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The Logic of Turing Progressions

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The Logic of Turing Progressions

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Behold the perfect pantheon of absence.

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Abstract

This dissertation is devoted to developing modal logical tools that can be used in the field of proof theory and ordinal analysis. More precisely, we focus on the relation between strictly positive modal logics and both Turing progressions and ordinal notation systems.

With respect to the former one, we introduce the system **TSC** that is tailored to generate exactly all relations that hold between different Turing progressions given a particular set of natural consistency notions. We also present an arithmetical interpretation for this modal system, named the *Formalized Turing progressions* interpretation. The logic is proven to be arithmetically sound and complete with respect to this interpretation.

After exploring the arithmetical semantics of **TSC**, we investigate the relational semantics of this system. For this purpose, we make use of the universal model of the closed fragment of *Gödel-Löb's Polymodal Logic (GLP)*, namely *Ignatiev's universal frame*. By slightly modifying the relations defined in this model, we obtain a new frame which is proven to be a universal model for **TSC**. Moreover, we show how the domain of this frame can be reduced to sequences with finite support while keeping the completeness of the system.

As for ordinal notations systems, we present the logic **BC** (for *Bracket Calculus*). Unlike other provability logics, **BC** is based on a purely modal signature that gives rise to an ordinal notation system instead of modalities indexed by some ordinal given *a priori*. Moreover, since the order between these notations can be established in terms of derivability within the calculus, the inferences in this system can be carried out without using any external property of ordinals. The presented logic is proven to be equivalent to *Reflection Calculus (RC_{R₀})*, that is, to the strictly positive fragment of **GLP_{R₀}**.

Resumen

El objetivo de esta tesis es desarrollar herramientas de lógica modal que puedan ser utilizadas en el campo de la teoría de la demostración y el análisis ordinal. Más precisamente, nos centramos en la relación entre las lógicas modales estrictamente positivas y las progresiones de Turing, y entre dichas lógicas y los sistemas de notación ordinal que surgen de ellas.

Con respecto a la primera parte, hemos introducido el sistema **TSC**, diseñado para generar exactamente todas las relaciones válidas entre las diferentes progresiones de Turing, dado un conjunto particular de nociones de consistencia naturales. También presentamos una interpretación aritmética para este sistema modal, denominada *interpretación de las Progresiones de Turing formalizadas*. Demostramos que la lógica es aritméticamente correcta y completa con respecto a esta interpretación.

Tras de estudiar la semántica aritmética de **TSC**, investigamos la semántica relacional de este sistema. Para este propósito, hacemos uso del modelo universal para el fragmento cerrado de *Gödel-Löb's Polymodal Logic (GLP)*, a saber, el *marco universal de Ignatiev*. Modificando ligeramente las relaciones definidas en este modelo, obtenemos un nuevo marco. Demostramos que éste es un modelo universal para **TSC**. Asimismo, mostramos cómo el dominio de este marco puede reducirse a secuencias con soporte finito manteniendo la completud del sistema.

Respecto a los sistemas de notación ordinal, presentamos la lógica **BC** (por *Bracket Calculus*). A diferencia de otras lógicas de la demostrabilidad, **BC** se basa en un lenguaje puramente modal que da lugar a un sistema de notación ordinal, en lugar de estar construido mediante modalidades indexadas por algún ordinal dado *a priori*. Además, ya que el orden entre estas notaciones puede establecerse en términos de derivabilidad dentro del cálculo, las inferencias en este sistema pueden llevarse a cabo sin usar ninguna propiedad externa de los ordinales. Demostramos que la lógica presentada es equivalente al *Reflection Calculus* (\mathbf{RC}_{Γ_0}), es decir, al fragmento estrictamente positivo de \mathbf{GLP}_{Γ_0} .

Introduction

Among other interesting features, modal logic provides us with a tool to study different mathematical phenomena. Usually, a modal system presents two modal operators, \Box and \Diamond , that are intended to be read as *It is necessary that* and *It is possible that*, respectively. However, there is an extensive range of interpretations for these modal operators that yields a wide family of modal systems. According to these interpretations we can speak about temporal logic, deontic logic, doxastic logic, etc.

One of the key ingredients of this dissertations is *provability logic*. In these modal systems the regular modalities are interpreted as provability predicates and consistency statements for arithmetical theories such as Peano Arithmetic (PA). The best known provability logic is **GL** (Gödel-Löb), obtained by extending the smallest normal modal logic **K** with a modal version of a formalization of Löb's theorem. This system was later generalized by Japaridze with **GLP** $_{\omega}$; a polymodal version of **GL** in which we have modal operators for each $n \in \omega$, and later on extended to a class-sized logic by Beklemishev where we have modalities for each ordinal.

The main aspect of this line of research is that we can use simple tools like these modal systems to perform complex analyses of arithmetical theories. In particular, Beklemishev showed that the closed fragment of **GLP** $_{\omega}$ was enough to perform a Π_1^0 -ordinal analysis of PA and similar arithmetical theories. The present dissertation is in the context of this line of research. The aim of this work is to contribute to the field of provability logic by providing new modal systems that can be used as tools in the area of ordinal analysis.

The first goal of this investigation is related to *Turing progressions*. In his PhD dissertation, Alan Turing introduced what he called systems of logics based on ordinals as a way to deal with Gödel's incompleteness results. Roughly speaking, the systems of logics were obtained by iteratedly extending a formal system L with one of these non-provable statements, such as the consistency of L . This hierarchy of theories is what we now call Turing progressions and can be defined (given an ordinal representation over which our ordinals range) by the following three clauses:

- T1. $(T)^0 := T$ where T is an initial or base theory;
- T2. $(T)^{\alpha+1} := (T)^\alpha \cup \{\text{Con}((T)^\alpha)\}$;
- T3. $(T)^\lambda := \bigcup_{\beta < \lambda} (T)^\beta$, for λ a limit ordinal.

This construction can be extended to different consistency notions that can be naturally defined in weak fragments of arithmetic. This yields different progressions that, in turn, can be used as the base theory of a new hierarchy based on a different consistency notion, and so on. Facing this situation, it is therefore natural to ask which principles govern the interactions between these different Turing progressions. The contribution of our work with respect to this question is to provide a modal system capable of generating all possible relations that hold between different Turing progressions and where each modal formula can be used to directly denote one of these theories.

Steps in this direction were already taken from both arithmetical and provability logic perspectives. From the arithmetical point view, a cornerstone in this investigation is the *fine-structure formula* theorem proven by Schmerl and later generalized by Beklemishev. This result contains two conservativity principles:

$$\begin{aligned} \text{S1} \quad & \forall \alpha \succ 0 \left((T)_{n+m}^\alpha \equiv_n (T)_n^{e^m(\alpha)} \right); \\ \text{S2} \quad & \forall \beta \succ 0 \left(((T)_{n+m}^\beta)_n^\alpha \equiv_n (T)_n^{e^m(\beta) \cdot (1+\alpha)} \right). \end{aligned}$$

Item **S1** is commonly known as *the reduction property* and states that the progression built from theory T using $n + m$ -consistency up to level α has the same Π_{n+1}^0 -consequences as the progression obtained by iterating n -consistency over the same base theory up to the ω -exponential tower

$$\omega^{\omega^{\dots^{\omega^\alpha}}}$$

of height m .

The system GLP_ω provides us with a complete characterization of the n -consistency notions, and so, we can give an approximation to the reduction property and analogous principles by using (the arithmetical interpretation of) modal formulas in this setting (see [32]). For example, by a simple application of **S1** we know that

$$(\text{PA})_1^1 \equiv_0 (\text{PA})_0^\omega.$$

Therefore, by considering the corresponding arithmetical interpretation, we can express this result in the language of GLP_ω by

$$\{\langle 1 \rangle_{\text{PA}} \top\} \equiv_0 \{\langle 0 \rangle_{\text{PA}}^{k+1} \top : k < \omega\} = \{\langle 0 \rangle_{\text{PA}} \top, \langle 0 \rangle_{\text{PA}} \langle 0 \rangle_{\text{PA}} \top, \dots\}.$$

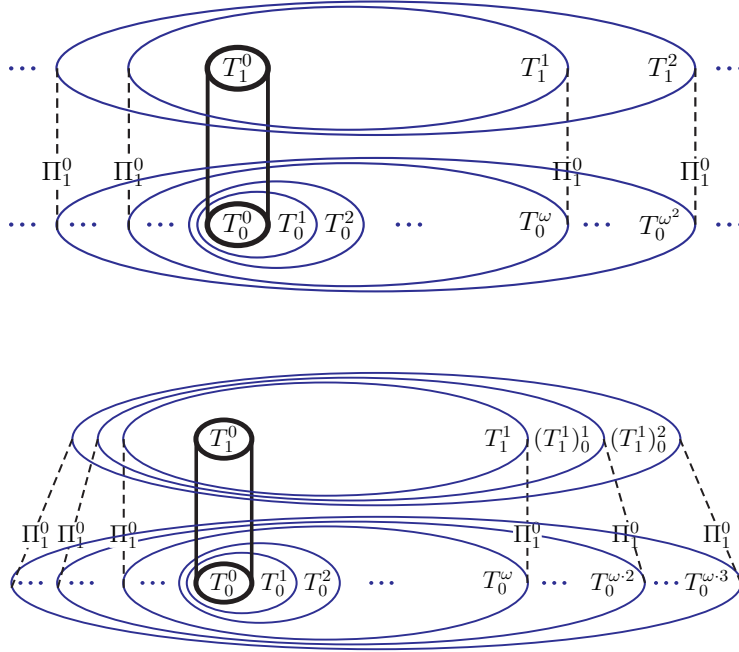


Figure 1: Graphical representations of some instances of **S1** and **S2**, respectively.

While $\langle 1 \rangle \top$ and each $\langle 0 \rangle^m \top$ for $m < \omega$ are formulas in the signature of \mathbf{GLP}_ω , $\langle 0 \rangle^\omega \top$ is not. Thus, we cannot directly denote $(\text{PA})_0^\omega$ with a single modal formula but with an infinite set of them. This way, results like the fine-structure formula can be approximated by using \mathbf{GLP}_ω but cannot be formulated as such within this modal system.

In Chapter 2 we present a way of dealing with this issue. We introduce *ordinal modalities*, modal operators of the form $\langle n^\alpha \rangle$ where $n < \omega$ and $\alpha < \Lambda$ for Λ a fixed recursive ε -number. The underlying idea of this operator is to encapsulate the iteration of the n -consistency notion up to level α . Thus, our previous example can be expressed in this setting by:

$$\langle 1^1 \rangle_{\text{PA}} \top \equiv_0 \langle 0^\omega \rangle_{\text{PA}} \top.$$

Moreover, the signature we define in this chapter is a strictly positive language (with no propositional variables), that is, we only consider formulas obtained by the following grammar: \top , conjunction and ordinal modalities. In this sense, the approach we follow is influenced by recent works papers by Dashkov and Beklemishev where they introduced the logic *Reflection Calculus* (**RC**). This system is weaker than \mathbf{GLP} , yet expressive enough to maintain its main proof-theoretic applications. More precisely, **RC** con-

sists of the fragment of **GLP** containing all implications of the form $\varphi \rightarrow \psi$ where φ and ψ are strictly positive formulas.

In general, by going to a strictly positive language, we gain a broader range of arithmetical interpretations. More precisely, the use of strictly positive formulas allows us to interpret modal formulas as arithmetical theories rather than individual arithmetical sentences. Therefore, in our setting, the modal operators are no longer conceived as consistency statements but theories that might be infinitely axiomatized.

With this handy signature, we also use Chapter 2 to introduce the system **TSC** for *Turing-Schmerl Calculus*. The derivable objects of this system are sequents of the form $\varphi \vdash \psi$ where φ and ψ are formulas in the above signature.

With all these ingredients, one can easily conjecture how the arithmetical semantics of this system are defined. First, we identified modal formulas with Turing progressions built by iterating different consistency notions over a weak fragment of arithmetic. In particular, we consider the conjunction of two formulas as the union of the respective theories. After this, the arithmetical interpretation of the sequents is defined as the (formalized) entailment between these theories. Thus, sequents are translated into arithmetical formulas expressing such entailment. This arithmetical interpretation is what we name the *Formalized Turing progressions* (FTP) interpretation. The details of this construction and the corresponding results of soundness and completeness are collected in Chapter 3.

Chapter 4 is devoted to study the relational semantics of **TSC**. To this end, we consider *Ignatiev's universal frame* for the variable-free fragment of **GLP**. The main drawback of \mathbf{GLP}_ω or any of its polymodal fragments is that it is incomplete with respect to its relational semantics. However, Ignatiev built a relational model for the closed fragment of the signature in which every consistent formula of variable-free \mathbf{GLP}_ω is satisfied [30]. This frame consists of a domain based on certain decreasing sequences of ordinals (ℓ -sequences) and a family of relations $\{T_n\}_{n < \omega}$.

In this chapter we define a modal model \mathcal{J} which is universal for our logic in the sense that any derivable sequent holds everywhere in the model whereas any non-derivable sequent is refuted by some point in the frame. This new frame is defined by preserving the domain given in Ignatiev's frame but consider a slightly different family of relations $\{R_n\}_{n < \omega}$. Apart from this, we define the auxiliary relations R_n^α to naturally model the behavior of the ordinal modalities $\langle n^\alpha \rangle$. After defining \mathcal{J} and these auxiliary relations, we use the remainder of the chapter to show the soundness and completeness results together with some useful tools about the definable sets in the frame and the characterization of the R_n^α -relations.

In the presence of ε -numbers, we can define a ℓ -sequence of the form

$\langle \alpha_0, \alpha_1, \dots, \alpha_n, \dots \rangle$ where for each α_i we have that $\alpha_i > 0$. On the one hand, these kind of sequences are not definable by means of any modal formula but, on the other hand, the branch of the frame where all these sequences live does not contain any additional information that is not present in any other part of the frame. Thus, by restricting our frame to sequences with finite support, that is, those such that eventually reach 0, we obtain a new frame for which we show that **TSC** is complete with respect to. In this new frame, every world becomes modally definable. The relevance of this feature with respect to the main aim of **TSC** is that we can use the frame as a road map of Turing progressions. Thanks to the modal definability, any sequence with finite support x can be translated to modal formula ψ_x and this formula itself can be translated into Turing progressions Th_{ψ_x} . Furthermore, by checking whether two sequences are related in the frame, we can derive an entailment principle between their corresponding Turing progressions. These results are collected in Chapter 5.

In view of Gödel's second incompleteness theorem, we know that the consistency of any sufficiently powerful formal theory cannot be established using purely 'finitary' means. Since then, the field of proof theory, and more specifically of ordinal analysis, has been successful in measuring the non-finitary assumptions required to prove consistency assertions via computable ordinals. Among the benefits of this work is the ability to linearly order natural theories of arithmetic with respect to notions such as their 'consistency strength' (e.g., their Π_1^0 ordinal) or their 'computational strength' (their Π_2^0 ordinal).

Nevertheless, the assignment of these proof-theoretic ordinals to formal theories depends on a choice of a 'natural' presentation for such ordinals, with well-known pathological examples having been presented by Kreisel [33] and Beklemishev [3]. These representation systems are what we know as *ordinal notation systems*. During my research stay in IRIT, Fernández-Duque suggested several questions regarding provability logics and the ordinal notation systems that arise from them. One of these problems is successfully solved in Chapter 6.

Some connections between ordinal notation systems and provability logics are already known and have been intensively studied (see [20], [7]). Within the closed fragment of \mathbf{GLP}_ω , there are some special formulas that are commonly known as *worms*; formulas obtained by iterating different diamonds over top. Thus, worms are formulas of the form $\langle n_0 \rangle \dots \langle n_k \rangle \top$. These formulas can be well ordered (modulo equivalence in \mathbf{GLP}_ω) according to their *consistency strength*, that is, given two worms A and B :

$$A <_0 B \quad :\iff \quad B \vdash \langle 0 \rangle A$$

Hence, it is possible to associate an ordinal $o(A)$ to any worm A . The set of \mathbf{GLP}_ω -worms provide us with an ordinal notation system for $\varepsilon_0 =$

$\sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \dots\} = \varphi_1(0)$ which measures the precise proof-theoretical strength of PA. We can go further by considering bigger fragments of **GLP** in which diamonds with transfinite ordinal entries are available. Extensions of **GLP** $_\omega$, denoted **GLP** $_\Lambda$, have been considered in cases where Λ is an ordinal [7, 19, 24] or even an arbitrary linear order [12]. Proof-theoretic interpretations for **GLP** $_\Lambda$ have been developed by Fernández-Duque and Joosten [25] for the case where Λ is a computable well-order.

In such cases, we are in need of a system of notation that can represent an ordinal $\lambda < \Lambda$. Fortunately we may ‘borrow’ this notation from finitary worms and represent λ itself as a worm. Iterating this process we obtain the *autonomous worms*, whose order types are exactly the ordinals below the Feferman-Schütte ordinal Γ_0 , that is, the smallest fixpoint of the Veblen functions.

$()$	$\langle 0 \rangle \top$	1
$(())$	$\langle 1 \rangle \top$	ω
$(()) (()) (())$	$\langle 1 \rangle \langle 2 \rangle \top$	$\omega^{\omega+1}$
$((()))$	$\langle \omega \rangle \top$	ε_0
$(()) (()) ((()))$	$\langle 1 \rangle \langle 0 \rangle \langle \omega \rangle \top$	$\varepsilon_0 + \omega$

Table 1: Some correspondences between brackets expressions, worms and ordinals.

By iterating this process we obtain a notation system for worms which uses only parentheses, as ordinals (including natural numbers) can be iteratively represented in this fashion. Thus a balanced brackets expression a can be used to denote an ordinal by first translating a into a worm A and then computing $o(A)$ (see Table 1). This is Beklemishev’s *brackets* notation system, introduced in [7], which provides a notation system for Γ_0 without any reference to an externally given ordinal.

The drawback of Beklemishev’s brackets system is that the actual computation of the ordering between different expressions is achieved via a translation into a traditional ordinal notation system. The goal of this investigation is to remove the need for such an intermediate step by providing an autonomous calculus for determining the ordering relation and, more generally, the logical consequence relation between bracket notations. To this end in Chapter 6 we present the *Bracket Calculus (BC)*.

The signature of **BC** is obtained by extending the balanced brackets grammar to a strictly positive modal signature in which we use the brackets expressions as modal operators. Therefore, the set of formulas is built from \top and propositional variables using conjunctions \wedge and modalities of the

form (a) where a is a balanced brackets expression. The main feature of this system is that, unlike other provability logics, **BC** is based on a purely modal signature that gives rise to an ordinal notation system instead of using modalities indexed by some ordinal given *a priori*. Moreover, since the order between these notations can be established in terms of derivability within the calculus, the inferences in this system can be carried out without using any external property of ordinals. To the best of our knowledge, this yields the first ordinal notation system presented as a purely modal deductive system. We prove that our calculus is sound and complete with respect to a natural translation into \mathbf{RC}_{Γ_0} .

Lastly, we would like to point out that the results contained in this dissertation have led to the following publications:

1. E. Hermo-Reyes and J. J. Joosten. The logic of Turing progressions. Accepted for publication in *Notre Dame Journal of Formal Logic*;
2. E. Hermo-Reyes and J. J. Joosten. Relational semantics for the Turing Schmerl calculus. In *Advances in Modal Logic*, volume 12, pages 327-346. College Publications, 2018;
3. D. Fernandez-Duque and E. Hermo-Reyes. A self-contained provability calculus for Γ_0 . In *Logic, Language, Information, and Computation - WoLLIC 2019*, volume 11541 of *Lecture Notes in Computer Science*, pages 195-207. Springer, 2019

Preliminaries

1.1. The Language of Arithmetic

Throughout this dissertation $\mathcal{L}_{\mathbb{N}}$ will denote the *first-order language of arithmetic*, that is, a first-order language whose non-logical symbols are the following: two binary function symbols $+$ and \cdot for addition and multiplication; two unary function symbols S and \exp , for the successor function and binary exponentiation; two binary relation symbols $=$ and \leq , for equality and “less than or equal to”; and a constant symbol 0 . The *standard model* of arithmetic is the $\mathcal{L}_{\mathbb{N}}$ -structure whose domain is the set of natural numbers \mathbb{N} and where the symbols of $\mathcal{L}_{\mathbb{N}}$ are given by their obvious interpretation.

Terms in $\mathcal{L}_{\mathbb{N}}$ are defined as usual. An occurrence of a quantifier is said to be *bounded* if it is of the form $\forall x \leq t \varphi(x)$ or $\exists x \leq t \varphi(x)$, where t is a term not containing x . These expressions are abbreviations for the formulas $\forall x (x \leq t \rightarrow \varphi(x))$ and $\exists x (x \leq t \wedge \varphi(x))$, respectively. Those formulas all of whose quantifiers are bounded are called *elementary formulas*. The set of elementary formulas is denoted by $\Delta_0^0(\exp)$.

The following definition allows us to classify arithmetical formulas according to their logical complexity. This classification is called the *arithmetical hierarchy*.

Definition 1.1.1. For $n \geq 0$, the classes of Σ_n^0 - and Π_n^0 -formulas are inductively defined as follows:

- i) $\Sigma_0^0 = \Pi_0^0 = \Delta_0^0$;
- ii) Σ_{n+1}^0 -formulas are of the form $\exists x_1 \dots \exists x_m \varphi(x_1, \dots, x_m)$ where φ is a Π_n^0 -formula. Analogously, a formula is in Π_{n+1}^0 if it is of the form $\forall x_1 \dots \forall x_m \varphi(x_1, \dots, x_m)$ where φ is a Σ_n^0 -formula.

Notice that, by the prenex normal form theorem, every formula is equivalent to a formula of the form $Qx_1 \dots Qx_m \varphi(x_1, \dots, x_m)$ where φ is quantifier-free, and so an elementary formula, and Q is either \forall or \exists for each i . Therefore, every formula is equivalent to some Σ_n^0 - or Π_n^0 -formula for some n . Naturally, these equivalences take place in some fixed theory T . Usually, T will be clear from the context. We shall sometimes call Σ_n^0 (Π_n^0) any formula that is logically equivalent to a Σ_n^0 -formula (Π_n^0 -formula).

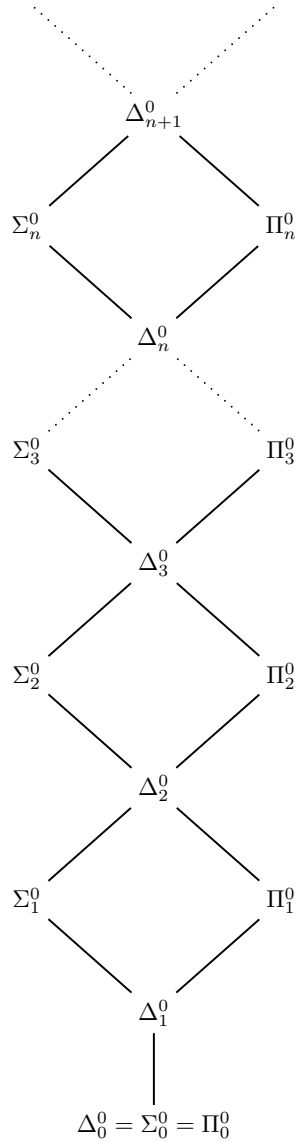


Figure 1.1: The arithmetical hierarchy.

We say that a formula is Δ_n^0 iff it is equivalent to both a Σ_n^0 -formula and a Π_n^0 -formula. As before, the theory T where these equivalences occur

will be clear. Since we might have empty blocks of quantifiers in the above definition, we obtain some obvious inclusions between these classes: $\Sigma_n^0 \subseteq \Delta_{n+1}^0 \subseteq \Sigma_{n+1}^0$ and $\Pi_n^0 \subseteq \Delta_{n+1}^0 \subseteq \Pi_{n+1}^0$ (see Figure 1.1). Moreover, we have that $\Sigma_n^0 \cup \Pi_n^0 \subseteq \Delta_{n+1}^0 \subseteq \Sigma_{n+1}^0 \cap \Pi_{n+1}^0$.

The classes Σ_n^0 and Π_n^0 are closed under \wedge and \vee . With respect to negation, if $\varphi(x)$ is a Σ_n^0 -formula, $\neg\varphi(x)$ is equivalent (in the predicate calculus) to a Π_n^0 -formula and vice-versa, $\neg\varphi(x)$ with $\varphi(x) \in \Pi_n^0$ is equivalent to a Σ_n^0 -formula. Hence, Δ_n^0 -formulas are closed under \wedge and \vee , but also under \neg .

It is worth mentioning that, from a computational perspective, the bottom of the hierarchy is particularly relevant. A set on \mathbb{N} is definable by a Σ_1^0 -formula iff it is *recursively enumerable* (r.e.). Thus, since the negation of a Σ_1^0 -formula is equivalent to a Π_1^0 formula, the Π_1^0 definable sets are the *co-r.e.* sets. The Δ_1^0 sets correspond exactly to the *recursive* sets (rec), and so, any Δ_0^0 definable set is decidable. Moreover, every Δ_0^0 definable set is primitive recursive (p.r.). In Figure 1.2 we can find a diagram of the bottom of the arithmetical hierarchy from a computational point of view.

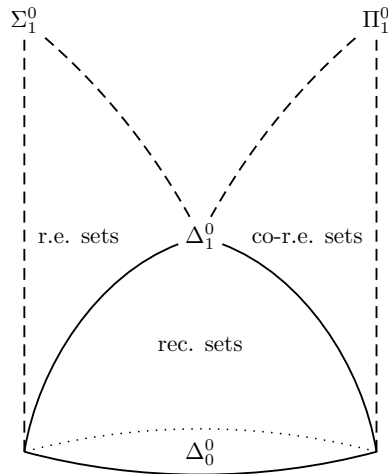


Figure 1.2: The bottom of the arithmetical hierarchy from a computational point of view.

Arithmetical theories

The need for a formal development of arithmetic was outlined during the second half of the nineteenth century in the work of Hermann Grassmann, Charles Sanders Peirce and Richard Dedekind among others. However, it was not until 1889, with the publication of *Arithmetices principia, nova methodo exposita* by Giuseppe Peano, that formal arithmetic began to take its current shape.

In his seminal work, Peano introduced his collection of principles for natural-number arithmetic that we nowadays know as *Peano Arithmetic* PA. In this subsection we shall introduce the axioms of PA together with some other weaker theories, all of them formulated in the arithmetical language $\mathcal{L}_{\mathbb{N}}$.

By PA we denote the theory containing all classical axioms of first order logic along with the following mathematical axioms:

- P1. $\neg S(a) = 0$;
- P2. $S(a) = S(b) \rightarrow a = b$;
- P3. $a + 0 = a$;
- P4. $a + S(b) = S(a + b)$;
- P5. $a \cdot 0 = 0$;
- P6. $a \cdot S(b) = a \cdot b + a$;
- P7. $\exp(0) = S(0)$;
- P8. $\exp(S(a)) = \exp(a) + \exp(a)$;
- P9. $a \leq 0 \leftrightarrow a = 0$;
- P10. $a \leq S(b) \leftrightarrow (a \leq b \vee a = S(b))$.

Apart from these axioms, PA also contains the induction axiom for all arithmetical formulas $\varphi(x)$:

$$I_{\varphi} : \quad \varphi(0) \wedge \forall x \left(\varphi(x) \rightarrow \varphi(S(x)) \right) \rightarrow \forall x \varphi(x).$$

By restricting the induction scheme we can obtain different fragments of PA. Of special interest is *Elementary Arithmetic* EA. This fragment of PA is obtained by restricting the induction axiom to Δ_0^0 -formulas and considering the axioms P1 to P10.

A natural extension of EA is EA^+ . This theory can be obtained by constructing a predicate Supexp which represents the graph of the super-exponential function $(x, y) \mapsto 2_y^x$ defined by $2_0^x := x$ and $2_{y+1}^x := 2_y^{2_y^x}$. Therefore, we will not introduce the function symbol but only work with the predicate. We define EA^+ as $EA + \text{SUP-EXP}$ where SUP-EXP is the axiom that states the totality of the super-exponential function using the predicate Supexp , that is, $\text{SUP-EXP} := \forall xy \exists z \text{Supexp}(x, y, z)$.

We can consider an equivalent formulation with a lower complexity by extending our arithmetical language by a function symbol supexp and thus, extending EA by adding to the set of axioms the corresponding recursive

defining equations for this super-exponential function. The totality of super-exponentiation can be easily proven by using induction for Δ_0^0 -formulas. This way, we obtain a Π_1^0 -conservative extension of EA^+ (see [32]). However, in this case, the notion of Δ_0 -presentable would not coincide with the more standard one from the literature.

The standard arithmetization of syntax of first order theories can be performed in EA^+ . Hence, arithmetical formulas can be one-to-one encoded by numbers. Given an expression ϵ , we refer to such a code as the *Gödel number* of ϵ , that will be denoted by $\ulcorner \epsilon \urcorner$. Different natural syntactical relations can be elementarily defined by using this encoding: we can code finite sequences of numbers and many of their basic properties like the length of the sequence, the i -st element of the sequence, concatenation, etc. Furthermore, we can also code properties about arithmetical formulas like being an arithmetical sentence, being a Π_n^0 -formula, being a logical axiom or that a certain formula can be obtained by the rules of generalization or modus ponens. All this machinery combined provide us with two important applications. First, with the help of the arithmetization of syntax, theories can be represented within weak fragments of arithmetic by elementary formulas numbering their axioms. These theories are the so-called *elementarily presented theories*:

Definition 1.1.2 (Elementarily presented theories). *A theory U is elementarily presented iff there is a bounded arithmetical formula $\sigma_U(x)$ that defines the sets of Gödel numbers coding the axioms of U in the standard model of arithmetic. Given elementary presented theories T and U , by $T + U$ we denote the theory whose set of axioms is defined by the formula $\sigma_T(x) \vee \sigma_U(x)$. By $\epsilon(x)$ we denote the canonical presentation of EA^+ .*

EA^+ and its extensions will be important throughout this thesis. However, we shall focus on extensions we can discuss from our base theory. Thus, by a *theory* we shall mean an elementary presented first order theory (with equality), formulated in $\mathcal{L}_{\mathbb{N}}$ and containing EA^+ . Via *Craig's trick*, any r.e. theory has an equivalent elementary presentation.

Another feature of the arithmetization of syntax is that allows us to define the predicate $\text{Prf}_T(x, y)$ meaning “ x codes a proof of a formula y in the theory T ”. Moreover, this predicate can be expressed by an elementary formula.

Gödel's provability predicate $\Box_T(x)$ is defined as $\exists y \text{Prf}_T(y, x)$; an arithmetical Σ_1^0 -formula expressing that there is x such that x codes a proof in T of a formula with code y . By \bar{n} we denote the *numeral* $S(S(\dots S(0)\dots))$ (n times). Often, we write $\Box_T(\varphi(\dot{x}))$ instead of $\Box_T(\ulcorner \varphi(\dot{x}) \urcorner)$, where $\ulcorner \varphi(\dot{x}_1, \dots, \dot{x}_j) \urcorner$ denotes the map sending n_1, \dots, n_k to the Gödel number $\ulcorner \varphi(\bar{n}_1, \dots, \bar{n}_j) \urcorner$.

The following proposition collects *Löb's derivability conditions* that provides an analysis of the basic properties of the provability predicate:

Proposition 1.1.3 (Löb's derivability conditions). *For any sentences φ, ψ and theory T :*

- L1. $T \vdash \varphi \implies \text{EA} \vdash \Box_T \varphi$;
- L2. $\text{EA} \vdash \Box_T(\varphi \rightarrow \psi) \rightarrow (\Box_T \varphi \rightarrow \Box_T \psi)$;
- L3. $\text{EA} \vdash \Box_T \varphi \rightarrow \Box_T \Box_T \varphi$.

Analogous conditions hold for formulas with free variables:

- L'1. $T \vdash \varphi(x) \implies \text{EA} \vdash \Box_T \varphi(\dot{x})$;
- L'2. $\text{EA} \vdash \Box_T(\varphi(\dot{x}) \rightarrow \psi(\dot{x})) \rightarrow (\Box_T \varphi(\dot{x}) \rightarrow \Box_T \psi(\dot{x}))$;
- L'3. $\text{EA} \vdash \Box_T \varphi(\dot{x}) \rightarrow \Box_T \Box_T \varphi(\dot{x})$;
- L'4. $\text{EA} \vdash \Box_T \forall x \varphi(x) \rightarrow \forall x \Box_T \varphi(\dot{x})$.

The following generalization of Gödel's famous second incompleteness theorem is known as Löb's theorem.

Theorem 1.1.4 (Löb's theorem). *For any elementarily presented theory T containing EA and any sentence φ :*

$$T \vdash \Box_T \varphi \rightarrow \varphi \iff T \vdash \varphi.$$

Moreover, Löb's theorem can be formalized within EA. This formalized version can be inferred from the non-formalized version:

Corollary 1.1.5 (Formalized Löb's theorem). *For any sentence φ :*

$$\text{EA} \vdash \Box_T(\Box_T \varphi \rightarrow \varphi) \rightarrow \Box_T \varphi.$$

Proof. Let $\chi := \Box_T(\Box_T \varphi \rightarrow \varphi) \rightarrow \Box_T \varphi$. We reason as follows:

$$\begin{aligned} \text{EA} \vdash & \left(\Box_T \chi \wedge \Box_T(\Box_T \varphi \rightarrow \varphi) \right) \rightarrow \Box_T \Box_T(\Box_T \varphi \rightarrow \varphi); \\ & \vdash \left(\Box_T \chi \wedge \Box_T(\Box_T \varphi \rightarrow \varphi) \right) \rightarrow \left(\Box_T \Box_T \varphi \wedge (\Box_T \Box_T \varphi \rightarrow \Box_T \varphi) \right); \\ & \vdash \left(\Box_T \chi \wedge \Box_T(\Box_T \varphi \rightarrow \varphi) \right) \rightarrow \Box_T \varphi. \end{aligned}$$

Thus, we have that

$$\text{EA} \vdash \Box_T \chi \rightarrow \chi.$$

By monotonicity we get that $T \vdash \Box_T \chi \rightarrow \chi$. Hence, by Löb's theorem $T \vdash \chi$ and so, $\text{EA} \vdash \Box_T \chi$. Thus, $\text{EA} \vdash \chi$. \square

By $\text{Con}(T)$ we denote the formula expressing the consistency of the theory T . This formula can be defined as $\text{Con}(T) := \neg \Box_T 0 = 1$ or equivalently, $\text{Con}(T) := \Box_T 0 = 1 \rightarrow 0 = 1$. Therefore, by substituting $0 = 1$ for φ in Löb's theorem, we obtain Gödel's second incompleteness theorem as a corollary (see [26]).

Theorem 1.1.6 (Gödel's second incompleteness theorem). *For any elementarily presented theory T containing EA:*

$$\text{If } T \text{ is consistent, then } T \not\vdash \text{Con}(T).$$

Moreover, if, in addition, T is Σ_1^0 -sound i.e. all Σ_1^0 theorems are true in the standard model of arithmetic, then $T \not\vdash \neg \text{Con}(T)$.

Up to this point, we have seen that provability and consistency have their own predicates that can be formalized within EA. This is not the case when it comes to truth. By the Gödel-Tarski undefinability theorem we know that there is no formula $\text{Tr}(x)$ such that for all sentences φ we have that:

$$\text{EA} \vdash \varphi \leftrightarrow \text{Tr}(\varphi).$$

However, it is well known that for the classes Σ_n^0 and Π_n^0 -formulas we can provide truth-definitions within EA.

Proposition 1.1.7 (Partial truth predicates). *For any $n < \omega$, there is an arithmetical Σ_n^0 -formula $\text{Tr}_{\Sigma_n^0}(x)$ such that for all Σ_n^0 -formulas $\varphi(x_1, \dots, x_n)$:*

$$\text{EA} \vdash \varphi(x_1, \dots, x_n) \leftrightarrow \text{Tr}_{\Sigma_n^0}(\varphi(\dot{x}_1, \dots, \dot{x}_n)).$$

The analogous result holds for Π_n^0 -formulas.

With this partial truth definitions, we can talk about n -consistency and so, about n -provability. For $n \geq 1$, by $\text{Tr}_{\Pi_n^0}(\mathbb{N})$ we denote the set of all true arithmetical Π_n^0 -sentences. For any arithmetical theory T , we shall say that T is n -consistent iff $T + \text{Tr}_{\Pi_n^0}(\mathbb{N})$ is consistent. Furthermore, this notion can be formalized within EA by the following Π_{n+1}^0 -formula:

$$\text{CON}_n(T) := \forall x \left(\text{Tr}_{\Pi_n^0}(x) \rightarrow \neg \Box_T \neg \text{Tr}_{\Pi_n^0}(x) \right).$$

The n -consistency of the theory T can equivalently (in EA) be expressed (see e.g. [6]) by the arithmetical formula:

$$\text{Con}_n(T) := \forall x \in \Pi_{n+1}^0 \left(\Box_T(x) \rightarrow \text{Tr}_{\Pi_{n+1}^0}(x) \right)$$

where $x \in \Pi_{n+1}^0$ expresses that x is the Gödel number of a Π_{n+1}^0 sentence.

The dual n -provability formula:

$$[n]_T \varphi := \neg \text{Con}_n(T + \neg \varphi)$$

expresses the provability of φ in T with the help of all true Π_n^0 -formulas. For these new provability predicates we have the analogous properties of the usual one. Thus, we have Löb's conditions and Löb's (formalized) theorem for $[n]_T$ (see [6]). As we shall see, the notion of n -consistency will be the key to graded Turing progressions and to introduce the motivation for the Gödel-Löb's polymodal logic (GLP).

1.2. Ordinals

In this section we shall introduce some basic features about ordinal numbers which are one of the key elements of this dissertation. Ordinal numbers are canonical representatives of well-orders. We can see ordinals as a generalization of the natural numbers into the transfinite, used to describe ordered collections of objects. In this sense, we can say that ordinal numbers measure the *length* of well-ordered sets.

Let us start by recalling the notion of *well-order*. Given a set X , a *preorder* on X is a reflexive and transitive relation on X . Moreover, a *linear order* is an antisymmetric and total preorder. A total preorder where every non-empty $Y \subseteq X$ has a minimal element is a *pre-well-order*. We say that $\langle X, \preceq \rangle$ is a *well-order* if \preceq is a linear pre-well-order.

The following definition introduces ordinal numbers as a particular case of a well-ordered set.

Definition 1.2.1. *We say that a set A is transitive if whenever $B \in A$, it follows that $B \subseteq A$. Then, a set ξ is an ordinal if ξ is transitive and $\langle \xi, \in \rangle$ is a well-order.*

From now on, we shall adopt the following notation: by *Ord* we denote the class of ordinal numbers. Furthermore, given ordinals α, β , we shall equally use $\alpha < \beta$ for $\alpha \in \beta$ and $\alpha \leq \beta$ if $\alpha < \beta$ or $\alpha = \beta$.

The class of all ordinals, *Ord*, is well-ordered by \in . Thus, for any ordinal α , we have that $\alpha = \{\beta \in \text{Ord} : \beta < \alpha\}$, and $0 = \emptyset$ is the least ordinal. There are two basic operations that we can define on *Ord*. For $\alpha \in \text{Ord}$ by $\alpha + 1$ we denote $\alpha \cup \{\alpha\}$; that is the least ordinal greater than α . Therefore, starting by 0 and iterating this operation, we can observe that any natural number is an ordinal. Moreover, new ordinals can be formed by taking unions. Hence, given Θ a set of ordinals, we have that $\lambda = \bigcup \Theta$ is an ordinal as well. Thus, we also have infinite ordinals; the union of the set of natural numbers is

itself an ordinal greater than each natural number. We denote this ordinal by ω .

The following lemma summarizes these observations:

Lemma 1.2.2. *For any $\alpha \in \text{Ord}$ exactly one of the following occurs:*

1. $\alpha = 0$;
2. *there exists β such that $\alpha = \beta + 1$, in which case we say that α is a successor;*
or
3. $\alpha = \bigcup_{\beta < \alpha} \beta$, *in which case we say that α is a limit.*

By *Succ* we shall denote the class of successor ordinals and analogously, *Lim* denotes the class of limit ordinals.

This generalization of the natural numbers also extends to the induction schema. *Transfinite induction* is an extension of regular mathematical induction to well-ordered sets, and so, it can be applied to ordinal numbers.

Definition 1.2.3 (Transfinite induction). *Let P be a property and X a well-ordered set. We define the transfinite induction axiom for P by*

$$\forall \alpha \in X (\forall \beta < \alpha P(\beta) \rightarrow P(\alpha)) \implies \forall \alpha \in X P(\alpha).$$

It is worth mentioning that the principle of transfinite induction also holds when considering the class *Ord* instead of X .

Order-types

As we have mentioned, a well-order $\mathfrak{X} = \langle X, \preceq \rangle$ can be canonically represented by using an ordinal. To do this, we define $o : X \rightarrow \text{Ord}$ such that:

$$o(x) = \bigcup_{y \prec x} (o(y) + 1).$$

We say that $o(x)$ is *order-type* of x . The rank of \mathfrak{X} is then defined as $\sup_{x \in X} (o(x) + 1)$.

The following lemma is useful in characterizing the rank function [20].

Lemma 1.2.4. *Let $\langle X, \preceq \rangle$ be a well-order. Then $o : X \rightarrow \text{Ord}$ is the unique function such that*

1. $x \prec y$ implies that $o(x) < o(y)$,
2. if $\zeta < o(x)$ then $\zeta = o(y)$ for some $y \in X$.

Ordinal arithmetic

In the previous section we showed how to define the successor of any given ordinal α . In this section we shall see how we can define the basic arithmetic operations and outline some of their properties.

We shall start defining the addition function for ordinals. We do this by transfinite recursion. As we can observe, this definition is analogous to the one for the natural numbers, with the exception that now we consider an extra step for limit ordinals.

Definition 1.2.5 (Ordinal addition). *Given ordinals α, β , we define $\alpha + \beta$ by recursion on β as follows:*

1. $\alpha + 0 = \alpha$;
2. $\alpha + (\beta + 1) = (\alpha + \beta) + 1$;
3. $\alpha + \beta = \bigcup_{\delta < \beta} (\alpha + \delta)$, for $\beta \in \text{Lim}$.

In a similar fashion we can define the product ($\alpha \cdot \beta$) and the exponentiation (α^β).

The following lemma collects some observations about these arithmetic operations.

Lemma 1.2.6. *For any $\alpha, \beta, \delta \in \text{Ord}$:*

1. $(\alpha + \beta) + \delta = \alpha + (\beta + \delta)$;
2. *if $\alpha + \beta = \alpha + \delta$, then $\beta = \delta$;*
3. $\alpha \cdot (\beta + \delta) = \alpha \cdot \beta + \alpha \cdot \delta$;
4. $\alpha^\beta \cdot \alpha^\delta = \alpha^{(\beta + \delta)}$;
5. $(\alpha^\beta)^\delta = \alpha^{\beta \cdot \delta}$.

As a particular case of the exponentiation function, we have the mapping $\alpha \mapsto \omega^\alpha$. This function is the basis of the *Cantor normal form* representation of ordinals:

Lemma 1.2.7 (Cantor normal form). *For any $\alpha \in \text{Ord}$ with $\alpha > 0$, there are $\alpha_0 \geq \alpha_1 \geq \dots \geq \alpha_k$ for some $k < \omega$ such that:*

$$\alpha = \omega^{\alpha_0} + \dots + \omega^{\alpha_k}.$$

A non-zero ordinal α such that for any $\delta, \gamma < \alpha$ we have that $\delta + \gamma < \alpha$ is called an *additively indecomposable ordinal*. These ordinals are exactly the ordinals of the form ω^β for some β .

Normal functions and hyper-exponentiation

The arithmetic operations defined in the previous section are examples of *normal functions*, that is, functions $f : \text{Ord} \rightarrow \text{Ord}$ which are strictly increasing and *continuous*, in the sense that if λ is a limit then $f(\lambda) = \sup_{\alpha < \lambda} f(\alpha)$. This notion can be formalized as follows:

Definition 1.2.8. *A function $f : \text{Ord} \rightarrow \text{Ord}$ is normal iff:*

1. $f(\alpha) < f(\beta)$ whenever $\alpha < \beta$, and
2. $f(\lambda) = \sup_{\alpha < \lambda} f(\alpha)$ for $\lambda \in \text{Lim}$.

When $f : X \rightarrow X$, it is natural and often useful to ask whether f has *fixed points*, i.e., solutions to the equation $x = f(x)$. In particular, normal functions have many fixed points:

Proposition 1.2.9. *Every normal function on Ord has arbitrarily large fixed points.*

The first ordinal α such that $\alpha = \omega^\alpha$ is the limit of the ω -sequence

$$\langle \omega, \omega^\omega, \omega^{\omega^\omega}, \dots \rangle,$$

and is usually denoted ε_0 . Every $\xi < \varepsilon_0$ can be written in terms of 0 using only addition and the function $\omega \mapsto \omega^\xi$ via its Cantor normal form. The hyper-exponential function is then a natural transfinite iteration of the ordinal exponential which remains normal after each iteration.

Definition 1.2.10 (Hyper-exponential functions). *The hyper-exponential functions $(e^\xi)_{\xi \in \text{Ord}}$ are the unique family of normal functions that satisfy*

1. $e^1 = e$,
2. $e^{\alpha+\beta} = e^\alpha \circ e^\beta$ for all α and β , and
3. if $(f^\xi)_{\xi \in \text{Ord}}$ is a family of functions satisfying 1 and 2, then for all $\alpha, \beta \in \text{Ord}$, $e^\alpha \beta \leq f^\alpha \beta$.

Fernández-Duque and Joosten proved that the hyperexponentials are well-defined [22]. In [20] it is also shown that the function $\xi \mapsto e^\xi 1$ is itself a normal function, hence it has a least non-zero fixed point: this fixed point is the Feferman-Schütte ordinal, Γ_0 . Just like ordinals below ε_0 may be written using 0, addition, and ω -exponentiation, every ordinal below Γ_0 may be written in terms of 0, 1, addition and the function $(\xi, \zeta) \mapsto e^\xi \zeta$.

As the reader might notice, there is a close link between hyper-exponential functions and the more standard Veblen functions. Notations using hyper-exponentials or Veblen functions can be translated from one to the other with the help of the following proposition that can be found in [20]:

Proposition 1.2.11. *Given ordinals α, β :*

1. $e^\alpha(0) = 0$,
2. $e^1(1 + \beta) = \varphi_0(1 + \beta)$,
3. $e^{\omega^{1+\alpha}}(1 + \beta) = \varphi_{1+\alpha}(\beta)$.

Ordinal notation systems

Broadly speaking, we can define an *ordinal notation system* for Λ as a way of encoding ordinals by using finite sequences of an alphabet A . For this presentation, we shall focus on representing ordinals as relations defined over \mathbb{N} . Furthermore, we are interested in ordinal notation systems with a low arithmetical complexity.

We say $\Lambda \in \text{Ord}$ is a *recursive ordinal* if there is $\mathcal{D} := \langle D, \prec \rangle$ such that both $D \subseteq \mathbb{N}$ and \prec are recursive, \prec is a well-order in D and the rank of \mathcal{D} is Λ . Along this thesis, we shall focus on recursive ordinals within EA^+ . To this purpose, we shall consider elementarily definable relations on \mathbb{N} . An *elementary linear ordering* is a structure $\mathcal{D} := \langle D, \prec \rangle$ where both, D and \prec are elementarily definable, $D \subseteq \mathbb{N}$ and EA^+ proves that \prec linearly orders D . An *elementary well-ordering* is an elementary linear ordering that is well-founded in the standard model.

For the remainder of this dissertation we shall furthermore assume that our elementary orders are nice in that the basic properties of and operations on the elements of the ordering are available in EA^+ . As such we can view the elements as ordinals, we can distinguish successor ordinals from limit ordinals and perform the basic operations like addition, multiplication and exponentiation.

1.3. Turing progressions

The well-known theorem of Gödel shows that every system of logic is in a certain sense incomplete, but at the same time it indicates means whereby from a system L of logic a more complete system L' may be obtained. By repeating the process we get a sequence $L, L_1 = L', L_2 = L'_1, \dots$ each more complete than the preceding. A logic L_ω may then be constructed in which the provable theorems are the totality of theorems provable with the help of the logics L, L_1, L_2, \dots . We may then form $L_{2\omega}$ related to L_ω the same way as L_ω was related to L . Proceeding in this way we can associate a system of logic with any constructive ordinal. It may be asked whether such a sequence of logics of this kind is complete in the sense that to any problem A there corresponds an ordinal α such that A is solvable by means of the logic L_α .

Alan Turing

After the impact of Gödel's incompleteness results, Alan Turing took the first endeavor to overcome this situation. This attempt was the topic of his PhD dissertation, published in 1939 under the title *Systems of logics based on ordinals* [36].

The method proposed by Turing to try to deal with Gödel's incompleteness is based on the natural idea of extending formal systems by iteratedly adding these non-provable statements, such as the consistency of the system. Furthermore, this construction can be carried out up to transfinite levels, being that given a sequence $\langle L^n \rangle_{n \in \mathbb{N}}$ of formal systems, we can consider a new system $L^\omega := \bigcup_{n \in \mathbb{N}} L_n$ by just taking the union of their axioms. Then this iteration can be continued to $L^{\omega+1}$ as we did for the construction of the L_n 's and so on. This process suggest a way of associating a formal system L^α to any recursive ordinal α .

In particular, if we start this construction by taking as our base formal system any sound r.e. arithmetical theory T that is sufficiently rich, we can build the successor step $T^{\alpha+1}$ by taking T^α together with the consistency statement for T^α , $\text{Con}(T^\alpha)$ and therefore, by Gödel's second incompleteness theorem, we obtain a hierarchy of theories of stronger strength. These hierarchies are known as *Turing progressions* which nowadays are widely used in proof theory and can serve the purpose of gauging the proof-theoretic strength of mathematical theories that contain or interpret arithmetic. (See e.g. [5].)

More formally, let us assume that we have fixed some recursive limit ordinal Λ together with a natural ordinal notation system for Λ that is elementary presented. Turing progressions are essentially defined by the following three clauses.

- T1. $(T)_n^0 := T$ where T is an initial or base theory;
- T2. $(T)_n^{\alpha+1} := (T)_n^\alpha \cup \{\text{Con}_n((T)_n^\alpha)\}$;
- T3. $(T)_n^\lambda := \bigcup_{\beta < \lambda} (T)_n^\beta$, for λ a limit ordinal not exceeding Λ .

There are various ways one can define/present a series of theories that satisfy these three clauses. Here we will consider *smooth Turing progressions*, studied by Beklemishev in among others [6], [] and [1]. We give a slightly different presentation.

Suppose we are given some elementary well-ordering (D, \prec) and an initial theory T . The conditions T1-T3 can be reformulated by the following unique clause:

$$(T)_n^\alpha := T \cup \{\text{Con}_n((T)_n^\beta) : \prec(\beta, \alpha), \beta \in D\} \quad \forall \alpha \in D, n < \omega.$$

Note that there appears to be a circularity in this definition. Of course, by means of the fixed point theorem (see [16]) we can find a formalization of the definition within arithmetic.

We say that $\tau_n^{\sigma(z)}(\alpha, x)$ enumerates the α -th theory of a progression based on iteration of n -consistency along (D, \prec) with base $\sigma(z)$ if:

$$\text{EA}^+ \vdash \tau_n^{\sigma(z)}(\alpha, x) \leftrightarrow \left((\epsilon(x) \vee \sigma(x)) \vee \exists \beta (\prec(\beta, \alpha) \wedge x = \ulcorner \text{Con}_n(\tau_n^{\sigma(z)}(\dot{\beta}, y)) \urcorner) \right).$$

The free variables of $\tau_n^{\sigma(z)}(\alpha, x)$ are x and α , where x is the Gödel number of a formula and α is the code of an ordinal. Recall that by our reading conventions we have that $\text{Con}_n(\tau_n^{\sigma(z)}(\dot{\beta}, y))$ denotes the n -consistency of the theory axiomatized by $\tau_n^{\sigma(z)}(\dot{\beta}, y)$. Since the equivalence does not directly refer to $\tau_n^{\sigma(z)}(\dot{\beta}, y)$ but rather to the Gödel number of its n -consistency (that is $\text{Con}_n(\tau_n^{\sigma(z)}(\dot{\beta}, y))$), the existence of such $\tau_n^{\sigma(z)}(\alpha, x)$ is guaranteed by the fixed point theorem. Furthermore, in [2] and [4] it is shown that given an elementary presented theory T and an elementary well-ordering (D, \prec) , there is a unique progression $(T)_n^\alpha$ for $n < \omega$ and $\alpha \in D$, modulo provable equivalence in EA^+ . The proof of this result makes use of a trick coming from the work of U. Schmerl [34], namely *reflexive induction* or sometimes called *reflexive transfinite induction*.

Proposition 1.3.1 (Reflexive induction). *For any p.r. well-ordering (D, \prec) , any theory T containing EA^+ is closed under the following reflexive induction rule:*

$$T \vdash \forall \alpha \left(\Box_T (\forall \beta \prec \dot{\alpha} \varphi(\beta)) \rightarrow \varphi(\alpha) \right) \implies T \vdash \forall \alpha \varphi(\alpha).$$

Reflexive induction is our main tool for proving facts about Turing progressions. It may seem very tricky but it has a very simple proof and is in a sense a direct consequence of Löb's theorem. Whenever we say that we are going to prove that T proves $\forall \alpha \varphi(\alpha)$ by reflexive induction, we implicitly say that we are going to show that in T we can prove

$$\forall \alpha \left(\Box_T (\forall \beta \prec \dot{\alpha} \varphi(\beta)) \rightarrow \varphi(\alpha) \right).$$

Further, we call the challenge of proving $T \vdash \varphi(0)$ the base case, and for given α , refer to the assumption in T that $\Box_T (\forall \beta \prec \dot{\alpha} \varphi(\beta))$, as the reflexive induction hypothesis (RIH).

1.4. Provability Logic

Provability logics are modal systems in which the modal operators \Box and \Diamond are interpreted as provability predicates and consistency statements for theories such as EA, EA^+ or PA. The basic provability logic **GL** (Gödel-Löb) is given in the language of propositional modal logic whose formulas

are constructed from propositional variables, standard Boolean connectives and a unary modal operator \Box . **GL** is a *normal modal logic*, that is, the theorems of **GL** contains all boolean tautologies and formulas of the form $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$. Moreover, the theorems of **GL** are closed under substitution and the inference rules of modus ponens and necessitation. The smallest logic satisfying these conditions is the modal logic **K**. **GL** extends **K** by adding the following axiom scheme:

$$(GL) \quad \Box(\Box\varphi \rightarrow \varphi) \rightarrow \Box\varphi.$$

We can observe that this axiom scheme is a modal version of the formalization of Löb's theorem as expressed in Corollary 1.1.5, which makes **GL** a suitable candidate to capture the behavior of the provability predicate. To show that this is the case, we need to define a way to interpret modal formulas as arithmetical statements. An *arithmetical realization* of the language of **GL** for the theory T is a function $*$ that maps propositional variables to arithmetical sentences. This can be naturally extended to any modal formula φ . In particular, the modal operator \Box is mapped to the Gödel provability predicate \Box_T . Hence, we have the following:

- $(\varphi \rightarrow \psi)^* = \varphi^* \rightarrow \psi^*$;
- $(\neg\varphi)^* = \neg\varphi^*$;
- $(\Box\varphi)^* = \Box_T\varphi^*$.

Clearly, we can see that for each modal formula φ and any arithmetical realization $*$, if **GL** $\vdash \varphi$ then $EA^+ \vdash (\varphi)^*$. The converse implication holds by the *arithmetical completeness theorem* for **GL** due to Solovay [35]:

Theorem 1.4.1. *For any elementarily presented theory T which is Σ_1^0 -sound and contains EA we have that:*

$$GL \vdash \varphi \iff T \vdash (\varphi)^*, \text{ for every realization } * \text{ of the variables in } \varphi.$$

Apart from arithmetical semantics, **GL** also has relational semantics which provide a powerful tool to help us in the study of provability logic.

The relational semantics we shall discuss in this dissertation are understood in the usual way. A *frame* is a pair $\mathcal{F} = \langle W, R \rangle$ consisting of a nonempty set W together with R , a binary relation on W known as the *accessibility relation*. The elements of W are called *worlds*. A frame $\mathcal{F} = \langle W, R \rangle$ is said to be *finite* iff W is finite. Moreover, sometimes \mathcal{F} is said to have some property of its corresponding accessibility relation e.g. \mathcal{F} is reflexive iff $R \subseteq W^2$ is reflexive. A *valuation* v on a set W is a function $v : W \times \text{Var} \rightarrow \{0, 1\}$ where Var is the set of propositional variables. A valuation assigns a truth value to each propositional variable at each world of W . A *Kripke*

model is a triple $\mathcal{M} := \langle W, R, v \rangle$ where $\langle W, R \rangle$ is a frame and v a valuation on W . We say that a model is finite iff the frame on which is based is finite. Analogously, we say that a model is reflexive, transitive, etc iff the the frame on which is based is reflexive, transitive, etc.

For each modal formula φ , each model $\mathcal{M} := \langle W, R, v \rangle$ and each $x \in W$, we inductively define the relation

$$\mathcal{M}, x \Vdash \varphi$$

as follows:

- $\mathcal{M}, x \Vdash \top$;
- $\mathcal{M}, x \Vdash p$ iff $v(x, p) = 1$, for $p \in \text{Var}$;
- $\mathcal{M}, x \Vdash \neg\varphi$ iff $\mathcal{M}, x \not\Vdash \varphi$;
- $\mathcal{M}, x \Vdash \varphi \wedge \psi$ iff $\mathcal{M}, x \Vdash \varphi$ and $\mathcal{M}, x \Vdash \psi$;
- $\mathcal{M}, x \Vdash \Box\varphi$ iff for all $y \in W$ such that xRy , we have that $\mathcal{M}, y \Vdash \varphi$.

The following theorem establishes the completeness of **GL** with respect to finite, transitive and irreflexive models.

Theorem 1.4.2. *For any formula φ :*

$$\mathbf{GL} \vdash \varphi \iff \forall x \in W, \mathcal{M}, x \Vdash \varphi,$$

for every finite, transitive and irreflexive model $\mathcal{M} := \langle W, R, v \rangle$.

GLP

As we saw in Section 1.1, apart from consistency statements and the standard provability predicate, we can define the notions of n -consistency with their dual n -provability predicate. The logic that fully describes these operators is **GLP** $_{\omega}$; a polymodal version of **GL** wich contains modalities $[n]$ and $\langle n \rangle$ for each $n < \omega$. This system was introduced by Japaridze in [31] and later, extended to a class-sized logic by Beklemishev in [7] where for each $\alpha \in \text{Ord}$ we have modalities $\langle \alpha \rangle$ and $[\alpha]$.

Given $\Lambda \in \text{Ord}$, we can consider fragments **GLP** $_{\Lambda}$ of **GLP** by restricting our signature to modalities $[\alpha]$ for $\alpha < \Lambda$. The logic **GLP** $_{\Lambda}$ is then given by the following rules and axioms:

1. all propositional tautologies,
2. $[\alpha](\phi \rightarrow \psi) \rightarrow ([\alpha]\phi \rightarrow [\alpha]\psi)$ for all $\alpha < \Lambda$,

3. $[\alpha]([\alpha]\phi \rightarrow \phi) \rightarrow [\alpha]\phi$ for all $\alpha < \Lambda$,
4. $[\beta]\phi \rightarrow [\alpha]\phi$ for $\beta < \alpha < \Lambda$,
5. $\langle \beta \rangle \phi \rightarrow [\alpha]\langle \beta \rangle \phi$ for $\beta < \alpha < \Lambda$,
6. modus ponens and
7. necessitation for each $[\alpha]$.

\mathbf{GLP}_ω is sound and complete with respect to its natural arithmetical realization in terms of n -provability. Moreover, one of the main applications of \mathbf{GLP} is that its closed fragment can be used to perform a Π_1^0 -ordinal analysis of PA and related systems [5]. On the other hand, the main disadvantage of \mathbf{GLP} or even the smallest polymodal fragment, \mathbf{GLP}_2 , is that it is incomplete for its relational semantics. Without propositional variables, Ignatiev built a relational model in which every consistent formula of variable-free \mathbf{GLP}_ω is satisfied [30], and Fernández-Duque and Joosten extended this to variable-free \mathbf{GLP} over the ordinals.

The Reflection Calculus

\mathbf{GLP} gained much interest due to Beklemishev's proof-theoretic applications [5]; however, from a modal logic point of view, it is not an easy system to work with. To this end, in [17, 8, 9] Beklemishev and Dashkov introduced the system called *Reflection Calculus*, \mathbf{RC} , that axiomatizes the fragment of \mathbf{GLP}_ω consisting of unnested implications of strictly positive formulas. This system is much simpler than \mathbf{GLP}_ω but yet expressive enough to maintain its main proof-theoretic applications.

Similar to \mathbf{GLP}_Λ , the signature of \mathbf{RC}_Λ contains modalities of the form $\langle \alpha \rangle$ for $\alpha \in \Lambda$. However, since this system only considers strictly positive formulas, the signature does not contain negation, disjunction or modalities $[\alpha]$. Thus, the set of formulas $\mathbb{F}_{\mathbf{RC}_\Lambda}$ in this signature is defined by the following grammar:

$$\varphi := \top \mid p \mid (\varphi \wedge \psi) \mid \langle \alpha \rangle \varphi \quad \text{for } \alpha \in \Lambda.$$

Next we define a consequence relation over $\mathbb{F}_{\mathbf{RC}_\Lambda}$. For the purposes of this presentation, a *deductive calculus* is a pair $\mathbf{X} = (\mathbb{F}_{\mathbf{X}}, \vdash_{\mathbf{X}})$ such that $\mathcal{F}_{\mathbf{X}}$ is some set, the *language* of \mathbf{X} , and $\vdash_{\mathbf{X}} \subseteq \mathbb{F}_{\mathbf{X}} \times \mathbb{F}_{\mathbf{X}}$. We write $\varphi \cong_{\mathbf{X}} \psi$ for $\varphi \vdash_{\mathbf{X}} \psi$ and $\psi \vdash_{\mathbf{X}} \varphi$. We will omit the subscript \mathbf{X} when this does not lead to confusion, including in the definition below, where \vdash denotes $\vdash_{\mathbf{RC}_\Lambda}$.

Definition 1.4.3. *Given an ordinal Λ , the calculus \mathbf{RC}_Λ over $\mathbb{F}_{\mathbf{RC}_\Lambda}$ is given by the following set of axioms and rules:*

Axioms:

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1. $\varphi \vdash \varphi, \quad \varphi \vdash \top;$
2. $\varphi \wedge \psi \vdash \varphi, \quad \varphi \wedge \psi \vdash \psi;$
3. $\langle \alpha \rangle \langle \alpha \rangle \varphi \vdash \langle \alpha \rangle \varphi;$
4. $\langle \alpha \rangle \varphi \vdash \langle \beta \rangle \varphi \quad \text{for } \alpha > \beta;$
5. $\langle \alpha \rangle \varphi \wedge \langle \beta \rangle \psi \vdash \langle \alpha \rangle (\varphi \wedge \langle \beta \rangle \psi)$
for $\alpha > \beta.$

Rules:

1. If $\varphi \vdash \psi$ and $\varphi \vdash \chi$, then $\varphi \vdash \psi \wedge \chi$;
2. If $\varphi \vdash \psi$ and $\psi \vdash \chi$, then $\varphi \vdash \chi$;
3. If $\varphi \vdash \psi$, then $\langle \alpha \rangle \varphi \vdash \langle \alpha \rangle \psi$;

The reflection calculus has natural arithmetical [25], Kripke [17, 9], algebraic [11] and topological [13, 19, 29, 30] interpretations for which it is sound and complete, but in this dissertation we will work exclusively with reflection calculi from a syntactical perspective. Other variants of the reflection calculus have been proposed, for example working exclusively with worms [18] or allowing additional conservativity operators [15, 10].

This suffices to define the consistency ordering between worms, a crucial ingredient in Beklemishev's proof-theoretic analysis.

Worms and the consistency ordering

We review the consistency ordering between worms, along with some of their basic properties.

Definition 1.4.4. Fix an ordinal $\Theta < \Lambda$. The set \mathbb{W}_Θ of worms in $\mathbb{F}_{\text{RC}_\Lambda}$ is recursively defined as follows:

1. $\top \in \mathbb{W}_\Theta$;
2. If $A \in \mathbb{W}_\Theta$ and $\alpha < \Theta$, then $\langle \alpha \rangle A \in \mathbb{W}_\Theta$.

Similarly, we inductively define for each $\alpha \in \Theta$ the set of worms $\mathbb{W}_\Theta^{\geq \alpha}$ where all ordinals are at least α :

1. $\top \in \mathbb{W}_\Theta^{\geq \alpha}$;
2. If $A \in \mathbb{W}_\Theta^{\geq \alpha}$ and $\beta \geq \alpha$, then $\langle \beta \rangle A \in \mathbb{W}_\Theta^{\geq \alpha}$.

Definition 1.4.5. Let $A = \langle \xi_1 \rangle \dots \langle \xi_n \rangle \top$ and $B = \langle \zeta_1 \rangle \dots \langle \zeta_m \rangle \top$ be worms. Then, define $AB = \langle \xi_1 \rangle \dots \langle \xi_n \rangle \langle \zeta_1 \rangle \dots \langle \zeta_m \rangle \top$. Given an ordinal λ , define $\lambda \uparrow A$ to be $\langle \lambda + \xi_1 \rangle \dots \langle \lambda + \xi_n \rangle \top$.

Often we will want to put an extra ordinal between two worms, and we write $B \langle \lambda \rangle A$ for $B(\langle \lambda \rangle A)$. Next, we define the consistency ordering between worms.

Definition 1.4.6. Given ordinals $\beta, \Theta < \Lambda$, we define a relation $<_\beta$ on $\mathbb{W}_\Theta^{\geq \beta}$ by $B <_\beta A$ if and only if $A \vdash \langle \beta \rangle B$. We also define $B \leq_\beta A$ if $B <_\beta A$ or $B \cong A$.

The ordering \leq_0 has some nice properties.

Theorem 1.4.7. For any ordinal Θ , the relation \leq_0 is a pre-well-order on \mathbb{W}_Θ .

Note that \leq_0 fails to be a linear order merely because it is not antisymmetric. To get around this, one may instead consider worms modulo provable equivalence. Alternately, as Beklemishev has done [7], one can choose a canonical representative for each worm.

Definition 1.4.8 (Beklemishev Normal Form). A worm $A \in \mathbb{W}$ is defined recursively to be in BNF if either

1. $A = \top$, or
2. $A := A_k \langle \alpha \rangle A_{k-1} \langle \alpha \rangle \dots \langle \alpha \rangle A_0$ with
 - $\alpha = \min \mathcal{S}(A)$;
 - $k \geq 1$;
 - $A_i \in \mathbb{W}_\Lambda^{\geq \alpha+1}$, for $i \leq k$;

such that $A_i \in \text{BNF}$ and $A_{i+1} \leq_{\alpha+1} A_i$ for each $i < k$.

This definition essentially mirrors that of Cantor normal forms for ordinals. The following was proven in [7] and [12].

Theorem 1.4.9. Given any worm A there is a unique $A' \in \text{BNF}$ such that $A \cong A'$.

Thus, with the help of BNF we have the following result:

Theorem 1.4.10. For any ordinal Θ , the relation \leq_0 is a well-order on $\mathbb{W}_\Theta \cap \text{BNF}$.

The consistency ordering between worms allows us to compute the order-type of “finite” worms i.e. \mathbb{W}_ω . The elements of \mathbb{W}_ω provide an alternative ordinal notation for ordinals up to ε_0 . Let us introduce first the following operations on worms: ‘promotion’ (\uparrow) and ‘demotion’ (\downarrow).

Definition 1.4.11. Let $\varphi \in \mathbb{F}_{\text{RC}_\Lambda}$ and λ an ordinal. We define $\lambda \uparrow \varphi$ to be the result of replacing every ordinal α appearing in φ by $\lambda + \alpha$. Similarly define $\lambda \downarrow \varphi$ by replacing every occurrence of α by $-\lambda + \alpha$.

Now we can formulate a simple correspondence between $\mathbb{W}_\omega \cap \text{BNF}$ and the ordinals below ε_0 .

Theorem 1.4.12. The map $o: \mathbb{W}_\omega \cap \text{BNF} \rightarrow \varepsilon_0$ is surjective and satisfies

1. $o(\top) = 0$, and;

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$$2. o((1 \uparrow A)\langle 0 \rangle B) = o(B) + \omega^{o(A)}.$$

We can go further and consider worms which contain modalities with transfinite ordinals as entries. Fortunately, Theorem 1.4.10 holds for any set of worms, even those which have transfinite entries. Thus, we can extend our previous theorem to all of \mathbb{W} obtaining a complete calculus for o .

Theorem 1.4.13. *Let $A, B \in \mathbb{W}$ and $\alpha \in \text{Ord}$:*

1. $o(\top) = 0$, and;
2. $o(A\langle 0 \rangle B) = o(B) + 1 + o(A)$;
3. $o(\alpha \uparrow A) = e^\alpha(o(A))$.

Finally, we mention a useful property of o proven in [20] where $\max A$ is the greatest ordinal appearing in A .

Lemma 1.4.14. *Let $A \neq \top$ be a worm and μ an ordinal. Then,*

1. *if $\mu \leq \max A$, then $o(\langle \mu \rangle \top) \leq o(A)$, and*
2. *if $\max A < \mu$, then $o(A) < o(\langle \mu \rangle \top)$.*

The Turing-Schmerl Calculus

In this Chapter we introduce a propositional modal language using *ordinal modalities*, which are modalities of the form $\langle n^\alpha \rangle$ where $\alpha \in \Lambda$ for some fixed ordinal Λ and $n \in \omega$ (named *exponent* and *base*, respectively). Also, we will introduce the system **TSC** (for **T**uring-**S**chmerl **C**alculus) in this language.

First rank inhabitants of **TSC** are the so-called *monomial normal forms* (MNF's for short). We will show that each formula is equivalent to such a MNF.

2.1. Signature and Ordinal Modalities

We shall work with the closed fragment of a strictly positive propositional modal signature. From now on, we use Λ to denote some fixed ordinal. Moreover, if we shall speak of Λ in a formalized setting, we shall tacitly assume that Λ is some elementary well-order the properties of which are readily provable in a weak base theory. Further, we require some properties of Λ that are collected in the following remark.

Remark 2.1.1. *As we shall see in the axioms of our logic, they only make sense if the ordinals occurring in them are available. As such, the usable closure conditions on Λ naturally suggest themselves. Since it suffices to require that for $n < \omega$ that $\alpha, \beta < \Lambda \Rightarrow \alpha + e^n(\beta) < \Lambda$, we shall for the remainder assume that Λ is an ε -number, that is, a fixpoint of e whence $e(\Lambda) = \Lambda = \omega^\Lambda$.*

The modal signature we are going to work with consists of one constant symbol \top , one logical connective \wedge and a set of modal connectives $\mathcal{M} := \{\langle n^\alpha \rangle : n < \omega \text{ and } \alpha < \Lambda\}$, named *ordinal modalities*. We call n the *base* and α the *exponent*. The set of formulas in this language is defined as follows:

Definition 2.1.2. By \mathbb{F}_{TSC} we denote the smallest set such that:

- I) $\top \in \mathbb{F}_{TSC}$;
- II) If $\varphi, \psi \in \mathbb{F}_{TSC}$ then $(\varphi \wedge \psi) \in \mathbb{F}_{TSC}$;
- III) If $\varphi \in \mathbb{F}_{TSC}$, $n < \omega$ and $\alpha < \Lambda$ then $\langle n^\alpha \rangle \varphi \in \mathbb{F}_{TSC}$.

In the next chapter we shall see how these formulas can be associated to arithmetical theories in a natural way. To have some control over the nature of these theories, for any formula ψ in the signature we define the following two functions returning the set of base and exponent elements respectively of any modality occurring in ψ . That is:

Definition 2.1.3. Given a formula ψ , by $N\text{-mod}(\psi)$ we denote the set of natural numbers corresponding to the base element in each modality that occurs in ψ , i.e.,

- I) $N\text{-mod}(\top) = \emptyset$;
- II) $N\text{-mod}(\varphi \wedge \psi) = N\text{-mod}(\varphi) \cup N\text{-mod}(\psi)$;
- III) $N\text{-mod}(\langle n^\alpha \rangle \psi) = \{n\} \cup N\text{-mod}(\psi)$.

Analogously, by $O\text{-mod}(\psi)$ we denote the set of ordinals below Λ corresponding to the exponent element of each modality in ψ , i.e.,

- I) $O\text{-mod}(\top) = \emptyset$;
- II) $O\text{-mod}(\varphi \wedge \psi) = O\text{-mod}(\varphi) \cup O\text{-mod}(\psi)$;
- III) $O\text{-mod}(\langle n^\alpha \rangle \psi) = \{\alpha\} \cup O\text{-mod}(\psi)$.

Ordinal worms and restricted sets of formulas

Within \mathbb{F}_{TSC} we can find some special formulas, named *Ordinal Worms* (OWs), which are defined as follows:

Definition 2.1.4 (Ordinal Worms, \mathbb{W}^{Ord}). The set of OWs denoted by \mathbb{W}^{Ord} is inductively defined as:

- I) $\top \in \mathbb{W}^{\text{Ord}}$;
- II) if $B \in \mathbb{W}^{\text{Ord}}$, then $\langle n^\alpha \rangle B \in \mathbb{W}^{\text{Ord}}$ for any $n < \omega$ and $\alpha < \Lambda$.

In order to control the complexity of the theories corresponding to our formulas, for any $n < \omega$ we define the set of formulas $\mathbb{F}_{< n}$ as follows:

Definition 2.1.5. $\mathbb{F}_{< n}$ is the smallest set such that:

- I) $\top \in \mathbb{F}_{< n}$;

- II) if $\varphi, \psi \in \mathbb{F}_{<n}$ then $(\varphi \wedge \psi) \in \mathbb{F}_{<n}$;
- III) if $\varphi \in \mathbb{F}_{<n}$ then $\langle m^\beta \rangle \varphi \in \mathbb{F}_{<n}$ for all $m < n$ and $\beta < \Lambda$.

This last set of formulas will have some special relevance when trying to capture some conservativity results.

From now on, we let Greek letters $\alpha, \beta, \gamma, \dots$ denote ordinals below Λ , Greek lower case in the middle of the alphabet like φ, ψ, χ denote formulas, and capital latin letters A, B, C, \dots denote OWs. Also, given $A \in \mathbb{W}^{Ord}$ of the form $\langle n_0^{\alpha_0} \rangle \dots \langle n_k^{\alpha_k} \rangle \top$, and $\varphi \in \mathbb{F}_{TSC}$ by $A\varphi$ we denote the formula ψ obtained by the concatenation $\langle n_0^{\alpha_0} \rangle \dots \langle n_k^{\alpha_k} \rangle \varphi$.

Monomial and increasing normal forms

In this subsection we consider two different kinds of special formulas named *monomial normal forms* and *increasing normal forms*. Formulas in monomial normal form are used in the axiomatization of the calculus TSC as introduced in the next section and play an important role in this thesis.

Later we will show that for every formula ψ in monomial normal form there is a unique equivalent (modulo TSC) OW A in increasing normal form and vice versa. Moreover, we shall show that for any formula φ in the above signature, there is a unique equivalent ψ in monomial normal form. These uniqueness claims shall be proved after proving the arithmetical soundness.

Definition 2.1.6. *The set Mon of monomials is defined as the set of all OWs obtained by a single iteration of an ordinal modality over \top . Moreover, we shall focus on those which have an exponent greater than zero i.e., $Mon := \{ \langle n^\alpha \rangle \top \in \mathbb{W}^{On} : \text{for some } n < \omega \text{ and } 0 < \alpha < \Lambda \}$.*

Monomial normal forms are conjunctions of monomials with an additional condition on the occurring exponents. To formulate this condition we make use of the *hyper-exponential* functions introduced in Definition 1.2.10 of Chapter 1, Section 1.2.

Definition 2.1.7. *The set of formulas in monomial normal form, MNF , is inductively defined as follows:*

- I) $\top \in MNF$;
- II) if $A \in Mon$ then $A \in MNF$;
- III) if (a) $\langle n_0^{\alpha_0} \rangle \top \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle \top \in MNF$;
- (b) $n < n_0$;
- (c) α is of the form $e^{n_0-n}(\alpha_0) \cdot (2 + \beta)$ for some $\beta < \Lambda$,

then $\langle n^\alpha \rangle \top \wedge \langle n_0^{\alpha_0} \rangle \top \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle \top \in MNF$.

Thus, monomial normal forms are conjunctions of monomials where the base of the modalities run in increasing order from left to right. The technical Item III) (c) is there to ensure that only monomials are included in the MNF if they add new information to the MNF that was not already implicit by one of the other monomials.

Remark 2.1.8. *We can observe that when we chop the left-most monomial of a MNF, the resulting formula remains a MNF. Hence if $\langle n^\alpha \rangle \top \wedge \psi \in MNF$, we have that $\psi \in MNF$.*

As we shall later see, there is another natural class of formulas that provides us with alternative normal forms. We define this set already here.

Definition 2.1.9. *The set of formulas in increasing normal form, INF, is inductively defined as follows:*

- I) $\top \in INF$;
- II) if $A \in Mon$ then $A \in INF$;
- III) if $\langle m^\beta \rangle A \in INF$, $n < m$ and $0 < \alpha$, then $\langle n^\alpha \rangle \langle m^\beta \rangle A \in INF$.

2.2. The Turing-Schmerl Calculus

In this section we introduce the logic TSC in this modal language whose main goal is to express valid relations that hold between the corresponding Turing progressions.

We will use the following notation: by $\varphi \equiv \psi$ we will denote that both $\varphi \vdash \psi$ and $\psi \vdash \varphi$ are derivable. Analogously, by $\varphi \equiv_n \psi$ we denote that for any formula $\chi \in \mathbb{F}_{<n+1}$, we have that $\varphi \vdash \chi$ iff $\psi \vdash \chi$, i.e., φ and ψ share the same $\mathbb{F}_{<n+1}$ consequences. Also, by convention we take that for any n , the formula $\langle n^0 \rangle \varphi$ is just φ .

Definition 2.2.1. *TSC is given by the following set of axioms and rules:*

Axioms:

1. $\varphi \vdash \varphi, \quad \varphi \vdash \top$;
2. $\varphi \wedge \psi \vdash \varphi, \quad \varphi \wedge \psi \vdash \psi$;
3. *Monotonicity axioms:* $\langle n^\alpha \rangle \varphi \vdash \langle n^\beta \rangle \varphi, \quad \text{for } \beta \leq \alpha$;
4. *Co-additivity axioms:* $\langle n^{\alpha+\beta} \rangle \varphi \equiv \langle n^\beta \rangle \langle n^\alpha \rangle \varphi$;

5. *Reduction axioms:* $\langle (m+n)^\alpha \rangle \varphi \vdash \langle m^{e^n(\alpha)} \rangle \varphi$;

6. *Schmerl axioms:*

$$\langle n^\alpha \rangle (\langle n_0^{\alpha_0} \rangle \top \wedge \psi) \equiv \langle n^{e^{n_0-n}(\alpha_0) \cdot (1+\alpha)} \rangle \top \wedge \langle n_0^{\alpha_0} \rangle \top \wedge \psi$$

for $n < n_0$ and $\langle n_0^{\alpha_0} \rangle \top \wedge \psi \in MNF$.

Rules:

1. If $\varphi \vdash \psi$ and $\varphi \vdash \chi$, then $\varphi \vdash \psi \wedge \chi$;

2. If $\varphi \vdash \psi$ and $\psi \vdash \chi$, then $\varphi \vdash \chi$;

3. If $\varphi \vdash \psi$, then $\langle n^\alpha \rangle \varphi \vdash \langle n^\alpha \rangle \psi$;

4. If $\varphi \vdash \psi$, then $\langle n^\alpha \rangle \varphi \wedge \langle m^{\beta+1} \rangle \psi \vdash \langle n^\alpha \rangle (\varphi \wedge \langle m^{\beta+1} \rangle \psi)$ for $m < n$.

It is worth mentioning the special character of Axioms 5 and 6 since both axioms are modal formulations of principles related to Schmerl's fine-structure theorem, also known as *Schmerl's formulas* (see [34] and [4]), that we already mentioned in the introduction. These results are formulated precisely in Proposition 3.2.5 of this thesis.

The following proposition expresses some useful properties that can be easily proved.

Proposition 2.2.2. *For any $A \in \mathbb{W}^{\text{Ord}}$ and $\varphi, \psi \in \mathbb{F}_{\text{TSC}}$:*

1. If $\varphi \vdash \psi$, then $A\varphi \vdash A\psi$;

2. $A\varphi \vdash \varphi \wedge A$;

3. $\langle (m+n)^\alpha \rangle \varphi \vdash \langle m^\alpha \rangle \varphi$;

4. $\langle n^\alpha \rangle (\varphi \wedge \psi) \wedge \langle m^{\beta+1} \rangle \varphi \equiv \langle n^\alpha \rangle (\langle m^{\beta+1} \rangle \varphi \wedge \psi)$ for $m < n$;

5. $\langle n^\alpha \rangle \varphi \wedge \psi \equiv \langle n^\alpha \rangle (\varphi \wedge \psi)$ for $\psi \in \mathbb{F}_{<n}$ and $O\text{-mod}(\psi) \subset \text{Succ}$.

Proof. The only non-trivial item is Item 5 which follows from a straightforward induction on the length of ψ using Rule 4. \square

2.3. Normal Forms

In this section we shall prove that any formula of \mathbb{F}_{TSC} is equivalent to a formula in monomial normal form, MNF, and also equivalent to a formula in increasing normal form, INF.

Monomial normal forms versus increasing normal forms

In this subsection we prove that monomial normal forms and increasing normal forms can easily be transformed into each other. Let us first observe that for each MNF there is an equivalent INF that holds some similarities with the original MNF. From now on we shall make use of the Euclidean algorithm for ordinals. More precisely, for $\beta, \delta \in \text{Lim}$, by $\frac{\beta}{\delta}$ we denote the unique α such that $\delta \cdot \alpha = \beta$, if this exists.

Theorem 2.3.1. *For every $\psi \in \text{MNF}$ with $\psi := \langle n_0^{\alpha_0} \rangle \top \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle \top$, there is an ordinal worm $A \in \text{INF}$ such that:*

1. $\psi \equiv A$;
2. $A := \langle n_0^{\beta_0} \rangle \dots \langle n_k^{\beta_k} \rangle \top$ where:
 - a) $\beta_k = \alpha_k$ and,
 - b) for all i , $0 \leq i < k$ we have

$$\beta_i = -1 + \frac{\alpha_i}{e^{n_{i+1}-n_i}(\alpha_{i+1})}.$$

Proof. By induction on k . The base case is trivial and the inductive case follows from the I.H. and the Schmerl axiom (Axiom 6). Note that by an easy induction we see that the division is well-defined in virtue of Item (iii) of Definition 2.1.7. \square

In particular, this theorem tells us that if we start out with a MNF ψ , there is an equivalent INF ordinal worm A where the modality bases are the same. The next theorem tells us that this feature of conserving the modality bases also holds if we go from an ordinal worm A in INF to an equivalent formula ψ in MNF.

Theorem 2.3.2. *Let $A := \langle n_0^{\beta_0} \rangle \dots \langle n_k^{\beta_k} \rangle \top \in \text{INF}$. There is $\psi \in \text{MNF}$ such that:*

1. $A \equiv \psi$;
2. $\psi := \langle n_0^{\alpha_0} \rangle \top \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle \top$ where
 - a) $\alpha_k = \beta_k$ and,
 - b) for all i , $0 \leq i < k$, $\alpha_i = e^{n_{i+1}-n_i}(\beta_{i+1}) \cdot (1 + \beta_i)$.

Proof. By induction on k , with the help of the Schmerl axiom (Axiom 6) for the inductive case. \square

The proofs of these theorems actually provide operations \mathcal{I} and \mathcal{M} to go from a formula ψ in MNF to a corresponding $\mathcal{I}(\psi)$ in INF and from an ordinal worm A in INF to a corresponding formula $\mathcal{M}(A)$ in MNF.

Corollary 2.3.3. *Given $\psi := \langle n_0^{\alpha_0} \rangle_{\top} \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle_{\top} \in \text{MNF}$ and A of the form $\langle n_0^{\beta_0} \rangle \dots \langle n_k^{\beta_k} \rangle_{\top} \in \text{INF}$, we have that:*

- $\mathcal{M} \circ \mathcal{I}(\psi) = \psi$;
- $\mathcal{I} \circ \mathcal{M}(A) = A$;
- If $\mathcal{I}(\psi) = A$, then $\mathcal{I}(\langle n_1^{\alpha_1} \rangle_{\top} \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle_{\top}) = \langle n_1^{\beta_1} \rangle \dots \langle n_k^{\beta_k} \rangle_{\top}$.

Monomial normal forms are closed under conjunctions

In this subsection we will prove that MNF's are closed under taking conjunctions. A first step in doing so is established by the following lemma.

Lemma 2.3.4. *Let $\psi := \langle n_0^{\alpha_0} \rangle_{\top} \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle_{\top} \in \text{MNF}$ and $n < n_0$. For any $\beta < \Lambda$, there is $\psi' \in \text{MNF}$ such that:*

$$\langle n^{\beta} \rangle_{\top} \wedge \psi \equiv \psi',$$

with $N\text{-mod}(\psi') \subseteq N\text{-mod}(\langle n^{\beta} \rangle_{\top} \wedge \psi)$.

Proof. If $\beta \leq e^{n_0-n}(\alpha_0)$, clearly we can take $\psi' = \psi$. Otherwise, let β_{CNF} be the Cantor normal form of β with base ω and we write $\beta_{\text{CNF}} := \beta_0 + \dots + \beta_i$ with each β_j non-zero and additively indecomposable and $\beta_0 \geq \dots \geq \beta_i$.

If for all j with $0 \leq j \leq i$ we have that $\beta_j \geq e^{n_0-n}(\alpha_0)$ (or equivalently, if $\beta_i \geq e^{n_0-n}(\alpha_0)$), then it is easy to see that $\langle n^{\beta} \rangle_{\top} \wedge \psi \in \text{MNF}$. If on the contrary, there is j with $0 < j \leq i$ such that $\beta_j < e^{n_0-n}(\alpha_0)$ and $\beta_{j-1} \geq e^{n_0-n}(\alpha_0)$, then we reason as follows:

$$\begin{aligned} \langle n^{\beta} \rangle_{\top} \wedge \psi &\equiv \langle n^{\beta} \rangle_{\top} \wedge \langle n_0^{\alpha_0} \rangle_{\top} \wedge \psi; \\ &\equiv \langle n^{\beta_0 + \dots + \beta_i} \rangle_{\top} \wedge \langle n_0^{\alpha_0} \rangle_{\top} \wedge \psi; \\ &\vdash \langle n^{\beta_0 + \dots + \beta_{j-1} + 1} \rangle_{\top} \wedge \langle n_0^{\alpha_0} \rangle_{\top} \wedge \psi; \\ &\vdash \langle n_0^{\alpha_0} \rangle \langle n^{\beta_0 + \dots + \beta_{j-1} + 1} \rangle_{\top} \wedge \psi; \\ &\vdash \langle n^{e^{n_0-n}(\alpha_0)} \rangle \langle n^{\beta_0 + \dots + \beta_{j-1} + 1} \rangle_{\top} \wedge \psi; \\ &\vdash \langle n^{\beta_0 + \dots + \beta_{j-1} + e^{n_0-n}(\alpha_0)} \rangle_{\top} \wedge \psi. \end{aligned}$$

However, by the monotonicity axiom (Axiom 3) we see that

$$\langle n^{\beta_0 + \dots + \beta_{j-1} + e^{n_0-n}(\alpha_0)} \rangle_{\top} \vdash \langle n^{\beta} \rangle_{\top}$$

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so that actually $\langle n^{\beta_0+\dots+\beta_{j-1}+e^{n_0-n}(\alpha_0)} \rangle_{\top} \wedge \psi \equiv \langle n^{\beta} \rangle_{\top} \wedge \psi$. Clearly we have

$$\langle n^{\beta_0+\dots+\beta_{j-1}+e^{n_0-n}(\alpha_0)} \rangle_{\top} \wedge \psi \in \text{MNF}$$

and we are done.

We can observe that, in any case, the modality bases occurring in ψ' are contained in $N\text{-mod}(\langle n^{\beta} \rangle_{\top} \wedge \psi)$, and so, $N\text{-mod}(\psi') \subseteq N\text{-mod}(\langle n^{\beta} \rangle_{\top} \wedge \psi)$. \square

We get the following simple corollaries.

Corollary 2.3.5. *Any conjunction of monomials χ is equivalent to some $\psi \in \text{MNF}$ with $N\text{-mod}(\psi) \subseteq N\text{-mod}(\chi)$.*

Proof. By the length of such a conjunction of monomials written with the modality bases from small to large, applying Lemma 2.3.4 at the inductive step. \square

We can recast this corollary as the first inductive step that each formula is equivalent to a MNF.

Corollary 2.3.6. *The monomials normal forms –MNF's– are closed under conjunctions.*

In the light of this corollary, it only remains to show that MNF's are closed under putting an ordinal-diamond up front of them.

Monomial normal forms are closed under ordinal diamonds

We shall now see that the collection of monomial normal forms is, modulo provable equivalence, closed under putting ordinal diamonds up front of them. Let us recall Schmerl's axiom scheme:

$$\langle n^{\alpha} \rangle \left(\langle n_0^{\alpha_0} \rangle_{\top} \wedge \psi \right) \equiv \langle n^{e^{n_0-n}(\alpha_0) \cdot (1+\alpha)} \rangle_{\top} \wedge \langle n_0^{\alpha_0} \rangle_{\top} \wedge \psi,$$

for $n < n_0$ and $\langle n_0^{\alpha_0} \rangle_{\top} \wedge \psi \in \text{MNF}$. We observe that this axiom tells us that putting a relatively small ($n < n_0$) non-trivial ($\alpha \geq 1$) ordinal modality up front a MNF $\langle n_0^{\alpha_0} \rangle_{\top} \wedge \psi$ yields a new MNF. To see that putting *any* ordinal modality up front a MNF is equivalent to some other MNF, we need to study two cases: $n = n_0$ and $n > n_0$. Let us start with the latter.

Proposition 2.3.7. *Let n, n_0 be natural numbers with $n > n_0$. Furthermore, let $\langle n_0^{\alpha_0} \rangle_{\top} \wedge \langle n_1^{\alpha_1} \rangle_{\top} \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle_{\top}$ with $k \geq 0$ be a conjunction of monomials not necessarily in MNF. The following principle (we use PS for Pseudo Schmerl) is derivable in TSC:*

$$\begin{aligned} \langle n^{\alpha} \rangle \left(\langle n_0^{\alpha_0} \rangle_{\top} \wedge \langle n_1^{\alpha_1} \rangle_{\top} \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle_{\top} \right) &\equiv & (PS1) \\ \langle n_0^{\alpha_0+e^{n-n_0}(\alpha)} \rangle_{\top} \wedge \langle n^{\alpha} \rangle \left(\langle n_1^{\alpha_1} \rangle_{\top} \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle_{\top} \right). & \end{aligned}$$

Note that for the case when $k = 0$, the formula $\langle n_1^{\alpha_1} \rangle_{\top} \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle_{\top}$ is to be read as the empty conjunction which is \top so that the principle becomes

$$\langle n^\alpha \rangle \langle n_0^{\alpha_0} \rangle_{\top} \equiv \langle n_0^{\alpha_0 + e^{n-n_0}(\alpha)} \rangle_{\top} \wedge \langle n^\alpha \rangle_{\top}.$$

Proof. For the right-to-left sequent, we apply the following reasoning:

$$\begin{aligned} \langle n_0^{\alpha_0 + e^{n-n_0}(\alpha)} \rangle_{\top} \wedge \langle n^\alpha \rangle & \left(\langle n_1^{\alpha_1} \rangle_{\top} \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle_{\top} \right) \\ & \vdash \langle n_0^{\alpha_0 + 1} \rangle_{\top} \wedge \langle n^\alpha \rangle \left(\langle n_1^{\alpha_1} \rangle_{\top} \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle_{\top} \right); \\ & \vdash \langle n^\alpha \rangle \left(\langle n_0^{\alpha_0 + 1} \rangle_{\top} \wedge \langle n_1^{\alpha_1} \rangle_{\top} \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle_{\top} \right); \\ & \vdash \langle n^\alpha \rangle \left(\langle n_0^{\alpha_0} \rangle_{\top} \wedge \langle n_1^{\alpha_1} \rangle_{\top} \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle_{\top} \right). \end{aligned}$$

For the left-to-right sequent we can reason as follows:

$$\begin{aligned} \langle n^\alpha \rangle & \left(\langle n_0^{\alpha_0} \rangle_{\top} \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle_{\top} \right) \\ & \vdash \langle n^\alpha \rangle \langle n_0^{\alpha_0} \rangle_{\top} \wedge \langle n^\alpha \rangle \left(\langle n_1^{\alpha_1} \rangle_{\top} \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle_{\top} \right); \\ & \vdash \langle n_0^{e^{n-n_0}(\alpha)} \rangle \langle n_0^{\alpha_0} \rangle_{\top} \wedge \langle n^\alpha \rangle \left(\langle n_1^{\alpha_1} \rangle_{\top} \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle_{\top} \right); \\ & \vdash \langle n_0^{\alpha_0 + e^{n-n_0}(\alpha)} \rangle_{\top} \wedge \langle n^\alpha \rangle \left(\langle n_1^{\alpha_1} \rangle_{\top} \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle_{\top} \right). \end{aligned}$$

□

We now turn to the remaining case when $n = n_0$, that is, when we take a Turing progression based on n_0 consistency of a theory that is axiomatized by a MNF whose smallest modality has base n_0 .

Proposition 2.3.8. *Let $\langle n_0^{\alpha_0} \rangle_{\top} \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle_{\top} \in \text{MNF}$ for $k \geq 1$. The following principle (that we shall call Pseudo Schmerl 2) is derivable in TSC:*

$$\begin{aligned} \langle n_0^\alpha \rangle & \left(\langle n_0^{\alpha_0} \rangle_{\top} \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle_{\top} \right) & (PS2) \\ & \equiv \langle n_0^{\alpha_0 + e^{n_1-n_0}(\alpha_1) \cdot \alpha} \rangle_{\top} \wedge \langle n_0^\alpha \rangle \left(\langle n_1^{\alpha_1} \rangle_{\top} \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle_{\top} \right); \\ & \equiv \langle n_0^{\alpha_0 + e^{n_1-n_0}(\alpha_1) \cdot \alpha} \rangle_{\top} \wedge \langle n_1^{\alpha_1} \rangle_{\top} \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle_{\top}. \end{aligned}$$

Proof. Note that by means of Theorem 2.3.1 together with the co-additivity axiom (Axiom 4) we have that

$$\langle n_0^\alpha \rangle \left(\langle n_0^{\alpha_0} \rangle_{\top} \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle_{\top} \right) \equiv \langle n_0^{\delta_0 + \alpha} \rangle \langle n_1^{\delta_1} \rangle \dots \langle n_k^{\alpha_k} \rangle_{\top}$$

with $\delta_i = -1 + \frac{\alpha_i}{e^{n_{i+1}-n_i}(\alpha_{i+1})}$ for $0 \leq i < k$. Thus, by Corollary 2.3.3 we get that:

$$\langle n_0^{\delta_0 + \alpha} \rangle \langle n_1^{\delta_1} \rangle \dots \langle n_k^{\alpha_k} \rangle_{\top} \equiv \langle n_0^{\delta_0 + \alpha} \rangle \left(\langle n_1^{\alpha_1} \rangle_{\top} \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle_{\top} \right).$$

Now, we apply the Schmerl axiom (Axiom 6) observing that

$$e^{n_1-n_0}(\alpha_1) \cdot (1 + \delta_0 + \alpha) = e^{n_1-n_0}(\alpha_1) \cdot \left(1 + (-1 + \frac{\alpha_0}{e^{n_1-n_0}(\alpha_1)}) + \alpha\right)$$

that is,

$$e^{n_1-n_0}(\alpha_1) \cdot (1 + \delta_0 + \alpha) = e^{n_1-n_0}(\alpha_1) \cdot \left(\frac{\alpha_0}{e^{n_1-n_0}(\alpha_1)} + \alpha\right) = \alpha_0 + e^{n_1-n_0}(\alpha_1) \cdot \alpha,$$

to conclude that

$$\langle n_0^{\delta_0+\alpha} \rangle \langle n_1^{\delta_1} \rangle \dots \langle n_k^{\alpha_k} \rangle \top \equiv \langle n_0^{\alpha_0+e^{n_1-n_0}(\alpha_1)\cdot\alpha} \rangle \top \wedge \langle n_1^{\alpha_1} \rangle \top \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle \top.$$

Thus,

$$\begin{aligned} \langle n_0^\alpha \rangle \left(\langle n_0^{\alpha_0} \rangle \top \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle \top \right) &\equiv \\ \langle n_0^{\alpha_0+e^{n_1-n_0}(\alpha_1)\cdot\alpha} \rangle \top \wedge \langle n_1^{\alpha_1} \rangle \top \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle \top. & \end{aligned}$$

Since we have that $\langle n_0^\alpha \rangle \left(\langle n_0^{\alpha_0} \rangle \top \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle \top \right) \vdash \langle n_0^\alpha \rangle \left(\langle n_1^{\alpha_1} \rangle \top \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle \top \right)$ and $\langle n_0^\alpha \rangle \left(\langle n_1^{\alpha_1} \rangle \top \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle \top \right) \vdash \langle n_1^{\alpha_1} \rangle \top \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle \top$, we also get that

$$\begin{aligned} \langle n_0^\alpha \rangle \left(\langle n_0^{\alpha_0} \rangle \top \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle \top \right) & \\ \vdash \langle n_0^{\alpha_0+e^{n_1-n_0}(\alpha_1)\cdot\alpha} \rangle \top \wedge \langle n_0^\alpha \rangle \left(\langle n_1^{\alpha_1} \rangle \top \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle \top \right); & \\ \vdash \langle n_0^{\alpha_0+e^{n_1-n_0}(\alpha_1)\cdot\alpha} \rangle \top \wedge \langle n_1^{\alpha_1} \rangle \top \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle \top, & \end{aligned}$$

which concludes the proof. \square

This proposition allows us to incorporate a small modality into a MNF that starts with the same modality. We note that the degenerate version of the proposition would be with $k = 0$ which is covered by the co-additivity axiom (Axiom 4): $\langle n_0^\alpha \rangle \langle n_0^{\alpha_0} \rangle \top \equiv \langle n_0^{\alpha_0+\alpha} \rangle \top$.

Theorem 2.3.9. *The set of MNF's is, modulo provable equivalence, closed under putting ordinal modalities up front.*

Proof. For an arbitrary MNF $\langle n_0^{\alpha_0} \rangle \top \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle \top$ and arbitrary ordinal modality $\langle n^\alpha \rangle$ we have to see that $\langle n^\alpha \rangle \left(\langle n_0^{\alpha_0} \rangle \top \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle \top \right)$ is equivalent to some MNF. The proof follows directly from our earlier results.

By Pseudo Schmerl 1, Proposition 2.3.7, we can 'pull' the ordinal modality $\langle n^\alpha \rangle$ over the smaller ordinal modalities of $\langle n_0^{\alpha_0} \rangle \top \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle \top$ until $n \leq n_i$ for some $0 \leq i \leq k$.

In case $n = n_i$ we can apply (PS2) (or the co-additivity axiom (Axiom 4) for the degenerate case) and we can apply directly Schmerl's axiom when $n < n_i$. The result yields us a conjunction of monomials which by Corollary 2.3.5 we know is equivalent to some MNF. \square

Each formula is equivalent to a normal form

In this subsection, we shall combine all the earlier results of this section to obtain the, by now, simple corollary that every formula φ is equivalent to a MNF formula. As a corollary we will obtain that φ is also equivalent to an ordinal worm A in INF. However, to establish the uniqueness of MNF's and INF's we need non-derivability results. And, without modal semantics, the arithmetical soundness of the system is required. Hence, the uniqueness will be discussed later.

Theorem 2.3.10. *For every formula φ , there is $\psi \in MNF$ such that $\varphi \equiv \psi$.*

Proof. By induction on φ applying Corollary 2.3.6 and Theorem 2.3.9 for the conjunctive and ordinal modality cases, respectively. \square

Consequently, we see that each formula also has an INF.

Corollary 2.3.11. *For every formula φ , there is $A \in INF$ such that $\varphi \equiv A$.*

Proof. By combining Theorems 2.3.1 and 2.3.10. \square

Arithmetical Semantics

We dedicate this chapter to introduce the arithmetical semantics of **TSC**. First we shall make the link between our logic and graded Turing progressions more precise by introducing the *Formalized Turing Progression (FTP) Interpretation* that defines an arithmetical translation between modal formulas and arithmetical formulas. This allows us to fix an interpretation for the modal sequents within EA^+ .

In Section 3.2 we prove the soundness of **TSC** with respect to this interpretation. In Section 3.3, we prove some conservativity results and a characterization for the derivability between MNF. With these results we prove the arithmetical completeness of the system.

The content of this chapter can be found in [27].

3.1. Arithmetical interpretation of modal formulas

As we already mentioned in the introduction, we want our modal formulas to directly denote arithmetical theories rather than individual arithmetical sentences. Therefore, we interpret our modal formulae in terms of the τ -formulae presented in Section 1.1 of Chapter 1. Moreover, the conjunction of modal formulas is intended to mean the union of theories. Hence, we map the interpretation of modal conjunctions to the disjunction of the respective interpretations.

Definition 3.1.1. *An arithmetical interpretation for \mathbb{F}_{TSC} is a map $* : \mathbb{F}_{TSC} \rightarrow \text{Form}_{\mathcal{L}_N}$ inductively defined as follows:*

1. $(\top)^*(x) = \epsilon(x)$;

2. $(\varphi \wedge \psi)^*(x) = (\varphi)^*(x) \vee (\psi)^*(x)$;
3. $(\langle n^\alpha \rangle \varphi)^*(x) = \tau_n^{\varphi^*}(\alpha, x)$.

Note that since \mathbb{F}_{TSC} has no propositional variables, we can identify a modal formula with its arithmetical interpretation unambiguously and so we will do in the remainder of this dissertation.

Definition 3.1.2. *Given a numeration σ , by Th_σ we denote the theory of σ . We say that $\chi \in \text{Th}_\sigma$ iff $\mathbb{N} \models \sigma(\ulcorner \chi \urcorner)$.*

For the sake of clarity, and since we are working in the closed fragment, we will use the following notation: given $\varphi \in \mathbb{F}_{\text{TSC}}$ by Th_φ we denote Th_σ where $\varphi^*(x) = \sigma(x)$, following Definition 3.1.1. If $\varphi^*(x) = \epsilon(x)$ we use just EA^+ instead of Th_ϵ . Also, if $\varphi := \langle n^\alpha \rangle \psi$ we write $(\text{Th}_\psi)_n^\alpha$ for the theory of $(\langle n^\alpha \rangle \psi)^*$.

The Formalized Turing Progression Interpretation (FTP) is the representation of provable **TSC** sequents as the entailment between the corresponding first order theories within EA^+ via the Π_2^0 -sentence expressing such derivability. Thus, given $\varphi, \psi \in \mathbb{F}_{\text{TSC}}$ the intended interpretation of $\varphi \vdash \psi$ is the arithmetical statement:

$$\text{EA}^+ \vdash \forall x (\Box_{\text{Th}_\psi}(x) \rightarrow \Box_{\text{Th}_\varphi}(x)).$$

Hereinafter, we will adopt the following notation: by $T \subseteq U$ we denote the arithmetical formula $\forall x (\Box_T(x) \rightarrow \Box_U(x))$. Likewise, by $T \equiv U$, we denote the formula $\forall x (\Box_T(x) \leftrightarrow \Box_U(x))$ and $T \equiv_n U$ stands for the formula $\forall x \in \Pi_{n+1}^0 (\Box_T(x) \leftrightarrow \Box_U(x))$.

3.2. Arithmetical Soundness of TSC

In this section we shall prove that the logic **TSC** adequately describes graded Turing progressions.

Before proving soundness of **TSC** under the FTP interpretation we will present some useful facts and tools in this subsection. In Lemma 2.2 of [2] a very simple yet very useful fact is shown: if a formula is provable in some Turing progression, then it is provable in the base theory together with a single consistency statement.

Lemma 3.2.1. *Provably in EA^+ we have that*

$$\forall \alpha \succ 0 \forall x \left(\Box_{T_n^\alpha}(x) \rightarrow \exists \gamma \prec \alpha \Box_T(\text{Con}_n((T)_n^\gamma) \rightarrow x) \right).$$

The proof of this lemma is elementary as holds for the following.

Lemma 3.2.2. *Let T and U be theories extending EA. For any $n, m < \omega$ and $\delta \preceq \alpha \prec \Lambda$, provably in EA^+ :*

1. $T \subseteq U \rightarrow (\text{Con}_n(U) \rightarrow \text{Con}_n(T));$
2. $U \equiv_n T \rightarrow (\text{Con}_n(T) \leftrightarrow \text{Con}_n(U));$
3. $\beta \preceq \gamma \rightarrow (T)_n^\beta \subseteq (T)_n^\gamma;$
4. $\text{Con}_n((T)_m^\alpha) \rightarrow \text{Con}_n((T)_m^\delta).$

Proof. All items allow for elementary proofs. For example, reasoning informally in EA^+ , for Item 1, assume $T \subseteq U$ and $\text{Con}_n(U)$. Now, let $\pi \in \Pi_{n+1}^0$ such that $\Box_T \pi$. By assumption, $\Box_U \pi$ and by $\text{Con}_n(U)$ we get that $\text{Tr}_{n+1}(\pi)$ as was to be shown. Item 2 goes analogously. Item 3 follows directly from Lemma 3.2.1 and Item 4 follows from Item 1 together with the fact that $(T)_m^\delta \subseteq (T)_m^\alpha$. \square

In Chapter 1, Section 1.3 we presented Schmerl's *reflexive induction*. With this tool at hand, we have access to some form of transfinite induction even if we cannot prove well-foundedness of the corresponding orderings within the base theory. Let us, by way of example, proof with detail that stronger theories yield stronger Turing progressions.

Lemma 3.2.3. *Let T and U be theories extending EA^+ , $n < \omega$ and $\alpha \prec \Lambda$. Provably in EA^+ :*

$$T \subseteq U \rightarrow (T)_n^\alpha \subseteq (U)_n^\alpha.$$

Proof. We reason informally in EA^+ , assume $T \subseteq U$ and use reflexive transfinite induction on α with the base case being trivial. Thus, for $\alpha \succ 0$, consider an arbitrary formula χ such that $\Box_{(T)_n^\alpha} \chi$. Then, there is some $\delta \prec \alpha$ such that $\Box_T (\text{Con}_n((T)_n^\delta) \rightarrow \chi)$. By the assumption that $T \subseteq U$ together with provable monotonicity of the provability operator, we get that $\Box_{(U)_n^\alpha} (\text{Con}_n((T)_n^\delta) \rightarrow \chi)$. Moreover, we have that $\Box_{(U)_n^\alpha} \text{Con}_n((U)_n^\delta)$. Now, by the RIH we know that $\Box_{\text{EA}^+} ((T)_n^\delta \subseteq (U)_n^\delta)$ so that along with Item 1 of Lemma 3.2.2 under the \Box_{EA^+} we obtain $\Box_{(U)_n^\alpha} \text{Con}_n((T)_n^\delta)$ and consequently $\Box_{(U)_n^\alpha} \chi$. \square

From now on we shall include less details in our proofs that employ reflexive induction.

Lemma 3.2.4. *Let T be any theory extending EA^+ and $\sigma \in \Sigma_{n+1}^0$. The following principles are provable in EA^+ :*

1. $\text{Con}_n(T) \wedge \sigma \rightarrow \text{Con}_n(T + \sigma);$
2. $(T + \sigma)_n^\alpha \equiv (T)_n^\alpha + \sigma;$

3. $T + \text{Con}_n((T)_n^\alpha) \equiv (T)_n^{\alpha+1}$;
4. $\text{Con}_{n+1}(T) \rightarrow \text{Con}_n(T)$;
5. $(T)_n^\alpha \subseteq (T)_{n+m}^\alpha$.

Proof. Items 1, 3 and 4 are easy to check. The right-to-left implication of Item 2 is straightforward.

For the left-to-right implication we proceed by transfinite reflexive induction with the base case being trivial. Thus, we reason in EA^+ , assume as the RIH that $\Box_{\text{EA}^+} \forall \beta \prec \alpha ((T + \sigma)_n^\beta \subseteq (T)_n^\beta + \sigma)$, and let $\Box_{(T+\sigma)_n^\alpha} \chi$ for some arbitrary χ . Thus, $\exists \delta \prec \alpha \Box_{T+\sigma} (\text{Con}_n((T + \sigma)_n^\delta) \rightarrow \chi)$, and by monotonicity

$$\exists \delta \prec \alpha \Box_{(T)_n^\alpha + \sigma} (\text{Con}_n((T + \sigma)_n^\delta) \rightarrow \chi). \quad (3.1)$$

Notice that $\Box_{(T)_n^\alpha + \sigma} (\text{Con}_n((T)_n^\delta) \wedge \sigma)$ so with the help of Item 1 we obtain that $\Box_{(T)_n^\alpha + \sigma} \text{Con}_n((T)_n^\delta + \sigma)$. By the RIH together with monotonicity, we get that $\Box_{(T)_n^\alpha + \sigma} ((T + \sigma)_n^\delta \subseteq (T)_n^\delta + \sigma)$. Combining this with Item 1 of Lemma 3.2.2 (under a box) we conclude that $\Box_{(T)_n^\alpha + \sigma} \text{Con}_n((T + \sigma)_n^\delta)$. With the help of 3.1 we finally get that $\Box_{(T)_n^\alpha + \sigma} \chi$.

Item 5 tells us that Turing progressions are monotone in the consistency notion and follows from Item 4 from a straightforward reflexive transfinite induction. \square

Our calculus will prove all the provable relations between the Turing progressions that are considered in this dissertation. A first step and cornerstone in this study is Proposition 3.2.5 below as was proven by Schmerl in [34] for the base theory of primitive recursive arithmetic. Beklemishev generalized this to the setting of EA^+ with variable base theories. We cite Beklemishev's formulation here (Theorem 3 and Theorem 4 of [4]).

Proposition 3.2.5 (Fine-structure formula). *Let T be an axiomatizable theory which is a Π_{n+2}^0 -axiomatized extension of EA , then provably in EA^+ :*

$$S1 \quad \forall \alpha \succ 0 \left((T)_{n+m}^\alpha \equiv_n (T)_n^{e^m(\alpha)} \right);$$

$$S2 \quad \forall \beta \succ 0 \left(((T)_{n+m}^\beta)_n^\alpha \equiv_n (T)_n^{e^m(\beta) \cdot (1+\alpha)} \right).$$

Soundness

In this subsection we shall show that **TSC** is sound for the FTP interpretation. We first present the main simple argument and will then fill out the missing details in the remainder of this subsection.

Theorem 3.2.6 (Soundness). *Given $\varphi, \psi \in \mathbb{F}_{TSC}$, if $\varphi \vdash \psi$ then*

$$\text{EA}^+ \vdash \forall x (\Box_{\text{Th}_\psi}(x) \rightarrow \Box_{\text{Th}_\varphi}(x)).$$

Proof. By induction on the length of a TSC proof of $\varphi \vdash \psi$. Thus, we see that all axioms are arithmetically sound and that all rules preserve provability.

The first three axioms and first two rules are easily seen to be arithmetically sound. In particular, Axiom 3 – $\langle n^\alpha \rangle \varphi \vdash \langle n^\beta \rangle \varphi$ for $\alpha \geq \beta$ – expresses the property that smooth Turing progressions are monotone in the ordinal as expressed by Item 3 of Lemma 3.2.2. On the other hand, Rule 3 –if $\varphi \vdash \psi$, then $\langle n^\alpha \rangle \varphi \vdash \langle n^\alpha \rangle \psi$ – expresses the monotonicity of smooth Turing progressions in the base theory as reflected in Lemma 3.2.3. For a proof of the soundness of the co-additivity axiom, Axiom 4, we refer to Lemma 2.6 of [2]. The remaining rule and the remaining two axioms are separately proven to be sound in the remainder of this subsection. \square

We will start by looking at the remaining rule. Note that in proving the soundness we may use the arithmetical counterpart of the modal axioms and rules that we already have proven to be sound. Instead of talking every time about the arithmetical counterpart of such rules and axioms, we will for the sake of presentation simply speak about the modal rules and axioms. As such, our proofs below will seem an amalgamate of modal and arithmetical reasoning.

Lemma 3.2.7. *Rule 4 is sound w.r.t. the FTP interpretation.*

Proof. Let us recall Rule 4:

$$\text{If } \varphi \vdash \psi, \text{ then } \langle n^\alpha \rangle \varphi \wedge \langle m^{\beta+1} \rangle \psi \vdash \langle n^\alpha \rangle (\varphi \wedge \langle m^{\beta+1} \rangle \psi) \text{ for } m < n.$$

To prove the arithmetical soundness, we assume that provably in EA^+ we have $\text{Th}_\psi \subseteq \text{Th}_\varphi$. Consequently $(\text{Th}_\psi + \text{Th}_\varphi) \subseteq \text{Th}_\varphi$ so that by Rule 3 we obtain

$$(\text{Th}_\psi + \text{Th}_\varphi)_n^\alpha \subseteq (\text{Th}_\varphi)_n^\alpha. \quad (3.2)$$

Also, we have the following provably in EA^+ :

$$\left(\text{Th}_\varphi + (\text{Th}_\psi)_m^{\beta+1} \right)_n^\alpha \subseteq \left(\text{Th}_\varphi + \text{Th}_\psi + \text{Con}_m((\text{Th}_\psi)_m^\beta) \right)_n^\alpha,$$

and since $\text{Con}_m((\text{Th}_\psi)_m^\beta) \in \Pi_{m+1}^0$ and $\Pi_{m+1}^0 \subset \Sigma_{n+1}^0$ we also have

$$\left(\text{Th}_\varphi + \text{Th}_\psi + \text{Con}_m((\text{Th}_\psi)_m^\beta) \right)_n^\alpha \subseteq (\text{Th}_\varphi + \text{Th}_\psi)_n^\alpha + \text{Con}_m((\text{Th}_\psi)_m^\beta).$$

Consequently, for any formula χ we have provably in EA^+ that

$$\Box \left(\text{Th}_\varphi + (\text{Th}_\psi)_m^{\beta+1} \right)_n^\alpha \chi \longrightarrow \Box (\text{Th}_\varphi + \text{Th}_\psi)_n^\alpha + \text{Con}_m((\text{Th}_\psi)_m^\beta) \chi.$$

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Now, by the formalized deduction theorem together with 3.2 we have that for any formula χ that if $\Box_{(\text{Th}_\varphi + \text{Th}_\psi)_n^\alpha + \text{Con}_m((\text{Th}_\psi)_m^\beta)} \chi$, then

$$\Box_{(\text{Th}_\varphi)_n^\alpha} (\text{Con}_m((\text{Th}_\psi)_m^\beta) \rightarrow \chi)$$

whence by monotonicity $\Box_{(\text{Th}_\varphi)_n^\alpha + (\text{Th}_\psi)_m^{\beta+1}} (\text{Con}_m((\text{Th}_\psi)_m^\beta) \rightarrow \chi)$. In particular we get $\Box_{(\text{Th}_\varphi)_n^\alpha + (\text{Th}_\psi)_m^{\beta+1}} \chi$ as was to be shown. \square

We will now consider Axiom 5: $\langle (m+n)^\alpha \rangle \varphi \vdash \langle m^{e^n(\alpha)} \rangle \varphi$. It is easy to see that we can reduce the arithmetical soundness of this axiom to the arithmetical soundness of $\langle (n+1)^\alpha \rangle \varphi \vdash \langle n^{e(\alpha)} \rangle \varphi$. Thus, we just check the soundness of this principle with the help of the following proposition:

Proposition 3.2.8. *Let $\varphi(x)$ be any arithmetical formula. By $I_\varphi(\bar{n})$ we denote the formula $\varphi(0) \wedge \forall x (\varphi(x) \rightarrow \varphi(s(x))) \rightarrow \varphi(\bar{n})$. For any extension T of EA , provably in EA^+ :*

$$\forall n \Box_T I_\varphi(\bar{n}).$$

Proof. Sketch: We consider the proof that combines $\varphi(0)$ with $\varphi(0) \rightarrow \varphi(1)$ to obtain $\varphi(1)$ and so forth till arriving at $\varphi(\bar{n})$ and observe that the Gödel number of this proof can be exponentially bounded in the Gödel number of φ . \square

The following lemma states a slightly stronger principle than the one captured by Axiom 5 and thus suits the purpose of proving soundness.

Lemma 3.2.9. *For any extension T of EA^+ , any ordinals $\alpha, \beta \prec \Lambda$ and natural number n , provably in EA^+ we have:*

$$((T)_n^\beta)^{e(\alpha)} \subseteq ((T)_n^\beta)_{n+1}^\alpha.$$

Proof. By reflexive transfinite induction on α with the base case being trivial. Thus, reasoning in EA^+ , we assume

$$\Box_{\text{EA}^+} \forall \gamma \prec \alpha \forall \beta \left(((T)_n^\beta)^{e(\gamma)} \subseteq ((T)_n^\beta)_{n+1}^\gamma \right)$$

as the RIH. We proceed by a case distinction on α .

If $\alpha \in \text{Succ}$, i.e., $\alpha := \delta + 1$, let χ be some arbitrary formula such that $\Box_{((T)_n^\beta)^{e(\delta+1)}} \chi$. Note that $e(\delta + 1) = e(\delta) \cdot \omega$. Therefore we have

$$\Box_{((T)_n^\beta)^{e(\delta+1)}} \chi \longrightarrow \exists m \prec \omega \Box_T \left(\text{Con}_n \left(((T)_n^\beta)^{e(\delta) \cdot m} \right) \rightarrow \chi \right),$$

and by monotonicity of provability we obtain

$$\exists m \prec \omega \square_{((T)_n^\beta)_{n+1}^{\delta+1}} \left(\text{Con}_n \left(((T)_n^\beta)^{e(\delta) \cdot m} \right) \rightarrow \chi \right).$$

Let $\theta(x) := \text{Con}_n \left(((T)_n^\beta)^{e(\delta) \cdot x} \right)$. We prove

$$\forall m \prec \omega \square_{((T)_n^\beta)_{n+1}^{\delta+1}} \theta(m) \tag{3.3}$$

by applying Proposition 3.2.8. Therefore, it suffices to show

$$\square_{((T)_n^\beta)_{n+1}^{\delta+1}} \left(\theta(0) \wedge \forall x \left(\theta(x) \rightarrow \theta(s(x)) \right) \right).$$

Clearly $\square_{((T)_n^\beta)_{n+1}^{\delta+1}} \text{Con}_n(T)$, that is, $\square_{((T)_n^\beta)_{n+1}^{\delta+1}} \theta(0)$. We only need to check that

$$\square_{((T)_n^\beta)_{n+1}^{\delta+1}} \forall x \left(\theta(x) \rightarrow \theta(s(x)) \right).$$

For this purpose, assume $\square_{((T)_n^\beta)_{n+1}^{\delta+1}} \text{Con}_n \left(((T)_n^\beta)^{e(\delta) \cdot m} \right)$ and set out to prove

$\square_{((T)_n^\beta)_{n+1}^{\delta+1}} \text{Con}_n \left(((T)_n^\beta)^{e(\delta) \cdot (m+1)} \right)$. Hence, we have that

$$\square_{((T)_n^\beta)_{n+1}^{\delta+1}} \left(\text{Con}_n \left(((T)_n^\beta)^{e(\delta) \cdot m} \right) \wedge \text{Con}_{n+1} \left(((T)_n^\beta)_{n+1}^\delta \right) \right).$$

By Lemmas 3.2.4 and 3.2.2 we can push the small consistency inside the bigger ones and obtain

$$\square_{((T)_n^\beta)_{n+1}^{\delta+1}} \text{Con}_{n+1} \left(\left(((T)_n^\beta)^{e(\delta) \cdot m} \right)_{n+1}^\delta \right). \tag{3.4}$$

We note that $((T)_n^\beta)^{e(\delta) \cdot m} \equiv (T)_n^{\beta+e(\delta) \cdot m}$ and by the RIH we have that

$$\square_{((T)_n^\beta)_{n+1}^{\delta+1}} \left(\left((T)_n^{\beta+e(\delta) \cdot m} \right)_n^{e(\delta)} \subseteq \left((T)_n^{\beta+e(\delta) \cdot m} \right)_{n+1}^\delta \right).$$

Thus, $\square_{((T)_n^\beta)_{n+1}^{\delta+1}} \text{Con}_{n+1} \left(\left((T)_n^{\beta+e(\delta) \cdot m} \right)_n^{e(\delta)} \right)$, and by monotonicity

$$\square_{((T)_n^\beta)_{n+1}^{\delta+1}} \text{Con}_n \left(\left((T)_n^{\beta+e(\delta) \cdot m} \right)_n^{e(\delta)} \right).$$

We can observe that $\left((T)_n^{\beta+e(\delta) \cdot m} \right)_n^{e(\delta)} \equiv \left((T)_n^\beta \right)_n^{e(\delta) \cdot (m+1)}$. Therefore, we have that $\square_{((T)_n^\beta)_{n+1}^{\delta+1}} \text{Con}_n \left(\left((T)_n^\beta \right)_n^{e(\delta) \cdot (m+1)} \right)$ as required.

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In the case that $\alpha \in \mathbf{Lim}$ we assume $\Box_{((T)_n^\beta)^{e(\alpha)}} \chi$. Since e is a continuous function, as before we have that

$$\exists \delta \prec \alpha \Box_T \left(\text{Con}_n \left(((T)_n^\beta)^{e(\delta)} \right) \rightarrow \chi \right).$$

By monotonicity we obtain for this δ that

$$\Box_{((T)_n^\beta)_{n+1}^\alpha} \left(\text{Con}_n \left(((T)_n^\beta)^{e(\delta)} \right) \rightarrow \chi \right).$$

Since $\Box_{((T)_n^\beta)_{n+1}^\alpha} \text{Con}_n \left(((T)_n^\beta)_{n+1}^\delta \right)$ by the RIH we get

$$\Box_{((T)_n^\beta)_{n+1}^\alpha} \text{Con}_n \left(((T)_n^\beta)^{e(\delta)} \right)$$

so that $\Box_{((T)_n^\beta)_{n+1}^\alpha} \chi$. □

Corollary 3.2.10. *Axiom 5 is sound.*

Proof. This follows directly from the previous lemma by taking $\beta = 0$. □

Before we are going to prove the soundness of the Schmerl axioms, we first need some observations to the extent that for sufficiently similar base theories, their corresponding Turing progressions will be similar as well.

Lemma 3.2.11. *For any theories T and U extending EA, $0 < m$ and $\alpha \prec \Lambda$, if, provably in EA^+ , $T \equiv_{n+m} U$, then*

$$\text{EA}^+ \vdash \text{Con}_n((T)_n^\alpha) \leftrightarrow \text{Con}_n((U)_n^\alpha).$$

Proof. For reasoning in EA^+ , suppose that $T \equiv_{n+m} U$. By reflexive transfinite induction on α . Since base case is trivial, we check the Lemma for $\alpha \succ 0$. Thus, assume as RIH that $\Box_{\text{EA}^+} \forall \beta \prec \alpha \left(\text{Con}_n((T)_n^\beta) \leftrightarrow \text{Con}_n((U)_n^\beta) \right)$.

For the left-to-right implication, assume $\text{Con}_n((T)_n^\alpha)$ and assume for $\pi \in \Pi_{n+1}^0$, $\Box_{(U)_n^\alpha} \pi$. Thus, $\Box_U \left(\text{Con}_n((U)_n^\delta) \rightarrow \pi \right)$ for $\delta \prec \alpha$. Since $\pi \in \Pi_{n+1}^0$ and $n < n + m$, by assumption, $\Box_T \left(\text{Con}_n((U)_n^\delta) \rightarrow \pi \right)$ and by monotonicity,

$$\Box_{(T)_n^\alpha} \left(\text{Con}_n((U)_n^\delta) \rightarrow \pi \right).$$

Observe that $\Box_{(T)_n^\alpha} \text{Con}_n((T)_n^\delta)$, and since by monotonicity the RIH holds provably in $(T)_n^\alpha$, we conclude that $\Box_{(T)_n^\alpha} \pi$. For the other other implication we reason analogously. □

With this previous result, we can establish the following corollary about the conservativity between Turing progressions built up from sufficiently similar base theories.

Corollary 3.2.12. *For any theories T and U extending EA, any $m > 0$ and any ordinal $\alpha < \Lambda$,*

$$\text{EA}^+ \vdash T \equiv_{n+m} U \implies \text{EA}^+ \vdash (T)_n^\alpha \equiv_n (U)_n^\alpha.$$

Proof. For reasoning in EA^+ , suppose that $T \equiv_{n+m} U$. Let $\pi \in \Pi_{n+1}^0$ and assume $\Box_{(T)_n^\alpha} \pi$ for some arbitrary α . Thus,

$$\exists \delta \prec \alpha \Box_T \left(\text{Con}_n((T)_n^\delta) \rightarrow \pi \right).$$

By assumption together with monotonicity, we have that

$$\Box_{(U)_n^\alpha} \left(\left(\text{Con}_n((T)_n^\delta) \rightarrow \pi \right) \wedge \text{Con}_n((U)_n^\delta) \right).$$

Therefore, by previous lemma, $\Box_{(U)_n^\alpha} \text{Con}_n((T)_n^\delta)$ and so $\Box_{(U)_n^\alpha} \pi$. For the other inclusion we reason analogously. \square

We can now prove that the smallest term of an MNF captures all the small consistency strength that is implied by the entire MNF.

Lemma 3.2.13. *Let $\psi := \langle n_0^{\alpha_0} \rangle \top \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle \top \in \text{MNF}$. We then have that provably in EA^+ :*

$$\text{Th}_\psi \equiv_{n_0} (\text{EA}^+)_{n_0}^{\alpha_0}.$$

Proof. By induction on k with base case being trivial. For the inductive step, let $\psi' := \langle n_1^{\alpha_1} \rangle \top \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle \top \in \text{MNF}$ and $\psi := \langle n_0^{\alpha_0} \rangle \top \wedge \psi' \in \text{MNF}$. Notice that the right-to-left inclusion is straightforward so we just check the left-to-right direction. Reasoning in EA^+ , suppose that $\Box_{\text{Th}_\psi} \pi$ for some $\pi \in \Pi_{n_0+1}^0$. Thus, $\Box_{\text{Th}_{\psi'}} \left(\text{Con}_{n_0}((\text{EA}^+)_{n_0}^\delta) \rightarrow \pi \right)$ for some $\delta \prec \alpha_0$. Notice that

$$\left(\text{Con}_{n_0}((\text{EA}^+)_{n_0}^\delta) \rightarrow \pi \right) \in \Pi_{n_0+2}^0$$

whence also in $\Pi_{n_1+1}^0$ so that by the external induction on k we get that

$$\Box_{(\text{EA}^+)_{n_1}^{\alpha_1}} \left(\text{Con}_{n_0}((\text{EA}^+)_{n_0}^\delta) \rightarrow \pi \right). \text{ By monotonicity it follows that}$$

$$\Box_{((\text{EA}^+)_{n_1}^{\alpha_1})_{n_0}^{\beta_0}} \left(\text{Con}_{n_0}((\text{EA}^+)_{n_0}^\delta) \rightarrow \pi \right)$$

for β_0 so that $\alpha_0 = e^{n_1-n_0}(\alpha_1) \cdot (1 + \beta_0)$. Notice that by Schmerl's fine-structure formula (Item 3.2.5 of Proposition 3.2.5) together with the facts that

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$\delta \prec \alpha_0$ and $\text{Con}_{n_0}((\text{EA}^+)_{n_0}^\delta) \in \Pi_{n_0+1}^0$, we get that $\square_{((\text{EA}^+)_{n_1}^{\alpha_1})_{n_0}^{\beta_0}} \text{Con}_{n_0}((\text{EA}^+)_{n_0}^\delta)$ and so, $\square_{((\text{EA}^+)_{n_1}^{\alpha_1})_{n_0}^{\beta_0}} \pi$. Applying Schmerl's fine-structure formula again we get that $\square_{(\text{EA}^+)_{n_0}^{\alpha_0}} \pi$. \square

The lemma can be easily generalized to conclude that initial truncations of an MNF provide optimal approximations of the entire MNF.

Lemma 3.2.14. *Let $\psi := \langle n_0^{\alpha_0} \rangle \top \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle \top \in \text{MNF}$. We then have for any $j \leq k$ that provably in EA^+ :*

$$\text{Th}_\psi \equiv_{n_j} (\text{EA}^+)_{n_0}^{\alpha_0} \wedge \dots \wedge (\text{EA}^+)_{n_j}^{\alpha_j}.$$

Proof. The right-to-left implication is straightforward. For the other implication, we make the following case distinction on j . If $j = 0$, then it follows from Lemma 3.2.13. If $j \neq 0$, for reasoning within EA^+ , let $\pi \in \Pi_{n_j+1}^0$ such that $\square_{\text{Th}_\psi} \pi$. By the formalized deduction theorem we get that

$$\square_{(\text{EA}^+)_{n_j}^{\alpha_j} + \dots + (\text{EA}^+)_{n_k}^{\alpha_k}} \left(\bigwedge_{0 \leq i < j} \text{Con}_{n_i}((\text{EA}^+)_{n_i}^{\beta_i}) \rightarrow \pi \right)$$

for some sequence of $\beta_i \prec \alpha_i$. Notice that each $\text{Con}_{n_i}((\text{EA}^+)_{n_i}^{\beta_i}) \in \Pi_{n_i+1}^0$ with $n_i < n_j$. Therefore, $\left(\bigwedge_{0 \leq i < j} \text{Con}_{n_i}((\text{EA}^+)_{n_i}^{\beta_i}) \rightarrow \pi \right) \in \Pi_{n_j+1}^0$.

Also, notice that $\langle n_j^{\alpha_j} \rangle \top \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle \top \in \text{MNF}$. Thus, applying Lemma 3.2.13, we obtain that

$$\square_{(\text{EA}^+)_{n_j}^{\alpha_j}} \left(\bigwedge_{0 \leq i < j} \text{Con}_{n_i}((\text{EA}^+)_{n_i}^{\beta_i}) \rightarrow \pi \right)$$

and by monotonicity,

$$\square_{(\text{EA}^+)_{n_0}^{\alpha_0} + \dots + (\text{EA}^+)_{n_j}^{\alpha_j}} \left(\bigwedge_{0 \leq i < j} \text{Con}_{n_i}((\text{EA}^+)_{n_i}^{\beta_i}) \rightarrow \pi \right).$$

Finally, we have that $\square_{(\text{EA}^+)_{n_0}^{\alpha_0} + \dots + (\text{EA}^+)_{n_j}^{\alpha_j}} \bigwedge_{0 \leq i < j} \text{Con}_{n_i}((\text{EA}^+)_{n_i}^{\beta_i})$ since $\beta_i \prec \alpha_i$ for each i , $0 \leq i < j$. Therefore, we can conclude that

$$\square_{(\text{EA}^+)_{n_0}^{\alpha_0} + \dots + (\text{EA}^+)_{n_j}^{\alpha_j}} \pi.$$

\square

We are now ready to prove the soundness of our more complicated axiom.

Lemma 3.2.15. *Schmerl's axioms –Axiom 6– is sound. That is,*

$$\langle n^\alpha \rangle \left(\langle n_0^{\alpha_0} \rangle \top \wedge \psi \right) \equiv \langle n^{e^{n_0-n}(\alpha_0) \cdot (1+\alpha)} \rangle \top \wedge \langle n_0^{\alpha_0} \rangle \top \wedge \psi$$

for $n < n_0$ and $\langle n_0^{\alpha_0} \rangle \top \wedge \psi \in MNF$.

Proof. Let $\psi' := \langle n_0^{\alpha_0} \rangle \top \wedge \psi$. Thus, by the previous lemma we have that

$$EA^+ \vdash Th_{\psi'} \equiv_{n_0} (EA^+)_{n_0}^{\alpha_0}$$

and then, since $n < n_0$, by Corollary 3.2.12 we get that

$$EA^+ \vdash (Th_{\psi'})_n^\alpha \equiv_n ((EA^+)_{n_0}^{\alpha_0})_n^\alpha.$$

Thus, applying Schmerl's fine-structure formula (Proposition 3.2.5, Item 3.2.5), we have that $EA^+ \vdash (Th_{\psi'})_n^\alpha \equiv_n (EA^+)_n^{e^{n_0-n}(\alpha_0) \cdot (1+\alpha)}$.

To check the soundness of Axioms 6, we need to see that provably in EA^+ : $(Th_{\psi'})_n^\alpha \equiv_n (EA^+)_n^{e^{n_0-n}(\alpha_0) \cdot (1+\alpha)} + Th_{\psi'}$. For the right-to-left inclusion, suppose that $\Box_{(EA^+)_n^{e^{n_0-n}(\alpha_0) \cdot (1+\alpha)} + Th_{\psi'}} \varphi$ for some formula φ of arbitrary complexity. Therefore, by the formalized deduction theorem,

$$\exists \delta \prec e^{n_0-n}(\alpha_0) \cdot (1+\alpha) \Box_{Th_{\psi'}} \left(Con_n((EA^+)_n^\delta) \rightarrow \varphi \right).$$

Hence, by monotonicity, $\Box_{(Th_{\psi'})_n^\alpha} \left(Con_n((EA^+)_n^\delta) \rightarrow \varphi \right)$. Since we have that provably in EA^+ , $(Th_{\psi'})_n^\alpha \equiv_n (EA^+)_n^{e^{n_0-n}(\alpha_0) \cdot (1+\alpha)}$, that $Con_n((EA^+)_n^\delta) \in \Pi_{n+1}^0$ and that $\delta \prec e^{n_0-n}(\alpha_0) \cdot (1+\alpha)$, we get that $\Box_{(Th_{\psi'})_n^\alpha} Con_n((EA^+)_n^\delta)$ and so, $\Box_{(Th_{\psi'})_n^\alpha} \varphi$.

For the left-to-right inclusion, assume $\Box_{(Th_{\psi'})_n^\alpha} \varphi$ for some φ . Then, by the formalized deduction theorem, $\Box_{Th_{\psi'}} \left(Con_n((Th_{\psi'})_n^\delta) \rightarrow \varphi \right)$ for some $\delta \prec \alpha$ and by monotonicity, $\Box_{(EA^+)_n^{e^{n_0-n}(\alpha_0) \cdot (1+\alpha)} + Th_{\psi'}} \left(Con_n((Th_{\psi'})_n^\delta) \rightarrow \varphi \right)$. Since we have that $Con_n((Th_{\psi'})_n^\delta) \in \Pi_{n+1}^0$, that $\Box_{(Th_{\psi'})_n^\alpha} Con_n((Th_{\psi'})_n^\delta)$ and that provably in EA^+ ,

$$(Th_{\psi'})_n^\alpha \equiv_n (EA^+)_n^{e^{n_0-n}(\alpha_0) \cdot (1+\alpha)}$$

and

$$(EA^+)_n^{e^{n_0-n}(\alpha_0) \cdot (1+\alpha)} \subseteq (EA^+)_n^{e^{n_0-n}(\alpha_0) \cdot (1+\alpha)} + Th_{\psi'},$$

then, we have that $\Box_{(EA^+)_n^{e^{n_0-n}(\alpha_0) \cdot (1+\alpha)} + Th_{\psi'}} Con_n((Th_{\psi'})_n^\delta)$ and this way, we get that $\Box_{(EA^+)_n^{e^{n_0-n}(\alpha_0) \cdot (1+\alpha)} + Th_{\psi'}} \varphi$. This concludes the proof of the lemma. \square

3.3. Completeness of TSC

In this section, we shall prove the uniqueness of MNF's (Theorem 3.3.6) together with some important facts that characterize the derivability in TSC. With these results we shall show the completeness of our system in Theorem 3.3.13.

Conservativity and uniqueness of MNF's

The following corollary is a result of combining arithmetical soundness of TSC together with Gödel's second incompleteness theorem.

Corollary 3.3.1. *For any formula $\varphi \in \mathbb{F}_{TSC}$, $n < \omega$ and $0 < \alpha < \Lambda$, the sequent $\varphi \vdash \langle n^\alpha \rangle \varphi$ is not derivable in TSC.*

As we shall see, this corollary is the only tool needed to obtain the required results on non-derivability. Many of the results that we will prove below can be proven in a shorter fashion using the soundness theorem. Moreover, Corollary 3.3.1 can also be obtained via relational semantics as we will show in the following chapter.

The following result is an application of Corollary 3.3.1.

Proposition 3.3.2. *Let $\psi := \langle n_0^{\alpha_0} \rangle \top \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle \top \in MNF$. For any i with $0 \leq i \leq k$ the sequent $\psi \vdash \langle n_i^{\alpha_i+1} \rangle \top$ is not derivable.*

Proof. If $k = 0$ then the result follows directly from a combination of Axiom 4 and Corollary 3.3.1. For $k > 0$, assume $\psi \vdash \langle n_i^{\alpha_i+1} \rangle \top$ and reason as follows:

$$\begin{aligned} \langle n_i^1 \rangle \psi &\vdash \langle n_0^{\alpha_0+1} \rangle \top \wedge \dots \wedge \langle n_i^1 \rangle \langle n_i^{\alpha_i+1} \rangle \top \text{ by (PS1) and monotonicity;} \\ &\vdash \langle n_i^{\alpha_i+2} \rangle \langle n_0^{\alpha_0+1} \rangle \top \wedge \dots \wedge \langle n_i^2 \rangle \langle n_i^{\alpha_i} \rangle \top \text{ by Axiom 4 and} \\ &\hspace{15em} \text{Proposition 2.2.2, Item 4;} \\ &\vdash \langle n_i^2 \rangle \langle n_0^{\alpha_0+1} \rangle \top \wedge \dots \wedge \langle n_i^2 \rangle \langle n_i^{\alpha_i} \rangle \top \text{ by monotonicity;} \\ &\vdash \langle n_0^{\alpha_0+e^{n_i-n_0}(2)} \rangle \top \wedge \dots \wedge \langle n_i^2 \rangle \langle n_i^{\alpha_i} \rangle \top \text{ by Axioms 5 and 4.} \end{aligned}$$

Therefore, in case $i = k$, by (PS1) we get that $\langle n_i^1 \rangle \psi \vdash \langle n_i^2 \rangle \psi$ which by means of Corollary 3.3.1 makes the sequent $\psi \vdash \langle n_k^{\alpha_k+1} \rangle \top$ not derivable. If

$i < k$, we can make the following observation:

$$\begin{aligned}
 \langle n_i^1 \rangle \psi &\vdash \langle n_i^1 \rangle \left(\langle n_i^{\alpha_i+1} \rangle \top \wedge \langle n_{i+1}^{\alpha_{i+1}} \rangle \top \right); \\
 &\vdash \langle n_i^1 \rangle \left(\langle n_{i+1}^{\alpha_{i+1}} \rangle \langle n_i^{\alpha_i+1} \rangle \top \wedge \langle n_{i+1}^{\alpha_{i+1}} \rangle \top \right) \\
 &\hspace{15em} \text{by Proposition 2.2.2, Item 4;} \\
 &\vdash \langle n_i^1 \rangle \left(\langle n_i^{\alpha_i+e^{n_{i+1}-n_i}(\alpha_{i+1})} \rangle \top \wedge \langle n_{i+1}^{\alpha_{i+1}} \rangle \top \right) \\
 &\hspace{15em} \text{by Axiom 5 and Axiom 4;} \\
 &\vdash \langle n_i^{\alpha_i+e^{n_{i+1}-n_i}(\alpha_{i+1})+1} \rangle \top \wedge \langle n_{i+1}^{\alpha_{i+1}} \rangle \top \text{ by Axiom 4;} \\
 &\vdash \langle n_{i+1}^{\alpha_{i+1}} \rangle \langle n_i^{\alpha_i+e^{n_{i+1}-n_i}(\alpha_{i+1})+1} \rangle \top \wedge \langle n_{i+1}^{\alpha_{i+1}} \rangle \top \\
 &\hspace{15em} \text{by Proposition 2.2.2, Item 4;} \\
 &\vdash \langle n_i^{\alpha_i+e^{n_{i+1}-n_i}(\alpha_{i+1}) \cdot 2} \rangle \top \wedge \langle n_{i+1}^{\alpha_{i+1}} \rangle \top \text{ by Axiom 5 and Axiom 4;} \\
 &\vdash \langle n_i^{\alpha_i+e^{n_{i+1}-n_i}(\alpha_{i+1}) \cdot 2} \rangle \top \wedge \langle n_{i+1}^{\alpha_{i+1}} \rangle \top \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle \top \\
 &\vdash \langle n_i^2 \rangle \left(\langle n_i^{\alpha_i} \rangle \top \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle \top \right) \text{ by (PS2).}
 \end{aligned}$$

Combining this result with previous reasoning we have that

$$\langle n_i^1 \rangle \psi \vdash \langle n_0^{\alpha_0+e^{n_i-n}(2)} \rangle \top \wedge \dots \wedge \langle n_i^2 \rangle \left(\langle n_i^{\alpha_i} \rangle \top \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle \top \right)$$

and then, by (PS1) we get that $\langle n_i^1 \rangle \psi \vdash \langle n_i^2 \rangle \psi$. Hence, by Corollary 3.3.1 we obtain that the sequent $\psi \vdash \langle n_i^{\alpha_i+1} \rangle \top$ is not derivable. \square

The next proposition tells us in a sense that we are justified to not necessarily enlist consecutive modalities in MNF's. For the sake of clarity let us introduce the following definition.

Definition 3.3.3. Let $\psi := \langle n_0^{\alpha_0} \rangle \top \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle \top \in \text{MNF}$ and $n \leq n_k$. By $\mu(n, \psi)$ we denote the least upper bound of n in $N\text{-mod}(\psi)$ defined as:

$$\mu(n, \psi) = \min\{m \in N\text{-mod}(\psi) : m \geq n\}.$$

Proposition 3.3.4. Let $\psi := \langle n_0^{\alpha_0} \rangle \top \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle \top \in \text{MNF}$. For any $n \leq n_k$ and $\mu(n, \psi) = n_J$, the following rules are derivable in TSC:

- (B1) If $\psi \vdash \langle m^\beta \rangle \top$ for $m > n_k$ and $\beta > 0$, then $\langle n_k^1 \rangle \psi \vdash \langle n_k^{e^{m-n_k}(\beta)} \rangle \psi$;
- (B2) If $\psi \vdash \langle n^\gamma \rangle \top$ for $\gamma > e^{n_J-n}(\alpha_J)$, then $\langle n_J^1 \rangle \psi \vdash \langle n_J^2 \rangle \psi$;

Proof. For a proof of (B1), suppose that $\psi \vdash \langle m^\beta \rangle \top$ for some $m > n_k$. Thus, we reason as follows:

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$$\begin{aligned}
\langle n_k^1 \rangle \psi &\vdash \langle n_k^1 \rangle \left(\langle n_0^{\alpha_0} \rangle \top \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle \top \right) \wedge \langle m^\beta \rangle \top; \\
&\vdash \langle n_0^{\alpha_0 + e^{n_k - n_0}(1)} \rangle \top \wedge \dots \wedge \langle n_k^1 \rangle \langle n_k^{\alpha_k} \rangle \top \wedge \langle m^\beta \rangle \top \text{ by (PS1)}; \\
&\vdash \langle n_0^{\alpha_0 + e^{n_k - n_0}(1)} \rangle \top \wedge \dots \wedge \langle n_k^{\alpha_k + 1} \rangle \top \wedge \langle m^\beta \rangle \top \text{ by Axiom 4}; \\
&\vdash \langle n_0^{\alpha_0 + 1} \rangle \top \wedge \dots \wedge \langle n_k^{\alpha_k + 1} \rangle \top \wedge \langle m^\beta \rangle \top \text{ by Axiom 3};
\end{aligned}$$

Notice that by Proposition 2.2.2, Item 4:

$$\begin{aligned}
\langle n_0^{\alpha_0 + 1} \rangle \top \wedge \dots \wedge \langle n_k^{\alpha_k + 1} \rangle \top \wedge \langle m^\beta \rangle \top &\vdash \\
&\langle n_0^{\alpha_0 + 1} \rangle \top \wedge \dots \wedge \langle m^\beta \rangle \langle n_k^{\alpha_k + 1} \rangle \top \wedge \langle m^\beta \rangle \top.
\end{aligned}$$

Thus, by a combination of Axioms 5 and 4 together with the previous fact we obtain that $\langle n_k^1 \rangle \psi \vdash \langle n_0^{\alpha_0 + 1} \rangle \top \wedge \dots \wedge \langle n_k^{\alpha_k + e^{m - n_k}(\beta)} \rangle \top \wedge \langle m^\beta \rangle \top$. Applying successively Proposition 2.2.2, Item 4 we get that

$$\langle n_k^1 \rangle \psi \vdash \langle n_k^{\alpha_k + e^{m - n_k}(\beta)} \rangle \langle n_0^{\alpha_0 + 1} \rangle \top \wedge \dots \wedge \langle n_k^{\alpha_k + e^{m - n_k}(\beta)} \rangle \top \wedge \langle m^\beta \rangle \top,$$

and combining Axioms 5 and 4:

$$\langle n_k^1 \rangle \psi \vdash \langle n_0^{\alpha_0 + e^{n_k - n_0}(\alpha_k + e^{m - n_k}(\beta))} \rangle \top \wedge \dots \wedge \langle n_k^{\alpha_k + e^{m - n_k}(\beta)} \rangle \top$$

and so, by monotonicity

$$\langle n_k^1 \rangle \psi \vdash \langle n_0^{\alpha_0 + e^{n_k - n_0}(e^{m - n_k}(\beta))} \rangle \top \wedge \dots \wedge \langle n_k^{\alpha_k + e^{m - n_k}(\beta)} \rangle \top.$$

Finally, by successively applying (PS1) we get that $\langle n_k^1 \rangle \psi \vdash \langle n_k^{e^{m - n_k}(\beta)} \rangle \psi$. Since $\beta > 0$ and $m > n_k$ we see that $e^{m - n_k}(\beta) \in \text{lim}$ so that $1 + e^{m - n_k}(\beta) = e^{m - n_k}(\beta)$ whence, by Axiom 4, $\langle n_k^{e^{m - n_k}(\beta)} \rangle \psi \equiv \langle n_k^{e^{m - n_k}(\beta)} \rangle \langle n_k^1 \rangle \psi$ and so,

$$\langle n_k^1 \rangle \psi \vdash \langle n_k^{e^{m - n_k}(\beta)} \rangle \langle n_k^1 \rangle \psi.$$

As pointed out, this yields a contradiction with Gödel II as reflected in Corollary 3.3.1.

As for (B2), for $n = n_j$ the rule is trivially admissible since the antecedent is not derivable because of Proposition 3.3.2. Thus, we just check the case where $n < n_j$.

Assume $\psi \vdash \langle n^\gamma \rangle \top$ for some $\gamma > e^{n_j - n}(\alpha_j)$. Therefore, by Axiom 3 we get that $\psi \vdash \langle n^{e^{n_j - n}(\alpha_j) + 1} \rangle \top \wedge \langle n_j^{\alpha_j} \rangle \top$ and so, by Proposition 2.2.2, Item 4,

$$\psi \vdash \langle n_j^{\alpha_j} \rangle \langle n^{e^{n_j - n}(\alpha_j) + 1} \rangle \top.$$

By a combination of Axioms 5 and 4, $\psi \vdash \langle n^{e^{nJ-n}(\alpha_J) \cdot 2} \rangle \top$ and hence,

$$\langle n^1 \rangle \psi \vdash \langle n^{e^{nJ-n}(\alpha_J) \cdot 2+1} \rangle \top \wedge \langle n_J^{\alpha_J} \rangle \top.$$

By Proposition 2.2.2, Item 4, combined with Axioms 5 and 4,

$$\langle n^1 \rangle \psi \vdash \langle n^{e^{nJ-n}(\alpha_J) \cdot 3} \rangle \top. \quad (3.5)$$

By (PS1) and Axiom 6, we have that

$$\langle n^1 \rangle \psi \vdash \langle n_0^{\alpha_0 + e^{n-n_0}(1)} \rangle \top \wedge \dots \wedge \langle n^{e^{nJ-n}(\alpha_J) \cdot 2} \rangle \top \wedge \langle n_J^{\alpha_J} \rangle \top \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle \top.$$

This combined with (3.5) give us that

$$\langle n^1 \rangle \psi \vdash \langle n_0^{\alpha_0 + e^{n-n_0}(1)} \rangle \top \wedge \dots \wedge \langle n^{e^{nJ-n}(\alpha_J) \cdot 3} \rangle \top \wedge \langle n_J^{\alpha_J} \rangle \top \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle \top.$$

Now we can reason as follows:

$$\begin{aligned} \langle n^1 \rangle \psi &\vdash \langle n_0^{\alpha_0 + e^{n-n_0}(1)} \rangle \top \wedge \dots \wedge \langle n^{e^{nJ-n}(\alpha_J) \cdot 3} \rangle \top \wedge \\ &\quad \langle n_J^{\alpha_J} \rangle \top \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle \top; \\ &\vdash \langle n_0^{\alpha_0+1} \rangle \top \wedge \dots \wedge \langle n^{e^{nJ-n}(\alpha_J) \cdot 3} \rangle \top \wedge \\ &\quad \langle n_J^{\alpha_J} \rangle \top \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle \top \text{ by monotonicity}; \\ &\vdash \langle n^{e^{nJ-n}(\alpha_J) \cdot 3} \rangle \langle n_0^{\alpha_0+1} \rangle \top \wedge \dots \wedge \langle n^{e^{nJ-n}(\alpha_J) \cdot 3} \rangle \top \wedge \\ &\quad \langle n_J^{\alpha_J} \rangle \top \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle \top \text{ by Proposition 2.2.2, Item 4}; \\ &\vdash \langle n^2 \rangle \langle n_0^{\alpha_0+1} \rangle \top \wedge \dots \wedge \langle n^{e^{nJ-n}(\alpha_J) \cdot 3} \rangle \top \wedge \\ &\quad \langle n_J^{\alpha_J} \rangle \top \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle \top \text{ by monotonicity}; \\ &\vdash \langle n^2 \rangle \langle n_0^{\alpha_0+1} \rangle \top \wedge \dots \wedge \\ &\quad \langle n^2 \rangle \left(\langle n_J^{\alpha_J} \rangle \top \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle \top \right) \text{ by Axiom 6}; \\ &\vdash \langle n_0^{\alpha_0 + e^{n-n_0}(2)} \rangle \top \wedge \dots \wedge \\ &\quad \langle n^2 \rangle \left(\langle n_J^{\alpha_J} \rangle \top \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle \top \right) \text{ by Axioms 5 and 4}; \\ &\vdash \langle n^2 \rangle \left(\langle n_0^{\alpha_0} \rangle \top \wedge \dots \wedge \langle n_J^{\alpha_J} \rangle \top \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle \top \right) \text{ by (PS1)}; \\ &\vdash \langle n^2 \rangle \psi. \end{aligned}$$

We may observe that this result yields a contradiction with Gödel II, that is, Corollary 3.3.1. Hence, we obtain that if $\psi \vdash \langle n^\gamma \rangle \top$, then $\gamma \leq e^{nJ-n}(\alpha_J)$. \square

The following corollary gathers some immediate consequences of Rules (B1) and (B2) together with Corollary 3.3.1.

Corollary 3.3.5. *Let $\psi := \langle n_0^{\alpha_0} \rangle \top \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle \top \in \text{MNF}$, $n \leq n_k$ and $\mu(n, \psi) = n_j$*

1. *If $\psi \vdash \langle m^\beta \rangle \top$ for some $\beta > 0$, then $m \leq n_k$;*
2. *If $\psi \vdash \langle n_i^\beta \rangle \top$ for $n_i \in N\text{-mod}(\psi)$, then $\beta \leq \alpha_i$;*
3. *If $\psi \vdash \langle n_i^\delta \rangle \top$ for $n_i \in N\text{-mod}(\psi)$, then $\langle n_i^{\alpha_i} \rangle \top \vdash \langle n_i^\delta \rangle \top$;*
4. *If $\psi \vdash \langle n^\gamma \rangle \top$ for $n \notin N\text{-mod}(\psi)$, then $\langle n_j^{\alpha_j} \rangle \top \vdash \langle n^\gamma \rangle \top$.*

With these results, we can now obtain uniqueness of our MNF's.

Theorem 3.3.6. *Given $\psi_1, \psi_2 \in \text{MNF}$, if $\psi_1 \equiv \psi_2$ then $\psi_1 = \psi_2$.*

Proof. This now follows by an easy induction on the number of monomials in an MNF by the previous results. \square

This last theorem can be combined with Theorem 2.3.10 obtaining the next corollary.

Corollary 3.3.7. *For every formula φ there is a unique $\psi \in \text{MNF}$ such that $\varphi \equiv \psi$.*

The uniqueness of MNF's will be key in proving completeness. It also has a couple of other nice corollaries.

Corollary 3.3.8. *Each formula has a unique INF.*

Corollary 3.3.9. *Let $\varphi \in \mathbb{F}_{<n}$ for some n , and $\psi \in \text{MNF}$ such that $\varphi \equiv \psi$. Then $\psi \in \mathbb{F}_{<n}$.*

Proof. As we have seen in Section 2.3, there is a procedure by which any formula φ can be reduced to an equivalent $\psi_\varphi \in \text{MNF}$ that only contains modality bases occurring in φ , and so $\psi_\varphi \in \mathbb{F}_{<n}$. Moreover, by uniqueness, for every $\psi \in \text{MNF}$ such that $\varphi \equiv \psi$, we obtain that $\psi = \psi_\varphi$ and thus $\psi \in \mathbb{F}_{<n}$. \square

In a next proposition we will prove that in a sense, initial truncations of MNF's are modally optimal approximations of the entire MNF. In order to precisely formulate what this means, we first need to recall that $\varphi \equiv_n \psi$ is defined as $\varphi \vdash \chi$ iff $\psi \vdash \chi$, for any $\chi \in \mathbb{F}_{<n+1}$.

We point out that the following proposition is a direct consequence of Lemma 3.2.14 through the arithmetical soundness and completeness of our calculus. However, since the current proof only hinges on purely modal facts together with 3.3.1 it is amenable to a setting where arithmetical completeness or soundness is not (yet) available.

Proposition 3.3.10. *Given $\psi := \langle n_0^{\alpha_0} \rangle_{\top} \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle_{\top} \in \text{MNF}$ and $n \leq n_k$, provably in TSC:*

$$\langle n_0^{\alpha_0} \rangle_{\top} \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle_{\top} \equiv_n \bigwedge_{0 \leq i \leq J} \langle n_i^{\alpha_i} \rangle_{\top}$$

where $n_J = \mu(n, \psi)$.

Proof. This follows from Corollary 3.3.9 together with Items 3 and 4 of Corollary 3.3.5. \square

Arithmetical Completeness

With the help of some of the results established in the previous subsection, we can state the following characterization of the derivability between formulas in MNF.

Proposition 3.3.11. *For any $\psi_0, \psi_1 \in \text{MNF}$, where $\psi_0 := \langle n_0^{\alpha_0} \rangle_{\top} \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle_{\top}$ and $\psi_1 := \langle m_0^{\beta_0} \rangle_{\top} \wedge \dots \wedge \langle m_j^{\beta_j} \rangle_{\top}$, $\psi_0 \vdash \psi_1$ iff the following holds:*

1. $m_j \leq n_k$;
2. For any $m_l \in \text{N-mod}(\psi_1)$ and $n_l = \mu(m_l, \psi_0)$, we have $\beta_l \leq e^{n_l - m_l}(\alpha_l)$.

Proof. Assume $\psi_0 \vdash \psi_1$. For a proof of Item 2, consider some $m_l \in \text{N-mod}(\psi_1)$. Since $\psi_0 \vdash \psi_1$ then, in particular, $\psi_0 \vdash \langle m_l^{\beta_l} \rangle_{\top}$ and so, by Corollary 3.3.5, Items 2 and 4, together with Proposition 3.3.2, we have that $\beta_l \leq e^{n_l - m_l}(\alpha_l)$. Moreover, since if $\psi_0 \vdash \psi_1$ then $\psi_0 \vdash \langle m_j^{\beta_j} \rangle_{\top}$. Thus, Item 1 follows from Corollary 3.3.5, Item 1.

For the other direction, assume Items 1 and 2 and consider any monomial $\langle m_l^{\beta_l} \rangle_{\top}$ occurring in ψ_1 . We can distinguish two cases: either $m_l \in \text{N-mod}(\psi_0)$ and then by Axiom 3 $\psi_0 \vdash \langle m_l^{\beta_l} \rangle_{\top}$ or $m_l \notin \text{N-mod}(\psi_0)$, but then by a combination of Axioms 5 and 3, $\langle n_l^{\alpha_l} \rangle_{\top} \vdash \langle m_l^{\beta_l} \rangle_{\top}$, and so $\psi_0 \vdash \langle m_l^{\beta_l} \rangle_{\top}$. Therefore, $\psi_0 \vdash \psi_1$. \square

Before proving the completeness of our system, we need to check the correctness of the derivable Rules (B1) and (B2) introduced in Proposition 3.3.4 of the previous subsection.

Lemma 3.3.12. *Rules (B1) and (B2) are sound w.r.t. the FTP interpretation.*

Proof. To check the soundness of (B1), assume $(\text{EA}^+)_m^{\beta} \subseteq \text{Th}_{\psi}$. Since for $m > n_k$, $e^{m-n_k}(\beta) \in \text{Lim}$, we can observe that $((\text{Th}_{\psi})_{n_k}^1)_{n_k}^{e^{m-n_k}(\beta)} \equiv (\text{Th}_{\psi})_{n_k}^{e^{m-n_k}(\beta)}$, and then, by correctness of Axiom 5:

$$(\text{Th}_{\psi})_{n_k}^{e^{m-n_k}(\beta)} \subseteq ((\text{Th}_{\psi})_{n_k}^1)_m^{\beta} \subseteq ((\text{Th}_{\psi})_{n_k}^1 + \text{EA}^+)_m^{\beta}.$$

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Since $m > n_k$, we get that $((\text{Th}_\psi)_{n_k}^1 + \text{EA}^+)_m^\beta \subseteq (\text{Th}_\psi)_{n_k}^1 + (\text{EA}^+)_m^\beta$ and by our assumption, $(\text{Th}_\psi)_{n_k}^1 + (\text{EA}^+)_m^\beta \subseteq (\text{Th}_\psi)_{n_k}^1$. Therefore, $(\text{Th}_\psi)_{n_k}^{e^{m-n_k}(\beta)} \subseteq (\text{Th}_\psi)_{n_k}^1$.

With respect to (B2), we can observe that it holds trivially since the assumption of the antecedent together with Lemma 3.2.14 and Proposition 3.2.5, Item 3.2.5 yields Th_ψ is inconsistent. \square

With these two results, we are ready now to prove completeness of TSC:

Theorem 3.3.13. *For any $\varphi, \psi \in \mathbb{F}_{\text{TSC}}$, if $\text{EA}^+ \vdash \text{Th}_\psi \subseteq \text{Th}_\varphi$, then the sequent $\varphi \vdash \psi$ is derivable in TSC.*

Proof. W.l.o.g., let $\varphi, \psi \in \text{MNF}$ be such that $\varphi := \langle n_0^{\alpha_0} \rangle \top \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle \top$ and $\psi := \langle m_0^{\beta_0} \rangle \top \wedge \dots \wedge \langle m_j^{\beta_j} \rangle \top$. By contraposition, assume $\varphi \not\vdash \psi$. Hence by Proposition 3.3.11, either:

- $n_k < m_j$ or,
- There is $m_l \in \text{N-mod}(\psi)$ and $n_l = \mu(m_l, \varphi)$, such that $\beta_l > e^{n_l - m_l}(\alpha_l)$.

Towards a contradiction assume that $\text{EA}^+ \vdash \text{Th}_\psi \subseteq \text{Th}_\varphi$. If $n_k < m_j$, notice that provably in EA^+ we have that $(\text{EA}^+)_{m_j}^{\beta_j} \subseteq \text{Th}_\psi$ and so, in particular we have that $(\text{EA}^+)_{m_j}^{\beta_j} \subseteq \text{Th}_\varphi$. Thus, applying (B1), together with Lemma 3.3.12 we obtain that

$$\text{EA}^+ \vdash (\text{Th}_\varphi)_{n_k}^{e^{m_j - n_k}(\beta)} \subseteq (\text{Th}_\varphi + \text{Con}_{n_k}(\text{Th}_\varphi)).$$

Therefore, $\text{EA}^+ \vdash \Box_{(\text{Th}_\varphi)_{n_k}^1} \text{Con}_{n_k}((\text{Th}_\varphi)_{n_k}^1)$ and then $\text{EA}^+ \vdash \Box_{(\text{Th}_\varphi)_{n_k}^1} \perp$. Hence, $\mathbb{N} \models \Box_{(\text{Th}_\varphi)_{n_k}^1} \perp$ and so, $(\text{Th}_\varphi)_{n_k}^1$ is inconsistent.

For the second case, we reason analogously but using (B2). \square

Relational Semantics for TSC

In this chapter we shall discuss relational semantics for **TSC**. We will start by defining a new model \mathcal{J} which is proven to be universal for our logic. That is, any derivable sequent will hold -in a precise sense to be defined below- everywhere in the model whereas any non-derivable sequent will be refuted somewhere in the model. This frame \mathcal{J} is a slight modification of *Ignatiev's universal model* \mathcal{I} .

The results of this chapter are included in [28].

4.1. A Variation on Ignatiev's Frame

In [30], K. Ignatiev introduced a frame which is a universal model for the variable-free fragment of **GLP** which has been intensively studied (see [14], [23], [29]).

This model will be based on special sequences of ordinals. In order to define them, we need the following central definition.

Definition 4.1.1. We define ordinal logarithm as $\ell(0) := 0$ and $\ell(\alpha + \omega^\beta) := \beta$.

Thus, the set of worlds of this frame is defined as follows.

Definition 4.1.2. By Ig^ω we denote the set of ℓ -sequences or Ignatiev sequences. That is, the set of sequences $x := \langle x_0, x_1, x_2, \dots \rangle$ where for $i < \omega$, $x_{i+1} \leq \ell(x_i)$.

Given a ℓ -sequence x , if all but finitely many of its elements are zero, we will write $\langle x_0, \dots, x_n, \vec{0} \rangle$ to denote such sequence or even simply $\langle x_0, \dots, x_n \rangle$

whenever $x_{n+1} = 0$.

We can define relations T_n in Ig^ω by:

$$xT_ny :\Leftrightarrow (\forall m < n \ x_m = y_m \wedge x_n > y_n).$$

The resulting structure $\mathcal{I} = \langle \text{Ig}^\omega, \{T_n\}_{n < \omega} \rangle$ is called the *Ignatiev's frame*.

To define \mathcal{J} we will make use of the same set of ℓ -sequences but we shall consider some minor modifications in the definition of the relations:

Definition 4.1.3. $\mathcal{J}_\Lambda := \langle I, \{R_n\}_{n < \omega} \rangle$ is defined as follows:

$$I := \{x \in \text{Ig}^\omega : x_m < \Lambda \text{ for } m < \omega\}$$

and

$$xR_ny :\Leftrightarrow (\forall m \leq n \ x_m > y_m \wedge \forall m > n \ x_m \geq y_m).$$

Since Λ was already fixed in the previous chapter, from now on we suppress the subindex Λ .

The observations collected in the next lemma all have elementary proofs. Basically, the lemma confirms that the R_n -relations are good to model provability logic and respect the increasing strength of the provability predicates $[n]$.

Lemma 4.1.4.

1. Each R_n for $n < \omega$ is transitive: $xR_ny \wedge yR_nz \Rightarrow xR_nz$;
2. Each R_n for $n < \omega$ is Noetherian: each non-empty $X \subseteq I$ has an R_n -maximal element $y \in X$, i.e., $\forall x \in X \ \neg yR_nx$;
3. The relations R_n are monotone in n in the sense that: $xR_ny \Rightarrow xR_my$ whenever $n > m$.

Note that Item 2 is equivalent to stating that there are no infinite ascending R_n chains. In other words, the converse of R_n is well-founded.

We can also make the following observation.

Lemma 4.1.5. For any $x, y, z \in I$ and $n, m < \omega$:

$$xR_nyR_mz \implies xR_nz \wedge xR_mz.$$

Proof. Assume that xR_ny and yR_mz . Then, we have that $x_i > y_i \geq z_i$ for all $i \leq n$ and for all $i > n$, we get that $x_i \geq y_i \geq z_i$. Hence, xR_nz . On the other hand, for any $i < \omega$, we have that $x_i \geq y_i$. Therefore, for all $i \leq m$, $x_i > z_i$ and for any $i > m$, $x_i \geq z_i$. Thus, xR_mz . \square

We define the auxiliary relations R_n^α for any $n < \omega$ and $\alpha < \Lambda$. The idea is that the R_n^α relation will model the $\langle n^\alpha \rangle$ modality.

Definition 4.1.6. Given $x, y \in I$ and R_n on I , we recursively define $xR_n^\alpha y$ as follows:

1. $xR_n^0 y \iff x = y$;
2. $xR_n^{1+\alpha} y \iff \forall \beta < 1+\alpha \exists z (xR_n z \wedge zR_n^\beta y)$.

In Definition 4.1.6, we could have given an equivalent formulation in terms of successor and limit ordinals as the next lemmata tell us.

Proposition 4.1.7. Given $x, y \in I$, $n < \omega$ and $\alpha < \Lambda$:

$$xR_n^{\alpha+1} y \iff \exists z (xR_n z \wedge zR_n^\alpha y).$$

Proof. We make a case distinction on α with $\alpha = 0$ being trivial. For $\alpha > 0$, we have that $\alpha = 1 + \gamma$ for some $\gamma \leq \alpha$. With the help of this fact, we can reason as follows:

$$\begin{aligned} xR_n^{\alpha+1} y &\iff xR_n^{1+\gamma+1} y; \\ &\iff \forall \beta < 1 + \delta + 1 \exists z (xR_n z \wedge zR_n^\beta y); \\ &\implies \exists z (xR_n z \wedge zR_n^\alpha y), \text{ in particular.} \end{aligned}$$

Thus, $xR_n^{\alpha+1} y \implies \exists z (xR_n z \wedge zR_n^\alpha y)$. For the right-to-left implication we proceed analogously:

$$\begin{aligned} \exists z (xR_n z \wedge zR_n^\alpha y) &\iff \exists z (xR_n z \wedge zR_n^{1+\gamma} y); \\ &\iff \exists z \left((xR_n z \wedge zR_n^{1+\gamma} y) \wedge \right. \\ &\quad \left. \forall \beta < 1 + \gamma \exists z' (zR_n z' \wedge z'R_n^\beta y) \right); \\ &\implies \forall \beta' < 1 + \gamma + 1 \exists u (xR_n u \wedge uR_n^{\beta'} y); \\ &\implies xR_n^{1+\gamma+1} y; \\ &\implies xR_n^{\alpha+1} y. \end{aligned}$$

□

We can also make a similar simple observation for limit ordinals.

Proposition 4.1.8. Let $x, y \in I$, $n < \omega$ and $\lambda < \Lambda$ such that $\lambda \in \text{Lim}$:

$$xR_n^\lambda y \iff \forall \beta < \lambda xR_n^{1+\beta} y \iff \forall \beta, 0 < \beta < \lambda, xR_n^\beta y.$$

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Proof. Since the last equivalence is straightforward, we just show that:

$$xR_n^\lambda y \iff \forall \beta < \lambda \ xR_n^{1+\beta} y.$$

For left-to-right implication, notice that if $xR_n^\lambda y$ then by Definition 4.1.6, we have that

$$\forall \beta < \lambda \ \exists u \ (xR_n u \wedge uR_n^\beta y).$$

Therefore, since $\lambda \in \text{Lim}$ we obtain that $\forall \beta < \lambda \ \exists u \ (xR_n u \wedge uR_n^{1+\beta} y)$ and so, by Definition 4.1.6, we have that

$$\forall \beta < \lambda \ \forall \delta < 1 + \beta \ \exists u, v \ (xR_n u \wedge uR_n v \wedge vR_n^\delta y).$$

By transitivity of R_n , we get that $\forall \beta < \lambda \ xR_n^{1+\beta} y$. For the other direction, if $\forall \beta < \lambda \ xR_n^{1+\beta} y$, then in particular, $\forall \beta < \lambda \ xR_n^{\beta+1} y$ and then, by Proposition 4.1.7, $\forall \beta < \lambda \ \exists u \ (xR_n u \wedge uR_n^\beta y)$, that is, $xR_n^\lambda y$. \square

It is easy to see that for example $\langle \omega, \vec{0} \rangle R_0^{1+n} \langle m, \vec{0} \rangle$ for each $n, m \in \omega$, so that also $\langle \omega, \vec{0} \rangle R_0^\omega \langle m, \vec{0} \rangle$ for each $m \in \omega$. Clearly, we do not have $\langle \omega, \vec{0} \rangle R_0^{\omega+1} \langle m, \vec{0} \rangle$ for any $m < \omega$ but we do have $\langle \omega + 1, \vec{0} \rangle R_0^{\omega+1} \langle m, \vec{0} \rangle$ for all $m < \omega$.

We also note that the dual definition $x\bar{R}_n^0 y := x = y$; and $x\bar{R}_n^{1+\alpha} y := \forall \beta < 1+\alpha \ \exists z \ (x\bar{R}_n^\beta z \wedge z\bar{R}_n^\alpha y)$ does not make much sense on our frames. For example we could have $\langle \omega, \vec{0} \rangle \bar{R}_0^\alpha \langle 0, \vec{0} \rangle$ for any ordinal $\alpha > 0$.

With the the auxiliary relations R_n^α , we give the following definition for a formula φ being true in a point x of \mathcal{J} .

Definition 4.1.9. Let $x \in I$ and $\varphi \in \mathbb{F}_{\text{TSC}}$. By $x \Vdash \varphi$ we denote the validity of φ in x that is recursively defined as follows:

- $x \Vdash \top$ for all $x \in I$;
- $x \Vdash \varphi \wedge \psi$ iff $x \Vdash \varphi$ and $x \Vdash \psi$;
- $x \Vdash \langle n^\alpha \rangle \varphi$ iff there is $y \in I$, $xR_n^\alpha y$ and $y \Vdash \varphi$.

In the following lemma we present some easy observations on the R_n^α relations which among others tell us that all the $R_n^{1+\alpha}$ serve the purpose of a provability predicate for any $n < \omega$ and $\alpha < \Lambda$.

Lemma 4.1.10.

1. Each $R_n^{1+\alpha}$ for $n < \omega$ and α an ordinal is transitive: $xR_n^{1+\alpha} y \wedge yR_n^{1+\alpha} z \Rightarrow xR_n^{1+\alpha} z$;
2. Each $R_n^{1+\alpha}$ for $n < \omega$ and α an ordinal is Noetherian: each non-empty $X \subseteq I$ has an $R_n^{1+\alpha}$ -maximal element $y \in X$, i.e., $\forall x \in X \neg yR_n^{1+\alpha} x$;

4.1. A Variation on Ignatiev's Frame

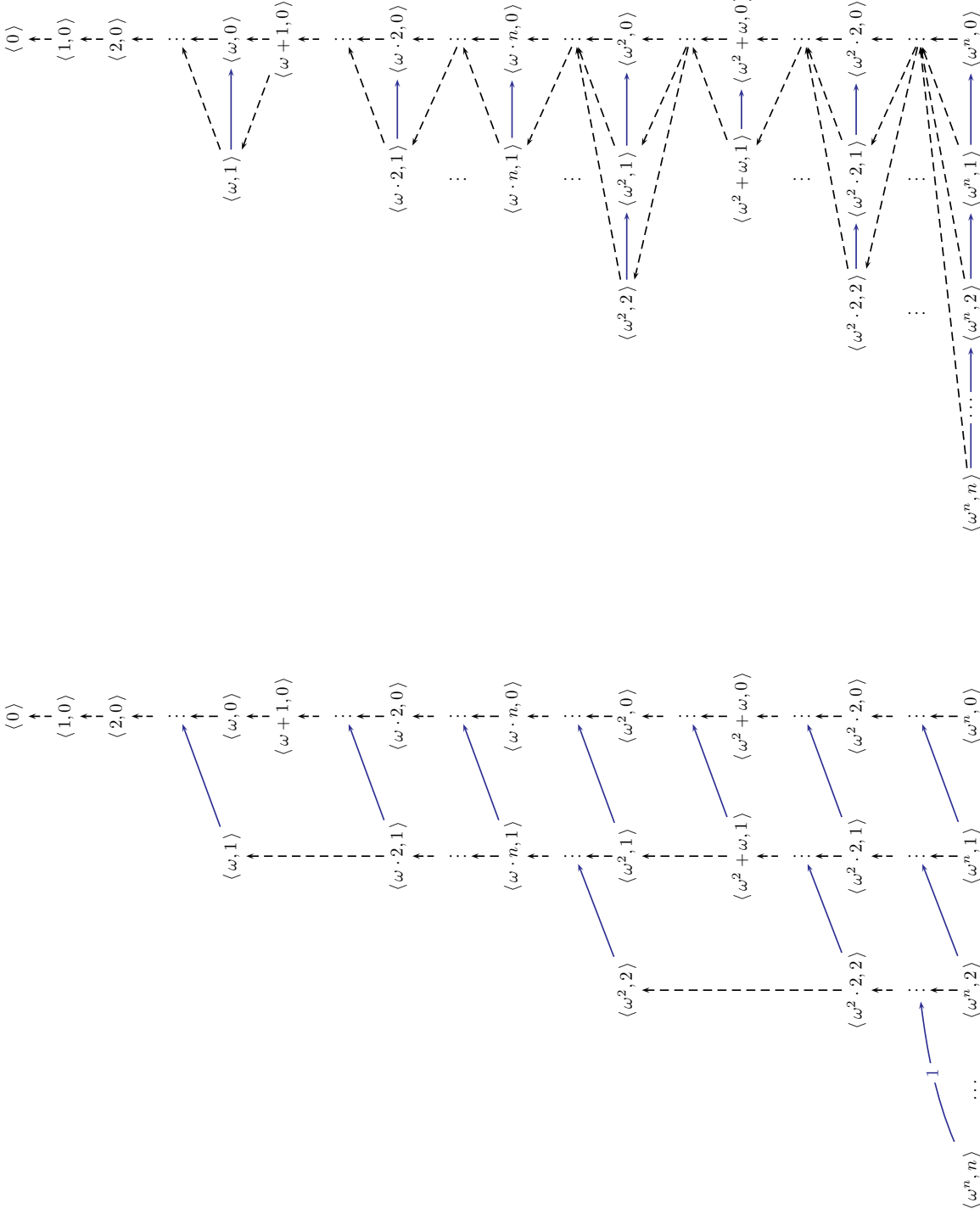


Figure 4.1: In the above pictures we can compare the behavior of the relations in \mathcal{J} and \mathcal{I} . The dashed arrows represent R_0 relations, while the continuous arrows represent R_1 relations.

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3. The relations $R_n^{1+\alpha}$ are monotone in n in the sense that: $xR_n^{1+\alpha}y \Rightarrow xR_m^{1+\alpha}y$ whenever $n > m$;
4. The relations $R_n^{1+\alpha}$ are monotone in $1 + \alpha$ in the sense that: $xR_n^{1+\alpha}y \Rightarrow xR_n^{1+\beta}y$ whenever $1 + \beta < 1 + \alpha$.

Proof. The first three items follow directly from Lemma 4.1.4 by an easy transfinite induction. The last item is also easy. \square

The following lemma states the polytransitivity condition for the R_n^α relations.

Lemma 4.1.11 (Polytransitivity). *For any $x, y, z \in I$, $n, m < \omega$ and $\alpha, \beta < \Lambda$ with $0 < \alpha, \beta$:*

$$xR_n^\alpha y R_m^\beta z \implies xR_n^\alpha z \wedge xR_m^\beta z.$$

Proof. We assume $xR_n^\alpha y$ and $yR_m^\beta z$ and set out to prove $xR_n^\alpha z$. We proceed by induction on α . For the base case, let $\alpha = 1$. We can observe that by Lemma 4.1.10, Item 4, since $yR_m^\beta z$, then $yR_m z$. Therefore, the implication follows from Lemma 4.1.5.

For $\alpha \in \mathbf{Succ}$, let $\alpha := \alpha' + 1$. By Proposition 4.1.7, if $xR_n^\alpha y$, then $\exists u \in I$ such that $xR_n u$ and $uR_n^{\alpha'} y$. Since $uR_n^{\alpha'} y$ and $yR_m^\beta z$, by the I.H. we have that $uR_n^{\alpha'} z$ and thus, $xR_n^\alpha z$.

For $\alpha \in \mathbf{Lim}$, by Proposition 4.1.8, we have that for all $\delta < \alpha$, $xR_n^{1+\delta} y$ and $yR_m^\beta z$. Thus, by the I.H. we get that $xR_n^{1+\delta} z$ for all $\delta < \alpha$. Hence, $xR_n^\alpha z$.

To prove that $xR_m^\beta z$ we proceed by induction on β . The base and limit cases are analogous to the ones of the induction on α . For the case $\beta \in \mathbf{Succ}$, let $\beta := \beta' + 1$. With the help of Proposition 4.1.7 we have that $\exists u \in I$ such that $yR_m u$ and $uR_m^{\beta'} z$. By Lemma 4.1.10, Item 4, since $xR_n^\alpha y$, then $xR_n y$ and by Lemma 4.1.5 $xR_m u$. Therefore, $xR_m^\beta z$. \square

A characterization for transfinite accessibility

The intuitive idea behind the $xR_n^\alpha y$ assertion, is that this tells us that there exists a chain of ‘length’ α of R_n steps leading from the point x up to the point y . The following useful lemma tries to capture this intuition.

Lemma 4.1.12. *For $x, y \in I$, $\alpha < \Lambda$ and $n < \omega$ we have that the following are equivalent*

1. $xR_n^{1+\alpha} y$
2. There exists a collection $c^{1+\alpha} := \{x_\delta^{1+\alpha}\}_{\delta < 1+\alpha}$ so that
 - a) $xR_n x_\delta^{1+\alpha}$ for any $x_\delta^{1+\alpha} \in c^{1+\alpha}$,

- b) $x_0^{1+\alpha} = y$ and,
 c) for any $\delta' < \delta < 1 + \alpha$ we have $x_\delta^{1+\alpha} R_n x_{\delta'}^{1+\alpha}$.

Proof. By induction on α . The base case is immediate if we consider the collection $\{y\}$. For $\alpha \in \mathbf{Succ}$, let $\alpha := \alpha' + 1$. By Proposition 4.1.7, we have that $xR_n^{1+\alpha'+1}y$ iff there is z such that $xR_n z$ and $zR_n^{1+\alpha'}y$. Thus, by the I.H. $xR_n^{1+\alpha'+1}y$ iff there is z such that $xR_n z$ and there is a collection $c^{1+\alpha'} := \{z_\delta^{1+\alpha'}\}_{\delta < 1+\alpha'}$ such that:

1. $zR_n z_\delta^{1+\alpha'}$ for any $z_\delta^{1+\alpha'} \in c^{1+\alpha'}$,
2. $z_0^{1+\alpha'} = y$ and,
3. for any $\delta' < \delta < 1 + \alpha'$ we have $z_\delta^{1+\alpha'} R_n z_{\delta'}^{1+\alpha'}$.

Thus, we just need to consider the collection $c^{1+\alpha'+1}$ defined as follows:

$$x_\delta^{1+\alpha'+1} = \begin{cases} z & \text{if } \delta = 1 + \alpha'; \\ z_\delta^{1+\alpha'} & \text{otherwise.} \end{cases}$$

To check the other implication, assume we have a collection $c^{1+\alpha'+1}$ satisfying the properties 2a, 2b and 2c. Then, we have that $xR_n x_{1+\alpha'}^{1+\alpha'+1}$ and applying the I.H. to the collection $\{x_\delta^{1+\alpha'+1}\}_{\delta < 1+\alpha'}$, we get that $x_{1+\alpha'}^{1+\alpha'+1} R_n^{1+\alpha'} y$. Hence, $xR_n^{1+\alpha'+1}y$.

For $\alpha \in \mathbf{Lim}$, let us first prove the implication from Item 1 to Item 2. By Proposition 4.1.8 we have that $xR_n^\alpha y$ iff for any $\gamma < \alpha$, $xR_n^{1+\gamma}y$. Thus, for each $\gamma < \alpha$, we have a collection $c^{1+\gamma} := \{z_\delta^{1+\gamma}\}_{\delta < 1+\gamma}$ satisfying the properties 2a, 2b and 2c. In order to build the collection $c^\alpha := \{x_\gamma^\alpha\}_{\gamma < \alpha}$, we set x_0^α to be y and define $x_{1+\gamma}^\alpha$ uniformly as follows:

$$(x_{1+\gamma}^\alpha)_m := \begin{cases} y_m & \text{in case } m > n, \\ \min\{(z_{1+\gamma}^\zeta)_m : z_{1+\gamma}^\zeta \in c^\zeta \text{ for } \zeta > 1 + \gamma\} & \text{in case } m = n, \\ e^{n-m}((x_{1+\gamma}^\alpha)_n) & \text{in case } m < n. \end{cases}$$

The reason why we consider the minimum value for $(x_{1+\gamma}^\alpha)_n$ is that minimum means highest in the R_n relation. Therefore, the minimum value guarantees that we are sufficiently down to reach y but, at the same time, that we still have room enough to build the remaining members of our collection.

First, we check that $c^\alpha \subseteq I$. Since $x_0^\alpha = y \in I$, we only check $x_\gamma^\alpha \in I$ for $\gamma > 0$. Since for $m > n$, $(x_\gamma^\alpha)_m = y_m$ and $y \in I$, it suffices to establish the following:

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- $\ell((x_\gamma^\alpha)_n) \geq (x_\gamma^\alpha)_{n+1}$. By definition, there is $\zeta^* > \gamma$, such that $(x_\gamma^\alpha)_n = (z_\gamma^{\zeta^*})_n$ and $z_\gamma^{\zeta^*} \in I$. Hence $\ell((z_\gamma^{\zeta^*})_n) \geq (z_\gamma^{\zeta^*})_{n+1}$. By the I.H. Items **2b** and **2c** $z_\gamma^{\zeta^*} R_n y$, and so $(z_\gamma^{\zeta^*})_{n+1} \geq y_{n+1} = (x_\gamma^\alpha)_{n+1}$. Thus, $\ell((x_\gamma^\alpha)_n) \geq (x_\gamma^\alpha)_{n+1}$.
- $\ell((x_\gamma^\alpha)_m) \geq (x_\gamma^\alpha)_{m+1}$ for $m < n$. By definition we have that $(x_\gamma^\alpha)_{m+1} = e^{n-(m+1)}((x_\gamma^\alpha)_n)$ and $(x_\gamma^\alpha)_m = e^{n-m}((x_\gamma^\alpha)_n)$. Thus,

$$\ell((x_\gamma^\alpha)_m) = \ell(e^{n-m}((x_\gamma^\alpha)_n)) = e^{n-(m+1)}((x_\gamma^\alpha)_n) = (x_\gamma^\alpha)_{m+1}.$$

With respect to the conditions **2a**, **2b** and **2c**, **Item 2b** follows by definition. Regarding **Item 2a**, pick an arbitrary $x_\delta^\alpha \in c^\alpha$. By the I.H. we know that for any point $z_\delta^{1+\gamma} \in c^{1+\gamma}$ for $\gamma < \alpha$ we have that $x R_n z_\delta^{1+\gamma}$. Then:

- $x R_n z_0^{1+\gamma}$, that is, $x R_n y$ and so, $x_m \geq y_m = (x_\delta^\alpha)_m$ for $m > n$;
- $x_n > \min\{(z_\delta^\zeta)_n : z_\delta^\zeta \in c^\zeta \text{ for } \zeta > \delta\} = (x_\delta^\alpha)_n$;
- For $m < n$ and $\zeta > \delta$, we have that $x_m > (z_\delta^\zeta)_m \geq e^{n-m}((z_\delta^\zeta)_n)$. Therefore, $x_m > e^{n-m}((x_\delta^\alpha)_n)$.

Hence, $x R_n x_\delta^\alpha$ for any $\delta < \alpha$. Concerning **Item 2c**, by definition of c^α , to obtain that for any $\delta' < \delta < \alpha$, we have that $x_\delta^\alpha R_n x_{\delta'}^\alpha$, it suffices to check that $(x_{\delta'}^\alpha)_n < (x_\delta^\alpha)_n$ considering that $(x_{\delta'}^\alpha)_m = (x_\delta^\alpha)_m$ for $m > n$ and $(x_{\delta'}^\alpha)_m < (x_\delta^\alpha)_m$ for $m \leq n$. We assume $\delta' \neq 0$ since that case is easy.

We can observe that there are ζ^* and ξ^* such that $\alpha > \zeta^* > \delta$ and $\alpha > \xi^* < \delta'$, satisfying the following:

$$(x_\delta^\alpha)_n = (z_\delta^{\zeta^*})_n \text{ and } (x_{\delta'}^\alpha)_n = (z_{\delta'}^{\xi^*})_n.$$

Since $\delta > \delta'$, there is $z_{\delta'}^{\xi^*} \in c^{\xi^*}$ and by the I.H. $z_{\delta'}^{\xi^*} R_n z_\delta^{\zeta^*}$. Thus, $(z_{\delta'}^{\xi^*})_n < (z_\delta^{\zeta^*})_n$. Since $(x_{\delta'}^\alpha)_n$ is obtained by considering the minimum, we have have that

$$(x_{\delta'}^\alpha)_n = (z_{\delta'}^{\xi^*})_n \leq (z_{\delta'}^{\zeta^*})_n < (z_\delta^{\zeta^*})_n = (x_\delta^\alpha)_n.$$

That is, $(x_{\delta'}^\alpha)_n < (x_\delta^\alpha)_n$.

We know prove the implication from **Item 2** to **Item 1**. Thus, we assume there is a collection c^α satisfying conditions **2a**, **2b** and **2c**. We will show that $x R_n^\alpha y$. First we prove the following by induction on γ :

$$\text{For any } x_\gamma^\alpha \in c^\alpha, x_\gamma^\alpha R_n^\gamma y.$$

The base case is immediate since we have that $x_0^\alpha = y$ and $y R_n^0 y$. For the successor step, let $\gamma = \gamma' + 1$. Since by assumption $x_{\gamma'}^\alpha R_n x_\gamma^\alpha$ and by the I.H.

$x_{\gamma}^{\alpha} R_n^{\gamma'} y$, then $x_{\gamma}^{\alpha} R_n^{\gamma'+1} y$. For the limit case, assume as the I.H. that for any $\delta < \gamma$, $x_{\delta}^{\alpha} R_n^{\delta} y$. By assumption we have that $x_{\gamma}^{\alpha} R_n x_{\delta}^{\alpha}$ for any $\delta < \gamma$. Thus, $x_{\gamma}^{\alpha} R_n x_{\delta}^{\alpha}$ and $x_{\delta}^{\alpha} R_n^{\delta} y$ for any $\delta < \gamma$, and so, $x_{\gamma}^{\alpha} R_n^{\gamma} y$.

Hence, for any $\gamma < \alpha$ we have that $x R_n x_{\gamma}^{\alpha}$ and $x_{\gamma}^{\alpha} R_n^{\gamma} y$, that is, $x R_n^{\alpha} y$. \square

We shall now provide another characterization of the $R_n^{1+\alpha}$ relations that is easier to work with. To this end, let us for convenience define

$$x R_{-1}^{\zeta} y \quad :\Leftrightarrow \quad \forall n > 0 \quad x_n \geq y_n.$$

With this notation the following theorem makes sense.

Theorem 4.1.13 (Characterization Theorem). *For $x, y \in I$ and n with $0 \leq n < \omega$ we have that the following are equivalent*

1. $x R_n^{1+\alpha} y$;
2. $x_n \geq y_n + (1 + e(y_{n+1})) \cdot (1 + \alpha)$ and $x R_{n-1}^{e(1+\alpha)} y$;

3.

$$\begin{aligned} x_n &\geq y_n + (1 + e(y_{n+1})) \cdot (1 + \alpha) && \text{and,} \\ x_m &> y_m \text{ for } m < n && \text{and,} \\ x_m &\geq y_m \text{ for } m > n. \end{aligned}$$

We dedicate the remainder of this subsection to prove this theorem and move there through a series of lemmas. The first lemma in this series is pretty obvious. It tells us that if we can move from x to y in α many steps, then the distance between x_n and y_n must allow α many steps; That is, they lie at least α apart.

Lemma 4.1.14. *For $x, y \in I$ and $n < \omega$ and any ordinal $\alpha < \Lambda$, if $x R_n^{\alpha} y$, then $x_n \geq y_n + \alpha$.*

Proof. By an easy induction on α . The base case is immediate since $x = y$. For $\alpha \in \text{Succ}$, we just need to apply Proposition 4.1.7 and the I.H. The case $\alpha \in \text{Lim}$ follows from Proposition 4.1.8 and the I.H. \square

However, how many R_n steps one can make is not entirely determined by the n coordinates of the points. For example, there is just a single R_0 step from the point $\langle \omega \cdot 2, 1 \rangle$ to the point $\langle \omega, 1 \rangle$ whereas these points lie ω apart on the '0 coordinate'. The following lemma tells us how for R_n steps, the n -th coordinates are affected by the values of the $n + 1$ -th coordinate.

Lemma 4.1.15. *For $x, y \in I$ and $n < \omega$ with $x R_n^{1+\alpha} y$, we have*

$$x_n \geq y_n + e(y_{n+1}) \cdot (1 + \alpha).$$

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In order to give a smooth presentation of this proof, we first give two simple technical lemmas with useful observations on the ordinals and ordinal functions involved.

Lemma 4.1.16. *For α, β and γ ordinals we have*

1. $\ell(\beta) \geq 1 + \alpha \iff \beta \in e(1 + \alpha) \cdot (1 + \text{Ord})$;
2. *If $(1 + \alpha) < \beta$ and $\gamma \in e(\beta) \cdot (1 + \text{Ord})$, then $\gamma \in e(1 + \alpha) \cdot (1 + \text{Ord})$;*
3. $e(\beta + (1 + \alpha)) = e(\beta) \cdot e(1 + \alpha)$;

Proof. The first two items can easily be seen by using a Cantor normal form expression with base ω . For Item 1, we use the fact that $\beta \in \text{Lim}$ together with that if $\ell(\beta) \geq 1 + \alpha$, then $\beta \geq e(\ell(\beta)) \geq e(1 + \alpha)$. For Items 2 and 3 we use that $e(1 + \omega) = \omega^{1+\omega} = \omega^1 \cdot \omega^\omega$. \square

Lemma 4.1.17. *For $x, y \in I$ and $n < \omega$, we have that $xR_n y \implies x_n \geq y_n + e(x_{n+1})$.*

Proof. For $y_n = 0$ the lemma is straightforward. Hence, we assume $y_n > 0$ and proceed by a case distinction on x_n . We can observe that since $xR_n y$, then $x_n > y_n$ and thus, $x_n \neq 0$. If $x_n \in \text{Succ}$ then the lemma is trivial since $e(x_{n+1}) = 0$. If x_n is an additively indecomposable limit ordinal, then for any $\alpha, \beta < x_n$, we have that $x_n > \alpha + \beta$. Thus, since $x_n > y_n$ and $x_n \geq e(x_{n+1})$, we can easily check that $x_n \geq y_n + e(x_{n+1})$.

Otherwise, we can rewrite x_n as $\alpha + e(\beta)$ for some $\beta \geq x_{n+1}$, and y_n as $\delta + \omega^\gamma$. If $y_n \leq \alpha$ then clearly $x_n \geq y_n + e(x_{n+1})$. For the remaining case, if $\alpha = \delta$ and $\gamma < \beta$, then we have that $\omega^\gamma + e(\beta) = e(\beta)$ and thus, $\alpha + e(\beta) = \delta + \omega^\gamma + e(\beta) \geq y_n + e(x_{n+1})$. \square

With these technical lemmas at hand we can now prove Lemma 4.1.15.

Proof. By induction on α . For $\alpha := \mathbf{0}$, we check that $x_n \geq y_n + e(y_{n+1})$. Note that since $xR_n y$, then $x_n \geq y_n + e(x_{n+1})$ and $x_{n+1} \geq y_{n+1}$, whence $x_n \geq y_n + e(y_{n+1})$.

For $\alpha := \beta + \mathbf{1}$, if $xR_n^{1+\beta+1} y$ then there is $z \in I$ such that $xR_n z$ and $zR_n^{1+\beta} y$. Thus, by the I.H. we have the following:

1. $x_n \geq z_n + e(z_{n+1})$;
2. $z_n \geq y_n + e(y_{n+1}) \cdot (1 + \beta)$.

Therefore, $x_n \geq y_n + e(y_{n+1}) \cdot (1 + \beta) + e(z_{n+1})$. Since $e(z_{n+1}) \geq e(y_{n+1})$ then $x_n \geq y_n + e(y_{n+1}) \cdot (1 + \beta) + e(y_{n+1})$, i.e., $x_n \geq y_n + e(y_{n+1}) \cdot (1 + \beta + \mathbf{1})$.

For $\alpha \in \text{Lim}$, we notice that by the I.H. we have that $x_n \geq y_n + e(y_{n+1}) \cdot (1 + \delta)$ for any $\delta < \alpha$. Thus, $x_n \geq y_n + e(y_{n+1}) \cdot (1 + \alpha)$. \square

Combining Lemma 4.1.15 and Lemma 4.1.14 we get the following.

Corollary 4.1.18. *For $x, y \in I$ and $n < \omega$ we have that*

$$xR_n^{1+\alpha}y \implies x_n \geq y_n + (1 + e(y_{n+1})) \cdot (1 + \alpha).$$

Proof. For $y_{n+1} > 0$, we have that $e(y_{n+1}) \in \text{Lim}$ and thus, $1 + e(y_{n+1}) = e(y_{n+1})$. Therefore, it follows immediately from Lemma 4.1.15. If $y_{n+1} = 0$, then $y_n + (1 + e(y_{n+1})) \cdot (1 + \alpha) = y_n + (1 + \alpha)$ and so, it follows from Lemma 4.1.14. \square

This corollary takes care of part of the implication from Item (1) to Item (2) in the characterization theorem (Theorem 4.1.13). We will now focus on the implication from Item (3) to Item (1) but before we do so, we first formulate a simple yet useful lemma.

Lemma 4.1.19. *For $x, y \in I$ and $m > 0$, if $xR_{n+m}y$, then $x_n \geq y_n + e(x_{n+1})$.*

Proof. Since R_{n+m} is contained in R_n , if $xR_{n+m}y$ then xR_ny and thus by Lemma 4.1.17, $x_n \geq y_n + e(x_{n+1})$. \square

With this technical lemma we can obtain the next step in the direction from Item (3) to Item (1) in the characterization theorem (Theorem 4.1.13).

Lemma 4.1.20. *For $x, y \in I$ and $n < \omega$ we have that if*

$$\begin{aligned} x_n &\geq y_n + (1 + e(y_{n+1})) \cdot (1 + \alpha) && \text{and,} \\ x_m &> y_m \text{ for } m < n && \text{and,} \\ x_m &\geq y_m \text{ for } m > n, \end{aligned}$$

then

$$xR_n^{1+\alpha}y.$$

Proof. First, we assume

1. $x_n \geq y_n + (1 + e(y_{n+1})) \cdot (1 + \alpha)$;
2. $x_m > y_m$ for $m < n$;
3. $x_m \geq y_m$ for $m > n$.

We use Lemma 4.1.12 whence are done if we can find a collection $c^{1+\alpha} = \{z_\delta^{1+\alpha}\}_{\delta < 1+\alpha}$ so that

1. $xR_n z_\delta^{1+\alpha}$ for any $z_\delta^{1+\alpha} \in c^{1+\alpha}$,
2. $z_0^{1+\alpha} = y$ and,
3. for any $\delta' < \delta < 1 + \alpha$ we have $z_\delta^{1+\alpha} R_n z_{\delta'}^{1+\alpha}$.

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We set $z_0^{1+\alpha} := y$ and for $\gamma > 0$, we define $z_\gamma^{1+\alpha}$ uniformly as follows:

$$(z_\gamma^{1+\alpha})_m := \begin{cases} y_m & \text{in case } m > n, \\ y_n + (1 + e(y_{n+1})) \cdot \gamma & \text{in case } m = n, \\ y_m + e((z_\gamma^{1+\alpha})_{m+1}) & \text{in case } m < n. \end{cases}$$

We make a collection of simple observations:

- I For any $\gamma < 1 + \alpha$ and $m < \omega$, we have that $(z_\gamma^{1+\alpha})_{m+1} \leq \ell((z_\gamma^{1+\alpha})_m)$ and so, each $z_\gamma^{1+\alpha}$ is an element of I ;
- II We now see that $xR_n z_\gamma^{1+\alpha}$ for each $\gamma < \alpha$. For $m > n$ we obviously have that $x_m \geq (z_\gamma^{1+\alpha})_m$ and also $x_n > (z_\gamma^{1+\alpha})_n$ is clear. For $m < n$, first we can observe that from assumptions 1, 2 and 3 we get that $xR_n y$. Using this fact, we check by induction on $n - m$ that $x_m > (z_\gamma^{1+\alpha})_m$. For the base case, since $xR_n y$, using Lemma 4.1.19 we have that

$$\begin{aligned} x_{n-1} &\geq y_{n-1} + e(x_n) \\ &\geq y_{n-1} + e(y_n + (1 + e(y_{n+1})) \cdot (1 + \alpha)) \\ &> y_{n-1} + e((z_\gamma^{1+\alpha})_n). \end{aligned}$$

For the inductive step, assume $x_{n-m} > (z_\gamma^{1+\alpha})_{n-m}$ as the I.H. Hence, by Lemma 4.1.19 $x_{n-(m+1)} \geq y_{n-(m+1)} + e(x_{n-m})$. Combining this with the I.H. we get that $x_{n-(m+1)} > (z_\gamma^{1+\alpha})_{n-(m+1)}$;

- III $x_0 = y$ by definition;
- IV By strict monotonicity of e , we see that for any $\gamma' < \gamma < \alpha$ we have $z_\gamma^{1+\alpha} R_n z_{\gamma'}^{1+\alpha}$.

□

We are now ready to combine all the above observations and obtain a proof for the characterization theorem (Theorem 4.1.13).

Proof. From Item 2 to Item 3 is easy and from Item 3 to Item 1 is Lemma 4.1.20 so we focus on the remaining implication, that is:

$$xR_n^{1+\alpha} y \implies x_n \geq y_n + (1 + e(y_{n+1})) \cdot (1 + \alpha) \text{ and } xR_{n-1}^{e(1+\alpha)} y$$

for any $x, y \in I$ and n with $0 \leq n < \omega$.

As mentioned before, half of this implication follows from Corollary 4.1.18 so that it remains to show that $xR_n^{1+\alpha} y \implies xR_{n-1}^{e(1+\alpha)} y$. For $n = 0$ this is trivial and in case $n \neq 0$ we reason as follows.

Since $xR_n^{1+\alpha}y$ we get in particular that $x_n \geq y_n + 1 + \alpha$. We proceed by a case distinction. If $y_n = 0$, then by Lemma 4.1.19 we see

$$x_{n-1} \geq y_{n-1} + e(x_n) \geq y_{n-1} + e(y_n + 1 + \alpha) = y_{n-1} + e(1 + \alpha).$$

Thus, $x_{n-1} \geq y_{n-1} + (1 + e(y_n)) \cdot (1 + e(1 + \alpha))$ and by Lemma 4.1.20 we are done. If $y_n > 0$, then $e(y_n) \in \text{Lim}$ and so, using the fact (Lemma 4.1.16) that $e(y_n + 1 + \alpha) = e(y_n) \cdot e(1 + \alpha)$ we get that

$$x_{n-1} \geq y_{n-1} + e(x_n) \geq y_{n-1} + e(y_n + 1 + \alpha) = y_{n-1} + (1 + e(y_n)) \cdot (1 + e(1 + \alpha)).$$

The result now follows from an application of Lemma 4.1.20. \square

Definable sets

In this subsection we shall define a map between formulas in MNF and ℓ -sequences as well as a way of characterizing definable subsets of I . Moreover, we shall see how some of these subsets of I can be related to the extensions of single formulas.

In virtue of the reduction axioms (Axiom 5), a formula $\psi \in \text{MNF}$ may bear implicit information on monomials $\langle n^\alpha \rangle \top$ for $n \notin N\text{-mod}(\psi)$. The next definition is made to retrieve this information.

Definition 4.1.21. Let $\psi := \langle n_0^{\alpha_0} \rangle \top \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle \top \in \text{MNF}$. By $\pi_{n_i}(\psi)$ we denote the corresponding exponent α_i . Moreover, for $m \notin N\text{-mod}(\psi)$, with $n_k > m$, $\pi_m(\psi)$ is set to be $e(\pi_{m+1}(\psi))$ and for $m' > n_k$, $\pi_{m'}(\psi)$ is defined to be 0.

We can use these functions to associate a sequence of ordinals to each formula in MNF.

Definition 4.1.22. Let $\psi := \langle n_0^{\alpha_0} \rangle \top \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle \top \in \text{MNF}$. By x_ψ we denote the sequence $\langle \pi_i(\psi) \rangle_{i < \omega}$.

On account of Definition 4.1.21, we can observe that for $\psi \in \text{MNF}$, we have that $x_\psi \in I$. Therefore, we shall refer x_ψ as *the corresponding world of ψ* . Furthermore, we shall see that x_ψ is the *first* point in I where ψ holds. First we make some simple observations.

Lemma 4.1.23.

1. For any $x \in I$, $x \Vdash \langle n^\alpha \rangle \top$ iff $x_n \geq \alpha$;
2. For any $\psi \in \text{MNF}$, $x_\psi \Vdash \psi$.

Proof. The second item follows from the first one and the definition of x_ψ (Definition 4.1.22). For the right-to-left implication of the first item, the case $\alpha = 0$ is straightforward. Hence, assume $x_n \geq \alpha > 0$. Therefore, for $i < n$, we

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have that $x_i > 0$ and for $i > n$, $x_i \geq 0$. Thus, by the characterization theorem (Theorem 4.1.13), $xR_n^\alpha \langle 0 \rangle$ and so $x \Vdash \langle n^\alpha \rangle \top$.

For the other direction, assume $x \Vdash \langle n^\alpha \rangle \top$ for $\alpha > 0$. Hence, there is $y \in I$ such that $xR_n^\alpha y$ and $y \Vdash \top$. By the characterization theorem, $x_n \geq y_n + (1 + e(y_{n+1})) \cdot \alpha$ and so, $x_n \geq \alpha$. \square

The following two definitions introduce the extension of ℓ -sequences and the extension of formulas, respectively.

Definition 4.1.24. Given $x \in I$, by $\llbracket x \rrbracket$ we denote the set of ℓ -sequences which are coordinate-wise at least as big as x . That is, we define $\llbracket x \rrbracket := \{y \in I : y_i \geq x_i \text{ for every } i < \omega\}$.

Definition 4.1.25. Let $\varphi \in \mathbb{F}_{TSC}$. By $\llbracket \varphi \rrbracket$ we denote the set of worlds where φ holds i.e. $\llbracket \varphi \rrbracket = \{x \in I : x \Vdash \varphi\}$.

The following lemma relates definitions 4.1.24 and 4.1.25.

Lemma 4.1.26. For any $\varphi \in \mathbb{F}_{TSC}$, there is $x := \langle x_0, \dots, x_k, 0 \rangle \in I$ such that $\llbracket \varphi \rrbracket = \llbracket x \rrbracket$.

Proof. The proof goes by induction on φ . The base case is trivial. For **the conjunctive case**, let $\varphi = \psi \wedge \chi$. By the I.H. we have that there are $y, z \in I$ such that $\llbracket \psi \rrbracket = \llbracket y \rrbracket$ and $\llbracket \chi \rrbracket = \llbracket z \rrbracket$. Moreover, by the I.H. we also have that $y := \langle y_0, \dots, y_j, 0 \rangle$ and $z := \langle z_0, \dots, z_k, 0 \rangle$. Let $n = \max(j, k)$, that is, the index of the rightmost non-zero component. Hence we can define x as follows:

- $x_i = \max(y_i, z_i)$ for $i \geq n$;
- $x_i = \min\{\delta : \delta \geq \max(y_i, z_i) \ \& \ l(\delta) \geq x_{i+1}\}$ for $i < n$.

We can easily check that $x \in I$. Next, we check that for any $x' \in I$, we have that

$$x' \Vdash \psi \wedge \chi \text{ iff } x' \in \llbracket x \rrbracket.$$

For the right-to-left implication, consider $x' \in \llbracket x \rrbracket$. Thus, for $i < \omega$, we have that both $x'_i \geq x_k \geq y_i$ and $x'_i \geq x_i \geq z_i$. Thus, $x' \in \llbracket y \rrbracket \cap \llbracket z \rrbracket$ and so by the I.H. $x' \Vdash \psi \wedge \chi$.

For the other direction, consider $x' \in I$ such that $x' \Vdash \psi \wedge \chi$. Clearly, for $i > n$, we have that $x'_i \geq x_i$. We check by induction on m that $x'_{n-m} \geq x_{n-m}$. For the base case, since $x' \Vdash \psi \wedge \chi$, then by the I.H. $x' \in \llbracket y \rrbracket \cap \llbracket z \rrbracket$ and so $x'_n \geq y_n$ and $x'_n \geq z_n$. Thus, $x'_n \geq \max(y_n, z_n) = x_n$. For the inductive step, by definition of the Ignatiev sequences together with the I.H., we have that $l(x'_{n-(m+1)}) \geq x'_{n-m} \geq x_{n-m}$ and since $x' \Vdash \psi \wedge \chi$, then we can conclude that $x'_{n-(m+1)} \geq \max(y_{n-(m+1)}, z_{n-(m+1)})$. Therefore, being $x_{n-(m+1)}$ the minimal

ordinal satisfying both conditions, we can conclude that $x'_{n-(m+1)} \geq x_{n-(m+1)}$. Hence, $\llbracket \psi \wedge \chi \rrbracket = \llbracket x \rrbracket$.

For **the modality case**, let $\varphi := \langle n^\alpha \rangle \psi$ with $\alpha > 0$. Thus, by the I.H. there is $y \in I$ such that $\llbracket \psi \rrbracket = \llbracket y \rrbracket$ and $y := \langle y_0, \dots, y_j, 0 \rangle$. We can define x as follows:

- $x_i = y_i$ for $i > n$;
- $x_n = y_n + (1 + e(y_{n+1})) \cdot \alpha$;
- $x_i = \min\{\delta : \delta > y_i \ \& \ l(\delta) \geq x_{i+1}\}$ for $i < n$.

As in the previous case, we can easily check that $x \in I$. Thus, we check that $\llbracket x \rrbracket = \llbracket \langle n^\alpha \rangle \psi \rrbracket$. To prove $\llbracket x \rrbracket \subseteq \llbracket \langle n^\alpha \rangle \psi \rrbracket$, let $x' \in \llbracket x \rrbracket$. By the characterization theorem (Theorem 4.1.13) we can see that $xR_n^\alpha y$. Hence, since $x'_i \geq x_i$ for $i < \omega$, $x'R_n^\alpha y$ and so $x' \Vdash \langle n^\alpha \rangle \psi$.

For the other inclusion, consider $x' \in I$ such that $x' \Vdash \langle n^\alpha \rangle \psi$. Thus, there is $z \in \llbracket y \rrbracket$ such that $x'R_n^\alpha z$. Hence, with the help of the characterization theorem we can establish that for $i > n$ we have that $x'_i \geq z_i \geq y_i = x_i$, that is, $x'_i \geq x_i$. For $i \leq n$, we proceed by an easy induction on k to see that $x'_{n-k} \geq x_{n-k}$. The base case follows directly from the characterization theorem. For the inductive step, by definition of Ignatiev sequences together with the I.H., we have that $l(x'_{n-(k+1)}) \geq x'_{n-k} \geq x_{n-k}$. Since $x' \Vdash \langle n^\alpha \rangle \psi$, then there is $z \in I$ such that $x'R_n^\alpha z$ and $z \Vdash \psi$. Thus, by the I.H., $z \in \llbracket y \rrbracket$, and so we have that $x'_{n-(k+1)} > z_{n-(k+1)} \geq y_{n-(k+1)}$. Therefore, we get that $l(x'_{n-(k+1)}) \geq x_{n-k}$ and $x'_{n-(k+1)} > y_{n-(k+1)}$. Thus, since $x_{n-(k+1)}$ is the least ordinal satisfying both conditions, we have that $x'_{n-(k+1)} \geq x_{n-(k+1)}$. \square

Some of the facts established in the proof of the previous lemma are collected in the following corollary.

Corollary 4.1.27. *For any $\varphi, \psi \in \mathbb{F}_{TSC}$:*

1. *If $\llbracket \langle x_0, \dots, x_j, 0 \rangle \rrbracket = \llbracket \varphi \rrbracket$ and $\llbracket \langle y_0, \dots, y_k, 0 \rangle \rrbracket = \llbracket \psi \rrbracket$, then there is $z \in I$ such that $\llbracket z \rrbracket = \llbracket \varphi \wedge \psi \rrbracket$ and z is defined as follows:*
 - $z_i = \max(x_i, y_i)$ for $i \geq \max(j, k)$;
 - $z_i = \min\{\delta : \delta \geq \max(x_i, y_i) \ \& \ l(\delta) \geq z_{i+1}\}$ for $i < \max(j, k)$.
2. *If $\llbracket x \rrbracket = \llbracket \varphi \rrbracket$, then there is $y \in I$ such that $\llbracket y \rrbracket = \llbracket \langle n^\alpha \rangle \varphi \rrbracket$ and y is defined as follows:*
 - $y_i = x_i$ for $i > n$;
 - $y_n = x_n + (1 + e(y_{n+1})) \cdot \alpha$;
 - $y_i = \min\{\delta : \delta > x_i \ \& \ l(\delta) \geq y_{i+1}\}$ for $i < n$.

The next lemma relates the extension of a formula ψ in MNF with the extension of its corresponding world x_ψ .

Lemma 4.1.28. *For any $\psi \in \text{MNF}$, we have that $\llbracket \psi \rrbracket = \llbracket x_\psi \rrbracket$.*

Proof. Let $\psi := \langle n_0^{\alpha_0} \rangle \top \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle \top$. We proceed by induction on k . For the base case, we start by checking $\llbracket \psi \rrbracket \subseteq \llbracket x_\psi \rrbracket$. Assume $x \Vdash \langle n_0^{\alpha_0} \rangle \top$. Then, there is $y \in I$ such that $xR_{n_0^{\alpha_0}}y$, and so, by the characterization theorem (Theorem 4.1.13) we have that $xR_m^{e^{n_0-m}(\alpha_0)}y$ for $m \leq n$, that is, $x \Vdash \langle m^{e^{n_0-m}(\alpha_0)} \rangle \top$. Therefore, with the help of Lemma 4.1.23, Item 1 we have that $x_m \geq e^{n_0-m}(\alpha_0)$, and so $x \in \llbracket x_\psi \rrbracket$.

For the other inclusion, let $x \in \llbracket x_\psi \rrbracket$. Thus, $x_{n_0} \geq (x_\psi)_{n_0}$ and so, by Lemma 4.1.23, Item 1 $x \Vdash \langle n_0^{\alpha_0} \rangle \top$. For the inductive step we reason analogously. \square

4.2. Modal Soundness and Completeness

We dedicate this section to prove the soundness and completeness of TSC w.r.t. \mathcal{J} .

Modal soundness

To prove the soundness of TSC, let us begin by semantically define the entailment between our modal formulas.

Definition 4.2.1. *For any formulas $\varphi, \psi \in \mathbb{F}_{\text{TSC}}$, we write $\varphi \models_{\mathcal{J}} \psi$ iff for all $x \in I$, if $x \Vdash \varphi$ then $x \Vdash \psi$. Analogously, we write $\varphi \equiv_{\mathcal{J}} \psi$ iff for any $x \in I$, we have that $x \Vdash \varphi$ iff $x \Vdash \psi$.*

With our notion of semantical entailment we can formulate our soundness theorem.

Theorem 4.2.2 (Soundness). *For any formulas $\varphi, \psi \in \mathbb{F}_{\text{TSC}}$,*

$$\varphi \vdash \psi \implies \varphi \models_{\mathcal{J}} \psi.$$

Proof. By induction on the length of a TSC proof of $\varphi \vdash \psi$. It is easy to see that the first three rules:

1. If $\varphi \vdash \psi$ and $\varphi \vdash \chi$, then $\varphi \vdash \psi \wedge \chi$;
2. If $\varphi \vdash \psi$ and $\psi \vdash \chi$, then $\varphi \vdash \chi$;
3. If $\varphi \vdash \psi$, then $\langle n^\alpha \rangle \varphi \vdash \langle n^\alpha \rangle \psi$;

preserve validity. With respect to the axioms, the first two axioms

1. $\varphi \vdash \varphi, \quad \varphi \vdash \top$;

2. $\varphi \wedge \psi \vdash \varphi$, $\varphi \wedge \psi \vdash \psi$;

are easily seen to be valid. The the correctness of reduction axiom

$$\langle (m+n)^\alpha \rangle \varphi \vdash \langle m^{e^n(\alpha)} \rangle \varphi$$

is given by the characterization theorem (Theorem 4.1.13). The remaining axioms and rules are separately proven to be sound in the remainder of this subsection. \square

We start by proving the soundness of co-additivity axiom i.e.

$$\langle n^\alpha \rangle \langle n^\beta \rangle \varphi \equiv_{\mathcal{J}} \langle n^{\beta+\alpha} \rangle \varphi.$$

Proposition 4.2.3. *For any $x, z \in I$, $n < \omega$ and $\alpha, \beta < \Lambda$,*

$$\exists y \in I (xR_n^\alpha y \text{ and } yR_n^\beta z) \iff xR_n^{\beta+\alpha} z.$$

Proof. We proceed by transfinite induction on α with the base case being trivial. For $\alpha \in \text{Succ}$, let $\alpha := \delta + 1$ for some δ . Therefore:

$$xR_n^\alpha y \text{ and } yR_n^\beta z \iff xR_n^{\delta+1} y \text{ and } yR_n^\beta z;$$

$$\iff \exists u (xR_n u \wedge uR_n^\delta y \wedge yR_n^\beta z);$$

$$\iff \exists u (xR_n u \wedge uR_n^{\beta+\delta} z), \text{ by the I.H. ;}$$

$$\iff xR_n^{\beta+\delta+1} z;$$

$$\iff xR_n^{\beta+\alpha} z.$$

For $\alpha \in \text{Lim}$, we have that $xR_n^\alpha y \text{ and } yR_n^\beta z \iff \forall \delta < \alpha (xR_n^{1+\delta} y \wedge yR_n^\beta z)$ by Proposition 4.1.8. By the I.H. we obtain $\forall \delta < \alpha xR_n^{\beta+1+\delta} z$ and so $xR_n^{\beta+\alpha} z$. \square

With this last result, we get the co-additivity of the R_n^α relations. This together with Definition 4.1.9 gives us the following corollary.

Corollary 4.2.4. *The co-additivity axiom is sound.*

Proof. By Definition 4.1.9, $x \Vdash \langle n^\alpha \rangle \langle n^\beta \rangle \varphi$ iff there are $y, z \in I$ such that $xR_n^\alpha y$, $yR_n^\beta z$ and $z \Vdash \varphi$. Thus, by Proposition 4.2.3, $x \Vdash \langle n^\alpha \rangle \langle n^\beta \rangle \varphi$ iff $xR_n^{\beta+\alpha} z$ and $z \Vdash \varphi$ i.e., $x \Vdash \langle n^{\beta+\alpha} \rangle \varphi$. \square

The following proposition establishes the soundness of the monotonicity axiom.

Proposition 4.2.5. *The monotonicity axiom is sound, that is:*

$$\langle n^\alpha \rangle \varphi \models_{\mathcal{J}} \langle n^\beta \rangle \varphi$$

for $\beta < \alpha$.

Proof. With the help of the monotonicity property of the $R_n^{1+\alpha}$ relations (Lemma 4.1.10, Item 4), we have that if $x \Vdash \langle n^\alpha \rangle \varphi$ then $x \Vdash \langle n^\beta \rangle \varphi$ for β , $0 < \beta < \alpha$. We check that if $x \Vdash \langle n^1 \rangle \varphi$ then $x \Vdash \varphi$ by induction on φ .

The base and the conjunctive cases are straightforward, so we consider $\varphi := \langle m^\delta \rangle \psi$ for some $\delta > 0$ and assume $x \Vdash \langle n^1 \rangle \langle m^\delta \rangle \psi$. Thus, there are $y, z \in I$ such that $xR_n yR_m^\delta z$ and $z \Vdash \psi$. Then, by the polytransitivity condition (Lemma 4.1.11) we have that $xR_m^\delta z$, and so $x \Vdash \langle m^\delta \rangle \psi$. \square

The following proposition establishes the correction of the Schmerl axiom by using the translation between formulas in monomial normal form and ℓ -sequences.

Proposition 4.2.6. *The Schmerl axiom is sound i.e.*

$$\langle n^\alpha \rangle (\langle n_0^{\alpha_0} \rangle \top \wedge \psi) \equiv_{\mathcal{J}} \langle n^{e^{n_0-n}(\alpha_0) \cdot (1+\alpha)} \rangle \top \wedge \langle n_0^{\alpha_0} \rangle \top \wedge \psi$$

for $n < n_0$ and $\langle n_0^{\alpha_0} \rangle \top \wedge \psi \in MNF$.

Proof. For the left-to-right direction, assume $x \Vdash \langle n^\alpha \rangle (\langle n_0^{\alpha_0} \rangle \top \wedge \psi)$. Thus, by soundness of monotonicity axiom, we have that $x \Vdash \langle n_0^{\alpha_0} \rangle \top \wedge \psi$. Therefore, we only need to check that $x \Vdash \langle n^{e^{n_0-n}(\alpha_0) \cdot (1+\alpha)} \rangle \top$. We can observe that $x \Vdash \langle n^\alpha \rangle \langle n_0^{\alpha_0} \rangle \top$ and so there are $y, z \in I$ such that $xR_n^\alpha yR_{n_0}^{\alpha_0} z$. By the characterization theorem (Theorem 4.1.13) we have that

$$x_n \geq y_n + (1 + e(y_{n+1})) \cdot \alpha. \quad (4.1)$$

Also since $yR_{n_0}^{\alpha_0} z$ then $yR_n^{e^{n_0-n}(\alpha_0)} z$ and $yR_{n+1}^{e^{n_0-(n+1)}(\alpha_0)} z$. Hence by the characterization theorem we get that $y_n \geq e^{n_0-n}(\alpha_0)$ and $y_{n+1} \geq e^{n_0-(n+1)}(\alpha_0)$. Combining this with 4.1 we get that

$$\begin{aligned} x_n &\geq e^{n_0-n}(\alpha_0) + \left(1 + e(e^{n_0-(n+1)}(\alpha_0))\right) \cdot \alpha \\ &= e^{n_0-n}(\alpha_0) + e(e^{n_0-(n+1)}(\alpha_0)) \cdot \alpha \\ &= e^{n_0-n}(\alpha_0) + e^{n_0-n}(\alpha_0) \cdot \alpha \\ &= e^{n_0-n}(\alpha_0) \cdot (1 + \alpha). \end{aligned}$$

Thus, we have that $xR_n^{e^{n_0-n}(\alpha_0) \cdot (1+\alpha)} \langle 0 \rangle$ and so, $x \Vdash \langle n^{e^{n_0-n}(\alpha_0) \cdot (1+\alpha)} \rangle \top$.

For the other direction, assume $x \Vdash \langle n^{e^{n_0-n}(\alpha_0) \cdot (1+\alpha)} \rangle \top \wedge \langle n_0^{\alpha_0} \rangle \top \wedge \psi$. Let $\psi' = \langle n^{e^{n_0-n}(\alpha_0) \cdot (1+\alpha)} \rangle \top \wedge \langle n_0^{\alpha_0} \rangle \top \wedge \psi$. In virtue of Definition 4.1.22 we can consider the corresponding worlds of $\langle n_0^{\alpha_0} \rangle \top \wedge \psi$ and ψ' , that are, $x_{\langle n_0^{\alpha_0} \rangle \top \wedge \psi}$ and $x_{\psi'}$. We set out to prove $xR_n^\alpha x_{\langle n_0^{\alpha_0} \rangle \top \wedge \psi}$.

Since $x \Vdash \psi'$ then by Lemma 4.1.28 we have that $x \in \llbracket \psi' \rrbracket = \llbracket x_{\psi'} \rrbracket$. Thus, we can make the following observations in order to apply the characterization theorem:

- By Definition 4.1.22, for all $m > n$, we have that $(x_{\psi'})_m = (x_{\langle n_0^{\alpha_0} \rangle_{\top \wedge \psi}})_m$. Combining this with the fact that $x \in \llbracket x_{\psi} \rrbracket$, we get that for all $m > n$, $x_m \geq (x_{\langle n_0^{\alpha_0} \rangle_{\top \wedge \psi}})_m$.
- Since $x \in \llbracket x_{\psi} \rrbracket$ we have that $x_n \geq (x_{\psi'})_n = e^{n_0-n}(\alpha_0) \cdot (1 + \alpha)$. Also, by Definition 4.1.22, we can observe that $(x_{\langle n_0^{\alpha_0} \rangle_{\top \wedge \psi}})_n = e^{n_0-n}(\alpha_0)$ and $(x_{\langle n_0^{\alpha_0} \rangle_{\top \wedge \psi}})_{n+1} = e^{n_0-(n+1)}(\alpha_0)$. Thus, we can reason as follows:

$$\begin{aligned}
 x_n &\geq e^{n_0-n}(\alpha_0) \cdot (1 + \alpha) \\
 &= e^{n_0-n}(\alpha_0) + e^{n_0-n}(\alpha_0) \cdot \alpha \\
 &= e^{n_0-n}(\alpha_0) + e(e^{n_0-(n+1)}(\alpha_0)) \cdot \alpha \\
 &= (x_{\langle n_0^{\alpha_0} \rangle_{\top \wedge \psi}})_n + \left(1 + e((x_{\langle n_0^{\alpha_0} \rangle_{\top \wedge \psi}})_{n+1})\right) \cdot \alpha;
 \end{aligned}$$

- For all $m < n$, we can observe that $x_n \geq e^{n-m}(e^{n_0-n}(\alpha_0) \cdot (1 + \alpha))$. On the other hand, $(x_{\langle n_0^{\alpha_0} \rangle_{\top \wedge \psi}})_m = e^{n-m}(e^{n_0-n}(\alpha_0))$. Therefore, for all $m < n$, we have that $x_m > (x_{\langle n_0^{\alpha_0} \rangle_{\top \wedge \psi}})_m$.

Hence, by the characterization theorem, we have that $xR_n^\alpha x_{\langle n_0^{\alpha_0} \rangle_{\top \wedge \psi}}$ and thus $x_{\langle n_0^{\alpha_0} \rangle_{\top \wedge \psi}}$. \square

Lastly, we check the soundness of Rule 4 by applying the relation between definable sets and the extension of ℓ -sequences proved in Lemma 4.1.26. This next result concludes the soundness proof of TSC.

Proposition 4.2.7. *If $\varphi \models_{\mathcal{J}} \psi$ then, for $m < n$:*

$$\langle n^\alpha \rangle \varphi \wedge \langle m^{\beta+1} \rangle \psi \models_{\mathcal{J}} \langle n^\alpha \rangle (\varphi \wedge \langle m^{\beta+1} \rangle \psi).$$

Proof. Assume $\varphi \models_{\mathcal{J}} \psi$ and let $x \in I$ such that $x \Vdash \langle n^\alpha \rangle \varphi \wedge \langle m^{\beta+1} \rangle \psi$. Since $\varphi \models_{\mathcal{J}} \psi$, by Lemma 4.1.26, there are $y, z \in I$ such that

$$\llbracket y \rrbracket = \llbracket \varphi \rrbracket \subseteq \llbracket \psi \rrbracket = \llbracket z \rrbracket.$$

Let $y', z' \in I$ such that $\llbracket y' \rrbracket = \llbracket \langle n^\alpha \rangle \varphi \rrbracket$ and $\llbracket z' \rrbracket = \llbracket \langle m^{\beta+1} \rangle \psi \rrbracket$, and $w \in I$ such that:

$$\llbracket w \rrbracket = \llbracket \varphi \wedge \langle m^{\beta+1} \rangle \psi \rrbracket = \llbracket \varphi \rrbracket \cap \llbracket \langle m^{\beta+1} \rangle \psi \rrbracket = \llbracket y \rrbracket \cap \llbracket z' \rrbracket.$$

First, we shall make some observations about w and x , respectively. We will use these facts together with the characterization theorem (Theorem 4.1.13) to conclude that $xR_n^\alpha w$. Since $y \in \llbracket z \rrbracket$, then for all $i < \omega$ we have that $y_i \geq z_i$.

Furthermore, by Corollary 4.1.27, Item 2 we have that for $i > m$, $z'_i = z_i$. Thus, we can observe that

- For $i > m$ we have that

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$$\begin{aligned} w_i &= \min\{\delta : \delta \geq \max(y_i, z'_i) \ \& \ l(\delta) \geq w_{i+1}\} \\ &= \min\{\delta : \delta \geq \max(y_i, z_i) \ \& \ l(\delta) \geq w_{i+1}\} \\ &= y_i; \end{aligned}$$

$$\begin{aligned} \blacksquare \ w_m &= \min\{\delta : \delta \geq \max(y_i, z'_i) \ \& \ l(\delta) \geq w_{m+1}\} \\ &= \min\{\delta : \delta \geq \max(y_i, z'_i) \ \& \ l(\delta) \geq y_{m+1}\}. \end{aligned}$$

On the other hand, since $x \Vdash \langle n^\alpha \rangle \varphi \wedge \langle m^{\beta+1} \rangle \psi$, we have the following:

- By Corollary 4.1.27, Item 1 $x_i \geq \min\{\delta : \delta \geq \max(y'_i, z'_i) \ \& \ l(\delta) \geq x_{i+1}\}$ for $i \leq m$;
- Since $x \Vdash \langle n^\alpha \rangle \varphi \wedge \langle m^{\beta+1} \rangle \psi$, then $x \Vdash \varphi$ and so, we have that $x_i \geq y_i$ for $i > n$;
- As $x \Vdash \langle n^\alpha \rangle \varphi$ we get that $x_n \geq y'_n$ and $x_i \geq \min\{\delta : \delta \geq y'_i \ \& \ l(\delta) \geq x_{i+1}\}$ for $i, m < i < n$.

It remains to be checked that $xR_n^\alpha w$. Recall that $m < n$ and that for $i > m$, we have that $w_i = y_i$. Thus, for $i > n$, we have that $x_i \geq y_i = w_i$. Also, since $w_n = y_n$, $w_{n+1} = y_{n+1}$ and $x_n \geq y'_n = y_n + (1 + e(y_{n+1})) \cdot \alpha$ we have that $x_n \geq w_n + (1 + e(w_{n+1})) \cdot \alpha$. Thus, we need to see that $x_i > w_i$ for $i < n$.

For $i, m < i < n$, we know that $y_i = w_i$ and by Corollary 4.1.27, Item 2, $y'_i > y_i$. Since $x_i \geq y'_i$ for $i, m < i < n$, then $x_i > w_i$. For $i \leq m$, we show by induction on k that $x_{m-k} > w_{m-k}$. For the base case, recall that by assumption $\varphi \models_{\mathcal{J}} \psi$ and so, $y \in \llbracket z \rrbracket$, that is, for $i \in \omega$, $y_i \geq z_i$. Thus, we can first observe that $z_{m+1} \leq y_{m+1} = w_{m+1} < x_{m+1}$ and $\max(y'_m, z'_m) \geq \max(y_m, z'_m)$. We make the following case distinction:

- If $w_m = y_m$, then $x_m \geq y'_m > y_m$. Therefore, $x_m > w_m$;
- If $w_m = z_m + (1 + e(z_{m+1})) \cdot (1 + \beta) + e(w_{m+1})$, then since $x_{m+1} > w_{m+1}$, we have that

$$\begin{aligned} x_m &\geq z_m + (1 + e(z_{m+1})) \cdot (1 + \beta) + e(x_{m+1}) \\ &> z_m + (1 + e(z_{m+1})) \cdot (1 + \beta) + e(w_{m+1}) \\ &= w_m. \end{aligned}$$

Thus, $x_m > w_m$.

For the inductive step, by the I.H. we have that $x_{m-k} > w_{m-k}$. We reason as follows:

- If $w_{m-(k+1)} = \min\{\delta : \delta \geq y_{m-(k+1)} \ \& \ l(\delta) \geq w_{m-k}\}$, then since $y'_m > y_m$ and by the I.H. we have that $x_{m-(k+1)} > w_{m-(k+1)}$;

- If $w_{m-(k+1)} = \min\{\delta : \delta \geq z'_{m-(k+1)} \ \& \ l(\delta) \geq w_{m-k}\}$, then we can observe that $w_{m-(k+1)} = z_{m-(k+1)} + (1 + e(z_{m-k})) \cdot (1 + \beta) + e(w_{m-k})$. Thus, by the I.H. we get that

$$\begin{aligned} x_{m-(k+1)} &\geq z_{m-(k+1)} + (1 + e(z_{m-k})) \cdot (1 + \beta) + e(x_{m-k}) \\ &> z_{m-(k+1)} + (1 + e(z_{m-k})) \cdot (1 + \beta) + e(w_{m-k}) \\ &= w_{m-(k+1)}, \end{aligned}$$

and so $x_{m-(k+1)} > w_{m-(k+1)}$.

Hence, in virtue of the characterization theorem we get that $xR_n^\alpha w$, and so we obtain that $x \Vdash \langle n^\alpha \rangle (\varphi \wedge \langle m^{\beta+1} \rangle \psi)$ as needed. \square

Modal completeness

In the previous chapter we made use of Corollary 3.3.1 to characterize the non-derivability in our system. As we mentioned, the same result can be obtained by using relational semantics since for any $\alpha > 0$ and $x \in I$, we have that $\neg xR_n^\alpha x$.

In this chapter we shall follow a different approach to establish the modal completeness of **TSC**. First we prove the following proposition that characterizes the derivability between formulas in monomial normal form in terms of the π_n -functions introduced in Definition 4.1.21. Therefore, this result provides a refinement of Proposition 3.3.11 by means of the π_n -functions.

Proposition 4.2.8. *Given $\psi_0, \psi_1 \in MNF$, we have that:*

$$\psi_0 \vdash \psi_1 \iff \pi_n(\psi_0) \geq \pi_n(\psi_1) \text{ for all } n \in N\text{-mod}(\psi_1).$$

Proof. For the left-to-right implication, assume $\psi_0 \vdash \psi_1$. By Theorem 3.3.11, Item 2 we have that $\pi_n(\psi_1) \leq e^{m-n}(\pi_m(\psi_0))$ for $m = \mu(n, \psi_0)$. We can observe that $e^{m-n}(\pi_m(\psi_0)) \leq \pi_n(\psi_0)$, and so, we get that $\pi_n(\psi_0) \geq \pi_n(\psi_1)$.

For the right-left-implication, assume that $\pi_n(\psi_0) \geq \pi_n(\psi_1)$ for for any $n \in N\text{-mod}(\psi_1)$. We make the following case distinction:

- If $n \in N\text{-mod}(\psi_0)$, then we have that $\psi_0 \vdash \langle n^{\pi_n(\psi_0)} \rangle \top$. Since by assumption $\pi_n(\psi_0) \geq \pi_n(\psi_1)$, then by monotonicity axiom (Axiom 3), $\langle n^{\pi_n(\psi_0)} \rangle \top \vdash \langle n^{\pi_n(\psi_1)} \rangle \top$. Thus, $\psi_0 \vdash \langle n^{\pi_n(\psi_1)} \rangle \top$.
- If $n \notin N\text{-mod}(\psi_0)$, then $\pi_n(\psi_0) = e^{m-n}(\pi_m(\psi_0))$ where $m = \mu(n, \psi_0)$. Hence, we have that $\psi_0 \vdash \langle n^{\pi_n(\psi_0)} \rangle \top$ and by reduction axiom (Axiom 5), $\psi_0 \vdash \langle m^{e^{m-n}(\pi_n(\psi_0))} \rangle \top$. Since $e^{m-n}(\pi_m(\psi_0)) = \pi_n(\psi_0) \geq \pi_n(\psi_1)$, with the help of monotonicity axiom, we get that

$$\langle n^{\pi_n(\psi_0)} \rangle \top \vdash \langle n^{\pi_n(\psi_1)} \rangle \top.$$

Therefore, $\psi_0 \vdash \langle n^{\pi_n(\psi_1)} \rangle \top$.

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Thus, for each monomial $\langle n^\alpha \rangle \top$ occurring in ψ_1 we have that $\psi_0 \vdash \langle n^\alpha \rangle \top$, and so, $\psi_0 \vdash \psi_1$. \square

The following corollary follows immediately from this characterization of the derivability between formulas in monomial normal form.

Corollary 4.2.9. *Given $\varphi, \psi \in \text{MNF}$, if $\varphi \not\vdash \psi$ then there is $m_I \in \text{N-mod}(\psi)$ such that $\pi_{m_I}(\varphi) < \pi_{m_I}(\psi)$.*

Now we are ready to prove the completeness of TSC.

Theorem 4.2.10 (Completeness). *Given formulas $\varphi, \psi \in \mathbb{F}_{\text{TSC}}$, if $\varphi \models_{\mathcal{J}} \psi$, then $\varphi \vdash \psi$.*

Proof. By Theorem 3.3.7, w.l.o.g. let $\varphi, \psi \in \text{MNF}$ such that $\varphi := \langle n_0^{\alpha_0} \rangle \top \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle \top$ and $\psi := \langle m_0^{\beta_0} \rangle \top \wedge \dots \wedge \langle m_j^{\beta_j} \rangle \top$. Reasoning by contraposition, suppose $\varphi \not\vdash \psi$. Therefore, by Corollary 4.2.9, we can conclude that for some $m_I \in \text{N-mod}(\psi)$, we have that $\pi_{m_I}(\varphi) < \pi_{m_I}(\psi)$. Thus, consider the Ignatiev sequence x_φ . By Lemma 4.1.23, Item (2), $x_\varphi \Vdash \varphi$ but $x_\varphi \not\vdash \langle m_I^{\beta_I} \rangle \top$. Hence, $x_\varphi \not\vdash \psi$ and so $\varphi \not\models_{\mathcal{J}} \psi$. \square

More on TSC

We dedicate this chapter to collecting some miscellaneous results about **TSC**. First, we shall consider a new universal frame for **TSC** by restricting the domain of \mathcal{J} to those ℓ -sequences with finite support. This new frame will provide us with some useful arithmetical applications by depicting a road map to consistency strength and conservativity results of Turing progressions. Secondly, we shall introduce a presentation of **TSC** that makes no use of MNF's, namely **TSC***. We prove the equivalence between both systems. Finally, we give a modal formulation of the Schmerl's formulas and provide a modal proof of them without requiring the completeness result.

5.1. A Finitely Supported Universal Frame

In this section we present a new universal frame \mathcal{H} , which is a slight modification of \mathcal{J} and allows for the modal definability of each world in the domain. More precisely, the set of worlds of \mathcal{H} is built from finitely supported ℓ -sequences, i.e., sequences of the form $\langle x_0, \dots, x_k, 0 \rangle \in I$.

Definition 5.1.1. $\mathcal{H} := \langle H, \{S_n\}_{n < \omega} \rangle$ is defined as follows:

$$H := \{x \in I : \exists n < \omega \ x_n = 0\}$$

and

$$xS_ny \Leftrightarrow (\forall m \leq n \ x_m > y_m \wedge \forall m > n \ x_m \geq y_m).$$

Remark 5.1.2. Since $H \subseteq I$ and $S_n = R_n \upharpoonright H$, clearly for any $x, y \in H$, we have that $xS_ny \iff xR_ny$. Moreover, we can easily check that if $x \in H$ and xR_ny for some $y \in I$, then $y \in H$ and xS_ny .

The next lemma is a reformulation of Theorem 4.1.4 for the S_n -relations. These observations confirm that the S_n -relations are good to model provability logic.

Lemma 5.1.3.

1. Each S_n for $n \in \omega$ is transitive: $xS_ny \wedge yS_nz \Rightarrow xS_nz$;
2. Each S_n for $n \in \omega$ is Noetherian: each non-empty $X \subseteq H$ has an S_n -maximal element $y \in X$, i.e., $\forall x \in X \neg yS_nx$;
3. The relations S_n are monotone in n in the sense that: $xS_ny \Rightarrow xS_my$ whenever $n > m$.

As we did for \mathcal{J} , we need to define the auxiliary relations S_n^α for any $n < \omega$ and $\alpha < \Lambda$ whose purpose is to model the $\langle n^\alpha \rangle$ modality.

Definition 5.1.4. Given $x, y \in H$ and S_n on H , we recursively define $xS_n^\alpha y$ as follows:

1. $xS_n^0 y \iff x = y$;
2. $xS_n^{1+\alpha} y \iff \forall \beta < 1+\alpha \exists z (xS_n z \wedge zS_n^\beta y)$.

We can also reformulate Definition 4.1.9 in terms of \mathcal{H} .

Definition 5.1.5. Let $x \in H$ and $\varphi \in \mathbb{F}_{\text{TSC}}$. By $x \Vdash \varphi$ we denote the validity of φ in x that is recursively defined as follows:

1. $x \Vdash \top$ for all $x \in H$;
2. $x \Vdash \varphi \wedge \psi$ iff $x \Vdash \varphi$ and $x \Vdash \psi$;
3. $x \Vdash \langle n^\alpha \rangle \varphi$ iff there is $y \in H$, $xS_n^\alpha y$ and $y \Vdash \varphi$.

The following proposition establishes the preservation between the S_n^α relations of \mathcal{H} and the R_n^α relations of \mathcal{J} when restricted to the elements in H .

Proposition 5.1.6. For any $x, y \in H$, $n < \omega$ and $\alpha < \Lambda$:

$$xS_n^\alpha y \iff xR_n^\alpha y.$$

Proof. We proceed by induction on α being the base case trivial. The successor and limit cases follows from Remark 5.1.2 and the I.H. \square

Preservation of validity

To prove the completeness of **TSC** w.r.t. \mathcal{H} , first we shall prove the preservation of validity between the frames \mathcal{H} and \mathcal{J} when restricted to ℓ -sequences with finite support.

Theorem 5.1.7. *For any $x \in H$ and $\varphi \in \mathbb{F}_{TSC}$,*

$$\mathcal{J}, x \Vdash \varphi \iff \mathcal{H}, x \Vdash \varphi.$$

Proof. Proof goes by induction on φ . The base and conjunctive cases are straightforward. For the case $\varphi := \langle n^\alpha \rangle \psi$, assume $\mathcal{J}, x \Vdash \langle n^\alpha \rangle \psi$. Thus, there is $y \in I$ such that $xR_n^\alpha y$ and $\mathcal{J}, y \Vdash \psi$. By Remark 5.1.2, we know $y \in H$ and by Proposition 5.1.6 we have that $xS_n^\alpha y$. This together with the I.H. gives us that there is $y \in H$ such that $xS_n^\alpha y$ and $\mathcal{H}, y \Vdash \psi$, that is, $\mathcal{H}, x \Vdash \langle n^\alpha \rangle \psi$. For the other implication, suppose that $\mathcal{H}, x \Vdash \langle n^\alpha \rangle \psi$ i.e. there is $y \in H$ such that $xS_n^\alpha y$ and $\mathcal{H}, y \Vdash \psi$. By Remark 5.1.2, we know $H \subset I$. Thus, by Proposition 5.1.6 and the I.H. we obtain that there is $y \in I$ such that $xR_n^\alpha y$ and $\mathcal{J}, y \Vdash \psi$ and so, $\mathcal{J}, x \Vdash \langle n^\alpha \rangle \psi$. \square

Now we can show the completeness of our system w.r.t. \mathcal{H} .

Theorem 5.1.8. *For any $\varphi, \psi \in \mathbb{F}_{TSC}$, we have that:*

$$\varphi \vdash \psi \iff \forall x \in H \left(\mathcal{H}, x \Vdash \varphi \implies \mathcal{H}, x \Vdash \psi \right).$$

Proof. For the left-to-right implication, assume $\varphi \vdash \psi$. Thus, by Theorem 4.2.2, $\varphi \models_{\mathcal{J}} \psi$. Assume $\mathcal{H}, x \Vdash \varphi$ for some $x \in H$. Then, by Theorem 5.1.7, $\mathcal{J}, x \Vdash \varphi$ and since $\varphi \models_{\mathcal{J}} \psi$, we get that $\mathcal{J}, x \Vdash \psi$. Hence, by Theorem 5.1.7, $\mathcal{H}, x \Vdash \psi$.

For the other implication, reasoning by contraposition, assume $\varphi \not\vdash \psi$ with $\varphi, \psi \in \text{MNF}$. Thus, by Corollary 4.2.9, we have that $\pi_m(\varphi) < \pi_m(\psi)$ for some $m \in \text{N-mod}(\psi)$. Therefore, by Lemma 4.1.23, Item 1, we have the following:

$$\mathcal{J}, x_\varphi \Vdash \varphi \text{ and } \mathcal{J}, x_\varphi \not\vdash \psi.$$

By Definitions 4.1.22 and 4.1.21 we can observe that x_φ is a ℓ -sequence with finite support. Thus, $x_\varphi \in H$. Hence, by means of Theorem 5.1.7, we obtain that $\mathcal{H}, x_\varphi \Vdash \varphi$ but $\mathcal{H}, x_\varphi \not\vdash \psi$ which concludes the proof. \square

Modal definability

In this subsection we shall see an application of the completeness of **TSC** w.r.t. \mathcal{H} . As established by some results proven in the previous chapter, some ℓ -sequences seemed to be modally definable, but having finite support was

essential. We show how this modal definability can be achieved in \mathcal{H} .

By $m : H \rightarrow \mathbb{F}_{\text{TSC}}$ we denote the map defined as follows:

$$m(\langle \alpha_0, \dots, \alpha_j, 0 \rangle) = \bigwedge_{0 \leq i \leq j} \langle i^{\alpha_i} \rangle_{\top}.$$

While it is not generally true that given $x \in H$, we have that $m(x) \in \text{MNF}$, by simple applications of the reduction axiom (Axiom 5) we can omit the superfluous terms contained in $m(x)$ and thus, obtain a formula ψ_x such that

$$m(x) \equiv \psi_x \in \text{MNF}.$$

We shall refer to ψ_x as *the corresponding MNF of x* . The following lemma establishes the natural relation between the notion of *corresponding world*, introduced in Definition 4.1.22, and the notion of *corresponding MNF*. The proof follows immediately from said definition.

Lemma 5.1.9. *For any $x \in H$ we have that $x_{\psi_x} = x$.*

With the following auxiliary definition we are ready to introduce our result about the modal definability of the worlds in \mathcal{H} .

Definition 5.1.10. *For each $x \in H$, by x^\downarrow we denote the subset of H such that $y \in x^\downarrow$ iff $y_n < x_n$ for some $n < \omega$.*

The following two results express the relation between the *downset* generated by a world x and the extension of its corresponding MNF.

Lemma 5.1.11. *For any $x, y \in H$:*

$$y \in x^\downarrow \iff \mathcal{H}, y \not\models \psi_x.$$

Proof. For the left-to-right implication, assume $y \in x^\downarrow$, that is, $y_n < x_n$ for some $n < \omega$. Thus, by Lemma 4.1.23, Item 1 together with Theorem 5.1.7, we have that $\mathcal{H}, y \not\models \langle n^{x_n} \rangle_{\top}$. On the other hand, since $\psi_x \equiv m(x)$ and $m(x) \vdash \langle n^{x_n} \rangle_{\top}$, we have that $\psi_x \vdash \langle n^{x_n} \rangle_{\top}$. Thus, by completeness (Theorem 5.1.8) we get that $\mathcal{H}, y \not\models \psi_x$.

For the other implication, reasoning by contraposition, assume $y \notin x^\downarrow$. Thus, for all $n < \omega$ we have that $y_n \geq x_n$ and so, in particular, for all $n \in \text{N-mod}(\psi_x)$, we have that $y_n \geq x_n = \pi_n(\psi_x)$. Then, by Lemma 4.1.23, Item 1 and Theorem 5.1.7, $\mathcal{H}, y \models \psi_x$. \square

With this last lemma we obtain the following corollary about the modal definability.

Corollary 5.1.12. *For any $x \in H$:*

$$\mathcal{H}, x \models \psi_x \ \& \ \forall y \in x^\downarrow, \ \mathcal{H}, y \not\models \psi_x.$$

Proof. Follows directly from previous lemma by observing that $x \notin x^\downarrow$. \square

One of the most useful features of the modal definability of \mathcal{H} is that it provides us with a road map to understand the inclusions that hold between different Turing progressions. The relation between Turing progressions and ℓ -sequences has already been explored in [32] where Ignatiev's universal frame is presented as a map to conservation results. However, the analysis we provide here allows us to compare both the conservation and the consistency strength of the Turing progressions depicted in the frame (see Figure 5.1).

To capture this idea, first we make the following observation.

Lemma 5.1.13. *For any $x, y \in H$, $n < \omega$ and $\alpha < \Lambda$, we have that:*

$$xS_n^\alpha y \iff \psi_x \vdash \langle n^\alpha \rangle \psi_y.$$

Proof. For the left-to-right implication, assume $xS_n^\alpha y$ and let $\mathcal{H}, z \Vdash \psi_x$ for some $z \in H$. We set out to prove $\mathcal{H}, z \Vdash \langle n^\alpha \rangle \psi_y$. If $\mathcal{H}, z \Vdash \psi_x$, then $z \in \llbracket \psi_x \rrbracket$. By Lemma 4.1.28, we have that $z \in \llbracket x_{\psi_x} \rrbracket$ and by Lemma 5.1.9, $z \in \llbracket x \rrbracket$. Thus, since $xS_n^\alpha y$ we have that $zS_n^\alpha y$ and by Theorem 5.1.12, we have that $\mathcal{H}, y \Vdash \psi_y$. Therefore, $\mathcal{H}, z \Vdash \langle n^\alpha \rangle \psi_y$, and so, by completeness (Theorem 5.1.8), $\psi_x \vdash \langle n^\alpha \rangle \psi_y$.

For the other implication, assume $\psi_x \vdash \langle n^\alpha \rangle \psi_y$. By completeness we have that

$$\forall z \in H (\mathcal{H}, z \Vdash \psi_x \implies \mathcal{H}, z \Vdash \langle n^\alpha \rangle \psi_y)$$

and by Theorem 5.1.12, $\mathcal{H}, x \Vdash \psi_x$. Hence, $\mathcal{H}, x \Vdash \langle n^\alpha \rangle \psi_y$, that is, $\exists z' \in H$ such that $xS_n^\alpha z'$ and $\mathcal{H}, z' \Vdash \psi_y$. Then, $x \in \llbracket z' \rrbracket$ and by a combination of Lemmas 4.1.28 and 5.1.9, $z' \in \llbracket \psi_y \rrbracket = \llbracket y \rrbracket$. Therefore, $xS_n^\alpha y$. \square

This last result together with the arithmetical completeness (Theorems 3.2.6 and 3.3.13) gives us the aforementioned road map.

Lemma 5.1.14. *For any $x, y \in H$, $n < \omega$ and $\alpha < \Lambda$, we have that:*

$$xS_n^\alpha y \iff \text{EA}^+ \vdash (Th_{\psi_y})_n^\alpha \subseteq Th_{\psi_x}.$$

\mathcal{H} also provide us with a road map to conservation results. This can be achieved by defining the following family of relations $\{\approx_n\}_{n < \omega}$ on H :

$$x \approx_n y :\Leftrightarrow \forall m \leq n x_m = y_m.$$

From an arithmetical point of view, the \approx_n -relations specify that (the theories of) x and y accumulate the same amount of m -consistency for $m \leq n$. Thus, we obtain the following result

Proposition 5.1.15. *For any $x, y \in H$ and $n < \omega$, we have that:*

$$x \approx_n y \iff \text{EA}^+ \vdash \text{Th}_{\psi_x} \equiv_n \text{Th}_{\psi_y}.$$

The proof of this last proposition follows from Lemma 3.2.14 and the following observation:

Lemma 5.1.16. *Let $\psi := \langle n_0^{\alpha_0} \rangle \top \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle \top \wedge \langle m^\beta \rangle \top \in \text{MNF}$ and $n, n_k < n < m$. Provably in EA^+ :*

$$\text{Th}_\psi \equiv_n (\text{EA}^+)_{n_0}^{\alpha_0} + \dots + (\text{EA}^+)_{n_k}^{\alpha_k} + (\text{EA}^+)_n^{e^{m-n}(\beta)}.$$

Proof. Since $\psi \vdash \langle n_0^{\alpha_0} \rangle \top \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle \top \wedge \langle n^{e^{m-n}(\beta)} \rangle \top$, in particular we have the right-to-left inclusion. For the left-to-right inclusion, reasoning in EA^+ , let $\pi \in \Pi_{n+1}^0$ such that

$$\square_{(\text{EA}^+)_{n_0}^{\alpha_0} + \dots + (\text{EA}^+)_{n_k}^{\alpha_k} + (\text{EA}^+)_n^{e^{m-n}(\beta)}} \pi.$$

By the formalized deduction theorem we get that

$$\square_{(\text{EA}^+)_n^{e^{m-n}(\beta)}} \left(\bigwedge_{0 \leq i \leq k} \text{Con}_{n_i}((\text{EA}^+)_{n_i}^{\beta_i}) \rightarrow \pi \right)$$

for some sequence of $\beta_i \prec \alpha_i$. Since each $\text{Con}_{n_i}((\text{EA}^+)_{n_i}^{\beta_i}) \in \Pi_{n_i+1}^0$ we have that $\left(\bigwedge_{0 \leq i < j} \text{Con}_{n_i}((\text{EA}^+)_{n_i}^{\beta_i}) \rightarrow \pi \right) \in \Pi_{n+1}^0$. Thus, by reduction property (Item 3.2.5 of Proposition 3.2.5), we have that

$$\square_{(\text{EA}^+)_m^\beta} \left(\bigwedge_{0 \leq i \leq k} \text{Con}_{n_i}((\text{EA}^+)_{n_i}^{\beta_i}) \rightarrow \pi \right)$$

and so, by monotonicity $\square_{\text{Th}_\psi} \pi$. \square

Now we are ready to give a detailed proof of Proposition 5.1.15.

Proof. We start by the left-to-right implication. Suppose $x \approx_n y$, and let $\psi_x := \langle n_0^{\alpha_0} \rangle \top \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle \top$ and $\psi_y := \langle m_0^{\beta_0} \rangle \top \wedge \dots \wedge \langle m_j^{\beta_j} \rangle \top$. Moreover, let $n^* = \mu(n, \psi_x)$ and $m^* = \mu(n, \psi_y)$. Thus, by a combination of Lemmas 3.2.14 and 5.1.16, we have that:

- a) $\text{Th}_{\psi_x} \equiv_{n^*} (\text{EA}^+)_{n_0}^{\alpha_0} + \dots + (\text{EA}^+)_{n^*}^{\alpha_{n^*}} \equiv_n (\text{EA}^+)_{n_0}^{\alpha_0} + \dots + (\text{EA}^+)_n^{e^{n^*-n}(\alpha_{n^*})}$;
- b) $\text{Th}_{\psi_y} \equiv_{m^*} (\text{EA}^+)_{m_0}^{\beta_0} + \dots + (\text{EA}^+)_{m^*}^{\beta_{m^*}} \equiv_n (\text{EA}^+)_{m_0}^{\beta_0} + \dots + (\text{EA}^+)_n^{e^{m^*-n}(\beta_{m^*})}$.

Furthermore, we can make the following observations:

$$\blacksquare e^{n^*-n}(\alpha_{n^*}) = x_{n^*} = y_{n^*} = e^{m^*-n}(\beta_{m^*});$$

- Since for all $m < n$ we have that $x_m = y_m$, for any $n_i < n$ such that $n_i \in \mathbb{N}\text{-mod}(\psi_x)$, provably in EA^+ :

$$(\text{EA}^+)_{n_i}^{\alpha_{n_i}} \subseteq (\text{EA}^+)_{m_0}^{\beta_0} + \dots + (\text{EA}^+)_n^{e^{m^*-n}(\beta_{m^*})}.$$

Analogously, for any $m_i < n$ such that $m_i \in \mathbb{N}\text{-mod}(\psi_y)$, we have that:

$$(\text{EA}^+)_{m_i}^{\beta_{m_i}} \subseteq (\text{EA}^+)_{n_0}^{\alpha_0} + \dots + (\text{EA}^+)_n^{e^{n^*-n}(\alpha_{n^*})}.$$

Thus, we have that

$$\begin{aligned} \text{Th}_{\psi_x} &\equiv_n (\text{EA}^+)_{n_0}^{\alpha_0} + \dots + (\text{EA}^+)_n^{e^{n^*-n}(\alpha_{n^*})} \\ &\equiv (\text{EA}^+)_{m_0}^{\beta_0} + \dots + (\text{EA}^+)_n^{e^{m^*-n}(\beta_{m^*})} \\ &\equiv_n \text{Th}_{\psi_y}. \end{aligned}$$

For the right-to-left implication, assume $\text{Th}_{\psi_x} \equiv_n \text{Th}_{\psi_y}$ and suppose towards a contradiction that there is $m < n$ such that $x_m \neq y_m$. W.l.o.g assume that $x_m > y_m$. Hence, we can observe that $\text{Con}_m((\text{EA}^+)_{m}^{y_m})$ is a Π_{n+1}^0 -formula that by Gödel's second incompleteness theorem is not provable in Th_{ψ_y} while remains provable in Th_{ψ_x} . This argument can be formalized in EA^+ which contradicts our assumption. \square

5.2. A calculus without normal forms

This section is dedicated to introduce a presentation of **TSC** that makes no use of formulas in monomial normal form. To this purpose, we shall make use of INF's, define the notion of m - β -ordinals and replace Schmerl's axiom by a new principle that establishes the derivability between INF's.

Definition 5.2.1. Let $\langle n^\alpha \rangle A \in \text{INF}$, $m < n$ and $\beta < \Lambda$. By $o_m^\beta(\langle n^\alpha \rangle A)$ we denote the m - β -ordinal of $\langle n^\alpha \rangle A$, that is recursively defined as follows:

- I) $o_m^\beta(\langle n^\alpha \rangle \top) = e^{n-m}(\alpha) \cdot (1 + \beta)$;
- II) $o_m^\beta(\langle n^\alpha \rangle A) = e^{n-m}(o_n^\alpha(A)) \cdot (1 + \beta)$.

For any $m < \omega$ and $\beta < \Lambda$, we set $o_m^\beta(\top)$ to be zero.

By **TSC*** we denote the system obtained by substituting in **TSC** the Schmerl axiom by the following principle:

$$\text{IW axioms: } \langle n^\alpha \rangle A \equiv \langle n^{o_n^\alpha(A)} \rangle \top \wedge A$$

for $\langle n^\alpha \rangle A \in \text{INF}$.

MNF's and INF's

To prove the equivalence of both systems, first we shall see how formulas in monomial normal form and increasing worms are related. Therefore, in the following lemmata we state how every formula in monomial normal form is equivalent to an increasing worm, modulo \mathbf{TSC}^* , and likewise, that every increasing worm is \mathbf{TSC} -equivalent to a formula in monomial normal form.

From now on we will use the following notation: given $\varphi, \psi \in \mathbb{F}_{\mathbf{TSC}}$, we write $\varphi \vdash_{\mathbf{TSC}} \psi$ ($\varphi \vdash_{\mathbf{TSC}^*} \psi$) to denote that the sequent $\varphi \vdash \psi$ is derivable in \mathbf{TSC} (\mathbf{TSC}^*). Analogously, we use $\varphi \equiv_{\mathbf{TSC}} \psi$ ($\varphi \equiv_{\mathbf{TSC}^*} \psi$) to denote that both $\varphi \vdash \psi$ and $\psi \vdash \varphi$ are derivable in \mathbf{TSC} (\mathbf{TSC}^*).

Lemma 5.2.2. *For every $\psi := \langle n_0^{\alpha_0} \rangle \top \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle \top \in \mathbf{MNF}$ there is an $A \in \mathbf{INF}$ such that:*

1. $A \equiv_{\mathbf{TSC}^*} \psi$;
2. $A := \langle n_0^{\beta_0} \rangle \dots \langle n_k^{\beta_k} \rangle \top$ where:
 - a) $\beta_k = \alpha_k$ and
 - b) for all i , $0 \leq i < k$ we have

$$\beta_i = -1 + \frac{\alpha_i}{e^{n_{i+1}-n_i}(\alpha_{i+1})}.$$

Proof. By induction on k . The base case is trivial and the inductive case follows from the I.H. and the IW axiom. \square

Recall that by Corollary 3.3.7, for each formula $\varphi \in \mathbb{F}_{\mathbf{TSC}}$ there is a unique $\psi \in \mathbf{MNF}$ such that $\varphi \equiv \psi$. Thus, in particular, any $A \in \mathbf{INF}$ has its corresponding $\psi \in \mathbf{MNF}$. This combined with Theorem 2.3.2 gives us the following lemma.

Lemma 5.2.3. *For any $A := \langle n_0^{\beta_0} \rangle \dots \langle n_k^{\beta_k} \rangle \top \in \mathbf{INF}$ there is a unique $\psi \in \mathbf{MNF}$ such that:*

1. $\psi \equiv_{\mathbf{TSC}} A$;
2. $\psi := \langle n_0^{\alpha_0} \rangle \top \wedge \dots \wedge \langle n_k^{\alpha_k} \rangle \top$ where:
 - a) $\alpha_k = \beta_k$ and
 - b) for all i , $0 \leq i < k$ we have

$$\alpha_i = o_{n_i}^{\beta_i}(\langle n_{i+1}^{\beta_{i+1}} \rangle \dots \langle n_k^{\beta_k} \rangle \top).$$

Equivalence between TSC* and TSC

We are ready to prove the equivalence of both systems by checking the interderivability of Schmerl and IW axioms.

Proposition 5.2.4. *The Schmerl axiom is derivable in TSC* i.e.*

$$\langle n^\alpha \rangle (\langle n_0^{\alpha_0} \rangle_{\top} \wedge \psi) \equiv_{\text{TSC}^*} \langle n^{e^{n_0-n}(\alpha_0) \cdot (1+\alpha)} \rangle_{\top} \wedge \langle n_0^{\alpha_0} \rangle_{\top} \wedge \psi$$

for $n < n_0$ and $\langle n_0^{\alpha_0} \rangle_{\top} \wedge \psi \in \text{MNF}$.

Proof. By Lemma 5.2.2 we have that:

$$\langle n^\alpha \rangle (\langle n_0^{\alpha_0} \rangle_{\top} \wedge \psi) \equiv_{\text{TSC}^*} \langle n^\alpha \rangle \langle n_0^{\beta_0} \rangle A$$

with $\langle n_0^{\alpha_0} \rangle_{\top} \wedge \psi \equiv_{\text{TSC}^*} \langle n_0^{\beta_0} \rangle A \in \text{INF}$. Thus, by the IW axiom, we get that

$$\langle n^\alpha \rangle (\langle n_0^{\alpha_0} \rangle_{\top} \wedge \psi) \equiv_{\text{TSC}^*} \langle n^{o_n^\alpha(\langle n_0^{\beta_0} \rangle A)} \rangle_{\top} \wedge \langle n_0^{\beta_0} \rangle A,$$

and so

$$\langle n^\alpha \rangle (\langle n_0^{\alpha_0} \rangle_{\top} \wedge \psi) \equiv_{\text{TSC}^*} \langle n^{o_n^\alpha(\langle n_0^{\beta_0} \rangle A)} \rangle_{\top} \wedge \langle n_0^{\alpha_0} \rangle_{\top} \wedge \psi.$$

Since $o_n^\alpha(\langle n_0^{\beta_0} \rangle A) = e^{n_0-n}(o_{n_0}^{\beta_0}(A)) \cdot (1+\alpha)$ and by Lemma 5.2.2, $o_{n_0}^{\beta_0}(A) = \alpha_0$, we can conclude that $o_n^\alpha(\langle n_0^{\beta_0} \rangle A) = e^{n_0-n}(\alpha_0) \cdot (1+\alpha)$ and therefore:

$$\langle n^\alpha \rangle (\langle n_0^{\alpha_0} \rangle_{\top} \wedge \psi) \equiv_{\text{TSC}^*} \langle n^{e^{n_0-n}(\alpha_0) \cdot (1+\alpha)} \rangle_{\top} \wedge \langle n_0^{\alpha_0} \rangle_{\top} \wedge \psi.$$

□

Proposition 5.2.5. *The IW axiom is derivable in TSC i.e.*

$$\langle n^\alpha \rangle A \equiv_{\text{TSC}} \langle n^{o_n^\alpha(A)} \rangle_{\top} \wedge A$$

for $\langle n^\alpha \rangle A \in \text{INF}$.

Proof. By induction on A with base case being trivial. For the inductive step, let $A := \langle n_0^{\beta_0} \rangle A'$. By Lemma 5.2.3, we have that

$$\langle n^\alpha \rangle A \equiv_{\text{TSC}} \langle n^\alpha \rangle (\langle n_0^{o_{n_0}^{\beta_0}(A')} \rangle_{\top} \wedge \psi)$$

for $\langle n_0^{\beta_0} \rangle A' \equiv_{\text{TSC}} \langle n_0^{o_{n_0}^{\beta_0}(A')} \rangle_{\top} \wedge \psi \in \text{MNF}$, and so by Schmerl's axiom:

$$\langle n^\alpha \rangle A \equiv_{\text{TSC}} \langle n^{e^{n_0-n}(o_{n_0}^{\beta_0}(A')) \cdot (1+\alpha)} \rangle \wedge \langle n_0^{o_{n_0}^{\beta_0}(A')} \rangle_{\top} \wedge \psi.$$

Thus,

$$\langle n^\alpha \rangle A \equiv_{\text{TSC}} \langle n^{o_n^\alpha(\langle n_0^{\beta_0} \rangle A')} \rangle \wedge \langle n_0^{o_{n_0}^{\beta_0}(A')} \rangle_{\top} \wedge \psi,$$

that is, $\langle n^\alpha \rangle A \equiv_{\text{TSC}} \langle n^{o_n^\alpha(\langle n_0^{\beta_0} \rangle A')} \rangle \wedge A$.

□

Corollary 5.2.6. For any $\varphi, \psi \in \mathbb{F}_{TSC}$,

$$\varphi \vdash_{TSC} \psi \iff \varphi \vdash_{TSC^*} \psi.$$

Proof. By induction on the length of the proof. It follows immediately from Propositions 5.2.4 and 5.2.5. \square

5.3. Modal Schmerl principles

In this subsection we give a modal formulation of the Schmerl's formulas introduced in Proposition 3.2.5. We provide a modal proof without using the completeness result.

Proposition 5.3.1. For any $\varphi \in \mathbb{F}_{<n+2}$, the following principles are derivable TSC:

1. $\langle (n+1)^\alpha \rangle \varphi \equiv_n \langle n^{e(\alpha)} \rangle \varphi$;
2. $\langle n^\alpha \rangle \langle (n+1)^\beta \rangle \varphi \equiv_n \langle n^{e(\beta) \cdot (1+\alpha)} \rangle \varphi$.

Proof. We only check Item 2 since Item 1 follows from Item 2 by taking $\alpha = 0$. Let $\psi \in \text{MNF}$ such that $\psi \equiv \varphi$. First we prove that the sequent

$$\langle n^\alpha \rangle \langle (n+1)^\beta \rangle \varphi \vdash \langle n^{e(\beta) \cdot (1+\alpha)} \rangle \varphi$$

is derivable. By Theorem 2.3.10 and monotonicity we have that:

$$\begin{aligned} \langle n^\alpha \rangle \langle (n+1)^\beta \rangle \varphi \vdash & \left(\bigwedge_{0 \leq i \leq k} \langle n_i^{\alpha_i + e^{n-n_i}(e(\beta) \cdot (1+\alpha))} \rangle \top \right) \wedge \\ & \langle n^{\gamma + e(\beta) \cdot (1+\alpha)} \rangle \top \wedge \langle (n+1)^\delta \rangle \top \end{aligned}$$

where each $\langle n_i^{\alpha_i} \rangle \top$ occurs in ψ with $n_k < n$, and γ and δ might be 0. Thus, by (PS1) we get that $\langle n^\alpha \rangle \langle (n+1)^\beta \rangle \varphi \vdash \langle n^{e(\beta) \cdot (1+\alpha)} \rangle \varphi$.

For the other direction, consider $\chi \in \mathbb{F}_{n+1}$ such that $\langle n^\alpha \rangle \langle (n+1)^\beta \rangle \varphi \vdash \chi$. Therefore, by (PS1) and (PS2) get the following:

$$\begin{aligned} & \left(\bigwedge_{0 \leq i \leq k} \langle n_i^{\alpha_i + e^{n+1-n_i}(\delta+\beta) + e^{n-n_i}(\alpha)} \rangle \top \right) \wedge \\ & \langle n^{\gamma + e(\delta+\beta) + e(\delta+\beta) \cdot \alpha} \rangle \top \wedge \langle (n+1)^{\delta+\beta} \rangle \top \vdash \chi \end{aligned}$$

where as before, each $\langle n_i^{\alpha_i} \rangle \top$ occurs in ψ with $n_k < n$, and γ and δ might be 0. Thus,

$$\langle n_0^{\kappa_0} \rangle \top \wedge \dots \wedge \langle n_k^{\kappa_k} \rangle \top \wedge \langle n^{\gamma + e(\delta+\beta) \cdot (1+\alpha)} \rangle \top \wedge \langle (n+1)^{\delta+\beta} \rangle \top \vdash \chi$$

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where $\kappa_k := \alpha_k + e^{n-n_k}(\gamma + e(\delta + \beta) \cdot (1 + \alpha))$ and $\kappa_i := \alpha_i + e^{n_{i+1}-n_i}(\kappa_{i+1})$. We can easily check that there is $\{n_0, \dots, n_j\} \subseteq \text{N-mod}(\langle n_0^{k_0} \rangle_{\top} \wedge \dots \wedge \langle n_k^{k_k} \rangle_{\top})$ ¹ such that:

$$\langle n_0^{k_0} \rangle_{\top} \wedge \dots \wedge \langle n_j^{k_j} \rangle_{\top} \wedge \langle n^{\gamma+e(\delta+\beta)\cdot(1+\alpha)} \rangle_{\top} \wedge \langle (n+1)^{\delta+\beta} \rangle_{\top} \vdash \chi$$

and $\langle n_0^{k_0} \rangle_{\top} \wedge \dots \wedge \langle n_j^{k_j} \rangle_{\top} \wedge \langle n^{\gamma+e(\delta+\beta)\cdot(1+\alpha)} \rangle_{\top} \wedge \langle (n+1)^{\delta+\beta} \rangle_{\top} \in \text{MNF}$. Let $\chi' \in \text{MNF}$ such that $\chi' \equiv \chi$. Then, with the help of Proposition 3.3.10 we get that:

$$\langle n_0^{k_0} \rangle_{\top} \wedge \dots \wedge \langle n_j^{k_j} \rangle_{\top} \wedge \langle n^{\gamma+e(\delta+\beta)\cdot(1+\alpha)} \rangle_{\top} \vdash \chi'.$$

Finally, notice that

$$\langle n^{e(\beta)\cdot(1+\alpha)} \rangle_{\psi} \vdash \langle n_0^{k_0} \rangle_{\top} \wedge \dots \wedge \langle n_k^{k_k} \rangle_{\top} \wedge \langle n^{\gamma+e(\delta+\beta)\cdot(1+\alpha)} \rangle_{\top}.$$

This way, $\langle n^{e(\beta)\cdot(1+\alpha)} \rangle_{\psi} \vdash \langle n_0^{k_0} \rangle_{\top} \wedge \dots \wedge \langle n_j^{k_j} \rangle_{\top} \wedge \langle n^{\gamma+e(\delta+\beta)\cdot(1+\alpha)} \rangle_{\top}$ and so

$$\langle n^{e(\beta)\cdot(1+\alpha)} \rangle_{\psi} \vdash \chi'$$

i.e. $\langle n^{e(\beta)\cdot(1+\alpha)} \rangle_{\varphi} \vdash \chi$. □

¹Observe that this subset may be empty.

The Bracket Calculus

The content of this chapter is focused on the relation between provability logics and the ordinal notation systems that emerge from them. As raised in the introduction, Beklemishev introduced an ordinal notation system for the Feferman-Schütte ordinal Γ_0 based on the *autonomous expansion* of provability algebras (see [7]). In this chapter we shall present the logic **BC** (for *Bracket Calculus*). The main feature of **BC** is that it is based on a signature that extends Beklemishev's brackets notation system to a strictly positive modal language. Thus, unlike other provability logics, **BC** does not rely on using modalities indexed by some ordinal given *a priori* but it is built up from a purely modal signature. Moreover, since the order between these notations can be established in terms of derivability within the calculus, the inferences in this system can be carried out without using any external property of ordinals. In this sense, we say that **BC** provides an autonomous provability calculus that to the best of our knowledge, yields the first ordinal notation system presented as a purely modal deductive system.

The presented logic is proven to be equivalent to \mathbf{RC}_{Γ_0} . The main results of this chapter can be found in [21].

6.1. Beklemishev's bracket notation system for Γ_0

Before we introduce the full bracket calculus, let us review Beklemishev's notation system from [7]. We start by defining the set of *bracket expressions*.

Definition 6.1.1. By \mathbb{W}_O we denote the smallest set such that:

1. $\top \in \mathbb{W}_O$;
2. if $a, b \in \mathbb{W}_O$, then $(a)b \in \mathbb{W}_O$.

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By convention we shall write $\langle \rangle a$, for $a \in \mathbb{W}_{\langle \rangle}$ to denote $\langle \top \rangle a \in \mathbb{W}_{\langle \rangle}$.

We can define a translation $*$: $\mathbb{W}_{\langle \rangle} \rightarrow \mathbb{W}$ in such a way that the elements in $\mathbb{W}_{\langle \rangle}$ will denote the ordinals indexing the modalities in \mathbb{W} :

1. $\top^* = \top$
2. $\langle (a)b \rangle^* = \langle o(a^*) \rangle b^*$.

Therefore, we can also define $o^* : \mathbb{W}_{\langle \rangle} \rightarrow \Gamma_0$ as $o^*(a) = o(a^*)$. Thus the worm $\langle 0 \rangle \top$ becomes $\langle \rangle$ and $o^*(\langle \rangle) = 1$, $\langle 1 \rangle \top$ becomes $\langle \langle \rangle \rangle$ with $o^*(\langle \langle \rangle \rangle) = \omega$, $\langle \omega \rangle \top$ becomes $\langle \langle \langle \rangle \rangle \rangle$ that corresponds to the ordinal ε_0 , etc.

Next we make some observations about how the ordinals represented by worms in $\mathbb{W}_{\langle \rangle}$ can be bounded in terms of the maximum number of nested brackets occurring in them. With this purpose, we introduce the following two definitions.

Definition 6.1.2. For $a \in \mathbb{W}_{\langle \rangle}$, we define the nesting of a , $N(a)$, as the maximum number of nested brackets. That is:

1. $N(\top) = 0$;
2. $N(\langle a \rangle b) = \max(N(a) + 1, N(b))$.

Definition 6.1.3. We recursively define the function $h : \mathbb{N} \rightarrow \Gamma_0$ as follows:

1. $h(0) = 0$;
2. $h(n + 1) = e^{h(n)} 1$.

Note that $\lim_{n \rightarrow \infty} h(n) = \Gamma_0$.

In the following proposition we can find upper and lower bounds for any ordinal $o^*(a)$, with $a \in \mathbb{W}_{\langle \rangle}$, according to the nesting of a .

Proposition 6.1.4. For $a \in \mathbb{W}_{\langle \rangle}$, if $N(a) = n$, then $h(n) \leq o^*(a) < h(n + 1)$.

Proof. By induction on n . If $n = 0$ then we must have $a = \top$, hence $h(0) = 0 = o^*(a) < 1 = h(1)$.

For $n = n' + 1$, we have that $a = \langle a_0 \rangle \dots \langle a_m \rangle$ for some $m \in \omega$. Moreover,

1. $N(a_i) \leq n'$ for $i, 0 \leq i \leq m$;
2. there is a_j such that $N(a_j) = n'$.

Thus by the I.H. we get that $a^* = \langle \alpha_0 \rangle \dots \langle \alpha_k \rangle \top$ such that:

1. For each i , $\alpha_i < h(n' + 1)$;
2. there is $\alpha_j \geq h(n')$.

By Lemma 1.4.14,

$$o(\langle h(n') \rangle \top) \leq o(a^*) < o(\langle h(n' + 1) \rangle \top);$$

but by Theorem 1.4.13 $o(\langle h(n') \rangle \top) = e^{h(n')} 1 = h(n)$, while $o(\langle h(n' + 1) \rangle \top) = e^{h(n+1)} 1 = h(n + 1)$, as needed. \square

As a consequence of this last proposition, we get the following corollaries.

Corollary 6.1.5. For $a \in \mathbb{W}_{\langle \rangle}$, if $N(a) = n$, then $a^* \in \mathbb{W}_{h(n)}$.

Corollary 6.1.6. For $a, b \in \mathbb{W}_{\langle \rangle}$, $o^*(a) \geq o^*(b) \Rightarrow N(a) \geq N(b)$.

Proof. We reason by contrapositive applying Proposition 6.1.4. \square

6.2. Signature

The set of **BC**-formulas, $\mathbb{F}_{\langle \rangle}$, is defined by extending $\mathbb{W}_{\langle \rangle}$ to a strictly positive signature.

Definition 6.2.1. By $\mathbb{F}_{\langle \rangle}$ we denote the set of formulas built-up by the following grammar:

$$\varphi := \top \mid p \mid \varphi \wedge \psi \mid (a) \varphi \text{ for } a \in \mathbb{W}_{\langle \rangle}.$$

Similarly to **RC**, **BC** is based on *sequents*, i.e. expressions of the form $\varphi \vdash \psi$, where $\varphi, \psi \in \mathbb{F}_{\langle \rangle}$. In addition to this, we will also use $a \succeq b$, for $a, b \in \mathbb{W}_{\langle \rangle}$, to denote that either $a \vdash () b$ or $a \vdash b$ are derivable. Analogously, we will use $a \succ b$ to denote that the sequent $a \vdash () b$ is derivable.

Definition 6.2.2. **BC** is given by the following set of axioms and rules:

Axioms:

1. $\varphi \vdash \varphi, \quad \varphi \vdash \top$;
2. $\varphi \wedge \psi \vdash \varphi, \quad \varphi \wedge \psi \vdash \psi$;

Rules:

1. If $\varphi \vdash \psi$ and $\varphi \vdash \chi$, then $\varphi \vdash \psi \wedge \chi$;

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2. If $\varphi \vdash \psi$ and $\psi \vdash \chi$, then $\varphi \vdash \chi$;
3. If $\varphi \vdash \psi$ and $a \succeq b$, then $(a) \varphi \vdash (b) \psi$ and $(a) (b) \varphi \vdash (b) \psi$;
4. If $a \succ b$, then $(a) \varphi \wedge (b) \psi \vdash (a) (\varphi \wedge (b) \psi)$.

Although **BC** and **RC** look quite similar, there is a main difference between them. As we can see, in **BC** there is no external reference to ordinals, and thus, every derivation is carried out within the calculus. Let us consider the following example.

Let $n \in \mathbb{N}$. The sequent $\langle \omega \rangle \top \vdash \langle n \rangle \top$ is derivable in \mathbf{RC}_{Γ_0} by a simple application of Axiom 4 together with the fact that $\omega > n$. However, this last fact is something that we check outside the calculus. If we would like to mimic this derivation within \mathbf{BC}^1 , we could make use of Rule 3. By Axiom 1, we have that $\top \vdash_{\mathbf{BC}} \top$, so it suffices to give a derivation of " $\omega > n$ " within **BC**.

For $n = 0$, we need to check that $(()) \vdash ()$. By Axiom 1, we obtain that the sequents $() \vdash \top$ and $\top \vdash \top$ are derivable in **BC**. Thus, we can combine this with the following reasoning:

$$\text{(Rule 3)} \frac{() \vdash \top \quad \top \vdash \top}{(()) \vdash ()}$$

If $n = 1$, we need to check that $(()) \vdash ()()$. First, we can extend the previous reasoning as follows:

$$\text{(**)} \frac{\text{(Rule 3)} \frac{() \vdash \top \quad \top \vdash \top}{(()) \vdash ()} \quad \text{(Rule 1)} \frac{(()) \vdash () \quad (()) \vdash (())}{(()) \vdash () \wedge (())}}{\text{(Rules 4 and 2)} \frac{(()) \vdash () \wedge (()) \quad () \vdash ()}{(()) \vdash (()) (() \wedge \top)}}$$

Moreover, we have that

$$\text{(Rule 3)} \frac{() \wedge \top \vdash () \quad () \vdash ()}{(()) (() \wedge \top) \vdash (()) ()}$$

Combining these two derivations we obtain that

$$\text{(Rule 2)} \frac{\text{(Rule 2)} \frac{(()) \vdash (()) (() \wedge \top) \quad (()) (() \wedge \top) \vdash (()) ()}{(()) \vdash (()) ()} \quad () \vdash \top}{\text{(Rule 3 and 2)} \frac{(()) \vdash (()) () \quad (()) \vdash (()) ()}{(()) \vdash () ()}}$$

¹in previous section we shall make precise how to interpret **BC**-formulas into \mathbf{RC}_{Γ_0} -formulas and vice versa

In general, by iterating n -times this last part of the derivation from where we made an introduction of the conjunction (**), we obtain that

$$((\)) \vdash \underbrace{(\) \dots (\)}_{n+1\text{-times}}$$

i.e., " $\omega > n$ ". In a similar fashion, applying Rule 3 together with this last derivation we get that

$$(((\))) \vdash \underbrace{(\) \dots (\)}_{n+1\text{-times}}$$

i.e., " $\varepsilon_0 > e^n(1)$ ". These examples illustrate how the required relations between ordinals can be derived within **BC** without any help of external references.

6.3. Translation and preservability

In this section we introduce a way of interpreting **BC**-formulas as \mathbf{RC}_{Γ_0} -formulas, and prove that under this translation, both systems can derive exactly the same sequents.

Definition 6.3.1. We define a translation τ between $\mathbb{F}_()$ and \mathbb{F}_{Γ_0} , $\tau : \mathbb{F}_() \rightarrow \mathbb{F}_{\Gamma_0}$, as follows:

1. $\top^\tau = \top$;
2. $p^\tau = p$;
3. $(\varphi \wedge \psi)^\tau = (\varphi^\tau \wedge \psi^\tau)$;
4. $(\langle a \rangle \varphi)^\tau = \langle o^*(a) \rangle \varphi^\tau$.

Note that for $a \in \mathbb{W}_()$, $a^\tau = a^*$. From this and a routine induction, the following can readily be verified.

Lemma 6.3.2. Given $\varphi \in \mathbb{F}_()$ and $\alpha \in \mathcal{S}(\varphi^\tau)$, there is a subformula $a \in \mathbb{W}_()$ of φ such that $\alpha = o^*(a)$.

The following lemma establishes the preservability of **BC** with respect to \mathbf{RC}_{Γ_0} , under τ .

Lemma 6.3.3. For any $\varphi, \psi \in \mathbb{F}_()$: $\varphi \vdash_{\mathbf{BC}} \psi \implies \varphi^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} \psi^\tau$.

Proof. By induction on the length of the derivation. We can easily check that the set of axioms of **BC** is preserved under τ . Likewise, the cases for a derivation ending on Rules 1 or 2 are straightforward. Thus, we only check Rules 3 and 4.

Regarding Rule 3, we need to prove that if $a \succeq b$ then both sequents $\langle o^*(a) \rangle \varphi^\tau \vdash \langle o^*(b) \rangle \psi^\tau$ and $\langle o^*(a) \rangle \langle o^*(b) \rangle \varphi^\tau \vdash \langle o^*(b) \rangle \psi^\tau$ are derivable in \mathbf{RC}_{Γ_0} . We can make the following observations by applying the I.H.:

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1. Since $a \succeq b$, we have that either $a^\tau \vdash \langle 0 \rangle b^\tau$ or $a^\tau \vdash b^\tau$ are derivable in \mathbf{RC}_{Γ_0} . Therefore, $o(a^\tau) \geq o(b^\tau)$. Since $o^*(a) = o(a^*) = o(a^\tau)$ and the same equality holds for b , we have that $o^*(a) \geq o^*(b)$.
2. We also have that $\varphi^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} \psi^\tau$ and thus, by Rule 3 of \mathbf{RC}_{Γ_0} we obtain that $\langle o^*(a) \rangle \varphi^\tau \vdash \langle o^*(a) \rangle \psi^\tau$ and $\langle o^*(a) \rangle \langle o^*(b) \rangle \varphi^\tau \vdash \langle o^*(a) \rangle \langle o^*(b) \rangle \psi^\tau$ are derivable in \mathbf{RC}_{Γ_0} .

On the one hand, by these two facts together with Axiom 4 we obtain that $\langle o^*(a) \rangle \varphi^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} \langle o^*(b) \rangle \psi^\tau$. On the other hand, we can combine Axioms 4 and 3 to get that $\langle o^*(a) \rangle \langle o^*(b) \rangle \varphi^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} \langle o^*(b) \rangle \psi^\tau$.

We follow an analogous reasoning in the case of Rule 4. By the I.H. we have that $a^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} \langle 0 \rangle b^\tau$. Therefore $o^*(a) > o^*(b)$ and by Axiom 5, $\langle o^*(a) \rangle \varphi \wedge \langle o^*(b) \rangle \psi \vdash_{\mathbf{RC}_{\Gamma_0}} \langle o^*(a) \rangle (\varphi \wedge \langle o^*(b) \rangle \psi)$. \square

With the following definition we fix a way of translating \mathbb{F}_{Γ_0} -formulas into formulas in $\mathbb{F}_{\langle \rangle}$. However, since different words in $\mathbb{W}_{\langle \rangle}$ might denote the same ordinal, we need a normal form theorem for $\mathbb{W}_{\langle \rangle}$.

Definition 6.3.4. We define $N \subset \mathbb{W}_{\langle \rangle}$ to be the smallest set of $\mathbb{W}_{\langle \rangle}$ -words such that $\top \in N$ and for any $(a)b \in \mathbb{W}_{\langle \rangle}$, if $a, b \in N$ and $((a)b)^* \in \text{BNF}$, then $(a)b \in N$.

Every element of $\mathbb{W}_{\langle \rangle}$ has a unique normal form, as shown by L. Beklemishev in [7].

Theorem 6.3.5 (Beklemishev). For each $\alpha \in \Gamma_0$ we can associate a unique $a_\alpha \in N$ such that $o^*(a_\alpha) = \alpha$.

Proposition 6.3.6 (Beklemishev). The ordering $(N, <_{\langle \rangle})$ is a well-ordering of order type Γ_0 .

Now we are ready to translate \mathbb{F}_{Γ_0} -formulas into $\mathbb{F}_{\langle \rangle}$ -formulas.

Definition 6.3.7. We define a translation ι between \mathbb{F}_{Γ_0} and $\mathbb{F}_{\langle \rangle}$, $\iota : \mathbb{F}_{\Gamma_0} \rightarrow \mathbb{F}_{\langle \rangle}$, as follows:

1. $\top^\iota = \top$;
2. $p^\iota = p$;
3. $(\varphi \wedge \psi)^\iota = (\varphi^\iota \wedge \psi^\iota)$;
4. $(\langle \alpha \rangle \varphi)^\iota = (a_\alpha) \varphi^\iota$.

The following remark follows immediately from the definitions of τ and ι .

Remark 6.3.8. For any $\varphi \in \mathbb{F}_{\Gamma_0}$, $(\varphi^\iota)^\tau = \varphi$.

With the next definition, we extend the nesting $N(a)$ of $a \in \mathbb{W}_{\langle \rangle}$ to $\mathbb{F}_{\langle \rangle}$ -formulas.

Definition 6.3.9. For $\varphi \in \mathbb{F}_{\langle \rangle}$, we define the nesting of φ , $Nt(\varphi)$, as the maximum number of nested brackets. That is:

1. $Nt(\top) = Nt(p) = N(\top)$;
2. $Nt(\varphi \wedge \psi) = \max(Nt(\varphi), Nt(\psi))$;
3. $Nt(\langle a \rangle \varphi) = \max(N(\langle a \rangle), Nt(\varphi)) = \max(N(a) + 1, Nt(\varphi))$.

The upcoming remark collects a useful observation concerning the nesting $Nt(\varphi)$ of a formula φ and its subformulas. This fact can be verified by an easy induction.

Remark 6.3.10. For any $\varphi \in \mathbb{F}_{\langle \rangle}$ with $\varphi \neq p$, there is a subformula $a \in \mathbb{W}_{\langle \rangle}$ of φ such that $Nt(\varphi) = Nt(a)$. Moreover, if $Nt(\varphi) \geq 1$, there is a subformula $a \in \mathbb{W}_{\langle \rangle}$ of φ such that $Nt(\varphi) = Nt(a) + 1$.

For each \mathbf{RC}_{Γ_0} -formula φ , we can define the signature of φ as the set of ordinals occurring in any of its modalities.

Definition 6.3.11. For any $\varphi \in \mathbb{F}_{\Gamma_0}$, we define the signature of φ , $\mathcal{S}(\varphi)$, as follows:

1. $\mathcal{S}(\top) = \mathcal{S}(p) = \emptyset$;
2. $\mathcal{S}(\varphi \wedge \psi) = \mathcal{S}(\varphi) \cup \mathcal{S}(\psi)$;
3. $\mathcal{S}(\langle \alpha \rangle \varphi) = \{\alpha\} \cup \mathcal{S}(\varphi)$.

With the help of this last definition we can make the following observation:

Lemma 6.3.12. For any $\varphi, \psi \in \mathbb{F}_{\Gamma_0}$:

1. If $\mathcal{S}(\psi) \neq \emptyset$ and $\varphi \vdash \psi$, then $\max \mathcal{S}(\varphi) \geq \max \mathcal{S}(\psi)$;
2. If $\mathcal{S}(\varphi) = \emptyset$ and $\varphi \vdash \psi$, then $\mathcal{S}(\psi) = \emptyset$.

Proof. By an easy induction on the length of the derivation of $\varphi \vdash \psi$. \square

The following lemma relates the derivability in \mathbf{RC}_{Γ_0} under τ , and the nesting of formulas in $\mathbb{F}_{\langle \rangle}$.

Lemma 6.3.13. For any $\varphi, \psi \in \mathbb{F}_{\langle \rangle}$:

$$\varphi^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} \psi^\tau \implies Nt(\varphi) \geq Nt(\psi).$$

Proof. Suppose that $\varphi^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} \psi^\tau$. If $\mathcal{S}(\psi^\tau) = \emptyset$ then it is easy to check that $\text{Nt}(\psi) = 0$ and there is nothing to prove, so assume otherwise. Then, by Lemma 6.3.12.1, $\max \mathcal{S}(\varphi^\tau) \geq \max \mathcal{S}(\psi^\tau)$. Using Lemma 6.3.2, let $a \in \mathbb{W}_\langle \rangle$ be a subformula of φ such that $o^*(a) = \max \mathcal{S}(\varphi^\tau)$. Moreover, since $\mathcal{S}(\psi^\tau) = \emptyset$, then $\text{Nt}(\psi) \geq 1$. Therefore, with the help of Remark 6.3.10 we can consider $b \in \mathbb{W}_\langle \rangle$, a subformula of ψ such that $\text{Nt}(\psi) = \text{N}(b) + 1$. If we had $\text{N}(a) < \text{N}(b)$ then it would follow from Corollary 6.1.6 that $o^*(a) < o^*(b)$, contradicting $\max \mathcal{S}(\varphi^\tau) \geq \max \mathcal{S}(\psi^\tau)$. Thus $\text{N}(a) \geq \text{N}(b)$ and $\text{Nt}(\varphi) \geq \text{N}(a) + 1 \geq \text{Nt}(\psi)$, as needed. \square

With the following theorem we conclude the proof of the preservability between **BC** and \mathbf{RC}_{Γ_0} .

Theorem 6.3.14. *For any $\varphi, \psi \in \mathbb{F}_\langle \rangle$:*

$$\varphi^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} \psi^\tau \iff \varphi \vdash_{\mathbf{BC}} \psi.$$

Proof. The right-to-left direction is given by Lemma 6.3.3, so we focus on the other. Proceed by induction on $\text{Nt}(\varphi)$. For the base case, assume $\text{Nt}(\varphi) = 0$ and $\varphi^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} \psi^\tau$. By a subsidiary induction on the length of the derivation of $\varphi^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} \psi^\tau$, we set to prove $\varphi \vdash_{\mathbf{BC}} \psi$. If the derivation has length one it suffices to check \mathbf{RC}_{Γ_0} -Axioms 1 and 2, which is immediate. If it has length greater than one it must end in a rule. The case for \mathbf{RC}_{Γ_0} -Rule 1 follows by the I.H.. For \mathbf{RC}_{Γ_0} -Rule 2, we have that there is $\chi \in \mathbb{F}_\langle \rangle$ such that $\varphi^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} \chi$ and $\chi \vdash_{\mathbf{RC}_{\Gamma_0}} \psi^\tau$. By Remark 6.3.8 and Lemma 6.3.13, we get that $\varphi^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} (\chi')^\tau$ and $(\chi')^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} \psi^\tau$ with $\text{Nt}(\chi') = 0$. Thus, by the subsidiary I.H., $\varphi \vdash_{\mathbf{BC}} \chi'$ and $\chi' \vdash_{\mathbf{BC}} \psi$ and by **BC**-Rule 2, $\varphi \vdash_{\mathbf{BC}} \psi$.

For the inductive step, let $\text{Nt}(\varphi) = n + 1$. We proceed by a subsidiary induction on the length of the derivation. If $\varphi^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} \psi^\tau$ is obtained by means of \mathbf{RC}_{Γ_0} -Axioms 1 and 2, then clearly $\varphi \vdash_{\mathbf{BC}} \psi$. If $\varphi^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} \psi^\tau$ is an instance of \mathbf{RC}_{Γ_0} -Axiom 3, then we have that $\varphi^\tau := \langle o^*(a) \rangle \langle o^*(b) \rangle \chi^\tau$ and $\psi^\tau := \langle o^*(c) \rangle \chi^\tau$ for some $\chi \in \mathbb{F}_\langle \rangle$ and $a, b, c \in \mathbb{W}_\langle \rangle$ such that $o^*(a) = o^*(b) = o^*(c)$. Hence, there are $A, B, C \in \mathbb{W}$ such that $a^* = A$, $b^* = B$ and $c^* = C$, and so $A \vdash_{\mathbf{RC}_{\Gamma_0}} B$ and $B \vdash_{\mathbf{RC}_{\Gamma_0}} C$. Since $\text{Nt}(w) < n + 1$ for $w \in \{a, b, c\}$, by the main I.H. we have that $a \vdash_{\mathbf{BC}} b$ and $b \vdash_{\mathbf{BC}} c$. Thus, we have the following **BC**-derivation:

$$\begin{array}{c} \text{(Rule 3)} \frac{\chi \vdash \chi \quad b \vdash c}{(b)\chi \vdash (c)\chi} \quad a \vdash b \quad \text{(Rule 3)} \frac{\chi \vdash \chi \quad b \vdash c}{(b)(c)\chi \vdash (c)\chi} \\ \text{(Rule 3)} \frac{(b)\chi \vdash (c)\chi \quad a \vdash b}{(a)(b)\chi \vdash (b)(c)\chi} \\ \text{(Rule 2)} \frac{(a)(b)\chi \vdash (b)(c)\chi}{(a)(b)\chi \vdash (c)\chi} \end{array}$$

If $\varphi^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} \psi^\tau$ is obtained by using \mathbf{RC}_{Γ_0} -Axiom 4, then $\varphi^\tau := \langle o^*(a) \rangle \chi^\tau$ and $\psi^\tau := \langle o^*(b) \rangle \chi^\tau$ for some $\chi \in \mathbb{F}_\langle \rangle$ and $a, b \in \mathbb{W}_\langle \rangle$ with $o^*(a) > o^*(b)$. Therefore, there are $A, B \in \mathbb{W}_{\Gamma_0}$ such that $A \vdash_{\mathbf{RC}_{\Gamma_0}} \langle 0 \rangle B$, $a^* = A$ and

$b^* = B$. Since $o^*(a) \geq o^*(\langle \rangle b)$, by Lemma 6.3.12, $\text{Nt}(\langle \rangle b) \leq \text{Nt}(a)$ and since $\varphi^\tau := \langle o^*(a) \rangle \chi^\tau$, we have that $\text{Nt}(a) < \text{Nt}(\varphi)$. Thus, by the main I.H. $a \vdash_{\mathbf{BC}} \langle \rangle b$ and by **BC-Rule 3**, $(a)\chi \vdash_{\mathbf{BC}} (b)\chi$. If $\varphi^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} \psi^\tau$ is an instance of **RC $_{\Gamma_0}$ -Axiom 5**, then we have that $\varphi^\tau := \langle o^*(a) \rangle \chi_0^\tau \wedge \langle o^*(b) \rangle \chi_1^\tau$ and $\psi^\tau := \langle o^*(a) \rangle (\chi_0^\tau \wedge \langle o^*(b) \rangle \chi_1^\tau)$, for some $\chi_0, \chi_1 \in \mathbb{F}_{\langle \rangle}$ and $a, b \in \mathbb{W}_{\langle \rangle}$ with $o^*(a) > o^*(b)$. Therefore, there are $A, B \in \mathbb{W}_{\Gamma_0}$ such that $a^* = A$, $b^* = B$ and $A \vdash_{\mathbf{RC}_{\Gamma_0}} \langle 0 \rangle B$. By Lemma 6.3.12 together with the main I.H. we obtain that $a \vdash_{\mathbf{BC}} \langle \rangle b$ and by applying **BC-Rule 4**, $(a)\chi_0 \wedge (b)\chi_1 \vdash (a)(\chi_0 \wedge (b)\chi_1)$. Regarding rules, **RC $_{\Gamma_0}$ -Rule 1** is immediate and **RC $_{\Gamma_0}$ -Rule 3** follows an analogous reasoning to that of Axiom 4. This way, we only check **RC $_{\Gamma_0}$ -Rule 2**. Assume $\varphi^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} \psi^\tau$ is obtained by an application of **RC $_{\Gamma_0}$ -Rule 2**. Then, there is $\chi \in \mathbb{F}_{\Gamma_0}$ such that $\varphi^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} \chi$ and $\chi \vdash_{\mathbf{RC}_{\Gamma_0}} \psi^\tau$. By Remark 6.3.8 together with Lemma 6.3.13 we obtain that $\varphi^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} (\chi')^\tau$ and $(\chi')^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} \psi^\tau$ with $\text{Nt}(\chi) \leq n + 1$. By the subsidiary I.H. $\varphi \vdash_{\mathbf{BC}} \chi'$ and $\chi' \vdash_{\mathbf{BC}} \psi$ and hence, by **BC-Rule 2**, $\varphi \vdash_{\mathbf{BC}} \psi$. \square

With this we obtain our main result: an autonomous calculus for representing ordinals below Γ_0 .

Theorem 6.3.15. *For $a, b \in N$ define $a \triangleleft b$ if and only if $a \vdash_{\mathbf{BC}} \langle \rangle b$. Then, \triangleleft is a strict linear order of order-type Γ_0 .*

Proof. By Theorem 6.3.14, $a \triangleleft b$ if and only if $a^\tau \vdash_{\mathbf{RC}_{\Gamma_0}} \langle 0 \rangle b^\tau$ if and only if $o^*(a) < o^*(b)$. Moreover if $\zeta < o^*(a)$ then there is some $B <_0 a^\tau$ such that $\zeta = o(B)$, hence $\zeta = o^*(B')$. Thus by Lemma 1.2.4 o^* is the order-type function on N . That the range of o^* is Γ_0 follows from Proposition 6.1.4 which tells us that $o^*(a) < h(N(a) + 1) < \Gamma_0$ for all $a \in \mathbb{W}_{\langle \rangle}$, while if we define recursively $a_0 = \top$ and $a_{n+1} = (a_n)$, Theorem 1.4.13 and an easy induction readily yield $\Gamma_0 = \lim_{n \rightarrow \infty} h(n) = \lim_{n \rightarrow \infty} o^*(a_n)$. \square

Future Work

In this last chapter we shall collect some of the future directions in which the work presented in this dissertation can be extended. For the most part, the ideas discussed in this last chapter are work in progress and so, they will not be presented in full detail. However, we deem the open problems that came up during the writing of the thesis to be worth mentioning.

Extensions of TSC

In Chapters 4 and 5, we examine the relations between MNF's and ℓ -sequences. As we saw, formulas in monomial normal form and ℓ -sequences of the form $\langle x_0, \dots, x_k, 0 \rangle$ can be related. Thus, having finite support is essential. However, when working in a signature with $\Lambda > \varepsilon_0$, we shall come across sequences that do not converge to zero e.g. the sequence $\langle \varepsilon_0, \varepsilon_0, \dots, \varepsilon_0, \dots \rangle$.

Non-finitely supported ℓ -sequences cannot be defined by means of a modal formula. However, we can provide a natural arithmetical interpretation of these sequences.

Given a ℓ -sequence of the form $\langle \varepsilon_\alpha, \varepsilon_\alpha, \dots, \varepsilon_\alpha, \dots \rangle$, it can be associated to the Turing progression

$$\bigcup_{n < \omega} (\text{EA}^+)_n^{\varepsilon_\alpha}.$$

Thus, the natural candidate to represent this theory in our modal setting would be

$$\bigwedge_{n < \omega} \langle n^{\varepsilon_\alpha} \rangle \top,$$

which requires to allow infinite conjunctions.

Another solution to this situation involves extending our current modal signature and making use of the conservativity results about transfinite induction and Turing progressions established by Schmerl in [34].

Let $\text{TI}(\alpha)$ be the transfinite induction scheme up to ordinal α and $\text{TI}_n(\alpha)$ the restriction of $\text{TI}(\alpha)$ to Π_{n+1}^0 -formulas. By $[\text{EA}^+]^{\varepsilon_\alpha}$ we denote the theory

$EA^+ + TI(\varepsilon_\alpha)$ and, analogously, by $[EA^+]_n^{\varepsilon_\alpha}$ we denote the theory $EA^+ + TI_n(\varepsilon_\alpha)$.

The following result captures how transfinite induction can be approximated by means of Turing progressions.

Proposition. *For any $n < \omega$, provably in EA^+ :*

1. $(EA^+)_n^{\varepsilon_\alpha} \subseteq [EA^+]^{\varepsilon_\alpha}$;
2. $[EA^+]_n^{\varepsilon_\alpha} \subseteq (EA^+)_{n+2}^{\varepsilon_\alpha}$;
3. $[EA^+]^{\varepsilon_\alpha} \equiv_n (EA^+)_n^{\varepsilon_\alpha}$.

Therefore, taking back our previous example, we have that

$$\bigcup_{n < \omega} (EA^+)_n^{\varepsilon_\alpha} \equiv [EA^+]^{\varepsilon_\alpha}.$$

A way of making **TSC** capable of capturing theories like these ones, but at same time, not requiring infinite conjunctions, would be by extending \mathbb{F}_{TSC} with a new modality $[\varepsilon_\alpha]$:

Definition. *By $\mathbb{F}_{\text{TSC}}^+$ we denote the smallest set such that:*

- I) $\top \in \mathbb{F}_{\text{TSC}}^+$;
- II) *If $\varphi, \psi \in \mathbb{F}_{\text{TSC}}^+$ then $(\varphi \wedge \psi) \in \mathbb{F}_{\text{TSC}}^+$;*
- III) *if $\varphi \in \mathbb{F}_{\text{TSC}}^+$, $n < \omega$ and $\alpha < \Lambda$ then $\langle n^\alpha \rangle \varphi \in \mathbb{F}_{\text{TSC}}^+$;*
- IV) *if $\varphi \in \mathbb{F}_{\text{TSC}}^+$ and $\varepsilon_\alpha < \Lambda$ then $[\varepsilon_\alpha]\varphi \in \mathbb{F}_{\text{TSC}}^+$;*

The goal of this new modality is to capture the transfinite induction schema up to some ε -number. Thus, formulas of the form $[\varepsilon_\alpha]\varphi$ are intended to be arithmetically interpreted as $\text{Th}_\varphi + TI(\varepsilon_\alpha)$.

The additional axioms and rules that should be added to **TSC** to obtain a complete calculus capable of generating all the principles that govern the behavior of such theories, remains as an open problem. However, there are some principles that should be derivable in this new system.

First of all, in this setting we are allowed to build progressions over base theories which might have transfinite induction available. To this end, the following conservativity result should be expressible within this new system:

Lemma. *For any $\varepsilon_\alpha, \beta \prec \Lambda$ and $n < \omega$, provably in EA^+ :*

$$([EA^+]^{\varepsilon_\alpha})_n^\beta \equiv_n (EA^+)_n^{\varepsilon_\alpha \cdot (1+\beta)}.$$

Proof. Reasoning in EA^+ , since $[\text{EA}^+]^{\varepsilon_\alpha} \equiv_{n+1} (\text{EA}^+)_{n+1}^{\varepsilon_\alpha}$, by a combination of Corollary 3.2.12, Proposition 3.2.5, Item 3.2.5 and the fact that $\omega^{\varepsilon_\alpha} = \varepsilon_\alpha$, we obtain that $([\text{EA}^+]^{\varepsilon_\alpha})_n^\beta \equiv_n (\text{EA}^+)_{n+1}^{\varepsilon_\alpha \cdot (1+\beta)}$. \square

Thus, there are some modal principles that should be derivable:

- $[\varepsilon_\alpha] \top \vdash \langle n^{\varepsilon_\alpha} \rangle \top$ for any $n < \omega$;
- $\langle n^\beta \rangle [\varepsilon_\alpha] \top \vdash \langle n^{\varepsilon_\alpha \cdot (1+\beta)} \rangle \top$;
- $[\varepsilon_\alpha][\varepsilon_\beta] \top \equiv [\varepsilon_{\max(\alpha, \beta)}] \top$.

On the other hand, since Schmerl Axioms are the keystone of **TSC**, we are confident that a modal version of the following formulation of the fine-structure formula for theories $[\text{EA}^+]^{\varepsilon_\alpha}$ might play the role of the Schmerl Axioms for this extended version of **TSC**.

Proposition. For any $\varepsilon_\alpha, \beta, \gamma \prec \Lambda$, and $n < \omega$, provably in EA^+ :

1. $([\text{EA}^+]^{\varepsilon_\alpha})_{n+1}^\beta \equiv_n ([\text{EA}^+]^{\varepsilon_\alpha})_n^{e^1(\varepsilon_\alpha \cdot \beta)}$;
2. $(([\text{EA}^+]^{\varepsilon_\alpha})_{n+1}^\beta)_n^\gamma \equiv_n ([\text{EA}^+]^{\varepsilon_\alpha})_n^{e^1(\varepsilon_\alpha \cdot \beta) \cdot (1+\gamma)}$.

Proof. Item 1 follows from Item 2 for $\gamma = 0$. Thus, we only prove Item 2. Reasoning in EA^+ we have that

$$([\text{EA}^+]^{\varepsilon_\alpha})_n^{e^1(\varepsilon_\alpha \cdot \beta) \cdot (1+\gamma)} \equiv_n (\text{EA}^+)_{n+1}^{\varepsilon_\alpha \cdot (e^1(\varepsilon_\alpha \cdot \beta) \cdot (1+\gamma))}.$$

We can observe that $e^1(\varepsilon_\alpha \cdot \beta) \cdot (1+\gamma) = \varepsilon_\alpha \beta \cdot (1+\gamma)$. Therefore,

$$\varepsilon_\alpha \cdot (e^1(\varepsilon_\alpha \cdot \beta) \cdot (1+\gamma)) = \varepsilon_\alpha \cdot (\varepsilon_\alpha \beta \cdot (1+\gamma)) = e^1(\varepsilon_\alpha \cdot (1+\beta)) \cdot (1+\gamma).$$

On the other hand, we have that:

$$([\text{EA}^+]^{\varepsilon_\alpha})_{n+1}^\beta \equiv_{n+1} (\text{EA}^+)_{n+1}^{\varepsilon_\alpha \cdot (1+\beta)}$$

and

$$(([\text{EA}^+]^{\varepsilon_\alpha})_{n+1}^\beta)_n^\gamma \equiv_n ((\text{EA}^+)_{n+1}^{\varepsilon_\alpha \cdot (1+\beta)})_n^\gamma \equiv_n (\text{EA}^+)_{n+1}^{e^1(\varepsilon_\alpha \cdot (1+\beta)) \cdot (1+\gamma)}.$$

Combining all these facts we get that

$$\begin{aligned} (([\text{EA}^+]^{\varepsilon_\alpha})_{n+1}^\beta)_n^\gamma &\equiv_n (\text{EA}^+)_{n+1}^{e^1(\varepsilon_\alpha \cdot (1+\beta)) \cdot (1+\gamma)}; \\ &\equiv_n (\text{EA}^+)_{n+1}^{\varepsilon_\alpha \cdot (e^1(\varepsilon_\alpha \cdot (1+\beta)) \cdot (1+\gamma))}; \\ &\equiv_n ([\text{EA}^+]^{\varepsilon_\alpha})_{n+1}^{e^1(\varepsilon_\alpha \cdot (1+\beta)) \cdot (1+\gamma)}. \end{aligned}$$

□

From the relational semantics point of view, this new modality seems to have a natural interpretation in our frame \mathcal{J} :

$$\mathcal{J}, x \Vdash [\varepsilon_\alpha]\varphi \text{ iff there is } y \in I \text{ such that } \mathcal{J}, y \Vdash \varphi \text{ and } \forall n < \omega \ x R_n^{\varepsilon_\alpha} y.$$

This interpretation allow us to relate non-finitely supported ℓ -sequences to modal formulas and ultimately, to arithmetical theories in the obvious way:

$\langle \varepsilon_0, \varepsilon_0, \dots, \varepsilon_0, \dots \rangle$	$[\varepsilon_0]\top$	$[\mathbf{EA}^+]^{\varepsilon_0}$
$\langle \varepsilon_0 \cdot 2, \varepsilon_0, \dots, \varepsilon_0, \dots \rangle$	$\langle 0^1 \rangle [\varepsilon_0]\top$	$([\mathbf{EA}^+]^{\varepsilon_0})_0^1$
$\langle \varepsilon_0^2, \varepsilon_0 \cdot 2, \dots, \varepsilon_0, \dots \rangle$	$\langle 1^1 \rangle [\varepsilon_0]\top$	$([\mathbf{EA}^+]^{\varepsilon_0})_1^1$
$\langle \varepsilon_0^2 \cdot 2, \varepsilon_0 \cdot 2, \dots, \varepsilon_0, \dots \rangle$	$\langle 0^1 \rangle \langle 1^1 \rangle [\varepsilon_0]\top$	$(([\mathbf{EA}^+]^{\varepsilon_0})_1^1)_0^1$

In a different vein, in Chapter 3, Section 3.3 we introduce Corollary 3.3.1 to be the only tool to obtain non-derivability in **TSC**. We have chosen to do so since Corollary 3.3.1 can also be obtained via relational semantics. Consequently, we shall be able to generalize the entire purely modal results of this dissertation to a setting where we work with ordinal modalities where the base elements are also transfinite ordinals.

BC

In Chapter 6 we gave a purely syntactical analysis of **BC** which leaves room for a semantical treatment. The equivalence between our system and \mathbf{RC}_{Γ_0} allows us to make use of the relational semantics for \mathbf{RC}_{Γ_0} presented in [17, 9], but we leave the question of whether it is possible to define natural semantics that work only with **BC** expressions and do not directly reference ordinals. Moreover, [20] suggests variants of the brackets notation for representing the Bachmann-Howard ordinal and beyond. Sound and complete calculi for these systems remain to be found.

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