language, and both require certain order, captured by the rejection of exchange. We will see their intuition in natural language in the comparison with the linear intensional conditional $\multimap$.

Some facts about conditionals and Modus Ponens Formulas in *LO* (as in *LL*) are used to get the conclusion. This is the case for atomic formulas, but also for conditionals (interpreted as the transformation of some A into some B). For this reason, the validity of Modus Ponens for each conditional depends on the position of the antecedent in each case with respect to the conditional. For $\bullet\!-$, the antecedent needs to be used *after* the conditional,

- $A, B \bullet\!- A \nvdash B$
- $B \bullet\!- A, A \vdash B$

While for $-\!\bullet$ the antecedent needs to be used *before* the conditional,

- $A, A -\!\bullet B \vdash B$
- $A -\!\bullet B, A \nvdash B$

We have just seen that $B \bullet\!- A$ represents the fact that A is the last step in a series of events for reaching B, and given that the connection between A and B is a transformation of A to get B, the sequent $A, B \bullet\!- A \nvdash B$ cannot be derived. Similar reasoning justifies the validity or invalidity of the other three inferences.

Extensional vs Intensional

We can distinguish in *LO* the following three reasons to assert 'if A then B" given Γ:

- OE-reasons: given Γ, we get $\neg A$ / Given Γ, we get B,
- OI-reasons: there is a connection between Γ, A, and B such that, given Γ, using A (after Γ), we get B.
- OI-reasons: there is a connection between Γ, A, and B such that given Γ, using A (before Γ), we get B as a result.

This stronger intensional connection between A and B is formalised in *LO* either with $-\!\bullet$ or $\bullet\!-$, and is explained by the lack of weakening, contraction, and exchange in $\vdash_O$,

- OE-conditional: $\Gamma \vdash_O \neg A \Rightarrow \Gamma \vdash_O A \rightsquigarrow B$ / $\Gamma \vdash_O B \Rightarrow \Gamma \vdash_O A \rightsquigarrow B$
- OI-conditional: $\Gamma, A \vdash_O B \Rightarrow \Gamma \vdash B \bullet\!- A$
- OI-conditional: $A, \Gamma \vdash_O B \Rightarrow \Gamma \vdash A -\!\bullet B$.

And in natural language, the examples for $\rightsquigarrow$ are similar to the previous ones (25), introduced for E-reasons, but we can express a causality with an order as in 26 and 27, introduced for OI reasons,

25. If you boil the water of a coffee maker then, if you have already filled it with coffee, the water will be 100°.

26. If you fill a coffee maker with coffee, then, if you boil the water, you get a coffee,

27. If you boil the water of a coffee maker then, if you have already filled it with coffee, you will get a coffee.

Intensional$\multimap\!\!\bullet$/$\bullet\!\!-$ vs Intensional$\multimap$

Again, an interesting dimension of intensional connectives are those inferences that distinguish the intensional conditional in *LO* from the intensional conditional in the other logics, and in particular linear logic *LL*. We can distinguish the two intensional reasons to assert 'if A then B' given Γ,

- LI-reasons: given Γ, using A we get B as a result (independently of the order of Γ and A),
- OI-reasons: given Γ, using A (after Γ), we get B.
- OI-reasons: given Γ, using A (before Γ), we get B.

The reasons are distinguished in the two linear logics:

- LI-conditional: $\Gamma, A \vdash_O B \Rightarrow \Gamma \vdash A \multimap B \;/\; A, \Gamma \vdash_O B \Rightarrow \Gamma \vdash A \multimap B$,
- OI-conditional: $\Gamma, A \vdash_O B \Rightarrow \Gamma \vdash B \bullet\!\!- A$,
- OI-conditional: $A, \Gamma \vdash_O B \Rightarrow \Gamma \vdash A -\!\!\bullet B$.

They can be distinguished in natural language. See the contrast between the previous two examples (repeated here as 29 and 30) introduced for LO-reasons, and the linear one, 28, which is introduced for LI-reasons:

28. If you order and pay for a coffee, you get a coffee.

29. If you fill a coffee maker with coffee, then, if you boil the water, you get a coffee.

30. If you boil the water of a coffee maker then, if you have already filled it with coffee, you will get a coffee.

The two conditionals validate different inferences. For instance, neither $A \mathrel{-\!\!\bullet} (B \mathrel{-\!\!\bullet} C) \vdash B \mathrel{-\!\!\bullet} (A \mathrel{-\!\!\bullet} C)$ nor $(C \mathrel{\bullet\!\!-} B) \mathrel{\bullet\!\!-} A \vdash (C \mathrel{\bullet\!\!-} A) \mathrel{\bullet\!\!-} B$ can be derived in LO. Note that if the first was a valid sequent, Exchange could be derived too (and a similar reasoning proves that the second is invalid too):

$$
\begin{prooftree}
\AxiomC{$A \mathrel{-\!\!\bullet} (B \mathrel{-\!\!\bullet} C) \vdash B \mathrel{-\!\!\bullet} (A \mathrel{-\!\!\bullet} C)$}
\AxiomC{$B, A \vdash C$}
\RightLabel{$-\!\!\bullet$R}
\UnaryInfC{$A \vdash B \mathrel{-\!\!\bullet} C$}
\RightLabel{$-\!\!\bullet$R}
\UnaryInfC{$\vdash A \mathrel{-\!\!\bullet} (B \mathrel{-\!\!\bullet} C)$}
\RightLabel{Cut}
\BinaryInfC{$\vdash B \mathrel{-\!\!\bullet} (A \mathrel{-\!\!\bullet} C)$}
\AxiomC{$B \vdash B$}
\AxiomC{$A \mathrel{-\!\!\bullet} C \vdash A \mathrel{-\!\!\bullet} C$}
\RightLabel{$-\!\!\bullet$L}
\BinaryInfC{$B, B \mathrel{-\!\!\bullet} (A \mathrel{-\!\!\bullet} C) \vdash A \mathrel{-\!\!\bullet} C$}
\RightLabel{Cut}
\BinaryInfC{$B \vdash A \mathrel{-\!\!\bullet} C$}
\AxiomC{$A \vdash A$}
\AxiomC{$C \vdash C$}
\RightLabel{$-\!\!\bullet$L}
\BinaryInfC{$A, A \mathrel{-\!\!\bullet} C \vdash C$}
\RightLabel{Cut}
\BinaryInfC{$A, B \vdash C$}
\end{prooftree}
$$

However, the sequent can be derived in LL, using Exchange:

$$
\begin{prooftree}
\AxiomC{$A \vdash A$}
\AxiomC{$B \vdash B$}
\AxiomC{$C \vdash C$}
\RightLabel{$\multimap$L}
\BinaryInfC{$B \multimap C, B \vdash C$}
\RightLabel{$\multimap$L}
\BinaryInfC{$A \multimap (B \multimap C), A, B \vdash C$}
\RightLabel{EL}
\UnaryInfC{$A, A \multimap (B \multimap C), B \vdash C$}
\RightLabel{$\multimap$R}
\UnaryInfC{$A \multimap (B \multimap C), B \vdash A \multimap C$}
\RightLabel{EL}
\UnaryInfC{$B, A \multimap (B \multimap C) \vdash A \multimap C$}
\RightLabel{$\multimap$R}
\UnaryInfC{$A \multimap (B \multimap C) \vdash B \multimap (A \multimap C)$}
\end{prooftree}
$$

Negation

Again, the role of negation is extremely relevant for disjunction, so let us first examine it before presenting the disjunction. Abrusci distinguishes two negations, which he calls post negation ($-^{\perp}$), and retro negation ($^{\perp}-$), introduced with the following rules,

$$
\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \Delta, {}^{\perp}A}\ {}^{\perp}\!-\!\mathrm{L} \qquad\qquad \frac{\Gamma \vdash A, \Delta}{{}^{\perp}A, \Gamma \vdash \Delta}\ {}^{\perp}\!-\!\mathrm{R}
$$

$$
\frac{A, \Gamma \vdash \Delta}{\Gamma \vdash A^{\perp}, \Delta}\ -^{\perp}\mathrm{L} \qquad\qquad \frac{\Gamma \vdash \Delta, A}{\Gamma, A^{\perp} \vdash \Delta}\ -^{\perp}\mathrm{R}
$$

As in the case for the conditional, the two negations are affected by the position that the premise or the conclusion that is negated occupies in a given series of premises or conclusions.

One important property of the ordered negations is that $A^{\perp\perp}$ cannot be identified with A, and neither does ${}^{\perp\perp}A$. However, A can be identified both with $({}^{\perp}A)^{\perp}$ and ${}^{\perp}(A^{\perp})$, as one can easily check from the Left and Right rules.

Our interpretation of the negations in LO is the same as in LL, interpreting $A^{\perp}$ and ${}^{\perp}A$ both as the avoidance of A. However, the difference in the symbols will be relevant for situating the avoidance in a particular order with respect to the rest of the formulas.

Disjunction

The rules in [Abrusci, 1991] for disjunction are presented as follows.

$$\frac{\Gamma, A, \Gamma' \vdash \Delta \qquad \Gamma, B, \Gamma' \vdash \Delta}{\Gamma, A \sqcup B, \Gamma' \vdash \Delta}\sqcup\mathrm{L} \qquad \frac{\Gamma \vdash \Delta, A, \Delta'}{\Gamma \vdash \Delta, A \sqcup B, \Delta'}\sqcup\mathrm{R} \quad \frac{\Gamma \vdash \Delta, B, \Delta'}{\Gamma \vdash \Delta, A \sqcup B, \Delta'}\sqcup\mathrm{R}$$

$$\frac{\Gamma, A, \Gamma' \vdash \Delta \qquad \Gamma'', B, \Gamma''' \vdash \Delta'}{\Gamma'', \Gamma, A \oplus B, \Gamma''', \Gamma' \vdash \Delta, \Delta'}\oplus\mathrm{L} \qquad \frac{\Gamma \vdash \Delta, A, B, \Delta'}{\Gamma \vdash \Delta, A \oplus B, \Delta'}\oplus\mathrm{R}$$

if $\Gamma' = \Gamma'' = \emptyset$ or $\Gamma' = \Delta = \emptyset$ or $\Gamma'' = \Delta' = \emptyset$[20]

Abrusci notes that the following equivalences hold: $A \multimap B \dashv\vdash A^{\perp} \oplus B$ and $B \multimapinv A \dashv\vdash B \oplus^{\perp} A$. Given that we are interested in the assertion of a disjunction, we will base our explanation on the following two derived equivalences:

- $A \oplus B \dashv\vdash {}^{\perp}A \multimap B$
- $A \oplus B \dashv\vdash A \multimapinv B^{\perp}$

Extensional vs Intensional

The ordered extensional disjunction adds side formulas at both sides of the formula to which the connective is added, but apart from this $\sqcup_O$ has the expected behaviour, similar to the $\sqcup_L$. The intensional disjunction expresses a priority of the first disjunct over the second one. We can distinguish the following kinds of reasons to infer a disjunction 'A or B" from a set of premises Γ:

- OE-reasons: using Γ one gets A / using Γ one gets B,

[20]This entails that the rule can be applied in the following three cases:

$$\frac{\Gamma, A \vdash \Delta \qquad B, \Gamma''' \vdash \Delta'}{\Gamma, A \oplus B, \Gamma''' \vdash \Delta, \Delta'} \qquad \frac{\Gamma, A \vdash \qquad \Gamma'', B, \Gamma''' \vdash \Delta'}{\Gamma'', \Gamma, A \oplus B, \Gamma''' \vdash \Delta'} \qquad \frac{\Gamma, A, \Gamma' \vdash \Delta \qquad B, \Gamma''' \vdash}{\Gamma, A \oplus B, \Gamma''', \Gamma' \vdash \Delta}$$

- OI-reasons: using Γ and avoiding A (before Γ), we get B / using Γ and avoiding B (after Γ), we get A.

The important characteristic of the linear disjunction is that the order of the disjuncts determines the order with respect to Γ. In particular, the first disjunct in the intensional disjunction is always situated before Γ while the second disjunct is always situated after Γ, a fact that is not captured by the extensional one:

- OE-conditional: $\Gamma \vdash_O A \Rightarrow \Gamma \vdash_O A \sqcup B$ / $\Gamma \vdash_O B \Rightarrow \Gamma \vdash_O A \sqcup B$
- OI-conditional: ${}^{\perp}A, \Gamma \vdash_O B \Rightarrow \Gamma \vdash A \oplus B$ / $\Gamma, B^{\perp} \vdash_O A \Rightarrow \Gamma \vdash A \oplus B$

In natural language, 31 is introduced extensionally, while 32 is introduced intensionally,

31. If I boil the water, then I either get water at 100° or I get a coffee,
32. If you push the button of the machine then either the machine is empty (before pushing) or you get a candy (after pushing).

Note that the two disjuncts are in a particular position with respect to Γ: the negation/avoidance of each one of them linearly entails (together with Γ) the other disjunct, but in a particular position with respect to Γ. In 32, there is a connection between the machine not being empty (before pushing the button) and getting a candy (after pushing the button). However, the fact that the machine might be empty *after* getting the last candy does not affect getting this last candy. This connection cannot be expressed in any of the substructural logics presented here except for *LO*.

Intensional$_o$ vs Intensional$_l$

Let us see the difference between $\oplus$ and $\varoplus$. We can distinguish those disjunctions in which there is a priority between the two disjuncts. Hence, *LO* introduces another reason to assert '*A* or *B*' given Γ:

- LI-reasons: there is a connection between Γ, A, and B such that using Γ then, either avoiding A, we get B, or avoiding B, we get A,
- OI-reasons: there is a connection between Γ, A, and B such that using Γ then, avoiding A (before Γ), we get B / avoiding B (after Γ), we get A.

This is a direct consequence of the connection between $\multimap$ and $\oplus$ and between $-\!\!\bullet / \bullet\!\!-$ and $\boldsymbol{\oplus}$:

- LI-disjunction: $\Gamma \vdash A \oplus B \Rightarrow \Gamma \vdash B \oplus A$
- OI-disjunction: $\Gamma \vdash A \boldsymbol{\oplus} B \nRightarrow \Gamma \vdash B \boldsymbol{\oplus} A$

As an illustration of the distinction between them, consider the following examples in natural language, asserted for LI-reasons:

31. If you order the menu at a restaurant then, either you get the menu or you refrain from paying,
32. If you order the menu at a restaurant then, either you refrain from paying or you get a menu.

They can be shown to be equivalent. However, the following are not, asserted for OI-reasons:

33. If you push the button of the machine then either the machine is empty (before pushing) or you get a candy (after pushing),
34. If you push the button then either you get a candy (before pushing) or the machine is empty (after pushing).

In effect, we use exchange to derive 31 from 32:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B^{\perp} \vdash B^{\perp} \qquad C \vdash C}{B^{\perp} \oplus C \vdash B^{\perp}, C}\,\oplus\mathrm{L}}{B^{\perp} \oplus C \vdash C, {}^{\perp}B}\,\mathrm{EL}}{B^{\perp} \oplus C \vdash C \oplus B^{\perp}}\,\oplus\mathrm{R} \qquad A \vdash A}{A, A \multimap (B^{\perp} \oplus C) \vdash (C \oplus B^{\perp})}\,\multimap\mathrm{L}}{A \multimap (B^{\perp} \oplus C) \vdash A \multimap (C \oplus B^{\perp})}\,\rightarrow\mathrm{R}$$

Conjunction

The role of conjunction in Ordered Logic, as in Linear Logic, is more straightforward than the previous connectives. Many of the examples seen in previous chapters (i.e. Chapter 3) were motivated by the enrichment captured by an ordered conjunction. The rules for conjunction presented in [Abrusci, 1991] are the following:

$$\frac{\Gamma \vdash \Delta, A, \Delta' \qquad \Gamma' \vdash \Delta'', B, \Delta'''}{\Gamma, \Gamma' \vdash \Delta'', \Delta, A \otimes B, \Delta''', \Delta'} \otimes R \qquad \frac{\Gamma, A, B, \Gamma' \vdash \Delta}{\Gamma, (A \otimes B), \Gamma' \vdash \Delta} \otimes L$$

if $\Delta' = \Delta'' = \emptyset$ or $\Delta' = \Gamma = \emptyset$ o $\Delta'' = \Gamma' = \emptyset$[21]

$$\frac{\Gamma \vdash \Delta, A, \Delta' \qquad \Gamma \vdash \Delta, B, \Delta'}{\Gamma \vdash \Delta, A \sqcap B, \Delta'} \sqcap R \qquad \frac{\Gamma, A, \Gamma' \vdash \Delta}{\Gamma, A \sqcap B, \Gamma' \vdash \Delta} \qquad \frac{\Gamma, B, \Gamma' \vdash \Delta}{\Gamma, A \sqcap B, \Gamma' \vdash \Delta} \sqcap L$$

Extensional vs Intensional

Before focusing on the differences in the enrichment of the conjunction in *LO* in comparison with *LR* and *LL*, let us see the difference between extensional and intensional conjunctions in *LO*. We can distinguish the following reasons to assert 'if A and B then C" within *LO*:

- OE-reasons: using A one gets C / using B one gets C,
- OI-reasons: using both A and B, in that order, we can get C.

This, again, is captured by *LO* language:

- OE-conjunction: $\Gamma \vdash A \multimap C \Rightarrow \Gamma \vdash (A \sqcap B) \multimap C$ / $\Gamma \vdash B \multimap C \Rightarrow \Gamma \vdash (A \sqcap B) \multimap C$ / $\Gamma \vdash C \multimapinv A \Rightarrow \Gamma \vdash C \multimapinv (A \sqcap B)$ / $\Gamma \vdash C \multimapinv B \Rightarrow \Gamma \vdash C \multimapinv (A \sqcap B)$
- OI-conjunction: $\Gamma \vdash A \multimap (B \multimap C) \Rightarrow \Gamma \vdash (A \otimes B) \multimap C$ / $\Gamma \vdash (C \multimapinv B) \multimapinv A \Rightarrow \Gamma \vdash C \multimapinv (A \otimes B)$

These can be distinguished in natural language:

35. If I have a croissant and a coffee, I have my dose of caffeine,
36. If the old king has died and a Republic has been declared, Tom will be content.

Intensional$_o$ vs Intensional$_l$

Let us see the distinction between the intensional conjunction in linear logic *LL* and in ordered logic *LO*. The intensional conjunction in the antecedent of a conditional expresses that the conjuncts need to be ordered for implying the consequent. Hence, the connection

[21] This implies that the rule can be applied in the following three cases:

$$\frac{\Gamma \vdash \Delta, A \qquad \Gamma' \vdash B, \Delta'''}{\Gamma, \Gamma' \vdash \Delta, A \times B, \Delta'''} \qquad \frac{\vdash \Delta, A \qquad \Gamma' \vdash \Delta'', B, \Delta'''}{\Gamma' \vdash \Delta'', \Delta, A \times B, \Delta'''} \qquad \frac{\Gamma \vdash \Delta, A, \Delta' \qquad \vdash B, \Delta'''}{\Gamma \vdash \Delta, A \times B, \Delta''', \Delta'}$$

for conjunction is stronger than in LR and LL. Whenever a conjunction is embedded as the antecedent of a conditional, both conjuncts are relevant for the consequent, but also in a certain order. Hence, we can add a further reason (OI) to assert 'if (A and B) then C':

- LI-reasons: using A and B, together, we get C,
- OI-reasons: using A and B, together and in this order, we get C.

The two logics distinguish them:

- LI-conjunction: $\Gamma \vdash (A \otimes B) \multimap C \Rightarrow \Gamma \vdash (B \otimes A) \multimap C$
- OI-conjunction: $\Gamma \vdash (A \boldsymbol{\otimes} B) \mathrel{-\!\!\bullet} C \not\Rightarrow \Gamma \vdash (B \boldsymbol{\otimes} A) \mathrel{-\!\!\bullet} C$

In natural language, 37 are asserted for LI-reasons while 38 for OI-reasons:

37. If one has 4 Euro and has 5 Euro, one can buy a cinema ticket in Barcelona.
38. If you close the door and open the door, the door will remain opened.

Note that if the conjunction could be exchanged in the antecedent of a conditional, exchange would be restored:

$$\dfrac{B, A \vdash B \boldsymbol{\otimes} A \qquad \dfrac{\dfrac{(A \boldsymbol{\otimes} B) \mathrel{-\!\!\bullet} C \vdash (B \boldsymbol{\otimes} A) \mathrel{-\!\!\bullet} C \qquad \dfrac{\dfrac{A, B \vdash C}{A \boldsymbol{\otimes} B \vdash C}\,\boldsymbol{\otimes}\mathrm{L}}{\vdash (A \boldsymbol{\otimes} B) \mathrel{-\!\!\bullet} C}\,\mathrel{-\!\!\bullet}\mathrm{R}}{\vdash (B \boldsymbol{\otimes} A) \mathrel{-\!\!\bullet} C}\,\mathrm{Cut} \qquad \dfrac{\dfrac{\dfrac{A \vdash A \qquad B \vdash B}{B, A \vdash B \boldsymbol{\otimes} A}\,\boldsymbol{\otimes}\mathrm{R}}{B \boldsymbol{\otimes} A \vdash B \boldsymbol{\otimes} A}\,\boldsymbol{\otimes}\mathrm{L} \qquad C \vdash C}{(B \boldsymbol{\otimes} A) \mathrel{-\!\!\bullet} C, B \boldsymbol{\otimes} A \vdash C}\,\mathrel{-\!\!\bullet}\mathrm{L}}{B \boldsymbol{\otimes} A \vdash C}\,\mathrm{Cut}}{B, A \vdash C}\,\mathrm{Cut}$$

Another interesting difference between the intensional linear and intensional ordered conjunction is whenever 'A and B' is derived from Γ, for which we can distinguish the following reasons:

- LI-reasons: given Γ one can get *both* A and B,
- OI-reasons: given Γ one can get *both* A and B in this order.

This difference is captured in the two different logics, noticing that the conjunction $A \otimes B$ is commutable while $A \boldsymbol{\otimes} B$ is not:

- LI-conjunction: $\Gamma \vdash A \otimes B \Rightarrow \Gamma \vdash B \otimes A$

- OI-conjunction: $\Gamma \vdash A \otimes B \nRightarrow \Gamma \vdash B \otimes A$

And they are clearly distinguished in natural language:

39. If you pay 10 Euro you get a pack of Camels and a pack of Marlboros (with no specific order),

40. If you pay 10 Euro you get a main dish and a dessert (in this order).

The two conjunctions validate different inferences, as $C \multimapdot (A \otimes B) \nvdash C \multimapdot (B \otimes A)$, while $C \multimap (A \otimes B) \vdash C \multimap (B \otimes A)$. The validity of the first sequent would validate Exchange, which can be shown substituting C by $A \otimes B$:

$$
\dfrac{\dfrac{\dfrac{\dfrac{(A\otimes B)\multimapdot(A\otimes B)\vdash(A\otimes B)\multimapdot(B\otimes A) \qquad \dfrac{A\otimes B\vdash A\otimes B}{\vdash A\otimes B\multimapdot A\otimes B}\multimapdot\mathrm{R}}{\vdash A\otimes B\multimapdot B\otimes A}\text{Cut} \qquad \dfrac{A\otimes B\vdash A\otimes B \qquad B\otimes A\vdash B\otimes A}{A\otimes B, A\otimes B\multimapdot B\otimes A\vdash B\otimes A}\multimapdot\mathrm{L}}{A\otimes B\vdash B\otimes A}\text{Cut} \qquad \dfrac{B,A\vdash C}{B\otimes A\vdash C}\otimes\mathrm{L}}{A\otimes B\vdash C}\text{Cut} \qquad \dfrac{A\vdash A \qquad B\vdash B}{A,B\vdash A\otimes B}\otimes R}{A,B\vdash C}\text{Cut}
$$

The following table summarizes the different reasons to assert the connectives in Ordered Logic:

Natural Language	Reasons to assert	LK	LO
if C, if A then B	if C, not A / if C, B using A (after C) one gets B using A (before C), one gets B	$\supset$	$\rightsquigarrow_O$ $\multimapdot$ $\multimapdotinv$
if C, A or B	using C, A / using C, B avoiding A (before C) one gets B avoiding B (after C) one gets A	$\vee$	$\sqcup_O$ $\oplus$
if (A and B) then C	if A then C / if B then C if A and B (in this order) then C	$\wedge$	$\sqcap_O$ $\otimes$
if C then (A and B)	if C then A and B (any) if C then A and B (both, in this order)	$\wedge$	$\sqcap_O$ $\otimes$

PART III

Virtues and Objections

CHAPTER 6

Virtues of Pragmatic Logical Pluralism

This chapter explains some advantages of Pragmatic Logical Pluralism over monist positions and other versions of pluralism. In particular, we will show how Pragmatic Logical Pluralism has a natural answer to the paradoxes of the material conditional, the Free Choice Permission Paradox (with a diagnostic that departs from [**?**] but suggests a different diagnostic), and the Lottery and Preface paradoxes (reinterpreting the diagnostic that Paoli [2005] and Zardini [2015a] give to some of these paradoxes pragmatically).

6.1 Paradoxes of the material conditional

Although we have seen how to explain the mismatch between natural and formal language, and this mismatch includes all the logical vocabulary, the case of the conditional is especially counterintuitive, in comparison with the conjunction and disjunction. How to formalize 'if...then' has been a prominent discussion in philosophical logic. In particular, it has been a major debate between the classical and relevantist traditions: the so-called paradoxes of the material conditional convinced some philosophers that there is a semantic mismatch between the natural connective and its classical formalization.

Before presenting how the pluralist perspective presented in this thesis solves simple and complex paradoxes, let us recall the main points explained in Chapters 3 and 4 that are used to solve these paradoxes.

6.1.1 Enrichment of 'if...then'

The acceptance that $\supset$ is the correct analysis of the indicative conditional has counterintuitive results, as certain derivations which are valid for $\supset$ do not seem to correspond to our use or even to our rules for the conditional. Consider the following derivations in natural language (already presented in Chapter 4):

1. The theory of evolution is correct. Hence, if God created all the animal species, then the theory of evolution is correct.
2. Santa Claus does not exist. Hence, if Santa Claus exists, he does not have a beard.

These conditionals seem false, as they seem to express a false connection between the antecedent and the consequent. However, the inferences are classically valid, as q can be arbitrary in the following derivations:

1c. $p \vdash q \supset p$

2c. $\neg p \vdash p \supset q$

If we want the connection to be part of the meaning of 'if...then', then $\supset$ cannot capture the meaning of the material conditional. In other words, if we agree that $\supset$ captures the meaning of 'if... then', the connection between the antecedent and the consequent of a conditional is not part of its meaning, contrary to our intuitions.

Embedded conditionals: difficulties for Grice The literature against the classical formalization of the conditional has focused on the counterintuitive results of embedding the material conditional under the scope of other operators. As we have already seen, in these cases, the Gricean analysis is not directly applicable.

Consider, for instance, the negation of a conditional:

3. It is false that if the Sun explodes darkness will cover the Earth in 28 minutes from now: I am not sure whether the Sun exploded, but darkness would cover the Earth in 8 minutes if that was the case.

Given the classical analysis ($\neg(A \supset B) \equiv A \wedge \neg B$) the negated conditional is equivalent to:

4. The Sun exploded, and darkness will not cover the Earth in 28 minutes' time from now.

However, this does not seem to be the intended meaning of the negation; that would make the first sentence in 3 contradictory with the second one. One seems to be able to negate a conditional without knowing the truth-value of its antecedent (or its consequent). Or consider a conditional embedded in a disjunction:

5. Either if the Sun explodes life on Earth will last one thousand years more or if life on Earth lasts one thousand years more, the Sun will explode.

In this case, the disjunction seems to be a disjunction of the connection between certain facts related to the Sun and life on Earth. Although the connections seem false in both cases, the classical formalization is a tautology, as $\vdash_K (A \supset B) \vee (B \supset A)$. Moreover, given that the conditionals are not asserted, it is not clear how the Gricean schema can solve this problem and restore the classical analysis of 'if...then'. In this case, the Gricean analysis is unavailable.

These kind of paradoxes are the reason why some philosophers abandon the material analysis of the indicative conditionals and endorse deviant meanings for 'if...then':

> [R]eflections about compounds support the conclusion that conditionals don't have truth conditions. [Edgington, 2000, 115]

One result of Pragmatic Logical Pluralism is that the reasons for asserting the logical connectives (which are the information that allows us to codify the pragmatic enrichments associated with them) can be codified. In particular, with relevant logic *LR* we distinguished two different reasons to assert the conditional:

a. E-reasons: those reasons that allow introducing the conditional but violating the Gricean relation maxims ($\rightsquigarrow$).

b. I-reasons: those reasons to assert the conditional without violating any Gricean relation maxim ($\rightarrow$).

6.1.2 The paradoxes solved

A general diagnostic

In this section, we will see how *LR* provides the necessary tools to diagnose the paradoxes in *LK*. For each paradox, we will see four things. First, we will introduce the defective inference in natural language, as presented in the literature, which motivates the intuition that we should endorse the premises but reject the conclusion. Second, we will see its classical formalization, which makes the defective inference valid, given that *LK* does not distinguish the reasons to assert the connectives. Third, we will present at least one relevant formalization with the intensional conditional (which expresses the intended meaning of the paradox) and show that it is invalid in *LR*. This explains why the paradoxes are counterintuitive: the intended meaning of the premises does not give reasons to assert the intended meaning of the consequent. Finally, we will see that there is at least one relevantly valid version of the paradox with an extensional conditional: if some conditional in the paradox is translated as 'there are E-reasons for asserting an utterance of the form 'if A then B' the derivation is valid.

Although the paradoxes can be of any syntactic complexity, we can simplify and divide them into the following main groups:

- Unambiguous paradoxes: paradoxes with just one relevantly valid formalization. These, in turn, can be divided into two groups:
  - Paradoxes with a conditional in the premises,
  - Paradoxes with a conditional in the conclusion.[1]
- Ambiguous paradoxes: Paradoxes with more than one relevantly correct formalization.

Let us see them in detail.

Unambiguous paradoxes

Paradoxes with a conditional in the premises In the first group of paradoxes, we find those in which a conditional is embedded in the premises. In these cases, the paradox emerges because we seem to endorse the premises but not the conclusion. This will be explained given that when a conditional is embedded in a derivation, we usually interpret it as its pragmatic enrichment and not its literal meaning. Hence, the classical formalization is usually a bad formalization of the intended meaning of the natural language presentation of the paradox.

Negation Consider the following argument in natural language:

1. It is not the case that if there is a good God the prayers of evil people will be answered. Hence, there is a good God. [Cooper, 1968]

In classical logic the negation of the consequent (and also the antecedent) follows from the negation of a conditional:

1c. $\neg(p \supset q) \vdash p$

[1]This group includes those tautologies from classical logic that are paradoxical, considering them as inferences without premises.

This validity shows that the argument is classically valid, and hence truth-preserving. However, our use of the conditional does not usually derive either the antecedent or the negation of the consequent of a conditional just from its negation. In fact, it seems misleading to do so, given that what we tend to express with 'it is not the case that 'if p then q' is something along the following lines: 'there is no I-reason to assert 'if p then q' (i.e., there is no connection between being a good God and the prayers of evil people being answered). That is, what we tend to negate is that there is a connection, and from this, the negation of the conclusion does not follow:

1r. $\neg(p \to q) \nvdash p$

However, the argument is classically valid, that is, it preserves truth. The reason is that $\supset$ does not distinguish between E and I reasons to assert a conditional, and the premise can express something else, from which the conclusion does follow. That is, if we just endorse the fact that there is no connection between being a good God and the prayers of evil people being answered, and we remain sceptic about the existence of God, then we do not really endorse that 'it is not the case that if there is a good God the prayers of evil people will be answered': there can be an E-reason to assert the conditional, and we do not endorse its negation. In *LR* it is valid to infer the antecedent of a negated conditional if the conditional is asserted with an E-reason.

1r'. $\neg(p \rightsquigarrow q) \vdash_R p$

This shows that there are cases in which one can derive p from the negation of a conditional $p \supset q$: when the negated conditional is asserted for E-reasons, for instance in the context of a cards game in which someone asserts the utterance 'It not the case that if I have 5 aces then pigs can fly', just to assert that one *does* have four aces.

Conditional nested in the antecedent Another case of a paradox with a conditional in the premises and which is not a good formalization of what we tend to express with 'if...then' is that of a conditional nested as an antecedent of another conditional:

2. If the cup breaks if dropped, then it is fragile; and the cup has not dropped. Hence, the cup is fragile. [Gibbard, 1980, p.237]

The classical formalization validates the argument:

2c. $(p \supset q) \supset r, \neg p \vdash r$

However, this derivation seems valid only if the embedded conditional is true due to the falsity of the antecedent and in no case it seems valid given a connection between antecedent and consequent. What we usually try to express when we embed a conditional as an antecedent is 'if there are I-reasons to assert $A \supset B$, then...', but from this and the falsity of the antecedent, the conclusion does not follow. The formal counterpart in LR confirms this:

2c. $(p \rightarrow q) \rightarrow r, \neg p \nvdash r$

This analysis is a generalization of the one given by Gibbard:

> [C]onsider first the antecedent 'if the cup was dropped, it broke'. Such an indicative conditional may have an obvious *basis*,
>
> > [*basis*] The cup was disposed to break on being dropped. [Gibbard, 1980, p.237]

The present solution subsumes Gibbard's view: 'basis' is just the expression of an I-reason to assert the conditional.

Nevertheless, the extensional reading of the conditional makes the derivation valid (but again, this is not what we tend to express with a conditional nested in the antecedent of another conditional),

2r. $(p \rightsquigarrow q) \rightarrow r, \neg p \vdash r$

The conclusion only follows if the reasons to assert the first conditional are E-reasons. This can be the case if one asserts something as the following 'If it is the case that If I have four aces, then pigs can fly, then I should abandon this round. And I don't have four aces. Hence, I should abandon this round'.

Paradoxes due to a conditional in the conclusion Let us turn to the second kind of paradoxes: those generated by an embedded conditional in the conclusion. In this case, the general diagnostic is that we endorse the premises, but they only provide an E-reason for a conditional in the conclusion. That is, the premises only give reasons to use in the conclusion some conditional for E-reasons, which means that the conditional in the conclusion could be substituted by the negation of the antecedent or by the consequent. However, in no case, do they imply a connection between them. This explains why we seem to reject the conclusion of these paradoxes.

Affirmed conditional Among this subgroup of paradoxes we find those in which a conditional is asserted, either as a simple conditional or embedded in a conjunction:

3. There are planets. Hence, if there are no planets anywhere, the solar system has at least eight planets. [Bennett, 2003]

The classical formalization of the argument is valid:

3c. $A \vdash \neg A \supset B$

But our intended meaning of the conclusion is that there is a connection between the fact that there are no planets and the fact that the solar system has at least eight planets. This is why it seems invalid: the connection seems plainly false. In *LR* we confirm that the premise does not give any reason to affirm a connection between these two facts:

3r. $A \nvdash \neg A \to B$

The only reason why we can assert $\neg A \supset B$ in the conclusion is that given the truth of A, that is, the truth of 'there are planets', we have an E-reason to affirm that 'if there are no planets then the solar system has at least eight planets': the known falsity of the antecedent assures that it is a true conditional, independently of what the consequent says:

3r′. $A \vdash \neg A \rightsquigarrow B$

Disjunction of conditionals (I) When a conditional is embedded under the scope of a disjunction, it can easily generate a paradox. Consider the following disjunction in natural language:

4. Either if it is raining the sun is shining or if the sun is shining it is snowing.

In *LK* the following derivation is valid:

4c. $\vdash (A \supset B) \vee (B \supset C)$

This expression is usually understood as, first, that either one or the other disjunct is the case. That is, there are I-reasons to assert the disjunction. Moreover, we also understand that the embedded conditionals are intensional, that is, that either there is an I-reason to assert 'if A then B' or there is an I-reason to assert 'if B then C'. However, this is invalid in *LR*, which shows that the disjunction of I-reasons is not a relevant tautology.

4r′. $\nvdash (A \to B) + (B \to C)$

Why, then, is this argument a tautology in classical logic? Given that $B \vee \neg B$ is a logical truth, either there are E-reasons to assert $A \supset B$ or there are E-reasons to assert $B \supset C$. From the relevant perspective, this fact is confirmed:

4r. $\vdash (A \rightsquigarrow B) + (B \rightsquigarrow C)$

Disjunction of conditionals (II) Consider the following inference:

5. If you close switch x and switch y the light will go on. Hence, it is the case either that if you close switch x the light will go on, or that if you close switch y the light will go on. [[Cooper, 1968] as cited in [Priest, 2008]]

The classical formalization validates the argument:

5c. $(A \wedge B) \supset C \vdash (A \supset C) \vee (B \supset C)$

However, the premise does not give reasons for the disjunction of intensional conditionals. That is, the premise does not imply that either there is a connection between A and C or a connection between B and C:

5r. $(A \times B) \to C \nvdash (A \to C) + (B \to C)$

Let us see a relevantly valid formalization of the argument:

5r′. $(A \times B) \to C \vdash (A \rightsquigarrow C) + (B \to C)$

Note that either A is true or false, that is, $A \vee \neg A$. If it is false, that gives an E-reasons to assert $A \supset C$, that is, $A \rightsquigarrow C$. However, if A is true, then there is a connection between B and C, that is, $B \to C$ (given that $(A \times B) \to C$). The connection is dependent on the truth of A, as the formalization suggests.

Ambiguous paradoxes

All the paradoxes seen so far have just one relevantly valid formalization. However, there are other paradoxes which have at least two relevantly valid formalizations. This means that there is not just one diagnostic, but more. This is the case because the reasons to assert some other connective in the derivation affect the reasons to assert the rest of the connectives. Let us see these dependencies in detail.

Interderivability In order to solve combined paradoxes, one needs to bear in mind the relation between the different connectives. Consider the following equivalences between the intensional/extensional disjunction and conjunction and the intensional/extensional conditional in *LR*:

	E-connectives	conditional equiv.	I-connectives	conditional equiv.
Disjunction/conditional	$\neg A \sqcup B$	$A \rightsquigarrow B$	$\neg A + B$	$A \to B$
Conjunction/conditional	$\neg(A \sqcap B)$	$A \rightsquigarrow \neg B$	$\neg(A \times B)$	$A \to \neg B$
Conjunction/conditional	$(A \sqcap B) \to C$	$A \rightsquigarrow (B \to C)$ $A \to (B \rightsquigarrow C)$	$(A \times B) \to C$	$A \to (B \to C)$

Given the equivalence relations between the different connectives in *LR*, it is clear that E or I reasons for asserting an utterance with a particular connective imply E or I reasons for asserting their equivalents. For instance:

6. It is false that $(2 + 2 = 5$ and dinosaurs existed). Hence, if $2 + 2 = 5$ then dinosaurs existed (both the negated conjunction and the conditional are asserted for E-reasons).

7. Either Paris is in France or Gödel was a physicist. Hence, if Paris is not in France then Gödel was a physicist (both the disjunction and the conditional are asserted for E-reasons).

This fact will be crucial for the solution to McGee's paradox.

Conditional in the consequent: McGee McGee's counterexamples to Modus Ponens are based on a conditional embedded as the consequent of another conditional. McGee presents three counterexamples, and we will develop the claim that only the first one ('The Election') allows for a formalization of the diagnostic in *LR*, although the diagnostic made is common to the three examples. Let us see them in detail:

The Election

> Opinion polls taken just before the 1980 election showed the Republican Ronald Reagan decisively ahead of the Democrat Jimmy Carter, with the other Republican in the race, John Anderson, a distant third. Those apprised of the polls believed, with good reason:

a If a Republican wins the election, then if it's not Reagan who wins it will be Anderson.

b A Republican will win the election.

Yet they did not have reason to believe:

c If it's not Reagan who wins, it will be Anderson.

Lungfish

I see what looks like a large fish writhing in a fisherman's net a ways off. I believe

a If that creature is a fish, then if it has lungs, it's a lungfish

That, after all, is what one means by 'lungfish'. Yet, even though I believe the antecedent of this conditional [(b) that creature is a fish], I do not conclude

c If that creature has lung, it's a lungfish.

Lungfishes are rare, oddly shaped, and, to my knowledge, appear only in freshwater. It is more likely that, even though it does not look like one, the animal in the net is a porpoise.

Uncle Otto

Having learned that gold and silver were both once mined in his region, Uncle Otto has dug a mine in his backyard. Unfortunately, it is virtually certain that he will find neither gold nor silver, and it is entirely certain that he will find nothing else of value. There are simple reasons to believe

a If Uncle Otto doesn't find gold, then if he strikes it rich, it will be by finding silver.

b Uncle Otto won't find gold.

Since, however, his chances of finding gold, though slim, are no slimmer than his chances of finding silver, there is no reason to suppose that

c If Uncle Otto strikes it rich, it will be by finding silver.

General structure The general classical formalization of McGee's counterexamples is:

- $A \supset (B \supset C), A \vdash B \supset C$

The analysis in *LR* shows that from an I-reason to assert a conditional in the premise follows an I-reason to assert the conditional in the conclusion:

- $A \rightarrow (B \rightarrow C), A \vdash B \rightarrow C$

Hence, the paradoxes seem to escape the above analysis, and it is necessary to give another explanation for why the conclusions of the arguments seem unacceptable while the premises do not.

We need to realize that the three examples have a common characteristic: in the three illustrations, there is a tension between the truth of the antecedent of the conclusion (B) and the minor premise (A). To be more specific, in the three vignettes, one seems to assert A given that one ignores whether B is the case or even assuming that B is *not* the case. Let us see this tension for each case: first, in the Election, the reason for saying A in the minor premise is that Reagan is winning, that is, $\neg B$. A is then in tension with B because knowing B would rule out the assertion of A in that particular scenario. Second, in the lungfish example, the reason for saying A is the shape of the fish. However, given that lungfishes are rare and it is much more probable that if the creature has lungs it is something different to a fish, the knowledge of B (the creature has lungs) would prevent one of asserting A (it is a fish) as a minor premise, again, in this particular scenario. Third, knowing that uncle Otto is rich (B) would make one refrain from asserting A, that is, that he has not found gold.

With this tension in mind, we can explain why the premises are highly assertable but the conclusion is not. In each case, C is a very special situation that only happens when A together with B happen, which is also a very special situation because B is in tension with A. That is, 'if B then C' is hardly assertible because A is also required but B rules it out. In other words, the conditional in the conclusion seems to be false given that the antecedent usually rules out one of the premises (A), which is necessary (together with B) for the truth of the consequent of the conclusion.

Consider an alternative argument with a similar structure in which this tension between the minor premise and the antecedent of the conclusion is not present:

a If a car runs for 1 hour, then, if it runs at 80 kph, it will cover a distance of 80 km,

b I have been driving my car for 1 hour,

c Hence, if my velocity was 80 kph then I have covered 80 km.

The consequent of the conditional in the conclusion does not seem to follow from the antecedent alone, but it requires that the minor premise is also true, that is, there is no reason to assert the last conditional unless the second premise is true. However, in this case, the antecedent of the conditional is not in tension with the minor premise, and the argument seems perfectly correct.

It is not surprising that one can transform McGee's vignettes in illustrations without the tension mentioned above: adding to the second illustration a DNA analysis of the creature which confirms (before knowing whether it has lungs or not) that the creature is a fish; an analysis of uncle Otto's terrain which confirms that there is no gold in his terrain; or the knowledge that the elections are manipulated by the Republican party in order to make one of the candidates the winner one. With these three modifications, the arguments seem as regular as the car example.

In sum, the premises of the arguments support a conditional in the conclusion that seems to be true because of the falsity of the antecedent rather than the connection between antecedent and consequent. However, we have seen that the general form of the inference is valid for the intensional conditional in *LR*. This is because the responsible for this are the grounds for saying A, which are not analysed in *LR* given that they are not related to the use of a logical connective of the propositional fragment of *LR*. Nonetheless, the Election paradox is an exception, and it allows for an analysis in *LR* in which this diagnostic is confirmed.

The Election: an ambiguous paradox While in the Uncle Otto and the Lungfish there is just one way of formalizing A (although there might be different grounds for it), in the case of the Election we can disambiguate these different grounds for saying 'a Republican will win the election', and give a similar diagnostic to the previous paradoxes. In other words, in the case of the Election *LR* will codify the different senses for saying the minor premise, and we will be able to justify better the general diagnostic that we have seen for the three variants of McGee's paradoxes.

In more detail, we will argue that the paradox is generated by the connection between the second premise and the conclusion in the following sense: the tension between the minor premise and the antecedent of the conditional in the conclusion is generated because the minor premise violates the Gricean maxim of quantity and this can only entail a conclusion which also violates the Gricean maxim of quantity, while our reading of the conclusion is usually pragmatically enriched.

This diagnostic rests on Paoli's solution to the counterexample in [2005], which is based on the distinction between an intensional and an extensional existential quantifier, which is also based on the distinction between an intensional and an extensional disjunction. While Paoli argues for the distinction in LL, we use LR for the same task[2]. In particular, Paoli's diagnostic is that the second premise is using a different quantifier ($\bigsqcup$) from the first (Σ). We reinterpret this ambiguity pragmatically, maintaining that the argument is classically valid but violates a Gricean maxim.[3]

Following Paoli, we can distinguish two reasons to assert an existential quantifier. The difference between them is that the intensional quantifier is pragmatically enriched with a Gricean maxim of Quantity. Let D be a finite domain of discourse $\{a_1 ... a_n\}$. There are two reasons to assert 'there is a $x \in D$ that Px':

a S knows one specific object $a \in D$ that Pa,

b S knows that there is an $x \in D$ such that Px but ignores which one.

Given that [a] violates the maxim of Quantity, 'there is an x that Px' is usually enriched with [b], in which case it is legitimate to conclude 'if not Pa_1 then, if not Pa_2 ... then Pa_n'. This enrichment is similar to the enrichment that a disjunction may have, according to which we assume that the speaker establishes a connection between disjuncts. Hence, equivalently to the rest of the connectives, we can distinguish two reasons to assert the existential quantifier $\exists$:

Natural Language	Reasons to assert	LK	LR
'There is a x that is P'	Pa	$\exists$	$\bigsqcup$
	connection: if not Pa_1 then, if not Pa_2 ... then Pa_n		Σ

A formalization for the two existential quantifiers can be that presented by Zardini [2011], accepting a restriction to finite domains for Σ:

$$\frac{\Gamma, \phi(a) \vdash \Delta}{\Gamma, \bigsqcup x\phi \vdash \Delta}\ \bigsqcup L \qquad \frac{\Gamma \vdash \phi(t), \Delta}{\Gamma \vdash \bigsqcup x\phi, \Delta}\ \bigsqcup R$$

$$\frac{\Gamma, \phi(t_1) \vdash \Delta \qquad \Gamma', \phi(t_2) \vdash \Delta' \qquad ...}{\Gamma, \Gamma', ..., \Sigma x\phi \vdash \Delta, \Delta', ...}\ \Sigma L \qquad \frac{\Gamma \vdash \phi(t_1), \phi(t_2), ..., \Delta}{\Gamma \vdash \Sigma x\phi, \Delta}\ \Sigma R$$

[2]Paoli's diagnostic based on LL, but we can consider the same interpretation of the quantifiers in LR, with the difference that the intensional implies the extensional, contrary to Paoli's view.

[3]Zardini [2015a] argues against Paoli's diagnostic of the paradox, in which the present solution is sustained, considering an equally problematic counterexample to the Election in which the minor premise is substituted by 'Carston is not going to win the Election'. We cannot formalize and analyze this version of the counterexample, but this is not surprising: in this case the example is similar to the uncle Otto or the Lungfish: the tension between the minor premise and the antecedent of the conditional in the conclusion is present but our formal language does not support a correct way of formalizing it.

Let us see how this applies to the Election case. First, note the different reasons to assert the premises in the argument: the first premise does not violate any Gricean maxim, while the second violates the maxim of Quantity: one should say 'Reagan will win the Election' instead of 'a Republican will win the Election'. Now recall that E-reasons to assert some connective entail E-reasons to assert their equivalences (an E-reason to assert a negated conjunction entails an E-reason to assert a conditional, and the same for disjunction and the other equivalences presented above). The same applies to the existential quantifier: an E-reason to assert an existential quantifier only entails an E-reason to assert a disjunction, which only entails an E-reason to assert the equivalent conditional. We can easily check that the two premises in the example do not entail an I-reason to assert the conditional in the conclusion:

6.1.2r. $\Sigma x Px \rightarrow (\neg Pa \rightarrow Pb), \sqcup x Px \nvdash \neg Pa \rightarrow Pb$

However, as in the above cases, the argument is valid for some alternative relevant formalization. In this case, we can give two relevantly valid arguments:

6.1.2r′. $\Sigma x Px \rightarrow (\neg Pa \rightarrow Pb), \Sigma x Px \vdash \neg Pa \rightarrow Pb$

6.1.2r″. $\sqcup x Px \rightarrow (\neg Pa \rightsquigarrow Pb), \sqcup x Px \vdash \neg Pa \rightsquigarrow Pb$

If the second premise is asserted without violating the Gricean maxim of quantity the conditional in the conclusion can be asserted for I-reasons. Consider the alternative scenario mentioned before in which the speaker is sure that a Republican is going to win the Election, but not on grounds of her knowledge that a particular one is the favorite one, but because the Election is manipulated by the Republican party. In that case, the existential quantifier in the second premise (asserted for I-reasons) gives I-reasons to assert the conditional in the conclusion.

Given that in the scenario presented by McGee the second premise is asserted for E-reasons, no connection between antecedent and consequent can be derived from it. In conclusion, the E-reasons to assert the existential quantifier only gives E-reasons to assert the conditional in the conclusion.

6.2 Free choice permission

Another criticism to *LK*, not from relevant logic but from deontic logic, is that a disjunction under the scope of 'may' is transformed into some kind of conjunction. This generates what is known as the *Free Choice Permission* paradox, and it can be solved if the language is formalised with linear logic. Lokhorst [1997] suggests first that linear logic is suitable for deontic logic, and **?** develops the idea. Following Barker, we will argue that the best formalisation for the paradox is linear logic, but we will apply the idea to a more general view on deontic logic, while solving certain problems that his framework poses.

6.2.1 The paradox

The Free Choice Permission paradox, presented by Kamp [1973] is generated by a conjunctive reading of 'or' in natural language, in the context of classical logic augmented with deontic operators. The paradox arises because from the truth of 3,

3. You may eat an apple or a biscuit,

one seems to be able to derive 4 and 5,

4. You may eat an apple,
5. You may eat a biscuit.

And what is even worse, in some formalisations for deontic logic, from 4, one can derive 3, given the introduction of a disjunction, so we would be able to derive the permissibility of any action (for instance 5) from any other permitted action (for instance 4). Moreover, this conjunctive reading of 'or' cannot be interpreted as a classical conjunction, given that one does not seem to have permission to perform *both* actions, but only one. That is, the 'or' is read as a limited conjunction.

In sum, a solution to the Free Choice Permission paradox needs to explain the conjunctive reading of the disjunction in 3, and also the limited reading of this conjunction. In other words, we need to explain why 'you can do A or B' is understood as 'you can do A and you can do B, but not necessarily both'.

Following **?** we will argue that linear logic, as a resource-sensitive logic, gives an accurate diagnostic of the paradox, and that 'or' and 'and' in natural language can be explained with their counterparts in linear logic, rather than the classical ones, in this specific context.

> [A] complete characterization of permission sentences must not only tell us whether permission exists and what type of permission it is (i.e., permission to eat an apple versus permission to eat a pear), it must also characterize how much permission has been granted. [**?**, p. 3]

We will argue for a solution to the paradox identifying an equivocation in its traditional formulation. The question why permission should be analysed with a resource-sensitive logic can be explained by observing that a given permission can be transformed once a permitted action has been fulfilled (that is, there is a reaction on permission, given the performance of one permitted action).

There is more than one formal framework for deontic logic, the Anderson-Kanger system and Modal Deontic Logic being the two main ones. **?** presents his solution in the first one, and it does not seem to work in the second one. So our job will be to extend Barker's diagnostic to Modal Deontic Logic.

6.2.2 Anderson-Kanger deontic logic

Anderson-Kanger in *LK*

Following Kanger [1970] and Anderson [1967], Lokhorst [1997] formalises deontic logic as a reduction of deontic operators to conditionals (and this is the deontic form chosen by Barker for solving the paradox [**?**, p.11]): We can define obligation and permission as follows - where δ represents something along the following lines: 'all things are as required':

- p is obligatory: $\delta \supset p$
- p is permitted: $p \supset \delta$

The paradox can be formalised as follows (in the deontic extension of classical logic *LK*):

$$(A \vee B) \supset \delta \vdash (A \supset \delta) \wedge (B \supset \delta)$$

This logic has a natural explanation for why the disjunction is transformed into a conjunction. However, it cannot explain limited permission (and this cannot be explained in its relevant formalization either).

Anderson and Kanger in *LL*

The problem of limited permission is solved by Barker and Lockhorst, as they both codify Deontic Logic in Linear Logic:

- $(A \sqcup B) \multimap \delta \vdash (A \multimap \delta) \sqcap (B \multimap \delta)$

Given the expressive power of $\sqcap$, it does not state that both conjuncts are possible together, but just that one of the two is (contrary to *LK*'s $\wedge$). Hence, the key element in Barker's explanation is that $\sqcap$ differs from the classical $\wedge$ in that it does not guarantee that both conjuncts are available together.

One problem with this solution to the paradox is that it is linked to a particular deontic system: the transformation of a disjunction into a conjunction is not explained by the adoption of linear logic, but because permission is formalised as a kind of conditional (recall that the disjunctive reading of 'and' was already explained by the classical formalization of the paradox).

A second observation is that this deontic framework does not support any kind of cancellation of the conjunctive reading of 'or'. Consider 6:

6. You may eat an apple or a pizza. *I don't know which.*

Schulz [2005] has argued for a pragmatic account of the free choice permission paradox, observing that adding 'I don't know which' cancels the conjunctive reading of 'or'. Zimmermann also makes this observation [2000]. However, there is no way of limiting the conjunctive reading of 'or' if the free choice permission is formalised in the Krager-Anderson system.

Moreover, and related to this problem, as we have seen in the introduction, some authors have argued that from 'you may go by boat' one can introduce a disjunction such as 'you may go by boat or by plane' [Zimmermann, 2000]. However, if we understand permission as a conditional this is not possible, as $A \multimap \delta \nvdash (A \sqcup B) \multimap \delta$.

Finally, we aim to diagnose the paradox in a different deontic system which embraces non-deontic free-choice sentences. The free-choice phenomenon seems to appear also in other modal contexts, for instance, consider the sentence 'this piece of wood makes four chairs or makes one bed'. Although there is no permission operator in the sentence, the modal operator which is implicit in it seems to behave in a similar manner as the modal operator of the free-choice permission paradox: the piece of wood can be transformed into any of the two options, but not both.

For all these reasons we suggest an alternative codification of deontic logic in general and of the paradox in particular.

6.2.3 Modal Deontic Logic

Deontic Classical Logic (DLK)

Deontic Logic is the result of adding obligation and permission operators to classical logic (Von Wright [1951]). There are many systems for deontic logic and it is outside the scope of this research to defend one of them. However, what the systems have in common is the adding of the following rule to the propositional fragment of classical logic LK, which will be enough for our purposes,[4]

$$\frac{\Gamma \vdash \Delta}{O\Gamma \vdash O\Delta}\ \mathrm{O}$$

And we can introduce the permission operator P via its relation with the operator O:

$$\frac{\Gamma \vdash \neg O \neg B}{\Gamma \vdash PB}\ \mathrm{PR} \qquad\qquad \frac{\neg O \neg A \vdash \Delta}{PA \vdash \Delta}\ \mathrm{PL}$$

The Free Choice Permission paradox in such a logic emerges because we cannot derive the conjunctive reading of the disjunction under the scope of 'may', as the following sequent is invalid (contrary to in the case of Anderson-Kragen system):

- $P(A \vee B) \nvdash PA \wedge PB$

However, we need an explanation for why our intuition is that 'you can eat an apple or a pear' entails that both options are permitted. One could suggest that in DLK we should add the sequent $P(A \vee B) \vdash PA \wedge PB$. However, if we add this, given that we can derive $PA \vdash P(A \vee \bot)$,

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{A \vdash A}{A \vdash A \vee \bot}\ \vee\mathrm{R}}{\neg(A \vee \bot) \vdash \neg A}\ \neg\mathrm{R}/\neg\mathrm{L}}{O\neg(A \vee \bot) \vdash O\neg A}\ \mathrm{O}}{\neg O \neg A \vdash \neg O \neg (A \vee \bot)}\ \neg\mathrm{R}/\neg\mathrm{L}}{PA \vdash P(A \vee \bot)}\ \mathrm{PR/PL}$$

we can derive $P\bot$ from PA:

[4] The generality of the rules for the obligation operator O, together with its transformation into permission P, which is sufficient for a good diagnostic of the paradox in LL, illustrates why the present solution can be extended to other free choice paradox which depends on other modal operators distinct to permission, such as possibility.

$$\dfrac{\dfrac{PA \vdash P(A \vee \bot) \qquad P(A \vee \bot) \vdash PA \wedge P\bot}{PA \vdash PA \wedge P\bot}\,\text{Cut} \qquad \dfrac{P\bot \vdash P\bot}{PA \wedge P\bot \vdash P\bot}\,\wedge\text{L}}{PA \vdash P\bot}\,\text{Cut}$$

The deontic extension of LK does not seem to encode our intuitions about the disjunction embedded in a deontic clause 'may'. So contrary to the Anderson-Kanger system, we cannot explain the disjunctive reading of the conjunction.

Deontic Linear Logic DLL

The problem is solved if, instead of classical logic, we use linear logic to model deontic logic. This option has four virtues: (i) it explains the conjunctive reading of 'or' (this was already solved by the classical and the linear interpretation of Anderson-Kanger deontic classical logic), (ii) it solves the limited reading of 'or' (this was already solved by Barker in the Linear Anderson-Kanger deontic logic), (iii) it explains how the conjunctive reading of 'or' is compatible with disjunction introduction, and with the 'I don't know which' clause (this is not solved in any of them), (iv) it can be extended to other free-choice phenomena which do not involve permission operators.

In DLL (as in DKL), from a disjunction one cannot derive a conjunction, for none of the following combinations between intensional and extensional disjunctions and conjunctions is valid in LL[5]:

- $P(A \sqcup B) \nvdash_{DLL} PA \otimes PB$
- $P(A \sqcup B) \nvdash_{DLL} PA \sqcap PB$
- $P(A \oplus B) \nvdash_{DLL} PA \otimes PB$
- $P(A \oplus B) \nvdash_{DLL} PA \sqcap PB$

However, we do not need to derive a conjunction from a disjunction in order to solve the paradox. The paradox has its origin in a fallacy of equivocation between a genuine disjunction and a conjunction that is sometimes read as 'or', which, as Barker correctly identified, is the extensional conjunction $\sqcap$.

The derivation from 3 to 4 and 5 is justified by the observation that in LL, contrary to LK, there is a conjunction, $\sqcap$, that in natural language is sometimes read as 'or', but which expresses the alternative reading between two available options:

[5]Note that $P(A \vee B) \vdash PA \wedge PB$ is invalid in DLK (page 154), and given that $DLL \Rightarrow DLK$, its invalidity in DLL is straightforward.

- $P(A \sqcap B) \vdash_{DLL} PA \sqcap PB$

See the following derivation of the Free Choice Permission paradox in DLL:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{A \sqcap B \vdash A}}{\neg A \vdash \neg(A \sqcap B)}}{O\neg A \vdash O\neg(A \sqcap B)}}{\neg O\neg(A \sqcap B) \vdash \neg O \neg A} \qquad \dfrac{\dfrac{\dfrac{\dfrac{B \vdash B}{A \sqcap B \vdash B}}{\neg B \vdash \neg(A \sqcap B)}}{O\neg B \vdash O\neg(A \sqcap B)}}{\neg O\neg(A \sqcap B) \vdash \neg O \neg B}}{\neg O\neg(A \sqcap B) \vdash \neg O\neg A \sqcap \neg O \neg B}}{P(A \sqcap B) \vdash PA \sqcap PB}$$

Moreover, there is another sense of 'or' which corresponds to the genuine disjunction, and which obeys the disjunction introduction:

- $PA \vdash_{DLL} P(A \sqcup B)$

Hence, the conjunctive reading of the disjunction in the free choice permission paradox is not a disjunction, but a linear conjunction. And there is an equivocation with the disjunction introduced from 4 to 3, which is a genuine disjunction. Hence, the 'or' in 3 has a conjunctive sense and should be formalised with $\sqcap$: The same sense of the conjunction that connected the various available desserts that one could choose from in the linear menu example in chapter 4. And the reason why a conjunction is read as a disjunction is that one needs to choose from the available options, given that the premises are treated as resources, and because one usually interprets 'and' in these contexts as $\otimes$ rather than $\sqcap$.

The Pragmatic interpretation

So far it is possible to argue that the solution given is compatible with monism in logic, and with a semantic interpretation of the substructural vocabulary: it is not clear yet how the formalization of 'or' as $\sqcap$ can be interpreted pragmatically and is compatible with the classical literal meaning of the connectives. However, we can provide arguments for the pragmatic reading of the Free Choice Permission paradox (other than those presented in Chapter 5).

Permissions should be formalised in LL given that performing any of the permitted actions has a *reaction* on the grounds from which the permission is granted. Hence, the formalization of 'or' depends on the instability of the derivation of any of the permitted actions with respect to the grounds that allow them.

Moreover, as was briefly mentioned before, the fact that 'or' should be formalised as $\sqcap$ in free-choice contexts has a very natural explanation if we interpret this substructural logic as enriched. This can be seen focusing on the fact that the 'or' can be disambiguated, as well as reinforced.

Disambiguation Whenever one says 'you may eat an apple or a pear, I don't know which', manifests that the 'or' is a disjunction rather than a conjunction, and the hearer understands that we are in a case of $P(a \sqcup b)$ rather than $P(a \sqcap b)$ - and this is why the free choice is blocked. Something similar would happen if the waiter at a restaurant, after saying 'for dessert we have Ice Cream, Bananas, or Souffle, *I don't know which*': the customer would automatically transform what was understood as $\sqcap$ into $\sqcup$.

Reinforcement Provided that permission is given in a resource context, the conjunction 'and' is usually enriched with the availability of the two conjuncts together given the premises. Saying 'you may eat an apple and a pear' expressed the availability of the two events, and hence is formalised with $\otimes$. The non-enriched 'and' in these contexts is usually read as 'or'. However, we can cancel this enriched scenario (that is, the enrichment of a reaction on the grounds for deriving the permission), for instance, saying 'but you don't need my permission'. In this case, the 'or' could be seen as a non-limited conjunction. Something similar would happen if in a restaurant in which the waiter says 'for desert, we have cake or apple pie', we discover afterwards that we are in a self-service buffet.

6.3 Epistemic Paradoxes

We will finish this chapter with a discussion about the Lottery paradox and the Preface paradox, which have been vindicated as paradoxes that can be solved using substructural logics. Moreover, this will give us the opportunity to present the pragmatic enrichment of the universal quantifier, just as McGee's paradox allowed us to present the pragmatic enrichment of the existential quantifier.

6.3.1 The paradoxes

The Lottery Paradox

Kyburg [1970] presents the Lottery paradox as follows:

> Consider a fair, 1,000,000-ticket lottery, with one prize. On almost any view of probability, the probability that a given ticket (say ticket n° 1) will win is

> 0.000001, and the probability that it will not in is 0.999999. Surely if a sheer probability is ever sufficient to warrant the acceptance of a hypothesis, this is a case. It is hard to think of grounds on which to base a distinction between this case and the cases of thoroughly acceptable statistical hypotheses. The same argument, however, goes through for ticket n° 7, ticket n° 156, etc. In fact for any *i* between 1 and 1,000,000 inclusive, the argument goes though and we should rationally be entitled - indeed obligated - to accept the statement 'ticket *i* will not win'. A commonly accepted principle of acceptability is that if *S* and *T* are acceptable statements, then their conjunction is also acceptable. But this means that in the lottery case, since each statement of the form 'ticket *i* will not win' is acceptable, so is the conjunction of 1,000,000 of these statements, which is equivalent to the statement that no ticket will win in the lottery, which contradicts the statement which we initially took to be acceptable that one ticket would win the lottery. [Kyburg, 1970, p. 197]

Kyburg observes that (i) and (ii) are the origin of the paradox:

i. if A has high probability, then it is rational to believe A,

ii. if A and A' are rational to be believed, so is $A \wedge A'$.

To put things more formal, and following [Douven, 2003] let t be some value between 0 and 1 such that 'rationally acceptable' means that the subjective probability of p (symbolised $P(p)$) is above or equal to t, $P(p) \geq t$. (ii) can be formalised with the following Conjunction Principle:

> Conjunction Principle (CP): If each of the propositions $\varphi_1, ..., \varphi_n$ are rationally acceptable, so is $\varphi_1 \wedge ... \wedge \varphi_n$.

Let $\forall$ generalize the conjunction $\wedge$. Now we can define a Lottery case as one which satisfies the following two facts, given a set of tickets in a lottery $Lot = \{x_1, ..., x_n\}$, and being Lx defined as 'x is a loosing ticket'. First,

i. for any ticket $x_i \in Lot$, $P(Lx_i) \geq t$

This, together with CP, which generalizes our belief about each particular ticket and says that this is true whatever ticket one chooses, amounts to accept that is rational to accept the following,

- $\forall x Lx$

We will refer to this as the *positive universal*. Second, it is also rational to believe, as we know that the following is false,

ii. $p(Lx_1 \wedge ... \wedge Lx_n) < t$

That is, a negation of the previously accepted conjunction,

∘ $\neg\forall x Lx$

Which we will refer to as the *negative universal*. The positive universal and the negative universal are inconsistent beliefs which we seem to endorse, so the paradox emerges.

Moreover, in lottery cases the probability that we give to the negative universal is 1, and this creates a connection between each ϕ such that:

iii. there is some $x_i \in Lot$ and some m such that $1 < m < n$: $p(Lx_i | Lx_1, ..., Lx_m) < t$

That is, it is not rational to believe, given a sufficiently high number of losing tickets, that another ticket is also a losing one.

Preface

The preface paradox [Makinson, 1965] considers the conjunction of all the sentences of a book, the author of which endorses. However, the author also rationally believes that the conjunction of all of them is false, given her fallibility, and by a simple induction on her previous research. In Makinson words,

> Suppose that in the course of his book a writer makes a great many assertions, which we shall call $s_1, ..., s_n$. Given each one of these, he believes that it is true. If he has already written other books, and received corrections from readers and reviewers, he may also believe that not everything he has written in his latest book is true. His approach is eminently rational; he has learnt from experience. The discovery of errors among statements which previously he believed to be true gives him good grounds for believing that there are undetected errors in his latest book.
>
> However, to say that not everything I assert in this book is true, is to say that at least one statement in this book is false. That is to say that at least one of $s_1, ..., s_n$ is false, where $s_1, ..., s_n$ are the statements in the book; that $(s_1 \wedge ... \wedge s_n)$ is false; that $\neg(s_1 \wedge ... \wedge s_n)$ is true. The author who writes and believes each of $s_1, ..., s_n$ and yet in a preface asserts and believes $\neg(s_1 \wedge ... \wedge s_n)$ is, it appears,

> behaving very rationally. Yet clearly he is holding logically incompatible beliefs: he believes each of $s_1, ..., s_n, \neg(s_1 \wedge ... \wedge s_n)$, which form an inconsistent set. The man is being rational though inconsistent. More that this: he is being rational even though he believes each of a certain collection of statements, which *he knows* are logically incompatible. [Makinson, 1965, p.205]

Again, we have a positive and a negative universal. However, there is a substantial difference between a lottery and a preface: there is no incompatibility in the truth of *all* the sentences being true together, contrary to the lottery. And in fact, it seems that our mental process might go in the opposite direction than the reaction in the Lottery: while in the Lottery, the confirmation that some amount of tickets are losing tickets increases our belief that another specific ticket is a winning one, in the case of the Preface, the discovery that some amount of sentences are true makes us more and more confident that the rest are true too. Hence, while (i)-(ii) can be reformulated as follows, being $Book = \{x_1, ..., x_n\}$, and Tx defined as 'x is true',

i. for any sentence $x_i \in Book$, $p(Tx_i) \geq t$

ii. $p(Tx_1 \wedge ... \wedge Tx_n) < t$

for all we said above, (iv) should substitute (iii):

iv. for all $x \in Book$ and all $\{x_j, ..., x_k\} \subseteq Book$ such that $1 \leq j \leq n$ and $1 \leq k \leq n$: $p(Tx|Tx_j \wedge ... \wedge Tx_k) \geq t$[6]

6.3.2 Substructural framework

The paradoxes in a logic without contraction

The universal quantifier is also split into two in the substructural logics endorsed here and, following Paoli [2005] and Zardini [2015a], we can distinguish two senses of 'all' in LL: the generalisation of the extensional conjunction $\sqcap$ and the generalisation of the intensional conjunction $\otimes$. This distinction gives us a framework for analysing the lottery and preface paradoxes. But while Zardini and Paoli respond to the paradoxes from a semantic perspective and arguing then that the distinction between the two senses of 'all' is semantic, we will argue that it is pragmatic.

In more detail, Paoli [2005] and Zardini [2015a] argue that the distinction between the two universal quantifiers capture the following two senses: on the one hand 'each' or

[6]The formulation is in [Douven, 2003] for characterizing what he calls *genuine paradox*

'any', formalised here as $\sqcap$, is the generalization of the extensional conjunction $\sqcap$. On the other hand, 'all' or 'every' formalised as $\bigwedge$, is a generalization on the intensional $\otimes$.[7] Again, with the restriction to finite domains, we can set the following rules for the quantifiers from [Zardini, 2011],

$$\frac{\Gamma, \phi(a) \vdash \Delta}{\Gamma, \sqcap x\phi \vdash \Delta}\sqcap L \qquad \frac{\Gamma \vdash \Delta, \phi(a)}{\Gamma \vdash \Delta, \sqcap x\phi}\sqcap R$$

$$\frac{\Gamma, \phi(t_1), \phi(t_2), ... \vdash \Delta}{\Gamma, \bigwedge x\phi \vdash \Delta}\bigwedge L \qquad \frac{\Gamma \vdash \Delta, \phi(t_1) \qquad \Gamma' \vdash \Delta, \phi(t_2) \qquad ...}{\Gamma, \Gamma', ... \vdash \Delta, \Delta', ..., \bigwedge x\phi}\bigwedge \mathrm{R}$$

It is natural that the two universal quantifiers distinguish two reasons to assert 'All x Px'. $\sqcap$ and $\bigwedge$ distinguish two reasons to assert $\forall xPx$:

- Px for any x: $\sqcap xPx$
- Px for every x: $\bigwedge xPx$

That is,

Natural Language	Reasons to assert	LK	LR
All x are P	for any arbitrary n, Pn	$\forall$	$\sqcap$
	for every n, Pn		$\bigwedge$

Hence the paradox is formalised thus: the positive universal is extensional,

$$\sqcap xLx$$

And the negative universal is intensional,

$$\neg\bigwedge xLx$$

Given that in a logic without contraction $\sqcap xPx \nvdash \bigwedge xPx$, one can endorse both the positive and the negative universal. Hence, Zardini's solution is the rejection that each particular belief entails 'every', but the claim that they entail 'any'; and given that 'any' does not entail 'every', there is no contradiction on one's beliefs in the case of the lottery.

Paoli has a similar diagnostic, as he argues that there is no contradiction between 'for any x, Lx' and 'there is an x, $\neg Lx$', which is equivalent to 'not for every x, Lx'. Paoli presents the paradox as a contradiction between the two existential quantifiers equivalent to the universals:

[7] We are presenting the quantifiers in LL, but the same rules are valid for LR.

- $\sqcap xLx \Leftrightarrow \neg\sqcup x\neg Lx$: expresses that there is no particular x such that $\neg Lx$,
- $\neg\Lambda xLx \Leftrightarrow \Sigma xLx$: expresses that there is a connection between each ticket x such that if the first is a loosing ticket then, if the second is a loosing ticket then... the last one is not a loosing ticket Lx.

Given that the negative universal is asserted because some ticket has to be the winning ticket ($\Sigma x\neg Lx$) and the positive universal is asserted because there is no particular ticket which is the winning ticket ($\sqcap x\neg Lx$), and given that $\sqcap x\neg Lx \nvdash \neg\Sigma x\neg Lx$, there is no contradiction between these two claims.

6.3.3 The solution in Pragmatic Logical Pluralism

Again, we will reinterpret the language pragmatically. Our solution rests on the notion of *reaction*, which is different for each paradox. Let us see the details.

Lotteries

Let Γ be the ground for deriving each Lx_i. First, we agree that given that we can derive each Lx_i from some background Γ the following holds:

- $\Gamma \vdash \sqcap xLx$

Moreover, given each Lx_i there is a reaction on Γ as it might be the case that it is not rational anymore to derive the rest of the conjuncts. In particular, the principle (iii) which holds for the lottery, $P(Lx|L_1, ..., L_m) < t$ for some m can be interpreted as the reaction on Γ that each L_i has. Given this reaction, the following also holds:

- $\Gamma \vdash \neg\Lambda xLx$

The reaction is something that we know about lotteries, but not something that follows from the meaning of $\vdash$ and neither from the literal meaning of 'and'. It is, indeed, an enrichment.

The fake lottery Contrast the lottery scenario with the following fake-lottery, in which there is no reaction on Γ given some L_i, and hence the dependency expressed by $\neg\Lambda Lx$ is not present, in order to see that $\Gamma \vdash \sqcap Lx$ and $\Gamma \vdash \Lambda xLx$. Consider the scenario of a lottery with 1,000,000 tickets, numbered with odd numbers. Imagine that we learn that *each* number in the lottery drum has been substituted by its immediate successor, and that we have learned this fact for each particular ticket and not as a general change. In this case, one can legitimately believe $\sqcap xLx$, and, contrary to the classical lottery, one should also believe that *all* tickets are losing tickets, that is, ΛxLx because of the lack of reaction on Γ.

The Preface Paradox

The solution to the lottery rests on the claim that we should not endorse the intensional universal given that there is a reaction on the grounds for asserting each conjunct which does not permit to endorse all of the conjuncts *together*. However, the Preface paradox mentioned above seems to escape this diagnostic because there is no reaction on Γ given one or more Tx.[8]

We argue that it is still the case that we should formalize the Preface case as the Lottery, with the extensional positive and the intensional negative universals:

- $\Gamma \vdash \sqcap Tx$
- $\Gamma \vdash \neg \Lambda xTx$.

Note that the difference between the Lottery and the Preface is that the negative universal of the Preface paradox has not the maxim probability 1, contrary to the negative universal of the Lottery. In the preface, the negative universal might be a rational belief to which we should give similar credence that we give to any of the particular propositions of the book has. Hence, $\neg \Lambda xTx$ can be added to the list of our rational beliefs, together with Tx_1, Tx_2, etc. In particular, our set of beliefs can be formalised as follows:

- $\Gamma \vdash \sqcap xTx \sqcap \neg \Lambda Tx$[9]

Or, to put things more simple,

- $\Gamma \vdash T1 \sqcap T2 \sqcap ... \sqcap Tn \sqcap \neg \Lambda xTx$

Given this, and contrary to the Lottery, after some amount of sentences are shown to be true, we become suspicious about the negative universal. That is, there is a reaction in believing the negative universal given a sufficiently great amount of True sentences. In other words, there is a reaction is on the negative universal:

[8]Note that a lottery can be transformed into a preface if it is a lottery in which there is *usually* but not always a winning ticket. And a preface can be transformed into a lottery if a colleague in which we trust tells us that she spotted one mistake in our manuscript.

[9]It is clear that we can also have a similar list of beliefs in the Lottery, with the difference that the negative universal in the Lottery has probability 1, which is not modified as a reaction of any of the tickets being Losing tickets.

- $P(\neg \Lambda x T x | T x_1, T x_2, ... T x_n) < t$

On the other hand, we might be completely convinced that there is some false sentence. Hence, we might put the negative universal in our list of confirmed sentences, and given that belief together with some amount of true sentences, might have some reaction over our credence of other sentences:

- $p(T x_i | \neg \Lambda x T x, T x_1, T x_2, ... T x_n) < t$

Again, none of these considerations about the reaction on Γ that the confirmation of different sentences of the book have is part of the meaning of $\vdash$, and neither part of the literal meaning of 'and'. Rather, it is a pragmatic enrichment of the vocabulary.

Chapter 7

Objections solved

In this chapter, we address some objections that the reader might have against Logical Pluralism in general and Pragmatic Logical Pluralism in particular. This gives us the opportunity of reinforcing some ideas defended along the thesis with the aid of the answer that Pragmatic Logical Pluralism has against these objections.

7.1 General objections against Logical Pluralism

7.1.1 Inferential role of logical constants

So far we have seen that there are different formal languages which encode 'if...then', 'and' and 'or' in different ways. However, there is still room for arguing that only one of the proposals (or none of them) captures the correct inferential role of logical constants. In particular, the question now is, how should we reason with 'if...then', 'and' and 'or'? Does the inferential role of a connective c coincide with the inferential role that some of the logics presented attribute to it? Our response, in brief, is that, given that we should observe quality, relation, and manner, the logical constants have a plurality of inferential roles captured by $\vdash_K$, $\vdash_R$, $\vdash_L$, $\vdash_O$, depending on the particular enrichments in each case.

The normative dimension of the pragmatic enrichment

The fact that the different logics presented here capture the normative role of logical constants depends on the normative dimension of the pragmatic enrichments. And although we have distinguished implicatures from enrichments, the normative dimension of implicatures can be extended to enrichments, given the role that the Gricean maxims have in both of them. The claim that the maxims have a normative dimension is observed by Grice,

> I would like to be able to think of the standard type of conversational practice not merely as something that all or most do IN FACT follow but as something that it is REASONABLE for us to follow, that we SHOULD NOT abandon. [Grice, 1989a]

However, one might argue that the fact that we should observe the maxims in our discourse is not yet an argument for endorsing the deviant inferential role of substructural logics, given the different role they have in deriving implicatures from enrichments. Another argument for the normative role of pragmatic enrichments emerges from two observations. First, the following necessary requirement for logic to be normative: that the truth-bearers of the sentences in our arguments are related to our mental states,

> Presumably, if logic is normative for thinking or reasoning, its normative force will stem, at least in part, from the fact that truth bearers which act as the relata of our consequence relation and the bearers of other logical properties are identical to (or at least are very closely related in some other way) to the objects of thinking or reasoning: the contents of one's mental states or acts such as the content of one's beliefs or inferences, for example. [Steinberger, 2017b]

Second, the fact that the pragmatic enrichment of logical vocabulary is part of our mental representation of certain utterances using 'if...then', 'and' and 'or': although it is outside the scope of this thesis to develop a theory of the mental states of the utterances containing logical connectives, it seems natural to assume that the pragmatic content that enriches certain utterances are part of our mental representations of those utterances.

The rules for reasoning

In a sense, we should infer '2+2=4' from 'either 2+2=4 or I am the Pope' and 'I am not the Pope'. In another sense, the inference seems defective, because we are not relevant (the reason for asserting the disjunction is the conclusion, so we are not inferring the conclusion *from* the premises). Hence, we should not only preserve truth, but we should also observe the Gricean maxims when reasoning. Consider the following dialogue from MacFarlane [2004b], with two possible (and intuitively legitimate) ramifications:

- Teacher: So does "2+2=4" follow logically from "Today is Tuesday"?
- Student: Huh?

(C)

- Teacher **C**: Well, could the conclusion be false if the premise was true?
- Student: I guess not, because the conclusion couldn't be false period. I guess it does follow.

(R)

- Teacher **R**: Well, could the conclusion be validly inferred from the premise?
- Student: No, that would be crazy. I guess it does not follow.

The two ramifications of the dialogue seem legitimate given the different normative forces we might give to the vocabulary and to the notion of 'follows from'. In this particular example, the two responses can be captured with classical and relevant logic, but the example could be generalised to other logics. Let us see now the general rules that should govern our reasoning.

The bridge principle

How to specify the connection between correct inferences in a logic and the normative constraint that it imposes on our reasoning is a major debate, and there is a plurality of normative constraints which can connect the two facts. Following MacFarlane [2004b] formulation of the Bridge Principle[1] and Ferrari and Moruzzi [2017][2], we suggest the following formulation:

> BRIDGE PRINCIPLE: If $B_1, ..., B_n$ follows from A_1, ..., A_n, then S ought to (if S judges all the $A_i s$, then S judges some B)

One problem with the formulation of the bridge principle is that it presupposes the classical vocabulary in the metalanguage: the conditional that connects the norm is material, and the conjunction and disjunction that connects premises and conclusions are classical. However, if we reject such identifications, we can have more precisifications, with different senses of 'ought': we can distinguish the following four rules for reasoning given a valid inference from $A_1, ..., A_n$ to $B_1, ..., B_n$:

[1] MacFarlane discusses some forms of Bridge Principles, being the following the best candidate: 'If schema S is formally valid and you apprehend the inference $A, B \Rightarrow_k C$ as an instance of S, then you ought to see to it that if you believe A and you believe B, you believe C.' [MacFarlane, 2004b]

[2] Ferrari and Moruzzi, following MacFarlane, also discuss some forms of Bridge Principles among which the following is considered as the best candidate: 'If B follows from $A_1, ..., A_n$, then S ought to (if S judges all the $A_i s$, then S judges that B).'[Ferrari and Moruzzi, 2017]

i. If $A_1, ..., A_n \vdash_K B_1, ..., B_n$ then S ought to$_{(for\ preserving\ truth)}$ (if S judges $A_1 \wedge ... \wedge A_n$ then ($\supset$) S judges that $B_1 \vee ... \vee B_n$).

ii. If $A_1, ..., A_n \vdash_R B_1, ..., B_n$ then S ought to$_{(for\ being\ relevant)}$ (if S judges $A_1 \times ... \times A_n$ then ($\rightarrow$) S judges that $B_1 + ... + B_n$).

iii. If $A_1, ..., A_n \vdash_L B_1, ..., B_n$ then S ought to$_{(for\ being\ perspicuous)}$ (if S judges $A_1 \otimes ... \otimes A_n$ then ($\multimap$) S judges that $B_1 \oplus ... \oplus B_n$).

iv. If $A_1, ..., A_n \vdash_O B_1, ..., B_n$ then S ought to$_{(for\ being\ ordered)}$ (if S judges $A_1 \otimes ... \otimes A_n$ then ($-\!\bullet / \bullet\!-$) S judges that $B_1 \oplus ... \oplus B_n$).

The interest of having a multiplicity of bridge principles lies in those cases in which they diverge.

Let us first see the divergence between (i) and the other principles, (ii)-(iv), with a particular case: (i) forces us to validate an inference such as explosion ($A \wedge \neg A \vdash B$), while (ii)-(iv) do not establish this normative constraint. On the one hand, $A \wedge \neg A \vdash_K B$ is true, and it is also true that one should not judge $A \wedge \neg A$. Hence, given that the conditional in the metalanguage for stating the bridge principle is material and the antecedent of the conditional is false, it validates the fact that one ought to reason according to explosion *for preserving truth*. However, this is not the case for any of the other bridge principles (ii)-(iv) given that the law of explosion is invalid, to begin with. A general principle that emerges from (i) is:

> PRESERVE TRUTH: Whenever the logical vocabulary is not pragmatically enriched, reason according to the rules of *LK*.

Second, what (ii)-(iv) state is that we should consider the connection between premises and conclusion, considering different senses of 'follows from'. First, relevantly valid inferences are linked to our reasoning when we strive to preserve relevance. Consider for instance $\neg(A \times B), A \vdash B$. If one judges $\neg(A \times B)$ and A, then one should, for being relevant, judge that B. However, if one judges $\neg(A \sqcap B)$ and A, one is not entitled to conclude that B follows from the premises. Given that those inferences that are classically valid but relevantly invalid fail to be relevant we should avoid reasoning, for being relevant, following relevantly invalid inferences:

> REASON RELEVANTLY: Do not reason with pragmatically enriched vocabulary (observing Relation maxims) following the rules of non-enriched vocabulary, and viceversa - that is, do not confuse extensional with intensional connectives.

Third, in the particular cases in which 'follows from' captures the *use* of the premises for deriving the conclusion, we should reason with $\vdash_L$ or $\vdash_O$. Consider $A \otimes B \vdash B \otimes A$ as an illustration. The inference is classically valid (in its classical form $A \wedge B \vdash B \wedge A$), relevantly valid (in its relevant form $A \times B \vdash B \times A$), linearly valid (in its linear form $A \otimes B \vdash B \otimes A$), but not orderly valid ($A \otimes B \nvdash B \otimes A$). Hence, in order to reason orderly, if S judges $A \otimes B$ (that is, one judges A and B enriching the meaning of 'and' with order), one is entitled, for being perspicuous (following (iv)), to judge its ordered conclusions, but one is not entitled to thereby judge $B \otimes A$.

In other words, and given that the linear enrichment is ignored in LR, and that the order enrichment is ignored in LL we should follow this third rule:

> REASON PERSPICUOUSLY: Do not reason with pragmatically enriched vocabulary observing Manner maxims following the rules of non-manner-enriched vocabulary, and viceversa - that is, do not confuse the intensional connectives of a linear logic L with the intensional connectives of a stronger logic L'.

Let us see some examples that illustrate how reasoning seems defective when the rules applied to an enriched logical constant do not correspond to the logic that encodes its pragmatic enrichment.

How to choose the best formalization

Let us see, for each logical connective, (i) a summary of the different reasons presented in the previous two chapters for using that logical connective, (ii) an illustration of the violation of the principle 'reason relevantly' and (iii) an illustration of a violation of the principle 'reasons perspicuously'.

Conditional How should we reason with a conditional? Each logic introduced different reasons to derive 'if A then B' given Γ, and each of them had different inferential roles. Let us recall the different reasons to derive a conditional from some background Γ:

- RE-reasons: Γ gives reasons to assert $\neg A$ / Γ gives reasons to assert B.
- LE/OE-reasons: using Γ one avoids A / using Γ one gets B.
- RI-reasons: given Γ, A cannot be true without B being true.
- LI-reasons: given Γ, using A we get B.
- OI-reasons: given Γ, using A (after Γ), we get B.

- OI-reasons: given Γ, using A (before Γ), we get B.

Now we can distinguish how the above rules for reasoning determine how we should reason with a conditional.

Reason relevantly Consider the reasoning in 1,

1. The sun will come up tomorrow. Hence, if it does not, it won't matter.

First, given that $A \vdash \neg A \supset B$ is LK-valid, if a subject S believes A, she should, to preserve truth, also believe $\neg A \supset B$ (in its literal meaning), given the bridge principle (i).

Second, given that $A \nvdash \neg A \rightarrow B$, the bridge principle (ii) does not force one to infer, for being relevant, $\neg A \rightarrow B$ from A. However, S should believe $\neg A \rightsquigarrow B$. In other words, S should not infer a connection between $\neg A$ and B, but should believe that the conditional is literally true.

Reason perspicuously Consider 2 asserted by a chemist:

2. If one combines H and H with O, then one has H_2O; and I have combined H and H with O. Hence, I will get H_2O, H, H and O.

First, given that $A \wedge (A \supset B) \vdash A \wedge B$, if one interprets the logical vocabulary in its literal sense, in order to preserve truth, one should believe the conclusion if the premises are believed, given the bridge principle (i). Second, even if the conditional is enriched with a relevant connection between A and B, given that $A \times (A \rightarrow B) \vdash A \times B$, if a subject believes $A \times (A \rightarrow B)$ she should, to be relevant, believe $A \times B$, given the bridge principle (ii).

However, the combination of the elements to form a molecule makes those elements no longer available, and in this case, the conditional of the premise is usually asserted for LI-reasons, that is, the conditional should be formalised as $\multimap$. Given that $(A \multimap B) \otimes A \nvdash A \otimes B$ and the bridge principle (iii), it is not the case that a subject should, for being perspicuous, infer the conclusion from the enriched sense of the premises. In other words, she should not reason with $\multimap$ as if it were $\rightarrow$.

Disjunction We have distinguished four reasons to assert a disjunction 'A or B' given some background Γ:

- RE-reasons: Γ gives reasons for asserting A / Γ gives reasons for asserting B,

- LE/OE-reasons: using Γ one gets A / using Γ one gets B,
- RI-reasons: given Γ there is a connection between A and B such that the negation of one disjunct implies the other.
- LI-reasons: given Γ there is a connection between A and B such that using $A^{\perp}$ (or avoiding A) we get B as a result.
- OI-reasons: given Γ, using the avoidance of A (before Γ), we get B / using the avoidance of B (after Γ), we get A.

Which determine different inferential roles for 'or'.

Be relevant Consider a disjunction with a relation enrichment, such as the following:

3. Emma lives in India or she lives in Mexico.

Again, the bridge principle (i) requires that if one believes the premises literally then, to preserve truth, one should believe the conclusion.

But suppose that 3 is asserted because the speaker believes that Emma lives in India (that is, the disjunction is asserted for RE-reasons) and consider the situation in which the speaker learns that Emma does not live in India anymore; should she derive that Emma lives in Mexico, from her previous assertion 'A or B' and the new information $\neg A$? It seems that this would be irrational[3], that is, it would be illegitimate to use disjunctive syllogism in this case: given that $A \sqcup B, \neg A \nvdash B$ and the bridge principle (ii), it is not the case that the conclusion should, to be relevant, be derived from the premises (contrary to the case in which the disjunction is believed for I-reasons, in which case the disjunctive syllogism is valid, and the premises mean that, to be relevant, one infers the conclusion from them).

Be perspicuous Consider the example 4,

4. Either I do not have ten coins in my pocket, or I can order a pizza; I have ten coins in my pocket. Hence, I can both have a pizza and ten coins in my pocket

First, given that $\neg A \vee B, A \vdash B \wedge A$, one should infer the conclusion from the literal meaning of the premises. Moreover, the inference is valid in relevant logic, so even if the disjunction is relevantly enriched, to preserve relevance, one should conclude $A \times B$ from $\neg A + B$ and $A \vdash B$.

[3]See [Paoli, 2007] for similar examples.

But the inference $\neg A \oplus B, A \vdash B \otimes A$ is invalid in linear and ordered logic. And if the disjunction pragmatically entails that the ten euros are wasted in the process of ordering a pizza, the disjunction in the antecedent is asserted for IL-reasons, that is, it should be formalised with $\oplus$. Hence, it is not the case that, to be perspicuous, one should infer $B \otimes A$ from the premises.

Conjunction Finally, the case of conjunction has been a bit more complex, as we have seen the differences between instances of 'and' whenever it is embedded under different connectives.

First, under the scope of a negation, we have distinguished the following reasons to assert 'not (A and B)' given a background Γ:

- RE-reasons: from Γ we derive not A / from Γ we derive not B
- LE/OE-reasons: using Γ we avoid A / using Γ we avoid B
- RI-reasons: from Γ we derive that there is some connection between A and B that makes them incompatible: i.e. one conjunct excludes the other, A entails $\neg B$.

Second, we have distinguished different reasons to derive Δ from 'A and B':

- RE-reasons: A implies Δ / B implies Δ,
- LE/OE-reasons: using A one gets Δ / using B one gets Δ
- RI-reasons: both A and B are required to derive Δ
- LI-reasons: A and B together are used to get Δ
- OI-reasons: using both A and B, in this order, we can infer Δ

Third, there are three reasons to assert 'A and B' given a background Γ:

- RE-reasons: from Γ one can derive *any* among A and B
- LE/OE-reasons: using Γ one gets *any* among A and B
- LI-reasons: using Γ one can get *both* A and B
- OI-reasons: using Γ one can get *both* A and B in this order

Be relevant Consider the situation in which one asserts the following:

5. If I have a coffee and a croissant, then I have my dose of caffeine. Hence, if I have a coffee, then if I have a croissant, I have my dose of caffeine.

First, given that $(A \wedge B) \supset C \vdash A \supset (B \supset C)$, from the literal meaning of the premises one should conclude, to preserve truth, the literal meaning of the conclusion.

However, if one asserts the first conditional because one of the conjuncts in the antecedent is sufficient for the consequent, the relevant enrichment is false in the antecedent of 5, and it should be formalised with $\sqcap$. Given that $(A \sqcap B) \rightarrow C \nvdash A \rightarrow (B \rightarrow C)$, by principle (ii), it is not the case that, to be relevant, one should derive the conclusion from the premises.

Be perspicuous Finally, consider again 6,

6. If the old king dies of a heart attack and a Republic is declared, Tom will be content. A Republic will be declared. So if the old king dies, Tom will be content.

The inference is valid in LK, LR and LL: $(A \otimes B) \multimap C, B \vdash A \multimap C$. Hence, to preserve truth, relevance, and even to be perspicuous (in one sense), one should derive the conclusion from the premises.

However, if the conjunction in the first premise is enriched with order, it is not the case that one should derive the conclusion, as $(A \boldsymbol{\otimes} B) \multimapdot C, B \nvdash C \multimapdotinv A$ and neither $C \multimapdotinv (A \boldsymbol{\otimes} B), B \nvdash C \multimapdotinv A$ (and it is clear that the conclusion should be formalised with $\multimapdotinv$ rather than with $\multimapdot$ given that the death of the king happens after the rest of the premises in this particular example). And this seems to be precisely the case in the example. Hence, it is not the case that one should derive the conclusion from the enriched sense of the premises. In other words, one should not reason with $\boldsymbol{\otimes}$ as if it were $\otimes$.

7.1.2 Collapse

Another problem related to the inferential role of the logical constants is the problem of collapse, which was already presented in Chapter 2 when discussing Beall and Restall's pluralism, and to which the present proposal must give an answer: given that we are not always aware of which framework or logic we are reasoning in, and given that the question of whether Δ follows from Γ is logic dependent, we need to give an answer to the apparent discrepancy between the positive and negative answer to the question about the validity of a certain inference.

The collapse problem is raised by Priest in [2001], [2006] and [2014], Read [2006] and Keefe [2014][4]. Consider the following formulation, which responds to Beall and Restall's version of pluralism (and hence refers to *situations*):

> Let s be some situation about which we are reasoning; suppose that s is in different classes of situations, say, K_1 and K_2. Should one use the notion of validity appropriate for K_1 or for K_2? We cannot give the answer 'both' here. Take some inference that is valid in K_1 but not K_2, $\alpha \vdash \beta$, and suppose that we know (or assume) α holds in s; are we, or are we not entitled to accept that β does? Either we are or we are not: there can be no pluralism about this. [Priest, 2006, p.203]

A more general presentation of the Collapse argument which does not depend on Beall and Restall's version of Logical Pluralism can be found in [Read, 2006]:

> [S]uppose there really are two equally good accounts of deductive validity, $K1$ and K_2, that b follows from a according to K_1 but not $K2$, and we know that a is true.... It follows K_1-ly that b is true, but not K_2-ly. Should we, or should we not conclude that b is true? [Read, 2006, p. 3]

Moreover, Stei [2017] presents an argument which shows that the collapse argument affects any version of logical pluralism that endorses the following four principles which are widely accepted by different versions of pluralism, including ours [Stei, 2017, p.3][5]:

- There is more than one correct logical consequence relation within one language,
- Logical consequence is global in scope,
- There is rivalry between different correct consequence relations,
- Logical consequence is normative.

Given that Pragmatic Logical Pluralism presented here endorses all of them, an answer to the argument is required.

[4]For a detailed presentation of the three versions see [Stei, 2017].

[5]We have seen in Chapter 2 a version of Logical Pluralism defended by Barrio et al. [2018] which avoids the Collapse problem by rejecting the third point. Versions of localism can also avoid the problem by rejecting the second point. In any case, our version of pluralism is included in Stei's classification, as well as many of the versions presented in Chapter 2.

The collapse challenge has two main different direct answers, both equally problematic: first, one may argue that, given that the inference from α to β is valid in K_1, one is entitled to accept β, independently of what K_2 states. Hence, K_1 is a better logic than K_2. Second, one might be interested in those inferences that are valid in *any* logic, and confer them some special property, such as being *super*-valid (using an analogy with supervaluationism[6]), and assign to those some special properties in comparison with the rest of validity relations. That would make K_2 a better logic than K_1. A similar reasoning is made by Bueno and Shalkowski [2009], as we have seen in Chapter 2, which reduces pluralism to nihilism.

Classical Collapse

The first kind of answer would make classical logic an overall better logic than any substructural one. Let us call this reasoning *classical collapse*. The previous quote by Read continues as follows,

> The answer seems clear: K_1 trumps K_2. After all, K_2 does not tell us that b is false; it simply fails to tell us whether it is true.... K_1 and K_2 are not equally good. K_1 answers a crucial question which K_2 does not. [This] question is the central question of logic. [Read, 2006, p. 3]

Pragmatic solution Our answer, in short, is that given that there is more than one 'should', related to the different normative forces of each maxim, the question 'should we, or should we not conclude that b is true?' does not have a yes/no answer [7].

In more detail, we have already seen, with the plurality of bridge principles and the illustrations above, that there is not a yes/no answer to the question 'should we infer Δ from Γ?'. It is evident that the target cases for the collapse challenge are those inferences

[6]See [Ferrari and Moruzzi, 2017] for an extensive discussion about this.

[7]Our solution can be seen as a particular case of Caret's [2017], who solves the collapse problem arguing for a contextualist account of validity, Collapse problems do not emerge because once a logic is salient for a given context it makes no sense to look at other logics that have incompatible answers about the validity claims in that context.

> The contextualist response should be clear: since deductive standards depend on context, there is no greater weight accorded to one logic or another as *such*. For the logical contextualist, each admissible consequence relation is simply an eligible content of validity attributions. [Caret, 2017, p.17]

The collapse is simply misleading in its formulation: it presupposes that all logics are equally correct in all contexts, while the pluralist thesis emerges from a different view: different logics seem correct in different contexts.

which are valid in *LK* but invalid in some substructural logic. Hence, the above examples (from the previous section) serve as an illustration of our response to the collapse challenge: first, if we ignore the pragmatic enrichment of the vocabulary and we focus on the literal meaning of premises and conclusion, given that we should observe the maxim of Quality we should believe that the conclusion is true, as truth is preserved. However, this is the case if we consider truth preservation between the literal meaning of the premises and the conclusion. But there are other normative requirements, and different ways of pragmatically enriching the vocabulary which change the truth conditions of the premises or the conclusions. Under these other readings of the vocabulary, it might be the case that the conclusion should not be derived from the premises.

In sum, we have a natural explanation of the plurality of inferential roles that the different logics endorsed in this dissertation attribute to the logical connectives. And this plurality is based on the pragmatic enrichment of the logical vocabulary for each inference, and the normative constraint that we observe in each case.

Ordered Collapse

The previous answer is sufficient for giving a response to the first possible solution to the collapse problem, according to which validity collapses into the validity of the stronger logic. But we still have the problem of considering the weakest logic as the overall best logic, given that none of its inferences is negated by any other logic. In our case, *LO* seems to provide the minimal class of valid inferences that are uncontroversial in all the endorsed logics. Let us call this reasoning *ordered collapse*.

Ordered collapse does not seem to be so problematic as classical collapse: in classical collapse, those logics that are weaker than classical logic *fail* to validate certain valid inferences, while in this case certain logics validate inferences that are problematic in a weaker logic. Hence, the observations presented here aim to dismiss the idea that there is anything interesting in the super-validity property, that is, in the valid inferences in *LO* in comparison to the remaining inferences.

First, the enrichments considered by the different formal languages embraced here do not guarantee that there cannot be other logics, rejecting other structural rules, and with legitimate formalizations of consequence and of the logical connectives that negate some valid inferences in *LO*. There is nothing in the inferences in *LO* that make them *immune* to suitable counterexamples with other formal languages which capture different pragmatic enrichments. However, this should not drive us towards nihilism, for the second reason.

Second, those inferences that are invalid in *LO* but valid in the rest of the logics are invalid given certain pragmatic enrichments of the logical vocabulary. However, their linear, relevant, or classical counterparts ignore those pragmatic enrichments - their validity, then, is not special in comparison to other valid inferences in those same logics, given that pragmatic enrichment is not present.

7.1.3 Meaning variance

The meaning-variance argument is a common objection launched against not only logical pluralism but also against logical deviance in general. The Pragmatic Logical Pluralism has a straightforward answer to the challenge, which emerges from what has been said in Chapter 3.

The argument in Quine

We will introduce and discuss Quine's version of the meaning-variance argument, as more recent versions share the same premises, and a satisfactory answer to the Quinean argument is already an answer to any meaning-variance argument. The argument is presented in the following quote, in which Quine discusses a dialogue between a classical logician and a deviant logician who rejects the law of explosion:

> My view of this dialogue is that neither party knows what he is talking about. They think they are talking about negation, '$\neg$', 'not'; but surely the notation ceased to be recognizable as negation when they took to regarding some conjunctions of the form '$p \wedge \neg p$' as true, and stopped regarding such sentences as implying all others. Here, evidently, is the deviant logician's predicament: when he tries to deny the doctrine he only changes the subject. [Quine, 1986, p.81]

The meaning variance argument that we find in the quote has the following three premises:

a. A change of logic is a change of formal language,

b. The meaning of a logical constant is given by its inferential role/truth conditions,

c. Classical formal language captures the inferential role/truth conditions of logical vocabulary.

The conclusion of [a] and [b] alone is that a change of logic is a change of the meaning of the logical vocabulary. With the addition of [c], which is justified by the implicit assumption of classical logic in the quotation above, the argument also concludes that a deviation of classical logic is a deviation from the genuine meaning of the logical constants.

A posible answer: revision of [c] The premise [c] has been rejected both by pluralists and by monists - deviant monists substitute classical logic by some other logic, while pluralists replace classical logic by a plurality of logics. In fact, the reasons to accept [c] are circular, as Quine himself notices:

> If anyone questions the meaningfulness of classical negation, we are tempted to say in defense that the negation of any given closed sentence is explained thus: it is true if and only if the given sentence is not true. This, we may feel, meets the charge of meaninglessness by providing meaning, and indeed a meaning that assures that any closed sentence or its negation is true. However, our defense here begs the question; let us give the dissident his due. In explaining the negation as true if and only if the given sentence is not true, we use the same classical 'not' that the dissident is rejecting. [Quine, 1986, p. 84]

Also Priest [2006] recognizes this circularity in Quine:

> [H]e assumes that the logical constructions of the vernacular *are* those of classical logic...'$\neg$' is a sign of a formal language with a certain semantics (classical for Quine), whereas negation is a notion from vernacular reasoning. [Priest, 2006, p.171]

Moreover, Quine concedes that there can be reasons for rejecting the classical negation,

> [W]hoever denies the law of excluded middle changes the subject. This is not to say that he is wrong in so doing. In repudiating 'p or not p' he is indeed giving up classical negation, or perhaps alternation, or both; and he may have his reasons. [Quine, 1986, p. 83]

However, the rejection of [c] is usually done by a substitution of classical logic by another logic, in which case we can formulate a more general form of the meaning variance argument.

General meaning-variance argument The revision of [c] can lead to what we can call the *general* meaning-variance argument, if one substitutes classical logic by any other logic, leading to the general meaning variance argument for monists:

a. A change of logic is a change of formal language,

b. The meaning of a logical constant is given by its truth conditions / inferential role,

c_m. There is one formal language that captures the truth conditions / inferential role of logical vocabulary.

This inference fixes a logic as the correct formalization of the logical vocabulary, which makes impossible to disagree with this logic without meaning variation. That is, it makes any disagreement about logic a disagreement about meanings.

Another possible substitution of [c] is the pluralist one, leading to the general meaning variance for pluralists:

a. A change of logic is a change of formal language,

b. The meaning of a logical constant is given by its truth conditions / inferential role,

c_p. There is more than one formal language that captures the truth conditions / inferential role of logical vocabulary.

However, this leads to a pluralism of meanings, according to which the endorsement of more than one logic is equivalent to the endorsement of more than one language. Although this might be a valid version of logical pluralism, it is not the preferred view for many logical pluralists, including us.

The diagnosis The rejection of [c] leads to general versions of the meaning variance argument. We suggest that the best dissolution of these versions of the argument is via the rejection or refinement of [b]. The strategy is to separate the inferential role or truth conditions from the meaning of logical constants. In other words, we will argue that a natural response to the argument is the rejection of the claim that a variation of the truth conditions or the inferential role of a logical connective entails a variation of its meaning: from a pluralist perspective one can adopt one logic as meaning determining and vary the truth condition or inferential role without a meaning variance.

This strategy is not original (although its particular motivation and explanation are). From a pluralist perspective, the strategy is similar to the suggestion in Beall and Restall [2006] (for the truth-conditional version) and Restall [2014] (for the inferential-role version). However, while in [Restall, 2014] and [Beall and Restall, 2006] [b] is *rejected*, we will *refine* it, allowing for one logic to be meaning determining, while other logics capture legitimate truth conditions and inferential roles.

Not only pluralists, but also from the monist perspective, the rejection of [b] has been explored to defend deviant logics. For instance, we share Putnam's [1962] intuition,

> The logical words 'or' and 'not' have a certain core meaning which is... *independent* of the principle of the excluded middle. Thus in a certain sense the

> meaning does not change if we go over to a three-valued logic or to Intuitionistic logic. Of course, if by saying that a change in the accepted logical principles is tantamount to a change in the meaning of the logical connectives, what one has in mind is the fact that changing the accepted logical principles will affect the global use of the logical connectives, then the thesis is tautological and hardly arguable. However, if the claim is that a change in the accepted logical principles would amount *merely* to redefining the logical connectives, then, in the case of Intuitionistic logic, this is demonstrably false. [Putnam, 1962, p. 377]

The crucial question is to determine what principles fix the meaning of logical connectives across logics, considering that such logics vary their truth conditions and their inferential role. We have defended that there is one logic (*LK*) which captures the truth conditions or inferential role for the logical constants, but that there are other legitimate logics that might vary these meaning determining principles. This idea is mentioned by Read [1988],

> The sense of 'not' is, let us suppose, fixed by reference to classical beliefs about the inferential properties of that particle. But once that fixed point is settled, any of those properties may be questioned. [Read, 1988, p.150]

And also by Restall [2002], who refers to this position as 'pick one consequence relation' [Restall, 2002],

> We can pick one of our plurality of consequence relations as determining the meaning of a connective. If we take the primitive classical negation rules as the ones used to determine the meaning of negation, then we take the weaker relevant rules tell us something else about the behavior of that connective. If any set of rules is sufficient to pick out a single meaning for the connective, take that set of rules, and accept those as meaning determining. The other rules are important when it comes to giving an account of a kind of logical consequence, but they are not used to determine meaning. [Restall, 2002, p. 11]

However this is not the preferred view for Restall, and we will try to motivate that this position is the most natural one for certain versions of logical pluralism, especially for Pragmatic Logical Pluralism.

The two versions The meaning-variance argument has two versions, depending on whether one identifies the meaning of a logical constant with its truth conditions or with its inferential role, and it is clear how both versions are included in Quine's quote above: the deviant logician changes the meaning of negation both when she changes its truth conditions ('they took to regarding some conjunctions of the form '$p \wedge \neg p$' as true') and its inferential role ('[they] stopped regarding such sentences as implying all others').

Independently of which is our preferred semantic theory about the logical vocabulary, each logical connective has an inferential role, and it also makes a contribution to truth conditions to the expression in which it appears; and this claim is neutral to any thesis about what constitutes their meaning. In fact, it is clear that one dimension affects the other, as many authors have studied. [8]

Inferential role version

As we have mentioned above, the original presentation of the argument in Quine encapsulates further uses of the same argument. The inferential role version, for instance, sustains the following argument of Williamson [2014] against logical pluralism:

> Schematically, we must distinguish three candidates for the inferential role of a connective C as used by a subject S: (i) how S ought to reason with C; (ii) how S thinks S ought to reason with C; (iii) how S actually does reason with C. (...) Of (i)-(iii), the best candidate for that identification is (i). But since all parties to the debate are to be held responsible to the public meaning of C, (i) is the same for all of them. S ought to reason with C according to P if and only if the defender of orthodoxy ought to reason with C according to P. Thus, on pain of implausibly individualistic consequences, inferential role semantics does not support the idea that deviant logicians use the logical connectives of their native language with deviant meanings. [Williamson, 2014, p. 225]

In order to reject the monists conclusions of the argument, we need to reject the claim that the meaning of a logical constant is fixed by its inferential role, that is, that there can be a change of the inferential role of a logical constant without a meaning variance of that connective.

Structural minimalism The defence of logical pluralism that embraces substructural logics needs a criterion to connect the meaning of the connectives in the different substructural logics. A prominent, and it seems a promising solution is the claim that the

[8] See for instance Hodes [2004], Hjortland [2014a] and Restall [2009]

meaning of a logical connective is fixed by its left and right rules. Following Gentzen [1964], Haack [1978] defends this thesis about the meaning of connectives:

> But now consider Gentzen's formulation of minimal logic (L_j): it differs from classical logic, not in respect of the introduction and elimination rules for the connectives, but in respect of the structural rules for deducibility; namely, it results from restricting the rules for classical logic (L_k) by disallowing multiple consequents. Since this restriction involves no essential reference to any connectives, it is hard to see how it could be explicable as arising from divergence of meaning of connectives. [Haack, 1978, p.10]

Also Restall has defended a version of the claim [Restall, 2014]. And from a monist position [Paoli, 2014] has also defended minimalism. Paoli distinguishes for each logical connective its global meaning from its operational meaning. The global meaning for c is specified by the provable sequents containing c, but it does not determine the genuine meaning of the connective, which is captured by its operational meaning, captured by its left and right rules.

> If we identify meaning *tout court* with operational meaning, therefore, we are in a position to claim that although the classes of provable sequents are different in each case (and therefore our logics are genuine competitors), the connectives' meanings do not change across this particular range of logics. A change of logic, *pace* Quine, does not entail a change of subject. Genuine rivalry between logics is possible after all. [Paoli, 2014, p.3]

This position is sufficient to reject the meaning-variance argument,

> This allows to counter Quine's meaning variance charge: since the operational meaning of negation remains unaltered, there is no such 'change of subject' as Quine adumbrates. Genuine disagreement arises whenever L and L' ascribe different properties to what we can plausibly identify as the *same* constant, given the invariance of its operational meaning across logics. [Paoli, 2007, p. 557]

Structural Maximalism We suggest a different refinement for [b]: there is one logic which captures the meaning of the logical constants, which is LK. In more detail, Left and Right rules, *together with weakening, contraction, and exchange*, determine the meaning of the logical constants (that is, the meaning of the logical connectives is determined by its inferential role in classical logic). The presence of structural rules *weakens* the expressive power of the logical connectives; however, if the richer expressive power of substructural logics can be explained pragmatically, this weakening is equivalent to the cancellation of certain pragmatic enrichments, which reestablishes the literal meaning of the constants.

Consider the linear distinction between $\sqcap$ and $\otimes$. Their different inferential roles in 2.b. and 2.c. is a consequence of the rejection of weakening and contraction, and linear logic captures the differences between the inferential role of $\sqcap$ and $\otimes$. We claim that despite these divergences, the meaning of 'and' is captured by its inferential role in classical logic, that is, it is captured by $\wedge$.

Truth-conditional version

The version of the meaning-variance argument based on the different truth conditions of the logical connectives identifies the meaning of the logical connectives with the particular contributions to the truth conditions that each connective makes. A variation of these truth conditions, if they are identified with the meaning of 'and', is tantamount to the variation of their meaning. And hence logical pluralism is just a case of ambiguity, as Priest argues:

> If we give different truth conditions for the connectives, we are giving the formal connectives different meanings. When we apply the logics to vernacular reasoning we are, therefore, giving different theories of the meanings of the vernacular connectives. We have a case of theoretical pluralism; and the theories cannot both be right - or if they are, we simply have a case of ambiguity. [Priest, 2006, p. 204]

We have argued against the conclusion of Priest's reasoning by arguing against the first sentence: we will illustrate how a logical connective can have different truth conditions without concluding that the connective has a different meaning and hence without ambiguity.

Semantic Maximalism If one is a pluralist, as [Beall and Restall, 2006], and the different logics that one endorses attribute different truth conditions to the logical connectives, one possible solution is that each clause is an incomplete dimension of the meaning. Discussing the divergence about $\neg$ on Classical, Relevant, and Intuitionistic logic, Beall and Restall reason as follows:

> The clauses can both be equally accurate in exactly the same way as different claims about a thing can be equally true: they can be equally true of one and

> the same object simply in virtue of being incomplete claims about the object. What is required is that such incomplete claims do not conflict, but the clauses governing negation do not conflict. [Beall and Restall, 2009, p.98]

It is clear that semantic maximalism would entail that the meaning of conjunction is determined by all the possible truth conditions in all the logics which one might endorse, as long as there is no conflict between them. This possibility is absurd from the point of view of the present proposal, given that the pragmatic contributions of any term are not part of the meaning of the term.

Semantic Minimalism Contributions to truth conditions can vary without meaning variance. If one opts for a truth-conditional semantics for logical constants, one wants to fix the meaning of some word that has a particular contribution to the truth conditions of those sentences in which it appears. And we have fixed the meaning of the logical connectives with their classical truth conditions. But once the meaning is fixed, the truth-conditional contribution of 'and' can vary, departing from its minimal truth-function, and one does not need to postulate a new meaning for the conjunction. We have seen how substructural logics vary the truth conditions of the logical constants: recall the two conjunctions in lineal logic, $\otimes$ and $\sqcap$, which make different contributions to the truth conditions of an utterance:

2.b. If you have 10 Euro you can have a pack of Camels and a pack of Marlboros (any of them: $\sqcap$)

2.c. If you have 10 Euro you can have a pack of Camels and a pack of Marlboros (both of them: $\otimes$)

The contribution of 'and' in 2.b. is that with 10 euros one can have any of the two (imagine that each one of them costs 7 Euros). The contribution of 'and' in 2.c. is that with 10 euros you can get both of them (imagine that each one of them costs 5 Euro or less). Although the truth conditions are different, it is not necessary to explain this divergence as a case of ambiguity, as it has a more natural explanation as a case of expansion of a core meaning of 'and'.

7.1.4 Mixed utterances and mixed inferences

A further question for the present and other versions of logical pluralism is the problem of mixed inferences: given our interpretation of divergent logical vocabulary, the problem emerges from the fact that nothing prevents Γ and Δ from including connectives with different enrichments.

Hence, some inferences with enriched vocabulary do not find a counterpart in one substructural logic, but in a *combination* of them. Consider, for instance, the following sentence:

1. If they marry and have a child, their parents will be happy.

Suppose that the parents are happy if both events happen in this particular order. Hence, it is natural to formalize the conjunction with the ordered $\otimes$. However, the conditional that connects the two events with the consequent does not seem to be one among $\multimap$ or $\multimapinv$: there does not seem to be a reaction given the happiness of the parents on the marriage and children of the couple. It rather seems to be a relevant conditional $\rightarrow$.

In order to better diagnose the problem of mixed inferences, let us see what the problem of formalizing the sentence purely in relevant or ordered logic is. First, if the utterance is formalised in relevant logic we have the following unwelcome consequence: the following inference is valid, although it is intuitively invalid,

2. If they marry and have a child, their parents will be happy. And they had a child. Hence, if they marry their parents will be happy

as the sequent in *LR* is valid,

- $(A \times B) \rightarrow C, B \vdash A \rightarrow C$

The inference is invalid in *LO*, so it might be a good candidate logic for formalising it. However, if we formalise it in ordered logic, given the reaction that $\multimap$ codifies, the following inference is invalid, although it seems intuitively valid:

3. If they marry and have a child, their parents will be happy. They are married. So if they have a child, their parents will be happy, and moreover, they are married.

- $C \multimapinv (A \otimes B), A \nvdash (C \multimapinv B) \otimes A$

In sum, it seems that we should formalize the initial utterance mixing the two logics as follows:

- $(A \otimes B) \rightarrow C$

The problem we are facing now is how to reason with these mixed utterances. In particular, the problem is whether we should use contraction and exchange when reasoning about the previous utterance.

Consider now a slightly more complex case, with the following reformulation of Zardini's [2016] example:

4. If this piece of wood makes 4 chairs, then it makes 1 bed, and if this piece of wood makes 4 chairs, I am going to buy it. And this piece of wood makes 4 chairs.

We seem to be talking about resources in the first conditional but not in the second. That is, the first conditional entails that there is a waste of the piece of wood in the process of making one bed, while the second conditional does not entail any reaction; at most, it expresses a relevant connection between the two facts. Let us see the problem of formalizing the sentence purely in *LR* and in *LL*.

On the one hand, relevant logic validates an inference that seems intuitively invalid:

5. If this piece of wood makes 4 chairs, then it makes 1 bed, and if this piece of wood makes 4 chairs, I am going to buy it. And this piece of wood makes 4 chairs. Hence, this piece of wood makes 1 bed and 4 chairs and I'm going to buy it.

As the following sequent is valid in *LR*

- $A \rightarrow B, A \rightarrow C, A \vdash B \times A \times C$

Linear logic invalidates the previous inference. However, it also invalidates an inference that seems intuitively valid:

6. If this piece of wood makes 4 chairs, then it makes 1 bed, and if this piece of wood makes 4 chairs, I am going to buy it. And this piece of wood makes 4 chairs. Hence, this piece of wood makes 1 bed, and I am going to buy it.

As the following is invalid in *LL*:

- $A \multimap B, A \multimap C, A \nvdash B \otimes C$

Hence, we have another case of mixed inference, with two conditionals that share the antecedent, and in which there is a waste of the antecedent in one of them but not in the other. We should formalize 4 mixing the two logics:

- $A \multimap B, A \rightarrow C, A$

The scope of this research

We have argued how the different substructural logics codify different pragmatic enrichments, and it is further work to see how these different pragmatic enrichments can be related to each other. This limitation does not undermine Pragmatic Logical Pluralism: as we have put forward on the previous objection, our aim was to explain each particular substructural logic as codifying pragmatically enriched senses of the logical connectives. However, we have not argued that *any* pragmatic enrichment is codified by a substructural logic. We can expand the answer given above to the present case: it is not the case that *any* inference with pragmatically enriched senses of the connectives is captured by some substructural language: we might need a combination of them. [9]

[9] In any case, the solution to mixed inferences is an interesting continuation and expansion of this research and of Pragmatic Logical Pluralism. Note that the primary problem of mixed inferences is that there will be cases in which weakening, contraction and exchange will be valid just for some part of a sequent. Structural rules primarily affect the comma of the sequents in a calculus, and this gives us the target of a possible solution: some commas in a sequent will be able to be contracted, weakened, or exchanged, and some of them will be not.

The combination of different kinds of commas, with different properties, in one calculus, is neither new nor original. Just to cite two important references, Belnap [1982] in his display calculus introduces different ways of combining premises in a sequent, and Read [1988] also distinguishes between intensional and extensional premise combination, which entails different structural rules for different kinds of commas.

Following this path gives us some interesting (although preliminar) results. Let first distinguish different commas, depending on whether they can be exchanged, contracted or weakened (consider, for the scope of this footnote, that ';' can be exchanged but not contracted and weakened, and that ',' can be exchanged, contracted but not weakened). And let us consider two auxiliary symbols, the brackets [,], which are introduced whenever a group of formulas connected with one specific comma, are all related to another formula with another kind of comma. These brackets just prevent that some formula relates to another formula in a way that is not permitted, but they do not behave like regular parenthesis. For instance, consider $[B; A], A$ - the A outside the brackets relates in a contractive way with any of the formulas inside the brackets, so a possible transformation would be $B; A$. Consider now $[B, A]; A$ - in this case, the A outside the brackets cannot be contracted with the A inside, given that it relates to the formulas inside the brackets in a non-contractive way.

Now, consider for instance the following valid derivation of 6, in which there is one use of contraction,

$$
\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A \qquad B \vdash B}{A \multimap B; A \vdash B}\,{\multimap}\mathrm{L} \qquad \dfrac{A \vdash A \qquad B \vdash C}{A \to C, A \vdash C}\,{\to}\mathrm{L}}{[A \multimap B; A], A \to C, A \vdash B \times C}\,{\times}\mathrm{R}}{[A \multimap B; A], A, A \to C \vdash B \times C}\,\mathrm{EL}}{[A \multimap B; A], A \to C \vdash B \times C}\,\mathrm{CL}
$$

Contrast the previous derivation with the one following invalid one that would be required for 5. Departing from the last sequent we have proven in the last derivation, we would need to contract the A introduced in the last step, which is incorrect given that the A is connected with the rest of the formulas in a non-contractive way:

7.1.5 Metalanguage

A common question raised against any version of logical pluralism is related to the metalanguage in which the thesis is defended. One might wonder which logic is the logical pluralist *using* in the metalanguage when arguing for logical pluralism. In [Sereni and Sforza Fogliani, 2017] we find that this question might be a key question which might restore monism. In particular, Sereni and Sforza Fogliani argue that the logical pluralist should answer the following:

> [W]hat logic, or logics, are logical pluralists adopting, which are they entitled to adopt, and which ought they to adopt, in arguing for their view? [Sereni and Sforza Fogliani, 2017, p.2]

The answer to this question is relevant as the justification for a particular logic may entail some kind of circularity which emerges both for rejecting and for justifying a logical law l, on grounds of some evidence x:

$$\begin{array}{l} \text{If } x \text{ then } l \text{ fails} \\ x \\ \hline \text{Therefore } l \text{ fails} \end{array} \qquad\qquad \begin{array}{l} \text{If } x \text{ then } l \text{ is justified} \\ x \\ \hline \text{Therefore } l \text{ is justified} \end{array}$$

In case the principle l which one wants to reject or justify is Modus Ponens, one enters into a circular argument in which the validity of MP is used for the very argument for accepting/rejecting it. For this and similar concerns, it is important, specially for the pluralist, to be aware of which logic is she using in her metalanguage and *why*.

Sereni and Sforza Fogliani discuss three options for the logical pluralist, arguing that each one of them puts some troubles for the pluralist. A first possibility is that *no logic* is used in arguing for LP: there are other criteria for endorsing a logic which are not a deductive argument. However, this answer puts some problems on the view that it is logic what makes rational reasoning rational. [Sereni and Sforza Fogliani, 2017, p. 6]

A second option is that there is *one single logic* which is required in arguing for Logical Pluralism. In this case the pluralist is required to justify why one logic is apt for justifying

$$\dfrac{\dfrac{\dfrac{[A \multimap B : A], A \to C \vdash B \times C \qquad A \vdash A}{[[A \multimap B; A], A \to C]; A \vdash (B \times C) \otimes A}\,\otimes R}{[A \to C, [A \multimap B; A]]; A \vdash (B \times C) \otimes A}\,EL}{A \to C, [A \multimap B; A] \vdash (B \times C) \otimes A}\,C^*$$

* Illegitimate use of Contraction.

the claim, contrary to another equally accepted logic by the view. Moreover, there are two main candidates for being the logic of the metalanguage, according to Sereni and Sforza Fogliani: (i) the logic which shares its valid inferences with all logics embraced by the pluralist; (ii) the strongest logic endorsed by the view. In the first case, the resulting logic becomes a very weak one (in the best case, if it does not become empty), and in the second, there will be at last one logic endorsed by the pluralist which fails to determine that some logic is a valid one, which also seems problematic.

A final option is that *more than one logic* is used in arguing for Logical Pluralism. In particular, this might mean that there are two arguments in two different logics which support logical pluralism, or that there is one argument which is valid in more than one logic, for logical pluralism. In this second case, we have seen (in the second option) how they argued that this is problematic allegedly the weakness of the common core of logics. And if each logic has its own argument, and they are embeddable (as in our case), that just amounts to having classical arguments for Logical Pluralism, which is a form of collapse into monism.

The pragmatic interpretation of the metalanguage

The worries which emerge given a plurality of logics in the metalanguage vanish whenever those logics are understood as interpreting different senses of the logical vocabulary, as it has been defended in the present thesis. It is very plausible that some of the logics presented here have been used in the metalanguage (in particular, *LK* and *LR*), while some other have been not (in particular, *LL* and *LO*). However, there is nothing *intrinsic* in these logics for being or not being used for this purpose, as this fact is just determined by the particular enrichments that the logical vocabulary has had along the dissertation. And it is precisely because it is not intrinsic in *LL* or *LO* that they should not be used in the metalanguage, or that *LR* should be used (*LK* has a special status, as we will see), that the worries expressed by Sereni and Sforza Fogliani vanish: first, although it is the case that there is one logic which should be observed along all this dissertation, which is *LK*: it has been vindicated as the logic of truth-preservation, and that gives it a predominance over the rest of the logics. However, given that we have explained why the other logics disagree with *LK*, this fact does not undermines their validity, as the second worry suggest, given that the notion of consequence that they codify does not coincide with the classical one. It is straightforward that the inferences in the metalanguage should preserve truth, but they might also be required to preserve relevance or manner whenever the vocabulary is so enriched. Hence, the rest of the logics should be also used in the metalanguage whenever the logical vocabulary in the metalanguage is pragmatically enriched, and this does not entail a minimal common core of valid inferences for defending logical pluralism. That is, it is not the case that more than one logic collapses into one, as the third worry claims, as the different logics which might be used for arguing for the different claims defended here vary from one case to another, depending on the particular enrichments in each case. In other words, we hope that each connective used in the dissertation, and each consequence derived from some premises, has been used in accordance with its enrichments.

7.2 Specific Objections to Pragmatic Logical Pluralism

7.2.1 Information transmission

The inferential role of a logical constant depends on the specific enrichment the connective has, and given that the enrichments can vary from one speaker to another, we need to answer the problem of information transmission that this version of pluralism has.

The problem affects the distinction between extensional and intensional connectives in the different substructural logics independently of the interpretation we give to them (that is, semantic or pragmatic) and was formulated by Burgess [1984]. It focuses on disjunction in *LR*, but the objection can be extended to the other connectives. Consider the following situation:

> By the regulation of a certain government agency, a citizen C is entitled to a pension if and only if C either satisfies certain age requirements or satisfies certain disability requirements. An employee E of the agency is presented with documents establishing that C is disabled. E transmits to fellow-employee F the information that C is entitled to a pension (i.e., is either aged or disabled). F subsequently receives from another source the information that C is not aged, and concludes that C must be disabled. [Burgess, 1984, p.218][10]

The problem is that E and F seem to reason correctly, and with the same connective, but the rules of their reasoning are for different disjunctions in *LR*. Let A be 'C is disabled' and B 'C is aged'. We can formalize the argument of E and of F as follows:

$$\text{E: } \frac{A}{A \sqcup B}\sqcup\text{I} \qquad\qquad \text{F: } \frac{A + B \qquad \text{not } B}{A}+\text{E}$$

Notice that both E and F seem to reason correctly, and that there is a transmission of information, i.e. 'A or B'. Hence, the 'or' seems to be the same for both, but in *LR* this

[10] Notation modified.

cannot be the case: if they are reasoning correctly then the disjunctions are different, E being extensional and F intensional.

The challenge seems to put the relevantist in a difficult position: she has to reject that the relevant connection between A and B is objective, a rather radical view of the meaning of logical constants (a view endorsed by Paoli):

> To the charge that this would make the distinction between fission [$+$] and join [$\sqcup$] subjective, and thus psychological or epistemological rather than logical, I just reply that from my perspective the grounds we have for asserting a sentence are constitutive of its meaning. [Paoli, 2007, p.567]

From the present pluralist perspective, Paoli's view about meaning does not follow. We can say both that (i) E and F are both using 'or' in natural language, whose literal meaning is captured by the classical $\vee$, so there is no change of connective in the transmission of information; and that (ii) E and F *should* reason differently with the disjunction: there is a change from $\sqcup$ to $+$ when the information is transmitted. The grounds to assert 'A or B' are different and subjective, and hence, the inferential role of the disjunction is different for E and F, while the truth conditions of the 'or' are the same, and captured by $\vee$.

In effect, E would violate a Relation maxim if she reasoned with the Disjunctive Syllogism: if, on the one hand, she discovers that $\neg A$ (that is, that C is not disabled), she should retract from 'A or B', and should also inform F about this. On the other hand, if she discovers that $\neg B$ (that is, that C is not aged), that would not be relevant for her inference of C being entitled to a pension, as she already knew that C is disabled, and that was enough. The situation is different for F: when she receives the information that 'A or B' from E, her grounds for the disjunction are not one of the disjuncts, but the knowledge that one of them is the case. Hence, it is legitimate for F to reason with the Disjunctive Syllogism. If, for instance, F sees C and checks that he is not aged, she must infer that C is disabled, and the inference would be, in her situation, completely legitimate.

We can define a similar situation with manner enrichment and linear logics. Consider the following situation: one is entitled to a compensation if one falls in the street and *then* gets injured. An agency is evaluating some cases, and the employee E sees that there is a citizen C who both fell and got injured in the street (she does not check in which order), and passes this information to F.

We can formalize the situation as follows:[11]

[11]In order to simplify we will formalize the situation entirely in *LO*, although this is probably a case of mixed inference.

E: $\dfrac{A \qquad B}{A \sqcap B}$ F: $\dfrac{A \otimes B \multimap\!\bullet\, C \qquad A \otimes B}{C}$

In this case F understands or derives the enrichment that A and B happened in the required order, so she accepts the compensation for citizen C. However, E is not legitimised to infer this, as the conjunction for her is not the linear one, and should check before whether the to events happened in the required order.

In general, when there is a transmission of information, even if the speaker asserts something with an extensional version of a connective, the receiver usually translates it using the intensional version, given that the grounds for asserting the utterance are not transmitted, and the enrichment is derived. One exception would be the Dutchman conditionals or Dutchman disjunctions, as the grounds are sufficiently clear for the hearer and the enrichment is not derived.

7.2.2 Why do enrichments restrict rather than expand?

The pragmatic enrichment of any utterance has the virtue of conveying more information than the literal meaning of the utterance. However, we have seen how substructural languages, which are pragmatically enriched, *restrict* rather than *expand* what one can infer from them. This seems to go against the very idea of pragmatic enrichment and requires some clarification[12].

Let us first draw a distinction between the enrichment of senses and the enrichment of consequences. It is not the case that if a sense (of a logical constant or of any other term in natural language) is enriched then the set of consequences which follows from it is expanded (consider for instance the inferential role that 'red' has, in comparison with the inferential role that 'crimson' has: it is clear that some consequences that one can derive from 'a is red' are invalid for 'a is crimson').

But in any case, and focusing on the enrichment of logical vocabulary, what the objection points to is the fact that substructural logics are weaker logics than classical logic. For instance, while permutation is valid for $\wedge$ in LK, it is valid for $\otimes$ in LO:

- $A \wedge B \vdash B \wedge A$
- $A \otimes B \nvdash B \otimes A$

However, once we fix a notion of consequence, it is not the case that non-enriched connectives have more consequences than their enriched counterparts. They just validate different inferences. Consider, for instance, Disjunctive Syllogism in LR:

[12] I want to thank Pablo Cobreros and Mar Alloza for observations that motivate this objection.

- $A \sqcup B, \neg A \nvdash B$
- $A + B, \neg A \vdash B$

Hence, let us then answer the following two questions in order to clarify the connection between enrichments and restrictions of consequence: (i) Why do substructural consequences restrict the notion of consequence?; (ii) fixing on a consequence relation, why do certain pragmatic enrichments restrict/expand with respect to the non-enriched counterpart?

First, as was explained in chapter 4, the rejection of structural rules from the full-structural logic *LK* enriches the sense of 'follows from'. And whenever it is enriched, certain structural rules are shown to be unacceptable. Substructural notions of 'follows from' restrict their consequence relation because the classical notion validates *too much* when it is pragmatically enriched.

Second, the intensional and the extensional connectives were introduced by distinguishing different reasons for asserting that connective, depending on whether they violated certain Gricean maxims. It is not surprising that once the consequence relation is also enriched there are certain utterances which follow from the E-reasons for asserting the connective (consider the examples in the previous chapter when discussing the paradoxes of the material conditional) and that there are other utterances which follow from the I-reasons for asserting it.

In other words, it is not the case that enriched senses of the connectives restrict what follows from them, but that substructural senses of 'follows from' restrict what follows from what. And whenever a consequence relation is fixed, it is not the case that intensional connectives are weaker than the extensional ones, as they just validate different inferences.

7.2.3 Is a connective enriched iff it is captured by a substructural logic?

The following concern is related to the direction of the connection between pragmatic enrichment and substructural logics, and runs as follows: does a logical connective expresses a pragmatic content iff it is captured by a substructural logic?[13] Let us see the details for each direction of the biconditional.

If a connective expresses a pragmatic enrichment then it is captured by a substructural language We argue that this direction of the biconditional is, in general, false. There are pragmatic enrichments which are not captured by substructural logics. On the one

[13] Question raised by Greg Restall.

hand, certain pragmatic content can be encoded by classical logic: consider the exclusive disjunction $\veebar$, which expresses that a disjunction is true, but it is not the case that both disjuncts are true together. The natural language particle 'or' in natural language many times expresses $\veebar$ rather than $\vee$, and the distinction between the two disjunctions is pragmatic (one can easily cancel the exclusive reading of 'or'). However, this sense of disjunction can be formalised in the full-structural logic *LK*, as follows:

- $A \veebar B :_{def} (A \vee B) \wedge \neg(A \wedge B)$

On the other hand, given the great variety of pragmatic enrichments that natural language can express, it is reasonable to think that many of them are not captured by any substructural language.

If a connective is captured by a substructural language then it expresses some pragmatic enrichment We argue that this direction is true for the substructural logics endorsed in this dissertation, and although we cannot commit to the general application of Pragmatic Logical Pluralism to all substructural logics (that is, the claim that for each substructural relation $\vdash$ there is a pragmatic interpretation for it and its language), we are sympathetic to the claim that the present proposal can be extended to other combinations of structural rules, and that there can be contexts in which other logics, different to the four embraced here are better suited. This expansion is open to further investigation.

There are at least two paths for expanding our research to other substructural logics, the first being more promising than the second.

Other combinations rejecting the same structural rules In chapters 4 and 5 we have learned how each structural rule affects the notion of consequence and the logical vocabulary: weakening adds irrelevancies in the sequent, contraction avoids the reaction on the premises, and exchange allows to ignore the order of the formulas. Moreover, we have considered these four logics to be related hierarchically: classical logic preserves truth, relevant logic preserves truth plus relevance, and linear logics also preserve manner. However, one might argue that the maxims are not hierarchical and that a connective might observe some manner maxim while flouting a relevant one.

Consider for instance *LK* without contraction, Affine logic, *LA*[14]: one possible interpretation of such logic is a connection between premises and conclusion in which the relation maxim might be flouted (there is not necessarily a relevant connection between premises and conclusion as seen for *LR*) but in which there might be nevertheless a reaction on the premises given the conclusion (as in *LL* and *LO*). It seems reasonable to

[14] I want to thank Elia Zardini for motivating this possibility in informal conversation.

imagine that such logic has also a pragmatic interpretation of the vocabulary, and we can imagine certain utterances and derivations in natural language in which both the affine notion of consequence, and its required enriched vocabulary, grasp certain pragmatic enrichments which the logics embraced in this dissertation fail to capture.

Rejecting other structural rules Another path in which our investigation could be expanded is rejecting two rules that have remained valid across all logics considered: Cut and the Axiom.

However, a first difficulty that we find in trying to expand the project in this direction is related a fact that we have learned in Chapter 2: we have seen a substructural logic, ST (by Cobreros et al. [2012] and discussed by Barrio et al. [2018]), which does not differ from LK in its set of valid inferences, although it diverges from LK in the meta-inferences. This makes the logical connectives behave as the classical ones (at least in the object language), and hence this might be a good counterexample to the general claim stated by this direction of the biconditional. In any case, we should find divergences in the behaviour of the logical connectives used in the metalanguage, which might be an interesting phenomena to explore.

7.2.4 Explicit enrichment of the vocabulary

There are cases in which the enrichment of the logical vocabulary is semantisiced and explicit. For instance, one can say 'they married and *then* they had a child'. This possibility of making explicit some particular content associated to the conjunction shows that the genuine conjunction does not include such content. That would reduce our pluralism to logical monism. [15]

However, it is clear that we can make explicit all the particular enrichments that are codified by substructural logics; and this fact is not different to the claim that we can cancel them without contradiction: it is a particularity of them being pragmatic rather than semantic.

One might wonder whether one should formalize 'and' with the classical $\wedge$ or the ordered $\otimes$ in those cases in which the order is not codified with the 'and' but with some additional particle in the sentence. A natural response to this is that the ordered $\otimes$ codifies 'and then' in these cases, so it should formalize the whole expression whenever the enrichment is explicit.

A similar diagnostic affects those cases in which one makes explicit the enrichment in LL saying 'if you have 5 Euros, you can get *both* A and B'. It seems that the 'and' in this

[15] I want to thank to Graham Priest, Ole Hjortland, Andrea Sereni, Sebastiano Moruzzi, and Jessica Wilson for a discussion that motivates this objection.

case is not enriched and the content that the linear $\otimes$ express is something different to the genuine 'and'. However, if we eliminate the particle 'both' in the expression, the same information is now codified in the 'and'. Hence, it seems clear that in both cases 'and' should be formalised with $\otimes$.

7.2.5 This is not a version of Logical Pluralism

Another objection to this project, and related to the previous objection, is the claim that it does not defend a genuine version of logical pluralism.[16] One reason to believe that this is not a version of logical pluralism is that the literal meaning of the logical constants is captured by classical logic and that classical logic is the only logic that captures 'literal'-truth preservation. A first reaction to such criticism is that whether this is a version of logical pluralism or not is a terminological question. However, there are reasons to believe that it is.

First, it is not entirely clear how a monist position can hold the view that among several logics, the only correct one (if the objection goes in the direction that the present proposal is a form of classical monism) is the logic with the *least* cases in which it *should* be used. Second, it makes compatible some rival logics, that is, it makes compatible $\Gamma \vdash \Delta$ and $\Gamma \nvdash \Delta$. Third, this compatibility is possible given different inferential roles for the same logical vocabulary, capturing an intuitive idea that there are inferences that are somehow correct and incorrect (as explosion), in the following intuitive notion of 'correctness' of a logic already seen in the Introduction:

> [A] logical system is correct if the formal arguments which are valid in that system correspond to informal arguments which are valid in the extra-systematic sense, and the wffs which are logically true in the system correspond to statements which are logically true in the extra-systematic sense. The monist holds that there is a unique logical system which is correct in this sense, the pluralist that there are several. [Haack, 1978, p.222]

Hence, if we succeed in explaining that the plurality of codifications of 'follows from' captured by $\vdash_K$, $\vdash_R$, $\vdash_L$, and $\vdash_O$, and also in justifying that the formalizations of the logical vocabulary in these different logics capture the normative role that logical connectives have in a correct reasoning, we have defended a version of Reasoning Logical Pluralism presented in the Introduction.

[16]Thanks to Graham Priest for an observation that motivates this objection.

CHAPTER 8

Conclusion

Our aim in this dissertation was to explain how different logics can be reconciled. In particular, we wanted to show that the formal languages of classical and three substructural logics (relevant, linear and ordered logic) codify different senses of logical consequence, and that their formalization of the logical connectives correctly codifies their inferential role, given that these weaker logics codify different pragmatic enrichments of logical vocabulary. Our guiding intuition was that there is more than one logic which should guide our reasoning.

Our first task has been to put forward the different attempts to endorse more than one correct logic, that is, to present the different versions of logical pluralism. First, we distinguished levels from versions of logical pluralism, and, on the one hand, we distinguished three different levels in which the pluralist thesis can be defended: a plurality of pure logics, a plurality of logics applied to different domains, and a plurality of logics for reasoning. On the other hand, we presented and classified some of the main attempts in the literature to defend a pluralist view on logic, distinguishing consequence-related logical pluralism and language-related logical pluralism. With the presentation of each version we have established which is the level in which the version of pluralism can be situated, and what are the main challenges that the version faces.

Finally, we have introduced Pragmatic Logical Pluralism. We have introduced the key ideas that have been developed in the following chapters: there is more than way of understanding the logical vocabulary, both the notion of logical consequence and logical constants. The aim of this short presentation in this first chapter is twofold: first, we want to connect the view with the rest of the presentations in the literature related to it. Second, we aim to introduce the intuition that guides the pluralist thesis, and which motivates the following chapter.

The possibility of endorsing more than one formalization for logical connectives is better studied in the tradition of the philosophy of language rather than in the philosophy of logic. In effect, turning our attention to recent debates on the division between semantics and pragmatics we found that the phenomena of pragmatic enrichment could very well explain how one and the same logical connective can vary its inferential role and even its truth conditions without implying a change of its meaning. All we needed to do is to put into work this idea and give a systematic explanation of why pragmatic enrichments could drive us to a plurality of logic.

The Gricean theory has been presented, but rejected or refined for a better version which distinguishes implicatures from pragmatic enrichments. These can be the phenomena that the logical connectives in substructural logics codify. What we need to develop is the particular mechanisms that explain such enrichments, and a general principle has been suggested: a connective is enriched depending on the reasons one has for using it in an utterance. This connects with the following two chapters, which will show how the different connectives in the four logics endorsed can be distinguished by the different reasons for introducing them.

With the ideas from the philosophy of language tradition in hand, we have justified that there is more than one legitimate formal language to encode logical vocabulary, and in particular that the four different logics endorsed in the dissertation diverge on the codification of certain pragmatic enrichments. In order to motivate this pluralist claim, we first need a notion of consequence for which there can be more than one instance and which depends on different legitimacies of deduction. The core definition of consequence is:

> Δ is a logical consequence$_x$ of Γ iff there is a deduction of Δ from Γ by a finite number of legitimate$_x$ rules of inference.

Now, the pluralist thesis emerged from the plurality of legitimate deduction rules for one language. A first and natural candidate for being considered *legitimate* is truth preservation,

> *LK*: a rule of inference from Γ to Δ is legitimate$_x$ if it preserves truth.

We have argued that all structural rules are valid for codifying this notion of consequence. It has been argued, moreover, that the truth-preserving connection requires a minimal or literal meaning of logical connectives, and so we have argued that this is captured by the classical codification of the constants. Moreover, given that substructural connectives are presented as encoding different reasons for introducing *a* logical connective, classical connectives fix this reference, and hence substructural connectives are presented as distinguishing reasons to assert classical connectives.

A second legitimate criterion for the legitimacy in the definition of logical consequence is the derivation of the consequence operating with the premises, giving rise to the relevant notion of consequence:

LR: a rule of inference from Γ to Δ is legitimate$_x$ if operating with Γ we get Δ.

Whenever logical consequence is understood relevantly, weakening is shown to be invalid. We saw that weakening is connected to relation enrichments, and we could justify that both logical consequence and the logical connectives in relevant logic are enriched with pragmatic information codified with the help of Relation and Quantity Gricean maxims.

A similar task has been developed for linear logic and ordered logic, whose legitimacy has been interpreted as follows,

LL/*LO*: a rule of inference from Γ to Δ is legitimate$_x$ if it uses Γ to get Δ.

With this idea of consequence in mind, we have justified that, in addition to rejecting weakening, contraction should be rejected for linear logic, and also exchange for ordered logic. We have justified the intuition behind both logics as a pragmatic enrichment of the logical connectives related to the Manner maxims.

Lastly, we wanted to conclude the dissertation with some virtues and objection. First, we have reviewed some important insights of Pragmatic Logical Pluralism, with the application of the theory to three different paradoxes of Classical Logic. The pragmatic interpretation in terms of enrichment of the logical vocabulary in the different substructural logics, which keeps the literal meaning encoded by classical logic, can naturally explain some of the problems raised against this logic. In particular, we have the necessary keys to argue that some defective inferences are truth-preserving, and at the same time, their counterintuitive reasoning can be naturally explained through pragmatic enrichments. Our pluralism gives good resources for analyzing these arguments. Note that these virtues are so not because (or *just* because) the different logics embraced here can give natural answers to some philosophical problems, but because the pragmatic interpretation (together with its pluralist consequence) can solve some problems that the monists proposals have.

First, the paradoxes of the material conditional have a very natural explanation. We could explain a multitude of counterintuitive inferences using the material conditional of classical logic, arguing that while the paradoxes of the conditional are truth-preserving, they do not capture our intended meaning of 'if...then' which is usually pragmatically enriched. A particularly interesting application was the solution to McGee counterexample to Modus Ponens.

Second, the application of the present pluralist thesis, and in particular the pragmatic interpretation of linear logic to deontic logic can also shed some light on the phenomenon of Free Choice Permission, interpreting the 'or' in the paradox as non-enriched conjunction in a linearly enriched framework.

Finally, we concluded the chapter with an analysis of the Lottery and the Preface paradox, focusing first on the interpretation of the paradoxes in the substructural literature, which illuminated our own diagnostic on the paradox, based on the pragmatic enrichment that the universal quantifier can manifest.

We have raised some objections against logical pluralism in general and against Pragmatic Logical Pluralism in particular, and from the answer to some of these objections we have learned about further developments and investigations that might depart from the connection between substructural logics and pragmatic enrichments of the logical connectives.

In particular, we have explored the plurality of inferential roles of the logical connectives. Given the plurality of consequence relations that this dissertation suggests for the logical connectives, and that it is argued that all logics range over the same logical connectives, we needed to give a satisfactory response to the question about how to reason, outside any particular framework, with a logical constant. And we gave a multiplicity of bridge principles that connect the plurality of logics with a plurality of norms for reasoning. With this idea in hand, we were also capable of answering some further problems that have generated controversy in the pluralist literature, such as the Collapse problem. And Pragmatic Logical Pluralism has also been shown to be capable of giving a satisfactory answer to the meaning variance objection, which finds a natural answer given the particular view on the meaning of logical connectives defended in the dissertation. Related to these problems, we have discussed a further related objection, which gives rise to interesting developments of the Pragmatic Logical Pluralism, which is the mixed inference problem: given that in natural language most utterance will be formalised with a combination of connectives from different logics, it is interesting to see how the different logics embraced here can be combined. Finally, we have answered the problem about metalanguage, that is, about the selection of one or more than one logics as the correct one to reason about logical pluralism.

At last, we have concluded with some particular objections raised against the particular view on logical pluralism defended in the dissertation, and related to the possible objections that one can raise against the pragmatic interpretation of logical vocabulary in substructural logics. We hope to have answered some of the most important reactions that one can have against this view.

A note for monists

Those who might not be convinced about the arguments for pluralism presented in this dissertation might still be convinced that the disagreement between the logics presented here can be explained pragmatically: consider for instance a monist relevant logician. First, she might disagree with the claim that classical logic is a correct logic, but this thesis can better clarify a new perspective on the place were to find the disagreement with classical logicians: certain contents which are semantic for the relevant logician (i.e. the connection between antecedent/consequent, conjuncts, or disjuncts) is observed by the classical logician as pragmatic, and not encoded by the logical vocabulary. Second, she might believe that linear logics are not genuine logics and, following the argument of this dissertation, consider that the disagreement with these logics is pragmatic, in the sense that certain content that the relevant logician considers being pragmatic rather than semantic, and which should be not captured by logical vocabulary, is captured by these logics, and this is where the divergence emerges.

Appendices

A. Notation

Notation for Relevant Logic

Negation	$\neg$
Extensional conditional	$\rightsquigarrow$
Intensional conditional	$\rightarrow$
Extensional disjunction	$\sqcup$
Intensional disjunction	$+$
Extensional conjunction	$\sqcap$
Intensional conjunction	$\times$

Notation for Linear Logic

Negation	$^{\perp}$
Extensional conditional	$\rightsquigarrow_L$
Intensional conditonal	$\multimap$
Extensional disjunction	$\sqcup_L$
Intensional disjunction	$\oplus$
Extensional conjunction	$\sqcap_L$
Intensional conjunction	$\otimes$

Notation for Ordered Logic

Post negation	$-^{\perp}$
Retro negation	$^{\perp}-$
Extensional conditional	$\rightsquigarrow_O$
Post conditional	$\multimapdot$
Retro conditional	$\multimapdotinv$
Extensional disjunction	$\sqcup_O$
Intensional disjunction	$\oplus$
Extensional conjunction	$\sqcap_O$
Intensional conjunction	$\otimes$

B. Calculus

Classical Logic *LK*

$$\frac{}{A \vdash A}$$

$$\frac{\Gamma \vdash \Delta, A \qquad A, \Gamma \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ \text{Cut}$$

$$\frac{\Gamma, A, B, \Gamma' \vdash \Delta}{\Gamma, B, A, \Gamma' \vdash \Delta}\ EL \qquad \frac{\Gamma \vdash \Gamma', A, B, \Delta}{\Gamma \vdash \Gamma', B, A, \Delta}\ ER$$

$$\frac{\Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}\ WL \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A}\ WR$$

$$\frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}\ CL \qquad \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}\ CR$$

$$\frac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta}\ \neg L \qquad \frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A}\ \neg R$$

$$\frac{A, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}\ \wedge L_1 \qquad \frac{B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}\ \wedge L_2 \qquad \frac{\Gamma \vdash \Delta, A \qquad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}\ \wedge R$$

$$\frac{A, \Gamma \vdash \Delta \qquad B, \Gamma \vdash \Delta}{A \vee B, \Gamma \vdash \Delta}\ \vee L \qquad \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \vee B}\ \vee R_1 \qquad \frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \vee B}\ \vee R_2$$

$$\frac{\Gamma \vdash \Delta, A \qquad B, \Gamma' \vdash \Delta'}{A \supset B, \Gamma, \Gamma' \vdash \Delta, \Delta'}\ \supset L \qquad \frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \supset B}\ \supset R$$

Table 1: Classical Logic

Relevant Logic *LR*

$$\frac{}{A \vdash A}$$

$$\frac{\Gamma \vdash \Delta, A \qquad A, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ \text{Cut}$$

$$\frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}\ CL \qquad \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}\ CR$$

$$\frac{\Gamma, A, B, \Gamma' \vdash \Delta}{\Gamma, B, A, \Gamma' \vdash \Delta}\ EL \qquad \frac{\Gamma \vdash \Delta, A, B, \Delta'}{\Gamma \vdash \Delta, B, A, \Delta'}\ ER$$

$$\frac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta}\ \neg L \qquad \frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A}\ \neg R$$

$$\frac{\Gamma \vdash \Delta, A \qquad B, \Gamma \vdash \Delta}{A \rightsquigarrow B, \Gamma \vdash \Delta}\ \rightsquigarrow L \qquad \frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, A \rightsquigarrow B}\ \rightsquigarrow R_1 \quad \frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \rightsquigarrow B}\ \rightsquigarrow R_2$$

$$\frac{\Gamma \vdash \Delta, A \qquad B, \Gamma' \vdash \Delta'}{A \to B, \Gamma, \Gamma' \vdash \Delta, \Delta'}\ \to L \qquad \frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \to B}\ \to R$$

$$\frac{A, \Gamma \vdash \Delta \qquad B, \Gamma' \vdash \Delta'}{A + B, \Gamma, \Gamma' \vdash \Delta, \Delta'}\ +L \qquad \frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A + B}\ +R$$

$$\frac{A, \Gamma \vdash \Delta \qquad B, \Gamma \vdash \Delta}{A \sqcup B, \Gamma \vdash \Delta}\ \sqcup L \qquad \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \sqcup B}\ \sqcup R_1 \quad \frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \sqcup B}\ \sqcup R_2$$

$$\frac{A, B, \Gamma \vdash \Delta}{A \times B, \Gamma \vdash \Delta}\ \times L \qquad \frac{\Gamma \vdash \Delta, A \qquad \Gamma' \vdash \Delta', B}{\Gamma, \Gamma' \vdash \Delta, \Delta', A \times B}\ \times R$$

$$\frac{A, \Gamma \vdash \Delta}{A \sqcap B, \Gamma \vdash \Delta}\ \sqcap L_1 \quad \frac{B, \Gamma \vdash \Delta}{A \sqcap B, \Gamma \vdash \Delta}\ \sqcap L_2 \qquad \frac{\Gamma \vdash \Delta, A \qquad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \sqcap B}\ \sqcap R$$

Table 2: Relevant Logic

Linear Logic *LL*

$$\frac{}{A \vdash A}$$

$$\frac{\Gamma \vdash \Delta, A \qquad A, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ \text{Cut}$$

$$\frac{\Gamma, A, B, \Gamma' \vdash \Delta'}{\Gamma, B, A, \Gamma' \vdash \Delta'}\ EL \qquad \frac{\Gamma \vdash \Delta, A, B, \Delta'}{\Gamma \vdash \Delta, B, A, \Delta'}\ ER$$

$$\frac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta}\ \neg L \qquad \frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A}\ \neg R$$

$$\frac{\Gamma \vdash \Delta, A \qquad B, \Gamma \vdash \Delta}{A \rightsquigarrow B, \Gamma \vdash \Delta} \rightsquigarrow L \qquad \frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, A \rightsquigarrow B} \rightsquigarrow R_1 \quad \frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \rightsquigarrow B} \rightsquigarrow R_2$$

$$\frac{\Gamma \vdash \Delta, A \qquad B, \Gamma' \vdash \Delta'}{A \multimap B, \Gamma, \Gamma' \vdash \Delta, \Delta'} \multimap L \qquad \frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \multimap B} \multimap R$$

$$\frac{A, \Gamma \vdash \Delta \qquad B, \Gamma \vdash \Delta}{A \sqcup B, \Gamma \vdash \Delta}\ \sqcup L \qquad \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \sqcup B}\ \sqcup R_1 \qquad \frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \sqcup B}\ \sqcup R_2$$

$$\frac{A, \Gamma \vdash \Delta \qquad B, \Gamma' \vdash \Delta'}{A \oplus B, \Gamma, \Gamma' \vdash \Delta, \Delta'}\ \oplus L \qquad \frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \oplus B}\ \oplus R$$

$$\frac{A, \Gamma \vdash \Delta}{A \sqcap B, \Gamma \vdash \Delta}\ \sqcap L_1 \qquad \frac{B, \Gamma \vdash \Delta}{A \sqcap B, \Gamma \vdash \Delta}\ \sqcap L_2 \qquad \frac{\Gamma \vdash \Delta, A \qquad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \sqcap B}\ \sqcap R$$

$$\frac{A, B, \Gamma \vdash \Delta}{A \otimes B, \Gamma \vdash \Delta}\ \otimes L \qquad \frac{\Gamma \vdash \Delta, A \qquad \Gamma' \vdash \Delta', B}{\Gamma, \Gamma' \vdash \Delta, \Delta', A \otimes B}\ \otimes R$$

Table 3: Linear Logic

Ordered Logic *LO*

$$\frac{\Gamma \vdash \Delta', A, \Delta'' \qquad \Gamma', A, \Gamma'' \vdash \Delta}{\Gamma', \Gamma, \Gamma'' \vdash \Delta', \Delta, \Delta''}\ \text{Cut}$$

if $\Delta' = \Gamma'' = \emptyset$ or $\Delta'' = \Gamma' = \emptyset$ or $\Gamma' = \Gamma'' = \emptyset$, or $\Delta' = \Delta'' = \emptyset$

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma', B, \Gamma'' \vdash \Delta'}{\Gamma', B \bullet\!\!- A, \Gamma, \Gamma'' \vdash \Delta', \Delta}\ \bullet\!\!-\text{L} \qquad \frac{\Gamma, A \vdash \Delta, B}{\Gamma \vdash \Delta, B \bullet\!\!- A}\ \bullet\!\!-\text{R}$$

$$\frac{\Gamma \vdash \Delta, A \qquad \Gamma', B, \Gamma'' \vdash \Delta'}{\Gamma', \Gamma, A -\!\!\bullet B, \Gamma'' \vdash \Delta, \Delta'}\ -\!\!\bullet\text{L} \qquad \frac{A, \Gamma \vdash B, \Delta}{\Gamma \vdash A -\!\!\bullet B, \Delta}\ -\!\!\bullet\text{R}$$

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \Delta, {}^{\perp}A}\ {}^{\perp}-\text{L} \qquad \frac{\Gamma \vdash A, \Delta}{{}^{\perp}A, \Gamma \vdash \Delta}\ {}^{\perp}-\text{R}$$

$$\frac{A, \Gamma \vdash \Delta}{\Gamma \vdash A^{\perp}, \Delta}\ -^{\perp}\text{L} \qquad \frac{\Gamma \vdash \Delta, A}{\Gamma, A^{\perp} \vdash \Delta}\ -^{\perp}\text{R}$$

$$\frac{\Gamma, A, \Gamma' \vdash \Delta \qquad \Gamma, B, \Gamma' \vdash \Delta}{\Gamma, A \sqcup B, \Gamma' \vdash \Delta}\ \sqcup\text{L} \qquad \frac{\Gamma \vdash \Delta, A, \Delta'}{\Gamma \vdash \Delta, A \sqcup B, \Delta'}\ \sqcup\text{R} \quad \frac{\Gamma \vdash \Delta, B, \Delta'}{\Gamma \vdash \Delta, A \sqcup B, \Delta'}\ \sqcup\text{R}$$

$$\frac{\Gamma, A, \Gamma' \vdash \Delta \qquad \Gamma'', B, \Gamma''' \vdash \Delta'}{\Gamma'', \Gamma, A \oplus B, \Gamma''', \Gamma' \vdash \Delta, \Delta'}\ \oplus\text{L} \qquad \frac{\Gamma \vdash \Delta, A, B, \Delta'}{\Gamma \vdash \Delta, A \oplus B, \Delta'}\ \oplus\text{R}$$

if $\Gamma' = \Gamma'' = \emptyset$ or $\Gamma' = \Delta = \emptyset$ or $\Gamma'' = \Delta' = \emptyset$

$$\frac{\Gamma \vdash \Delta, A, \Delta' \qquad \Gamma' \vdash \Delta'', B, \Delta'''}{\Gamma, \Gamma' \vdash \Delta'', \Delta, A \otimes B, \Delta''', \Delta'}\ \otimes\text{R} \qquad \frac{\Gamma, A, B, \Gamma' \vdash \Delta}{\Gamma, (A \otimes B), \Gamma' \vdash \Delta}\ \otimes\text{L}$$

if $\Delta' = \Delta'' = \emptyset$ or $\Delta' = \Gamma = \emptyset$ o $\Delta'' = \Gamma' = \emptyset$

$$\frac{\Gamma \vdash \Delta, A, \Delta' \qquad \Gamma \vdash \Delta, B, \Delta'}{\Gamma \vdash \Delta, A \sqcap B, \Delta'}\ \sqcap\text{R} \qquad \frac{\Gamma, A, \Gamma' \vdash \Delta}{\Gamma, A \sqcap B, \Gamma' \vdash \Delta} \quad \frac{\Gamma, B, \Gamma' \vdash \Delta}{\Gamma, A \sqcap B, \Gamma' \vdash \Delta}\ \sqcap\text{L}$$

Table 4: Ordered Logic

C. Reasons (classified by logic)

Natural Language	Reasons to assert	*LK*	*LR*
if A then B	not A / B	$\supset$	$\rightsquigarrow$
	given A, B		$\rightarrow$
A or B	A / B	$\vee$	$\sqcup$
	the rejection of A entails B, or viceversa		$+$
not (A and B)	not A / not B	$\wedge$	$\sqcap$
	incompatibility between A and B		$\times$
if (A and B) then C	if A then C / if B then C	$\wedge$	$\sqcap$
	both A and B are required for C		$\times$

Table 5: Reasons in Relevant Logic

Natural Language	Reasons to assert	*LK*	*LL*
if A then B	not A / B	$\supset$	$\rightsquigarrow_L$
	using A one gets B		$\multimap$
A or B	A / B	$\vee$	$\sqcup_L$
	avoiding A one gets B, or viceversa		$\oplus$
if (A and B), then and C	if A then C / if B then C	$\wedge$	$\sqcap_L$
	if both A and B, then C		$\otimes$
if A then (B and C)	if A then (B and C) (any)	$\wedge$	$\sqcap_L$
	if A then (B and C) (both)		$\otimes$

Table 6: Reasons Linear Logic

Natural Language	Reasons to assert	*LK*	*LO*
	if C, not A / if C, B		$\rightsquigarrow_O$
if C, if A then B	using A (after C) one gets B	$\supset$	$-\!\bullet$
	using A (before C), one gets B		$\bullet\!-$
	A / B		$\sqcup_O$
if C, A or B	avoiding A (before C) one gets B	$\vee$	$\oplus$
	avoiding B (after C) one gets A		
if (A and B) then C	if A then C / if B then C	$\wedge$	$\sqcap_O$
	if A and B (in this order) then C		$\otimes$
if C then (A and B)	if C then A and B (any)	$\wedge$	$\sqcap_O$
	if C then A and B (both, in this order)		$\otimes$

Table 7: Reasons in Ordered Logic

D. Reasons (classified by connective)

Conditional

Given Γ, if A then B

- RE-reasons: Γ gives reasons to assert $\neg A$ / Γ gives reasons to assert B.
- LE/OE-reasons: using Γ one avoids A / using Γ one gets B.
- RI-reasons: given Γ, A cannot be true without B being true.
- LI-reasons: given Γ, using A we get B.
- OI-reasons: given Γ, using A (after Γ), we get B.
- OI-reasons: given Γ, using A (before Γ), we get B.

Disjunction

Given Γ, A or B

- RE-reasons: Γ gives reasons for asserting A / Γ gives reasons for asserting B,
- LE/OE-reasons: using Γ one gets A / using Γ one gets B,
- RI-reasons: given Γ there is a connection between A and B such that the negation of one disjunct implies the other.
- LI-reasons: given Γ there is a connection between A and B such that using $A^{\perp}$ (or avoiding A) we get B as a result.
- OI-reasons: given Γ, using the avoidance of A (before Γ), we get B / using the avoidance of B (after Γ), we get A.

Conjunction

Given Γ, not A and B

- RE-reasons: from Γ we derive not A / from Γ we derive not B
- LE/OE-reasons: using Γ we avoid A / using Γ we avoid B
- RI-reasons: from Γ we derive that there is some connection between A and B that makes them incompatible: i.e. one conjunct excludes the other, A entails $\neg B$.

Given A and B, then Δ

- RE-reasons: A implies Δ / B implies Δ,
- LE/OE-reasons: using A one gets Δ / using B one gets Δ,
- RI-reasons: both A and B are required to derive Δ.
- LI-reasons: A and B together are used to get Δ.
- OI-reasons: using both A and B, in this order, we can infer Δ.

Given Γ, A and B

- RE-reasons: from Γ one can derive *any* among A and B
- LE/OE-reasons: using Γ one gets *any* among A and B
- LI-reasons: from Γ one can get *both* A and B
- OI-reasons: from Γ one can get *both* A and B in this order

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