# Appendix A

# Plasma geometrical description

## A.1 General parametric equations

The last closed magnetic surface (LCMS) may be modelled by the following parametric equations:

$$\xi = \xi_1(\theta), \qquad \zeta = \zeta_1(\theta),$$

with $\xi = x/a$ , $\zeta = z/a$ where $a$ is the plasma horizontal minor radius. The coordinate origin is taken at the middle of the horizontal axis of the LMCS, and $\theta$ is a poloidal angle which varies between $0$ and $2\pi$.

The other magnetic surfaces, characterized by the similarity factor $\rho$, are supposed to be similar to the LCMS. Note that we have $\rho = 0$ at the magnetic axis and $\rho = 1$ at the plasma boundary. The parametric equations of the magnetic surface are then

$$\xi = \rho\xi_1(\theta), \qquad \zeta = \rho\zeta_1(\theta),$$

where the variables $\rho, \theta$ are the integration variables. The corresponding Jacobian determinant is found to be

$$J = \rho a^2 J_1^*(\theta) \quad \text{with} \quad J_1^*(\theta) = \xi_1\zeta_1' - \zeta_1\xi_1'.$$

Using the parametric equations for the magnetic surface we obtain the following poloidal surface integration of an arbitrary function F:

$$\frac{1}{S_p}\int_{S_p} F(\rho)\, dS_p = \frac{1}{S_p}\pi a^2 I_1^* \int_0^1 F(\rho)\, 2\rho d\rho, \tag{A.1}$$

where

$$I_1^* = \frac{1}{2\pi}\int_0^{2\pi} J_1^*(\theta)\, d\theta. \tag{A.2}$$

In the particular case $F(\rho) = 1$ the poloidal surface is deduced from Eq. (A.1), resulting in $S_p = \pi a^2 I_1^*$.

In the same way, the volume integration of an arbitrary function F leads to

$$\int_V F(\rho)\, dV = 2\pi R \times \pi a^2 I_1^* \int_0^1 F(\rho) \left(1 + \frac{\rho}{A} \frac{I_2^*}{I_1^*}\right) 2\rho d\rho, \tag{A.3}$$

where

$$I_2^* = \frac{1}{2\pi} \int_0^{2\pi} \xi_1(\theta)\, J_1^*(\theta)\, d\theta. \tag{A.4}$$

Introducing

$$\Theta_1 = -\frac{2}{3} \frac{I_2^*}{I_1^*} \tag{A.5}$$

we finally obtain

$$\frac{1}{V} \int_V F(\rho) dV = \frac{1}{1 - \Theta_1/A} \int_0^1 F(\rho) \left(1 - \frac{3}{2} \frac{\Theta_1}{A} \rho\right) 2\rho d\rho, \tag{A.6}$$

and the plasma volume can be expressed as

$$V = 2\pi R \times \pi a^2 I_1^* \left(1 + \frac{2}{3} \frac{I_2^*}{I_1^*} \frac{1}{A}\right) \tag{A.7}$$

or

$$V = 2\pi R \times \pi a^2 I_1^* \left(1 - \frac{\Theta_1}{A}\right). \tag{A.8}$$

## A.2 Geometry description with 8 parameters

We consider the plasma magnetic surfaces are described by 8 geometrical parameters $\kappa_X^{(1)}$, $\kappa_X^{(2)}$, $\delta_X^{(1)}$, $\delta_X^{(2)}$, $\psi^{-(1)}$, $\psi^{+(1)}$, $\psi^{-(2)}$, and $\psi^{+(2)}$, where $\kappa_X$, $\delta_X$ are the elongation and triangularity at the X-point, and $\psi^-$, $\psi^+$ are the angles between the last closed magnetic surface and the horizontal plane at the internal and external sides of the torus, respectively. The superscripts $(1)$, $(2)$ refer to the upper and lower parts of the poloidal cross–section, respectively (see Fig. 2.8). In addition to this, we suppose that the contact points of the vertical sides of the rectangular envelope with the poloidal section are situated on the same horizontal axis.

### A.2.1 Parametric expressions for the LCMS

Considering the above assumptions the LCMS may be described by one curve for each of the four segments of the surface: the inner-upper part, the inner-lower part, the outer-upper part, and the outer-lower. Depending of the parameter values such curves will be ellipses, parabolas, hyperbolas, or straight lines.

**Inner side of the LCMS:**

The upper and lower part of the inner last closed magnetic surface, which is represented by the $-$ sign, may be processed together by considering the corresponding superscripts $(1)$ or $(2)$, respectively.

- If

$$\tan\psi^{-(i)} < \frac{\kappa_X^{(i)}}{2(1-\delta_X^{(i)})} \quad \text{i.e.} \quad t^{-(i)} < \frac{1}{2} \quad \text{with} \quad t^{-(i)} = \frac{1-\delta_X^{(i)}}{\kappa_X^{(i)}}\tan\psi^{-(i)}$$

  the upper or lower inner part of the poloidal section of the last closed magnetic surface may be modelled with a portion of ellipse with the following parametric equations ($\xi = x/a$, $\zeta = z/a$), where $x$ is the distance measured outwards from the major radius $R$ :

$$\begin{cases} \xi = \alpha_0^{-(i)} + \alpha^{-(i)}\cos\theta \\ \zeta = \beta^{-(i)}\sin\theta \end{cases} \qquad \text{for} \qquad \begin{cases} \theta_X^{-(1)} < \theta < \pi & \text{(upper part)} \\ \pi < \theta < 2\pi - \theta_X^{-(2)} & \text{(lower part)} \end{cases}$$

  with

$$\theta_X^{-(i)} = \frac{\pi}{2} + \arcsin\frac{t^{-(i)}}{1-t^{-(i)}} = \pi - \arcsin\frac{\left(1-2t^{-(i)}\right)^{1/2}}{1-t^{-(i)}},$$

$$\cos\theta_X^{-(i)} = -\frac{t^{-(i)}}{1-t^{-(i)}}, \quad \sin\theta_X^{-(i)} = \frac{\left(1-2t^{-(i)}\right)^{1/2}}{1-t^{-(i)}},$$

$$\alpha_0^{-(i)} = -\frac{\delta_X^{(i)} - (1+\delta_X^{(i)})t^{-(i)}}{1-2t^{-(i)}}, \quad \alpha^{-(i)} = (1-\delta_X^{(i)})\frac{1-t^{-(i)}}{1-2t^{-(i)}},$$

$$\beta^{-(i)} = \kappa_X^{(i)}\frac{1-t^{-(i)}}{(1-2t^{-(i)})^{1/2}}.$$

- If

$$\tan\psi^{-(i)} = \frac{\kappa_X^{(i)}}{2(1-\delta_X^{(i)})} \quad \text{i.e.} \quad t^{-(i)} = \frac{1}{2}$$

  the upper or lower inner part of the poloidal section of the last closed magnetic surface may be modelled with a portion of parabola with the following equation

$$\xi = -1 + \frac{1-\delta_X^{(i)}}{\kappa_X^{(i)2}}\zeta^2.$$

- If

$$\tan\psi^{-(i)} > \frac{\kappa_X^{(i)}}{2(1-\delta_X^{(i)})} \quad \text{i.e.} \quad t^{-(i)} > \frac{1}{2}$$

the upper or lower inner part of the poloidal section of the last closed magnetic surface may be modelled with a portion of hyperbola with the following parametric equations

$$\left\{ \begin{array}{l} \xi = \alpha_0^{-(i)} + \alpha^{-(i)}\cosh\varphi \\ \zeta = \beta^{-(i)}\sinh\varphi \end{array} \right. \quad \text{for} \quad \left\{ \begin{array}{ll} 0 < \varphi < \varphi_X^{-(1)} & \text{(upper part)} \\ -\varphi_X^{-(2)} < \varphi < 0 & \text{(lower part)} \end{array} \right.$$

with

$$\varphi_X^{-(i)} = \text{arcsinh}\frac{\left(2t^{-(i)}-1\right)^{1/2}}{1-t^{-(i)}},$$

$$\cosh\varphi_X^{-(i)} = \frac{t^{-(i)}}{1-t^{-(i)}}, \quad \sinh\varphi_X^{-(i)} = \frac{\left(2t^{-(i)}-1\right)^{1/2}}{1-t^{-(i)}},$$

$$\alpha_0^{-(i)} = -\frac{(1+\delta_X^{(i)})t^{-(i)} - \delta_X^{(i)}}{2t^{-(i)}-1}, \quad \alpha^{-(i)} = (1-\delta_X^{(i)})\frac{1-t^{-(i)}}{2t^{-(i)}-1},$$

$$\beta^{-(i)} = \kappa_X^{(i)}\frac{1-t^{-(i)}}{\left(2t^{-(i)}-1\right)^{1/2}}.$$

- If

$$\tan\psi^{-(i)} = \frac{\kappa_X^{(i)}}{1-\delta_X^{(i)}} \quad \text{i.e.} \quad t^{-(i)} = 1$$

the upper or lower inner part of the poloidal section of the last closed magnetic surface may be modelled with a portion of straight line with the following equation

$$\xi = -1 + \frac{1-\delta_X^{(i)}}{\kappa_X^{(i)}}\zeta.$$

**Outer side of the LCMS:**

The upper and lower part of the outer last closed magnetic surface, which is represented by the $+$ sign, may be processed together by considering the corresponding superscripts $(1)$ or $(2)$, respectively.

- If

$$\tan\psi^{+(i)} < \frac{\kappa_X^{(i)}}{2(1+\delta_X^{(i)})} \quad \text{i.e.} \quad t^{+(i)} < \frac{1}{2} \quad \text{with} \quad t^{+(i)} = \frac{1+\delta_X^{(i)}}{\kappa_X^{(i)}}\tan\psi^{+(i)}$$

the upper or lower outer part of the poloidal section of the last closed magnetic surface may be modelled with a portion of ellipse with the following parametric equations

$$\begin{cases} \xi = \alpha_0^{+(i)} + \alpha^{+(i)} \cos\theta \\ \zeta = \beta^{+(i)} \sin\theta \end{cases} \quad \text{for} \quad \begin{cases} 0 < \theta < \theta_X^{+(1)} & \text{(upper part)} \\ -\theta_X^{+(2)} < \theta < 0 & \text{(lower part)} \end{cases}$$

with

$$\theta_X^{+(i)} = \frac{\pi}{2} - \arcsin\frac{t^{+(i)}}{1 - t^{+(i)}} = \arcsin\frac{\left(1 - 2t^{+(i)}\right)^{1/2}}{1 - t^{+(i)}},$$

$$\cos\theta_X^{+(i)} = \frac{t^{+(i)}}{1 - t^{+(i)}}, \quad \sin\theta_X^{+(i)} = \frac{\left(1 - 2t^{+(i)}\right)^{1/2}}{1 - t^{+(i)}},$$

$$\alpha_0^{+} = -\frac{\delta_X^{(i)} + (1 - \delta_X^{(i)})t^{+(i)}}{1 - 2t^{+(i)}}, \quad \alpha^{+(i)} = (1 + \delta_X^{(i)})\frac{1 - t^{+(i)}}{1 - 2t^{+(i)}},$$

$$\beta^{+(i)} = \kappa_X^{(i)} \frac{1 - t^{+(i)}}{\left(1 - 2t^{+(i)}\right)^{1/2}}.$$

- If

$$\tan\psi^{+(i)} = \frac{\kappa_X^{(i)}}{2(1 + \delta_X^{(i)})} \quad \text{i.e.} \quad t^{+(i)} = \frac{1}{2}$$

the upper or lower outer part of the poloidal section of the last closed magnetic surface may be modelled with a portion of parabola with the following equation

$$\xi = 1 - \frac{1 + \delta_X^{(i)}}{\kappa_X^{(i)2}}\zeta^2.$$

- If

$$\tan\psi^{+(i)} > \frac{\kappa_X^{(i)}}{2(1 + \delta_X^{(i)})} \quad \text{i.e.} \quad t^{+(i)} > \frac{1}{2}$$

the upper or lower outer part of the poloidal section of the last closed magnetic surface may be modelled with a portion of hyperbola with the following parametric equations

$$\begin{cases} \xi = \alpha_0^{+(i)} + \alpha^{+(i)} \cosh\varphi \\ \zeta = \beta^{+(i)} \sinh\varphi \end{cases} \quad \text{for} \quad \begin{cases} 0 < \varphi < \varphi_X^{+(1)} & \text{(upper part)} \\ -\varphi_X^{+(2)} < \varphi < 0 & \text{(lower part)} \end{cases}$$

with

$$\varphi_X^{+(i)} = \text{arcsinh}\frac{\left(2t^{+(i)} - 1\right)^{1/2}}{1 - t^{+(i)}},$$

$$\cosh\varphi_X^{+(i)} = \frac{t^{+(i)}}{1-t^{+(i)}}, \quad \sinh\varphi_X^{+(i)} = \frac{\left(2t^{+(i)}-1\right)^{1/2}}{1-t^{+(i)}},$$
$$\alpha_0^{+(i)} = \frac{(1-\delta_X^{(i)})t^{+(i)}+\delta_X^{(i)}}{2t^{+(i)}-1}, \quad \alpha^{+(i)} = -(1+\delta_X^{(i)})\frac{1-t^{+(i)}}{2t^{+(i)}-1},$$
$$\beta^{+(i)} = \kappa_X^{(i)}\frac{1-t^{+(i)}}{\left(2t^{+(i)}-1\right)^{1/2}}.$$

- If

$$\tan\psi^{+(i)} = \frac{\kappa_X^{(i)}}{1+\delta_X^{(i)}} \quad \text{i.e.} \quad t^{+(i)} = 1$$

  the upper or lower outer part of the poloidal section of the last closed magnetic surface may be modelled with a portion of straight line with the following equation

$$\xi = 1 - \frac{1+\delta_X^{(i)}}{\kappa_X^{(i)}}\zeta.$$

### A.2.2 Poloidal surface integration

For all the above cases and using the plasma geometrical description with 8 parameters, the function $I_1^*$ introduced in Eq. (A.2) can be expressed as

$$I_1^* = \frac{\kappa_X^{(1)}+\kappa_X^{(2)}}{2}\Theta_{Sp}, \tag{A.9}$$

and the poloidal plasma surface becomes

$$S_p = \pi a^2\frac{\kappa_X^{(1)}+\kappa_X^{(2)}}{2}\Theta_{S_p}. \tag{A.10}$$

On the other hand, the poloidal surface integration of an arbitrary function $F$ defined in Eq. (A.1) leads to

$$\frac{1}{S_p}\int_{S_p} F(\rho)dS_p = \int_0^1 F(\rho)2\rho d\rho. \tag{A.11}$$

The coefficient $\Theta_{Sp}\left(\kappa_X^{(1)},\kappa_X^{(2)},\delta_X^{(1)},\delta_X^{(2)},\psi^{-(1)},\psi^{+(1)},\psi^{-(2)},\psi^{+(2)}\right)$ is the correction with respect to the pure elliptical plasma case. It may be related to the partial correction coefficients of the upper and lower parts of the LCMS as

$$\Theta_{Sp} = \frac{\kappa_X^{(1)}\Theta_{Sp}^{(1)}+\kappa_X^{(2)}\Theta_{Sp}^{(2)}}{\kappa_X^{(1)}+\kappa_X^{(2)}}, \tag{A.12}$$

where $\Theta_{Sp}^{(i)}$ may be again related to the inner and outer part of the LCMS by splitting the poloidal surface into the inner-upper part, the inner-lower part, the outer-upper part, and the outer-lower part. Hence,

$$S_p = S_p^{-(1)} + S_p^{+(1)} + S_p^{-(2)} + S_p^{+(2)} \tag{A.13}$$

with

$$S_p^{-(i)} = \frac{1}{2}\pi a^2 \kappa_X^{(i)} \Theta_{Sp}^{-(i)}, \quad S_p^{+(i)} = \frac{1}{2}\pi a^2 \kappa_X^{(i)} \Theta_{Sp}^{+(i)}, \tag{A.14}$$

giving

$$\Theta_{Sp}^{(i)} = \Theta_{Sp}^{-(i)} + \Theta_{Sp}^{+(i)}. \tag{A.15}$$

**Inner side:**

We have the following expressions for the upper or lower correction coefficients of the inner side depending on the kind of curve of the corresponding portion.

- If

$$t^{-(i)} < \frac{1}{2} \quad : \quad \Theta_{Sp}^{-(i)} = \frac{1}{2}\left(1 - \delta_X^{(i)}\right) \frac{(1 - t^{-(i)})^2}{(1 - 2t^{-(i)})^{3/2}} \operatorname{arctan}(t^{-(i)}),$$

  where

$$\operatorname{arctan}(t^{-(i)}) = 1 - \frac{2}{\pi} \arcsin \frac{t^{-(i)}}{1 - t^{-(i)}} - \frac{2}{\pi} \frac{t^{-(i)}(1 - 2t^{-(i)})^{1/2}}{(1 - t^{-(i)})^2}. \tag{A.16}$$

- If

$$t^{-(i)} = \frac{1}{2} \quad : \quad \Theta_{Sp}^{-(i)} = \frac{4\left(1 - \delta_X^{(i)}\right)}{3\pi}.$$

- If

$$\frac{1}{2} < t^{-(i)} < 1 \quad : \quad \Theta_{Sp}^{-(i)} = \frac{1 - \delta_X^{(i)}}{\pi} \frac{(1 - t^{-(i)})^2}{(2t^{-(i)} - 1)^{3/2}} \operatorname{arctanh}(t^{-(i)}),$$

  where

$$\operatorname{arctanh}(t^{-(i)}) = \frac{t\left(2t^{-(i)} - 1\right)^{1/2}}{\left(1 - t^{-(i)}\right)^2} - \operatorname{arcsinh} \frac{\left(2t^{-(i)} - 1\right)^{1/2}}{1 - t^{-(i)}}. \tag{A.17}$$

- If

$$t^{-(i)} = 1 \quad : \quad \Theta_{Sp}^{-(i)} = \frac{1 - \delta_X^{(i)}}{\pi}.$$

**Outer side:**

We have the following expressions for the upper or lower correction coefficients of the outer side depending on the kind of curve of the corresponding portion.

- If

$$t^{+(i)} < \frac{1}{2} \quad : \quad \Theta_{Sp}^{+(i)} = \frac{1}{2}\left(1 + \delta_X^{(i)}\right) \frac{(1 - t^{+(i)})^2}{(1 - 2t^{+(i)})^{3/2}} \, \mathrm{arctan}(t^{+(i)}),$$

where

$$\mathrm{arctan}(t^{+(i)}) = 1 - \frac{2}{\pi} \arcsin \frac{t^{+(i)}}{1 - t^{+(i)}} - \frac{2}{\pi} \frac{t^{+(i)}(1 - 2t^{+(i)})^{1/2}}{(1 - t^{+(i)})^2}. \qquad \text{(A.18)}$$

- If

$$t^{+(i)} = \frac{1}{2} \quad : \quad \Theta_{Sp}^{+(i)} = \frac{4\left(1 + \delta_X^{(i)}\right)}{3\pi}.$$

- If

$$\frac{1}{2} < t^{+(i)} < 1 \quad : \quad \Theta_{Sp}^{+(i)} = \frac{1 + \delta_X^{(i)}}{\pi} \frac{(1 - t^{+(i)})^2}{(2t^{+(i)} - 1)^{3/2}} \mathrm{arctanh}(t^{+(i)}),$$

where

$$\mathrm{arctanh}(t^{+(i)}) = \frac{t\left(2t^{+(i)} - 1\right)^{1/2}}{\left(1 - t^{+(i)}\right)^2} - \mathrm{arcsinh}\frac{\left(2t^{+(i)} - 1\right)^{1/2}}{1 - t^{+(i)}}. \qquad \text{(A.19)}$$

- If

$$t^{+(i)} = 1 \quad : \quad \Theta_{Sp}^{+(i)} = \frac{1 + \delta_X^{(i)}}{\pi}.$$

In the special case $\psi^- = \psi^+ = 0$ (no X-points), we obtain $\Theta_{Sp} = 1$ and the poloidal surface, as well as the poloidal surface integration of an arbitrary function $F$, are independent of the plasma triangularity $\delta_X^{(i)}$.

### A.2.3 Volume integration

In the same way, the plasma volume is expressed as

$$V = 2\pi R \times \pi a^2 \frac{\kappa_X^{(1)} + \kappa_X^{(2)}}{2} \Theta_V, \qquad \text{(A.20)}$$

where $\Theta_V\left(\kappa_X^{(1)},\kappa_X^{(2)},\delta_X^{(1)},\delta_X^{(2)},\psi^{-(1)},\psi^{+(1)},\psi^{-(2)},\psi^{+(2)}\right)$ is the correction coefficient with respect to the pure elliptical plasma case. It may be expressed as

$$\Theta_V=\frac{\kappa_X^{(1)}\Theta_V^{(1)}+\kappa_X^{(2)}\Theta_V^{(2)}}{\kappa_X^{(1)}+\kappa_X^{(2)}}. \tag{A.21}$$

Splitting the plasma volume into the inner-upper part, the inner-lower part, the outer-upper part, and the outer-lower part, we obtain

$$V=V^{-(1)}+V^{-(2)}+V^{+(1)}+V^{+(2)}, \tag{A.22}$$

with

$$V^{-(i)}=\pi R\times\pi a^2\kappa_X^{(i)}\Theta_V^{-(i)},\quad V^{+(i)}=\pi R\times\pi a^2\kappa_X^{(i)}\Theta_V^{+(i)}, \tag{A.23}$$

giving

$$\Theta_V^{(i)}=\Theta_V^{-(i)}+\Theta_V^{+(i)}. \tag{A.24}$$

The following expressions for the correction coefficients for the plasma volume are deduced depending on the kind of portion of curve, for the inner and outer side.

**Inner side:**

- If $t^{-(i)}<\frac{1}{2}$ :

$$\Theta_V^{-(i)}=\frac{1}{2}\left(1-\delta_X^{(i)}\right)\left[\left(1-\frac{1}{A}\frac{\delta_X^{(i)}-(1+\delta_X^{(i)})t^{-(i)}}{1-2t^{-(i)}}\right)\frac{\left(1-t^{-(i)}\right)^2}{\left(1-2t^{-(i)}\right)^{3/2}}\text{arctan}(t^{-(i)})-\frac{4}{3\pi}\frac{1}{A}\left(1-\delta_X^{(i)}\right)\frac{1}{1-2t^{-(i)}}\right],$$

  where $\arctan(t^{-(i)})$ is given by Eq. (A.16).

- If $t^{-(i)}=\frac{1}{2}$ :

$$\Theta_V^{-(i)}=\frac{4\left(1-\delta_X^{(i)}\right)}{3\pi}\left(1-\frac{1}{A}\frac{2+3\delta_X^{(i)}}{5}\right).$$

- If $\frac{1}{2}<t^{-(i)}<1$ :

$$\Theta_V^{-(i)}=\frac{1-\delta_X^{(i)}}{\pi}\left[\left(1-\frac{1}{A}\frac{(1+\delta_X^{(i)})t^{-(i)}-\delta_X^{(i)}}{2t^{-(i)}-1}\right)\frac{\left(1-t^{-(i)}\right)^2}{\left(2t^{-(i)}-1\right)^{3/2}}\text{arctanh}(t^{-(i)})+\frac{2}{3}\frac{1}{A}\left(1-\delta_X^{(i)}\right)\frac{1}{2t^{-(i)}-1}\right],$$

  where arctanh$(t^{-(i)})$ is given by Eq. (A.17).

- If $t^{-(i)}=1$ :

$$\Theta_V^{-(i)}=\frac{1-\delta_X^{(i)}}{\pi}\left(1-\frac{1}{A}\frac{1+2\delta_X^{(i)}}{3}\right).$$

**Outer side:**

- If $\quad t^{+(i)} < \frac{1}{2}$ :

$$\Theta_V^{+(i)} = \tfrac{1}{2}\left(1+\delta_X^{(i)}\right)\left[\left(1-\tfrac{1}{A}\tfrac{\delta_X^{(i)}+(1-\delta_X^{(i)})t^{+(i)}}{1-2t^{+(i)}}\right)\tfrac{\left(1-t^{+(i)}\right)^2}{\left(1-2t^{+(i)}\right)^{3/2}}\text{arctan}(t^{+(i)}) + \tfrac{4}{3\pi}\tfrac{1}{A}\left(1+\delta_X^{(i)}\right)\tfrac{1}{1-2t^{+(i)}}\right],$$

  where $\arctan(t^{+(i)})$ is given by Eq. (A.18).

- If $\quad t^{+(i)} = \frac{1}{2}$ :

$$\Theta_V^{-(i)} = \frac{4\left(1+\delta_X^{(i)}\right)}{3\pi}\left(1+\frac{1}{A}\frac{2-3\delta_X^{(i)}}{5}\right).$$

- If $\quad \frac{1}{2} < t^{+(i)} < 1$ :

$$\Theta_V^{+(i)} = \tfrac{1+\delta_X^{(i)}}{\pi}\left[\left(1+\tfrac{1}{A}\tfrac{(1-\delta_X^{(i)})t^{+(i)}+\delta_X^{(i)}}{2t^{+(i)}-1}\right)\tfrac{\left(1-t^{+(i)}\right)^2}{\left(2t^{+(i)}-1\right)^{3/2}}\text{arctanh}(t^{+(i)}) - \tfrac{2}{3}\tfrac{1}{A}\left(1+\delta_X^{(i)}\right)\tfrac{1}{2t^{+(i)}-1}\right],$$

  where arctanh$(t^{+(i)})$ is given by Eq. (A.19).

- If $\quad t^{+(i)} = 1$ :

$$\Theta_V^{-(i)} = \frac{1+\delta_X^{(i)}}{\pi}\left(1+\frac{1}{A}\frac{1-2\delta_X^{(i)}}{3}\right).$$

In the special case $\psi^- = \psi^+ = 0$, we obtain

$$\Theta_V = 1-\left(1-\frac{8}{3\pi}\right)\frac{\delta_X}{A}.$$

Finally, from Eq. A.5 the following expression for the fuction $I_2^*$ is deduced

$$I_2^* = -\frac{3}{2}\frac{\kappa_X^{(1)}+\kappa_X^{(2)}}{2}\Theta_{Sp}\Theta_1.$$

Comparing the definitions of $\Theta_{Sp}$ and $\Theta_V$ in Eqs. (A.10) and (A.20) with the definitions of $I_1^*$, $I_2^*$ and $\Theta_1$ in Eqs. (A.2), (A.4), and (A.5) we obtain the relation (A.25) between $\Theta_1$, $\Theta_{Sp}$, $\Theta_V$.

$$\frac{\Theta_1}{A} = 1-\frac{\Theta_V}{\Theta_{Sp}}. \qquad \text{(A.25)}$$

In the special case $\psi^{-(1)} = \psi^{+(1)} = \psi^{-(2)} = \psi^{+(2)} = 0$, $\kappa_X^{(1)} = \kappa_X^{(2)}$, $\delta_X^{(1)} = \delta_X^{(2)} = \delta_X$ we obtain

$$\Theta_1 = \left(1 - \frac{8}{3\pi}\right)\delta_X.$$

### A.2.4 Plasma surface integration

Proceeding in the same way of the previous poloidal surface and volume integrations, the surface $S$ of the plasma may be written as

$$S = 2\pi R \times 2\pi a \mathbf{E}_1 \left(\frac{\kappa_X^{(1)} + \kappa_X^{(2)}}{2}\right)\Theta_S \tag{A.26}$$

with

$$\Theta_S = \frac{\frac{1}{2}\left[\mathbf{E}_1\left(\kappa_X^{(1)}\right)\Theta_S^{(1)} + \mathbf{E}_1\left(\kappa_X^{(2)}\right)\Theta_S^{(2)}\right]}{\mathbf{E}_1\left(\frac{\kappa_X^{(1)}+\kappa_X^{(2)}}{2}\right)}, \tag{A.27}$$

where

$$\mathbf{E}_1(\kappa_X) = \frac{2}{\pi}\kappa_X \mathbf{E}\left[(1 - 1/\kappa_X^2)^{1/2}\right], \tag{A.28}$$

and $\mathbf{E}(x)$ is the complete elliptic integral of the second kind.

Splitting into the inner and the outer sides, and into the upper and lower parts, we have

$$S = S^{-(1)} + S^{-(2)} + S^{+(1)} + S^{+(2)} \tag{A.29}$$

with

$$S^{-(i)} = \pi R \times 2\pi a \mathbf{E}_1(\kappa_X^{(i)})\Theta_S^{-(i)}, \quad S^{+(i)} = \pi R \times 2\pi a \mathbf{E}_1(\kappa_X^{(i)})\Theta_S^{+(i)}, \tag{A.30}$$

giving

$$\Theta_S^{(i)} = \Theta_S^{-(i)} + \Theta_S^{+(i)}. \tag{A.31}$$

Introducing

$$\kappa^{-(i)} = \frac{\kappa_X^{(i)}}{1 - \delta_X^{(i)}}, \quad \kappa^{+} = \frac{\kappa_X^{(i)}}{1 + \delta_X^{(i)}},$$

the following expressions for the partial correction coefficients for the surface integrations are derived.

**Inner side:**

- If

$$0 \le t^{-(i)} < \frac{1}{2} - \frac{1}{2\kappa^{-(i)^2}}$$

then

$$\begin{aligned}\Theta_S^{-(i)} = {} & \frac{\kappa_X^{(i)}}{\pi \mathbf{E}_1(\kappa_X^{(i)})} \frac{1-t^{-(i)}}{\left(1-2t^{-(i)}\right)^{1/2}} \left\{ \left[1 - \frac{1}{A} \frac{\delta_X^{(i)}-(1+\delta_X^{(i)})t^{-(i)}}{1-2t^{-(i)}}\right] \right. \\ & \times E\left[\pi - \theta_X^{-(i)}, \left(\frac{1-2t^{-(i)}-1/\kappa^{-(i)^2}}{1-2t^{-(i)}}\right)^{1/2}\right] - \frac{1}{A}\frac{1-\delta_X^{(i)}}{2}\frac{1}{\left(1-2t^{-(i)}\right)^{1/2}} \\ & \left. \times \left[\frac{\left(t^{-(i)2}+1/\kappa^{-(i)2}\right)^{\frac{1}{2}}}{1-t^{-(i)}} + \frac{1-t^{-(i)}}{\left(1-2t^{-(i)}-1/\kappa^{-(i)2}\right)^{1/2}} \arcsin \frac{\left(1-2t^{-(i)}-1/\kappa^{-(i)2}\right)^{\frac{1}{2}}}{1-t^{-(i)}}\right]\right\}.\end{aligned}$$

- If

$$t^{-(i)} = \frac{1}{2} - \frac{1}{2\kappa^{-(i)^2}}$$

then

$$\begin{aligned}\Theta_S^{-(i)} = {} & \frac{1}{2\pi \mathbf{E}_1(\kappa_X^{(i)})} \frac{\kappa_X^2 + \left(1-\delta_X^{(i)}\right)^2}{1-\delta_X^{(i)}} \\ & \times \left[\left(1 + \frac{1}{2A}\frac{\kappa_X^{(i)2}+\delta_X^{(i)2}-1}{1-\delta_X^{(i)}}\right) \arcsin \frac{2\kappa_X^{(i)}\left(1-\delta_X^{(i)}\right)}{\kappa_X^{(i)2}+\left(1-\delta_X^{(i)}\right)^2} - \frac{\kappa_X^{(i)}}{A}\right]\end{aligned}$$

- If

$$\frac{1}{2} - \frac{1}{2\kappa^{-(i)^2}} < t^{-(i)} < \frac{1}{2}$$

then

$$\begin{aligned}\Theta_S^{-(i)} = {} & \frac{1-\delta_X^{(i)}}{\pi \mathbf{E}_1(\kappa_X^{(i)})} \frac{1-t^{-(i)}}{1-2t^{-(i)}} \left\{ \left[1 - \frac{1}{A}\frac{\delta_X^{(i)}-(1+\delta_X^{(i)})t^{-(i)}}{1-2t^{-(i)}}\right] \left[\mathbf{E}\left\{\left[1-\left(1-2t^{-(i)}\right)\kappa^{-(i)2}\right]^{\frac{1}{2}}\right\}\right.\right. \\ & \left. - E\left\{\theta_X^{-(i)} - \frac{\pi}{2}, \left[1-\left(1-2t^{-(i)}\right)\kappa^{-(i)2}\right]^{\frac{1}{2}}\right\}\right] - \frac{\kappa_X^{(i)}}{2A}\left[\frac{\left(t^{-(i)^2}+1/\kappa^{-(i)^2}\right)^{1/2}}{1-t^{-(i)}}\right. \\ & \left.\left. + \frac{1-t^{-(i)}}{\left(2t^{-(i)}-1+1/\kappa^{-(i)^2}\right)^{1/2}} \ln \frac{\left(t^{-(i)^2}+1/\kappa^{-(i)2}\right)^{1/2}+\left(2t^{-(i)}-1+1/\kappa^{-(i)^2}\right)^{1/2}}{1-t^{-(i)}}\right]\right\}.\end{aligned}$$

- If
$$t^{-(i)} = \frac{1}{2}$$
then
$$\Theta_S^{-(i)} = \tfrac{1}{2\pi \mathbf{E}_1(\kappa_X^{(i)})} \left\{ \sqrt{\kappa_X^{(i)2} + 4\left(1-\delta_X^{(i)}\right)^2} \left[1 - \tfrac{1}{A} \tfrac{8\left(1-\delta_X^{(i)2}\right)-\kappa_X^{(i)2}}{16\left(1-\delta_X^{(i)}\right)}\right] \right.$$
$$\left. + \tfrac{\kappa_X^{(i)2}}{2\left(1-\delta_X^{(i)}\right)} \left[1 - \tfrac{1}{A} \tfrac{16\left(1-\delta_X^{(i)}\right)+\kappa_X^{(i)2}}{16\left(1-\delta_X^{(i)}\right)}\right] \text{arcsinh} \tfrac{2\left(1-\delta_X^{(i)}\right)}{\kappa_X^{(i)}} \right\}.$$

- If
$$\frac{1}{2} < t^{-(i)} < 1$$
then
$$\Theta_S^{-(i)} = \tfrac{\kappa_X^{(i)}}{\pi \mathbf{E}_1(\kappa_X^{(i)})} \left\{ \left[1 - \tfrac{1}{A} \tfrac{(1+\delta_X^{(i)})t^{-(i)}-\delta_X^{(i)}}{2t^{-(i)}-1}\right] \left[ \tfrac{\left(t^{-(i)2}+1/\kappa^{-(i)2}\right)^{\frac{1}{2}}}{t^{-(i)}} \right. \right.$$
$$\left. + \tfrac{1-t^{-(i)}}{\left(2t^{-(i)}-1\right)^{1/2}} \left(F^*\left(\varphi^{-(i)}, k^{-(i)}\right) - E^*\left(\varphi^{-(i)}, k^{-(i)}\right)\right) \right]$$
$$+ \tfrac{1}{A} \tfrac{1-\delta_X^{(i)}}{2} \tfrac{1-t^{-(i)}}{2t^{-(i)}-1} \left[ \tfrac{\left(t^{-(i)2}+1/\kappa^{-(i)2}\right)^{1/2}}{1-t^{-(i)}} + \tfrac{1-t^{-(i)}}{\left(2t^{-(i)}-1+1/\kappa^{-(i)2}\right)^{1/2}} \right.$$
$$\left. \left. \times \ \ln \tfrac{\left(t^{-(i)2}+1/\kappa^{-(i)2}\right)^{1/2}+\left(2t^{-(i)}-1+1/\kappa^{-(i)2}\right)^{1/2}}{1-t^{-(i)}} \right] \right\}.$$
where
$$\varphi^{-(i)} = \arcsin \frac{\left(2t^{-(i)}-1\right)^{1/2}}{t^{-(i)}}, \quad k^{-(i)} = \frac{1}{\kappa^{-(i)}\left(2t^{-(i)}-1\right)^{1/2}},$$
$$F^*(\varphi, k) = \int_0^{\varphi} \frac{1}{\sqrt{1+k^2\sin^2\theta}} d\theta = \frac{1}{\sqrt{1+k^2}} \left[\mathbf{K}\left(\tfrac{k}{\sqrt{1+k^2}}\right) - F\left(\frac{\pi}{2} - \varphi, \tfrac{k}{\sqrt{1+k^2}}\right)\right],$$
$$E^*(\varphi, k) = \int_0^{\varphi} \sqrt{1+k^2\sin^2\theta}\, d\theta = \sqrt{1+k^2} \left[\mathbf{E}\left(\tfrac{k}{\sqrt{1+k^2}}\right) - E\left(\frac{\pi}{2} - \varphi, \tfrac{k}{\sqrt{1+k^2}}\right)\right].$$

- If $t^{-(i)} = 1$ then
$$\Theta_S^{-(i)} = \frac{1}{\pi \mathbf{E}_1(\kappa_X^{(i)})} \sqrt{\kappa_X^{(i)2} + \left(1-\delta_X^{(i)}\right)^2} \left(1 - \frac{1}{A} \frac{1+\delta_X^{(i)}}{2}\right).$$

In the Special cases $\psi^{-(i)} = 0$, $\delta_X^{(i)} = 0$, we have
$$\Theta_S^{-(i)} = \frac{1}{2} \left[1 - \frac{1}{A} \frac{1}{\pi \mathbf{E}_1(\kappa_X^{(i)})} \left(1 + \kappa_X^{(i)} \frac{\arcsin \sqrt{1-1/\kappa_X^{(i)2}}}{\sqrt{1-1/\kappa_X^{(i)2}}}\right)\right].$$

**Outer side:**

- If
$$0 \leq t^{+(i)} < \frac{1}{2} - \frac{1}{2\kappa^{+(i)2}}$$
then
$$\begin{aligned}\Theta_S^{+(i)} = &\frac{\kappa_X^{(i)}}{\pi \mathbf{E}_1(\kappa_X^{(i)})} \frac{1-t^{+(i)}}{\left(1-2t^{+(i)}\right)^{1/2}} \left\{ \left[ 1 - \frac{1}{A} \frac{\delta_X^{(i)} + (1-\delta_X^{(i)}) t^{+(i)}}{1-2t^{+(i)}} \right] \right.\\ &\times E \left[ \theta_X^{+(i)}, \left( 1 - \frac{1}{\left(1-2t^{+(i)}\right)\kappa^{+(i)2}} \right)^{1/2} \right] + \frac{1}{A} \frac{1+\delta_X^{(i)}}{2} \frac{1}{\left(1-2t^{+(i)}\right)^{1/2}} \\ &\times \left. \left[ \frac{\left(t^{+(i)2} + 1/\kappa^{+(i)2}\right)^{\frac{1}{2}}}{1-t^{+(i)}} + \frac{1-t^{+(i)}}{\left(1-2t^{+(i)} - 1/\kappa^{+(i)2}\right)^{\frac{1}{2}}} \arcsin \frac{\left(1-2t^{+(i)} - 1/\kappa^{+(i)2}\right)^{\frac{1}{2}}}{1-t^{+(i)}} \right] \right\}.\end{aligned}$$

- If
$$t^{+(i)} = \frac{1}{2} - \frac{1}{2\kappa^{+(i)2}}$$
then
$$\begin{aligned}\Theta_S^{+(i)} = &\frac{1}{2\pi \mathbf{E}_1(\kappa_X)} \frac{\kappa_X^2 + (1+\delta_X)^2}{1+\delta_X} \\ &\times \left[ \left( 1 - \frac{1}{2A} \frac{\kappa_X^2 + \delta_X^2 - 1}{1+\delta_X} \right) \arcsin \frac{2\kappa_X (1+\delta_X)}{\kappa_X^2 + (1+\delta_X)^2} + \frac{\kappa_X}{A} \right].\end{aligned}$$

- If
$$\frac{1}{2} - \frac{1}{2\kappa^{+2}} < t^{+} < \frac{1}{2}$$
then
$$\begin{aligned}\Theta_S^{+(i)} = &\frac{1+\delta_X^{(i)}}{\pi \mathbf{E}_1(\kappa_X^{(i)})} \frac{1-t^{+(i)}}{1-2t^{+(i)}} \left\{ \left[ 1 - \frac{1}{A} \frac{\delta_X^{(i)} + (1-\delta_X^{(i)}) t^{+(i)}}{1-2t^{+(i)}} \right] \left[ \mathbf{E} \left\{ \left[ 1 - \left(1-2t^{+(i)}\right) \kappa^{+(i)2} \right]^{1/2} \right\} \right. \right.\\ &\left. - E \left\{ \frac{\pi}{2} - \theta_X^{+(i)}, \left[ 1 - (1-2t^{+(i)}) \kappa^{+(i)2} \right]^{1/2} \right\} \right] + \frac{\kappa_X^{(i)}}{2A} \left[ \frac{\left(t^{+(i)2} + 1/\kappa^{+(i)2}\right)^{1/2}}{1-t^{+(i)}} \right. \\ &+ \left. \left. \frac{1-t^{+(i)}}{\left(2t^{+(i)} - 1 + 1/\kappa^{+(i)2}\right)^{1/2}} \ln \frac{\left(t^{+(i)2} + 1/\kappa^{+(i)2}\right)^{1/2} + \left(2t^{+(i)} - 1 + 1/\kappa^{+(i)2}\right)^{1/2}}{1-t^{+(i)}} \right] \right\}.\end{aligned}$$

- If

$$t^{+(i)} = \frac{1}{2}$$

then

$$\Theta_S^{+(i)} = \frac{1}{2\pi \mathbf{E}_1(\kappa_X^{(i)})} \left\{ \sqrt{\kappa_X^{(i)2} + 4\left(1+\delta_X^{(i)}\right)^2} \left[1 + \frac{1}{A}\frac{8\left(1-\delta_X^{(i)2}\right)-\kappa_X^{(i)2}}{16\left(1+\delta_X^{(i)}\right)}\right] + \frac{\kappa_X^{(i)2}}{2\left(1+\delta_X^{(i)}\right)} \left[1 + \frac{1}{A}\frac{16\left(1+\delta_X^{(i)}\right)+\kappa_X^{(i)2}}{16\left(1+\delta_X^{(i)}\right)}\right] \text{arcsinh}\frac{2\left(1+\delta_X^{(i)}\right)}{\kappa_X^{(i)}} \right\}.$$

- If

$$\frac{1}{2} < t^{+(i)} < 1$$

then

$$\Theta_S^{+(i)} = \frac{\kappa_X^{(i)}}{\pi \mathbf{E}_1(\kappa_X^{(i)})} \left\{ \left[1 + \frac{1}{A}\frac{(1-\delta_X^{(i)})t^{+(i)}+\delta_X^{(i)}}{2t^{+(i)}-1}\right] \times \left[\frac{\left(t^{+(i)2}+1/\kappa^{+(i)2}\right)^{1/2}}{t^{+(i)}} + \frac{1-t^{+(i)}}{\left(2t^{+(i)}-1\right)^{1/2}}\left(F^*\left(\varphi^{+(i)}, k^{+(i)}\right) - E^*\left(\varphi^{+(i)}, k^{+(i)}\right)\right)\right] - \frac{1}{A}\frac{1+\delta_X^{(i)}}{2}\frac{1-t^{+(i)}}{2t^{+(i)}-1}\left[\frac{\left(t^{+(i)2}+1/\kappa^{+(i)2}\right)^{1/2}}{1-t^{+(i)}} + \frac{1-t^{+(i)}}{\left(2t^{+(i)}-1+1/\kappa^{+(i)2}\right)^{1/2}} \times \ln\frac{\left(t^{+(i)2}+1/\kappa^{+(i)2}\right)^{1/2}+\left(2t^{+(i)}-1+1/\kappa^{+(i)2}\right)^{1/2}}{1-t^{+(i)}}\right]\right\},$$

where

$$\varphi^{+(i)} = \arcsin\frac{\left(2t^{+(i)}-1\right)^{1/2}}{t^{+(i)}}, \quad k^{+(i)} = \frac{1}{\kappa^{+(i)}\left(2t^{+(i)}-1\right)^{1/2}}.$$

- If

$$t^{+(i)} = 1$$

then

$$\Theta_S^{+(i)} = \frac{1}{\pi \mathbf{E}_1(\kappa_X^{(i)})}\sqrt{\kappa_X^{(i)2} + \left(1+\delta_X^{(i)}\right)^2}\left(1 + \frac{1}{A}\frac{1-\delta_X^{(i)}}{2}\right).$$