UNIVERSITAT DE BARCELONA

LOCAL COHOMOLOGY MODULES SUPPORTED ON MONOMIAL IDEALS

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CERTIFICA:

Que la present memòria ha estat realitzada sota la seva direcció per Josep Àlvarez Montaner i que constitueix la tesi d'aquest per a aspirar al grau de Doctor en Matemàtiques.

Barcelona, Març de 2002

Signat: Santiago Zarzuela Armengou.

Als meus pares i germans

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Introduction

The aim of this work is the study of local cohomology modules supported on monomial ideals. Let us first introduce the subject and main problems. After this we will expose some known related results and, finally, we will give a summary of the results obtained in this work.

Local cohomology was introduced by A. Grothendieck as an algebraic cohomology theory analogous to the classical relative cohomology. This analogy comes from the fact that many results on projective varieties can be reformulated in terms of graded rings or complete local rings. The notes published by R. Hartshorne based on a course given by A. Grothendieck [38] will be the starting point for introducing this theory.

Let X be a topological space. Given a locally closed subspace $Z \subseteq X$ we consider the **functor of sections with support in** Z, that will be denoted by $\Gamma_Z(X, \mathcal{F})$ where $\mathcal{F}$ is a sheaf of abelian groups over X. This is a left exact functor. Since the category of sheaves of abelian groups on X has enough injectives, one may consider the right derived functors of $\Gamma_Z(X, \mathcal{F})$ that will be called **local cohomology groups of** X **with coefficients in** $\mathcal{F}$ **and support in** Z, and will be denoted by $H^r_Z(X, \mathcal{F}) := \mathbb{R}^r\Gamma_Z(X, \mathcal{F})$.

A first interpretation of these groups is given by the existence of the long exact sequence of cohomology groups

$$0 \longrightarrow H^0_Z(X, \mathcal{F}) \longrightarrow H^0(X, \mathcal{F}) \longrightarrow H^0(U, \mathcal{F}_{|U}) \longrightarrow H^1_Z(X, \mathcal{F}) \longrightarrow \cdots$$

where $U = X \setminus Z$ is the complement of Z in X. Then, $H^1_Z(X, \mathcal{F})$ is nothing but the obstruction to the extension of sections of $\mathcal{F}$ over U to the whole space X.

Even though the geometric point of view of the notes by A. Grothendieck, local cohomology quickly became an indispensable tool in the theory of Noetherian commutative rings. In particular, R. Y. Sharp [83], described local cohomology in the framework of Commutative Algebra. Throughout this work, we will consider the following situation:

Let R be a Noetherian commutative ring. Given an ideal $I \subseteq R$ we consider the **functor of I-torsion**, that is defined for any R-module M as $\Gamma_I(M) := \{x \in M \mid I^n x = 0 \text{ for any } n \geq 1\}$. This is an additive, covariant and left exact functor. Since the category of R-modules has enough injectives, one may consider the right derived functors of Γ_I that will be called **local cohomology modules of M with support the ideal** I and will be denoted by $H^r_I(M) := \mathbb{R}^r\Gamma_I(M)$.

Whether R is Noetherian, the sheaf associated to an injective R-module is flasque. From this fact, one may check that the notion of local cohomology defined by using the theory of sheaves coincide with the local cohomology defined in algebraic terms. More precisely, let $X = \operatorname{Spec} R$ be an affine space, $\mathcal{F} = \widetilde{M}$ the sheaf associated to a R-module M and Z the subvariety defined by an ideal $I \subseteq R$. Then, one has isomorphisms $H^r_Z(X, \widetilde{M}) \cong H^r_I(M)$ for all r.

Despite the effort of many authors in the study of these modules, their structure is still quite unknown. Following C. Huneke's criteria [45], the basic problems concerning local cohomology modules are:

- Annihilation of local cohomology modules.
- Finitely generation of local cohomology modules.
- Artinianity of local cohomology modules.
- Finiteness of the associated primes set of local cohomology modules.

In general, one can not even say when they vanish. Moreover, when they do not vanish they are rarely finitely generated. However, in some situations these modules verify some finiteness properties that provide a better understanding of their structure.

Next we will expose some results one may find in the literature related to these problems and are best related with the contents of this work. Moreover, we will comment some applications both in Algebraic Geometry and Commutative Algebra.

Annihilation of local cohomology modules

A. Grothendieck obtained the first basic results on annihilation by giving bounds for the possible integers r such that $H_I^r(M) \neq 0$ in terms of the dimension and the grade. Namely, let M be a finitely generated R-module, then one has $\operatorname{grade}(I, M) \leq r \leq \dim R$ for all r such that $H_I^r(M) \neq 0$.

Whether $(R, \mathfrak{m})$ is a Noetherian local ring, this result may be completed in order to provide a cohomological characterization of the Krull dimension. Namely, $H_{\mathfrak{m}}^r(M)$ vanishes for all $r > \dim M$ and it is different from zero for $r = \dim M$.

Even though the lower bound is sharp, it does not occur for the upper bound. In order to precise this bound, we introduce the integer $\operatorname{cd}(R, I) := \max\{r \mid H_I^r(M) \neq 0 \ \forall M\}$, that will be named the cohomological dimension of the ideal I with respect to R. It is worthwhile to point out that one only has to study the case $M = R$ since the cohomological dimension equals to the biggest integer such that $H_I^r(R) \neq 0$.

Among the results that we may found on the annihilation of local cohomology modules we detach the Hartshorne-Lichtenbaum vanishing theorem [40]:

Theorem *Let* $(R, \mathfrak{m})$ *be a complete local domain of dimension* d. *Let* $I \subseteq R$ *be an ideal, then* $\operatorname{cd}(R, I) < d$ *if and only if* $\dim R/I > 0$.

The following theorem was proved by R. Hartshorne [40] in the geometric case, by A. Ogus [74] in characteristic zero and by C. Peskine-L. Szpiro [76] and R. Hartshorne-R. Speiser [43] in characteristic $p > 0$. A characteristic free proof was given by C. Huneke-G. Lyubeznik [47].

Theorem *Let* $(R, \mathfrak{m})$ *be a complete local domain of dimension* d *with separately closed residue field. Let* $I \subseteq R$ *be an ideal, then* $\operatorname{cd}(R, I) < d - 1$ *if and only if* $\dim R/I > 1$ *and* $\operatorname{Spec}(R/I) \setminus \{\mathfrak{m}\}$ *is connected.*

Unfortunately, there is not a simple extension of these result to lower cohomological dimensions. However, one may found bounds for the cohomological dimension in some special cases. From this point of view, we detach the work of G. Faltings [24] and Huneke-Lyubeznik [47].

To illustrate the utility of local cohomology modules we are going to announce some applications of the results described above.

• Let $\mathrm{ara}(I)$ be the minimum number of generators that are required in order to define the ideal I but radical. Then, one has $\mathrm{cd}\,(R, I) \leq \mathrm{ara}(I)$. We have to point out that this integer has a key role in the study of the connectivity of algebraic varieties (see [15]).

• In the case of subvarieties of projective spaces, the local cohomology annihilation theorems have topological applications (see [40], [74], [47]). For example, one may obtain the following generalization of the Lefschetz theorem:

Let $I \subseteq R = \mathbb{C}[x_0, \ldots, x_n]$ be the defining ideal of a closed subvariety $Y \subseteq \mathbb{P}^n_{\mathbb{C}}$. Then, the morphism

$$H^r_{\mathrm{dR}}(\mathbb{P}^n_{\mathbb{C}}) \longrightarrow H^r_{\mathrm{dR}}(Y),$$

where $H^r_{\mathrm{dR}}(\centerdot)$ denote the de Rham cohomology groups, is an isomorphism for all $r < n - \mathrm{cd}\,(R, I)$ and is a monomorphism for $r = n - \mathrm{cd}\,(R, I)$ ([42, Theorem III.7.1]).

Finitely generation of local cohomology modules

In general, the modules $H^r_I(M)$ are rarely finitely generated, even if the module M is. G. Faltings [23] gave a criteria to determine the finitely generation of local cohomology modules. This criteria depends on the numbers $s(I, M) := \min\{\mathrm{depth}\,(M_{\mathfrak{p}}) + \mathrm{ht}\,((I + \mathfrak{p})/\mathfrak{p}) \mid I \not\subseteq \mathfrak{p},\ \mathfrak{p} \in \mathrm{Spec}\,(R)\}$.

Theorem *Let R be a Noetherian ring, $I \subseteq R$ an ideal and M a finitely generated R-module. Then, $H^r_I(M)$ is finitely generated for all $r < s(I, M)$ and is not for $r = s(I, M)$.*

A. Grothendieck conjectured that even though the local cohomology modules $H^r_I(R)$ are not finitely generated, the modules $\mathrm{Hom}_R(R/I, H^r_I(R))$ are. R. Hartshorne showed that this conjecture is false [41]. However, there has been a big effort in the study of the cofiniteness of local cohomology modules, where we say that a R-module M is I-cofinite if $\mathrm{Supp}_R(M) \subseteq V(I)$ and $\mathrm{Ext}^r_R(R/I, M)$ is finitely generated for all $r \geq 0$. From this point of view, we detach the work of Huneke-Koh [46], D. Delfino [18] and Delfino-Marley [19].

Finiteness properties of local cohomology modules

Although the local cohomology modules are in general not finitely generated, under certain conditions they satisfy some finiteness properties that provide a better understanding of their structure. Following this path, we remak the following result:

Let R be a unramified regular local ring. Then, for any ideal $I \subseteq R$, any prime ideal $\mathfrak{p} \subseteq R$ and any $r \geq 0$ one has:

· The set of associated primes of $H_I^r(R)$ is finite.

· The Bass numbers $\mu_p(\mathfrak{p}, H_I^r(R))$ are finite.

This result has been proved by Huneke-Sharp [48] in the positive characteristic case and by G. Lyubeznik in the zero characteristic [55] and mixed characteristic case [57].

We have to point out that G. Lyubeznik uses the algebraic theory of $\mathcal{D}$-modules due to the fact that local cohomology modules are finitely generated as $\mathcal{D}$-modules.

R. Hartshorne [41] gave an example of a local cohomology module whose Bass numbers may be infinite if R is not regular. The finiteness of the associated primes set of $H_I^r(R)$ for any Noetherian ring R and any ideal I was an open question until A. Singh [84] (non local case) and M. Katzman [51] (local case) have given examples of local cohomology modules having infinite associated primes.

By using the finiteness of Bass numbers, G. Lyubeznik defined a new set of numerical invariants for local rings A containing a field, that are denoted as $\lambda_{p,i}(A)$. More precisely:

Let $(R, \mathfrak{m}, k)$ be a regular local ring of dimension n containing a field k, and let A be a local ring that admits an epimorphism of rings $\pi : R \longrightarrow A$. Let $I = \operatorname{Ker} \pi$. Then one defines $\lambda_{p,i}(A)$ as the Bass number $\mu_p(\mathfrak{m}, H_I^{n-i}(R)) = \dim_k \operatorname{Ext}_R^p(k, H_I^{n-i}(R))$. These numbers depend on A, i and p, but neither on R nor on π.

They have an interesting topological interpretation, as it was pointed out by G. Lyubeznik.

• Let V be a scheme of finite type over $\mathbb{C}$ of dimension d and let A be the local ring of V at a closed point $q \in V$. If q is an isolated singular point of V then, from a theorem of A. Ogus [74] relating local cohomology and algebraic de Rham cohomology, and the comparison theorem between algebraic de Rham cohomology and singular cohomology proved by R. Hartshorne in [42], one gets

$$\lambda_{0,i}(A) = \dim_{\mathbb{C}} H^i_q(V, \mathbb{C}) \text{ for } \ 1 \leq i \leq d-1,$$

where $H^i_q(V, \mathbb{C})$ is the i-th singular local cohomology group of V with support in q and with coefficients in $\mathbb{C}$.

R. Garcia and C. Sabbah [29] generalize this result for the pure dimensional case by using the theory of $\mathcal{D}$-modules. In particular, they express these Lyubeznik numbers in terms of Betti numbers of the associated real link.

Graded structure of local cohomology modules

Whether the ring R and the ideal I are graded, the local cohomology modules $H^r_I(M)$ have a graded structure too for any graded R-module M. We have to point out that the graded version of the main results on local cohomology modules remain true (see [15]).

Graded local cohomology has interesting applications in projective Algebraic Geometry. In particular, the Castelnuovo-Mumford regularity $\operatorname{reg}(M)$, is an invariant of the R-module M, determined by the graded local cohomology. Whether M is a finitely generated module, this invariant provides some information on the resolution of the module. For example, let $M = I_{\mathbb{P}^n_k}(V)$ be the defining ideal of a projective variety $V \subseteq \mathbb{P}^n_k$, where k is an algebraically closed field. Then, the grades of the homogeneous polynomials that define $I_{\mathbb{P}^n_k}(V)$ can not be larger than $\operatorname{reg}(I_{\mathbb{P}^n_k}(V))$.

Some different applications of graded local cohomology may be found in the study of graded rings associated to filtrations of a commutative ring R, mainly the Rees algebra and the associated graded ring of an ideal $I \subseteq R$. Recall that these rings play an essential role in the study of singularities, since they become the algebraic realization of the classical notion of blowing up along a subvariety.

Local cohomology modules supported on monomial ideals

Let $R = k[x_1, \ldots, x_n]$, where k is a field, be the polynomial ring on the variables $x_1, \ldots, x_n$. Let $\mathfrak{m} := (x_1, \ldots, x_n) \subseteq R$ be the homogeneous maximal ideal and $I \subseteq R$ a squarefree monomial ideal. Then, the local cohomology modules $H^r_{\mathfrak{m}}(R/I)$ and $H^r_I(R)$ have a natural $\mathbb{Z}^n$-structure, where the corresponding graded pieces are k-vector spaces of finite dimension.

From the Taylor resolution of the quotient ring R/I, G. Lyubeznik [54] gave a description of the local cohomology modules $H_I^r(R)$, determined their annihilation and described the cohomological dimension of I with respect to R.

By the Stanley-Reisner correspondence, one may associate to any squarefree monomial ideal $I \subseteq R$ a simplicial complex Δ defined on the set of vertices $\{x_1, \dots, x_n\}$. In [86] one may found a result of M. Hochster where a description of the graded Hilbert series of the local cohomology modules $H_{\mathfrak{m}}^r(R/I)$ is given in terms of the reduced simplicial cohomology of certain subcomplexes of Δ. More precisely, given a face $\sigma_\alpha := \{x_i \mid \alpha_i = 1\} \in \Delta$, we define:

- Link of σ_α in Δ: $\mathrm{link}_\alpha \Delta := \{\tau \in \Delta \mid \sigma_\alpha \cap \tau = \emptyset,\ \sigma_\alpha \cup \tau \in \Delta\}$.
- Restriction to σ_α: $\Delta_\alpha := \{\tau \in \Delta \mid \tau \in \sigma_\alpha\}$.

Let $\Delta^\vee$ be the Alexander dual simplicial complex of Δ. Then, from the equality of complexes $\Delta^\vee_{\mathbf{1}-\alpha} = (\mathrm{link}_\alpha \Delta)^\vee$ and Alexander duality we get the isomorphisms

$$\widetilde{H}_{n-r-|\alpha|-1}(\mathrm{link}_\alpha \Delta; k) \cong \widetilde{H}^{r-2}(\Delta^\vee_{\mathbf{1}-\alpha}; k).$$

Moreover, we have to point out that the inclusion $\Delta^\vee_{\mathbf{1}-\alpha-\varepsilon_i} \subseteq \Delta^\vee_{\mathbf{1}-\alpha}$ induces the morphisms:

$$\widetilde{H}_{n-r-|\alpha-\varepsilon_i|-1}(\mathrm{link}_{\alpha-\varepsilon_i} \Delta; k) \longrightarrow \widetilde{H}_{n-r-|\alpha|-1}(\mathrm{link}_\alpha \Delta; k),$$

M. Hochster's result is then as follows:

Theorem *The graded Hilbert series of $H_{\mathfrak{m}}^r(R/I)$ is:*

$$H(H_{\mathfrak{m}}^r(R/I); \mathbf{x}) = \sum_{\sigma_\alpha \in \Delta} \dim_k \widetilde{H}^{r-|\alpha|-1}(\mathrm{link}_\alpha \Delta; k) \prod_{\alpha_i = 1} \frac{x_i^{-1}}{1 - x_i^{-1}}.$$

From M. Hochster's formula one may deduce that the multiplication by the variable x_i establishes an isomorphism between the pieces $H_{\mathfrak{m}}^r(R/I)_\beta$ and $H_{\mathfrak{m}}^r(R/I)_{\beta+\varepsilon_i}$ for all $\beta \in \mathbb{Z}^n$ such that $\beta_i \neq -1$, where $\varepsilon_1, \dots, \varepsilon_n$ is the natural basis of $\mathbb{Z}^n$. Notice then that, in order to determine the graded structure of this module, we only have to determine the multiplication by x_i on the pieces $H_{\mathfrak{m}}^r(R/I)_{-\alpha}$, $\alpha \in \{0,1\}^n$.

H. G. Gräbe [35], gave a topological interpretation of these multiplications by using the isomorphisms:

$$H^r_{\mathfrak{m}}(R/I)_\beta \cong \widetilde{H}^{r-|\alpha|-1}(\mathrm{link}_\alpha\Delta;k),\ \ \forall\beta\in\mathbb{Z}^n \text{ such that } \sigma_\alpha=\sup_-(\beta),$$

where $\sup_-(\beta) := \{x_i \mid \beta_i < 0\}$.

Theorem *For all $\alpha\in\{0,1\}^n$ such that $\sigma_\alpha\in\Delta$, the morphism of multiplication by the variable x_i:*

$$\cdot x_i : H^r_{\mathfrak{m}}(R/I)_{-\alpha} \longrightarrow H^r_{\mathfrak{m}}(R/I)_{-(\alpha-\varepsilon_i)}$$

corresponds to the morphism

$$\widetilde{H}^{r-|\alpha|-1}(\mathrm{link}_\alpha\Delta;k) \longrightarrow \widetilde{H}^{r-|\alpha-\varepsilon_i|-1}(\mathrm{link}_{\alpha-\varepsilon_i}\Delta;k),$$

induced by the inclusion $\Delta^\vee_{\mathbf{1}-\alpha-\varepsilon_i} \subseteq \Delta^\vee_{\mathbf{1}-\alpha}$.

Inspired by M. Hochster's formula, N. Terai [92] gave a description of the graded Hilbert series of the local cohomology modules $H^r_I(R)$, expressed in terms of the reduced simplicial homology of the links $\mathrm{link}_\alpha\Delta$ such that $\sigma_\alpha\in\Delta$, $\alpha\in\{0,1\}^n$.

Theorem *The graded Hilbert series of $H^r_I(R)$ is:*

$$H(H^r_I(R);\mathbf{x}) = \sum_{\alpha\in\{0,1\}^n} \dim_k \widetilde{H}_{n-r-|\alpha|-1}(\mathrm{link}_\alpha\Delta;k)\ \prod_{\alpha_i=0}\frac{x_i^{-1}}{1-x_i^{-1}}\ \prod_{\alpha_j=1}\frac{1}{1-x_j}.$$

From N. Terai's formula one also may deduce that the multiplication by the variable x_i establishes an isomorphism between the pieces $H^r_I(R)_\beta$ and $H^r_I(R)_{\beta+\varepsilon_i}$ for all $\beta\in\mathbb{Z}^n$ such that $\beta_i\neq -1$.

Independently, M. Mustaţă [72] has also described the pieces of the local cohomology modules $H^r_I(R)$ and, moreover, he has given a topological interpretation of the morphism of multiplication by x_i on the pieces $H^r_I(R)_{-\alpha}$, $\alpha\in\{0,1\}^n$. We have to point out that these results have been used for the computation of cohomology of coherent sheaves on toric varieties (see [22]).

Theorem *For all $\alpha\in\{0,1\}^n$ such that $\sigma_\alpha\in\Delta$, the morphism of multiplication by the variable x_i:*

$$\cdot x_i : H^r_I(R)_{-\alpha} \longrightarrow H^r_I(R)_{-(\alpha-\varepsilon_i)}$$

corresponds to the morphism

$$\widetilde{H}_{n-r-|\alpha|-1}(\mathrm{link}_\alpha \Delta; k) \longrightarrow \widetilde{H}_{n-r-|\alpha-\varepsilon_i|-1}(\mathrm{link}_{\alpha-\varepsilon_i}\Delta; k),$$

induced by the inclusion $\Delta^\vee_{\mathbf{1}-\alpha-\varepsilon_i} \subseteq \Delta^\vee_{\mathbf{1}-\alpha}$.

We remark that the formulas of M. Hochster and N. Terai are equivalent by using the Čech hull and Alexander duality (see [70]). The same happens with the formulas of H. G. Gräbe and M. Mustaţă.

Finally, K. Yanagawa [97] has introduced the category of straight modules, that are $\mathbb{Z}^n$-graded modules such that the multiplication by the variables x_i between their pieces satisfy certain conditions. In this framework, K. Yanagawa can study the local cohomology modules since, from the results of N. Terai and M. Mustaţă one may check that the shifted modules $H^r_I(R)(-1,\ldots,-1)$ are straight.

Algorithmic computation of local cohomology modules

Let k be a field of characteristic zero, $R = k[x_1,\ldots,x_n]$ the ring of polynomials over k in n variables and $\mathcal{D}$ be the corresponding ring of differential operators. Recently, it has been a great effort to provide an effective computation of local cohomology modules.

F. Barkats [3] gave an algorithm to compute a presentation of local cohomology modules $H^r_I(R)$ supported on monomial ideals $I \subseteq R$ by using the Taylor resolution of the quotient ring R/I. This algorithm was implemented for ideals contained in $R = k[x_1,\ldots,x_6]$.

By using the theory of Gröbner bases over the ring $\mathcal{D}$ we can find two different methods for computing local cohomology modules. The first one is due to U. Walther [93] and is based on the construction of the Čech complex of holonomic $\mathcal{D}$-modules. In particular, let $\mathfrak{m}$ be the homogeneous maximal ideal and $I \subseteq R$ be any ideal. Then, U. Walther determines the structure of the modules $H^r_I(R)$, $H^p_{\mathfrak{m}}(H^r_I(R))$ and compute the Lyubeznik numbers $\lambda_{p,i}(R/I)$. The second method is due to T. Oaku and N. Takayama [73]. It relies in their algorithm for computing the derived restriction modules of holonomic $\mathcal{D}$-modules.

Computations can be done in the computer algebra system Macaulay 2 [37] by using the package D-modules for Macaulay 2 [53].

Objectives

In the sequel, we are going to introduce the problems considered in this work.

Let R be a regular ring containing a field of characteristic zero and $I \subseteq R$ an ideal. Our intend is, following the path opened by G. Lyubeznik in [55], to make a deeper use of the theory of $\mathcal{D}$-modules in order to study the local cohomology modules $H_I^r(R)$. We are mainly interested on an effective description of the annihilation and finiteness properties of these modules.

The main tool we will use is the **characteristic cycle**. This is an invariant that one may associate to any holonomic $\mathcal{D}$-module M (e.g. local cohomology modules) which, in the cases we will consider in this work, is described as a sum

$$CC(M) = \sum m_i \ T^*_{X_i} X,$$

where $m_i \in \mathbb{Z}$, $X = \operatorname{Spec}(R)$ and $T^*_{X_i} X$ is the conormal bundle relative to the subvariety $X_i \subseteq X$. Notice that the information given by this invariant is a set of subvarieties $X_i \subseteq X$ and a set of multiplicities m_i.

The usefulness of the characteristic cycle in our study is reflected in the following examples:

• The support as R-module of a $\mathcal{D}$-module M is described by the subvarieties that appear in the characteristic cycle.

• The Lyubeznik numbers $\lambda_{p,i}(R/I)$ may be computed as the multiplicities of the characteristic cycle of the module $H^p_{\mathfrak{m}}(H^r_I(R))$.

More precisely, let $R = k[[x_1, \dots, x_n]]$ be the formal power series ring with coefficients in a field k of characteristic zero. Let $\mathfrak{m} = (x_1, \dots, x_n) \subseteq R$ be the maximal ideal and $I \subseteq R$ be any ideal. Then, from the results of [55], one may see that

$$H^p_{\mathfrak{m}}(H^r_I(R)) \cong \bigoplus H^n_{\mathfrak{m}}(R)^{\oplus \lambda_{p,i}(R/I)},$$

where $\lambda_{p,i}(R/I)$ are the Lyubeznik numbers. By computing the characteristic cycle, one may check that these invariants of the quotient ring R/I are nothing but the corresponding multiplicities.

Observe then that the characteristic cycles of the local cohomology modules $H^r_I(R)$ and $H^p_{\mathfrak{m}}(H^r_I(R))$ provide information for both the module and the quotient ring R/I as well.

Throughout this work, we will be interested in the interpretation of the information given by the characteristic cycle and make explicit computations. The main problems we want to treat are:

• **Problem 1**. Let I be an ideal contained in a regular local ring R. Then, we want to study the invariance of the multiplicities of the characteristic cycle of the local cohomology modules $H_I^r(R)$ with respect to the quotient ring R/I.

Let R be the polynomial ring or the formal power series ring with coefficients in a field of characteristic zero and $I \subseteq R$ be a squarefree monomial ideal. Recall that these ideals can be interpreted in the following ways:

· Stanley-Reisner ideal of a simplicial complex.

· Defining ideal of an arrangement of linear varieties.

Then, the problems we will consider in this case are:

• **Problem 2**. Explicit computation of the characteristic cycle of local cohomology modules.

From this computation we will turn our interest in the study of:

- **Problem 2.1** Study the support of local cohomology modules. In particular:
    · Annihilation of local cohomology modules.
    · Cohomological dimension.
    · Description of the support of local cohomology modules.
    · Krull dimension of local cohomology modules.
    · Artinianity of local cohomology modules.
- **Problem 2.2** Arithmetical properties of the quotient rings R/I. Namely, we want to determine the following properties:
    · Cohen-Macaulay property.
    · Buchsbaum property.
    · Gorenstein property.
    · The type of Cohen-Macaulay rings.

- **Problem 2.3** Interpretation of the multiplicities of the characteristic cycle of the local cohomology modules $H_I^r(R)$. In particular:

    - Study the topological and algebraic invariants of the Stanley-Reisner simplicial complexes associated to the rings R/I and $R/I^\vee$, where $I^\vee$ is the Alexander dual ideal of I.

    - Study the topological invariants of the complement of the arrangement of linear varieties defined by the ideal I.

- **Problem 2.4** Explicit computation of the Bass numbers of the local cohomology modules. From these computations we will consider:

    - Annihilation of Bass numbers.

    - Injective dimension of local cohomology modules.

    - Associated primes of local cohomology modules.

    - Small support of local cohomology modules.

Even though it provides many information, the characteristic cycle does not describe completely the structure of the local cohomology modules. This fact is reflected in the work of A. Galligo, M. Granger and Ph. Maisonobe [27], where a description of the category of regular holonomic $\mathcal{D}$-modules with support a normal crossing (e.g. local cohomology modules supported on squarefree monomial ideals), is given by using the Riemann-Hilbert correspondence. Then, we will also consider the following question:

• **Problem 3**. Study the structure of local cohomology modules by using the following points of view:

- Study the $\mathbb{Z}^n$-graded structure of local cohomology modules.

- Study the regular holonomic $\mathcal{D}$-modules with support a normal crossing and Riemann-Hilbert correspondence.

Conclusions

The contents of this thesis are the following:

• In **Chapter 1** we introduce the definitions and notation we will use throughout this work.

In a first section we give the definition of **local cohomology module** and we state some of the basic properties we will use. Finally, we introduce some of the tools that will allow us to compute these modules: The Čech complex, the long exact sequence of local cohomology, the Mayer-Vietoris sequence, the Brodmann's sequence and the Grothendieck's spectral sequence.

In a second section we are focused in the study of **squarefree monomial ideals**. A natural framework for these ideals is the category of $\mathbb{Z}^n$-graded modules. Is for this reason that we recall the notions of free and injective resolutions in this category. Recently, it has been a big effort in the study of these resolutions in order to make them treatable, so have found convenient to introduce the notions of cel.lular matrices, cel.lular resolutions and Čech hull.

To any squarefree monomial ideal one can associate a simplicial complex by means of the Stanley-Reisner correspondence. For this reason we review some topological notions, with an especial attention to Alexander duality. On the other side we can refer to any squarefree monomial ideal as the defining ideal of an arrangement of linear subvarieties. From this point of view, we recall Goresky-MacPherson's formula for the computation of the cohomology of the complement of these type of arrangements.

We also review the theory of **$\mathcal{D}$-modules**. We start giving the basic definitions of regular holonomic $\mathcal{D}$-modules since local cohomology modules are of this type. To these modules one may associate an invariant, the characteristic cycle, that allows us to compute the support of these modules. After describing some geometric properties of these characteristic cycles, we develop some examples and computations.

Finally, we introduce the notion of solution of a $\mathcal{D}$-module. We recall that the solutions of a regular holonomic $\mathcal{D}$-module are Nilsson class functions and that the solutions functor restricted to the category of regular holonomic $\mathcal{D}$-modules establishes a categorical equivalence with the category of perverse sheaves that is named the Riemann-Hilbert correspondence.

• In **Chapter 2** we prove that the multiplicities of the characteristic cycle of local cohomology modules are invariants of the corresponding quotient ring. In particular, these invariants generalize Lyubeznik numbers.

Given a field k of characteristic zero we consider the ring of formal power series $R = k[[x_1, \dots, x_n]]$, where $x_1, \dots, x_n$ are independent variables. Let $I \subseteq R$ be any ideal, $\mathfrak{p} \subseteq R$ a prime ideal and $\mathfrak{m} = (x_1, \dots, x_n)$ the maximal ideal. First we prove that Lyubeznik numbers $\lambda_{p,i}(R/I)$ are nothing but multiplicities of the characteristic cycle of the local cohomology modules $H^p_{\mathfrak{m}}(H^{n-i}_I(R))$. Following the path given in the proof of G. Lyubeznik, we prove that the following multiplicities are also invariants of the ring R/I:

- The multiplicities of the characteristic cycle of $H^{n-i}_I(R)$.
- The multiplicities of the characteristic cycle of $H^p_{\mathfrak{p}}(H^{n-i}_I(R))$.

Among these multiplicities one may found:

  - The Bass numbers $\mu_p(\mathfrak{p}, H^{n-i}_I(R))$.
  - The Lyubeznik numbers $\lambda_{p,i}(R/I) := \mu_p(\mathfrak{m}, H^{n-i}_I(R))$.

More precisely, in this chapter we prove the following results:

Theorem 2.2.2 *Let A be a local ring which admits a surjective ring homomorphism $\pi : R \longrightarrow A$, where $R = k[[x_1, \dots, x_n]]$ is the formal power series ring. Set $I = \ker \pi$ and let*

$$CC(H^{n-i}_I(R)) = \sum m_{i,\alpha}\ T^*_{X_\alpha} X,$$

be the characteristic cycle of the local cohomology modules $H^{n-i}_I(R)$. Then, the multiplicities $m_{i,\alpha}$ depends only on A, i and α but neither on R nor on π.

Collecting these multiplicities by the dimension of the corresponding irreducible varieties we obtain some other invariants. Even though these invariants are coarser, in some situations they will be better suited in order to make a precise description of the support of the local cohomology modules

Definition 2.2.4 *Let $I \subseteq R$ be an ideal. If $CC(H^{n-i}_I(R)) = \sum m_{i,\alpha}\ T^*_{X_\alpha} X$ is the characteristic cycle of the local cohomology modules $H^{n-i}_I(R)$ then we define:*

$$\gamma_{p,i}(R/I) := \{\sum m_{i,\alpha} \mid \dim X_\alpha = p\}.$$

These invariants have the same properties as Lyubeznik numbers. Namely:

Proposition 2.2.5 *Let* $d = \dim R/I$. *The invariants* $\gamma_{p,i}(R/I)$ *have the following properties:*

i) $\gamma_{p,i}(R/I) = 0$ if $i > d$.

ii) $\gamma_{p,i}(R/I) = 0$ if $p > i$.

iii) $\gamma_{d,d}(R/I) \neq 0$.

Then, we can collect these invariants in a triangular matrix that will be denoted by $\Gamma(R/I)$.

On the other side, for the multiplicities of the characteristic cycle of the local cohomology modules $H^p_{\mathfrak{p}}(H^{n-i}_I(R))$ we obtain the following result:

Theorem 2.2.6 *Let* A *be a local ring which admits a surjective ring homomorphism* $\pi : R \longrightarrow A$, *where* $R = k[[x_1, \dots, x_n]]$ *is the formal power series ring. Set* $I = \ker \pi$, *let* $\mathfrak{p} \subseteq A$ *be a prime ideal and let*

$$CC(H^p_{\mathfrak{p}}(H^{n-i}_I(R))) = \sum \lambda_{\mathfrak{p},p,i,\alpha} \; T^*_{X_\alpha} X,$$

be the characteristic cycle of the local cohomology modules $H^p_{\mathfrak{p}}(H^{n-i}_I(R))$. *Then, the multiplicities* $\lambda_{\mathfrak{p},p,i,\alpha}$ *depends only on* A, $\mathfrak{p}$, p, i *and* α *but neither on* R *nor on* π.

As a consequence of this result, we prove that the Bass numbers, in particular the Lyubeznik numbers, are invariants of R/I since they are multiplicities of the characteristic cycle. More precisely:

Proposition 2.2.8 *Let* $I \subseteq R$ *be an ideal,* $\mathfrak{p} \subseteq R$ *a prime ideal and*

$$CC(H^p_{\mathfrak{p}}(H^{n-i}_I(R))) = \sum \lambda_{\mathfrak{p},p,i,\alpha} \; T^*_{X_\alpha} X$$

be the characteristic cycle of the local cohomology module $H^p_{\mathfrak{p}}(H^{n-i}_I(R))$. *Then, the Bass numbers with respect to* $\mathfrak{p}$ *of* $H^{n-i}_I(R)$ *are*

$$\mu_p(\mathfrak{p}, H^{n-i}_I(R)) = \lambda_{\mathfrak{p},p,i,\alpha_{\mathfrak{p}}},$$

where $X_{\alpha_{\mathfrak{p}}}$ *is the subvariety of* $X = \operatorname{Spec}(R)$ *defined by* $\mathfrak{p}$.

• Let k be a field of characteristic zero. In **Chapter 3** we consider any of the following rings:

- $R = k[[x_1, \ldots, x_n]]$ the formal power series ring.
- $R = k\{x_1, \ldots, x_n\}$ the convergent power series ring.
- $R = k[x_1, \ldots, x_n]$ the polynomial ring.

The main result of this chapter is the computation of the characteristic cycle of the local cohomology modules $H_I^r(R)$ with support a monomial ideal $I \subseteq R$. First, we have to find among the tools introduced in Chapter 1, which one will be better suited for our purposes.

A first approach is to use the **Čech complex**. Let $I = (\mathbf{x}^{\alpha_1}, \ldots, \mathbf{x}^{\alpha_s})$ be a squarefree monomial ideal. Consider the Čech complex

$$\check{C}_I^\bullet : \quad 0 \longrightarrow R \xrightarrow{d_0} \bigoplus_{1 \leq i_1 \leq r} R[\frac{1}{\mathbf{x}^{\alpha_{i_1}}}] \xrightarrow{d_1} \cdots \xrightarrow{d_{s-1}} R[\frac{1}{\mathbf{x}^{\alpha_1} \cdots \mathbf{x}^{\alpha_s}}] \longrightarrow 0$$

The characteristic cycle of the localizations $R_r := \bigoplus R[\frac{1}{\mathbf{x}^{\alpha_{i_1}} \cdots \mathbf{x}^{\alpha_{i_r}}}]$ have been computed in Chapter 1. Then, since the cohomology of the Čech complex gives the local cohomology modules, i.e. $H_I^r(R) = H^r(\check{C}_I^\bullet)$, we get $CC(H_I^r(R)) = CC(\operatorname{Ker} d_r) - CC(\operatorname{Im} d_{r-1})$ by using the additivity of the characteristic cycle with respect to exact sequences.

When the quotient ring R/I is Cohen-Macaulay, due to the fact that the cohomological dimension equals the height of the ideal I, there is only a non vanishing local cohomology module (**Proposition** 3.1.1). Then, its characteristic cycle is easy to compute.

Proposition 3.1.2 *Let $I \subseteq R$ be an ideal of height h generated by squarefree monomials. If R/I is Cohen-Macaulay then:*

$$CC(H_I^h(R)) = CC(R_h) - CC(R_{h+1}) + \cdots + (-1)^{s-h} CC(R_s)$$
$$-CC(R_{h-1}) + \cdots + (-1)^h CC(R_0).$$

Whether the quotient ring R/I is not Cohen-Macaulay, the computation of the characteristic cycles $CC(\operatorname{Ker} d_r)$ and $CC(\operatorname{Im} d_r)$ is more involved due to the fact that the characteristic variety of the localizations $R_r := \bigoplus R[\frac{1}{\mathbf{x}^{\alpha_{i_1}} \cdots \mathbf{x}^{\alpha_{i_r}}}]$ are not irreducible.

In the general case we will use the **Mayer-Vietoris exact sequence**. Basically, the process we will use is the following: Consider a presentation $I = U \cap V$ of the ideal I as the intersection of two simple ideals. Then, if we split the sequence

$$\cdots \longrightarrow H^r_{U+V}(R) \longrightarrow H^r_U(R) \oplus H^r_V(R) \longrightarrow H^r_I(R) \longrightarrow H^{r+1}_{U+V}(R) \longrightarrow \cdots ,$$

into short exact sequences of kernels and cokernels

$$0 \longrightarrow B_r \longrightarrow H^r_U(R) \oplus H^r_V(R) \longrightarrow C_r \longrightarrow 0$$

$$0 \longrightarrow C_r \longrightarrow H^r_I(R) \longrightarrow A_{r+1} \longrightarrow 0$$

$$0 \longrightarrow A_{r+1} \longrightarrow H^{r+1}_{U+V}(R) \longrightarrow B_{r+1} \longrightarrow 0,$$

we get that the characteristic cycle of the local cohomology module $H^r_I(R)$ is

$$CC(H^r_I(R)) = CC(C_r) + CC(A_{r+1}) =$$

$$= (CC(H^r_U(R) \oplus H^r_V(R)) - CC(B_r)) + (CC(H^{r+1}_{U+V}(R)) - CC(B_{r+1})).$$

Note that we have reduced our problem to the computation of the characteristic cycles $CC(H^r_U(R) \oplus H^r_V(R))$, $CC(H^{r+1}_{U+V}(R))$ and $CC(B_r)$ $\forall r$.

In order to compute the characteristic cycle of the local cohomology modules $H^r_U(R)$, $H^r_V(R))$ and $H^{r+1}_{U+V}(R)$, we have to consider a decomposition of the ideals U, V and $U+V$. Then, we split the corresponding Mayer-Vietoris sequences in the same way we did before. Although we are working with some different sequences at the same time, the ideals U, V and $U+V$ we are considering are simpler than ideal I. We will repeat this process until we get a situation where one may compute the characteristic cycles $CC(H^r_U(R) \oplus H^r_V(R))$, $CC(H^{r+1}_{U+V}(R))$ and $CC(B_r)$ $\forall r$ that appear in all the Mayer-Vietoris sequences involved.

In this chapter we give an inductive process that allows to choose the ideals U and V in a systematical way, i.e. independently of the ideal's complexity, by taking advantage of the good properties that satisfy the minimal primary decomposition $I = I_{\alpha_1} \cap \cdots \cap I_{\alpha_m}$ of a squarefree monomial ideal. By means of this process, we reach a situation where the local cohomology modules $H^r_U(R)$,

$H_V^r(R))$ and $H_{U+V}^{r+1}(R)$ that appear in the different Mayer-Vietoris sequences are exactly one of the following $2^m - 1$ modules:

$$H^r_{I_{\alpha_{i_1}}+\cdots+I_{\alpha_{i_j}}}(R), \quad 1 \leq i_1 < \cdots < i_j \leq m, \quad j = 1, \ldots, m.$$

In order to organize this information, we introduce the partially ordered set (poset) $\mathcal{I} = \{\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_m\}$, formed by all the sums of face ideals that appear in the minimal primary decomposition of I, i.e. $\mathcal{I}_j := \{I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} \mid 1 \leq i_1 < \cdots < i_j \leq m\}$.

Roughly speaking, we have broken the local cohomology module $H_I^r(R)$ into smaller pieces, the modules $H^r_{I_{\alpha_{i_1}}+\cdots+I_{\alpha_{i_j}}}(R)$, that are labeled by the poset $\mathcal{I}$. These modules will be denoted initial pieces. The characteristic cycle of these pieces may be computed since $R/I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}}$ is Cohen-Macaulay. Then, it only remains to compute the characteristic cycle of the modules B_r that appear in the different Mayer-Vietoris sequences.

From the fact that the characteristic varieties of the initial pieces are irreducible, we prove that $CC(B_r)$ is nothing but the sum of the characteristic cycles of local cohomology modules supported on sums of face ideals in the minimal primary decomposition of I satisfying the following property:

Definition 3.2.9 *We say that* $I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} \in \mathcal{I}_j$ *and* $I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} + I_{\alpha_{i_{j+1}}} \in \mathcal{I}_{j+1}$ *are paired if*

$$I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} = I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} + I_{\alpha_{i_{j+1}}}$$

Finally, once we control all the pieces of the Mayer-Vietoris sequences as well the kernels and cokernels, we use the additivity of the characteristic cycle with respect to exact sequences to compute $CC(H_I^r(R))$. More precisely, $CC(H_I^r(R))$ is the sum of the characteristic cycle of the initial pieces labeled by a subposet $\mathcal{P} \subseteq \mathcal{I}$, that it is calculated by means of an algorithm and is formed by the sums of face ideals $I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}}$ such that are not paired and whose height is $r + (j-1)$. Namely, if we consider the sets

$$\mathcal{P}_{j,r} := \{I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} \in \mathcal{P}_j \mid \operatorname{ht}(I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}}) = r + (j-1)\}$$

then we have:

Theorem 3.2.11 *Let $I \subseteq R$ be an ideal generated by squarefree monomials and let $I = I_{\alpha_1} \cap \cdots \cap I_{\alpha_m}$ be its minimal primary decomposition. Then :*

$$CC(H^r_I(R)) = \sum_{I_{\alpha_i} \in \mathcal{P}_{1,r}} CC(H^r_{I_{\alpha_i}}(R)) + \sum_{I_{\alpha_{i_1}} + I_{\alpha_{i_2}} \in \mathcal{P}_{2,r}} CC(H^{r+1}_{I_{\alpha_{i_1}} + I_{\alpha_{i_2}}}(R)) + \cdots +$$
$$+ \sum_{I_{\alpha_1} + \cdots + I_{\alpha_m} \in \mathcal{P}_{m,r}} CC(H^{r+(m-1)}_{I_{\alpha_1} + \cdots + I_{\alpha_m}}(R)).$$

The information we obtain from the characteristic cycle of the local cohomology modules will be treated from different points of view. First, the varieties that appear in the characteristic cycle describe the support of the module $H^r_I(R)$. On the other side, the multiplicities allow us to describe some properties of the quotient ring R/I since they are invariants.

Annhilation and support of local cohomology modules: A first reading of the formula for the characteristic cycle of the modules $H^r_I(R)$ allows us to obtain the following results:

· Annihilation of local cohomology modules. **Proposition 3.3.3**

· Cohomological dimension. **Corollary 3.3.5**

· Support of local cohomology modules. **Proposition 3.3.8**

· Krull dimension of local cohomology modules. **Corollary 3.3.9**

· Artinianity of local cohomology modules. **Corollary 3.3.10**

We have to point out that these results are expressed in terms of the ideals in the minimal primary decomposition of the ideal I.

Arithmetical properties of the ring R/I: The multiplicities of the characteristic cycle of local cohomology modules allow us to give some criteria to study the following properties:

· Cohen-Macaulay property. **Propositions 3.3.11** and **3.3.12**

· Buchsbaum property. **Propositions 3.3.13** and **3.3.14**

· Gorenstein property. **Propositions 3.3.15 and 3.3.19**

· The type of Cohen-Macaulay rings. **Proposition 3.3.18**

In the literature one may find some criteria to determine these properties, that use the topological properties of the Stanley-Reisner simplicial complex associated to the ring R/I, see [86] for more details. Our criteria are based on the annihilation of certain multiplicities in the Cohen-Macaulay and Buchsbaum case. In the Gorenstein case we ask whether certain multiplicities are exactly 1. Anyway, they are expressed in terms of the ideals in the minimal primary decomposition of the ideal I.

Once we have computed the multiplicities of the characteristic cycle of the local cohomology modules we have seen that they are useful to describe properties of the modules $H_I^r(R)$ and the ring R/I as well. Now, we want to give an interpretation of these invariants and compare them with other known invariants.

Combinatorics of the Stanley-Reisner rings and multiplicities: From the Stanley-Reisner correspondence, one may associate a simplicial complex Δ to any squarefree monomial ideal $I \subseteq R$. In this section we give a description of the topological invariants described by the f-vector and the h-vector of Δ in terms of the invariants

$$\mathcal{B}_j := \sum_{i=0}^{d-j} (-1)^i \gamma_{j,j+i}(R/I),$$

i.e. the alternating sum of invariants $\gamma_{p,i}(R/I)$ in a row of the matrix $\Gamma(R/I)$.

Proposition 3.3.21 *Let $I \subseteq R$ be a squarefree monomial ideal. The f-vector and the h-vector of the corresponding simplicial complex Δ are described as follows:*

i) $f_k = \sum_{j=k+1}^{d} \binom{j}{k+1} \mathcal{B}_j.$

ii) $h_k = (-1)^k \sum_{j=0}^{d-k} \binom{d-j}{k} \mathcal{B}_j.$

By using these descriptions we also determine some invariants as the Euler characteristic of Δ or the Hilbert series of R/I (**Corollary 3.3.23**). In general, the invariants $\gamma_{p,i}(R/I)$ are finer than the f-vector and the h-vector. They are equivalent whether R/I is Cohen-Macaulay (**Corollary 3.3.22**).

A different interpretation of the multiplicities may be given by means of Alexander duality.

Betti numbers and multiplicities: Let $I^\vee \subseteq R$ be the Alexander dual ideal of a squarefree monomial ideal $I \subseteq R$. The Taylor complex $\mathbb{T}_\bullet(I^\vee)$ is a cellular free resolution $I^\vee$ with support on a simplicial complex whose vertices are labeled by the poset $\mathcal{I}$ that we used in order to label the initial pieces of the local cohomology modules $H_I^r(R)$.

Recall that the characteristic cycle of these modules is described by means of the poset $\mathcal{P} \subseteq \mathcal{I}$ that we compute by using an algorithm. From the correspondence given by Alexander duality, this algorithm may be interpreted as an algorithm that allows to minimize the free resolution of $I^\vee$ given by the Taylor complex. Then, we can describe the multiplicities of the characteristic cycle $CC(H_I^r(R))$ in terms of the Betti numbers of $I^\vee$.

Proposition 3.3.25 *Let $I^\vee \subseteq R$ be Alexander dual ideal of a squarefree monomial ideal $I \subseteq R$. Then we have:*

$$\beta_{j,\alpha}(I^\vee) = m_{n-|\alpha|+j,\,\alpha}(R/I).$$

In particular, the multiplicities $m_{i,\alpha}$ of a fixed module $H_I^{n-i}(R)$ describe the Betti numbers of the $(n-i)$-lineal strand of $I^\vee$. Whether R/I is Cohen-Macaulay there is only a non vanishing local cohomology, and we may recover the following result of J. A. Eagon and V. Reiner [20].

Corollary 3.3.28 *Let $I^\vee \subseteq R$ be the Alexander dual ideal of a squarefree monomial ideal $I \subseteq R$. Then, R/I is Cohen-Macaulay if and only if $I^\vee$ has a linear free resolution.*

A generalization of that result expressed in terms of the projective dimension of R/I and the Castelnuovo-Mumford regularity of $I^\vee$ is given by N. Terai in [90]. We can give a different approach by using the previous results.

Corollary 3.3.29 *Let $I^\vee \subseteq R$ be the Alexander dual ideal of a squarefree monomial ideal $I \subseteq R$. Then we have:*

$$\mathrm{pd}(R/I) = \mathrm{reg}\,(I^\vee).$$

• Let k be a field of characteristic zero. In **Chapter 4** we consider any of the following rings:

- $R = k[[x_1, \dots, x_n]]$ the formal power series ring.
- $R = k\{x_1, \dots, x_n\}$ the convergent power series ring.
- $R = k[x_1, \dots, x_n]$ the polynomial ring.

The main result of this chapter is the computation of the characteristic cycle of the local cohomology modules $H^p_{\mathfrak{p}_\gamma}(H^r_I(R))$ where $I \subseteq R$ is a monomial ideal and $\mathfrak{p}_\gamma \subseteq R$ is a face ideal. The techniques we will use throughout this chapter are a natural continuation of those used in the previous chapter. Namely, consider the short exact sequence $0 \longrightarrow C_r \longrightarrow H^r_I(R) \longrightarrow A_{r+1} \longrightarrow 0$, obtained by splitting the Mayer-Vietoris sequence

$$\cdots \longrightarrow H^r_{U+V}(R) \longrightarrow H^r_U(R) \oplus H^r_V(R) \longrightarrow H^r_I(R) \longrightarrow H^{r+1}_{U+V}(R) \longrightarrow \cdots .$$

Then, if we split the long exact sequence of local cohomology

$$\cdots \longrightarrow H^p_{\mathfrak{p}_\gamma}(C_r) \longrightarrow H^p_{\mathfrak{p}_\gamma}(H^r_I(R)) \longrightarrow H^p_{\mathfrak{p}_\gamma}(A_{r+1}) \longrightarrow H^{p+1}_{\mathfrak{p}_\gamma}(C_r) \longrightarrow \cdots ,$$

into short exact sequences of kernels and cokernels

$$0 \longrightarrow Z_{p-1} \longrightarrow H^p_{\mathfrak{p}_\gamma}(C_r) \longrightarrow X_p \longrightarrow 0$$

$$0 \longrightarrow X_p \longrightarrow H^p_{\mathfrak{p}_\gamma}(H^r_I(R)) \longrightarrow Y_p \longrightarrow 0$$

$$0 \longrightarrow Y_p \longrightarrow H^p_{\mathfrak{p}_\gamma}(A_{r+1}) \longrightarrow Z_p \longrightarrow 0$$

we get that the characteristic cycle of the local cohomology module $H^p_{\mathfrak{p}_\gamma}(H^r_I(R))$ is

$$CC(H^p_{\mathfrak{p}_\gamma}(H^r_I(R))) = CC(X_p) + CC(Y_p) =$$
$$= (CC(H^p_{\mathfrak{p}_\gamma}(C_r)) - CC(Z_{p-1})) + (CC(H^p_{\mathfrak{p}_\gamma}(A_{r+1})) - CC(Z_p)).$$

Applying the long exact sequence of local cohomology to the sequences we have obtained in the process described in the previous chapter, we divide the local cohomology module $H^p_{\mathfrak{p}_\gamma}(H^r_I(R))$ into smaller pieces, the modules $H^p_{\mathfrak{p}_\gamma}(H^r_{I_{\alpha_{i_1}}+\cdots+I_{\alpha_{i_j}}}(R))$ that are labeled by the poset $\mathcal{P}$ due to Theorem 3.2.11.

The characteristic cycle of these pieces may be computed by means of the Grothendieck's spectral sequence since $R/I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}}$ is Cohen-Macaulay (**Proposition 4.1.1**). More precisely,

$$CC(H^p_{\mathfrak{p}_\gamma}(H^r_{I_{\alpha_{i_1}}+\cdots+I_{\alpha_{i_j}}}(R)) = CC(H^{p+r}_{\mathfrak{p}_\gamma+I_{\alpha_{i_1}}+\cdots+I_{\alpha_{i_j}}}(R)).$$

In order to organize the information given by these pieces we define the sets $\mathcal{P}_{\gamma,j,\alpha} := \{I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} \in \mathcal{P}_j \mid \mathfrak{p}_\gamma + I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} = I_\alpha\}$.

Notice that it only remains to compute the characteristic cycle of all the modules Z_p that appear in the different long exact sequences of local cohomology. The characteristic variety of these initial pieces is also irreducible, so we prove now that the characteristiccycle $CC(Z_p)$ is nothing but the sum of characteristic cycles of the initial pieces whose corresponding sums of face ideals that appear in the minimal primary decomposition of I satisfy the following property:

For any face ideal $I_\alpha \subseteq R$ we define the subsets

$$\mathcal{P}_{\gamma,j,\alpha} := \{I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} \in \mathcal{P}_j \mid \mathfrak{p}_\gamma + I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} = I_\alpha\}.$$

Definition 4.2.8 *We say that* $I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} \in \mathcal{P}_{\gamma,j,\alpha}$ *and* $I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} + I_{\alpha_{i_{j+1}}} \in \mathcal{P}_{\gamma,j+1,\alpha}$ *are almost paired if*

$$\mathrm{ht}\,(I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}}) + 1 = \mathrm{ht}\, I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} + I_{\alpha_{i_{j+1}}}$$

Finally, once we control all the pieces in the long exact sequences of local cohomology as well the kernels and cokernels, we use the additivity of the characteristic cycle with respect to exact sequences in order to compute $CC(H^p_{\mathfrak{p}_\gamma}(H^r_I(R)))$. More precisely, $CC(H^p_{\mathfrak{p}_\gamma}(H^r_I(R)))$ is the sum of the characteristic cycle of the pieces labeled by a poset $\mathcal{Q} \subseteq \mathcal{P}$, that it is calculated by means of an algorithm and is formed by the sums of face ideals $I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}}$ that are not almost paired and whose height is $r + (j-1)$. Namely, if we consider the sets

$$\mathcal{Q}_{\gamma,j,r,\alpha} := \{I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} \in \mathcal{Q}_{\gamma,j,\alpha} \mid \mathrm{ht}\,(I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}}) = r + (j-1)\},$$

then we have:

Theorem 4.2.9 *Let $I = I_{\alpha_1} \cap \cdots \cap I_{\alpha_m}$ be the minimal primary decomposition of a squarefree monomial ideal $I \subseteq R$ and $\mathfrak{p}_\gamma \subseteq R$ a face ideal. Then:*

$$CC(H^p_{\mathfrak{p}_\gamma}(H^r_I(R))) = \sum_{\alpha \in \{0,1\}^n} \lambda_{\gamma,p,n-r,\alpha} \; CC(H^{|\alpha|}_{I_\alpha}(R)),$$

where $\lambda_{\gamma,p,n-r,\alpha} = \sharp\, \mathcal{Q}_{\gamma,j,r,\alpha}$ such that $|\alpha| = p + (r + (j-1))$.

Whether $\mathfrak{p}_\gamma = \mathfrak{m}$ is the homogeneous maximal ideal we have the equality of sets $\mathcal{P}_{j,r} = \mathcal{P}_{\alpha_\mathfrak{m},j,r,\alpha_\mathfrak{m}}$ $\forall j$, $\forall r$. Applying the algorithm of cancellation of almost pairs and ordering by the number of summands and the height, we obtain the sets $\mathcal{Q}_{j,r} = \mathcal{Q}_{\alpha_\mathfrak{m},j,r,\alpha_\mathfrak{m}}$. Then we get the following result:

Corollary 4.2.12 *Let $I = I_{\alpha_1} \cap \cdots \cap I_{\alpha_m}$ be the minimal primary decomposition of a squarefree monomial ideal $I \subseteq R$ and $\mathfrak{m} \subseteq R$ be the homogeneous maximal ideal. Then:*

$$CC(H^p_{\mathfrak{m}}(H^r_I(R))) = \lambda_{p,n-r} \; T^*_{X_{\alpha_\mathfrak{m}}} X,$$

where $\lambda_{p,n-r} = \sharp \mathcal{Q}_{j,r}$ such that $n = p + (r + (j-1))$.

Among the information given by the characteristic cycle of these local cohomology modules we will focus on the multiplicities. More precisely, we will turn our attention to the invariants of the ring R/I described by the Bass numbers of the modules $H^r_I(R)$.

Bass numbers and local cohomology modules: Let $I \subseteq R$ be an ideal generated by squarefree monomials, $\mathfrak{p}_\gamma \subseteq R$ a face ideal and $\mathfrak{m} \subseteq R$ the homogeneous maximal ideal. Recall that in Chapter 1 we have seen that the Bass numbers may be computed in the following way:

Proposition 4.3.3 *Let $I \subseteq R$ be an ideal generated by squarefree monomials and $\mathfrak{p}_\gamma \subseteq R$ be a face ideal. If*

$$CC(H^p_{\mathfrak{p}_\gamma}(H^{n-i}_I(R))) = \sum \lambda_{\gamma,p,i,\alpha} \; T^*_{X_\alpha} X,$$

is the characteristic cycle of the local cohomology module $H^p_{\mathfrak{p}_\gamma}(H^{n-i}_I(R))$ then

$$\mu_p(\mathfrak{p}_\gamma, H^{n-i}_I(R)) = \lambda_{\gamma,p,i,\gamma}.$$

In particular, the Lyubeznik numbers are:

Corollary 4.3.5 *Let $I \subseteq R$ be an ideal generated by squarefree monomials and $\mathfrak{m} \subseteq R$ be the homogeneous maximal ideal. If*

$$CC(H^p_{\mathfrak{m}}(H^{n-i}_I(R))) = \lambda_{p,i}\ T^*_{X_{\alpha_{\mathfrak{m}}}} X,$$

is the characteristic cycle of the local cohomology module $H^p_{\mathfrak{m}}(H^{n-i}_I(R))$ then

$$\mu_p(\mathfrak{m}, H^{n-i}_I(R)) = \lambda_{p,i}.$$

Once they are computed, we use the Bass numbers in order to describe more accurately the prime ideals that appear in the support of the modules $H^r_I(R)$ studied in the previous chapter. In particular we obtain the following results:

· Annihilation of Bass numbers. **Proposition 4.3.11**

· Injective dimension of local cohomology modules. **Corollary 4.3.12**

· Associated primes of local cohomology modules. **Proposition 4.3.14**

· Small suport of local cohomology modules. **Proposition 4.3.16**

Once again, we point out that the results are expressed in terms of the face ideals in the minimal primary decomposition of the ideal I. To illustrate these computations we give some examples of modules $H^r_I(R)$ satisfying:

· $\mathrm{id}_R H^r_I(R) = \dim_R H^r_I(R)$.

· $H^r_I(R)$ has no embedded primes.

· $\mathrm{Min}_R(H^r_I(R)) = \mathrm{Ass}_R(H^r_I(R))$.

· $\mathrm{supp}_R(H^r_I(R)) = \mathrm{Supp}_R(H^r_I(R))$.

We also give examples of modules $H^r_I(R)$ satisfying:

· $\mathrm{id}_R H^r_I(R) < \dim_R H^r_I(R)$.

· $H^r_I(R)$ has embedded primes.

· $\mathrm{Min}_R(H^r_I(R)) \subsetneq \mathrm{Ass}_R(H^r_I(R))$.

· $\mathrm{supp}_R(H^r_I(R)) \subsetneq \mathrm{Supp}_R(H^r_I(R))$.

• Let $\mathbb{A}_k^n$ denote the affine space of dimension n over a field k, let $X \subset \mathbb{A}_k^n$ be an arrangement of linear subvarieties. Set $R = k[x_1, \ldots, x_n]$ and let $I \subset R$ denote an ideal which defines X. In **Chapter 5**, we study the local cohomology modules $H_I^i(R)$ with special regard of the case where the ideal I is generated by monomials. However, in this chapter we will pay more attention to the structure of these modules instead of their numerical invariants.

Even though the tools we will use are independent of the characteristic of the field k, whether $\mathrm{char}(k) = 0$ we will keep in mind the structure as $\mathcal{D}$-module of the modules $H_I^r(R)$. On the other side, if $\mathrm{char}(k) > 0$ we will use the notion of F-module introduced by G. Lyubeznik in [56, Definition 1.1].

We have to point out that an arrangement of linear varieties X determine a poset $P(X)$ formed by the intersections of the irreducible components of X and the order given by the inclusion. For example, if X is defined by a squarefree monomial ideal $I \subseteq R$, $P(X)$ is nothing but the poset $\mathcal{I}$ defined in Chapter 3 identifying the sums of face ideals in the minimal primary decomposition of I whether they describe the same ideal.

First of all, in an analogous way to the construction of the Mayer-Vietoris spectral sequences for singular cohomology and ℓ-adic cohomology introduced by A. Björner and T. Ekedahl [11], we prove the existence of a **Mayer–Vietoris spectral sequence** for local cohomology:

$$E_2^{-i,j} = \varinjlim_{P(X)}^{(i)} H_{I_p}^j(R) \Rightarrow H_I^{j-i}(R)$$

where p is an element of the poset $P(X)$, I_p is the defining (radical) ideal of the irreducible variety corresponding to p, and $\varinjlim_{P(X)}^{(i)}$ is the i-th left derived functor of the direct limit functor in the category of direct systems indexed by the poset $P(X)$.

Studying in detail this spectral sequence, we observe that the E_2-term is defined by the reduced homology of the simplicial complex associated to the poset $P(X)$ which has as vertices the elements of $P(X)$ and where a set of vertices $p_0, \ldots, p_r$ determines a r-dimensional simplex if $p_0 < \cdots < p_r$.

Proposition 5.1.4 *Let $X \subset \mathbb{A}_k^n$ be an arrangement of linear varieties. Let $K(> p)$ be the simplicial complex attached to the subposet $\{q \in P(X) \mid q > p\}$ of $P(X)$. Then, there are R-module isomorphisms*

$$\varinjlim_{P(X)}^{(i)} H_{I_p}^j(R) \simeq \bigoplus_{h(p)=j} [\, H_{I_p}^j(R) \otimes_k \tilde{H}_{i-1}(K(> p); k) \,],$$

where $\tilde{H}(-;k)$ denotes reduced simplicial homology.

We agree that the reduced homology with coefficients in k of the empty simplicial complex is k in degree -1 and zero otherwise. Moreover, if $\mathrm{char}(k) = 0$ they are isomorphisms of $\mathcal{D}$–modules and if $\mathrm{char}(k) > 0$ they are isomorphisms of F–modules.

The main result of this section is the degeneration of this spectral sequence in the E_2-term. The main ingredient of the proof is the Matlis–Gabriel structure theorem on the injective modules. This contrasts with the fact that the proof of the degeneration of the Mayer-Vietoris spectral sequence for ℓ-adic or singular cohomology relies on the strictness of Deligne's weight filtration.

Theorem 5.1.6 *Let $X \subset \mathbb{A}^n_k$ be an arrangement of linear varieties defined by an ideal $I \subset R$. Then, the Mayer–Vietoris spectral sequence*

$$E_2^{-i,j} = \varinjlim\nolimits^{(i)}_{P(X)} H^j_{I_p}(R) \Rightarrow H^{j-i}_I(R)$$

degenerates at the E_2–term.

The degeneration of the Mayer–Vietoris spectral sequence provides a filtration of the local cohomology modules, where the successive quotients are given by the E_2-term.

Corollary 5.1.7 *Let $X \subset \mathbb{A}^n_k$ be an arrangement of linear varieties defined by an ideal $I \subset R$. Then, for all $r \geq 0$ there is a filtration $\{F^r_j\}_{r\leq j\leq n}$ of $H^r_I(R)$ by R–submodules such that*

$$F^r_j/F^r_{j-1} \cong \bigoplus\nolimits_{h(p)=j} [\, H^j_{I_p}(R) \otimes_k \tilde{H}_{h(p)-r-1}(K(> p);k)\,].$$

Moreover, if char$(k) = 0$ it is a filtration by holonomic $\mathcal{D}$–modules and if char$(k) > 0$ it is a filtration by F-modules.

For any $0 \leq j \leq n$, we have an exact sequence:

$$(s_j): \qquad 0 \to F_{j-1} \to F_j \to F_j/F_{j-1} \to 0,$$

that defines an element of ${}^*\mathrm{Ext}^1_R(F_j/F_{j-1}, F_{j-1})$. In general, not all the **extension problems** associated to this filtration have a trivial solution. This is a major difference between the case we consider here and the cases considered by Björner and Ekedahl. Namely, in the analogous situation for the ℓ-adic cohomology of an arrangement defined over a finite field the extensions appearing are trivial not only as extensions of $\mathbb{Q}_\ell$-vector spaces but also as Galois representations, and for the singular cohomology of a complex arrangement the extensions appearing are trivial as extensions of mixed Hodge structures (cf. [11, pg. 179]).

Whether k is a field of characteristic zero, we can compute the characteristic cycle of the modules $H^r_I(R)$ from the short exact sequences determined by the filtration and the additivity of the characteristic cycle with respect to exact sequences. In particular, in the case of squarefree monomial ideals, we give a different approach to the formula obtained in Theorem 3.2.11.

Corollary 5.1.9 *Let k be a field of characteristic zero and I be the defining ideal of an arrangement $X \subseteq \mathbb{A}^n_k$ of linear varieties. Then, the characteristic cycle of the holonomic $\mathcal{D}$-module $H^r_I(R)$ is*

$$CC(H^r_I(R)) = \sum m_{n-r,p}\, T^*_{X_p}\mathbb{A}^n_k\,,$$

where $m_{n-r,p} = \dim_k \tilde{H}_{h(p)-r-1}(K(> p); k)$ and $T^*_{X_p}\mathbb{A}^n_k$ denotes the relative conormal subspace of $T^*\mathbb{A}^n_k$ attached to X_p.

Whether k is the field of real or complex numbers, these characteristic cycles determine the Betti numbers of the complement of the arrangement X in $\mathbb{A}^n_k$.

Corollary 5.1.10 *With the above notations, we have:*

If $k = \mathbb{R}$ is the field of real numbers, the Betti numbers of the complement of the arrangement X in $\mathbb{A}^n_{\mathbb{R}}$ can be computed in terms of the multiplicities $\{m_{n-r,p}\}$ as

$$\dim_{\mathbb{Q}} \tilde{H}_r(\mathbb{A}^n_{\mathbb{R}} - X; \mathbb{Q}) = \sum_p m_{n-(r+1),\,p}.$$

If $k = \mathbb{C}$ is the field of complex numbers, then one has

$$\dim_{\mathbb{Q}} \tilde{H}_r(\mathbb{A}^n_{\mathbb{C}} - X; \mathbb{Q}) = \sum_p m_{n-(r+1-h(p)),\,p}.$$

This formula follows from a theorem of Goresky–MacPherson ([34, III.1.3. Theorem A]). We have to point out that we give an algebraic approach to these Betti numbers by using local cohomology while, in the result of Goresky–MacPherson, the Betti numbers are dimensions of certain Morse groups.

Once we have described the consequences of the filtration of local cohomology modules we will give a solution to the extension problems in the case of arrangements of linear varieties defined by squarefree monomial ideals. More precisely, we will solve these extension problems in the framework of the ε-**straight** modules, which is a slight variation of a category introduced by K. Yanagawa in [97]. These modules are characterized by the following property:

Proposition 5.2.4 *A $\mathbb{Z}^n$-graded R-module M is ε-straight if and only if there are integers $m_\alpha \geq 0$, $\alpha \in \{0,1\}^n$ and an increasing filtration $\{F_j\}_{0\leq j\leq n}$ of M by graded submodules such that for all $1 \leq j \leq n$ one has graded isomorphisms*

$$F_j/F_{j-1} \simeq \bigoplus_{\substack{\alpha\in\{0,1\}^n \\ |\alpha|=j}} (H^{|\alpha|}_{\mathfrak{p}_\alpha}(R))^{\oplus m_\alpha}.$$

The multiplicities m_α that appear in the filtration of a ε-straight module have the following interpretation:

Proposition 5.2.5 *Let M be a ε-straight module with an increasing filtration as in Proposition 5.2.4. Then:*

$$m_\alpha = \dim_k M_{-\alpha},$$

i.e. the integer m_α, $\alpha \in \{0,1\}^n$, is the dimension of the piece of M of degree $-\alpha$.

Local cohomology modules supported on squarefree monomial ideals have a natural structure of ε-straight module by means of Corollary 5.1.7. We have to point out that, from the results of M. Mustaţă [72], the description of the multiplicities of the filtration given by Corollary 5.1.7 and the previous proposition we obtain:

Corollary 5.2.6 *Let $I^\vee \subseteq R$ be Alexander dual ideal of a squarefree monomial ideal $I \subseteq R$. Then we have:*

$$\beta_{i,\alpha}(I^\vee) = m_{n-|\alpha|+i,\,\alpha} = \dim_k \tilde{H}_{i-1}(K(>\alpha);k).$$

Whether k is a field of characteristic zero, ε-straight modules have a $\mathcal{D}$-module structure. Their characteristic cycle may be easily described from the filtration (**Proposition** 5.2.7). In the case of local cohomology modules, the previous results provide a different approach to the interpretation of the multiplicities.

Corollary 5.2.8 *Let $CC(H_I^r(R)) = \sum m_{n-r,\alpha}\ T^*_{X_\alpha}X$, be the characteristic cycle of a local cohomology module $H_I^r(R)$ supported on a squarefree monomial ideal $I \subseteq R$. Then:*

i) $m_{n-r,\alpha}(R/I) = \dim_k \tilde{H}_{|\alpha|-r-1}(K(>\alpha);k)$.

ii) $m_{n-r,\alpha}(R/I) = \dim_k (H_I^r(R))_{-\alpha}$.

iii) $m_{n-r,\alpha}(R/I) = \beta_{|\alpha|-r,\alpha}(I^\vee)$.

These results allow to compute the Hilbert series of a ε-straight module by using the multiplicities of the characteristic cycle (**Theorem 5.2.9**). In the case of local cohomology modules supported on squarefree monomial ideals, we give a different approach to the formula given by N. Terai [92] (see also [98]):

Corollary 5.2.10 *Let $CC(H_I^r(R)) = \sum m_{n-r,\alpha}\ T^*_{X_\alpha}X$, be the characteristic cycle of a local cohomology module $H_I^r(R)$ supported on a squarefree monomial ideal $I \subseteq R$. Then, its Hilbert series is in the form:*

$$H(H_I^r(R);\mathbf{x}) = \sum_{\alpha\in\{0,1\}^n} m_{n-r,\alpha} \prod_{\alpha_i=0} \frac{x_i^{-1}}{1-x_i^{-1}} \prod_{\alpha_j=1} \frac{1}{1-x_j}.$$

Let M be a ε-straight module and $\{F_j\}_{0\leq j\leq n}$ be the filtration of M obtained in Proposition 5.2.4. For each $0 \leq j \leq n$ consider the exact sequence

$$(s_j): \qquad 0 \to F_{j-1} \to F_j \to F_j/F_{j-1} \to 0,$$

that corresponds to an element of ${}^*\mathrm{Ext}^1_R(F_j/F_{j-1}\,, F_{j-1})$. By using **Lemma 5.3.1**, there is a morphism between the class of the extension of (s_j) and the class of the extension of the sequence

$$(s'_j): \qquad 0 \to F_{j-1}/F_{j-2} \to F_j/F_{j-2} \to F_j/F_{j-1} \to 0.$$

Working with this extension, we observe that it is determined by the k-linear maps

$$\delta^\alpha_{-\alpha} : H^0_{\mathfrak{p}_\alpha}(F_j/F_{j-1})_{-\alpha} \longrightarrow H^1_{\mathfrak{p}_\alpha}(F_{j-1}/F_{j-2})_{-\alpha}, \qquad |\alpha| = j.$$

By using the Čech complex, we finally prove that the k-linear map $\delta^{\alpha}_{-\alpha}$ is determined by the multiplication:

$$\begin{array}{rcl} M_{-\alpha} & \longrightarrow & \oplus_{\alpha_i=1} M_{-\alpha+\varepsilon_i} \\ m & \mapsto & \oplus(x_i \cdot m). \end{array}$$

Then, the solution to the extension problem of the ε-straight modules is as follows:

Proposition 5.3.5 *The extension class* (s_j) *is uniquely determined by the* k*-linear maps* $\cdot x_i : M_{-\alpha} \to M_{-\alpha+\varepsilon_i}$ *where* $\mid \alpha \mid= j$ *and* $\alpha_i = 1$.

M. Mustaţă proved in [72], that in the case of local cohomology modules supported on squarefree monomial ideals, the k-linear maps $\cdot x_i : H^j_I(R)_{-\alpha} \to H^j_I(R)_{-\alpha+\varepsilon_i}$ may be computed in terms of the simplicial cohomology of a Stanley–Reisner complex associated to the ideal I (see also Section 6.4).

Finally, we have to point out that the very definition of ε-straight modules shows that they are determined as graded modules by the vector spaces $M_{-\alpha}$, $\alpha \in \{0,1\}^n$ and the multiplication maps $\cdot x_i : M_{-\alpha} \to M_{-\alpha+\varepsilon_i}$, $\alpha_i = 1$. However, this fact alone is not very enlightening if one wishes to know how the extension problems arising from the Mayer-Vietoris sequence are related to these data. We have chosen a more algebraic approach because it seems us that it might be better suited to extend our results to local cohomology modules supported at more general types of arrangements

• In **Chapter 6** we again study the structure of local cohomology modules supported on squarefree monomial ideals by using the theory of $\mathcal{D}$-modules. More precisely, we will study the category of n-hypercubes introduced by A. Galligo, M. Granger and Ph. Maisonobe in [27]. Let $X = \mathbb{C}^n$ and $R = \mathbb{C}[x_1, \dots, x_n]$. In this chapter we will use the following notations:

· $\mathcal{O}_X =$ the the sheaf of holomorphic functions in X.

· $\mathcal{D}_X =$ the sheaf of differential operators in X with holomorphic coefficients.

· $T =$ the union of the coordinate hyperplanes in X, endowed with the stratification given by the intersections of its irreducible components.

Let $Perv^T(\mathbb{C}^n)$ be the category of complexes of sheaves of finitely dimensional vector spaces on X which are perverse relatively to the given stratification of T ([27, I.1]). We denote $\mathrm{Mod}(\mathcal{D}_X)^T_{hr}$ the full abelian subcategory of the category of regular holonomic modules $\mathcal{M}$ in $\mathbb{C}^n$ such that their solution complex $\mathbb{R}\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$ is an object of $Perv^T(\mathbb{C}^n)$. By the Riemann-Hilbert correspondence, the functor of solutions is then an equivalence of categories between $\mathrm{Mod}(\mathcal{D}_X)^T_{hr}$ and $Perv^T(\mathbb{C}^n)$.

The category $Perv^T(\mathbb{C}^n)$ has been linearized in [27] as follows: Let $\mathcal{C}_n$ be the category whose objects are families $\{\mathcal{M}_\alpha\}_{\alpha\in\{0,1\}^n}$ of finitely dimensional complex vector spaces endowed with linear maps

$$\mathcal{M}_\alpha \xrightarrow{u_i} \mathcal{M}_{\alpha+\varepsilon_i} \quad , \quad \mathcal{M}_\alpha \xleftarrow{v_i} \mathcal{M}_{\alpha+\varepsilon_i}$$

for all $\alpha \in \{0,1\}^n$ such that $\alpha_i = 0$. These linear maps are called canonical (resp., variation) maps, and they are required to satisfy the conditions:

$$u_i u_j = u_j u_i,\ \ v_i v_j = v_j v_i,\ \ u_i v_j = v_j u_i \ \text{ i } \ v_i u_i + id \ \text{ is invertible.}$$

Such an object will be called an n-hypercube, the vector spaces $\mathcal{M}_\alpha$ will be called its vertices. A morphism between two n-hypercubes $\{\mathcal{M}_\alpha\}_\alpha$ and $\{\mathcal{N}_\alpha\}_\alpha$ is a set of linear maps $\{f_\alpha : \mathcal{M}_\alpha \to \mathcal{N}_\alpha\}_\alpha$, commuting with the canonical and variation maps (see [27]). It is proved in [loc. cit.] that there is an equivalence of categories between $Perv^T(\mathbb{C}^n)$ and $\mathcal{C}_n$.

For our purposes, we introduce the full abelian subcategory $\mathcal{D}^T_{v=0}$ of the category $\mathrm{Mod}(\mathcal{D}_X)^T_{hr}$ formed by the modules that satisfy the following property:

Definition 6.2.1 *We say that an object $\mathcal{M}$ of $\mathrm{Mod}(\mathcal{D}_X)^T_{hr}$ has* **variation zero** *if the morphisms $v_i : \mathrm{Hom}_{\mathcal{D}_{X,0}}(\mathcal{M}_0, \mathcal{S}_{\alpha+\varepsilon_i,0}) \longrightarrow \mathrm{Hom}_{\mathcal{D}_{X,0}}(\mathcal{M}_0, \mathcal{S}_{\alpha,0})$ are zero for all $1 \le i \le n$ and all $\alpha \in \{0,1\}^n$ with $\alpha_i = 0$.*

The objects of $\mathcal{D}^T_{v=0}$ are characterized by the following particular filtration:

Proposition 6.2.3 *An object $\mathcal{M}$ of $\mathrm{Mod}(\mathcal{D}_X)^T_{hr}$ has variation zero if and only if there is a increasing filtration $\{\mathcal{F}_j\}_{0\le j\le n}$ of $\mathcal{M}$ by objects of $\mathcal{D}^T_{hr}$ and there are integers $m_\alpha \ge 0$ for $\alpha \in \{0,1\}^n$ such that for all $1 \le j \le n$ one has $\mathcal{D}$-module isomorphisms*

$$\mathcal{F}_j/\mathcal{F}_{j-1} \simeq \bigoplus_{|\alpha|=j} (\mathcal{H}^j_{X_\alpha}(\mathcal{O}_X))^{\oplus m_\alpha}.$$

To illustrate this category of modules with variation zero we give a full description of the following objects:

· Projective modules. **Propositions 6.2.4 and 6.2.6**

· Injective modules. **Propositions 6.2.8 and 6.2.10**

· Simple modules. **Propositions 6.2.12 and 6.2.14**

Let $R = \mathbb{C}[x_1, \ldots, x_n]$, be the polynomial ring with coefficients in $\mathbb{C}$ and $\mathcal{D} = D(R, \mathbb{C})$ be the corresponding ring of differential operators. If M is a $\mathcal{D}$-module, then $\mathcal{M}^{an} := \mathcal{O}_X \otimes_R M$ has a natural structure of $\mathcal{D}_X$-module. This allows to define a functor

$$\begin{array}{rcl} (-)^{an} : Mod(\mathcal{D}) & \longrightarrow & Mod(\mathcal{D}_X). \\ M & \longrightarrow & \mathcal{M}^{an} \\ f & \longrightarrow & id \otimes f \end{array}$$

By using Propositions 5.2.4 and 6.2.3, we observe that if M is a ε-straight module then, $\mathcal{M}^{an}$ is an object of $\mathcal{D}^T_{v=0}$. The main result of this chapter is:

Theorem 6.3.1 *The functor*

$$(-)^{an} : \varepsilon - \mathbf{Str} \longrightarrow \mathcal{D}^T_{v=0}$$

is an equivalence of categories.

We also give a more explicit description of this equivalence by describing the n-hypercube of a module $\mathcal{M} \in \mathcal{D}^T_{v=0}$ from the structure of the corresponding ε-straight module M. Namely, the vertices of the n-hypercube are described as:

Proposition 6.3.4 *Let $\mathcal{M} \in \mathcal{D}^T_{v=0}$ be a regular holonomic $\mathcal{D}_X$-module with variation zero and $M \in \varepsilon - \mathbf{Str}$ be the corresponding ε-straight module. Then we have the isomorphism:*

$$\mathcal{M}_\alpha \cong (M_{-\alpha})^*,$$

where $(M_{-\alpha})^*$ denotes the dual of the $\mathbb{C}$-vector space defined by the piece of M of multidegree $-\alpha$, for all $\alpha \in \{0,1\}^n$.

The linear maps u_i of the n-hypercube are described as:

Proposition 6.3.5 *Let $\mathcal{M} \in \mathcal{D}^T_{v=0}$ be a regular holonomic $\mathcal{D}_X$-module with variation zero and $M \in \varepsilon - \mathbf{Str}$ be the corresponding ε-straight module. Then, the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{M}_\alpha & \xrightarrow{u_i} & \mathcal{M}_{\alpha+\varepsilon_i} \\ \cong \uparrow & & \cong \uparrow \\ (M_{-\alpha})^* & \xrightarrow{(x_i)^*} & (M_{-\alpha-\varepsilon_i})^* \end{array}$$

where $(x_i)^*$ is the dual of the multiplication by x_i.

The local cohomology modules $H^r_I(R)$ with support squarefree monomial ideals $I \subseteq R$ have a natural structure of modules with variation zero. In the last section of this chapter we compute the corresponding n-hypercubes by using the Čech complex and studying the cellular structure of the complexes obtained at each vertex. Let Δ be the full simplicial complex whose vertices are labeled by a minimal system of generators of I. Let $T_\alpha := \{\sigma_{\mathbf{1}-\beta} \in \Delta \mid \beta \not\geq \alpha\}$ be a subcomplex of Δ. Then, we obtain the following result:

Proposition 6.4.3 *The local cohomology modules $H^r_I(R)$ supported on squarefree monomial ideals $I \subseteq R$, have variation zero. The corresponding n-hypercubes $GGM(H^r_I(R))$ are:*

- **Vertices:** $(\mathcal{H}^r_I(\mathcal{R}))_\alpha \cong \widetilde{H}_{r-2}(T_\alpha; \mathbb{C})$.
- **Linear maps:** *We have the following commutative diagram:*

$$\begin{array}{ccc} (\mathcal{H}^r_I(\mathcal{R}))_\alpha & \xrightarrow{u_i} & (\mathcal{H}^r_I(\mathcal{R}))_{\alpha+\varepsilon_i} \\ \downarrow \cong & & \downarrow \cong \\ \widetilde{H}_{r-2}(T_\alpha; \mathbb{C}) & \xrightarrow{\nu_i} & \widetilde{H}_{r-2}(T_{\alpha+\varepsilon_i}; \mathbb{C}) \end{array}$$

where ν_i is induced by the inclusion $T_\alpha \subseteq T_{\alpha+\varepsilon_i}$.

We have to point out that we recover the result on the structure of local cohomology modules $H^r_I(R)$ supported on squarefree monomial ideals given by M. Mustaţă [72]. Namely, the graded pieces of the module are:

$$[H^r_I(R)]_{-\alpha} \cong (\widetilde{H}_{r-2}(T_\alpha; \mathbb{C}))^* \cong \widetilde{H}^{r-2}(T_\alpha; \mathbb{C}), \quad \alpha \in \{0,1\}^n,$$

and the morphism of multiplication $x_i : [H^r_I(R)]_{-\alpha-\varepsilon_i} \longrightarrow [H^r_I(R)]_{-\alpha}$ is determined by the commutative diagram:

$$\begin{array}{ccc}
([H_I^r(R)]_{-\alpha})^* & \xrightarrow{(x_i)^*} & ([H_I^r(R)]_{-\alpha-\varepsilon_i})^* \\
\cong\uparrow & & \uparrow\cong \\
(\mathcal{H}_I^r(\mathcal{R}))_\alpha & \xrightarrow{u_i} & (\mathcal{H}_I^r(\mathcal{R}))_{\alpha+\varepsilon_i} \\
\cong\downarrow & & \downarrow\cong \\
(\widetilde{H}^{r-2}(T_\alpha;\mathbb{C}))^* & \xrightarrow{(\nu_i)^*} & (\widetilde{H}^{r-2}(T_{\alpha+\varepsilon_i};\mathbb{C}))^*
\end{array}$$

where ν_i is induced by the inclusion $T_\alpha \subseteq T_{\alpha+\varepsilon_i}$.

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· J. Àlvarez Montaner, R. García López and S. Zarzuela, *Local cohomology, arrangements of subspaces and monomial ideals,* to appear in Adv. in Math.

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Chapter 1

Preliminaries

In this chapter we collect some basic definitions and facts needed in the chapters that follow. At the beginning of each section some references will be given for detailed proofs and additional information. Of course, those are not the only possible references, they are just the ones we have mainly used.

1.1 Local Cohomology Modules

Our main reference for local cohomology modules will be the book of M.P. Brodmann and R.Y. Sharp [15]. The reader might also see [21], [38].

Let R be a commutative Noetherian ring. Given an ideal $I \subseteq R$ we consider **the functor of I-torsion** over the category $\mathrm{Mod}(R)$ of R-modules $\Gamma_I : \mathrm{Mod}(R) \longrightarrow \mathrm{Mod}(R)$ defined by

$$\Gamma_I(M) := \{x \in M \mid I^n x = 0 \text{ for some } n \geq 1\},$$

for any R-module M. $\mathrm{Mod}(R)$ is an abelian category with enough injectives and the I-torsion functor Γ_I is additive, covariant and left-exact so it makes sense to consider the right derived functors of Γ_I. These are called **the local cohomology modules of M with respect to I** and are denoted by

$$H_I^r(M) := \mathbb{R}^r \Gamma_I(M).$$

Some other equivalent definitions of the local cohomology modules of M with respect to I are the following:

- The functor of I-torsion can be expressed as

$$\Gamma_I(M) = \bigcup_{n\geq 0}(0:_M I^n) = \varinjlim \operatorname{Hom}_R(R/I^n, M),$$

so we have

$$H^r_I(M) = \varinjlim \operatorname{Ext}^r_R(R/I^n, M).$$

- **Local cohomology and Čech cohomology:** An useful approach to local cohomology is by using Čech cohomology. Let $f_1, \ldots, f_s \in R$. For any R-module M we may consider the Čech complex of M with respect to $f_1, \ldots, f_s$:

$$0 \longrightarrow \check{C}^0(f_1,\ldots,f_s;M) \xrightarrow{d^0} \cdots \xrightarrow{d^{s-1}} \check{C}^s(f_1,\ldots,f_s;M) \longrightarrow 0\ ,$$

where

$$\check{C}^p(f_1,\ldots,f_s;M) = \bigoplus_{1\leq i_1<\cdots<i_p\leq s} M[\frac{1}{f_{i_1}\cdots f_{i_p}}], \quad \text{for } 0 \leq p \leq s$$

and the differentials d^p are defined by using the canonical localization morphism on every component $M[\frac{1}{f_{i_1}\cdots f_{i_p}}]\longrightarrow M[\frac{1}{f_{j_1}\cdots f_{j_{p+1}}}]$ as follows:

$$d^p(m) = \begin{cases} (-1)^k\ \frac{m}{1} & \text{if } \{i_1,\ldots,i_p\} = \{j_1,\ldots,\widehat{j_k},\ldots,j_{p+1}\}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $I \subseteq R$ be the ideal generated by $f_1, \ldots, f_s$. Then we have

$$H^r_I(M) = H^r(\check{C}^\bullet_I(M)),$$

where we denote $\check{C}^\bullet_I(M) := \check{C}^\bullet(f_1,\ldots,f_s;M)$ for simplicity. In the case $M = R$ we will only denote $\check{C}^\bullet_I$.

Remark 1.1.1. Čech cohomology can also be described as an inductive limit of cohomology of Koszul complexes. For example, see [87] for details.

The following basic properties of local cohomology modules will be often used without further mention:

- $H_I^r(M) = H_{\mathrm{rad}(I)}^r(M),$ for all $r \geq 0$.
- Let $\{M_j\}_{j\in J}$ be an inductive system of R-modules. Then:
$$H_I^r(\varinjlim M_j) = \varinjlim H_I^r(M_j).$$
- **Invariance with respect to base ring:** Let $R \longrightarrow S$ be a homomorphism of rings. Let $I \subseteq R$ be an ideal and M an S-module. Then:
$$H_{IS}^r(M) \cong H_I^r(M).$$
- **Flat base change:** Let $R \longrightarrow S$ be a flat homomorphism of rings. Let $I \subseteq R$ be an ideal and M an R-module. Then:
$$H_{IS}^r(M \otimes_R S) \cong H_I^r(M) \otimes_R S.$$

The local cohomology modules are in general not finitely generated as R-modules, so they are difficult to treat. In order to extract some properties of these modules the usual method is to use several exact sequences or spectral sequences involving these modules. Enumerated are some examples we will use in this work, for details we refer to [15] .

1) **Long exact sequence of local cohomology:** Let $I \subseteq R$ be an ideal and $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$ an exact sequence of R-modules. Then we have the exact sequence:
$$\cdots \longrightarrow H_I^r(M_1) \longrightarrow H_I^r(M_2) \longrightarrow H_I^r(M_3) \longrightarrow H_I^{r+1}(M_1) \longrightarrow \cdots$$
2) **Mayer-Vietoris sequence:** Let $I, J \subseteq R$ be ideals and M an R-module. Then we have the exact sequence:
$$\cdots \longrightarrow H_{I+J}^r(M) \longrightarrow H_I^r(M) \oplus H_J^r(M) \longrightarrow H_{I\cap J}^r(M) \longrightarrow H_{I+J}^{r+1}(M) \longrightarrow \cdots$$
3) **Brodmann's sequence:** Let $I \subseteq R$ be an ideal and M an R-module. For any element $x \in R$ we have the exact sequence:
$$\cdots \longrightarrow H_{I+(x)}^r(M) \longrightarrow H_I^r(M) \longrightarrow H_I^r(M)[\frac{1}{x}] \longrightarrow H_{I+(x)}^{r+1}(M) \longrightarrow \cdots$$
4) **Grothendieck's spectral sequence:** Let $I, J \subseteq R$ be ideals and M an R-module. Then we have the spectral sequence:
$$E_2^{p,q} = H_J^p(H_I^q(M)) \Longrightarrow H_{I+J}^{p+q}(M).$$

1.2 Squarefree monomial ideals

The reader might see [86], [14], [88], among many others, for more information. A good source for some topics treated in this work are the notes from a course given by B. Sturmfels at the COCOA VI Summer School [71].

Let $R = k[x_1, \ldots, x_n]$, where k is a field of any characteristic, be the polynomial ring in n independent variables $x_1, \ldots, x_n$. A **monomial** in R is a product:

$$\mathbf{x}^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \text{ where } \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n.$$

An ideal $I \subseteq R$ is said to be a **monomial ideal** if it may be generated by monomials.

The minimal primary decomposition of a monomial ideal can be easily described by means of the following property:

· **Splitting lemma:** Let $\mathbf{x}^{\boldsymbol{\beta_1}}, \mathbf{x}^{\boldsymbol{\beta_2}}$ be two relatively prime monomials, i.e. $\mathrm{GCD}\,(\mathbf{x}^{\boldsymbol{\beta_1}}, \mathbf{x}^{\boldsymbol{\beta_2}}) = 1$. Then, for any monomials $\mathbf{x}^{\boldsymbol{\alpha_1}}, \ldots, \mathbf{x}^{\boldsymbol{\alpha_r}}$, we have the following decomposition:

$$(\mathbf{x}^{\boldsymbol{\beta_1}} \cdot \mathbf{x}^{\boldsymbol{\beta_2}}, \mathbf{x}^{\boldsymbol{\alpha_1}}, \ldots, \mathbf{x}^{\boldsymbol{\alpha_r}}) = (\mathbf{x}^{\boldsymbol{\beta_1}}, \mathbf{x}^{\boldsymbol{\alpha_1}}, \ldots, \mathbf{x}^{\boldsymbol{\alpha_r}}) \cap (\mathbf{x}^{\boldsymbol{\beta_2}}, \mathbf{x}^{\boldsymbol{\alpha_1}}, \ldots, \mathbf{x}^{\boldsymbol{\alpha_r}}).$$

As a consequence, the minimal primary decomposition of a monomial ideal is given in terms of **monomial primary ideals**:

$$\mathfrak{p}_\alpha := (x_i^{\alpha_i} \mid \alpha_i \neq 0) \quad \text{where } \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n.$$

Among all monomial ideals we are interested on **squarefree monomial ideals**, i.e. ideals $I \subseteq R$ generated by monomials of the form

$$\mathbf{x}^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \text{ where } \alpha = (\alpha_1, \ldots, \alpha_n) \in \{0,1\}^n.$$

In particular the minimal primary decomposition of a squarefree monomial ideal is described in terms of **squarefree monomial prime ideals**

$$\mathfrak{p}_\alpha := (x_i^{\alpha_i} \mid \alpha_i \neq 0) \quad \text{where } \alpha = (\alpha_1, \ldots, \alpha_n) \in \{0,1\}^n,$$

that we will call **face ideals**. For simplicity, when $\alpha = (1, \ldots, 1) \in \{0,1\}^n$ we will denote

$$\mathfrak{m} := \mathfrak{p}_{(1,\ldots,1)} = (x_1, \ldots, x_n).$$

1.2.1 $\mathbb{Z}^n$-grading

The theory of multigraded rings and modules is analogous to that of graded rings and modules. The main sources we will use are [14], [33], [44]. Nevertheless, we have to point out that in this work, we will only treat the special case of modules over the polynomial ring $R = k[x_1, \dots, x_n]$, where k is a field of any characteristic and $x_1, \dots, x_n$ are independent variables.

Let $\varepsilon_1, \dots, \varepsilon_n$ be the canonical basis of $\mathbb{Z}^n$. Then, the ring R has a natural $\mathbb{Z}^n$-graduation given by $\deg(x_i) = \varepsilon_i$. Henceforth, the term graded will always mean $\mathbb{Z}^n$-graded. If $M = \oplus_{\alpha\in\mathbb{Z}^n} M_\alpha$ is a graded R-module and $\beta \in \mathbb{Z}^n$, as usual we denote by $M(\beta)$ the shifted graded R-module whose underlying R-module structure is the same as that of M and where the grading is given by $(M(\beta))_\alpha = M_{\beta+\alpha}$.

Let $M = \oplus_{\alpha\in\mathbb{Z}^n} M_\alpha$ and $N = \oplus_{\alpha\in\mathbb{Z}^n} N_\alpha$ be graded R-modules. A morphism $f : M \to N$ is said to be graded if $f(M_\alpha) \subseteq N_\alpha$ for all $\alpha \in \mathbb{Z}^n$. The abelian group of graded morphisms from M to N will be denoted by ${}^*\mathrm{Hom}_R(M, N)_{\mathbf{0}}$.

From now on, ${}^*\mathrm{Mod}(R)$ will denote the category which has as objects the graded R-modules and as morphisms the graded morphisms. Some facts about ${}^*\mathrm{Mod}(R)$ which will be relevant for us are:

. **Homogeneous elements:** The monomials $\mathbf{x}^\alpha$, $\alpha \in \mathbb{Z}^n$, are the homogeneous elements of R.

. **Homogeneous ideals:** Monomial ideals $I \subseteq R$ are the homogeneous ideals (or graded ideals) of R. In particular, the face ideals $\mathfrak{p}_\alpha$, $\alpha \in \{0,1\}^n$, are the homogeneous prime ideals of R and $\mathfrak{m}$ is the homogeneous maximal ideal of R.

. **Some particular graded modules:** The quotients R/I, where $I \subseteq R$ is a monomial ideal, and the localizations $R[\frac{1}{\mathbf{x}^\alpha}]$, $\alpha \in \mathbb{N}^n$, are objects of ${}^*\mathrm{Mod}(R)$.

. **Graded free modules:** The shifted polynomial rings $R(\beta)$, where $\beta \in \mathbb{Z}^n$, are free objects of ${}^*\mathrm{Mod}(R)$ and every graded free module is isomorphic to a unique (up to order) direct sum of modules of this type.

. **Graded injective modules:** If M is a graded module one can define its $*$-injective envelope ${}^*E(M)$ (in particular, ${}^*\mathrm{Mod}(R)$ is a category with enough injectives). Then, a graded version of the Matlis–Gabriel theorem holds: The indecomposable injective objects of ${}^*\mathrm{Mod}(R)$ are the shifted injective envelopes ${}^*E(R/\mathfrak{p}_\alpha)(\beta)$, where $\mathfrak{p}_\alpha$ is a face ideal of R and $\beta \in \mathbb{Z}^n$, and every graded injective module is isomorphic to a unique (up to order) direct sum of indecomposable injectives.

. **Graded local cohomology modules:** Injective objects of ${}^*\mathrm{Mod}(R)$ are usually not injective as objects of $\mathrm{Mod}(R)$. However, if $I \subseteq R$ is a monomial ideal, $*$-injective objects are acyclic with respect to the functor $\Gamma_I(\,\cdot\,)$. It follows that the local cohomology modules $H_I^r(R)$ are objects of ${}^*\mathrm{Mod}(R)$.

. **The functors *Hom and *Ext:** Let M, N be graded R-modules. Then, we set

$$ {}^*\mathrm{Hom}_R(M,N) = \bigoplus_{\beta\in\mathbb{Z}^n} {}^*\mathrm{Hom}_R(M, N(\beta))_{\mathbf{0}}. $$

Note that ${}^*\mathrm{Hom}_R(M,N)$ has a natural graded structure, where the piece ${}^*\mathrm{Hom}_R(M,N)_\beta$ of multidegree $\beta \in \mathbb{Z}^n$, is nothing but the abelian group of R-module morphisms $f : M \to N$ such that $f(M_\alpha) \subseteq N_{\alpha+\beta}$ for all $\alpha \in \mathbb{Z}^n$.

The abelian group ${}^*\mathrm{Hom}_R(-,-)$ should not be confused with the usual $\mathrm{Hom}_R(-,-)$ in the category $\mathrm{Mod}(R)$, in particular $\mathrm{Hom}_R(-,-)$ is in general not a graded R-module. However, we have ${}^*\mathrm{Hom}_R(M,N) = \mathrm{Hom}_R(M,N)$ when M is a finitely generated graded R-module.

Since the category ${}^*\mathrm{Mod}(R)$ has enough injectives, one can define the derived functors of ${}^*\mathrm{Hom}_R(-,-)$ that will be denoted ${}^*\mathrm{Ext}_R^i(-,-),\ i \geq 0$.

1.2.2 Free resolutions

The theory of graded free resolutions is analogous to that of free resolutions. A comprehensive source for graded resolutions is the thesis of E. Miller [70].

Let M be a graded R-module. By using the description of graded free modules given in the previous section, one can construct graded free resolutions of M. Among them, we are interested in the minimal graded free resolution, i.e. the shortest one, unique up to isomorphism. The length of a minimal graded free resolution is the same as the length of a minimal free resolution. This length is the projective dimension.

The minimal graded free resolution of M is a sequence of free graded R-modules:

$$ \mathbb{F}_\bullet(M): \quad 0 \longrightarrow F_m \xrightarrow{d_m} \cdots \longrightarrow F_1 \xrightarrow{d_1} F_0 \longrightarrow 0\,, $$

exact everywhere except the 0-th step such that $M = \mathrm{Coker}\,(d_1)$. The j-th term is of the form

$$F_j = \bigoplus_{\alpha \in \mathbb{Z}^n} R(-\alpha)^{\beta_{j,\alpha}(M)},$$

and it holds that the matrices of the morphisms $d_j : F_j \longrightarrow F_{j-1}$ do not contain invertible elements. From this expression we can get the following:

• The **projective dimension** of M, denoted $\mathrm{pd}(M)$, is the greatest homological degree in the resolution. Namely

$$\mathrm{pd}(M) := \max\{j \mid F_j \neq 0\}.$$

• The **Betti numbers** of M are the invariants defined by $\beta_{j,\alpha}(M)$. Betti numbers can also be computed from another way. Namely, if M, N are graded R-modules, then $M \otimes_R N$ as well its derived functors $\mathrm{Tor}_j^R(M, N)$ have a natural graded structure. Then we have:

$$\beta_{j,\alpha}(M) = \dim_k \mathrm{Tor}_j^R(M, k)_\alpha.$$

• The **Castelnuovo-Mumford regularity** of M denoted $\mathrm{reg}\,(M)$ is

$$\mathrm{reg}\,(M) := \max\{|\alpha| - j \mid \beta_{j,\alpha}(M) \neq 0\}.$$

The i-linear strand: Given an integer i, the i-linear strand of $\mathbb{F}_\bullet(M)$ is the complex:

$$\mathbb{F}_\bullet^{<i>}(M) : \quad 0 \longrightarrow F_m^{<i>} \xrightarrow{d_m^{<i>}} \cdots \longrightarrow F_1^{<i>} \xrightarrow{d_1^{<i>}} F_0^{<i>} \longrightarrow 0\ ,$$

where

$$F_j^{<i>} = \bigoplus_{|\alpha| = j+i} R(-\alpha)^{\beta_{j,\alpha}(M)},$$

and the differentials $d_j^{<i>} : F_j^{<i>} \longrightarrow F_{j-1}^{<i>}$ are the corresponding components of d_j. We say that M has a **linear free resolution** if there exists i such that $\mathbb{F}_\bullet(M) = \mathbb{F}_\bullet^{<i>}(M)$.

Taylor resolution: Let $I \subseteq R$ be a monomial ideal. An useful free resolution for R/I is given by D. Taylor in [89]. In general, this resolution is nonminimal and can be viewed as a generalization to monomial ideals of the free resolution of $R/\mathfrak{m}$ given by the Koszul complex. For the case of squarefree monomial ideals we have the following description:

Let $\{\mathbf{x}^{\alpha_1}, \dots, \mathbf{x}^{\alpha_m}\}$ be a set of generators of I. Let F be the free R-module of rank m generated by $e_1, \dots, e_m$. The **Taylor complex** $\mathbb{T}_\bullet(R/I)$ is of the form:

$$\mathbb{T}_\bullet(R/I): \quad 0 \longrightarrow F_m \xrightarrow{d_m} \cdots \longrightarrow F_1 \xrightarrow{d_1} F_0 \longrightarrow 0 \,,$$

where $F_j = \bigwedge^j F$ is the j-th exterior power of F and if LCM denotes the least common multiple, then the differentials d_j are defined as

$$d_j(e_{i_1}\wedge\cdots\wedge e_{i_j}) = \sum_{1\leq k\leq j} (-1)^k \frac{\mathrm{LCM}\,(\mathbf{x}^{\alpha_{i_1}}, \dots, \mathbf{x}^{\alpha_{i_j}})}{\mathrm{LCM}\,(\mathbf{x}^{\alpha_{i_1}}, \dots, \widehat{\mathbf{x}^{\alpha_{i_k}}}, \dots, \mathbf{x}^{\alpha_{i_j}})} \; e_{i_1}\wedge\cdots\wedge\widehat{e_{i_k}}\wedge\cdots\wedge e_{i_j}.$$

1.2.3 Injective resolutions

The main reference for graded injective resolutions of modules will be [33]. See also [70].

Let M be a graded R-module. By using the description of graded injective modules given in the previous section, one can construct graded injective resolutions of M. Among them, we are interested in the minimal graded injective resolution, i.e. the shortest one, unique up to isomorphism.

The minimal graded injective resolution of M is a sequence:

$$\mathbb{I}^\bullet(M): \quad 0 \longrightarrow I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} \cdots \longrightarrow I^m \xrightarrow{d^m} \cdots \,,$$

exact everywhere except the 0-th step such that $M = \mathrm{Ker}\,(d^0)$, the j-th term is

$$I^j = \bigoplus_{\alpha\in\mathbb{Z}^n} {}^*E(R/\mathfrak{p}_\alpha)(\beta)^{\mu_j(\mathfrak{p}_\alpha, M)},$$

and I^j is the injective envelope of $\mathrm{Ker}\, d^j$. From this expression we can get the following:

• The **graded injective dimension** of M, denoted ${}^*\mathrm{id}(M)$, is the greatest cohomological degree in the minimal graded injective resolution. Namely

$${}^*\mathrm{id}(M) = \max\{j \mid I^j \neq 0\}.$$

• The **Bass numbers** of M are the invariants defined by $\mu_j(\mathfrak{p}_\alpha, M)$. By using the results of [33], these numbers are equal to the usual Bass numbers that appear in the minimal injective resolution of M. So, they can also be computed as

$$\mu_j(\mathfrak{p}_\alpha, M) = \dim_{k(\mathfrak{p}_\alpha)} \operatorname{Ext}^j_R(k(\mathfrak{p}_\alpha), M_{\mathfrak{p}_\alpha}).$$

Moreover, the Bass numbers with respect to any prime ideal can be computed from the Bass numbers with respect to homogeneous prime ideals [33, Theorem 1.2.3]. In particular, we have ${}^*\mathrm{id}(M) \leq \mathrm{id}(M)$, where $\mathrm{id}(M)$ denotes the usual injective dimension of M. If $M \neq 0$ is a finitely generated R-module then the equality holds.

1.2.4 Further considerations

Graded free resolutions are easily treatable due to the fact that graded free modules are finite. In particular, the matrices corresponding to the differentials are well described. On the other hand, graded injective modules are not finitely generated so graded injective resolutions are more difficult to deal with. Recently, it has been a great effort in the study of these resolutions in order to make them as treatable as graded free resolutions. We will introduce some notions that will be useful in our work. We refer to [69] for details.

• **Monomial matrices:** A map between graded free R-modules is described by means of a matrix with monomial entries. In [69], a convenient modification of the notations of the source and the target of such a matrix is introduced in order to work with injective resolutions.

• **Čech hull:** The Taylor complex $\mathbb{T}_\bullet(R/I)$ and the Čech complex $\check{\mathrm{C}}^\bullet_I$ are closely related. Roughly speaking, their corresponding monomial matrices have the same entries. In [69], a bridge among both complexes is given by means of the so-called Čech hull functor. In fact, this transition works for all free resolutions of R/I so we can get some different complexes which can replace the Čech complex.

It has been also developed another approach to the study of graded resolutions by giving them a topological interpretation.

• **Cellular and cocellular resolutions:** The notion of cellular free resolution attached to a cellular complex is introduced in [6]. A monomial matrix whose scalar entries constitute a augmented oriented chain (cochain) complex up to a homological shift for some regular cellular complex X is called a **cellular (cocellular) monomial matrix** supported on X.

A cellular (cocellular) free resolution is a free resolution where the differentials are described by cellular (cocellular) monomial matrices. This construction is generalized [69] in order to describe cellular (cocellular) injective complexes.

1.2.5 Hilbert series

Let M be a $\mathbb{Z}^n$-graded module of dimension d such that $\dim_k(M_\alpha)$ is finite for all $\alpha \in \mathbb{Z}^n$. The $\mathbb{Z}^n$-graded Hilbert series of M is:

$$H(M;\mathbf{x}) = \sum_{\alpha \in \mathbb{Z}^n} \dim_k(M_\alpha)\ \mathbf{x}^\alpha.$$

The usual $\mathbb{Z}$-graded Hilbert series of M:

$$H(M;t) = \sum_{n \in \mathbb{Z}} \dim_k(M_n)\ t^n,$$

where M_n is the piece of degree $n \in \mathbb{Z}$ of M, can be obtained from $H(M;\mathbf{x})$ by substituting $x_i = t$ for all i.

The $\mathbb{Z}$-graded Hilbert series can be expressed as $H(M;t) = Q_M(t)/(1-t)^d$, where $Q_M(t) \in \mathbb{Z}[t,t^{-1}]$ is such that $Q_M(1) \neq 0$. In particular, the following invariants are defined:

· The multiplicity of M: $e(M) := Q_M(1)$

· The generalized multiplicities of M: $e_i(M) := \frac{Q_M^{(i)}(1)}{i!}$, for $i = 0, \dots, d-1$

1.2.6 Stanley-Reisner ring

Our main reference for Stanley-Reisner ring will be the book of R. P. Stanley [86]. The reader might see also [14], [88].

Let Δ be a simplicial complex on the set of vertices $\{x_1, \dots, x_n\}$. Given any field k, we define the **Stanley-Reisner** ring of Δ with respect to k as the homogeneous k-algebra:

$$k[\Delta] := k[x_1, \dots, x_n]/I_\Delta,$$

where $I_\Delta = (\mathbf{x}^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha \in \{0,1\}^n,\ \{x_i^{\alpha_i} \mid \alpha_i = 1\} \notin \Delta)$.

The ideal I_Δ is generated by squarefree monomials and its minimal primary decomposition is of the form

$$I_\Delta = \bigcap_\alpha \mathfrak{p}_\alpha,$$

where the intersection is taken over all face ideals $\mathfrak{p}_\alpha := (x_i^{\alpha_i} \mid \alpha_i \neq 0)$, $\alpha \in \{0,1\}^n$, such that $\sigma_{\mathbf{1}-\alpha} := \{x_1, \dots, x_n\} \setminus \{x_i \mid \alpha_i = 1\}$ is a facet of Δ, where we denote $\mathbf{1} = (1, \dots, 1) \in \mathbb{Z}^n$. In particular $\dim k[\Delta] = \dim \Delta + 1$.

Conversely, to a squarefree monomial ideal I one can associate a simplicial complex Δ such that $I = I_\Delta$.

Alexander duality : The Alexander dual ideal of I_Δ is the ideal

$$I_\Delta^\vee = (\ \mathbf{x}^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \mathbf{x}^{\mathbf{1}-\alpha} \notin I_\Delta).$$

The minimal primary decomposition of $I_\Delta^\vee$ can be easily described from I_Δ. Namely, let $\{\mathbf{x}^{\alpha_1}, \dots, \mathbf{x}^{\alpha_r}\}$ be a minimal system of generators of I_Δ. Then, the minimal primary decomposition of $I_\Delta^\vee$ is of the form

$$I_\Delta^\vee = \bigcap_{i=1}^r \mathfrak{p}_{\alpha_i},$$

and we have $I_\Delta^{\vee\vee} = I_\Delta$.

The most usual form of Alexander duality can be found in a topological context. It gives an isomorphism between the homology of a closed topological subspace X of an $(n-2)$-sphere S^{n-2} with coefficients in a group G and the cohomology of the complement. Namely we have

$$\widetilde{H}_{r-1}(X; G) \cong \widetilde{H}^{n-2-r}(S^{n-2} \backslash X; G)$$

The context we are interested in we will take coefficients in the field k. A simplicial complex Δ is a closed subcomplex of the $(n-2)$-sphere S^{n-2} obtained as the boundary of the simplex spanned by $\{x_1, \dots, x_n\}$. The complement $S^{n-2} \backslash \Delta$ has a deformation retract to the Alexander dual simplicial complex $\Delta^\vee$ which consists of the complements of the nonfaces of Δ, i.e.

$$\Delta^\vee := \{\overline{\sigma} \mid \sigma \notin \Delta\},$$

where $\overline{\sigma} := \{x_1, \dots, x_n\} \setminus \sigma$. It is straightforward to check that $I_\Delta^\vee = I_{\Delta^\vee}$ and the previous result gives

$$\widetilde{H}_{r-1}(\Delta; k) \cong \widetilde{H}^{n-2-r}(\Delta^\vee; k).$$

The following example will be very useful in our work. We refer to [70] for more details.

Example: Let Δ be a simplicial complex on the set of vertices $\{x_1, \dots, x_n\}$ and $\sigma_\alpha \in \Delta$, $\alpha \in \{0,1\}^n$ be a face. Then, considering the subcomplexes

· **restriction to σ_α:** $\Delta_\alpha := \{\tau \in \Delta \mid \tau \in \sigma_\alpha\}$

· **link of σ_α:** $\text{link}_\alpha \Delta := \{\tau \in \Delta \mid \sigma_\alpha \cap \tau = \emptyset, \ \sigma_\alpha \cup \tau \in \Delta\}$

we have the isomorphism:

$$\widetilde{H}_{n-|\alpha|-r-1}(\text{link}_\alpha \Delta; k) \cong \widetilde{H}^{r-2}(\Delta^\vee_{\mathbf{1}-\alpha}; k)$$

given by the equality of complexes $\Delta^\vee_{\mathbf{1}-\alpha} = (\text{link}_\alpha \Delta)^\vee$ and Alexander duality.

1.2.7 Some particular posets

Let $I = I_{\alpha_1} \cap \cdots \cap I_{\alpha_m}$ be the minimal primary decomposition of a squarefree monomial ideal $I \subseteq R$. In our work it will be very useful to use the following posets in order to encode the information given by the minimal primary decomposition.

- **The poset $\mathcal{I}$:** We consider $\mathcal{I} = \{\mathcal{I}_1, \mathcal{I}_2, \dots, \mathcal{I}_m\}$, where:

$\mathcal{I}_1 = \{I_{\alpha_{i_1}} \mid \ 1 \leq i_1 \leq m\}$,

$\mathcal{I}_2 = \{I_{\alpha_{i_1}} + I_{\alpha_{i_2}} \mid \ 1 \leq i_1 < i_2 \leq m\}$,

$\vdots$

$\mathcal{I}_m = \{I_{\alpha_1} + I_{\alpha_2} + \cdots + I_{\alpha_m}\}$.

Namely, the sets $\mathcal{I}_j$ are formed by the sums of j face ideals in the minimal primary decomposition of I. Notice that every sum of face ideals in the poset are treated as different elements even if they describe the same ideal.

Let $\{\mathbf{x}^{\beta_1}, \mathbf{x}^{\beta_2}, \ldots, \mathbf{x}^{\beta_r}\}$ be a minimal set of generators of a squarefree monomial ideal $I \subseteq R$. It will also be very useful to use the following poset:

- **The poset $\mathcal{J}$:** We consider $\mathcal{J} = \{\mathcal{J}_1, \mathcal{J}_2, \ldots, \mathcal{J}_r\}$, where:

$\mathcal{J}_1 = \{\mathbf{x}^{\beta_{i_1}} | \ \ 1 \leq i_1 \leq r\}$,

$\mathcal{J}_2 = \{\mathrm{LCM}\,(\mathbf{x}^{\beta_{i_1}}, \mathbf{x}^{\beta_{i_2}}) | \ \ 1 \leq i_1 < i_2 \leq r\}$,

$\vdots$

$\mathcal{J}_r = \{\mathrm{LCM}\,(\mathbf{x}^{\beta_1}, \mathbf{x}^{\beta_2}, \ldots, \mathbf{x}^{\beta_r})\}$.

Namely, the sets $\mathcal{J}_j$ are formed by the LCM of j generators in the minimal set of generators of I. Every LCM of monomials in the poset are treated as different elements even they describe the same monomial.

Remark 1.2.1. By identifying the elements of the poset $\mathcal{J}$ when they describe the same monomial we get, except for the bottom element, the LCM-lattice introduced in [30].

If we consider the corresponding posets $\mathcal{I}^\vee$ and $\mathcal{J}^\vee$ associated to the Alexander dual ideal $I^\vee$ then we have $\forall j$:

$$\mathfrak{p}_\alpha \in \mathcal{I}_j \Leftrightarrow \mathbf{x}^\alpha \in \mathcal{J}_j^\vee, \quad \mathbf{x}^\alpha \in \mathcal{J}_j \Leftrightarrow \mathfrak{p}_\alpha \in \mathcal{I}_j^\vee.$$

Finally we present an example of how this poset can be used in order to describe the cellular structure of the free resolution given by the Taylor complex.

Example: Let $\{\mathbf{x}^{\boldsymbol{\beta}_\mathbf{1}}, \ldots, \mathbf{x}^{\boldsymbol{\beta}_\mathbf{r}}\}$ be a minimal system of generators of a squarefree monomial ideal $I \subseteq R$. The Taylor complex $\mathbb{T}_\bullet(R/I)$ is a cellular free resolution supported on the simplex Δ whose vertices are labelled by the generators of the ideal. More precisely, the vertices are labelled by the elements of $\mathcal{J}_1$ and the faces of Δ of dimension $j-1$ are labelled by the elements of $\mathcal{J}_j$ $\forall j$.

1.2.8 Arrangement of linear subvarieties

The study of **arrangement of linear subvarieties**, i.e. the study of a finite collection of affine subspaces in $\mathbb{A}_k^n$, has had a great development from different points of view such as combinatorics, geometry and topology. See [10] for a complete survey. An important case that has been deeply studied is the case of **hyperplane arrangements**, i.e. arrangements defined by linear varieties of codimension 1, for this case we also refer to the book of Orlik and Terao [75].

In this work we are interested on the **Betti numbers** of the complement of an arrangement of linear subvarieties X in the real or complex affine space, i.e. the ranks of the reduced singular cohomology groups $\tilde{H}_i(\mathbb{A}^n_k - X; \mathbb{Z})$ where $k = \mathbb{R}$ or $k = \mathbb{C}$, with a special attention to the arrangements such that their defining ideal is a monomial ideal. A nice formula for these Betti numbers follows from a theorem of Goresky–MacPherson ([34, III.1.3. Theorem A]) that gives the structure of these cohomology groups. Our aim in this section is to present this formula, so we will start introducing the notations we will use.

The order complex: Let $X \subset \mathbb{A}^n_k$, be an arrangement of linear subvarieties. The arrangement X defines a partially ordered set $(P(X), <)$ whose elements correspond to the intersections of irreducible components of X and where the order $<$ is given by inclusion.

The order complex associated to the poset $(P(X), <)$ is the simplicial complex $K(P(X))$ which has as vertexs the elements of $P(X)$ and where a set of vertexs $p_0, \dots, p_r$ determines a r-dimensional simplex if $p_0 < \cdots < p_r$.

Similarly, we will also consider the simplicial complexes $K(> p)$ and $K(\geq p)$ attached to the subposets $\{q \in P(X) \mid q > p\}$ and $\{q \in P(X) \mid q \geq p\}$ of $P(X)$.

Goresky-MacPherson's result: Let X_p be the irreducible variety defined by an element $p \in P(X)$ and denote by $h(p)$ the height of the ideal of definition of X_p. The theorem of Goresky–MacPherson describes the reduced singular cohomology groups of the complement $\mathbb{A}^n_k - X$ in terms of the cohomology of the pair $(K(\geq p), K(> p))$. Namely it states that:

$$\tilde{H}_r(\mathbb{A}^n_{\mathbb{R}} - X; \mathbb{Z}) \cong \bigoplus_p H^{h(p)-r-1}(K(\geq p), K(> p); \mathbb{Z}).$$

Regarding a complex arrangement in $\mathbb{A}^n_{\mathbb{C}}$ as a real arrangement in $\mathbb{A}^{2n}_{\mathbb{R}}$, a formula for the Betti numbers of the complement of a complex arrangement follows from the formula for real arrangements. See [34, III.Corollary 1.4]).

1.3 D-Modules

The purpose of this section is to provide some basic foundations in the theory of modules over the ring of differential operators. The main references we will use are [8], [12] and [17].

1.3.1 Ring of differential operators

Let $R = k[x_1, \ldots, x_n]$, where k is a field of characteristic zero, be the polynomial ring in the independent variables $x_1, \ldots, x_n$. The ring of differential operators $D(R, k)$ is the subring of $\mathrm{End}_k(R)$ generated by the k-linear derivations and the multiplications by elements of R.

One may see that $D(R, k)$ coincide with the Weyl algebra, i.e. the non commutative R-algebra generated by the partial derivatives $\partial_i = \frac{d}{dx_i}$, with the relations given by:

i) $\partial_i \partial_j = \partial_j \partial_i$.

ii) $\partial_i r - r\partial_i = \frac{dr}{dx_i}$, where $r \in R$.

The ring of differential operators $D(R, k)$, that we will denote $\mathcal{D}$ for simplicity, is a left and right Noetherian ring. By a $\mathcal{D}$-module we mean a left $\mathcal{D}$-module.

Any element $P \in \mathcal{D}$ can be uniquely written as

$$P = \sum_{\alpha \in I} c_\alpha \partial^\alpha,$$

where I is a finite set of $\mathbb{N}^n$, $c_\alpha \in R$ and we use the multidegree notation $\partial^\alpha := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$.

We call **order** of P the integer $o(P) = \sup \{|\alpha| \mid c_\alpha \neq 0\}$, and **principal symbol** of P the element of the polynomial ring $R[\xi_1, \ldots, \xi_n]$ in the independent variables $\xi_1, \ldots, \xi_n$ defined by:

$$\sigma(P) = \sum_{|\alpha| = o(P)} c_\alpha \xi^\alpha,$$

where we use again the multidegree notation $\xi^\alpha := \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$.

Order filtration on $\mathcal{D}$: The ring $\mathcal{D}$ has a natural increasing filtration given by the order. The sets of differential operators of order less than m,

$$\mathcal{D}(m) = \{P \in \mathcal{D} \mid o(P) \leq m\},$$

form an increasing sequence of finitely generated R-submodules $\mathcal{D}(0) \subseteq \mathcal{D}(1) \subseteq \cdots \subseteq \mathcal{D}$ satisfying $\forall k, m \geq 0$:

i) $\mathcal{D} = \bigcup_{m \in \mathbb{N}} \mathcal{D}(m)$,

ii) $\mathcal{D}(m)\mathcal{D}(k) = \mathcal{D}(m+k)$.

The principal symbol allows us to give an isomorphism between the associated graded ring $gr(\mathcal{D}) = \bigoplus_{m \geq 0} \mathcal{D}(m)/\mathcal{D}(m-1)$ and the polynomial ring $R[\xi_1, \ldots, \xi_n]$. Namely, the map:

$$gr(\mathcal{D}) \longrightarrow R[\xi_1, \ldots, \xi_n]$$

$$\textstyle\sum_m \overline{P_m} \longrightarrow \sum_m \sigma(P_m)$$

is an isomorphism of commutative rings.

Filtration on a finitely generated $\mathcal{D}$-module: A finitely generated $\mathcal{D}$-module M has a **good** filtration, i.e. M has an increasing sequence of finitely generated R-submodules $M(0) \subseteq M(1) \subseteq \cdots \subseteq M$ satisfying :

i) $M = \bigcup M(k)$,

ii) $\mathcal{D}(m)M(k) = M(m+k) \quad k >> 0, \forall m \geq 0$,

such that the associated graded module $gr(M) = \bigoplus_{m \geq 0} M(m)/M(m-1)$ is a finitely generated $gr(\mathcal{D})$-module.

Dimension theory: Let $(x, \xi) = (x_1, \ldots, x_n, \xi_1, \ldots, \xi_n) \in R[\xi_1, \ldots, \xi_n]$ be the homogeneous maximal ideal. The Hilbert series of the graded module $gr(M)$:

$$H(gr(M); t) = \sum_{j \geq 0} \dim_k[(x, \xi)^j gr(M)/(x, \xi)^{j+1} gr(M)] \; t^j,$$

is of the form $H(gr(M); t) = q(t)/(1-t)^d$, where $q(t) \in \mathbb{Z}[t, t^{-1}]$ is such that $q(1) \neq 0$. The Krull dimension of $gr(M)$ is d and the multiplicity of $gr(M)$ is $q(1)$. These integers are independent of the good filtration on M

and are called the dimension and the multiplicity of M. We will denote them $d(M)$ and $e(M)$ respectively. The following result is a deep theorem, proved by M. Sato, T. Kawai and M. Kashiwara in [81] (see also [59]), by using microlocal techniques. Later, O. Gabber [26] found a purely algebraic proof:

. **Bernstein's inequality:** Let M be a non-zero finitely generated $\mathcal{D}$-module. Then $d(M) \geq n$.

Holonomic $\mathcal{D}$-modules: Now we single out the important class of $\mathcal{D}$-modules having the minimal possible dimension.

Let M be a finitely generated $\mathcal{D}$-module. One says that M is **holonomic** if $M = 0$ or $d(M) = n$. The class of holonomic modules has many good properties. Among them we found:

- Holonomic modules form a full abelian subcategory of the category of $\mathcal{D}$-modules. In particular if $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$ is an exact sequence of $\mathcal{D}$-modules, then M_2 is holonomic if and only if M_1 and M_3 are both holonomic.

- If M is holonomic if and only if M is of finite length as a $\mathcal{D}$-module.

Regular holonomic $\mathcal{D}$-modules: One meets several equivalent definitions of the notion of regularity for holonomic $\mathcal{D}$-modules in the literature. We present here the one given by M. Kashiwara and T. Kawai [50]:

Let M be a holonomic $\mathcal{D}$-module. One says that M is **regular** if there exists a good filtration on M such that the annihilator of $gr(M)$ is a radical ideal in $gr(\mathcal{D})$. We also have:

- Regular holonomic modules form a full abelian subcategory of the category of $\mathcal{D}$-modules.

Characteristic cycle: Let M be a finitely generated $\mathcal{D}$-module equipped with a good filtration.

- The **characteristic ideal** of M is the ideal in $gr(\mathcal{D}) = R[\xi_1, \ldots, \xi_n]$ given by:

$$J(M) := \mathrm{rad}\,(\mathrm{Ann}_{\,gr(\mathcal{D})}(gr(M))).$$

One can prove that $J(M)$ is independent of the good filtration on M.

• The **characteristic variety** of M is the closed algebraic set given by:

$$C(M) := V(J(M)) \subseteq \mathrm{Spec}\,(gr(\mathcal{D})) = \mathrm{Spec}\,(R[\xi_1, \ldots, \xi_n]).$$

The characteristic variety $C(M)$ of a finitely generated $\mathcal{D}$-module M is a conical variety, i.e. it has the following property: if $(\mathbf{x}, \xi) \in C(M)$ then $(\mathbf{x}, \lambda\xi) \in C(M)$ for any $\lambda \in k$.

The characteristic variety of a finitely generated $\mathcal{D}$-module provides a geometric description of its dimension. Namely, we have $\dim C(M) = d(M)$. In particular $C(M) = 0$ if and only if $M = 0$.

The characteristic variety also allows us to describe the support of a finitely generated $\mathcal{D}$-module as R-module. Let

$$\pi : \mathrm{Spec}\,(R[\xi_1, \ldots, \xi_n]) \longrightarrow \mathrm{Spec}\,(R)$$

be the map defined by $\pi(x, \xi) = x$. Then:

$$\mathrm{Supp}_R(M) = \pi(C(M)).$$

• The **characteristic cycle** of M is defined as:

$$CC(M) = \sum m_i V_i$$

where the sum is taken over all the irreducible components V_i of the characteristic variety $C(M)$ and the $m_i' s$ are the following multiplicities:

Let $V_i = V(\mathfrak{p}_i) \subseteq C(M)$ be an irreducible component, where $\mathfrak{p}_i \in \mathrm{Spec}\,(gr(\mathcal{D}))$. Then m_i is the multiplicity of the module $gr(M)_{\mathfrak{p}_i}$. Namely, the Hilbert series:

$$H(gr(M)_{\mathfrak{p}_i}; t) = \sum_{j \geq 0} \dim_k[\mathfrak{p}_i^j gr(M)_{\mathfrak{p}_i} / \mathfrak{p}_i^{j+1} gr(M)_{\mathfrak{p}_i}] \; t^j$$

is in the form $H(gr(M)_{\mathfrak{p}_i}; t) = q_i(t)/(1-t)^{d_i}$, where $q_i(t) \in \mathbb{Z}[t, t^{-1}]$ is such that $q_i(1) \neq 0$, so the multiplicity in the characteristic cycle of the irreducible component V_i is then $m_i = q_i(1)$.

• A fundamental property we will use throughout this work is the **additivity of the characteristic cycle with respect to exact sequences.** Namely, let $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$ be an exact sequence of holonomic $\mathcal{D}$-modules. Then, we have $CC(M_2) = CC(M_1) + CC(M_3)$.

1.3.2 Inverse and direct image

Some geometrical operations as inverse and direct image have a key role in the theory of $\mathcal{D}$-modules. Our aim in this section is to give a brief survey of these operations in the case of the embedding:

$$\begin{array}{rcl} i : \mathbb{A}_k^n & \longrightarrow & \mathbb{A}_k^{n+1} \\ \mathbf{x} & \longmapsto & (\mathbf{x}, 0). \end{array}$$

In our work we will only use this case and it corresponds to the projection from $R' = k[x_1, \ldots, x_n, t]$ to $R = k[x_1, \ldots, x_n]$ given by $t = 0$. The rings of differential operators corresponding to R' and R will be denoted by $\mathcal{D}_{n+1}$ and $\mathcal{D}_n$ respectively.

The main reference we will use in this section is [17]. For general considerations on inverse and direct images we refer to [65], [12].

• **Inverse image functor:** Let N be a $\mathcal{D}_{n+1}$-module. Its inverse image is the $\mathcal{D}_n$-module $i^*(N)$ defined as

$$i^*(N) = R \widehat{\otimes}_{R'} N \cong N/(t)N,$$

where $\widehat{\otimes}$ denotes the external product with the usual actions. It establishes a right exact functor $i^* : \mathrm{Mod}(\mathcal{D}_{n+1}) \longrightarrow \mathrm{Mod}(\mathcal{D}_n)$. The corresponding left derived functors are well determined. Namely we have:

Lemma 1.3.1. *Let N be a $\mathcal{D}_{n+1}$-module. Let $t : N \longrightarrow N$ be the homomorphism given by the multiplication by the variable t. Then we have:*

$\mathbb{L}^0 i^*(N) = \mathrm{Coker}\,(t),$

$\mathbb{L}^{-1} i^*(N) = \mathrm{Ker}\,(t),$

and the other left derived functors $\mathbb{L}^{-j} i^*(N)$ vanish $\forall j$.

• **Direct image functor:** Let M be a $\mathcal{D}_n$-module. Its direct image is the $\mathcal{D}_{n+1}$-module $i_+(M)$ defined as

$$i_+(M) = k[\partial_t] \widehat{\otimes}_k M = M[\partial_t],$$

where $\widehat{\otimes}$ denotes the external product with the usual actions. It establishes an exact functor $i_+ : \mathrm{Mod}(\mathcal{D}_n) \longrightarrow \mathrm{Mod}(\mathcal{D}_{n+1})$.

• **Direct image and characteristic variety:** The characteristic variety of $i_+(M)$ can be computed from the characteristic variety of M. Namely, we have:

$$C(i_+(M)) = \{(\mathbf{x}, 0, \xi, \tau) \mid (\mathbf{x}, \xi) \in C(M)\} \subseteq \mathrm{Spec}\,(R'[\xi_1, \ldots, \xi_n, \tau]),$$

where we have considered $C(M) \subseteq \mathrm{Spec}\,(R[\xi_1, \ldots, \xi_n])$.

• **Kashiwara's theorem:** Let $\mathrm{Mod}_{fg}(\mathcal{D}_n)$ be the category of finitely generated $\mathcal{D}_n$-modules and $\mathrm{Mod}_{fg,\{t=0\}}(\mathcal{D}_{n+1})$ the category of finitely generated $\mathcal{D}_{n+1}$-modules with support contained in the variety defined by $t = 0$. The direct image functor i_+ establishes an equivalence of categories

$$i_+ : \mathrm{Mod}_{fg}(\mathcal{D}_n) \longrightarrow \mathrm{Mod}_{fg,\{t=0\}}(\mathcal{D}_{n+1}).$$

Its inverse is $\mathbb{L}^{-1}i^*$. Namely, for all $N \in \mathrm{Mod}_{fg,\{t=0\}}(\mathcal{D}_{n+1})$ we have:

$$i_+(\mathbb{L}^{-1}i^*(N)) = N.$$

• **Direct image and local cohomology modules:** Local cohomology modules and direct image are closely related by means of Kashiwara's theorem. The following result will be very useful in our work.

Lemma 1.3.2. *Let $I \subseteq R$ be an ideal. The direct image of the local cohomology module $H^{n-i}_I(R)$ is:*

$$i_+(H^{n-i}_I(R)) = H^1_{(t)}(H^{n-i}_{IR'}(R')).$$

PROOF: For simplicity we will denote the local cohomology module $H^{n-i}_{IR'}(R')$ by N. It is a (t)-torsion free module so we have the exact sequence:

$$0 \longrightarrow N \longrightarrow N[\tfrac{1}{t}] \longrightarrow H^1_{(t)}(N) \longrightarrow 0\ .$$

Consider the long exact sequence of left derived functors of i^*:

$$\cdots \longrightarrow \mathbb{L}^{-1}i^*(H^1_{(t)}(N)) \longrightarrow \mathbb{L}^0 i^*(N) \longrightarrow \mathbb{L}^0 i^*(N[\tfrac{1}{t}]) \longrightarrow \mathbb{L}^0 i^*(H^1_{(t)}(N)) \longrightarrow 0\ .$$

Applying Lemma 1.3.1 to the modules $N[\frac{1}{t}]$ and $H^1_{(t)}(N)$ we get:

$$\mathbb{L}^0 i^*(N[\tfrac{1}{t}]) = \mathbb{L}^{-1} i^*(N[\tfrac{1}{t}]) = 0 \quad \text{and} \quad \mathbb{L}^0 i^*(H^1_{(t)}(N)) = 0.$$

So we have:

$$\mathbb{L}^{-1}i^*(H^1_{(t)}(N)) \cong \mathbb{L}^0 i^*(N)$$

Notice that $H^1_{(t)}(N) = H^1_{(t)}(H^{n-i}_{IR'}(R')) \in \operatorname{Mod}_{fg,\{t=0\}}(\mathcal{D}_{n+1})$, so we get the desired result by Kashiwara's equivalence. Namely we have:

$$\begin{aligned} H^1_{(t)}(H^{n-i}_{IR'}(R')) = i_+(\mathbb{L}^{-1}i^*(H^1_{(t)}(H^{n-i}_{IR'}(R')))) = i_+(\mathbb{L}^0 i^*(H^{n-i}_{IR'}(R'))) = \\ = i_+(H^{n-i}_{IR'}(R)). \end{aligned}$$

□

1.3.3 Further considerations on $\mathcal{D}$-module theory

Although we are mostly interested on modules over the Weyl algebra we can also consider rings of differential operators over different base rings as the ring of convergent series or the formal power series ring. These theories are very similar and one can easily extend the results considered in section 1.3.1. For details and examples we refer to [8], [36], [52], [77].

Rings of differential operators over different base rings: Let k be a field of characteristic zero and $x_1, \dots, x_n$ be independent variables. Consider any of the following rings:

- $R = k[[x_1, \dots, x_n]]$, the formal power series ring,
- $R = k\{x_1, \dots, x_n\}$, the ring of convergent series,

instead of the polynomial ring $R = k[x_1, \dots, x_n]$. In any case the ring of differential operators $D(R, k)$ will be denoted by $\mathcal{D}$.

If one mimics the constructions given in section 1.3.1 for the Weyl algebra, one can check that the results we have considered before, conveniently reformulated, remain true for the rings of differential operators we are working with in this section.

For inverse and direct images one should be careful with the details (see [60], [61], [65]), but the particular results formulated in section 1.3.2 remain true for the cases we are considering.

It is worthwhile to point out that although these theories are closely related they have some important differences when one goes deeply into the study of rings of differential operators. For example the Fourier transform is specific to the case of the Weyl algebra (see [17], [65]).

Nevertheless our main interest in this work is the study of the characteristic cycle of some regular holonomic $\mathcal{D}$-modules for any of the rings of differential operators considered before. Notice that

$$D(k[[x_1,\ldots,x_n]],k) = D(k[x_1,\ldots,x_n],k) \otimes_{k[x_1,\ldots,x_n]} k[[x_1,\ldots,x_n]],$$

$$D(k\{x_1,\ldots,x_n\},k) = D(k[x_1,\ldots,x_n],k) \otimes_{k[x_1,\ldots,x_n]} k\{x_1,\ldots,x_n\}.$$

So, it will be important to study the behavior of the characteristic cycle by flat base change in these cases.

Characteristic cycle and flat base change: Let $\varphi : R_1 \longrightarrow R_2$ be a flat homomorphism of rings, where we are considering any of the following situations:

- $\varphi : k[x_1,\ldots,x_n] \longrightarrow k[[x_1,\ldots,x_n]]$
- $\varphi : k[x_1,\ldots,x_n] \longrightarrow k\{x_1,\ldots,x_n\}$

Given a good filtration $\{M(m)\}_{m\geq 0}$ on a finitely generated $D(R_1,k)$-module M, one may check that the extended filtration $\{M(m)\otimes_{R_1} R_2\}_{m\geq 0}$ is then a good filtration on the $D(R_2,k)$-module $M\otimes_{R_1} R_2$. The characteristic ideal of $M\otimes_{R_1} R_2$ is the extension of the characteristic ideal of M to $gr(D(R_2,k)) = gr(D(R_1,k))\otimes_{R_1} R_2$. So if the characteristic cycle of M is of the form:

$$CC(M) = \sum m_i V_i, \quad \text{where} \quad V_i = V(\mathfrak{p}_i) \quad \text{for} \quad \mathfrak{p}_i \in \operatorname{Spec}(gr(D(R_1,k))),$$

then, the characteristic cycle of $M\otimes_{R_1} R_2$ is:

$$CC(M\otimes_{R_1} R_2) = \sum m_i W_i, \quad \text{where} \quad W_i = V(\mathfrak{p}_i \otimes_{R_1} R_2).$$

We have to point out that in the situations we are considering, one can easily check that $\mathfrak{p}_i\otimes_{R_1} R_2 \in \operatorname{Spec}(gr(D(R_2,k)))$.

This result will be very useful for computing the characteristic cycle of $\mathcal{D}$-modules that have a good behavior with respect to flat base change. In particular we state that, roughly speaking, the formulas of the characteristic cycle of the base ring R, the localization $R[\frac{1}{f}]$, where $f\in k[x_1,\ldots,x_n]$ and the local cohomology modules $H^r_I(R)$, where $I \subseteq k[x_1, \ldots, x_n]$ are the same for R being any of the rings introduced in this section.

Example: Let $H^r_I(R_1)$ be a local cohomology module supported on a ideal $I \subseteq R_1 = k[x_1, \ldots, x_n]$. Let R_2 be any of the following rings:

- $k[[x_1, \ldots, x_n]]$, the formal power series ring.
- $k\{x_1, \ldots, x_n\}$, the ring of convergent series.

If the characteristic cycle of $H^r_I(R_1)$ is of the form:

$$CC(H^r_I(R_1)) = \sum m_i V_i, \quad \text{where} \quad V_i = V(\mathfrak{p}_i) \quad \text{for} \quad \mathfrak{p}_i \in \operatorname{Spec}(gr(D(R_1, k))),$$

then the characteristic cycle of $H^r_{IR_2}(R_2)$ is of the form:

$$CC(H^r_{IR_2}(R_2)) = \sum m_i W_i, \quad \text{where} \quad W_i = V(\mathfrak{p}_i \otimes_{R_1} R_2).$$

Characteristic cycle and coefficients: The results given in the previous section do not hold in general when we change the field of coefficients. This is due to the fact that the irreducible components of the characteristic variety may differ depending on the coefficients we are using.

Nevertheless, throughout this work we will mainly work with characteristic varieties defined by monomial ideals. In particular, their irreducible components do not depend on the field of coefficients. More precisely, let k and k' be fields of characteristic zero and consider a flat homomorphism of rings

$$\varphi : R_1 = k[x_1, \ldots, x_n] \longrightarrow R_2 = k'[x_1, \ldots, x_n].$$

Let M be a $D(R_1, k)$-module whose characteristic variety is defined by a monomial ideal. If the characteristic cycle of M is of the form:

$$CC(M) = \sum m_i V_i, \quad \text{where} \quad V_i = V(\mathfrak{p}_i) \quad \text{for} \quad \mathfrak{p}_i \in \operatorname{Spec}(gr(D(R_1, k))),$$

then, the characteristic cycle of $M \otimes_{R_1} R_2$ is:

$$CC(M \otimes_{R_1} R_2) = \sum m_i W_i, \quad \text{where} \quad W_i = V(\mathfrak{p}_i \otimes_{R_1} R_2).$$

Examples of $\mathcal{D}$-modules having characteristic varieties defined by monomial ideals include the base ring R, the localization $R[\frac{1}{\mathbf{x}^\alpha}]$, where $\mathbf{x}^\alpha$ is a squarefree monomial and the local cohomology modules $H^r_I(R)$, where $I \subseteq R$ is a squarefree monomial ideal.

Sheaf of differential operators: Sometimes, as it is usual in Algebraic and Analytic Geometry, it will be necessary to introduce the sheaf-like version of the $\mathcal{D}$-module theories introduced before. For details and examples we refer to [36], [52], [77].

We will not make a systematic use of the theory of sheaves of differential operators but the usual framework that we can find in the literature for many results (that will be important in our work), are these theories. For this reason we will fix some notations in order to give the precise statement of these results as they can be found in the literature.

Consider any of the following sheaves:

• $\mathcal{O}_X$, sheaf of holomorphic functions, where X is a complex analytic variety of dimension n.

• $\mathcal{O}_X$, sheaf of regular functions, where X is an algebraic variety of dimension n over a field of characteristic zero.

We can define the sheaf of differential operators in the obvious way. In any case it will be denoted by $\mathcal{D}_X$. Being careful at some points, one can give the sheaf-like version of the results given in sections 1.3.1 and 1.3.2.

1.3.4 Geometry of the characteristic cycle

An accurate study of the geometric properties of the irreducible varieties that appear in the characteristic cycle of a holonomic $\mathcal{D}_X$-module may be found in [77], also [52] and [68]. Our aim in this section is to explain these results. For this purpose we consider:

• $\mathcal{O}_X$, sheaf of holomorphic functions, where X is a complex analytic variety of dimension n.

Let $\pi : T^*X \longrightarrow X$ denote the canonical projection from the cotangent bundle. The characteristic variety of a coherent $\mathcal{D}_X$-module is a subvariety of T^*X due to the fact that the cotangent bundle T^*X may be identified with $\operatorname{Spec}(gr(\mathcal{D}_X))$.

The cotangent bundle T^*X is endowed with a global 1-form θ such that in local coordinates $(x_1, \dots, x_n, \xi_1, \dots, \xi_n)$ it is written $\theta = \sum_{i=1}^n \xi_i dx_i$. It makes T^*X into a symplectic manifold with a canonical 2-form $\omega = d\theta$, locally written $\omega = \sum_{i=1}^{n} d\xi_i \wedge dx_i$.

For any point $(x,\xi) \in T^*X$, ω defines an antisymmetric and non degenerate bilinear form on $T_{(x,\xi)}(T^*X)$. We denote $(-)^{\perp}$ the orthogonal with respect to ω. Let $V \subseteq T^*X$ be a closed variety. V is called **involutive** (resp. **lagrangian**) if for any smooth point $a \in V$ $(T_aV)^{\perp} \subseteq T_aV$ (resp. $(T_aV)^{\perp} = T_aV$).

• The characteristic variety $C(\mathcal{M})$ of a coherent $\mathcal{D}_X$-module $\mathcal{M} \neq 0$ is involutive. In particular we have $\dim C(\mathcal{M}) \geq n$ (Bernstein's inequality).

• The characteristic variety $C(\mathcal{M})$ of a holonomic $\mathcal{D}_X$-module $\mathcal{M} \neq 0$ is lagrangian. In particular we have $\dim C(\mathcal{M}) = n$.

Conical lagrangian varieties are completely determined. Namely, let $X_i \subseteq X$ be an analytic irreducible subset. The conormal bundle to X_i in X, denoted by $T^*_{X_i}X$, is an analytic conical and lagrangian irreducible subset of T^*X. It can be computed as follows:

Let X_i^0 be the smooth part of $X_i \subseteq X$. Set:

$$Z = \{v \in T^*X \mid p = \pi(v) \in X_i^0 \text{ and } v \text{ annihilates } T_pX\}.$$

Then, $T^*_{X_i}X$ is the closure of Z in T^*X.

Conversely, if $V_i \subseteq T^*X$ is an analytic conical and lagrangian irreducible subset of T^*X, then $V_i = T^*_{X_i}X$, where $X_i = \pi(V_i)$.

Thus the characteristic cycle of a holonomic $\mathcal{D}_X$-module $\mathcal{M}$ is in the form:

$$CC(\mathcal{M}) = \sum m_i \; T^*_{X_i}X.$$

In particular we can easily describe the support of $\mathcal{M}$ due to the fact that $\pi(T^*_{X_i}X) = X_i$, namely we have $\mathrm{Supp}_{\mathcal{O}_X}(\mathcal{M}) = \bigcup X_i$.

Finally we present the example of conormal bundle to a subvariety that we will mostly use during our work.

Example: Let $X_\alpha \subseteq X$ be the analytic subset defined by the face ideal $\mathfrak{p}_\alpha \subseteq \mathbb{C}[x_1,\dots,x_n]$, $\alpha \in \{0,1\}^n$. If $(x_1,\dots,x_n,\xi_1,\dots,\xi_n)$ is a local coordinates system of T^*X then, the conormal bundle to X_α in X is the subvariety defined by the equations $\{x_i = 0 \mid \alpha_i = 1\}$ and $\{\xi_i = 0 \mid \alpha_i = 0\}$, i.e.

$$T^*_{X_\alpha}X = V(\{x_i \mid \alpha_i = 1\}, \{\xi_i \mid \alpha_i = 0\}).$$

As usual we denote $T^*_XX = V(\xi_1,\dots,\xi_n)$ for the case $\alpha = (0,\dots,0) \in \{0,1\}^n$.

Remark 1.3.3. For any ring R we use throughout this work, we will make the following abuse of notations:

Let $X = \mathrm{Spec}\,(R)$, $T^*X = \mathrm{Spec}\,(R[\xi_1, \dots, \xi_n])$ and $\pi : T^*X \longrightarrow X$ be the canonical projection. Let V_i be a subvariety of T^*X and $\pi(V_i) = X_i$ be its projection. If V_i is irreducible it will be denoted by $T^*_{X_i}X$.

For simplicity, we will also refer to $T^*_{X_i}X$ as the conormal bundle to the subvariety X_i in X.

Example: Let k be a field of characteristic zero. Let $X_\alpha \subseteq X$ be the subset defined by the face ideal $\mathfrak{p}_\alpha \subseteq k[x_1, \dots, x_n]$, $\alpha \in \{0,1\}^n$. Then:

$$T^*_{X_\alpha}X = V(\{x_i \mid \alpha_i = 1\}, \{\xi_i \mid \alpha_i = 0\}).$$

1.3.5 Some examples and computations

Let k be a field of characteristic zero. We will consider any of the following rings:

- $R = k[[x_1, \dots, x_n]]$ the formal power series ring.
- $R = k\{x_1, \dots, x_n\}$ the convergent power series ring.
- $R = k[x_1, \dots, x_n]$ the polynomial ring.

In any case the ring of differential operators will be denoted by $\mathcal{D}$. The main examples of regular holonomic $\mathcal{D}$-modules we will use in this work are:

1) The ring R has a natural structure of regular holonomic $\mathcal{D}$-module.

2) Let M be a regular holonomic $\mathcal{D}$-module and $f \in R$. Then, the localization $M[\frac{1}{f}]$ is a regular holonomic $\mathcal{D}$-module.

3) Let M be a regular holonomic $\mathcal{D}$-module and $I \subseteq R$ the ideal generated by $\{f_1, \dots, f_r\}$. By using the Čech complex

$$\check{\mathrm{C}}^\bullet_I(M) : 0 \longrightarrow M \xrightarrow{d_0} \bigoplus_{1 \le i_1 \le r} M[\frac{1}{f_{i_1}}] \xrightarrow{d_1} \cdots \xrightarrow{d_{r-1}} M[\frac{1}{f_1 \cdots f_r}] \longrightarrow 0$$

and the properties of regular holonomic $\mathcal{D}$-modules with respect to exact sequences we may conclude that the local cohomology modules $H^r_I(M)$ are regular holonomic $\mathcal{D}$-modules for all r.

The characteristic cycle of R can be easily described. In the case of the localization $M[\frac{1}{f}]$ its characteristic cycle can also be described by means of the characteristic cycle of M, while for the case of the local cohomology modules $H^r_I(M)$ not much can be found in the literature except some particular cases.

1) **The characteristic cycle of R:** $CC(R) = T^*_X X$.

To compute the characteristic cycle of R we only have to consider the filtration $\Gamma^0 = \Gamma^1 = \cdots = R$. Then we have $gr_\Gamma(R) = R = R[y_1, \dots, y_n]/(y_1, \dots, y_n)$, so Γ is a good filtration, and the characteristic ideal is

$$J(R) = Ann_{gr_\Sigma(\mathcal{D})}(gr_\Gamma(R)) = (y_1, \dots, y_n).$$

Now it is easy to see that $C(R) = T^*_X X$, in particular the characteristic variety is irreducible and we get the desired result.

2) **The characteristic cycle of $M[\frac{1}{f}]$:** Even in many cases the characteristic cycle of $M[\frac{1}{f}]$ can be computed directly by means of an explicit presentation, the general answer can be found in [13] for the following situation:

Let $\mathcal{O}_X$ be the sheaf of holomorphic functions, where X is a complex analytic variety of dimension n. Starting from a result of V. Ginsburg [32, Theorem 3.3], J. Briançon, P. Maisonobe and M. Merle [13] give a geometric formula of the characteristic cycle of $\mathcal{M}[\frac{1}{f}]$ in terms of the characteristic cycle of $\mathcal{M}$, where $\mathcal{M}$ is a regular holonomic $\mathcal{D}_X$-module and $f \in \mathcal{O}_X$.

First we will recall how to compute the conormal bundle relative to f. Let Y^0 be the smooth part of an analytic subset $Y \subseteq X$ where $f|_Y$ is a submersion. Set:

$$W = \{v \in T^*X \ \mid \ p = \pi(v) \in Y^0 \ \text{ and } \ v \ \text{ annihilates } \ T_p(f|_Y)^{-1}(f(p))\}.$$

The conormal bundle relative to f, $T_{f|_Y}$, is then the closure of W in $T^*X|_Y$

Theorem 1.3.4 ([13]). *Let $CC(\mathcal{M}) = \sum m_i \ T^*_{X_i} X$ be the characteristic cycle of a regular holonomic $\mathcal{D}_X$-module $\mathcal{M}$.*

Considering the divisor defined by f on the conormal bundle relative to f $T_{f|_{X_i}}$ and the irreducible components, $\Gamma_{i,j}$, of this divisor with $m_{i,j}$ the multiplicity of the ideal defined by $\pi(\Gamma_{i,j})$, let:

$$\Gamma_i = \sum m_{i,j} \ \Gamma_{i,j}.$$

Then the characteristic cycle of $\mathcal{M}[\frac{1}{f}]$ *is:*

$$CC(\mathcal{M}[\frac{1}{f}]) = \sum\nolimits_{f(X_i)\neq 0} m_i \ (\Gamma_i + T^*_{X_i}X).$$

Example: Let R be any of the rings considered at the beginning of this section. Let $\mathbf{x}^\alpha$, $\alpha \in \{0,1\}^n$, be a squarefree monomial on the variables $x_1, \dots, x_n$. To compute the characteristic cycle of $R[\frac{1}{\mathbf{x}^\alpha}]$ we can reduce by flat base change to the case $R = \mathbb{C}\{x_1, \dots, x_n\}$ and then apply the results of Theorem 1.3.4.

With the notations given in the previous section we have the following formula:

The characteristic cycle of $R[\frac{1}{\mathbf{x}^\alpha}]$**:** $CC(R[\frac{1}{\mathbf{x}^\alpha}]) = \sum_{\beta\leq\alpha} \ T^*_{X_\beta}X.$

We will proceed by induction on $|\alpha|$ the number of variables of the monomial $f = \mathbf{x}^\alpha$:

Let $\alpha \in \{0,1\}^n$ such that $|\alpha| = 1$, i.e. $\mathbf{x}^\alpha = x_{i_1}$, where $1 \leq i_1 \leq n$. We want to compute the characteristic cycle of $R[\frac{1}{x_{i_1}}]$. We have $CC(R) = T^*_X X$ so by using the notations in Theorem 1.3.4, we get $Y = X$ and $f(X) \neq 0$. By definition, Y^0 is the nonsingular part of X where f is a submersion. We must look for the points in X such that the gradient of f is different from zero. Since $\nabla f = \varepsilon_{i_1}$ and X is nonsingular, we have $Y^0 = X$. Denote by $\mathcal{C}$ the hypersurface defined by $(f|_X)^{-1}(f(x_1, x_2, \dots, x_n))$. Then $T_{f|_X} \subseteq \ T^*X$ is the closure of:

$$\{v \in T^*X \ \mid \ z = \pi(v) \in X \ \text{ and } \ v \ \text{ annihilates } \ T_z\mathcal{C}\}.$$

Since $T_z\mathcal{C} =< -\varepsilon_j \ \mid \ j \neq i_1 >$ we have:

$$T_{f|_X} = \{(x_1, x_2, \dots, x_n, \xi_1, \xi_2, \dots, \xi_n) \in T^*X \ \mid \ \xi_i = 0 \ \text{ if } \ i \neq i_1\}.$$

So the divisor defined by f on $T_{f|_X}$ is:

$$\Gamma = \{(x_1, x_2, \dots, x_n, \xi_1, \xi_2, \dots, \xi_n) \in T^*X \ \mid \ x_{i_1} = 0 \ \text{ and } \ \xi_i = 0 \ \text{ if } \ i \neq i_1\}.$$

It has only a component with multiplicity 1, and it is easy to prove that $\Gamma = T^*_{X_\alpha}X$. So we get the desired result

$$CC(R[\frac{1}{x_{i_1}}]) = T^*_X X + T^*_{X_\alpha}X.$$

Let $\alpha \in \{0,1\}^n$ such that $|\alpha| = j > 1$, i.e. $\mathbf{x}^\alpha = x_{i_1} \cdots x_{i_j}$, where $1 \leq i_1 < \cdots < i_j \leq n$. To compute the characteristic cycle of $R[\frac{1}{x_{i_1} \cdots x_{i_j}}]$ we will consider the localization of the module $R[\frac{1}{x_{i_1} \cdots x_{i_{j-1}}}]$ by the monomial x_{i_j}. By induction we have

$$CC(R[\frac{1}{x_{i_1} \cdots x_{i_{j-1}}}]) = \sum_{\beta \leq \alpha - \varepsilon_{i_j}} T^*_{X_\beta} X.$$

Note that $f(X_\beta) \neq 0$ for all $\beta \leq \alpha - \varepsilon_{i_j}$ so for every component of the characteristic cycle we have $Y = Y^0 = X_\beta$. In every case we obtain:

$$T_{f|X_\beta} = \{(x_1, x_2, \dots, x_n, \xi_1, \xi_2, \dots, \xi_n) \in T^*X \mid \begin{array}{c} \xi_i = 0 \text{ if } \beta_i = 0 \text{ and } i \neq i_j \\ x_i = 0 \text{ if } \beta_i = 1 \end{array} \}.$$

The divisor defined by f on $T_{f|X_\beta}$ is:

$$\Gamma = \{(x_1, x_2, \dots, x_n, \xi_1, \xi_2, \dots, \xi_n) \in T^*X \mid \begin{array}{c} \xi_i = 0 \text{ if } \beta_i = 0 \text{ and } i \neq i_j \\ x_i = 0 \text{ if } \beta_i = 1 \text{ or } i = i_j \end{array} \}$$

It has only a component with multiplicity 1, and it is easy to prove that $\Gamma = T^*_{X_{\beta+\varepsilon_{i_j}}} X$. So we get the desired result

$$CC(R[\frac{1}{x_{i_1} \cdots x_{i_j}}]) = \sum_{\beta \leq \alpha - \varepsilon_{i_j}} (T^*_{X_\beta} X + T^*_{X_{\beta+\varepsilon_{i_j}}} X) = \sum_{\beta \leq \alpha} T^*_{X_\beta} X.$$

Remark 1.3.5. We have also proved the following:

Let M be a regular holonomic $\mathcal{D}$-module such that the corresponding characteristic cycle is $CC(M) = T^*_{X_\alpha} X$. Then, the characteristic cycle of the localization $M[\frac{1}{x_i}]$, where $\alpha_i = 0$, is:

$$CC(M[\frac{1}{x_i}]) = T^*_{X_\alpha} X + T^*_{X_{\alpha+\varepsilon_i}} X.$$

Remark 1.3.6. The characteristic cycle of $R[\frac{1}{x_1}]$ is computed in [36] by using the presentation:

$$R\,[\frac{1}{x_1}] \cong \frac{\mathcal{D}}{\mathcal{D}(x_1\partial_1 + 1, \partial_2, \dots, \partial_n)}.$$

In an analogous way, the characteristic cycle of $R[\frac{1}{\mathbf{x}^\alpha}]$ can also be computed by considering the following presentation:

$$R\,[\frac{1}{\mathbf{x}^\alpha}] \cong \frac{\mathcal{D}}{\mathcal{D}(\{x_i\partial_i + 1 \mid \alpha_i = 1\}, \{\partial_j \mid \alpha_j = 0\})}.$$

3) **The characteristic cycle of $H^r_I(M)$:** In the following chapters we will compute the characteristic cycle of the following local cohomology modules:

- $H^r_I(R)$, where $I \subseteq R$ is a monomial ideal.
- $H^p_{\mathfrak{p}_\alpha}(M)$, where $\mathfrak{p}_\alpha \subseteq R$ is a face ideal and $M = H^r_I(R)$ is a local cohomology module supported on a monomial ideal.

For the moment we present the following:

Example: Let R be any of the rings considered at the beginning of this section. Let $\mathfrak{p}_\alpha \subseteq R$, $\alpha \in \{0,1\}^n$, be a face ideal. By using the Čech complex

$$\check{\mathrm{C}}^\bullet_{\mathfrak{p}_\alpha} : \ 0 \longrightarrow R \xrightarrow{d_0} \bigoplus_{\alpha_i=1} R[\frac{1}{x_i}] \xrightarrow{d_1} \cdots \xrightarrow{d_{|\alpha|-1}} R[\frac{1}{\mathbf{x}^\alpha}] \longrightarrow 0,$$

the results of the previous example and the additivity of the characteristic cycle with respect to exact sequences we get:

The characteristic cycle of $H^{|\alpha|}_{\mathfrak{p}_\alpha}(R)$: $\quad CC(H^{|\alpha|}_{\mathfrak{p}_\alpha}(R)) = T^*_{X_\alpha}X.$

In particular, the characteristic variety of any local cohomology modules supported on a face ideal is irreducible.

Remark 1.3.7. This result can also be found, for example in [36], by using the following presentation:

$$H^{|\alpha|}_{\mathfrak{p}_\alpha}(R) \cong \frac{R[\frac{1}{\mathbf{x}^\alpha}]}{\sum_{\alpha_i=1} R[\frac{1}{\mathbf{x}^{\alpha-\varepsilon_i}}]} \cong \frac{\mathcal{D}}{\mathcal{D}(\{x_i \mid \alpha_i = 1\}, \{\partial_j \mid \alpha_j = 0\})}.$$

1.3.6 Solutions of a $\mathcal{D}_X$-module

We start considering a linear homogeneous ordinary differential equation:

(1) $\qquad c_m(x)\partial_x^m(u) + \cdots + c_1(x)\partial_x(u) + c_0(x)u = 0, \quad c_i(x) \in \mathbb{C}\{x\}.$

Let $P = c_m\partial_x^m + \cdots + c_1\partial_x + c_0 \in \mathcal{D} = D(\mathbb{C}\{x\}, \mathbb{C})$ be a differential operator. The equation (1) can be written as $P \cdot u = 0$ so given a $\mathcal{D}$-module F viewed as a space of functions, the solutions of P in F is the $\mathbb{C}$-vector space:

$$\mathrm{Sol}\,(P, F) = \{u \in F \mid P \cdot u = 0 \text{ in } F\}.$$

From the exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{D}}(\mathcal{D}/\mathcal{D}P, F) \longrightarrow F \xrightarrow{P} F \longrightarrow \mathrm{Ext}^1_{\mathcal{D}}(\mathcal{D}/\mathcal{D}P, F) \longrightarrow 0\ ,$$

we get an isomorphism of $\mathbb{C}$-vector spaces $\mathrm{Hom}_{\mathcal{D}}(\mathcal{D}/\mathcal{D}P, F) \cong \mathrm{Sol}\,(P, F)$ given by $\varphi \longrightarrow \varphi(\overline{1})$. In this case, the module $\mathrm{Ext}^1_{\mathcal{D}}(\mathcal{D}/\mathcal{D}P, F)$ can be understood as the obstruction to the resolution of the equation (1). Since $\mathrm{Sol}\,(P, F)$ as well the obstruction depend on the $\mathcal{D}$-module $M = \mathcal{D}/\mathcal{D}P$ instead of the operator P it will be more precise to consider the bounded complex:

$$\mathbb{S}\mathrm{ol}(M, F) := \mathbb{R}\mathrm{Hom}_{\mathcal{D}}(M, F).$$

We can generalize this construction to systems of linear partial differential equations

$$(2) \qquad \begin{cases} P_{11} \cdot u_1 + \cdots + P_{1m} \cdot u_m = 0 \\ \quad \vdots \qquad\qquad\qquad \vdots \qquad \vdots \\ P_{n1} \cdot u_1 + \cdots + P_{nm} \cdot u_m = 0 \end{cases}$$

where $P_{ij} \in \mathcal{D} = D(\mathbb{C}\{x_1, \ldots, x_n\}, \mathbb{C})$ are differential operators. The solutions of the system (2) in a $\mathcal{D}$-module F depend on the $\mathcal{D}$- module M described by the presentation:

$$\mathcal{D}^m \xrightarrow{(P_{ij})} \mathcal{D}^n \longrightarrow M \longrightarrow 0\ .$$

In this case we will also consider the bounded complex of $\mathbb{C}$-vector spaces $\mathbb{S}\mathrm{ol}(M, F) := \mathbb{R}\mathrm{Hom}_{\mathcal{D}}(M, F)$.

This construction also extends to sheaf theory. Namely, let X be an analytic variety of dimension n, $\mathcal{O}_X$ be the sheaf of holomorphic functions on X and $\mathcal{D}_X$ be the sheaf of differential operators. Let $\mathcal{M}, \mathcal{F}$ be $\mathcal{D}_X$-modules. The complex of solutions of $\mathcal{M}$ in $\mathcal{F}$ is the complex of sheaves of of $\mathbb{C}$-vector spaces $\mathbb{S}\mathrm{ol}(\mathcal{M}, \mathcal{F}) := \mathbb{R}\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{F})$.

In the next section we will be interested on the solutions in the sheaf $\mathcal{O}_X$. In this case we will denote the **complex of solutions** of a $\mathcal{D}_X$-module $\mathcal{M}$ by

$$\mathbb{S}\mathrm{ol}(\mathcal{M}) := \mathbb{R}\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X).$$

Nilsson class functions: If M is a regular holonomic $\mathcal{D}$-module, for $\mathcal{D} = D(\mathbb{C}\{x_1,\ldots,x_n\},\mathbb{C})$, the solutions of M in some space of functions are determined. First we will introduce some notation:

Consider $X = \mathbb{C}^n = \prod_{i=1}^n \mathbb{C}_i$, let $K_i = \mathbb{R}^+ \subset \mathbb{C}_i$ and set $V_i = \mathbb{C}_i \setminus K_i$. We define:

$$B_\epsilon = \{\mathbf{x} = (x_1,\ldots,x_n) \in \mathbb{C}^n \mid \|\mathbf{x}\| < \epsilon\}$$

$$\dot{B}_\epsilon = \{\mathbf{x} = (x_1,\ldots,x_n) \in \mathbb{C}^n \mid 0 < \|\mathbf{x}\| < \epsilon\}$$

$$B_\epsilon - \mathbb{R}^+ = \{\mathbf{x} = (x_1,\ldots,x_n) \in \textstyle\prod_{i=1}^n V_i \mid \|\mathbf{x}\| < \epsilon\}$$

If $\mathcal{O}_X(B_\epsilon - \mathbb{R}^+)$ is the set of holomorphic functions on $B_\epsilon - \mathbb{R}^+$ we denote by $\widetilde{\mathcal{O}}_X(B_\epsilon - \mathbb{R}^+)$ the set of holomorphic functions in $B_\epsilon - \mathbb{R}^+$ that are multi-valued holomorphic functions in $\dot{B}_\epsilon$. Then the following spaces of functions:

$$\mathcal{A} = \varinjlim_\epsilon \mathcal{O}_X(B_\epsilon - \mathbb{R}^+)$$

$$\widetilde{\mathcal{A}} = \varinjlim_\epsilon \widetilde{\mathcal{O}}_X(B_\epsilon - \mathbb{R}^+)$$

have a natural structure of $\mathcal{D}$-module. A finite sum

$$f = \sum_{\alpha,m} \varphi_{\alpha,m}(\mathbf{x})(\log \mathbf{x})^m \mathbf{x}^\alpha, \quad \text{where} \quad \varphi_{\alpha,m}(\mathbf{x}) \in \mathbb{C}\{\mathbf{x}\}$$

is called a **Nilsson class function**. If we denote by $\mathcal{N}$ the set of Nilsson class functions, then we have $\mathbb{C}\{\mathbf{x}\} \subset \mathcal{N} \subset \widetilde{\mathcal{A}} \subset \mathcal{A}$ due to the fact that

$$\mathbf{x}^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in \widetilde{\mathcal{A}}, \qquad \text{for } \alpha = (\alpha_1,\ldots,\alpha_n) \in \mathbb{C}^n.$$

$$(\log \mathbf{x})^m = (\log x_1)^{m_1} \cdots (\log x_n)^{m_n} \in \widetilde{\mathcal{A}}, \quad \text{for } m = (m_1,\ldots,m_n) \in (\mathbb{Z}^+)^n.$$

The solutions of a regular holonomic $\mathcal{D}$-module in these spaces of functions are determined. Namely we have the following equivalency:

i) M is a regular holonomic $\mathcal{D}$-module.

ii) The solutions $\mathrm{Hom}_{\mathcal{D}}(M,\mathcal{A}) = \mathrm{Hom}_{\mathcal{D}}(M,\widetilde{\mathcal{A}})$ are Nilsson class functions.

iii) The solutions $\mathrm{Hom}_{\mathcal{D}}(M,\mathcal{A}/\mathbb{C}\{\mathbf{x}\}) = \mathrm{Hom}_{\mathcal{D}}(M,\widetilde{\mathcal{A}}/\mathbb{C}\{\mathbf{x}\})$ are Nilsson class functions.

This result is a generalization of a classical result of Fuchs for ordinary differential equations. See [8] for details.

Kashiwara's constructibility theorem :

Let $\Sigma = \{\Sigma_j\}_{0\leq j\leq n}$ be a stratification of X. We say that a sheaf $\mathcal{F}$ over X of complex vector spaces is constructible with respect to Σ if the restriction to each stratum $\mathcal{F}|_{\Sigma_j}$ is a locally constant sheaf of finitely generated complex vector spaces.

More generally, a complex of sheaves $\mathcal{F}^\bullet$ over X of complex vector spaces is constructible with respect to Σ if the i-th homology group $h^i(\mathcal{F}^\bullet)$ is constructible for all i.

Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. M. Kashiwara [49] proves that the solution complex $\mathbb{S}\mathrm{ol}(\mathcal{M})$ is a constructible complex, bounded in degrees $[0, n]$, such that verifies the support condition:

$$\dim h^i(\mathbb{S}\mathrm{ol}(\mathcal{M})) \leq n - i, \quad \forall i \geq 0.$$

Kashiwara even proves that there exists a Whitney stratification $\Sigma = \{\Sigma_j\}_{0\leq j\leq n}$ such that $CC(\mathcal{M}) \subseteq \bigcup_j T^*_{\Sigma_j}X$ and $\mathbb{S}\mathrm{ol}(\mathcal{M})$ is constructible with respect to this stratification.

For a more geometric proof of Kashiwara's constructibility theorem the reader might see [65],[67].

1.3.7 Riemann-Hilbert correspondence

On dealing with complexes it will be convenient to work in the derived category $\mathbb{D}^b_{hr}(\mathcal{D}_X)$ of bounded complexes of $\mathcal{D}_X$-modules with regular holonomic cohomology. The solution functor $\mathbb{S}\mathrm{ol}$ restricted to $\mathbb{D}^b_{hr}(\mathcal{D}_X)$ establishes an equivalence of categories, known as the **Riemann-Hilbert correspondence**:

$$\mathbb{S}\mathrm{ol} : \mathbb{D}^b_{hr}(\mathcal{D}_X) \longrightarrow \mathbb{D}^b_c(\mathbb{C}_X),$$

where $\mathbb{D}^b_c(\mathbb{C}_X)$ is the derived category of bounded complexes of sheaves of complex vector spaces over X with constructible cohomology. It has been proved independently by Z. Mebkhout [63], [64] and M. Kashiwara–T. Kawai [50].

The constructible complexes of sheaves that correspond via $\mathbb{S}\mathrm{ol}$ to an object of $\mathrm{Mod}(\mathcal{D}_X)_{hr}$ can be nicely characterized.

An object $\mathcal{F}^\bullet \in \mathbb{D}^b_c(X)$ is called a **perverse sheaf** with respect to the stratification Σ if it satisfies:

i) $h^i(\mathcal{F}^\bullet) = 0$ for all $i \notin \{1, \dots, n\}$.

ii) The support of the sheaf $h^i(\mathcal{F}^\bullet)$ is contained in $\bigcup_{0 \le j \le n-i} \Sigma_j$.

iii) The Verdier dual $(\mathcal{F}^\bullet)^* = \mathbb{R}\mathcal{H}om_{\mathbb{C}}(\mathcal{F}^\bullet, \mathbb{C}_X)$ satisfies *i)* and *ii)*.

The category of perverse sheaves is denoted Perv (X). Note that a perverse sheaf is in general a complex of sheaves. For many reasons the category of perverse sheafs is important. See e.g. the survey article [58], (see also [7]).

Perverse sheaves are precisely those constructible complexes of sheaves that correspond to the objects of $\mathrm{Mod}(\mathcal{D}_X)_{hr}$ by the Riemann-Hilbert correspondence. Moreover, the solution functor $\mathbb{S}\mathrm{ol}$ restricted to $\mathrm{Mod}(\mathcal{D}_X)_{hr}$ establishes an equivalence of categories

$$\mathbb{S}\mathrm{ol} : \mathrm{Mod}(\mathcal{D}_X)_{hr} \longrightarrow \mathrm{Perv}\,(X).$$

A proof of this result can be found in [16].

Chapter 2

Some Numerical Invariants of Local Rings

Let $R = k[[x_1, \ldots, x_n]]$ be the formal power series ring in the independent variables $x_1, \ldots, x_n$ over a field k of characteristic zero. Set $\mathfrak{m} = (x_1, \ldots, x_n) \subseteq R$ and let $I \subseteq R$ be an ideal.

In Section 2.1 we recall the notion of Lyubeznik numbers and we describe them as the multiplicities of the characteristic cycle of the local cohomology modules $H^p_{\mathfrak{m}}(H^{n-i}_I(R))$.

In Section 2.2 we introduce some numerical invariants of local rings that generalize Lyubeznik numbers by using the multiplicities of the characteristic cycle of the modules $H^{n-i}_I(R)$ and $H^p_{\mathfrak{p}}(H^{n-i}_I(R))$, for any prime ideal $\mathfrak{p} \subseteq R$.

As we pointed out in Remark 1.3.3, for any ring R, we make the following abuse of notations:

Let $X = \mathrm{Spec}\,(R)$, $T^*X = \mathrm{Spec}\,(R[\xi_1, \ldots, \xi_n])$ and $\pi : T^*X \longrightarrow X$ be the canonical projection. Let V_i be a subvariety of T^*X and $\pi(V_i) = X_i$ be its projection. If V_i is irreducible it is denoted by $T^*_{X_i}X$. For simplicity, we refer to $T^*_{X_i}X$ as the conormal bundle to the subvariety X_i in X.

2.1 Lyubeznik numbers

In his paper [55] G. Lyubeznik uses the theory of algebraic $\mathcal{D}$-modules to study local cohomology modules. He proves, in particular, that if R is any regular ring containing a field of characteristic zero and $I \subseteq R$ an ideal, all the Bass numbers of the local cohomology modules $H_I^i(R)$ are finite. By using this property he defines a new set of numerical invariants for any local ring A containing a field, denoted by $\lambda_{p,i}(A)$.

Namely, let $(R, \mathfrak{m}, k)$ be a regular local ring of dimension n containing the field k, and A a local ring which admits a surjective ring homomorphism $\pi : R \longrightarrow A$. Set $I = \operatorname{Ker} \pi$. Then, $\lambda_{p,i}(A)$ is defined as the Bass number $\mu_p(\mathfrak{m}, H_I^{n-i}(R)) = \dim_k \operatorname{Ext}_R^p(k, H_I^{n-i}(R))$. This invariant depends only on A, i and p, but neither on R nor on π.

Completion does not change $\lambda_{p,i}(A)$ so if A contains a field but it is not necesarily the quotient of a regular local ring then one can define $\lambda_{p,i}(A) = \lambda_{p,i}(\hat{A})$ where $\hat{A}$ is the completion of A with respect to the maximal ideal. Therefore one can assume $R = k[[x_1, \dots, x_n]]$, with $x_1, \dots, x_n$ independent variables.

Lyubeznik also gives some properties of these numbers:

Proposition 2.1.1 ([55], Section 4). *Let $d = \dim A$. The Lyubeznik numbers $\lambda_{p,i}(A)$ have the following properties:*

i) $\lambda_{p,i}(A) = 0$ *if* $i > d$.

ii) $\lambda_{p,i}(A) = 0$ *if* $p > i$.

iii) $\lambda_{d,d}(A) \neq 0$.

U. Walther [95] defines the type of A as the tringular matrix given by $\lambda_{p,i}(A)$:

$$\Lambda(A) = \begin{pmatrix} \lambda_{0,0} & \cdots & \lambda_{0,d} \\ & \ddots & \vdots \\ & & \lambda_{d,d} \end{pmatrix}.$$

These invariants carry interesting topological information of the local ring A as it is pointed out by Lyubeznik. Namely, the following result is stated in [55]:

Let V be a scheme of finite type over $\mathbb{C}$ of dimension d and let A be the local ring of V at a closed point $q \in V$. If q is an isolated singular point of V then, from a theorem of A. Ogus [74] relating local cohomology and algebraic de Rham cohomology, and the comparison theorem between algebraic de Rham cohomology and singular cohomology proved by R. Hartshorne in [42], one gets

$$\lambda_{0,i}(A) = \dim_{\mathbb{C}} H^i_q(V,\mathbb{C}) \text{ for } \ 1 \leq i \leq d-1,$$

where $H^i_q(V,\mathbb{C})$ is the i-th singular local cohomology group of V with support in q and with coefficients in $\mathbb{C}$.

R. Garcia and C. Sabbah [29], generalize this result for the pure dimensional case by using the theory of $\mathcal{D}$-modules. Namely they prove:

Theorem 2.1.2 ([29]). *Let V be a scheme of finite type over $\mathbb{C}$ of pure dimension $d \geq 2$ and let A be the local ring of V at a closed point $q \in V$. If q is an isolated singular point of V then:*

i) $\lambda_{0,i}(A) = \dim_{\mathbb{C}} H^i_q(V,\mathbb{C})$ *for* $1 \leq i \leq d-1$.

ii) $\lambda_{p,d}(A) = \dim_{\mathbb{C}} H^{p+d}_q(V,\mathbb{C})$ *for* $2 \leq i \leq d$.

iii) All other $\lambda_{p,i}(A)$ vanish.

And for $d = 1$ all $\lambda_{p,i}(A)$ vanish except $\lambda_{1,1}(A)$ which is equal to 1.

They also express these Lyubeznik numbers in terms of Betti numbers of the associated real link.

When $d = 2$ these results also follow from the results of U. Walther [95]. In this paper he describes all the possible values of Lyubeznik numbers for any local ring A of $\dim(A) \leq 2$. By using the theory of Gröbner basis on $\mathcal{D}$-modules, the same author [93] gives some algorithms to determine a presentation as a $\mathcal{D}$-module of $H^i_I(R)$ and $H^p_{\mathfrak{m}}(H^i_I(R))$ for arbitrary i, p and to find $\lambda_{p,i}(R/I)$.

2.1.1 Lyubeznik numbers and characteristic cycles

Let $R = k[[x_1, \ldots, x_n]]$, where k is a field of characteristic zero, be the formal power series ring in the independent variables $x_1, \ldots, x_n$ and $\mathcal{D} = D(R, k)$ be its corresponding ring of differential operators. Set $\mathfrak{m} = (x_1, \ldots, x_n) \subseteq R$, $X_{\alpha_\mathfrak{m}} \subseteq \mathrm{Spec}\,(R)$ its corresponding variety and let $I \subseteq R$ be an ideal.

The Lyubeznik numbers $\lambda_{p,i}(R/I) := \mu_p(\mathfrak{m}, H_I^{n-i}(R))$ can be computed by using the multiplicities of the characteristic cycle of $H_\mathfrak{m}^p(H_I^{n-i}(R))$ as follows:

Lemma 2.1.3. *The characteristic cycle of $H_\mathfrak{m}^p(H_I^{n-i}(R))$ is of the form:*

$$CC(H_\mathfrak{m}^p(H_I^{n-i}(R))) = \lambda_{p,i}(R/I)\ T^*_{X_{\alpha_\mathfrak{m}}}X,$$

where $\lambda_{p,i}(R/I)$ is the corresponding Lyubeznik number.

Proof: By [55, Lemma 1.4] we have:

$$\mu_p(\mathfrak{m}, H_I^{n-i}(R)) = \mu_0(\mathfrak{m}, H_\mathfrak{m}^p(H_I^{n-i}(R))).$$

On the other side we have [55, Theorem 3.4]:

$$H_\mathfrak{m}^p(H_I^{n-i}(R)) = E_R(R/\mathfrak{m})^{\mu_0(\mathfrak{m}, H_\mathfrak{m}^p(H_I^{n-i}(R)))} = E_R(R/\mathfrak{m})^{\lambda_{p,i}(R/I)},$$

where $E_R(R/\mathfrak{m})$ is the injective hull of the residue field of R.

Notice that the characteristic cycle of $E_R(R/\mathfrak{m})$ can be computed by using the isomorphism $H_\mathfrak{m}^n(R) \cong E_R(R/\mathfrak{m})$ and the results of Section 1.3.5. So, by the additivity of the characteristic cycle we have

$$CC(H_\mathfrak{m}^p(H_I^{n-i}(R))) = \lambda_{p,i}(R/I)\ T^*_{X_{\alpha_\mathfrak{m}}}X.$$

□

2.2 Multiplicities of the Characteristic Cycle

Our aim in this section is to introduce some new numerical invariants by using the multiplicities of the characteristic cycle of certain local cohomology modules. From now on in this section, given a field of characteristic zero k we will consider the formal power series ring $R = k[[x_1, \ldots, x_n]]$ on the independent variables $x_1, \ldots, x_n$.

Let $I \subseteq R$ be an ideal, $\mathfrak{p} \subseteq R$ be a prime ideal and $\mathfrak{m} = (x_1, \ldots, x_n)$ be the maximal ideal. We have seen in the previous section that Lyubeznik numbers are multiplicities of the characteristic cycle of the local cohomology modules $H^p_{\mathfrak{m}}(H^{n-i}_I(R))$. We will generalize these invariants by proving that the following multiplicities are also invariants of the local ring R/I:

1) The multiplicities of the characteristic cycle of $H^{n-i}_I(R)$.

2) The multiplicities of the characteristic cycle of $H^p_{\mathfrak{p}}(H^{n-i}_I(R))$. Among these multiplicities we will find:

   i) The Bass numbers $\mu_p(\mathfrak{p}, H^{n-i}_I(R))$.

   ii) The Lyubeznik numbers $\lambda_{p,i}(R/I) := \mu_p(\mathfrak{m}, H^{n-i}_I(R))$ for the case $\mathfrak{p} = \mathfrak{m}$.

Remark 2.2.1. It is worthwhile to point out that the multiplicities of the characteristic cycle of $H^{n-i}_I(R)$ and $H^p_{\mathfrak{p}}(H^{n-i}_I(R))$ will be also invariants of the ring R/I in the following cases:

- $R = k\{x_1, \ldots, x_n\}$ the convergent power series ring.
- $R = k[x_1, \ldots, x_n]$ the polynomial ring.

This can be easily proved by using the results in Section 1.3.3 on characteristic cycle and flat base change.

2.2.1 Multiplicities of $CC(H^{n-i}_I(R))$

The main goal of this section is the following:

Theorem 2.2.2. *Let A be a local ring which admits a surjective ring homomorphism $\pi : R \longrightarrow A$, where $R = k[[x_1, \ldots, x_n]]$ is the formal power series ring. Set $I = \ker \pi$ and let*

$$CC(H^{n-i}_I(R)) = \sum m_{i,\alpha} \; T^*_{X_\alpha} X,$$

be the characteristic cycle of the local cohomology modules $H^{n-i}_I(R)$. Then the multiplicities $m_{i,\alpha}$ depends only on A, i and α but neither on R nor on π.

The proof of the theorem is analogous to the proof of [55, Theorem 4.1], but here we must be careful with the behavior of the characteristic cycle so instead of [55, Lemma 4.3] we will use the following:

Lemma 2.2.3. *Let $g : R' \longrightarrow R$ be a surjective ring homomorphism, where R' is a formal power series ring of dimension n'. Set $I' = \ker \pi g$ and let*

$$CC(H_I^{n-i}(R)) = \sum m_{i,\alpha} \; T^*_{X_\alpha} X,$$

be the characteristic cycle of the local cohomology modules $H_I^{n-i}(R)$. Then the characteristic cycle of $H_{I'}^{n'-i}(R')$ is

$$CC(H_{I'}^{n'-i}(R')) = \sum m_{i,\alpha} \; T^*_{X'_\alpha} X',$$

where X'_α is the subvariety of $X' = \operatorname{Spec} R'$ defined by the defining ideal of X_α contracted to R'.

Proof: R is regular so $\operatorname{Ker} g$ is generated by $n'-n$ elements that form part of a minimal system of generators of the maximal ideal $\mathfrak{m}' \subseteq R'$. By induction on $n'-n$ we are reduced to the case $n'-n=1$, so $\operatorname{Ker} g$ is generated by one element $f \in \mathfrak{m}' \setminus \mathfrak{m}'^2$. By Cohen's structure theorem $R' = k[[x_1, \dots, x_n, t]]$ where we assume $f = t$ by a change of variables. We identify R with the subring $k[[x_1, \dots, x_n]]$ of R'. In particular we have to consider $I' = IR' + (t)$.

In this situation we can apply the results of Section 1.3.2 on the direct image functor i_+. By Lemma 1.3.2 we have:

$$i_+(H_I^{n-i}(R)) = H^1_{(t)}(H^{n-i}_{IR'}(R')).$$

The spectral sequence $E_2^{p,q} = H^p_{(t)}(H^q_{IR'}(R')) \Longrightarrow H^{p+q}_{IR'+(t)}(R')$ degenerates at the E_2-term due to the fact that $H^q_{IR'}(R')$ is a (t)-torsion free module. In particular, we get

$$i_+(H_I^{n-i}(R)) = H^{n-i+1}_{IR'+(t)}(R').$$

Now if $CC(H_I^{n-i}(R)) = \sum m_{i,\alpha} \; T^*_{X_\alpha} X$, then the characteristic cycle of $i_+(H_I^{n-i}(R))$ can be computed by using the results of Section 1.3.2. Namely, we get the desired result:

$$CC(H^{n-i+1}_{IR'+(t)}(R')) = \sum m_{i,\alpha} \; T^*_{X'_\alpha} X'.$$

Notice that if X_α is the subvariety of $X = \operatorname{Spec} R$ defined by the ideal $\mathfrak{p}_\alpha$ then X'_α is the subvariety of $X' = \operatorname{Spec} R'$ defined by the ideal $\mathfrak{p}_\alpha R' + (t)$.

□

Now we continue the proof of Theorem 2.2.2.

PROOF: Let $\pi' : R' \longrightarrow A$ and $\pi'' : R'' \longrightarrow A$ be surjections with $R' = k[[y_1, \dots, y_{n'}]]$ and $R'' = k[[z_1, \dots, z_{n''}]]$. Let $I' = \ker \pi'$ and let $I'' = \ker \pi''$. Let $R''' = R' \widehat{\otimes}_k R''$ be the external tensor product, $\pi''' = \pi' \widehat{\otimes}_k \pi'' : R' \widehat{\otimes}_k R'' \longrightarrow A$ and $I''' = \ker \pi'''$.

By Lemma 2.2.3, if the characteristic cycle of $H^{n'-i}_{I'}(R')$ is

$$CC(H^{n'-i}_{I'}(R')) = \sum m'_{i,\alpha} \; T^*_{X'_\alpha} X',$$

then the characteristic cycle of $H^{n'+n''-i}_{I'''}(R''')$ is

$$CC(H^{n'+n''-i}_{I'''}(R')) = \sum m'_{i,\alpha} \; T^*_{X'''_\alpha} X''',$$

where X'''_α is the subvariety of $X''' = \operatorname{Spec} R'''$ defined by the defining ideal of X'_α contracted to R'''.

By Lemma 2.2.3, if the characteristic cycle of $H^{n''-i}_{I''}(R')$ is

$$CC(H^{n''-i}_{I''}(R'')) = \sum m''_{i,\alpha} \; T^*_{X''_\alpha} X'',$$

then the characteristic cycle of $H^{n'+n''-i}_{I'''}(R''')$ is

$$CC(H^{n'+n''-i}_{I'''}(R')) = \sum m''_{i,\alpha} \; T^*_{X'''_\alpha} X''',$$

where X'''_α is the subvariety of $X''' = \operatorname{Spec} R'''$ defined by the defining ideal of X''_α contracted to R'''.

In particular we have $m'_{i,\alpha} = m''_{i,\alpha}$ for all i and α.

□

Collecting the multiplicities of the characteristic cycle of the local cohomology modules $H^{n-i}_I(R)$ by the dimension of the corresponding irreducible varieties we define the following invariants:

Definition 2.2.4. *Let $I \subseteq R$ be an ideal. If $CC(H_I^{n-i}(R)) = \sum m_{i,\alpha}\ T^*_{X_\alpha}X$ is the characteristic cycle of the local cohomology modules $H_I^{n-i}(R)$ then we define:*

$$\gamma_{p,i}(R/I) := \{\sum m_{i,\alpha} \mid \dim X_\alpha = p\}.$$

These invariants have the same properties as Lyubeznik numbers. Namely:

Proposition 2.2.5. *Let $d = \dim R/I$. The invariants $\gamma_{p,i}(R/I)$ have the following properties*

i) $\gamma_{p,i}(R/I) = 0$ *if* $i > d$.

ii) $\gamma_{p,i}(R/I) = 0$ *if* $p > i$.

iii) $\gamma_{d,d}(R/I) \neq 0$.

PROOF:

i) Since $H_I^{n-i}(R)$ vanishes for $n - i < \operatorname{ht} I$, i.e. $i > d$, the corresponding multiplicities $\gamma_{p,i}(R/I)$ vanish for $i > d$.

ii) The statement is equivalent to $\dim \operatorname{Supp}_R H_I^{n-i}(R) \leq i$ by using the description of the support as R-module of a $\mathcal{D}$-module given in Section 1.3.1.

Suppose there exists a prime ideal $\mathfrak{p} \in \operatorname{Supp} H_I^{n-i}(R)$ such that $\operatorname{ht} \mathfrak{p} < n - i$. Then $(H_I^{n-i}(R))_{\mathfrak{p}} = H_{IR_{\mathfrak{p}}}^{n-i}(R_{\mathfrak{p}}) \neq 0$ but we get a contradiction because in this case we have $\operatorname{ht}(IR_{\mathfrak{p}}) \geq n - i > \operatorname{ht} \mathfrak{p} = \dim R_{\mathfrak{p}}$.

iii) Let $I = I_{\alpha_1} \cap \cdots \cap I_{\alpha_m}$ be a minimal primary decomposition such that $\operatorname{ht} I_{\alpha_m} = \operatorname{ht} I = h$. Notice that $\operatorname{ht}((I_{\alpha_1} \cap \cdots \cap I_{\alpha_{m-1}}) + I_{\alpha_m}) > h$ so, by using the Mayer-Vietoris sequence, we get an inclusion

$$0 \longrightarrow \begin{array}{c} H^h_{I_{\alpha_1}\cap\cdots\cap I_{\alpha_{m-1}}}(R) \\ \oplus \\ H^h_{I_{\alpha_m}}(R) \end{array} \longrightarrow H_I^h(R) \longrightarrow H^{h+1}_{(I_{\alpha_1}\cap\cdots\cap I_{\alpha_{m-1}})+I_{\alpha_m}}(R) \longrightarrow \cdots.$$

Let $\mathfrak{p} = \operatorname{rad}(I_{\alpha_m})$. Since $H^h_{I_{\alpha_m}R_{\mathfrak{p}}}(R_{\mathfrak{p}}) \neq 0$ we have $\mathfrak{p} \in \operatorname{Supp} H^h_{I_{\alpha_m}}(R)$. By the additivity of the characteristic cycle we have $CC(H^h_{I_{\alpha_m}}(R)) \subseteq CC(H_I^h(R))$ so $\mathfrak{p} \in \operatorname{Supp} H_I^h(R)$ and we get the desired result.

□

From now on we will denote by $\Gamma(A)$ the tringular matrix given by the invariants $\gamma_{p,i}(R/I)$.

$$\Gamma(A) = \begin{pmatrix} \gamma_{0,0} & \cdots & \gamma_{0,d} \\ & \ddots & \vdots \\ & & \gamma_{d,d} \end{pmatrix}.$$

We will see in Chapter 3 that these invariants $\gamma_{p,i}(R/I)$ are finer than the Lyubeznik numbers $\lambda_{p,i}(R/I)$.

2.2.2 Multiplicities of $CC(H^p_{\mathfrak{p}}(H^{n-i}_I(R)))$

Let A be a ring that admits a presentation $A \cong R/I$, where $I \subseteq R$ is an ideal. Recall that we have $\operatorname{Spec}(A) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid I \subseteq \mathfrak{p}\}$. Throughout this section, a prime ideal of A will also mean the corresponding prime ideal of R that contains I.

The main goal of this section is the following:

Theorem 2.2.6. *Let A be a local ring which admits a surjective ring homomorphism $\pi : R \longrightarrow A$, where $R = k[[x_1, \dots, x_n]]$ is the formal power series ring. Set $I = \ker \pi$, let $\mathfrak{p} \subseteq A$ be a prime ideal and let*

$$CC(H^p_{\mathfrak{p}}(H^{n-i}_I(R))) = \sum \lambda_{\mathfrak{p},p,i,\alpha}\ T^*_{X_\alpha}X,$$

be the characteristic cycle of the local cohomology modules $H^p_{\mathfrak{p}}(H^{n-i}_I(R))$. Then the multiplicities $\lambda_{\mathfrak{p},p,i,\alpha}$ depends only on A, $\mathfrak{p}$, p, i and α but neither on R nor on π.

The proof of the theorem is analogous to the proof of Theorem 2.2.2. First we have to introduce the notation for the lemma we will use. Let R/I and R'/I' be two different presentations of the local ring A. Then, for any prime ideal of A, we will denote $\mathfrak{p}' \in \operatorname{Spec}(R')$ the prime ideal that corresponds to $\mathfrak{p} \in \operatorname{Spec}(R)$ by the isomorphism $\operatorname{Spec}(R/I) \cong \operatorname{Spec}(R'/I')$

Lemma 2.2.7. *Let $g : R' \longrightarrow R$ be a surjective ring homomorphism, where R' is a formal power series ring of dimension n'. Set $I' = \ker \pi g$ and let*

$$CC(H^p_{\mathfrak{p}}(H^{n-i}_I(R))) = \sum \lambda_{\mathfrak{p},p,i,\alpha}\ T^*_{X_\alpha}X,$$

be the characteristic cycle of the local cohomology modules $H^p_{\mathfrak{p}}(H^{n-i}_I(R))$. Then, the characteristic cycle of $H^p_{\mathfrak{p}'}(H^{n'-i}_{I'}(R'))$ is

$$CC(H^p_{\mathfrak{p}'}(H^{n'-i}_{I'}(R'))) = \sum \lambda_{\mathfrak{p},p,i,\alpha}\ T^*_{X'_\alpha}X',$$

where X'_α is the subvariety of $X' = \operatorname{Spec} R'$ defined by the defining ideal of X_α contracted to R'.

Proof: As in the proof of Lemma 2.2.3 we only have to consider the case $R' = k[[x_1, \dots, x_n, t]]$, identifying R with the subring $k[[x_1, \dots, x_n]]$ of R' and considering $I' = IR' + (t)$. In this case we also have $\mathfrak{p}' = \mathfrak{p}R' + (t)$

Let $M = H^{n-i}_I(R)$ be the local cohomology module and let $\mathfrak{p} = (f_1, \dots, f_r) \subseteq R$ be the prime ideal. In order to compute $H^p_{\mathfrak{p}}(H^{n-i}_I(R))$ we use the Čech complex:

$$\check{\mathrm{C}}^\bullet_{\mathfrak{p}}(M): \quad 0 \longrightarrow M \longrightarrow \oplus M[\frac{1}{f_i}] \longrightarrow \oplus M[\frac{1}{f_i f_j}] \longrightarrow \cdots \longrightarrow M[\frac{1}{f_1 \cdots f_r}] \longrightarrow 0.$$

Let i_+ be the direct image functor given in Section 1.3.2. Now we want to compute $H^p_{\mathfrak{p}R'+(t)}(i_+(M))$, where $i_+(M) = H^{n+1-i}_{IR'+(t)}(R')$.

Notice that we only have to compute $H^p_{\mathfrak{p}R'}(i_+(M))$ due to the fact that $i_+(M)$ is a (t)-torsion module. Namely, by using the Grothendieck's spectral sequence $E_2^{p,q} = H^p_{\mathfrak{p}R'}(H^q_{(t)}(i_+(M))) \Longrightarrow H^{p+q}_{\mathfrak{p}R'+(t)}(i_+(M))$ we have:

$$H^p_{\mathfrak{p}R'+(t)}(i_+(M)) = H^p_{\mathfrak{p}R'}(H^0_{(t)}(i_+(M))) = H^p_{\mathfrak{p}R'}(i_+(M)).$$

We will compute the local cohomology module $H^p_{\mathfrak{p}R'}(i_+(M))$ by means of the Čech complex:

$$\check{\mathrm{C}}^\bullet_{\mathfrak{p}R'}(i_+(M)): \quad 0 \longrightarrow i_+(M) \longrightarrow \oplus i_+(M)[\frac{1}{f_i}] \longrightarrow \cdots \longrightarrow i_+(M)[\frac{1}{f_1 \cdots f_r}] \longrightarrow 0.$$

We point out that this complex is obtained by applying i_+ to the complex $\check{\mathrm{C}}^\bullet_{\mathfrak{p}}(M)$ and observing that $i_+(M[\frac{1}{f}]) = i_+(M)[\frac{1}{f}]\ \forall f \in R$. Roughly speaking, we state that in order to compute $H^p_{\mathfrak{p}R'}(i_+(M))$ we have to make the same computations as for $H^p_{\mathfrak{p}}(M)$ but using $i_+(M)$ instead of M.

Now, let $CC(H^p_{\mathfrak{p}}(H^{n-i}_I(R))) = \sum \lambda_{\mathfrak{p},p,i,\alpha}\ T^*_{X_\alpha}X$ be the characteristic cycle of the local cohomology module $H^p_{\mathfrak{p}}(H^{n-i}_I(R))$. Then, by using the results of Section 1.3.2 on the characteristic cycle of direct images, we get the desired result:

$$CC(H^p_{\mathfrak{p}R'}(H^{n+1-i}_{IR'+(t)}(R'))) = \sum \lambda_{\mathfrak{p},p,i,\alpha}\ T^*_{X'_\alpha}X'.$$

□

Now we continue the proof of Theorem 2.2.6.

Proof: Let $\pi' : R' \longrightarrow A$ and $\pi'' : R'' \longrightarrow A$ be surjections with $R' = k[[y_1, \dots, y_{n'}]]$ and $R'' = k[[z_1, \dots, z_{n''}]]$. Let $I' = \ker \pi'$ and let $I'' = \ker \pi''$. Let $R''' = R'\widehat{\otimes}_k R''$ be the external tensor product, $\pi''' = \pi'\widehat{\otimes}_k \pi'' : R'\widehat{\otimes}_k R'' \longrightarrow A$ and $I''' = \ker \pi'''$. Let $\mathfrak{p}', \mathfrak{p}'', \mathfrak{p}'''$ be the prime ideals of R', R'', R''' respectively that correspond to a prime ideal $\mathfrak{p}$ of A.

By Lemma 2.2.7, if the characteristic cycle of $H^p_{\mathfrak{p}'}(H^{n'-i}_{I'}(R'))$ is

$$CC(H^p_{\mathfrak{p}'}(H^{n'-i}_{I'}(R'))) = \sum \lambda'_{\mathfrak{p},p,i,\alpha}\ T^*_{X'_\alpha}X',$$

then the characteristic cycle of $H^p_{\mathfrak{p}'''}(H^{n'+n''-i}_{I'''}(R'''))$ is

$$CC(H^p_{\mathfrak{p}'''}(H^{n'+n''-i}_{I'''}(R'''))) = \sum \lambda'_{\mathfrak{p},p,i,\alpha}\ T^*_{X'''_\alpha}X''',$$

where X'''_α is the subvariety of $X''' = \operatorname{Spec} R'''$ defined by the defining ideal of X'_α contracted to R'''.

By Lemma 2.2.7, if the characteristic cycle of $H^p_{\mathfrak{p}''}(H^{n''-i}_{I''}(R''))$ is

$$CC(H^p_{\mathfrak{p}''}(H^{n''-i}_{I''}(R''))) = \sum \lambda''_{\mathfrak{p},p,i,\alpha}\ T^*_{X''_\alpha}X'',$$

then the characteristic cycle of $H^p_{\mathfrak{p}'''}(H^{n'+n''-i}_{I'''}(R'''))$ is

$$CC(H^p_{\mathfrak{p}'''}(H^{n'+n''-i}_{I'''}(R'''))) = \sum \lambda''_{\mathfrak{p},p,i,\alpha}\ T^*_{X'''_\alpha}X''',$$

where X'''_α is the subvariety of $X''' = \operatorname{Spec} R'''$ defined by the defining ideal of X''_α contracted to R'''.

In particular we have $\lambda'_{\mathfrak{p},p,i,\alpha} = \lambda''_{\mathfrak{p},p,i,\alpha}$ for all p, i and α.

□

Among the multiplicities of the characteristic cycle of $H^p_{\mathfrak{p}}(H^{n-i}_I(R))$ we can find the Bass numbers $\mu_p(\mathfrak{p}, H^{n-i}_I(R))$ of the local cohomology modules $H^{n-i}_I(R)$. Namely we have:

Proposition 2.2.8. *Let $I \subseteq R$ be an ideal, $\mathfrak{p} \subseteq R$ a prime ideal and*

$$CC(H^p_{\mathfrak{p}}(H^{n-i}_I(R))) = \sum \lambda_{\mathfrak{p},p,i,\alpha}\ T^*_{X_\alpha} X$$

be the characteristic cycle of the local cohomology module $H^p_{\mathfrak{p}}(H^{n-i}_I(R))$. Then, the Bass numbers with respect to $\mathfrak{p}$ of $H^{n-i}_I(R)$ are

$$\mu_p(\mathfrak{p}, H^{n-i}_I(R)) = \lambda_{\mathfrak{p},p,i,\alpha_{\mathfrak{p}}},$$

where $X_{\alpha_{\mathfrak{p}}}$ is the subvariety of $X = \operatorname{Spec}(R)$ defined by $\mathfrak{p}$.

Proof: Let $\widehat{R_{\mathfrak{p}}}$ be the completion with respect to the maximal ideal $\mathfrak{p}R_{\mathfrak{p}}$ of the localization $R_{\mathfrak{p}}$. Notice that $\widehat{R_{\mathfrak{p}}}$ is a formal power series ring of dimension $\operatorname{ht}\mathfrak{p}$. Since Bass numbers are invariant by completion we have:

$$\mu_p(\mathfrak{p}, H^{n-i}_I(R)) = \mu_0(\mathfrak{p}\widehat{R_{\mathfrak{p}}}, H^{n-i}_{I\widehat{R_{\mathfrak{p}}}}(\widehat{R_{\mathfrak{p}}})) = \mu_0(\mathfrak{p}\widehat{R_{\mathfrak{p}}}, H^p_{\mathfrak{p}\widehat{R_{\mathfrak{p}}}}(H^{n-i}_{I\widehat{R_{\mathfrak{p}}}}(\widehat{R_{\mathfrak{p}}}))),$$

where the last assertion follows from [55, Lemma 1.4]. As we did in Lemma 2.1.3 for Lyubeznik numbers, it will be enough to compute the characteristic cycle of $H^p_{\mathfrak{p}\widehat{R_{\mathfrak{p}}}}(H^{n-i}_{I\widehat{R_{\mathfrak{p}}}}(\widehat{R_{\mathfrak{p}}}))$. Namely, by using [55, Theorem 3.4] we have:

$$H^p_{\mathfrak{p}\widehat{R_{\mathfrak{p}}}}(H^{n-i}_{I\widehat{R_{\mathfrak{p}}}}(\widehat{R_{\mathfrak{p}}})) = E_{\widehat{R_{\mathfrak{p}}}}(\widehat{R_{\mathfrak{p}}}/\mathfrak{p}\widehat{R_{\mathfrak{p}}})^{\mu_0(\mathfrak{p}\widehat{R_{\mathfrak{p}}}, H^p_{\mathfrak{p}\widehat{R_{\mathfrak{p}}}}(H^{n-i}_{I\widehat{R_{\mathfrak{p}}}}(\widehat{R_{\mathfrak{p}}})))}.$$

So, its characteristic cycle is:

$$CC(H^p_{\mathfrak{p}\widehat{R_{\mathfrak{p}}}}(H^{n-i}_{I\widehat{R_{\mathfrak{p}}}}(\widehat{R_{\mathfrak{p}}}))) = \mu_p(\mathfrak{p}, H^{n-i}_I(R))\ T^*_{X'_{\alpha_{\mathfrak{p}}}} X',$$

where $X'_{\alpha_{\mathfrak{p}}}$ is the subvariety of $X' = \operatorname{Spec}\widehat{R_{\mathfrak{p}}}$ defined by the ideal $\mathfrak{p}\widehat{R_{\mathfrak{p}}}$. Notice that we have used the following fact:

$$CC(E_{\widehat{R_{\mathfrak{p}}}}(\widehat{R_{\mathfrak{p}}}/\mathfrak{p}\widehat{R_{\mathfrak{p}}})) = CC(H^{\operatorname{ht}\mathfrak{p}}_{\mathfrak{p}\widehat{R_{\mathfrak{p}}}}(\widehat{R_{\mathfrak{p}}})) = T^*_{X'_{\alpha_{\mathfrak{p}}}} X'.$$

Finally, to get the desired result, we only have to point out that this characteristic cycle can be obtained from the characteristic cycle of $H^p_{\mathfrak{p}}(H^{n-i}_I(R))$ by flat base change. Namely, if

$$CC(H^p_{\mathfrak{p}}(H^{n-i}_I(R))) = \sum \lambda_{\mathfrak{p},p,i,\alpha}\ T^*_{X_\alpha} X$$

is the characteristic cycle of the module $H^p_{\mathfrak{p}}(H^{n-i}_I(R))$, then we have

$$CC(H^p_{\mathfrak{p}\widehat{R_{\mathfrak{p}}}}(H^{n-i}_{I\widehat{R_{\mathfrak{p}}}}(\widehat{R_{\mathfrak{p}}}))) = \lambda_{\mathfrak{p},p,i,\alpha_{\mathfrak{p}}} \; T^*_{X'_{\alpha_{\mathfrak{p}}}} X'.$$

□

As we have seen, Bass numbers $\mu_p(\mathfrak{p}, H^{n-i}_I(R))$ of the local cohomology modules $H^{n-i}_I(R)$ are multiplicities of the characteristic cycle of $H^p_{\mathfrak{p}}(H^{n-i}_I(R))$. Then, by Theorem 2.2.6, we obtain the following:

Corollary 2.2.9. *Let A be a ring which admits a surjective ring homomorphism $\pi : R \longrightarrow A$, where $R = k[[x_1, \ldots, x_n]]$ is the formal power series ring. Set $I = \ker \pi$ and let $\mathfrak{p} \subseteq A$ be a prime ideal. The Bass numbers $\mu_p(\mathfrak{p}, H^{n-i}_I(R))$ depends only on A, $\mathfrak{p}$, p, i and α but neither on R nor on π.*

When $\mathfrak{p} = \mathfrak{m}$ is the maximal ideal, as a consequence of Lemma 2.1.3 and Theorem 2.2.6 we get the invariance of the Lyubeznik numbers:

Corollary 2.2.10 ([55], Theorem 4.1). *Let A be a ring which admits a surjective ring homomorphism $\pi : R \longrightarrow A$, where $R = k[[x_1, \ldots, x_n]]$ is the formal power series ring. Set $I = \ker \pi$ and let $\mathfrak{m} \subseteq R$ be the homogeneous maximal ideal. The Lyubeznik numbers $\lambda_{p,i}(A) := \mu_p(\mathfrak{m}, H^{n-i}_I(R))$ depends only on A, p and i but neither on R nor on π.*

Chapter 3

Characteristic cycle of local cohomology modules

Let k be a field of characteristic zero. Throughout this chapter we will consider any of the following rings:

- $R = k[[x_1, \dots, x_n]]$ the formal power series ring.
- $R = k\{x_1, \dots, x_n\}$ the convergent power series ring.
- $R = k[x_1, \dots, x_n]$ the polynomial ring.

In Section 3.1 we will describe the characteristic cycle of the local cohomology modules $H_I^r(R)$ for a given ideal $I \subseteq R$ generated by monomials such that the quotient ring R/I is Cohen-Macaulay. We will pay some special attention to the case of local cohomology modules supported on face ideals since they will play a key role in the rest of the chapter.

In Section 3.2 we present the main result of this chapter. Namely we will give a formula for the characteristic cycle of the local cohomology modules $H_I^r(R)$ for a given ideal $I \subseteq R$ generated by squarefree monomials.

Finally, we will provide several consequences of the formula for the characteristic cycle $CC(H_I^r(R))$ in Section 3.3. In particular we will study the following topics:

· Annihilation and support of local cohomology modules $H_I^r(R)$.

· Arithmetical properties of the quotient ring R/I.

· Some combinatorial relations between the multiplicities of $CC(H_I^r(R))$ and some invariants of the Stanley-Reisner ring R/I, e.g. the f-vector and the h-vector.

· The relation between the multiplicities of $CC(H_I^r(R))$ and the Betti numbers of the Alexander dual ideal $I^\vee$.

3.1 Cohen-Macaulay Case

The aim of this section is to compute explicitly the characteristic cycle of local cohomology modules supported on squarefree monomial ideals such that the corresponding quotient ring is Cohen-Macaulay, with special attention to the case of face ideals. First, we observe that, as a consequence of [54, Theorem 1], there is only one local cohomology module which does not vanish in this case.

Proposition 3.1.1. *For any ideal $I \subseteq R$ generated by squarefree monomials, the following are equivalent:*

i) R/I is Cohen-Macaulay.

ii) $H_I^r(R) = 0$ for any $r \neq \operatorname{ht} I$.

PROOF: By flat base change we only have to consider the case $R = k[[x_1, \dots, x_n]]$. We have $\inf\ \{r \mid H_I^r(R) \neq 0\} = \operatorname{grade} I = \operatorname{ht} I$. On the other hand by [54, Theorem 1 (iv)] we have $\operatorname{cd}(R, I) = \max\ \{r \mid H_I^r(R) \neq 0\} = n - \operatorname{depth}_R(R/I) = \operatorname{ht} I$.

□

Let $I = (\mathbf{x}^{\alpha_1}, \dots, \mathbf{x}^{\alpha_s}) \subseteq R$ be a squarefree monomial ideal of height h. We want to use the Čech complex:

$$\check{C}_I^\bullet : \quad 0 \longrightarrow R \xrightarrow{d_0} \bigoplus_{1 \leq i_1 \leq s} R[\frac{1}{\mathbf{x}^{\alpha_{i_1}}}] \xrightarrow{d_1} \cdots \xrightarrow{d_{s-1}} R[\frac{1}{\mathbf{x}^{\alpha_1} \cdots \mathbf{x}^{\alpha_s}}] \longrightarrow 0$$

and the additivity of the characteristic cycle in order to compute the characteristic cycle of the local cohomology module $H_I^h(R)$.

The characteristic cycle of the modules in the Čech complex have been computed in Section 1.3.5. For completeness, we recall the precise statements:

The characteristic cycle of R: $CC(R) = T_X^* X$.

The characteristic cycle of $R[\frac{1}{\mathbf{x}^\alpha}]$: $CC(R[\frac{1}{\mathbf{x}^\alpha}]) = \sum_{\beta \leq \alpha} T_{X_\beta}^* X$.

In order to compute the $CC(H_I^h(R))$ it remains to give a precise description for the characteristic cycle of the kernels and the images of the differentials d_i. This will be very difficult in general, mostly due to the fact that the characteristic varieties of the localizations $R[\frac{1}{\mathbf{x}^\alpha}]$ are not irreducible.

If I is a squarefree monomial ideal of height h such that R/I is Cohen-Macaulay then $H_I^h(R)$ is the unique non zero local cohomology module. This allows us to describe the characteristic cycle of the kernels and the images of the differentials and compute $CC(H_I^h(R))$. More precisely, let

$$R_j = \bigoplus_{1 \leq i_1 < \cdots < i_j \leq s} R[\frac{1}{\mathbf{x}^{\alpha_{i_1}} \cdots \mathbf{x}^{\alpha_{i_j}}}] \quad \text{for} \quad j = 0, \dots, s,$$

be the modules in the Čech complex, then we have:

Proposition 3.1.2. *Let $I \subseteq R$ be an ideal of height h generated by squarefree monomials. If R/I is Cohen-Macaulay then:*

$$CC(H_I^h(R)) = CC(R_h) - CC(R_{h+1}) + \cdots + (-1)^{s-h} CC(R_s)$$
$$-CC(R_{h-1}) + \cdots + (-1)^h CC(R_0).$$

Proof: By Proposition 3.1.1 $H_I^r(R) = 0$ for any $r \neq h$.

We have $0 = H_I^0(R) = \operatorname{Ker} d_0$ so we get $CC(\operatorname{Im} d_0) = CC(R_0)$. Similarly, if $h > 1$, we have $0 = H_I^1(R) = \operatorname{Ker} d_1 / \operatorname{Im} d_0$ and so $CC(\operatorname{Im} d_1) = CC(R_1) - CC(R_0)$. By repeating this argument we obtain

$$CC(Im \ d_{h-1}) = CC(R_{h-1}) - CC(R_{h-2}) \cdots + (-1)^h CC(R_0).$$

On the other hand, if $r > h$, we have $0 = H_I^r(R) = \operatorname{Ker} d_r / \operatorname{Im} d_{r-1}$, so we get $CC(\operatorname{Ker} d_r) = CC(R_r)$. Similarly as before we obtain

$$CC(\operatorname{Ker} d_h) = CC(R_h) - CC(R_{h+1}) + \cdots + (-1)^{s-h} CC(R_s).$$

Since $H_I^h(R) = \operatorname{Ker} d_h / \operatorname{Im} d_{h-1}$, we get the formula.

□

If R/I is a complete intersection then $s = h$ so we have:

Corollary 3.1.3. *Let $I \subseteq R$ be an ideal of height h generated by squarefree monomials. If R/I is a complete intersection then:*

$$CC(H_I^h(R)) = CC(R_h) - CC(R_{h-1}) + \cdots + (-1)^h CC(R_0).$$

A particular case of a complete intersection is when the ideal I is a face ideal. The characteristic cycle of local cohomology modules supported on face ideals have been computed in Section 1.3.5. By using the previous corollary we get the same result. The precise statement is:

The characteristic cycle of $H_{\mathfrak{p}_\alpha}^{|\alpha|}(R)$: $\quad CC(H_{\mathfrak{p}_\alpha}^{|\alpha|}(R)) = T_{X_\alpha}^* X.$

These kind of modules will have a special significance in the rest of the chapter due to the fact that their characteristic varieties are irreducible and that the minimal primary decomposition of any squarefree monomial ideal is expressed in terms of face ideals.

Remark 3.1.4. Corollary 3.1.3 is also true for any complete intersection ideal due to the fact that there is only one non vanishing local cohomology module supported on ideals of this class.

3.2 Main Result

In this section we will give a formula for the characteristic cycle of the local cohomology modules $H_I^r(R)$ for a given ideal $I = (\mathbf{x}^{\alpha_1}, \ldots, \mathbf{x}^{\alpha_s}) \subseteq R$ generated by squarefree monomials. A first approach is to use the Čech complex:

$$\check{C}_I^\bullet : \quad 0 \longrightarrow R \xrightarrow{d_0} \bigoplus_{1 \le i_1 \le s} R[\frac{1}{\mathbf{x}^{\alpha_{i_1}}}] \xrightarrow{d_1} \cdots \xrightarrow{d_{s-1}} R[\frac{1}{\mathbf{x}^{\alpha_1} \cdots \mathbf{x}^{\alpha_s}}] \longrightarrow 0$$

and the additivity of the characteristic cycle with respect to exact sequences. The reducibility of the characteristic variety of the localizations $R[\frac{1}{\mathbf{x}^\alpha}]$ that appear in the complex is the main inconvenient for a precise description of the characteristic cycle of the kernels and the images of the differentials d_i.

The approach we will give in this section is by using some Mayer-Vietoris sequences in a systematical way. For this purpose, we will apply the good properties of the minimal primary decomposition of a squarefree monomial ideal $I = I_{\alpha_1} \cap \cdots \cap I_{\alpha_m} \subseteq R$. The main goal of this procedure will be to reduce the problem to a description of the kernels and the images of morphisms between local cohomology modules supported on face ideals, i.e. modules with irreducible characteristic varieties.

In Subsection 3.2.1 we describe the process that allows us to study the local cohomology modules in a systematical way, i.e. independently of the ideal's complexity. To shed some light we present some examples with few face ideals in the minimal primary decomposition of I.

In Subsection 3.2.2 we explain how to use the process described in the previous subsection in order to compute the characteristic cycle of $H_I^r(R)$. In particular, we will see that it is necessary to describe the characteristic cycle of the kernels and the images of morphisms between certain local cohomology modules supported on face ideals. This computation will be done in Subsection 3.2.3 by means of an algorithm.

Finally, a formula for the characteristic cycle of the local cohomology modules $H_I^r(R)$ is given in Subsection 3.2.4. For the moment, we point out that the formula is described in terms of the minimal primary decomposition $I = I_{\alpha_1} \cap \cdots \cap I_{\alpha_m} \subseteq R$.

3.2.1 Mayer-Vietoris process

Let $I \subseteq R$ be an ideal. The usual method to determine the annihilation or the support of the local cohomology modules $H_I^r(R)$ is to find a representation of I as the intersection of two simpler ideals U and V and then apply the Mayer-Vietoris sequence

$$\cdots \longrightarrow H^r_{U+V}(R) \longrightarrow H^r_U(R) \oplus H^r_V(R) \longrightarrow H^r_{U\cap V}(R) \longrightarrow H^{r+1}_{U+V}(R) \longrightarrow \cdots$$

In general, there are several choices for such a representation but in the case of local cohomology modules supported on squarefree monomial ideals we can avoid this problem by using the good properties of a primary decomposition

$I = I_{\alpha_1} \cap \cdots \cap I_{\alpha_m}$ of a squarefree monomial ideal. Our aim in this subsection is to develop a method that will allow us to study these local cohomology modules in a systematical way, i.e. independently of the ideal's complexity.

Although our method will work out for any primary decomposition we will start by studying some examples with a simple minimal primary decomposition, that is, with few face ideals in the minimal primary decomposition of I. Note that the case $m = 1$ has already been studied in Section 1.3.5.

The case m=2: Let $I = I_{\alpha_1} \cap I_{\alpha_2}$ be the minimal primary decomposition of a squarefree monomial ideal. To study the local cohomology modules $H^r_I(R)$ we use a Mayer-Vietoris sequence with:

$$U = I_{\alpha_1}, \qquad U \cap V = I = I_{\alpha_1} \cap I_{\alpha_2},$$
$$V = I_{\alpha_2}, \qquad U + V = I_{\alpha_1} + I_{\alpha_2}.$$

We get the long exact sequence:

$$\cdots \longrightarrow H^r_{I_{\alpha_1}}(R) \oplus H^r_{I_{\alpha_2}}(R) \longrightarrow H^r_I(R) \longrightarrow H^{r+1}_{I_{\alpha_1}+I_{\alpha_2}}(R) \longrightarrow \cdots$$

In order to describe the local cohomology modules $H^r_I(R)$ we have to study the modules $H^r_{I_{\alpha_1}}(R)$, $H^r_{I_{\alpha_2}}(R)$, $H^r_{I_{\alpha_1}+I_{\alpha_2}}(R)$ and the homomorphisms of the Mayer-Vietoris sequence. We state that these modules are the **initial pieces** that allows us to describe $H^r_I(R)$.

The case m=3 Let $I = I_{\alpha_1} \cap I_{\alpha_2} \cap I_{\alpha_3}$ be the minimal primary decomposition of a squarefree monomial ideal. To study the local cohomology modules $H^r_I(R)$ we first use a Mayer-Vietoris sequence with:

$$U = I_{\alpha_1} \cap I_{\alpha_2}, \qquad U \cap V = I = I_{\alpha_1} \cap I_{\alpha_2} \cap I_{\alpha_3},$$
$$V = I_{\alpha_3}, \qquad U + V = (I_{\alpha_1} \cap I_{\alpha_2}) + I_{\alpha_2}.$$

We get the long exact sequence:

$$\cdots \longrightarrow H^r_{I_{\alpha_1} \cap I_{\alpha_2}}(R) \oplus H^r_{I_{\alpha_3}}(R) \longrightarrow H^r_I(R) \longrightarrow H^{r+1}_{(I_{\alpha_1} \cap I_{\alpha_2})+I_{\alpha_3}}(R) \longrightarrow \cdots$$

• The ideal $I_{\alpha_1} \cap I_{\alpha_2}$ is not a face ideal but we can describe the modules $H^r_{I_{\alpha_1} \cap I_{\alpha_2}}(R)$ by using a Mayer-Vietoris sequence with:

$$U = I_{\alpha_1}, \qquad U \cap V = I_{\alpha_1} \cap I_{\alpha_2},$$
$$V = I_{\alpha_2}, \qquad U + V = I_{\alpha_1} + I_{\alpha_2}.$$

• In general, the ideal $(I_{\alpha_1} \cap I_{\alpha_2}) + I_{\alpha_3}$ is not either a face ideal but we can describe the modules $H^r_{(I_{\alpha_1} \cap I_{\alpha_2})+I_{\alpha_3}}(R)$ by using a Mayer-Vietoris sequence with:

$$U = I_{\alpha_1} + I_{\alpha_3}, \qquad U \cap V = (I_{\alpha_1} \cap I_{\alpha_2}) + I_{\alpha_3},$$
$$V = I_{\alpha_2} + I_{\alpha_3}, \qquad U + V = I_{\alpha_1} + I_{\alpha_2} + I_{\alpha_3}.$$

We can reflect the above process in the following diagram:

$$
\begin{array}{ccccccc}
 & \vdots & & & \vdots & & \\
 & \uparrow & & & \uparrow & & \\
 & H^{r+1}_{I_{\alpha_1}+I_{\alpha_2}}(R) & & & H^{r+2}_{I_{\alpha_1}+I_{\alpha_2}+I_{\alpha_3}}(R) & & \\
 & \uparrow & & & \uparrow & & \\
\cdots \longrightarrow & H^r_{I_{\alpha_1}\cap I_{\alpha_2}}(R) \oplus H^r_{I_{\alpha_3}}(R) & \longrightarrow & H^r_I(R) \longrightarrow & H^{r+1}_{(I_{\alpha_1}\cap I_{\alpha_2})+I_{\alpha_3}}(R) & \longrightarrow \cdots & \\
 & \uparrow & & & \uparrow & & \\
 & H^r_{I_{\alpha_1}}(R) \oplus H^r_{I_{\alpha_2}}(R) & & & H^{r+1}_{I_{\alpha_1}+I_{\alpha_3}}(R) \oplus H^{r+1}_{I_{\alpha_2}+I_{\alpha_3}}(R) & & \\
 & \uparrow & & & \uparrow & & \\
 & H^r_{I_{\alpha_1}+I_{\alpha_2}}(R) & & & H^{r+1}_{I_{\alpha_1}+I_{\alpha_2}+I_{\alpha_3}}(R) & & \\
 & \uparrow & & & \uparrow & & \\
 & \vdots & & & \vdots & &
\end{array}
$$

Notice that the vertical Mayer-Vietoris sequences have been studied in this subsection for the case $m = 2$ since the corresponding ideals U's and V's are face ideals.

In order to describe the local cohomology modules $H^r_I(R)$ we have to study the modules:

$$H^r_{I_{\alpha_1}}(R), \qquad H^r_{I_{\alpha_2}}(R), \qquad H^r_{I_{\alpha_3}}(R)$$
$$H^r_{I_{\alpha_1}+I_{\alpha_2}}(R), \qquad H^r_{I_{\alpha_1}+I_{\alpha_3}}(R), \qquad H^r_{I_{\alpha_2}+I_{\alpha_3}}(R),$$
$$H^r_{I_{\alpha_1}+I_{\alpha_2}+I_{\alpha_3}}(R).$$

and the homomorphisms of the corresponding Mayer-Vietoris sequences. These modules are the local cohomology modules supported on all the face ideals we can construct as sums of face ideals in the minimal primary decomposition of I. We state that these are the **initial pieces** that allows us to describe the modules $H^r_I(R)$.

This method can be generalized as follows:

General case: Let $I = I_{\alpha_1} \cap \cdots \cap I_{\alpha_m}$ be the minimal primary decomposition of a squarefree monomial ideal $I \subseteq R$. To study the local cohomology modules $H^r_I(R)$ we use the Mayer-Vietoris sequence with:

$$U = I_{\alpha_1} \cap \cdots \cap I_{\alpha_{m-1}}, \qquad U \cap V = I = I_{\alpha_1} \cap \cdots \cap I_{\alpha_m},$$
$$V = I_{\alpha_m}, \qquad U + V = (I_{\alpha_1} \cap \cdots \cap I_{\alpha_{m-1}}) + I_{\alpha_m}.$$

We get the long exact sequence:

$$\cdots \longrightarrow \begin{array}{c} H^r_{I_{\alpha_1}\cap\cdots\cap I_{\alpha_{m-1}}}(R) \\ \oplus \\ H^r_{I_{\alpha_m}}(R) \end{array} \longrightarrow H^r_I(R) \longrightarrow H^{r+1}_{(I_{\alpha_1}\cap\cdots\cap I_{\alpha_{m-1}})+I_{\alpha_m}}(R) \longrightarrow \cdots.$$

The ideal $V = I_{\alpha_m}$ is a face ideal, so the local cohomology modules $H^r_{I_{\alpha_m}}(R)$ have been computed in Section 1.3.5, but in general neither $U = I_{\alpha_1} \cap \cdots \cap I_{\alpha_{m-1}}$ nor $U + V = (I_{\alpha_1} \cap \cdots \cap I_{\alpha_{m-1}}) + I_{\alpha_m}$ are face ideals, so we will describe the local cohomology modules

$$H^r_{I_{\alpha_1}\cap\cdots\cap I_{\alpha_{m-1}}}(R) \quad \text{and} \quad H^r_{(I_{\alpha_1}\cap\cdots\cap I_{\alpha_{m-1}})+I_{\alpha_m}}(R)$$

by using a Mayer-Vietoris sequence again.

Our gain is that the minimal primary decomposition of U is the intersection of $m-1$ prime ideals while $U + V$ has a primary decomposition of the form $U + V = (I_{\alpha_1} + I_{\alpha_m}) \cap \cdots \cap (I_{\alpha_{m-1}} + I_{\alpha_m})$, i.e. with $m-1$ terms too.

• To study $H^r_{I_{\alpha_1}\cap\cdots\cap I_{\alpha_{m-1}}}(R)$ we will use the Mayer-Vietoris sequence with:

$$U = I_{\alpha_1}\cap\cdots\cap I_{\alpha_{m-2}}, \qquad U\cap V = I_{\alpha_1}\cap\cdots\cap I_{\alpha_{m-1}},$$
$$V = I_{\alpha_{m-1}}, \qquad U+V = (I_{\alpha_1}\cap\cdots\cap I_{\alpha_{m-2}}) + I_{\alpha_{m-1}}.$$

• To study $H^r_{(I_{\alpha_1}\cap\cdots\cap I_{\alpha_{m-1}})+I_{\alpha_m}}(R)$ we will use the Mayer-Vietoris sequence with:

$$U = (I_{\alpha_1}\cap\cdots\cap I_{\alpha_{m-2}}) + I_{\alpha_m}, \qquad U\cap V = (I_{\alpha_1}\cap\cdots\cap I_{\alpha_{m-1}}) + I_{\alpha_m},$$
$$V = I_{\alpha_{m-1}+I_{\alpha_m}}, \qquad U+V = (I_{\alpha_1}\cap\cdots\cap I_{\alpha_{m-2}}) + I_{\alpha_{m-1}} + I_{\alpha_m}.$$

In both Mayer-Vietoris sequences the corresponding ideals V's are face ideals but neither the ideals U's nor the $U+V$'s are face ideals, so we will repeat the process until the corresponding ideals U's, V's and $U+V$'s are face ideals.

Notice that at each step of the process the number of ideals in the primary decomposition of the ideals U's and $U+V$'s decreases by one, so we get the desired result after $m-1$ steps. In particular the process finishes after getting $2^{m-1}-1$ Mayer-Vietoris sequences with only face ideals involved in. From now on we will call it **the Mayer-Vietoris process**.

Once we have developed the Mayer-Vietoris process, in order to describe the local cohomology modules $H^r_I(R)$ we have to study the local cohomology modules supported on all possible face ideals we can construct as sums of face ideals in the minimal primary decomposition of I (i.e. ideals of the form $I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}}$, $1 \le i_1 < \cdots < i_j \le m$) and the homomorphisms of the corresponding Mayer-Vietoris sequences. We state that these local cohomology modules are the **initial pieces** that allows us to describe the modules $H^r_I(R)$.

The initial pieces are labelled by the poset $\mathcal{I}$ introduced in Section 1.2.8. In particular, there are the following $2^m - 1$ initial pieces:

$$H^r_{I_{\alpha_{i_1}}}(R) \qquad I_{\alpha_{i_1}} \in \mathcal{I}_1,$$
$$H^r_{I_{\alpha_{i_1}}+I_{\alpha_{i_2}}}(R) \qquad I_{\alpha_{i_1}} + I_{\alpha_{i_2}} \in \mathcal{I}_2,$$
$$\vdots$$
$$H^r_{I_{\alpha_1}+I_{\alpha_2}+\cdots+I_{\alpha_m}}(R) \qquad I_{\alpha_1} + I_{\alpha_2} + \cdots + I_{\alpha_m} \in \mathcal{I}_m.$$

Notice that the initial pieces are treated as different elements even if they describe the same local cohomology module.

As we see, the Mayer-Vietoris process allows us, in some sense, to break the local cohomology modules $H_I^r(R)$ into simpler pieces, i.e. local cohomology modules supported on face ideals, that are easier to study. But, in order to give a full description of the characteristic cycle of the modules $H_I^r(R)$ it remains to study the homomorphisms of the sequences that appear during the process. This will be done in the next sections.

3.2.2 Mayer-Vietoris process and characteristic cycle

By using the Mayer-Vietoris process described before and the additivity of the characteristic cycle with respect to exact sequences we will describe the characteristic cycle of the local cohomology modules $H_I^r(R)$ in terms of the characteristic cycle of its initial pieces. Roughly speaking, our method for computing the characteristic cycle $CC(H_I^r(R))$ is as follows:

We start by splitting the Mayer-Vietoris sequences obtained in the last step of the Mayer-Vietoris process, i.e. the sequences

$$\cdots \longrightarrow H^r_{U+V}(R) \longrightarrow H^r_U(R) \oplus H^r_V(R) \longrightarrow H^r_{U\cap V}(R) \longrightarrow H^{r+1}_{U+V}(R) \longrightarrow \cdots,$$

where the corresponding ideals U's and V's are face ideals, into short exact sequences of kernels and cokernels:

$$0 \longrightarrow B_r \longrightarrow H^r_U(R) \oplus H^r_V(R) \longrightarrow C_r \longrightarrow 0,$$

$$0 \longrightarrow C_r \longrightarrow H^r_{U\cap V}(R) \longrightarrow A_{r+1} \longrightarrow 0,$$

$$0 \longrightarrow A_{r+1} \longrightarrow H^{r+1}_{U+V}(R) \longrightarrow B_{r+1} \longrightarrow 0.$$

This allows us to compute the characteristic cycle of the local cohomology modules $H^r_{U\cap V}(R)$. Namely we have:

$$CC(H^r_{U\cap V}(R)) = CC(C_r) + CC(A_{r+1}).$$

The key point is now to compute the characteristic cycle of the modules B_r $\forall r$. At this step of the process we can give a complete description of these modules in terms of the minimality of the decomposition $U \cap V$ by using the irreducibility of the characteristic variety of the initial pieces. Namely we have the following:

Remark 3.2.1. Since the characteristic varieties of $H^r_U(R)$, $H^r_V(R)$ and $H^r_{U+V}(R)$ are irreducible and

$$C(B_r) \subseteq C(H^r_U(R) \oplus H^r_V(R)),$$
$$C(B_r) \subseteq C(H^r_{U+V}(R)),$$

we get:

• $B_r = 0 \; \forall r$ if and only if $U+V$ is different from U and V. This occurs when $I = U \cap V$ is a minimal primary decomposition.

• $B_r \cong H^r_{U+V}(R) \neq 0$ for $r = \mathrm{ht}\,(U+V)$ if and only if $U+V=U$ or $U+V=V$. This occurs when the primary decomposition $I = U \cap V$ is not minimal. (Notice that in this case the local cohomology module $H^r_I(R)$ could be described in a simpler way.)

We continue our method by splitting the Mayer-Vietoris sequences obtained in the second last step of the Mayer-Vietoris process, i.e. the sequences

$$\cdots \longrightarrow H^r_{(U\cap V)+W}(R) \longrightarrow \begin{array}{c} H^r_{U\cap V}(R) \\ \oplus \\ H^r_W(R) \end{array} \longrightarrow H^r_{U\cap V\cap W}(R) \longrightarrow H^{r+1}_{(U\cap V)+W}(R) \longrightarrow \cdots ,$$

where the corresponding ideals U's, V's and W's are face ideals, into short exact sequences of kernels and cokernels:

$$0 \longrightarrow B_r \longrightarrow H^r_{U\cap V}(R) \oplus H^r_W(R) \longrightarrow C_r \longrightarrow 0,$$

$$0 \longrightarrow C_r \longrightarrow H^r_{U\cap V\cap W}(R) \longrightarrow A_{r+1} \longrightarrow 0,$$

$$0 \longrightarrow A_{r+1} \longrightarrow H^{r+1}_{(U\cap V)+W}(R) \longrightarrow B_{r+1} \longrightarrow 0.$$

We have:

$$CC(H^r_{U\cap V\cap W}(R)) = CC(C_r) + CC(A_{r+1}).$$

The characteristic cycle of the modules $H^r_W(R)$, $H^r_{U\cap V}(R)$ and $H^r_{(U\cap V)+W}(R)$ have been computed in the previous step so it only remains to compute the characteristic cycle of the modules B_r $\forall r$. This computation is more involved at this step and we will not give a complete description of them by now. For the moment we only point out that they will be described in terms of the initial pieces.

The process continues by computing the characteristic cycle of the intermediate local cohomology modules in the Mayer-Vietoris process. To do this we split the corresponding Mayer-Vietoris sequence into short exact sequences of kernels and cokernels and we use the results obtained in the previous steps of the process.

The process finishes when we compute the characteristic cycle of the local cohomology modules $H^r_I(R)$ by splitting the Mayer-Vietoris sequence that involves this module. Namely, let $I = I_{\alpha_1} \cap \cdots \cap I_{\alpha_m}$ be the minimal primary decomposition of the squarefree monomial ideal I, then we split the Mayer-Vietoris sequence:

$$\cdots \longrightarrow \begin{array}{c} H^r_{I_{\alpha_1}\cap\cdots\cap I_{\alpha_{m-1}}}(R) \\ \oplus \\ H^r_{I_{\alpha_m}}(R) \end{array} \longrightarrow H^r_I(R) \longrightarrow H^{r+1}_{(I_{\alpha_1}\cap\cdots\cap I_{\alpha_{m-1}})+I_{\alpha_m}}(R) \longrightarrow \cdots ,$$

into short exact sequences of kernels and cokernels:

$$0 \longrightarrow B_r \longrightarrow H^r_{I_{\alpha_1}\cap\cdots\cap I_{\alpha_{m-1}}}(R) \oplus H^r_{I_{\alpha_m}}(R) \longrightarrow C_r \longrightarrow 0,$$

$$0 \longrightarrow C_r \longrightarrow H^r_I(R) \longrightarrow A_{r+1} \longrightarrow 0,$$

$$0 \longrightarrow A_{r+1} \longrightarrow H^{r+1}_{(I_{\alpha_1}\cap\cdots\cap I_{\alpha_{m-1}})+I_{\alpha_m}}(R) \longrightarrow B_{r+1} \longrightarrow 0.$$

The characteristic cycle of the local cohomology modules $H^r_I(R)$ is then:

$$CC(H^r_I(R)) = CC(C_r) + CC(A_{r+1}).$$

In the next section we will compute the characteristic cycle of the modules B_r $\forall r$ for all the Mayer-Vietoris sequences obtained in the Mayer-Vietoris process in a systematical way, i.e. independently of the ideal's complexity.

We will illustrate this process for the cases with few face ideals in the minimal primary decomposition of I.

The case m=2: Let $I = I_{\alpha_1} \cap I_{\alpha_2}$ be the minimal primary decomposition of a squarefree monomial ideal. By Remark 3.2.1 we have $B_r = 0\ \forall r$. In particular we can describe easily the local cohomology modules $H_I^r(R)$ and compute their characteristic cycle. Namely, denote $h_1 := \operatorname{ht} I_{\alpha_1}$, $h_2 := \operatorname{ht} I_{\alpha_2}$ and $h_{12} := \operatorname{ht}(I_{\alpha_1} + I_{\alpha_2})$ and suppose $h_1 \leq h_2$. The non zero local cohomology modules $H_I^r(R)$ are:

1) If $h_1 < h_2 < h_{12} - 1$ then:
   $H_I^{h_1}(R) \cong H_{I_{\alpha_1}}^{h_1}(R)$,
   $H_I^{h_2}(R) \cong H_{I_{\alpha_2}}^{h_2}(R)$ and
   $H_I^{h_{12}-1}(R) \cong H_{I_{\alpha_1}+I_{\alpha_2}}^{h_{12}}(R)$,

2) If $h_1 = h_2 < h_{12} - 1$ then:
   $H_I^{h_1}(R) \cong H_{I_{\alpha_1}}^{h_1}(R) \oplus H_{I_{\alpha_2}}^{h_2}(R)$ and
   $H_I^{h_{12}-1}(R) \cong H_{I_{\alpha_1}+I_{\alpha_2}}^{h_{12}}(R)$,

3) If $h_1 < h_2 = h_{12} - 1$ then:
   $H_I^{h_1}(R) \cong H_{I_{\alpha_1}}^{h_1}(R)$ and
   $0 \longrightarrow H_{I_{\alpha_2}}^{h_2}(R) \longrightarrow H_I^{h_2}(R) \longrightarrow H_{I_{\alpha_1}+I_{\alpha_2}}^{h_{12}}(R) \longrightarrow 0$,

4) If $h_1 = h_2 = h_{12} - 1$ then:
   $0 \longrightarrow H_{I_{\alpha_1}}^{h_1}(R) \oplus H_{I_{\alpha_2}}^{h_2}(R) \longrightarrow H_I^{h_1}(R) \longrightarrow H_{I_{\alpha_1}+I_{\alpha_2}}^{h_{12}}(R) \longrightarrow 0$,

By using the additivity of the characteristic cycle with respect to exact sequences, the characteristic cycles for the non zero local cohomology modules $H_I^r(R)$ are:

1) If $h_1 < h_2 < h_{12} - 1$ then:
   $CC(H_I^{h_1}(R)) = CC(H_{I_{\alpha_1}}^{h_1}(R))$,
   $CC(H_I^{h_2}(R)) = CC(H_{I_{\alpha_2}}^{h_2}(R))$ and
   $CC(H_I^{h_{12}-1}(R)) = CC(H_{I_{\alpha_1}+I_{\alpha_2}}^{h_{12}}(R))$.

2) If $h_1 = h_2 < h_{12} - 1$ then:
$CC(H_I^{h_1}(R)) = CC(H_{I_{\alpha_1}}^{h_1}(R)) + CC(H_{I_{\alpha_2}}^{h_2}(R))$ and
$CC(H_I^{h_{12}-1}(R)) = CC(H_{I_{\alpha_1}+I_{\alpha_2}}^{h_{12}}(R))$.

3) If $h_1 < h_2 = h_{12} - 1$ then:
$CC(H_I^{h_1}(R)) = CC(H_{I_{\alpha_1}}^{h_1}(R))$ and
$CC(H_I^{h_2}(R)) = CC(H_{I_{\alpha_2}}^{h_2}(R)) + CC(H_{I_{\alpha_1}+I_{\alpha_2}}^{h_{12}}(R))$.

4) If $h_1 = h_2 = h_{12} - 1$ then:
$CC(H_I^{h_1}(R)) = CC(H_{I_{\alpha_1}}^{h_1}(R)) + CC(H_{I_{\alpha_2}}^{h_2}(R)) + CC(H_{I_{\alpha_1}+I_{\alpha_2}}^{h_{12}}(R))$.

The case m=3 Let $I = I_{\alpha_1} \cap I_{\alpha_2} \cap I_{\alpha_3}$ be the minimal primary decomposition of a squarefree monomial ideal. Recall that the Mayer-Vietoris process is reflected in the diagram:

$$
\begin{array}{ccccccc}
 & \vdots & & & & \vdots & \\
 & \uparrow & & & & \uparrow & \\
 & H^{r+1}_{I_{\alpha_1}+I_{\alpha_2}}(R) & & & & H^{r+2}_{I_{\alpha_1}+I_{\alpha_2}+I_{\alpha_3}}(R) & \\
 & \uparrow & & & & \uparrow & \\
\cdots \longrightarrow & H^r_{I_{\alpha_1}\cap I_{\alpha_2}}(R) \oplus H^r_{I_{\alpha_3}}(R) & \longrightarrow & H^r_I(R) & \longrightarrow & H^{r+1}_{(I_{\alpha_1}\cap I_{\alpha_2})+I_{\alpha_3}}(R) & \longrightarrow \cdots \\
 & \uparrow & & & & \uparrow & \\
 & H^r_{I_{\alpha_1}}(R) \oplus H^r_{I_{\alpha_2}}(R) & & & & H^{r+1}_{I_{\alpha_1}+I_{\alpha_3}}(R) \oplus H^{r+1}_{I_{\alpha_2}+I_{\alpha_3}}(R) & \\
 & \uparrow & & & & \uparrow & \\
 & H^r_{I_{\alpha_1}+I_{\alpha_2}}(R) & & & & H^{r+1}_{I_{\alpha_1}+I_{\alpha_2}+I_{\alpha_3}}(R) & \\
 & \uparrow & & & & \uparrow & \\
 & \vdots & & & & \vdots &
\end{array}
$$

The vertical Mayer-Vietoris sequences are the sequences obtained in the last step of the Mayer-Vietoris process so we will start computing the characteristic cycles of the local cohomology modules $H^r_{I_{\alpha_1}\cap I_{\alpha_2}}(R)$ and $H^r_{(I_{\alpha_1}\cap I_{\alpha_2})+I_{\alpha_3}}(R)$ by splitting these vertical sequences into short exact sequences of kernels and cokernels, and then computing the corresponding modules B_r by using Remark 3.2.1:

• The characteristic cycle of $H^r_{I_{\alpha_1}\cap I_{\alpha_2}}(R)$: The primary decomposition $I_{\alpha_1}\cap I_{\alpha_2}$ is minimal so the corresponding modules B_r are zero for all r, in particular we can use the previous results obtained in this section for the case $m=2$.

• The characteristic cycle of $H^r_{(I_{\alpha_1}\cap I_{\alpha_2})+I_{\alpha_3}}(R)$: The primary decomposition $(I_{\alpha_1}\cap I_{\alpha_2})+I_{\alpha_3}=(I_{\alpha_1}+I_{\alpha_3})\cap(I_{\alpha_2}+I_{\alpha_3})$ is not necessarily minimal so we have to distinguish the following cases:

i) If the primary decomposition is minimal then the corresponding modules B_r are zero for all r and we can use the previous results obtained in this section for the case $m=2$.

ii) If the primary decomposition is not minimal then $B_r\cong H^r_{I_{\alpha_1}+I_{\alpha_2}+I_{\alpha_3}}(R)\neq 0$ for $r=\operatorname{ht}(I_{\alpha_1}+I_{\alpha_2}+I_{\alpha_3})$. (Note that in this case the local cohomology modules $H^r_{(I_{\alpha_1}\cap I_{\alpha_2})+I_{\alpha_3}}(R)$ could be described in a simplest way, in particular we could use the result obtained in Section 1.3.5 for the case $m=1$.)

Once we have computed the characteristic cycles of the local cohomology modules $H^r_{I_{\alpha_1}\cap I_{\alpha_2}}(R)$ and $H^r_{(I_{\alpha_1}\cap I_{\alpha_2})+I_{\alpha_3}}(R)$ we split the horizontal Mayer-Vietoris sequence in the diagram into short exact sequences of kernels and cokernels.

A full description of the corresponding modules B_r $\forall r$ will also depend on the minimality of the primary decomposition of the ideal $(I_{\alpha_1}\cap I_{\alpha_2})+I_{\alpha_3}$. Namely we have the following cases:

i) If the primary decomposition is minimal then:

• $B_r=H^r_{I_{\alpha_1}+I_{\alpha_2}+I_{\alpha_3}}(R)\neq 0$ for $r=\operatorname{ht}(I_{\alpha_1}+I_{\alpha_2}+I_{\alpha_3})$ if and only if $I_{\alpha_1}+I_{\alpha_2}=I_{\alpha_1}+I_{\alpha_2}+I_{\alpha_3}$.

• $B_r=0$ otherwise.

ii) If the primary decomposition is not minimal then B_r is always zero.

We will not give a complete description for all possible local cohomology modules with support on $I = I_{\alpha_1} \cap I_{\alpha_2} \cap I_{\alpha_3}$ as we have done for the case $m = 2$. We will only present the following significant examples:

The first one has the particularity that all the modules B_r that appear in the process vanish.

Example: Let $I = (x_1, x_2) \cap (x_3, x_4) \cap (x_5, x_6, x_7) \subseteq R = k[x_1, x_2, x_3, x_4, x_5, x_6, x_7]$.

We start the process with the Mayer-Vietoris sequence:

$$\cdots \longrightarrow \begin{array}{c} H^r_{(x_1,x_2)\cap(x_3,x_4)}(R) \\ \oplus \\ H^r_{(x_5,x_6,x_7)}(R) \end{array} \longrightarrow H^r_I(R) \longrightarrow H^{r+1}_{((x_1,x_2)\cap(x_3,x_4))+(x_5,x_6,x_7)}(R) \longrightarrow \cdots$$

The next step of the process is to study the local cohomology modules not supported on face ideals that appear in that sequence.

• We describe the local cohomology modules $H^r_{(x_1,x_2)\cap(x_3,x_4)}(R)$ by using the results for the case $m = 2$ because the corresponding modules B_r vanish for all r. Namely, we have:

$$H^2_{(x_1,x_2)\cap(x_3,x_4)}(R) \cong H^2_{(x_1,x_2)}(R) \oplus H^2_{(x_3,x_4)}(R) \quad \text{and}$$

$$H^3_{(x_1,x_2)\cap(x_3,x_4)}(R) \cong H^4_{(x_1,x_2)+(x_3,x_4)}(R).$$

• We describe the local cohomology modules $H^r_{((x_1,x_2)\cap(x_3,x_4))+(x_5,x_6,x_7)}(R)$ by using the results for the case $m = 2$ because the primary decomposition

$$((x_1, x_2) \cap (x_3, x_4)) + (x_5, x_6, x_7) = ((x_1, x_2) + (x_5, x_6, x_7)) \cap ((x_3, x_4) + (x_5, x_6, x_7))$$

is minimal so the corresponding modules B_r vanish for all r. Namely, we have:

$$H^5_{((x_1,x_2)\cap(x_3,x_4))+(x_5,x_6,x_7)}(R) \cong H^5_{(x_1,x_2)+(x_5,x_6,x_7)}(R) \oplus H^5_{(x_3,x_4)+(x_5,x_6,x_7)}(R) \quad \text{and}$$

$$H^6_{((x_1,x_2)\cap(x_3,x_4))+(x_5,x_6,x_7)}(R) \cong H^7_{(x_1,x_2)+(x_3,x_4)+(x_5,x_6,x_7)}(R).$$

Since the initial pieces that describe the modules $H^r_{(x_1,x_2)\cap(x_3,x_4)}(R) \oplus H^r_{(x_5,x_6,x_7)}(R)$ are different from the initial pieces of $H^r_{((x_1,x_2)\cap(x_3,x_4))+(x_5,x_6,x_7)}(R)$ the corresponding modules B_r for the Mayer-Vietoris sequence vanish for all r so we get:

$$H_I^2(R) \cong H^2_{(x_1,x_2)\cap(x_3,x_4)}(R)$$
$$H_I^3(R) \cong H^3_{(x_1,x_2)\cap(x_3,x_4)}(R) \oplus H^3_{(x_5,x_6,x_7)}(R)$$
$$H_I^4(R) \cong H^5_{((x_1,x_2)\cap(x_3,x_4))+(x_5,x_6,x_7)}(R)$$
$$H_I^5(R) \cong H^6_{((x_1,x_2)\cap(x_3,x_4))+(x_5,x_6,x_7)}(R)$$

By using the additivity of the characteristic cycle we get:

$$CC(H_I^2(R)) = CC(H^2_{(x_1,x_2)}(R)) + CC(H^2_{(x_3,x_4)}(R)).$$
$$CC(H_I^3(R)) = CC(H^3_{(x_5,x_6,x_7)}(R)) + CC(H^4_{(x_1,x_2,x_3,x_4)}(R)).$$
$$CC(H_I^4(R)) = CC(H^5_{(x_1,x_2,x_5,x_6,x_7)}(R)) + CC(H^5_{(x_3,x_4,x_5,x_6,x_7)}(R)).$$
$$CC(H_I^5(R)) = CC(H^7_{(x_1,x_2,x_3,x_4,x_5,x_6,x_7)}(R)).$$

Remark 3.2.2. In this case the characteristic cycle of the local cohomology modules $H_I^r(R)$ is described in terms of the characteristic cycle of **all** its initial pieces because all the modules B_r that appear in the process vanish.

The second example have a non zero module B_r in the process.

Example: Let $I = (x_1, x_2) \cap (x_1, x_3) \cap (x_2, x_3) \subseteq R = k[x_1, x_2, x_3]$.

We start the process with the Mayer-Vietoris sequence:

$$\cdots \longrightarrow \begin{matrix} H^r_{(x_1,x_2)\cap(x_1,x_3)}(R) \\ \oplus \\ H^r_{(x_2,x_3)}(R) \end{matrix} \longrightarrow H_I^r(R) \longrightarrow H^{r+1}_{((x_1,x_2)\cap(x_1,x_3))+(x_2,x_3)}(R) \longrightarrow \cdots$$

The next step of the process is to compute the characteristic cycle of the local cohomology modules not supported on face ideals that appear in that sequence.

• We compute the characteristic cycle of $H^r_{(x_1,x_2)\cap(x_1,x_3)}(R)$ by using the results for the case $m=2$. Namely, we get:

$$CC(H^2_{(x_1,x_2)\cap(x_1,x_3)}(R)) = CC(H^2_{(x_1,x_2)}(R)) + CC(H^2_{(x_1,x_3)}(R)) + CC(H^3_{(x_1,x_2,x_3)}(R)).$$

• We compute the characteristic cycle of $H^r_{((x_1,x_2)\cap(x_1,x_3))+(x_2,x_3)}(R)$ by using the results for the case $m=1$ because $((x_1,x_2)\cap(x_1,x_3))+(x_2,x_3)=(x_1,x_2,x_3)$. Namely, we get:

$$CC(H^3_{((x_1,x_2)\cap(x_1,x_3))+(x_2,x_3)}(R)) = CC(H^3_{(x_1,x_2,x_3)}(R)).$$

Notice that by using Remark 3.2.1 we get in the corresponding Mayer-Vietoris sequence the module $B_3 = H^3_{(x_1,x_2)+(x_1,x_3)+(x_2,x_3)}(R)$.

Putting both results together one obtains the short exact sequence:

$$0 \longrightarrow \begin{array}{c} H^2_{(x_1,x_2)\cap(x_1,x_3)}(R) \\ \oplus \\ H^2_{(x_2,x_3)}(R) \end{array} \longrightarrow H^2_I(R) \longrightarrow H^3_{(x_1,x_2,x_3)}(R) \longrightarrow 0\,.$$

By the additivity of the characteristic cycle we get:

$$\begin{aligned} CC(H^2_I(R)) = {} & CC(H^2_{(x_1,x_2)}(R)) + CC(H^2_{(x_1,x_3)}(R)) + CC(H^2_{(x_2,x_3)}(R)) + \\ & +2\ CC(H^3_{(x_1,x_2,x_3)}(R)). \end{aligned}$$

Remark 3.2.3. In this case the characteristic cycle of the local cohomology modules $H^r_I(R)$ is **not** described in terms of the characteristic cycle of **all** its initial pieces because there is a non vanishing module B_r in the process.

3.2.3 Optimization of the Mayer-Vietoris process

In the previous subsection we have used the Mayer-Vietoris process in order to compute the characteristic cycle of the local cohomology modules $H^r_I(R)$. In particular, the local cohomology module could be described in a simplest way if a module B_r appearing in the process is different from zero. This is reflected in the fact that the corresponding Mayer-Vietoris sequence obtained in the process provides superfluous information, that we have to cancel.

Optimal Mayer-Vietoris process

The easiest case to study is when all the modules B_r that appear during the process vanish. This is reflected in the Mayer-Vietoris sequences obtained in the process by means of the following property:

Definition 3.2.4. *We say that the Mayer-Vietoris sequence*

$$\cdots \longrightarrow H^r_{U+V}(R) \longrightarrow H^r_U(R) \oplus H^r_V(R) \longrightarrow H^r_{U\cap V}(R) \longrightarrow H^{r+1}_{U+V}(R) \longrightarrow \cdots$$

is minimal if it splits for all r into short exact sequences

$$0 \longrightarrow H^r_U(R) \oplus H^r_V(R) \longrightarrow H^r_{U\cap V}(R) \longrightarrow H^{r+1}_{U+V}(R) \longrightarrow 0.$$

If all the Mayer-Vietoris sequences that appear in the Mayer-Vietoris process are minimal then the characteristic cycle of the local cohomology modules $H^r_I(R)$ is described in terms of the characteristic cycle of **all** its initial pieces. We will say then that **the Mayer-Vietoris process is optimal**, in the sense that it gives the simplest way to describe the local cohomology modules.

Ideals with disjoint faces: Let $I = I_{\alpha_1} \cap \cdots \cap I_{\alpha_m}$ be the minimal primary decomposition of a squarefree monomial ideal $I \subseteq R$. The characteristic varieties of the initial pieces are irreducible, so in order to get an optimal Mayer-Vietoris process one should have the following property:

- The initial pieces that describe $H^r_{U+V}(R)$ are different from those that describe $H^r_U(R) \oplus H^r_V(R)$ $\forall r$.

This is clearly equivalent to the fact that all the sums of face ideals in the minimal primary decomposition $I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}}$, $1 \leq i_1 < \cdots < i_j \leq m$, are different, which may be reformulated by means of following definition:

Definition 3.2.5. *Let $I = I_{\alpha_1} \cap \cdots \cap I_{\alpha_m}$ be the minimal primary decomposition of a squarefree monomial ideal $I \subseteq R$. We say that I has disjoint faces if* $\operatorname{ht}(I_{\alpha_i} + I_{\alpha_j}) = \operatorname{ht} I_{\alpha_i} + \operatorname{ht} I_{\alpha_j}$ *for all $1 \leq i < j \leq m$.*

The characteristic cycle of local cohomology modules supported on squarefree monomial ideals with disjoint faces can be easily described. Roughly speaking one has to fit the initial pieces into the right place. Namely, let

$$\mathcal{I}_{j,r} := \{I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} \in \mathcal{I}_j \mid \operatorname{ht}(I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}}) = r + (j-1)\}.$$

Then we have:

Proposition 3.2.6. *Let $I = I_{\alpha_1} \cap \cdots \cap I_{\alpha_m}$ be the minimal primary decomposition of a squarefree monomial ideal with disjoint faces $I \subseteq R$. Then:*

$$CC(H^r_I(R)) = \sum_{I_{\alpha_i} \in \mathcal{I}_{1,r}} CC(H^r_{I_{\alpha_i}}(R)) + \sum_{I_{\alpha_{i_1}} + I_{\alpha_{i_2}} \in \mathcal{I}_{2,r}} CC(H^{r+1}_{I_{\alpha_{i_1}} + I_{\alpha_{i_2}}}(R)) + \cdots +$$
$$+ \sum_{I_{\alpha_1} + \cdots + I_{\alpha_m} \in \mathcal{I}_{m,r}} CC(H^{r+(m-1)}_{I_{\alpha_1} + \cdots + I_{\alpha_m}}(R)).$$

Proof: We shall proceed by induction on m, the number of ideals in the minimal primary decomposition, being the case $m = 1$ trivial. To do it we will start the Mayer-Vietoris process with the minimal sequence:

$$0 \longrightarrow H^r_U(R) \oplus H^r_V(R) \longrightarrow H^r_{U\cap V}(R) \longrightarrow H^{r+1}_{U+V}(R) \longrightarrow 0,$$

where:

$$U = I_{\alpha_1} \cap \cdots \cap I_{\alpha_{m-1}}, \qquad U \cap V = I = I_{\alpha_1} \cap \cdots \cap I_{\alpha_m},$$
$$V = I_{\alpha_m}, \qquad U + V = (I_{\alpha_1} \cap \cdots \cap I_{\alpha_{m-1}}) + I_{\alpha_m}.$$

Assume we have proved the formula for ideals with less terms than m in the minimal primary decomposition. We have $CC(H^r_V(R)) = CC(H^r_{I_{\alpha_m}}(R))$.

Denote by $\mathcal{I}_{j,r}(U)$ the set of face ideals obtained as a sum of face ideals in the minimal primary decomposition of $U = I_{\alpha_1} \cap \cdots \cap I_{\alpha_{m-1}}$ ordered by the number of summands and the height of the ideals. By induction, the characteristic cycle of $H^r_U(R)$ is

$$CC(H^r_{I_{\alpha_1} \cap \cdots \cap I_{\alpha_{m-1}}}(R)) = \sum_{I_{\alpha_{i_1}} \in \mathcal{I}_{1,r}(U)} CC(H^r_{I_{\alpha_{i_1}}}(R)) + \cdots +$$
$$+ \sum_{I_{\alpha_1} + \cdots + I_{\alpha_{m-1}} \in \mathcal{I}_{m-1,r}(U)} CC(H^{r+(m-2)}_{I_{\alpha_1} + \cdots + I_{\alpha_{m-1}}}(R)).$$

Denote by $\mathcal{I}_{j,r+1}(U+V)$ the set of face ideals obtained as a sum of face ideals in the minimal primary decomposition of

$$U + V = (I_{\alpha_1} \cap \cdots \cap I_{\alpha_{m-1}}) + I_{\alpha_m} = (I_{\alpha_1} + I_{\alpha_m}) \cap \cdots \cap (I_{\alpha_{m-1}} + I_{\alpha_m})$$

ordered by the number of summands and the height of the ideals. Notice that the primary decomposition of $U+V$ is minimal because the ideal I has disjoint faces. By induction, the characteristic cycle of $H^{r+1}_{U+V}(R)$ is

$$CC(H^{r+1}_{(I_{\alpha_1}\cap\cdots\cap I_{\alpha_{m-1}})+I_{\alpha_m}}(R)) = \sum_{I_{\alpha_{i_1}}+I_{\alpha_m}\in\mathcal{I}_{1,r+1}(U+V)} CC(H^{r+1}_{I_{\alpha_{i_1}}+I_{\alpha_m}}(R)) + \cdots +$$

$$+ \sum_{I_{\alpha_1}+\cdots+I_{\alpha_m}\in\mathcal{I}_{m-1,r+1}(U+V)} CC(H^{r+1+(m-2)}_{I_{\alpha_1}+\cdots+I_{\alpha_m}}(R)).$$

So by using the additivity of the characteristic cycle we get the desired result.

□

Non optimal Mayer-Vietoris process

In general the Mayer-Vietoris process is not optimal since some of the modules B_r appearing in the Mayer-Vietoris sequences may not vanish. Besides the characteristic cycle, a full description of the modules B_r for the case of ideals with few ideals in the minimal primary decomposition has been given in the previous subsection. Namely we have:

The case m=2: Let $I = I_{\alpha_1} \cap I_{\alpha_2}$ be the minimal primary decomposition of a squarefree monomial ideal. The corresponding modules B_r in the process for computing the characteristic cycle of the local cohomology modules $H^r_I(R)$ vanish.

The case m=3: Let $I = I_{\alpha_1}\cap I_{\alpha_2}\cap I_{\alpha_3}$ be the minimal primary decomposition of a squarefree monomial ideal. The corresponding non vanishing modules B_r in each step of the process for computing the characteristic cycle of the local cohomology modules $H^r_I(R)$ are:

• First step: $B_r = H^r_{I_{\alpha_1}+I_{\alpha_2}+I_{\alpha_3}}(R) \neq 0$ for $r = \operatorname{ht}(I_{\alpha_1} + I_{\alpha_2} + I_{\alpha_3})$ if and only if $I_{\alpha_1} + I_{\alpha_3} = I_{\alpha_1} + I_{\alpha_2} + I_{\alpha_3}$ or $I_{\alpha_2} + I_{\alpha_3} = I_{\alpha_1} + I_{\alpha_2} + I_{\alpha_3}$.

• Second step: $B_r = H^r_{I_{\alpha_1}+I_{\alpha_2}+I_{\alpha_3}}(R) \neq 0$ for $r = \operatorname{ht}(I_{\alpha_1} + I_{\alpha_2} + I_{\alpha_3})$ if and only if $I_{\alpha_1} + I_{\alpha_2} = I_{\alpha_1} + I_{\alpha_2} + I_{\alpha_3}$ and the modules B_r obtained in the first step vanish.

In particular we see that the extra information for describing the local cohomology modules comes from the initial pieces corresponding to sums of face ideals in the minimal primary decomposition of I that satisfy one of the following conditions:

- $I_{\alpha_1} + I_{\alpha_3} = I_{\alpha_1} + I_{\alpha_2} + I_{\alpha_3}$,
- $I_{\alpha_2} + I_{\alpha_3} = I_{\alpha_1} + I_{\alpha_2} + I_{\alpha_3}$,
- $I_{\alpha_1} + I_{\alpha_2} = I_{\alpha_1} + I_{\alpha_2} + I_{\alpha_3}$.

For the general case one can check that the characteristic cycle of the corresponding modules B_r will have a similar behavior. The precise statement will be proved in Theorem 3.2.11 but for the moment we point out the following:

Remark 3.2.7. Let $I = I_{\alpha_1} \cap \cdots \cap I_{\alpha_m}$ be the minimal primary decomposition of a squarefree monomial ideal. The characteristic cycle of the modules B_r is described in terms of the initial pieces corresponding to sums of face ideals in the minimal primary decomposition of I that satisfy

$$I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} = I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} + I_{\alpha_{i_{j+1}}}, \quad 1 \leq i_1 < \cdots < i_{j+1} \leq m.$$

Now we want to determine which initial pieces determine precisely the characteristic cycle of the local cohomology modules $H_I^r(R)$. To shed some light at this point we will study in more detail the second example in the previous subsection.

Example: Let $I = (x_1, x_2) \cap (x_1, x_3) \cap (x_2, x_3) \subseteq R = k[x_1, x_2, x_3]$. Recall that we have computed the characteristic cycle of the unique non zero local cohomology module $H_I^2(R)$ and we have obtained

$$CC(H_I^2(R)) = CC(H^2_{(x_1,x_2)}(R)) + CC(H^2_{(x_1,x_3)}(R)) + CC(H^2_{(x_2,x_3)}(R)) + \\ +2\ CC(H^3_{(x_1,x_2,x_3)}(R)).$$

We have not used the Mayer-Vietoris process step by step. Instead, we have computed directly the module $H^r_{((x_1,x_2)\cap(x_1,x_3))+(x_2,x_3)}(R) = H^3_{(x_1,x_2,x_3)}(R)$. If we want to follow closely the process we must use the Mayer-Vietoris sequences as they are indicated in the diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \uparrow & & & & & & \\
 & & H^3_{(x_1,x_2)+(x_1,x_3)}(R) & & & & 0 & & \\
 & & \uparrow & & & & \uparrow & & \\
0 & \longrightarrow & H^2_{(x_1,x_2)\cap(x_1,x_3)}(R)\oplus H^2_{(x_2,x_3)}(R) & \longrightarrow & H^2_I(R) & \longrightarrow & H^3_{((x_1,x_2)\cap(x_1,x_3))+(x_2,x_3)}(R) & \longrightarrow & 0 \\
 & & \uparrow & & & & \uparrow & & \\
 & & H^2_{(x_1,x_2)}(R)\oplus H^2_{(x_1,x_3)}(R) & & & & \begin{array}{c} H^3_{(x_1,x_2)+(x_2,x_3)}(R) \\ \oplus \\ H^3_{(x_1,x_3)+(x_2,x_3)}(R) \end{array} & & \\
 & & \uparrow & & & & \uparrow & & \\
 & & 0 & & & & H^3_{(x_1,x_2)+(x_1,x_3)+(x_2,x_3)}(R) & & \\
 & & & & & & \uparrow & & \\
 & & & & & & 0 & &
\end{array}$$

In particular the initial pieces that would allow us to compute the characteristic cycle of $H^2_I(R)$ are the local cohomology modules

$$\begin{array}{lll}
H^2_{(x_1,x_2)}(R), & H^3_{(x_1,x_2)+(x_1,x_3)}(R), & H^3_{(x_1,x_2)+(x_1,x_3)+(x_2,x_3)}(R), \\
H^2_{(x_1,x_3)}(R), & H^3_{(x_1,x_2)+(x_2,x_3)}(R), & \\
H^2_{(x_2,x_3)}(R), & H^3_{(x_1,x_3)+(x_2,x_3)}(R), &
\end{array}$$

and there is *extra information* that comes from the non minimal Mayer-Vietoris sequence:

$$0 \longrightarrow H^3_{(x_1,x_2)+(x_1,x_3)+(x_2,x_3)}(R) \longrightarrow \begin{array}{c} H^3_{(x_1,x_2)+(x_2,x_3)}(R) \\ \oplus \\ H^3_{(x_1,x_3)+(x_2,x_3)}(R) \end{array} \longrightarrow H^3_{((x_1,x_2)\cap(x_1,x_3))+(x_2,x_3)}(R) \longrightarrow 0 \,.$$

This sequence is not minimal due to the fact that the primary decomposition $((x_1,x_2)\cap(x_1,x_3))+(x_2,x_3) = ((x_1,x_2)+(x_2,x_3))\cap((x_1,x_3)+(x_2,x_3))$ is not minimal. In particular we have

$$B_3 = H^3_{(x_1,x_2)+(x_1,x_3)+(x_2,x_3)}(R).$$

To make this sequence minimal we will consider $((x_1, x_2) \cap (x_1, x_3)) + (x_2, x_3) = (x_1, x_2) + (x_2, x_3)$, so the characteristic cycle of $H^2_I(R)$ can be obtained using only the following pieces:

$H^2_{(x_1,x_2)}(R)$, $\quad H^3_{(x_1,x_2)+(x_1,x_3)}(R)$,

$H^2_{(x_1,x_3)}(R)$, $\quad H^3_{(x_1,x_2)+(x_2,x_3)}(R)$,

$H^2_{(x_2,x_3)}(R)$.

We have removed the information coming from the initial pieces corresponding to the ideals $(x_1, x_3) + (x_2, x_3)$ and $(x_1, x_2) + (x_1, x_3) + (x_2, x_3)$. We will say that both initial pieces have been **canceled**.

Remark 3.2.8. Once the minimal primary decomposition of the squarefree monomial ideal I has been fixed, **the cancellation order** is given in a natural way by the Mayer-Vietoris process. However, recall that the computation of the local cohomology modules $H^r_I(R)$ is independent of the Mayer-Vietoris sequence used.

In the previous example, if we consider the minimal primary decomposition $I = (x_1, x_3) \cap (x_1, x_2) \cap (x_2, x_3)$ instead of $I = (x_1, x_2) \cap (x_1, x_3) \cap (x_2, x_3)$ then, we cancellate the initial pieces corresponding to the ideals $(x_1, x_2) + (x_2, x_3)$ and $(x_1, x_2) + (x_1, x_3) + (x_2, x_3)$ in order to make minimal the Mayer-Vietoris sequence appearing in the process. Notice that we get the same result, i.e. the characteristic cycle of $H^2_I(R)$ can be obtained using the pieces:

$H^2_{(x_1,x_2)}(R)$, $\quad H^3_{(x_1,x_2)+(x_1,x_3)}(R)$,

$H^2_{(x_1,x_3)}(R)$, $\quad H^3_{(x_1,x_3)+(x_2,x_3)}(R)$,

$H^2_{(x_2,x_3)}(R)$.

Algorithm for canceling initial pieces

Let $I = I_{\alpha_1} \cap \cdots \cap I_{\alpha_m}$ be the minimal primary decomposition of a squarefree monomial ideal $I \subseteq R$. The initial pieces that allow us to compute the characteristic cycle of $H^r_I(R)$ are the local cohomology modules supported on sums of face ideals included in the set $\mathcal{I} = \{\mathcal{I}_1, \mathcal{I}_2, \dots, \mathcal{I}_m\}$. In order to describe the superfluous information, see Remark 3.2.7, we introduce the following:

Definition 3.2.9. *We say that* $I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} \in \mathcal{I}_j$ *and* $I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} + I_{\alpha_{i_{j+1}}} \in \mathcal{I}_{j+1}$ *are paired if* $I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} = I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} + I_{\alpha_{i_{j+1}}}$, *i.e., a generator of* $I_{\alpha_{i_{j+1}}}$ *is also a generator of* $I_{\alpha_{i_k}}$ *for some* $k = 1, \ldots, j$.

Once the Mayer-Vietoris process be concluded we will get an expression of the characteristic cycle of $H_I^r(R)$ in terms of the characteristic cycle of local cohomology modules supported on face ideals that are contained in a subset of $\mathcal{I}$ obtained by canceling all possible pairs with the order given by the Mayer-Vietoris process. We will illustrate this order for the case of few face ideals in the minimal primary decomposition of I.

The case m=2: Let $I = I_{\alpha_1} \cap I_{\alpha_2}$ be the minimal primary decomposition of a squarefree monomial ideal. To compute $CC(H^r_{I_{\alpha_1} \cap I_{\alpha_2}}(R))$ we consider the sets of sums of face ideals that give the initial pieces:

$\mathcal{I}_1 = \{I_{\alpha_1} I_{\alpha_2}\}$ and

$\mathcal{I}_2 = \{I_{\alpha_1} + I_{\alpha_2}\}$.

We will organize this information in the following diagram

$$I_{\alpha_2} \quad \begin{cases} I_{\alpha_1} \\ I_{\alpha_1} + I_{\alpha_2} \end{cases}$$

We will call it **the diagram of** $I = I_{\alpha_1} \cap I_{\alpha_2}$. Now, as we pointed out in the previous example, see Remark 3.2.8, the canceling order is as follows:

- Compare the ideals I_{α_2} and $I_{\alpha_1} + I_{\alpha_2}$ and remove them if they are paired.

  Otherwise

- Compare the ideals I_{α_1} and $I_{\alpha_1} + I_{\alpha_2}$ and remove them if they are paired.

The canceling order is reflected in the diagram of I as follows:

- Compare the ideals: $\quad I_{\alpha_2} \begin{cases} * \\ I_{\alpha_1} + I_{\alpha_2} \end{cases}$

- If they do not cancellate, compare the ideals: $\quad * \begin{cases} I_{\alpha_1} \\ I_{\alpha_1} + I_{\alpha_2} \end{cases}$

The case m=3: Let $I = I_{\alpha_1} \cap I_{\alpha_2} \cap I_{\alpha_3}$ be the minimal primary decomposition of a squarefree monomial ideal. To compute $CC(H^r_{I_{\alpha_1} \cap I_{\alpha_2} \cap I_{\alpha_3}}(R))$ we consider the sets of ideals that give the initial pieces:

$$\begin{aligned}
&\mathcal{I}_1 = \{I_{\alpha_1}, I_{\alpha_2}, I_{\alpha_3}\},\\
&\mathcal{I}_2 = \{I_{\alpha_1}+I_{\alpha_2}, I_{\alpha_1}+I_{\alpha_3}, I_{\alpha_2}+I_{\alpha_3}\} \quad \text{and}\\
&\mathcal{I}_3 = \{I_{\alpha_1}+I_{\alpha_2}+I_{\alpha_3}\}.
\end{aligned}$$

We will organize this information in the diagram of I.

$$I_{\alpha_3} \left\{ \begin{array}{ll} I_{\alpha_2} & \left\{ \begin{array}{l} I_{\alpha_1} \\ I_{\alpha_1}+I_{\alpha_2} \end{array} \right. \\ & \\ I_{\alpha_2}+I_{\alpha_3} & \left\{ \begin{array}{l} I_{\alpha_1}+I_{\alpha_3} \\ I_{\alpha_1}+I_{\alpha_2}+I_{\alpha_3} \end{array} \right. \end{array} \right.$$

Now we will give the canceling order for the ideal I and how it is reflected in the diagram of I. To do this we will use the same notation as in the previous case.

The canceling order starts with the second step in the Mayer-Vietoris process:

- Compare the ideals:

$$* \left\{ \begin{array}{ll} I_{\alpha_2} & \left\{ \begin{array}{l} * \\ I_{\alpha_1}+I_{\alpha_2} \end{array} \right. \\ * & \left\{ \begin{array}{l} * \\ * \end{array} \right. \end{array} \right. \qquad \text{and} \qquad * \left\{ \begin{array}{ll} * & \left\{ \begin{array}{l} * \\ * \end{array} \right. \\ I_{\alpha_2}+I_{\alpha_3} & \left\{ \begin{array}{l} * \\ I_{\alpha_1}+I_{\alpha_2}+I_{\alpha_3} \end{array} \right. \end{array} \right.$$

- If they do not cancellate, compare the ideals:

$$* \left\{ \begin{array}{ll} * & \left\{ \begin{array}{l} I_{\alpha_1} \\ I_{\alpha_1}+I_{\alpha_2} \end{array} \right. \\ * & \left\{ \begin{array}{l} * \\ * \end{array} \right. \end{array} \right. \qquad \text{and} \qquad * \left\{ \begin{array}{ll} * & \left\{ \begin{array}{l} * \\ * \end{array} \right. \\ * & \left\{ \begin{array}{l} I_{\alpha_1}+I_{\alpha_3} \\ I_{\alpha_1}+I_{\alpha_2}+I_{\alpha_3} \end{array} \right. \end{array} \right.$$

Now, the canceling order continues with the first step of the Mayer-Vietoris process:

- Compare the ideals:

$$I_{\alpha_3} \left\{ \begin{array}{ll} * & \left\{ \begin{array}{l} * \\ * \end{array} \right. \\ * & \left\{ \begin{array}{l} I_{\alpha_1}+I_{\alpha_3} \\ * \end{array} \right. \end{array} \right. \qquad \text{and} \qquad I_{\alpha_3} \left\{ \begin{array}{ll} * & \left\{ \begin{array}{l} * \\ * \end{array} \right. \\ I_{\alpha_2}+I_{\alpha_3} & \left\{ \begin{array}{l} * \\ * \end{array} \right. \end{array} \right.$$

- If they do not cancellate, compare the ideals:

$$
* \left\{ \begin{array}{l} * \left\{ \begin{array}{l} I_{\alpha_1} \\ * \end{array} \right. \\ * \left\{ \begin{array}{l} I_{\alpha_1} + I_{\alpha_3} \\ * \end{array} \right. \end{array} \right. , \quad * \left\{ \begin{array}{l} * \left\{ \begin{array}{l} * \\ I_{\alpha_1} + I_{\alpha_2} \end{array} \right. \\ * \left\{ \begin{array}{l} * \\ I_{\alpha_1} + I_{\alpha_2} + I_{\alpha_3} \end{array} \right. \end{array} \right. \quad \text{and} \quad * \left\{ \begin{array}{l} I_{\alpha_2} \quad \left\{ \begin{array}{l} * \\ * \end{array} \right. \\ I_{\alpha_2} + I_{\alpha_3} \left\{ \begin{array}{l} * \\ * \end{array} \right. \end{array} \right.
$$

General case: Let $I = I_{\alpha_1} \cap \cdots \cap I_{\alpha_m}$ be the minimal primary decomposition of a squarefree monomial ideal. To compute $CC(H_I^r(R))$ we consider the set of sums of face ideals that gives the initial pieces $\mathcal{I} = \{\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_m\}$. We use the Mayer-Vietoris process to organize this information in the diagram of I. Then, the cancellation order is given by the following:

Algorithm :

Let m be the number of ideals in the minimal primary decomposition $I = I_{\alpha_1} \cap \cdots \cap I_{\alpha_m}$.

- For j from 1 to $m-1$, incrementing by 1

  - For all $k \leq j$ **COMPARE** the ideals

    $I_{\alpha_{j+1}} + (I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_r}})$ and $I_{\alpha_k} + I_{\alpha_{j+1}} + (I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_r}})$.

  - For all $k \leq j$ **COMPARE** the ideals

    $I_{\alpha_k} + (I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_r}})$ and $I_{\alpha_k} + I_{\alpha_{j+1}} + (I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_r}})$.

The following remarks will be very useful to prove our main result and further computations.

Remark 3.2.10. *i)* The diagram of $(I_{\alpha_1} \cap \cdots \cap I_{\alpha_{m-1}}) + I_{\alpha_m}$ is the same as the diagram of $I_{\alpha_1} \cap \cdots \cap I_{\alpha_{m-1}}$ adding I_{α_m} to every sum of face ideals.

ii) Every time we add a new face ideal to the minimal primary decomposition of an ideal, the diagram duplicates and a new vertex is included.

iii) Let I, J be squarefree monomial ideals. If the ideals in the diagram of J are included in the diagram of I we say that they form a subdiagram of I. In particular, the algorithm for canceling paired ideals in the diagram of J is compatible with the algorithm for canceling paired ideals in the diagram of I.

To get the formula in Theorem 3.2.11, we consider:

INPUT: The set $\mathcal{I}$ of all the sums of face ideals obtained as a sum of face ideals in the minimal primary decomposition of I.

We apply the algorithm where **COMPARE** means remove both ideals in case they are paired.

OUTPUT: The set $\mathcal{P}$ of all the non paired sums of face ideals obtained as a sum of face ideals in the minimal primary decomposition of I.

We order $\mathcal{P}$ by the number of summands $\mathcal{P} = \{\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_m\}$, in such a way that no sum in $\mathcal{P}_j$ is paired with a sum in $\mathcal{P}_{j+1}$. Observe that some of these sets can be empty. Finally we define the sets of non paired sums of face ideals with a given height

$$\mathcal{P}_{j,r} := \{I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} \in \mathcal{P}_j \mid \operatorname{ht}(I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}}) = r + (j-1)\}.$$

The formula we will give in Theorem 3.2.11 will be expressed in terms of these sets of non paired sums of face ideals.

3.2.4 Main result

The main result of this chapter is a closed formula that describes the characteristic cycle of the local cohomology modules supported on squarefree monomial ideals in terms of the sets $\mathcal{P}_{j,r}$ of face ideals in the minimal primary decomposition of the ideal. In particular, we give a complete description of the characteristic cycle of the modules B_r that appear in the Mayer-Vietoris process and we determine which initial pieces describe precisely the characteristic cycle of these local cohomology modules.

Theorem 3.2.11. *Let $I \subseteq R$ be an ideal generated by squarefree monomials and let $I = I_{\alpha_1} \cap \cdots \cap I_{\alpha_m}$ be its minimal primary decomposition. Then:*

$$CC(H_I^r(R)) = \sum_{I_{\alpha_i} \in \mathcal{P}_{1,r}} CC(H^r_{I_{\alpha_i}}(R)) + \sum_{I_{\alpha_{i_1}} + I_{\alpha_{i_2}} \in \mathcal{P}_{2,r}} CC(H^{r+1}_{I_{\alpha_{i_1}} + I_{\alpha_{i_2}}}(R)) + \cdots +$$

$$+ \sum_{I_{\alpha_1} + \cdots + I_{\alpha_m} \in \mathcal{P}_{m,r}} CC(H^{r+(m-1)}_{I_{\alpha_1} + \cdots + I_{\alpha_m}}(R)).$$

The following remarks will be very useful for the proof of the theorem.

Remark 3.2.12. Theorem 3.2.11 is also true if we consider a non minimal primary decomposition of I, the lack of minimality will only increase the number of cancellations in the algorithm.

Remark 3.2.13. From the formula it is easy to see the following:

- If $CC(H^{r+(j-1)}_{I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}}}(R)) \in CC(H_I^r(R))$ then

$$CC(H^{r+(j-1)}_{I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}}}(R)) \notin CC(H_I^s(R)), \ \forall s \neq r.$$

PROOF: We shall proceed by induction on m, the number of ideals in the minimal primary decomposition, being the case $m = 1$ trivial. To do it we will start the Mayer-Vietoris process with the sequence:

$$\cdots \longrightarrow H^r_{U+V}(R) \longrightarrow H^r_U(R) \oplus H^r_V(R) \longrightarrow H^r_{U\cap V}(R) \longrightarrow H^{r+1}_{U+V}(R) \longrightarrow \cdots,$$

where:

$$U = I_{\alpha_1} \cap \cdots \cap I_{\alpha_{m-1}}, \qquad U \cap V = I = I_{\alpha_1} \cap \cdots \cap I_{\alpha_m},$$
$$V = I_{\alpha_m}, \qquad U + V = (I_{\alpha_1} \cap \cdots \cap I_{\alpha_{m-1}}) + I_{\alpha_m}.$$

Assume we have proved the formula for ideals with less terms than m in the minimal primary decomposition.

We have $CC(H_V^r(R)) = CC(H^r_{I_{\alpha_m}}(R))$. Recall that I_{α_m} is the vertex of the diagram of I.

To describe $CC(H_U^r(R))$ we will denote by $\mathcal{I}(U)$ the set of sums of face ideals in the minimal primary decomposition $U = I_{\alpha_1} \cap \cdots \cap I_{\alpha_{m-1}}$. Ordering this set by the number of summands we get:

$$\mathcal{I}_1(U) = \{I_{\alpha_{i_1}} \mid 1 \leq i_1 \leq m-1\},$$
$$\mathcal{I}_2(U) = \{I_{\alpha_{i_1}} + I_{\alpha_{i_2}} \mid 1 \leq i_1 < i_2 \leq m-1\},$$
$$\vdots$$
$$\mathcal{I}_{m-1}(U) = \{I_{\alpha_1} + I_{\alpha_2} + \cdots + I_{\alpha_{m-1}}\}.$$

Observe that $\mathcal{I}(U)$ is the subset of $\mathcal{I}$ formed by the sums of face ideals in the upper half of the diagram of I. Applying the algorithm of cancellation to $\mathcal{I}(U)$ we obtain the poset $\mathcal{P}(U) := \{\mathcal{P}_1(U), \dots, \mathcal{P}_{m-1}(U)\}$. Ordering the ideals by heights we obtain the sets:

$$\mathcal{P}_{j,r}(U) := \{I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} \in \mathcal{P}_j(U) \mid \operatorname{ht}(I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}}) = r + (j-1)\}.$$

By induction we have:

$$CC(H^r_{I_{\alpha_1} \cap \cdots \cap I_{\alpha_{m-1}}}(R)) = \sum_{I_{\alpha_{i_1}} \in \mathcal{P}_{1,r}(U)} CC(H^r_{I_{\alpha_{i_1}}}(R)) + \cdots +$$
$$+ \sum_{I_{\alpha_1} + \cdots + I_{\alpha_{m-1}} \in \mathcal{P}_{m-1,r}(U)} CC(H^{r+(m-2)}_{I_{\alpha_1} + \cdots + I_{\alpha_{m-1}}}(R)).$$

To compute $CC(H^{r+1}_{U+V}(R))$ we will consider the non necessarily minimal primary decomposition $U+V = (I_{\alpha_1} \cap \cdots \cap I_{\alpha_{m-1}}) + I_{\alpha_m} = (I_{\alpha_1} + I_{\alpha_m}) \cap \cdots \cap (I_{\alpha_{m-1}} + I_{\alpha_m})$, (see Remark 3.2.12). We will denote by $\mathcal{I}(U+V)$ the set of sums of face ideals in the primary decomposition. Ordering this set by the number of summands we get:

$$\mathcal{I}_1(U+V) = \{I_{\alpha_{i_1}} + I_{\alpha_m} \mid 1 \leq i_1 \leq m-1\},$$
$$\mathcal{I}_2(U+V) = \{I_{\alpha_{i_1}} + I_{\alpha_{i_2}} + I_{\alpha_m} \mid 1 \leq i_1 < i_2 \leq m-1\},$$
$$\vdots$$
$$\mathcal{I}_{m-1}(U+V) = \{I_{\alpha_1} + I_{\alpha_2} + \cdots + I_{\alpha_{m-1}} + I_{\alpha_m}\}.$$

Observe that $\mathcal{I}(U+V)$ is the subset of $\mathcal{I}$ formed by the sums of face ideals in the lower half of the diagram of I (there is a shift in the number of summands of $\mathcal{I}(U+V)$ because I_{α_m} appears as a summand in every ideal of the primary decomposition). Applying the algorithm of cancellation to $\mathcal{I}(U+V)$ we obtain the poset $\mathcal{P}(U+V) := \{\mathcal{P}_1(U+V), \dots, \mathcal{P}_{m-1}(U+V)\}$. Ordering the ideals by heights we obtain the sets:

$$\mathcal{P}_{j,r+1}(U+V) := \{I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} + I_{\alpha_m} \in \mathcal{P}_j(U+V) \mid \\ \operatorname{ht}(I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} + I_{\alpha_m}) = r+1+(j-1)\}.$$

By induction we have:

$$CC(H^{r+1}_{(I_{\alpha_1}\cap\cdots\cap I_{\alpha_{m-1}})+I_{\alpha_m}}(R)) = \sum_{I_{\alpha_{i_1}}+I_{\alpha_m}\in\mathcal{P}_{1,r+1}(U+V)} CC(H^{r+1}_{I_{\alpha_{i_1}}+I_{\alpha_m}}(R)) + \cdots + \\ + \sum_{I_{\alpha_1}+\cdots+I_{\alpha_m}\in\mathcal{P}_{m-1,r+1}(U+V)} CC(H^{r+1+(m-2)}_{I_{\alpha_1}+\cdots+I_{\alpha_m}}(R)).$$

Now we split the Mayer-Vietoris sequence into short exact sequences of kernels and cokernels:

$$0\longrightarrow B_r\longrightarrow H^r_U(R)\oplus H^r_V(R)\longrightarrow C_r\longrightarrow 0,$$

$$0\longrightarrow C_r\longrightarrow H^r_{U\cap V}(R)\longrightarrow A_{r+1}\longrightarrow 0,$$

$$0\longrightarrow A_{r+1}\longrightarrow H^{r+1}_{U+V}(R)\longrightarrow B_{r+1}\longrightarrow 0,$$

so, by additivity we have:

$$CC(H^r_{U\cap V}(R)) = CC(C_r) + CC(A_{r+1}) =$$

$$= (CC(H^r_U(R)\oplus H^r_V(R)) - CC(B_r)) + (CC(H^{r+1}_{U+V}(R)) - CC(B_{r+1})).$$

By induction, we have computed the characteristic cycle of the local cohomology modules $H^r_U(R)\oplus H^r_V(R)$ and $H^{r+1}_{U+V}(R)$ so we only have to describe $CC(B_r)$ $\forall r$.

Notice that, during the process of computation of $CC(H^r_U(R)\oplus H^r_V(R))$, we have canceled the pairs coming from the upper half of the diagram of I. On the other side, in order to compute $CC(H^{r+1}_{U+V}(R))$, we have canceled the pairs coming from the lower half of the diagram.

To get the desired formula, it remains to cancel the possible pairs formed by ideals $I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} \in \mathcal{P}_{j,r}(U)$, i.e those coming from the computation of $CC(H^r_U(R))$ and $I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} + I_{\alpha_m} \in \mathcal{P}_{j,r}(U+V)$, coming from the computation of $CC(H^r_{U+V}(R))$. It also remains to cancel the possible pair formed by the ideal I_{α_m} coming from the computation of $CC(H^r_V(R))$ and some $I_{\alpha_i}+I_{\alpha_m} \in \mathcal{P}_{1,r}(U+V)$, coming from the computation of $CC(H^r_{U+V}(R))$.

So, we only have to prove that the initial pieces coming from these paired ideals describe the modules $CC(B_r)$ $\forall r$. It will be done by means of the following:

Claim:

$$CC(B_r) = \sum CC(H^{|\alpha|}_{I_\alpha}(R)),$$

where the sum is taken over the cycles that come from almost pairs of the form

- $I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} \in \mathcal{P}_{j,r}(U)$ and $I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} + I_{\alpha_m} \in \mathcal{P}_{j,r+1}(U+V)$,

  where $I_\alpha = I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} = I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} + I_{\alpha_m}$, and $|\alpha| = r + (j-1)$.

- I_{α_m} and $I_{\alpha_{i_1}} + I_{\alpha_m} \in \mathcal{P}_{1,r+1}(U+V)$,

  where $I_\alpha = I_{\alpha_m} = I_{\alpha_{i_1}} + I_{\alpha_m}$, and $|\alpha| = r$.

The inclusion $\subseteq$ is obvious because $CC(B_r)$ belongs to $CC(H^r_U(R) \oplus H^r_V(R))$ and $CC(H^r_{U+V}(R))$. To prove the other one, let $T^*_{X_\alpha}X = CC(H^{|\alpha|}_{I_\alpha}(R))$ be a cycle that come from a pair and suppose that does not belong to $CC(B_r)$. Consider a Mayer-Vietoris sequence

$$\cdots \longrightarrow H^r_{U'+V'}(R) \longrightarrow H^r_{U'}(R) \oplus H^r_{V'}(R) \longrightarrow H^r_{U'\cap V'}(R) \longrightarrow H^{r+1}_{U'+V'}(R) \longrightarrow \cdots,$$

where:

- If $T^*_{X_\alpha}X$ belongs to $CC(H^r_U(R))$ and $CC(H^r_{U+V}(R))$ we choose:

$U' = I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}}$, $\qquad U' \cap V' = I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} \cap I_{\alpha_m}$,

$V' = I_{\alpha_m}$, $\qquad U' + V' = I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} + I_{\alpha_m}$.

- If $T^*_{X_\alpha}X$ belongs to $CC(H^r_V(R))$ and $CC(H^r_{U+V}(R))$ we choose:

$U' = I_{\alpha_i}$, $\qquad U' \cap V' = I_{\alpha_i} \cap I_{\alpha_m}$,

$V' = I_{\alpha_m}$, $\qquad U' + V' = I_{\alpha_i} + I_{\alpha_m}$.

Notice that in the corresponding short exact sequences

$$0 \longrightarrow B'_r \longrightarrow H^r_{U'}(R) \oplus H^r_{V'}(R) \longrightarrow C'_r \longrightarrow 0,$$

$$0 \longrightarrow C'_r \longrightarrow H^r_{U' \cap V'}(R) \longrightarrow A'_{r+1} \longrightarrow 0,$$

$$0 \longrightarrow A'_{r+1} \longrightarrow H^{r+1}_{U'+V'}(R) \longrightarrow B'_{r+1} \longrightarrow 0,$$

we have:

$T^*_{X_\alpha} X \in CC(H^r_{U'}(R) \oplus H^r_{V'}(R))$,

$T^*_{X_\alpha} X \in CC(H^r_{U'+V'}(R))$,

$T^*_{X_\alpha} X \notin CC(B'_r)$.

The last statement is due to the fact that the diagram of the ideal $U' \cap V'$ is a subdiagram of the diagram of I, i.e. the corresponding ideals are contained in the diagram of I (see Remark 3.2.10). In particular, if the cycle $T^*_{X_\alpha} X$ has not been canceled during the computation of $H^r_I(R)$, i.e. the cycle does not belong to $CC(B_r)$ then, it can not be canceled in the computation of $H^r_{U' \cap V'}(R)$, i.e. the cycle does not belong to any $CC(B'_r)$ of the corresponding short exact sequences we have considered.

Finally, by induction and using Remark 3.2.13 there is a contradiction due to the fact that in this case we get:

$$T^*_{X_\alpha} X \in CC(H^{r-1}_{U' \cap V'}(R)) \quad \text{and} \quad T^*_{X_\alpha} X \in CC(H^r_{U' \cap V'}(R)).$$

□

Remark 3.2.14. The sets $\mathcal{P}_{j,r}$ that describe precisely the characteristic cycle of the local cohomology modules $H^r_I(R)$ are independent of the order given to the face ideals in the minimal primary decomposition of I. This order only changes the way we represent the ideals in $\mathcal{P}_{j,r}$ as a sum of j face ideals.

Example: Let $R = k[x_1, x_2, x_3, x_4]$. Consider the ideal:

- $I = (x_1, x_2) \cap (x_2, x_3) \cap (x_3, x_4)$.

We organize the information given by the poset $\mathcal{I}$ that label the initial pieces in the diagram of I:

$$I_{\alpha_3} = (x_3, x_4) \begin{cases} I_{\alpha_2} = (x_2, x_3) & \begin{cases} I_{\alpha_1} = (x_1, x_2) \\ I_{\alpha_1} + I_{\alpha_2} = (x_1, x_2, x_3) \end{cases} \\ \\ I_{\alpha_2} + I_{\alpha_3} = (x_2, x_3, x_4) & \begin{cases} I_{\alpha_1} + I_{\alpha_3} = (x_1, x_2, x_3, x_4) \\ I_{\alpha_1} + I_{\alpha_2} + I_{\alpha_3} = (x_1, x_2, x_3, x_4) \end{cases} \end{cases}$$

We start to cancel the paired sums of face ideals by using the algorithm.

Step 1:

- The ideals $I_{\alpha_2} \neq I_{\alpha_1} + I_{\alpha_2}$ and $I_{\alpha_2} + I_{\alpha_3} \neq I_{\alpha_1} + I_{\alpha_2} + I_{\alpha_3}$ do not cancellate.
- The ideals $I_{\alpha_1} \neq I_{\alpha_1} + I_{\alpha_2}$ do not cancellate, but the ideals $I_{\alpha_1} + I_{\alpha_3}$ and $I_{\alpha_1} + I_{\alpha_2} + I_{\alpha_3}$ describe the same ideal so we cancellate them.

Step 2:

- The ideals $I_{\alpha_3} \neq I_{\alpha_2} + I_{\alpha_3}$ do not cancellate.
- The ideals $I_{\alpha_2} \neq I_{\alpha_2} + I_{\alpha_3}$ do not cancellate.

Note that these are the pairs of sums of face ideals that remain to compare and have not been canceled in the previous step.

We organize the information given by the poset $\mathcal{P}$ in the subdiagram of I:

$$I_{\alpha_3} = (x_3, x_4) \quad \begin{cases} I_{\alpha_2} = (x_2, x_3) & \begin{cases} I_{\alpha_1} = (x_1, x_2) \\ I_{\alpha_1} + I_{\alpha_2} = (x_1, x_2, x_3) \end{cases} \\ \\ I_{\alpha_2} + I_{\alpha_3} = (x_2, x_3, x_4) & \end{cases}$$

Collecting the ideals by the height we get the sets $\mathcal{P}_{1,2} = \{I_{\alpha_1}, I_{\alpha_2}, I_{\alpha_3}\}$ and $\mathcal{P}_{2,2} = \{I_{\alpha_1} + I_{\alpha_2}, I_{\alpha_2} + I_{\alpha_3}\}$. Then we have:

$$\begin{aligned} CC(H^2_I(R)) = {} & CC(H^2_{I_{\alpha_1}}(R)) + CC(H^2_{I_{\alpha_2}}(R)) + CC(H^2_{I_{\alpha_3}}(R)) + \\ & + CC(H^3_{I_{\alpha_1}+I_{\alpha_2}}(R)) + CC(H^3_{I_{\alpha_2}+I_{\alpha_3}}(R)). \end{aligned}$$

In terms of conormal bundles relative to a subvariety we have:

$$\begin{aligned} CC(H^2_I(R)) = {} & T^*_{X_{(1,1,0,0)}}X + T^*_{X_{(0,1,1,0)}}X + T^*_{X_{(0,0,1,1)}}X + T^*_{X_{(1,1,1,0)}}X \\ & + T^*_{X_{(0,1,1,1)}}X. \end{aligned}$$

3.2.5 Characteristic cycle and restriction

The aim of this section is to provide an useful tool, the functor of restriction, that we will use throughout this work. For details and further considerations see [70].

Definition 3.2.15. *Depending on the ring R we are working with, we define the restriction of R to a face ideal $\mathfrak{p}_\alpha \subseteq R$, $\alpha \in \{0,1\}^n$ as any of the following rings:*

- $R_{[\mathfrak{p}_\alpha]} = k[[x_i \mid \alpha_i = 1]]$.
- $R_{[\mathfrak{p}_\alpha]} = k\{x_i \mid \alpha_i = 1\}$.
- $R_{[\mathfrak{p}_\alpha]} = k[x_i \mid \alpha_i = 1]$.

Let $I = I_{\alpha_1} \cap \cdots \cap I_{\alpha_m}$ be the minimal primary decomposition of a squarefree monomial ideal $I \subseteq R$. Then, the restriction of I to the face ideal $\mathfrak{p}_\alpha$ is the squarefree monomial ideal $I_{[\mathfrak{p}_\alpha]} \subseteq R_{[\mathfrak{p}_\alpha]}$ whose face ideals in the minimal primary decomposition are those face ideals I_{α_j} contained in $\mathfrak{p}_\alpha$. Namely

$$I_{[\mathfrak{p}_\alpha]} = \bigcap_{\alpha_j \leq \alpha} I_{\alpha_j}.$$

The algorithm of canceling paired sums of face ideals that allows to obtain the formula Theorem 3.2.11 has a good behavior with respect to the exact functor of restriction. In particular, we can easily describe the characteristic cycle of the local cohomology modules $H^{n-i}_{I_{[\mathfrak{p}_\alpha]}}(R_{[\mathfrak{p}_\alpha]})$.

Corollary 3.2.16. *Let $CC(H^{n-i}_I(R)) = \sum m_{i,\beta}\ T^*_{X_\beta}X$ be the characteristic cycle of a local cohomology module $H^{n-i}_I(R)$ supported on a face ideal $I \subseteq R$. Let $\mathfrak{p}_\alpha \subseteq R$, $\alpha \in \{0,1\}^n$, be a face ideal. Then we have:*

$$CC(H^{n-i}_{I_{[\mathfrak{p}_\alpha]}}(R_{[\mathfrak{p}_\alpha]})) = \sum_{\beta \leq \alpha} m_{i,\beta}\ T^*_{X'_\beta}X',$$

where $X' = \operatorname{Spec} R_{[\mathfrak{p}_\alpha]}$ and $X'_\beta \subseteq X'$ is the subvariety defined by the face ideal $\mathfrak{p}_\beta \subseteq R_{[\mathfrak{p}_\alpha]}$.

3.3 Consequences

Let $CC(H_I^{n-i}(R)) = \sum m_{i,\alpha}\ T^*_{X_\alpha}X$ be the characteristic cycle of a local cohomology module $H_I^{n-i}(R)$. It provides many information on the module $H_I^{n-i}(R)$ as well on the ring R/I due to the fact that the multiplicities $m_{i,\alpha}$ are invariants (see Chapter 2).

The formula obtained in Theorem 3.2.11 is given in terms of the face ideals in the minimal primary decomposition of a squarefree monomial ideal I, so the results in this section will be described in these terms.

In a first reading of the characteristic cycle we may decide when the module $H_I^{n-i}(R)$ vanishes, in particular we will compute the cohomological dimension $\operatorname{cd}(R,I)$. A deeper reading of the formula will allow us to determine the support and the dimension of $H_I^{n-i}(R)$.

On the other hand we will study the ring R/I by using the invariants given by the multiplicities $m_{i,\alpha}$. In particular we will compare them with some other invariants and give new characterizations of several arithmetical properties of R/I.

Sometimes, it will be enough to consider the coarser invariants $\gamma_{p,i}(R/I)$ introduced in Section 2.2.1. In particular, some results will be expressed in terms of the triangular matrix:

$$\Gamma(R/I) = \begin{pmatrix} \gamma_{0,0} & \cdots & \gamma_{0,d} \\ & \ddots & \vdots \\ & & \gamma_{d,d} \end{pmatrix}.$$

Theorem 3.2.11 gives an easy interpretation of these invariants in terms of the face ideals in the minimal primary decomposition of the monomial ideal I. More precisely, the invariants $\gamma_{p,i}(R/I)$ are described in terms of the sets $\mathcal{P}_{j,n-i}$ of non paired sums of j face ideals obtained in the optimization of the Mayer-Vietoris process. Namely,

Proposition 3.3.1. *Let $I \subseteq R$ be a squarefree monomial ideal. Let $\mathcal{P}$ be the poset of sums of face ideals obtained from the poset $\mathcal{I}$ by means of the algorithm of canceling paired ideals. Then, we have the following description:*

$$\gamma_{p,i}(R/I) = \sharp\ \mathcal{P}_{i-p+1,n-i}.$$

3.3.1 Annihilation and support of local cohomology modules

Annihilation of local cohomology modules: A holonomic $\mathcal{D}$-module is zero if and only if the corresponding characteristic cycle vanish, so the formula given in Theorem 3.2.11 for the characteristic cycle of the local cohomology modules provides a criterion to decide the vanishing of $H_I^{n-i}(R)$.

Proposition 3.3.2. *For any ideal $I \subseteq R$ generated by squarefree monomials, the following are equivalent:*

i) $H_I^{n-i}(R) \neq 0$.

ii) There exists j such that $\gamma_{j,i}(R/I) \neq 0$.

We can interpret this result as follows: The local cohomology module $H_I^{n-i}(R)$ does not vanish if the corresponding column in the matrix $\Gamma(R/I)$ is different from zero. Since these invariants are described in terms of the face ideals in the minimal primary decomposition of I we obtain the following precise statement:

Proposition 3.3.3. *For any ideal $I \subseteq R$ generated by squarefree monomials, the following are equivalent:*

i) $H_I^{n-i}(R) \neq 0$.

ii) There exists $I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} \in \mathcal{P}_j$ such that $\mathrm{ht}\,(I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}}) = n - i + (j-1)$.

Remark 3.3.4. By using [54, Theorem 1,(iii)] this criterion allows us to determine the vanishing of $\mathrm{Ext}_R^{n-i}(R/I, R)$.

Cohomological dimension: Once the annihilation of the local cohomology modules is determined we can compute the cohomological dimension of R with respect to I:

$$\mathrm{cd}\,(R, I) = \max\{r \mid H_I^r(R) \neq 0\}.$$

Corollary 3.3.5. *Let $I \subseteq R$ be a squarefree monomial ideal. Then, the cohomological dimension of R with respect to I is:*

$$\mathrm{cd}\,(R, I) = \max\,\{n - i \mid \gamma_{p,i}(R/I) \neq 0\}.$$

In terms of the face ideals in the minimal primary decomposition of I we have:

$$\mathrm{cd}\,(R, I) = \max\,\{\mathrm{ht}\,(I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}}) - (j-1) \mid I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} \in \mathcal{P}_j\}.$$

Remark 3.3.6. Let $I \subseteq R$ be a squarefree monomial ideal. By using [54, Theorem 1,(iv)], we have $\mathrm{cd}\,(R, I) = \mathrm{pd}(R/I)$. In Section 3.3.4 we will give another approach to this result.

In the literature one can find some important results on the cohomological dimension of R with respect to I. Our aim is to give a brief sketch of how they can be deduced from Theorem 3.2.11 for the case of squarefree monomial ideals.

Let $I = I_{\alpha_1} \cap \cdots \cap I_{\alpha_m}$ be the minimal primary decomposition of a squarefree monomial ideal $I \subseteq R$. Set $b = \mathrm{bight}\, I := \max\{\mathrm{ht}\, I_{\alpha_j} \mid j = 1, \ldots, m\}$ the big height of I.

• **[40] Hartshorne-Lichtenbaum vanishing theorem:** The following conditions are equivalent:

i) $\mathrm{cd}\,(R, I) < n$.

ii) $\dim R/I > 0$

Proof: From Theorem 3.2.11 it is easy to see that $H^n_I(R) = 0$, i.e. $\gamma_{0,0} = 0$, if and only if $I \neq \mathfrak{m}$ so we are done. □

This following theorem was proved by R. Hartshorne [40] in the geometric case, by A. Ogus [74] in characteristic zero and by C. Peskine-L. Szpiro [76] and R. Hartshorne-R. Speiser [43] in characteristic $p > 0$. A characteristic free proof was given by C. Huneke-G. Lyubeznik [47].

• **[47, Theorem 1.1]:** The following conditions are equivalent:

i) $\mathrm{cd}\,(R, I) < n - 1$.

ii) $\dim R/I > 1$ and $\mathrm{Spec}\,(R/I) \setminus \{\mathfrak{m}\}$ is connected.

PROOF: From Theorem 3.2.11 we have $\gamma_{1,1} = 0$ if and only if $b < n-1$, i.e. $\dim R/I > 1$. On the other side $\gamma_{0,1} = 0$ if and only if for any sum $I_{\alpha_j} + I_{\alpha_k} = \mathfrak{m}$ there exists I_{α_i} such that $I_{\alpha_i} + I_{\alpha_j} + I_{\alpha_k} = \mathfrak{m}$. Notice that the last assertion is equivalent to $\operatorname{Spec}(R/I) \setminus \{\mathfrak{m}\}$ is connected. □

In the same paper Huneke and Lyubeznik gave a sufficient condition for $\operatorname{cd}(R, I) < n-2$ expressed in terms of sums of ideals in the minimal primary decomposition.

- **[47, Theorem 5.8] :** If the following conditions are satisfied:

i) $2b + 2 \leq n$ and,

ii) $I_{\alpha_i} + I_{\alpha_j} + I_{\alpha_k} \neq \mathfrak{m}$ for any $\{i, j, k\}$.

Then, we have $\operatorname{cd}(R, I) < n-2$.

Recall that Theorem 3.2.11 gives a necessary and sufficient condition for $\operatorname{cd}(R, I) < n-2$, so it provides a more precise statement than conditions i) and ii). Nevertheless, in terms of the invariants $\gamma_{p,i}(R/I)$, we should notice that condition i) assures that $b < n-2$ so we get $\gamma_{2,2} = 0$. It also states that $\operatorname{ht}(I_{\alpha_j} + I_{\alpha_k}) < n-1$ for any $\{j, k\}$ so $\gamma_{1,2} = 0$. From ii) we get $\gamma_{0,2} = 0$.

To shed some light, we provide an example where $\operatorname{cd}(R, I) < n-2$ but conditions i) and ii) are not satisfied.

Example: Let $R = k[x_1, x_2, x_3, x_4, x_5]$. Consider the ideal:

- $I = (x_1, x_2) \cap (x_3, x_4) \cap (x_2, x_3) \cap (x_5)$.

By using Theorem 3.2.11 we compute the characteristic cycle of the corresponding local cohomology modules. Collecting the multiplicities we obtain the triangular matrix:

$$\Gamma(R/I) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 2 & 0 \\ & & 0 & 5 & 0 \\ & & & 3 & 0 \\ & & & & 1 \end{pmatrix}.$$

In particular, we have $\operatorname{cd}(R, I) = 2 < 5 - 2$, but this ideal does not satisfy condition ii). Namely, we have $(x_1, x_2) + (x_3, x_4) + (x_5) = \mathfrak{m}$.

After this brief sketch we continue with the consequences of Theorem 3.2.11.

Support and dimension of local cohomology modules: A further reading of the formula for the characteristic cycle of the local cohomology modules $H_I^{n-i}(R)$ allows us to describe precisely their support. Namely, we only have to check out the minimal prime ideals that appear in the formula.

Let $CC(H_I^{n-i}(R)) = \sum m_{i,\alpha}\ T^*_{X_\alpha}X$ be the characteristic cycle of the local cohomology modules $H_I^{n-i}(R)$ supported on a squarefree monomial ideal $I \subseteq R$. First we will see that the face ideals in the minimal primary decomposition of I only appear in the support of the local cohomology module corresponding to their height.

Proposition 3.3.7. *Let $I = I_{\alpha_1} \cap \cdots \cap I_{\alpha_m}$ be the minimal primary decomposition of the squarefree monomial ideal $I \subseteq R$. Then, for $j = 1, \dots, m$ the following are equivalent:*

i) $I_{\alpha_j} \in \mathrm{Supp}_R(H_I^{n-i}(R))$.

ii) $T^*_{X_{\alpha_j}}X \in CC(H_I^{n-i}(R))$.

iii) $|\alpha_j| = n - i$.

In this case, I_{α_j} is a minimal prime ideal of $H_I^{n-i}(R)$.

In general there can be other minimal prime ideals of $H_I^{n-i}(R)$ than the face ideals in the minimal primary decomposition, see the example given in this section. So the precise statement for computing these minimal prime ideals is the following:

Proposition 3.3.8. *Let $I \subseteq R$ be a squarefree monomial ideal, $\mathfrak{p}_\alpha \subseteq R$ a face ideal and $m_{i,\alpha}$ the corresponding multiplicity in the characteristic cycle of $H_I^{n-i}(R)$. Then, the following are equivalent:*

i) $\mathfrak{p}_\alpha \in \mathrm{Supp}_R(H_I^{n-i}(R))$ *is minimal.*

ii) $m_{i,\alpha} \neq 0$ *and* $m_{i,\beta} = 0$ *for all* $\beta \in \{0,1\}^n$ *such that* $\beta < \alpha$.

In particular, let $\Omega := \{\alpha \in \{0,1\}^n \mid \alpha \text{ satisfies } ii)\ \}$. Then we have:

$$\mathrm{Supp}_R(H_I^{n-i}(R)) = \bigcup_{\alpha \in \Omega} V(\mathfrak{p}_\alpha).$$

Once the support has been described we can compute the dimension of the local cohomology modules.

Corollary 3.3.9. *Let $I \subseteq R$ be a squarefree monomial ideal. Then, the Krull dimension of the local cohomology module $H_I^{n-i}(R)$ is:*

$$\dim H_I^{n-i}(R) = n - \min \{|\alpha| \mid T^*_{X_\alpha}X \in CC(H_I^{n-i}(R))\}.$$

In terms of the invariants $\gamma_{p,i}(R/I)$ we have:

$$\dim H_I^{n-i}(R) = \max \{p \mid \gamma_{p,i}(R/I) \neq 0\}.$$

For the case of zero dimensional local cohomology modules we can give the following characterization:

Corollary 3.3.10. *For any ideal $I \subseteq R$ generated by squarefree monomials, the following are equivalent:*

i) $\dim H_I^{n-i}(R) = 0$.

ii) $\gamma_{j,i}(R/I) = 0$ *for all* $j > 0$.

iii) $H_I^{n-i}(R) \cong E(R/\mathfrak{m})^{\oplus \gamma_{0,i}(R/I)}$.

iv) $H_I^{n-i}(R)$ *is an Artinian R-module .*

Proof: By [55, Theorem 2.4], a zero dimensional $\mathcal{D}$-module is a direct sum of copies of $E(R/\mathfrak{m})$, so it is an Artinian R-module. For the case of the local cohomology module $H_I^{n-i}(R)$ it has to be a direct sum of $\gamma_{0,i}$ copies due to the formula given in Theorem 3.2.11. □

The previous results are used in the following example in order to determine the annihilation and support of the local cohomology modules $H_I^{n-i}(R)$ supported on a squarefree monomial ideal $I \subseteq R$. This example has the particularity that a minimal face ideal in the support of $H_I^3(R)$ is not a face ideal in the minimal primary decomposition of I.

Example: Let $R = k[x_1, x_2, x_3, x_4, x_5]$. Consider the ideal:

- $I = (x_1, x_4) \cap (x_2, x_5) \cap (x_1, x_2, x_3)$.

By using Theorem 3.2.11 we compute the characteristic cycle of the corresponding local cohomology modules. Namely, we get:

$$CC(H_I^2(R)) = T^*_{X_{(1,0,0,1,0)}}X + T^*_{X_{(0,1,0,0,1)}}X.$$

$$CC(H_I^3(R)) = T^*_{X_{(1,1,1,0,0)}}X + T^*_{X_{(1,1,1,1,0)}}X + T^*_{X_{(1,1,1,0,1)}}X + T^*_{X_{(1,1,0,1,1)}}X + \\ + T^*_{X_{(1,1,1,1,1)}}X.$$

Collecting the multiplicities we obtain the triangular matrix:

$$\Gamma(R/I) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ & 0 & 3 & 0 \\ & & 1 & 0 \\ & & & 2 \end{pmatrix}.$$

There are two local cohomology modules different from zero, $H_I^2(R)$ and $H_I^3(R)$, in particular $\operatorname{cd}(R,I) = 3$. The matrix $\Gamma(R/I)$ also allows us to compute the dimension of these modules. Namely we have:

$\gamma_{3,3} \neq 0$ so $\dim H_I^2(R) = 3$.

$\gamma_{2,2} \neq 0$ so $\dim H_I^3(R) = 2$.

Studying in detail the support of these modules we obtain:

$\operatorname{Supp} H_I^2(R) = V(x_1, x_4) \cup V(x_2, x_5)$.

$\operatorname{Supp} H_I^3(R) = V(x_1, x_2, x_3) \cup V(x_1, x_2, x_4, x_5)$.

Note that (x_1, x_2, x_4, x_5) is not a face ideal in the minimal primary decomposition of I.

3.3.2 Arithmetical properties

Cohen-Macaulay property: In Proposition 3.1.1 we reduce the Cohen-Macaulay property of the quotient ring R/I to a question on the vanishing of the local cohomology modules $H_I^{n-i}(R)$. By using the previous results we can give new characterizations of this arithmetical property in terms of the face ideals in the minimal primary decomposition of the monomial ideal I.

Proposition 3.3.11. *For any ideal $I \subseteq R$ generated by squarefree monomials, the following are equivalent:*

i) *R/I is Cohen-Macaulay.*

ii) $\gamma_{j,i}(R/I) = 0$ *, when* $i \neq n - \operatorname{ht} I$.

More precisely, the matrix $\Gamma(R/I)$ has only one column different from zero. In terms of the face ideals in the minimal primary decomposition of I we obtain the following precise statement:

Proposition 3.3.12. *For any ideal $I \subseteq R$ generated by squarefree monomials, the following are equivalent:*

i) R/I *is Cohen-Macaulay.*

ii) $\operatorname{ht}(I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}}) = \operatorname{ht} I + (j-1)$, *for all* $I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} \in \mathcal{P}_j$.

Buchsbaum property: By [86, Theorem 8.1], for any pure squarefree monomial ideal I, the Buchsbaum property of R/I is equivalent to the Cohen-Macaulayness of the localized rings $(R/I)_{\mathfrak{p}}$ for any prime ideal $\mathfrak{p} \neq \mathfrak{m}$. By using the previous results and the localization of the characteristic cycle we get:

Proposition 3.3.13. *For any pure ideal $I \subseteq R$ generated by squarefree monomials, the following are equivalent:*

i) R/I *is Buchsbaum.*

ii) $\gamma_{j,i}(R/I) = 0$, *when* $i \neq n - \operatorname{ht} I$ *and* $j > 0$.

This last assertion states that the local cohomology modules $H_I^{n-i}(R)$ have dimension zero when $i \neq n - \operatorname{ht} I$, so by using Corollary 3.3.10 we get a different approach to the following result, that can be found in [92].

Proposition 3.3.14. *For any pure ideal $I \subseteq R$ generated by squarefree monomials, the following are equivalent:*

i) R/I *is Buchsbaum.*

ii) $H_I^{n-i}(R)$ *is an Artinian R-module for all* $i \neq n - \operatorname{ht} I$.

Gorenstein property: The characterization of the Gorenstein property of R/I is more involved and we point out that we will need some results that appear in Section 3.3.4. These results give a relation between the multiplicities $m_{i,\alpha}$ of the characteristic cycle of the local cohomology modules $H_I^{n-i}(R)$ and the Betti numbers of the Alexander dual ideal $I^{\vee}$. Nevertheless, we present the following results here for completeness of the arithmetical properties of R/I.

Proposition 3.3.15. *Let $I = I_{\alpha_1} \cap \cdots \cap I_{\alpha_m}$ be the minimal primary decomposition of any ideal $I \subseteq R$ generated by squarefree monomials. Then, the following are equivalent:*

i) R/I is Gorenstein such that x_i is a zero divisor in R/I for all i.

ii) R/I is Cohen-Macaulay and $m_{n-\mathrm{ht}\, I,\alpha} = 1$ for all $\alpha \geq \alpha_j$, $j = 1, \dots, m$.

Remark 3.3.16. The condition x_i is a zero divisor in R/I for all i means that all the variables x_i appear in the ideal I.

PROOF: R/I is Gorenstein if and only if it is isomorphic to its canonical module. Then, we have to check when there exists a graded isomorphism $\mathrm{Ext}_R^{\mathrm{ht}\, I}(R/I, R(-\mathbf{1})) \cong R/I$. Since both modules have a natural structure of squarefree module (see [96] for details), we have to study the graded pieces of these modules in degrees $\beta \in \{0,1\}^n$ and the morphism of multiplication by the variables x_i between these pieces.

First, note that the dimensions of the pieces of R/I are:

$\dim_k (R/I)_\beta = 1$ for all $\beta \leq \mathbf{1} - \alpha_j$, $j = 1, \dots, m$.

By using the results that appear in Section 3.3.4 we will see that the following conditions are equivalent:

i) $\dim_k \mathrm{Ext}_R^{\mathrm{ht}\, I}(R/I, R(-\mathbf{1}))_\beta = 1$ for all $\beta \leq \mathbf{1} - \alpha_j$, $j = 1, \dots, m$.

ii) R/I is Cohen-Macaulay and $m_{n-\mathrm{ht}\, I,\alpha} = 1$ for all $\alpha \geq \alpha_j$, $j = 1, \dots, m$.

First, we have to point out that R/I is Cohen-Macaulay if and only if the modules $\mathrm{Ext}_R^j(R/I, R(-\mathbf{1}))$ vanish for all $j \neq \mathrm{ht}\, I$.

More precisely, by Corollary 3.3.28, R/I is Cohen-Macaulay if and only if $\beta_{j,\alpha}(I^\vee)$ for all $j \neq |\alpha| - \mathrm{ht}\, I$, where $\beta_{j,\alpha}(I^\vee)$ denotes the j-th Betti number of the Alexander dual ideal $I^\vee$. On the other side, by [72, Corollary 3.1] we have:

$$\dim_k \mathrm{Ext}_R^{|\alpha|-j}(R/I, R(-\mathbf{1}))_{\mathbf{1}-\alpha} = \dim_k \mathrm{Ext}_R^{|\alpha|-j}(R/I, R)_{-\alpha} = \beta_{j,\alpha}(I^\vee)$$

so, in order to compute the dimension of the graded pieces of the module $\mathrm{Ext}_R^{\mathrm{ht}\, I}(R/I, R(-\mathbf{1}))$ we have to describe the Betti numbers $\beta_{|\alpha|-\mathrm{ht}\, I,\alpha}(I^\vee)$.

Finally, the proof follows from the fact that these Betti numbers are related to the multiplicities $m_{j,\alpha}$ of the characteristic cycle of the local cohomology modules $H_I^{n-i}(R)$ by means of Proposition 3.3.25. Namely we have:

$$\beta_{|\alpha|-\mathrm{ht}\, I,\alpha}(I^\vee) = m_{n-\mathrm{ht}\, I,\alpha}.$$

Once we have checked that the graded pieces of both modules coincide we have to study the morphism of multiplication by the variables x_i between these pieces.

Let $\varepsilon_1, \dots, \varepsilon_n$ be the natural basis of $\mathbb{Z}^n$. The morphism of multiplication $x_i : (R/I)_\beta \longrightarrow (R/I)_{\beta+\varepsilon_i}$ between the graded pieces of the quotient ring R/I is the identity if both pieces are different from zero and $\beta_i = 0$. On the other side, a topological interpretation of the multiplication by the variables x_i between the pieces of the canonical module $\mathrm{Ext}_R^{\mathrm{ht}\, I}(R/I, R(-\mathbf{1}))$ has been given in [72]. Namely, letΔ be the full simplicial complex whose vertices are labelled by the minimal system of generators of I. Let $T_\alpha := \{\sigma_{\mathbf{1}-\beta} \in \Delta \mid \beta \not\geq \alpha\}$ be a simplicial subcomplex of Δ (see also Chapter 6 for details). Then, the morphism of multiplication is determined by the relative simplicial cohomology morphism

$$\nu_i : \widetilde{H}^{r-2}(T_{\beta+\varepsilon_i}; k) \longrightarrow \widetilde{H}^{r-2}(T_\beta; k),$$

induced by the inclusion $T_\beta \subseteq T_{\beta+\varepsilon_i}$. In the case we are considering, it is easy to check that this morphism is the identity if both pieces are different from zero and $\beta_i = 0$.

□

Remark 3.3.17. The isomorphism between the $\mathbb{Z}^n$-graded pieces of the quotient ring R/I and the canonical module $\mathrm{Ext}_R^{\mathrm{ht}\, I}(R/I, R(-\mathbf{1}))$ states that the corresponding $\mathbb{Z}^n$-graded Hilbert series coincide.

R. P. Stanley [85] studied the Gorenstein property by using the coarser $\mathbb{Z}$-graded Hilbert series. In particular he gave the following example of non Gorenstein ring R/I having the same Hilbert series as its canonical module:

Example: Let $R = k[x_1, x_2, x_3, x_4]$. Consider the ideal:

- $I = (x_1, x_2) \cap (x_1, x_4) \cap (x_2, x_4) \cap (x_3, x_4)$.

The $\mathbb{Z}^n$-graded Hilbert series of the quotient ring and the canonical module are different due to the fact that the characteristic cycle of the local cohomology module $H_I^2(R)$ is:

$$CC(H^2_I(R)) = T^*_{X_{(1,1,0,0)}}X + T^*_{X_{(1,0,0,1)}}X + T^*_{X_{(0,1,0,1)}}X + T^*_{X_{(0,0,1,1)}}X+ \\ +2\,T^*_{X_{(1,1,0,1)}}X + T^*_{X_{(1,0,1,1)}}X + T^*_{X_{(0,1,1,1)}}X + T^*_{X_{(1,1,1,1)}}X.$$

Notice that the corresponding multiplicities do not satisfy $m_{2,\alpha} = 1$ for all α bigger than the subindices corresponding to the face ideals in the minimal primary decomposition of I. Nevertheless, even though the quotient ring R/I is not Gorenstein, the $\mathbb{Z}$-graded Hilbert series of the quotient ring and the canonical module coincide (see [85]).

To illustrate the criteria described in this section to study the arithmetical properties of the quotient ring R/I, we present the following example.

Example: Let $R = k[x_1, x_2, x_3, x_4]$. Consider the ideals:

- $I_1 = (x_1, x_2) \cap (x_3, x_4)$.
- $I_2 = (x_1, x_2) \cap (x_2, x_3) \cap (x_3, x_4)$.
- $I_3 = (x_1, x_2) \cap (x_1, x_4) \cap (x_2, x_3) \cap (x_3, x_4)$.

Computing the corresponding matrices of multiplicities we have:

$$\Gamma(R/I_1) = \begin{pmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 2 \end{pmatrix}, \quad \Gamma(R/I_2) = \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 2 \\ & & 3 \end{pmatrix}, \quad \Gamma(R/I_3) = \begin{pmatrix} 0 & 0 & 1 \\ & 0 & 4 \\ & & 4 \end{pmatrix}.$$

• R/I_1 is Buchsbaum but it is not Cohen-Macaulay.

• R/I_2 is Cohen-Macaulay but it is not Gorenstein due to the fact that the multiplicity $m_{2,\alpha_{\mathfrak{m}}} = 0$.

• R/I_3 is Cohen-Macaulay and we can also prove that it is Gorenstein by computing the characteristic cycle. Namely we have:

$$CC(H^2_{I_3}(R)) = T^*_{X_{(1,1,0,0)}}X + T^*_{X_{(1,0,0,1)}}X + T^*_{X_{(0,1,1,0)}}X + T^*_{X_{(0,0,1,1)}}X+ \\ +T^*_{X_{(1,1,0,1)}}X + T^*_{X_{(1,1,1,0)}}X + T^*_{X_{(1,0,1,1)}}X + T^*_{X_{(0,1,1,1)}}X+ \\ +T^*_{X_{(1,1,1,1)}}X,$$

so the corresponding multiplicities are $m_{2,\alpha} = 1$ for all α bigger than the subindices corresponding to the face ideals in the minimal primary decomposition of I_3 .

Recall that **the type** of a Cohen-Macaulay quotient ring R/I of dimension d is the Bass number

$$r(R/I) := \mu_d(\mathfrak{m}, R/I).$$

By duality (see [21, Corollary 21.16]), it is the nonzero total Betti number of highest homological degree. We will give a description of the type by using the results that appear in Section 3.3.4. To this purpose we introduce the following notation:

Let $\mathcal{P}$ (resp. $\mathcal{P}^\vee$) be the poset of non paired sums of face ideals obtained from $\mathcal{I}$ (resp. $\mathcal{I}^\vee$). In an analogous way, we will denote by $\mathcal{R}$ (resp. $\mathcal{R}^\vee$) the poset of non paired LCM of generators obtained from the posets $\mathcal{J}$ (resp. $\mathcal{J}^\vee$) introduced in Section 1.2.8.

Notice that $\forall j$:

$$\mathfrak{p}_\alpha \in \mathcal{P}_j \Leftrightarrow \mathbf{x}^\alpha \in \mathcal{R}_j^\vee, \quad \mathbf{x}^\alpha \in \mathcal{R}_j \Leftrightarrow \mathfrak{p}_\alpha \in \mathcal{P}_j^\vee.$$

The type of a Cohen-Macaulay quotient ring R/I will be expressed in terms of these posets.

In the sequel, $CC(H_{I^\vee}^{n-i}(R)) = \sum m_{i,\alpha}(R/I^\vee)\ T^*_{X_\alpha}X$ will be the characteristic cycle of a local cohomology module $H_{I^\vee}^{n-i}(R)$, where $I^\vee$ is the Alexander dual ideal of a squarefree monomial ideal $I \subseteq R$.

Proposition 3.3.18. *Let $I \subseteq R$ be a squarefree monomial ideal of height $\operatorname{ht} I = h$ such that the quotient ring R/I is Cohen-Macaulay. Then, the type of R/I is:*

$$r(R/I) = \sum_{\{\alpha \mid \mathfrak{p}_\alpha \in \mathcal{P}_h^\vee\}} m_{n-|\alpha|+h-1,\alpha}(R/I^\vee).$$

PROOF: The projective dimension of R/I is $h = \operatorname{ht} I$ so we have $r(R/I) = \beta_h(R/I)$. This total Betti number is obtained as a sum of the graded Betti numbers $\beta_{h,\alpha}(R/I)$ labelled by the set $\mathcal{R}_h$ so we have:

$$r(R/I) := \mu_d(\mathfrak{m}, R/I) = \sum_{\{\alpha \mid \mathbf{x}^\alpha \in \mathcal{R}_h\}} \beta_{h,\alpha}(R/I) = \sum_{\{\alpha \mid \mathbf{x}^\alpha \in \mathcal{R}_h\}} \beta_{h-1,\alpha}(I).$$

By Proposition 3.3.25 we get the desired result. □

If R/I is a Cohen-Macaulay ring of type 1 then, it is Gorenstein. By using the previous description, we can give the following criteria:

Proposition 3.3.19. *Let $I \subseteq R$ be a squarefree monomial ideal of height $\operatorname{ht} I = h$ such that the quotient ring R/I is Cohen-Macaulay. Then, the following are equivalent:*

i) R/I *is Gorenstein.*

ii) $\displaystyle\sum_{\{\alpha \mid \mathfrak{p}_\alpha \in \mathcal{P}_h^\vee\}} m_{n-|\alpha|+h-1,\alpha}(R/I^\vee) = 1.$

3.3.3 Combinatorics of the Stanley-Reisner ring and multiplicities

Let $R = k[x_1, \dots, x_n]$ be the polynomial ring in n variables over a field k of characteristic zero. We want to study the combinatorics of the Stanley-Reisner ring R/I, where $I \subseteq R$ is a squarefree monomial ideal. More precisely we will describe the f-vector and the h-vector of the simplicial complex Δ associated to the Stanley-Reisner ring R/I in terms of the invariants $\gamma_{p,i}(R/I)$.

Recall that if we denote by f_k the number of k-dimensional faces of Δ then the f-vector is $f(\Delta) = (f_{-1}, \dots, f_{d-1})$. Once the f-vector is known we can describe the h-vector $h(\Delta) = (h_0, \dots, h_d)$ by using the relation:

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{k-i} f_{i-1}.$$

For our purposes, it will be useful to introduce the following invariant:

Definition 3.3.20. *Let $\Gamma(R/I)$ be the triangular matrix of invariants $\gamma_{p,i}(R/I)$ corresponding to a squarefree monomial ideal $I \subseteq R$. Then, we define:*

$$\mathcal{B}_j := \sum_{i=0}^{d-j} (-1)^i \gamma_{j,j+i}(R/I).$$

Namely, $\mathcal{B}_j$ is the alternating sum of invariants $\gamma_{p,i}(R/I)$ in a row of the matrix $\Gamma(R/I)$.

Proposition 3.3.21. *Let $I \subseteq R$ be a squarefree monomial ideal. The f-vector and the h-vector of the corresponding simplicial complex Δ are described as follows:*

i) $f_k = \sum_{j=k+1}^{d} \binom{j}{k+1} \mathcal{B}_j.$

ii) $h_k = (-1)^k \sum_{j=0}^{d-k} \binom{d-j}{k} \mathcal{B}_j.$

Proof: *Proof of i)* In order to get the invariant f_k associated to a simplicial complex Δ we have to count the number of k-faces in any maximal face of Δ and eliminate the excess given by the intersections of maximal faces. More precisely, let $\sigma_{\alpha_1}, \dots, \sigma_{\alpha_m}$ be the maximal faces of Δ and

$$f_{k,i} := \sharp \ \{\sigma \in \Delta \ | \ \sigma \in \sigma_{\alpha_{j_1}} \cap \cdots \cap \sigma_{\alpha_{j_i}} \ , \ \dim \sigma = k\}$$

be the number of k-faces in the intersection of i maximal faces. The invariant f_k is then the alternating sum:

$$f_k = f_{k,1} - f_{k,2} + f_{k,3} - \cdots + (-1)^{m-1} f_{k,m}.$$

By the Stanley-Reisner correspondence the maximal faces $\sigma_{\alpha_1}, \dots, \sigma_{\alpha_m}$ of the simplicial complex Δ correspond to the face ideals in the minimal primary decomposition $I = I_{\alpha_1} \cap \cdots \cap I_{\alpha_m}$ of a squarefree monomial ideal $I \subseteq R$. In terms of the poset $\mathcal{I}$, a sum $I_{\alpha_{j_1}} + \cdots + I_{\alpha_{j_i}} \in \mathcal{I}_i$ corresponds to a face that is intersection of i maximal faces of Δ. If $\operatorname{ht}(I_{\alpha_{j_1}} + \cdots + I_{\alpha_{j_i}}) = n - (j+1)$ then the dimension of the corresponding face is j.

The number of k-faces in a face of dimension j is $\binom{j+1}{k+1}$. Then:

$$f_{k,i} = \sum_{j=k}^{d-1} \binom{j+1}{k+1} \sharp \{I_{\alpha_{j_1}} + \cdots + I_{\alpha_{j_i}} \in \mathcal{I}_i \ | \ \operatorname{ht}(I_{\alpha_{j_1}} + \cdots + I_{\alpha_{j_i}}) = n - (j+1)\} =$$

$$= \sum_{j=k+1}^{d} \binom{j}{k+1} \sharp \{I_{\alpha_{j_1}} + \cdots + I_{\alpha_{j_i}} \in \mathcal{I}_i \ | \ \operatorname{ht}(I_{\alpha_{j_1}} + \cdots + I_{\alpha_{j_i}}) = n - j\}.$$

It remains to give an interpretation of this invariants in terms of the invariants $\gamma_{p,i}(R/I)$. First we consider the following particular case:

• When $I \subseteq R$ is a squarefree monomial ideal with disjoint faces, due to the fact that the characteristic cycle of $H_I^r(R)$ is expressed in terms of all the sums of face ideals contained in the poset $\mathcal{I}$, we have the following interpretation:

. The invariants in the main diagonal of the matrix $\Gamma(R/I)$ are:

$\gamma_{j,j}(R/I) = \sharp\ \{I_{\alpha_1} \in \mathcal{I}_1 \mid \operatorname{ht} I_{\alpha_1} = n-j\}$

. The invariants in the subdiagonals of the matrix $\Gamma(R/I)$ are:

$\gamma_{j,j+(i-1)}(R/I) = \sharp\ \{I_{\alpha_{j_1}} + \cdots + I_{\alpha_{j_i}} \in \mathcal{I}_i \mid \operatorname{ht}(I_{\alpha_{j_1}} + \cdots + I_{\alpha_{j_i}}) = n-j\}$

Then we get the desired result by using the fact that $\gamma_{j,j+i} = 0$ for $i > d-j$. Namely:

$$f_k = \sum_{i=0}^{m-1} (-1)^i f_{k,i+1} =$$

$$= \sum_{i=0}^{m-1} (-1)^i \Big(\sum_{j=k+1}^{d} \binom{j}{k+1} \sharp\,\{I_{\alpha_{j_1}} + \cdots + I_{\alpha_{j_i}} \in \mathcal{I}_i \mid \operatorname{ht}(I_{\alpha_{j_1}} + \cdots + I_{\alpha_{j_i}}) = n-j\}\Big) =$$

$$= \sum_{j=k+1}^{d} \binom{j}{k+1} \Big(\sum_{i=0}^{m-1} (-1)^i\, \sharp\,\{I_{\alpha_{j_1}} + \cdots + I_{\alpha_{j_i}} \in \mathcal{I}_i \mid \operatorname{ht}(I_{\alpha_{j_1}} + \cdots + I_{\alpha_{j_i}}) = n-j\}\Big) =$$

$$= \sum_{j=k+1}^{d} \binom{j}{k+1} \Big(\sum_{i=0}^{d-j} (-1)^i \gamma_{j,j+i}\Big) = \sum_{j=k+1}^{d} \binom{j}{k+1} \mathcal{B}_j .$$

• In general, for any squarefree monomial ideal $I \subseteq R$ we have

$\gamma_{j,j+(i-1)}(R/I) \neq \sharp\,\{I_{\alpha_{j_1}} + \cdots + I_{\alpha_{j_i}} \in \mathcal{I}_i \mid \operatorname{ht}(I_{\alpha_{j_1}} + \cdots + I_{\alpha_{j_i}}) = n-j\}$,

so we can not give the same interpretation as before. Nevertheless, notice that in order to compute the f-vector we only have to describe the alternate sum:

$$\sum_{i=0}^{m-1} (-1)^i\, \sharp\,\{I_{\alpha_{j_1}} + \cdots + I_{\alpha_{j_i}} \in \mathcal{I}_i \mid \operatorname{ht}(I_{\alpha_{j_1}} + \cdots + I_{\alpha_{j_i}}) = n-j\} = (\bigstar)$$

This sum can also be expressed in terms of the poset $\mathcal{P}$ due to the fact that the cancellation of two paired sums of face ideals does not have an effect on the sum. Then we are done because we can consider:

$$(\bigstar) = \sum_{i=0}^{m-1} (-1)^i \sharp \{I_{\alpha_{j_1}} + \cdots + I_{\alpha_{j_i}} \in \mathcal{P}_i \mid \mathrm{ht}\,(I_{\alpha_{j_1}} + \cdots + I_{\alpha_{j_i}}) = n - j\} =$$

$$= \textstyle\sum_{i=0}^{d-j} (-1)^i \gamma_{j,j+i} = \mathcal{B}_j.$$

Proof of ii) By using the relation between the h-vector and the f-vector we have:

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{k-i} f_{i-1} = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{k-i} \ (\sum_{j=0}^{d-(k+1)} \binom{d-j}{k+1} \mathcal{B}_{d-j}).$$

Reorganizing the terms given in this last expression we get:

$$\begin{aligned} h_k = \ & \mathcal{B}_d \ (\binom{d-k}{0}\binom{k}{k} - \cdots + (-1)^k \binom{d}{k}\binom{d}{0}) \ + \cdots + \\ & + \ \mathcal{B}_k \ (\binom{d-k}{0}\binom{k}{k} - \cdots + (-1)^k \binom{d}{k}\binom{k}{0}) + \\ & + \ \mathcal{B}_{k-1} \ (-\binom{d-k+1}{1}\binom{k-1}{k-1} - \cdots + (-1)^k \binom{d}{k}\binom{k-1}{0}) \ + \cdots + \\ & + \ \mathcal{B}_0 \ ((-1)^k \binom{d}{k}\binom{0}{0}). \end{aligned}$$

Namely we have:

$$h_k = \sum_{j=0}^{d} \ \mathcal{B}_j \ (\sum_{i=0}^{k} (-1)^i \binom{d-k+i}{i}\binom{j}{k-i}).$$

By using the identity

$$\sum_{i=0}^{k} (-1)^{k-i} \binom{d-k+i}{i}\binom{j}{k-i} = (-1)^k \ \binom{j-(d-k+1)}{k},$$

we get the desired result, i.e.:

$$h_k = \sum_{j=0}^{d} \mathcal{B}_j \ ((-1)^k \sum_{i=0}^{k} (-1)^{k-i} \binom{d-k+i}{i} \binom{j}{k-i}) =$$

$$= \sum_{j=0}^{d} \mathcal{B}_j \binom{j-(d-k+1)}{k} = (-1)^k \sum_{j=0}^{d-k} \binom{d-j}{k} \mathcal{B}_j,$$

where the last assertion comes from the following fact:

$$\binom{j-(d-k+1)}{k} = \begin{cases} 0 & \text{if } j > d-k, \\ (-1)^k \binom{d-j}{k} & \text{if } j \leq d-k. \end{cases}$$

□

Let $I \subseteq R$ be a squarefree monomial ideal such that the corresponding quotient ring R/I is Cohen-Macaulay. Since $\mathcal{B}_j = (-1)^{d-j} \gamma_{j,d}(R/I)$, we obtain the following result:

Corollary 3.3.22. *Let $I \subseteq R$ be a squarefree monomial ideal such that R/I is Cohen-Macaulay. The f-vector and the h-vector of the corresponding simplicial complex Δ are described as follows:*

i) $f_k = \sum_{j=k+1}^{d} \binom{j}{k+1} (-1)^{d-j} \gamma_{j,d}(R/I).$

ii) $h_k = (-1)^k \sum_{j=0}^{d-k} \binom{d-j}{k} (-1)^{d-j} \gamma_{j,d}(R/I).$

Once the f-vector and the h-vector have been described we can use them to obtain the following characterizations in terms of the invariants $\gamma_{p,i}(R/I)$.

Corollary 3.3.23. *We have:*

i) $\sum (-1)^{p+i} \gamma_{p,i}(R/I) = \mathcal{B}_d + \cdots + \mathcal{B}_0 = f_{-1} = 1.$

ii) The Euler characteristic of Δ is $\chi(\Delta) = 1 - \mathcal{B}_0$.

iii) The Hilbert series of the Stanley-Reisner ring R/I is:

$$H(R/I;t) = \sum_{i=0}^{d} \frac{\mathcal{B}_i}{(1-t)^i}$$

In particular the generalized multiplicities are $e_i(R/I) = (-1)^i \mathcal{B}_{d-i}$.

From Proposition 3.3.21 we can see that the invariants $\gamma_{p,i}(R/I)$ are finer than the f-vector or the h-vector. They are equivalent when R/I is Cohen-Macaulay (see Corollary 3.3.22). In the following example we will find two simplicial complexes with the same f-vector but with different invariants $\gamma_{p,i}(R/I)$.

Example: Let $R = k[x_1, x_2, x_3, x_4, x_5]$. Consider the ideals:

- $I_1 = (x_1, x_3) \cap (x_1, x_4) \cap (x_1, x_5) \cap (x_2, x_3) \cap (x_2, x_4)$.
- $I_2 = (x_1, x_4) \cap (x_1, x_5) \cap (x_2, x_5) \cap (x_3, x_5) \cap (x_4, x_5) \cap (x_1, x_2, x_3)$.

Both simplicial complexes have the same f-vector, namely $f(\Delta_1) = f(\Delta_2) = (1, 5, 9, 5)$, but the corresponding invariants $\gamma_{p,i}(R/I)$ are different:

$$\Gamma(R/I_1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ & 0 & 0 & 2 \\ & & 0 & 6 \\ & & & 5 \end{pmatrix}, \qquad \Gamma(R/I_2) = \begin{pmatrix} 0 & 0 & 1 & 1 \\ & 0 & 2 & 4 \\ & & 1 & 7 \\ & & & 5 \end{pmatrix}.$$

Note that R/I_1 is Cohen-Macaulay but R/I_2 is not Cohen-Macaulay.

3.3.4 Betti numbers and multiplicities

Let $R = k[x_1, \ldots, x_n]$ be the polynomial ring in n variables over a field k of characteristic zero. Let $I \subseteq R$ be a squarefree monomial ideal. If $I = I_{\alpha_1} \cap \cdots \cap I_{\alpha_m}$ is its minimal primary decomposition then the Alexander dual ideal $I^\vee$ is of the form $I^\vee = (\mathbf{x}^{\alpha_1}, \ldots, \mathbf{x}^{\alpha_m})$.

In this section, our aim is to explain how the optimization of the Mayer-Vietoris process for computing the characteristic cycle of the local cohomology modules $H^r_I(R)$ is related to the process of minimize the free resolution of the ideal $I^\vee$ given by the Taylor complex.

The Taylor complex: Let F be the free R-module of rank m generated by $e_1,\dots,e_m$. By using the Taylor complex we have a resolution of the ideal $I^\vee$:

$$\mathbb{T}_\bullet(I^\vee):\quad 0 \longrightarrow F_m \xrightarrow{d_m} \cdots \longrightarrow F_1 \xrightarrow{d_1} I^\vee \longrightarrow 0\,,$$

where $F_j = \bigwedge^j F$ is the j-th exterior power of F and the differentials d_j are defined

$$d_j(e_{i_1}\wedge\cdots\wedge e_{i_j}) = \sum_{1\leq k\leq j} (-1)^k \frac{\mathrm{LCM}\,(\mathbf{x}^{\alpha_{i_1}},\dots,\mathbf{x}^{\alpha_{i_j}})}{\mathrm{LCM}\,(\mathbf{x}^{\alpha_{i_1}},\dots,\widehat{\mathbf{x}^{\alpha_{i_k}}},\dots,\mathbf{x}^{\alpha_{i_j}})}\; e_{i_1}\wedge\cdots\wedge\widehat{e_{i_k}}\wedge\cdots\wedge e_{i_j}.$$

We have seen in Section 1.2.8 that the Taylor complex $\mathbb{T}_\bullet(R/I^\vee)$ is a cellular free resolution of $R/I^\vee$ with support a simplex Δ labelled by the poset $\mathcal{J}^\vee$. Namely, the vertices are labelled by the elements of $\mathcal{J}_1^\vee$ and the faces of Δ of dimension $j-1$ are labelled by the elements of $\mathcal{J}_j^\vee$ $\forall j$. By Alexander duality, the poset $\mathcal{I}$ provides as well this information.

On the other side the Taylor complex $\mathbb{T}_\bullet(I^\vee)$ constitute a non augmented oriented chain complex up to a homological shift for the same simplex Δ. In particular, the poset $\mathcal{J}^\vee$ (as well the poset $\mathcal{I}$) label the Taylor complex.

From now on we will consider the labels given by the poset $\mathcal{I}$ because it gives the initial pieces that we use in the Mayer-Vietoris process in order to compute the characteristic cycle of the local cohomology modules $H_I^r(R)$.

Minimality of the Taylor complex: In general, the free resolution of $I^\vee$ given by the Taylor complex is not minimal. Precisely, it is not minimal when there exist j and k such that

$$\frac{\mathrm{LCM}\,(\mathbf{x}^{\alpha_{i_1}},\dots,\mathbf{x}^{\alpha_{i_j}})}{\mathrm{LCM}\,(\mathbf{x}^{\alpha_{i_1}},\dots,\widehat{\mathbf{x}^{\alpha_{i_k}}},\dots,\mathbf{x}^{\alpha_{i_j}})} = 1.$$

By the Alexander duality correspondence this is equivalent to the equality $I_{\alpha_{i_1}}+\cdots+I_{\alpha_{i_j}} = I_{\alpha_{i_1}}+\cdots+\widehat{I_{\alpha_{i_k}}}+\cdots+I_{\alpha_{i_j}}$ of sums of face ideals in the minimal primary decomposition of I, i.e. it is equivalent to say that

$$I_{\alpha_{i_1}}+\cdots+I_{\alpha_{i_j}} \text{ and } I_{\alpha_{i_1}}+\cdots+\widehat{I_{\alpha_{i_k}}}+\cdots+I_{\alpha_{i_j}} \text{ are paired ideals.}$$

An algorithm (expressed in terms of the poset $\mathcal{I}$) for minimize the free resolution of the Alexander dual ideal $I^\vee$ given by the Taylor complex is described as follows:

Algorithm: Let m be the number of ideals in the minimal primary decomposition $I = I_{\alpha_1} \cap \cdots \cap I_{\alpha_m}$.

- For j from 1 to $m-1$, incrementing by 1

  - For all $k \leq j$ **COMPARE** the ideals

    $I_{\alpha_{j+1}} + (I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_r}})$ and $I_{\alpha_k} + I_{\alpha_{j+1}} + (I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_r}})$

  - For all $k \leq j$ **COMPARE** the ideals

    $I_{\alpha_k} + (I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_r}})$ and $I_{\alpha_k} + I_{\alpha_{j+1}} + (I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_r}})$

We want to get a minimal free resolution of $I^\vee$ from the Taylor resolution, so we will apply the algorithm in the following way:

INPUT: The free resolution of $I^\vee$ given by the Taylor complex. Equivalently, the set $\mathcal{I}$ of all the sums of face ideals obtained as a sum of face ideals in the minimal primary decomposition of I.

In this case **COMPARE** means that if both ideals are paired then:

1) From the image of the generator labeled by $I_{\alpha_k} + I_{\alpha_{j+1}} + (I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_r}})$ one finds a relation

$$I_{\alpha_{j+1}} + (I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_r}}) = \sum \lambda_{j_1,\dots,j_{r+1}} \; I_{\alpha_{j_1}} + \cdots + I_{\alpha_{j_{r+1}}}.$$

2) Replace the previous equality in all the relations where $I_{\alpha_{j+1}} + (I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_r}})$ appear.

3) Remove the generators labeled by both ideals $I_{\alpha_{j+1}} + (I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_r}})$ and $I_{\alpha_k} + I_{\alpha_{j+1}} + (I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_r}})$.

OUTPUT: A minimal free resolution of $I^\vee$. In particular we get the set $\mathcal{P}$ of all the non paired sums of face ideals obtained as a sum of face ideals in the minimal primary decomposition of I.

Remark 3.3.24. i) We can interpret our algorithm for canceling paired ideals in order to compute the characteristic cycle of local cohomology modules supported on a monomial ideal I as an algorithm for minimize the free resolution of the Alexander dual ideal $I^\vee$ given by the Taylor complex.

ii) The poset $\mathcal{P}$ that describes the characteristic cycles $CC(H_I^r(R))$ also describes the modules and the Betti numbers of the minimal free resolution of $I^\vee$. Notice that the description of the differentials of this minimal free resolution is more involved than the description of the differentials of the Taylor complex.

Example: Let $R = k[x_1, x_2, x_3, x_4]$. Consider the ideal:

- $I = (x_1, x_2) \cap (x_1, x_4) \cap (x_2, x_3) \cap (x_2, x_4)$.

By the Alexander duality correspondence we have

- $I^\vee = (x_1x_2, x_1x_4, x_2x_3, x_2x_4)$.

Consider the poset $\mathcal{I}$ of sums of face ideals in the minimal primary decomposition of I:

$$\mathcal{I}_1 = \{I_{\alpha_1}, \quad I_{\alpha_2}, \quad I_{\alpha_3}, \quad I_{\alpha_4}\} = \{(x_1, x_2), (x_1, x_4), (x_2, x_3), (x_2, x_4)\}$$

$$\mathcal{I}_2 = \{I_{\alpha_1} + I_{\alpha_2}, \quad I_{\alpha_1} + I_{\alpha_3}, \quad I_{\alpha_1} + I_{\alpha_4}, \quad I_{\alpha_2} + I_{\alpha_3}, \quad I_{\alpha_2} + I_{\alpha_4}, \quad I_{\alpha_3} + I_{\alpha_4}\} =$$
$$= \{(x_1, x_2, x_4), (x_1, x_2, x_3), (x_1, x_2, x_4), (x_1, x_2, x_3, x_4), (x_1, x_2, x_4), (x_2, x_3, x_4)\}$$

$$\mathcal{I}_3 = \{I_{\alpha_1} + I_{\alpha_2} + I_{\alpha_3}, \quad I_{\alpha_1} + I_{\alpha_2} + I_{\alpha_4}, \quad I_{\alpha_1} + I_{\alpha_3} + I_{\alpha_4}, \quad I_{\alpha_2} + I_{\alpha_3} + I_{\alpha_4}\} =$$
$$= \{(x_1, x_2, x_3, x_4), (x_1, x_2, x_4), (x_1, x_2, x_3, x_4), (x_1, x_2, x_3, x_4)\}$$

$$\mathcal{I}_4 = \{I_{\alpha_1} + I_{\alpha_2} + I_{\alpha_3} + I_{\alpha_4}\} = \{(x_1, x_2, x_3, x_4)\}$$

This poset allows us to describe the free resolution of the ideal $I^\vee$ given by the Taylor complex. Namely, let F be the free R-module of rank 4 labelled by $I_{\alpha_1}, I_{\alpha_2}, I_{\alpha_3}, I_{\alpha_4}$. Then we have:

$$0 \longrightarrow F_4 \xrightarrow{d_4} F_3 \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} I^\vee \longrightarrow 0\ ,$$

where

$$F_1 = R(-1,-1,0,0) \oplus R(-1,0,0,-1) \oplus R(0,-1,-1,0) \oplus R(0,-1,0,-1)$$

$$F_2 = R(-1,-1,0,-1) \oplus R(-1,-1,-1,0) \oplus R(-1,-1,0,-1) \oplus R(-1,-1,-1,-1) \oplus$$
$$R(-1,-1,0,-1) \oplus R(0,-1,-1,-1)$$

$$F_3 = R(-1,-1,-1,-1) \oplus R(-1,-1,0,-1) \oplus R(-1,-1,-1,-1) \oplus R(-1,-1,-1,-1)$$

$$F_4 = R(-1,-1,-1,-1)$$

and the matrices of the differentials are

$$d_1 = \begin{pmatrix} x_1x_2 & x_1x_4 & x_2x_3 & x_2x_4 \end{pmatrix}$$

$$d_2 = \begin{pmatrix} x_4 & x_3 & x_4 & 0 & 0 & 0 \\ -x_2 & 0 & 0 & x_2x_3 & x_2 & 0 \\ 0 & -x_1 & 0 & -x_1x_4 & 0 & x_4 \\ 0 & 0 & -x_1 & 0 & -x_1 & -x_3 \end{pmatrix}$$

$$d_3 = \begin{pmatrix} -x_3 & -1 & 0 & 0 \\ x_4 & 0 & -x_4 & 0 \\ 0 & 1 & x_3 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & x_3 \\ 0 & 0 & -x_1 & -x_1 \end{pmatrix}$$

$$d_4 = \begin{pmatrix} 1 \\ -x_3 \\ 1 \\ -1 \end{pmatrix}$$

In order to compute the characteristic cycle of the local cohomology modules $H_I^r(R)$ by using Theorem 3.2.11, we apply the algorithm of canceling paired ideals to the poset $\mathcal{I}$. In this case we have:

Step 1: Cancel the paired ideals $I_{\alpha_2} + I_{\alpha_3}$ and $I_{\alpha_1} + I_{\alpha_2} + I_{\alpha_3}$.

Step 2: Cancel the paired ideals $I_{\alpha_2} + I_{\alpha_4}$ and $I_{\alpha_1} + I_{\alpha_2} + I_{\alpha_4}$.

Step 3: Cancel the paired ideals $I_{\alpha_2} + I_{\alpha_3} + I_{\alpha_4}$ and $I_{\alpha_1} + I_{\alpha_2} + I_{\alpha_3} + I_{\alpha_4}$.

We obtain the poset $\mathcal{P}$ of non paired sums of face ideals in the minimal primary decomposition of I:

$$\mathcal{P}_1 = \{I_{\alpha_1}, \quad I_{\alpha_2}, \quad I_{\alpha_3}, \quad I_{\alpha_4}\} = \{(x_1,x_2),(x_1,x_4),(x_2,x_3),(x_2,x_4)\}.$$

$$\mathcal{P}_2 = \{I_{\alpha_1} + I_{\alpha_2}, \quad I_{\alpha_1} + I_{\alpha_3}, \quad I_{\alpha_1} + I_{\alpha_4}, \quad I_{\alpha_3} + I_{\alpha_4}\} =$$
$$= \{(x_1,x_2,x_4),(x_1,x_2,x_3),(x_1,x_2,x_4),(x_2,x_3,x_4)\}.$$

$$\mathcal{P}_3 = \{I_{\alpha_2} + I_{\alpha_3} + I_{\alpha_4}\} = \{(x_1,x_2,x_3,x_4)\}.$$

Applying Theorem 3.2.11 we get a local cohomology module different from zero and its characteristic cycle is:

$$CC(H_I^2(R)) = T^*_{X_{(1,1,0,0)}}X + T^*_{X_{(1,0,0,1)}}X + T^*_{X_{(0,1,1,0)}}X + T^*_{X_{(0,1,0,1)}}X +$$
$$+ T^*_{X_{(1,1,1,0)}}X + 2\ T^*_{X_{(1,1,0,1)}}X + T^*_{X_{(0,1,1,1)}}X + T^*_{X_{(1,1,1,1)}}X.$$

On the other side we apply the algorithm for canceling paired ideals in order to obtain a minimal free resolution of $I^\vee$.

Step 1: The ideals $I_{\alpha_2}+I_{\alpha_3}$ and $I_{\alpha_1}+I_{\alpha_2}+I_{\alpha_3}$ are paired

1) From the image of the generator labeled by $I_{\alpha_1}+I_{\alpha_2}+I_{\alpha_3}$ we get the relation
$$I_{\alpha_2}+I_{\alpha_3} = -x_3\ (I_{\alpha_1}+I_{\alpha_2}) + x_4\ (I_{\alpha_1}+I_{\alpha_3}).$$
2) Replace the previous equality in the relation
$$I_{\alpha_2}+I_{\alpha_3} = x_3\ (I_{\alpha_2}+I_{\alpha_4}) - x_1\ (I_{\alpha_3}+I_{\alpha_4}).$$
3) Remove the generators labeled by both ideals $I_{\alpha_2}+I_{\alpha_3}$ and $I_{\alpha_1}+I_{\alpha_2}+I_{\alpha_3}$.

We obtain the free resolution:

$$0 \longrightarrow F_4 \xrightarrow{d_4} F_3 \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} I^\vee \longrightarrow 0\ ,$$

where

$$F_1 = R(-1,-1,0,0)\oplus R(-1,0,0,-1)\oplus R(0,-1,-1,0)\oplus R(0,-1,0,-1)$$

$$F_2 = R(-1,-1,0,-1)\oplus R(-1,-1,-1,0)\oplus R(-1,-1,0,-1)\oplus R(-1,-1,0,-1)\oplus R(0,-1,-1,-1)$$

$$F_3 = R(-1,-1,0,-1)\oplus R(-1,-1,-1,-1)\oplus R(-1,-1,-1,-1)$$

$$F_4 = R(-1,-1,-1,-1)$$

and the matrices of the differentials are

$$d_1 = \begin{pmatrix} x_1x_2 & x_1x_4 & x_2x_3 & x_2x_4 \end{pmatrix}$$

$$d_2 = \begin{pmatrix} x_4 & x_3 & x_4 & 0 & 0 \\ -x_2 & 0 & 0 & x_2 & 0 \\ 0 & -x_1 & 0 & 0 & x_4 \\ 0 & 0 & -x_1 & -x_1 & -x_3 \end{pmatrix}$$

$$d_3 = \begin{pmatrix} -1 & 0 & x_3 \\ 0 & -x_4 & -x_4 \\ 1 & x_3 & 0 \\ -1 & 0 & x_3 \\ 0 & -x_1 & -x_1 \end{pmatrix}$$

$$d_4 = \begin{pmatrix} -x_3 \\ 1 \\ -1 \end{pmatrix}$$

Step 2: The ideals $I_{\alpha_2} + I_{\alpha_4}$ and $I_{\alpha_1} + I_{\alpha_2} + I_{\alpha_4}$ are paired

1) From the image of the generator labeled by $I_{\alpha_1} + I_{\alpha_2} + I_{\alpha_4}$ we get the relation
$$I_{\alpha_2} + I_{\alpha_4} = - \ (I_{\alpha_1} + I_{\alpha_2}) + (I_{\alpha_1} + I_{\alpha_4}).$$

2) Replace the previous equality in the relation
$$x_3 \ (I_{\alpha_2} + I_{\alpha_4}) = -x_3 \ (I_{\alpha_1} + I_{\alpha_2}) + x_4 \ (I_{\alpha_1} + I_{\alpha_3}) - x_1 \ (I_{\alpha_3} + I_{\alpha_4}).$$

3) Remove the generators labeled by both ideals $I_{\alpha_2}+I_{\alpha_4}$ and $I_{\alpha_1}+I_{\alpha_2}+I_{\alpha_4}$.

We obtain the free resolution:

$$0 \longrightarrow F_4 \xrightarrow{d_4} F_3 \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} I^{\vee} \longrightarrow 0 \ ,$$

where

$$F_1 = R(-1,-1,0,0) \oplus R(-1,0,0,-1) \oplus R(0,-1,-1,0) \oplus R(0,-1,0,-1)$$
$$F_2 = R(-1,-1,0,-1) \oplus R(-1,-1,-1,0) \oplus R(-1,-1,0,-1) \oplus R(0,-1,-1,-1)$$
$$F_3 = R(-1,-1,-1,-1) \oplus R(-1,-1,-1,-1)$$
$$F_4 = R(-1,-1,-1,-1)$$

and the matrices of the differentials are

$$d_1 = \begin{pmatrix} x_1x_2 & x_1x_4 & x_2x_3 & x_2x_4 \end{pmatrix}$$

$$d_2 = \begin{pmatrix} x_4 & x_3 & x_4 & 0 \\ -x_2 & 0 & 0 & 0 \\ 0 & -x_1 & 0 & x_4 \\ 0 & 0 & -x_1 & -x_3 \end{pmatrix}$$

$$d_3 = \begin{pmatrix} 0 & 0 \\ -x_4 & -x_4 \\ x_3 & x_3 \\ -x_1 & -x_1 \end{pmatrix}$$

$$d_4 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Step 3: The ideals $I_{\alpha_2} + I_{\alpha_3} + I_{\alpha_4}$ and $I_{\alpha_1} + I_{\alpha_2} + I_{\alpha_3} + I_{\alpha_4}$ are paired

1) From the image of the generator labeled by $I_{\alpha_1} + I_{\alpha_2} + I_{\alpha_3} + I_{\alpha_4}$ we get the relation
$$I_{\alpha_2} + I_{\alpha_3} + I_{\alpha_4} = I_{\alpha_1} + I_{\alpha_3} + I_{\alpha_4}.$$

2) In this case we do not have to replace the previous equality.

3) Remove the generators labeled by both ideals $I_{\alpha_2}+I_{\alpha_3}$ and $I_{\alpha_1}+I_{\alpha_2}+I_{\alpha_3}$.

We obtain the free resolution:

$$0 \longrightarrow F_3 \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} I^\vee \longrightarrow 0\ ,$$

where

$$F_1 = R(-1,-1,0,0) \oplus R(-1,0,0,-1) \oplus R(0,-1,-1,0) \oplus R(0,-1,0,-1)$$
$$F_2 = R(-1,-1,0,-1) \oplus R(-1,-1,-1,0) \oplus R(-1,-1,0,-1) \oplus R(0,-1,-1,-1)$$
$$F_3 = R(-1,-1,-1,-1)$$

and the matrices of the differentials are

$$d_1 = \begin{pmatrix} x_1x_2 & x_1x_4 & x_2x_3 & x_2x_4 \end{pmatrix}$$
$$d_2 = \begin{pmatrix} x_4 & x_3 & x_4 & 0 \\ -x_2 & 0 & 0 & 0 \\ 0 & -x_1 & 0 & x_4 \\ 0 & 0 & -x_1 & -x_3 \end{pmatrix}$$
$$d_3 = \begin{pmatrix} 0 \\ -x_4 \\ x_3 \\ -x_1 \end{pmatrix}$$

This free resolution is minimal because there are no invertible elements as entries for the matrices corresponding to the differentials. Note that the poset $\mathcal{P}$ allows us to compute the modules in the free resolution but the description of the corresponding differentials is not immediate. In particular the Betti numbers of $I^\vee$ are:

$$\begin{array}{lll} \beta_{0,(1,1,0,0)}(I^\vee) = 1 & \beta_{1,(1,1,1,0)}(I^\vee) = 1 & \beta_{2,(1,1,1,1)}(I^\vee) = 1 \\ \beta_{0,(1,0,0,1)}(I^\vee) = 1 & \beta_{1,(1,1,0,1)}(I^\vee) = 2 & \\ \beta_{0,(0,1,1,0)}(I^\vee) = 1 & \beta_{1,(0,1,1,1)}(I^\vee) = 1 & \\ \beta_{0,(0,1,0,1)}(I^\vee) = 1 & & \end{array}$$

Recall that the multiplicities of the characteristic cycle of the local cohomology module $H_I^2(R)$ are:

$$\begin{array}{lll} m_{2,(1,1,0,0)} = 1 & m_{2,(1,1,1,0)} = 1 & m_{2,(1,1,1,1)} = 1 \\ m_{2,(1,0,0,1)} = 1 & m_{2,(1,1,0,1)} = 2 & \\ m_{2,(0,1,1,0)} = 1 & m_{2,(0,1,1,1)} = 1 & \\ m_{2,(0,1,0,1)} = 1 & & \end{array}$$

Betti numbers: Recall that the minimal free resolution of $I^\vee$ obtained from the Taylor complex $\mathbb{T}_\bullet(I^\vee)$ by using the algorithm is labeled by the elements in the poset $\mathcal{P}$, see Remark 3.3.24. More precisely, it is in the form:

$$0 \longrightarrow F_m \xrightarrow{d_m} \cdots \longrightarrow F_1 \xrightarrow{d_1} I^\vee \longrightarrow 0 \,,$$

where

$$F_j = \bigoplus_{\{\alpha | \mathfrak{p}_\alpha \in \mathcal{P}_j\}} R(-\alpha)^{\beta_{j-1,\alpha}(I^\vee)}.$$

Then, we can easily describe the Betti numbers of $I^\vee$ from the multiplicities of the characteristic cycle of the local cohomology modules supported on I. Our main result in this section is the following:

Proposition 3.3.25. *Let $I^\vee \subseteq R$ be Alexander dual ideal of a squarefree monomial ideal $I \subseteq R$. Then we have:*

$$\beta_{j,\alpha}(I^\vee) = m_{n-|\alpha|+j,\,\alpha}(R/I).$$

A different approach to this result is also given in Section 5.3 (see Corollary 5.2.8).

Remark 3.3.26. Observe that the multiplicities $m_{i,\alpha}$ of the characteristic cycle of a fixed local cohomology module $H_I^{n-i}(R)$ describe the modules and the Betti numbers of the $(n-i)$-linear strand of $I^\vee$.

Collecting the graded Betti numbers we get the total Betti numbers:

Corollary 3.3.27. *Let $I^\vee \subseteq R$ be the Alexander dual ideal of a squarefree monomial ideal $I \subseteq R$. The total Betti number $\beta_j(I^\vee)$ is in the form:*

$$\beta_j(I^\vee) = \sum_{\{\alpha | \mathfrak{p}_\alpha \in \mathcal{P}_j\}} m_{n-|\alpha|+j,\,\alpha}(R/I).$$

If R/I is Cohen-Macaulay then there is only a non vanishing local cohomology module, so by Remark 3.3.26 we recover the following fundamental result of J. A. Eagon and V. Reiner [20].

Corollary 3.3.28. *Let $I^\vee \subseteq R$ be the Alexander dual ideal of a squarefree monomial ideal $I \subseteq R$. Then, R/I is Cohen-Macaulay if and only if $I^\vee$ has a linear free resolution.*

A generalization of that result expressed in terms of the projective dimension of R/I and the Castelnuovo-Mumford regularity of $I^\vee$ is given by N. Terai in [90]. We can easily give a different approach by using the previous results.

Corollary 3.3.29. *Let $I^\vee \subseteq R$ be the Alexander dual ideal of a squarefree monomial ideal $I \subseteq R$. Then we have:*

$$\mathrm{pd}(R/I) = \mathrm{reg}\,(I^\vee).$$

Proof: By using Proposition 3.3.25 we have:

$$\mathrm{reg}\,(I^\vee) := \max\ \{|\alpha| - j \mid \beta_{j,\alpha}(I^\vee) \neq 0\} = \max\{|\alpha| - j \mid m_{n-|\alpha|+j,\,\alpha}(R/I) \neq 0\}.$$

Then, by Corollary 3.3.5 we get the desired result since:

$$\mathrm{pd}(R/I) = \mathrm{cd}\,(R, I) = \max\ \{|\alpha| - j \mid m_{n-|\alpha|+j,\,\alpha}(R/I) \neq 0\},$$

where the first assertion comes from [54]. □

Chapter 4

Bass numbers of local cohomology modules

Let k be a field of characteristic zero. Throughout this chapter we will consider any of the following rings:

- $R = k[[x_1, \ldots, x_n]]$ the formal power series ring.
- $R = k\{x_1, \ldots, x_n\}$ the convergent power series ring.
- $R = k[x_1, \ldots, x_n]$ the polynomial ring.

In Section 4.1 we will describe the characteristic cycle of the local cohomology modules $H^p_{\mathfrak{p}}(H^r_I(R))$ for a given ideal $I \subseteq R$ generated by squarefree monomials such that the quotient ring R/I is Cohen-Macaulay and $\mathfrak{p} \subseteq R$ is a prime ideal. We will also pay some special attention to the case of I and $\mathfrak{p} = \mathfrak{p}_\gamma$ being face ideals.

In Section 4.2 we present the main result of this chapter. Namely we will give a formula for the characteristic cycle of the local cohomology modules $H^p_{\mathfrak{p}_\gamma}(H^r_I(R))$ for a given ideal $I \subseteq R$ generated by monomials and a face ideal $\mathfrak{p}_\gamma \subseteq R$.

We will provide several consequences of the formula for the characteristic cycle $CC(H^p_{\mathfrak{p}_\gamma}(H^r_I(R)))$ in Section 4.3. In particular we will study the following topics:

· Bass numbers of local cohomology modules $H^r_I(R)$. In particular, Lyubeznik numbers $\lambda_{p,i}(R/I)$.

· Injective dimension of $H^r_I(R)$.

· Associated primes and small support of $H^r_I(R)$.

4.1 Cohen-Macaulay Case

Let $I \subseteq R$ be an ideal generated by squarefree monomials and $\mathfrak{p} \subseteq R$ a prime ideal. The aim of this section is to compute explicitly the characteristic cycle of local cohomology modules $H^p_{\mathfrak{p}}(H^r_I(R))$ when R/I is Cohen-Macaulay, with special attention to the case of I and $\mathfrak{p} = \mathfrak{p}_\gamma$ being face ideals. As a consequence of Proposition 3.1.1 there is only one non vanishing local cohomology module $H^r_I(R)$ so the Grothendieck spectral sequence:

$$E_2^{p,q} = H^p_{\mathfrak{p}}(H^q_I(R)) \Longrightarrow H^{p+q}_{\mathfrak{p}+I}(R)$$

degenerates at the E_2-term. In particular we get the following:

Proposition 4.1.1. *Let $I \subseteq R$ be an ideal of height h generated by squarefree monomials and $\mathfrak{p} \subseteq R$ a prime ideal. If R/I is Cohen-Macaulay then:*

$$CC(H^p_{\mathfrak{p}}(H^h_I(R))) = CC(H^{p+h}_{\mathfrak{p}+I}(R)).$$

If $\mathfrak{p} = \mathfrak{p}_\gamma$ is a face ideal, this last characteristic cycle can be computed by using Theorem 3.2.11 with the non necessarily minimal primary decomposition

$$\mathfrak{p}_\gamma + I = (\mathfrak{p}_\gamma + I_{\alpha_1}) \cap \cdots \cap (\mathfrak{p}_\gamma + I_{\alpha_m}),$$

where $I = I_{\alpha_1} \cap \cdots \cap I_{\alpha_m}$ is the minimal primary decomposition of I.

A particular case of interest for us is when the ideal I is a face ideal. This kind of ideals will have a special significance in the rest of this chapter.

Corollary 4.1.2. *Let $I_\alpha \subseteq R$ be a face ideal of height $|\alpha| = h$ and $\mathfrak{p}_\gamma \subseteq R$ be another face ideal. Then we have:*

$$CC(H^p_{\mathfrak{p}_\gamma}(H^h_{I_\alpha}(R))) = CC(H^{p+h}_{\mathfrak{p}_\gamma+I_\alpha}(R))) \neq 0 \quad \textit{iff} \quad p = \mathrm{ht}\,(\mathfrak{p}_\gamma + I_\alpha) - h.$$

Since $\mathfrak{p}_\gamma + I_\alpha$ is a face ideal we can use the results of Section 3.1 in order to compute this last characteristic cycle. In particular, the characteristic variety of $H^p_{\mathfrak{p}_\gamma}(H^h_{I_\alpha}(R))$ is irreducible.

Another case of interest for us is when $\mathfrak{p} = \mathfrak{m}$ is the homogeneous maximal ideal. Then we have:

Corollary 4.1.3. *Let $I \subseteq R$ be an ideal of height h generated by squarefree monomials. If R/I is Cohen-Macaulay then we have:*

$$CC(H^p_{\mathfrak{m}}(H^h_I(R))) = CC(H^{p+h}_{\mathfrak{m}}(R)) \neq 0 \quad \textit{iff} \quad p = n - h.$$

4.2 Main result

Let $I \subseteq R$ be an ideal generated by squarefree monomials and $\mathfrak{p}_\gamma \subseteq R$ a face ideal. In this section we will give a formula for the characteristic cycle of the local cohomology modules $H^p_{\mathfrak{p}_\gamma}(H^r_{I_\alpha}(R))$. The process we will use to obtain the formula involves the Mayer-Vietoris process used in Chapter 3 and the long exact sequence of local cohomology with support the ideal $\mathfrak{p}_\gamma \subseteq R$. In particular, the techniques we will use are very similar to those used in the previous chapter.

In Subsection 4.2.1 we describe the process that allows us to study the local cohomology modules $H^p_{\mathfrak{p}_\gamma}(H^r_{I_\alpha}(R))$ in a systematical way. To shed some light we present some examples with few face ideals in the minimal primary decomposition of I.

In Subsection 4.2.2 we explain how to use the process described in the previous subsection in order to compute the characteristic cycle of $H^p_{\mathfrak{p}_\gamma}(H^r_{I_\alpha}(R))$. In particular, only the description of the characteristic cycle of the kernels and the images of morphisms between certain local cohomology modules supported on face ideals is needed.

Subsection 4.2.3 is devoted to the computation of the characteristic cycle of the kernels and the images introduced previously by means of an algorithm. Finally, a formula for the characteristic cycle of the local cohomology modules $H^p_{\mathfrak{p}_\gamma}(H^r_{I_\alpha}(R))$ is given in Subsection 3.2.4.

4.2.1 Double process

The usual method to determine the modules $H^p_{\mathfrak{p}_\gamma}(H^r_I(R))$ is to find a suitable short exact sequence involving the local cohomology module $H^r_I(R)$ and then apply the long exact sequence of local cohomology with support the ideal $\mathfrak{p}_\gamma$. Again, we will study these modules in a systematical way, i.e. independently of the ideal's complexity.

Although our method will work out for any primary decomposition we will start studying again the case of only two face ideals in the minimal primary decomposition of I. Note that the case $m = 1$ has already been studied in Section 4.1.

The case m=2: Let $I = I_{\alpha_1} \cap I_{\alpha_2}$ be the minimal primary decomposition of a squarefree monomial ideal. We denote $h_i := \operatorname{ht} I_{\alpha_i}$ $i = 1, 2$ and $h_{12} := \operatorname{ht}(I_{\alpha_1} + I_{\alpha_2})$ and we suppose $h_1 \leq h_2$. Then we get the following non vanishing cases:

1) If $h_1 < h_2 < h_{12} - 1$ then:
$H^{h_1}_I(R) \cong H^{h_1}_{I_{\alpha_1}}(R)$, $H^{h_2}_I(R) \cong H^{h_2}_{I_{\alpha_2}}(R)$ and $H^{h_{12}-1}_I(R) \cong H^{h_{12}}_{I_{\alpha_1}+I_{\alpha_2}}(R)$.

   We get:

$$H^p_{\mathfrak{p}_\gamma}(H^{h_1}_I(R)) \cong H^p_{\mathfrak{p}_\gamma}(H^{h_1}_{I_{\alpha_1}}(R)),$$
$$H^p_{\mathfrak{p}_\gamma}(H^{h_2}_I(R)) \cong H^p_{\mathfrak{p}_\gamma}(H^{h_2}_{I_{\alpha_2}}(R)) \quad \text{and}$$
$$H^p_{\mathfrak{p}_\gamma}(H^{h_{12}-1}_I(R)) \cong H^p_{\mathfrak{p}_\gamma}(H^{h_{12}}_{I_{\alpha_1}+I_{\alpha_2}}(R)).$$

2) If $h_1 = h_2 < h_{12} - 1$ then:
$H^{h_1}_I(R) \cong H^{h_1}_{I_{\alpha_1}}(R) \oplus H^{h_2}_{I_{\alpha_2}}(R)$ and $H^{h_{12}-1}_I(R) \cong H^{h_{12}}_{I_{\alpha_1}+I_{\alpha_2}}(R)$.

   We get:

$$H^p_{\mathfrak{p}_\gamma}(H^{h_1}_I(R)) \cong H^p_{\mathfrak{p}_\gamma}(H^{h_1}_{I_{\alpha_1}}(R)) \oplus H^p_{\mathfrak{p}_\gamma}(H^{h_2}_{I_{\alpha_2}}(R)) \quad \text{and}$$
$$H^p_{\mathfrak{p}_\gamma}(H^{h_{12}-1}_I(R)) \cong H^p_{\mathfrak{p}_\gamma}(H^{h_{12}}_{I_{\alpha_1}+I_{\alpha_2}}(R)).$$

3) If $h_1 < h_2 = h_{12} - 1$ then:
$H^{h_1}_I(R) \cong H^{h_1}_{I_{\alpha_1}}(R)$ and $0 \longrightarrow H^{h_2}_{I_{\alpha_2}}(R) \longrightarrow H^{h_2}_I(R) \longrightarrow H^{h_{12}}_{I_{\alpha_1}+I_{\alpha_2}}(R) \longrightarrow 0.$

So we get $H^p_{\mathfrak{p}_\gamma}(H^{h_1}_I(R)) \cong H^p_{\mathfrak{p}_\gamma}(H^{h_1}_{I_{\alpha_1}}(R))$ and the exact sequence:

$$\cdots \longrightarrow H^p_{\mathfrak{p}_\gamma}(H^{h_2}_{I_{\alpha_2}}(R)) \longrightarrow H^p_{\mathfrak{p}_\gamma}(H^{h_2}_I(R)) \longrightarrow H^p_{\mathfrak{p}_\gamma}(H^{h_{12}}_{I_{\alpha_1}+I_{\alpha_2}}(R)) \longrightarrow \cdots .$$

4) If $h_1 = h_2 = h_{12} - 1$ then:
$0 \longrightarrow H^{h_1}_{I_{\alpha_1}}(R) \oplus H^{h_2}_{I_{\alpha_2}}(R) \longrightarrow H^{h_1}_I(R) \longrightarrow H^{h_{12}}_{I_{\alpha_1}+I_{\alpha_2}}(R) \longrightarrow 0.$

We get the exact sequence:

$$\cdots \longrightarrow H^p_{\mathfrak{p}_\gamma}(H^{h_1}_{I_{\alpha_1}}(R) \oplus H^{h_2}_{I_{\alpha_2}}(R)) \longrightarrow H^p_{\mathfrak{p}_\gamma}(H^{h_1}_I(R)) \longrightarrow H^p_{\mathfrak{p}_\gamma}(H^{h_{12}}_{I_{\alpha_1}+I_{\alpha_2}}(R)) \longrightarrow \cdots .$$

In order to describe the local cohomology modules $H^p_{\mathfrak{p}_\gamma}(H^r_I(R))$ we have to study the modules

$$H^p_{\mathfrak{p}_\gamma}(H^{h_1}_{I_{\alpha_1}}(R)),\ \ H^p_{\mathfrak{p}_\gamma}(H^{h_2}_{I_{\alpha_2}}(R))) \text{ and } H^p_{\mathfrak{p}_\gamma}(H^{h_{12}}_{I_{\alpha_1}+I_{\alpha_2}}(R)).$$

and the homomorphisms of the long exact sequence when it is necessary. Notice that these modules have been computed in Section 4.1. We state that these are the **initial pieces** that allows us to describe the modules $H^p_{\mathfrak{p}_\gamma}(H^r_I(R))$.

This method can be generalized as follows:

General case: Let $I = I_{\alpha_1} \cap \cdots \cap I_{\alpha_m}$ be the minimal primary decomposition of a squarefree monomial ideal $I \subseteq R$.

We split all the Mayer-Vietoris sequences

$$\cdots \longrightarrow H^r_{U+V}(R) \longrightarrow H^r_U(R) \oplus H^r_V(R) \longrightarrow H^r_{U\cap V}(R) \longrightarrow H^{r+1}_{U+V}(R) \longrightarrow \cdots$$

obtained at each step of the Mayer-Vietoris process into short exact sequences of kernels and cokernels:

$$0 \longrightarrow B_r \longrightarrow H^r_U(R) \oplus H^r_V(R) \longrightarrow C_r \longrightarrow 0,$$

$$0 \longrightarrow C_r \longrightarrow H^r_{U\cap V}(R) \longrightarrow A_{r+1} \longrightarrow 0,$$

$$0 \longrightarrow A_{r+1} \longrightarrow H^{r+1}_{U+V}(R) \longrightarrow B_{r+1} \longrightarrow 0.$$

In order to compute the characteristic cycle of the modules $H^p_{\mathfrak{p}_\gamma}(H^r_{U\cap V}(R))$ we will apply the long exact sequence of local cohomology with support the face ideal $\mathfrak{p}_\gamma$, to the corresponding short exact sequences

$$0\longrightarrow C_r\longrightarrow H^r_{U\cap V}(R)\longrightarrow A_{r+1}\longrightarrow 0$$

Namely, we get the long exact sequences

$$\cdots\longrightarrow H^p_{\mathfrak{p}_\gamma}(C_r)\longrightarrow H^p_{\mathfrak{p}_\gamma}(H^r_{U\cap V}(R))\longrightarrow H^p_{\mathfrak{p}_\gamma}(A_{r+1})\longrightarrow H^{p+1}_{\mathfrak{p}_\gamma}(C_r)\longrightarrow\cdots .$$

Since at each step of the Mayer-Vietoris process the number of ideals in the primary decomposition of the ideals $U's$ and $U+V's$ decreases by one, the process finishes when we apply the long exact sequence of cohomology to short exact sequences that only involve face ideals. From now on we will call it **the double process**.

Recall that we have described the characteristic cycle of the modules C_r and A_{r+1} in Theorem 3.2.11 in terms of the characteristic cycle of the initial pieces that describe precisely the local cohomology modules $H^r_{U\cap V}(R)$. In the following remark we recall which are these initial pieces and how they are reflected in the diagram of the ideal $U\cap V$

Remark 4.2.1. i) The inclusion $CC(C_r)\subseteq CC(H^r_U(R)\oplus H^r_V(R))$ is reflected in the diagram of $U\cap V$ as follows:

The initial pieces that describe the characteristic cycle of $H^r_U(R)$ are those corresponding to the non paired ideals in the upper half of the diagram of $U\cap V$.

The initial pieces that describe the characteristic cycle of C_r are those, among the initial pieces of $H^r_U(R)$ and the vertex of the diagram corresponding to $H^r_V(R)$, that are not paired to any ideal in the lower half of the diagram of $U\cap V$.

ii) The inclusion $CC(A_{r+1})\subseteq CC(H^r_{U+V}(R))$ is reflected in the diagram of $U\cap V$ as follows:

The initial pieces that describe the characteristic cycle of $H^r_{U+V}(R)$ are those corresponding to the non paired ideals in the lower half of the diagram of $U\cap V$.

The initial pieces that describe the characteristic cycle of A_{r+1} are those, among the initial pieces of $H^r_{U+V}(R)$, that are not paired to any ideal in the upper half of the diagram of $U\cap V$.

Once we have developed the double process, we will see that in order to describe the local cohomology modules $H^p_{\mathfrak{p}_\gamma}(H^r_I(R))$ we only have to study the modules $H^p_{\mathfrak{p}_\gamma}(H^r_{I_{\alpha_{i_1}}+\cdots+I_{\alpha_{i_j}}}(R))$, where the sums $I_{\alpha_{i_1}}+\cdots+I_{\alpha_{i_j}}$, $1\leq i_1<\cdots<i_j\leq m$ of face ideals in the minimal primary decomposition of I are not paired, and the homomorphisms of the corresponding long exact sequences.

We state that these local cohomology modules are the **initial pieces** that allow us to describe the modules $H^p_{\mathfrak{p}_\gamma}(H^r_I(R))$. Namely, consider the set of non paired sums of face ideals $\mathcal{P}=\{\mathcal{P}_1,\mathcal{P}_2,\dots,\mathcal{P}_m\}$. Then, there are the following initial pieces:

$$\begin{array}{ll} H^p_{\mathfrak{p}_\gamma}(H^r_{I_{\alpha_{i_1}}}(R)) & I_{\alpha_{i_1}}\in\mathcal{P}_1,\\ H^p_{\mathfrak{p}_\gamma}(H^r_{I_{\alpha_{i_1}}+I_{\alpha_{i_2}}}(R)) & I_{\alpha_{i_1}}+I_{\alpha_{i_2}}\in\mathcal{P}_2,\\ \vdots & \\ H^p_{\mathfrak{p}_\gamma}(H^r_{I_{\alpha_1}+I_{\alpha_2}+\cdots+I_{\alpha_m}}(R)) & I_{\alpha_1}+I_{\alpha_2}+\cdots+I_{\alpha_m}\in\mathcal{P}_m. \end{array}$$

The double process allows us, in some sense, to break the local cohomology modules $H^p_{\mathfrak{p}_\gamma}(H^r_I(R))$ into simpler pieces that are easier to study. But, in order to give a full description of the characteristic cycle of the modules $H^p_{\mathfrak{p}_\gamma}(H^r_I(R))$ it remains to study the homomorphisms of the sequences that appear in the process. This will be done in the next sections.

4.2.2 Double process and characteristic cycle

Our method for computing the characteristic cycle of the local cohomology modules $H^p_{\mathfrak{p}_\gamma}(H^r_I(R))$ will be as follows:

We start by splitting the long exact sequences obtained in the last step of the double process, i.e. the sequences

$$\cdots\longrightarrow H^p_{\mathfrak{p}_\gamma}(C_r)\longrightarrow H^p_{\mathfrak{p}_\gamma}(H^r_{U\cap V}(R))\longrightarrow H^p_{\mathfrak{p}_\gamma}(A_{r+1})\longrightarrow H^{p+1}_{\mathfrak{p}_\gamma}(C_r)\longrightarrow\cdots$$

obtained from

$$0\longrightarrow C_r\longrightarrow H^r_{U\cap V}(R)\longrightarrow A_{r+1}\longrightarrow 0,$$

where the corresponding ideals U's and V's are face ideals, into short exact sequences of kernels and cokernels:

$$0\longrightarrow Z_{p-1}\longrightarrow H^p_{\mathfrak{p}_\gamma}(C_r)\longrightarrow X_p\longrightarrow 0$$

$$0 \longrightarrow X_p \longrightarrow H^p_{\mathfrak{p}_\gamma}(H^r_{U\cap V}(R)) \longrightarrow Y_p \longrightarrow 0$$

$$0 \longrightarrow Y_p \longrightarrow H^p_{\mathfrak{p}_\gamma}(A_{r+1}) \longrightarrow Z_p \longrightarrow 0.$$

This allows us to compute the characteristic cycle of the local cohomology modules $H^p_{\mathfrak{p}_\gamma}(H^r_{U\cap V}(R))$. Namely we have:

$$CC(H^p_{\mathfrak{p}_\gamma}(H^r_{U\cap V}(R))) = CC(X_p) + CC(Y_p).$$

The computation of the characteristic cycles of the modules Z_p $\forall p$ is more involved than the analogous one for the modules B_r in Section 3.2, and we will not give a complete description of them by now. For the moment, we only point out that they will be described in terms of the initial pieces of the modules $H^p_{\mathfrak{p}_\gamma}(H^r_I(R))$, and that we will be able to give a necessary condition for their annihilation by using the irreducibility of the characteristic variety of the initial pieces. Namely, we have the following:

Remark 4.2.2. If U and V are face ideals such that $U \cap V$ is a minimal primary decomposition then, by Remark 3.2.1, we have:

$$C_r = H^r_U(R) \oplus H^r_V(R) \quad \text{and} \quad A_{r+1} = H^{r+1}_{U+V}(R).$$

Since

$$C(Z_{p-1}) \subseteq C(H^p_{\mathfrak{p}_\gamma}(H^r_U(R) \oplus H^r_V(R))),$$
$$C(Z_{p-1}) \subseteq C(H^{p-1}_{\mathfrak{p}_\gamma}(H^r_{U+V}(R)))$$

and the characteristic varieties of $H^p_{\mathfrak{p}_\gamma}(H^r_U(R))$, $H^p_{\mathfrak{p}_\gamma}(H^r_V(R))$ and $H^{p-1}_{\mathfrak{p}_\gamma}(H^r_{U+V}(R))$ are irreducibles we get

- $Z_{p-1} = 0$ $\forall p$ when one of the following conditions is satisfied:
  - $\mathfrak{p}_\gamma + U + V$ is different from $\mathfrak{p}_\gamma + U$ and $\mathfrak{p}_\gamma + V$.
  - $\mathfrak{p}_\gamma + U + V$ is equal to $\mathfrak{p}_\gamma + U$ (resp. $\mathfrak{p}_\gamma + V$) but $p + \operatorname{ht} U \neq p - 1 + \operatorname{ht}(U+V)$ (resp. $p + \operatorname{ht} V \neq p - 1 + \operatorname{ht}(U+V)$).

The process continues by computing the characteristic cycle of the intermediate local cohomology modules in the double process. To do this we split the corresponding long exact sequence into short exact sequences of kernels and cokernels and we use the results obtained in the previous steps of the process.

The process finishes when we compute the characteristic cycle of the local cohomology modules $H^p_{\mathfrak{p}_\gamma}(H^r_I(R))$ by splitting the long exact sequence that involves this module. Namely, let $I = I_{\alpha_1} \cap \cdots \cap I_{\alpha_m}$ be the minimal primary decomposition of the squarefree monomial ideal I. Consider the short exact sequence $0 \longrightarrow C_r \longrightarrow H^r_I(R) \longrightarrow A_{r+1} \longrightarrow 0$ obtained by splitting the Mayer-Vietoris sequence

$$\cdots \longrightarrow \begin{array}{c} H^r_{I_{\alpha_1}\cap\cdots\cap I_{\alpha_{m-1}}}(R) \\ \oplus \\ H^r_{I_{\alpha_m}}(R) \end{array} \longrightarrow H^r_I(R) \longrightarrow H^{r+1}_{(I_{\alpha_1}\cap\cdots\cap I_{\alpha_{m-1}})+I_{\alpha_m}}(R) \longrightarrow \cdots ,$$

into short exact sequences of kernels and cokernels. Applying the long exact sequence of local cohomology with support the ideal $\mathfrak{p}_\gamma \subseteq R$ we have:

$$\cdots \longrightarrow H^p_{\mathfrak{p}_\gamma}(C_r) \longrightarrow H^p_{\mathfrak{p}_\gamma}(H^r_I(R)) \longrightarrow H^p_{\mathfrak{p}_\gamma}(A_{r+1}) \longrightarrow H^{p+1}_{\mathfrak{p}_\gamma}(C_r) \longrightarrow \cdots .$$

Then by splitting this sequence we get:

$$0 \longrightarrow Z_{p-1} \longrightarrow H^p_{\mathfrak{p}_\gamma}(C_r) \longrightarrow X_p \longrightarrow 0$$

$$0 \longrightarrow X_p \longrightarrow H^p_{\mathfrak{p}_\gamma}(H^r_I(R)) \longrightarrow Y_p \longrightarrow 0$$

$$0 \longrightarrow Y_p \longrightarrow H^p_{\mathfrak{p}_\gamma}(A_{r+1}) \longrightarrow Z_p \longrightarrow 0.$$

The characteristic cycle of the local cohomology modules $H^p_{\mathfrak{p}_\gamma}(H^r_I(R))$ is then:

$$CC(H^p_{\mathfrak{p}_\gamma}(H^r_I(R))) = CC(X_p) + CC(Y_p).$$

In the next section we will compute the characteristic cycle of the modules Z_p $\forall p$ for all the long exact sequences obtained in the double process in a systematical way, i.e. independently of the ideal's complexity.

We will illustrate this process for the cases with two face ideals in the minimal primary decomposition of I.

The case m=2: Let $I = I_{\alpha_1} \cap I_{\alpha_2}$ be the minimal primary decomposition of a squarefree monomial ideal. Denote $h_i := \operatorname{ht} I_{\alpha_i}$ $i = 1, 2$ and $h_{12} := \operatorname{ht}(I_{\alpha_1} + I_{\alpha_2})$ and suppose $h_1 \leq h_2$. Then we get the following cases:

1) If $h_1 < h_2 < h_{12}-1$ then:
$CC(H^p_{\mathfrak{p}_\gamma}(H^{h_1}_I(R))) = CC(H^{p+h_1}_{\mathfrak{p}_\gamma+I_{\alpha_1}}(R))$,
$CC(H^p_{\mathfrak{p}_\gamma}(H^{h_2}_I(R))) = CC(H^{p+h_2}_{\mathfrak{p}_\gamma+I_{\alpha_2}}(R))$ and
$CC(H^p_{\mathfrak{p}_\gamma}(H^{h_{12}-1}_I(R))) = CC(H^{p+h_{12}}_{\mathfrak{p}_\gamma+I_{\alpha_1}+I_{\alpha_2}}(R))$.

2) If $h_1 = h_2 < h_{12}-1$ then:
$CC(H^p_{\mathfrak{p}_\gamma}(H^{h_1}_I(R))) = CC(H^{p+h_1}_{\mathfrak{p}_\gamma+I_{\alpha_1}}(R)) \oplus CC(H^{p+h_2}_{\mathfrak{p}_\gamma+I_{\alpha_2}}(R))$ and
$CC(H^p_{\mathfrak{p}_\gamma}(H^{h_{12}-1}_I(R))) = CC(H^{p+h_{12}}_{\mathfrak{p}_\gamma+I_{\alpha_1}+I_{\alpha_2}}(R))$.

3) If $h_1 < h_2 = h_{12}-1$ then:
$CC(H^p_{\mathfrak{p}_\gamma}(H^{h_1}_I(R))) = CC(H^{p+h_1}_{\mathfrak{p}_\gamma+I_{\alpha_1}}(R))$
and from the exact sequence

$$\cdots \longrightarrow H^p_{\mathfrak{p}_\gamma}(H^{h_2}_{I_{\alpha_2}}(R)) \longrightarrow H^p_{\mathfrak{p}_\gamma}(H^{h_2}_I(R)) \longrightarrow H^p_{\mathfrak{p}_\gamma}(H^{h_{12}}_{I_{\alpha_1}+I_{\alpha_2}}(R)) \longrightarrow \cdots$$

we have to determine the characteristic cycle of $H^p_{\mathfrak{p}_\gamma}(H^{h_2}_I(R))$.

Let Z_{p-1} be the kernel of the morphism $H^p_{\mathfrak{p}_\gamma}(H^{h_2}_{I_{\alpha_2}}(R)) \longrightarrow H^p_{\mathfrak{p}_\gamma}(H^{h_2}_I(R))$. Since

$$CC(Z_{p-1}) \subseteq CC(H^{p-1}_{\mathfrak{p}_\gamma}(H^{h_{12}}_{I_{\alpha_1}+I_{\alpha_2}}(R)))$$

$$CC(Z_{p-1}) \subseteq CC(H^p_{\mathfrak{p}_\gamma}(H^{h_2}_{I_{\alpha_2}}(R)))$$

we have to study the equality between the initial pieces $H^{p-1}_{\mathfrak{p}_\gamma}(H^{h_{12}}_{I_{\alpha_1}+I_{\alpha_2}}(R))$ and $H^p_{\mathfrak{p}_\gamma}(H^{h_2}_{I_{\alpha_2}}(R))$ in order to compute these modules. Notice that this is equivalent to study the equality between the sums of face ideals $\mathfrak{p}_\gamma + I_{\alpha_1} + I_{\alpha_2}$ and $\mathfrak{p}_\gamma + I_{\alpha_2}$. We have to consider the following cases:

i) $\mathfrak{p}_\gamma + I_{\alpha_2} \neq \mathfrak{p}_\gamma + I_{\alpha_1} + I_{\alpha_2}$.

Let $\operatorname{ht}(\mathfrak{p}_\gamma + I_{\alpha_2}) = p + h_2$. In this case $\operatorname{ht}(\mathfrak{p}_\gamma + I_{\alpha_1} + I_{\alpha_2}) = p + h_{12}$ so we get the short exact sequence:

$$0 \longrightarrow H^p_{\mathfrak{p}_\gamma}(H^{h_2}_{I_{\alpha_2}}(R)) \longrightarrow H^p_{\mathfrak{p}_\gamma}(H^{h_2}_I(R)) \longrightarrow H^p_{\mathfrak{p}_\gamma}(H^{h_{12}}_{I_{\alpha_1}+I_{\alpha_2}}(R)) \longrightarrow 0.$$

By the additivity of the characteristic cycle

$$CC(H^p_{\mathfrak{p}_\gamma}(H^{h_2}_I(R)) = CC(H^{p+h_2}_{\mathfrak{p}_\gamma+I_{\alpha_2}}(R)) + CC(H^{p+h_{12}}_{\mathfrak{p}_\gamma+I_{\alpha_1}+I_{\alpha_2}}(R)).$$

ii) $\mathfrak{p}_\gamma + I_{\alpha_2} = \mathfrak{p}_\gamma + I_{\alpha_1} + I_{\alpha_2}$.

Let $\mathrm{ht}\,(\mathfrak{p}_\gamma + I_{\alpha_2}) = p + h_2$. We have an exact sequence:

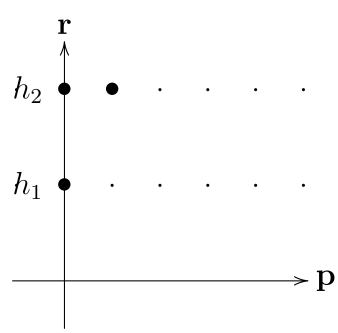
$$0 \longrightarrow H^{p-1}_{\mathfrak{p}_\gamma}(H^{h_2}_I(R)) \longrightarrow H^{p-1}_{\mathfrak{p}_\gamma}(H^{h_{12}}_{I_{\alpha_1}+I_{\alpha_2}}(R)) \longrightarrow H^p_{\mathfrak{p}_\gamma}(H^{h_2}_{I_{\alpha_2}}(R)) \longrightarrow H^p_{\mathfrak{p}_\gamma}(H^{h_2}_I(R)) \longrightarrow 0.$$

We will prove that the local cohomology modules $H^p_{\mathfrak{p}_\gamma}(H^{h_2}_I(R))$ vanish for all p, so $Z_{p-1} = H^{p-1}_{\mathfrak{p}_\gamma}(H^{h_{12}}_{I_{\alpha_1}+I_{\alpha_2}}(R)) \neq 0$.
Let $x_{i_1} \in I_{\alpha_1}$ be an independent variable such that we have the equality $I_{\alpha_1} + I_{\alpha_2} = I_{\alpha_2} + (x_{i_1})$. Since $\mathfrak{p}_\gamma + I_{\alpha_2} = \mathfrak{p}_\gamma + I_{\alpha_1} + I_{\alpha_2}$ we have $x_{i_1} \in \mathfrak{p}_\gamma$. So let $\mathfrak{p}_\gamma = (x_{i_1}, x_{i_2}, \cdots, x_{i_s})$ be the homogeneous prime ideal.
First we will compute $H^p_{(x_{i_1})}(H^{h_2}_I(R))$ by using the spectral sequence:

$$E_2^{p,r} = H^p_{(x_{i_1})}(H^r_I(R)) \Longrightarrow H^{p+r}_{I+(x_{i_1})}(R).$$

The spectral sequence degenerates at the E_2-term because we have:
- $H^p_{(x_{i_1})}(H^{h_2}_I(R)) = 0 \quad \forall p \geq 2.$
- $H^p_{(x_{i_1})}(H^{h_1}_I(R)) = 0 \quad \forall p \geq 1$, due to the fact that $H^{h_1}_I(R) \cong H^{h_1}_{I_{\alpha_1}}(R)$ and $x_{i_1} \in I_{\alpha_1}$.

The degeneration of the spectral sequence can be easily seen on the diagram:



As a consequence we have:

$$H^p_{(x_{i_1})}(H^{h_2}_I(R)) = H^{p+h_2}_{I+(x_{i_1})}(R) = H^{p+h_2}_{I_{\alpha_1}}(R) = 0 \quad \forall p$$

because $I + (x_{i_1}) = I_{\alpha_1}$ and $\operatorname{ht} I_{\alpha_1} < h_2$.

Now by using Brodmann's long exact sequence

$$\cdots \longrightarrow H^p_{(x_{i_1},x_{i_2})}(H^{h_2}_I(R)) \longrightarrow H^p_{(x_{i_1})}(H^{h_2}_I(R)) \longrightarrow (H^p_{(x_{i_1})}(H^{h_2}_I(R)))_{x_{i_2}} \longrightarrow \cdots ,$$

we get $H^p_{(x_{i_1},x_{i_2})}(H^{h_2}_I(R)) = 0 \ \ \forall p$, and iterating this procedure on the generators of $\mathfrak{p}_\gamma$ we get $H^p_{\mathfrak{p}_\gamma}(H^{h_2}_I(R)) = 0 \ \ \forall p$.

4) If $h_1 = h_2 = h_{12} - 1$ then we get the following exact sequence

$$\cdots \longrightarrow H^p_{\mathfrak{p}_\gamma}(H^{h_1}_{I_{\alpha_1}}(R) \oplus H^{h_2}_{I_{\alpha_2}}(R)) \longrightarrow H^p_{\mathfrak{p}_\gamma}(H^{h_1}_I(R)) \longrightarrow H^p_{\mathfrak{p}_\gamma}(H^{h_{12}}_{I_{\alpha_1}+I_{\alpha_2}}(R)) \longrightarrow \cdots .$$

There is only a non vanishing local cohomology module so R/I is Cohen-Macaulay. Then, we can compute the characteristic cycle of the modules $H^p_{\mathfrak{p}_\gamma}(H^r_I(R))$ by using the results of Section 4.1. Namely we get

$$CC(H^p_{\mathfrak{p}_\gamma}(H^{h_1}_I(R))) = CC(H^{p+h_1}_{\mathfrak{p}_\gamma+I}(R)).$$

This last characteristic cycle can be computed by using the results of Chapter 3. So we have:

i) If the primary decomposition $\mathfrak{p}_\gamma + I = (\mathfrak{p}_\gamma + I_{\alpha_1}) \cap (\mathfrak{p}_\gamma + I_{\alpha_2})$ is minimal then:

$$CC(H^p_{\mathfrak{p}_\gamma}(H^{h_1}_I(R))) = CC(H^{p+h_1}_{\mathfrak{p}_\gamma+I_{\alpha_1}}(R)) + CC(H^{p+h_2}_{\mathfrak{p}_\gamma+I_{\alpha_2}}(R)))+$$
$$+CC(H^{p+h_{12}}_{\mathfrak{p}_\gamma+I_{\alpha_1}+I_{\alpha_2}}(R)))$$

In particular the corresponding modules Z_p vanish for all p.

ii) If the primary decomposition $\mathfrak{p}_\gamma + I = (\mathfrak{p}_\gamma + I_{\alpha_1}) \cap (\mathfrak{p}_\gamma + I_{\alpha_2})$ is not minimal then we have $\mathfrak{p}_\gamma + I = \mathfrak{p}_\gamma + I_{\alpha_1}$ or $\mathfrak{p}_\gamma + I = \mathfrak{p}_\gamma + I_{\alpha_2}$ so:

$$CC(H^p_{\mathfrak{p}_\gamma}(H^{h_1}_I(R))) = CC(H^{p+h_1}_{\mathfrak{p}_\gamma+I_{\alpha_1}}(R))$$

or

$$CC(H^p_{\mathfrak{p}_\gamma}(H^{h_1}_I(R))) = CC(H^{p+h_2}_{\mathfrak{p}_\gamma+I_{\alpha_2}}(R)))$$

In particular $Z_p = H^{p+h_{12}}_{\mathfrak{p}_\gamma+I_{\alpha_1}+I_{\alpha_2}}(R)) \neq 0$ for $p = \operatorname{ht}(\mathfrak{p}_\gamma + I_{\alpha_1} + I_{\alpha_2}) - h_{12}$.

4.2.3 Optimization of the process

In the previous section we have seen how to use the double process in order to compute the characteristic cycle of the local cohomology modules $H^p_{\mathfrak{p}_\gamma}(H^r_I(R))$. In particular, we have seen in the case of two ideals in the minimal primary decomposition of I (see the cases 3)ii) and 4)ii)), that if a module Z_p is different from zero then the local cohomology module could be described in a simplest way. This is reflected in the fact that the corresponding long exact sequence obtained in the process provides superfluous information, that we have to cancel.

Optimal double process

The easiest case to study is when all the modules Z_p that appear during the process vanish. This is reflected in the long exact sequences obtained in the double process by means of the following property:

Definition 4.2.3. *We say that the long exact sequence*

$$\cdots \longrightarrow H^p_{\mathfrak{p}_\gamma}(C_r) \longrightarrow H^p_{\mathfrak{p}_\gamma}(H^r_{U\cap V}(R)) \longrightarrow H^p_{\mathfrak{p}_\gamma}(A_{r+1}) \longrightarrow H^{p+1}_{\mathfrak{p}_\gamma}(C_r) \longrightarrow \cdots ,$$

is minimal if it splits for all p into short exact sequences

$$0 \longrightarrow H^p_{\mathfrak{p}_\gamma}(C_r) \longrightarrow H^p_{\mathfrak{p}_\gamma}(H^r_{U\cap V}(R)) \longrightarrow H^p_{\mathfrak{p}_\gamma}(A_{r+1}) \longrightarrow 0.$$

If all the long exact sequences that appear in the double process are minimal then the characteristic cycle of the local cohomology module $H^p_{\mathfrak{p}_\gamma}(H^r_I(R))$ is described in terms of the characteristic cycle of **all** the initial pieces of the module $H^p_{\mathfrak{p}_\gamma}(H^r_I(R))$, i.e. the initial pieces corresponding to non paired sums of face ideals in the minimal primary decomposition of I.

If the Mayer-Vietoris process is optimal and the corresponding long exact sequences are minimal we will say that **the double process is optimal** in the sense that the characteristic cycle of the local cohomology modules $H^p_{\mathfrak{p}_\gamma}(H^r_I(R))$ is described in terms of the characteristic cycle of **all** the initial pieces of the module $H^r_I(R)$, i.e. the initial pieces corresponding to all the sums of face ideals in the minimal primary decomposition of I.

Ideals with disjoint faces: Let $I = I_{\alpha_1} \cap \cdots \cap I_{\alpha_m}$ be the minimal primary decomposition of a squarefree monomial ideal $I \subseteq R$. The characteristic varieties of the initial pieces are irreducible, so in order to get an optimal double process one should have one of the following properties:

- The initial pieces that describe $H^p_{\mathfrak{p}_\gamma}(H^r_{U+V}(R))$ are different from those that describe $H^p_{\mathfrak{p}_\gamma}(H^r_U(R))\oplus H^p_{\mathfrak{p}_\gamma}(H^r_V(R))$ $\forall p$.

- In the case that any of the initial pieces that describe $H^p_{\mathfrak{p}_\gamma}(H^r_{U+V}(R))$ is equal to some of the initial piece that describe $H^p_{\mathfrak{p}_\gamma}(H^r_U(R))\oplus H^p_{\mathfrak{p}_\gamma}(H^r_V(R))$, then the heights of the corresponding non paired sums of face ideals in the minimal primary decomposition of I differs at least by two.

For the case of local cohomology modules supported on squarefree monomial ideals I with disjoint faces such that $\operatorname{ht} I \geq 2$ the double process is optimal. In order to compute its characteristic cycle one has to fit the initial pieces into the right place. Namely, let

$$\mathcal{I}_{\gamma,j,r,\alpha} := \{I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} \in \mathcal{I}_{j,r} \mid I_\alpha = \mathfrak{p}_\gamma + I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}}\}.$$

Then we have:

Proposition 4.2.4. *Let $I = I_{\alpha_1} \cap \cdots \cap I_{\alpha_m}$ be the minimal primary decomposition of a squarefree monomial ideal with disjoint faces $I \subseteq R$ such that* $\operatorname{ht} I \geq 2$ *and $\mathfrak{p}_\gamma \subseteq R$ a face ideal. Then:*

$$CC(H^p_{\mathfrak{p}_\gamma}(H^r_I(R))) = \sum_{\alpha\in\{0,1\}^n} \lambda_{\gamma,p,n-r,\alpha}\ CC(H^{|\alpha|}_{I_\alpha}(R)),$$

where $\lambda_{\gamma,p,n-r,\alpha} = \sharp\, \mathcal{I}_{\gamma,j,r,\alpha}$ such that $r + (j-1) = |\alpha| - p$.

Proof: We will proceed by induction on m, the number of ideals in the minimal primary decomposition, being the case $m = 1$ trivial. Since the ideal I has disjoint faces the Mayer-Vietoris sequences in the process are minimal. We start by applying the long exact sequence of local cohomology supported on $\mathfrak{p}_\gamma$ to the minimal Mayer-Vietoris sequence with

$$U = I_{\alpha_1} \cap \cdots \cap I_{\alpha_{m-1}}, \qquad U \cap V = I = I_{\alpha_1} \cap \cdots \cap I_{\alpha_m},$$
$$V = I_{\alpha_m}, \qquad U + V = (I_{\alpha_1} \cap \cdots \cap I_{\alpha_{m-1}}) + I_{\alpha_m}.$$

Since $\operatorname{ht} I \geq 2$ we get the minimal long exact sequence:

$$0 \longrightarrow H^p_{\mathfrak{p}_\gamma}(H^r_U(R)\oplus H^r_V(R)) \longrightarrow H^p_{\mathfrak{p}_\gamma}(H^r_{U\cap V}(R)) \longrightarrow H^p_{\mathfrak{p}_\gamma}(H^{r+1}_{U+V}(R)) \longrightarrow 0.$$

Assume we have proved the formula for ideals with less terms than m in the minimal primary decomposition. We have $CC(H^p_{\mathfrak{p}_\gamma}(H^r_V(R))) = CC(H^{p+r}_{\mathfrak{p}_\gamma + I_{\alpha_m}}(R))$.

Denote by $\mathcal{I}_{\gamma,j,r,\alpha}(U)$ the set of face ideals obtained as a sum of face ideals in the minimal primary decomposition of $U = I_{\alpha_1} \cap \cdots \cap I_{\alpha_{m-1}}$ ordered by the number of summands, the height of the ideals and such they give the face ideal I_α when $\mathfrak{p}_\gamma$ is added. By induction, the characteristic cycle of $H^p_{\mathfrak{p}_\gamma}(H^r_U(R))$ is

$$CC(H^p_{\mathfrak{p}_\gamma}(H^r_{I_{\alpha_1}\cap\cdots\cap I_{\alpha_{m-1}}}(R))) = \sum_{\alpha\in\{0,1\}^n} \lambda_{\gamma,p,n-r,\alpha}(U)\ CC(H^{|\alpha|}_{I_\alpha}(R)),$$

where $\lambda_{\gamma,p,n-r,\alpha}(U) = \sharp\mathcal{I}_{\gamma,j,r,\alpha}(U)$ such that $r + (j-1) = |\alpha| - p$.

Denote by $\mathcal{I}_{\gamma,j,r,\alpha}(U+V)$ the set of face ideals obtained as a sum of face ideals in the minimal primary decomposition of $U+V = (I_{\alpha_1} \cap \cdots \cap I_{\alpha_{m-1}}) + I_{\alpha_m}$ ordered by the number of summands, the height of the ideals and such they give the face ideal I_α when $\mathfrak{p}_\gamma$ is added. By induction, the characteristic cycle of $H^p_{\mathfrak{p}_\gamma}(H^{r+1}_{U+V}(R))$ is

$$CC(H^p_{\mathfrak{p}_\gamma}(H^{r+1}_{(I_{\alpha_1}\cap\cdots\cap I_{\alpha_{m-1}})+I_{\alpha_m}}(R))) = \sum_{\alpha\in\{0,1\}^n} \lambda_{\gamma,p,n-r,\alpha}(U+V)\ CC(H^{|\alpha|}_{I_\alpha}(R)),$$

where $\lambda_{\gamma,p,n-r,\alpha}(U+V) = \sharp\mathcal{I}_{\gamma,j,r,\alpha}(U+V)$ such that $r + (j-1) = |\alpha| - p$.

So by using the additivity of the characteristic cycle we get the desired result.

□

Let $\mathfrak{m} \subseteq R$ be the homogeneous maximal ideal. For any sum of face ideals $I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} \in \mathcal{I}_{j,r}$ we have $\mathfrak{m} = \mathfrak{m} + I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}}$ so $\mathcal{I}_{j,r} = \mathcal{I}_{\alpha_\mathfrak{m},j,r,\alpha_\mathfrak{m}}$ $\forall j$, $\forall r$. Since $CC(H^n_\mathfrak{m}(R)) = T^*_{X_{\alpha_\mathfrak{m}}}X$ we get:

Corollary 4.2.5. *Let $I = I_{\alpha_1} \cap \cdots \cap I_{\alpha_m}$ be the minimal primary decomposition of a squarefree monomial ideal with disjoint faces $I \subseteq R$ such that* $\operatorname{ht} I \geq 2$ *and $\mathfrak{m} \subseteq R$ the homogeneous maximal ideal. Then:*

$$CC(H^p_\mathfrak{m}(H^r_I(R))) = \lambda_{p,n-r}\ T^*_{X_{\alpha_\mathfrak{m}}}X,$$

where $\lambda_{p,n-r} = \sharp\mathcal{I}_{j,r}$ such that $r + (j-1) = n - p$.

Example: Let $I = (x_1, x_2)\cap(x_3, x_4)\cap(x_5, x_6, x_7) \subseteq R = k[x_1, x_2, x_3, x_4, x_5, x_6, x_7]$ and let $\mathfrak{m} = (x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ be the homogeneous maximal ideal. Then:

$CC(H^5_{\mathfrak{m}}(H^2_I(R))) = 2\ CC(H^7_{\mathfrak{m}}(R))$.

$CC(H^4_{\mathfrak{m}}(H^3_I(R))) = CC(H^7_{\mathfrak{m}}(R))$ and $CC(H^3_{\mathfrak{m}}(H^3_I(R))) = CC(H^7_{\mathfrak{m}}(R))$.

$CC(H^2_{\mathfrak{m}}(H^4_I(R))) = 2\ CC(H^7_{\mathfrak{m}}(R))$.

$CC(H^0_{\mathfrak{m}}(H^5_I(R))) = CC(H^7_{\mathfrak{m}}(R))$.

Non optimal double process

In general some of the modules Z_p appearing in the long exact sequences may not vanish. Besides the characteristic cycle, a full description of the modules Z_p for the case of ideals with two ideals in the minimal primary decomposition was given in the previous section. Namely we have:

The case m=2: Let $I = I_{\alpha_1} \cap I_{\alpha_2}$ be the minimal primary decomposition of a squarefree monomial ideal. The corresponding non vanishing modules Z_p in the process for computing the characteristic cycle of the local cohomology modules $H^p_{\mathfrak{p}_\gamma}(H^r_I(R))$ are:

• $Z_p = H^{p+h_{12}}_{\mathfrak{p}_\gamma + I_{\alpha_1} + I_{\alpha_2}}(R)) \neq 0$ for $p = \operatorname{ht}(\mathfrak{p}_\gamma + I_{\alpha_1} + I_{\alpha_2}) - h_{12}$ if and only if one of the following conditions is satisfied:

- $\mathfrak{p}_\gamma + I_{\alpha_1} = \mathfrak{p}_\gamma + I_{\alpha_1} + I_{\alpha_2}$ and $\operatorname{ht} I_{\alpha_1} = \operatorname{ht}(I_{\alpha_1} + I_{\alpha_2}) - 1$
- $\mathfrak{p}_\gamma + I_{\alpha_2} = \mathfrak{p}_\gamma + I_{\alpha_1} + I_{\alpha_2}$ and $\operatorname{ht} I_{\alpha_2} = \operatorname{ht}(I_{\alpha_1} + I_{\alpha_2}) - 1$

For the general case one can check that the characteristic cycle of the corresponding modules Z_p will have a similar behavior. The precise statement will be proved in Theorem 3.2.11 but for the moment we point out the following:

Remark 4.2.6. Let $I = I_{\alpha_1} \cap \cdots \cap I_{\alpha_m}$ be the minimal primary decomposition of a squarefree monomial ideal. The characteristic cycle of the modules Z_p is described in terms of the initial pieces corresponding to sums of face ideals of the form:

$$\mathfrak{p}_\gamma + I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} = \mathfrak{p}_\gamma + I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} + I_{\alpha_{i_{j+1}}}$$

that satisfy

$$\operatorname{ht}(I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}}) + 1 = \operatorname{ht}(I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} + I_{\alpha_{i_{j+1}}}), \quad 1 \leq i_1 < \cdots < i_{j+1} \leq m.$$

Now we want to determine which initial pieces determine precisely the characteristic cycle of the local cohomology modules $H^p_{\mathfrak{p}_\gamma}(H^r_I(R))$. To shed some light at this point we will study in more detail the case $m = 2$.

The case m=2: Let $I = I_{\alpha_1} \cap I_{\alpha_2}$ be the minimal primary decomposition of a squarefree monomial ideal. We denote $h_i := \operatorname{ht} I_{\alpha_i}$ $i = 1, 2$ and $h_{12} := \operatorname{ht}(I_{\alpha_1} + I_{\alpha_2})$ and we suppose $h_1 \leq h_2$ then we get the following cases:

1) If $h_1 < h_2 < h_{12} - 1$ then:

$$CC(H^p_{\mathfrak{p}_\gamma}(H^{h_1}_I(R))) = CC(H^{p+h_1}_{\mathfrak{p}_\gamma + I_{\alpha_1}}(R)),$$

$$CC(H^p_{\mathfrak{p}_\gamma}(H^{h_2}_I(R))) = CC(H^{p+h_2}_{\mathfrak{p}_\gamma + I_{\alpha_2}}(R)) \quad \text{and}$$

$$CC(H^p_{\mathfrak{p}_\gamma}(H^{h_{12}-1}_I(R))) = CC(H^{p+h_{12}}_{\mathfrak{p}_\gamma + I_{\alpha_1} + I_{\alpha_2}}(R)).$$

In this case the double process is optimal, i.e. all the modules Z_p that appear in the process vanish.

2) If $h_1 = h_2 < h_{12} - 1$ then:

$$CC(H^p_{\mathfrak{p}_\gamma}(H^{h_1}_I(R))) = CC(H^{p+h_1}_{\mathfrak{p}_\gamma + I_{\alpha_1}}(R)) \oplus CC(H^{p+h_2}_{\mathfrak{p}_\gamma + I_{\alpha_2}}(R)) \quad \text{and}$$

$$CC(H^p_{\mathfrak{p}_\gamma}(H^{h_{12}-1}_I(R))) = CC(H^{p+h_{12}}_{\mathfrak{p}_\gamma + I_{\alpha_1} + I_{\alpha_2}}(R)).$$

In this case the double process is optimal, i.e. all the modules Z_p that appear in the process vanish.

3) If $h_1 < h_2 = h_{12} - 1$ then:

$$CC(H^p_{\mathfrak{p}_\gamma}(H^{h_1}_I(R))) = CC(H^{p+h_1}_{\mathfrak{p}_\gamma + I_{\alpha_1}}(R))$$

and

i) If $\mathfrak{p}_\gamma + I_{\alpha_2} \neq \mathfrak{p}_\gamma + I_{\alpha_1} + I_{\alpha_2}$ then:

$$CC(H^p_{\mathfrak{p}_\gamma}(H^{h_2}_I(R)) = CC(H^{p+h_2}_{\mathfrak{p}_\gamma + I_{\alpha_2}}(R)) + CC(H^{p+h_{12}}_{\mathfrak{p}_\gamma + I_{\alpha_1} + I_{\alpha_2}}(R)).$$

In this case the double process is optimal, i.e. all the modules Z_p that appear in the process vanish.

ii) If $\mathfrak{p}_\gamma + I_{\alpha_2} = \mathfrak{p}_\gamma + I_{\alpha_1} + I_{\alpha_2}$ then:

$CC(H^p_{\mathfrak{p}_\gamma}(H^{h_2}_I(R)) = 0 \ \forall p$

In this case the double process is not optimal, in particular we have $Z_{p-1} = H^{p-1+h_{12}}_{\mathfrak{p}_\gamma+I_{\alpha_1}+I_{\alpha_2}}(R) \neq 0$ for $p-1 = \operatorname{ht}(\mathfrak{p}_\gamma + I_{\alpha_1} + I_{\alpha_2})$.

In terms of initial pieces, we state that there is *extra information* that comes from the pieces

$$H^{p+h_2}_{\mathfrak{p}_\gamma+I_{\alpha_2}}(R) \quad \text{and} \quad H^{p-1+h_{12}}_{\mathfrak{p}_\gamma+I_{\alpha_1}+I_{\alpha_2}}(R).$$

So we have to remove the information coming from the initial pieces corresponding to the ideals $\mathfrak{p}_\gamma + I_{\alpha_2}$ and $\mathfrak{p}_\gamma + I_{\alpha_1} + I_{\alpha_2}$ in order to get the correct result. We will say that the ideals I_{α_2} and $I_{\alpha_1} + I_{\alpha_2}$ have been **canceled**.

4) If $h_1 = h_2 = h_{12} - 1$ then:

$CC(H^p_{\mathfrak{p}_\gamma}(H^{h_1}_I(R))) = CC(H^{p+h_1}_{\mathfrak{p}_\gamma+I}(R)).$

If we want to follow closely the double process we have to consider the following cases:

i) $\mathfrak{p}_\gamma + I_{\alpha_1} \neq \mathfrak{p}_\gamma + I_{\alpha_1} + I_{\alpha_2} \quad$ and $\quad \mathfrak{p}_\gamma + I_{\alpha_2} \neq \mathfrak{p}_\gamma + I_{\alpha_1} + I_{\alpha_2}$.

Let $\operatorname{ht}(\mathfrak{p}_\gamma + I_{\alpha_1}) = p + h_1$. Notice that $\operatorname{ht}(\mathfrak{p}_\gamma + I_{\alpha_2}) = p + h_2$ and $\operatorname{ht}(\mathfrak{p}_\gamma + I_{\alpha_1} + I_{\alpha_2}) = p + h_{12}$ so we get the following exact sequence

$$0 \longrightarrow H^p_{\mathfrak{p}_\gamma}(H^{h_1}_{I_{\alpha_1}}(R) \oplus H^{h_2}_{I_{\alpha_2}}(R)) \longrightarrow H^p_{\mathfrak{p}_\gamma}(H^{h_1}_I(R)) \longrightarrow H^p_{\mathfrak{p}_\gamma}(H^{h_{12}}_{I_{\alpha_1}+I_{\alpha_2}}(R)) \longrightarrow 0.$$

In particular:

$$CC(H^p_{\mathfrak{p}_\gamma}(H^{h_1}_I(R))) = CC(H^{p+h_1}_{\mathfrak{p}_\gamma+I_{\alpha_1}}(R)) + CC(H^{p+h_2}_{\mathfrak{p}_\gamma+I_{\alpha_2}}(R))) + \\ + CC(H^{p+h_{12}}_{\mathfrak{p}_\gamma+I_{\alpha_1}+I_{\alpha_2}}(R)))$$

In this case the double process is optimal, i.e. all the modules Z_p that appear in the process vanish.

ii) $\mathfrak{p}_\gamma + I_{\alpha_1} \neq \mathfrak{p}_\gamma + I_{\alpha_1} + I_{\alpha_2}$ and $\mathfrak{p}_\gamma + I_{\alpha_2} = \mathfrak{p}_\gamma + I_{\alpha_1} + I_{\alpha_2}$.

Let $\mathrm{ht}\,(\mathfrak{p}_\gamma + I_{\alpha_1}) = p+h_1$. Notice that $\mathfrak{p}_\gamma = \mathfrak{p}_\gamma + I_{\alpha_1} \subsetneq \mathfrak{p}_\gamma + I_{\alpha_2} = \mathfrak{p}_\gamma + I_{\alpha_1} + I_{\alpha_2}$. In particular $\mathrm{ht}\,(\mathfrak{p}_\gamma + I_{\alpha_2}) = p+1+h_2$ and $\mathrm{ht}\,(\mathfrak{p}_\gamma + I_{\alpha_1} + I_{\alpha_2}) = p+h_{12}$, so we get the exact sequences

$$0 \longrightarrow H^p_{\mathfrak{p}_\gamma}(H^{h_1}_{I_{\alpha_1}}(R)) \longrightarrow H^p_{\mathfrak{p}_\gamma}(H^{h_1}_{I}(R)) \longrightarrow H^p_{\mathfrak{p}_\gamma}(H^{h_{12}}_{I_{\alpha_1}+I_{\alpha_2}}(R)) \longrightarrow Z_p \longrightarrow 0$$

$$\text{and} \qquad 0 \longrightarrow Z_p \longrightarrow H^{p+1}_{\mathfrak{p}_\gamma}(H^{h_2}_{I_{\alpha_2}}(R)) \longrightarrow H^{p+1}_{\mathfrak{p}_\gamma}(H^{h_1}_{I}(R)) \longrightarrow 0.$$

By using the degeneration of the spectral sequence and the fact that $\mathfrak{p}_\gamma + I = \mathfrak{p}_\gamma$ we get $Z_p = H^p_{\mathfrak{p}_\gamma}(H^{h_{12}}_{I_{\alpha_1}+I_{\alpha_2}}(R))$ so:

$$CC(H^p_{\mathfrak{p}_\gamma}(H^{h_1}_I(R))) = CC(H^{p+h_1}_{\mathfrak{p}_\gamma + I_{\alpha_1}}(R)).$$

In terms of the initial pieces we state that the *extra information* comes from the pieces

$$H^{p+h_2}_{\mathfrak{p}_\gamma + I_{\alpha_2}}(R) \quad \text{and} \quad H^{p-1+h_{12}}_{\mathfrak{p}_\gamma + I_{\alpha_1} + I_{\alpha_2}}(R).$$

So we have to **cancel** the ideals I_{α_2} and $I_{\alpha_1} + I_{\alpha_2}$ in order to get the correct result.

iii) $\mathfrak{p}_\gamma + I_{\alpha_1} = \mathfrak{p}_\gamma + I_{\alpha_1} + I_{\alpha_2}$ and $\mathfrak{p}_\gamma + I_{\alpha_2} \neq \mathfrak{p}_\gamma + I_{\alpha_1} + I_{\alpha_2}$.

This case is analogous to the previous one. We get

$$CC(H^p_{\mathfrak{p}_\gamma}(H^{h_1}_I(R))) = CC(H^{p+h_2}_{\mathfrak{p}_\gamma + I_{\alpha_2}}(R)).$$

In terms of the initial pieces we state that the *extra information* comes from the pieces

$$H^{p+h_1}_{\mathfrak{p}_\gamma + I_{\alpha_1}}(R) \quad \text{and} \quad H^{p-1+h_{12}}_{\mathfrak{p}_\gamma + I_{\alpha_1} + I_{\alpha_2}}(R).$$

So we have to **cancel** the ideals I_{α_1} and $I_{\alpha_1} + I_{\alpha_2}$ in order to get the correct result.

iv) $\mathfrak{p}_\gamma + I_{\alpha_1} = \mathfrak{p}_\gamma + I_{\alpha_1} + I_{\alpha_2}$ and $\mathfrak{p}_\gamma + I_{\alpha_2} = \mathfrak{p}_\gamma + I_{\alpha_1} + I_{\alpha_2}$.

Let $\mathrm{ht}\,(\mathfrak{p}_\gamma + I_{\alpha_1}) = p + h_1$. Notice that $\mathfrak{p}_\gamma = \mathfrak{p}_\gamma + I_{\alpha_1} = \mathfrak{p}_\gamma + I_{\alpha_2} = \mathfrak{p}_\gamma + I_{\alpha_1} + I_{\alpha_2}$ so we get the exact sequences

$$0 \longrightarrow H^{p-1}_{\mathfrak{p}_\gamma}(H^{h_1}_I(R)) \longrightarrow H^{p-1}_{\mathfrak{p}_\gamma}(H^{h_{12}}_{I_{\alpha_1}+I_{\alpha_2}}(R)) \longrightarrow Z_{p-1} \longrightarrow 0$$

and

$$0 \longrightarrow Z_{p-1} \longrightarrow H^{p}_{\mathfrak{p}_\gamma}(H^{h_1}_{I_{\alpha_1}}(R) \oplus H^{h_2}_{I_{\alpha_2}}(R)) \longrightarrow H^{p}_{\mathfrak{p}_\gamma}(H^{h_1}_I(R)) \longrightarrow 0.$$

By using the degeneration of the spectral sequence we get

$$H^{p}_{\mathfrak{p}_\gamma}(H^{h_1}_I(R)) = H^{p+h_1}_{\mathfrak{p}_\gamma}(R)),$$

so $Z_{p-1} = H^{p-1}_{\mathfrak{p}_\gamma}(H^{h_{12}}_{I_{\alpha_1}+I_{\alpha_2}}(R))$. To make this sequence minimal we will consider $H^{p}_{\mathfrak{p}_\gamma}(H^{h_1}_I(R)) = H^{p+h_1}_{\mathfrak{p}_\gamma+I_{\alpha_1}}(R)$ so we get

$CC(H^{p}_{\mathfrak{p}_\gamma}(H^{h_1}_I(R))) = CC(H^{p+h_1}_{\mathfrak{p}_\gamma+I_{\alpha_1}}(R))$.

In terms of the initial pieces we state that the *extra information* comes from the pieces

$$H^{p+h_2}_{\mathfrak{p}_\gamma+I_{\alpha_2}}(R) \quad \text{and} \quad H^{p-1+h_{12}}_{\mathfrak{p}_\gamma+I_{\alpha_1}+I_{\alpha_2}}(R).$$

So we have to **cancel** the ideals I_{α_2} and $I_{\alpha_1} + I_{\alpha_2}$ in order to get the correct result.

Remark 4.2.7. Once the minimal primary decomposition of the squarefree monomial ideal I has been fixed, **the cancellation order** is given in a natural way by the double process.

In the case 4) iv) stated above, if we consider the minimal primary decomposition $I = I_{\alpha_2} \cap I_{\alpha_1}$ of the ideal I instead of $I = I_{\alpha_1} \cap I_{\alpha_2}$ then we cancellate the initial pieces corresponding to the ideals I_{α_1} and $I_{\alpha_1} + I_{\alpha_2}$ in order to get the correct result. Notice that we get the same result, i.e. the characteristic cycle of $H^{p}_{\mathfrak{p}_\gamma}(H^{h_1}_I(R))$ can be obtained using the piece $H^{p+h_2}_{\mathfrak{p}_\gamma+I_{\alpha_2}}(R) = H^{p+h_1}_{\mathfrak{p}_\gamma+I_{\alpha_1}}(R)$.

Algorithm for canceling initial pieces

Let $I = I_{\alpha_1} \cap \cdots \cap I_{\alpha_m}$ be the minimal primary decomposition of a squarefree monomial ideal $I \subseteq R$. By the double process we have seen that the initial pieces that allows us to compute the characteristic cycle of $H^p_{\mathfrak{p}_\gamma}(H^r_I(R))$ are the local cohomology modules $H^p_{\mathfrak{p}_\gamma}(H^r_{I_{\alpha_{i_1}}+\cdots+I_{\alpha_{i_j}}}(R)) = H^{p+r}_{\mathfrak{p}_\gamma+I_{\alpha_{i_1}}+\cdots+I_{\alpha_{i_j}}}(R)$, where the face ideals $I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}}$, $1 \leq i_1 < \cdots < i_{j+1} \leq m$, are included in the sets $\mathcal{P}_j$.

In these sets there is extra information that we must remove. For this purpose we define for every face ideal $I_\alpha \subseteq R$ the subsets

$$\mathcal{P}_{\gamma,j,\alpha} := \{I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} \in \mathcal{P}_j \mid \mathfrak{p}_\gamma + I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} = I_\alpha\}$$

and we introduce the following:

Definition 4.2.8. *We say that* $I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} \in \mathcal{P}_{\gamma,j,\alpha}$ *and* $I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} + I_{\alpha_{i_{j+1}}} \in \mathcal{P}_{\gamma,j+1,\alpha}$ *are almost paired if* $\operatorname{ht}(I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}}) + 1 = \operatorname{ht} I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} + I_{\alpha_{i_{j+1}}}$*, i.e., there is a generator of* $I_{\alpha_{i_{j+1}}}$ *which is not a generator of* $I_{\alpha_{i_k}}$ *for all* $k = 1, \ldots, j$.

The extra information comes from these almost paired ideals (see Remark 4.2.6), so the formula for the characteristic cycle of $H^p_{\mathfrak{p}_\gamma}(H^r_I(R))$ is expressed in terms of the characteristic cycle of local cohomology modules supported on face ideals that are obtained as the sum of $\mathfrak{p}_\gamma$ and a face ideal contained in a subset of $\mathcal{P}$ obtained by canceling all possible almost pairs, with the order given by the double process.

In order to compute the initial pieces that describe precisely the the characteristic cycle of the local cohomology modules $H^p_{\mathfrak{p}_\gamma}(H^r_I(R))$ we can reflect all the information in the diagram of I. Namely, we have to cancel almost paired ideals in the subdiagram of the diagram of I formed by the non paired ideals, with the order given by the process.

General case: Let $I = I_{\alpha_1} \cap \cdots \cap I_{\alpha_m}$ be the minimal primary decomposition of a squarefree monomial ideal. To compute $CC(H^p_{\mathfrak{p}_\gamma}(H^r_I(R)))$ we consider the sets of ideals that give the initial pieces $\mathcal{P}_{\gamma,\alpha} = \{\mathcal{P}_{\gamma,1,\alpha}, \mathcal{P}_{\gamma,2,\alpha}, \ldots, \mathcal{P}_{\gamma,m,\alpha}\}$ $\forall\alpha$. We use the double process to organize this information in **the subdiagram of** I. Then, the cancellation order is given by the same algorithm as in Subsection 3.2.3:

Algorithm :

Let m be the number of ideals in the minimal primary decomposition $I = I_{\alpha_1} \cap \cdots \cap I_{\alpha_m}$.

- For j from 1 to $m-1$, incrementing by 1
  - For all $k \leq j$ **COMPARE** the ideals
    $I_{\alpha_{j+1}} + (I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_r}})$ and $I_{\alpha_k} + I_{\alpha_{j+1}} + (I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_r}})$
  - For all $k \leq j$ **COMPARE** the ideals
    $I_{\alpha_k} + (I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_r}})$ and $I_{\alpha_k} + I_{\alpha_{j+1}} + (I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_r}})$

But now we consider:

INPUT: The sets $\mathcal{P}_{\gamma,\alpha}$ of all the non paired sums of face ideals obtained as a sum of face ideals in the minimal primary decomposition of I such that they give the face ideal I_α when they are added to the ideal $\mathfrak{p}_\gamma$.

We apply the algorithm where **COMPARE** means remove both ideals in case they are almost paired.

OUTPUT: The sets $\mathcal{Q}_{\gamma,\alpha}$ of all the non paired and non almost paired sums of face ideals obtained as a sum of face ideals in the minimal primary decomposition of I such that they give the face ideal I_α when they are added to the ideal $\mathfrak{p}_\gamma$.

We order $\mathcal{Q}_{\gamma,\alpha}$ by the number of summands $\mathcal{Q}_{\gamma,\alpha} = \{\mathcal{Q}_{\gamma,1,\alpha}, \mathcal{Q}_{\gamma,2,\alpha}, \ldots, \mathcal{Q}_{\gamma,m,\alpha}\}$, in such a way that no sum in $\mathcal{Q}_{\gamma,j,\alpha}$ is almost paired with a sum in $\mathcal{Q}_{\gamma,j+1,\alpha}$. Observe that some of these sets can be empty. Finally we define the sets of non paired and non almost paired sums of face ideals with a given height

$$\mathcal{Q}_{\gamma,j,r,\alpha} := \{I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} \in \mathcal{Q}_{\gamma,j,\alpha} \mid \mathrm{ht}\,(I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}}) = r + (j-1)\}.$$

The formula we will give in Theorem 4.2.9 will be expressed in terms of these sets of non paired and non almost paired sums of face ideals.

4.2.4 Main result

The main result of this chapter is a closed formula that describes the characteristic cycle of the local cohomology modules $H^p_{\mathfrak{p}_\gamma}(H^r_I(R))$, where $I \subseteq R$ is any squarefree monomial ideal and $\mathfrak{p}_\gamma \subseteq R$ is a face ideal, in terms of the sets $\mathcal{Q}_{\gamma,j,r,\alpha}$ of face ideals in the minimal primary decomposition of the ideal. In particular we will give a complete description of the characteristic cycle of the modules Z_p that appear in the double process and we will determine which initial pieces describe precisely the characteristic cycle of these local cohomology modules.

Theorem 4.2.9. *Let $I = I_{\alpha_1} \cap \cdots \cap I_{\alpha_m}$ be the minimal primary decomposition of a squarefree monomial ideal $I \subseteq R$ and $\mathfrak{p}_\gamma \subseteq R$ a face ideal. Then:*

$$CC(H^p_{\mathfrak{p}_\gamma}(H^r_I(R))) = \sum_{\alpha \in \{0,1\}^n} \lambda_{\gamma,p,n-r,\alpha} \; CC(H^{|\alpha|}_{I_\alpha}(R)),$$

where $\lambda_{\gamma,p,n-r,\alpha} = \sharp \, \mathcal{Q}_{\gamma,j,r,\alpha}$ such that $|\alpha| = p + (r + (j-1))$.

The following remark will be very useful for the proof of the theorem.

Remark 4.2.10. From the formula it is easy to see the following:

- If $CC(H^p_{\mathfrak{p}_\gamma}(H^{r+(j-1)}_{I_{\alpha_{i_1}}+\cdots+I_{\alpha_{i_j}}}(R))) \in CC(H^p_{\mathfrak{p}_\gamma}(H^r_I(R)))$, then

$$CC(H^p_{\mathfrak{p}_\gamma}(H^{r+(j-1)}_{I_{\alpha_{i_1}}+\cdots+I_{\alpha_{i_j}}}(R))) \notin CC(H^q_{\mathfrak{p}_\gamma}(H^r_I(R))), \;\; \forall q \neq p.$$

PROOF: We are going to use similar ideas as in the proof of Theorem 3.2.11 and we will use the same notation. We proceed by induction on m, the number of ideals in the minimal primary decomposition, being the case $m = 1$ trivial.

We start the Mayer-Vietoris process with the sequence:

$$\cdots \longrightarrow H^r_{U+V}(R) \longrightarrow H^r_U(R) \oplus H^r_V(R) \longrightarrow H^r_{U\cap V}(R) \longrightarrow H^{r+1}_{U+V}(R) \longrightarrow \cdots,$$

where

$$U = I_{\alpha_1} \cap \cdots \cap I_{\alpha_{m-1}}, \qquad U \cap V = I = I_{\alpha_1} \cap \cdots \cap I_{\alpha_m},$$

$$V = I_{\alpha_m}, \qquad U + V = (I_{\alpha_1} \cap \cdots \cap I_{\alpha_{m-1}}) + I_{\alpha_m}.$$

Then we have:

$$0 \longrightarrow C_r \longrightarrow H^r_{U\cap V}(R) \longrightarrow A_{r+1} \longrightarrow 0.$$

Applying the long exact sequence of local cohomology we get:

$$\cdots \longrightarrow H^p_{\mathfrak{p}_\gamma}(C_r) \longrightarrow H^p_{\mathfrak{p}_\gamma}(H^r_{U\cap V}(R)) \longrightarrow H^p_{\mathfrak{p}_\gamma}(A_{r+1}) \longrightarrow H^{p+1}_{\mathfrak{p}_\gamma}(C_r) \longrightarrow \cdots .$$

Assume we have proved the formula for ideals with less terms than m in the minimal primary decomposition. This allows to compute the characteristic cycles of $H^p_{\mathfrak{p}_\gamma}(C_r)$ and $H^p_{\mathfrak{p}_\gamma}(A_{r+1})$.

To describe $CC(H^p_{\mathfrak{p}_\gamma}(C_r))$ we will denote by $\mathcal{P}(C)$ the set of face ideals that give the initial pieces that allows us to compute the characteristic cycle of C_r. By using Remark 4.2.1 we state that this set is formed by the vertex, if it is necessary, and the ideals in the upper half of the subdiagram of I.

Applying the algorithm of cancellation to $\mathcal{P}(C)$ we obtain the poset $\mathcal{Q}(C) := \{\mathcal{Q}_1(C), \dots, \mathcal{Q}_{m-1}(C)\}$. Ordering the ideals by heights and checking the sums with the ideal $\mathfrak{p}_\gamma$, we obtain the sets

$$\mathcal{Q}_{\gamma,j,r,\alpha}(C) := \{I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} \in \mathcal{Q}_{\gamma,j,\alpha}(C) \mid \operatorname{ht}(I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}}) = r + (j-1)\}.$$

By induction we have:

$$CC(H^p_{\mathfrak{p}_\gamma}(C_r)) = \sum_{\alpha\in\{0,1\}^n} \lambda_{\gamma,p,n-r,\alpha}(C)\ CC(H^{|\alpha|}_{I_\alpha}(R)),$$

where $\lambda_{\gamma,p,n-r,\alpha}(C) = \sharp\ \mathcal{Q}_{\gamma,j,r,\alpha}(C)$ such that $|\alpha| = p + (r + (j-1))$.

To describe $CC(H^p_{\mathfrak{p}_\gamma}(A_{r+1}))$ we will denote by $\mathcal{P}(A)$ the set of face ideals that give the initial pieces that allows us to compute the characteristic cycle of A_{r+1}. By using Remark 4.2.1 we state that this set is formed by the ideals in the lower half of the subdiagram of I.

Applying the algorithm of cancellation to $\mathcal{P}(A)$ we obtain the poset $\mathcal{Q}(A) := \{\mathcal{Q}_1(A), \dots, \mathcal{Q}_{m-1}(A)\}$. Ordering the ideals by heights and checking the sums with the ideal $\mathfrak{p}_\gamma$, we obtain the sets

$$\mathcal{Q}_{\gamma,j,r+1,\alpha}(A) := \{I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}+I_{\alpha_m}} \in \mathcal{Q}_{\gamma,j,\alpha}(A) \mid$$
$$\mid \operatorname{ht}(I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} + I_{\alpha_m}) = r + 1 + (j-1)\}.$$

By induction we have:

$$CC(H^p_{\mathfrak{p}_\gamma}(A_{r+1})) = \sum_{\alpha\in\{0,1\}^n} \lambda_{\gamma,p,n-(r+1),\alpha}(A)\ \ CC(H^{|\alpha|}_{I_\alpha}(R)),$$

where $\lambda_{\gamma,p,n-(r+1),\alpha}(A) = \ \sharp\ \mathcal{Q}_{\gamma,j,r+1,\alpha}(A)$ such that $|\alpha| = p + (r + 1 + (j-1))$.

Now we split the long exact sequence into short exact sequences of kernels and cokernels:

$$0 \longrightarrow Z_{p-1} \longrightarrow H^p_{\mathfrak{p}_\gamma}(C_r) \longrightarrow X_p \longrightarrow 0$$

$$0 \longrightarrow X_p \longrightarrow H^p_{\mathfrak{p}_\gamma}(H^r_{U\cap V}(R)) \longrightarrow Y_p \longrightarrow 0$$

$$0 \longrightarrow Y_p \longrightarrow H^p_{\mathfrak{p}_\gamma}(A_{r+1}) \longrightarrow Z_p \longrightarrow 0.$$

So, by additivity we have:

$$CC(H^p_{\mathfrak{p}_\gamma}(H^r_{U\cap V}(R))) = CC(X_p) + CC(Y_p) =$$

$$= (CC(H^p_{\mathfrak{p}_\gamma}(C_r)) - CC(Z_{p-1})) + (CC(H^p_{\mathfrak{p}_\gamma}(A_{r+1})) - CC(Z_p)).$$

By induction we have computed the characteristic cycle of the local cohomology modules $H^p_{\mathfrak{p}_\gamma}(C_r)$ and $H^p_{\mathfrak{p}_\gamma}(A_{r+1})$ so we only have to describe $CC(Z_p)$ $\forall p$.

Notice that, during the process of computation of $H^p_{\mathfrak{p}_\gamma}(C_r)$, we have canceled the almost pairs coming from the upper half of the diagram of I. On the other side, in order to compute $H^p_{\mathfrak{p}_\gamma}(A_{r+1})$, we have canceled the almost pairs coming from the lower half of the diagram.

To get the desired formula, it remains to cancel the possible almost pairs formed by ideals $I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} \in \mathcal{Q}_{\gamma,j,r,\alpha}(C)$, i.e those coming from the computation of $CC(H^p_{\mathfrak{p}_\gamma}(C_r))$ and $I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} + I_{\alpha_m} \in \mathcal{Q}_{\gamma,j,r+1,\alpha}(A)$, coming from the computation of $CC(H^p_{\mathfrak{p}_\gamma}(A_{r+1}))$.

It also remains to cancel the possible almost pair formed by the ideal $I_{\alpha_m} \in \mathcal{Q}_{\gamma,1,r,\alpha}(C)$ coming from the computation of the modules $CC(H^p_{\mathfrak{p}_\gamma}(C_r))$ and some $I_{\alpha_i} + I_{\alpha_m} \in \mathcal{Q}_{\gamma,1,r+1,\alpha}(A)$, coming from the computation of the modules $CC(H^p_{\mathfrak{p}_\gamma}(A_{r+1}))$.

So, we only have to prove that the initial pieces coming from these almost paired ideals describe the modules $CC(Z_p)$ $\forall p$. It will be done by means of the following:

Claim:

$$CC(Z_{p-1}) = \sum CC(H^{|\alpha|}_{I_\alpha}(R)),$$

where the sum is taken over the cycles that come from almost pairs of the form

- $I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} \in \mathcal{Q}_{\gamma,j,r,\alpha}(C)$ and $I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} + I_{\alpha_m} \in \mathcal{Q}_{\gamma,j,r+1,\alpha}(A)$,
  where $I_\alpha = \mathfrak{p}_\gamma + I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} = \mathfrak{p}_\gamma + I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} + I_{\alpha_m}$,
  and $|\alpha| = p + (r + (j-1))$.
- $I_{\alpha_m} \in \mathcal{Q}_{1,r,\alpha}(C)$ and $I_{\alpha_{i_1}} + I_{\alpha_m} \in \mathcal{Q}_{1,r+1,\alpha}(A)$,
  where $I_\alpha = \mathfrak{p}_\gamma + I_{\alpha_m} = \mathfrak{p}_\gamma + I_{\alpha_{i_1}} + I_{\alpha_m}$, and $|\alpha| = p + r$.

The inclusion $\subseteq$ is obvious because $CC(Z_{p-1})$ belongs to $CC(H^{p-1}_{\mathfrak{p}_\gamma}(A_{r+1}))$ and $CC(H^p_{\mathfrak{p}_\gamma}(C_r))$. To prove the other one let $T^*_{X_\alpha}X = CC(H^{|\alpha|}_{I_\alpha}(R))$ be a cycle that come from an almost pair and suppose that does not belong to $CC(Z_{p-1})$. Consider a Mayer-Vietoris sequence

$$\cdots \longrightarrow H^r_{U'+V'}(R) \longrightarrow H^r_{U'}(R) \oplus H^r_{V'}(R) \longrightarrow H^r_{U'\cap V'}(R) \longrightarrow H^{r+1}_{U'+V'}(R) \longrightarrow \cdots,$$

where:

- If $T^*_{X_\alpha}X = CC(H^p_{\mathfrak{p}_\gamma}(H^{r+(j-1)}_{I_{\alpha_{i_1}}+\cdots+I_{\alpha_{i_j}}}(R))) = CC(H^{p-1}_{\mathfrak{p}_\gamma}(H^{r+1+(j-1)}_{I_{\alpha_{i_1}}+\cdots+I_{\alpha_{i_j}}+I_{\alpha_m}}(R)))$ belongs to $CC(H^p_{\mathfrak{p}_\gamma}(C_r))$ and $CC(H^{p-1}_{\mathfrak{p}_\gamma}(A_{r+1}))$ we choose:

$$U' = I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}}, \qquad U' \cap V' = I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} \cap I_{\alpha_m},$$
$$V' = I_{\alpha_m}, \qquad U' + V' = I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} + I_{\alpha_m}.$$

• If $T^*_{X_\alpha}X = CC(H^p_{\mathfrak{p}_\gamma}(H^r_{I_{\alpha_m}}(R))) = CC(H^{p-1}_{\mathfrak{p}_\gamma}(H^{r+1}_{I_{\alpha_{i_1}}+I_{\alpha_m}}(R)))$ belongs to $CC(H^p_{\mathfrak{p}_\gamma}(C_r))$ and $CC(H^{p-1}_{\mathfrak{p}_\gamma}(A_{r+1}))$ we choose:

$$U' = I_{\alpha_i}, \qquad U' \cap V' = I_{\alpha_i} \cap I_{\alpha_m},$$
$$V' = I_{\alpha_m}, \qquad U' + V' = I_{\alpha_i} + I_{\alpha_m}.$$

Notice that in the corresponding short exact sequences

$$0 \longrightarrow Z'_{p-1} \longrightarrow H^p_{\mathfrak{p}_\gamma}(C'_r) \longrightarrow X'_p \longrightarrow 0$$

$$0 \longrightarrow X'_p \longrightarrow H^p_{\mathfrak{p}_\gamma}(H^r_{U' \cap V'}(R)) \longrightarrow Y'_p \longrightarrow 0$$

$$0 \longrightarrow Y'_p \longrightarrow H^p_{\mathfrak{p}_\gamma}(A'_{r+1}) \longrightarrow Z'_p \longrightarrow 0.$$

we have:

$T^*_{X_\alpha}X \in CC(H^p_{\mathfrak{p}_\gamma}(C'_r))$,

$T^*_{X_\alpha}X \in CC(H^{p-1}_{\mathfrak{p}_\gamma}(A'_{r+1}))$,

$T^*_{X_\alpha}X \notin CC(Z'_{p-1})$.

The last statement is due to the fact that the diagram of the ideal $U' \cap V'$ is a subdiagram of the diagram of I, i.e. the corresponding ideals are contained in the diagram of I (see Remark 3.2.10). In particular, if the cycle $T^*_{X_\alpha}X$ has not been canceled during the computation of $H^p_{\mathfrak{p}_\gamma}(H^r_I(R))$, i.e. the cycle does not belong to $CC(Z_p)$ then, it can not be canceled in the computation of $H^p_{\mathfrak{p}_\gamma}(H^r_{U' \cap V'}(R))$, i.e. the cycle does not belong to any $CC(Z'_p)$ of the corresponding short exact sequences we have considered.

Finally, by induction and using Remark 4.2.10 there is a contradiction due to the fact that in this case we get:

$$T^*_{X_\alpha}X \in CC(H^{p-1}_{\mathfrak{p}_\gamma}(H^r_{U' \cap V'}(R))) \quad \text{and} \quad T^*_{X_\alpha}X \in CC(H^p_{\mathfrak{p}_\gamma}(H^r_{U' \cap V'}(R))).$$

□

Remark 4.2.11. The sets $\mathcal{Q}_{\gamma,j,r,\alpha}$ that describe precisely the characteristic cycle of the local cohomology modules $H^p_{\mathfrak{p}_\gamma}(H^r_I(R))$ are independent of the order given to the minimal primary decomposition of I. This order only changes the way we represent the ideals in $\mathcal{Q}_{\gamma,j,r,\alpha}$ as a sum of j face ideals.

Let $\mathfrak{m} \subseteq R$ be the homogeneous maximal ideal. For any sum of face ideals $I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} \in \mathcal{P}_{j,r}$, we have $\mathfrak{m} = \mathfrak{m} + I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}}$ and so $\mathcal{P}_{j,r} = \mathcal{P}_{\alpha_\mathfrak{m},j,r,\alpha_\mathfrak{m}}$ $\forall j$, $\forall r$. Applying the algorithm of cancellation of almost pairs to these sets and ordering by the number of summands and the height we get the sets $\mathcal{Q}_{j,r} = \mathcal{Q}_{\alpha_\mathfrak{m},j,r,\alpha_\mathfrak{m}}$. As a consequence we have:

Corollary 4.2.12. *Let $I = I_{\alpha_1} \cap \cdots \cap I_{\alpha_m}$ be the minimal primary decomposition of a squarefree monomial ideal $I \subseteq R$ and $\mathfrak{m} \subseteq R$ be the homogeneous maximal ideal. Then:*

$$CC(H^p_\mathfrak{m}(H^r_I(R))) = \lambda_{p,n-r}\ T^*_{X_{\alpha_\mathfrak{m}}}X,$$

where $\lambda_{p,n-r} = \sharp\mathcal{Q}_{j,r}$ such that $n = p + (r + (j-1))$.

Example: Let $R = k[x_1, x_2, x_3, x_4]$. Consider the ideal:

- $I = (x_1, x_2) \cap (x_2, x_3) \cap (x_3, x_4)$.

By applying Theorem 3.2.11 we have:

$$CC(H^2_I(R)) = CC(H^2_{I_{\alpha_1}}(R)) + CC(H^2_{I_{\alpha_2}}(R)) + CC(H^2_{I_{\alpha_3}}(R)) + \\ + CC(H^3_{I_{\alpha_1}+I_{\alpha_2}}(R)) + CC(H^3_{I_{\alpha_2}+I_{\alpha_3}}(R)).$$

We organize the information given by the poset $\mathcal{P}$ in the subdiagram of I:

$$I_{\alpha_3} = (x_3, x_4) \left\{ \begin{array}{l} I_{\alpha_2} = (x_2, x_3) \qquad \left\{ \begin{array}{l} I_{\alpha_1} = (x_1, x_2) \\ I_{\alpha_1} + I_{\alpha_2} = (x_1, x_2, x_3) \end{array} \right. \\ \\ I_{\alpha_2} + I_{\alpha_3} = (x_2, x_3, x_4) \end{array} \right.$$

In order to compute the characteristic cycle of the local cohomology modules $H^p_\mathfrak{m}(H^2_I(R))$, where $\mathfrak{m} = (x_1, x_2, x_3, x_4)$ is the maximal ideal, we have to cancel the almost paired sums of face ideals by using the algorithm.

Step 1:

- We cancellate the ideals $I_{\alpha_2} \neq I_{\alpha_1} + I_{\alpha_2}$.

Step 2:

- We cancellate the ideals $I_{\alpha_3} \neq I_{\alpha_2} + I_{\alpha_3}$.

We only have the set $\mathcal{Q}_{1,2} = \{I_{\alpha_1}\}$ so $\lambda_{2,2} = 1$ and the other multiplicities vanish. Then we have:

$$CC(H^2_{\mathfrak{m}}(H^2_I(R))) = T^*_{X_{(1,1,1,1,1)}}X.$$

4.3 Consequences

Let $CC(H^p_{\mathfrak{p}_\gamma}(H^{n-i}_I(R))) = \sum \lambda_{\gamma,p,i,\alpha}\ T^*_{X_\alpha}X$, be the characteristic cycle of a local cohomology module $H^p_{\mathfrak{p}_\gamma}(H^{n-i}_I(R))$. It provides many information on the modules $H^p_{\mathfrak{p}_\gamma}(H^{n-i}_I(R))$ $H^{n-i}_I(R)$ as well on the ring R/I due to the fact that the multiplicities $\lambda_{\gamma,p,i,\alpha}$ are invariants (see Chapter 2).

In Theorem 4.2.9 we have given a formula for the characteristic cycle of these local cohomology modules supported on monomial ideals, so we will study in this case the consequences of the formula.

The annihilation of the modules $H^p_{\mathfrak{p}_\gamma}(H^{n-i}_I(R))$ as well as a full description of their support can be described in the same way we did for the modules $H^{n-i}_I(R)$. Instead, our effort in this section will be a more detailed study of the Bass numbers of these local cohomology modules, in particular the Lyubeznik numbers. This study will provide a description of the injective dimension, the associated primes and the small support of the local cohomology modules $H^{n-i}_I(R)$.

First we recall that the multiplicities $\lambda_{\gamma,p,i,\alpha}(R/I)$ of the characteristic cycle of $H^p_{\mathfrak{p}_\gamma}(H^{n-i}_I(R))$ are described in terms of the sets $\mathcal{Q}_{\gamma,j,n-i,\alpha}$ of non paired and non almost paired sums of j face ideals obtained in the optimization of the double process. Namely we have:

Proposition 4.3.1. *Let $I \subseteq R$ be a squarefree monomial ideal. Let $\mathcal{Q}$ be the poset of sums of face ideals obtained from the poset $\mathcal{P}$ by means of the algorithm* *of canceling almost paired ideals. Then, we have the following description:*

$$\lambda_{\gamma,p,i,\alpha}(R/I) = \sharp\, \mathcal{Q}_{\gamma,|\alpha|+1-p-(n-i),n-i,\alpha}.$$

Similarly, the multiplicities $\lambda_{p,i}(R/I)$ of the characteristic cycle of the local cohomology module $H^p_{\mathfrak{m}}(H^{n-i}_I(R))$ are described in terms of the sets $\mathcal{Q}_{j,n-i}$:

Corollary 4.3.2. *Let $I \subseteq R$ be a squarefree monomial ideal. Let $\mathcal{Q}$ be the poset of sums of face ideals obtained from the poset $\mathcal{P}$ by means of the algorithm of canceling almost paired ideals. Then, we have the following description:*

$$\lambda_{p,i}(R/I) = \sharp\, \mathcal{Q}_{i-p+1,n-i}.$$

4.3.1 Bass numbers of local cohomology modules

Let $I \subseteq R$ be an ideal generated by squarefree monomials and $\mathfrak{p}_\gamma \subseteq R$ be a face ideal. In Theorem 4.2.9 we have computed the characteristic cycle of $H^p_{\mathfrak{p}_\gamma}(H^{n-i}_I(R))$. Then, by using Proposition 2.2.8, we can compute the Bass numbers of the local cohomology modules $H^{n-i}_I(R)$ with respect to $\mathfrak{p}_\gamma$. For completeness, we recall the precise statement.

Proposition 4.3.3. *Let $I \subseteq R$ be an ideal generated by squarefree monomials and $\mathfrak{p}_\gamma \subseteq R$ be a face ideal. If*

$$CC(H^p_{\mathfrak{p}_\gamma}(H^{n-i}_I(R))) = \sum \lambda_{\gamma,p,i,\alpha}\ T^*_{X_\alpha}X,$$

is the characteristic cycle of the local cohomology module $H^p_{\mathfrak{p}_\gamma}(H^{n-i}_I(R))$ then

$$\mu_p(\mathfrak{p}_\gamma, H^{n-i}_I(R)) = \lambda_{\gamma,p,i,\gamma}.$$

The Bass numbers $\mu_p(\mathfrak{p}, H^{n-i}_I(R))$ of the local cohomology modules $H^{n-i}_I(R)$ with respect to any prime ideal $\mathfrak{p} \subseteq R$, can be computed by means of the following remark:

Remark 4.3.4. For any prime ideal $\mathfrak{p} \subseteq R$, $\mathfrak{p}^*$ denote the largest face ideal contained in $\mathfrak{p}$. Let d be the Krull dimension of $R_{\mathfrak{p}}/\mathfrak{p}^* R_{\mathfrak{p}}$. Then, by using [33, Theorem 1.2.3], we have:

$$\mu_p(\mathfrak{p}, H^{n-i}_I(R)) = \mu_{p-d}(\mathfrak{p}^*, H^{n-i}_I(R)),\ \ \forall p.$$

In order to compute Lyubeznik numbers we have to consider the characteristic cycle of $H^p_{\mathfrak{m}}(H^{n-i}_I(R))$. Namely, we have:

Corollary 4.3.5. *Let $I \subseteq R$ be an ideal generated by squarefree monomials and $\mathfrak{m} \subseteq R$ be the homogeneous maximal ideal. If*

$$CC(H^p_{\mathfrak{m}}(H^{n-i}_I(R))) = \lambda_{p,i}\ T^*_{X_{\alpha_{\mathfrak{m}}}}X,$$

is the characteristic cycle of the local cohomology module $H^p_{\mathfrak{m}}(H^{n-i}_I(R))$ then

$$\mu_p(\mathfrak{m}, H^{n-i}_I(R)) = \lambda_{p,i}.$$

Remark 4.3.6. G. Lyubeznik asked [55, Question 4.5] if $\lambda_{d,d}(R/I) = 1$ for all R/I where $d = \dim R/I$. U. Walther [95] gave a negative answer when $d = 2$. We may give counterexamples for any dimension d as follows:

Let $I = I_{\alpha_1} \cap \cdots \cap I_{\alpha_m}$ be the minimal primary decomposition of a squarefree monomial ideal with disjoint faces $I \subseteq R$ such that $\operatorname{ht} I \geq 2$. Then all the sum of face ideals are non paired and non almost paired, so $\lambda_{d,d}(R/I)$ is the number of face ideals in the minimal primary descomposition of height $n - d$.

Example: Let $R = k[x_1, x_2, x_3, x_4, x_5, x_6, x_7]$. Consider the ideal:

- $I = (x_1, x_2) \cap (x_3, x_4) \cap (x_5, x_6, x_7)$.

By using Corollary 4.2.12 we compute the characteristic cycle of $H^p_{\mathfrak{m}}(H^r_I(R))$ and we get the type of R/I, i.e. the triangular matrix:

$$\Lambda(R/I) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 2 & 0 & 0 \\ & & & 0 & 1 & 0 \\ & & & & 1 & 0 \\ & & & & & 2 \end{pmatrix}$$

4.3.2 Bass numbers and restriction

Let $\widehat{R_{\mathfrak{p}}}$ be the completion with respect to the maximal ideal of the localization of R to a prime ideal $\mathfrak{p} \subseteq R$. Then we have:

$$\mu_p(\mathfrak{p}, H^{n-i}_I(R)) = \mu_p(\mathfrak{p}\widehat{R_{\mathfrak{p}}}, H^{n-i}_{I\widehat{R_{\mathfrak{p}}}}(\widehat{R_{\mathfrak{p}}})).$$

Let $\mathfrak{p} = \mathfrak{p}_\gamma$ be a face ideal. Due to the flatness of the morphism $R_{[\mathfrak{p}_\gamma]} \longrightarrow \widehat{R_{\mathfrak{p}_\gamma}}$, where $R_{[\mathfrak{p}_\gamma]}$ is the restriction of R to $\mathfrak{p}_\gamma$, and the invariance of the multiplicities of the characteristic cycle with respect to flat base change, we also have:

$$\mu_p(\mathfrak{p}_\gamma, H_I^{n-i}(R)) = \mu_p(\mathfrak{p}_{\gamma_{[\mathfrak{p}_\gamma]}}, H^{n-i}_{I_{[\mathfrak{p}_\gamma]}}(R_{[\mathfrak{p}_\gamma]})).$$

In particular, the characteristic cycles $CC(H^p_{\mathfrak{p}_{\gamma_{[\mathfrak{p}_\gamma]}}}(H^{n-i}_{I_{[\mathfrak{p}_\gamma]}}(R_{[\mathfrak{p}_\gamma]})))$ allow us to compute these Bass numbers. Notice that in this way we reduce the number of computations because we are not considering all the invariants $\lambda_{\gamma,p,i,\alpha}$ given by the characteristic cycle of $H^p_{\mathfrak{p}_\gamma}(H_I^{n-i}(R))$.

As an easy application of this restriction procedure we will calculate the Bass numbers of $H_I^{n-i}(R)$ with respect to a face ideal in the minimal primary decomposition of I.

Proposition 4.3.7. *Let $I = I_{\alpha_1} \cap \cdots \cap I_{\alpha_m}$ be the minimal primary decomposition of a squarefree monomial ideal $I \subseteq R$. Then:*

$$\mu_p(I_{\alpha_j}, H_I^r(R)) = \begin{cases} 1 & \text{if } p = 0, r = |\alpha_j|, \\ 0 & \text{otherwise.} \end{cases}$$

PROOF: Any face ideal I_{α_j} belongs only to the support of $H_I^{|\alpha_j|}(R)$ by Proposition 3.3.7. On the other side $I_{[I_{\alpha_j}]} = I_{\alpha_j}$ so, applying Corollary 4.3.5, we get the desired result. □

Another easy application of the restriction procedure allows us to compute the Bass numbers of $H_I^{n-i}(R)$, when R/I is Cohen-Macaulay. Recall that in Section 4.1 we have computed the characteristic cycle of the local cohomology modules $H^p_{\mathfrak{p}_\gamma}(H^r_I(R))$ when R/I is Cohen-Macaulay by using the degeneration of the Grothendieck spectral sequence

$$E_2^{p,q} = H^p_{\mathfrak{p}_\gamma}(H^q_I(R)) \Longrightarrow H^{p+q}_{\mathfrak{p}_\gamma+I}(R).$$

In particular we have $CC(H^p_{\mathfrak{p}_\gamma}(H^q_I(R))) = CC(H^{p+q}_{\mathfrak{p}_\gamma+I}(R))$ by Proposition 4.1.1. Then, applying the restriction functor we get:

Proposition 4.3.8. *Let $I \subseteq R$ be a squarefree monomial ideal of height h such that R/I is Cohen-Macaulay. Let $\mathfrak{p}_\gamma \subseteq R$ be a face ideal such that $I \subseteq \mathfrak{p}_\gamma$, then:*

$$\mu_p(\mathfrak{p}_\gamma, H_I^r(R)) = \begin{cases} 1 & \text{if } p = |\gamma| - h, r = h, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 4.3.9. If $I \subseteq R$ is any ideal such that there is only a non vanishing local cohomology module $H_I^h(R)$, where $h = \operatorname{ht} I$, then for any prime ideal $\mathfrak{p} \subseteq R$ such that $I \subseteq \mathfrak{p}$, the same result holds, i.e.

$$\mu_p(\mathfrak{p}, H_I^h(R)) = \begin{cases} 1 & \text{if } p = \operatorname{ht}\mathfrak{p} - h, \\ 0 & \text{otherwise.} \end{cases}$$

4.3.3 Some particular cases

When the dimension of the local cohomology module $H_I^{n-i}(R)$ is small enough one can easily describe its Bass numbers. For example:

- If $\dim R/I = 0$, i.e. $I = \mathfrak{m}$, then we have $H_{\mathfrak{m}}^n(R) = E(R/\mathfrak{m})$.
- If $\dim R/I = 1$ then R/I is Cohen-Macaulay, so the Bass numbers of the module $H_I^{n-1}(R)$ can be described by using Proposition 4.3.8.

In this section we will study the module $H_I^{n-1}(R)$ for the case $\dim R/I > 1$. Recall that $\dim H_I^{n-1}(R) \leq 1$.

Proposition 4.3.10. *Let $I \subseteq R$ be a squarefree monomial ideal such that $\dim R/I > 1$. If $H_I^{n-1}(R)$ is different from zero then it is an injective module.*

Proof: Let $I = I_{\alpha_1} \cap \cdots \cap I_{\alpha_m}$ be the minimal primary decomposition of I. Then:

- If $\dim H_I^{n-1}(R) = 0$ then $H_I^{n-1}(R) \cong E(R/\mathfrak{m})^{\oplus \gamma_{0,1}}$ by Corollary 3.3.10.
- If $\dim H_I^{n-1}(R) = 1$ then, for all the face ideals I_{α_j} such that $|\alpha_j| = n-1$ we have

$$\mu_p(I_{\alpha_j}, H_I^{n-1}(R)) = \begin{cases} 1 & \text{if } p = 0, \\ 0 & \text{otherwise} \end{cases}$$

by Proposition 4.3.7 so, it only remains to prove $\mu_p(\mathfrak{m}, H_I^{n-1}(R)) = 0$ for $p > 0$.

Consider the decomposition $I = I_1 \cap I_2$, where I_1 is an ideal such that $\operatorname{bight} I_1 < n-1$ and I_2 is a pure ideal of height $n-1$. We have $I_1 + I_2 = \mathfrak{m}$ so, by using the Mayer-Vietoris sequence we get $H_I^i(R) = H_{I_1}^i(R)$ for all $i < n-1$ and the short exact sequence

$$0 \longrightarrow H_{I_1}^{n-1}(R) \oplus H_{I_2}^{n-1}(R) \longrightarrow H_I^{n-1}(R) \longrightarrow H_{\mathfrak{m}}^n(R) \longrightarrow 0\,.$$

Splitting the corresponding long exact sequence of local cohomology with respect to the maximal ideal $\mathfrak{m}$ into short exact sequences of kernels and cokernels we get

$$0 \longrightarrow H^0_{\mathfrak{m}}(H^{n-1}_{I_1}(R)) \oplus H^0_{\mathfrak{m}}(H^{n-1}_{I_2}(R)) \longrightarrow H^0_{\mathfrak{m}}(H^{n-1}_{I}(R)) \longrightarrow Y_0 \longrightarrow 0$$

$$0 \longrightarrow Y_0 \longrightarrow H^0_{\mathfrak{m}}(H^{n}_{\mathfrak{m}}(R)) \longrightarrow Z_0 \longrightarrow 0$$

$$0 \longrightarrow Z_0 \longrightarrow H^1_{\mathfrak{m}}(H^{n-1}_{I_1}(R)) \oplus H^1_{\mathfrak{m}}(H^{n-1}_{I_2}(R)) \longrightarrow H^1_{\mathfrak{m}}(H^{n-1}_{I}(R)) \longrightarrow 0$$

The ring R/I_2 is Cohen-Macaulay, so we have:

$$H^1_{\mathfrak{m}}(H^{n-1}_{I_2}(R)) = E(R/\mathfrak{m}) \text{ and } H^0_{\mathfrak{m}}(H^{n-1}_{I_2}(R)) = 0.$$

On the other side $\dim H^{n-1}_{I_1}(R) \leq 0$, so we have:

$$H^0_{\mathfrak{m}}(H^{n-1}_{I_1}(R)) = E(R/\mathfrak{m})^{\oplus\beta} \text{ and } H^1_{\mathfrak{m}}(H^{n-1}_{I_1}(R)) = 0,$$

where β may vanish. Following closely the proof of Theorem 4.2.9 applied to this particular case we get $Y_0 = 0$ and $Z_0 = E(R/\mathfrak{m})$. So we get the desired result since we have:

$H^0_{\mathfrak{m}}(H^{n-1}_{I}(R)) = E(R/\mathfrak{m})^{\oplus\beta}$ where β is such that $H^{n-1}_{I_1}(R) = E(R/\mathfrak{m})^{\oplus\beta}$.

$H^1_{\mathfrak{m}}(H^{n-1}_{I}(R)) = 0$.

In particular, we get $H^{n-1}_{I}(R) \cong (\bigoplus_{|\alpha_j|=n-1} E(R/I_{\alpha_j})) \oplus E(R/\mathfrak{m})^{\oplus\beta}$.

□

4.3.4 Injective dimension of local cohomology modules

The first goal of this section is to give an annihilation criteria for the Bass numbers $\mu_p(\mathfrak{p}_\gamma, H^{n-i}_I(R))$ of the local cohomology modules $H^{n-i}_I(R)$. We remark that this criteria will be described in terms of the face ideals in the minimal primary decomposition of the monomial ideal I.

Proposition 4.3.11. *Let $I \subseteq R$ be an ideal generated by squarefree monomials and $\mathfrak{p}_\gamma \subseteq R$ be a face ideal. The following are equivalent:*

i) $\mu_p(\mathfrak{p}_\gamma, H_I^{n-i}(R)) \neq 0$.

ii) $T^*_{X_\gamma} X \in CC(H^p_{\mathfrak{p}_\gamma}(H_I^{n-i}(R)))$.

iii) *There exists* $I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} \in \mathcal{Q}_{\gamma,j,\gamma}$ *such that* $\operatorname{ht}(I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}}) = n - i + (j-1) = |\gamma| - p$.

PROOF: By Proposition 4.3.3 the Bass number $\mu_p(\mathfrak{p}_\gamma, H_I^{n-i}(R))$ does not vanish if and only if the corresponding face ideal $\mathfrak{p}_\gamma \subseteq R$ belongs to the support of the module $H^p_{\mathfrak{p}_\gamma}(H_I^{n-i}(R))$. Then, we are done by Proposition 4.3.1. □

Once the annihilation of Bass numbers has been described, we can compute the graded injective dimension in the same terms.

Corollary 4.3.12. *Let $I \subseteq R$ be an ideal generated by squarefree monomials. The graded injective dimension of $H_I^{n-i}(R)$ is:*

$$^*\mathrm{id}\, H_I^{n-i}(R) = \max_{\gamma,p}\{p \mid \lambda_{\gamma,p,i,\gamma} \neq 0 \}.$$

In terms of the face ideals in the minimal primary decomposition of I we have:

$$^*\mathrm{id}\, H_I^{n-i}(R) = \max_{\gamma,j}\{|\gamma| - \operatorname{ht}(I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}}) \mid I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} \in \mathcal{Q}_{\gamma,j,\gamma} \}.$$

Remark 4.3.13. The injective dimension $\mathrm{id}\, H_I^{n-i}(R)$ can be computed from the graded injective dimension $^*\mathrm{id}\, H_I^{n-i}(R)$ by using [33, Theorem 1.2.3] (see Remark 4.3.4).

4.3.5 Associated prime ideals of local cohomology modules

Bass numbers allow us to describe the associated primes of a R-module M. Namely we have:

$$\mathrm{Ass}_R(M) := \{\mathfrak{p} \in \operatorname{Spec} R \mid \mu_0(\mathfrak{p}, M) \neq 0\}.$$

In Section 3.3 we have studied the support of the local cohomology modules $H_I^{n-i}(R)$. Now, as a consequence of Proposition 4.3.11, we will distinguish the associated prime ideals among those prime ideals in the support.

Proposition 4.3.14. *Let $I \subseteq R$ be an ideal generated by squarefree monomials and $\mathfrak{p}_\gamma \subseteq R$ be a face ideal. The following are equivalent:*

i) $\mathfrak{p}_\gamma \in \mathrm{Ass}_R H_I^{n-i}(R)$.

ii) $T^*_{X_\gamma} X \in CC(H^0_{\mathfrak{p}_\gamma}(H_I^{n-i}(R)))$.

iii) There exists $I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} \in \mathcal{Q}_{\gamma,j,\gamma}$ *such that* $\mathrm{ht}\,(I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}}) = n - i + (j-1) = |\gamma|$.

The minimal primes in the support of any R-module are associated primes. For the case of local cohomology modules we can compute all the Bass numbers with respect to their minimal primes.

Proposition 4.3.15. *Let $I \subseteq R$ be an ideal generated by squarefree monomials and $\mathfrak{p}_\gamma \subseteq R$ be a face ideal. If $\mathfrak{p}_\gamma$ is a minimal prime in the support of the local cohomology module $H_I^{n-i}(R)$ then:*

$\mu_0(\mathfrak{p}_\gamma, H_I^{n-i}(R)) \neq 0$, *and*

$\mu_p(\mathfrak{p}_\gamma, H_I^{n-i}(R)) = 0$ *for all* $p > 0$.

Moreover, if $CC(H_I^{n-i}(R)) = \sum m_{i,\alpha}\ T^*_{X_\alpha} X$ *is the corresponding characteristic cycle, then:*

$\mu_0(\mathfrak{p}_\gamma, H_I^{n-i}(R)) = m_{i,\gamma}$.

Proof: We only have to apply the restriction functor at the face ideal $\mathfrak{p}_\gamma$ and apply Corollary 4.3.5. □

Finally we present an example of the existence of embedded prime ideals. From the proof of Proposition 4.3.10 it suffices to look for a precise local cohomology module $H_I^{n-1}(R)$ where $\dim R/I > 1$.

Example: Let $R = k[x_1, x_2, x_3, x_4, x_5]$. Consider the ideal:

- $I = (x_1, x_2, x_5) \cap (x_3, x_4, x_5) \cap (x_1, x_2, x_3, x_4)$.

By using Theorem 3.2.11 we compute the characteristic cycle of the corresponding local cohomology modules. Namely, we get:

$CC(H_I^2(R)) = T^*_{X_{(1,1,0,0,1)}} X + T^*_{X_{(0,0,1,1,1)}} X$.

$CC(H_I^3(R)) = T^*_{X_{(1,1,1,1,0)}} X + 2\ T^*_{X_{(1,1,1,1,1)}} X$.

Collecting the multiplicities we obtain the triangular matrix:

$$\Gamma(R/I) = \begin{pmatrix} 0 & 2 & 0 \\ & 1 & 0 \\ & & 2 \end{pmatrix}$$

If we compute the Bass numbers with respect to the maximal ideal, i.e. the Lyubeznik numbers, we obtain the triangular matrix:

$$\Lambda(R/I) = \begin{pmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 2 \end{pmatrix}$$

Then:

- For the module $H^2_I(R)$ we have:
  - $\cdot$ $\mathrm{id} H^2_I(R) = \dim H^2_I(R) = 2$.
  - $\cdot$ $\mathrm{Min}_R(H^2_I(R)) = \mathrm{Ass}_R(H^2_I(R))$.
- For the module $H^3_I(R)$ we have:
  - $\cdot$ $0 = \mathrm{id} H^3_I(R) < \dim H^3_I(R) = 1$.
  - $\cdot$ $\mathrm{Min}_R(H^3_I(R)) \subsetneq \mathrm{Ass}_R(H^3_I(R))$.

More precisely, $H^3_I(R) \cong E(R/(x_1, x_2, x_3, x_4)) \oplus E(R/(x_1, x_2, x_3, x_4, x_5))$ so, the maximal ideal $\mathfrak{m} = (x_1, x_2, x_3, x_4, x_5)$ is an embedded associated prime ideal of this local cohomology module.

4.3.6 Small support of local cohomology modules

The small support of a R-module M was introduced by H. B. Foxby in [25]. Namely, it is defined as:

$$\mathrm{supp}_R(M) := \{\mathfrak{p} \in \mathrm{Spec}\, R \mid \exists\, p \geq 0 \text{ such that } \mu_p(\mathfrak{p}, M) \neq 0\}.$$

As a consequence of Proposition 4.3.11 we can describe the small support of the local cohomology modules $H^{n-i}_I(R)$ in terms of the face ideals in the minimal primary decomposition of the monomial ideal I.

Proposition 4.3.16. *Let $I \subseteq R$ be an ideal generated by squarefree monomials and $\mathfrak{p}_\gamma \subseteq R$ be a face ideal. The following are equivalent:*

i) $\mathfrak{p}_\gamma \in \mathrm{supp}_R H_I^{n-i}(R)$.

ii) $T^*_{X_\gamma} X \in CC(H^p_{\mathfrak{p}_\gamma}(H_I^{n-i}(R)))$ *for some* p.

iii) There exists $I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} \in \mathcal{Q}_{\gamma,j,\gamma}$ *such that* $\mathrm{ht}\,(I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}}) = |\gamma| - p$ *for some* p.

It is clear that $\mathrm{Ass}_R M \subseteq \mathrm{supp}_R M \subseteq \mathrm{Supp}_R M$ for any R-module M. If M is finitely generated then $\mathrm{supp}_R M = \mathrm{Supp}_R M$, but in general they are not equal. Now we present an example of a local cohomology module $H_I^{n-i}(R)$ that satisfies an strictly inequality between the small support and the support.

Example: Let $R = k[x_1, x_2, x_3, x_4, x_5]$. Consider the ideal:

- $I = (x_1, x_4) \cap (x_2, x_5) \cap (x_1, x_2, x_3)$.

By using Theorem 3.2.11 we compute the characteristic cycle of the corresponding local cohomology modules. Namely, we get:

$$CC(H_I^2(R)) = T^*_{X_{(1,0,0,1,0)}} X + T^*_{X_{(0,1,0,0,1)}} X.$$

Then, $\mathrm{Supp}_R(H_I^2(R)) = V(x_1, x_4) \cup V(x_2, x_5)$. Computing the Bass numbers of the face ideals contained in the support of $H_I^2(R)$ we get:

$\mathfrak{p}_\gamma$	μ_0	μ_1	μ_2	μ_3
(x_1, x_4)	1	-	-	-
(x_2, x_5)	1	-	-	-
(x_1, x_4, x_i)	-	1	-	-
(x_2, x_5, x_i)	-	1	-	-
(x_1, x_4, x_i, x_j)	-	-	1	-
(x_2, x_5, x_i, x_j)	-	-	1	-
$(x_1, x_2, x_3, x_4, x_5)$	-	-	-	1

In this case we have:

$\cdot$ $\mathrm{id} H_I^2(R) = \dim H_I^2(R) = 3$.

$\cdot$ $\mathrm{Min}_R(H_I^2(R)) = \mathrm{Ass}_R(H_I^2(R))$.

$\cdot$ $\mathrm{supp}_R(H_I^2(R)) = \mathrm{Supp}_R(H_I^2(R))$.

On the other side we have:

$$CC(H_I^3(R)) = T^*_{X_{(1,1,1,0,0)}}X + T^*_{X_{(1,1,1,1,0)}}X + T^*_{X_{(1,1,1,0,1)}}X + T^*_{X_{(1,1,0,1,1)}}X+$$
$$+T^*_{X_{(1,1,1,1,1)}}X.$$

Then, $\mathrm{Supp}_R(H_I^3(R)) = V(x_1, x_2, x_3) \cup V(x_1, x_2, x_4, x_5)$. Computing the Bass numbers of the face ideals contained in the support of $H_I^3(R)$ we get:

$\mathfrak{p}_\gamma$	μ_0	μ_1	μ_2
(x_1, x_2, x_3)	1	-	-
(x_1, x_2, x_3, x_4)	-	-	-
(x_1, x_2, x_3, x_5)	-	-	-
(x_1, x_2, x_4, x_5)	1	-	-
$(x_1, x_2, x_3, x_4, x_5)$	-	1	-

In this case we have:

· $1 = \mathrm{id} H_I^3(R) < \dim H_I^3(R) = 2$.

· $\mathrm{Min}_R(H_I^3(R)) = \mathrm{Ass}_R(H_I^3(R))$.

· $\mathrm{supp}_R(H_I^3(R)) \subsetneq \mathrm{Supp}_R(H_I^3(R))$.

In particular the face ideals (x_1, x_2, x_3, x_4) and (x_1, x_2, x_3, x_5) do not belong to $\mathrm{supp}_R(H_I^3(R))$.

Chapter 5

Local cohomology, arrangements of subspaces and monomial ideals

Let $\mathbb{A}_k^n$ denote the affine space of dimension n over a field k, let $X \subset \mathbb{A}_k^n$ be an arrangement of linear subvarieties. Set $R = k[x_1, \ldots, x_n]$ and let $I \subset R$ denote an ideal which defines X. In this chapter we study the local cohomology modules $H_I^r(R)$ with special regard of the case where the ideal I is generated by monomials.

Even though the tools we will use are independent of the characteristic of the field k, whether $\mathrm{char}(k) = 0$ we will keep in mind the structure as $\mathcal{D}$-module of the modules $H_I^r(R)$. On the other side, if $\mathrm{char}(k) > 0$ we will use the notion of F-module introduced by G. Lyubeznik in [56, Definition 1.1].

We have to point out that an arrangement of linear varieties X determine a poset $P(X)$ formed by the intersections of the irreducible components of X and the order given by the inclusion. For example, if X is defined by a squarefree monomial ideal $I \subseteq R$, $P(X)$ is nothing but the poset $\mathcal{I}$ defined in Chapter 3 identifying the sums of face ideals in the minimal primary decomposition of I whether they describe the same ideal.

In Section 5.1, in an analogous way to the construction of the Mayer-Vietoris stectral sequences for singular cohomology and ℓ-adic cohomology introduced by A. Björner and T. Ekedahl [11] we prove the existence of a

Mayer–Vietoris spectral sequence for local cohomology:

$$E_2^{-i,j} = \varinjlim\nolimits^{(i)}_{P(X)} H^j_{I_p}(R) \Rightarrow H_I^{j-i}(R)$$

where p is an element of the poset $P(X)$, I_p is the defining (radical) ideal of the irreducible variety corresponding to p, and $\varinjlim^{(i)}_{P(X)}$ is the i-th left derived functor of the direct limit functor in the category of direct systems indexed by the poset $P(X)$.

Studying in detail this spectral sequence, we observe that the E_2-term is defined by the reduced homology of the simplicial complex associated to the poset $P(X)$ which has as vertexs the elements of $P(X)$ and where a set of vertexs $p_0, \dots, p_r$ determines a r-dimensional simplex if $p_0 < \cdots < p_r$.

The main result of this section is the degeneration of this spectral sequence in the E_2-term. The main ingredient of the proof is the Matlis–Gabriel structure theorem on the injective modules. This contrasts with the fact that the proof of the degeneration of the Mayer-Vietoris spectral sequence for ℓ-adic or singular cohomology relies on the strictness of Deligne's weight filtration.

The degeneration of the Mayer–Vietoris spectral sequence provides a filtration $\{F_j\}_{j\geq 0}$ of the local cohomology modules, where the successive quotients F_j/F_{j-1} are given by the E_2-term. In particular, for any $0 \leq j \leq n$, we have an exact sequence:

$$(s_j): \qquad 0 \to F_{j-1} \to F_j \to F_j/F_{j-1} \to 0,$$

that defines an element of ${}^*\mathrm{Ext}^1_R(F_j/F_{j-1}, F_{j-1})$. In general, not all the **extension problems** associated to this filtration have a trivial solution. This is a major difference between the case we consider here and the cases considered by Björner and Ekedahl. Namely, in the analogous situation for the ℓ-adic cohomology of an arrangement defined over a finite field the extensions appearing are trivial not only as extensions of $\mathbb{Q}_\ell$-vector spaces but also as Galois representations, and for the singular cohomology of a complex arrangement the extensions appearing are trivial as extensions of mixed Hodge structures (cf. [11, pg. 179]).

Whether k is a field of characteristic zero, we can compute the characteristic cycle of the modules $H^r_I(R)$ from the short exact sequences determined by the filtration and the additivity of the characteristic cycle with respect to exact sequences. If k is the field of real or complex numbers, these characteristic cycles determine the Betti numbers of the complement of the arrangement X in $\mathbb{A}_k^n$.

In Section 5.2 we recall some basic facts on the category of ε**-straight** modules, which is a slight variation of a category introduced by K. Yanagawa in [97]. We will also pay attention to the $\mathcal{D}$-module structure of ε-straight modules as well to the relation between the multiplicities of the characteristic cycle and the Hilbert series of these modules, specially the case of local cohomology modules supported on monomial ideals.

In Section 5.3 we will give a solution to the extension problems associated to the filtration of local cohomology modules supported on monomial ideals (the result stated is actually more general, in that we will work in the category of ε-straight modules). It turns out that these extensions can be described by a finite set of linear maps, which for local cohomology modules supported on monomial ideals can be effectively computed in combinatorial terms from certain Stanley-Reisner simplicial complexes (using the results in [72]).

5.1 Filtrations on local cohomology modules

Throughout this chapter, if M is an R-module endowed with a filtration $\{F_k\}_{k\geq 0}$, it will be always assumed that the filtration is exhaustive, i.e. $M = \cup_k F_k$, and we will agree that $F_{-1} = \{0\}$.

5.1.1 The Roos complex

Let S be an inductive system of R–modules. J.E. Roos introduced in [78] a complex which has as i-th cohomology the i-th left derived functor of the inductive limit functor evaluated at S (and the dual notion for projective systems as well, this is actually the case treated by Roos in more detail). We recall his definition in the case of interest for us:

Let $(P, \leq)$ be a partially ordered set (poset), let $\mathcal{A}b$ be an abelian category with enough projectives and such that the direct sum functors are exact (usually, $\mathcal{A}b$ will be a category of modules, sometimes with enhanced structure: a $\mathcal{D}$-module structure, a F-module structure or a grading). We will regard P as a small category which has as objects the elements of P and, given $p, q \in P$, there is one morphism $p \to q$ if $p \leq q$. A diagram over P of objects of the category $\mathcal{A}b$ is by definition a covariant functor $F : P \to \mathcal{A}b$. Notice that the image of F is an inductive system of objects of $\mathcal{A}b$ indexed by P. The category which has as objects the diagrams of objects of $\mathcal{A}b$ and as functors the natural transformations is abelian and will be denoted $\mathrm{Diag}(P, \mathcal{A}b)$.

If $(P, \leq)$ is a poset, we will denote by $K(P)$ the simplicial complex which has as vertexs the elements of P and where a set of vertexs $p_0, \ldots, p_r$ determines a r-dimensional simplex if $p_0 < \cdots < p_r$. If K is a simplicial complex and E is a k-vector space, we will denote by $\mathcal{C}_\bullet(K; E)$ the complex of simplicial chains of K with coefficients in E.

Definition 5.1.1. *The Roos complex of a covariant functor $F : P \to \mathcal{A}b$ is the homological complex of objects of $\mathcal{A}b$ defined by*

$$Roos_k(F) := \bigoplus_{p_0 < \cdots < p_k} F_{p_0 \ldots p_k} \ ,$$

where $F_{p_0 \ldots p_k} = F(p_0)$ and, if $i > 0$ and we denote by $\pi_{p_0 \ldots \widehat{p_i} \ldots p_k}$ the projection from $\bigoplus_{p_0 < \cdots < p_k} F_{p_0 \ldots p_k}$ onto $F_{p_0 \ldots \widehat{p_i} \ldots p_k}$, the differential on $F_{p_0 \ldots p_k}$ is given by

$$F(p_0 \to p_1) + \sum_{i=1}^{k} (-1)^i \pi_{p_0 \ldots \widehat{p_i} \ldots p_k}.$$

This construction defines a functor $\mathrm{Roos}_*(\cdot) : \mathrm{Diag}(P, \mathcal{A}b) \to \mathrm{C}(\mathcal{A}b)$, where $\mathrm{C}(\mathcal{A}b)$ denotes the category of chain complexes of objects of $\mathcal{A}b$. It is easy to see that this functor is exact and commutes with direct sums. We also point out that the i-th cohomology of the Roos complex is the i-th left derived functor of the inductive limit functor evaluated at P. In particular, we have the augmented Roos complex:

$$\mathrm{Roos}_*(F) \to \varinjlim_P F \to 0.$$

5.1.2 The Roos complex of an arrangement of linear varieties

Let $X \subset \mathbb{A}_k^n$ be an arrangement defined by an ideal $I \subset R = k[x_1, \ldots, x_n]$. Given $p \in P(X)$, we will denote by X_p the linear affine variety in $\mathbb{A}_k^n$ corresponding to p and by $I_p \subset R$ the radical ideal which defines X_p in $\mathbb{A}_k^n$. Notice that the poset $P(X)$ is isomorphic to the poset of ideals $\{I_p\}_p$, ordered by reverse inclusion. We denote by $h(p)$ the k-codimension of X_p in $\mathbb{A}^n_k$ (that is, $h(p)$ equals the height of the ideal I_p). If $J_1 \subseteq J_2$ are ideals of R, we set $\mathrm{ht}\,(J_2/J_1) := \mathrm{ht}\,(J_2) - \mathrm{ht}\,(J_1)$.

Let M be a R-module, $i \geq 0$ an integer. Then one can define a diagram of R–modules $H^i_{[*]}(M)$ on the poset $P(X)$ by

$$H^i_{[*]}(M): \quad p \mapsto H^i_{I_p}(M).$$

This defines a functor $H^i_{[*]}(\cdot) : \mathrm{Mod}(R) \to \mathrm{Diag}(P(X), \mathrm{Mod}(R))$.

Our aim in this section is to construct a spectral sequence by using the Roos functor applied to the diagrams $H^i_{[*]}(\cdot)$. Notice that in this case, the augmented Roos complex is in the form $\mathrm{Roos}_*(H^0_{[*]}(\cdot)) \to H^0_I(\cdot) \to 0$. For details on the construction and properties of spectral sequences we refer to [79].

The following lemma will play a key role in the construction of the spectral sequence we want to obtain.

Lemma 5.1.2. *If E is an injective R–module, then the augmented Roos complex*

$$Roos_*(H^0_{[*]}(E)) \to H^0_I(E) \to 0$$

is exact.

PROOF: Since both $\mathrm{Roos}_*(\cdot)$ and $H^0_{[*]}(\cdot)$ commute with direct sums, by the Matlis–Gabriel theorem on the structure of injective modules over Noetherian rings, we can assume that there is a prime ideal $\mathfrak{p} \subset R$ such that $E = E_R(R/\mathfrak{p})$, the injective envelope of $R/\mathfrak{p}$ in the category of R-modules. Notice also that for any ideal $J \subseteq R$, $H^0_J(E_R(R/\mathfrak{p})) = E_R(R/\mathfrak{p})$ if $\mathfrak{p} \subseteq J$ and it is zero otherwise. It will be enough to prove that if $\mathfrak{m} \subseteq R$ is a maximal ideal, then the complex

$$(\mathrm{Roos}_*(H^0_{[*]}(E)))_{\mathfrak{m}} \to (H^0_I(E))_{\mathfrak{m}} \to 0$$

is exact. If $I \not\subseteq \mathfrak{m}$, this complex is zero. Otherwise, it is a chain complex that equals the augmented complex

$$\mathcal{C}_\bullet(K, E_{R_\mathfrak{m}}(R_\mathfrak{m}/\mathfrak{p}R_\mathfrak{m})) \to E_{R_\mathfrak{m}}(R_\mathfrak{m}/\mathfrak{p}R_\mathfrak{m}) \to 0,$$

where K is the simplicial complex attached to the subposet of $P(X)$ which has as vertexs those linear subspaces X_p such that $I_p \subseteq \mathfrak{m}$. As K has a unique maximal element in it, K is contractible, and then the lemma follows. $\square$

Fix now an injective resolution $0 \to R \to E^*$ of R in the category of R–modules. Each of the modules E^j $(j \geq 0)$ defines a diagram $H^0_{[*]}(E^j)$ over $P(X)$ and one obtains a double complex

$$\mathrm{Roos}_{-i}(H^0_{[*]}(E^j)), \quad i \leq 0\ ,\ j \geq 0.$$

(the change of sign on the indexing of the Roos complex is because we prefer to work with a double complex which is cohomological in both degrees). This is a second quadrant double complex with only a finite number of non-zero columns, so it gives rise to a spectral sequence that converges to $H^*_I(R)$ (because of the lemma above). More precisely, we have

$$E_1^{-i,j} = \mathrm{Roos}_i(H^0_{[*]}(E^j)) \Rightarrow H^{j-i}_I(R).$$

The differential d_1 is that of the Roos complex, since this complex computes the i-th left derived functor of the inductive limit, the E_2 term will be

$$E_2^{-i,j} = \varinjlim{}_P^{(i)} H^j_{[*]}(R) \Rightarrow H^{j-i}_I(R)$$

(we write P for $P(X)$ in order to simplify the writing of our formulas). Hereafter this sequence will be called **Mayer-Vietoris spectral sequence**.

If the base field k is of characteristic zero, we can choose an injective resolution of R in the category of modules over the ring of differential operators $\mathcal{D}$. Since $\mathcal{D}$ is free as R-module, it follows (see e.g. [9, II.2.1.2]) that this is also an injective resolution of R in the category of R-modules. If the base field k is of characteristic $p > 0$, the ring R has a natural F-module structure and its minimal injective resolution is a complex of F-modules and F-module homomorphisms (see [56, (1.2.b")]). Therefore, the spectral sequence above may be regarded as a spectral sequence in the category of $\mathcal{D}$–modules (respectively, of F-modules).

Remark 5.1.3. Instead of an injective resolution of R in the category of R-modules, one could take as well an acyclic resolution with respect to the functors $\Gamma_J(\cdot)$, $J \subset R$ an ideal (recall that $\Gamma_J(M) := \{m \in M \mid \exists r \geq 0$ such that $J^r m = 0\}$). This fact will be used in the next sections.

Before proving the main result of this section, i.e. the **degeneration of the Mayer-Vietoris spectral sequence**, we describe the E_2-terms of the spectral sequence. Namely we have:

Proposition 5.1.4. *Let $X \subset \mathbb{A}^n_k$ be an arrangement of linear varieties. Let $K(>p)$ be the simplicial complex attached to the subposet $\{q \in P(X) \mid q > p\}$ of $P(X)$. Then, there are R-module isomorphisms*

$$\varinjlim_P^{(i)} H^j_{[*]}(R) \simeq \bigoplus_{h(p)=j} [\, H^j_{I_p}(R) \otimes_k \tilde{H}_{i-1}(K(>p);k)\,],$$

where $\tilde{H}$ denotes reduced simplicial homology. Moreover, if char$(k) = 0$ they are isomorphisms of $\mathcal{D}$–modules and if char$(k) > 0$ they are isomorphisms of F–modules.

Remark 5.1.5. Let $\emptyset$ be the empty simplicial complex. Then, we agree that

$$\tilde{H}_{-1}(K(>p);M) = \begin{cases} 0 & \text{if } K(>p) \neq \emptyset, \\ M & \text{if } K(>p) = \emptyset. \end{cases}$$

Proof: (c.f. [11, Proposition 4.5]): Given $p \in P(X)$ and a R–module M we consider the following three diagrams:

$F_{M,\geq p}$, defined by $F_{M,\geq p}(q) = M$ if $q \geq p$ and $F_{M,\geq p}(q) = 0$ otherwise.

$F_{M,>p}$, defined by $F_{M,>p}(q) = M$ if $q > p$ and $F_{M,>p}(q) = 0$ otherwise.

$F_{M,p}$, defined by $F_{M,p}(q) = M$ if $q = p$ and $F_{M,p}(q) = 0$ otherwise

(in all three cases $F(p \to q) = id$ if $F(p) = F(q)$ and it is zero otherwise). In the category of diagrams of R–modules over $P(X)$ we have an exact sequence

$$0 \to F_{M,>p} \to F_{M,\geq p} \to F_{M,p} \to 0.$$

Let $K(\geq p)$ be the simplicial complex attached to the subposet $\{q \in P(X) \mid q \geq p\}$ of $P(X)$. Then one has

$$\mathrm{Roos}_*(F_{M,\geq p}) = \mathcal{C}_\bullet(K(\geq p), M) \quad \text{and} \quad \mathrm{Roos}_*(F_{M,>p}) = \mathcal{C}_\bullet(K(>p), M).$$

Since the complex $K(\geq p)$ is contractible (to the vertex corresponding to p), the long exact homology sequence obtained from the sequence of complexes

$$0 \to \mathrm{Roos}_*(F_{M,>p}) \to \mathrm{Roos}_*(F_{M,\geq p}) \to \mathrm{Roos}_*(F_{M,p}) \to 0$$

gives

$$\varinjlim_P^{(i)} F_{M,p} \cong \tilde{H}_{i-1}(K(> p); M),$$

where the tilde denotes reduced homology.

On the other hand, for any $p \in P(X)$ the module $H^j_{I_p}(R)$ vanishes unless $h(p) = j$, so one has an isomorphism of diagrams $H^j_{[*]}(R) \simeq \bigoplus_{h(p)=j} F_{H^j_{I_p}(R),\, p}$. Thus,

$$\varinjlim_P^{(i)} H^j_{[*]}(R) \cong \bigoplus_{h(p)=j} \varinjlim_P^{(i)} F_{H^j_{I_p}(R),\, p} \cong \bigoplus_{h(p)=j} \tilde{H}_{i-1}(K(> p); H^j_{I_p}(R))$$

By the universal coefficient theorem, for $i > 0$

$$\tilde{H}_{i-1}(K(> p); H^j_{I_p}(R)) \cong H^j_{I_p}(R) \otimes_k \tilde{H}_{i-1}(K(> p), k).$$

Although the isomorphism given by the universal coefficient theorem is a priori only an isomorphism of k-vector spaces, it is easy to check that in our case is also an isomorphism of $\mathcal{D}$–modules (if $\operatorname{char}(k) = 0$) and of F-modules (if $\operatorname{char}(k) > 0$). In particular, it is always an isomorphism of R–modules.

□

The description of the E_2-term allows us to check that the differentials of the spectral sequence at the E_2 level are zero. As a consequence, we obtain the following:

Theorem 5.1.6. *Let $X \subset \mathbb{A}^n_k$ be an arrangement of linear varieties defined by an ideal $I \subset R$. Then, the Mayer–Vietoris spectral sequence*

$$E_2^{-i,j} = \varinjlim_P^{(i)} H^j_{[*]}(R) \Rightarrow H^{j-i}_I(R)$$

degenerates at the E_2–term.

Proof: Observe first that if $I \subset R$ is an ideal and we set $h = \operatorname{ht}(I)$, then all associated primes of $H^h_I(R)$ are minimal primes of I (this is well-known and due to the structure of the minimal injective resolution of R).

It follows that if $\mathfrak{p}, \mathfrak{q} \subset R$ are prime ideals such that $\mathfrak{p} \not\subset \mathfrak{q}$ and we set $i = \operatorname{ht}(\mathfrak{p})$, $j = \operatorname{ht}(\mathfrak{q})$, then $\operatorname{Hom}_R(H^i_{\mathfrak{p}}(R), H^j_{\mathfrak{q}}(R)) = 0$. From this last fact and Proposition 5.1.4 above follows that the Mayer–Vietoris sequence degenerates at the E_2–term. □

The main consequence of this result is the existence of the following **filtration on local cohomology modules**:

Corollary 5.1.7. *Let $X \subset \mathbb{A}^n_k$ be an arrangement of linear varieties defined by an ideal $I \subset R$. Then, for all $r \geq 0$ there is a filtration $\{F^r_j\}_{r \leq j \leq n}$ of $H^r_I(R)$ by R–submodules such that*

$$F^r_j / F^r_{j-1} \cong \bigoplus_{h(p)=j} [\, H^j_{I_p}(R) \otimes_k \tilde{H}_{h(p)-r-1}(K(> p); k)\,].$$

Moreover, if $char(k) = 0$ it is a filtration by holonomic $\mathcal{D}$–modules and if $char(k) > 0$ it is a filtration by F-modules.

Proof: The filtration $\{F^r_j\}$ is the one given by the degeneration of the Mayer–Vietoris spectral sequence. It is a filtration by $\mathcal{D}$-modules (respectively, F-modules) by Theorem 5.1.61.2.iii). □

Remark 5.1.8. The filtration given in Corollary 5.1.7 is functorial with respect to affine transformations.

5.1.3 Characteristic cycle of local cohomology modules supported on arrangements of linear varieties

From the filtration given in Corollary 5.1.7, we can compute the characteristic cycle of local cohomology modules supported on the defining ideal I of an arrangement $X \subseteq \mathbb{A}^n_k$ of linear varieties in the case when $\mathrm{char}(k) = 0$. In particular, we give a different approach to Theorem 3.2.11.

Corollary 5.1.9. *Let k be a field of characteristic zero and I be the defining ideal of an arrangement $X \subseteq \mathbb{A}^n_k$ of linear varieties. Then, the characteristic cycle of the holonomic $\mathcal{D}$-module $H^r_I(R)$ is*

$$CC(H^r_I(R)) = \sum m_{n-r,p}\, T^*_{X_p} \mathbb{A}^n_k \,,$$

where $m_{n-r,p} = \dim_k \tilde{H}_{h(p)-r-1}(K(> p); k)$ and $T^*_{X_p} \mathbb{A}^n_k$ denotes the relative conormal subspace of $T^* \mathbb{A}^n_k$ attached to X_p.

Proof: The formula for the characteristic cycle follows from the fact that if $\mathrm{ht}\,(I_p) = h$, then $CC(H^h_{I_p}(R)) = T^*_{X_p}\, \mathbb{A}^n_k$ and the additivity of the characteristic cycle with respect to short exact sequences. □

Let k be the field of real or complex numbers. By using a result of Goresky–MacPherson (see Section 1.2.8), the characteristic cycles determine the Betti numbers of the complement of the arrangement X in $\mathbb{A}^n_k$.

Corollary 5.1.10. *With the above notations, we have:*

If $k=\mathbb{R}$ is the field of real numbers, the Betti numbers of the complement of the arrangement X in $\mathbb{A}^n_{\mathbb{R}}$ can be computed in terms of the multiplicities $\{m_{n-r,p}\}$ as

$$\dim_{\mathbb{Q}} \tilde{H}_r(\mathbb{A}^n_{\mathbb{R}} - X;\mathbb{Q}) = \sum_p m_{n-(r+1),\,p}.$$

If $k=\mathbb{C}$ is the field of complex numbers, then one has

$$\dim_{\mathbb{Q}} \tilde{H}_r(\mathbb{A}^n_{\mathbb{C}} - X;\mathbb{Q}) = \sum_p m_{n-(r+1-h(p)),\,p}.$$

Proof: The formula for the Betti numbers of the complement $\mathbb{A}^n_{\mathbb{R}} - X$ follows from a theorem of Goresky–MacPherson ([34, III.1.3. Theorem A]), which states (slightly reformulated) that

$$\tilde{H}_r(\mathbb{A}^n_{\mathbb{R}} - X;\mathbb{Z}) \cong \bigoplus\nolimits_p H^{h(p)-r-1}(K(\geq p), K(>p);\mathbb{Z}).$$

Regarding a complex arrangement in $\mathbb{A}^n_{\mathbb{C}}$ as a real arrangement in $\mathbb{A}^{2n}_{\mathbb{R}}$, the formula for the Betti numbers of the complement of a complex arrangement follows from the formula for real arrangements. □

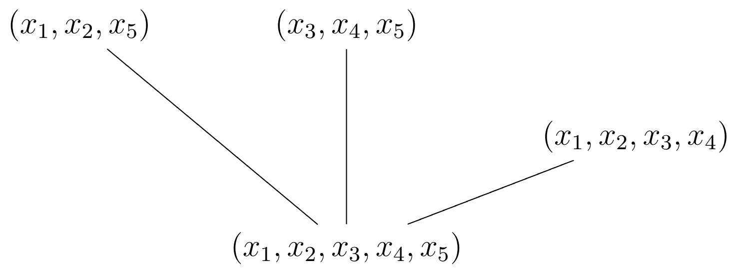
We remark that the integers that appear in Goresky–MacPherson' s formula are dimensions of certain Morse groups (cf. [34, Part III, Theorems 1.3 and 3.5]), while we can give a purely algebraic interpretation of them in terms of local cohomology.

To illustrate the computations we present the following:

Example: Let $R=k[x_1,x_2,x_3,x_4,x_5]$. Consider the arrangement X of linear varieties defined by the monomial ideal:

- $I=(x_1,x_2,x_5)\cap(x_3,x_4,x_5)\cap(x_1,x_2,x_3,x_4)$.

The poset of ideals defined by the arrangement ordered by reverse inclusion is:



Computing the order complexes $K(> p)$ and their corresponding reduced homologies we get:

p	I_p	$K(> p)$	$\dim_k \tilde{H}_{-1}$	$\dim_k \tilde{H}_0$	$\dim_k \tilde{H}_1$
p_1	(x_1, x_2, x_5)	$\emptyset$	1	-	-
p_2	(x_3, x_4, x_5)	$\emptyset$	1	-	-
p_3	(x_1, x_2, x_3, x_4)	$\emptyset$	1	-	-
q	$(x_1, x_2, x_3, x_4, x_5)$	$\bullet\bullet\bullet$	-	2	-

If $\operatorname{char}(k) = 0$ then by using Corollary 5.1.7 we get the characteristic cycle of the corresponding local cohomology modules:

$CC(H_I^3(R)) = T^*_{X_{p_1}} \mathbb{A}^5_k + T^*_{X_{p_2}} \mathbb{A}^5_k.$

$CC(H_I^4(R)) = T^*_{X_{p_3}} \mathbb{A}^5_k + 2\ T^*_{X_q} \mathbb{A}^5_k.$

The Betti numbers of the complementary of the arrangement are:

- If $k = \mathbb{R}$ is the field of real numbers, then one has:

  $\dim_{\mathbb{Q}} \tilde{H}_2(\mathbb{A}^5_{\mathbb{R}} - X; \mathbb{Q}) = 2.$

  $\dim_{\mathbb{Q}} \tilde{H}_3(\mathbb{A}^5_{\mathbb{R}} - X; \mathbb{Q}) = 3.$

- If $k = \mathbb{C}$ is the field of complex numbers, then one has:

  $\dim_{\mathbb{Q}} \tilde{H}_5(\mathbb{A}^5_{\mathbb{C}} - X; \mathbb{Q}) = 2.$

  $\dim_{\mathbb{Q}} \tilde{H}_7(\mathbb{A}^5_{\mathbb{C}} - X; \mathbb{Q}) = 1.$

  $\dim_{\mathbb{Q}} \tilde{H}_8(\mathbb{A}^5_{\mathbb{C}} - X; \mathbb{Q}) = 2.$

Let $X \subset \mathbb{A}_k^n$ be an arrangement of linear varieties defined by an ideal $I \subset R$. Consider the filtration $\{F_j^r\}_{r\le j\le n}$ of the local cohomology modules $H_I^r(R)$ given in Corollary 5.1.7. For each $0 \le j \le n$, one has an exact sequence

$$(s_j): \qquad 0 \to F_{j-1} \to F_j \to F_j/F_{j-1} \to 0,$$

which defines an element of ${}^*\mathrm{Ext}_R^1(F_j/F_{j-1}, F_{j-1})$. In general, not all the **extension problems** attached to the filtration have a trivial solution.

Example: Set $R = k[x,y]$, consider the ideal $I = (x \cdot y) \subset k[x,y]$ and denote $I_1 = (x)$, $I_2 = (y)$, $\mathfrak{m} = (x,y)$. For the filtration of $H_I^1(R)$ introduced above one has $F_1 \simeq H_{I_1}^1(R) \oplus H_{I_2}^1(R)$, $F_2 = H_I^2(R)$, and the exact sequence $0 \to F_1 \to F_2 \to F_2/F_1 \to 0$ is nothing but the Mayer - Vietoris exact sequence

$$0 \to H_{I_1}^1(R) \oplus H_{I_2}^1(R) \to H_I^1(R) \to H_{\mathfrak{m}}^2(R) \to 0.$$

This sequence is not split, e.g. because the maximal ideal $\mathfrak{m}$ is not a minimal prime of I and so cannot be an associated prime of $H_I^1(R)$.

This is a major difference between the case we consider here and the cases considered by Björner and Ekedahl. Namely, in the analogous situation for the ℓ-adic cohomology of an arrangement defined over a finite field the extensions appearing are trivial not only as extensions of $\mathbb{Q}_\ell$-vector spaces but also as Galois representations, and for the singular cohomology of a complex arrangement the extensions appearing are trivial as extensions of mixed Hodge structures (cf. [11, pg. 179]).

In section 5.4. we will study in detail the extension problems when I is a squarefree monomial ideal in order to describe the structure of the local cohomology modules $H_I^r(R)$.

Remark 5.1.11. The formalism of Mayer–Vietoris sequences can be applied to functors other than $H_{[*]}^i(\cdot)$ (and other than those considered in [11]). For example, one can consider the diagrams

$$\mathrm{Ext}_R^i\,(R/[*], R) : p \mapsto \mathrm{Ext}_R^i\,(R/I_p, R)$$

and, similarly as for the local cohomology modules, one has a spectral sequence

$$E_2^{-i,j} = \varinjlim_P{}^{(i)}\, \mathrm{Ext}_R^j\,(R/I_p, R) \Rightarrow \mathrm{Ext}_R^{j-i}\,(R/I, R)$$

which degenerates at the E_2-term. Therefore, one can endow the module $\mathrm{Ext}^r_R(R/I, R)$ with a filtration $\{G^r_j\}_{r\leq j\leq n}$ such that

$$G^r_j/G^{r-1}_j \simeq \bigoplus_{h(p)=j} [\mathrm{Ext}^j_R(R/I_p, R) \otimes_k \tilde{H}_{h(p)-r-1}(K(> p); k)].$$

The natural morphism $\mathrm{Ext}^r_R(R/I, R) \to H^r_I(R)$ is filtered due to the functoriality of the construction.

5.2 Straight modules

Let M be a graded R-module and $\alpha \in \mathbb{Z}^n$. As usual, we denote by $M(\alpha)$ the graded R-module whose underlying R-module structure is the same as that of M and where the grading is given by $(M(\alpha))_\beta = M_{\alpha+\beta}$. If $\alpha \in \mathbb{Z}^n$, we set $\mathrm{supp}(\alpha) = \{i \mid \alpha_i > 0\}$. We recall the following definition of K. Yanagawa:

Definition 5.2.1. ([97, 2.7]) *A $\mathbb{Z}^n$-graded module M is said to be* **straight** *if the following two conditions are satisfied:*

i) $\dim_k M_\alpha < \infty$ *for all* $\alpha \in \mathbb{Z}^n$.

ii) The multiplication map $M_\alpha \ni y \mapsto \mathbf{x}^\beta y \in M_{\alpha+\beta}$ *is bijective for all* $\alpha, \beta \in \mathbb{Z}^n$ *with* $\mathrm{supp}(\alpha+\beta) = \mathrm{supp}(\alpha)$.

The full subcategory of the category ${}^*\mathrm{Mod}(R)$ which has as objects the straight modules will be denoted **Str**. Let $\mathbf{1} = \sum_{i=1}^n \varepsilon_i = (1, \dots, 1) \in \mathbb{Z}^n$. In order to avoid shiftings in local cohomology modules, we will consider instead the following (equivalent) category:

Definition 5.2.2. *We will say that a graded module M is ε-straight if $M(-\mathbf{1})$ is straight in the above sense. We denote ε-*$\mathbf{Str}$ *the full subcategory of* ${}^*Mod(R)$ *which has as objects the ε-straight modules.*

A slightly modified transposition of [97, Proposition 2.12] to the category ε-**Str** gives a characterization of these modules in terms of a filtration such that the corresponding quotients are local cohomology modules supported on face ideals. Before announcing the precise statement, we will prove the following useful lemma:

Lemma 5.2.3. *Let $\mathfrak{p}_\alpha, \mathfrak{p}_\beta \subseteq R$ be face ideals. Then we have:*

$$^*\mathrm{Ext}^1_R(H^{|\alpha|}_{\mathfrak{p}_\alpha}(R), H^{|\beta|}_{\mathfrak{p}_\beta}(R)) = \begin{cases} k & \text{if } \ \mathfrak{p}_\beta \subseteq \mathfrak{p}_\alpha, \quad h(\mathfrak{p}_\alpha/\mathfrak{p}_\beta) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

PROOF: The minimal $*$-injective resolution of $H^{|\beta|}_{\mathfrak{p}_\beta}(R)$ is

$$0 \to H^{|\beta|}_{\mathfrak{p}_\beta}(R) \to^* \mathrm{E}_R(R/\mathfrak{p}_\beta)(\mathbf{1}) \to \bigoplus_{i|\beta_i=0} {}^*\mathrm{E}_R(R/\mathfrak{p}_{\beta-\varepsilon_i})(\mathbf{1}) \to \dots$$

(see e.g. [97, 3.12]). Then we only have to prove:

$$^*\mathrm{Hom}_R(H^{|\alpha|}_{\mathfrak{p}_\alpha}(R),^* \mathrm{E}_R(R/\mathfrak{p}_\beta)(\mathbf{1})) = \begin{cases} k & \text{if } \ \mathfrak{p}_\beta = \mathfrak{p}_\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

If $\mathfrak{p}_\alpha \nsubseteq \mathfrak{p}_\beta$ the statement follows easily from the fact that $\mathfrak{p}_\beta$ is the only associated prime of ${}^*\mathrm{E}_R(R/\mathfrak{p}_\beta)(\mathbf{1})$, so we will assume $\mathfrak{p}_\alpha \subseteq \mathfrak{p}_\beta$. By the equivalence of categories proved in [97, 2.8], there is a bijection between the groups ${}^*\mathrm{Hom}(H^{|\alpha|}_{\mathfrak{p}_\alpha}(R)(-\mathbf{1}),^* \mathrm{E}_R(R/\mathfrak{p}_\beta))$ and ${}^*\mathrm{Hom}_R(R/\mathfrak{p}_\alpha(\alpha - \mathbf{1}), R/\mathfrak{p}_\beta)$. A graded morphism $\varphi : R/\mathfrak{p}_\alpha(\alpha - \mathbf{1}) \to R/\mathfrak{p}_\beta$ is determined by $\varphi(1) \in (R/\mathfrak{p}_\beta)_{\mathbf{1}-\alpha}$. But

$$(R/\mathfrak{p}_\beta)_{\mathbf{1}-\alpha} = \begin{cases} k & \text{if } \ \mathfrak{p}_\beta \subseteq \mathfrak{p}_\alpha \\ 0 & \text{otherwise} \end{cases}$$

so we are done.

□

The modules in the category ε-**Str** are characterized as follows:

Proposition 5.2.4. *A graded R-module M is ε-straight if and only if there are integers $m_\alpha \geq 0$, $\alpha \in \{0,1\}^n$ and an increasing filtration $\{F_j\}_{0 \leq j \leq n}$ of M by graded submodules such that for all $1 \leq j \leq n$ one has graded isomorphisms*

$$F_j/F_{j-1} \simeq \bigoplus_{\substack{\alpha \in \{0,1\}^n \\ |\alpha| = j}} (H^{|\alpha|}_{\mathfrak{p}_\alpha}(R))^{\oplus m_\alpha}.$$

PROOF: The existence of an increasing filtration $\{G_j\}_j$ of M by ε-straight submodules such that all quotients G_j/G_{j-1} are isomorphic to local cohomology modules supported at homogeneous prime ideals is an immediate transposition to ε-straight modules of [97, 2.12] (which relies on [96, 2.5]). Inspection of Yanagawa's proof shows that, in order to prove the existence of a filtration $\{F_j\}_j$ satisfying the condition of the proposition, it is enough to show that if $\mathfrak{p}_\alpha, \mathfrak{p}_\beta$ are homogeneous prime ideals with $|\alpha| = |\beta|$, then ${}^*\mathrm{Ext}^1_R(H^{|\alpha|}_{\mathfrak{p}_\alpha}(R), H^{|\beta|}_{\mathfrak{p}_\beta}(R)) = 0$. So we are done by Lemma 5.2.3

□

The integers m_α that appear in the filtration of a ε-straight module have the following interpretation:

Proposition 5.2.5. *Let M be a ε-straight module with an increasing filtration as in Proposition 5.2.4. Then:*

$$m_\alpha = \dim_k M_{-\alpha},$$

i.e. the integer m_α, $\alpha \in \{0,1\}^n$, is the dimension of the piece of M of degree $-\alpha$.

PROOF: The equality is proved by using the fact that the pieces of a local cohomology module supported on a face ideal are well determined (see [97]), namely for $\beta \in \{0,1\}^n$ we have

$$(H^{|\alpha|}_{\mathfrak{p}_\alpha}(R))_{-\beta} = \begin{cases} 0 & \text{if } \beta \neq \alpha, \\ k & \text{if } \beta = \alpha, \end{cases}$$

and the exactness of the functors $\begin{array}{ccc} {}^*\mathrm{Mod}(R) & \longrightarrow & \mathrm{Vect}_k \\ M & \longmapsto & M_\alpha \end{array}$.

□

Local cohomology modules case: Local cohomology modules $H^r_I(R)$ supported on a squarefree monomial ideal $I \subseteq R$ are ε-straight modules by Proposition 5.2.4 and Corollary 5.1.7. In general, the submodules F_j of a filtration $\{F_j\}_{0\le j\le n}$ of $H^r_I(R)$ as in the definition above are not necessarily local cohomology modules. This is one of the reasons for us to consider the category ε-**Str**.

Consider an increasing filtration $\{F_j^r\}_{0\leq j\leq n}$ of a local cohomology module $H_I^r(R)$ as in Corollary 5.1.7, i.e. such that for all $1 \leq j \leq n$ one has

$$F_j^r/F_{j-1}^r \cong \bigoplus_{|\alpha|=j} (H^j_{\mathfrak{p}_\alpha}(R))^{\oplus\, m_{n-r,\,\alpha}}.$$

By Proposition 5.2.5 we have for all $\alpha \in \{0,1\}^n$

$$m_{n-r,\,\alpha} = \dim_k (H_I^r(R))_{-\alpha}.$$

On the other hand, M. Mustaţă has shown in [72] that if $\alpha \in \{0,1\}^n$, and $I^\vee$ is the Alexander dual ideal of I, then one has

$$\beta_{i,\alpha}(I^\vee) = \dim_k (H_I^{|\alpha|-i}(R))_{-\alpha}\ ,$$

and all other graded Betti numbers vanish. As a consequence we obtain:

Corollary 5.2.6. *Let $I^\vee \subseteq R$ be Alexander dual ideal of a squarefree monomial ideal $I \subseteq R$. Then we have:*

$$\beta_{i,\alpha}(I^\vee) = m_{n-|\alpha|+i,\,\alpha} = \dim_k \tilde{H}_{i-1}(K(>\alpha); k).$$

$\mathcal{D}$-module structure of ε-straight modules: Let $R = k[x_1, \ldots, x_n]$ be the polynomial ring, where k is a field of characteristic zero. A ε-straight module M can be endowed with a functorial $\mathcal{D}$ - module structure extending its R-module structure as follows: If $\alpha \in \mathbb{Z}^n$ and $m \in M_\alpha$, then $\partial_i \cdot m := \alpha_i\, x_i^{-1}\, m$ (see [97, Remark 2.14]). By using Proposition 5.2.5 and the additivity of characteristic cycle we have:

Proposition 5.2.7. *Let M be a ε-straight module. Then, the characteristic cycle of M is:*

$$CC(M) = \sum_{\alpha\in\{0,1\}^n} m_\alpha\ T^*_{X_\alpha}X,$$

where $m_\alpha = \dim_k M_{-\alpha}$.

The characteristic cycle of a local cohomology module $H_I^r(R)$ supported on a squarefree monomial ideal $I \subseteq R$ has been computed in Theorem 3.2.11. Now, recollecting the results given in Corollary 5.1.7, Proposition 5.2.5 and Corollary 5.2.6, we can give equivalent descriptions of the multiplicities of $CC(H_I^r(R))$. In particular we give a different approach to Proposition 3.3.25.

Corollary 5.2.8. *Let* $CC(H_I^r(R)) = \sum m_{n-r,\alpha}\ T^*_{X_\alpha}X$, *be the characteristic cycle of a local cohomology module* $H_I^r(R)$ *supported on a squarefree monomial ideal* $I \subseteq R$. *Then:*

i) $m_{n-r,\alpha}(R/I) = \dim_k \tilde{H}_{|\alpha|-r-1}(K(>\alpha); k)$.

ii) $m_{n-r,\alpha}(R/I) = \dim{}_k(H_I^r(R))_{-\alpha}$.

iii) $m_{n-r,\alpha}(R/I) = \beta_{|\alpha|-r,\alpha}(I^\vee)$.

Hilbert series of ε-straight modules: By using the previous results we can give an approach to the Hilbert series of a ε-straight module M by using the multiplicities of the characteristic cycle of M.

Let the graded Hilbert series of M be:

$$H(M; \mathbf{x}) = \sum_{\alpha\in\mathbb{Z}^n} \dim_k(M_\alpha)\ \mathbf{x}^\alpha.$$

Then, by using the properties of ε-straight modules we only have to compute $\dim_k(M_{-\alpha})$ for $\alpha \in \{0,1\}^n$ and these dimensions are the multiplicities of $CC(M)$. More precisely, let $\mathrm{supp}(\beta) := \{i \mid \beta_i \neq 0\}$ be the support of a vector $\beta \in \mathbb{Z}^n$. The sum of all Laurent monomials $\mathbf{x}^\beta$, $\beta \in \mathbb{Z}^n$, whose negative parts $\beta_- := (\beta_i \mid \beta_i < 0)$ have support precisely $\mathbf{1} - \alpha$ is

$$\prod_{\alpha_i=0} \frac{x_i^{-1}}{1-x_i^{-1}} \prod_{\alpha_j=1} \frac{1}{1-x_j}.$$

Then we have:

Theorem 5.2.9. *Let* $CC(M) = \sum m_\alpha\ T^*_{X_\alpha}X$, *be the characteristic cycle of a* ε*-straight module* M. *Then, its Hilbert series is in the form:*

$$H(M; \mathbf{x}) = \sum_{\alpha\in\{0,1\}^n} m_\alpha \prod_{\alpha_i=0} \frac{x_i^{-1}}{1-x_i^{-1}} \prod_{\alpha_j=1} \frac{1}{1-x_j}.$$

In this way, when $M = H_I^r(R)$ is a local cohomology module supported on a monomial ideal $I \subseteq R$, we can give a different approach to the formula given by N. Terai [92], by using the multiplicities of the characteristic cycle of local cohomology modules (see also [98]). Namely, we have:

Corollary 5.2.10. *Let $CC(H^r_I(R)) = \sum m_{n-r,\alpha}\ T^*_{X_\alpha}X$, be the characteristic cycle of a local cohomology module $H^r_I(R)$ supported on a squarefree monomial ideal $I \subseteq R$. Then, its Hilbert series is in the form:*

$$H(H^r_I(R);\mathbf{x}) = \sum_{\alpha\in\{0,1\}^n} m_{n-r,\alpha} \prod_{\alpha_i=0} \frac{x_i^{-1}}{1-x_i^{-1}} \prod_{\alpha_j=1} \frac{1}{1-x_j}.$$

5.3 Extension problems

Henceforth we will assume that a ε-straight module M, together with an increasing filtration $\{F_j\}_{0\leq j\leq n}$ of M as in Proposition 5.2.4, have been fixed. For each $0 \leq j \leq n$, one has then an exact sequence

$$(s_j): \qquad 0 \to F_{j-1} \to F_j \to F_j/F_{j-1} \to 0,$$

which defines an element of ${}^*\mathrm{Ext}^1_R(F_j/F_{j-1}\,, F_{j-1})$. In this section, our aim is to show that this element is determined, in a sense that will be made precise below, by the k-linear maps $\cdot x_i : M_{-\alpha} \to M_{-\alpha+\varepsilon_i}$, where $\alpha \in \{0,1\}^n$, $|\alpha| = j$ and i is such that $\alpha_i = 1$. In particular, the sequence (s_j) splits if and only if $x_i M_{-\alpha} = 0$ for α, i in this range. This is in general *not* the case for local cohomology modules (see Section 5.2), which contrasts with the situations considered in [11], where the extensions appearing are always split.

Reduction to the case of local cohomology modules: Even F_j/F_{j-1} are sums of local cohomology modules F_j are not. So first we will reduce our study to extensions of local cohomology modules by local cohomology modules. Namely, the extension class of (s_j) maps to the extension class of the sequence

$$(s'_j): \qquad 0 \to F_{j-1}/F_{j-2} \to F_j/F_{j-2} \to F_j/F_{j-1} \to 0.$$

by using the following result:

Lemma 5.3.1. *The natural maps*

$${}^*\mathrm{Ext}(F_j/F_{j-1}\,, F_{j-1}) \to^* \mathrm{Ext}(F_j/F_{j-1}\,, F_{j-1}/F_{j-2})$$

are injective for all $j \geq 1$.

Proof: Applying ${}^*\mathrm{Hom}(F_j/F_{j-1}, \cdot)$ to the short exact sequence (s_{j-1}), we obtain the exact sequence

$$ {}^*\mathrm{Ext}(F_j/F_{j-1}, F_{j-2}) \to^* \mathrm{Ext}(F_j/F_{j-1}, F_{j-1}) \to^* \mathrm{Ext}(F_j/F_{j-1}, F_{j-1}/F_{j-2}). $$

It suffices to prove that ${}^*\mathrm{Ext}(F_j/F_{j-1}, F_{j-2}) = 0$. Applying again the functor ${}^*\mathrm{Hom}(F_j/F_{j-1}, \cdot)$ to the exact sequences (s_l) for $l \le j-2$, and descending induction, the assertion reduces to the statement ${}^*\mathrm{Ext}(F_j/F_{j-1}, F_l/F_{l-1}) = 0$ for $l \le j-2$. By using Lemma 5.2.3 we are done. $\square$

Remark 5.3.2. If $I \subset R$ is an ideal defining an arbitrary arrangement of linear varieties in $\mathbb{A}_k^n$, one can prove an analogous lemma for the filtration of the local cohomology module $H_I^r(R)$ introduced in Section 5.2.

Determination of the extensions: Once we have reduced to the case of local cohomology modules we will see first that the extensions are determined by means of a pull-back.

Consider the extension class of the sequence:

$$ (s'_j): \qquad 0 \to F_{j-1}/F_{j-2} \to F_j/F_{j-2} \to F_j/F_{j-1} \to 0. $$

Let

$$ 0 \longrightarrow F_{j-1}/F_{j-2} \longrightarrow {}^*\mathrm{E}^0 \xrightarrow{d^0} {}^*\mathrm{E}^1 \longrightarrow \dots $$

be the minimal $*$-injective resolution of F_{j-1}/F_{j-2}. Given a graded morphism $F_j/F_{j-1} \to \mathrm{Im}\, d^0$, one obtains an extension of F_{j-1}/F_{j-2} by F_j/F_{j-1} taking the following pull-back:

$$ \begin{array}{ccccccccc} 0 & \longrightarrow & F_{j-1}/F_{j-2} & \longrightarrow & {}^*\mathrm{E}^0 & \longrightarrow & \mathrm{Im}\; d^0 & \longrightarrow & 0 \\ & & \uparrow = & & \uparrow & \square & \uparrow \varphi & & \\ 0 & \longrightarrow & F_{j-1}/F_{j-2} & \longrightarrow & E_\varphi & \longrightarrow & F_j/F_{j-1} & \longrightarrow & 0\,, \end{array} $$

and all extensions of F_{j-1}/F_{j-2} by F_j/F_{j-1} are obtained in this way.

Take $\alpha \in \{0,1\}^n$ with $\mid \alpha \mid = j$. Applying the functor $H^*_{\mathfrak{p}_\alpha}(\cdot)$ to this diagram, we obtain a commutative square

$$\begin{array}{ccc}
H^0_{\mathfrak{p}_\alpha}(\operatorname{Im} d^0) & \xrightarrow{\ \sim\ } & H^1_{\mathfrak{p}_\alpha}(F_{j-1}/F_{j-2}) \\
\uparrow{\scriptstyle \varphi^\alpha} & & \uparrow{\scriptstyle \sim} \\
H^0_{\mathfrak{p}_\alpha}(F_j/F_{j-1}) & \xrightarrow{\ \delta^\alpha\ } & H^1_{\mathfrak{p}_\alpha}(F_{j-1}/F_{j-2})
\end{array}$$

where

i) The upper horizontal arrow is an isomorphism because ${}^*E^0 \simeq \oplus_{|\beta|=j-1} {}^*(\mathrm{E}_R(R/\mathfrak{p}_\beta))^{\oplus m_\beta}(\mathbf{1})$, and then $H^i_{\mathfrak{p}_\alpha}({}^*E^0) = 0$ for all $i \geq 0$.

ii) The morphism δ^α is the connecting homomorphism of the given extension.

Remark 5.3.3. By using the description of F_j/F_{j-1}, F_{j-1}/F_{j-2} and the degeneracy of the Grothendieck spectral sequence

$$E_2^{p,q} = H^p_{\mathfrak{p}_\alpha}(H^q_{\mathfrak{p}_\beta}(R)) \Longrightarrow H^{p+q}_{\mathfrak{p}_\alpha+\mathfrak{p}_\beta}(R)$$

we have:

i) $H^0_{\mathfrak{p}_\alpha}(F_j/F_{j-1}) = (H^{|\alpha|}_{\mathfrak{p}_\alpha}(R))^{\oplus m_\alpha}$.

ii) $H^1_{\mathfrak{p}_\alpha}(F_{j-1}/F_{j-2}) = \bigoplus_{\alpha_i=1} (H^{|\alpha|}_{\mathfrak{p}_\alpha}(R))^{\oplus m_{\alpha-\varepsilon_i}}$.

Since $F_j/F_{j-1} = \oplus_{|\alpha|=j} H^0_{\mathfrak{p}_\alpha}(F_j/F_{j-1})$, the morphism φ is determined by the morphisms φ^α for $\mid \alpha \mid = j$, and these are in turn determined by the connecting homomorphisms δ^α via the commutative square above.

Finally, we will show that, because of Remark 5.3.3, it suffices to consider the restriction of δ^α to the k-vector space of homogeneous elements of degree $-\alpha$.

Lemma 5.3.4. *Let $k_1, k_2 \geq 0$ be integers, let $M_1 = H^{|\alpha|}_{\mathfrak{p}_\alpha}(R)^{\oplus k_1}$, $M_2 = H^{|\alpha|}_{\mathfrak{p}_\alpha}(R)^{\oplus k_2}$. The restriction map*

$${}^*Hom_R(M_1, M_2) \longrightarrow Hom_{Vect_k}((M_1)_\alpha, (M_2)_\alpha)$$

is a bijection. Moreover, these bijections are compatible with composition of graded maps.

Proof: Taking the components of a graded map $M_1 \to M_2$, it will enough to prove that all graded endomorphisms of $N := H^{|\alpha|}_{\mathfrak{p}_\alpha}(R)$ are multiplications by constants of the base field k. Let $\varphi : N \to N$ a graded endomorphism. Notice that φ can be regarded also as an endomorphism of the graded module $N(-\mathbf{1})$. By [97], $N(-\mathbf{1})$ is a straight module and φ is determined by its restriction to the $\mathbb{N}^n$-graded part of $N(-\mathbf{1})$, which is $R/\mathfrak{p}_\alpha(\alpha-\mathbf{1})$. A graded R-module map $R/\mathfrak{p}_\alpha(\alpha-\mathbf{1}) \to R/\mathfrak{p}_\alpha(\alpha-\mathbf{1})$ is determined by the image of $1 \in (R/\mathfrak{p}_\alpha(\alpha-\mathbf{1}))_{\mathbf{1}-\alpha}$. Since

$$(R/\mathfrak{p}_\alpha(\alpha-\mathbf{1}))_{\mathbf{1}-\alpha} = (R/\mathfrak{p}_\alpha)_0 = k,$$

φ must be the multiplication by some constant, as was to be proved. □

Conclusion: Any extension class of the sequence:

$$(s'_j): \qquad 0 \to F_{j-1}/F_{j-2} \to F_j/F_{j-2} \to F_j/F_{j-1} \to 0.$$

is determined by the k-linear maps

$$\delta^\alpha_{-\alpha} : H^0_{\mathfrak{p}_\alpha}(F_j/F_{j-1})_{-\alpha} \longrightarrow H^1_{\mathfrak{p}_\alpha}(F_{j-1}/F_{j-2})_{-\alpha}.$$

By using Čech complexes we determine these connecting homomorphisms obtained applying $H^*_{\mathfrak{p}_\alpha}(-)$ to the exact sequence (s'_j):

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \bigoplus(F_{j-1}/F_{j-2}[\frac{1}{x_ix_j}])_{-\alpha} & \longrightarrow & \bigoplus(F_j/F_{j-2}[\frac{1}{x_ix_j}])_{-\alpha} & \longrightarrow & \bigoplus(F_j/F_{j-1}[\frac{1}{x_ix_j}])_{-\alpha} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \bigoplus(F_{j-1}/F_{j-2}[\frac{1}{x_i}])_{-\alpha} & \longrightarrow & \bigoplus(F_j/F_{j-2}[\frac{1}{x_i}])_{-\alpha} & \longrightarrow & \bigoplus(F_j/F_{j-1}[\frac{1}{x_i}])_{-\alpha} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & (F_{j-1}/F_{j-2})_{-\alpha} & \longrightarrow & (F_j/F_{j-2})_{-\alpha} & \longrightarrow & (F_j/F_{j-1})_{-\alpha} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

We have:

$$H^0_{\mathfrak{p}_\alpha}(F_j/F_{j-1})_{-\alpha} = \operatorname{Ker}[(F_j/F_{j-1})_{-\alpha} \longrightarrow \oplus_{\alpha_i=1}(F_j/F_{j-1}[\tfrac{1}{x_i}])_{-\alpha}]$$

$$H^1_{\mathfrak{p}_\alpha}(F_{j-1}/F_{j-2})_{-\alpha} = \operatorname{Ker}[\oplus((F_{j-1}/F_{j-2})[\tfrac{1}{x_i}])_{-\alpha} \longrightarrow \oplus((F_{j-1}/F_{j-2})[\tfrac{1}{x_i x_j}])_{-\alpha}]$$

where the last assertion comes from $(F_{j-1}/F_{j-2})_{-\alpha} = 0$. On the other side, one can easily check that:

$$H^0_{\mathfrak{p}_\alpha}(F_j/F_{j-1})_{-\alpha} \simeq M_{-\alpha}$$

$$H^1_{\mathfrak{p}_\alpha}(F_{j-1}/F_{j-2})_{-\alpha} \simeq \bigoplus_{\alpha_i=1} M_{-\alpha+\varepsilon_i}$$

where the last isomorphism is given by multiplication by x_i. It turns out that, via the isomorphisms above, the corresponding map $\delta^\alpha_{-\alpha}$ is in this case precisely the map

$$\begin{array}{ccc} M_{-\alpha} & \longrightarrow & \oplus_{\alpha_i=1} M_{-\alpha+\varepsilon_i} \\ m & \mapsto & \oplus(x_i \cdot m). \end{array}$$

In conclusion,

Proposition 5.3.5. *The extension class (s_j) is uniquely determined by the k-linear maps $\cdot x_i : M_{-\alpha} \to M_{-\alpha+\varepsilon_i}$ where $|\alpha| = j$ and $\alpha_i = 1$.*

Remark 5.3.6. M. Mustaţă has proved in [72] that for local cohomology modules supported at monomial ideals, the linear maps $\cdot x_i : H^j_I(R)_{-\alpha} \to H^j_I(R)_{-\alpha+\varepsilon_i}$ can be explicitly computed in terms of the simplicial cohomology of certain Stanley–Reisner complexes attached to I. A different approach to this result is given in Section 6.4.

Chapter 6

Modules with variation zero

In Chapters 3 and 4 we have seen that the characteristic cycle of the local cohomology modules provides many information of these modules. However, it does not describe completely their structure as we have seen in Chapter 5.

The aim of this chapter is to give an approach to the study of the structure of local cohomology modules supported on squarefree monomial ideals by using the theory of $\mathcal{D}$-modules. More precisely, we will use a description of the category of regular holonomic $\mathcal{D}$-modules with support a normal crossing (e.g. local cohomology modules supported on squarefree monomial ideals) given by A. Galligo, M. Granger and Ph. Maisonobe [27] by using the Riemann-Hilbert correspondence.

In Section 6.1 we introduce the main tools we will use throughout this chapter. We have to point out that the work of A. Galligo, M. Granger and Ph. Maisonobe reflects the fact that the characteristic cycle of a regular holonomic $\mathcal{D}$-module with support a normal crossing does not describe completely its structure.

In Section 6.2 we introduce an appropriate full abelian subcategory of the category of regular holonomic $\mathcal{D}$-modules with support a normal crossing. This provides an useful framework to study local cohomology modules supported on squarefree monomial ideals.

The main result of this chapter is given in Section 6.3, where we establish an equivalence of categories between the category of ε-straight modules and the category introduced in the previous section. An explicit description of this equivalence is also given.

Finally, in Section 6.4 we study in detail the case of local cohomology modules supported on squarefree monomial ideals.

6.1 The category of n-hypercubes

Let T be the union of the coordinate hyperplanes in $X = \mathbb{C}^n$, endowed with the stratification given by the intersections of its irreducible components. We denote $Perv^T(\mathbb{C}^n)$ the category of complexes of sheaves of finitely dimensional vector spaces on $\mathbb{C}^n$ which are perverse relatively to the given stratification of T ([27, I.1]). We denote $\mathrm{Mod}(\mathcal{D}_X)^T_{hr}$ the full abelian subcategory of the category of regular holonomic modules $\mathcal{M}$ in $\mathbb{C}^n$ such that their solution complex $\mathbb{R}\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$ is an object of $Perv^T(\mathbb{C}^n)$. By the Riemann-Hilbert correspondence, the functor $\mathbb{S}\mathrm{ol}$ of solutions establishes an equivalence of categories between $\mathrm{Mod}(\mathcal{D}_X)^T_{hr}$ and $Perv^T(\mathbb{C}^n)$.

In [27], the category $Perv^T(\mathbb{C}^n)$ has been linearized as follows: Let $\mathcal{C}_n$ be the category whose objects are families $\{\mathcal{M}_\alpha\}_{\alpha\in\{0,1\}^n}$ of finitely dimensional complex vector spaces, endowed with linear maps

$$\mathcal{M}_\alpha \xrightarrow{u_i} \mathcal{M}_{\alpha+\varepsilon_i} \quad , \quad \mathcal{M}_\alpha \xleftarrow{v_i} \mathcal{M}_{\alpha+\varepsilon_i}$$

for each $\alpha \in \{0,1\}^n$ such that $\alpha_i = 0$. These maps are called canonical (resp., variation) maps, and they are required to satisfy the conditions:

$$u_i u_j = u_j u_i, \;\; v_i v_j = v_j v_i, \;\; u_i v_j = v_j u_i \;\text{ and }\; v_i u_i + id \text{ is invertible.}$$

Such an object will be called an n-hypercube. A morphism between two n-hypercubes $\{\mathcal{M}_\alpha\}_\alpha$ and $\{\mathcal{N}_\alpha\}_\alpha$ is a set of linear maps $\{f_\alpha : \mathcal{M}_\alpha \to \mathcal{N}_\alpha\}_\alpha$, commuting with the canonical and variation maps (see [28]). It is proved in [loc.cit.] that there is an equivalence of categories between $Perv^T(\mathbb{C}^n)$ and $\mathcal{C}_n$.

6.1.1 The functor GGM

Throughout this chapter we will denote by GGM the functor obtained as the composition of the categorical equivalences

$$\mathrm{Mod}(\mathcal{D}_X)^T_{hr} \longrightarrow Perv^T(\mathbb{C}^n) \longrightarrow \mathcal{C}_n.$$

The aim of this section is to describe explicitely the n-hypercube $GGM(\mathcal{M})$ corresponding to an object $\mathcal{M}$ of $\mathrm{Mod}(\mathcal{D}_X)^T_{hr}$ (see [28]).

Consider $\mathbb{C}^n = \prod_{i=1}^n \mathbb{C}_i$, let $K_i = \mathbb{R}^+ \subset \mathbb{C}_i$ and set $V_i = \mathbb{C}_i \setminus K_i$. For any $\alpha = (\alpha_1, \ldots, \alpha_n) \in \{0,1\}^n$ denote

$$\mathcal{S}_\alpha := \frac{\Gamma_{\prod_{i=1}^n V_i} \mathcal{O}_X}{\sum_{\alpha_k=1} \Gamma_{\mathbb{C}_k \times \prod_{i\neq k} V_i} \mathcal{O}_X}.$$

Denoting with a subscript 0 the stalk at the origin, one has:

- The vertices of the n-hypercube associated to $\mathcal{M}$ are the vector spaces $\mathcal{M}_\alpha := \mathrm{Hom}_{\mathcal{D}_{X,0}}(\mathcal{M}_0, S_{\alpha,0})$.
- The linear maps u_i are those induced by the natural quotient maps $\mathcal{S}_\alpha \to \mathcal{S}_{\alpha+\varepsilon_i}$.
- The linear maps v_i are the partial variation maps around the coordinate hyperplanes, i.e. for any $\varphi \in \mathrm{Hom}_{\mathcal{D}_{X,0}}(\mathcal{M}_0, \mathcal{S}_{\alpha,0})$ and $m \in \mathcal{M}_0$, $(v_i \circ u_i)(\varphi)(m) = (\Phi_i(\varphi) - \varphi)(m)$, where Φ_i is the partial monodromy around the hyperplane $x_i = 0$.

The following is proved as well in [27], [28]:

- If $CC(\mathcal{M}) = \sum m_\alpha\, T^*_{X_\alpha}\mathbb{C}^n$ is the characteristic cycle of $\mathcal{M}$, then for all $\alpha \in \{0,1\}^n$ one has the equality $\dim_{\mathbb{C}}\mathcal{M}_\alpha = m_\alpha$.
- Let $\alpha, \beta \in \{0,1\}^n$ be such that $\alpha_i\beta_i = 0$ for $1 \leq i \leq n$. For each j with $\beta_j = 1$ choose any $\lambda_j \in \mathbb{C} \setminus \mathbb{Z}$, set $\lambda_\beta = \{\lambda_j\}_j$ and let $I_{\alpha,\beta,\lambda_\beta}$ be the left ideal in $\mathcal{D}_X$:

  $$I_{\alpha,\beta,\lambda_\beta} = (\{x_i \,|\, \alpha_i = 1\}, \{\partial_k \,|\, \alpha_k = \beta_k = 0\}, \{x_j\partial_j - \lambda_j \,|\, \beta_j = 1\}).$$

  Then the simple objects of the category $\mathrm{Mod}(\mathcal{D}_X)^T_{hr}$ are the quotients:

  $$\frac{\mathcal{D}_X}{I_{\alpha,\beta,\lambda_\beta}}.$$

Remark 6.1.1. Since $\mathcal{M}_0$ is a regular holonomic $\mathcal{D}_{X,0}$-module in order to determine the solutions $\mathrm{Hom}_{\mathcal{D}_{X,0}}(\mathcal{M}_0, \mathcal{S}_{\alpha,0})$ we only have to consider those Nilsson class functions in $S_{\alpha,0}$ so if $\mathcal{N}$ denotes the set of Nilsson class functions then:

$$\mathcal{M}_\alpha := \mathrm{Hom}_{\mathcal{D}_{X,0}}(\mathcal{M}_0, \mathcal{N}_\alpha), \quad \text{where} \quad \mathcal{N}_\alpha = \mathcal{N} \cap \mathcal{S}_{\alpha,0}.$$

The functor $GGM : \mathrm{Mod}(\mathcal{D}_X)^T_{hr} \longrightarrow \mathcal{C}_n$ is a contravariant exact functor. Then we get the following result:

Lemma 6.1.2. *Let* $\mathcal{M}_\bullet : 0 \longrightarrow \mathcal{M}_0 \longrightarrow \mathcal{M}_1 \longrightarrow \cdots \longrightarrow \mathcal{M}_n \longrightarrow 0$ *be a bounded complex of* $\mathrm{Mod}(\mathcal{D}_X)^T_{hr}$*-modules. Then* $\forall r$*:*

$$GGM(H^r(\mathcal{M}_\bullet)) = H_r(GGM(\mathcal{M}_\bullet))$$

6.2 Modules with variation zero

Among the objects of the category $\mathrm{Mod}(\mathcal{D}_X)^T_{hr}$ we will be interested on those having the following property:

Definition 6.2.1. *We say that an object* $\mathcal{M}$ *of* $\mathrm{Mod}(\mathcal{D}_X)^T_{hr}$ *has* **variation zero** *if the morphisms* $v_i : \mathrm{Hom}_{\mathcal{D}_{X,0}}(\mathcal{M}_0, \mathcal{S}_{\alpha+\varepsilon_i,0}) \longrightarrow \mathrm{Hom}_{\mathcal{D}_{X,0}}(\mathcal{M}_0, \mathcal{S}_{\alpha,0})$ *are zero for all* $1 \leq i \leq n$ *and all* $\alpha \in \{0,1\}^n$ *with* $\alpha_i = 0$*.*

Modules with variation zero form a full abelian subcategory of $\mathrm{Mod}(\mathcal{D}_X)^T_{hr}$ that will be denoted $\mathcal{D}^T_{v=0}$. In the sequel, we will denote by $\mathcal{C}_{n,v=0}$ the corresponding full abelian subcategory of $\mathcal{C}_n$ of n-hypercubes having variation zero.

A simple object $\frac{\mathcal{D}_X}{I_{\alpha,\beta,\lambda_\beta}}$ of the category $\mathrm{Mod}(\mathcal{D}_X)^T_{hr}$ has variation zero if and only if $\beta_k = 0$ for $1 \leq k \leq n$. Thus, the simple objects of $\mathcal{D}^T_{v=0}$ are of the form:

$$\frac{\mathcal{D}_X}{\mathcal{D}_X(\{x_i \,|\, \alpha_i = -1\}, \{\partial_j \,|\, \alpha_j = 0\})}.$$

This module is isomorphic to the local cohomology module $\mathcal{H}^{|\alpha|}_{X_\alpha}(\mathcal{O}_X)$ (see Remark 1.3.7).

Every holonomic module has finite length, so if $\mathcal{M} \in \mathcal{D}^T_{v=0}$ then, there exists a finite increasing filtration $\{\mathcal{F}_j\}_{j\geq 0}$ of $\mathcal{M}$ by objects of $\mathcal{D}^T_{v=0}$ such that for all $j \geq 1$ one has $\mathcal{D}_X$-module isomorphisms

$$\mathcal{F}_j/\mathcal{F}_{j-1} \simeq \mathcal{H}^{|\alpha|}_{X_\alpha}(\mathcal{O}_X), \quad \alpha \in \{0,1\}^n.$$

Remark 6.2.2. The category $\mathcal{D}^T_{v=0}$, regarded as a subcategory of $\mathrm{Mod}(\mathcal{D}_X)^T_{hr}$, is not closed under extensions. For example, consider the following exact sequence of n-hypercubes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{C} & \xrightarrow{\begin{pmatrix}1\\0\end{pmatrix}} & \mathbb{C}^2 & \xrightarrow{(0\ \ 1)} & \mathbb{C} & \longrightarrow & 0 \\
 & & \Big\uparrow {\scriptstyle v_i''=0} & & \Big\uparrow {\scriptstyle v_i=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}} & & \Big\uparrow {\scriptstyle v_i'=0} & & \\
0 & \longrightarrow & \mathbb{C} & \xrightarrow[\begin{pmatrix}1\\0\end{pmatrix}]{} & \mathbb{C}^2 & \xrightarrow[(0\ \ 1)]{} & \mathbb{C} & \longrightarrow & 0
\end{array}$$

Even though the category $\mathcal{D}^T_{v=0}$ is not closed under extensions, its objects can be characterized by the following particular filtration:

Proposition 6.2.3. *An object $\mathcal{M}$ of $\mathcal{D}^T_{hr}$ has variation zero if and only if there is a increasing filtration $\{\mathcal{F}_j\}_{0\leq j\leq n}$ of $\mathcal{M}$ by objects of $\mathcal{D}^T_{hr}$ and there are integers $m_\alpha \geq 0$ for $\alpha \in \{0,1\}^n$ such that for all $1 \leq j \leq n$ one has $\mathcal{D}$-module isomorphisms*

$$\mathcal{F}_j/\mathcal{F}_{j-1} \simeq \bigoplus\nolimits_{|\alpha|=j} (\mathcal{H}^j_{X_\alpha}(\mathcal{O}_X))^{\oplus m_\alpha}.$$

Proof: If $\mathcal{M}$ is an object of $\mathcal{D}^T_{v=0}$, then the submodules $\mathcal{F}_j$ of $\mathcal{M}$ corresponding to the hypercube:

$$(\mathcal{F}_j)_\beta = \begin{cases} \mathcal{M}_\beta & \text{if } \mid \beta \mid \leq j \\ 0 & \text{otherwise,} \end{cases}$$

(the canonical and variation maps being either zero or equal to those in $\mathcal{M}$), give a filtration which satisfies the conditions of the theorem. This follows easily from the fact that the hypercube corresponding to $\mathcal{H}^{|\alpha|}_{X_\alpha}(\mathcal{O}_X)$ (see Section 6.2.1) is:

$$\mathcal{H}^{|\alpha|}_{X_\alpha}(\mathcal{O}_X)_\delta = \begin{cases} \mathbb{C} & \text{if } \delta = \alpha \\ 0 & \text{otherwise.} \end{cases}$$

Conversely, assume that $\mathcal{M}$ is an object of $\mathrm{Mod}(\mathcal{D}_X)^T_{hr}$ endowed with such a filtration $\{\mathcal{F}_j\}_{0\leq j\leq n}$. For all $1 \leq j \leq n$, we have exact sequences

$$(s_j): \qquad 0 \to \mathcal{F}_{j-1} \to \mathcal{F}_j \to \mathcal{F}_j/\mathcal{F}_{j-1} \to 0.$$

And from them we get corresponding exact sequences in the category of n-hypercubes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & [\mathcal{F}_j/\mathcal{F}_{j-1}]_\delta & \longrightarrow & [\mathcal{F}_j]_\delta & \longrightarrow & [\mathcal{F}_{j-1}]_\delta & \longrightarrow & 0 \\ & & \uparrow v_i'' & & \uparrow v_i & & \uparrow v_i' & & \\ 0 & \longrightarrow & [\mathcal{F}_j/\mathcal{F}_{j-1}]_{\delta-\varepsilon_i} & \longrightarrow & [\mathcal{F}_j]_{\delta-\varepsilon_i} & \longrightarrow & [\mathcal{F}_{j-1}]_{\delta-\varepsilon_i} & \longrightarrow & 0\,, \end{array}$$

for $\delta \in \{0,1\}^n$ with $\delta_i = 0$. From the description of the n-hypercube corresponding to $\mathcal{H}^{|\alpha|}_{X_\alpha}(\mathcal{O}_X)$ given above, it follows that we only have to consider those vertices $\delta \in \{0,1\}^n$ with $|\delta - \varepsilon_i| = j$. We have $[\mathcal{F}_j/\mathcal{F}_{j-1}]_\delta = [\mathcal{F}_{j-1}]_{\delta-\varepsilon_i} = 0$, so by induction all variations vanish, as was to be proved. □

6.2.1 Projective, Injective and Simple modules

By computing their corresponding n-hypercubes we can give some examples of modules with variation zero which have an essential role in the theory:

Projective modules

We have the following description:

Proposition 6.2.4. *Let $\mathbf{x}^\alpha \in \mathbb{C}[x_1,\dots,x_n]$ be a monomial. Then, the localizations $\mathcal{O}_X\left[\frac{1}{\mathbf{x}^\alpha}\right]$ are $\mathcal{D}_X$-modules with variation zero for all $\alpha \in \{0,1\}^n$. The corresponding n-hypercubes $GGM(\mathcal{O}_X)\left[\frac{1}{\mathbf{x}^\alpha}\right]$ are:*

- **Vertices:**

$$(\mathcal{O}_X\,[\frac{1}{\mathbf{x}^\alpha}])_\gamma = \begin{cases} \mathbb{C} & \textit{if } \gamma_i \leq \alpha_i \quad \textit{for all} \quad 1 \leq i \leq n\,, \\ \\ 0 & \textit{otherwise.} \end{cases}$$

- **Linear maps:** *The map* $u_i : (\mathcal{O}_X\,[\frac{1}{\mathbf{x}^\alpha}])_\gamma \longrightarrow (\mathcal{O}_X\,[\frac{1}{\mathbf{x}^\alpha}])_{\gamma+\varepsilon_i}$ *is the identity if both vertices are different from zero and it is zero otherwise.*

PROOF: The characteristic cycle of $\mathcal{O}_X\,[\frac{1}{\mathbf{x}^\alpha}]$ is $CC(\mathcal{O}_X[\frac{1}{\mathbf{x}^\alpha}]) = \sum_{\gamma\leq\alpha} \; T^*_{X_\gamma}X$, so we get the vertices of the corresponding n-hypercube. In order to compute the linear maps u_i we consider the presentation:

$$\mathcal{O}_X\,[\frac{1}{\mathbf{x}^\alpha}] \cong \frac{\mathcal{D}_X}{\mathcal{D}_X(\{x_i\partial_i + 1 \,|\, \alpha_i = 1\}, \{\partial_j \,|\, \alpha_j = 0\})}.$$

By solving the corresponding system of differential equations we get that the non zero vector spaces $(\mathcal{O}_X\,[\frac{1}{\mathbf{x}^\alpha}])_\gamma$ are the $\mathbb{C}$-vector spaces generated by $\frac{1}{\mathbf{x}^\alpha}$. In particular the variation is zero and the linear maps $u_i : (\mathcal{O}_X\,[\frac{1}{\mathbf{x}^\alpha}])_\gamma \longrightarrow (\mathcal{O}_X\,[\frac{1}{\mathbf{x}^\alpha}])_{\gamma+\varepsilon_i}$ are the identity when both vertices are different from zero.

$\square$

Corollary 6.2.5. *The sheaf* $\mathcal{O}_X$ *is a* $\mathcal{D}_X$*-module with variation zero. The corresponding* n*-hypercube* $GGM(\mathcal{O}_X)$ *is:*

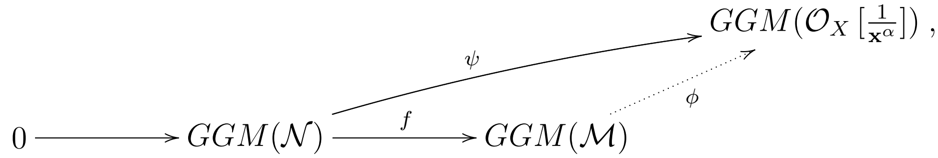
- **Vertices:**

$$(\mathcal{O}_X)_\gamma = \begin{cases} \mathbb{C} & \textit{if } \gamma = \mathbf{0} \\ \\ 0 & \textit{otherwise.} \end{cases}$$

- **Linear maps:** *The maps* $u_i : (\mathcal{O}_X)_\gamma \longrightarrow (\mathcal{O}_X)_{\gamma+\varepsilon_i}$ *are zero* $\forall i$.

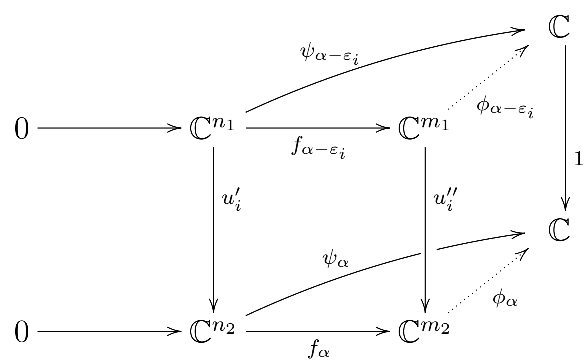
Proposition 6.2.6. *Let* $\mathbf{x}^\alpha \in \mathbb{C}[x_1, \ldots, x_n]$ *be a monomial. Then, the localizations* $\mathcal{O}_X\,[\frac{1}{\mathbf{x}^\alpha}]$ *are projective modules in the category* $\mathcal{D}^T_{v=0}$ *for all* $\alpha \in \{0,1\}^n$.

PROOF: The functor GGM is exact and contravariant so we have to prove that the n-hypercubes $GGM(\mathcal{O}_X\,[\frac{1}{\mathbf{x}^\alpha}])$ are injective objects of the category $\mathcal{C}_{n,v=0}$. Consider the diagram:



where $\mathcal{M}, \mathcal{N} \in \mathcal{D}^T_{v=0}$, f is a monomorphism and $\psi : GGM(\mathcal{N}) \longrightarrow GGM(\mathcal{O}_X[\frac{1}{\mathbf{x}^\alpha}])$ is any morphism. We will prove the existence of a morphism of n-hypercubes $\phi : GGM(\mathcal{M}) \longrightarrow GGM(\mathcal{O}_X[\frac{1}{\mathbf{x}^\alpha}])$ that makes the diagram commutative.

Since $(\mathcal{O}_X[\frac{1}{\mathbf{x}^\alpha}])_\gamma = 0$ if there exists $\gamma_i > \alpha_i$ for $1 \leq i \leq n$, we will first describe the linear map ϕ_α . Consider the corresponding n-hypercube:



Let $x \in \mathbb{C}^{m_2} = \mathcal{M}_\alpha$. We define $\phi_\alpha(x)$ as follows:

- If $x = f_\alpha(e) \in \mathrm{Im} f_\alpha$, then $\phi_\alpha(x) := \psi_\alpha(e)$.
- If $x \notin \mathrm{Im} f_\alpha$, then $\phi_\alpha(x) := 0$.

We have $\phi_\alpha \circ f_\alpha = \psi_\alpha$, so the map is well defined. Then, for all i such that $\alpha_i = 1$ we define:

$$\phi_{\alpha-\varepsilon_i} := \phi_\alpha \circ u''_i.$$

This map is also well defined. Namely we have:

$$\phi_{\alpha-\varepsilon_i} \circ f_{\alpha-\varepsilon_i} = \phi_\alpha \circ u''_i \circ f_{\alpha-\varepsilon_i} = \phi_\alpha \circ f_\alpha \circ u'_i = \psi_\alpha \circ u'_i = \psi_{\alpha-\varepsilon_i}.$$

In an analogous way we can define the linear maps ϕ_γ for all $\gamma \in \{0,1\}^n$.

□

Remark 6.2.7. One may prove that any projective module in the category $\mathcal{D}^T_{v=0}$ is a direct sum of modules of the form $\mathcal{O}_X[\frac{1}{\mathbf{x}^\alpha}]$ for $\alpha \in \{0,1\}^n$.

Injective modules

We have the following description:

Proposition 6.2.8. *The modules*

$$\mathcal{E}_\alpha = \frac{\mathcal{O}_X[\frac{1}{x_1\cdots x_n}]}{\sum_{\alpha_k=1}\mathcal{O}_X[\frac{1}{x_1\cdots\widehat{x_k}\cdots x_n}]},$$

have variation zero for all $\alpha \in \{0,1\}^n$. *The corresponding* n*-hypercubes* $GGM(\mathcal{E}_\alpha)$ *are:*

- **Vertices:**

$$(\mathcal{E}_\alpha)_\gamma = \begin{cases} \mathbb{C} & \text{if } \gamma_i \geq \alpha_i \text{ for all } 1 \leq i \leq n\,, \\ \\ 0 & \text{otherwise.} \end{cases}$$

- **Linear maps:** *The map* $u_i : (\mathcal{E}_\alpha)_\gamma \to (\mathcal{E}_\alpha)_{\gamma+\varepsilon_i}$ *is the identity if both vertices are different from zero and it is zero otherwise.*

PROOF: Modules with variation zero form a full abelian subcategory of $\mathrm{Mod}(\mathcal{D}_X)^T_{hr}$ so we are done by Proposition 6.2.4. □

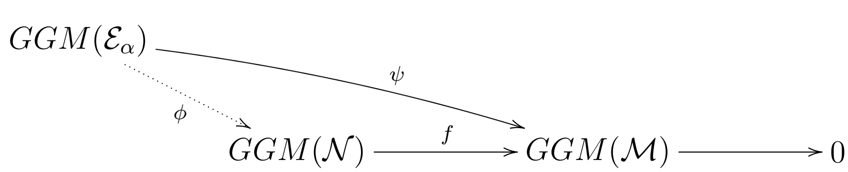
Remark 6.2.9. We have a presentation

$$\mathcal{E}_\alpha = \frac{\mathcal{D}_X}{\mathcal{D}_X(\{x_i \,|\, \alpha_i = 1\}, \{x_j\partial_j + 1 \,|\, \alpha_j = 0\})},$$

due to the fact that both modules have the same n-hypercube. By solving the corresponding system of differential equations we get that the non zero vertices of the n-hypercube are $\mathbb{C}$-vector spaces generated by $\frac{1}{x_1\cdots x_n}$.

Proposition 6.2.10. *The modules* $\mathcal{E}_\alpha$ *are injective modules in the category* $\mathcal{D}^T_{v=0}$ *for all* $\alpha \in \{0,1\}^n$.

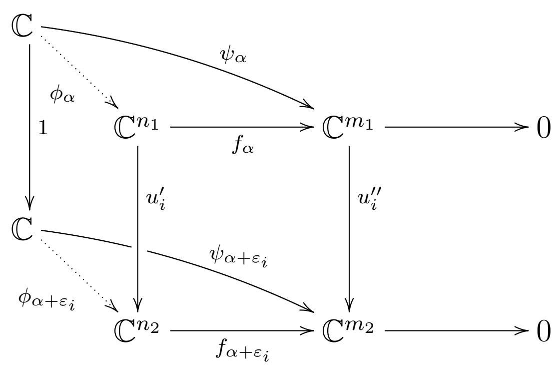
PROOF: The functor GGM is exact and contravariant so we have to prove that the n-hypercubes $GGN(\mathcal{E}_\alpha)$ are projective objects of the category $\mathcal{C}_{n,v=0}$. Consider the diagram:



,

where f is an epimorphism and $\psi : GGM(\mathcal{E}_\alpha) \longrightarrow GGM(\mathcal{M})$ is any morphism. We have to prove that there exists $\phi : GGM(\mathcal{E}_\alpha) \longrightarrow GGM(\mathcal{N})$ such that the diagram commute.

Since $(\mathcal{E}_\alpha)_\gamma = 0$ if there exists $\gamma_i < \alpha_i$ for $1 \leq i \leq n$ we will first describe the linear map ϕ_α . For that, consider the corresponding n-hypercube:



Let $\psi_\alpha(e) \in \mathbb{C}^{m_1} = \mathcal{M}_\alpha$ for a given $e \in \mathbb{C} = (\mathcal{E}_\alpha)_\alpha$. There exists $n \in \mathbb{C}^{n_1} = \mathcal{N}_\alpha$ such that $f_\alpha(n) = \psi_\alpha(e)$. We define $\phi_\alpha(e) := n$. We have $f_\alpha \circ \phi_\alpha = \psi_\alpha$, so the map is well defined. Then, for all i such that $\alpha_i = 0$ we define:

$$\phi_{\alpha+\varepsilon_i} := u'_i \circ \phi_\alpha.$$

This map is also well defined. Namely we have:

$$f_{\alpha+\varepsilon_i} \circ \phi_{\alpha+\varepsilon_i} = f_{\alpha+\varepsilon_i} \circ u'_i \circ \phi_\alpha = u''_i \circ f_\alpha \circ \phi_\alpha = u''_i \circ \psi_\alpha = \psi_{\alpha+\varepsilon_i}.$$

In an analogous way we can define the linear maps ϕ_γ for all $\gamma \in \{0,1\}^n$.

□

Remark 6.2.11. One may prove that any injective module in the category $\mathcal{D}^T_{v=0}$ is a sum of modules of the form $\mathcal{E}_\alpha$ for $\alpha \in \{0,1\}^n$.

Simple modules

Simple modules in the category $\mathcal{D}^T_{v=0}$ have been described. However, we present the precise statements here for completeness.

Proposition 6.2.12. *The local cohomology modules supported on face ideals*

$$\mathcal{H}^{|\alpha|}_{X_\alpha}(\mathcal{O}_X) = \frac{\mathcal{O}_X[\frac{1}{x_1^{\alpha_1}\cdots x_n^{\alpha_n}}]}{\sum_{\alpha_i=1} \mathcal{O}_X[\frac{1}{x_1^{\alpha_1}\cdots \widehat{x_i} \cdots x_n^{\alpha_n}}]},$$

have variation zero for all $\alpha \in \{0,1\}^n$. The n-hypercubes $GGM(\mathcal{H}^{|\alpha|}_{X_\alpha}(\mathcal{O}_X))$ are:

- **Vertices:**

$$(\mathcal{H}^{|\alpha|}_{X_\alpha}(\mathcal{O}_X))_\gamma = \begin{cases} \mathbb{C} & \text{if } \gamma = \alpha\,, \\ 0 & \text{otherwise.} \end{cases}$$

- **Linear maps:** *The map $u_i : (\mathcal{H}^{|\alpha|}_{X_\alpha}(\mathcal{O}_X))_\gamma \to (\mathcal{H}^{|\alpha|}_{X_\alpha}(\mathcal{O}_X))_{\gamma+\varepsilon_i}$ is zero $\forall i$.*

Proof: Modules with variation zero form a full abelian subcategory of $\mathrm{Mod}(\mathcal{D}_X)^T_{hr}$ so we are done by Proposition 6.2.4. □

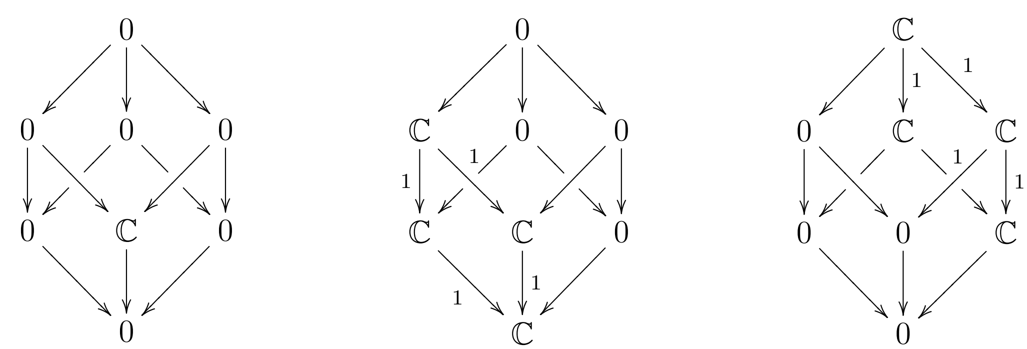
Remark 6.2.13. From the presentation (see Remark 1.3.7):

$$\mathcal{H}^{|\alpha|}_{X_\alpha}(\mathcal{O}_X) = \frac{\mathcal{D}_X}{\mathcal{D}_X(\{x_i \mid \alpha_i = 1\}, \{\partial_j \mid \alpha_j = 0\})}$$

we get that the non zero vertices of the n-hypercube is the $\mathbb{C}$-vector space generated by $\frac{1}{\mathbf{x}^\alpha}$.

Proposition 6.2.14. *The local cohomology modules supported on face ideals $\mathcal{H}^{|\alpha|}_{X_\alpha}(\mathcal{O}_X)$ are the simple modules in the category $\mathcal{D}^T_{v=0}$ for all $\alpha \in \{0,1\}^n$.*

Example: Let $\mathcal{O}_X$ be the sheaf of holomorphic functions over $X = \mathbb{C}^3$. The n-hypercubes corresponding to the modules with variation zero $\mathcal{H}^2_{X_{(1,0,1)}}(\mathcal{O}_X)$, $\mathcal{E}_{(1,0,0)}$ and $\mathcal{O}_X[\frac{1}{\mathbf{x}^{(0,1,1)}}]$ are:



$GGM(\mathcal{H}^2_{X_{(1,0,1)}}(\mathcal{O}_X))$ $GGM(\mathcal{E}_{(1,0,0)})$ $GGM(\mathcal{O}_X[\frac{1}{\mathbf{x}^{(0,1,1)}}])$

6.3 An Equivalence of Categories

Let $R = \mathbb{C}[x_1, \ldots, x_n]$, be the polynomial ring with coefficients in $\mathbb{C}$ and $\mathcal{D} = D(R, \mathbb{C})$ the corresponding ring of differential operators. If M is a $\mathcal{D}$-module, then $\mathcal{M}^{an} := \mathcal{O}_X \otimes_R M$ has a natural $\mathcal{D}_X$-module structure. This allows to define a functor

$$\begin{array}{rcl} (-)^{an}: Mod(\mathcal{D}) & \longrightarrow & Mod(\mathcal{D}_X). \\ M & \longrightarrow & \mathcal{M}^{an} \\ f & \longrightarrow & id \otimes f \end{array}$$

On the other hand, any ε-straight module M can be endowed with a functorial $\mathcal{D}$-module structure extending its R-module structure (see Section 5.3).

Example: Since the morphism $R \to \mathcal{O}_X$ is flat, one has isomorphisms:

i) $\mathcal{O}_X \otimes_R R[\frac{1}{\mathbf{x}^\alpha}] \cong \mathcal{O}_X[\frac{1}{\mathbf{x}^\alpha}]$ for all $\alpha \in \{0,1\}^n$.

ii) $\mathcal{O}_X \otimes_R {}^*\mathrm{E}_R(R/\mathfrak{p}_\alpha)(\mathbf{1}) \cong \mathcal{E}_\alpha$ for all $\alpha \in \{0,1\}^n$.

To see this isomorphism, we may use the description of $*$-injective envelopes in [33, 3.1.5] (or, alternatively, the equivalence of categories proved in [97, 2.8]) to get the presentation

$${}^*\mathrm{E}_R(R/\mathfrak{p}_\alpha)(\mathbf{1}) \cong \frac{R[\frac{1}{x_1 \cdots x_n}]}{\sum_{\alpha_k=1} R[\frac{1}{x_1 \cdots \widehat{x_k} \cdots x_n}]}.$$

iii) $\mathcal{O}_X \otimes_R H^{|\alpha|}_{\mathfrak{p}_\alpha}(R) \cong \mathcal{H}^{|\alpha|}_{X_\alpha}(\mathcal{O}_X)$ for all $\alpha \in \{0,1\}^n$.

From 5.2.4 and 6.2.3, it follows that if M is a ε-straight module then $\mathcal{M}^{an}$ is an object of $\mathcal{D}^T_{v=0}$. The main result of this section is the following :

Theorem 6.3.1. *The functor*

$$(-)^{an} : \varepsilon - \mathbf{Str} \longrightarrow \mathcal{D}^T_{v=0}$$

is an equivalence of categories.

We will prove first the following lemma (which in particular gives the fully faithfulness of $(-)^{an}$):

Lemma 6.3.2. *Let M, N be ε-straight modules. For all $i \geq 0$, we have functorial isomorphisms*

$$^*\mathrm{Ext}^i_R(M,N) \cong \mathrm{Ext}^i_{\mathcal{D}^T_{v=0}}(\mathcal{M}^{an}, \mathcal{N}^{an}).$$

PROOF: It has been proved by Yanagawa that the category of straight modules has enough injectives. It will follow from our proof that the category $\mathcal{D}^T_{v=0}$ has enough injectives as well, so that both Ext functors are defined and can be computed using resolutions.

By induction on the length we can suppose that M and N are simple objects, i.e. $M = H^{|\alpha|}_{\mathfrak{p}_\alpha}(R)$ and $N = H^{|\beta|}_{\mathfrak{p}_\beta}(R)$. The minimal $*$-injective resolution of N is:

$$0 \to H^{|\beta|}_{\mathfrak{p}_\beta}(R) \to^* \mathrm{E}_R(R/\mathfrak{p}_\beta)(\mathbf{1}) \to \bigoplus_{i|\beta_i=0} {}^*\mathrm{E}_R(R/\mathfrak{p}_{\beta+\varepsilon_i})(\mathbf{1}) \to \cdots \qquad (6.1)$$

From flatness of $R \to \mathcal{O}_X$ and the injectivity of the modules $\mathcal{E}_\alpha$ proved in Proposition 6.2.10, it follows that one has the following injective resolution of $\mathcal{N}^{an} = \mathcal{H}^{|\beta|}_{X_\beta}(\mathcal{O}_X)$ in $\mathcal{D}^T_{v=0}$:

$$0 \to \mathcal{H}^{|\beta|}_{X_\beta}(\mathcal{O}_X) \to \mathcal{E}_\beta \to \bigoplus_{i|\beta_i=0} \mathcal{E}_{\beta+\varepsilon_i} \to \cdots \qquad (6.2)$$

Let $K_1^\bullet$ be the complex obtained applying $^*\mathrm{Hom}_R(H^{|\alpha|}_{\mathfrak{p}_\alpha}(R), -)$ to the resolution (6.1) and let $K_2^\bullet$ be the one obtained applying $\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{H}^{|\alpha|}_{X_\alpha}(\mathcal{O}_X), -)$ to (6.2). We have an injection $K_1^\bullet \hookrightarrow K_2^\bullet$ and we want to show that it is an isomorphism. We have

$$^*\mathrm{Hom}_R(H^{|\alpha|}_{\mathfrak{p}_\alpha}(R), {}^*\mathrm{E}_R(R/\mathfrak{p}_\gamma)(\mathbf{1})) = \begin{cases} \mathbb{C} & \text{if } \alpha = \gamma \\ 0 & \text{otherwise.} \end{cases} \qquad (6.3)$$

This can be seen taking the positively graded parts and using [97, Proposition 2.8], as done before in similar situations. The same equality holds replacing the left hand side in (6.3) by $\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{H}^{|\alpha|}_{X_\alpha}(\mathcal{O}_X), \mathcal{E}_\gamma)$, this is easily proved considering the corresponding n-hypercubes. It follows that $K_1^\bullet \cong K_2^\bullet$, and then we are done.

□

Now we may prove Theorem 6.3.1:

PROOF: By Lemma 6.3.2 the functor $(-)^{an}$ is fully faithful, so it remains to prove that it is dense. Let $\mathcal{N}$ be an object of $\mathcal{D}^T_{v=0}$, and let $\mathcal{N}' \subseteq \mathcal{N}$ be a submodule such that $\mathcal{N}'' := \mathcal{N}/\mathcal{N}'$ is simple. By induction on the length, there are ε-straight R-modules M' and M'' such that $\mathcal{N}' \cong (\mathcal{M}')^{an}$ and $\mathcal{N}'' \cong (\mathcal{M}'')^{an}$. The extension $0 \to \mathcal{N}' \to \mathcal{N} \to \mathcal{N}'' \to 0$ correspond to an element ξ of $\mathrm{Ext}^1_{\mathcal{D}^T_{v=0}}(\mathcal{N}'', \mathcal{N}')$. Let $0 \to M' \to M \to M'' \to 0$ be an extension such that its class in ${}^*\mathrm{Ext}^1_R(M'', M')$ maps to ξ via the isomorphism of Lemma 6.3.2. One can check that $\mathcal{N} \cong (M)^{an}$, and then the theorem is proved.

□

Remark 6.3.3. Despite the category $\mathcal{D}^T_{v=0}$, regarded as a subcategory of $\mathrm{Mod}(\mathcal{D}_X)^T_{hr}$, is not closed under extensions (see Section 6.1.2), the category of ε-straight modules, regarded as a subcategory of the category of $\mathbb{Z}^n$-graded modules, is closed under extensions (see [97, Lemma 2.10]).

6.3.1 The graded structure of $\mathcal{D}_X$-modules with variation zero

Let $\mathcal{M} \in \mathcal{D}^T_{v=0}$ be a regular holonomic $\mathcal{D}_X$-module with variation zero and $M \in \varepsilon - \mathbf{Str}$ be the corresponding ε-straight module. Our aim in this section is to describe the n-hypercube $GGM(\mathcal{M})$ from the ε-straight module structure of M.

Note that a Nilsson class function

$$f = \sum_{\alpha,m} \varphi_{\alpha,m}(\mathbf{x})(\log \mathbf{x})^m \mathbf{x}^\alpha,$$

where $\varphi_{\alpha,m}(\mathbf{x}) \in \mathbb{C}\{\mathbf{x}\}$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$ and $m = (m_1, \ldots, m_n) \in (\mathbb{Z}^+)^n$, is a solution of a module with variation zero if and only if $\alpha \in \mathbb{Z}^n$, i.e. $f \in \mathcal{O}_X[\frac{1}{x_1 \cdots x_n}]$.

So, in order to determine the vertices of $GGM(\mathcal{M})$ for any module $\mathcal{M} \in \mathcal{D}^T_{v=0}$, we only have to consider the following spaces of solutions :

$$\mathcal{E}_\alpha := \frac{\mathcal{O}_X[\frac{1}{x_1 \cdots x_n}]}{\sum_{\alpha_k=1} \mathcal{O}_X[\frac{1}{x_1 \cdots \widehat{x_k} \cdots x_n}]} \subseteq \frac{\Gamma_{\prod_{i=1}^n V_i} \mathcal{O}_X}{\sum_{\alpha_k=1} \Gamma_{\mathbb{C}_k \times \prod_{i \neq k} V_i} \mathcal{O}_X} = \mathcal{S}_\alpha, \quad \alpha \in \{0,1\}^n.$$

In particular, we can give another description of the vertices of the n-hypercube of a module with variation zero. They are the vector spaces

$$\mathcal{M}_\alpha := \mathrm{Hom}_{\mathcal{D}_{X,0}}(\mathcal{M}_0, \mathcal{E}_{\alpha,0}).$$

This description makes the vertices of the n-hypercube more treatable due to the fact that $\mathcal{E}_\alpha$ are also modules with variation zero. In the sequel we will denote for simplicity E_α the corresponding ε-straight module, i.e.

$$E_\alpha := {}^*\mathrm{E}_R(R/\mathfrak{p}_\alpha)(\mathbf{1}) \cong \frac{R[\frac{1}{x_1\cdots x_n}]}{\sum_{\alpha_k=1} R[\frac{1}{x_1\cdots\widehat{x_k}\cdots x_n}]}.$$

We have to point out that, by using Proposition 5.2.7, the vertices $\mathcal{M}_\alpha$ of $GGM(\mathcal{M})$ are isomorphic to the graded pieces $M_{-\alpha}$ of the corresponding ε-straight module M for all $\alpha \in \{0,1\}^n$. But, in order to reflect the ε-straight module structure of M, we will consider the following description:

Proposition 6.3.4. *Let $\mathcal{M} \in \mathcal{D}^T_{v=0}$ be a regular holonomic $\mathcal{D}_X$-module with variation zero and $M \in \varepsilon - \mathbf{Str}$ be the corresponding ε-straight module. Then we have the isomorphism:*

$$\mathcal{M}_\alpha \cong (M_{-\alpha})^*,$$

where $(M_{-\alpha})^*$ denotes the dual of the $\mathbb{C}$-vector space defined by the piece of M of multidegree $-\alpha$, for all $\alpha \in \{0,1\}^n$.

PROOF: By using the isomorphism given in Lemma 6.3.2, we only have to describe the $\mathbb{C}$-vector space ${}^*\mathrm{Hom}_R(M, E_\alpha)$.

Any map $f \in {}^*\mathrm{Hom}_R(M, E_\alpha)$ is determined by the pieces $f_{-\beta}$, $\beta \in \{0,1\}^n$ due to the fact that M and E_α are ε-straight modules. Notice that $f_{-\beta} = 0$, for all $\beta < \alpha$, due to the fact that $\mathfrak{p}_\alpha$ is the unique associated prime of E_α. On the other side, for all i such that $\alpha_i = 0$ we have

$$f_{-\alpha-\varepsilon_i} : M_{-\alpha-\varepsilon_i} \xrightarrow{x_i} M_{-\alpha} \xrightarrow{f_{-\alpha}} [E_\alpha]_{-\alpha} \xrightarrow{\frac{1}{x_i}} [E_\alpha]_{-\alpha-\varepsilon_i} \ ,$$

i.e. $f_{-\alpha-\varepsilon_i} := \frac{1}{x_i} \circ f_{-\alpha} \circ x_i$. By using this construction in an analogous way we can describe $f_{-\beta}$ for all $\beta \geq \alpha$. In particular, the map $f \in {}^*\mathrm{Hom}_R(M, E_\alpha)$ is determined by the piece $f_{-\alpha}$. Namely, we have the isomorphism:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{C}}(M_{-\alpha}, [E_\alpha]_{-\alpha}) & \longrightarrow & {}^*\mathrm{Hom}_R(M, E_\alpha) \\ f_{-\alpha} & \dashrightarrow & f \end{array}.$$

Finally, since $[E_\alpha]_{-\alpha}$ is the $\mathbb{C}$-vector space spanned by $\frac{1}{\mathbf{x}^\alpha}$, the multiplication by $\mathbf{x}^\alpha$ gives an isomorphism:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{C}}(M_{-\alpha}, [E_\alpha]_{-\alpha}) & \longrightarrow & \mathrm{Hom}_{\mathbb{C}}(M_{-\alpha}, \mathbb{C}) = (M_{-\alpha})^* \\ f_{-\alpha} & \dashrightarrow & \mathbf{x}^\alpha f_{-\alpha} \end{array},$$

where we consider $\mathbb{C}$ as the $\mathbb{C}$-vector space spanned by 1.

□

Once the vertices of the n-hypercube are determined, the linear maps

$$u_i : \mathrm{Hom}_{\mathcal{D}_{X,0}}(\mathcal{M}_0, \mathcal{E}_{\alpha,0}) \longrightarrow \mathrm{Hom}_{\mathcal{D}_{X,0}}(\mathcal{M}_0, \mathcal{E}_{\alpha+\varepsilon_i,0})$$

induced by the natural quotient maps $\mathcal{E}_\alpha \longrightarrow \mathcal{E}_{\alpha+\varepsilon_i}$, may be described as follows:

Proposition 6.3.5. *Let $\mathcal{M} \in \mathcal{D}^T_{v=0}$ be a regular holonomic $\mathcal{D}_X$-module with variation zero and $M \in \varepsilon-\mathbf{Str}$ be the corresponding ε-straight module. Then, the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{M}_\alpha & \xrightarrow{u_i} & \mathcal{M}_{\alpha+\varepsilon_i} \\ \cong \uparrow & & \cong \uparrow \\ (M_{-\alpha})^* & \xrightarrow{(x_i)^*} & (M_{-\alpha-\varepsilon_i})^* \end{array}$$

where $(x_i)^*$ is the dual of the multiplication by x_i.

Proof: By using the isomorphism given in Lemma 6.3.2, we only have to describe the linear map

$$u_i : {}^*\mathrm{Hom}_R(M, E_\alpha) \longrightarrow {}^*\mathrm{Hom}_R(M, E_{\alpha+\varepsilon_i})$$

induced by the natural quotient maps $E_\alpha \longrightarrow E_{\alpha+\varepsilon_i}$.

Let $f \in {}^*\mathrm{Hom}_R(M, E_\alpha)$ be a morphism described, in the sense given in the proof of Proposition 6.3.4, by the linear map $f_{-\alpha} \in \mathrm{Hom}_{\mathbb{C}}(M_{-\alpha}, [E_\alpha]_{-\alpha})$. Then, the corresponding morphism $\overline{f} \in {}^*\mathrm{Hom}_R(M, E_{\alpha+\varepsilon_i})$ induced by the quotient map $E_\alpha \to E_{\alpha+\varepsilon_i}$ is described by $f_{-\alpha-\varepsilon_i} \in \mathrm{Hom}_{\mathbb{C}}(M_{-\alpha-\varepsilon_i}, [E_{\alpha+\varepsilon_i}]_{-\alpha-\varepsilon_i})$ defined as:

$$\begin{array}{ccccccc} M_{-\alpha-\varepsilon_i} & \xrightarrow{x_i} & M_{-\alpha} & \xrightarrow{f_{-\alpha}} & [E_\alpha]_{-\alpha} & \xrightarrow{\frac{1}{x_i}} & [E_\alpha]_{-\alpha-\varepsilon_i} \\ & \searrow^{f_{-\alpha-\varepsilon_i}} & & & & & \downarrow id \\ & & & & & & [E_{\alpha+\varepsilon_i}]_{-\alpha-\varepsilon_i} \end{array}.$$

In particular we have the commutative diagram:

$$\begin{array}{ccc}
{}^*\mathrm{Hom}_R(M, E_\alpha) & \longrightarrow & {}^*\mathrm{Hom}_R(M, E_{\alpha+\varepsilon_i}) \\
f & \cdots\cdots\to & \overline{f} \\
\cong \uparrow \quad \vdots & & \vdots \quad \uparrow \cong \\
f_{-\alpha} & \cdots\cdots\to & f_{-\alpha-\varepsilon_i} \\
\mathrm{Hom}_{\mathbb{C}}(M_{-\alpha}, [E_\alpha]_{-\alpha}) & \longrightarrow & \mathrm{Hom}_{\mathbb{C}}(M_{-\alpha-\varepsilon_i}, [E_{\alpha+\varepsilon_i}]_{-\alpha-\varepsilon_i})
\end{array}$$

Then we are done because the following diagram is commutative:

$$\begin{array}{ccc}
(M_{-\alpha})^* & \xrightarrow{(x_i)^*} & (M_{-\alpha-\varepsilon_i})^* \\
\mathbf{x}^\alpha f_{-\alpha} & \cdots\cdots\to & \mathbf{x}^{\alpha+\varepsilon_i} f_{-\alpha-\varepsilon_i} \\
\cong \uparrow \quad \vdots & & \vdots \quad \uparrow \cong \\
f_{-\alpha} & \cdots\cdots\to & f_{-\alpha-\varepsilon_i} \\
\mathrm{Hom}_k(M_{-\alpha}, [E_\alpha]_{-\alpha}) & \longrightarrow & \mathrm{Hom}_k(M_{-\alpha-\varepsilon_i}, [E_{\alpha+\varepsilon_i}]_{-\alpha-\varepsilon_i})
\end{array}$$

Namely, we have:

$$(x_i)^*(\mathbf{x}^\alpha f_{-\alpha}) = \mathbf{x}^\alpha f_{-\alpha} x_i = \mathbf{x}^{\alpha+\varepsilon_i} \frac{1}{x_i} f_{-\alpha} x_i = \mathbf{x}^{\alpha+\varepsilon_i} f_{-\alpha-\varepsilon_i}.$$

□

6.4 Local cohomology modules

The aim of this section is to compute the n-hypercube corresponding to local cohomology modules supported on squarefree monomial ideals.

Let $H_I^r(R)$ be a local cohomology module supported on a squarefree monomial ideal $I \subseteq R = \mathbb{C}[x_1, \ldots, x_n]$. By using the equivalence of categories given in Section 6.3, the corresponding n-hypercube may be described as follows:

Vertices: $(\mathcal{H}_I^r(\mathcal{R}))_\alpha = {}^*\mathrm{Hom}_R(H_I^r(R), E_\alpha)$

Linear maps: $u_i : \ {}^*\mathrm{Hom}_R(H_I^r(R), E_\alpha) \longrightarrow {}^*\mathrm{Hom}_R(H_I^r(R), E_{\alpha+\varepsilon_i})$, induced by the quotient map $E_\alpha \longrightarrow E_{\alpha+\varepsilon_i}$.

Notice that, by using Corollary 6.1.2, we only have to compute the homology of the complex $GGM(\check{C}_I^\bullet)$ obtained by applying the functor GGM to the Čech complex $\check{C}_I^\bullet$.

Namely, let $I = (\mathbf{x}^{\alpha_1}, \dots, \mathbf{x}^{\alpha_s}) \subseteq R$ be a minimal system of generators of the squarefree monomial ideal I. Consider the Čech complex:

$$\check{C}_I^\bullet : \; 0 \longrightarrow R \xrightarrow{d_0} \bigoplus_{1 \le i_1 \le s} R[\frac{1}{\mathbf{x}^{\alpha_{\mathbf{i_1}}}}] \xrightarrow{d_1} \cdots \xrightarrow{d_{s-1}} R[\frac{1}{\mathbf{x}^{\alpha_1} \cdots \mathbf{x}^{\alpha_s}}] \longrightarrow 0.$$

Then we have:

Proposition 6.4.1. *Local cohomology modules $H_I^r(R)$ supported on squarefree monomial ideals $I \subseteq R$, have variation zero. The corresponding n-hypercubes $GGM(H_I^r(R))$ are:*

- **Vertices:** $(\mathcal{H}_I^r(\mathcal{R}))_\alpha = H_r({}^*\mathrm{Hom}_R(\check{C}_I^\bullet, E_\alpha))$.
- **Linear maps:** $u_i \; : \; H_r({}^*\mathrm{Hom}_R(\check{C}_I^\bullet, E_\alpha)) \longrightarrow H_r({}^*\mathrm{Hom}_R(\check{C}_I^\bullet, E_{\alpha+\varepsilon_i}))$, *induced by the quotient map $E_\alpha \longrightarrow E_{\alpha+\varepsilon_i}$.*

To illustrate this computations we present the following:

Example: Let $R = \mathbb{C}[x_1, x_2, x_3]$. Consider the ideal:

- $I = (x_1x_2, x_1x_3, x_2x_3) = (x_1, x_2) \cap (x_1, x_3) \cap (x_2, x_3)$.

Applying Theorem 3.2.11 there is a local cohomology module different from zero and its characteristic cycle is:

$$CC(H_I^2(R)) = T^*_{X_{(1,1,0)}}X + T^*_{X_{(1,0,1)}}X + T^*_{X_{(0,1,1)}}X + 2 \; T^*_{X_{(1,1,1)}}X,$$

in particular we only have to study the vertices of the n-hypercube $(\mathcal{H}_I^2(\mathcal{R}))_\alpha$ for $\alpha = (1,1,0), (1,0,1), (0,1,1,), (1,1,1)$.

We have the Čech complex:

$$\check{C}_I^\bullet : \quad 0 \longrightarrow R \xrightarrow{d_0} \begin{matrix} R[\frac{1}{x_1x_2}] \\ \oplus \\ R[\frac{1}{x_1x_3}] \\ \oplus \\ R[\frac{1}{x_2x_3}] \end{matrix} \xrightarrow{d_1} \begin{matrix} R[\frac{1}{x_1x_2x_3}] \\ \oplus \\ R[\frac{1}{x_1x_2x_3}] \\ \oplus \\ R[\frac{1}{x_1x_2x_3}] \end{matrix} \xrightarrow{d_2} R[\frac{1}{x_1x_2x_3}] \longrightarrow 0$$

• The complex ${}^*\mathrm{Hom}_R(\check{C}_I^\bullet, E_{(1,1,1)})$ is of the form:

$$0 \longleftarrow 0 \xleftarrow{d_0} 0 \xleftarrow{d_1} \mathbb{C}^3 \xleftarrow{d_2=\begin{pmatrix}1\\-1\\1\end{pmatrix}} \mathbb{C} \longleftarrow 0$$

Then, we get the vertex:

$$(\mathcal{H}_I^2(\mathcal{R}))_{(1,1,1)} = H_2({}^*\mathrm{Hom}_R(\check{C}_I^\bullet, E_{(1,1,1)})) = \mathbb{C}^2.$$

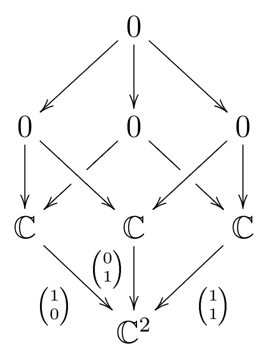
• The complexes ${}^*\mathrm{Hom}_R(\check{C}_I^\bullet, E_\alpha)$, for $\alpha = (1,1,0), (1,0,1), (0,1,1,)$, are of the form:

$$0 \longleftarrow 0 \xleftarrow{d_0} \mathbb{C} \xleftarrow{d_1=(1\ \ 0\ \ -1)} \mathbb{C}^3 \xleftarrow{d_2=\begin{pmatrix}1\\-1\\1\end{pmatrix}} \mathbb{C} \longleftarrow 0$$

Then, we get the vertices:

$$(\mathcal{H}_I^2(\mathcal{R}))_\alpha = H_2({}^*\mathrm{Hom}_R(\check{C}_I^\bullet, E_\alpha)) = \mathbb{C}, \text{ for } \alpha = (1,1,0), (1,0,1), (0,1,1,).$$

Computing the linear maps among these vertices in an adequate basis we get the n-hypercube:



6.4.1 Topological interpretation

The n-hypercubes corresponding to local cohomology modules supported on squarefree monomial ideals can be better described in topological terms by using the cellular structure of the complexes used in their computation.

As in the previous section, let $I = (\mathbf{x}^{\alpha_1}, \dots, \mathbf{x}^{\alpha_s}) \subseteq R$ be a minimal system of generators of a squarefree monomial ideal I. Consider the Čech complex:

$$\check{C}_I^{\bullet}: \ 0 \longrightarrow R \xrightarrow{d_0} \bigoplus_{1 \le i_1 \le s} R[\frac{1}{\mathbf{x}^{\alpha_{\mathbf{i_1}}}}] \xrightarrow{d_1} \cdots \xrightarrow{d_{s-1}} R[\frac{1}{\mathbf{x}^{\alpha_{\mathbf{1}}} \cdots \mathbf{x}^{\alpha_{\mathbf{s}}}}] \longrightarrow 0.$$

Vertices: To describe the vertices of the n-hypercubes corresponding to the local cohomology modules $H_I^r(R)$ we only have to notice the following:

• Applying the functor ${}^*\mathrm{Hom}_R(-, E_\alpha)$ to the Čech complex $\check{C}_I^{\bullet}$ in the case $\alpha = \mathbf{0} = (0, \dots, 0) \in \{0,1\}^n$, we obtain the complex:

$${}^*\mathrm{Hom}_R(\check{C}_I^{\bullet}, E_{\mathbf{0}}): \quad 0 \longleftarrow \mathbb{C} \xleftarrow{d_0} \mathbb{C}^s \xleftarrow{d_1} \cdots \xleftarrow{d_{s-1}} \mathbb{C} \longleftarrow 0\,.$$

This complex may be identified with the augmented relative simplicial chain complex $\widetilde{\mathcal{C}}_{\bullet}(\Delta; \mathbb{C})$, where Δ is the full simplicial complex whose vertices are labelled by the minimal system of generators of I.

Remark 6.4.2. The faces of Δ are labelled by the posets $\mathcal{I}, \mathcal{J}$ introduced in section 1.2.7. Namely, we have:

$$\sigma_{\mathbf{1}-\alpha} \in \Delta \Longleftrightarrow \mathfrak{p}_\alpha \in \mathcal{I}^\vee \Longleftrightarrow \mathbf{x}^\alpha \in \mathcal{J}.$$

• In general, for any $\alpha \in \{0,1\}^n$, the terms of the complex ${}^*\mathrm{Hom}_R(\check{C}_I^{\bullet}, E_\alpha)$ are:

$${}^*\mathrm{Hom}_R(R[\frac{1}{\mathbf{x}^\beta}], E_\alpha) = \begin{cases} \mathbb{C} & \text{if } \beta \ge \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Namely, from the augmented relative simplicial chain complex $\widetilde{\mathcal{C}}_{\bullet}(\Delta; \mathbb{C})$, we are taking out the pieces corresponding to the faces $\sigma_{\mathbf{1}-\beta} \in \Delta$ such that $\beta \not\ge \alpha$.

Let $T_\alpha := \{\sigma_{\mathbf{1}-\beta} \in \Delta \mid \beta \not\ge \alpha\}$ be a simplicial subcomplex of Δ. Then, the complex ${}^*\mathrm{Hom}_R(\check{C}_I^{\bullet}, E_\alpha)$ may be identified with the augmented relative simplicial chain complex $\widetilde{\mathcal{C}}_{\bullet}(\Delta, T_\alpha; \mathbb{C})$ associated to the pair (Δ, T_α). By taking homology, the vertices of the n-hypercubes corresponding to the local cohomology modules $H_I^r(R)$ are:

$$(\mathcal{H}_I^r(\mathcal{R}))_\alpha = H_r({}^*\mathrm{Hom}_R(\check{C}_I^{\bullet}, E_\alpha)) = \widetilde{H}_{r-1}(\Delta, T_\alpha; \mathbb{C}) = \widetilde{H}_{r-2}(T_\alpha; \mathbb{C}),$$

where the last assertion comes from the fact that Δ is contractible.

Linear maps: By using the description of the vertices we notice the following:

• The morphism of complexes ${}^*\mathrm{Hom}_R(\check{C}_I^{\bullet}, E_\alpha)) \longrightarrow^* \mathrm{Hom}_R(\check{C}_I^{\bullet}, E_{\alpha+\varepsilon_i})$, induced by the quotient map $E_\alpha \longrightarrow E_{\alpha+\varepsilon_i}$ is nothing but the morphism of complexes $\widetilde{\mathcal{C}}_{\bullet}(\Delta, T_\alpha; \mathbb{C}) \longrightarrow \widetilde{\mathcal{C}}_{\bullet}(\Delta, T_{\alpha+\varepsilon_i}; \mathbb{C})$ induced by the inclusion $T_\alpha \subseteq T_{\alpha+\varepsilon_i}$. By taking homology, the linear maps u_i of the n-hypercubes corresponding to the local cohomology modules $H_I^r(R)$ are:

$$\nu_i : \widetilde{H}_{r-2}(T_\alpha; \mathbb{C}) \longrightarrow \widetilde{H}_{r-2}(T_{\alpha+\varepsilon_i}; \mathbb{C}),$$

induced by the inclusion $T_\alpha \subseteq T_{\alpha+\varepsilon_i}$.

Collecting the previous results we get:

Proposition 6.4.3. *Local cohomology modules $H_I^r(R)$ supported on squarefree monomial ideals $I \subseteq R$, have variation zero. The corresponding n-hypercubes $GGM(H_I^r(R))$ are:*

- **Vertices:** $(\mathcal{H}_I^r(\mathcal{R}))_\alpha \cong \widetilde{H}_{r-2}(T_\alpha; \mathbb{C})$.
- **Linear maps:** *We have the following commutative diagram:*

$$\begin{array}{ccc} (\mathcal{H}_I^r(\mathcal{R}))_\alpha & \xrightarrow{u_i} & (\mathcal{H}_I^r(\mathcal{R}))_{\alpha+\varepsilon_i} \\ \downarrow \cong & & \downarrow \cong \\ \widetilde{H}_{r-2}(T_\alpha; \mathbb{C}) & \xrightarrow{\nu_i} & \widetilde{H}_{r-2}(T_{\alpha+\varepsilon_i}; \mathbb{C}) \end{array}$$

where ν_i is induced by the inclusion $T_\alpha \subseteq T_{\alpha+\varepsilon_i}$.

Remark 6.4.4. By using the previous result, Proposition 6.3.4 and Proposition 6.3.5 we recover the result on the module structure of the local cohomology modules $H_I^r(R)$ given by M. Mustaţă [72]. Namely, if $(-)^*$ denotes the dual of a $\mathbb{C}$-vector space, the graded pieces of $H_I^r(R)$ are:

$$[H_I^r(R)]_{-\alpha} \cong (\widetilde{H}_{r-2}(T_\alpha; \mathbb{C}))^* \cong \widetilde{H}^{r-2}(T_\alpha; \mathbb{C}), \quad \alpha \in \{0,1\}^n,$$

and the multiplication map $x_i : [H_I^r(R)]_{-\alpha-\varepsilon_i} \longrightarrow [H_I^r(R)]_{-\alpha}$ is determined by the following commutative diagram:

$$\begin{array}{ccc}
([H_I^r(R)]_{-\alpha})^* & \xrightarrow{(x_i)^*} & ([H_I^r(R)]_{-\alpha-\varepsilon_i})^* \\
\cong \uparrow & & \uparrow \cong \\
(\mathcal{H}_I^r(\mathcal{R}))_\alpha & \xrightarrow{u_i} & (\mathcal{H}_I^r(\mathcal{R}))_{\alpha+\varepsilon_i} \\
\cong \downarrow & & \downarrow \cong \\
(\widetilde{H}^{r-2}(T_\alpha;\mathbb{C}))^* & \xrightarrow{(\nu_i)^*} & (\widetilde{H}^{r-2}(T_{\alpha+\varepsilon_i};\mathbb{C}))^*
\end{array}$$

where ν_i is induced by the inclusion $T_\alpha \subseteq T_{\alpha+\varepsilon_i}$.

Chapter 7

Resum en català

7.1 Introducció

L'objectiu d'aquesta memòria és l'estudi dels mòduls de cohomologia local suportats en ideals monomials. En primer lloc, anem a introduir el tema i els principals problemes que tractarem. A continuació, exposarem els resultats coneguts sobre aquests problemes i finalment farem un resum dels resultats obtinguts en aquesta memòria.

La cohomologia local va ser introduïda per A. Grothendieck com una teoria cohomològica algebraica anàloga a la cohomologia relativa clàssica. Aquesta analogia ve motivada pel fet que molts resultats sobre varietats projectives es poden reformular en termes d'anells graduats o d'anells locals complets. Les notes publicades per R. Hartshorne d'un curs impartit per A. Grothendieck [38] ens serviran com a punt de partida per a introduir aquesta teoria.

Sigui X un espai topològic. Donat un subespai localment tancat $Z \subseteq X$ considerem el **functor de seccions amb suport en** Z, que denotem per $\Gamma_Z(X, \mathcal{F})$ on $\mathcal{F}$ és un feix de grups abelians sobre X. Aquest functor és exacte per l'esquerra. Com que la categoria de feixos de grups abelians sobre X té suficients injectius, podem considerar els functors derivats per la dreta de $\Gamma_Z(X, \mathcal{F})$ que anomenem **grups de cohomologia local de** $\mathcal{F}$ **en** X **amb suport en** Z, que notem per $H^r_Z(X, \mathcal{F}) := \mathbb{R}^r\Gamma_Z(X, \mathcal{F})$.

Una primera interpretació d'aquests grups ve donada per l'existència de la successió exacta llarga de grups de cohomologia

$$0 \longrightarrow H^0_Z(X,\mathcal{F}) \longrightarrow H^0(X,\mathcal{F}) \longrightarrow H^0(U,\mathcal{F}_{|U}) \longrightarrow H^1_Z(X,\mathcal{F}) \longrightarrow \cdots$$

on $U = X \setminus Z$ és el complementari de Z en X. Així doncs, $H^1_Z(X,\mathcal{F})$ no és res més que l'obstrucció a l'extensió de seccions de $\mathcal{F}$ sobre U a tot l'espai X.

Tot i el punt de vista geomètric de les notes d'A. Grothendieck, la cohomologia local va esdevenir ràpidament una eina indispensable en la teoria d'anells commutatius Noetherians. En particular, R. Y. Sharp [83] va descriure la cohomologia local en el context de l'Àlgebra Commutativa. Al llarg d'aquesta memòria, anem a considerar la següent situació:

sigui R un anell commutatiu Noetherià. Donat un ideal $I \subseteq R$ anem a considerar **el functor de I-torsió**, que està definit per a qualsevol R-mòdul M com $\Gamma_I(M) := \{x \in M \mid I^n x = 0 \text{ per algun } n \geq 1\}$. Aquest functor és additiu, covariant i exacte per l'esquerra. Com que la categoria de R-mòduls té suficients injectius, podem considerar els functors derivats per la dreta de Γ_I que anomenem **mòduls de cohomologia local de M amb suport l'ideal** I i que notem per $H^r_I(M) := \mathbb{R}^r\Gamma_I(M)$.

Si R és Noetherià aleshores el feixificat d'un R-mòdul injectiu és flàccid. A partir d'aquest fet, es pot comprovar que la noció de cohomologia local definida usant la teoria de feixos coincideix amb la cohomologia local definida en termes algebraics. Més concretament, sigui $X = \operatorname{Spec} R$ un espai afí, $\mathcal{F} = \widetilde{M}$ el feixificat d'un R-mòdul M i Z la varietat definida per un ideal $I \subseteq R$. Aleshores, es tenen isomorfismes $H^r_Z(X,\widetilde{M}) \cong H^r_I(M)$ per a tot r.

Tot i l'esforç d'un gran nombre d'autors en l'estudi d'aquests mòduls, la seva estructura és en molts casos desconeguda. Tot seguint el criteri de C. Huneke [45], els problemes bàsics sobre els mòduls de cohomologia local són:

- Anul.lació dels mòduls de cohomologia local.
- Finita generació dels mòduls de cohomologia local.
- Artinianitat dels mòduls de cohomologia local.
- Finitud del conjunt de primers associats dels mòduls de cohomologia local.

En general, ni tan sols podem dir quan aquests mòduls s'anul.len. A més, quan no s'anul.len difícilment són finit generats. Tot i això, en algunes situacions aquests mòduls compleixen algunes propietats de finitud que permeten entendre millor la seva estructura.

A continuació exposarem alguns dels resultats que es poden trobar en la literatura sobre aquests problemes i que tenen més relació amb els continguts d'aquesta memòria. A més, comentarem algunes de les seves aplicacions tant en Geometria Algebraica com en Àlgebra Commutativa.

Anul.lació dels mòduls de cohomologia local

A. Grothendieck ja va obtenir els primers resultats bàsics sobre anul.lació en donar cotes pels possibles enters r tals que $H^r_I(M) \neq 0$ en termes de la dimensió i el grau. Més concretament, sigui M un R-mòdul finit generat, aleshores es té $\text{grade}\,(I, M) \leq r \leq \dim R$ per a tot r tal que $H^r_I(M) \neq 0$.

En el cas que $(R, \mathfrak{m})$ sigui un anell local Noetherià, aquest resultat es pot completar i ens permet donar una caracterització cohomològica de la dimensió de Krull. Més concretament, $H^r_{\mathfrak{m}}(M)$ s'anul.la per a $r > \dim M$ i és diferent de zero per a $r = \dim M$.

Tot i que la cota inferior donada anteriorment és ajustada, la cota superior no ho és. Per a fixar aquesta cota, introduïm l'enter $\text{cd}\,(R, I) := \max\{r \mid H^r_I(M) \neq 0 \;\forall M\}$, que anomenem dimensió cohomològica de l'ideal I respecte de R. Cal remarcar que només cal estudiar el cas $M = R$ ja que la dimensió cohomològica és igual al més gran dels enters tals que $H^r_I(R) \neq 0$.

Entre els resultats obtinguts sobre l'anul.lació dels mòduls de cohomologia local, destaquem el teorema d'anul.lació de Hartshorne-Lichtenbaum [40]:

Teorema *Sigui* $(R, \mathfrak{m})$ *un domini local complet de dimensió* d*. Sigui* $I \subseteq R$ *un ideal, aleshores* $\text{cd}\,(R, I) < d$ *si i només si* $\dim R/I > 0$.

El següent resultat va ser provat per R. Hartshorne [40] en el cas geomètric, per A. Ogus [74] en característica zero i per Peskine-Szpiro [76] i Hartshorne-Speiser [43] en característica $p > 0$. Una demostració independent de la característica va ser donada per Huneke-Lyubeznik [47].

Teorema *Sigui* $(R, \mathfrak{m})$ *un domini local complet de dimensió* d *amb cos residual separablement tancat. Sigui* $I \subseteq R$ *un ideal, aleshores* $\text{cd}\,(R, I) < d-1$ *si i només si* $\dim R/I > 1$ *i* $\text{Spec}\,(R/I) \setminus \{\mathfrak{m}\}$ *és connexe.*

Malhauradament, no hi ha una extensió simple d'aquests resultats per dimensions cohomològiques més baixes. El que si que trobem són cotes per a la dimensió cohomològica en casos especials. En aquest sentit, destaquem els treballs de G. Faltings [24] i Huneke-Lyubeznik [47].

Per tal d'il.lustrar la utilitat de la cohomologia local en diversos camps, anem a enunciar algunes de les aplicacions dels resultats descrits anteriorment.

• Sigui $\mathrm{ara}(I)$ el nombre mínim de generadors necessaris per definir l'ideal I llevat de radical. Aleshores, es té $\mathrm{cd}\,(R, I) \leq \mathrm{ara}(I)$. Cal remarcar que aquest enter juga un paper destacat en l'estudi de la connectivitat de varietats algebraiques (vegeu [15]).

• En el cas de subvarietats d'espais projectius, els teoremes d'anul.lació de cohomologia local tenen aplicacions topològiques (vegeu [40], [74], [47]). Per exemple, es pot obtenir la següent generalització del teorema de Lefschetz:

Sigui $I \subseteq R = \mathbb{C}[x_0, \dots, x_n]$ l'ideal de definició d'una subvarietat tancada $Y \subseteq \mathbb{P}^n_{\mathbb{C}}$. Aleshores, el morfisme

$$H^r_{\mathrm{dR}}(\mathbb{P}^n_{\mathbb{C}}) \longrightarrow H^r_{\mathrm{dR}}(Y),$$

on $H^r_{\mathrm{dR}}(\centerdot)$ denoten els grups de cohomologia de de Rham, és un isomorfisme per $r < n - \mathrm{cd}\,(R, I)$ i és un monomorfisme per $r = n - \mathrm{cd}\,(R, I)$ ([42, Theorem III.7.1]).

Finita generació dels mòduls de cohomologia local

En general, els mòduls de cohomologia local $H^r_I(M)$ no són finit generats encara que el R-mòdul M ho sigui. Un criteri per a determinar la finita generació d'aquests mòduls va ser donat per G. Faltings [23]. Aquest criteri depèn dels nombres $s(I, M) := \min\{\mathrm{depth}\,(M_{\mathfrak{p}}) + \mathrm{ht}\,((I + \mathfrak{p})/\mathfrak{p}) \mid I \not\subseteq \mathfrak{p},\ \mathfrak{p} \in \mathrm{Spec}\,(R)\}$.

Teorema *Sigui R un anell noetherià, $I \subseteq R$ un ideal i M un R-mòdul finit generat. Aleshores, $H^r_I(M)$ és finit generat per a tot $r < s(I, M)$ i no ho és si $r = s(I, M)$.*

A. Grothendieck va conjecturar que $\mathrm{Hom}_R(R/I, H^r_I(R))$ és un mòdul finit generat encara que els mòduls de cohomologia local $H^r_I(R)$ no ho siguin. R. Hartshorne va demostrar que aquesta conjectura és falsa [41]. Tot i això, hi ha hagut un gran esforç per estudiar la cofinitud dels mòduls de cohomologia local, on diem que un R-mòdul M és I-cofinit si $\mathrm{Supp}_R(M) \subseteq V(I)$ i $\mathrm{Ext}^r_R(R/I, M)$ és finit generat per a tot $r \geq 0$. En aquest sentit, destaquem els treballs de Huneke-Koh [46], D. Delfino [18] i Delfino-Marley [19].

Propietats de finitud dels mòduls de cohomologia local

Tot i que els mòduls de cohomologia local no són finit generats, sota certes condicions compleixen algunes propietats de finitud que fan més comprensible la seva estructura. En aquest sentit destaquem el següent resultat:

Sigui R un anell local regular no ramificat. Aleshores, per a tot ideal $I \subseteq R$, tot ideal primer $\mathfrak{p} \subseteq R$ i tot $r \geq 0$ es té:

· El conjunt de primers associats de $H^r_I(R)$ és finit.

· Els nombres de Bass $\mu_p(\mathfrak{p}, H^r_I(R))$ són finits.

Aquest resultat va ser provat per Huneke-Sharp [48] en el cas de característica positiva i per G. Lyubeznik en el casos de característica zero [55] i de característica mixta [57].

Cal destacar que en els treballs de G. Lyubeznik s'utilitza la teoria algebraica de $\mathcal{D}$-mòduls ja que els mòduls de cohomologia local són finit generats com a $\mathcal{D}$-mòdul.

R. Hartshorne [41] va donar un exemple en el qual els nombres de Bass d'un mòdul de cohomologia local poden ser infinits si R no és regular. La finitud dels primers associats de $H^r_I(R)$ per a qualsevol anell Noetherià R i qualsevol ideal I era una qüestió oberta fins que A. Singh [84] (cas no local) i M. Katzman [51] (cas local) han donat exemples de mòduls de cohomologia local amb infinits primers associats.

Utilitzant la finitud dels nombres de Bass, G. Lyubeznik va definir un nou conjunt d'invariants numèrics per a anells locals A que contenen un cos, que notem $\lambda_{p,i}(A)$. Més precisament, sigui $(R, \mathfrak{m}, k)$ un anell local regular de dimensió n que conté un cos k, i sigui A un anell local que admet un epimorfisme d'anells $\pi : R \longrightarrow A$. Sigui $I = \operatorname{Ker} \pi$, aleshores definim $\lambda_{p,i}(A)$ com el nombre de Bass $\mu_p(\mathfrak{m}, H^{n-i}_I(R)) = \dim_k \operatorname{Ext}^p_R(k, H^{n-i}_I(R))$. Aquest invariant depen de A, i i p, però no depen ni de R ni de π.

Aquests invariants tenen una interessant interpretació topològica:

• Sigui V un esquema de tipus finit sobre $\mathbb{C}$ de dimensió d i sigui A l'anell local de V en el punt $q \in V$. Si q és un punt singular aïllat de V aleshores, degut a un teorema d'A. Ogus [74] que relaciona la cohomologia local i la cohomologia de de Rham, i el teorema de comparació entre cohomologia de de

Rham i cohomologia singular provat per R. Hartshorne [42], s'obté

$$\lambda_{0,i}(A) = \dim_{\mathbb{C}} H^i_q(V, \mathbb{C}) \text{ per } \; 1 \leq i \leq d-1,$$

on $H^i_q(V, \mathbb{C})$ és el i-èssim grup de cohomologia local singular de V amb suport en q i coeficients en $\mathbb{C}$.

Aquest resultat va ser generalitzat per R. Garcia i C. Sabbah [29] pel cas de dimensió pura tot utilitzant la teoria de $\mathcal{D}$-mòduls. En particular van expressar aquests nombres de Lyubeznik a partir dels nombres de Betti del nus real associat.

Estructura graduada dels mòduls de cohomologia local

En el cas que l'anell R i l'ideal I siguin graduats, els mòduls de cohomologia local $H^r_I(M)$, també tenen una estructura graduada per a tot R-mòdul graduat M. Cal remarcar que la versió graduada dels principals resultats sobre cohomologia local continuen essent vàlids (vegeu [15]).

Algunes de les raons de l'interès en la cohomologia local graduada és deguda a les aplicacions que té en la geometria algebraica projectiva. En particular, la regularitat de Castelnuovo-Mumford $\operatorname{reg}(M)$, és un invariant del R-mòdul M, que ve determinat per la cohomologia local graduada. En el cas que M sigui un mòdul finit generat, aquest invariant ens proporciona informació sobre la resolució del mòdul. Per exemple, sigui $M = I_{\mathbb{P}^n_k}(V)$ l'ideal de definició d'una varietat projectiva $V \subseteq \mathbb{P}^n_k$, on k és un cos algebraicament tancat. Aleshores, els graus dels polinomis homogenis que defineixen $I_{\mathbb{P}^n_k}(V)$ no poden ser majors que $\operatorname{reg}(I_{\mathbb{P}^n_k}(V))$.

Altres aplicacions de la cohomologia local graduada es troben en l'estudi dels anells graduats associats a filtracions d'un anell commutatiu R, especialment l'àlgebra de Rees i l'anell graduat associat a un ideal $I \subseteq R$. Recordem que aquests anells juguen un paper fonamental en l'estudi de singularitats, ja que suposen una realització algebraica de la noció clàssica d'explosió al llarg d'una subvarietat.

Mòduls de cohomologia local amb suport ideals monomials

Sigui $R = k[x_1, \ldots, x_n]$, on k és un cos, l'anell de polinomis en les variables $x_1, \ldots, x_n$. Sigui $\mathfrak{m} := (x_1, \ldots, x_n) \subseteq R$ el maximal homogeni i $I \subseteq R$ un ideal monomial lliure de quadrats. Observem que els mòduls de cohomologia local $H^r_{\mathfrak{m}}(R/I)$ i $H^r_I(R)$ tenen una estructura $\mathbb{Z}^n$-graduada.

A partir de la resolució de Taylor de l'anell quocient R/I, G. Lyubeznik [54] va descriure els mòduls de cohomologia local $H^r_I(R)$, va determinar la seva anul.lació i va descriure la dimensió cohomològica de I respecte de R.

Via la correspondència de Stanley-Reisner, a tot ideal monomial lliure de quadrats $I \subseteq R$ li podem associar un complexe simplicial Δ definit sobre el conjunt de vèrtexs $\{x_1, \ldots, x_n\}$. A [86] trobem un resultat de M. Hochster on dóna una descripció de la sèrie de Hilbert graduada dels mòduls de cohomologia local $H^r_{\mathfrak{m}}(R/I)$ en funció de la cohomologia simplicial reduïda d'uns subcomplexes de Δ. Més concretament, donada una cara $\sigma_\alpha := \{x_i \mid \alpha_i = 1\} \in \Delta$, definim:

- Nus de σ_α en Δ: $\mathrm{link}_\alpha \Delta := \{\tau \in \Delta \mid \sigma_\alpha \cap \tau = \emptyset,\ \sigma_\alpha \cup \tau \in \Delta\}$.
- Restricció a σ_α: $\Delta_\alpha := \{\tau \in \Delta \mid \tau \in \sigma_\alpha\}$.

Sigui $\Delta^\vee$ el complexe simplicial dual d'Alexander de Δ. Aleshores, la igualtat entre complexes $\Delta^\vee_{\mathbf{1}-\alpha} = (\mathrm{link}_\alpha \Delta)^\vee$ i la dualitat d'Alexander ens donen els isomorfismes

$$\widetilde{H}_{n-r-|\alpha|-1}(\mathrm{link}_\alpha \Delta; k) \cong \widetilde{H}^{r-2}(\Delta^\vee_{\mathbf{1}-\alpha}; k).$$

A més, cal remarcar que la inclusió $\Delta^\vee_{\mathbf{1}-\alpha-\varepsilon_i} \subseteq \Delta^\vee_{\mathbf{1}-\alpha}$ indueix els morfismes:

$$\widetilde{H}_{n-r-|\alpha-\varepsilon_i|-1}(\mathrm{link}_{\alpha-\varepsilon_i} \Delta; k) \longrightarrow \widetilde{H}_{n-r-|\alpha|-1}(\mathrm{link}_\alpha \Delta; k),$$

El resultat de M. Hochster es el següent:

Teorema *La sèrie de Hilbert graduada de $H^r_{\mathfrak{m}}(R/I)$ és:*

$$H(H^r_{\mathfrak{m}}(R/I); \mathbf{x}) = \sum_{\sigma_\alpha \in \Delta} \dim_k \widetilde{H}^{r-|\alpha|-1}(\mathrm{link}_\alpha \Delta; k) \prod_{\alpha_i = 1} \frac{x_i^{-1}}{1 - x_i^{-1}}.$$

De la fórmula de M. Hochster deduïm que la multiplicació per x_i estableix un isomorfisme entre les peces $H^r_{\mathfrak{m}}(R/I)_\beta$ i $H^r_{\mathfrak{m}}(R/I)_{\beta+\varepsilon_i}$ per a tot $\beta \in \mathbb{Z}^n$ tal que $\beta_i \neq -1$, on $\varepsilon_1, \ldots, \varepsilon_n$ és la base natural de $\mathbb{Z}^n$. Observem doncs, que per acabar de determinar l'estructura graduada d'aquest mòdul, només cal determinar la multiplicació per x_i sobre les peces $H^r_{\mathfrak{m}}(R/I)_{-\alpha}$, $\alpha \in \{0,1\}^n$.

Per a tot $\beta \in \mathbb{Z}^n$ denotem $\sup_-(\beta) := \{x_i \mid \beta_i < 0\}$. H. G. Gräbe [35], va donar una interpretació topològica d'aquestes multiplicacions a partir dels isomorfismes:

$$H^r_{\mathfrak{m}}(R/I)_\beta \cong \widetilde{H}^{r-|\alpha|-1}(\mathrm{link}_\alpha \Delta; k), \ \ \forall \beta \in \mathbb{Z}^n \text{ tal que } \sigma_\alpha = \sup_-(\beta).$$

Teorema *Per a tot $\alpha \in \{0,1\}^n$ tal que $\sigma_\alpha \in \Delta$, el morfisme de multiplicació per la variable x_i:*

$$\cdot x_i : H^r_{\mathfrak{m}}(R/I)_{-\alpha} \longrightarrow H^r_{\mathfrak{m}}(R/I)_{-(\alpha-\varepsilon_i)}$$

es correspon amb el morfisme

$$\widetilde{H}^{r-|\alpha|-1}(\mathrm{link}_\alpha \Delta; k) \longrightarrow \widetilde{H}^{r-|\alpha-\varepsilon_i|-1}(\mathrm{link}_{\alpha-\varepsilon_i}\Delta; k),$$

induït per la inclusió $\Delta^\vee_{\mathbf{1}-\alpha-\varepsilon_i} \subseteq \Delta^\vee_{\mathbf{1}-\alpha}$.

Inspirat per la fórmula de M. Hochster, N. Terai [92] donà una fórmula per a la sèrie de Hilbert graduada dels mòduls de cohomologia local $H^r_I(R)$, expressada també en funció de la cohomologia simplicial reduïda dels nusos $\mathrm{link}_\alpha\Delta$ tals que $\sigma_\alpha \in \Delta$, $\alpha \in \{0,1\}^n$.

Teorema *La sèrie de Hilbert graduada de $H^r_I(R)$ és:*

$$H(H^r_I(R); \mathbf{x}) = \sum_{\alpha\in\{0,1\}^n} \dim_k \widetilde{H}_{n-r-|\alpha|-1}(\mathrm{link}_\alpha\Delta; k) \prod_{\alpha_i=0} \frac{x_i^{-1}}{1-x_i^{-1}} \prod_{\alpha_j=1} \frac{1}{1-x_j}.$$

De la fórmula de N. Terai també es dedueix que la multiplicació per la variable x_i estableix un isomorfisme entre les peces $H^r_I(R)_\beta$ i $H^r_I(R)_{\beta+\varepsilon_i}$ per a tot $\beta \in \mathbb{Z}^n$ tal que $\beta_i \neq -1$.

De manera independent, M. Mustaţă [72] també va descriure les peces dels mòduls de cohomologia local $H^r_I(R)$ i, a més, va donar una interpretació topològica de les multiplicacions per x_i sobre les peces $H^r_I(R)_{-\alpha}$, $\alpha \in \{0,1\}^n$. Aquests resultats han estat utilitzats per al càlcul de la cohomologia de feixos coherents sobre varietats tòriques (vegeu [22]).

Teorema *Per a tot $\alpha \in \{0,1\}^n$ tal que $\sigma_\alpha \in \Delta$, el morfisme de multiplicació per la variable x_i:*

$$\cdot x_i : H^r_I(R)_{-\alpha} \longrightarrow H^r_I(R)_{-(\alpha-\varepsilon_i)}$$

es correspon amb el morfisme

$$\widetilde{H}_{n-r-|\alpha-\varepsilon_i|-1}(\mathrm{link}_{\alpha-\varepsilon_i}\Delta; k) \longrightarrow \widetilde{H}_{n-r-|\alpha|-1}(\mathrm{link}_\alpha\Delta; k),$$

induït per la inclusió $\Delta^\vee_{\mathbf{1}-\alpha-\varepsilon_i} \subseteq \Delta^\vee_{\mathbf{1}-\alpha}$.

Cal destacar que la fórmules de M. Hochster i de N. Terai es poden obtenir una a partir de l'altra utilitzat la clausura de Čech i la dualitat d'Alexander (veure [70]). El mateix passa amb les fórmules de H. G. Gräbe i M. Mustaţă.

Finalment, K. Yanagawa [97] introdueix la categoria de mòduls straight, que són mòduls $\mathbb{Z}^n$-graduats tals que la multiplicació per les variables x_i entre les seves peces compleixen certes condicions. En aquest ambient, K. Yanagawa estudia els mòduls de cohomologia local degut al fet que a partir dels resultats de N. Terai i de M. Mustaţă es pot comprovar que els mòduls torçats $H_I^r(R)(-1,\dots,-1)$ són straight.

Algoritmes de computació dels moduls de cohomologia local

Sigui k un cos de característica zero, $R = k[x_1,\dots,x_n]$ l'anell de polinomis sobre k i $\mathcal{D}$ el corresponent anell d'operadors diferencials. Recentment, hi ha hagut un gran esforç en el càlcul efectiu dels moduls de cohomologia local.

F. Barkats [3] va donar un algoritme per a calcular una presentació dels mòduls $H_I^r(R)$ amb suport ideals monomials $I \subseteq R$ tot utilitzant la resolució de Taylor de l'anell quocient R/I. Aquest algoritme va ser implementat de manera efectiva en el cas d'ideals continguts en $R = k[x_1,\dots,x_6]$.

Utilitzant la teoria de les bases de Gröbner sobre l'anell $\mathcal{D}$ podem trobar dos mètodes diferents per calcular moduls de cohomologia local. El primer és degut a U. Walther [93] i està basat en la construcció del complexe de Čech de $\mathcal{D}$-mòduls holònoms. En particular, sigui $\mathfrak{m}$ l'ideal maximal homogeni i $I \subseteq R$ un ideal qualsevol. Aleshores, U. Walther determina l'estructura dels mòduls $H_I^r(R)$, $H_\mathfrak{m}^p(H_I^r(R))$ i calcula els nombres de Lyubeznik $\lambda_{p,i}(R/I)$. El segon mètode és degut a T. Oaku i N. Takayama [73]. Es basa en el seu algoritme per calcular els mòduls derivats de la restricció de $\mathcal{D}$-mòduls holònoms.

Aquests càlculs es poden fer tot utilitzant el paquet 'D-modules' [53] que ha estat implementat per al sistema de computació algebraica Macaulay 2 [37].

7.2 Objectius

A continuació, anem a motivar i situar els problemes considerats en aquesta memòria.

Sigui R un anell regular que conté un cos de característica zero i $I \subseteq R$ un ideal. El nostre objectiu és, tot seguint la linia de recerca encetada per G. Lyubeznik en [55], utilitzar en profunditat la teoria de $\mathcal{D}$-mòduls per tal d'estudiar els mòduls de cohomologia local $H_I^r(R)$. Ens interessa especialment, descriure de forma efectiva l'anul.lació i les propietats de finitud d'aquests mòduls.

La principal eina que utilitzarem és el **cicle característic**. Aquest és un invariant que podem associar a tot $\mathcal{D}$-mòdul holònom M (e.g. els mòduls de cohomologia local) i que, en els casos que considerem en aquesta memòria, està descrit com una suma

$$CC(M) = \sum m_i \ V_i,$$

on $m_i \in \mathbb{Z}$, $X = \operatorname{Spec}(R)$ i $T^*_{X_i}X$ és el conormal relatiu a una subvarietat $X_i \subseteq X$. Observem doncs, que la informació que ens proporciona aquest invariant ve codificada per una col.lecció de subvarietats $X_i \subseteq X$ i per unes multiplicitats m_i.

El motius que ens porten a considerar l'estudi dels cicles característics venen reflexats bàsicament en els següents exemples:

• Les varietats que apareixen en el cicle característic descriuen el suport de M com a R-mòdul.

• Els nombres de Lyubeznik $\lambda_{p,i}(R/I)$ es poden calcular com la multiplicitat del cicle caracteristic del mòdul $H^p_{\mathfrak{m}}(H^r_I(R))$.

Més concretament, sigui $R = k[[x_1, \dots, x_n]]$ l'anell de sèries formals amb coeficients un cos k de característica zero. Sigui $\mathfrak{m} = (x_1, \dots, x_n) \subseteq R$ l'ideal maximal i $I \subseteq R$ un ideal qualsevol. Aleshores, a partir dels resultats de [55], es pot veure que

$$H^p_{\mathfrak{m}}(H^r_I(R)) \cong \bigoplus H^n_{\mathfrak{m}}(R)^{\oplus \lambda_{p,i}(R/I)},$$

on $\lambda_{p,i}(R/I)$ són els nombres de Lyubeznik. Calculant el cicle característic, podem veure que aquests invariants de R/I no són res més que les corresponents multiplicitats.

Observem doncs que el cicle característic dels mòduls de cohomologia local $H_I^r(R)$ i $H_{\mathfrak{m}}^p(H_I^r(R))$ ens proporciona informació, tant dels propis mòduls com de l'anell quocient R/I.

Al llarg d'aquesta memòria, ens interessarà interpretar tota la informació que podem extreure del cicle característic i a més fer els càlculs de forma explícita. Els principals problemes que volem tractar són:

• **Problema 1**. Sigui I un ideal contingut en un anell local regular R. Aleshores, volem fer un estudi de l'invariància de les multiplicitats del cicle caracteristic dels mòduls de cohomologia local $H_I^r(R)$ respecte de l'anell quocient R/I.

Sigui R l'anell de polinomis o l'anell de sèries formals amb coeficients en un cos de característica zero i $I \subseteq R$ un ideal monomial lliure de quadrats. Recordem que aquests ideals els podem interpretar de les següents formes:

· Ideal de Stanley-Reisner d'un complexe simplicial.

· Ideal de definició d'un arranjament de varietats lineals.

Aleshores, els problemes considerats en aquest cas són:

• **Problema 2**. Càlcul explícit del cicle característic dels mòduls de cohomologia local.

A partir d'aquest càlcul ens interessa estudiar:

- **Problema 2.1** Estudi del suport dels mòduls de cohomologia local. En particular:

  · Anul.lació dels mòduls de cohomologia local.

  · Dimensió cohomològica.

  · Descripció del suport dels mòduls de cohomologia local.

  · Dimensió de Krull dels mòduls de cohomologia local.

  · Artinianitat dels mòduls de cohomologia local.

- **Problema 2.2** Propietats aritmètiques dels anells R/I. Més concretament, ens interessa determinar les següents propietats:

- · Propietat Cohen-Macaulay.
- · Propietat Buchsbaum.
- · Propietat Gorenstein.
- · El tipus dels anells Cohen-Macaulay.

- **Problema 2.3** Interpretació de les multiplicitats del cicle característic dels mòduls de cohomologia local $H^r_I(R)$. En particular:

  - · Estudi dels invariants topològics i algebraics dels complexes simplicials de Stanley-Reisner associats als anells R/I i $R/I^\vee$, on $I^\vee$ és l'ideal dual d'Alexander de l'ideal I.
  - · Estudi dels invariants topològics del complementari de l'arranjament de varietats lineals determinat per l'ideal I.

- **Problema 2.4** Càlcul explícit dels nombres de Bass dels mòduls de cohomologia local. A partir d'aquest càlcul considerem:

  - · Anul.lació dels nombres de Bass.
  - · Dimensió injectiva dels mòduls de cohomologia local.
  - · Primers associats dels mòduls de cohomologia local.
  - · Suport petit dels mòduls de cohomologia local.

Tot i que ens aporta molta informació, el cicle característic no ens descriu completament l'estructura dels mòduls de cohomologia local. Aquest fet queda reflexat per exemple en el treball d'A. Galligo, M. Granger i Ph. Maisonobe [27], on es dóna una descripció de la categoria de $\mathcal{D}$-mòduls holònoms regulars amb suport un creuament normal (e.g. els mòduls de cohomologia local amb suport ideals monomials lliures de quadrats), a partir de la correspondència de Riemann-Hilbert. Així doncs, encara se'ns planteja la següent qüestió:

• **Problema 3**. Estudi de l'estructura dels mòduls de cohomologia local des dels següents punts de vista:

- · Estudi de la $\mathbb{Z}^n$-graduació dels mòduls de cohomologia local.
- · Estudi dels $\mathcal{D}$-mòduls holònoms regulars amb suport un creuament normal i correspondència de Riemann-Hilbert.

7.3 Conclusions

A continuació, anem a descriure els resultats obtinguts en la memòria.

• En el **Capítol 1** introduïm les notacions i definicions que necessitarem al llarg de la memòria.

En un primer apartat donem la definició de **mòdul de cohomologia local** i enunciem algunes de les propietats bàsiques que utilitzarem al llarg de la memòria. Finalment, introduïm algunes de les eines que ens permeteran calcular aquests mòduls, com ara el complexe de Čech, la successió llarga de cohomologia local, la successió de Mayer-Vietoris, la successió de Brodmann i la successió espectral de Grothendieck.

En un segon apartat ens centrem en l'estudi dels **ideals monomials lliures de quadrats**. Un ambient natural per aquests ideals és la categoria de mòduls $\mathbb{Z}^n$-graduats, és per això que recordem les nocions de resolucions lliures i injectives en aquesta categoria. Recentment, hi ha hagut un gran esforç en l'estudi d'aquestes resolucions per tal de fer-les manejables, per tant hem cregut convenient introduïr també les nocions de matrius monomials, resolucions cel.lulars i clausura de Čech.

Per la correspondència de Stanley-Reisner, podem associar un complexe simplicial a un ideal monomial lliure de quadrats. És per aquest motiu que fem un repàs d'algunes nocions topològiques, fent un especial esment a la dualitat d'Alexander. D'altra banda, un ideal monomial lliure de quadrats també el podem pensar com un ideal de definició d'un arranjament de subvarietats lineals. Seguint en aquesta línia de recerca, recordem la fórmula de Goresky-MacPherson per al càlcul de la cohomologia del complementari d'aquest tipus d'arranjaments.

Finalment donem un repàs de la teoria de **$\mathcal{D}$-mòduls**. Comencem donant les definicions bàsiques de $\mathcal{D}$-mòdul holònom regular ja que els mòduls de cohomologia local són mòduls d'aquest tipus. A aquests mòduls els hi podem associar un invariant, el cicle característic, que ens permet calcular el suport d'aquests mòduls. Després de recordar quina és la geometria d'aquests cicles característics, fem uns quants exemples i computacions.

Per acabar, introduïm la noció de solucions d'un $\mathcal{D}$-mòdul, recordem que les solucions d'un $\mathcal{D}$-mòdul holònom regular són funcions de classe de Nilsson i que el functor de solucions restringit a la categoria de $\mathcal{D}$-mòduls holònoms regulars estableix una equivalència amb la categoria de feixos perversos que denotem per correspondència de Riemann-Hilbert.

• En el **Capítol 2** demostrem que les multiplicitats del cicle característic dels mòduls de cohomologia local són invariants del corresponent anell quocient. En particular, aquests invariants generalitzen els nombres de Lyubeznik.

Donat un cos k de característica zero considerem l'anell de sèries formals de potències $R = k[[x_1, \ldots, x_n]]$, on $x_1, \ldots, x_n$ són variables independents. Sigui $I \subseteq R$ un ideal qualsevol, $\mathfrak{p} \subseteq R$ un ideal primer i $\mathfrak{m} = (x_1, \ldots, x_n)$ l'ideal maximal. En primer lloc provem que els nombres de Lyubeznik $\lambda_{p,i}(R/I)$ no són res més que les multiplicitats del cicle característic dels mòduls de cohomologia local $H^p_{\mathfrak{m}}(H^{n-i}_I(R))$. Seguint en la línia de la demostració de Lyubeznik, demostrem que les següents multiplicitats també són invariants de l'anell R/I:

- Les multiplicitats del cicle característic de $H^{n-i}_I(R)$.
- Les multiplicitats del cicle característic de $H^p_{\mathfrak{p}}(H^{n-i}_I(R))$.

Entre aquestes multiplicitats podem trobar:

· Els nombres de Bass $\mu_p(\mathfrak{p}, H^{n-i}_I(R))$.

· Els nombres de Lyubeznik $\lambda_{p,i}(R/I) := \mu_p(\mathfrak{m}, H^{n-i}_I(R))$.

Més concretament, en aquest capítol demostrem els següents resultats:

Teorema 2.2.2 *Sigui A un anell local que admet un epimorfisme d'anells $\pi : R \longrightarrow A$, on $R = k[[x_1, \ldots, x_n]]$ és l'anell de sèries formals de potències. Sigui $I = \ker \pi$ i*

$$CC(H^{n-i}_I(R)) = \sum m_{i,\alpha} \; T^*_{X_\alpha} X$$

el cicle característic dels mòduls de cohomologia local $H^{n-i}_I(R)$. Aleshores, les multiplicitats $m_{i,\alpha}$ depenen d'A, i i α però no depenen ni de R ni de π.

Agrupant aquestes multiplicitats per la dimensió de la corresponent varietat obtenim uns altres invariants. Encara que aquests invariants siguin més grollers, en moltes situacions ens seran útils per a fer una descripció acurada del suport dels mòduls de cohomologia local.

Definició 2.2.4 *Sigui $I \subseteq R$ un ideal. Si $CC(H_I^{n-i}(R)) = \sum m_{i,\alpha}\ T^*_{X_\alpha}X$ és el cicle característic dels mòduls de cohomologia local $H_I^{n-i}(R)$ aleshores definim:*

$$\gamma_{p,i}(R/I) := \{\sum m_{i,\alpha} \mid \dim X_\alpha = p\}.$$

Aquests invariants tenen les mateixes propietats que els nombres de Lyubeznik. Més precisament:

Proposició 2.2.5 *Sigui $d = \dim R/I$. Els invariants $\gamma_{p,i}(R/I)$ tenen les següents propietats:*

i) $\gamma_{p,i}(R/I) = 0$ si $i > d$.

ii) $\gamma_{p,i}(R/I) = 0$ si $p > i$.

iii) $\gamma_{d,d}(R/I) \neq 0$.

Així doncs, podem emmagatzemar aquests invariants en una matriu triangular que denotarem per $\Gamma(R/I)$.

D'altra banda, per a les multiplicitats del cicle característic dels mòduls de cohomologia local $H^p_{\mathfrak{p}}(H_I^{n-i}(R))$ tenim el següent resultat:

Teorema 2.2.6 *Sigui A un anell local que admet un epimorfisme d'anells $\pi : R \longrightarrow A$, on $R = k[[x_1, \ldots, x_n]]$ és l'anell de sèries formals de potències. Sigui $I = \ker \pi$, $\mathfrak{p} \subseteq R$ un ideal primer que conté a I i*

$$CC(H^p_{\mathfrak{p}}(H_I^{n-i}(R))) = \sum \lambda_{\mathfrak{p},p,i,\alpha}\ T^*_{X_\alpha}X,$$

el cicle característic dels mòduls de cohomologia local $H^p_{\mathfrak{p}}(H_I^{n-i}(R))$. Aleshores, les multiplicitats $\lambda_{\mathfrak{p},p,i,\alpha}$ depenen d'A, $\mathfrak{p}$, p, i i α però no depenen ni de R ni de π.

Com a conseqüència d'aquest resultat, demostrem que els nombres de Bass, en particular els nombres de Lyubeznik, són invariants de R/I ja que són multiplicitats del cicle característic. Més concretament:

Proposició 2.2.8 *Si el cicle característic del mòdul de cohomologia local $H^p_{\mathfrak{p}}(H_I^{n-i}(R))$ és*

$$CC(H^p_{\mathfrak{p}}(H_I^{n-i}(R))) = \sum \lambda_{\mathfrak{p},p,i,\alpha}\ T^*_{X_\alpha}X,$$

aleshores $\mu_p(\mathfrak{p}, H_I^{n-i}(R)) = \lambda_{\mathfrak{p},p,i,\alpha_{\mathfrak{p}}}$, on $\alpha_{\mathfrak{p}}$ és tal que la varietat definida per $\mathfrak{p}$ és $X_{\alpha_{\mathfrak{p}}}$.

• Sigui k un cos de característica zero. En el **Capítol 3** considerem qualsevol dels anells següents:

- $R = k[[x_1, \ldots, x_n]]$ l'anell de sèries formals de potències.
- $R = k\{x_1, \ldots, x_n\}$ l'anell de sèries convergents.
- $R = k[x_1, \ldots, x_n]$ l'anell de polinomis.

El resultat principal d'aquest capítol és el càlcul del cicle característic dels mòduls de cohomologia local $H_I^r(R)$ amb suport un ideal monomial $I \subseteq R$. Una primera qüestió que se'ns planteja és la de trobar d'entre les eines que hem introduït en el Capítol 1, quina ens serà més útil per als nostres càlculs.

Una primera aproximació és utilitzar el **complexe de Čech**. Sigui $I = (\mathbf{x}^{\alpha_1}, \ldots, \mathbf{x}^{\alpha_s})$ un ideal monomial. Considerem el complexe de Čech

$$\check{C}_I^\bullet: \quad 0 \longrightarrow R \xrightarrow{d_0} \bigoplus_{1 \leq i_1 \leq r} R[\frac{1}{\mathbf{x}^{\alpha_{i_1}}}] \xrightarrow{d_1} \cdots \xrightarrow{d_{s-1}} R[\frac{1}{\mathbf{x}^{\alpha_1} \cdots \mathbf{x}^{\alpha_s}}] \longrightarrow 0$$

Els mòduls de cohomologia local es poden calcular com la cohomologia d'aquest complexe, i.e. $H_I^r(R) = H^r(\check{C}_I^\bullet)$. Recordem que el cicle característic de les localitzacions $R_r := \bigoplus R[\frac{1}{\mathbf{x}^{\alpha_{i_1}} \cdots \mathbf{x}^{\alpha_{i_r}}}]$ que apareixen en el complexe de Čech, ja han estat calculats al Capítol 1. Així doncs, utilitzant l'additivitat del cicle característic respecte de successions exactes, tenim que $CC(H_I^r(R)) = CC(\operatorname{Ker} d_r) - CC(\operatorname{Im} d_{r-1})$.

Quan l'anell quocient R/I és Cohen-Macaulay, degut a que la dimensió cohomològica és igual a l'alçada de l'ideal I, només hi ha un mòdul de cohomologia local diferent de zero (**Proposició 3.1.1**). Això fa que el seu cicle característic sigui senzill de calcular.

Proposició 3.1.2 *Sigui $I \subseteq R$ un ideal d'alçada h generat per monomis lliures de quadrats. Si R/I és Cohen-Macaulay aleshores:*

$$CC(H_I^h(R)) = CC(R_h) - CC(R_{h+1}) + \cdots + (-1)^{s-h} CC(R_s)$$
$$-CC(R_{h-1}) + \cdots + (-1)^h CC(R_0).$$

Si l'anell quocient R/I no és Cohen-Macaulay, el càlcul dels cicles característics $CC(\operatorname{Ker} d_r)$ i $CC(\operatorname{Im} d_r)$ no és immediat. Tot i que no pretenem donar una descripció explícita d'aquests nuclis i imatges, el fet que la varietat característica de les localitzacions $R_r := \bigoplus R[\frac{1}{\mathbf{x}^{\alpha_{i_1}} \cdots \mathbf{x}^{\alpha_{i_r}}}]$ no sigui irreductible fa que el càlcul del cicle característic es compliqui.

Pel cas general utilitzarem la **successió de Mayer-Vietoris**. Bàsicament, el procediment és el següent: considerem una representació $I = U \cap V$ de l'ideal I com a intersecció de dos ideals més simples. Aleshores, si trenquem la successió

$$\cdots \longrightarrow H^r_{U+V}(R) \longrightarrow H^r_U(R) \oplus H^r_V(R) \longrightarrow H^r_I(R) \longrightarrow H^{r+1}_{U+V}(R) \longrightarrow \cdots ,$$

en successions exactes curtes de nuclis i conuclis

$$0 \longrightarrow B_r \longrightarrow H^r_U(R) \oplus H^r_V(R) \longrightarrow C_r \longrightarrow 0$$

$$0 \longrightarrow C_r \longrightarrow H^r_I(R) \longrightarrow A_{r+1} \longrightarrow 0$$

$$0 \longrightarrow A_{r+1} \longrightarrow H^{r+1}_{U+V}(R) \longrightarrow B_{r+1} \longrightarrow 0,$$

tenim que el cicle característic del mòdul de cohomologia local $H^r_I(R)$ és

$$CC(H^r_I(R)) = CC(C_r) + CC(A_{r+1}).$$

Observem doncs, que hem reduït el problema al càlcul dels cicles característics $CC(H^r_U(R) \oplus H^r_V(R))$, $CC(H^{r+1}_{U+V}(R))$ i $CC(B_r)$ $\forall r$.

Per a calcular el cicle característic dels mòduls de cohomologia local $H^r_U(R)$, $H^r_V(R))$ i $H^{r+1}_{U+V}(R)$, hem de considerar una descomposició dels ideals U, V i $U+V$. Després, trenquem les corresponents successions de Mayer-Vietoris de la mateixa manera que hem fet abans. Tot i que ara hem de treballar amb varies successions a la vegada, els ideals U, V i $U+V$ que estem considerant són més simples que l'ideal I. Aquest procediment l'hem d'anar repetint fins que arribem a una situació en la qual es pugui calcular els cicles característics $CC(H^r_U(R) \oplus H^r_V(R))$, $CC(H^{r+1}_{U+V}(R))$ i $CC(B_r)$ $\forall r$ que apareixen en totes les successions de Mayer-Vietoris involucrades.

En aquest capítol donem un procediment inductiu per escollir els ideals U i V de manera sistemàtica, i.e. independent de la complexitat de l'ideal, tot aprofitant les bones propietats que té la descomposició primària minimal $I = I_{\alpha_1} \cap \cdots \cap I_{\alpha_m}$ d'un ideal monomial lliure de quadrats. Mitjançant aquest procés, arribem a una situació en la qual els mòduls de cohomologia local $H^r_U(R)$, $H^r_V(R))$ i $H^{r+1}_{U+V}(R)$ que apareixen en les diferents successions de Mayer-Vietoris són exactament algun dels següents $2^m - 1$ mòduls:

$$H^r_{I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}}}(R), \quad 1 \leq i_1 < \cdots < i_j \leq m, \quad j = 1, \ldots, m.$$

Per organitzar tota aquesta informació, introduïm el poset $\mathcal{I} = \{\mathcal{I}_1, \mathcal{I}_2, \dots, \mathcal{I}_m\}$, format per totes les sumes d'ideals cara que apareixen en la descomposició primària minimal de I, i.e. $\mathcal{I}_j := \{I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} \mid 1 \leq i_1 < \cdots < i_j \leq m\}$.

Així doncs, el que hem fet és 'trencar' el mòdul de cohomologia local $H^r_I(R)$ en peces més petites, els mòduls $H^r_{I_{\alpha_{i_1}}+\cdots+I_{\alpha_{i_j}}}(R)$ que estan etiquetades pel poset $\mathcal{I}$ i anomenarem peces inicials. El cicle característic d'aquestes peces ja el coneixem degut a que $R/I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}}$ és Cohen-Macaulay. El que queda per calcular és el cicle característic de tots els mòduls B_r que apareixen en les diferents successions de Mayer-Vietoris.

Utilitzant el fet que la varietat característica de les peces inicials és irreductible, provem que el cicle característic $CC(B_r)$ és la suma de cicles característics de mòduls de cohomologia local amb suport sumes d'ideals cara que apareixen en la descomposició primària minimal de I i que compleixen la següent propietat:

Definició 3.2.9 *Diem que les sumes d'ideals cares* $I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} \in \mathcal{I}_j$ *i* $I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} + I_{\alpha_{i_{j+1}}} \in \mathcal{I}_{j+1}$ *estan aparellades si*

$$I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} = I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} + I_{\alpha_{i_{j+1}}}.$$

Finalment, un cop controlades totes les peces de les successions de Mayer-Vietoris així com els nuclis i conuclis, utilitzem la additivitat del cicle característic respecte de successions exactes curtes per calcular $CC(H^r_I(R))$. De forma més precisa, $CC(H^r_I(R))$ serà la suma del cicle característic de les peces inicials etiquetades per un subposet $\mathcal{P} \subseteq \mathcal{I}$, que calculem mitjançant un algoritme i que està format per les sumes d'ideals cara $I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}}$ que no estan aparellades i que tenen alçada $r + (j-1)$. Més concretament, si un cop calculat el poset $\mathcal{P}$, considerem els conjunts

$$\mathcal{P}_{j,r} := \{I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} \in \mathcal{P}_j \mid \operatorname{ht}(I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}}) = r + (j-1)\}$$

aleshores es té

Teorema 3.2.11 *Sigui $I \subseteq R$ un ideal generat per monomis lliures de quadrats i sigui $I = I_{\alpha_1} \cap \cdots \cap I_{\alpha_m}$ la seva descomposició primària minimal. Aleshores:*

$$CC(H^r_I(R)) = \sum_{I_{\alpha_i} \in \mathcal{P}_{1,r}} CC(H^r_{I_{\alpha_i}}(R)) + \sum_{I_{\alpha_{i_1}}+I_{\alpha_{i_2}} \in \mathcal{P}_{2,r}} CC(H^{r+1}_{I_{\alpha_{i_1}}+I_{\alpha_{i_2}}}(R)) + \cdots +$$

$$+ \sum_{I_{\alpha_1}+\cdots+I_{\alpha_m} \in \mathcal{P}_{m,r}} CC(H^{r+(m-1)}_{I_{\alpha_1}+\cdots+I_{\alpha_m}}(R)).$$

La informació que podem extreure del càlcul del cicle característic dels mòduls de cohomologia local la dividirem en dues parts. D'una banda, les varietats que apareixen en el cicle característic, i.e. les varietats amb multiplicitat no nul.la, ens descriuen el suport del mòdul $H_I^r(R)$. De l'altra, les multiplicitats ens permeten descriure propietats de l'anell quocient R/I ja que són invariants.

Anul.lació i suport dels mòduls de cohomologia local: Una lectura del cicle característic dels mòduls $H_I^r(R)$ ens permet obtindre els següents resultats:

· Anul.lació dels mòduls de cohomologia local. **Proposició 3.3.3**

· Dimensió cohomològica. **Corol.lari 3.3.5**

· Suport dels mòduls de cohomologia local. **Proposició 3.3.8**

· Dimensió de Krull dels mòduls de cohomologia local. **Corol.lari 3.3.9**

· Artinianitat dels mòduls de cohomologia local. **Corol.lari 3.3.10**

Cal remarcar que aquests resultats estan expressats en termes de les multiplicitats del cicle característic, que a la vegada han estat calculades a partir dels ideals de la descomposició primària minimal de l'ideal I.

Propietats aritmètiques dels anells R/I: Les multiplicitats del cicle característic dels mòduls de cohomologia local ens permeten donar criteris per estudiar les següents propietats:

· Propietat Cohen-Macaulay. **Proposicions 3.3.11** i **3.3.12**

· Propietat Buchsbaum. **Proposicions 3.3.13** i **3.3.14**

· Propietat Gorenstein. **Proposició 3.3.15**

· El tipus dels anells Cohen-Macaulay. **Proposició 3.3.18**

En la literatura podem trobar diferents criteris que utilitzen les propietats topològiques dels complexes simplicials de Stanley-Reisner associats a l'anell R/I, per més detalls vegeu [86]. Els nostres criteris en canvi, es basen en l'anul.lació de certes multiplicitats pels casos Cohen-Macaulay i Buchsbaum, o bé que certes multiplicitats siguin exactament 1 en el cas Gorenstein. Per tant estan expressats en termes dels ideals de la descomposició primària minimal de l'ideal I.

Un cop calculades les multiplicitats del cicle característic dels mòduls de cohomologia local hem vist que ens són útils per descriure propietats dels mòduls $H_I^r(R)$ així com de l'anell R/I. Ara, ens interessa donar una interpretació d'aquests invariants tot comparant-los amb d'altres ja coneguts.

Combinatòria de l'anell de Stanley-Reisner i multiplicitats: Via la correspondència de Stanley-Reisner, podem associar a cada ideal monomial lliure de quadrats $I \subseteq R$ un complexe simplicial Δ. En aquesta secció donem una descripció dels invariants topològics descrits pel f-vector i el h-vector de Δ a partir dels invariants

$$\mathcal{B}_j := \sum_{i=0}^{d-j} (-1)^i \gamma_{j,j+i}(R/I),$$

i.e. la suma alternada dels invariants $\gamma_{p,i}(R/I)$ continguts en una fila de la matriu $\Gamma(R/I)$.

Proposició 3.3.21 *Sigui $I \subseteq R$ un ideal monomial lliure de quadrats. El f-vector i el h-vector del complexe simplicial Δ estan descrits de la següent forma:*

i) $f_k = \sum_{j=k+1}^{d} \binom{j}{k+1} \mathcal{B}_j.$

ii) $h_k = (-1)^k \sum_{j=0}^{d-k} \binom{d-j}{k} \mathcal{B}_j.$

Usant aquestes descripcions també determinem altres invariants com ara la característica d'Euler de Δ o la sèrie de Hilbert de R/I (**Corol.lari 3.3.23**). En general, els invariants $\gamma_{p,i}(R/I)$ són més fins que el f-vector i el h-vector. Són equivalents quan R/I és Cohen-Macaulay (**Corol.lari 3.3.22**).

Una forma més natural d'interpretar les multiplicitats és utilitzant la dualitat d'Alexander.

Nombres de Betti i multiplicitats: Sigui $I^\vee \subseteq R$ l'ideal dual d'Alexander d'un ideal monomial lliure de quadrats $I \subseteq R$. El complexe de Taylor $\mathbb{T}_\bullet(I^\vee)$ és una resolució lliure cel.lular de $I^\vee$ amb suport un complexe simplicial etiquetat pel poset $\mathcal{I}$ que utilitzavem per etiquetar les peces inicials dels mòduls de cohomologia local $H_I^r(R)$.

Recordem que el cicle característic d'aquests mòduls està descrit a partir d'un poset $\mathcal{P} \subseteq \mathcal{I}$ que calculem a partir d'un algoritme. Via la correspondència donada per la dualitat d'Alexander, aquest algoritme el podem interpretar com un algoritme per minimitzar la resolució lliure de $I^\vee$ donada pel complexe de Taylor. Això ens permet descriure les multiplicitats del cicle característic $CC(H_I^r(R))$ a partir dels nombres de Betti de $I^\vee$.

Proposició 3.3.25 *Sigui $I^\vee \subseteq R$ l'ideal dual d'Alexander d'un ideal monomial lliure de quadrats $I \subseteq R$. Aleshores,* $\beta_{j,\alpha}(I^\vee) = m_{n-|\alpha|+j,\,\alpha}(R/I)$.

En particular, observem que les multiplicitats $m_{i,\alpha}$ d'un mòdul fixat $H_I^{n-i}(R)$ descriuen els nombres de Betti de l'estrat $(n-i)$-lineal de $I^\vee$. Quan R/I és Cohen-Macaulay només hi ha un mòdul de cohomologia local no nul, això ens permet recuperar el següent resultat de J. A. Eagon i V. Reiner [20].

Corol.lari 3.3.28 *Sigui $I^\vee \subseteq R$ l'ideal dual d'Alexander d'un ideal monomial lliure de quadrats $I \subseteq R$. Aleshores, R/I és Cohen-Macaulay si i només si $I^\vee$ té una resolució lliure lineal.*

N. Terai [90], dóna una generalització d'aquest resultat expressada en termes de la dimensió projectiva de R/I i la regularitat de Castelnuovo-Mumford de $I^\vee$. Usant els resultats previs en podem donar una aproximació diferent.

Corol.lari 3.3.29 *Sigui $I^\vee \subseteq R$ l'ideal dual d'Alexander d'un ideal monomial lliure de quadrats $I \subseteq R$. Aleshores,* $\operatorname{pd}(R/I) = \operatorname{reg}(I^\vee)$.

• Sigui k un cos de característica zero. En el **Capítol 4** també considerem qualsevol dels anells següents:

- $R = k[[x_1, \dots, x_n]]$ l'anell de sèries formals de potències.
- $R = k\{x_1, \dots, x_n\}$ l'anell de sèries convergents.
- $R = k[x_1, \dots, x_n]$ l'anell de polinomis.

El resultat principal d'aquest capítol és el càlcul del cicle característic dels mòduls de cohomologia local $H_{\mathfrak{p}_\gamma}^p(H_I^r(R))$ on $I \subseteq R$ és un ideal monomial i $\mathfrak{p}_\gamma \subseteq R$ és un ideal primer homogeni. Les tècniques que farem servir en aquest capítol són una continuació natural de les del capítol anterior. Més precisament, considerem la successió exacta curta $0 \longrightarrow C_r \longrightarrow H_I^r(R) \longrightarrow A_{r+1} \longrightarrow 0$, que s'obté al trencar la successió de Mayer-Vietoris

$$\cdots \longrightarrow H_{U+V}^r(R) \longrightarrow H_U^r(R) \oplus H_V^r(R) \longrightarrow H_I^r(R) \longrightarrow H_{U+V}^{r+1}(R) \longrightarrow \cdots .$$

Aleshores, si trenquem la successió llarga de cohomologia local

$$\cdots \longrightarrow H^p_{\mathfrak{p}_\gamma}(C_r) \longrightarrow H^p_{\mathfrak{p}_\gamma}(H^r_{U\cap V}(R)) \longrightarrow H^p_{\mathfrak{p}_\gamma}(A_{r+1}) \longrightarrow H^{p+1}_{\mathfrak{p}_\gamma}(C_r) \longrightarrow \cdots ,$$

en successions exactes curtes de nuclis i conuclis

$$0 \longrightarrow Z_{p-1} \longrightarrow H^p_{\mathfrak{p}_\gamma}(C_r) \longrightarrow X_p \longrightarrow 0$$

$$0 \longrightarrow X_p \longrightarrow H^p_{\mathfrak{p}_\gamma}(H^r_{U\cap V}(R)) \longrightarrow Y_p \longrightarrow 0$$

$$0 \longrightarrow Y_p \longrightarrow H^p_{\mathfrak{p}_\gamma}(A_{r+1}) \longrightarrow Z_p \longrightarrow 0$$

tenim que el cicle característic del mòdul de cohomologia local $H^p_{\mathfrak{p}_\gamma}(H^r_I(R))$ és

$$CC(H^p_{\mathfrak{p}_\gamma}(H^r_I(R))) = CC(X_p) + CC(Y_p).$$

A partir de la descomposició primària minimal $I = I_{\alpha_1} \cap \cdots \cap I_{\alpha_m}$ i aplicant la successió llarga de cohomologia a les successions que hem trobat en el procés utilitzat en el capítol anterior, aconseguim 'trencar' el mòdul de cohomologia local $H^p_{\mathfrak{p}_\gamma}(H^r_I(R))$ en peces més petites, els mòduls $H^p_{\mathfrak{p}_\gamma}(H^r_{I_{\alpha_{i_1}}+\cdots+I_{\alpha_{i_j}}}(R))$ que, degut al Teorema 3.2.11, estan etiquetades pel poset $\mathcal{P}$. El cicle característic d'aquestes peces es pot calcular utilitzant la successió espectral de Grothendieck ja que $R/I_{\alpha_{i_1}}+\cdots+I_{\alpha_{i_j}}$ és Cohen-Macaulay (**Proposició 4.1.1**). Més concretament,

$$CC(H^p_{\mathfrak{p}_\gamma}(H^r_{I_{\alpha_{i_1}}+\cdots+I_{\alpha_{i_j}}}(R)) = CC(H^{p+r}_{\mathfrak{p}_\gamma+I_{\alpha_{i_1}}+\cdots+I_{\alpha_{i_j}}}(R)).$$

Per organitzar millor tota la informació donada per aquestes peces definim els conjunts $\mathcal{P}_{\gamma,j,\alpha} := \{I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} \in \mathcal{P}_j \mid \mathfrak{p}_\gamma + I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} = I_\alpha\}$.

El que queda per calcular és el cicle característic de tots els mòduls Z_p que apareixen en les diferents successions llargues de cohomologia. La varietat característica d'aquestes peces inicials és també irreductible, d'aquesta manera ara provem que el cicle característic $CC(Z_p)$ és la suma de cicles característics de mòduls de cohomologia local amb suport sumes d'ideals cara que apareixen en la descomposició primària minimal de I i que compleixen la següent propietat:

Definició 4.2.8 *Diem que les sumes d'ideals cara* $I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} \in \mathcal{P}_{\gamma,j,\alpha}$ *i* $I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} + I_{\alpha_{i_{j+1}}} \in \mathcal{P}_{\gamma,j+1,\alpha}$ *estan quasi aparellades si*

$$\mathrm{ht}\,(I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}}) + 1 = \mathrm{ht}\,(I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} + I_{\alpha_{i_{j+1}}})$$

Finalment, un cop controlades totes les peces de les successions llargues de cohomologia així com els nuclis i conuclis, utilitzem la additivitat del cicle característic respecte de successions exactes curtes per calcular $CC(H^p_{\mathfrak{p}_\gamma}(H^r_I(R)))$. De forma més precisa, $CC(H^p_{\mathfrak{p}_\gamma}(H^r_I(R)))$ serà la suma del cicle característic de les peces inicials etiquetades per un subposet $\mathcal{Q} \subseteq \mathcal{P}$, que calculem mitjançant un algoritme i que està format per les sumes d'ideals cara $I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}}$ que no estan quasi aparellades i que tenen alçada $r + (j-1)$. Més concretament, si un cop calculat el poset $\mathcal{Q}$, considerem els conjunts

$$\mathcal{Q}_{\gamma,j,r,\alpha} := \{I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}} \in \mathcal{Q}_{\gamma,j,\alpha} \mid \mathrm{ht}\,(I_{\alpha_{i_1}} + \cdots + I_{\alpha_{i_j}}) = r + (j-1)\},$$

aleshores es té:

Teorema 4.2.9 *Sigui* $I = I_{\alpha_1} \cap \cdots \cap I_{\alpha_m}$ *la descomposició primària minimal d'un ideal* $I \subseteq R$ *generat per monomis lliures de quadrats. Sigui* $\mathfrak{p}_\gamma \subseteq R$ *un ideal cara, aleshores:*

$$CC(H^p_{\mathfrak{p}_\gamma}(H^r_I(R))) = \sum_{\alpha \in \{0,1\}^n} \lambda_{\gamma,p,n-r,\alpha}\ CC(H^{|\alpha|}_{I_\alpha}(R)),$$

on $\lambda_{\gamma,p,n-r,\alpha} = \sharp \mathcal{Q}_{\gamma,j,r,\alpha}$ *tal que* $|\alpha| = p + (r + (j-1))$.

En el cas que $\mathfrak{p}_\gamma = \mathfrak{m}$ sigui l'ideal maximal homogeni tenim la igualtat entre conjunts $\mathcal{P}_{j,r} = \mathcal{P}_{\alpha_\mathfrak{m},j,r,\alpha_\mathfrak{m}}$ $\forall j$, $\forall r$. Aplicant l'algoritme de cancel.lació de quasi parelles a aquests conjunts i ordenant-los pel nombre de sumands i l'alçada obtenim els conjunts $\mathcal{Q}_{j,r} = \mathcal{Q}_{\alpha_\mathfrak{m},j,r,\alpha_\mathfrak{m}}$. Per tant:

Corol.lari 4.2.12 *Sigui* $I = I_{\alpha_1} \cap \cdots \cap I_{\alpha_m}$ *la descomposició primària minimal d'un ideal* $I \subseteq R$ *generat per monomis lliures de quadrats. Sigui* $\mathfrak{m} \subseteq R$ *l'ideal maximal homogeni, aleshores:*

$$CC(H^p_{\mathfrak{m}}(H^r_I(R))) = \lambda_{p,n-r} \ T^*_{X_{\alpha_{\mathfrak{m}}}}X,$$

on $\lambda_{p,n-r} = \sharp\mathcal{Q}_{j,r}$ *tal que* $n = p + (r + (j-1))$.

De la informació que podem extreure del cicle característic d'aquestes composicions de mòduls de cohomologia local ens centrarem en les multiplicitats. Més concretament, ens fixarem en els invariants dels anells R/I descrits pels nombres de Bass dels mòduls $H^r_I(R)$.

Nombres de Bass dels mòduls de cohomologia local: Sigui $I \subseteq R$ un ideal generat per monomis lliures de quadrats, $\mathfrak{p}_\gamma \subseteq R$ un ideal cara i $\mathfrak{m} \subseteq R$ l'ideal maximal homogeni. Recordem que en el Capítol 1 hem vist que els nombres de Bass els calculem de la següent forma:

Proposició 4.3.3 *Sigui*

$$CC(H^p_{\mathfrak{p}_\gamma}(H^{n-i}_I(R))) = \sum \lambda_{\gamma,p,i,\alpha} \ T^*_{X_\alpha}X$$

el cicle característic del mòdul de cohomologia local $H^p_{\mathfrak{p}_\gamma}(H^{n-i}_I(R))$. *Aleshores*

$$\mu_p(\mathfrak{p}_\gamma, H^{n-i}_I(R)) = \lambda_{\gamma,p,i,\gamma}.$$

En particular, els nombres de Lyubeznik són:

Corol.lari 4.3.5 *Sigui*

$$CC(H^p_{\mathfrak{m}}(H^{n-i}_I(R))) = \lambda_{p,i} \ T^*_{X_{\alpha_{\mathfrak{m}}}}X$$

el cicle característic del mòdul de cohomologia local $H^p_{\mathfrak{m}}(H^{n-i}_I(R))$. *Aleshores,*

$$\mu_p(\mathfrak{m}, H^{n-i}_I(R)) = \lambda_{p,i}.$$

Un cop calculats, utilitzem els nombres de Bass per a donar una descripció més acurada dels primers que apareixen en el suport dels mòduls $H^r_I(R)$ estudiats en el capítol anterior. En particular obtenim els següents resultats:

· Anul.lació dels nombres de Bass. **Proposició 4.3.11**

· Dimensió injectiva dels mòduls de cohomologia local. **Corol.lari 4.3.12**

· Primers associats dels mòduls de cohomologia local. **Proposició 4.3.14**

· Suport petit dels mòduls de cohomologia local. **Proposició 4.3.16**

Una vegada més, remarquem que els resultats estan expressats a partir dels ideals de la descomposició primària minimal de l'ideal I. Per il.lustrar aquests càlculs donem exemples de mòduls $H_I^r(R)$ que compleixen:

· $\mathrm{id}_R H_I^r(R) = \dim_R H_I^r(R)$.

· $H_I^r(R)$ no té primers immersos.

· $\mathrm{Min}_R(H_I^r(R)) = \mathrm{Ass}_R(H_I^r(R))$.

· $\mathrm{supp}_R(H_I^r(R)) = \mathrm{Supp}_R(H_I^r(R))$.

També donem exemples de mòduls $H_I^r(R)$ que compleixen:

· $\mathrm{id}_R H_I^r(R) < \dim_R H_I^r(R)$.

· $H_I^r(R)$ té primers immersos.

· $\mathrm{Min}_R(H_I^r(R)) \subsetneq \mathrm{Ass}_R(H_I^r(R))$.

· $\mathrm{supp}_R(H_I^r(R)) \subsetneq \mathrm{Supp}_R(H_I^r(R))$.

• Sigui $\mathbb{A}_k^n$ l'espai afí de dimensió n sobre un cos k de característica qualsevol, sigui $X \subset \mathbb{A}_k^n$ un arranjament de subvarietats lineals. Sigui $I \subset R$ l'ideal de definició de X, on $R = k[x_1, \ldots, x_n]$. En el **Capítol 5**, estudiem els mòduls de cohomologia local $H_I^r(R)$, amb una especial atenció al cas en que I estigui generat per monomis. A diferència dels capítols anteriors, ens centrarem més en l'estructura d'aquests mòduls que no pas en els seus invariants numèrics.

Tot i que les eines que utilitzarem no depenen de la característica del cos k, en el cas que $\mathrm{char}(k) = 0$ tindrem en compte l'estructura de $\mathcal{D}$-mòdul dels mòduls $H_I^r(R)$. D'altra banda, si $\mathrm{char}(k) > 0$ utilitzarem la noció de F-mòdul introduïda per G. Lyubeznik en [56, Definition 1.1].

Cal remarcar que un arranjament de varietats lineals X determina un poset $P(X)$ format per les interseccions de les components irreductibles de X i l'ordre donat per la inclusió. Per exemple, si X està definit per un ideal monomial $I \subseteq R$, $P(X)$ no és res més que el poset $\mathcal{I}$ del Capítol 3 on hem identificat les sumes d'ideals cara de la descomposició primària minimal de I que defineixen el mateix ideal.

En primer lloc, de manera anàloga a la construcció de les successions espectrals de Mayer-Vietoris per a cohomologia singular i per a cohomologia ℓ-àdica introduïdes per A. Björner i T. Ekedahl [11] provem l'existència d'una

successió espectral de Mayer–Vietoris per a cohomologia local:

$$E_2^{-i,j} = \varinjlim{}^{(i)}_{P(X)} H^j_{I_p}(R) \Rightarrow H^{j-i}_I(R)$$

on p és un element del poset $P(X)$, I_p és l'ideal (radical) de definició de la varietat irreductible corresponent a p, i $\varinjlim{}^{(i)}_{P(X)}$ és l'i-èssim functor derivat per l'esquerra del functor límit inductiu en la categoria de sistemes inductius indexats pel poset $P(X)$.

Si estudiem en detall aquesta successió espectral, veiem que el terme E_2 està determinat per la homologia d'un complexe simplicial associat al poset $P(X)$.

Proposició 5.1.4 *Sigui $X \subset \mathbb{A}^n_k$ un arranjament de varietats lineals definit per un ideal $I \subset R$. Sigui $K(> p)$ el complexe simplicial associat al subposet $\{q \in P(X) \mid q > p\}$ of $P(X)$. Aleshores, tenim els isomorfismes de R-mòduls*

$$\varinjlim{}^{(i)}_{P(X)} H^j_{I_p}(R) \simeq \bigoplus_{h(p)=j} [\, H^j_{I_p}(R) \otimes_k \tilde{H}_{i-1}(K(> p); k)\,],$$

on $\tilde{H}(-; k)$ denota la homologia simplicial reduïda. Per conveni considerem que la homologia reduïda del complexe simplicial buit és k en grau -1 i és zero en els altres graus .

El resultat principal d'aquesta secció és la degeneració d'aquesta successió espectral en el terme E_2. L'ingredient principal de la demostració és el teorema d'estructura dels mòduls injectius de Matlis–Gabriel. Això contrasta amb la degeneració de les successions espectrals per a cohomologia singular o per a cohomologia ℓ-àdica de Björner–Ekedahl, on s'utilitza la filtració per pesos de Deligne.

Teorema 5.1.6 *Sigui $X \subset \mathbb{A}^n_k$ un arranjament de varietats lineals definit per un ideal $I \subset R$. Aleshores, la successió espectral de Mayer–Vietoris*

$$E_2^{-i,j} = \varinjlim{}^{(i)}_{P(X)} H^j_{I_p}(R) \Rightarrow H^{j-i}_I(R)$$

degenera en el terme E_2.

La degeneració de la successió espectral de Mayer–Vietoris ens proporciona una filtració de cada mòdul de cohomologia local, on els quocients successius estan donats pel terme E_2.

Corol.lari 5.1.7 *Sigui $X \subset \mathbb{A}^n_k$ un arranjament de varietats lineals definit per un ideal $I \subset R$. Aleshores, per a tot $r \geq 0$ existeix una filtració $\{F^r_j\}_{r \leq j \leq n}$ de $H^r_I(R)$ per R–submòduls tals que:*

$$F^r_j / F^r_{j-1} \cong \bigoplus_{h(p)=j} [\, H^j_{I_p}(R) \otimes_k \tilde{H}_{h(p)-r-1}(K(> p); k)\,].$$

Aquesta filtració és functorial respecte de transformacións afins. A més, és una filtració per $\mathcal{D}$–mòduls holònoms si char$(k) = 0$ i és una filtració per F-mòduls si char$(k) > 0$.

Per a cada $0 \leq j \leq n$, tenim una successió exacta:

$$(s_j): \qquad 0 \to F_{j-1} \to F_j \to F_j/F_{j-1} \to 0,$$

que defineix un element de ${}^*\mathrm{Ext}^1_R(F_j/F_{j-1}\,, F_{j-1})$. En general, no tots els **problemes d'extensió** que podem associar a aquesta filtració tenen una solució trivial. Aquesta és la diferència més gran entre els casos que considerem aquí i els casos considerats per Björner i Ekedahl. Més concretament, en la situació anàloga per a la cohomologia ℓ-àdica d'un arranjament definit sobre un cos finit, les extensions que apareixen són trivials, no només com a extensions d'espais vectorials sobre $\mathbb{Q}_\ell$ sinó que també ho són com a representacions de Galois. Pel cas de la cohomologia singular d'un arranjament complexe, les extensions que apareixen són trivials com a extensions d'estructures de Hodge mixtes (cf. [11, pg. 179]).

En el cas que k sigui un cos de característica zero, podem calcular el cicle característic dels mòduls $H^r_I(R)$ a partir de les successions curtes determinades per la filtració i l'additivitat del cicle característic respecte de successions exactes. En particular, en el cas d'ideals monomials, donem una aproximació diferent a la fórmula obtinguda en el Teorema 3.2.11.

Corol.lari 5.1.9 *Sigui k un cos de característica zero. Aleshores, el cicle característic del $\mathcal{D}$-mòdul holònom $H^r_I(R)$ és*

$$CC(H^r_I(R)) = \sum m_{n-r,p}\, T^*_{X_p}\mathbb{A}^n_k\,,$$

on $m_{n-r,p} = \dim_k \tilde{H}_{h(p)-r-1}(K(> p); k)$ i $T^*_{X_p}\mathbb{A}^n_k$ denota el subespai conormal relatiu de $T^*\mathbb{A}^n_k$ associat a X_p.

En el cas que k sigui el cos dels nombres reals o bé el cos dels nombres complexes, aquests cicles característics determinen els nombres de Betti del complement de l'arranjament X en $\mathbb{A}^n_k$.

Corol.lari 5.1.10 *Amb les notacions anteriors tenim:*

Si $k = \mathbb{R}$ és el cos dels nombres reals, els nombres de Betti del complement de l'arranjament X en $\mathbb{A}^n_{\mathbb{R}}$ es pot calcular en termes de les multiplicitats $\{m_{n-r,p}\}$ com

$$\dim_{\mathbb{Q}} \tilde{H}_r(\mathbb{A}^n_{\mathbb{R}} - X; \mathbb{Q}) = \sum_p m_{n-(r+1),\, p}.$$

Si $k = \mathbb{C}$ és el cos dels nombres complexes, aleshores tenim

$$\dim_{\mathbb{Q}} \tilde{H}_r(\mathbb{A}^n_{\mathbb{C}} - X; \mathbb{Q}) = \sum_p m_{n-(r+1-h(p)),\, p}.$$

Aquesta fórmula s'obté com a conseqüència d'un teorema de Goresky–MacPherson ([34, III.1.3. Theorem A]). Remarquem que nosaltres podem donar una aproximació completament algebraica d'aquestes multiplicitats en termes de cohomologia local, mentre que en el resultat de Goresky–MacPherson, les multiplicitats són dimensions d'uns certs grups de Morse.

Un cop hem descrit les conseqüències de la filtració dels mòduls de cohomologia local donarem una solució dels problemes d'extensió en el cas que l'arranjament de varietats lineals estigui definit per un ideal monomial. Per a ser més concrets, resoldrem aquests problemes d'extensió en el marc dels mòduls ε-**straight**, que és una petita variació dels mòduls straight introduïts per K. Yanagawa [97]. Aquests mòduls estan caracteritzats per la següent propietat:

Proposició 5.2.4 *Un mòdul $\mathbb{Z}^n$-graduat M és ε-straight si i només si existeixen enters $m_\alpha \geq 0$, $\alpha \in \{0,1\}^n$ i una filtració creixent $\{F_j\}_{0\leq j\leq n}$ de M per submòduls graduats tals que per a tot $1 \leq j \leq n$ es tenen isomorfismes graduats*

$$F_j/F_{j-1} \simeq \bigoplus_{\substack{\alpha\in\{0,1\}^n \\ |\alpha|=j}} (H^{|\alpha|}_{\mathfrak{p}_\alpha}(R))^{\oplus m_\alpha}.$$

Les multiplicitats m_α que apareixen en la filtració tenen la següent interpretació:

Proposició 5.2.5 *Sigui M un mòdul ε-straight amb una filtració com en la Proposició 5.2.4. Aleshores:*

$$m_\alpha = \dim_k M_{-\alpha},$$

i.e. l'enter m_α, $\alpha \in \{0,1\}^n$, és la dimensió de la peça de M de grau $-\alpha$.

Els mòduls de cohomologia local suportats en ideals monomials tenen una estructura natural de mòdul ε-straight degut al Corollary 5.1.7. Cal destacar que a partir dels els resultats de M. Mustaţă [72] i de les descripcions de les multiplicitats de la filtració donades pel Corol.lari 5.1.7 i la proposició anterior obtenim:

Corol.lari 5.2.6 *Sigui $I^\vee \subseteq R$ l'ideal dual d'Alexander d'un ideal monomial lliure de quadrats $I \subseteq R$. Aleshores:*

$$\beta_{i,\alpha}(I^\vee) = m_{n-|\alpha|+i,\,\alpha} = \dim_k \tilde{H}_{i-1}(K(>\alpha);k).$$

Si k és un cos de característica zero, els mòduls ε-straight tenen estructura de $\mathcal{D}$-mòdul. El seu cicle característic es pot calcular fàcilment a partir de la filtració (**Proposició 5.2.7**). En el cas de mòduls de cohomologia local, els resultats anteriors ens permenten donar una nova interpretació de les multiplicitats.

Corol.lari 5.2.8 *Sigui $CC(H_I^r(R)) = \sum m_{n-r,\alpha}\ T^*_{X_\alpha}X$, el cicle característic d'un mòdul de cohomologia local $H_I^r(R)$ suportat en un ideal monomial lliure de quadrats $I \subseteq R$. Aleshores:*

i) $m_{n-r,\alpha}(R/I) = \dim_k \tilde{H}_{|\alpha|-r-1}(K(>\alpha);k)$.

ii) $m_{n-r,\alpha}(R/I) = \dim_k (H_I^r(R))_{-\alpha}$.

iii) $m_{n-r,\alpha}(R/I) = \beta_{|\alpha|-r,\alpha}(I^\vee)$.

Aquests resultats ens permeten calcular la sèrie de Hilbert d'un mòdul ε-straight utilitzant les multiplicitats del cicle característic (**Teorema 5.2.9**). En el cas dels mòduls de cohomologia local amb suport un ideal monomial lliure de quadrats, donem una aproximació diferent a la fórmula de N. Terai [92] (vegeu també [98]):

Corol.lari 5.2.10 *Sigui $CC(H_I^r(R)) = \sum m_{n-r,\alpha}\ T^*_{X_\alpha}X$, el cicle característic d'un mòdul de cohomologia local $H_I^r(R)$ suportat en un ideal monomial lliure de quadrats $I \subseteq R$. Aleshores, la seva sèrie de Hilbert és de la forma:*

$$H(H_I^r(R);\mathbf{x}) = \sum_{\alpha\in\{0,1\}^n} m_{n-r,\alpha} \prod_{\alpha_i=0} \frac{x_i^{-1}}{1-x_i^{-1}} \prod_{\alpha_j=1} \frac{1}{1-x_j}.$$

Sigui M un mòdul ε-straight i $\{F_j\}_{0\leq j\leq n}$ la filtració de M obtinguda en la Proposició 5.2.4. Considerem per a cada $0\leq j\leq n$, la successió exacta

$$(s_j): \qquad 0\to F_{j-1}\to F_j\to F_j/F_{j-1}\to 0,$$

que correspon a un element de ${}^*\mathrm{Ext}^1_R(F_j/F_{j-1}, F_{j-1})$. Usant el **Lema 5.3.1**, hi ha un morfisme entre la classe de l'extensió de (s_j) i la classe de l'extensió de la successió

$$(s'_j): \qquad 0\to F_{j-1}/F_{j-2}\to F_j/F_{j-2}\to F_j/F_{j-1}\to 0.$$

Treballant sobre aquesta nova extensió, podem veure que està determinada per les aplicacions k-lineals

$$\delta^\alpha_{-\alpha}: H^0_{\mathfrak{p}_\alpha}(F_j/F_{j-1})_{-\alpha}\longrightarrow H^1_{\mathfrak{p}_\alpha}(F_{j-1}/F_{j-2})_{-\alpha}, \qquad |\alpha|=j.$$

Utilitzant el complexe de Čech, acabem demostrant que l'aplicació k-lineal $\delta^\alpha_{-\alpha}$ queda determinada pel morfisme de multiplicació:

$$\begin{array}{ccc} M_{-\alpha} & \longrightarrow & \oplus_{\alpha_i=1}M_{-\alpha+\varepsilon_i} \\ m & \mapsto & \oplus(x_i\cdot m). \end{array}$$

Per tant, la solució del problema d'extensió dels mòduls ε-straight és la següent:

Proposició 5.3.5 *La classe d'extensió (s_j) està unívocament determinada per les aplicacions k-lineals $\cdot x_i: M_{-\alpha}\to M_{-\alpha+\varepsilon_i}$ on $\mid\alpha\mid = j$ i $\alpha_i=1$.*

Cal remarcar que M. Mustaţă ha provat en [72], que pels mòduls de cohomologia local amb suport ideals monomials lliures de quadrats, les aplicacions lineals $\cdot x_i: H^j_I(R)_{-\alpha}\to H^j_I(R)_{-\alpha+\varepsilon_i}$ es poden calcular explícitament en termes de la cohomologia simplicial d'un complexe de Stanley–Reisner associat a l'ideal I (vegeu també Seccio 6.4).

Finalment, observem que la definició original dels mòduls ε-straight ens indica que estan determinats com a mòduls graduats pels espais vectorials $M_{-\alpha}$, $\alpha\in\{0,1\}^n$ i el morfisme de multiplicació $\cdot x_i: M_{-\alpha}\to M_{-\alpha+\varepsilon_i}$, $\alpha_i=1$. De totes maneres, aquest fet no és suficientment il.lustratiu si el que volem és entendre com aquestes dades determinen els problemes d'extensió que provenen de la successió espectral de Mayer-Vietoris. Nosaltres hem escollit aquesta aproximació més algebraica perquè ens sembla més apropiada per poder extendre els resultats a mòduls de cohomologia local amb suport altres tipus d'arranjaments.

• En el **Capítol 6** també estudiem l'estructura dels mòduls de cohomologia local amb suport ideals monomials lliures de quadrats utilitzant amb més profunditat la teoria de $\mathcal{D}$-mòduls. Més concretament, estudiarem la categoria de n-hipercubs introduïda per A. Galligo, M. Granger i Ph. Maisonobe en [27]. Sigui $X = \mathbb{C}^n$ i $R = \mathbb{C}[x_1, \dots, x_n]$, en aquest capítol usarem les següents notacions:

· $\mathcal{O}_X =$ el feix de funcions holomorfes en X.

· $\mathcal{D}_X =$ el feix d'operadors diferencials en X amb coeficients holomorfs.

· $T =$ la reunió dels hiperplans coordenats de X, amb l'estratificació donada per les interseccions de les seves components irreductibles.

Sigui $Perv^T(\mathbb{C}^n)$ la categoria de complexes de feixos d'espais vectorials de dimensió finita sobre $\mathbb{C}^n$ que són perversos respecte de la estratificació donada de T ([27, I.1]). Notem per $\mathrm{Mod}(\mathcal{D}_X)^T_{hr}$ la subcategoria abeliana plena de la categoria de mòduls holònoms regulars $\mathcal{M}$ en $\mathbb{C}^n$ tals que el seu complexe de solucions $\mathbb{R}\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$ és un objecte de $Perv^T(\mathbb{C}^n)$. Per la correspondència de Riemann-Hilbert, el functor de solucions estableix una equivalència de categories entre $\mathrm{Mod}(\mathcal{D}_X)^T_{hr}$ i $Perv^T(\mathbb{C}^n)$.

La categoria $Perv^T(\mathbb{C}^n)$ va ser linealitzada en [27] de la següent forma: Sigui $\mathcal{C}_n$ la categoria que té per objectes families $\{\mathcal{M}_\alpha\}_{\alpha \in \{0,1\}^n}$ d'espais vectorials de dimensió finita sobre els complexes, juntament amb aplicacions lineals

$$\mathcal{M}_\alpha \xrightarrow{u_i} \mathcal{M}_{\alpha+\varepsilon_i} \quad , \quad \mathcal{M}_\alpha \xleftarrow{v_i} \mathcal{M}_{\alpha+\varepsilon_i}$$

per a cada $\alpha \in \{0,1\}^n$ tal que $\alpha_i = 0$. Aquestes aplicacions lineals s'anomenen aplicació canònica (resp., variació), i han de satisfer les condicions:

$$u_i u_j = u_j u_i, \;\; v_i v_j = v_j v_i, \;\; u_i v_j = v_j u_i \;\text{ i }\; v_i u_i + id \;\text{ és invertible.}$$

Un objecte d'aquest tipus s'anomena n-hipercub, i els espais vectorials $\mathcal{M}_\alpha$ són els seus vèrtexs. Un morfisme entre dos n-hipercubs $\{\mathcal{M}_\alpha\}_\alpha$ i $\{\mathcal{N}_\alpha\}_\alpha$ és un conjunt d'aplicacions lineals $\{f_\alpha : \mathcal{M}_\alpha \to \mathcal{N}_\alpha\}_\alpha$, que commuten amb les aplicacions canòniques i variacions (vegeu [27]). En el mateix paper es demostra que hi ha una equivalència categòrica entre $Perv^T(\mathbb{C}^n)$ i $\mathcal{C}_n$.

Per als nostres propòsits, introduïm la subcategoria abeliana plena $\mathcal{D}^T_{v=0}$ de $\mathrm{Mod}(\mathcal{D}_X)^T_{hr}$ formada pels mòduls que compleixen la següent propietat:

Definició 6.2.1 *Diem que un objecte $\mathcal{M}$ de* $\mathrm{Mod}(\mathcal{D}_X)^T_{hr}$ *té* **variació zero** *si en el corresponent n-hipercub, els morfismes de variació* $v_i : \mathcal{M}_{\alpha+\varepsilon_i} \longrightarrow \mathcal{M}_\alpha$ *són zero per a tot* $1 \leq i \leq n$ *i tot* $\alpha \in \{0,1\}^n$ *amb* $\alpha_i = 0$.

Els objectes de $\mathcal{D}^T_{v=0}$ estan caracteritzats per la següent filtració:

Proposició 6.2.3 *Un objecte $\mathcal{M}$ de* $\mathrm{Mod}(\mathcal{D}_X)^T_{hr}$ *té variació zero si i només si existeix una filtració creixent* $\{\mathcal{F}_j\}_{0\leq j\leq n}$ *de $\mathcal{M}$ per objectes de* $\mathcal{D}^T_{hr}$ *i enters* $m_\alpha \geq 0$ *per* $\alpha \in \{0,1\}^n$ *tals que per a tot* $1 \leq j \leq n$ *es tenen isomorfismes de* $\mathcal{D}_X$*-mòduls*

$$\mathcal{F}_j/\mathcal{F}_{j-1} \simeq \bigoplus_{|\alpha|=j} (\mathcal{H}^j_{X_\alpha}(\mathcal{O}_X))^{\oplus\, m_\alpha}.$$

Per a il.lustrar millor aquesta categoria de mòduls amb variació zero donem una descripció completa dels següents objectes:

· Mòduls projectius. **Proposicions 6.2.4** i **6.2.6**

· Mòduls injectius. **Proposicions 6.2.8** i **6.2.10**

· Mòduls simples. **Proposicions 6.2.12** i **6.2.14**

Sigui $R = \mathbb{C}[x_1,\ldots,x_n]$, l'anell de polinomis amb coeficients en $\mathbb{C}$ i $\mathcal{D} = D(R,\mathbb{C})$ el corresponent anell d'operadors diferencials. Si M és un $\mathcal{D}$-mòdul, aleshores $\mathcal{M}^{an} := \mathcal{O}_X \otimes_R M$ té una estructura natural de $\mathcal{D}_X$-mòdul. Això ens permet definir el functor

$$\begin{array}{rcl} (-)^{an}:\ Mod(\mathcal{D}) & \longrightarrow & Mod(\mathcal{D}_X). \\ M & \longrightarrow & \mathcal{M}^{an} \\ f & \longrightarrow & id\otimes f \end{array}$$

Utilitzant les Proposicions 5.2.4 i 6.2.3, veiem que si M és un mòdul ε-straight, en particular és un $\mathcal{D}$-mòdul, aleshores $\mathcal{M}^{an}$ és un objecte de $\mathcal{D}^T_{v=0}$. El resultat principal d'aquest capítol és:

Teorema 6.3.1 *El functor*

$$(-)^{an} : \varepsilon - \mathbf{Str} \longrightarrow \mathcal{D}^T_{v=0}$$

estableix una equivalència de categories.

Per a fer més explícita aquesta equivalència, donem una descripció del n-hipercub d'un mòdul $\mathcal{M} \in \mathcal{D}^T_{v=0}$ a partir de l'estructura del corresponent mòdul ε-straight M. Més concretament, els vèrtexs del n-hipercub estan descrits com:

Proposició 6.3.4 *Sigui $\mathcal{M} \in \mathcal{D}^T_{v=0}$ un $\mathcal{D}_X$-mòdul holònom regular amb variació zero i $M \in \varepsilon - \mathbf{Str}$ el corresponent mòdul ε-straight. Aleshores tenim l'isomorfisme:*

$$\mathcal{M}_\alpha \cong (M_{-\alpha})^*,$$

on $(M_{-\alpha})^*$ denota l'espai vectorial sobre $\mathbb{C}$ dual de l'espai definit per la peça de M de grau $-\alpha$, per a tot $\alpha \in \{0,1\}^n$.

Les aplicacions lineals u_i del n-hipercub estan descrites com:

Proposició 6.3.5 *Sigui $\mathcal{M} \in \mathcal{D}^T_{v=0}$ un $\mathcal{D}_X$-mòdul holònom regular amb variació zero i $M \in \varepsilon - \mathbf{Str}$ el corresponent mòdul ε-straight. Aleshores, es té el següent diagrama commutatiu:*

$$\begin{array}{ccc} \mathcal{M}_\alpha & \xrightarrow{\quad u_i \quad} & \mathcal{M}_{\alpha+\varepsilon_i} \\ \cong \uparrow & & \cong \uparrow \\ (M_{-\alpha})^* & \xrightarrow{\quad (x_i)^* \quad} & (M_{-\alpha-\varepsilon_i})^* \end{array}$$

on $(x_i)^*$ és l'aplicació dual de la multiplicació per x_i.

Els mòduls de cohomologia local $H^r_I(R)$ amb suport ideals monomials lliures de quadrats $I \subseteq R$ tenen una estructura natural de mòdul amb variació zero. En la darrera secció d'aquest capítol calculem el corresponent n-hipercub a partir de la computació del n-hipercub del complexe de Čech i estudiant la estructura cel.lular dels complexes que obtenim a cada vèrtex. Sigui Δ el complexe simplicial complet tal que els seus vèrtexs estan etiquetats per un sistema minimal de generadors de I. Sigui $T_\alpha := \{\sigma_{\mathbf{1}-\beta} \in \Delta \mid \beta \not\geq \alpha\}$ un subcomplexe de Δ. Aleshores, el resultat obtingut és el següent:

Proposició 6.4.3 *Els mòduls de cohomologia local $H^r_I(R)$ amb suport ideals monomials lliures de quadrats $I \subseteq R$, tenen variació zero. El corresponent n-hipercub és de la forma:*

- **Vèrtexs:** $(\mathcal{H}_I^r(\mathcal{R}))_\alpha \cong \widetilde{H}_{r-2}(T_\alpha;\mathbb{C})$.
- **Aplicacions lineals:** *Tenim el diagrama commutatiu:*

$$\begin{array}{ccc} (\mathcal{H}_I^r(\mathcal{R}))_\alpha & \xrightarrow{u_i} & (\mathcal{H}_I^r(\mathcal{R}))_{\alpha+\varepsilon_i} \\ \downarrow\cong & & \downarrow\cong \\ \widetilde{H}_{r-2}(T_\alpha;\mathbb{C}) & \xrightarrow{\nu_i} & \widetilde{H}_{r-2}(T_{\alpha+\varepsilon_i};\mathbb{C}) \end{array}$$

on ν_i *està induït per la inclusió* $T_\alpha \subseteq T_{\alpha+\varepsilon_i}$.

Cal remarcar que utilitzant els resultats anteriors, recuperem el resultat sobre l'estructura dels mòduls de cohomologia local $H_I^r(R)$ amb suport ideals monomials lliures de quadrats donat per M. Mustaţă [72]. Més concretament, les peces graduades del mòdul són:

$$[H_I^r(R)]_{-\alpha} \cong (\widetilde{H}_{r-2}(T_\alpha;\mathbb{C}))^* \cong \widetilde{H}^{r-2}(T_\alpha;\mathbb{C}), \quad \alpha \in \{0,1\}^n,$$

i el morfisme de multiplicació $x_i : [H_I^r(R)]_{-\alpha-\varepsilon_i} \longrightarrow [H_I^r(R)]_{-\alpha}$ està determinat pel diagrama commutatiu:

$$\begin{array}{ccc} ([H_I^r(R)]_{-\alpha})^* & \xrightarrow{(x_i)^*} & ([H_I^r(R)]_{-\alpha-\varepsilon_i})^* \\ \cong\uparrow & & \uparrow\cong \\ (\mathcal{H}_I^r(\mathcal{R}))_\alpha & \xrightarrow{u_i} & (\mathcal{H}_I^r(\mathcal{R}))_{\alpha+\varepsilon_i} \\ \cong\downarrow & & \downarrow\cong \\ (\widetilde{H}^{r-2}(T_\alpha;\mathbb{C}))^* & \xrightarrow{(\nu_i)^*} & (\widetilde{H}^{r-2}(T_{\alpha+\varepsilon_i};\mathbb{C}))^* \end{array} .$$

on ν_i està induït per la inclusió $T_\alpha \subseteq T_{\alpha+\varepsilon_i}$.

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