Appendix B

Dielectric tensor

B.1 Anti-Hermitian part of the dielectric tensor

Taking for the electrons a Maxwellian relativistic distribution, the anti-Hermitian part of the dielectric tensor $\epsilon_{ij}^{"}$ can be expressed from Eq. (3.15) as

$$\epsilon_{jl}^{"} = \frac{\pi\omega_{pe}^2\mu^2}{\omega N_\perp^2\left(1-N_\parallel^2\right)K_2\left(\mu\right)}\sum_{n>n_0}^{\infty}\epsilon_{jl,n}^{"}\left(Y_n^2+N_\parallel^2-1\right)^{1/2}Y_n^2\exp\left(-\frac{\mu Y_n}{1-N_\parallel^2}\right) \tag{B.1}$$

with

$$n_0 = \frac{\omega}{\omega_{ce}}\left(1-N_\parallel^2\right)^{1/2}$$

where the frame of reference and parametersY_n, μ introduced in Chapter 3 (Section 3.2.4), $K_2\left(\mu\right)$ is the modified Bessel functions of the second kind (with $\mu = m_e c^2/\left(kT_e\right)$), and n is the harmonic number. The standard Larmor radius expansions of Bessel functions in the $\epsilon_{jl,n}^{"}$ term are avoided by using the compact representation which has been proposed by Granata and Fidone [Gra91], giving the following expressions:

$$\epsilon_{11,n}^{"} = \frac{\pi}{2\upsilon}g_n J_{n+\frac{1}{2}}^{+}J_{n+\frac{1}{2}}^{-}, \tag{B.2}$$

$$\epsilon_{12,n}^{"} = -i\epsilon_{11,n}^{"} + i\frac{\pi}{2n}g_n\left[\frac{\upsilon}{\left(4\upsilon^2-w^2\right)^{\frac{1}{2}}}\sum_{\pm}J_{n+\frac{1}{2}}^{\pm}J_{n+\frac{3}{2}}^{\mp}\right], \tag{B.3}$$

$$\epsilon^{"}_{22,n} = \epsilon^{"}_{11,n} + \frac{\pi \upsilon}{n^2 \left(4\upsilon^2 - w^2\right)} \left\{ \upsilon^2 \left(t_n + u_n\right) \left(J^+_{n+\frac{3}{2}} J^-_{n+\frac{3}{2}} - J^+_{n+\frac{1}{2}} J^-_{n+\frac{1}{2}} \right) \right.$$
$$+ \frac{1}{2} \sum_{\pm} \left[u_n z^{\mp} \left(2n + 3 \pm \frac{iw}{\left(4\upsilon^2 - w^2\right)^{\frac{1}{2}}} \right) \right.$$
$$\left.\left. -t_n z^{\pm} \left(2n + 1 \pm \frac{iw}{\left(4\upsilon^2 - w^2\right)^{\frac{1}{2}}} \right) J^{\mp}_{n+\frac{1}{2}} J^{\pm}_{n+\frac{3}{2}} \right] \right\}, \tag{B.4}$$

$$\epsilon^{"}_{13,n} = sy\epsilon^{"}_{11,n} + \frac{i\pi s}{2\upsilon \left(4\upsilon^2 - w^2\right)^{\frac{1}{2}}} g_n \sum_{\pm} \left[(\pm)\, z^{\pm} J^{\mp}_{n+\frac{1}{2}} J^{\pm}_{n+\frac{3}{2}} \right], \tag{B.5}$$

$$\epsilon^{"}_{23,n} = sy\epsilon^{"}_{12,n} + \frac{\pi s}{2\upsilon \left(4\upsilon^2 - w^2\right)^{\frac{1}{2}}} g_n \left\{ - \sum_{\pm} (\pm)\, z^{\pm} J^{\mp}_{n+\frac{1}{2}} J^{\pm}_{n+\frac{3}{2}} \right.$$
$$+ \frac{\upsilon^2}{n \left(4\upsilon^2 - w^2\right)^{\frac{1}{2}}} \left[2\left(n+1\right) \sum_{\pm} (\pm)\, J^{\mp}_{n+\frac{1}{2}} J^{\pm}_{n+\frac{3}{2}} \right.$$
$$\left.\left. + \frac{iw}{\left(4\upsilon^2 - w^2\right)^{\frac{1}{2}}} \sum_{\pm} J^{\mp}_{n+\frac{1}{2}} J^{\pm}_{n+\frac{3}{2}} + iw \left(J^+_{n+\frac{3}{2}} J^-_{n+\frac{3}{2}} - J^+_{n+\frac{1}{2}} J^-_{n+\frac{1}{2}} \right) \right] \right\}, \tag{B.6}$$

$$\epsilon^{"}_{33,n} = s^2 \left(y^2 + \frac{2\upsilon^2 - w^2}{4\upsilon^2 - w^2} \right) \epsilon^{"}_{11,n} + \frac{\pi s^2}{\upsilon \left(4\upsilon^2 - w^2\right)^{\frac{1}{2}}} g_n$$
$$\times \left\{ iy \sum_{\pm} (\pm)\, z^{\pm} J^{\mp}_{n+\frac{1}{2}} J^{\pm}_{n+\frac{3}{2}} + \frac{\upsilon}{\left(4\upsilon^2 - w^2\right)^{\frac{1}{2}}} \right.$$
$$\left. \times \left[\upsilon J^+_{n+\frac{3}{2}} J^-_{n+\frac{3}{2}} + \sum_{\pm} \left(\frac{\upsilon}{\left(4\upsilon^2 - w^2\right)^{\frac{1}{2}}} - \frac{n+1}{\upsilon} z^{\pm} \right) J^{\mp}_{n+\frac{1}{2}} J^{\pm}_{n+\frac{3}{2}} \right] \right\}, \tag{B.7}$$

where

$$\upsilon = \frac{N_\perp}{Y_1} \left(\frac{Y_n^2 + N_\parallel^2 - 1}{1 - N_\parallel^2} \right)^{\frac{1}{2}}, \quad w = \frac{\mu N_\parallel \left(Y_n^2 + N_\parallel^2 - 1 \right)^{\frac{1}{2}}}{1 - N_\parallel^2},$$

$$s = \frac{N_\perp \left(Y_n^2 + N_\parallel^2 - 1 \right)^{\frac{1}{2}}}{Y_n \left(1 - N_\parallel^2 \right)}, \quad y = \frac{N_\parallel Y_n}{\left(Y_n^2 + N_\parallel^2 - 1 \right)^{\frac{1}{2}}},$$

and

$$g_n = \frac{(2n-1)!!}{(2n)!!}, \quad t_n = \frac{n}{(n+1)} g_n, \quad u_n = \frac{(2n+3)}{(n+1)\,(n+2)} g_n.$$

The Bessel function of the first kind $J_r^\pm$ is given by

$$J_r^\pm = J_r\left(z^\pm\right), \quad z^\pm = \frac{1}{2}\left[\left(4v^2 - w^2\right)^{\frac{1}{2}} \pm iw\right].$$

B.1.1 Asymptotic expansion for the modified Bessel function

The Bessel function of the first kind $J_r^\pm$ is related to the modified Bessel function $I_r^\pm$ using

$$J_r^\pm = I_r^\pm \exp\left(\pm ir\frac{\pi}{2}\right),$$

where

$$I_r^\pm = I_r^\pm\left(y^\pm\right), \quad y^\pm = \left(\frac{w \pm \sqrt{w^2 - 4v^2}}{2}\right).$$

When $4v^2 < w^2$, which corresponds to ray path parallel or almost parallel to the magnetic field (angles θ close to 0 or π), we obtain $y^+ \gg 1$ and the Bessel function become undefined. In this case, we use the following Bessel asymptotic expansion [Abr72]

$$I_r(y^+) \sim \frac{\exp\left(y^+\right)}{\sqrt{2\pi y^+}} O_r^+, \tag{B.8}$$

where O_r^+ is the expansion function given by

$$O_r^+ = \left\{1 - \sum_{k=1}^{\infty} (-1)^k \frac{\prod_{l=1}^{k}\left[\eta - (2l-1)^2\right]}{k!\left(8y^+\right)^k}\right\} \tag{B.9}$$

with $\eta = 4r^2$.

As seen in Fig.B.1, this asymptotic expansion is valable for $y^+ > r$. To obtain a precision ε with respect to the exact Bessel function value, the remainder of Eq. (B.9) in absolute value after the addition of k_{up} terms must not exceed ε, i.e.

$$\left|(-1)^{k_{\text{up}}} \frac{(\eta-1)(\eta-9)(\eta-25)\cdots\left(\eta - \left[2k_{\text{up}}-1\right]^2\right)}{k_{\text{up}}!\left(8y^+\right)^{k_{\text{up}}}}\right| < \varepsilon.$$

In addition to this, the precision ε will be only achieved if the remainder in absolute value after $k+1$ terms is lower than the remainder in absolute value after k terms. Note however that the latter condition is satisfied for $y^+ > 20$ and $\varepsilon \sim 10^{-6}$ or 10^{-7}.

Applying Eq (B.8) in Eq. (B.2), we obtain

$$\epsilon^{"}_{11,n} = \frac{\pi}{2v}\frac{g_n}{\sqrt{\pi\left(w + \sqrt{w^2 - 4v^2}\right)}} I^-_{n+\frac{1}{2}} O^+_{n+\frac{1}{2}} \exp\left(\frac{w + \sqrt{w^2 - 4v^2}}{2}\right),$$

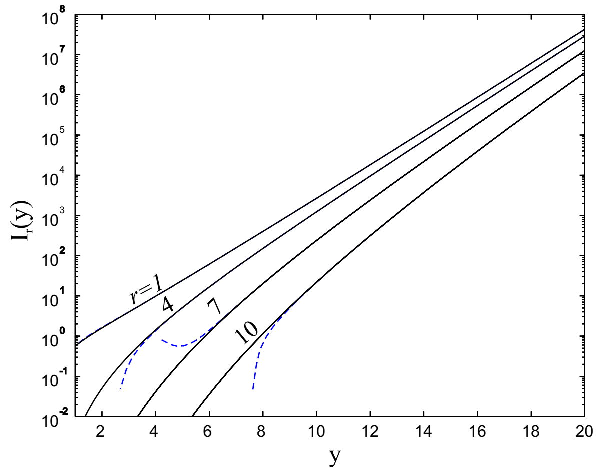

Figure B.1: Modified Bessel function of order $r = 1, 4, 7, 10$ (solid lines), and the corresponding asymptotic expansions (dashed lines).

and making the substitution in Eq. (B.1), the first component of the anti-Hermitian part of the dielectric tensor in the parallel or almost parallel case is given by

$$\epsilon_{11}^{"} = \frac{\pi^{3/2}}{2} \frac{\omega_{ce}\omega_{pe}^2\mu^2}{\omega^2 N_\perp^3 K_2(\mu) \sqrt{\mu N_\parallel + \sqrt{\mu^2 N_\parallel^2 - \frac{4N_\perp^2}{Y_1^2}\left(1 - N_\parallel^2\right)}}}$$
$$\times \sum_{n>n_0}^{\infty} \frac{g_n Y_n^2}{\left(Y_n^2 + N_\parallel^2 - 1\right)^{\frac{1}{2}}} I_{n+\frac{1}{2}}^{-} O_{n+\frac{1}{2}}^{+} \exp\left\{ \frac{\left(Y_n^2 + N_\parallel^2 - 1\right)^{1/2}}{2\left(1 - N_\parallel^2\right)} \right.$$
$$\left. \times \left[\mu N_\parallel + \sqrt{\mu^2 N_\parallel^2 - \frac{4N_\perp^2}{Y_1^2}\left(1 - N_\parallel^2\right)}\right] - \frac{\mu Y_n}{1 - N_\parallel^2} \right\}.$$

Note that this expression converges if

$$N_\parallel^2 \left(Y_n^2 + N_\parallel^2 - 1\right) < Y_n,$$

but this condition is always satisfied because $N_\parallel < 1$ in the range of parameters of interest for the synchrotron losses.

The same procedure is carried out for calculating $\epsilon_{12}^{"}$, $\epsilon_{22}^{"}$, $\epsilon_{13}^{"}$, $\epsilon_{23}^{"}$, and $\epsilon_{33}^{"}$ in the parallel or almost parallel case.

B.2 Appleton-Hartree equation

In the range of parameters of interest for the synchrotron losses problem ($\omega > 2\omega_{ce}$), the Hermitian part of the dielectric tensor is well described by the cold plasma approximation in the high frequency limit, which can be expressed as [Que68]

$$\boldsymbol{\epsilon}^h = \begin{pmatrix} 1 - \dfrac{\omega_{pe}^2}{\omega^2 - \omega_{ce}^2} & i\dfrac{\omega_{pe}^2 \omega_{ce}}{\omega\left(\omega^2 - \omega_{ce}^2\right)} & 0 \\ -i\dfrac{\omega_{pe}^2 \omega_{ce}}{\omega\left(\omega^2 - \omega_{ce}^2\right)} & 1 - \dfrac{\omega_{pe}^2}{\omega^2 - \omega_{ce}^2} & 0 \\ 0 & 0 & 1 - \dfrac{\omega_{pe}^2}{\omega^2} \end{pmatrix}$$

For every angle of propagation θ and wave frequency ω, the Appleton-Hartree equation gives the square value of the cold refraction index

$$N_{o,x}^2 = 1 - \frac{x}{1 - t \pm \text{sign}\,(1-x)\,\left(y\cos^2\theta + t^2\right)^{1/2}}, \tag{B.10}$$

where

$$t = \frac{y\sin^2\theta}{2\,(1-x)}, \quad x = \left(\frac{\omega_{pe}}{\omega}\right)^2,$$

and

$$y = \left(\frac{\omega_{ce}}{\omega}\right)^2.$$

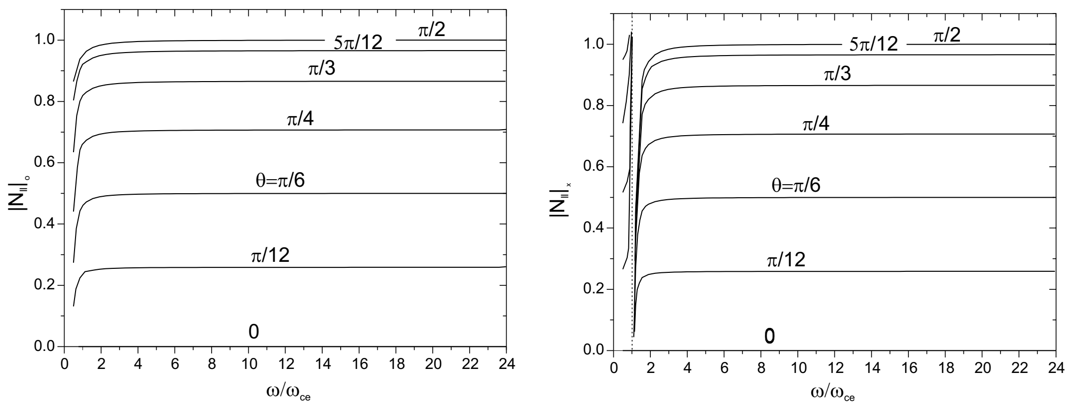

Figure B.2: Parallel refraction index for the ordinary (left side) and extraordinary (right side) modes of propagation versus the normalized frequency, for different propagation angles (θ=0°, 15°, 30°, 45°, 60°, 75°, 90°).

Note that when $N = 0$ we obtain the called cut-off frequency, whose value in the cold plasma approximation (see Eq. (3.61)) is given by the following equation:

$$1 - t \pm \left(y\cos^2\theta + t^2\right)^{1/2} - x = 0.$$

In Fig. B.2, we show the parallel refractive index

$$N_{||} = N \cos\theta$$

for the ordinary and extraordinary modes of propagation (o, x) resulting of the application of the Appleton-Hartree equation by the European commercial reactor parameters [Coo99] with $B_t = 6.8$ T, $n_{e_0} = 1.28 \times 10^{20}$ m^{-3}, giving $\omega_{ce_0}/2\pi = 190.3$ GHz, $\omega_{pe_0}/2\pi = 100.5$ GHz, and with $T_{e_0} = 30$ keV.