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ON THE SLOPE AND GEOGRAPHY OF FIBRED SURFACES AND THREEFOLDS

por

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CERTIFICA:

Que la present memòria ha estat realitzada sota la seva direcció per Miguel Ángel Barja Yáñez i que constitueix la tesi d'aquest per a aspirar al grau de Doctor en Matemàtiques.

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a Ana y Adrià

a Fernando

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Introducción[1]

1. Geografía de superficies y sólidos

En la clasificación birracional de variedades proyectivas complejas, aparece un primer invariante discreto que es la dimensión de Kodaira. Si X es una variedad y K_X es su divisor canónico, se define la dimensión de Kodaira de X como el máximo de las dimensiones de las imágenes de X por las aplicaciones racionales inducidas por los sistemas pluricanónicos. Tenemos una primera acotación

$$0 \leq \mathrm{kod}(X) \leq \dim(X).$$

Las variedades de dimensión de Kodaira máxima se denominan variedades de *tipo general*. La clasificación por dimensión de Kodaira es muy grosera en su clase superior: la mayoría de las variedades son de tipo general. Para acometer un estudio más detallado de éstas se puede establecer el siguiente programa. En primer lugar se identifica un elemento distinguido de la clase de equivalencia birracional: es el denominado modelo minimal. Viene caracterizado por la positividad del divisor canónico (ha de ser *nef*) y por poseer a lo sumo una cierta clase de singularidades. Se conoce su existencia para el caso de superficies y sólidos. En segundo lugar, podemos hacer una clasificación fijando ciertos invariantes numéricos de las variedades. Los primeros candidatos son los que se pueden calcular a partir de las clases de Chern de la variedad.

[1] En cumplimiento de la normativa vigente de la Universitat de Barcelona para la realización de tesis doctorales en lengua no oficial, incluimos en esta introducción un resumen de los contenidos de la memoria así como una sección dedicada a la descripción de los resultados originales obtenidos más importantes.

El problema de clasificación se divide en dos

1.- Fijados unos productos entre clases de Chern $c_{ij} = c_i(X)c_j(X) \in \mathbb{Z}$ $(i + j = \dim X)$ construir un espacio de moduli para las variedades poseyendo esos invariantes.

2.- Determinar qué invariantes c_{ij} son posibles.

En el caso de superficies para el primer problema disponemos del siguiente teorema de Gieseker: fijados $c_1^2(X)$ y $c_2(X)$, existe un espacio de moduli grosero quasi-proyectivo para las superficies de tipo general minimales con esos invariantes. A partir de aquí el estudio de este espacio de moduli (número de componentes, dimensiones,...) constituye todo un campo dentro del estudio de las superficies de tipo general.

Nosotros vamos a interesarnos en la segunda cuestión, que básicamente consiste en decir cuándo el anterior espacio de moduli es no vacío. Este problema es el que se conoce genéricamente como el de la *geografía* de las variedades de tipo general. Vamos a hacer una pequeña descripción de la situación en el caso de las superficies y los sólidos.

En el caso de las superficies los invariantes a estudiar son $c_1^2 = K_X^2$ y c_2. Si llamamos $p_g(X) = h^0(X, \mathcal{O}_X(K_X))$ al género geométrico de X, $q(X) = h^1(X, \mathcal{O}_X)$ a la irregularidad y $\chi\mathcal{O}_X = p_g(X) - q(X) + 1$, una fórmula clásica de Noether nos dice que

$$12\chi\mathcal{O}_X = K_X^2 + c_2.$$

Así es indiferente estudiar la pareja (K_X^2, c_2) a estudiar $(K_X^2, \chi\mathcal{O}_X)$. En ocasiones, de hecho, es interesante considerar la pareja $(K_X^2, p_g(X))$ o incluso la terna $(K_X^2, p_g(X), q(X))$. Nos va a interesar determinar la región del plano en la que se mueven estos invariantes (o de $\mathbb{R}^3$ en el caso de incluir la irregularidad). Esta región está delimitada por rectas correspondientes a desigualdades lineales entre los invariantes de X. Entenderemos el problema geográfico, en un contexto más general, como el problema de encontrar acotaciones entre los invariantes numéricos de X. Las primeras acotaciones que se tienen para el caso de superficies de tipo general son

1.- $K_X^2 + c_2 \equiv 0 \pmod{12}$ (Fórmula de Noether)

2.- $K_X^2 > 0$, $c_2 > 0$

3.- $K_X^2 \leq 3c_2$ o bien $K_X^2 \leq 9\chi\mathcal{O}_X$ (Desigualdad de Miyaoka-Yau)

4.- $K_X^2 \geq 2p_g(X) - 4$ (Desigualdad de Noether).

Se sabe que la mayoría de los puntos de coordenadas enteras en el plano encerrados por estas desigualdades corresponden a clases de Chern de superficies minimales.

Podemos restringir el problema a ciertas clases de superficies de tipo general. Una superficie de tipo general tiene una aplicación pluricanónica con imagen de dimensión 2. De hecho, usualmente la aplicación canónica es birracional. En este caso decimos que X es una superficie canónica. Para éstas puede mejorarse la cota inferior de K_X^2: un resultado clásico de Castelnuovo afirma que

$$K_X^2 \geq 3p_g(X) - 7.$$

De hecho, si incluimos la irregularidad también tenemos (cf. [21], [52])

$$K_X^2 \geq 3p_g(X) + q(X) - 7.$$

La primera fórmula es fina y existen ejemplos para todos los valores de $p_g(F)$ (por supuesto, todos ellos con $q(X) = 0$).

En el caso de los sólidos la situación es similar pero menos conocida. Se tienen las desigualdades (bajo la hipótesis $K_X = -c_1$ amplio):

1.- $c_1c_2 \equiv 0(\mathrm{mod}24)$

2.- $c_1^3 < 0$, $c_1c_2 < 0$

3.- $-c_1^3 \leq \frac{8}{3}(-c_1c_2)$ (Miyaoka-Yau).

Bajo hipótesis adicionales puede demostrarse que aquí también se verifica $K_X^3 \geq 2p_g(X) - 4$. Cuando el sólido es canónico tenemos (cf. [43])

$$K_X^3 \geq 4p_g(X) + c \quad (c = \textit{constante}).$$

El problema geográfico tiene una especial interrelación con el estudio de las fibraciones. Una fibración entre dos variedades proyectivas $f : X \longrightarrow Y$ es un morfismo exhaustivo con fibras conexas. En el caso de superficies toda fibración no trivial es necesariamente sobre una curva. En primer lugar hemos de observar que toda superficie puede ser fibrada birracionalmente sobre $\mathbb{P}^1$ simplemente resolviendo los puntos base de un sistema lineal. De todas formas, con esta construcción puede perderse la minimalidad de la superficie si hemos de explotar para resolver algún punto base. Por otro lado, la mayoría de los ejemplos de superficies de tipo general rellenando el área descrita anteriormente provienen de superficies fibradas (de hecho Persson cubre prácticamente todo el área de existencia desde la cota inferior de K_X^2 hasta $K_X^2 \leq 8\chi\mathcal{O}_X$ con superficies fibradas por curvas de género 2). Así pues, aparece de forma natural el problema del estudio de las fibraciones desde un punto de vista geográfico, que podríamos describir así: dada una fibración $f : X \longrightarrow Y$ con fibra general F estudiar cómo se relacionan los invariantes numéricos de X, Y y F.

2. Variedades fibradas

Sean X e Y dos variedades proyectivas lisas y $f : X \longrightarrow Y$ una fibración. En el estudio de variedades fibradas juega un papel relevante el haz dualizante relativo $\omega_{X/Y}$ y las imágenes directas de sus potencias $R^i f_* \omega_{X/Y}^{\otimes r}$. Estos últimos no son en general haces localmente libres, aunque sí lo son bajo hipótesis adicionales sobre la fibración (que el divisor de ramificación sea a cruzamienntos normales, por ejemplo). En el caso de ser localmente libres, resultan ser *nef*, es decir, cualquier cociente suyo es de grado mayor o igual que cero. Si la fibra general F de f es una variedad de tipo general, entonces además $\omega_{X/Y}$ es un haz inversible *nef*, lo cual se traduce equivalentemente en que su restricción a cualquier curva en X es de grado mayor o igual que cero. El caso sobre el que se tiene más información ocurre cuando la variedad Y es una curva. En este caso, si $i = 0$ y $r \geq 2$, los fibrados considerados resultan ser amplios (y por tanto cualquier cociente suyo tiene grado estrictamente positivo). Cuando $i = 0$, $r = 1$ el fibrado $\mathcal{E} = f_* \omega_{X/Y}$ puede descomponerse segun un clásico teorema de Fujita

$$\mathcal{E} = f_* \omega_{X/Y} = \mathcal{A} \oplus \mathcal{E}_1 \oplus \ldots \oplus \mathcal{E}_r$$

donde los fibrados $\mathcal{E}_i$ son fibrados estables de grado cero sobre Y y $\mathcal{A}$ es un haz amplio. El número de fibrados triviales en la anterior descomposición corresponde a la diferencia entre la irregularidad $q(X) = h^1(X, \mathcal{O}_X)$ de X y el género $b = g(Y)$ de Y. Hay una conjetura del propio Fujita que afirma que tales fibrados deberían ser semiamplios, lo cual equivale a decir que con un cambio de base se converitirían en suma directa de triviales.

La importancia de la positividad del fibrado $\mathcal{E}$ aparece en el estudio de conjeturas $C_{n,m}$ que relacionan las dimensiones de Kodaira de X, Y y F, así como en la construcción de espacios de moduli quasiproyectivos para variedades polarizadas (cf. [89]).

Como ya hemos observado, desde un punto de vista *geográfico* es interesante conocer la relación entre los invariantes numéricos de las variedades que aparecen en una fibración: la variedad base Y, la variedad de partida X y la fibra general F. Las técnicas generales disponibles se basan en ciertas propiedades de positividad (*nefitud*) tanto de $f_*\omega_{X/Y}$ como de $\omega_{X/Y}$ y funcionan sólo en el caso en que Y sea una curva. El caso más estudiado es desde luego el más sencillo, el de las superficies fibradas sobre curvas. Es también el que más puede aportar al más desarrollado tema de la geografía de las superficies de tipo general. Recientemente se ha dado el primer paso en el estudio de los sólidos fibrados sobre curvas, si bien la casuística se hace más complicada.

3. Superficies fibradas

3.1. Invariantes

Sea S una superficie lisa, B una curva lisa y $f : S \longrightarrow B$ una fibración. Supondremos siempre que la fibración es relativamente minimal, es decir, que no hay (-1)-curvas contenidas en las fibras. Llamemos F a la fibra general, $g = g(F)$ y $b = g(B)$. También llamaremos como arriba $\mathcal{E} = f_*\omega_{S/B}$.

Una primera relación sencilla entre los invariantes de la fibración es la siguiente: si consideramos la irregularidad de S $q(S) = h^1(S, \mathcal{O}_S)$ se tiene que

$$b \leq q(S) \leq b + g.$$

Tenemos aún más: las igualdades extremas están completamente caracterizadas. La igualdad $b = q(S)$ ocurre si y sólo si el morfismo de Albanese de S factoriza a través de B (situación a la que nos referiremos, si $b \geq 1$, diciendo que f *es una fibración de Albanese*). Por otro lado la igualdad $b + g = q(S)$ se verifica si y sólo si S es birracionalmente equivalente a un producto.

La forma más adecuada de estudiar otros invariantes numéricos en esta situación es considerar los invariantes *relativos*

$$K_{S/B}^2 = (K_S - f^*K_B)^2 = K_S^2 - 8(b-1)(g-1)$$

$$\chi_f = (-1)^{\dim S}(\chi\mathcal{O}_S - \chi\mathcal{O}_B\chi\mathcal{O}_F) = \chi\mathcal{O}_S - (b-1)(g-1)$$
$$e_f = e(S) - e(B)e(F) = e(S) - 4(b-1)(g-1)$$

donde $K_{S/B}$ es el divisor canónico relativo y e es la característica de Euler topológica. Respecto a ésta, hay un resultado clásico que la determina

$$e_f = \sum(e(F_i) - e(F))$$

donde la suma está tomada sobre todas las fibras singulares de f y donde $e(F_i) - e(F) \geq 0$ (de hecho $e(F_i) = e(F)$ ocurre si y sólo si $g = 1$ y F_i es un múltiplo de una curva elíptica lisa).

Un cálculo inmediato vía Riemann-Roch en S y B y la sucesión espectral de Leray nos dice que

$$\chi_f = \deg\mathcal{E}.$$

El primer resultado que relaciona las cantidades anteriores es la fórmula de Noether, que en versión relativa afirma

$$12\chi_f = K_{S/B}^2 + e_f.$$

Restrinjámonos a partir de ahora al caso en que F es una curva de tipo general, lo cual se traduce en pedir que $g \geq 2$ (es una condición necesaria, aunque no suficiente, para que S sea de tipo general). La nefitud de $K_{S/B}$ es aquí un resultado de Arakelov que, junto a la fórmula de Noether, nos permite afirmar que

$$K_{S/B}^2 \geq 0, \quad \chi_f \geq 0, \quad e_f \geq 0$$

y caracterizar la igualdad: si $K_{S/B}^2 = 0$ entonces f es *isotrivial* (es decir, todas las fibras lisas de f son isomorfas entre sí), si $e_f = 0$ entonces f es *lisa* (es decir, f no tiene fibras singulares) y finalmente si $\chi_f = 0$ entonces f es a la vez lisa e isotrivial (en cuyo caso a f se la denomina *localmente trivial*).

Supongamos que f no es localmente trivial. Entonces podemos definir la *pendiente* de la fibración f como el cociente

$$\lambda(f) = \frac{K^2_{S/B}}{\chi_f}$$

en términos de la cual las anteriores desigualdades se leen así:

$$0 \leq \lambda(f) \leq 12.$$

A la vista de los resultado conocidos sobre invariantes de fibraciones que resumiremos más adelante, el concepto de la pendiente de una fibración aparece como relevante. Una de las primeras propiedades de la pendiente es que es estable por operaciones naturales con fibraciones. La pendiente es invariante por cambios de base

$$\widetilde{S}[r]^{\widetilde{\sigma}}[d]^{\widetilde{f}}S[d]^{f}\widetilde{B}[r]^{\sigma}B$$

siempre y cuando σ no ramifique sobre la imagen de alguna fibra no semiestable de f. También es invariante por recubrimientos étales de S.

3.2. Pendiente y moduli

Tal vez la forma más sencilla de entender la adecuación del concepto de pendiente al estudio de las superficies fibradas es a través de una interpretación alternativa de la pendiente en el caso particular en el que la fibración sea *semiestable*, es decir, cuando todas las fibras de f son semiestables en el sentido de Deligne-Mumford. Esta interpretación nos sugerirá también el comportamiento esperado de la pendiente.

Sea $\overline{\mathcal{M}}$ la compactificación de Deligne-Mumford del espacio de moduli $\mathcal{M}_g$ de curvas lisas de género g, adjuntando las curvas singulares *estables*. Dada una fibracion f de curvas semiestables, tras la contracción de (-2)-curvas contenidas en fibras (que producen singularidades muy sencillas) producimos una fibración por curvas estables. Tenemos entonces un morfismo bien definido de B en $\overline{\mathcal{M}}$ cuya imagen es una curva $\overline{B}$ si la fibración no es isotrivial. Recíprocamente dada una curva $\overline{B}$ en $\overline{\mathcal{M}}$ podemos producir una fibración semiestable sobre un cambio de base de una desingularización de $\overline{B}$. Ambas construcciones no son exactamente inversas mutuas, pero están dominadas por un cambio de base común. Dado que trabajamos con fibraciones semiestables y la pendiente es entonces invariante por cambios de base, es indiferente trabajar con una u otra fibración. Así pues confundiremos en lo que sigue $\overline{B}$ con B y supondremos que la fibración f está directamente inducida sobre B.

La pendiente tiene una buena interpretación en términos de productos intersección de B con ciertos divisores destacados en $\overline{\mathcal{M}}$. Hay así un trasvase entre problemas geográficos (acotaciones) de la pendiente y problemas clásicos sobre la geometría de $\mathcal{M}_g$.

Llamemos h a la clase del divisor de Hodge en $\overline{\mathcal{M}}$ y δ a la clase del divisor que representa curvas singulares en $\overline{\mathcal{M}}$. En este contexto se tiene que

$$\delta B = e_f$$

$$hB = \mathrm{deg} f_* \omega_{S/B} = \chi_f$$

y usualmente se define la *pendiente de* B como

$$s(B) := \frac{\delta B}{hB}.$$

Por tanto se tiene a través de la fórmula de Noether que

$$\lambda(f) + s(B) = 12.$$

Como un ejemplo de la relación de la pendiente con la geometría de $\overline{\mathcal{M}}$ podemos mencionar el siguiente: es bien conocido que $K_{\overline{\mathcal{M}}} \sim_{\mathbb{Q}} 13h - 2\delta$ y que $\overline{\mathcal{M}}$ es de tipo general si $g \geq 24$. Tendremos que toda fibración semiestable con $\lambda(f) < \frac{11}{2}$ ($s(B) > \frac{13}{2}$) verifica

$$(mK_{\overline{\mathcal{M}}})B < 0$$

y por tanto B está en el lugar base de todos los sistemas pluricanónicos de $\overline{\mathcal{M}}$.

Por un teorema de Xiao para fibraciones arbitrarias (y de Cornalba- Harris para fibraciones semiestables) se tiene que

$$s(B) \leq 8 + \frac{4}{g}$$

pero cuyo límite máximo sólo es alcanzado por ciertas fibraciones hiperelípticas (donde la fibra general es una curva hiperelíptica). Si B pasa por un punto general de $\overline{\mathcal{M}}$ esta cota puede mejorarse sustancialmente. Por un resultado de Mumford, Harris y Eisenbud obtenemos

$$s(B) \leq 6 + o\left(\frac{1}{g}\right)$$

y que las fibraciones que no verifican esta desigualdad están contenidas en la clausura $\overline{\mathcal{D}}_k$ de un cierto lugar $k-$gonal. Por ejemplo, si g es impar, entonces se tiene que

$$s(B) \leq 6 + \frac{12}{g+1}$$

$$k = \frac{g+1}{2}$$

que corresponde a la cota $\lambda(f) \geq 6\frac{g-1}{g+1}$.

Aparece así una pregunta natural: ¿existe una estratificacion $\mathcal{J}_1 \cup \mathcal{J}_2 \cup ... \cup \mathcal{J}_l = \overline{\mathcal{M}}$ tal que $\mathcal{J}_1$ es el lugar hiperelíptico y $\mathcal{J}_{l-1} = \mathcal{D}_k$ (descrito por el resultado de Mumford, Harris y Eisenbud), $\mathcal{J}_l = \overline{\mathcal{M}} \setminus \overline{\mathcal{J}_{l-1}}$ para la cual tengamos sucesivamente mayores pendientes mínimas λ_j en $\overline{\mathcal{M}} \setminus \overline{\mathcal{J}_j}$?. Esta estratificación debería de contemplar al menos la gonalidad (o el índice de Clifford) de las curvas.

3.3. Resultados conocidos. Problemas

Hemos de tener en cuenta que el problema de estudiar la pendiente de fibraciones arbitrarias no puede reducirse al problema de estudiar fibraciones semiestables. En efecto, dada una fibración cualquiera, podemos obtener una semiestable básicamente a través de un cambio de base. Desafortunadamente ese cambio de base ramifica necesariamente sobre la imagen de las fibras no semiestables. En este proceso la variación de $K^2_{S/B}$, χ_f y e_f es bien conocida por un trabajo reciente de Tan (cf. [86], [87]) pero esta información no permite controlar cómo varía la pendiente durante el cambio de base. De todas formas, la pregunta anterior sobre la estratificación de $\overline{\mathcal{M}}$ tiene en el contexto de las fibraciones generales una traducción inmediata: ¿cómo afecta la gonalidad de la fibra general F a la pendiente de la fibración?.

Estudiar la pendiente $\lambda(f)$ de las superficies fibradas consiste en dar buenas cotas para su variación en términos de la geometría de las variedades implicadas (la propia superficie, la curva base y las fibras). Muy poco se sabe de las cotas superiores de $\lambda(f)$ aparte del mencionado resultado general $\lambda(f) \leq 12$. No se disponen de técnicas generales y únicamente es conocido que

Xiao ([94]), Matsusaka ([68]). Si f no es lisa y la fibra general de f es hiperelíptica de género g, entonces

$$\lambda(f) \leq \frac{4(g-1)(3g+1)}{g^2} \quad \textit{si} \quad g \quad \textit{par}$$

$$\lambda(f) \leq \frac{4(3g^2-2g+2)}{g^2+1} \quad \textit{si} \quad g \quad \textit{impar.}$$

En general se han concentrado los esfuerzos en el estudio de cotas inferiores de $\lambda(f)$. El resultado básico en esta dirección es el mencionado anteriormente y debido a Xiao:

Xiao (cf. [92]). Si $g \geq 2$ y f no es localmente trivial

$$\lambda(f) \geq 4\frac{g-1}{g}.$$

Posteriormente Konno (cf. [63]) demuestra que la igualdad sólo pueden verificarla ciertas fibraciones hiperelípticas. Así pues aparece de manera natural el problema de estudiar las fibraciones no hiperelípticas. El paso siguiente es claramente estudiar las fibraciones en las cuales la fibra general es trigonal. El resultado es

Konno ([65]). Si F es una curva trigonal y $g \geq 6$

$$\lambda(f) \geq \frac{14(g-1)}{3g+1}.$$

También se tiene en un caso particular

Stankova-Frenkel([85]). Si F es una curva trigonal y f es semiestable

$$\lambda(f) \geq \frac{24(g-1)}{5g+1};$$

si, además, la curva F tiene el invariante de Maroni general

$$\lambda(f) \geq 5 - \frac{6}{g}.$$

Un intento de atacar el problema más en general ha sido llevado a cabo recientemente por Konno (cf. [66]), pero las fórmulas obtenidas contienen términos no fácilmente calculables y dependen fuertemente de una conjetura de M. Green sobre las syzygias de las curvas canónicas. De todas formas, cabe destacar un caso particular que coincide con la cota dada por Mumford, Harris y Eisenbud en el caso semiestable

Konno([66]). Si F tiene índice de Clifford máximo (es decir, es general en moduli), g=g(F) es impar y la conjetura de Green es cierta, entonces

$$\lambda(f) \geq 6\frac{g-1}{g+1}.$$

Cabe destacar también los resultados de Konno (cf. [63]) para fibraciones de género bajo ($g = 3, 4$ y 5), obtenidos independientemente por Chen ([17]) para $g = 4$, perfilando la cota general de Xiao. También es importante notar que apenas se conocen fibraciones no hiperelípticas con $\lambda(f) < 4$ (sólo para $g \leq 6$).

Todos estos resultados estudian cómo se ve afectada la pendiente de una fibración por la geometría de la fibra general. Cabe preguntarse también por la influencia en la pendiente de otras propiedades, *globales*, de S. En este sentido tenemos

Xiao ([92]). Si f no es una fibración de Albanese (es decir, si $q = q(S) > b = g(B)$), entonces

$$\lambda(f) \geq 4$$

y la igualdad se verifica sólo si $q = b + 1$.

En la misma referencia, Xiao observa tambien que la positividad del fibrado $\mathcal{E} = f_*\omega_{S/B}$ influye en la pendiente y da evidencias de que si la pendiente es menor que 4 entonces $\mathcal{E}$ debería ser amplio.

A la vista de las anteriores consideraciones, cabe plantearse tres problemas (los dos primeros mucho más generales que el tercero) en el estudio de las superficies fibradas desde un punto de vista geográfico:

PROBLEMA 1: Estudio de la influencia de la geometría de la fibra general en la pendiente. En particular, cómo afecta la existencia de series lineales especiales. Aquí se encuadra el problema de la estratificación planteado en el conjunto de las fibraciones semiestables. Tal y como ya aparece en el resultado sobre fibraciones trigonales de Stankova-Frenkel, el comportamiento no dependerá sólo de la gonalidad, sino de posibles invariantes discretos adicionales.

PROBLEMA 2: ¿Cómo influyen invariantes globales de S (básicamente la irregularidad) en la pendiente de f?. El resultado anterior de Xiao, sugiere que la pendiende mínima de f depende crecientemente de la diferencia $s = q(S) - b$.

PROBLEMA 3: ¿Cómo son de especiales las fibraciones con pendiente baja ($\lambda(f) < 4$)?. El comportamiento de estas fibraciones está sugerido por un estudio más detallado de los resultados anteriores. El fibrado $\mathcal{E} = f_*\omega_{S/B}$ debería ser amplio y la fibra general F no hiperelíptica para $g >> 0$.

4. Sólidos fibrados sobre curvas

Sea ahora T un sólido y $f : T \longrightarrow B$ una fibración relativamente minimal sobre una curva lisa B (aquí el hecho de que la fibración sea relativamente minimal puede introducir ciertas singularidades en T). Como en el caso de las superficies fibradas se pretende relacionar los invariantes numéricos de T, B y la fibra general F. Aquí los invariantes relativos a considerar pueden ser

$$K_{T/B}^3 = K_T^3 - 6K_F^2(b-1)$$
$$\Delta_f = \deg\mathcal{E}$$
$$\chi_f = (-1)^{\dim_T}(\chi\mathcal{O}_T - \chi\mathcal{O}_B\chi\mathcal{O}_F) = \chi\mathcal{O}_F\chi\mathcal{O}_B - \chi\mathcal{O}_T$$
$$e_f = e(T) - e(B)e(F).$$

Tenemos también una desigualdad para la irregularidad

$$b \le q = q(T) \le b + q(F).$$

Aparecen no obstante comportamientos diferentes. Así por ejemplo ya no es cierto que la igualdad $q(T) = b + q(F)$ implique la trivialidad de f. Pero lo que será más importante para nosotros: no es cierto ahora que $\chi_f = \Delta_f$. Además, aunque Δ_f es positivo al ser el grado de un fibrado nef, no es conocido que χ_f sea positivo, siendo de los dos el invariante que acarrea información geográfica más detallada.

El único resultado general conocido en este contexto es el siguiente (damos una version simplificada del teorema; en Theorem 5.1 podemos encontrar una versión completa)

Ohno ([73]). Si F es una superficie de tipo general y $p_g(F) \geq 3$, entonces

$$K_{T/B}^3 \geq (4 - \varepsilon(p_g(F)))\Delta_f \geq (4 - \varepsilon(p_g(F)))\chi_f$$

donde $\varepsilon(p_g(F)) \sim o(\frac{1}{p_g(F)})$.

Observemos que la segunda desigualdad no proporciona ninguna información si $\chi_f < 0$ (pues $K_{T/B}^3 \geq 0$ al ser un divisor *nef*).

Los tres problemas enumerados en el caso de superficies fibradas pueden ser trasladados con cambios menores a este contexto. Aparece un problema añadido: demostrar que χ_f no es negativo. En ese caso, podríamos definir una pendiente para el caso de sólidos fibrados sobre curvas en términos de la cual el teorema de Ohno tiene una traducción inmediata.

5. Contenidos y resultados de esta memoria

Esta memoria está dedicada al estudio de la geografía de superficies y sólidos canónicos y de las superficies y sólidos fibrados sobre curvas.

5.1. Contenidos

Capítulo 1. Este primer capítulo está dedicado a la exposición y desarrollo de las principales técnicas que utilizaremos a lo largo de la memoria. Algunos resultados son originales, otros son técnicas bien conocidas y en otros casos extendemos métodos o técnicas conocidos y desarrollados en dimensiones bajas a dimensión superior. La sección §1.4 es fruto de un trabajo conjunto con Juan Carlos Naranjo.

En §1.1 damos las definiciones y notaciones básicas así como como una serie de construcciones elementales y bien conocidas en variedades fibradas sobre curvas $f : X \longrightarrow B$; en particular se construye el morfismo canónico relativo de una fibración. La sección §1.2 está dedicada a explicar el método más importante en el estudio de los invariantes de una variedad fibrada sobre una curva, debido básicamente a Xiao. Este método permite dar una cota inferior a la autointersección de un divisor *nef* D en X usando información de sistemas lineales en las fibras del morfismo y la información numérica de la filtración de Harder-Narasimhan del fibrado imagen directa $f_*\mathcal{O}_X(D)$ en B. En la sección §1.3 se introduce el método de las hipercuádricas relativas. Se trata de un método mucho más modesto en cuanto a capacidad de aplicaciones pero que funciona bien para valores bajos del género geométrico de las fibras (cuando la imagen canónica de la variedad fibra está contenida en pocas cuádricas). Introducido inicialmente por Catanese y Ciliberto en [19] y por Reid en [79] para el caso de superficies fibradas, ha sido utilizado para el estudio de superficies fibradas con género de la fibra bajo por Konno (cf. [63]). Aquí damos una construcción general para variedades X de dimensión arbitraria (no necesariamente lisas) y obtenemos las primeras aplicaciones para sólidos fibrados sobre curvas.

En §1.4 demostramos varios resultados de extensión de aplicaciones definidas en fibras de superficies fibradas a toda la variedad. Aunque el estudio se divide en tres partes, considerando el caso de automorfismos, de series lineales y el resto, debido a métodos diferentes de prueba, la conclusión es básicamente la misma siempre: tales morfismos extienden a un morfismo global de la variedad tras un cambio de base (y bajo ciertas hipótesis de unicidad sin necesidad de un cambio de base). Incluimos también un resultado sobre subesquemas abelianos relativos que nos permiten extender algunos resultados a sólidos fibrados sobre curvas en capítulos posteriores.

Finalmente, en la sección §1.5 hacemos un muy breve repaso a la teoría de Anulación Genérica de Green y Lazarsfeld. En especial remarcamos un resultado de Simpson sobre la estructura de los lugares excepcionales de no anulación de la cohomología que será clave en el estudio de una conjetura de Fujita en el Capítulo 3.

Capítulo 2. Este capítulo está básicamente dedicado al estudio de sólidos canónicos. A partir de una desigualdad general válida en dimensión arbitraria se obtienen, vía el teorema de Riemann-Roch, desigualdades para superficies y sólidos canónicos. En el caso de superficies la desigualdad obtenida incluyendo la irregularidad es bien conocida, pero utilizando un análisis de la caracterización de la igualdad en el caso límite se demuestra que la cota conocida no es fina en la mayoría de los casos. Se demuestran algunas cotas mejores bajo hipótesis adicionales.

La desigualdad obtenida para sólidos canónicos es (hasta donde sabemos) la primera conocida que incluye la irregularidad y se obtiene no sólo del anterior análisis sino de un estudio detallado del caso en el que la variedad está fibrada sobre una curva, utilizando fuertemente el método de Xiao y el método de las hipercuádricas relativas del Capítulo 1.

Capítulo 3. En este breve capítulo estudiamos la conjetura de Fujita sobre la semiamplitud de la imagen directa del haz canónico relativo. Si $f : X \longrightarrow Y$ es una fibración y consideramos el haz (localmente libre bajo ciertas condiciones) $\mathcal{E} = f_*\omega_{X/Y}$, el hecho de que este haz sea semiamplio implica que cualquier cociente suyo con determinante topológicamente trivial ha de tener determinante de torsión. Nosotros demostramos que este hecho es así a través de la utilización de un resultado clave de Simpson descrito en §1.5. Esto en particular resuelve completamente la conjetura de Fujita para fibraciones sobre curvas elípticas (sobre curvas racionales es trivial). Como consecuencia obtenemos también en el caso de fibraciones sobre curvas una descomposición de $\mathcal{E}$, tras un cambio de base no ramificado, con unas propiedades de positividad ligeramente mejores a las de la descomposición original de Fujita. Observamos que una repuesta afirmativa a la conjetura de Fujita para fibraciones sobre curvas tendría consecuencias importantes en las cotas obtenidas en los capítulos 4 y 5.

Capítulo 4. En este capítulo estudiamos la pendiente de superficies fibradas sobre curvas, en términos de los problemas 1,2 y 3 relacionados más arriba. Parte de los resultados contenidos en las secciones §4.2, §4.3 y §4.5 son consecuencia de un trabajo conjunto con Francesco Zucconi.

En §4.1 y §4.2 estudiamos las fibraciones que son recubrimiento doble de otra fibración. Hay dos motivos básicos para considerar tales fibraciones. En primer lugar, cuando se aplica el método de Xiao en el estudio de superficies fibradas, han de estudiarse ciertos sistemas lineales en la fibra general F. Más concretamente, interesa dar una cota inferior del grado de estos sistemas lineales en términos de su rango. Esta cota viene dada aproximadamente por un factor 2 según el lema de Clifford. Pero en realidad puede mejorarse a un factor 3 si dejamos de lado los casos en los que la curva F es un recubrimiento doble de otra curva E. Así pues, el estudio detallado del caso en el que la superficie es un recubrimiento doble permite dejar de lado esta posibilidad al atacar situaciones más generales como ocurrirá en la sección §4.3.

El segundo motivo aparece al entender el estudio de los recubrimientos dobles como una posible generalización del bien conocido caso de las fibraciones hiperelípticas (el caso más sencillo de recubrimiento doble). Un estudio detallado del caso bielíptico (recubrimiento doble de curvas elípticas) nos hace observar que posiblemente las fibraciones que son recubrimientos dobles jueguen un papel especial en el estudio del Problema 1. Las curvas que son recubrimiento doble tienen gonalidad especial y posiblemente hayan de ser un caso aparte en el estudio de éstas, de la misma forma que en el estudio de las fibraciones trigonales aparecían fibraciones trigonales especiales.

En §4.1 damos una cota inferior fina para la pendiente de las fibraciones bielípticas. También damos una caracterización de las fibraciones que están en el caso límite de la desigualdad. En otro orden de cosas, damos un teorema de estructura para las fibraciones bielípticas y demostramos que toda fibración bielíptica lisa es necesariamente isotrivial, lo que, hasta donde sabemos, sólo se conoce para fibraciones hiperelípticas y elípticas.

En §4.2 damos una cota inferior para la pendiente de las fibraciones que son recubrimientos dobles. En especial demostramos que estas fibraciones tienen pendiente al menos 4 bajo hipótesis adicionales muy generales. Ambos resultados han de entenderse como una contribución a los problemas 1 y 3.

En §4.3 estudiamos el problema de encontrar una cota inferior para la pendiente en términos de la irregularidad de la superficie (Problema 2) y de hecho encontramos una cota inferior estrictamente creciente como función de $s = q(S) - b$. En particular encontramos una influencia sobre la pendiente de f de la existencia de otras fibraciones en S sobre curvas de género al menos 2.

La sección §4.4 la dedicamos a estudiar el Problema 3. A partir de los cálculos anteriores y de los resultados del Capítulo 3 podemos concluir que el comportamiento conjeturado para las fibraciones de pendiente menor que 4 es cierto bajo hipótesis adicionales (fibras no hiperelípticas o género de la fibra o de la base bajos).

Finalmente en §4.5 incluimos una serie de familias de ejemplos para contrastar las cotas encontradas en las secciones anteriores. No encontramos que las cotas sean finas, pero sí que son asintóticamente finas (para valores grandes de los invariantes considerados en cada caso). Éste es un fenómeno que no ocurría en cotas ya existentes.

Capítulo 5. En este capítulo consideramos fibraciones de sólidos sobre curvas $f : T \longrightarrow B$ con fibra general F de tipo general, desde un punto de vista geográfico. En primer lugar en §5.1 demostramos que el invariante relativo $\chi_f = \chi\mathcal{O}_F\chi\mathcal{O}_B - \chi\mathcal{O}_T$ es ciertamente no negativo bajo hipótesis muy generales. Esto nos permite definir el concepto de pendiente para una clase muy general de fibraciones. De hecho definimos dos pendientes diferentes y estudiamos su comportamiento ante operaciones naturales. Damos también ejemplos en los cuales $\chi_f < 0$.

En §5.2 damos una cota inferior para estas pendientes que mejora substancialmente los resultados de Ohno para valores grandes del género de la fibra: si en el caso de Ohno las pendientes son asintóticamente 4, en nuestros caso aparece 9 como valor límite. Aparece también como fenómeno interesante que la existencia o no de sistemas lineales especiales en la fibra general F (hiperelípticos, trigonales o tetragonales) afecta de manera importante a la pendiente. Este fenómeno ha de entenderse como la manifestación del Problema 1 a nivel de sólidos fibrados.

En la sección §5.3 estudiamos el problema de dar una cota inferior de la pendiente en función de la irregularidad del sólido. El resultado es sorprendentemente mucho mejor que en el caso de las superficies fibradas. Esa cota inferior es 4, pero podemos afirmar que en realidad es 9 salvo en un número muy limitado de casos que son completamente descritos y para los cuales damos un resultado de estructura. Este fenómeno no aparecía en el caso de las superficies fibradas, en el cual ninguna constante superior a 4 parece jugar un papel destacado (las cotas encontradas en el Capítulo 4 son crecientes y varían entre 4 y 5, pero no sugieren la existencia de una cota inferior constante bajo ninguna hipótesis adicional).

En §5.4 estudiamos el análogo al Problema 3 en superficies: el estudio de sólidos fibrados con pendiente menor que 4. Estas fibraciones existen. De hecho Ohno da una lista de 7 posibles familias, la primera de las cuales está formada por fibraciones con fibra general fibrada a su vez por un sistema lineal de curvas de género 2. Nosotros demostramos que éste es el único caso posible si $p_g(F) \geq 15$ y damos restricciones adicionales a los casos $p_g(F) \leq 14$ (en los cuales aparecen también fibraciones hiperelípticas o trigonales). Demostramos además que estas fibraciones con pendiente baja verifican $q(T) = b$ (es decir, si $b \geq 1$ son fibraciones de Albanese) y que el haz $\mathcal{E} = f_*\omega_{T/B}$ es amplio si

$b \leq 1$.

5.2. Resumen de resultados

Pasamos a dar una lista de los principales resultados originales obtenidos en esta memoria. Los resultados de enunciado largo o necesitados de notación previa son resumidos y remitidos a los enunciados en la memoria.

Capítulo 1

(i) *Teorema.* Si $f : T \longrightarrow B$ es un sólido fibrado sobre una curva, damos una cota inferior para la autointersección del divisor canónico relativo en los casos en los que la imagen canónica de la superficie fibra general esté contenida en pocas cuádricas o bien el fibrado $\mathcal{E} = f_* \omega_{T/B}$ sea semiestable (Corollary 1.21).

(ii) *Teorema.* Sea $f : S \longrightarrow B$ una superficie fibrada tal que una cantidad no numerable de fibras F_t posee un morfismo sobre otra curva E_t; fijemos uno en cada una de ellas; entonces existe una aplicación racional globalmente definida en S, tras un cambio de base, sobre otra fibración. Esta aplicación restringida a una cantidad no numerable de las fibras coincide con los morfismos fijados. En caso de unicidad de los morfismos, fijados invariantes discretos, el cambio de base no es necesario (teoremas 1.23, 1.27 y 1.30).

(iii) *Teorema.* Si $f : \mathcal{A} \longrightarrow B$ es un esquema abeliano sobre B y una cantidad no numerable de fibras posee una subvariedad abeliana de dimensión fijada, entonces tras un cambio de base (evitable si hay unicidad) existe un subesquema abeliano $\mathcal{B} \hookrightarrow \mathcal{A}$ sobre B que restringido a una cantidad no numerable de fibras coincide con los previamente fijados (Theorem 1.28).

Aunque no es un resultado propiamente original, también queremos remarcar

(iv) En §1.3 hacemos una construcción del método de las hipercuádricas relativas para fibraciones sobre curvas de variedades de dimensión arbitraria, no necesariamente lisas, y para divisores de Weil. En particular generalizamos un lema técnico debido a Konno para la acotación de los términos negativos que aparecen en el método (Lemma 1.20).

Capítulo 2

(v) *Teorema.* Si T es un sólido canónico, entonces $K_T^3 \geq 4p_g(F)+6q(F)-32$ (Theorem 2.6).

También obtenemos en el caso de superficies

(vi) No existen superficies canónicas con $K_S^2 = 3p_g(S)+q(S)-7$ y $p_g(F) = 6$ o $p_g(F) \geq 8$ (Theorem 2.4).

Capítulo 3

(vii) *Teorema.* Si $f : X \longrightarrow Y$ es una fibración con divisor de ramificación a cruzamientos normales, $\mathcal{E} = f_*\omega_{X/Y}$ y $\mathcal{F}$ es un cociente de $\mathcal{E}$ con determinante topológicamente trivial, entonces $\det(\mathcal{F})$ es de torsión (Theorem 3.4).

Como casos particulares se tienen

(viii) Si $Y = B$ es una curva elíptica la conjetura de Fujita es cierta (Corollary 3.6).

(ix) Si $Y = B$ es una curva cualquiera, tras un cambio de base no ramificado el fibrado $\mathcal{E}$ es suma directa de un fibrado amplio, un fibrado trivial y un fibrado *estrictamente nef* (Corollary 3.7).

Capítulo 4

(x) *Teorema.* Si $f : S \longrightarrow B$ es una fibración bielíptica entonces es un recubrimiento doble de una fibración elíptica tras un cambio de base. Si el género de la fibra es al menos 6, entonces el cambio de base no es necesario. Si además la fibración es lisa, entonces es isotrivial (Proposition 4.1).

(xi) *Teorema.* Una fibración bielíptica de género al menos 6 tiene pendiente mayor o igual a cuatro. Esta cota es fina y las fibraciones que verifican la igualdad son recubrimientos dobles de superficies elípticas localmente triviales, ramificados sobre divisores con singularidades a lo sumo no esenciales (Theorem 4.3).

(xii) *Teorema.* Si $f : S \longrightarrow B$ es una fibración de género g que es recubrimiento doble de una fibración de género h, damos una cota inferior para la pendiente de f en función de h (función estrictamente creciente) si $g \geq 4h+1$ (Theorem 4.11).

(xiii) *Teorema.* Si f es una fibración que es recubrimiento doble como en (xii), entonces la pendiente de f es al menos 4 si $g \geq 2h+11$ y la fibra general F no es tetragonal (Theorem 4.13).

(xiv) *Teorema.* Si $f : S \longrightarrow B$ es una superficie fibrada que no es una fibración de Albanese (es decir, $q(S) > b$), damos una cota inferior para la pendiente de f que es una función creciente de $s = q(S) - b$ (Theorem 4.16 y Theorem 4.19).

(xv) *Teorema.* Si $f : S \longrightarrow B$ es una superficie fibrada, damos una cota inferior para la pendiente de f en función de la existencia de otras fibraciones en S sobre curvas de género al menos 2 (Theorem 4.24).

(xvi) *Teorema.* Si $f : S \longrightarrow B$ es una superficie fibrada con pendiente menor que 4, entonces el haz $\mathcal{E} = f_*\omega_{S/B}$ es amplio si la fibra general F es no hiperelíptica, o el género de F es menor o igual que 3 o el género de B es menor o igual que 1 (Theorem 4.27).

Además como complemento obtenemos

(xvii) Si $f : S \longrightarrow B$ es una superficie fibrada con pendiente exactamente 4, entonces $\mathcal{E}$ (como en (xvi)) tiene a lo sumo un cociente de grado cero y rango uno. Además, si lo tiene, $\mathcal{E}$ es la suma directa de ese haz invertible de grado cero y de un fibrado semiestable (Proposition 4.28).

Por último quisiéramos poner de manifiesto diversas familias de ejemplos construidas en §4.5 para contrastar las cotas encontradas en fibraciones de superficies sobre curvas.

También, como complemento a (x):

(xviii) Existe una fibración bielíptica en la cual la fibra general tiene exactamente dos involuciones bielípticas y que sólo es recubrimiento doble de una superficie elíptica tras un cambio de base (Example 4.2).

Capítulo 5

(xix) *Teorema.* Si $f : T \longrightarrow B$ es una fibración de un sólido sobre una curva, con fibra general F una superficie de tipo general, entonces $\chi_f = \chi\mathcal{O}_F\chi\mathcal{O}_B - \chi\mathcal{O}_T$ es no negativo, bajo hipótesis muy generales, si la dimensión de la imagen del morfismo de Albanese de T no es 1 (Theorem 5.7).

(xx) *Teorema.* Si $f : T \longrightarrow B$ es una fibración relativamente minimal de un sólido sobre una cuva, con fibra F de tipo general y $p_g(F) \geq 3$ entonces la pendiente de f es al menos $(9 - \varepsilon(p_g(F))$ (con $\varepsilon(p_g(F)) \sim o(\frac{1}{p_g(F)}))$ si F no está fibrada por curvas hiperelípticas, trigonales o tetragonales y $|K_F|$ no está compuesto con un pencil. En estos casos excepcionales damos cotas alternativas que dependen del género de la fibración que posee F (Theorem 5.11).

(xxi) *Teorema.* Si $f : T \longrightarrow B$ es una fibración relativamente minimal de un sólido sobre una curva, que no es una fibración de Albanese (es decir, $q(T) > b$)), entonces la pendiente de f es al menos 9 salvo que F esté fibrada por curvas hiperelípticas, trigonales o tetragonales y f factorice por un morfismo de T en una superficie S. Para estos casos excepcionales también damos cotas alternativas. En particular la pendiente siempre es al menos 4 (Theorem 5.14).

(xxii) *Teorema.* Si $f : T \longrightarrow B$ es una fibración relativamente minimal de un sólido sobre una curva, con pendiente menor que 4, entonces $\mathcal{E} = f_*\omega_{T/B}$ es amplio si $b = g(B) \leq 1$. Además $q(T) = b$ y la fibra general F está fibrada por un sistema lineal de curvas de género 2 si $p_g(F) \geq 15$ (por curvas hiperelípticas o trigonales si $p_g(F) \leq 14$, salvo en un caso excepcional) (Theorem 5.20).

Los siguientes resultados son posiblemente bien conocidos, pero desconocemos una referencia.

(xxiii) Si F es una curva que es un recubrimiento cíclico no ramificado de grado n de otra curva E y posee una única serie lineal g^1_d, entonces $n|d$ (Lemma 5.12).

(xxiv) Si F y $\widetilde{F}$ son superficies de tipo general y $\widetilde{F}$ es un recubrimiento cíclico no ramificado, de orden primo $n >> 0$, entonces $\widetilde{F}$ no posee fibraciones hiperelípticas, trigonales o tetragonales diferentes de las construidas por cambio de base de fibraciones en F, salvo que la fibración sea de curvas bielípticas (Lemma 5.13).

Chapter 1

Technical results

The main objects we are interested in along this work are the so called *fibrations* between projective varieties. In this chapter we collect the main techniques and results we will use everywhere.

After some preliminary definitions and constructions in §1.1 we give the two main methods to study the numerical invariants (*geography*) of varieties fibred over curves: the method of Xiao and the relative hyperquadrics method. The first one is the most powerful and being introduced by Xiao in [92] has been successfully used and generalized in several works ([65],[63],[73]). The method of relative hyperquadrics is more modest but is very useful in order to study lowest cases of the invariants; it appears in the study of fibred surfaces in several works ([19],[63],[79]). In §1.3 we generalize it to arbitrary fibrations over curves.

In section §1.4 we prove some facts on the extension of maps defined on fibres to the whole variety. Although most of the results are stated for fibred surfaces, we include a result on relative abelian subschemes that allows us to conclude similar results in higher dimensions (cf. Chapter 5).

Finally we include in §1.5 a brief account of Generic Vanishing theory. Although it is not a topic directly concerned with fibrations, the basic results of Green and Lazarsfeld on the structure of *exceptional loci* allows us to use it frequently. It is worth to mention the results of Simpson on the torsion nature of the exceptional locus when it is zero-dimensional; this will be the main tool we will need in Chapter 3 to study a conjecture of Fujita.

1.1 Preliminaries

1.1.1 Notations, conventions and basic definitions

All throughout this memory we work over the field of complex numbers $\mathbb{C}$.

A variety will be an integral, separated scheme of finite type over $\mathbb{C}$. Unless otherwise stated, they will be assumed to be projective. We will call it a *curve*, a *surface* or a *threefold* according to whether its dimension is 1, 2 or 3 respectively.

From now on let X be a normal variety of dimension d. We will set $\mathrm{Div}(X)$ for the set of Cartier divisors and $Z_{d-1}(X)$ for the set of Weil divisors. An element of $\mathrm{Div}(X) \otimes \mathbb{Q}$ will be called a $\mathbb{Q} - Cartier$ divisor. We will use the following notations for the equivalence relations

$\sim$: linear equivalence in $\mathrm{Div}(X)$ or in $Z_{d-1}(X)$

$\equiv$: numerical equivalence in $\mathrm{Div}(X)$ or in $Z_{d-1}(X)$

$\sim_{\mathbb{Q}}$: $\mathbb{Q}$-linear equivalence in $\mathrm{Div}(X) \otimes \mathbb{Q}$ (i.e., $D_1 \sim_{\mathbb{Q}} D_2$ if and only if there exists $r \in \mathbb{N}$ such that $rD_1 \sim rD_2$).

We note $\mathrm{Pic}(X)$ for the group of line bundles on X. Given a coherent sheaf $\mathcal{F}$ and a divisor D on X we note

$$\mathcal{F}^* = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$$
$$\mathcal{F}(D) = \mathcal{F} \otimes \mathcal{O}_X(D)$$

If $Z \subseteq X$ is a subscheme, we will denote $\mathcal{J}_{Z,X}$ for the ideal sheaf of Z in X.

Notation $f : X \longrightarrow Y$ will stand for morphisms while $f : X - - - - \rightarrow Y$ will stand for rational maps. If f is a morphism, the definition of f^* for Cartier divisors is the usual. It can be extended to $\mathbb{Q}$- Cartier divisors by linearity. For instance, if $E \in \mathrm{Div}(X) \otimes \mathbb{Q}$ and $r \in \mathbb{N}$ verifies that $rE \in \mathrm{Div}(X)$, we define $f^*(E) = \frac{1}{r} f^*(rE)$. Given a surjective morphism $f : X \longrightarrow Y$ and $D \in \mathrm{Div}(X) \otimes \mathbb{Q}$ we say that D is $f-$*exceptional* if $\mathrm{codim}_Y(f(D)) \geq 2$.

If X is smooth and $D \in \mathrm{Div}(X)$, we say that D is a *normal crossings divisor* if it is reduced and its irreducible components are smooth, meeting transversally.

There is a bijective correspondence between Weil divisors on X modulo linear equivalence and rank one reflexive sheaves on X modulo isomorphism. If D is a Weil divisor we note $\mathcal{O}_X(D)$ its associated sheaf. Under this correspondence $\mathcal{O}_X(D_1+D_2) \cong (\mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2))^{**}$. We will note $\mathcal{O}_X(D)^{[m]} = \mathcal{O}_X(mD)$.

There is a well defined Weil divisor on X, the canonical divisor K_X, such that, if $X^0 \subseteq X$ is the smooth part of X, $K_{X|X^0} = \det\Omega^1_{X_0}$. We note $w_X = \mathcal{O}_X(K_X)$, and $w_X^{[m]} = \mathcal{O}(mK_X)$.

X is said to be *factorial* if $Z_{d-1}(X) = \mathrm{D}iv(X)$, $\mathbb{Q}$-*factorial* if $Z_{d-1}(X) \otimes \mathbb{Q} = \mathrm{Div}(X) \otimes \mathbb{Q}$, Gorenstein if X is Cohen-Macaulay and $K_X \in \mathrm{Div}(X)$. X is said to be $\mathbb{Q}$-Gorenstein if $K_X \in Div(X) \otimes \mathbb{Q}$. If X is $\mathbb{Q}$-Gorenstein, the *index* of X is the least integer r such that $rK_X \in \mathrm{D}iv(X)$.

X is said to have only *canonical singularities* (respectively *terminal singularities*) if X is $\mathbb{Q}$-Gorenstein and there exists a resolution of singularities $f : Y \longrightarrow X$ such that $rK_Y = f^*(rK_X) + \sum a_i E_i$ with $a_i \geq 0$ (respectively $a_i > 0$) and E_i f-exceptional.

Let $D \in \mathrm{Div}(X) \otimes \mathbb{Q}$. D is said to be *nef* if $DC \geq 0$ for any curve $C \subseteq X$.

If $\mathcal{F}$ is a locally free sheaf on X and $\varphi : \mathbb{P} := \mathbb{P}_X(\mathcal{F}) \longrightarrow X$ is the associated projective bundle, we note $L_{\mathcal{F}} \in \mathrm{Div}(\mathbb{P})$ for a *tautological divisor* on $\mathbb{P}$, which is characterized by $\varphi_*\mathcal{O}_{\mathbb{P}}(L_{\mathcal{F}}) = \mathcal{F}$. We will use indifferently the notations $\mathcal{O}_{\mathbb{P}}(k)$ and $\mathcal{O}_{\mathbb{P}}(kL_{\mathcal{F}})$. $\mathcal{F}$ is called *nef* if $L_{\mathcal{F}}$ is nef in $\mathbb{P}_X(\mathcal{F})$. Equivalently, if and only if for any smooth curve C, and map $f : C \longrightarrow X$, any quotient of $f^*\mathcal{F}$ has nonnegative degree (cf. [89]).

X is called a *minimal variety* if X has only terminal singularities and K_X is nef.

1.1.2 General setting on fibrations

We quote here the general setting on fibrations we need for the development of the methods we are interested in along this chapter. More concrete results on fibred surfaces and threefolds are postponed to the introduction of Chapters 4 and 5 respectively. Also in Chapter 3 we quote general properties of arbitrary fibrations. All the results we state here are well known. We refer to [55], [71] and [73] for references.

Let W be a smooth projective variety and X any projective variety . We say that a morphism $f : X \longrightarrow W$ is a *fibration* if f is surjective and has connected fibres. The *branch locus* of f is the set of points in W over which f is non-smooth. If $g : W' \longrightarrow W$ is a morphism we sometimes will denote $X_{W'} = X \times_W W'$.

Given a fibration with X normal we have a natural Weil divisor $K_{X/W} = K_X - f^*K_W$ and its associated reflexive sheaf $\omega_{X/W}$, the *relative canonical divisor* and *sheaf* respectively.

Assume from now on that $W = B$ is a smooth curve and that X is a normal $\mathbb{Q}$-factorial variety of dimension n. Then $f_*\omega_{X/B}$ is a *nef* locally free sheaf on B (cf. [30])(for more general properties of this sheaf we refer to the introductions of Chapters 3,4 and 5). If the general fibre F of f is a variety of *general type* (see Chapter 2 for definition), $K_{X|F_t}$ is *nef* for every $t \in B$ and X has only terminal singularities then $\omega_{X/B}$ is also *nef* on X (cf. [73] theorem 1.4).

Let D be a Weil divisor. We can then consider $f_*\mathcal{O}_X(D)$ which is torsion free, and hence locally free, on B of rank $h^0(F, \mathcal{O}_F(D_{|F}))$ ($F = f^{-1}(t)$, $t \in B$, general fibre of f). Assume $h^0(F, \mathcal{O}_F(D_{|F})) \neq 0$, and take a nonzero subsheaf $\mathcal{F} \subseteq f_*\mathcal{O}_X(D)$. The inclusion induces a natural map $f^*\mathcal{F} \longrightarrow f^*f_*\mathcal{O}_X(D) \longrightarrow \mathcal{O}_X(D)$. Let $D_1 \in Z_{d-1}(X)$ be such that $\mathrm{Im}(f^*\mathcal{F} \longrightarrow \mathcal{O}_X(D))^{**} = \mathcal{O}_X(D - D_1)$. D_1 is called the *fixed locus of* $\mathcal{F}$ *in* X and it is the unique effective Weil divisor defined by this property.

The map $f^*\mathcal{F} \longrightarrow \mathcal{O}_X(D)$ induces a rational map $\psi : X - - - \rightarrow \mathbb{P}_B(\mathcal{F}) = \mathbb{P}$ over B which can be resolved in the following way.

Lemma 1.1 ([73] Lemma 1.1) *With the above assumptions, there exists a desingularization* $\mu : \widetilde{X} \longrightarrow X$ *such that* $\lambda = \psi \circ \mu : \widetilde{X} \longrightarrow \mathbb{P}$ *is everywhere defined and* $\lambda^*L_{\mathcal{F}} \sim_{\mathbb{Q}} \mu^*(D - D_1) - E$ *where* E *is an effective* μ*-exceptional* $\mathbb{Q}$*-divisor on* $\widetilde{X}$*.*

We call the *moving part of* $\mathcal{F}$ *on* $\widetilde{X}$ to be $M = \lambda^*L_{\mathcal{F}} \in \mathrm{Div}(\widetilde{X})$ (not necessarily effective) and *the fixed part of* $\mathcal{F}$ *on* $\widetilde{X}$ to be $Z = \mu^*(D_1) + E \in \mathrm{Div}(\widetilde{X}) \otimes \mathbb{Q}$. If $\mathcal{F} = f_*\mathcal{O}_X(D)$ we call them the *moving and fixed part of* D *on* $\widetilde{X}$, respectively.

If we take $D = K_{X/B}$, the rational map $\psi : X - - - \rightarrow \mathbb{P}_B(f_*\omega_{X/B})$ is called the *relative canonical map* of X over B.

The following are some easy but useful results about fibrations over curves that we will use frequently.

Lemma 1.2 *Let X be a normal variety, B a smooth curve of genus b and $f : X \longrightarrow B$ a fibration. Let X_t be the fibre of f over $t \in B$.*

For any coherent sheaf $\mathcal{G}$ on X and for any $\mathfrak{a} \in Pic(B)$ let $\mathcal{G}(\mathfrak{a}) = \mathcal{G} \otimes f^*(\mathfrak{a})$.

Suppose that $\mathfrak{a}$ is ample enough and that $\mathcal{G}$ satisfies the following condition: for general $t \in B$ the sequence

$$0 \longrightarrow \mathcal{G}(-t) \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}_{|X_t} \longrightarrow 0$$

is exact.

Then, the natural morphism

$$H^0(X, \mathcal{G}(\mathfrak{a})) \longrightarrow H^0(X_t, \mathcal{G}_{|X_t})$$

is surjective for general $t \in B$.

PROOF: Consider the exact sequence

$$0 \longrightarrow \mathcal{G}(\mathfrak{a} - t) \longrightarrow \mathcal{G}(\mathfrak{a}) \longrightarrow \mathcal{G}_{|X_t} \longrightarrow 0$$

for general $t \in B$. Taking cohomology we get

$$0 \longrightarrow H^0(X, \mathcal{G}(\mathfrak{a} - t)) \longrightarrow H^0(X, \mathcal{G}(\mathfrak{a})) \xrightarrow{m_t} H^0(X_t, \mathcal{G}_{|X_t})$$

If $\mathfrak{a}$ is ample enough then $h^1(B, (f_*\mathcal{G}) \otimes (\mathfrak{a} - t)) = 0$ and then $h^0(X, \mathcal{G}(\mathfrak{a} - t)) = h^0(B, (f_*\mathcal{G}) \otimes (\mathfrak{a} - t))$ does not depend on t by the Hirzebruch-Riemann-Roch Theorem for coherent sheaves on B.

Furthermore

$$\begin{aligned} \dim \mathrm{Im}\ (m_t) &= h^0(X, \mathcal{G}(\mathfrak{a})) - h^0(X, \mathcal{G}(\mathfrak{a} - t)) = \\ &= h^0(B, (f_*\mathcal{G}) \otimes \mathfrak{a}) - h^0(B, (f_*\mathcal{G}) \otimes (\mathfrak{a} - t)) = \\ &= d + r(a + 1 - b) - (d + r(a - 1 + 1 - b)) = r \end{aligned}$$

where

$$\begin{aligned} d &= \deg f_*\mathcal{G} \\ a &= \deg \mathfrak{a} \\ r &= \mathrm{rank}\ (f_*\mathcal{G}) = h^0(X_t, \mathcal{G}_{|X_t}) \quad \text{for } t \in B \text{ general} \end{aligned}$$ ■

Lemma 1.3 *Let $f : X \longrightarrow B$ be a fibration as above. Let $\mathcal{L} \in PicX$. If $\mathfrak{a} \in PicB$ is ample enough, then the natural map*

$$h : f^* f_* \mathcal{L}(\mathfrak{a}) \longrightarrow \mathcal{L}(\mathfrak{a})$$

is an epimorphism just except at the base points of the linear system $|\mathcal{L}(\mathfrak{a})|$. Moreover, if for a general fibre F of f, the linear system $|\mathcal{L}_{|F}|$ is base-point free, then such base points are concentrated on a finite number of fibres.

PROOF: From the sequence of maps $X \xrightarrow{f} B \xrightarrow{\rho} \mathrm{Spec}\mathbb{C}$ we can consider the following natural commutative diagram

$$f^*\rho^*\rho_*f_*\mathcal{L}(\mathfrak{a}) = (\rho \circ f)^*(\rho \circ f)_*\mathcal{L}(\mathfrak{a}) = H^0(S, \mathcal{L}(\mathfrak{a})) \otimes_{\mathbb{C}} \mathcal{O}_S[d]_k[r]\mathcal{L}(\mathfrak{a})f^*f_*\mathcal{L}(\mathfrak{a})[ur]^h$$

Since k is surjective for $\mathfrak{a}$ ample enough, it follows that surjectivity of h is equivalent to surjectivity of $e = h \circ k$. This fails to be an epimorphism precisely at the base points of $|\mathcal{L}(\mathfrak{a})|$. Finally, using Lemma 1.2 one has that $|\mathcal{L}(\mathfrak{a})|$ has no base points on a general fibre F of f. ∎

We frecuently will use this results in the following way

Remark 1.4 Let Σ be $\overline{\psi(X)}$ in $\mathbb{P}$, following the notation of Lemma 1.1. Let $\varphi : \mathbb{P} \longrightarrow B$ be the natural projection. Consider the sheaf $\mathcal{G} = \mathcal{J}_{\Sigma,\mathbb{P}} \otimes \mathcal{O}_{\mathbb{P}}(k)$, where $\mathcal{J}_{\Sigma,\mathbb{P}}$ is the ideal sheaf of Σ in $\mathbb{P}$. For $t \in B$ we have $\mathcal{G}_{|\mathbb{P}_t} = \mathcal{J}_{F_t,\mathbb{P}_t}(k)$. We claim that $\mathcal{G}$ verifies the hypotheses of Lemma 1.2 and so that for $\mathfrak{a}$ ample enough we have an epimorphism

$$H^0(\mathbb{P}, \mathcal{J}_{\Sigma,\mathbb{P}} \otimes \mathcal{O}_{\mathbb{P}}(k) \otimes \varphi^*(\mathfrak{a})) \longrightarrow H^0(\mathbb{P}^{g-1}, \mathcal{J}_{F_t,\mathbb{P}_t}(k))\,.$$

Indeed, just consider (we confuse F_t with $\overline{\psi(F_t)} \subseteq \Sigma$)

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{J}_{\Sigma,\mathbb{P}} \otimes \varphi^*(-t) \arrow[r] \arrow[d] & \mathcal{O}_{\mathbb{P}}(\varphi^*(-t)) \arrow[r] \arrow[d] & \mathcal{O}_{\Sigma}(\varphi^*(-t)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{J}_{\Sigma,\mathbb{P}} \arrow[r] \arrow[d] & \mathcal{O}_{\mathbb{P}} \arrow[r] \arrow[d] & \mathcal{O}_{\Sigma} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{J}_{F_t,\mathbb{P}_t} \arrow[r] \arrow[d] & \mathcal{O}_{\mathbb{P}_t} \arrow[r] \arrow[d] & \mathcal{O}_{F_t} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

with the three rows and the two right columns trivially exact. Then the snake Lemma makes the left hand side column exact. Now just tensor with $\mathcal{O}_{\mathbb{P}}(k)$, which is locally free.

1.2 Xiao's method

Let X be as in §1.1.2 a normal $\mathbb{Q}$-factorial projective variety of dimension n and D a nef Weil divisor. If $f : X \longrightarrow B$ is a fibration onto a smooth curve, Xiao's method consists in giving a lower bound for D^n in terms of $\deg f_*\mathcal{O}_X(D)$ via the use of the numerical information of the Harder-Narasimhan filtration of $\mathcal{E} = f_*\mathcal{O}_X(D)$, and the numerical invariants of some linear systems on the fibres of f.

The main idea is given by Xiao ([92]) where he uses the method in the case of fibred surfaces. Later on Ohno ([73]) gives a version for fibred threefolds, but not in complete generality. Finally Konno ([65]) gives a general version for any variety fibred over a curve. We give here an outline of the main results.

First of all we recall two basic results on vector bundles over curves.

Definition 1.5 Let $\mathcal{F}$ be a locally free sheaf over a smooth curve B. The *slope* of $\mathcal{F}$ is the ratio $\mu(\mathcal{F}) = \dfrac{\deg_{\mathcal{F}}}{\mathrm{rk}_{\mathcal{F}}}$. $\mathcal{F}$ is called *stable* (respectively *semistable*) if for every subsheaf $\mathcal{G} \subseteq \mathcal{F}$, $\mu(\mathcal{G}) < \mu(\mathcal{F})$ (respectively $\mu(\mathcal{G}) \leq \mu(\mathcal{F})$).

Theorem 1.6 (Harder-Narasimhan [38]) *Let $\mathcal{F}$ be a locally free sheaf over a smooth curve B. Then there exists a uniquely determined Harder-Narasimhan filtration by locally free sheaves*

$$0 = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \ldots \subseteq \mathcal{F}_\ell = \mathcal{F}$$

such that:

(i) for every i, $\mathcal{F}_i/\mathcal{F}_{i-1}$ is a semistable vector bundle

(ii) $\mu_1 > \mu_2 > \ldots > \mu_\ell$, where $\mu_i := \mu(\mathcal{F}_i/\mathcal{F}_{i-1})$.

Moreover, if $\mathcal{G} \subseteq \mathcal{F}$ verifies that $\mu(\mathcal{G}) = \mu_1$, then $\mathcal{G} \subseteq \mathcal{F}_1$.

Remark 1.7 If we put $r_i = \mathrm{rk}\mathcal{F}_i$ then $\deg\mathcal{F} = \sum_{r=0}^{\ell} r_i(\mu_i - \mu_{i+1})$ where $\mu_{\ell+1} = 0$ by definition.

Usually we call $\{\mu_1, \ldots, \mu_\ell\}$ the Harder-Narasimhan slopes of $\mathcal{F}$.

From now on we set $\mu_-(\mathcal{F}) = \mu_\ell(\mathcal{F}) = \mu(\mathcal{F}/\mathcal{F}_{\ell-1})$.

Sometimes is useful to consider the *virtual slopes* $\{\nu_1, \ldots, \nu_{\mathrm{rk}\mathcal{F}}\}$ of $\mathcal{F}$ as $\nu_j = \mu_i$ for $r_{i-1} < j \leq r_i$, $j = 1, \ldots, \mathrm{rk}\mathcal{F}$.

Theorem 1.8 (Miyaoka, [70]) *Let $\mathcal{F}$ be a locally free sheaf on B. Let $E \in Div(\mathbb{P}_B(\mathcal{F})) \otimes \mathbb{Q}$, $E \equiv L_{\mathcal{F}} - xG$, where G is a fibre of $\mathbb{P}_B(\mathcal{F}) \longrightarrow B$. Then E is nef if and only if $x \leq \mu_-(\mathcal{F})$.*

In particular, $\mathcal{F}$ is semistable if and only if $L_{\mathcal{F}} - \mu(\mathcal{F})G$ is nef.

Remark 1.9 Let $X, D \in Z_{d-1}(X)$, $f : X \longrightarrow B$ and $\mathcal{F} \subseteq f_*\mathcal{O}_X(D)$ as in §1.1.2. The induced map γ

$$\widetilde{X}[d]^{\mu}[r]^{\gamma}\mathbb{P}[ddl]^{\varphi}X[d]^{f}B$$

produces a divisor on $\widetilde{X}$, $\widetilde{H} = \gamma^* L_{\mathcal{F}}$ such that $N = \widetilde{H} - \mu_-(\mathcal{F})F$, is a nef $\mathbb{Q}$-divisor by the previous theorem. We have

$$\mu^* D \equiv N + \mu_-(\mathcal{F})F + Z$$

where $Z, N \in \mathrm{Div}(\widetilde{X}) \otimes \mathbb{Q}$ ($Z \in \mathrm{Div}(\widetilde{X})$ if D is Cartier) and Z is effective. Note that if $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq f_*\mathcal{O}_X(D)$ then by construction we have

$$N_1 + \mu_-(\mathcal{F}_1)F + Z_1 \equiv N_2 + \mu_-(\mathcal{F}_2)F + Z_2$$

where $Z_1 \geq Z_2$.

Proposition 1.10 (Xiao, [92]; Konno, [65]) *Let Y be a nonsingular projective variety of dimension n. Let $f : Y \longrightarrow B$ be a fibration onto a smooth curve and let F be the general fibre. Assume that there are rational numbers ν_1, ν_2, effective $\mathbb{Q}$-divisors Z_1, Z_2 and nef $\mathbb{Q}$-divisors N_1, N_2 on Y such that $\nu_1 \geq \nu_2$, $Z_1 \geq Z_2$ and $N_1 + \nu_1 F + Z_1 \equiv N_2 + \nu_2 F + Z_2$. Then for any integer k with $0 < k \leq n$ and any nef $\mathbb{Q}$-divisors $A_1, \ldots, A_{n-k}$,*

$$\left(N_2^k - N_1^k - (\nu_1 - \nu_2) F \sum_{i=0}^{k-1} N_1^i N_2^{k-1-i} \right) A_1 \ldots A_{n-k} \geq 0$$

holds.

Proof: Note that $N_2^k - N_1^k = (N_2 - N_1) \sum_{i=0}^{k-1} N_1^i N_2^{k-1-i}$ and $N_2 - N_1 \equiv (\nu_1 - \nu_2)F + (Z_1 - Z_2)$. Then use the nefness of N_i, A_j and F and that $(\nu_1 - \nu_2)F$, $(Z_1 - Z_2) \geq 0$. ■

Consider as before X a normal variety, $D \in Z_{d-1}(X)$, $f : X \longrightarrow B$ a fibration onto a smooth curve and $\mathcal{E} = f_*\mathcal{O}_X(D)$. Let $\mathcal{E}_i$ be a piece of the Harder-Narasimhan filtration of $\mathcal{E}$. Let $\mu : \widetilde{X} \longrightarrow X$ be a nonsingular model that resolves all the rational maps $X - - - \rightarrow \mathbb{P}_B(\mathcal{E}_i)$. By Theorem 1.8 and Remark 1.9 we have a sequence of rational numbers $\mu_1 > \mu_2 > \ldots > \mu_\ell$ and of effective $\mathbb{Q}$-divisors on Y $Z_1 \geq Z_2 \geq \ldots \geq Z_\ell$ such that if we set

$$H_i = \gamma^* L_{\mathcal{E}_i}$$

$$N_i := H_i - \mu_i \widetilde{F}$$

then for every i N_i is nef and for every i, j

$$N_i + \mu_i \widetilde{F} + Z_i \equiv \mu^*(D) \equiv N_j + \mu_j \widetilde{F} + Z_j$$

where $\widetilde{F}$ is a general fibre of $\widetilde{X} \longrightarrow B$.

Put $P_i = N_{i|\widetilde{F}} = H_{i|\widetilde{F}}$. Observe that P_i is a Cartier divisor on $\widetilde{F}$ and that $|P_i|$ is a linear subsystem of $|P_{i+1}|$ of dimension at least $r_i - 1$. Note also that by construction of $\widetilde{X}$ the linear systems $|P_i|$ are base point free.

Observe that if $\mathcal{E}$ is a nef vector bundle then $\mu_\ell \geq 0$. We can define $\mu_{\ell+1} = 0$, $Z_{\ell+1} = 0$, $N_{\ell+1} = H_{\ell+1} = N_\ell$ coherently with the above properties for $i \leq l$. If furthermore D is a nef Weil divisor in X, we have an alternative: $N_{\ell+1} = H_{\ell+1} = \mu^* D$.

We can then apply Proposition 1.10 and get

Proposition 1.11 *With the above notations, if D and $\mathcal{E}$ are nef and $\{i_1, \ldots, i_q\}$ is a sequence of indices $1 \leq i_1 < \ldots < i_q < i_{q+1} := \ell + 1$ we have*

(i) If $n = 2$, $D^2 \geq \sum_{p=1}^{q} (d_{i_p} + d_{i_{p+1}})(\mu_{i_p} - \mu_{i_p+1})$ where $d_i = degP_i = N_i F$.

(ii) If $n = 3$ and $1 \leq m \leq q+1$

$$D^3 \geq \sum_{p=1}^{m-1} (P_{i_p} + P_{i_{p+1}}) P_{i_m} (\mu_{i_p} - \mu_{i_{p+1}}) + \sum_{p=m}^{q} (P_{i_p}^2 + P_{i_p} P_{i_{p+1}} + P_{i_{p+1}}^2)(\mu_{i_p} - \mu_{i_{p+1}})$$

(iii) If $n = 4$ and $1 \leq m_1 \leq m_2 \leq q+1$

$$\begin{aligned} D^4 \quad \geq & \sum_{p=1}^{m_1-1} (P_{i_p} + P_{i_{p+1}}) P_{i_{m_1}} P_{i_{m_2}} (\mu_{i_p} - \mu_{i_{p+1}}) + \\ & + \sum_{p=m_1}^{m_2-1} (P_{i_p}^2 + P_{i_p} P_{i_{p+1}} + P_{i_{p+1}}^2) P_{i_{m_2}} (\mu_{i_p} - \mu_{i_{p+1}}) + \\ & + \sum_{p=m_2}^{q} (P_{i_p}^3 + P_{i_p}^2 P_{i_{p+1}} + P_{i_p} P_{i_{p+1}}^2 + P_{i_{p+1}}^3)(\mu_{i_p} - \mu_{i_{p+1}}) \end{aligned}$$

Proof: All these work in the same way. We prove (iii) . Since D is nef, so is $\mu^*(D)$ and we define $\mu^*(D) \cong H_{\ell+1} = N_{\ell+1}$. We have

$$D^4 = (\mu^*(D))^4 = N_{\ell+1}^4 \geq N_{i_{m_2}}^4 + \sum_{p=m_2}^{q} (P_{i_p}^3 + P_{i_p}^2 P_{i_{p+1}} + P_{i_p} P_{i_{p+1}}^2 + P_{i_{p+1}}^3)(\mu_{i_p} - \mu_{i_{p+1}})$$

taking $k = 4$ in Proposition 1.10.

Take $k = 3$ and use that $N_{i_{m_2}}$ is nef to get

$$N_{i_{m_2}}^4 \geq N_{i_{m_1}}^3 N_{i_{m_2}} + \sum_{p=m_1}^{m_2-1} (P_{i_p}^2 + P_{i_p}P_{i_{p+1}} + P_{i_{p+1}}^2) P_{i_{m_2}} (\mu_{i_p} - \mu_{i_{p+1}})$$

and finally take $k = 2$ and use the nefness of $N_{i_{m_1}}$

$$N_{i_{m_2}} N_{i_{m_1}}^3 \geq \sum_{p=1}^{m_1-1} (P_{i_p} + P_{i_{p+1}}) P_{i_{m_2}} P_{i_{m_1}} (\mu_{i_p} - \mu_{i_{p+1}})$$

■

Remark 1.12 As a matter of convenience for further references we concrete the formulas of propositions 1.10 and 1.11 in some cases.

Assume $f : S \longrightarrow B$ is a fibred surface with fibre of genus $g \geq 2$; taking $D = K_{S/B}$ and $\{i_1, ..., i_q\} = \{1, ..., \ell\}$ and $\{1, \ell\}$ respectively we get

$$K_{S/B}^2 \geq \sum_{i=1}^{\ell} (d_i + d_{i+1})(\mu_i - \mu_{i+1})$$

$$K_{S/B}^2 \geq (d_1 + d_\ell)(\mu_1 - \mu_\ell) + (d_\ell + d_{\ell+1})\mu_\ell \geq d_\ell(\mu_1 + \mu_\ell) = (2g - 2)(\mu_1 + \mu_\ell)$$

Assume now $f : T \longrightarrow B$ is a fibred threefold as in Chapter 5 with fibres of general type. Take $D = K_{T/B}$. If $\{i_1, ..., i_q\} = \{1, ..., \ell\}$ and $1 \leq m \leq \ell + 1$ we get

$$K_{T/B}^3 \geq \sum_{i=1}^{m-1} (P_i + P_{i+1}) P_{i_m} (\mu_i - \mu_{i+1}) + \sum_{p=m}^{\ell} (P_i^2 + P_i P_{i+1} + P_{i+1}^2)(\mu_i - \mu_{i+1})$$

In the particular case of $m = \ell + 1$, $P_{\ell+1} = \tau^* K_F$ (see notations in Chapter 5) we obtain

$$K_{T/B}^3 \geq \sum_{i=1}^{\ell} (P_i + P_{i+1})(\tau^* K_F)(\mu_i - \mu_{i+1}).$$

Finally, taking $\{i_1, ..., i_q\} = \{1, \ell\}$ we have

$$K^3_{T/B} \geq (P_1 + P_\ell)P_\ell(\mu_1 - \mu_\ell) + (P^2_\ell + P_\ell P_{\ell+1} + P^2_{\ell+1})\mu_\ell \geq P^2_\ell(\mu_1 + 2\mu_\ell).$$

1.3 The relative hyperquadrics method

The method of counting relative hyperquadrics, originated in [79] and [19] was successfully applied by Konno in [63] to study the slope of fibred surfaces with small fibre genus. Here we construct the fundamental sequence and prove the conclusions which are needed in the next chapters.

Proposition 1.13 *Let T be a normal, Q-factorial, projective variety of dimension n with only canonical singularities and let $f : T \longrightarrow B$ be a relatively minimal fibration onto a smooth curve of genus b. Let D be a Weil divisor on T and let $m \geq 2$. Let $\mathcal{E} = f_*\mathcal{O}_T(D)$ and $\varphi : \mathbb{P} = \mathbb{P}_B(\mathcal{E}) \longrightarrow B$. Let Y be the adherence of the image of T by the natural map over B $\psi : T - - - \rightarrow \mathbb{P}$. Then we have a natural exact sequence*

$$0 \longrightarrow \varphi_*\mathcal{J}_{Y,\mathbb{P}}(m) \longrightarrow S^m f_*\mathcal{O}_T(D) \longrightarrow f_*\mathcal{O}_T(D)^{[m]}$$

(the generalized Max-Noether sequence associated to f).

Proof: From the exact sequence

$$0 \longrightarrow \mathcal{J}_{Y,\mathbb{P}}(m) \longrightarrow \mathcal{O}_{\mathbb{P}}(m) \longrightarrow i_*\mathcal{O}_Y \otimes \mathcal{O}_{\mathbb{P}}(m) \longrightarrow 0$$

we have

$$0 \longrightarrow \varphi_*\mathcal{J}_{Y,\mathbb{P}}(m) \longrightarrow S^m f_*\mathcal{O}_T(D) \longrightarrow \varphi_*(i_*\mathcal{O}_Y \otimes \mathcal{O}_{\mathbb{P}}(m))$$

where $i : Y \hookrightarrow \mathbb{P}$ is the natural inclusion.

In order to finish the proof it is enough to prove that there is an inclusion

$$0 \rightarrow \varphi_*(i_*\mathcal{O}_Y \otimes \mathcal{O}_{\mathbb{P}}(m)) \longrightarrow f_*\mathcal{O}_T(D)^{[m]}$$

Let $\mu : \widetilde{T} \longrightarrow T$ be a desingularization of T such that the map $\gamma = \mu \circ \psi : \widetilde{T} \longrightarrow T$ is a morphism (see Lemma 1.1). Decompose $\gamma = i \circ \overline{\gamma}$ with $\overline{\gamma} : \widetilde{T} \longrightarrow Y$. Then we have

$$i_*\mathcal{O}_Y \otimes \mathcal{O}_{\mathbb{P}}(m) = i_*i^*\mathcal{O}_{\mathbb{P}}(m) \hookrightarrow i_*(\overline{\gamma}_*\mathcal{O}_{\widetilde{T}} \otimes i^*\mathcal{O}_{\mathbb{P}}(m)) = i_*\overline{\gamma}_*\overline{\gamma}^*i^*\mathcal{O}_{\mathbb{P}}(m) = \gamma_*\gamma^*\mathcal{O}_{\mathbb{P}}(m)$$

and hence

$$0 \to \varphi_*(i_*\mathcal{O}_Y \otimes \mathcal{O}_{\mathbb{P}}(m)) \hookrightarrow \varphi_*\gamma_*\gamma^*\mathcal{O}_{\mathbb{P}}(m) = f_*\mu_*\gamma^*\mathcal{O}_{\mathbb{P}}(m)$$

So it suffices to show an inclusion

$$\mu_*(\gamma^*\mathcal{O}_{\mathbb{P}}(m)) \hookrightarrow \mathcal{O}_T(mD)$$

If U is the open set of smooth points of T (hence $T \setminus U$ has codimension 2 since T has canonical singularities) and $V = \mu^{-1}(U)$, then it is clear from Lemma 1.1 that $\gamma^*\mathcal{O}_{\mathbb{P}}(m)$ and $\mu^*\mathcal{O}_T(mD)$ differ in an effective Cartier divisor on V. Hence we have an inclusion

$$\gamma^*\mathcal{O}_{\mathbb{P}}(m)_{|V} \hookrightarrow \mu^*\mathcal{O}_T(mD)_{|V}$$

which gives

$$\mu_*(\gamma^*\mathcal{O}_{\mathbb{P}}(m))_{|U} \hookrightarrow \mathcal{O}_T(mD)_{|U}$$

since $\mathcal{O}_T(mD)_{|U}$ is locally free and hence projection formula holds. If we set $j : U \hookrightarrow T$ for the natural inclusion then

$$\mu^*(\gamma^*\mathcal{O}_{\mathbb{P}}(m)) \hookrightarrow j_*(\mu_*(\gamma^*\mathcal{O}_{\mathbb{P}}(m))_{|U}) \hookrightarrow j_*(\mathcal{O}_T(mD)_{|U}) = \mathcal{O}_T(mD)$$

where the first natural map is injective since $\mu_*(\gamma^*\mathcal{O}_{\mathbb{P}}(m)$ is torsion free ($\gamma^*\mathcal{O}_{\mathbb{P}}(m)$ is locally free) and the last equality holds since $\mathcal{O}_T(mD)$ is reflexive and $T \setminus U$ has codimension at least two.

In fact the map we have obtained

$$\delta : S^m f_*\mathcal{O}_T(D) \longrightarrow f_*\mathcal{O}_T(D)^{[m]}$$

comes from the natural map $f^*f_*\mathcal{O}_T(D) \longrightarrow \mathcal{O}_T(D)$ which induces

$$S^m f^* f_*\mathcal{O}_T(D) = f^*(S^m f_*\mathcal{O}_T(D)) \longrightarrow (\mathcal{O}_T(D)^{\otimes m})^{**} = \mathcal{O}_T(D)^{[m]}$$

and hence δ by taking f_* and projection formula. Our approach allows us to identify Kerδ as $\varphi_*\mathcal{J}_{Y,\mathbb{P}}(mL_{\mathcal{E}})$. ■

From now on, let T be a $\mathbb{Q}$-Gorenstein threefold and $D = K_{T/B}$ the relative canonical divisor. The general construction associated to $f^*f_*\mathcal{O}_T(K_{T/B}) \longrightarrow$

$\mathcal{O}_T(K_{T/B})$ given in §1.1.2 produces here the *relative canonical map* of f. We also assume $p_g(F) \geq 2$ in order to get a nontrivial relative canonical map.

Corollary 1.14 *Under the above hypotheses, if F is a general fibre of f we have*

$$K^3_{T/B} \geq (2p_g(F) - 4)(\chi\mathcal{O}_B\chi\mathcal{O}_F - \chi\mathcal{O}_T) - 2\text{deg}K - 2\ell(2) \tag{1.1}$$

where $K = \varphi_*\mathcal{J}_{Y,\mathbb{P}}(2L_{\mathcal{E}})$ and $\ell(2)$ is the Reid-Fletcher second order correction to the plurigenera of T (cf. [28]).

Proof: Let $D = K_{T/B}$, $m = 2$ and take degrees in the generalized Max-Noether sequence. Use

$$d = \text{deg}f_*\omega_{T/B} \geq (\chi\mathcal{O}_B\chi\mathcal{O}_F - \chi\mathcal{O}_T) \qquad \text{([73] p. 656)}$$

$$\text{deg}f_*\omega^{[2]}_{T/B} = \frac{1}{2}K^3_{T/B} + 3(\chi\mathcal{O}_B\chi\mathcal{O}_F - \chi\mathcal{O}_T) + \ell(2) \qquad \text{([73] Lemma 2.8)}$$

On the other hand, if $\mathcal{F}$ is a rank r, degree d locally free sheaf on B, standard formulas give $\text{rk}S^2\mathcal{F} = \binom{r+1}{2}$, $\text{deg}S^2\mathcal{F} = (r+1)d$ and so

$$\text{deg}S^2 f_*\omega_{T/B} = (p_g(F) + 1)d$$

$$\text{rk}S^2 f_*\omega_{T/B} = \binom{p_g(F) + 1}{2}.$$

Finally note that, if $\mathcal{C} = \text{Coker}(S^2 f_*\mathcal{O}_T(K_{T/B}) \longrightarrow f_*\mathcal{O}_T(2K_{T/B}))$, $\text{deg}\mathcal{C} \geq 0$ since $f_*\mathcal{O}_T(2K_{T/B})$ is nef (cf. [53] and Chapter 3). ■

Remark 1.15 For small values of the invariants $p_g(F)$, $q(F)$, K^2_F it could be interesting to consider $D = sK_{T/B}$ for $s \geq 1$. We obtain then bounds for $K^3_{T/B}$ which are better than (1.1).

In general, $\text{deg}K$ is difficult to be computed or bounded. Note that the study of K is the study of the relative hyperquadrics that contain the relative canonical image of T, hence the name of the method.

First of all we need to bound the rank. Note that $\text{rk}K = h^0(\mathcal{J}_{\Sigma,\mathbb{P}^r}(2))$, where Σ is the canonical image of F and $r = p_g(F) - 1$.

Lemma 1.16 ([77], p. 195; see Lemma 2.2 for a proof) *Let $W \subseteq \mathbb{P}^N$ be an irreducible variety spanning $\mathbb{P}^N$ of dimension w. Then*

$$h^0\mathcal{J}_{W,\mathbb{P}^N}(2) \leq \binom{N-w+2}{2} - \min\{\deg W, 2(N-w)+1\}$$

Then we get

Corollary 1.17 *With the assumptions of Corollary 1.14,*

$$h^0(\mathcal{J}_{\Sigma,\mathbb{P}^r}(2)) \leq \tfrac{(r-2)(r-3)}{2} \qquad \text{if } \Sigma \text{ is a non ruled surface}$$

$$h^0(\mathcal{J}_{\Sigma,\mathbb{P}^r}(2)) \leq \tfrac{(r-1)(r-2)}{2} - q(\Sigma) \qquad \text{if } \Sigma \text{ is a ruled surface}$$

$$h^0(\mathcal{J}_{\Sigma,\mathbb{P}^r}(2)) \leq \tfrac{r(r-1)}{2} \qquad \text{if } \Sigma \text{ is a curve}$$

PROOF: For this just use the well known fact (cf. [10]) that if $\Sigma \subseteq \mathbb{P}^{r-1}$ is a non degenerate curve, then $\deg\Sigma \geq r-1$ and if it is a nondegenerate surface, then $\deg\Sigma \geq r-2+q(\Sigma)$ if Σ is ruled, $\deg\Sigma \geq 2r-4$ otherwise. Apply then Lemma 1.16 ∎

As for the degree of K we have

Proposition 1.18 *(i) If $p_g(F) \geq 2$ and $\mathcal{E} = f_*\omega_{T/B}$ is semistable then*

$$\deg K \leq 2\frac{\mathrm{rk}K}{p_g(F)}d.$$

(ii) If $K = \mathcal{L}_1 \oplus \ldots \oplus \mathcal{L}_s$ (s=rk$K$) then

$$\deg K \leq (\mathrm{rk}K)\frac{2}{3}d$$

(in particular this happens if $s \leq 1$ or $b = 0$).

PROOF: (i) If $\mathcal{E}$ is semistable then so it is $S^2\mathcal{E}$. Then we use the natural inclusion $K \hookrightarrow S^2\mathcal{E}$ and use the formulas in the proof of Corollary 1.14.

(ii) If $x_i = \deg\mathcal{L}_i$ then there exists a section $s \in H^0(K \otimes \mathcal{L}_i^{-1}) \cong H^0(\mathcal{J}_{Y,\mathbb{P}}(2L_{\mathcal{E}}) \otimes \mathcal{O}_{\mathbb{P}}(\varphi^*(\mathcal{L}_i^{-1})) \hookrightarrow H^0(\mathbb{P}, \mathcal{O}(2L_{\mathcal{E}}) \otimes \varphi^*(\mathcal{L}_i^{-1}))$ so there exists a relative hyperquadric $Q_i \equiv 2L_{\mathcal{E}} - x_i\varphi^{-1}(t)$. The result follows then from the following lemma which is a slight refinement of [63] Remark 1.7, and the fact that for every i, $\mathrm{rk}Q_i \geq 3$. ■

Definition 1.19 Let $Q \equiv 2L_{\mathcal{E}} - x\varphi^{-1}(t)$. We say that Q has rank p if $Q_{|\varphi^{-1}(t)}$ is a quadric in $\mathbb{P}^{\mathrm{rk}(\mathcal{E})-1}$ of rank p for $t \in B$ general.

Lemma 1.20 *Let $Q \equiv 2L_{\mathcal{E}} - x\varphi^{-1}(t)$ be a relative hyperquadric of rank p. Let $\nu_1 \geq \nu_2 \geq \ldots \geq \nu_k$ the virtual slopes of $\mathcal{E} = f_*\omega_{T/B}$. Then*

(i) If $p > r_{i-1}$ then $x \leq \mu_1 + \mu_i$ (in particular, if Q is smooth, $x \leq \mu_1 + \mu_\ell$).

(ii) $x \leq \min_{0\leq j\leq p-1}\{\nu_{j+1} + \nu_{p-j}\} \leq \frac{2}{p}\deg\mathcal{E}$

PROOF: (i) Cf. [63] Lemma 1.4.

(ii) Let μ_α such that $\nu_p = \mu_\alpha$ (i.e., $r_{\alpha-1} < p \leq r_\alpha$). Then by (i) $x \leq \mu_1 + \mu_\alpha = \nu_1 + \nu_p$ which is the case $j = 0$.

Take now an index $j \in \{1, \ldots, \left[\frac{p-1}{2}\right]\}$. Assume that there exists a piece $\mathcal{E}_i$ of the Harder-Narasimhan filtration of $\mathcal{E}$ such that $j = r_i$ (then $r_i + 1 \leq p - r_i$).

Consider now the Harder-Narasimhan filtration of $\mathcal{E}/\mathcal{E}_i$:

$$0 \subseteq \mathcal{E}_{i+1}/\mathcal{E}_i \subseteq \ldots \subseteq \mathcal{E}_\ell/\mathcal{E}_i$$

with slopes $\overline{\mu}_t = \mu_{i+t}$ $(t = 1, \ldots, \ell - i)$, $\overline{\nu}_s = \nu_{s+r_i}$ $(s = 1, \ldots, g - r_i)$. Consider now

$$\mathbb{P}_B(\mathcal{E})[d]_\varphi \mathbb{P}_B(\mathcal{E}/\mathcal{E}_i) =: B_i[l][dl]^{\varphi_i} B$$

and let $Q_i =: Q_{|B_i} \equiv 2L_{\mathcal{E}/\mathcal{E}_i} - x\varphi^{-1}(t)$. We have

$$p = \mathrm{rk}Q_{|\varphi^{-1}(t)} \geq \mathrm{rk}Q_{|\varphi_i^{-1}(t)} \geq \mathrm{rk}Q_{|\varphi^{-1}(t)} - 2r_i = p - 2r_i$$

by [2] p. 143. Then the $j = 0$ argument for Q_i reads

$$x \leq \overline{\nu}_1 + \overline{\nu}_{p-2r_i} = \nu_{r_i+1} + \nu_{p-r_i} = \nu_{j+1} + \nu_{p-j}$$

Assume finally that $j \neq r_i$ for every piece $\mathcal{E}_i$ of the Harder-Narasimhan filtration of $\mathcal{E}$. Let α be such that $r_{\alpha-1} < j \leq r_\alpha$ (i.e. $\nu_j = \mu_\alpha$). Since by hypothesis $j \neq r_\alpha$ we have $\nu_{j+1} = \nu_j = \nu_{r_\alpha} = \mu_\alpha$.

Then $\nu_{j+1}+\nu_{p-j}=\nu_j+\nu_{p-j}\geq \nu_{r_{\alpha-1}+1}+\nu_{p-r_{\alpha-1}}\geq x$ applying the previous step, since $r_{\alpha-1}+1\leq j$, $p-r_{\alpha-1}>p-j$ and the sequence of the ν_i is decreasing.

We have then $x\leq \min_{0\leq j\leq p-1}\{\nu_{j+1}+\nu_{p-j}\}$. Finally observe that

$$p\cdot \min_{0\leq j\leq p-1}\{\nu_{j+1}+\nu_{p-j}\}\leq \sum_{j=0}^{p-1}(\nu_{j+1}+\nu_{p-j})=2\sum_{s=1}^{p}\nu_s\leq 2\sum_{s=1}^{\mathrm{rk}\mathcal{E}}\nu_s=\deg\mathcal{E}$$

■

We can conclude

Corollary 1.21 *With the same notations as above, assume $p_g(F)\geq 2$.*

(i) If $\mathcal{E}=f_*\omega_{T/B}$ is semistable then

$$K^3_{T/B}\geq\left(10-\tfrac{24}{p_g(F)}\right)(\chi\mathcal{O}_B\chi\mathcal{O}_F-\chi\mathcal{O}_T)-2\ell(2)\qquad \text{if } \Sigma \text{ is a non-ruled surface}$$
$$K^3_{T/B}\geq\left(6-\tfrac{12}{p_g(F)}\right)(\chi\mathcal{O}_B\chi\mathcal{O}_F-\chi\mathcal{O}_T)-2\ell(2)\qquad \text{if } \Sigma \text{ is a ruled surface}$$
$$K^3_{T/B}\geq\left(2-\tfrac{4}{p_g(F)}\right)(\chi\mathcal{O}_B\chi\mathcal{O}_F-\chi\mathcal{O}_T)-2\ell(2)\qquad \text{if } \Sigma \text{ is a curve}$$

(ii) If $h^0(\mathcal{J}_{\Sigma,\mathbb{P}^r}(2))=0$ then

$$K^3_{T/B}\geq(2p_g(F)-4)(\chi\mathcal{O}_F\chi\mathcal{O}_B-\chi\mathcal{O}_T)-2\ell(2)$$

(iii) If $h^0(\mathcal{J}_{\Sigma,\mathbb{P}^r}(2))=1$ then

$$K^3_{T/B}\geq(2p_g(F)-\frac{16}{3})(\chi\mathcal{O}_F\chi\mathcal{O}_B-\chi\mathcal{O}_T)-2\ell(2)$$

Proof: Use Corollary 1.17 and Proposition 1.18 in Corollary 1.14. ■

1.4 Extension of maps

Let S be a surface and let $\pi: S\longrightarrow B$ be a fibration of curves of genus $g\geq 2$ and assume that for non countably many $t\in B$, the fibre F_t is endowed with a non-constant morphism $\varphi_t: F_t\longrightarrow D_t$ into a smooth curve. The goal of this paragraph is to show that the existence of these maps φ_t implies the existence of another fibration $T \longrightarrow B$ and of a rational map over B from S to T reflecting the properties of many φ_t. In fact we recover the original maps φ_t only for non countably many values of t (even in the case that one applies this under the hypothesis of existence of φ_t for a general t, one can not get better results as simple examples show).

To obtain the surface T we shall need base change in general. However under some hypotheses of unicity this base change can be avoided.

We consider three cases: first we assume that the maps φ_t are automorphisms; secondly we suppose that $1 \leq g(D_t) < g$, and finally we study linear series. We also obtain a similar result for abelian schemes with abelian subvarieties in the general fibres (Theorem 1.28).

1.4.1 Glueing automorphisms

Let $\pi : S \longrightarrow B$ be a fibred surface and denote by F_t the fibre of π in $t \in B$. The aim of this section is to prove that the existence of automorphisms on many fibres of π induces, up to base change, the existence of a birational automorphism of S. To prove this, we shall need some standard facts on Hilbert schemes that we recall now.

We fix a relatively very ample sheaf $\mathcal{O}_X(1)$ on $X := S \times_B S \longrightarrow B$. Following Grothendieck ([37]), we can consider the scheme $\underline{\mathcal{A}ut}_{S/B}$ as an open subscheme of $\underline{\mathcal{H}ilb}_{S/B}$ representing the functor

$$\begin{array}{rrcl} Aut: & B-\text{schemes} & \longrightarrow & \text{Groups} \\ & T & \longmapsto & \mathrm{Aut}_T(S_T). \end{array}$$

Then, giving a section σ of the natural map $\underline{\mathcal{A}ut}_{S/B} \longrightarrow B$ corresponds, via the identification

$$\mathrm{Hom}_B(B, \underline{\mathcal{A}ut}_{S/B}) = Aut_B(S),$$

to an automorphism Φ of S over B such that for $t \in B$, $\Phi_{|F_t} = \sigma(t) \in \mathrm{Aut}(F_t)$. We recall that $\underline{\mathcal{A}ut}_{S/B}$ is a group-scheme over B and, as in the case of Hilbert schemes, decomposes as a disjoint union of schemes $\underline{\mathcal{A}ut}_{S/B}$ obtained by fixing the Hilbert polynomial $p(t) \in \mathbb{Q}[t]$.

Choosing a suitable Hilbert polynomial and considering only elements of m-torsion one constructs a B-scheme $\underline{\mathcal{A}ut}^{m,r}_{S/B}$ parametrizing the automorphisms of the fibres of π of order m and with r fixed points.

Definition 1.22 Let F be a smooth curve of genus $g \geq 2$ and let φ be an automorphism of F. Assume $\varphi \neq \mathrm{Id}$. Consider the curve $G := F/ < \varphi >$ and the map $f : F \longrightarrow G$. If n is the order of φ we call the *type of* φ to the following data:

$$\Lambda = \{(k, a_k)\}_{k|n,\, k\neq 1}$$

where a_k is the number of ramification points of f of index k. Equivalently one can give $\Lambda' = \{(k, b_k)\}_{k|n,\, k\neq 1}$ where b_k is the number of fixed points of $\varphi^{\frac{n}{k}}$; notice that

$$b_k = \sum_{k|k'|n} a_{k'}.$$

Observe that one can consider the B-scheme of automorphisms of type Λ defining:

$$\underline{\mathcal{A}ut}^{\Lambda}_{S/B} := \bigcap_{k|n, k\neq 1} H_k^{-1}(\underline{\mathcal{A}ut}^{\frac{n}{k}, b_k}_{S/B}),$$

where $H_k : \underline{\mathcal{A}ut}_{S/B} \longrightarrow \underline{\mathcal{A}ut}_{S/B}$ sends x to x^k.

The main result of this section is the following:

Theorem 1.23 *Let $\pi : S \longrightarrow B$ be a fibred surface of genus $g \geq 2$. Assume that for non countably many $t \in B$, $Aut(F_t) \neq Id$. Then:*

(i) There exist a type Λ, a base change $B' \longrightarrow B$ and a birational automorphism Φ' of a non singular model $\widetilde{S}$ of $S_{B'}$ such that the restriction of Φ' to a general fibre of $\widetilde{S} \longrightarrow B'$ is an automorphism of type Λ.

(ii) The type Λ of (i) can be chosen previously, provided that for non countably many $t \in B$ the fibre F_t has an automorphism of type Λ.

(iii) If, furthermore, for non countably many $t \in B$, the automorphism of type Λ is unique, then base change is not needed.

PROOF: As $g \geq 2$, the fibres of $\underline{\mathcal{A}ut}_{S/B} \longrightarrow B$ are finite. Since B is a curve, there exists an irreducible component B' of $\underline{\mathcal{A}ut}_{S/B}$, dominating B. We can assume B' to be complete; otherwise the construction that follows will extend in a natural way to a compactification of B'. Since $\underline{\mathcal{A}ut}_{S/B}$ is a finite union of subschemes $\underline{\mathcal{A}ut}^{\Lambda}_{S/B}$, an infinite number of points (and hence a Zariski open set) of B' correspond to automorphisms of the same type Λ. Then

$$\underline{\mathcal{A}ut}_{S_{B'}/B'} \longrightarrow \underline{\mathcal{A}ut}_{S/B} \times_B B' \longrightarrow B',$$

has an obvious section which produces a relative automorphism Φ' of $S_{B'}$. Then Φ' determines a birational automorphism of $\widetilde{S}$. This proves (i) .

Observe that the same argument works for (ii) just fixing Λ and $\underline{\mathcal{A}ut}_{S/B}$ from the beginning. Finally, under the hypothesis of (iii) , $\underline{\mathcal{A}ut}^{\Lambda}_{S/B} \longrightarrow B$ is generically one-to-one and has a section. ■

Remark 1.24 In Example 4.2 we give an example of a bielliptic fibration of genus 5 for which the general fibre has two different bielliptic involutions and such that a non-trivial base change is needed in order to glue them into a global birational involution. So, in general, (iii) does not hold without the hypothesis of unicity.

1.4.2 Glueing morphisms of curves

In this section we consider a fibration $\pi : S \longrightarrow B$ such that for non countably many $t \in B$, the fibres F_t have a map onto a non-rational smooth curve. The aim is to produce, perhaps after base change, a rational map over B, $S - - \rightarrow T$, such that for non countably many fibres we recover the original morphisms, perhaps (if $T \longrightarrow B$ is an elliptic fibration) up to automorphisms on the image curve. The main point of the construction is to observe that a morphism from F_t onto a curve of genus ≥ 1 induces an endomorphism of $J(F_t)$, the Jacobian variety of F. Then we prove that endomorphisms on many fibres of an abelian scheme produce an endomorphism of the abelian scheme and from this the result follows quickly. As a by-product we find that the existence of non-trivial abelian subvarieties on the fibres of an abelian scheme implies the existence of a non-trivial abelian subscheme.

Let $\pi : \mathcal{A} \longrightarrow \mathcal{U}$ be an abelian scheme and let σ be the zero section. The theory of Hilbert schemes ensure the existence of a $\mathcal{U}$-scheme $\mathcal{E}nd_{\mathcal{U}}(\mathcal{A})$ parametrizing the endomorphisms of the fibres of π. By taking the kernel of the following map of group schemes over $\mathcal{U}$,

$$\mathcal{E}nd_{\mathcal{U}}(\mathcal{A}) \longrightarrow \mathcal{E}_{\mathcal{U}}(\mathcal{A}) \times \mathcal{U} \stackrel{\mathrm{Id}\times\sigma}{\longrightarrow} \mathcal{E}_{\mathcal{U}}(\mathcal{A}) \times \mathcal{A} \stackrel{\text{evaluation}}{\longrightarrow} \mathcal{A}$$

one obtains the existence of a $\mathcal{U}$-scheme $\underline{\mathcal{E}nd}_{\mathcal{U}}(\mathcal{A})$ parametrizing the endomorphisms of the fibres of π as abelian varieties. By construction one identifies the global sections of this scheme with the group of endomorphisms of $\mathcal{A}$ over $\mathcal{U}$ as abelian scheme.

To stay the main Theorem we shall need the following definition:

Definition 1.25 Let $f : C \longrightarrow D$ be a non-constant map of complete smooth curves. We say that f is *indecomposable* if it does not exist a factorization of f through a cyclic étale covering of D of degree $n \geq 2$.

Now Proposition (4.3) in [14], p. 337 reads:

Proposition 1.26 *Let $f : C \longrightarrow D$ as above. The map $f^* : JD \longrightarrow JC$ is injective if and only if f is indecomposable.*

The main result of this paragraph is the following theorem:

Theorem 1.27 *Let $\pi : S \longrightarrow B$ be a fibred surface of curves of genus g and fibres F_t, $t \in B$. Assume that for non countably many $t \in B$ there exist a non-constant map $\varphi_t : F_t \longrightarrow D_t$ on a curve of positive genus $q < g$. Then the following statements hold:*

(i) There exist a base change $B' \longrightarrow B$, an integer q, $0 < q < g$, a fibration $T' \longrightarrow B'$ of curves of genus q, and a rational map over B'

$$\Psi : S_{B'} - - - \rightarrow T'.$$

(ii) For non countably many $t \in B$ one has $\Psi_t = \varphi_t$ (up to automorphisms of D_t if $q = 1$).

(iii) If the map φ_t is unique for non countably many $t \in B$, then base change is not needed.

PROOF: After base change and taking a suitable open subset $\mathcal{U}$ of B we can assume the existence of a diagram

$$S_{|\mathcal{U}}[r]^{\varepsilon}[d]_{\pi^0}\mathcal{J}[d]\mathcal{U}[r]\mathcal{U}$$

where $\mathcal{J} \longrightarrow \mathcal{U}$ is the Jacobian fibration. The map ε is defined with the aid of a section of π^0 and fibre to fibre gives an inclusion of each F_t in its Jacobian variety.

Let us denote by $\mathcal{E}\mathcal{J}$ the $\mathcal{U}$-scheme $\underline{\mathcal{E}nd_{\mathcal{U}}}\mathcal{J}$ introduced above. It is well-known that an abelian variety has countably many endomorphisms, hence fibres of $\mathcal{E}J \longrightarrow \mathcal{U}$ are countable. Define $\rho_t : JF_t \longrightarrow JF_t$ the endomorphism $\varphi_t^* \circ \mathrm{Nm}_{\varphi_t}$, where Nm_{φ_t} is the norm map

$$\begin{aligned} \mathrm{Nm}_{\varphi_t} : JF_t &\longrightarrow JD_t \\ [\sum n_i P_i] &\longrightarrow [\sum n_i \varphi_t(P_i)] \end{aligned}$$

By hypothesis one has an irreducible component $W \subset \mathcal{E}\mathcal{J}$ dominating $\mathcal{U}$. By using again base change and the functoriality of $\mathcal{E}\mathcal{J}$, we can assume the existence of a section of $\mathcal{E}\mathcal{J} \longrightarrow \mathcal{U}$, providing an endomorphism λ of the $\mathcal{U}$-scheme $\mathcal{J}$.

Define T' to be a desingularization of the closure of the image of

$$S_{|U} \xrightarrow{\varepsilon} \mathcal{J} \xrightarrow{\lambda} \mathcal{J}$$

in some compactification of $\mathcal{J}$. By construction, one has the rational map we were looking for and part (i) is proved.

Observe that for non countably many $t \in B$, we recover the map $F_t \longrightarrow \varphi_t^*(D_t)$. If φ_t is an indecomposable map for non countably many t, then (ii) is clear from above. Otherwise, we can write for a non countably many $t \in B'$ $\varphi_t = \alpha_t \circ \beta_t$, where β_t is indecomposable and $\alpha_t : D'_t \longrightarrow D_t$ is an étale cyclic covering of degree $n_t \geq 2$. For non countably many t the degree n_t is constant and hence we can assume constant the genus of the curves D'_t. Notice that the morphisms α_t are determined by automorphisms on the curves D'_t. Therefore we can apply the indecomposable case proved above to glue the maps β_t and then the part (ii) of the Theorem 1.23 finishes the proof of (ii) (perhaps after a new base change).

Assume now that φ_t is unique for non countably many $t \in B$. In particular the curves D_t have not automorphisms (this forces $q \geq 2$) and the automorphism of F_t permute the fibres of φ_t.

As in (ii) we suppose first that the φ_t is indecomposable. The results of (i) and (ii) give a base change $B' \longrightarrow B$, a fibration T' of curves of genus q over B' and a rational map from $S_{B'}$ to T'. The first step is to observe that there exists a fibration $T \longrightarrow B$ such that T' is obtained from T by base change (at least on an open set of B'). Indeed, we consider the image B_0 of B' in the moduli space of curves of genus q. If $\dim B_0 = 0$, then the fibration $T' \longrightarrow B'$ is isotrivial and (doing again base change) we can assume that T' is the product $B' \times D$. In this case one simply defines T to be $B \times D$.

If $\dim B_0 = 1$, then one can construct a universal family of curves over an open set of B_0 (recall that D_t has not automorphisms). Fix a point $t \in B$ such that φ_t is unique and denote by $t_1, \ldots, t_r$ the preimages in B'. Since the curves F_{t_i} are isomorphic and by the unicity, one obtains that the fibres of $T' \longrightarrow B'$ at $t_1, \ldots, t_r$ are isomorphic. From this one easily proves that the modular morphism from B' to B_0 factorizes through the morphism $B' \longrightarrow B$. The pull-back of the universal family over B_0 to B allows to construct a surface T with a fibration over an open set of B. One can assume as usual that T is fibred over B.

Now, by the existence of such a fibration $T \longrightarrow B$ and the hypothesis of unicity, one checks that the graph of the rational map from S' to T' descents to a graph of a rational map from S to T.

As in part (ii) we divide the proof of the general case into two parts. One glue first the indecomposable maps and after one uses the part (iii) of Theorem 1.23. Observe that the curves D'_t (with the notations of (ii)) have a unique automorphism of this type due to the unicity of φ_t. ∎

Consider now a polarized abelian variety (A, L) of dimension a and let B be an abelian subvariety of dimension b. Let $\hat{A}$ be the dual abelian variety of A (i.e. the Picard variety of A) and call $\lambda : A \longrightarrow \hat{A}$ to the isogeny induced by the polarization. Consider the map

$$\alpha : A[r]^{\lambda} \hat{A}[r]^{j} \hat{B}$$

where j is the dual of the inclusion $B \subset A$. It is easy to see that the variety $P := \mathrm{Ker}\,(\alpha)^0 \subset A$ is an abelian subvariety of dimension $a - b$ and such that $I := B \cap P$ is finite. In other words, the addition map $s : B \times P \longrightarrow A$ is an isogeny. Let $r \in \mathbb{N}$ such that $rI = 0$. Then for any pair of integers (m, n) such that $r|(m - n)$ the endomorphism

$$\begin{array}{rl} B \times P & \longrightarrow B \times P \\ (x, y) & \mapsto (mx, ny) \end{array}$$

produces an endomorphism $\psi_{m,n}$ of A. Observe that $\psi_{r,0}(A) = B$ and $\psi_{0,r}(A) = P$.

The same arguments used above allow to prove the following theorem:

Theorem 1.28 *Let $\mathcal{A} \longrightarrow \mathcal{U}$ be an abelian scheme such that $\dim\mathcal{U}$=1. Assume that for non countably many $t \in \mathcal{U}$ the abelian variety A_t has a non-trivial* *abelian subvariety B_t of dimension b_t. Then there exist a base change $\mathcal{U}' \longrightarrow \mathcal{U}$ and a constant b such that $\mathcal{A}_{\mathcal{U}'}$ has an abelian subscheme $\mathcal{B} \longrightarrow \mathcal{U}'$ of relative dimension a and $(\mathcal{B})_t = B_t$ for non countably many $t \in \mathcal{U}'$.*

PROOF: We fix a relative polarization on $\mathcal{A}$. As above, the existence of an abelian subvariety in A_t induces the existence of an endomorphism $\psi_{r_t,0}$ of A_t. Arguing as in Theorem 1.27 we glue, up to base change, these endomorphisms to obtain an endomorphism of $\mathcal{A}$ over $\mathcal{U}$. The image of this endomorphism gives the abelian subscheme. ∎

Remark 1.29 One easily checks that base change is not needed if for non countably many $t \in \mathcal{U}$ there is a unique abelian subvariety of a given dimension b. If there is more than one subvariety this is not true: consider the fibration of bielliptic curves with two bielliptic maps constructed in Example 4.2. The corresponding Jacobian fibration (on an open set of the base) gives a counterexample.

1.4.3 Glueing linear series

Let $\pi : S \longrightarrow B$ be a fibration such that for non countably many t, the fibre F_t is d-gonal (i.e. F_t possesses a base point free g^1_d). As in previous sections we want to extend, after base change, the corresponding morphisms $F_t \longrightarrow \mathbb{P}^1$. More precisely, we want to prove:

Theorem 1.30 *Let $\pi : S \longrightarrow B$ be a fibred surface such that for non countably many $t \in B$, F_t is d-gonal; then*

(i) there exist a base change $B' \longrightarrow B$, a ruled surface R' over B' and a rational map $\Phi : S' - - - \to R'$ over B' such that $deg\,\Phi = d$.

(ii) If for non countably many $t \in B$, F_t has a unique g^1_d (hence complete), then base change is not needed.

Remark 1.31 This theorem is classical when $d = 2$ (cf., e.g., [10]). Recall that, for hyperelliptic curves, all base point free g^1_d with $d \leq 2g - 2$ are obtained by composing the hyperelliptic map with a Segre embedding of degree $d/2$ and then projecting. From this one obtains immediately the theorem for hyperelliptic curves and $d \leq 2g - 2$.

PROOF: We consider the cases $d \geq g+1$, $d=g$ and $d \leq g$ separately.

Assume $d \geq g+1$. After a base change we obtain a d-section D of π. By Riemann-Roch $h^0(F_t, D_t) \geq 2$ for all smooth F_t. We can choose a rank 2 vector bundle $\mathcal{F} \subset \pi_* \mathcal{O}_S(D) \otimes A$ generically generated by two global sections, where $A \in Pic(B)$ is of degree big enough. The natural map $\pi^*(\mathcal{F}) \longrightarrow \mathcal{O}_S(D) \otimes \pi^*(A)$ induces a rational map $\Phi : S - - - \rightarrow R = \mathbb{P}(\mathcal{E})$ over B such that $\Phi^* \mathcal{O}_{\mathbb{P}}(1)$ is the image of $\pi^* \mathcal{E} \longrightarrow \mathcal{O}_S(D) \otimes \pi^*(A)$. Fixing previously that D_{t_0} is base point free for some $t_0 \in B$ (it is possible by hypothesis) and that the two sections generating $\mathcal{E}$ have no base point at t_0, we can conclude that such image has no horizontal base component (see Lemma 1.3). Hence Φ has degree d and we are done.

Assume $d=g$. Note that the same proof of the case above works if we have a d-section D such that for $t \in B$ general $h^0(F_t, D_t) \geq 2$ and for some $t_0 \in B$, D_{t_0} is a base point free g^1_d. Take D' a $(g-2)$-section of π (after base change if necessary) such that $K_{F_{t_0}} - D'_{t_0}$ is base point free. Then, if $A \in \operatorname{Pic} B$ is ample enough, take a global section $D \in |K_S - D' + \pi^*(A)|$. For $t \in B$ general we have $h^0(F_t, D_t) = h^0(F_t, D'_t) + 1 \geq 2$ by Riemann-Roch.

Assume now that $d \leq g-1$. By Remark 1.31 we can assume that the generic fibre is not hyperelliptic.

We consider the Brill-Noether loci $W^r_d(F)$ of a fixed smooth d-gonal fibre F. Since $W^1_d(F)$ is not contained in $W^2_d(F)$ (cf. [1],p.182) we can assume that the linear series g^1_d is complete. Given a complete $L \in W^1_d(F)$, the projectivized tangent cone $W_{F,L}$ of $W^0_d(F)$ at L is a minimal degree variety in $\mathbb{P}^{g-1} = \mathbb{P}(H^0(F, K_F)^*)$ of dimension $d-1$ ruled by $(d-2)$-planes generated by the images of the divisors of $|L|$ by the canonical embedding ([2], p. 241). The singular locus of $W_{F,L}$ is the linear variety $H_{F,L} = \mathbb{P}(T_L W^1_d(F))$ intersection of such $(d-2)$-planes. Let us denote by $e(L)-1$ the dimension of $H_{F,L}$ and call $\widetilde{W}_{F,L} = Bl_{H_{F,L}} W_{F,L}$. Observe that $\widetilde{W}_{F,L}$ is a smooth rational scroll ruled by $(d-2)$-planes and with an endowed map $\beta_{F,L} : \widetilde{W}_{F,L} \longrightarrow \mathbb{P}^1$. If L is base point free we have $F \hookrightarrow \widetilde{W}_{F,L}$ and the composition of this inclusion with the map $\beta_{F,L}$ determines L.

Recall that, being $W_{F,L}$ a scroll of dimension $(d-1)$, $W_{F,L}$ has more than one system of $(d-2)$-planes if and only if $W_{F,L}$ is a rank 4 quadric (see [6], pp 49,51) and in this case it has two systems. We have then that $d = g-1$ and $W_{F,L} = W_{F,K_F-L}$, $L \neq K_F - L$. Finally we note that every $(d-2)-$plane contained in $W_{F,L}$ must be a fibre of one of the rulings.

In order to consider the above constructions relatively, we apply base change. Then we obtain the existence of enough sections and this fills-up the hypotheses in [88]. Hence, there exists a variety $W^r_d(\pi)$ over B, such that, for smooth F_t, $W^r_d(F_t) \cong W^r_d(\pi)_t$. By hypothesis, $\alpha : W^1_d(\pi) \longrightarrow B$ is dominant.

Since the construction of $W^1_d(\pi)$ is functorial, base change guarantees the existence of a section η of α such that $\eta(t)$ is, for t general, a complete base point free g^1_d over F_t. Let us call D to the image of η and W_D to the projectivized tangent cone of $W^0_d(\pi)$ at D. Note that W_D contains the relative canonical image of $S \dashrightarrow \mathbb{P}_B(\pi_*\omega_{S/B})$ (see [2]).

We can consider that, after some blow-ups, $\mu : S \longrightarrow W_D$ is a morphism (generically of degree one onto the image). Let

$$\begin{aligned}
&\mathbb{G} = \mathrm{Grass}^{d-2}_B(\mathbb{P}(\pi_*\omega_{S/B})),\\
&B^0 = \{t \in B \mid \pi \text{ is smooth at } t\},\\
&\varphi : W_D \longrightarrow B\\
&\mathcal{U} = \{(p,[R]) \in W_D \times_B \mathbb{G} \mid p \in R,\ R \subset W_D,\ \varphi(p) \in B^0\},\\
&\alpha_1 : \mathcal{U} \longrightarrow W_D,\ \alpha_2 : \mathcal{U} \longrightarrow \mathbb{G} \text{ the natural projections},\\
&M = \alpha_2(U) \text{ and}\\
&\widetilde{S} = S \times_W \mathcal{U}.
\end{aligned}$$

Note that α_1 is birational if $d < g-1$ and generically of degree ≤ 2 if $d = g-1$. Moreover, if $d < g-1$ and $t \in B^0$ then $\alpha_1^{-1}(W_t) \cong \widetilde{W}_{F_t,\eta(t)}$. The map $\alpha_2 : \mathcal{U} \longrightarrow M$ is generically a $\mathbb{P}^{d-2}$-bundle.

Let N be an irreducible horizontal component of a relative multihyperplane section of W_D over B of dimension two. Note that N is a ruled surface, and meets every $(d-2)$-plane of a general fibre of W_D exactly at one point. Then, if $\widetilde{N}$ is the pull-back of N in $\mathcal{U}$, we have that $\alpha_{2|\widetilde{N}} : \widetilde{N} \longrightarrow M$ is birational. Then M is a (possibly singular) ruled surface over B if $\deg \alpha_1 = 1$ or is a ruled surface over a double cover of B if $\deg \alpha_1 = 2$.

More precisely, take $\overline{M}$ a horizontal irreducible component of a desingularization of M and let $\overline{M} \longrightarrow \overline{B} \longrightarrow B$ the Stein factorization of $\overline{M} \longrightarrow B$. If we pull-back $\widetilde{S} \longrightarrow \mathcal{U} \longrightarrow \mathcal{M}$ to $\overline{S} \longrightarrow \overline{\mathcal{U}} \longrightarrow \overline{M}$ we have a rational map $\psi : \overline{S} \dashrightarrow \overline{M}$ over a ruled surface over $\overline{B}$. If $\deg \alpha_1 = 1$ then $\overline{B} = B$ and for $t \in B^0$, the map ψ_{F_t} corresponds to $\eta(t)$. If the map $\overline{B} \longrightarrow B$ has degree 2 and $\bar{t}_1$ and $\bar{t}_2$ are the preimages of $t \in B^0$, then $\psi_{\overline{F}_{\bar{t}_i}}$ corresponds to $\eta(t)$ or

$K_{F_t} - \eta(t)$. Finally note that, by construction, $\overline{S}$ is birational to $S \times_B \overline{B}$.

In order to prove (ii) we only have to prove that the existence of $\widetilde{W}_{B'}$ over a base extension B' of π implies the existence of $\widetilde{W}$ over B when the base point free linear series is unique.

Let δ be a base change

$$\widetilde{S}[r]^{\gamma'}[dr]_{\widetilde{\pi}}S'[r]^{\gamma}[d]_{\pi'}S[d]_{\pi}B'[r]^{\delta}B$$

where $\widetilde{S}$ is a minimal desingularization of S'. Denote by $\widetilde{\pi}$ to $\pi'\gamma'$. Then, from [71] 4.10 we get an inclusion

$$0 \longrightarrow \widetilde{\pi}_*(\omega_{\widetilde{S}}) \longrightarrow \delta^*(\pi_*\omega_S).$$

Since both are locally free sheaves of the same rank we get a birational map given by a sequence of elementary transformations on suitable fibres

$$\mathbb{P}_{B'}(\widetilde{\pi}_*\omega_{\widetilde{S}}) \; - - - \rightarrow \mathbb{P}_{B'}(\delta^*\pi_*\omega_S),$$

which produces

$$\rho : \mathbb{P}_{B'}(\widetilde{\pi}_*\omega_{\widetilde{S}/B'}) \cong \mathbb{P}_{B'}(\widetilde{\pi}_*\omega_{\widetilde{S}}) - - \rightarrow \mathbb{P}_{B'}(\delta^*\pi_*\omega_S) \longrightarrow \mathbb{P}_B(\pi_*\omega_S) \cong \mathbb{P}_B(\pi_*\omega_{S/B}).$$

This map ρ is linear on fibres and restricts to the natural map from the relative canonical image of $\widetilde{\pi} : \widetilde{S} \longrightarrow B'$ onto the relative canonical image of $\pi : S \longrightarrow B$. Fix a general $t \in B$ and consider $\delta^{-1}(t) = \{t_1, \ldots, t_k\}$. Then $\rho(\widetilde{W}_{t_i})$ is a variety of minimal degree containing the canonical image of F_t, ruled by $(d-2)$-planes producing the linear series on F_t. By the unicity we have that all the images $\rho(\widetilde{W}_{t_i})$ agree. Hence $(\rho(\widetilde{W}_{B'}))_t = \widetilde{W}_{F_t}$ for general $t \in B$. Then $\rho(\widetilde{W}_{B'})$ is the variety $\widetilde{W}$ we were seeking. Note that, if the g^1_d is unique we are not in the case where $W_{F,L}$ is a rank 4 quadric and then no new base change is needed in the proof of (i) . ■

Remark 1.32 Observe that the hypothesis of having a unique g^1_d is general for small values of d. Indeed, according to [1], Theorem 2.6 a general d-gonal curve with $2 \leq d < \frac{g}{2} + 1$ has a unique g^1_d.

1.5 Generic vanishing

We state two fundamental results due to Green-Lazarsfeld and Simpson on the generic vanishing of cohomology (see also [13]).

Let X be a smooth complex projective variety of dimension n. We set $alb : X \longrightarrow Alb(X)$ the Albanese map of X and $a = \dim alb(X)$.

Set

$$S^i_m(X) = \{\mathcal{L} \in Pic^0(X) | h^i(X, \mathcal{L}) \geq m\}$$

Let Y be another smooth projective variety and $f : Y \longrightarrow X$ a surjective morphism. Set

$$\widetilde{S^i}_m(X, f) = \{\mathcal{L} \in Pic^0(X) | h^i(Y, f^*\mathcal{L}) \geq m\}$$

Green and Lazarsfeld obtained a very concrete description of the sets $S^i_m(X)$, as a translated of tori in $Pic^0(X)$. Later, Simpson ([84]) proves part of the same statement on the more general sets $\widetilde{S^i}_m(X, f)$ (and others) and, which is more interesting for our purposes, he proves that the translation is by *torsion* points, which was previously conjectured by Beauville and Catanese.

Theorem 1.33 ([35],[36],[84]) *With the above hypotheses we have*

(i) The sets $S^i_m(X)$, $\widetilde{S^i}_m(X, f) \subseteq Pic^0(X)$ *are closed analytic subvarieties. If* Z *is an irreducible component of one of them, then* Z *is a translation of a subtorus of* $Pic^0(X)$ *by a torsion point.*

(ii) $dim S^i_m(X) \leq q(X) - a + i$.

(iii) If Z *is an irreducible component of* $S^i_m(X)$ *then there exists an analytic variety* W*, of dimension less or equal than i and maximal Albanese dimension, and an analytic dominant map* $g : X \longrightarrow W$ *such that* $Z \subseteq x + g^*(Pic^0(W))$ *for* $x \in Pic^0(X)$.

More concretely, in the case of H^1 we have

Theorem 1.34 ([13]) *Let* X *be a smooth complex projective variety and* $\{p_i : X \longrightarrow B_i\}_{i \in I}$ *the finite family of fibrations over curves of genus* $g \geq 2$. *Then* $S^1(X) = \cup_{i \in I} p_i^*(Pic^0 B_i) \cup \{$*finite number of (torsion) points*$\}$.

Chapter 2

Numerical Bounds of Canonical Varieties

Let X be a normal $\mathbb{Q}$-Gorenstein projective variety and let $K_X \in \mathrm{Div}(X) \otimes \mathbb{Q}$ be its canonical Weil divisor. A first invariant associated to X is its Kodaira dimension, which is defined as

$$kod(X) = \max\{\dim\varphi_m(X) | \varphi_m \text{ is the rational map induced by } |mK_X|\}.$$

The general behaviour corresponds to those that have maximal Kodaira dimension, $kod(X) = \dim X$, the so called varieties of *general type.* We can consider a classification of these according to the behaviour of the *first* canonical map φ_1. Then we can define

Definition 2.1 Let X be a normal, Gorenstein, projective variety, with canonical Cartier divisor $K_X \in \mathrm{Div}(X)$. We say that X is *canonical* if the rational map induced by the canonical system $|K_X|$ is birational.

Note that the word *canonical* is used also as a class of singularities. Nevertheless we always will use *canonical variety* in the above meaning and so no confusion will arrive.

We will assume from now on in this chapter that X is *minimal.* Then, if $\sigma : Y \longrightarrow X$ is any desingularization, by the definition of terminal singularities (canonical singularities is enough) we have that $H^0(X, K_X) \cong H^0(Y, K_Y)$ and hence is equivalent to say that X or Y are canonical.

We are interested in the *geography* of canonical varieties. More concretely in lower bounds of K_X^n depending on birational invariants of X. For this is necessary to consider X minimal since K_X^n varies in its birational class.

The first known result is related to canonical surfaces. A classical inequality of Castelnuovo asserts that $K_S^2 \geq 3p_g(S) - 7$. After this, Harris ([39]) proves that, if X is of dimension n and $|K_X|$ is very ample, then $K_X^n \geq (n+1)p_g(X) + d_n$ (d_n constant) although the proof works without the assumption of very ampleness (birationality is enough).

In the case of canonical surfaces we can consider inequalities relating also the irregularity $q(S) = h^1(S, \mathcal{O}_S)$. The first result is that for such surfaces $K_S^2 \geq 3\chi\mathcal{O}_X$ (cf. [12]) in the case they are irregular. A classical result of Jongmans gives the more accurate $K_S^2 \geq 3p_g + q - 7$ (cf. [52] and [21]).

Our aim is to give a three-dimensional version of the inequality of Jongmans as far as an analysis of other possible bounds in the case of surfaces.

First of all we give a general version of the method that produces Jongmans inequality, relying on a basic result on quadrics containing a variety due to Reid. From this we analyze the border case $K_S^2 = 3p_g + q - 7$ and prove that, in fact, equality holds only when $q = 0$ (if $p_g(S) \neq 4, 5, 7$). This indicates that the bound of Jongmans is not sharp for $q \neq 0$ (surfaces verifying $K_S^2 = 3p_g - 7$ are known to exist and completely classified ([3], [61])).

The general result given in §2.1 allows, via Riemann-Roch Theorem in X, to give lower bounds of K_X^n depending on other birational invariants. Unfortunately, this expression is not very interesting for $\dim X \geq 4$ since it contains negative terms which are not easy to bound.

Then, in the case of canonical threefolds we prove that $K_X^3 \geq 4p_g(X) + 6q(X) - 32$. For this we need not only the basic result of §2.1 but also extra information on fibred threefolds over curves. The *relative hyperquadrics method* of §1.3 and Xiao's method (§1.2) are essential in the argument.

2.1 A general inequality

The main ingredient we will use is the result on quadrics containing nondegenerate irreducible varieties given in Lemma 1.16 and due to Reid. We include a brief idea of proof since we will need it later.

Lemma 2.2. ([77], p. 195) *Let $\Sigma \subseteq \mathbb{P}^N$ be an irreducible variety spanning $\mathbb{P}^N$ of dimension w. Then*

$$h^0\mathcal{J}_{\Sigma,\mathbb{P}^N}(2) \leq \binom{N-w+2}{2} - \min\{\deg\Sigma, 2(N-w)+1\}.$$

PROOF: Take $\Sigma_0 \subseteq \Sigma_1 \subseteq \ldots \subseteq \Sigma_w = \Sigma$ general hyperplane sections. Then Σ_0 is a set of points in uniform position with respect hypersurfaces.

Since for $k = 0, \ldots, w$ Σ_k generates $\mathbb{P}^{N-w+k}$ we have an inclusion

$$H^0\mathcal{J}_{\Sigma_k}(2) \hookrightarrow H^0\mathcal{J}_{\Sigma_{k-1}}(2)$$

and so it is enough to prove the theorem for Σ_0. Let $d = \deg\Sigma = \#\Sigma_0$ and consider

$$0 \longrightarrow H^0\mathcal{J}_{\Sigma_0,\mathbb{P}^{N-w}}(2) \longrightarrow H^0\mathcal{O}_{\mathbb{P}^{N-w}}(2) \xrightarrow{f_0} H^0\mathcal{O}_{\Sigma_0}(2)$$

By a classical result of Castelnuovo we have that if $d \leq 2(N-w)+1$ then $\dim \mathrm{Im} f_0 = d$ and that if $d \geq 2(N-w)+3$ and $\dim \mathrm{Im} f_0 \leq 2(N-w)+1$ then Σ_0 is contained in a rational normal curve, intersection of quadrics containing it. Since $h^0\mathcal{J}_{\Sigma,\mathbb{P}^N}(2) \leq h^0\mathcal{J}_{\Sigma_0,\mathbb{P}^{N-w}} = \binom{N-w+2}{2} - \dim \mathrm{Im} f_0$, the result follows immediately. ■

Then we have an immediate consequence.

Proposition 2.3. *Let X be a normal projective variety of general type and dimension n. Let $L \in Div(X)$, $\mathcal{L} = \mathcal{O}_X(L) \in PicX$ and φ the rational map associated to $\mathcal{L}$. Assume φ is birational; then*

(i) $h^0(X, \mathcal{L}^{\otimes 2}) \geq (n+2)[h^0(X, \mathcal{L}) - \frac{n+1}{2}]$

(ii) *If equality holds in* (i) *then*

(a) $\Sigma := \varphi(X)$ *is contained in a minimal degree variety of* $\mathbb{P}^{h^0(X,\mathcal{L})-1}$ *of dimension* $n+1$ *obtained as the intersection of quadrics containing* Σ.

(b) $\Sigma \subseteq \mathbb{P}^{h^0(X,\mathcal{L})-1}$ *is linearly and quadratically normal.*

(c) $Bs|L| = Bs|2L|$.

(d) *If* $Bs|L| = \emptyset$ *and* $p, q \in X$ *then* $|L|$ *separates* p *and* q *if and only if so does* $|2L|$.

PROOF:

We can consider

$$\overline{X}[d]_\sigma @>>[dr]^{\overline{\varphi}} X @-- >[r]_\varphi \Sigma \subseteq \mathbb{P}^r$$

where $r = h^0(\mathcal{L}) - 1$, $\overline{X}$ is smooth, σ is birational and $\bar{\varphi}$ is defined by the moving part M of the linear system $|\sigma^*(L)|$, which has no base point.

By construction we have $\bar{\varphi}^*\mathcal{O}_{\mathbb{P}^r}(1) = \mathcal{O}_{\overline{X}}(M)$ and $2M \leq$ moving part of $|\sigma^*(2L)|$. Then, since X is normal and σ has connected fibres

$$\begin{aligned} h^0(X, \mathcal{O}_X(2L)) &= h^0(X, \sigma_*\sigma^*\mathcal{O}_X(2L)) = \\ &= h^0(\overline{X}, \sigma^*\mathcal{O}_X(2L)) \geq h^0(\overline{X}, \mathcal{O}_{\overline{X}}(2M)) = \\ &= h^0(\overline{X}, \overline{\varphi}^*\mathcal{O}_{\mathbb{P}^r}(2)) = \\ &= h^0(\Sigma, \overline{\varphi}_*\overline{\varphi}^*\mathcal{O}_{\mathbb{P}^r}(2)) \geq h^0(\Sigma, \mathcal{O}_\Sigma(2)) \end{aligned} \tag{2.1}$$

Now if we consider

$$0 \longrightarrow H^0\mathcal{J}_{\Sigma,\mathbb{P}^r}(2) \longrightarrow H^0\mathcal{O}_{\mathbb{P}^r}(2) \xrightarrow{f} H^0\mathcal{O}_\Sigma(2)$$

Lemma 2.2 gives

$$h^0(\Sigma, \mathcal{O}_\Sigma(2)) \geq \dim \mathrm{Im} f \geq \binom{r+2}{2} - \binom{r+2-n}{2} + \min\{\deg\Sigma, 2(r-n)+1\} \tag{2.2}$$

If H_i $(i = 1, \ldots, n)$ are general hyperplanes in $\mathbb{P}^r$ and $\Sigma_k = \Sigma \cap H_1 \cap \ldots \cap H_{n-k}$ is a general section of Σ of dimension k we have that Σ_2 is an irreducible surface of general type and then ([10], p. 115):

$$\deg\Sigma = \deg\Sigma_2 \geq 2(r-n+2) - 1 > 2(r-n) + 1$$

and hence (2.2) reads

$$h^0(\Sigma, \mathcal{O}_\Sigma(2)) \geq (n+2)[r - \frac{n-1}{2}] = (n+2)[h^0(\mathcal{O}_X(L)) - \frac{n+1}{2}].$$

This proves (i).

Assume from now on that equality holds in (i). In particular equality must hold at every step of (2.1) and (2.2). Then f is an epimorphism and $h^1\mathcal{J}_{\Sigma,\mathbb{P}^r}(2) = 0$. Since $h^1\mathcal{J}_{\Sigma,\mathbb{P}^r}(1)$ is always zero we have (b). Moreover we have

$$\begin{array}{ccccccc}
S^2H^0\mathcal{O}_{\mathbb{P}^r}(1) & \longrightarrow & S^2H^0\mathcal{O}_\Sigma(1) \cong & S^2H^0(\overline{X}, \sigma^*\mathcal{L}) \cong & S^2H^0(X, \mathcal{L}) \\
\downarrow & & & & \downarrow \alpha \\
H^0\mathcal{O}_{\mathbb{P}^r}(2) & \xrightarrow{f} & H^0\mathcal{O}_\Sigma(2) \cong & H^0(\overline{X}, \sigma^*\mathcal{O}_X(2L)) \cong & H^0(X, \mathcal{O}_X(2L))
\end{array}$$

and hence α is an epimorphism and (c) follows immediately.

In order to prove (d), consider local trivializations of $\mathcal{L}$ at p and q. For $\alpha, \beta \in H^0(\mathcal{L})$ we confuse α, β with their local expressions at these trivializations.

We need

Claim. If $Bs|L| = \emptyset$ then $|L|$ does not separate p and q if and only if for all $\alpha, \beta \in H^0(L)$, $\begin{vmatrix} \alpha(p) & \beta(p) \\ \alpha(q) & \beta(q) \end{vmatrix} = 0.$

Proof of the Claim:

Let $\beta \in H^0(\mathcal{L})$ be such that $\beta(p) = 0$. Since p is not a base point of $|L|$ there exists $\alpha \in H^0(\mathcal{L})$ such that $\alpha(p) \neq 0$. Then, from $\begin{vmatrix} \alpha(p) & 0 \\ \alpha(q) & \beta(q) \end{vmatrix} = 0$ we get $\beta(q) = 0$ and then β does not separate p and q.

Assume there exist $\alpha, \beta \in H^0(\mathcal{L})$ such that $\alpha(p) = a$, $\alpha(q) = b$, $\beta(p) = \overline{a}$, $\beta(q) = \overline{b}$ and $a\overline{b} - b\overline{a} \neq 0$. Let $\sigma = a\beta - \overline{a}\alpha \in H^0(\mathcal{L})$. Then clearly σ separates p and q. ■

If $|2L|$ does not separate p and q then trivially so does not $|L|$.

Assume $|L|$ does not separate p and q. Since $S^2H^0(\mathcal{L}) \longrightarrow H^0(\mathcal{L}^{\otimes 2})$ is surjective for every $\alpha, \beta \in H^0(\mathcal{L}^{\otimes 2})$, $\alpha = \sum a_{ij}s_is_j$, $\beta = \sum b_{ij}s_is_j$, $s_i \in H^0(\mathcal{L})$.

Since $|L|$ has no base point and does not separate p and q we can take $\overline{s} \in H^0(\mathcal{L})$ such that $\overline{s}(p) = a \neq 0$, $\overline{s}(q) = b \neq 0$. Since by the claim we have $\begin{vmatrix} s_i(p) & a \\ s_i(q) & b \end{vmatrix} = 0$ for every s_i we can define $\lambda_i = \frac{s_i(p)}{a} = \frac{s_i(q)}{b}$. Then $\alpha(p) = \sum a_{ij}\lambda_i\lambda_j a^2$, $\alpha(q) = \sum a_{ij}\lambda_i\lambda_j b^2$, $\beta(p) = \sum b_{ij}\lambda_i\lambda_j a^2$, $\beta(q) = \sum b_{ij}\lambda_i\lambda_j b^2$, and then

$$\begin{vmatrix} \alpha(p) & \beta(p) \\ \alpha(q) & \beta(q) \end{vmatrix} = a^2b^2 \begin{vmatrix} \sum a_{ij}\lambda_i\lambda_j & \sum b_{ij}\lambda_i\lambda_j \\ \sum a_{ij}\lambda_i\lambda_j & \sum b_{ij}\lambda_i\lambda_j \end{vmatrix} = 0$$

and hence, by the claim, $|2L|$ does not separate p and q.

For the proof of (a) we refer again to the proof of Lemma 2.2. If we call $\Sigma_0 = \Sigma \cap H_1 \cap \ldots \cap H_n$ we have that Σ_0 is a set of $d = \deg\Sigma \geq 2(r-n)+3$ points in $\mathbb{P}^{r-n}$. Proof of Lemma 2.2 shows that if we consider

$$H^0\mathcal{O}_{\mathbb{P}^r}(2) \xrightarrow{f} H^0\mathcal{O}_{\Sigma}(2)$$

$$H^0\mathcal{O}_{\mathbb{P}^{r-n}}(2) \xrightarrow{f_0} H^0\mathcal{O}_{\Sigma_0}(2)$$

then we have $\dim \operatorname{Im} f \geq \binom{r+2}{2} - \binom{r+2-n}{2} + \dim \operatorname{Im} f_0 \geq \binom{r+2}{2} - \binom{r+2-n}{2} + \min\{d, 2(r-n)+1\}$. Under our hypotheses equality holds and then we have that Σ_0 is a set of d points in $\mathbb{P}^{r-n}$ imposing exactly $2(r-n)+1$ conditions on quadrics. Then Σ_0 is contained in a rational normal curve Γ intersection of the quadrics containing Σ_0. Let T_k be the intersection of quadrics of $\mathbb{P}^{r-n+k}$ containing $\Sigma_k = \Sigma \cap H_1 \cap \ldots \cap H_k$. We have $T_k \subseteq T_{k+1} \cap H_{n-k}$ and hence $\Gamma = T_0 \subseteq T_n \cap H_1 \cap \ldots \cap H_n$. Then T_n has an irreducible component W containing Σ of dimension at least $n+1$. So

$$h^0\mathcal{J}_{W,\mathbb{P}^r}(2) = h^0\mathcal{J}_{\Sigma,\mathbb{P}^r}(2) = \binom{r-n}{2}$$

since $\Sigma \subseteq W \subseteq T_n$. Applying now Lemma 2.2 to W, if $w = \dim W \geq n+2$:

$$h^0\mathcal{J}_{W,\mathbb{P}^r}(2) \leq \binom{r-n}{2} - 1.$$

So $\dim W = n+1$ and, since $W \cap H_1 \cap \ldots \cap H_n = \Gamma$, W is a variety of minimal degree in $\mathbb{P}^r$. Since such varieties are always intersection of quadrics we have in particular $W = T_n$. ∎

2.2 Canonical surfaces

As a consequence of Proposition 2.3 we get the following result for minimal canonical surfaces. The first part is a well known fact (cf. [21], [52]).

Theorem 2.4. *Let S be a minimal canonical surface. Then*

(i) $K_S^2 \geq 3p_g + q - 7$.

(ii) *Assume $p_g(S) \geq 8$ or $p_g(S) = 6$. If $K_S^2 = 3p_g + q - 7$ then $q = 0$.*

Proof:

(i) Inequality $K_S^2 \geq 3p_g + q - 7$ follows immediately from Proposition 2.3 since $h^0(S, \omega_S^{\otimes 2}) = K_S^2 + \chi\mathcal{O}_S$.

(ii) In order to prove the statement we need first some properties of surfaces lying on the border line; let $\Sigma = \overline{\varphi(S)} \subseteq \mathbb{P}^{p_g - 1}$, where φ is the canonical map of S.

Claim 1. If $K_S^2 = 3p_g + q - 7$ and $q > 0$ then

(i) Σ lies in a threefold Z of minimal degree.

(ii) $|K_S|$ is base point free.

(iii) $|K_S|$ does not separate $p, q \in S$ (possibly infinitely near) if and only if so does not $|2K_S|$.

(iv) $q \geq 3$.

(v) If $\dim Sing\Sigma = 1$ and $K_S^2 \geq 10$ then the one dimensional components of $Sing\Sigma$ are double lines.

Proof of Claim 1:

(i), (ii) and (iii) are direct consequence of Proposition 2.3 and the fact that $|2K_S|$ has no base points if $p_g \geq 4$ ([15]).

(iv) If $q = 1, 2$ and $K_S^2 = 3p_g + q - 7$, then $K_S^2 < 3\chi\mathcal{O}_S$ and the canonical map of S can not be birational (cf. [21]).

(v) Assume $\dim Sing\Sigma = 1$. Let D be a one dimensional component of $Sing\Sigma$. The canonical map φ is not an embedding over D. Since $K_S^2 \geq 10$ and since, by (iii), points which are not separated by $|K_S|$ are those which are not separated by $|2K_S|$ we can apply Reider's Theorem (see [80]). Let $q \in D$

be a general point of D and let $p_1, p_2 \in S$ (possibly infinitely near) such that $\varphi(p_1) = \varphi(p_2) = q$. By Reider's Theorem there exists an effective divisor E passing through p_1, p_2 and verifying

$$0 \leq K_S E \leq 2 \qquad -2 \leq E^2 \leq 0$$

Since irreducible curves with trivial intersection with K_S are contracted by φ and $q \in D$ is general we can consider that irreducible components of E have positive intersection with K_S. Then only two possibilities may occur:

- E irreducible $\quad K_S E = 2 \quad E^2 = 0$
- $E = E_1 + E_2 \quad K_S E_1 = K_S E_2 = 1 \quad E^2 = 0,\ E_1^2 = E_2^2 = -3,\ E_1 E_2 = 3$

Note that moving $q \in D$ the curve E can not move because then we would have the surface S covered by curves of genus at most two and this is impossible since S is canonical. So we must have $\varphi(E) = D$ (set-theoretically) and $\deg\varphi_{|E} = 2$ since φ contracts at least two points over the general point of D. Then we have that, in both cases, D is a line in $\mathbb{P}^{p_g - 1}$.

Let us check that it is a double line. Assume that for $q \in D$ general we have three points p_1, p_2, p_3 contracted by φ over q. For any pair $\{p_i, p_j\}$ we must have E_{ij} passing through them verifying the above conditions. If we consider the irreducible curves that lie over D by φ it is clear that three curves E_1, E_2, E_3, $E_i^2 = -3$, $E_i E_j = 3$ $(i \neq j)$ must exist. Consider a hyperplane in $\mathbb{P}^{p_g-1}$ containing D (it is possible since D is a line) and consider the section $C \in |K_S|$ that it produces. We have

$$C = E_1 + E_2 + E_3 + \overline{C}.$$

But

$$3 = K_S(E_1+E_2+E_3) = (E_1+E_2+E_3)^2 + \overline{C}(E_1+E_2+E_3) = 9 + \overline{C}(E_1+E_2+E_3)$$

which contradicts the conectness of the canonical divisor. ∎

It is a well known fact that the only possibilities for a threefold Z of minimal degree in $\mathbb{P}^{p_g-1}$ are

(A) $Z = \mathbb{P}^3$ $(p_g = 4)$.

(B) Z is a cone over the Veronese surface $(p_g = 7)$.

(C) Z is a smooth quadric in $\mathbb{P}^4$ ($p_g = 5$).

(D) Z is a scroll of type $\mathbb{P}_{a,b,c}$, $0 \leq a \leq b \leq c$, $2 \leq a+b+c = p_g - 3$.

The following claim finishes the proof of (ii) in Theorem 2.4.

Claim 2. If $K_S^2 = 3p_g + q - 7$ and $q > 0$ then if Case (D) happens, $p_g \leq 5$.

Proof of Claim 2:

Our argument relies on constructing on (a birational model of) S a rational pencil of curves of geometric genus at most 4. Then we can apply a fundamental result of Xiao (cf. [93]) on the irregularity of S: if S has a rational pencil of curves of genus g, then $q(S) \leq \frac{1}{2}(g+1)$. If $g \leq 4$ then $q(S) \leq 2$, impossible by Claim 1 (iv).

Assume Z is a scroll. Consider

$$\overline{S}[d]_\sigma[r]\overline{\Sigma} \subseteq \overline{Z}[d][r]\mathbb{P}^1 S[r]\Sigma \subseteq Z$$

where $\overline{Z}$ is the desingularization of Z. Let $\overline{\alpha} : \overline{S} \longrightarrow \mathbb{P}^1$ be the induced fibration and $\overline{G}$ be a general fibre. Note that, by construction $(\varphi \circ \sigma)_{|\overline{G}} : \overline{G} \longrightarrow \mathbb{P}^{p_g - 1}$ induces on $\overline{G}$ a base point free linear subsystem of $|K_{\overline{G}}|$ and that $(\varphi \circ \sigma)(\overline{G}) \subseteq \mathbb{P}^2 \cong T$, where T is a general ruling of Z.

Note that the singularities of $(\varphi \circ \sigma)(\overline{G})$, for $\overline{G}$ general, lie on $SingZ$ (produced by the base points of $|\alpha(\overline{G})|$ on S) or on $Sing\Sigma \cap T$. If $a+b+c \geq 2$ (we only exclude the case $Z = \mathbb{P}^3$ which is Case (A)) then $p_g \geq 5$ and $K_S^2 \geq 8 + q \geq 11$ if $q \neq 0$. Then, if $Sing\Sigma$ has one dimensional components, they must be lines by Claim 1. Moreover we can assume that they are transversal to the general ruling. Since any such line in Z corresponds to an epimorphism $\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b) \oplus \mathcal{O}_{\mathbb{P}^1}(c) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(1)$, under the assumption $a+b+c \geq 4$ ($p_g \geq 7$) we have that the lines transversal to the ruling (if there are more than one) fit in a ruled surface of kind $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \hookrightarrow Z$ and so cut a general plane T in points which are on a line $\ell \subseteq T$, corresponding to the ruling of $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ in T. Then we can proceed as follows.

Assume first Z is smooth, i.e. $1 \leq a \leq b \leq c$. We have then that $S = \overline{S}$, $\sigma(\overline{G}) = G$ and $\varphi(G)$ is a plane curve of degree $d = 2g(G) - 2$ with only double points as singularities, lying all of them on a line. A simple computation shows that $d \leq 5$ and hence $g(G) \leq 3$. By Xiao's result $q \leq 2$, which contradicts Claim 1.

Assume $\dim SingZ = 1$, i.e. $0 = a = b < c$. Take a general section Γ of Σ containing $SingZ$. Γ corresponds to a section $|K_S| \ni C = cG + L$ where L is the component of the linear subsystem containing $SingZ$ (possibly $L = 0$).

We have then, since $p_g \geq 2q - 3$ (cf. [12])

$$\frac{7}{2}c + 5 \geq 3p_g + q - 7 = K_S^2 = cKG + KL \geq cKG.$$

Then, using $c \geq 3$, $cG^2 \leq K_SG$ and evenness of $K_SG + G^2$ we get that, in any case $2p_a(G) - 2 = K_SG + G^2 \leq 6$. Then $g(\overline{G}) \leq p_a(G) \leq 4$. Again by Xiao's result $q \leq 2$ again a contradiction.

Finally assume $\dim SingZ = 0$, i. e. $0 = a < b \leq c$. Take a general hyperplane section of Σ and Z. We get an irreducible curve $\overline{C}$ lying on a smooth ruled surface V of minimal degree in $\mathbb{P}^{p_g-2}$. Let h, l be the hyperplane divisor class and the fibre divisor class in V. We have that $h^2 = \deg V = p_g - 3$ and that $\overline{C} = \alpha h + \beta l$ with $\alpha \geq 1$, $\beta \geq 0$. Let $C \in |K_S|$ be the smooth curve lying over $\overline{C}$. Using that $K_V = -2h + (p_g - 5)l$ we get

$$\begin{aligned} 2K_S^2 &= 2g(C) - 2 \leq 2p_a(\overline{C}) - 2 = \alpha(\alpha - 1)(p_g - 3) + \beta(\alpha - 2) + \alpha(\beta - 2) \\ K_S^2 &= \deg(\overline{C}) = \overline{C}h = \alpha(p_g - 3) + \beta \\ K_S^2 &= 3p_g + q - 7 \end{aligned}$$

Using that $q \geq 3$ and that $p_g \geq 2q - 3$ one gets that, if $p_g \geq 6$ and $\alpha \geq 5$ then $q = 0$. Then we have $\alpha \leq 4$. But $\alpha = \overline{C}l$ is the degree of $(\varphi \circ \sigma)(\overline{G})$ in $T \cong \mathbb{P}^2$, so $p_a(\overline{G}) \leq \frac{1}{2}(\alpha - 1)(\alpha - 2) \leq 3$ and hence $q \leq 2$. We get then that the only possibilities for S in Case (D) with $q \neq 0$ occur when $p_g \leq 5$. ∎

Remark 2.5. Part (ii) of Theorem 2.4 shows that inequality $K_S^2 \geq 3p_g + q - 7$ is not sharp if $p_g \gg 0$. Since surfaces with $K_S^2 = 3p_g - 7$ are known to exist (and are completely understood, see [3]), it seems that a sharp bound should look like $K_S^2 \geq 3p_g + aq - 7$, with $a > 1$. Nevertheless probably it is not possible to find such an easy linear expression for a sharp lower bound. We can find some linear lower bounds under additional hypotheses. There are several partial results in this direction:

(i) Let $alb : S \longrightarrow alb(S)$ be the Albanese map of S. As a direct consequence of the study of the slope of fibrations, Konno ([63]) shows that, if $\dim alb(S) = 1$ then $K_S^2 \geq 3p_g + 7q - 7$.

(ii) In the same paper Konno proves that if the cotangent sheaf of S is nef then $K_S^2 \geq 6\chi\mathcal{O}_S = 6p_g - 6q + 6$ which is better than $K_S^2 \geq 3p_g + q - 7$ if $p_g \gg q$.

(iii) Note that even if $\dim alb(S) = 2$ but there exists a fibration $\pi : S \longrightarrow B$ with $b = g(B) \geq 2$ we have $K_S^2 \geq 3p_g + 2q - 7$. Indeed, for a general fibred surface (see Chapter 4) we have $K_S^2 \geq \lambda\chi\mathcal{O}_S + (8-\lambda)(b-1)(g-1)$ ($g = g(F)$, F smooth fibre of π). Note that if S is canonical $g \geq 3$. Under our hypothesis $\pi \neq alb$ and then Xiao (cf. [92] and §4.3) proves that $\lambda \geq 4$. Finally note that since $b + g \geq q$ ([12]) we have

$$\begin{array}{rll} (b-1)(g-1) & \geq (b-1)+(g-1) \geq q-2 & \text{if } b \geq 3 \\ \text{and } (b-1)(g-1) & \geq (b-1)+(g-1)-1 \geq q-3 & \text{if } b = 2 \end{array}$$

If $b = 2$ and $(b-1)(g-1) = q-3$ we have $q = b+g$. Again by [12] we can say that $S = B \times F$ with $b = g(B) = 2$. This is not possible if S is canonical. Finally we can apply that for a surface of general type $p_g \geq 2q-4$ and $p_g \geq 2q-3$ if it is canonical ([12]) and we get the desired bound.

(iv) Let $C \in |K_S|$; then we have

$$0 \to H^0(\mathcal{O}_S) \to H^0(\omega_S) \to H^0(C, \omega_{S|C}) \to H^1(\mathcal{O}_S) \stackrel{\rho_C}{\to} H^1(\omega_S) \to \ldots$$

Note that the above sequence is self-dual and then we can consider $\rho_C \in Sym\,\mathbb{C}^q$. The correspondence $H^0(S,\omega_S) \stackrel{\Phi}{\longrightarrow} Sym\,\mathbb{C}^q$ is clearly linear since it is induced by the natural map $H^0(S,\omega_S) \otimes H^1(\mathcal{O}_S) \longrightarrow H^1(S,\omega_S)$. Then, if $p_g \geq \binom{q+1}{2}$ there must exist $C \in |K_S|$ such that $\rho_C = 0$. For such C we have $h^0(C,\omega_{S|C}) = p_g + q - 1$.

Assume we can find such a C irreducible. Since the linear system $|K_S|_{|C}$ is birational we can apply "Clifford plus" ([77] p. 195) and get

$$p_g + q - 1 = h^0(C, \omega_{S|C}) \leq \frac{1}{3}(K_S^2 + 4)$$

and hence $K_S^2 \geq 3p_g + 3q - 7$.

2.3 Canonical threefolds

The case of canonical threefolds is not so straightforward from Proposition 2.3 as the case of surfaces is, since we have an extra numerical invariant which is $h^2\mathcal{O}_T$. Following a classical idea of Castelnuovo-De Francis ([18], [23]) in a general version ([13]) we reduce the proof to the study of a *fibred* canonical threefold over a curve. General results for fibred threefolds are given in [73] and improved in Chapter 5. Nevertheless we must study with more care the cases of low values of the invariants. For this we use the relative hyperquadrics method (see §1.3) and an extra construction of the relative quadric hull of hyperquadrics containing the relative canonical image of the fibration.

Theorem 2.6 *Let T be a canonical threefold. Then*

$$K_T^3 \geq 4p_g + 6q - 32$$

PROOF: Since T is canonical and minimal, K_T is nef and big and hence by the general Kawamata-Viehweg Theorem ([60] Thm. 2.17), the Reid-Fletcher's plurigenera formula ([28], Thm. 2.17) and Proposition 2.3 we get

$$\frac{1}{2}K_T^3 - 3\chi\mathcal{O}_T = \chi(\omega_T^{\otimes 2}) = h^0(T, \omega_T^{\otimes 2}) \geq 5(h^0(T, \omega_T) - 2) \tag{2.3}$$

since, being T Gorenstein, the Reid-Fletcher correction $\ell(m)$ for the plurigenera of T vanishes (cf. [28]).

Hence

$$K_T^3 \geq 4p_g + 6(h^2(\mathcal{O}_T) - h^1(\mathcal{O}_T)) - 14$$

Assume $h^2(\mathcal{O}_T) \geq 2h^1(\mathcal{O}_T) - 3$; then we get

$$K_T^3 \geq 4p_g + 6q - 32 \tag{2.4}$$

and then the Theorem is proved under this hypothesis.

From now on we assume $h^2(\mathcal{O}_T) \leq 2h^1(\mathcal{O}_T) - 4$. Let $\widetilde{T}$ be a desingularization of T. Since terminal singularities are rational, we have $h^0(\widetilde{T}, \Omega^i_{\widetilde{T}}) = h^i\mathcal{O}_{\widetilde{T}} = h^i\mathcal{O}_T$ and hence, if we consider the natural map $\Lambda^2 H^0(\widetilde{T}, \Omega^1_{\widetilde{T}}) \longrightarrow H^0(\widetilde{T}, \Omega^2_{\widetilde{T}})$ we can apply [10] Lemma X.7 and get the existence of two independent sections $\alpha_1, \alpha_2 \in H^0(\widetilde{T}, \Omega^1_{\widetilde{T}})$ such that $\alpha_1 \wedge \alpha_2 = 0$. We apply then [13] Proposition 1 and get a fibration $\widetilde{\pi} : \widetilde{T} \longrightarrow B$ onto a smooth curve of genus $b = g(B) \geq 2$. We observe that $\widetilde{\pi}$ descends to a fibration $\pi : T \longrightarrow B$ since $b \geq 2$ and the exceptional locus of $\widetilde{T} \longrightarrow T$ is rational.

Let F be a general fibre of π. Since $K_T + F_{|F} = K_F$ we have that the general fibre is a smooth canonical minimal surface (note that K_T is nef so in particular it is nef restricted to any divisor).

Then we can apply the results of Ohno ([73] Main Theorem 2) and state that (see Theorem 5.19):

$$K_T^3 - 6(b-1)K_F^2 = K_{T/B}^3 \geq 4(\chi\mathcal{O}_B\chi\mathcal{O}_F - \chi\mathcal{O}_T) \tag{2.5}$$

except for a explicit and finite number of exceptions which will be considered later. Starting from (2.5) we have

$$\begin{aligned} K_T^3 &\geq 2(b-1)[3K_F^2 - 2\chi\mathcal{O}_F] + 4p_g - 4(h^2(\mathcal{O}_T) - h^1(\mathcal{O}_T)) - 4 \geq \\ &\geq 2(b-1)[3K_F^2 - 2\chi\mathcal{O}_F] + 4p_g - 4q + 12 \end{aligned}$$

since we are assuming $h^2\mathcal{O}_T \leq 2h^1(\mathcal{O}_T) - 4$. Note that being F canonical we have $p_g(F) \geq 4$ and $K_F^2 \geq 3p_g(F) + q(F) - 7$ (se §2.2) and so

$$3K_F^2 - 2\chi\mathcal{O}_F \geq 7p_g(F) + 5q(F) - 23 \geq 5(q(F) + 1)$$

and

$$2(b-1)[3K_F^2 - 2\chi\mathcal{O}_F] \geq 10(b-1)(q(F)+1) \geq 10(q(F)+b) - 10$$

since $b \geq 2$, $q(F) \geq 0$.

Note also that from the Albanese maps associated to $F \hookrightarrow T \longrightarrow B$ we get $q(F) + b \geq q(T) = q$ (see §5.1) and so

$$K_T^3 \geq 4p_g + 6q + 2$$

which is stronger than we wanted.

Finally we must deal with the exceptions of Main Theorem 2 in [73] (see Theorem 5.19). Being $p_g(F) \geq 4$ and F canonical we get the following possibilities

- $K_F^2 \leq 2p_g(F) - 1$
- $K_F^2 = 2p_g(F)$, $q(F) \leq 1$
- $K_F^2 = 9$, $p_g(F) = 4$, $q(F) \leq 1$

By §2.1, $K_F^2 \geq 3p_g(F) + q(F) - 7$. Moreover, if $q(F) = 1$ then the Albanese map of F produces a fibration onto a curve of genus 1 and hence $K_F^2 \geq 3p_g(F) + 7q(F) - 7 = 3p_g(F)$ (cf. [62]). Also note that if $q(F) \neq 0$ and $K_F^2 < 3\chi\mathcal{O}_F$, F can not be canonical (cf. [21]). Putting all this together we have that only few cases occur:

(A) $p_g(F) = 4, 5$

(B) $p_g(F) = 6$, $K_F^2 = 11$ $(= 3p_g(F) - 7)$, $q(F) = 0$

(C) $p_g(F) = 6$, $K_F^2 = 12$ $(= 3p_g(F) - 6)$, $q(F) = 0$

(D) $p_g(F) = 7$, $K_F^2 = 14$ $(= 3p_g(F) - 7)$, $q(F) = 0$

Minimal surfaces of general type with $K_F^2 = 3p_g(F) - 7$ or $3p_g(F) - 6$ have being completely studied and classified in [3], [61]. From there we get that when $p_g(F) = 6, 7$ and $K_F^2 = 3p_g(F) - 7$, the canonical image of F is contained in a threefold of minimal degree, intersection of the quadrics containing it. When $p_g(F) = 6$, $K_F^2 = 3p_g(F) - 6 = 12$ the intersection of such quadrics is a threefold of Δ-genus 0 or 1 (i.e., of minimal degree or of minimal degree plus one). We divide then case C in two subcases:

(C.1) The canonical image of F lies in a threefold of $\Delta-$genus 0.

(C.2) The canonical image of F lies in a threefold of $\Delta-$genus 1.

We divide the study of the four cases in three different approaches of proof. In all of them we will prove $K_{T/B}^3 \geq 4(\chi\mathcal{O}_B\chi\mathcal{O}_F - \chi\mathcal{O}_T)$. Then the argument started in (2.5) works.

Case 1.- Case (A)

We use the results of the relative hyperquadrics method of §1.3. Let $\mathcal{E} = \pi_*\omega_{T/B}$ and consider the relative canonical image of T:

$$T@-->[r]^{-}\psi[d]Y \subseteq \mathbb{P}_B(\mathcal{E}) = \mathbb{P}[dl]^{\varphi}B$$

From now on we confuse F and $\psi(F)$.

Then Corollary 1.14 gives

$$K^3_{T/B} \geq (2p_g(F)-4)(\chi\mathcal{O}_B\chi\mathcal{O}_F - \chi\mathcal{O}_T) - 2\deg K - 2\ell(2)$$

where $K = \varphi_*\mathcal{J}_{Y,\mathbb{P}}(2)$. Note that since T is Gorenstein, $\ell(2) = 0$ ([28]).

If $p_g(F) = 4$, $K = 0$ since K is locally free (being a torsion free sheaf on a smooth curve) and $\mathrm{rk}K = h^0(\mathbb{P}^3, \mathcal{J}_{F,\mathbb{P}^3}(2)) = 0$. Then

$$K^3_{T/B} \geq 4(\chi\mathcal{O}_B\chi\mathcal{O}_F - \chi\mathcal{O}_T).$$

If $p_g(F) = 5$ then $\mathrm{rk}K \leq 1$. If $\mathrm{rk}K = 0$ we are done. Assume K is invertible and let $\deg K = x$. We have then

$$H^0(B,\mathcal{O}_B) = H^0(\mathcal{J}_{Y,\mathbb{P}}(2L_{\mathcal{E}}) \otimes \varphi^*(K^{-1})) \hookrightarrow H^0(\mathbb{P}, \mathcal{O}_Z(2L_{\mathcal{E}}) \otimes \varphi^*(K^{-1}))$$

and so there exists a relative hyperquadric $Q \equiv 2L_{\mathcal{E}} - x\varphi^{-1}(t)$ containing Y. Necessarily $\mathrm{rk}Q \geq 3$ since $Y_t \subseteq \mathbb{P}^4$ is nondegenerate. Then Lemma 1.20 applies and we get $\deg K = x \leq \frac{2}{3}\deg\mathcal{E}$. Taking degrees in the generalized Max-Noether sequence of Proposition 1.13, with $D = K_{T/B}$ and $m = 2$ as in the proof of Corollary 1.14 and using that $\deg\mathcal{E} \geq (\chi\mathcal{O}_B\chi\mathcal{O}_F - \chi\mathcal{O}_T)$ we obtain

$$\begin{aligned} K^3_{T/B} &\geq 2(p_g(F)+1)\deg\mathcal{E} - 6(\chi\mathcal{O}_B\chi\mathcal{O}_F - \chi\mathcal{O}_T) - 2\deg K \geq \\ &\geq \tfrac{32}{3}\deg\mathcal{E} - 6(\chi\mathcal{O}_B\chi\mathcal{O}_F - \chi\mathcal{O}_T) \geq \\ &\geq \tfrac{14}{3}(\chi\mathcal{O}_B\chi\mathcal{O}_F - \chi\mathcal{O}_T) \geq \\ &\geq 4(\chi\mathcal{O}_B\chi\mathcal{O}_F - \chi\mathcal{O}_T). \end{aligned}$$

Case 2.- Cases (B), (C.1) and (D)

Consider again the relative canonical image of T.

$$T@-- >[r]^{-}\psi[d]_{\pi}Y \subseteq \mathbb{P}_B(\mathcal{E}) =: \mathbb{P}[dl]^{\varphi}B$$

If $A \in PicB$ is ample enough we have, following Lemma 1.2 and Remark 1.4, an epimorphism

$$H^0(\mathcal{J}_{Y,\mathbb{P}}(2L_{\mathcal{E}}) \otimes \varphi^*(A))@- >>[r]H^0(\mathcal{J}_{F,\mathbb{P}^{p_g-1}}(2))$$

Let W be the horizontal irreducible component of the base locus of the linear system on $\mathbb{P}$ given by the sections of $H^0(\mathcal{J}_{Y,\mathbb{P}}(2L_{\mathcal{E}} \otimes \varphi^*(A)))$. Since under our hypotheses the intersection of quadrics containing F is a threefold of minimal degree (see [3] and [61]) W is a fourfold fibred over B by threefolds of minimal degree. Let $\widetilde{W}$ be a desingularization of W.

We want to relate the invariants of $\pi : T \longrightarrow B$ with those of $\Phi : \widetilde{W} \longrightarrow B$. We extend the ideas given by Konno in [65] for studying this situation. Let H be the pull-back of the tautological divisor of $\mathbb{P}$ to $\widetilde{W}$.

Lemma 2.7

(i) $\Phi_*\mathcal{O}_{\widetilde{W}}(H) = \pi_*\omega_{T/B}$.

(ii) $\deg\Phi_*\mathcal{O}_{\widetilde{W}}(2H) = H^4 + 4\deg\pi_*\omega_{T/B}$.

(iii) $K^3_{T/B} \geq 2H^4 + 2(\chi\mathcal{O}_B\chi\mathcal{O}_F - \chi\mathcal{O}_T)$.

PROOF:

(i) Both are locally free sheaves of the same rank. Consider the following exact sequences

$$0 \longrightarrow \mathcal{J}_{W,\mathbb{P}}(L_{\mathcal{E}}) \longrightarrow \mathcal{O}_{\mathbb{P}}(L_{\mathcal{E}}) \longrightarrow \mathcal{O}_W(L_{\mathcal{E}}) \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O}_W(L_{\mathcal{E}}) \longrightarrow \sigma_*\mathcal{O}_{\widetilde{W}} \otimes \mathcal{O}_{\mathbb{P}}(L_{\mathcal{E}}) = \sigma_*\mathcal{O}_{\widetilde{W}}(H)$$

where $\sigma : \widetilde{W} \longrightarrow W$ is the desingularization. Taking φ_* we have, since $\varphi_*\mathcal{J}_{W,\mathbb{P}}(L_{\mathcal{E}}) = 0$, an inclusion

$$\pi_*\omega_{T/B} = \mathcal{E} = \varphi_*\mathcal{O}_{\mathbb{P}}(L_{\mathcal{E}}) \hookrightarrow \Phi_*\mathcal{O}_{\widetilde{W}}(H)$$

On the other hand if we consider the induced map on a suitable desingularization of T

$$\widetilde{T}[d]^{\eta}[r]^{\widetilde{\psi}}\widetilde{W}[ddl]^{\Phi}T[d]^{\pi}B$$

we have by construction

$$\widetilde{\psi}^*(H) \leq \eta^* K_{T/B}$$

and

$$0 \longrightarrow \mathcal{J}_{\widetilde{\psi}(\widetilde{T}),\widetilde{W}}(H) \longrightarrow \mathcal{O}_{\widetilde{W}}(H) \longrightarrow \mathcal{O}_{\widetilde{\psi}(\widetilde{T})}(H) \longrightarrow 0$$

Since again $\Phi_*\mathcal{O}_{\widetilde{\psi}(\widetilde{T}),\widetilde{W}}(H) = 0$ this produces the opposite inclusion

$$\Phi_*\mathcal{O}_{\widetilde{W}}(H) \hookrightarrow \Phi_*\mathcal{O}_{\widetilde{\psi}(\widetilde{T})}(H)) \hookrightarrow \Phi_*(\widetilde{\psi}_*\mathcal{O}_{\widetilde{T}} \otimes \mathcal{O}_{\widetilde{W}}(H)) = \Phi_*(\widetilde{\psi}_*\widetilde{\psi}^*\mathcal{O}_{\widetilde{W}}(H)) =$$
$$\pi_*\eta_*(\widetilde{\psi}^*\mathcal{O}_{\widetilde{W}}(H)) \hookrightarrow \pi_*\eta_*\eta^*\omega_{T/B} = \pi_*\omega_{T/B}$$

where the last equality holds by projection formula since $\omega_{T/B}$ is locally free being T Gorenstein.

(ii) Note that the formula we want to prove is invariant under the change of H by $H + \Phi^*(A)$, $A \in \mathrm{Div}B$. Indeed, let $a = \deg A$; then

$$\deg\Phi_*\mathcal{O}_{\widetilde{W}}(2H + 2\Phi^*(A)) - (H + \Phi^*(A))^4 - 4\deg(\pi_*\omega_{T/B} \otimes \mathcal{O}_B(A)) =$$

$$\deg\Phi_*\mathcal{O}_{\widetilde{W}}(2H) - H^4 - 4\deg\pi_*\omega_{T/B} + a(2h^0(W_t, \mathcal{O}_{W_t}(2)) - 4\deg W_t - 4h^0(F_t, \omega_{F_t}))$$

If $r = p_g(F) - 1$ we have

$$\begin{aligned} \deg W_t &= r - 2 \qquad \text{since } W_t \text{ is a 3-fold of minimal degree in } \mathbb{P}^r \\ h^0(F_t, \omega_{F_t}) = p_g(F) &= r + 1 \\ h^0(W_t, \mathcal{O}_{W_t}(2)) &= \binom{r+2}{2} - h^0(\mathcal{J}_{W_t,\mathbb{P}^r}(2)) = \binom{r+2}{2} - \binom{r-2}{2} = 4r - 2 \end{aligned}$$

(cf. [2], p. 100) and then the coefficient of a vanishes.

So we can assume $|H|$ is base point free and hence get a smooth ladder $\widetilde{W} = W_4 \supseteq W_3 \supseteq W_2 \supseteq W_1 \supseteq W_0$ (i.e., W_i is smooth and $W_i \in |H_{|W_{i+1}}|$). Notice that W_2 is a ruled surface over B. By successive induction on i, n and m (in this order) one easily proves that

$$\forall i \geq 0 \; \forall m \geq 1 \; \forall n \geq 0 \quad R^m\Phi_*\mathcal{O}_{W_i}(nH_{|W_i}) = 0.$$

For this note first that the result is true for $i = 1$ and that $R^m\Phi_*\mathcal{O}_{W_i} = 0$ for every $m \geq 1$ and $i \geq 0$ since they are locally free ([58]) and $h^m(\mathcal{O}_{(W_i)_t}) = 0$

($(W_i)_t$ is rational). Then use inductively the following exact sequence after taking Φ_*

$$0 \longrightarrow \mathcal{O}_{W_i}((n-1)H_{|W_i}) \longrightarrow \mathcal{O}_{W_i}(nH_{|W_i}) \longrightarrow \mathcal{O}_{W_{i-1}}(nH_{|W_{i-1}}) \longrightarrow 0.$$

Hence

$$\deg\Phi_*\mathcal{O}_{W_i}(2H_{|W_i}) = \deg\Phi_*\mathcal{O}_{\widetilde{W}}(H) + \deg\Phi_*\mathcal{O}_{W_{i-1}}(2H_{i-1}).$$

Finally note that $\deg\Phi_*\mathcal{O}_{W_0}(2H) = H^4$ since W_0 is a reduced 0-dimensional scheme of degree H^4.

(iii) The natural map (as in (i)) $0 \longrightarrow \Phi_*\mathcal{O}_{\widetilde{W}}(2H) \longrightarrow \pi_*\omega_{T/B}^{\otimes 2}$ is injective and has a torsion cokernel since it is an isomorphism at a general fibre. Then the result follows calculating $\deg\pi_* w_{T/B}^{\otimes 2}$ as in proof of Corollary 1.14 and applying (ii). ■

In order to finish *Case 2* note that, since part (iii) of lemma holds, it is enough to prove that $H^4 \geq \deg\Phi_*\mathcal{O}_{\widetilde{W}}(H)$.

Claim: Let X be a smooth variety and $f : X \longrightarrow B$ a fibration onto a smooth curve. Let $D \in \mathrm{Div}(X)$ be a nef divisor and let $\mathcal{E} = f_*\mathcal{O}_X(D)$. Assume $\mathcal{E}$ is nef. Then $D^n \geq \deg f_*\mathcal{O}_X(D)$.

Proof of the Claim: We follow the notations of §1.2. Consider the Harder-Narasimhan data associated to $\mathcal{E} = f_*\mathcal{O}_X(D)$. For every $i = 1, \ldots, \ell+1$ let $\varphi_i : \widetilde{F} \longrightarrow \mathbb{P}^{r_i-1}$ the map induced by the nef base point free linear system P_i on a birational model $\widetilde{F}$ of $F = f^{-1}(t)$.

We construct a partition of $\{1, \ldots, \ell\}$ in the following way: for $i = 0, \ldots, n-1 = \dim F$ consider the (possibly empty) set

$$A_i = \{k = 1, \ldots, \ell | \dim\varphi_k(\widetilde{F}) = i\}$$

and define

$$a_i = \begin{cases} \min A_i & \text{if } A_i \neq \emptyset \\ a_{i+1} & \text{otherwise} \end{cases}$$

Note that if $j \in A_i$, $\deg\varphi_j(\widetilde{F}) \geq r_j - i$ since $\varphi_j(\widetilde{F})$ is a nondegenerate irreducible variety of dimension i in $\mathbb{P}^{r_j-1}$. Moreover, since $P_1 \leq P_2 \leq \ldots \leq P_{\ell+1}$

and they are nef, we have that

$$\left(\prod_{k>i} P_{a_k}\right) \sum_{r=0}^{i} P_j^{i-r} P_{j+1}^{r} \geq \left(\prod_{k>i} P_{a_k}\right) (i+1) P_j^i$$

Finally, since for $k > i$ $\dim \varphi_{a_k}(\widetilde{F}) > i$ and P_{a_k} is base point free, we have that

$$\left(\prod_{k>i} P_{a_k}\right) P_j^i \geq \deg \varphi_j(\widetilde{F})$$

Using then Proposition 1.10 as in Proposition 1.11 we have

$$D^n \geq \sum_{i=0}^{n} \sum_{j\in A_i} \left(\prod_{k>i} P_{a_k}\right) \left(\sum_{r=0}^{i} P_j^{i-r} P_{j+1}^{r}\right) (\mu_j - \mu_{j+1})$$

and hence by the previous remarks

$$D^n \geq \sum_{i=0}^{n} \sum_{j\in A_i} (i+1)(r_j - i)(\mu_j - \mu_{j+1}) \geq \sum_{i=0}^{n} \sum_{j\in A_i} r_j (\mu_j - \mu_{j+1}) = \deg \mathcal{E}$$

since $r_j \geq i+1$, being $\varphi_j(\widetilde{F})$ an i-dimensional variety in $\mathbb{P}^{r_j - 1}$. ∎

Case 3.- Case (C.2)

In this case (see [61]) the canonical image of F is a complete intersection of two quadrics and a cubic. We follow the notations of *Case 2.* Denote $H_i = H_{|W_i}$. Now $\widetilde{W} = W_4$ is fibred over B by threefolds of degree four in $\mathbb{P}^5$, complete intersections of two quadrics, and $W_2 \longrightarrow B$ is an elliptic surface over B.

Then we have

Lemma 2.8

(i) $\Phi_*\mathcal{O}_{\widetilde{W}}(H) = \pi_*\omega_{T/B}$.

(ii) $\deg\Phi_*\mathcal{O}_{\widetilde{W}}(2H) \geq \frac{1}{2}H^4 + 5\deg\Phi_*\mathcal{O}_{\widetilde{W}}(H)$.

(iii) $K^3_{T/B} \geq H^4 + 4(\chi\mathcal{O}_B\chi\mathcal{O}_F - \chi\mathcal{O}_T)$.

PROOF:

(i) Follows as in Case 2.

(ii) Note that, as in Case 2, formula (ii) is invariant under changing H by $H + \Phi^*(A)$ so we can construct a smooth ladder of $(\widetilde{W}, H)$. Indeed, as in Case 2, the coefficient of $a = \deg A$ is

$$2h^0(W_t, \mathcal{O}_{W_t}(2)) - 2\deg W_t - 5h^0(F_t, \omega_{F_t}) = 2\left[\binom{5+2}{2} - 2\right] - 8 - 30 = 0$$

since W_t is the complete intersection of two quadrics in $\mathbb{P}^5$. For $i \geq 2$ and $t \in B$ general $(W_i)_t \subseteq \mathbb{P}^{i+1}$, being a complete intersection, is projectively normal. On the other side $R^1\Phi_*\mathcal{O}_{W_i}$ is locally free for $i \geq 1$ (see [58]) and in fact $R^1\Phi_*\mathcal{O}_{W_i} = 0$ except for $i = 2$, for which it is a line bundle of degree $-\chi\mathcal{O}_{W_2}$ (observe that W_2 is an elliptic fibration over B). For this just use the exact sequences

$$0 \longrightarrow \mathcal{J}_{(W_i)_t, \mathbb{P}^{i+1}} \longrightarrow \mathcal{O}_{\mathbb{P}^{i+1}} \longrightarrow \mathcal{O}_{(W_i)_t} \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^{i+1}}(-4) \longrightarrow \mathcal{O}_{\mathbb{P}^{i+1}}(-2) \oplus \mathcal{O}_{\mathbb{P}^{i+1}}(-2) \longrightarrow \mathcal{J}_{(W_i)_t, \mathbb{P}^{i+1}} \longrightarrow 0$$

and [7] Proposition 12.2.

Finally we want to prove that $R^1\Phi_*\mathcal{O}_{W_2}(H) = 0$. Let $E = (W_2)_t$ any fibre of $\Phi : W_2 \longrightarrow B$. If E is smooth $h^1(E, \mathcal{O}_E(4)) = 0$ so $R^1\Phi_*\mathcal{O}_{W_2}(H)$ is torsion. We need to prove that $h^1(E, \mathcal{O}_E(4)) = 0$ for any fibre E. Since $H_2 + E$ is nef and big on W_2 we have from Kawamata-Viehweg vanishing and the exact sequence

$$0 \to H^0(W_2, -E-H_2) \to H^0(W_2, -H_2) \to H^0(E, \mathcal{O}_E(-1)) \to H^1(W_2, -E-H_2)$$

that $h^1(E, \mathcal{O}_E(1)) = h^0(E, \mathcal{O}_E(-1)) = 0$ (recall that $K_E = \mathcal{O}_E$ since W_2 is elliptic) and hence that $R^1\Phi_*\mathcal{O}_{W_2}(H) = 0$. Then again by induction as in Case

2 we have

$$\forall i \geq 0 \quad \forall n \geq 1 \qquad R^1\Phi_*\mathcal{O}_{W_i}(nH) = 0$$

$$\forall i \neq 2 \qquad R^1\Phi_*\mathcal{O}_{W_i} = 0$$

Therefore we have exact sequences

$$0 \longrightarrow \Phi_*\mathcal{O}_{W_{i+1}}(H) \longrightarrow \Phi_*\mathcal{O}_{W_{i+1}}(2H) \longrightarrow \Phi_*\mathcal{O}_{W_i}(2H) \longrightarrow 0 \quad \text{for } i \geq 0$$

$$0 \longrightarrow \Phi_*\mathcal{O}_{W_{i+1}} \longrightarrow \Phi_*\mathcal{O}_{W_{i+1}}(H) \longrightarrow \Phi_*\mathcal{O}_{W_i}(H) \longrightarrow 0 \quad \text{for } i \neq 1$$

$$0 \longrightarrow \Phi_*\mathcal{O}_{W_2} \longrightarrow \Phi_*\mathcal{O}_{W_2}(H) \longrightarrow \Phi_*\mathcal{O}_{W_1}(H) \longrightarrow R^1\Phi_*\mathcal{O}_{W_2} \longrightarrow 0$$

Denote $d = \deg\Phi_*\mathcal{O}_{\widetilde{W}}(H) = \deg\pi_*\omega_{T/B}$. Then we have

$$\begin{aligned} d &= \deg\Phi_*\mathcal{O}_{W_4}(H) = \deg\Phi_*\mathcal{O}_{W_3}(H) = \deg\Phi_*\mathcal{O}_{W_2}(H) = \\ &= \deg\Phi_*\mathcal{O}_{W_1}(H) - \deg R^1\Phi_*\mathcal{O}_{W_2} \end{aligned}$$

and then

$$\deg\Phi_*\mathcal{O}_{\widetilde{W}}(2H) = 4d + H^4 + \deg R^1\Phi_*\mathcal{O}_{W_2} = 4d + H^4 - \chi\mathcal{O}_{W_2}.$$

Note that, since $\Phi : W_2 \longrightarrow B$ is an elliptic fibration, we have that $K_{W_2} \cong \Phi^*(L)+M$, where $M \geq 0$ and is contained in fibres and $\deg L = \chi\mathcal{O}_{W_2}+2(b-1)$. So, Riemann-Roch on W_2 and Leray spectral sequence yields

$$\begin{aligned} d - 4(b-1) &= \chi\Phi_*\mathcal{O}_{W_2}(H) = \chi\mathcal{O}_{W_2}(H) = \chi\mathcal{O}_{W_2} + \tfrac{1}{2}H_2^2 - \tfrac{1}{2}H_2K_{W_2} \leq \\ &\leq -\chi\mathcal{O}_{W_2} - 4(b-1) + \tfrac{1}{2}H^4 \end{aligned}$$

since $H_2^2 = H^4$, H_2 is nef and $H\Phi^{-1}(t) = 4$ for $t \in B$. Then $-\chi\mathcal{O}_{W_2} \geq d - \frac{1}{2}H^4$ and hence $\deg\Phi_*\mathcal{O}_{\widetilde{W}}(2H) \geq 5d + \frac{1}{2}H^4$.

(iii) The same argument as in *Case 2* works. ■

Now we only have to use $H^4 \geq 0$ and get $K_{T/B}^3 \geq 4(\chi\mathcal{O}_B\chi\mathcal{O}_F - \chi\mathcal{O}_T)$ as needed. Note that using good lower bounds for H^4 as in *Case 2* we can obtain stronger bounds for $K_{T/B}^3$ in this case. ■

Remark 2.9 The bounds obtained in Theorem 3.1 for fibred canonical threefolds hold when simply $|K_F|$ induces a birational map (it is not necessary that T be canonical).

Chapter 3

On a conjecture of Fujita

Let Y be a smooth projective variety. Let $\mathcal{M} = \mathcal{O}_Y(D)$ be an invertible sheaf. There are several notions of positivity for $\mathcal{M}$. $\mathcal{M}$ is said to be *nef* (or semipositive) if $DC \geq 0$ for every curve $C \subseteq Y$; $\mathcal{M}$ is said to be *strictly nef* (cf. [81]) if $DC > 0$. On the other hand $\mathcal{M}$ is *ample* if for any dimension k subvariety $E \subseteq Y$ we have $D^k E > 0$. Alternatively $\mathcal{M}$ is ample if and only if $\mathcal{M}^{\otimes r}$ is very ample for some $r \geq 1$. From this, the definition of semiampleness comes naturally. $\mathcal{M}$ is said to be *semiample* if $\mathcal{M}^{\otimes r}$ is generated by global sections. Note that the notion of semiampleness is not numerical (for instance, $\mathcal{M} = \mathcal{O}_Y$ is semiample).

Let $\mathcal{F}$ be a locally free sheaf on Y. Consider $\mathcal{M} = \mathcal{O}_{\mathbb{P}}(L_{\mathcal{F}})$ the tautological line bundle on $\mathbb{P} = \mathbb{P}_Y(\mathcal{F})$. We can extend the above notions of positivity for line bundles to $\mathcal{F}$ via $\mathcal{M}$. Note that then $\mathcal{F}$ is semiample if and only if $S^r\mathcal{F}$ is generated by global sections on Y for some $r \geq 1$; indeed, just consider the natural isomorphism $H^0(Y, S^r\mathcal{F}) \cong H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(r))$.

Remember that given a smooth variety Z is equivalent to give a non constant map $f : Z \longrightarrow Y$ and a quotient line bundle $f^*\mathcal{F}@->>[r]\mathcal{L}$ on Z, to give a map $\varphi : Z \longrightarrow \mathbb{P}$ such that $\varphi^*\mathcal{O}_{\mathbb{P}}(L_{\mathcal{F}}) = \mathcal{L}$ (cf. [43] II.7.12). Then, when Y is a curve we can give alternative equivalent definitions for the numerical properties. Observe that if Z is a curve then $\deg\mathcal{L} = \deg\varphi^*\mathcal{O}_{\mathbb{P}}(L_{\mathcal{F}}) = \varphi(Z).L_{\mathcal{F}}$.

Following [26] we can define the *lower degree* of $\mathcal{F}$ as

$$\ell d(\mathcal{F}) = \min\{\deg\mathcal{L} \mid \mathcal{L} \text{ quotient line bundle of } \mathcal{F}\}$$

and the *stable lower degree* of $\mathcal{F}$ as

$$s\ell d(\mathcal{F}) = \inf\left\{\frac{\ell d(f^*\mathcal{F})}{\deg f} \mid f : Z \longrightarrow Y \text{ a finite map of nonsingular curves}\right\}$$

Then it comes out immediately from the definition that $\mathcal{F}$ is nef if and only if $\ell d(\mathcal{F}) \geq 0$ and that $\mathcal{F}$ is strictly nef if and only if $\ell d(\mathcal{F}) > 0$. Moreover $\mathcal{F}$ is nef if and only if any quotient of $\mathcal{F}$ has nonnegative degree. Obviously if $\mathcal{F}$ is ample any quotient of $\mathcal{F}$ has positive degree. A not so trivial fact based on Seshadri criterion for ampleness is that $\mathcal{F}$ is ample if and only if $s\ell d(\mathcal{F}) > 0$ (cf. [26] 1.2).

We have the obvious relations ample $\Rightarrow$ semiample $\Rightarrow$ nef, ample $\Rightarrow$ strictly nef $\Rightarrow$ nef.

Let X, Y be smooth projective varieties of dimensions n and m respectively and $f : X \longrightarrow Y$ a fibration (cf. §1.1). Consider the relative dualizing sheaf $\omega_{X/Y}$ and the relative canonical divisor $K_{X/Y}$ ($\omega_{X/Y} = \mathcal{O}_X(K_{X/Y})$). It turns out that the sheaves $R^i f_* \omega_{X/Y}^{\otimes r}$ have certain properties of positivity. In general these sheaves are only torsion free sheaves but not locally free. The notion of nefness can be extended to torsion free sheaves and to quasi-projective varieties, and we get the so called *weakly positive* sheaves (cf. [89]). Under our hypothesis both notions coincide and so we will not define this new notion of positivity.

The following proposition gives a brief account of the properties we need.

Proposition 3.1 *Let $f : X \longrightarrow Y$ be a fiber space between smooth projective varieties of dimensions n and m respectively. Assume the branch locus of f is contained in a normal crossings divisor. Then*

(i) ([59] 2.6) *$R^i f_* \omega_{X/Y}$ and $R^j f_* \mathcal{O}_X$ are locally free.*

(ii) ([59] Relative duality) *For $0 \leq i \leq d = n - m$, $(R^i f_* \omega_{X/Y})^* \cong R^{d-i} f_* \mathcal{O}_X$.*

(iii) ([89]) *For $k \geq 1$, $f_* \omega_{X/Y}^{\otimes k}$ is nef.*

(iv) ([26]) *For $k \geq 2$, $m = 1$, $f_* \omega_{X/Y}^{\otimes k}$ is ample.*

(v) ([30], [31]) *If $m = 1$, we have a decomposition $\mathcal{E} = f_* \omega_{X/Y} = \mathcal{A} \oplus \mathcal{E}_1 \oplus \ldots \oplus \mathcal{E}_r$, where $\mathcal{A}$ is ample, for $1 \leq i \leq r$, $\mathcal{E}_i$ are stable, degree zero and, if $s = q(X) - g(Y)$, $\mathcal{E}_1 = \ldots = \mathcal{E}_s = \mathcal{O}_Y$, $\mathcal{E}_j \neq \mathcal{O}_Y$ for $s + 1 \leq j \leq r$.*

Moreover, if $\mathcal{F}$ is a stable degree zero sheaf such that there exists a surjective map $\mathcal{E}@- >>[r]\mathcal{F}$, then $\mathcal{F}$ is a direct summand of $\mathcal{E}$ and hence $\mathcal{F} = \mathcal{E}_i$ for some $i \in \{1, \ldots, r\}$.

Note that if Y is a curve, then from (iii) and (v) we get that $\mathcal{E} = f_*\omega_{X/Y}$ is nef and not ample, provided $q(X) > g(Y)$.

In [32] p. 600, Fujita propose the following

Conjecture. Given a fibration $f : X \longrightarrow Y$, is there a birational model $f' : X' \longrightarrow Y'$ such that $\mathcal{E}' = f_*\omega_{X'/Y'}$ is semiample?

Observe that given a fibration f we can always get a birational model $f' : X' \longrightarrow Y'$ with branch locus contained in a normal crossings divisor via Hironaka's Theorem.

Before giving an alternative interpretation in case Y is a curve let us state some well known results on semiample sheaves. A good reference is [34].

Proposition 3.2 ([33], [34]) *Let $\mathcal{F}$ be a locally free sheaf on a smooth variety Y.*

(i) If X is smooth and $g : X \longrightarrow Y$ dominating, then $\mathcal{F}$ is semiample if and only if $g^*\mathcal{F}$ is semiample.

(ii) If $\mathcal{F} = \mathcal{F}_1 \oplus \ldots \oplus \mathcal{F}_k$, $\mathcal{F}$ is semiample if and only if $\mathcal{F}_i$ is semiample for $1 \leq i \leq k$.

(iii) If $\mathcal{F}$ is semiample then $\det(\mathcal{F})$ *is semiample.*

(iv) If $\mathcal{F}$ is semiample and $\operatorname{kod}(Y, \det(\mathcal{F})) = 0$ *then there exists an étale cover $g : \widetilde{Y} \longrightarrow Y$ such that $g^*\mathcal{F}$ is trivial.*

If Y is a curve, Fujita's conjecture is equivalent to ask whether the degree zero, non trivial summands $\{\mathcal{E}_i\}_{s+1 \leq i \leq r}$ become trivial after an étale base change.

$$\widetilde{X}[r][d]_{\widetilde{f}} X[d]^f \widetilde{Y}[r]Y$$

and hence $\widetilde{f}_*\omega_{\widetilde{X}/\widetilde{Y}} = \widetilde{\mathcal{A}} \oplus \mathcal{O}_{\widetilde{Y}}^{\oplus(q(\widetilde{X})-g(\widetilde{Y}))}$ with $\widetilde{\mathcal{A}}$ ample.

This phenomenon will be useful to us in the study of the slope of surfaces and threefolds fibred over curves (see Chapters 4 and 5).

We prove that Fujita's conjecture is true if Y is an elliptic curve. In fact we prove much more: for arbitrary X and Y any locally free quotient $\mathcal{F}$ of $\mathcal{E}$ with $\det(\mathcal{F}) \in Pic^0(Y)$ verifies that $\det(\mathcal{F})$ is a torsion line bundle. In particular, when Y is a curve $\det\mathcal{E}_i$ are torsion. If $\mathcal{E}_i$ has rank one this proves Fujita's conjecture for this piece. In any case note that according to Proposition 3.2 (ii) the torsion nature of $\det(\mathcal{E}_i)$ is a necessary condition in order Fujita's conjecture holds.

Let $\mathcal{H} = \mathcal{O}_Y(H)$ be an ample line bundle on Y. Recall that a locally free sheaf $\mathcal{F}$ on Y is called $\mathcal{H}$-stable if for any $\mathcal{G} \subseteq \mathcal{F}$ we have

$$\frac{H^{m-1}c_1(\mathcal{G})}{\mathrm{rk}\mathcal{G}} < \frac{H^{m-1}c_1(\mathcal{F})}{\mathrm{rk}\mathcal{F}}$$

Clearly $\mathcal{F}$ is $\mathcal{H}$-stable if and only if it is $\mathcal{H}^{\otimes r}$-stable for any $r \geq 1$ so we can assume $\mathcal{H}$ is very ample. Then if $C = H_1 \cap \ldots \cap H_{m-1}$ ($H_i \sim H$) is a smooth curve, it is equivalent to say that $\mathcal{F}$ is $\mathcal{H}$-stable to say that $\mathcal{F}_{|C}$ is stable in the usual sense.

Proposition 3.3 *Let Y be a smooth curve, X a smooth projective variety of dimension n and $f : X \longrightarrow Y$ a fibration. Let $\mathcal{F}$ be a stable, degree zero, locally free sheaf on Y. Then*

(i) A map $\mathcal{E} = f_*\omega_{X/Y} \longrightarrow \mathcal{F}$ is non trivial if and only if it is surjective

(ii) There exists a non zero map $\mathcal{E} = f_*\omega_{X/Y} \longrightarrow \mathcal{F}$ if and only if $h^0(Y, (R^{n-1}f_*\mathcal{O}_X) \otimes \mathcal{F}) \neq 0$

(iii) For any $1 \leq i \leq n-1$ there exists a finite number of stable, degree zero vector bundles $\mathcal{F}$ on Y such that $h^0(Y, (R^i f_*\mathcal{O}_X) \otimes \mathcal{F}) \neq 0$

PROOF:

(i) Since $\mathcal{E}$ is nef by Proposition 3.1, the image of $\mathcal{E} \longrightarrow \mathcal{F}$ has nonnegative degree. Since $\mathcal{F}$ is stable and of degree zero, $\mathcal{F}$ can not have any proper subsheaf of nonpositive degree. Hence if the map is non trivial it is surjective.

(ii) By relative duality we have

$$\mathrm{Hom}_Y(\mathcal{E}, \mathcal{F}) \cong H^0(Y, \mathcal{E}^* \otimes \mathcal{F}) = H^0(Y, (R^{n-1}f_*\mathcal{O}_X) \otimes \mathcal{F})$$

(iii) Assume first $i = n-1$. From (i) and (ii) we obtain that if $h^0(Y, (R^{n-1}f_*\mathcal{O}_X) \otimes \mathcal{F}) \neq 0$, then there is an epimorphism $\mathcal{E}@- >>[r]\mathcal{F}$. According to Fujita's decomposition in Proposition 3.1 (v), we know that there exists i such that $\mathcal{E}_i \cong \mathcal{F}$.

Assume now $1 \leq i \leq n-2$. If we consider Z a general linear section of codimension i in X and $g : Z \longrightarrow Y$ the induced fibration, if follows from [59] 2.34 that $R^{n-1-i} f_*\omega_{X/Y}$ is a direct summand of $g_*\omega_{Z/Y}$. Hence by relative duality

$$\begin{aligned}\mathrm{Hom}_Y(R^{n-1-i} f_*\omega_{X/Y}, \mathcal{F}) &= H^0(Y, (R^{n-1-i} f_*\omega_{X/Y})^* \otimes \mathcal{F}) = \\ &= H^0(Y, (R^i f_*\mathcal{O}_X) \otimes \mathcal{F}) \neq 0\end{aligned}$$

and hence there exists a non trivial map

$$g_*\omega_{Z/Y} @- >>[r] R^{n-1-i} f_*\omega_{X/Y} [r] \mathcal{F}$$

and the argument finishes arguing as in the case $i = n-1$. ■

Then we can state the main result.

Theorem 3.4 *Let X, Y be smooth projective varieties of dimension n and m respectively. Let $f : X \longrightarrow Y$ be a fibration with branch locus contained in a normal crossings divisor of Y.*

Fix a very ample line bundle $\mathcal{H} = \mathcal{O}_Y(H)$ on Y. Let $\mathcal{E} = f_*\omega_{X/Y}$ and let $\mathcal{F}$ be a $\mathcal{H}$-stable locally sheaf on Y such that $det(\mathcal{F}) \in Pic^0(Y)$.

If there exists a non-trivial map $\mathcal{E} \longrightarrow \mathcal{F}$, then $det(\mathcal{F})$ is torsion.

Proof: First of all note that it is enough to prove the theorem for $m = 1$. Indeed, let $Z \in |H|$ be a general smooth member. By Bertini's Theorem $T = f^*(Z) = X \times_Y Z$ is again smooth. Let $g = f_{|T} : T \longrightarrow Z$. By adjunction we have that $g_*\omega_{T/Z} = i^*(f_*\omega_{X/Y})$, where $i : Z \hookrightarrow X$ is the natural inclusion.

Since $\mathcal{F}$ is locally free, $\mathrm{Im}(\mathcal{E} \longrightarrow \mathcal{F})$ is torsion free and non trivial, hence the induced map $g_*\omega_{T/Z} = i^*(f_*\omega_{X/Y}) \longrightarrow i^*\mathcal{F}$ is non-trivial; we also have $\det(i^*\mathcal{F}) = i^*(\det\mathcal{F}) \in Pic^0(Z)$ and $i^*(\mathcal{F})$ is $\mathcal{H}$-stable on Z. By induction we have for some $r \in \mathbb{N}$ $(i^*\mathcal{L})^{\otimes r} = i^*(\mathcal{L}^{\otimes r}) = \mathcal{O}_Z$. Kodaira's vanishing gives $h^0(Y, \mathcal{L}^{\otimes r}) = h^0(Z, i^*(\mathcal{L}^{\otimes r})) = 1$ and hence $\mathcal{L}^{\otimes r} = \mathcal{O}_Y$; indeed, consider the exact sequence

$$\begin{aligned}0 \longrightarrow H^0(Y, \mathcal{O}_Y(-Z) \otimes \mathcal{L}^{\otimes r}) \longrightarrow H^0(Y, \mathcal{L}^{\otimes r}) \longrightarrow \\ \longrightarrow H^0(Z, i^*(\mathcal{L}^{\otimes r})) \longrightarrow H^1(Y, \mathcal{O}_Y(-Z) \otimes \mathcal{L}^{\otimes r})\end{aligned}$$

and that

$$H^0(Y, \mathcal{O}_Y(-Z) \otimes \mathcal{L}^{\otimes r}) = H^1(Y, \mathcal{O}_Y(-Z) \otimes \mathcal{L}^{\otimes r}) = 0$$

since $\mathcal{O}_Y(Z) \otimes \mathcal{L}^{-\otimes r}$ is ample ($\mathcal{O}_Y(Z)$ is ample, $\mathcal{L}$ is numerically trivial and ampleness is a numerical condition).

From now on we assume Y to be a smooth curve of genus b and $\mathcal{F}$ a stable, degree zero locally free sheaf on Y. Let $d = n - m = n - 1$ the relative dimension of f.

Consider, from Leray's spectral sequence $E_2^{p,q} = H^p(R^q f_* \mathcal{F}) \Rightarrow H^{p+q}(f_* \mathcal{F})$

$$0 \longrightarrow H^1(Y, (R^{d-1} f_* \mathcal{O}_X) \otimes \mathcal{F}) \longrightarrow H^{n-1}(X, f^*(\mathcal{F})) \longrightarrow$$

$$\longrightarrow H^0(Y, (R^d f_* \mathcal{O}_X) \otimes \mathcal{F}) \longrightarrow 0 \tag{3.1}$$

By Proposition 3.3, $h^0(Y, (R^{d-1} f_* \mathcal{O}_X) \otimes \mathcal{F}) = h^0(Y, (R^d f_* \mathcal{O}_X) \otimes \mathcal{F}) = 0$ except for a finite number of such $\mathcal{F}$. We have then that $h^{n-1}(X, f^*(\mathcal{F}))$ is constant, say a, except for a finite number of $\mathcal{F}$. Let $s = \mathrm{rank}\mathcal{F}$.

If $s = 1$ then Proposition 3.3 asserts that $\mathcal{F} \in A = \{\mathcal{M} \in Pic^0(Y) \mid h^{n-1}(X, f^*(\mathcal{M})) \geq a + 1\}$ which is finite. Then we can apply the remarkable result of Simpson in [84](see Theorem 1.33): the irreducible components of A are translations of tori by torsion points. Finiteness of A implies that in fact its points are torsion.

Assume $s \geq 2$. Following Viehweg ([89], [71] 4.11) we can consider $X^{(s)}$ a resolution of the component of $X^s = X \times_Y \overset{s)}{\ldots} \times_Y X$ dominating Y and $f^{(s)} : X^{(s)} \longrightarrow Y$ the induced fibration. We have then an inclusion

$$\left(f_*^{(s)} \omega_{X^{(s)}/Y}\right)^{**} \hookrightarrow \left(\overset{s}{\otimes} f_* \omega_{X/Y}\right)^{**}$$

Note that since $\dim Y = 1$ both are locally free sheaves (hence reflexive) and so we have an inclusion of vector bundles of the same rank

$$j : f_*^{(s)} \omega_{X^{(s)}/Y} \hookrightarrow \overset{s}{\otimes} f_* \omega_{X/Y}$$

Now consider the projection π

$$\overset{s}{\otimes} f_* \omega_{X/Y} @- >>[r] \overset{s}{\otimes} \mathcal{F} @- >>[r] \det \mathcal{F}$$

Note that $\pi \circ j$ is non trivial since the first two vector bundles have the same rank and j is injective, and then we can apply the argument of the rank one case. ■

Remark 3.5 A similar result is given independently in [95] for the very particular case of a surface S of Albanese dimension 1, where $q(S) = g(Alb(S)) = 1$ and the canonical map of S is composed with the Albanese fibration.

Once this memory was written, the author was informed by Professor Takao Fujita that the result in Theorem 3.4 also follows from an argument of Hodge Theory following [22] §4.2.

Corollary 3.6 *Let $f : X \longrightarrow B$ be a fibration of a smooth projective variety X onto an smooth curve of genus b.*

If $b \leq 1$ then $f_*\omega_{X/B}$ is semiample.

PROOF: We just need to apply that on an elliptic curve any stable degree zero sheaf has rank one. ■

If $Y = B$ is any smooth curve, Theorem 3.4 says that Fujita's conjecture is true for any rank one degree zero summand of $\mathcal{E}$. Hence the only open question is whether the degree zero summands of rank at least two in Fujita's decomposition are semiample. We can not prove this but we prove that if they are not semiample they are more positive than being nef, they are *strictly nef*. More concretely

Corollary 3.7 *Let $f : X \longrightarrow B$ be as above. Let $\mathcal{F}$ be a stable degree zero vector bundle on B.*

If there exists a non-trivial map $\mathcal{E} = f_*\omega_{X/B} \longrightarrow \mathcal{F}$ then there is a base change $\sigma : \widetilde{B} \longrightarrow B$ such that $\sigma^*\mathcal{F} = \mathcal{F}_0 \oplus \mathcal{O}_{\widetilde{B}}^{\oplus r}$, where $\mathcal{F}_0$ is a strictly nef vector bundle with trivial determinant.

PROOF: Assume $\mathcal{F}$ is not strictly nef. Then there exists a smooth curve C and a map $\sigma : C \longrightarrow B$ such that $\sigma^*\mathcal{F}@- >>[r]\mathcal{L}$, $\deg\mathcal{L} = 0$, where $\mathcal{L} = \widehat{\sigma}^*\mathcal{O}_{\mathbb{P}}(1)$ and $\widehat{\sigma}$ is a map $\widehat{\sigma} : C \longrightarrow \mathbb{P} = \mathbb{P}_B(\mathcal{F})$.

Since σ is flat we have ([71] 4.10)

$$0 \longrightarrow \widetilde{f}_*\omega_{\widetilde{S}/C} \longrightarrow \sigma^* f_*\omega_{S/B}$$

where $\widetilde{S}$ is a desingularization of $S \times_B C$.

Since $\sigma^*\mathcal{E}$@$- >>$[r]$\sigma^*\mathcal{F}$@$- >>$[r]$\mathcal{L}$ and the induced map $\widetilde{f}\omega_{\widetilde{S}/C} \longrightarrow \mathcal{L}$ can not be zero ($\widetilde{f}_*\omega_{\widetilde{S}/C}$ and $\sigma^* f_*\omega_{S/B}$ are of the same rank), by Theorem 3.4 $\mathcal{L}$ is torsion and hence, up to a new base change, we can assume is trivial.

Then we argue by induction on the rank of $\mathcal{F}$. ∎

Remark 3.8 If $\mathcal{F}_0$ is as in the previous result, note that for every $\widehat{\sigma} : C \longrightarrow \mathbb{P}_B(\mathcal{F}_0)$, $\mathcal{O}_{\mathbb{P}}(1)\widehat{\sigma}(C) > 0$ but $\mathcal{F}_0$ is not ample. In our case $s\ell d(\mathcal{F}_0) = 0$ but we can not achieve this infimum.

Chapter 4

The slope of fibred surfaces

In this chapter we consider fibrations $f : S \longrightarrow B$ from a smooth projective surface onto a smooth projective curve, a *fibred surface* for short. We call F the general smooth fibre of f and $b = g(B), g = g(F)$. After contracting (-1)-curves in fibres we can always get a relatively minimal model of the fibration. From now on we will always assume that the fibration is relatively minimal.

Invariants

From a geographical point of view it is interesting to know how the numerical invariants of F, B and S relate. For this it is useful to define the relative invariants of f as

$$
\begin{aligned}
&K_{S/B}^2 = (K_S - f^*K_B)^2 = K_S^2 - 8(b-1)(g-1) \\
&\chi_f = (-1)^{\dim S}(\chi\mathcal{O}_S - \chi\mathcal{O}_B\chi\mathcal{O}_F) = \chi\mathcal{O}_S - (b-1)(g-1) \\
&e_f = e(S) - e(B)e(F) = e(S) - 4(b-1)(g-1)
\end{aligned}
$$

where e denotes the topological Euler characteristic.

It is worth to mention that for e_f we have the classical result (cf. [10])

$$e_f = \sum(e(F_i) - e(F))$$

where the sum runs over the singular fibres F_i of f and where $e(F_i) - e(F) \geq 0$

and $e(F_i) = e(F)$ if and only if $g = 1$, and F_i is a multiple of a smooth elliptic curve.

Naturally associated to f we get $\mathcal{E} = f_*\omega_{S/B}$, a rank g, nef vector bundle on B (see Chapter 3). A standard computation via Riemann-Roch on B and S, and Leray's spectral sequence shows that

$$\chi_f = \deg \mathcal{E}.$$

There are several classical inequalities relating all these invariants. The first one is Noether's formula, which in relative version reads

$$12\chi_f = K^2_{S/B} + e_f.$$

When $g \leq 1$ we get ruled and elliptic surfaces. These are well known and never can be surfaces of general type. From now on we assume $g \geq 2$. Then a classical result of Arakelov and the previous inequalities show that

$$K^2_{S/B} \geq 0, \quad \chi_f \geq 0, \quad e_f \geq 0.$$

Moreover we have that if $K^2_{S/B} = 0$ then f is isotrivial (i.e., all smooth fibres are mutually isomorphic), if $e_f = 0$ then f is smooth (i.e., f has no singular fibres) and $\chi_f = 0$ if and only if f is isotrivial and smooth (we will call this a *locally trivial* fibration). When f is isotrivial it becomes trivial (a product) after a birational transformation and a base change (cf. [82]). We will see that, in fact, if $K^2_{S/B} = 0$ then f is also locally trivial.

Consider for $t \in B$ such that F_t is smooth the natural diagram of Albanese maps

$$F_t[r]^{\mathrm{alb}_{F_t}}[d]^{i_t} \mathrm{Alb}(F_t)[d]_{(i_t)_*} S[r]^{\mathrm{alb}_S}[d]^f \mathrm{Alb}(S)[d]_{f_*} B[r]^{\mathrm{alb}_B} \mathrm{Alb}(B)$$

As t varies, the abelian subvariety $(i_t)_*\mathrm{Alb}(F_t)$ remains constant, say A, by the rigidity property of abelian varieties. Let $a = \dim A$. From this we get

$$b \leq q(S) = b + a \leq b + g$$

Moreover, from the diagram and the universal property of Albanese varieties it is immediate to check that, when $b \geq 1$, $b = q$ if and only if $\mathrm{alb}_S(S) = B$.

In this case we say that f is an *Albanese fibration*. On the other hand, $q = b+g$ if and only if S is trivial (i.e., $S = B \times F$, being S minimal)(cf. [12]).

Slope

When f is not locally trivial, we can define the *slope* of f as

$$\lambda(f) = \frac{K^2_{S/B}}{\chi_f}$$

According to the previous results we have

$$0 \leq \lambda(f) \leq 12$$

and that if $\lambda(f) = 0$ then f is isotrivial (we will see in a moment after a result of Xiao that $\lambda(f) > 0$ whenever f non locally trivial). On the other hand, $\lambda(f) = 12$ if and only if f is smooth.

Given a fixed fibred surface $f : S \longrightarrow B$ there are two general ways of producing new fibrations with the same slope (cf. [92]). The first one is to consider base changes

$$\widetilde{S}[r]^{\bar{\sigma}}[d]^{\widetilde{f}} S[d]^{f} \widetilde{B}[r]^{\sigma} B$$

such that σ does not ramify on images of non-semistable fibres of f; in this case we have $\lambda(f) = \lambda(\widetilde{f})$. The second one consists in taking a torsion element or order n $\mathcal{L} \in \mathrm{Pic}^0(S)$ such that for $1 \leq i \leq n-1$ $\mathcal{L}^{\otimes i}_{|F} \neq \mathcal{O}_F$. Then $\mathcal{L}$ determines an étale cover $\sigma : \widetilde{S} \longrightarrow S$ such that $\widetilde{f} = f \circ \sigma$ has connected fibres and hence is a fibration over B. Again in this case we have $\lambda(f) = \lambda(\widetilde{f})$.

As for base changes that do ramify over the images of non-semistable fibres of f, it is worth to mention several works of Tan (cf. [86], [87]). There the contribution of the non-semistable fibres to $K^2_{\widetilde{S}/\widetilde{B}}$ and $\chi_{\widetilde{f}}$ is well understood but it is not known whether the slope $\lambda(\widetilde{f})$ increases or decreases with respect to $\lambda(f)$. We want to stress that hence we can not reduce the study of the slope of fibred surfaces to the slope of semistable fibrations; indeed, in the process of semistable reduction (cf. [7]) we must ramify on the images of non-semistable fibres and hence we lose the control on the slope.

Slope and moduli

In the particular case of semistable fibrations we can give an alternative definition of the slope which suggest some nice questions on its behaviour. Let $\overline{\mathcal{M}}$ be the Deligne-Mumford compactification of the moduli space $\mathcal{M}_g$ of smooth curves of genus g. Then $\overline{\mathcal{M}} = \mathcal{M}_g \cup \Delta$ where Δ is the divisor representing stable singular curves of genus g. Let h be the class of the Hodge bundle in $\overline{\mathcal{M}}$ and δ the class of certain divisor with support on Δ (we do not need to concrete this). In the geometry of $\overline{\mathcal{M}}$ divisors of type $ah-b\delta$ $(a,b>0)$ play a special role. For instance we have $K_{\overline{\mathcal{M}}} \sim_{\mathbb{Q}} 13h - 2\delta$. In [41] Harris and Morrison define the slope of $\mathcal{M}_g$ as

$$s_g = \inf\left\{\frac{a}{b} | \quad ah - b\delta \quad \text{is effective,} \quad a, b > 0\right\}.$$

Observe that the exact value of s_g has importance in order to get information on the Kodaira dimension of $\overline{\mathcal{M}}$ (cf. [41]).

Let $f : S \longrightarrow B$ be a *semistable* fibration of genus g. Up to contracting (-2)-curves (which produce mild singularities) we get a fibration of stable curves. We consider $\overline{B} \subset \overline{\mathcal{M}}$ its modular image. If f is not isotrivial, then $\overline{B}$ is a curve. For simplicity we assume $\overline{B} = B$. It turns out that

$$\delta B = e_f$$

$$hB = \deg f_* \omega_{S/B} = \chi_f.$$

In this context, the *slope of* f (or *of* B) is defined as

$$s(B) = \frac{\delta B}{hB}$$

and hence, by Noether's formula we get

$$\lambda(f) + s(B) = 12$$

and the problem of finding a lower bound for $\lambda(f)$ translates into the problem of finding an upper bound for $s(B)$. Observe that if B passes through a *general* point of $\mathcal{M}_g$ and $D = ah - b\delta$ if effective, then

$$0 \leq DB = ahB - b\delta B$$

and hence we get lower bounds for

$$\frac{a}{b} \geq \frac{\delta B}{hB} = s(B)$$
$$s_g \geq s(B).$$

Another consequence: using that $\overline{\mathcal{M}}$ is of general type if $g \geq 24$ and that $K_{\overline{\mathcal{M}}} = 13h - 2\delta$ we have that if a semistable fibration verifies that $\lambda(f) < \frac{11}{2}$ ($s(B) > \frac{13}{2}$), then:

$$(mK_{\overline{\mathcal{M}}})B < 0$$

and hence B is in the base locus of all the pluricanonical linear systems in $\overline{\mathcal{M}}$.

By a general result of Xiao in [92] (and Cornalba-Harris for semistable fibrations (cf. [20])) we have the general bound

$$s(B) \leq 8 + \frac{4}{g}$$

which is attained only for some hyperelliptic fibrations. If B passes through a general point of $\overline{\mathcal{M}}$, then Mumford-Harris-Eisenbud ([40], [42], [24]) give

$$s(B) \leq 6 + o(\frac{1}{g})$$

Moreover, fibrations which do not verify this inequality are contained in the closure $\overline{\mathcal{D}}_k$ of certain k-gonal locus. For instance, if g is odd, then

$$s(B) \leq 6 + \frac{12}{g+1}$$
$$k = \frac{g+1}{2}$$

which corresponds to $\lambda(f) \geq 6\frac{g-1}{g+1}$.

Having this description and results in mind, a natural question arises: find a stratification $\mathcal{J}_1 \cup \mathcal{J}_2 \cup ... \cup \mathcal{J}_l = \overline{\mathcal{M}}$ (such that $\mathcal{J}_1$ is the hyperelliptic locus and $\mathcal{J}_{l-1} = \mathcal{D}_k, \mathcal{J}_l = \overline{\mathcal{M}} \setminus \overline{\mathcal{J}_{l-1}}$ described by Mumford-Harris-Eisenbud result) for which we have successively bigger minimal slopes λ_j in $\overline{\mathcal{M}} \setminus \overline{\mathcal{J}_j}$. This stratification should at least consider the gonality (or the Clifford index) of the curves. Of course this problem, stated for semistable fibrations, has a natural translation for general fibrations.

Known results on slopes

The study of the slope of a fibred surface consists in giving sharp bounds for its variation depending on the geometry of the fibration. The upper bound of $\lambda(f)$ is not very well known and there is no general techniques. The general result $\lambda(f) \leq 12$ is improved only in the following result

Xiao ([94]), Matsusaka ([68]). If f is not smooth and the general fibre is hyperelliptic then

$$\lambda(f) \leq \frac{4(g-1)(3g+1)}{g^2} \quad \textit{if} \quad g \quad \textit{even}$$

$$\lambda(f) \leq \frac{4(3g^2-2g+2)}{g^2+1} \quad \textit{if} \quad g \quad \textit{odd.}$$

The lower bound of the slope has been more studied. The main result is

Xiao (cf. [92]); if $g \geq 2$ and f is not locally trivial

$$\lambda(f) \geq 4\frac{g-1}{g} \tag{4.0.1}$$

Later on Konno (cf. [63]) completes the proof begun by Xiao and proves that if the lower bound is achieved, then the general fibre F is hyperelliptic. Then the problem of finding a lower bound for non-hyperelliptic fibrations appears. Cases of low genus of the fibre have been considered in the literature (cf. Konno in [63] and independently Chen in [17] for $g = 4$). There we obtain that $\lambda \geq 3$ if $g = 3$, $\lambda \geq 24/7$ or $7/2$ (according to whether F has one or two g^1_3) if $g = 4$ and that $\lambda \geq 40/11$ or 4 (according to whether F is trigonal or not) if $g = 5$.

The next natural step to consider is the case of trigonal fibrations (i.e., when the general fibre F is a trigonal curve). We have

Konno [65]. If F is a trigonal curve and $g \geq 6$ then

$$\lambda(f) \geq \frac{14(g-1)}{3g+1} \tag{4.0.2}$$

although it is not known to be a sharp bound. Independently, and only for semistable fibrations,

Stankova-Frenkel ([85]). If F is trigonal and f semistable

$$\lambda(f) \geq \frac{24(g-1)}{5g+1} \tag{4.0.3}$$

Curiously enough this bound can be improved when the general trigonal fibre F has general *Maroni invariant* (i.e., the canonical image of F fits in $\mathbb{P}^1 \times \mathbb{P}^1$ or the blow-up of $\mathbb{P}^2$ at a point; this is the general behaviour of a trigonal curve),

Stankova-Frenkel ([85]). If F is trigonal with general Maroni invariant and f is semistable

$$\lambda(f) \geq 5 - \frac{6}{g} \tag{4.0.4}$$

Observe that when $g = 4$ these bounds coincide with those previously found by Konno and Chen.

According to Harris-Mumford-Eisenbud result quoted above for stable fibrations, it is reasonable to think in lower bounds for the slope depending on the existence of special linear series on the general fibre. In the case of stable fibrations this would correspond to the conjectured stratification of $\overline{\mathcal{M}}$. There is a recent attempt to this: in [66] Konno gets a formula for the slope depending on the Clifford index of the general fibre (which is a measure of the existence of special linear series). Unfortunately his formula contains a term which is not easily computed or bounded and depends strongly on a conjecture of Green on the syzygies of canonical curves. Nevertheless as a particular case he obtains

Konno ([66]). If F has maximal Clifford index, $g = g(F)$ is odd and Green's conjecture holds then

$$\lambda(f) \geq 6\frac{g-1}{g+1} \tag{4.0.5}$$

Observe that this bound coincides with that of Harris-Mumford-Eisenbud for stable fibrations.

All the results from (4.0.1) to (4.0.5) study the influence of the geometry of the general fibre over the lower bound of the slope. Another different kind of results deal with the influence on the slope of *global* properties of S. We have

Xiao ([92]).If f is not an Albanese fibration (i.e. $q = q(S) > b$) then

$$\lambda \geq 4 \tag{4.0.6}$$

and equality holds only if $q = b + 1$.

According to Fujita's decomposition of $\mathcal{E}$ this problem is very related to the positivity properties of $\mathcal{E}$. Indeed, in the same paper Xiao proves that $\mathcal{E}$ has no locally free quotient of degree zero and rank at least 2 when F is non hyperelliptic and $\lambda < 4$. In fact he conjectures that $\mathcal{E}$ must be ample when $\lambda < 4$.

Problems and results in this chapter

From the previous considerations we can state three general problems in the study of the slope of fibred surfaces:

PROBLEM 1: How does the special properties of the general fibre influence over the slope?. In particular, how does the existence of special linear series influence?.

PROBLEM 2: How does global numerical invariants of S (other than K_S^2 and $\chi\mathcal{O}_S$) influence on the slope?. In particular how does the irregularity of S influence?.

PROBLEM 3: How special are fibrations with low slope ($\lambda(f) < 4$)?.

As for the third problem is very clear which is the expected behaviour. Note that, as mentioned above, f is necessary an Albanese fibration; on the other hand only few examples of non-hyperelliptic fibrations with $\lambda < 4$ are known: for $g = 4$ Chen gives one example (cf. [17]) and for trigonal fibrations, examples on the border line in Stankova-Frenkel's bound provide further examples as far as $g \leq 6$. It seems that the only possibility of finding examples for all g arises only in the hyperelliptic case. Finally, according to Xiao's conjecture, the sheaf $\mathcal{E}$ should be ample. Summing up we can state

Conjecture. Let $f : S \longrightarrow B$ be a non-locally trivial, relatively minimal fibred surface. If $\lambda(f) < 4$ then $\mathcal{E} = f_*\omega_{S/B}$ is ample and (if $g >> 0$) F is hyperelliptic.

Along this chapter we contribute to the study of these three problems. In §4.1 and §4.2 we study the slope of double cover fibrations. Roughly speaking, these are fibrations for which the general fibre has an involution inherited from the surface. The reason for studying this kind of fibrations is twice. The first one is technical and arises when applying Xiao's method (see Chapter 1) for the study of the slope of arbitrary fibred surfaces; this method relies in the study of certain linear systems on the fibres. If we call d and r the degree and rank of one of these linear systems, the greatest is the ratio $\frac{d}{r}$ the best results we get from the method (see computations in this chapter). The worst possibility for this ratio, according to Clifford's lemma, is that it be near 2. This happens as far as F is a double cover of another smooth curve induced by the linear system. For this reason it is interesting to deal with this possibility independently, as will happen specially in §4.3.

The second reason comes out after understanding the case of double covers as a generalization of the case of hyperelliptic curves (which are double covers of curves of genus zero). The first step is to consider bielliptic curves (i.e., double covers of elliptic curves). In §4.1 we study the structure and slope of fibrations with general fibre a bielliptic curve. The structure is very simple: they are double covers of elliptic fibrations as far as $g \geq 6$. Which is more surprising: they are not necessarily so if $g \leq 5$. An example is given for which a base change is previously needed. Moreover, when a bielliptic fibration is smooth (i.e., all the fibres are smooth) then it is isotrivial; as far as we know this is only known to happen for hyperelliptic and elliptic fibrations. As for the slope we prove that $\lambda \geq 4$ and characterize the border cases; in particular $\lambda(f) = 4$ is attained for any genus. Observe that a bielliptic curve is tetragonal (it has infinitely many g^1_4 just composing the double cover with the infinitely many g^1_2 on the elliptic curve). Hence a new phenomenon appears with respect to Problem 1: we can always find tetragonal fibrations with slope 4 (which is less than the lower bound for trigonal fibrations). Certainly bielliptic curves are very special tetragonal curves. If F is tetragonal and non bielliptic it has finitely many g_4^1 (in fact only one if $g >> 0$). Probably what happens is just the phenomenon described by Stankova-Frenkel for trigonal curves: the general tetragonal fibration has a different behaviour from the special (bielliptic) one. This phenomenon can perfectly happen again when studying fibrations of higher non-general Clifford index; indeed, if a curve F is a double cover it never has general gonality. So the double cover fibrations could play an special role in the study of Problem 1.

In §4.2 we give a lower bound for double cover fibrations as an increasing function of h (if $F \longrightarrow E$ is the double cover, we call $h = g(E)$), provided $g \geq 4h + 1$. This contributes to Problem 1. As for Problem 3, we also prove that in fact $\lambda(f) \geq 4$ provided $g \geq 2h + 11$ (and F is not tetragonal).

In §4.3 we study Problem 2. We give bounds $\lambda(f) \geq H(g, q)$ in the case the fibration is not a double cover and $\lambda(f) \geq \bar{H}(g, q, h)$ otherwise, such that the functions H and $\bar{H}$ are increasing functions of q and which coincide with Xiao's bound (4.0.6) when $q - b = 2$. This is then a generalization of Xiao's result for the cases $q - b > 1$. It is worth to mention that we find through these inequalities, an influence over the slope of f of the existence of other fibrations on S over curves of genus at least 2 (see Theorem 4.24 below).

Section §4.4 is devoted to the study of Problem 3. Basically we prove that the conjectured behaviour for fibrations with low slope holds, provided the general fibre is non hyperelliptic or $b \leq 1$ or $g \leq 3$.

Finally in §4.5 we construct several families of fibred surfaces. They provide examples to check that the bounds we found in §4.2 and §4.3 are, at least, assimptotically sharp.

4.1 The slope of bielliptic fibrations

4.1.1 Bielliptic fibrations

Let F be a smooth curve of genus g. F is called *bielliptic* if F admits a 2-to-1 map onto an elliptic smooth curve E. Such a map is always given by the quotient by an involution $\iota \in \mathrm{Aut}(F)$, called a *bielliptic involution* on F. It is a well known fact that such an involution is unique if $g \geq 6$ (see [2]).

Let $f : S \longrightarrow B$ be a fibration of genus g. We say that f is *bielliptic* if so it is the general fibre F of f. The following result clarifies the structure of such fibrations. Recall that the fibration f is said to be *smooth* if every fibre is smooth and it is said to be *isotrivial* if all the smooth fibres are mutually isomorphic.

Proposition 4.1 *Let $f : S \longrightarrow B$ be a bielliptic fibration of genus g. Then*

(i) Up to base change, S is a rational double cover of an elliptic surface over the base curve.

(ii) If $g \geq 6$ the same is true without base change.

(iii) If f is smooth, then f is isotrivial.

PROOF: (i) and (ii) are consequence of general results given in §1.4.

(iii) Isotriviality can be checked after base change. Following §1.4.2 we can consider after base change

$$S[dr]_f@^{(}- >[r]^i J(f)[d]@-- >[r]^\ell J(f)[dl]B$$

where $J(f)$ is the relative Jacobian variety of S over B and ℓ is a rational relative endomorphism of $J(f)$ such that $\ell \circ i$ produces a bielliptic map on the general fibre of f. Let $\overline{V} = \overline{(\ell \circ i)(S)}$. Note that $\overline{V}$ is an elliptic surface over B (possibly singular). Let $\varepsilon_1 : V \longrightarrow \overline{V}$ be the normalization of V, and $\varepsilon_2 : \widehat{V} \longrightarrow V$ a minimal desingularization. Classification of singular fibres of smooth elliptic surfaces (cf. [7]) show that if $\widehat{V} \longrightarrow B$ has any fibre which is non smooth, then either every component of such fibre is rational or the fibre is a multiple of a nonsingular smooth elliptic curve. The second possibility can be transformed in a smooth fibre after an étale base change. If the first possibility holds then any component of the respective fibre of $V \longrightarrow B$ and of $\overline{V} \longrightarrow B$ must be rational, which is impossible since by construction for every $t \in B$ $J(\pi)_t = J(F_t)$ is an abelian variety and hence does not contain any rational curve. Then $\widehat{V} = V$ and the map $\tau : V \longrightarrow B$ is smooth. Moreover, since S is smooth, the map $\overline{\pi} = \ell \circ i : S - - - \rightarrow \overline{V}$ factors through a map $\pi : S - - - \rightarrow V$. This can be solved after some blow-ups but then exceptional curves must be contracted since V has no rational curve on fibres.

So we have that S is a double cover of a smooth elliptic fibration (perhaps after base change). In particular *every* fibre of f is bielliptic.

Consider now the double cover $\pi : S \longrightarrow V$. Since π has degree two the branching divisor of π must be smooth and hence it is étale over B. After new base changes we can assume that the irreducible components of the branching divisor D are sections of τ. Moreover, since $\tau : V \longrightarrow B$ is a smooth elliptic fibration it is isotrivial (cf. [7]) and then, after base change, we can assume $V = B \times E$ (E: elliptic smooth curve, cf. [82]). Let D_1 be an irreducible component of D. If D_1 is a trivial section of τ then so must be the other components (otherwise they must intersect, which is impossible since D is étale over B) and then f is clearly isotrivial. Assume D_1 is not a trivial section. Then $D_1 = \{(b, \alpha(b)) \in B \times E \mid \alpha : B \longrightarrow E \text{ non constant map}\}$. Consider a fixed structure of group on $E = (E, 0)$ and consider the automorphism of V over B: $\beta(b, x) = (b, x + \alpha(b))$. Note that $\beta^{-1}(D_1) = B \times \{0\}$ and, hence, $\beta^{-1}(D)$ is composed of trivial horizontal sections. If we change the base

$$S@[dr]|\otimes[r]^{\simeq}[d]_{\widetilde{\pi}}S[d]^{\pi}B \times E[r]_{\beta}B \times E$$

the branching divisor of $\widetilde{\pi}$ is just $\beta^{-1}(D)$ which is trivial. Then any fibre F_t of f is a double cover of the same elliptic curve with the same ramification locus δ. Any such double cover is determined by $\mathcal{L} \in Pic(E)$ such that $\mathcal{L}^{\otimes 2} = \mathcal{O}_E(\delta)$. There are obviously a finite number of such $\mathcal{L}$ and hence F_t belongs to a finite number of isomorphism classes. By continuity the isomorphism class of F_t is constant and hence f is isotrivial. ■

A bielliptic curve of genus $g \leq 5$ can have more than one bielliptic involution; the number of such involutions are in correspondence with the elliptic components of $W^1_4(F)$, the Brill-Noether locus of linear series on F of type g^1_4 (cf. [2]). We give an example which shows that these involutions do not glue independently for a general fibration.

Example 4.2 Take a genus five curve F with *exactly* two bielliptic involutions $\sigma_i : F \longrightarrow E_i$ such that $E_1 \ncong E_2$, with E_i having no exceptional automorphisms. Note that this is always possible. Indeed, consider two different lines in $\mathbb{P}^2$ and a general cubic. Let Q be the quintic they produce. Let W be the admissible double cover of Q ramified at its double points (cf. [9]). Now the Prym variety of the admissible double cover (W, Q) is the Jacobian of a smooth genus 5 curve F (cf. [9]). Moreover $W^1_4(F) \cong W$ and hence F has exactly two bielliptic involutions corresponding to the two elliptic components of W (the two double covers of the lines in Q).

We have that $\sigma_1 \times \sigma_2 : F \longrightarrow E_1 \times E_2$ embeds F as a smooth curve, $F \in |\ell_1^*(2p_1) \otimes \ell_2^*(2p_2)|$, being $\ell_i : E_1 \times E_2 \longrightarrow E_i$ the projections and $(p_1, p_2) \in E_1 \times E_2$. Since Aut $(E_1 \times E_2)$ acts transitively on $E_1 \times E_2$ we have that for every $(q_1, q_2) \in E_1 \times E_2$ there exists $\widetilde{F} \in |\ell_1^*(2q_1) \otimes \ell_2^*(2q_2)|$, $\widetilde{F} \cong F$.

Let B be any smooth curve having an involution ι and let $h : B \longrightarrow \overline{B} = B_{/<\iota>}$. Consider a morphism $\kappa : B \longrightarrow \mathbb{P}^1$ with no factorization through $\overline{B}$. Take a fixed $\bar{t} \in \overline{B}$ such that if $h^{-1}(\bar{t}) = \{t_1, t_2\}$ then $\kappa(t_1) \neq \kappa(t_2)$. After an automorphism of $\mathbb{P}^1$ we can suppose that $\kappa(t_i)$ is the modular invariant of E_i in $\mathbb{C} \subseteq \mathbb{P}^1$.

Then, by [7] p.160, there exists an elliptic fibration $\tau : V \longrightarrow B$ with a section, such that $\tau^{-1}(t_i) \cong E_i$. Let B' be the image in V of the section of τ. Consider the following pull-back

$$@C = 1.5truecm@R = 1.5truecmZ := V \times_B V[d]_{\xi_1}[r]^{\xi_2}[dr]^{\xi}V[d]_{\tau}V[r]_{\iota\circ\tau}B$$

Then, for $t \in B$ we have $Z_t = \xi^{-1}(t) = E_{\iota(t)} \times E_t$, where $E_m = \tau^{-1}(m)$. The natural involution on $V \times_{\mathbb{C}} V$ induces a commutative diagram

$$Z[r]^{\bar{\iota}}[d]^{\xi}Z[d]^{\xi}B[r]^{\iota}B$$

and then

$$@C = 1truecm@R = 1truecmZ[r]^{\overline{h}}[d]^{\xi}\overline{Z} := Z_{/<\bar{\iota}>}[d]^{\bar{\xi}}B[r]^{h}\overline{B}$$

Note that $\overline{Z}$ is a threefold fibred over $\overline{B}$ and the fibre over $h(t) \in \overline{B}$ general is $E_{\iota(t)} \times E_t$. We can assume $\overline{Z}$ is already smooth.

Let $B'' = \overline{h}(\xi_2^{-1}(B'))$ and $\mathcal{L} = \mathcal{O}_{\overline{Z}}(2B'')$. We have that $\mathcal{L}_{|\overline{Z}_{\bar{t}}} \cong \ell_1^*(2q_1) \otimes \ell_2^*(2q_2)$ for some $(q_1, q_2) \in E_1 \times E_2$. Note that if $\mathfrak{a} \in Pic(\overline{B})$ is ample enough we have, by Lemma 1.2, an epimorphism

$$H^0(\overline{Z}, \mathcal{L} \otimes \bar{\xi}^*(\mathfrak{a})) \longrightarrow H^0(E_1 \times E_2, \mathcal{L}_{|\overline{Z}_{\bar{t}}}) .$$

Since by hypothesis there exists $F \in |\mathcal{L}_{|\overline{Z}_{\bar{t}}}|$ we get $\overline{S} \in |\mathcal{L} \otimes \bar{\xi}^*(\mathfrak{a})|$ a surface fibred over $\overline{B}$, smooth at a general fibre and such that $\overline{S}_{\bar{t}} = F$. Again, we can suppose $\overline{S}$ is already smooth. Let $\overline{f} : \overline{S} \longrightarrow \overline{B}$ and $F_{\overline{m}} = \overline{f}^{-1}(\overline{m})$. For $\overline{m} \in \overline{B}$

general we have that $F_{\overline{m}}$ is an smooth curve of genus 5 having *at least* two bielliptic involutions given by the inclusion $F_{\overline{m}} \subseteq E_{\iota(m)} \times E_m$ (if $g(m) = \overline{m}$) as a $(2,2)$-divisor. We claim that for general $\overline{m} \in \overline{B}$, $F_{\overline{m}}$ has exactly two bielliptic involutions. Since this is the case for $F = F_{\overline{t}}$ we only have to prove that having *at most* two of them is an open condition. Consider $W^1_4(\overline{f}) \longrightarrow \overline{B}$, the relative Brill-Noether locus of $\overline{f}$ (at least over an open set of B, see [88]), after a base change if necessary. The number of bielliptic involutions of $F_{\overline{m}}$ is given by the number of elliptic components of $W^1_4(F_{\overline{m}}) \cong W^1_4(\overline{f})_{\overline{m}}$. Then, having at most two of such components is obviously an open condition.

We claim that $\overline{S}$ is not a (birational) double cover of any elliptic fibration $\bar{\tau} : \overline{V} \longrightarrow \overline{B}$. Indeed, assume we have a double cover $\overline{\pi} : \overline{S} \longrightarrow \overline{V}$ (we can suppose $\overline{\pi}$ everywhere defined after some blow-ups). Consider the base change diagram

$$@C = 1truecm @R = 1truecm Z[r] \overline{Z} S @(->[u][r][d]_{\widetilde{\pi}} \overline{S} @(->[u][d]^{\overline{\pi}} \widetilde{V}[r][d]_{\widetilde{\tau}} \overline{V}[d]^{\bar{\tau}} B[r] \overline{B}$$

For S we have three double covers of elliptic fibrations over B:

$$\begin{array}{ll} \widetilde{\pi} : S \longrightarrow \widetilde{V} & \\ \pi_i : S \longrightarrow V & \qquad f_i = \xi_{i|S} \quad i = 1,2 \end{array}$$

Set $U = \{m \in B \,|\, E_m \not\cong E_{\iota(m)};\ E_m, E_{\iota(m)}$ and $\widetilde{E}_m$ are smooth and F_m has exactly two bielliptic involutions$\}$ (where $\widetilde{E}_m = \widetilde{\tau}^{-1}(m)$). We have that U is a non-empty open set of B. Since $\pi_{1|F_m}$, $\pi_{2|F_m}$, $\widetilde{\pi}_{|F_m}$ are double covers of $E_{\iota(m)}$, E_m and $\widetilde{E}_m$ respectively we have that for every $m \in U$, $\widetilde{E}_m \cong E_{\iota(m)}$ or $\widetilde{E}_m \cong E_m$.

If $h_1 = h \circ \iota_{|U} : U \longrightarrow \mathbb{P}^1$, $h_2 = h_{|U} : U \longrightarrow \mathbb{P}^1$ and $\widetilde{h} : U \longrightarrow \mathbb{P}^1$ are the modular morphisms induced by $\iota \circ \tau$, τ and $\widetilde{\tau}$ over U respectively we have that $\widetilde{h} = h_1$ or $\widetilde{h} = h_2$. Assume $\widetilde{h} = h_2$.

As we have $t_1, t_2 \in U$ and $\iota(t_1) = t_2$ we get

$$E_{t_1} = \tau^{-1}(t_1) = \widetilde{\tau}^{-1}(t_1) \cong \widetilde{\tau}^{-1}(t_2) = \tau^{-1}(t_2) = E_{t_2}$$

since $\widetilde{\tau}$ is induced by $\bar{\tau} : \overline{V} \longrightarrow \overline{B}$ and then $\widetilde{\tau}^{-1}(m) \cong \widetilde{\tau}^{-1}(\iota(m))$ for all $m \in B$. But this is impossible since by hypothesis $E_{t_1} = E_1 \not\cong E_2 = E_{t_2}$. $\blacksquare$

4.1.2 The slope of bielliptic fibrations

We recall some basic facts about double covers (see [44], [7]).

By a double cover we mean a finite, degree two map between surfaces, $f_0 : S_0 \longrightarrow V_0$. This map is determined by a divisor R_0 on V_0 (the branch divisor) and a line bundle $\mathcal{L}_0$ such that $\mathcal{L}_0^{\otimes 2} = \mathcal{O}_{V_0}(R_0)$. If V_0 is smooth, S_0 is normal (respectively smooth) if and only if R_0 is reduced (respectively smooth).

Consider a double cover as above with S_0 normal and V_0 smooth. Then there exists a *canonical resolution of singularities* for S_0 which consists on a finite sequence of maps

$$\begin{tikzcd} S_k \arrow[r, "\sigma_k"] \arrow[d, "\pi_k"'] & S_{k-1} \arrow[r] \arrow[d, "\pi_{k-1}"'] & \ldots \arrow[r] & S_1 \arrow[r, "\sigma_1"] \arrow[d, "\pi_1"] & S_0 \arrow[d, "\pi_0"] \\ V_k \arrow[r, "\alpha_k"'] & V_{k-1} \arrow[r] & \ldots \arrow[r] & V_1 \arrow[r, "\alpha_1"'] & V_0 \end{tikzcd}$$

satisfying:

(i) α_j is the blow-up of V_{j-1} at a singular point p_{j-1} of R_{j-1} (the branching divisor of π_{j-1}).

(ii) π_j is the double cover of V_j defined by $\mathcal{L}_j^{\otimes 2} \cong \mathcal{O}(R_j)$, with $R_j = \alpha_j^*(R_{j-1}) - 2m_{j-1}E_j$, $\mathcal{L}_j = \alpha_j^*(\mathcal{L}_{j-1}) \otimes \mathcal{O}_{V_j}(-m_{j-1}E_j)$, where E_j is the exceptional divisor of α_j and p_{j-1} is a singular point of R_{j-1} of multiplicity $2m_{j-1}$ or $2m_{j-1}+1$.

(iii) σ_j is a birational morphism induced by the cartesian diagram of α_j and π_{j-1}.

(iv) R_k is smooth and, hence, S_k is a smooth surface.

Now we can use this as follows. Recall from section §4.1.1 that we have obtained $\pi : \widetilde{S} \longrightarrow V$ a generically 2-to-1 morphism from a blow-up of S onto an elliptic fibration V. We can assume the elliptic fibration is relatively minimal. Suppose also that f is relatively minimal. Then consider

$$@C = 1truecm@R = 1truecm\widetilde{S}[dd]^\sigma[drrr]^u\overline{S} = S_k[dl][dd]_{\pi_k}[r]\ldots[r]S_0[dd]_{\pi_0}S[dd]^f\ldots\overline{V} = V_k[r]\ldots[$$

where:

• $\pi = \pi_0 \circ u$ is the Stein factorization of π, with u birational, π_0 finite (so it is a double cover) and S_0 normal.

• $\pi_k : S_k \longrightarrow V_k$ is the canonical resolution of singularities of $\pi_0 : S_0 \longrightarrow V_0$.

• $\bar{\sigma} : S_k \longrightarrow S$ is the birational morphism defined by the relative minimality of f.

Theorem 4.3 *Let $f : S \longrightarrow B$ be a relatively minimal bielliptic fibration of genus $g \geq 6$. Let V be the relative minimal model of the elliptic fibration obtained in section §4.1.1. Then*

(i) $K^2_{S/B} - 4\chi_f \geq 2(g-5)\mathcal{X}\mathcal{O}_V$. In particular, if f is not locally trivial

$$\lambda(f) \geq 4 + \frac{2(g-5)\mathcal{X}\mathcal{O}_V}{\chi_f} \geq 4$$

(ii) $\lambda(f) = 4$ if and only if S is the minimal desingularization of a double cover $S_0 \longrightarrow V$ of a smooth elliptic surface such that

- *All the fibres of the elliptic fibration $\tau : V \longrightarrow B$ are smooth and isomorphic.*
- *The branch divisor of the double cover has only* negligeable *singularities (i.e., all the multiplicities m_j in the above process are* 2 *or* 3 *(see [13], [17])).*

In particular, the bound is sharp.

Proof:

(i) First of all we have

$$K^2_{S/B} - 4\chi_f = (K^2_S - 4\mathcal{X}\mathcal{O}_S) - 4(b-1)(g-1) \geq (K^2_{\overline{S}} - 4\mathcal{X}\mathcal{O}_{\overline{S}}) - 4(b-1)(g-1)\,. \tag{4.1}$$

For smooth double covers $\pi_k : \overline{S} \longrightarrow \overline{V}$ we have (see [3] p.183):

$$\begin{aligned} \mathcal{X}\mathcal{O}_{\overline{S}} &= 2\mathcal{X}\mathcal{O}_{\overline{V}} + \frac{1}{2}\mathcal{L}_k K_{\overline{V}} + \frac{1}{2}\mathcal{L}_k\mathcal{L}_k \\ K^2_{\overline{S}} &= 2K^2_{\overline{V}} + 4\mathcal{L}_k K_{\overline{V}} + 2\mathcal{L}_k\mathcal{L}_k \end{aligned}$$

so we have

$$K_{\overline{S}}^2 - 4\mathcal{X}\mathcal{O}_{\overline{S}} = 2[K_{V_k}^2 - 4\mathcal{X}\mathcal{O}_{V_k}] + 2\mathcal{L}_k K_{V_k}\,. \tag{4.2}$$

Moreover, in each blow-up $\alpha_j : V_j \longrightarrow V_{j-1}$ we get

$$\mathcal{X}\mathcal{O}_{V_j} = \mathcal{X}\mathcal{O}_{V_{j-1}}; \quad K_{V_j} = \alpha_j^* K_{V_{j-1}} + E_j; \quad \mathcal{L}_j = \alpha_j^* \mathcal{L}_{j-1} - m_{j-1}E_j\,.$$

Then

$$\begin{aligned} 2[K_{V_j}^2 - 4\mathcal{X}\mathcal{O}_{V_j}] + 2\mathcal{L}_j K_{V_j} &= 2[K_{V_{j-1}}^2 - 4\mathcal{X}\mathcal{O}_{V_{j-1}}] + \\ 2\mathcal{L}_{j-1}K_{V_{j-1}} + 2(m_{j-1}-1) &\geq 2[K_{V_{j-1}}^2 - 4\mathcal{X}\mathcal{O}_{V_{j-1}}] + 2\mathcal{L}_{j-1}K_{V_{j-1}} \end{aligned} \tag{4.3}$$

Finally as $\tau : V \longrightarrow B$ is an elliptic minimal fibration, numerically we have $K_V \equiv \left[2(b-1) + \mathcal{X}\mathcal{O}_V + \sum_i \frac{(n_i-1)}{n_i}\right] E$ ([7] p. 162) where E denotes a smooth fibre of τ and $\{n_i\}$ are the multiplicities of singular fibres of τ. In particular $K_V^2 \equiv 0$.

As $\mathcal{L}_0^{\otimes 2} = \mathcal{O}_{V_0}(R_0)$ and R_0 is the branch divisor of π_0 we get $\mathcal{L}_0 E = (g-1)$ by Hurwitz formula. So

$$\begin{aligned} 2[K_{V_0}^2 - 4\mathcal{X}\mathcal{O}_{V_0}] + 2\mathcal{L}_0 K_{V_0} &= -8\mathcal{X}\mathcal{O}_{V_0} + 2\mathcal{L}_0 E \left[2(b-1) + \mathcal{X}\mathcal{O}_{V_0} + \sum_i \frac{(n_i-1)}{n_i}\right] \\ &\geq 4(b-1)(g-1) + 2(g-5)\mathcal{X}\mathcal{O}_V \end{aligned}$$

Then (i) follows from this, (4.1), (4.2), (4.3) and from the fact that $\mathcal{X}\mathcal{O}_V \geq 0$ for elliptic fibrations.

(ii) Looking at the proof of (i) we see that $\lambda = 4$ if and only if $\mathcal{X}\mathcal{O}_V = 0$ and equality holds in all the above inequalities. So we have $\lambda = 4$ if and only if S is the minimal desingularization of a double cover of an elliptic, relatively minimal, fibration $\tau : V \longrightarrow B$ such that:

- τ has no multiple fibres ($\forall\, i \quad n_i = 1$).
- $\mathcal{X}\mathcal{O}_V = 0$.
- The branch divisor R_0 of the double cover has only *negligeable singularities* (see [13], [17]), i.e. all the multiplicities of the singularities of the branch divisors in the process of canonical resolution are 2 or 3.

But the first two conditions are equivalent to the fact that τ is smooth and isotrivial ([83] thms. 6,7 Ch.IV). This allows us to construct examples with $\lambda(f) = 4$ which are essentially the same as in [92] Example 4.3. So the bound is sharp. ■

Remark 4.4 Although we cannot use double covers for the case of bielliptic fibrations of genus 5 we already know that $\lambda \geq 4$ also holds for such fibrations (see [63] Thm.5.1, [66]).

4.2 The slope of double covers

4.2.1 Slope

Definition 4.5 Let $f : S \longrightarrow B$ be a relatively minimal fibration of genus g. We say that f is a *double cover fibration* if there exists a relatively minimal fibration $\tau : V \longrightarrow B$ and a rational map $\pi : S - - - \rightarrow V$ over B which is a generically two to one map.

We will call F, E the fibres of f and τ respectively and $g = g(F)$, $h = g(E)$.

Remark 4.6 If f is a double cover fibration of τ, then F is a double cover of E for general F and E, but the converse is not true as pointed out in Example 4.2. Nevertheless the converse is true if the involution that produces the double cover is unique for a fixed h (see Theorem 1.23). The following lemma shows that this always happens if $g \gg h$.

Lemma 4.7 *Let F be a smooth curve of genus g. Let $h \in \mathbb{N}$ such that $g \geq 4h+2$. Then F has at most one involution ι such that the genus of $E = F/ < \iota >$ is h.*

PROOF: Assume there exist two involutions ι_1 and ι_2 as in the statement. Let $E_i = F/ < \iota_i >$. Consider the natural maps $\sigma_i : F \longrightarrow E_i$, $\sigma = \sigma_1 \times \sigma_2 : F \longrightarrow E_1 \times E_2$. Let $\pi_i : E_1 \times E_2 \longrightarrow E_i$ the projections. Let $\overline{F} = \sigma(F)$. Since $\deg(\pi_i \circ \sigma) = 2$, we have that $\deg\sigma = 1$ or 2.

Assume $\deg\sigma = 2$. Then $E_1 \cong \overline{F} \cong E_2$ and $\overline{F}$ is the graph of an automorphism φ of $E_1 = E_2$. Then clearly $\sigma_1 = \sigma_2 \circ \varphi$ and $\iota_1 = \iota_2$.

Assume $\deg\sigma = 1$. Then F is the desingularization of $\overline{F}$. Note that $K_{E_1 \times E_2} \equiv (2h-2)(L_1 + L_2)$ where $L_i = \pi_i^{-1}(t_i)$ for $t_i \in E_i$. We have $\overline{F}L_i = 2$;

hence

$$\overline{F}^2 \leq \frac{((L_1+L_2)\overline{F})^2}{(L_1+L_2)^2} = 8$$

by [43] V 1.9 (b). Now adjunction yields

$$2g-2 \leq 2p_a(\overline{F})-2 = (K_{E_1\times E_2}+\overline{F})\overline{F} = 8h-8+8 = 8h$$

which is impossible if $g \geq 4h+2$. ■

Let f be a double cover fibration. Following notations of §4.1.2 we have

$$\widetilde{S}[r]^{\widetilde{\pi}}[d]^{\sigma}\widetilde{V}[d]^{\eta}S[r]^{\pi}[d]^{f}V[dl]^{\tau}B$$

where τ and π exist by definition, π is a generically 2-to-1 rational map and τ is a relatively minimal fibration; $\eta : \widetilde{V} \longrightarrow V$ and $\sigma : \widetilde{S} \longrightarrow S$ are any birational maps such that the induced rational map $\widetilde{\pi} = \eta^{-1}\circ\pi\circ\sigma$ is a morphism. It is important to remark that η and σ can be chosen to resolve other maps of S or V if necessary. Let $\widetilde{f} = f\circ\sigma$ and $\widetilde{\tau} = \tau\circ\eta$. Note that at general $t\in B$ $f^{-1}(t) = \widetilde{f}^{-1}(t)$, $\tau^{-1}(t) = \widetilde{\tau}^{-1}(t)$.

The map $\eta\circ\widetilde{\pi}$ factorizes by Stein Theorem as $\eta\circ\widetilde{\pi} = \pi_0\circ u$, where π_0 is finite and u is birational. Let D be the branching divisor of π_0 and $\mathcal{L}\in Pic(V)$ such that $\mathcal{L}^{\otimes 2} = \mathcal{O}_V(D)$. By standard theory of double coverings (cf. [7], p.182) we have that

$$\begin{aligned}\mathcal{E} &= f_*\omega_{S/B} = \widetilde{f}_*\omega_{\widetilde{S}/B} = \tau_*((\eta\circ\widetilde{\pi})_*\omega_{\widetilde{S}/B}) = \tau_*(\omega_{V/B}\oplus(\omega_{V/B}\otimes\mathcal{L})) = \\ &= \tau_*\omega_{V/B}\oplus\tau_*(\omega_{V/B}\otimes\mathcal{L})\end{aligned}$$

From now on we set

$$\mathcal{F} = \tau_*\omega_{V/B}$$

$$\mathcal{G} = \tau_*(\omega_{V/B}\otimes\mathcal{L})$$

and call $\{\mu_i, r_i, d_i\}_{1\leq i\leq \ell_1}$, $\{\overline{\mu}_i, \overline{r}_i, \overline{d}_i\}_{1\leq i\leq \ell_2}$, the Harder-Narshimann data of $\mathcal{F}$ and $\mathcal{G}$ respectively (see §1.2). Note that $r_{\ell_1} = \mathrm{rk}\mathcal{F} = h = g(E)$, $\overline{r}_{\ell_2} = \mathrm{rk}\mathcal{G} = g-h$. Call

$$\chi_1 = \deg\mathcal{F}$$

$$\chi_2 = \deg\mathcal{G}.$$

We have $\chi_f = \deg\mathcal{E} = \chi_1+\chi_2$.

The following is a useful result that relates the Harder-Narasimhan filtrations of decompositions as above.

Proposition 4.8 *Let $\mathcal{F}, \mathcal{G}, \mathcal{E}$ locally free sheaves on a smooth curve B. Let $0 = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \ldots \subseteq \mathcal{F}_{\ell_1} = \mathcal{F}$, $0 = \mathcal{G}_0 \subseteq \mathcal{G}_1 \subseteq \ldots \subseteq \mathcal{G}_{\ell_2} = \mathcal{G}$, $0 = \mathcal{E}_0 \subseteq \mathcal{E}_1 \subseteq \ldots \subseteq \mathcal{E}_\ell = \mathcal{E}$ its Harder-Narasimhan filtrations. Let $\{\mu_i\}_{1\leq i\leq \ell_1}$, $\{\overline{\mu}_i\}_{1\leq i\leq \ell_2}$, $\{\gamma_i\}_{1\leq i\leq \ell}$ its Harder-Narshimann slopes. Assume $\mathcal{E} = \mathcal{F} \oplus \mathcal{G}$. Then we can define $\psi : \{0, \ldots, \ell\} \longrightarrow \{0, \ldots, \ell_1\}$, $\varphi : \{0, \ldots, \ell\} \longrightarrow \{0, \ldots, \ell_2\}$ such that*

(i) $\psi(0) = \varphi(0) = 0$; for $1 \leq i \leq \ell \quad \psi(i) = \psi(i-1)$ if $\mu_k \neq \gamma_i$ for every $k \in \{1, \ldots, \ell_1\}$ (respectively, $\varphi(i) = \varphi(i-1)$ if $\overline{\mu}_k \neq \gamma_i$ for every $k \in \{1, \ldots, \ell_2\}$) and $\psi(i) = k$ if $\gamma_i = \mu_k$ (respectively, $\varphi(i) = k$ if $\gamma_i = \overline{\mu}_k$).

(ii) $\mathcal{E}_i = \mathcal{F}_{\psi(i)} \oplus \mathcal{G}_{\varphi(i)}$

Proof: Call $\pi_1 : \mathcal{E} \longrightarrow \mathcal{F}$, $\pi_2 : \mathcal{E} \longrightarrow \mathcal{G}$ the natural projections. Let $\mathcal{E}_1^1 = \pi_1(\mathcal{E}_1)$, $\mathcal{E}_1^2 = \pi_2(\mathcal{E}_1)$; both are locally free since they are torsion free ($\mathcal{E}_1^1 \subseteq \mathcal{F}$, $\mathcal{E}_1^2 \subseteq \mathcal{G}$). We have $\mathcal{E}_1 \subseteq \mathcal{E}_1^1 \oplus \mathcal{E}_1^2$.

Assume $\mathcal{E}_1^1 \neq 0$. Since $\mathcal{E}_1$ is semistable and $\mathcal{E}_1^1$ is a quotient, we have that $\mu(\mathcal{E}_1^1) \geq \mu(\mathcal{E}_1) = \gamma_1$. From the inclusions $\mathcal{E}_1^1 \subseteq \mathcal{F} \subseteq \mathcal{E}$ we get $\gamma_1 \leq \mu(\mathcal{E}_1^1) \leq \mu_1 \leq \gamma_1$ since $\mathcal{F}_1, \mathcal{E}_1$ are the maximal unstabilizing sheaves in $\mathcal{F}$ and $\mathcal{E}$ respectively. Hence $\gamma_1 = \mu_1$ and $\mathcal{E}_1^1 \subseteq \mathcal{F}_1 \subseteq \mathcal{E}_1$ by the maximality of $\mathcal{F}_1$ and of $\mathcal{E}_1$ (see Theorem 1.6). The same argument works if $\mathcal{E}_1^2 \neq 0$.

Assume $\overline{\mu}_1 \neq \gamma_1$. Then necessarily $\mathcal{E}_1^2 = 0$ and $\mathcal{E}_1^1 \neq 0$. Hence $\mu_1 = \gamma_1$, $\mathcal{E}_1 \subseteq \mathcal{E}_1^1 \subseteq \mathcal{F}_1$ and then $\mathcal{E}_1 = \mathcal{F}_1 = \mathcal{F}_1 \oplus \mathcal{G}_0$ by maximality. The same argument works if $\mu_1 \neq \gamma_1$.

Assume $\mu_1 = \overline{\mu}_1 = \gamma_1$. Then $\mathcal{E}_1 \subseteq \mathcal{E}_1^1 \oplus \mathcal{E}_1^2 \subseteq \mathcal{F}_1 \oplus \mathcal{G}_1$ with $\mu(\mathcal{F}_1 \oplus \mathcal{G}_1) = \gamma_1$. Again by maximality of $\mathcal{E}_1$ we conclude $\mathcal{E}_1 = \mathcal{F}_1 \oplus \mathcal{G}_1$.

The proof concludes by induction dealing with $\mathcal{E}/\mathcal{E}_1 = \mathcal{F}/\mathcal{F}_{\psi(1)} \oplus \mathcal{G}/\mathcal{G}_{\varphi(1)}$. ■

Corollary 4.9 *With the above notations we have*

$$\max\{\mu_1, \overline{\mu}_1\} = \gamma_1$$
$$\min\{\mu_{\ell_1}, \overline{\mu}_{\ell_2}\} = \gamma_\ell$$

Proof: Obvious. ■

We are interested in finding a lower bound of the slope of double covers depending on g and h. In order to get this note first that if the cover is étale $\lambda(f)=\lambda(\tau)$ (cf. [92]). Since then $\mathcal{L}\in Pic^0(V)$ and $\mathcal{L}\notin\tau^*Pic^0(B)$ (otherwise the double cover with data $(\mathcal{L},\mathcal{O}_V)$ will not have connected fibres over B) we have $q(V)>b$ and hence $\lambda(f)=\lambda(\tau)\geq 4$ (see §4.3) and so $\lambda(f)=4$ can always be achieved. From now on we assume the double cover is not étale.

For $h=0$ (hyperelliptic case) the general bound $\lambda(f)\geq 4\frac{g-1}{g}$ holds and is attained (cf. [92]). For $h=1$ (bielliptic case) we got in §4.1 $\lambda(f)\geq 4=4\frac{g-1}{g-1}$ and also the bound is attained. Note that in both cases formula $\lambda(f)\geq 4\frac{g-1}{g-h}$ holds. Moreover in both cases fibrations verifying $\lambda(f)=4\frac{g-1}{g-h}$ are the easiest ones: they are double covers of isotrivial fibrations ramified along a divisor with negligeable singularities. In [92] Xiao constructs examples of this easy double covers for any h and proves that for them, equality $\lambda(f)=4\frac{g-1}{g-h}$ holds (see Example 4.29).

From all these considerations it seems natural to ask whether $\lambda(f)\geq 4\frac{g-1}{g-h}$ holds for double cover fibrations. There is even an additional reason to consider $4\frac{g-1}{g-h}$. Starting with an arbitrary double cover fibration we can construct a family of new double cover fibrations with the same general fibre (i.e., the modular image coincides) and slope as close to $4\frac{g-1}{g-h}$ as wanted. In fact it is easy to prove that the sequence of slopes $\{\lambda_n\}$ is monotonous, so we could prove $\lambda(f)\geq 4\frac{g-1}{g-h}$ if we prove that the sequence is decreasing.

Indeed, consider a double cover fibration

$$\widetilde{S}[r]^u[d]^\sigma S_0[r]^{\pi_0}V[ddll]^\tau S[d]^f B$$

where S_0 is normal and π_0 finite. Note that $\omega_{S_0/B}=\pi_0^*(\omega_{V/B}\otimes\mathcal{L})$ and $\omega_{\widetilde{S}/B}=\widetilde{\pi}^*(\omega_{V/B}\otimes\mathcal{L})\otimes\mathcal{O}_{\widetilde{S}}(E)$, where E is an effective divisor contracted by u on the singularities of S_0. Remember also that the singularities of S_0 lie over the singularities of the branching divisor D.

Then we have

$$\begin{aligned}
K^2_{S/B} &\geq K^2_{\widetilde{S}/B}=K^2_{S_0/B}+E^2=2(K_{V/B}+\tfrac{1}{2}D)^2+E^2\\
\chi\mathcal{O}_S &=\chi\mathcal{O}_{\widetilde{S}}=\chi\mathcal{O}_{S_0}-h^0(S_0,R^1u_*\mathcal{O}_{\widetilde{S}})=\chi(V,\mathcal{O}_V\oplus\mathcal{L}^{-1})-h^0(S_0,R^1u_*\mathcal{O}_{\widetilde{S}})=\\
&=\chi\mathcal{O}_V+\chi(V,\mathcal{L}^{-1})-h^0(S_0,R^1u_*\mathcal{O}_{\widetilde{S}})
\end{aligned}$$

where $h^0(S_0, R^1u_*\mathcal{O}_{\widetilde{S}})$ depends only on the non-rational contracted curves and $R^1u_*\mathcal{O}_{\widetilde{S}}$ is torsion, with support on the singularities of S_0.

We are going to associate to f a new fibration f_n (for $n \gg 0$) which is again a double cover of τ. For this take $n \gg 0$ such that $|2np|$ moves in B ($p \in B$), and let $p_1 + \ldots + p_{2n} \in |2np|$ a smooth member. Set $\mathcal{L}_n = \mathcal{L} \otimes \mathcal{O}_V(nE_p)$ and consider $D_n = D + E_1 + \ldots + E_{2n}$ such that $\mathcal{L}_n^{\otimes 2} = \mathcal{O}_V(D_n)$. We can consider that the fibres $E_1, \ldots, E_{2n}$ cut transversally the divisor D.

We can then consider the surface $S_{0,n}$ which is the double cover of V with the data $(\mathcal{L}_n, D_n)$. Let $\widetilde{S}_n$ be its minimal desingularization and let S_n be a relatively minimal model over B.

Observe that, by construction, the divisor D_n has the singularities of D plus the double points produced by the transversal intersection of D with E_i. Double points are negligeable singularities and hence, they do not have numerical contribution for $K^2_{\widetilde{S}_n}$ and $\chi\mathcal{O}_{\widetilde{S}_n}$.

Then we have

$$\begin{aligned} K^2_{S_n/B} &\geq K^2_{\widetilde{S}_n/B} = K^2_{S_{0,n}/B} + E^2 = 2(K_{V/B} + \tfrac{1}{2}D_n)^2 + E^2 = \\ &= 2(K_{V/B} + \tfrac{1}{2}D)^2 + E^2 + n(2ED + 4K_{V/B}E) \end{aligned}$$

$$\begin{aligned} \chi\mathcal{O}_{S_n} &= \chi\mathcal{O}_{\widetilde{S}_n} = \chi\mathcal{O}_{S_{0,n}} - h^0(S_0, R^1u_*\mathcal{O}_{\widetilde{S}}) = \\ &= \chi(V, \mathcal{O}_V \oplus \mathcal{L}_n^{-1}) - h^0(S_0, R^1u_*\mathcal{O}_{\widetilde{S}}) = \chi\mathcal{O}_S + n(\tfrac{1}{2}ED + \tfrac{1}{2}K_VE) \end{aligned}$$

Let $f_n : S_n \longrightarrow B$ be the induced fibration. Note that f_n and f have the same general fibre. Hence $\chi_{f_n} = \chi\mathcal{O}_{S_n} - (b-1)(g-1)$. Assume f is not locally trivial. Hence f_n is not locally trivial and we have a well defined slope $\lambda_n = \lambda(f_n)$. Define

$$\lambda_\infty = \lim_{n\to\infty}\lambda_n = \frac{2ED + 4K_{V/B}E}{\frac{1}{2}ED + \frac{1}{4}K_VE} = \frac{2ED + 4K_VE}{\frac{1}{2}ED + \frac{1}{2}K_VE}$$

Observe that F is a double cover of E ramified over $D_{|E}$ so $DE = 2g - 2 - 4(h-1) = 2g - 4h + 2$. Since $K_{V|E} = K_E$ by adjunction on V we have $K_VE = 2h - 2$. Hence

$$\lambda_\infty = 4\frac{g-1}{g-h}.$$

Unfortunately it is not true that $\lambda(f) \geq 4\frac{g-1}{g-h}$ as we will see in two families of examples in §4.5 (see Example 4.30 and Example 4.31). Nevertheless the counterexamples found seem to be very special, since they are double covers of double covers and always verify $g \leq 4h$. We ignore whether the *expected bound* $\lambda\mathrm{exp} = 4\frac{g-1}{g-h}$ holds under additional (and, of course, general) hypotheses.

Instead, we can get *assimptotically* sharp bounds (when $g >> h$) for double cover fibrations. In general we can at least prove that $\lambda(f) \geq 4$ under some mild extra hypotheses.

As a first step consider the following result. The first part is due to Konno (cf. [67]).

Proposition 4.10 *Let $f : S \longrightarrow B$ be a genus g, relatively minimal, non isotrivial fibration, which is a double cover of a genus $h \geq 1$ fibration $\tau : V \longrightarrow B$. With the preceding notations, we have*

(i) $K^2_{S/B} \geq 4\chi_f - 4(\mu_1 + \mu_{\ell_1}) + 2(g-2h+1)max\{\frac{\mu_1}{h}, \mu_{\ell_1}\}$

(ii) If $g \geq 2h+1$ *then* $K^2_{S/B} \geq 8\frac{g(g-1)}{g^2+g-1}\chi_1$

(iii) If $g \geq 2h+1$ *then* $K^2_{S/B} \geq 4\frac{g-1}{g-h}\chi_2$

Proof: (i) (Cf. [67]) Keeping the notations of §4.1.2 we have from (4.1), (4.2), and (4.3) that

$$K^2_{S/B} - 4\chi_f \geq 2(K^2_{V/B} - 4\chi_1) + K_{V/B}R$$

where $R = R_0$ is the branch divisor of $S_0 \longrightarrow V_0 = V$.

By Proposition 1.10 we have a nef $\mathbb{Q}$-divisor N_1 and an effective divisor Z_1 in V such that $K_{V/B} \equiv N_1 + \mu_1 E + Z_1$. Let $R = R_h + R_v$ the decomposition of R in its horizontal and vertical part respectively. Let $R_h = C_1 + \ldots + C_m$ the decomposition in irreducible components (note that R is reduced since S_0 is normal). Let n_i be the multiplicity of C_i in Z_1. Then

$$\sum_{i=1}^{m} n_i C_i E \leq Z_1 E \leq 2(h-1) \tag{4.4}$$

since E is nef and $Z_1 \leq K_{V/B}$.

Hurwitz formula yields

$$2(g-2h+1) = R_h E = \sum_{i=1}^{m} C_i E \tag{4.5}$$

By construction

$$(n_i+1)K_{V/B}-\mu_1 E \equiv n_i(K_{V/B}+C_i)+N_1+(Z_1-n_iC_i)$$

an so we have that

$$((n_i+1)K_{V/B}-\mu_1 E)C_i \geq 0 \tag{4.6}$$

since $(K_{V/B}+C_i)C_i \geq 0$ (Hurwitz formula), $N_1C_i \geq 0$ (N_i is nef) and $(Z_1-n_iC_i)C_i \geq 0$ (C_i is not a component of $Z_1-n_iC_i$).

Claim. $K_{V/B}R \geq \frac{2(g-2h+1)}{h}\mu_1$

Proof of the Claim. We can assume $n_1 \geq n_2 \geq \ldots \geq n_m \geq 0$.

If $h-1 \geq n_1$ ($\geq n_i$ for all i) we have that $(hK_{V/B}-\mu_1E)C_i \geq 0$ by (4.6) since $K_{V/B}$ is nef.

Assume $h \leq n_1$. Since $n_1C_1E \leq 2(h-1)$ we must have $C_1E=1$. Note that (4.4) gives $n_i \leq 2h-2-n_1$ for $i \geq 2$. Hence, using (4.5) and (4.6) we have

$$\begin{aligned} K_{V/B}R_h \geq \mu_1 \sum_{i=1}^{m} \tfrac{1}{n_i+1}C_iE \quad &\geq \mu_1\left(\tfrac{C_1E}{n_1+1}+\tfrac{(R_h-C_1)E}{2h-1-n_1}\right)= \\ &=\mu_1\left(\tfrac{1}{n_1+1}+\tfrac{2g-4h+1}{2h-1-n_1}\right) \geq \mu_1\tfrac{2(g-2h+1)}{h} \end{aligned}$$

since $n_1 \geq h$. This proves the Claim.

Finally, since $K_{V/B}-\mu_nE$ is nef (Theorem 1.8) we have by (4.5)

$$K_{V/B}R \geq 2(g-2h+1)\mu_{\ell_1}$$

In [92] p. 460 Xiao gives the following bound for any fibration

$$K^2_{V/B} \geq 4\chi_1 - 2(\mu_1+\mu_{\ell_1})$$

So

$$K^2_{S/B}-4\chi_f \geq -4(\mu_1+\mu_{\ell_1})+2(g-2h+1)\max\{\frac{\mu_1}{h},\mu_{\ell_1}\}.$$

(ii) Consider the Harder-Narasimhan decomposition of $\mathcal{F}=\tau_*\omega_{V/B}$

$$0=\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \ldots \subseteq \mathcal{F}_{\ell_1}=\mathcal{F}$$

with slopes $\mu_1 > \mu_2 > \ldots > \mu_{\ell_1} \geq 0$ and ranks $\{r_i\}$. By Remark 1.9, since $\mathcal{F} \subseteq \mathcal{E}$, each $\mathcal{F}_i$ induces a nef $\mathbb{Q}$-divisor N_i and an effective divisor Z_i on a suitable birational model $\sigma : \widehat{S} \longrightarrow S$, such that for $i \neq j$,

$$N_i + \mu_i F + Z_i \equiv N_j + \mu_j F + Z_j \equiv \sigma^* K_{S/B}$$

Every N_i induces a linear system $|P_i|$ on F of degree e_i and (projective) dimension at least $r_i - 1$. Then, by Proposition 1.10 and arguing as in Proposition 1.11 we get

$$K_{S/B}^2 = (\sigma^* K_{S/B})^2 \geq \sum_{i=1}^{\ell_1} (e_i + e_{i+1})(\mu_i - \mu_{i+1}) \tag{4.7}$$

where we define $N_{\ell_1+1} = \sigma^* K_{S/B}$, $\mu_{\ell_1+1} = 0$, $Z_{\ell_1+1} = 0$ and hence $e_{\ell_1+1} = 2g - 2$.

By a result of Konno (cf. [64], p.680) we have

$$K_{S/B}^2 \geq \frac{4g(g-1)}{2g-1} \gamma_1 \tag{4.8}$$

Let F and E be the general smooth fibres of f and τ respectively (note that f and $\widetilde{f} = f \circ \sigma$ have the same general fibre). Let $\alpha : F \longrightarrow E$ be the induced double cover. We have a natural decomposition

$$H^0(F, \omega_F) = H^0(E, \omega_E) \oplus H^0(E, \omega_E \otimes \mathcal{L}_{|E})$$

For $i \leq \ell_1$ every $\mathcal{F}_i$ induces as above a linear system $|Q_i|$ on E of dimension greater or equal to $r_i - 1$ and degree d_i. By construction we have in fact that $|P_i| = |\alpha^* Q_i|$ and hence that $e_i = 2d_i$.

Since $|Q_i|$ is a linear subsystem of $|K_E|$ we can apply Clifford's Lemma and get that $d_i \geq 2r_i - 2$. Hence, for $i \leq \ell_1 - 1$ we have $e_i + e_{i+1} \geq 8r_i - 4$. For $i = \ell_1$, $e_{\ell_1} + e_{\ell_1+1} \geq (4r_{\ell_1} - 4) + 2g - 2 = 4h + 2g - 6 \geq 8h - 4$ since $g \geq 2h + 1$ by hypothesis.

Hence (4.7) reads

$$K_{S/B}^2 \geq 8 \sum_{i=1}^{\ell_1} r_i(\mu_i - \mu_{i+1}) - 4\mu_1 = 8\chi_1 - 4\mu_1 \geq 8\chi_1 - 4\gamma_1$$

since $\mu_1 = \mu_1(\mathcal{F}) \leq \gamma_1$. Eliminating γ_1 from (4.8) we finally get

$$K^2_{S/B} \geq 8\frac{g(g-1)}{g^2+g-1}\chi_1$$

(iii) As in (ii) consider now the Harder-Narasimhan decomposition of $\mathcal{G} = \tau_*(\omega_{V/B} \otimes \mathcal{L})$

$$0 = \mathcal{G}_0 \subseteq \mathcal{G}_1 \subseteq \ldots \subseteq \mathcal{G}_{\ell_2} = \mathcal{G}$$

with slopes $\overline{\mu}_1 > \overline{\mu}_2 > \ldots \geq \overline{\mu}_{\ell_2} \geq 0$ and ranks $\{\overline{r}_i\}$. we can reproduce exactly the same argument since $\mathcal{G} \subseteq \mathcal{E} = f_*\omega_{S/B}$, and get

$$K^2_{W/B} \geq 2\sum_{i=1}^{\ell_2}(\overline{d}_i + \overline{d}_{i+1})(\overline{\mu}_i - \overline{\mu}_{i+1}) \tag{4.9}$$

where $\overline{d}_i$ is the degree of a linear system $|\overline{Q}_i|$ on E of dimension greater or equal to $\overline{r}_i - 1$. The only difference is that now $|\overline{Q}_i|$ is a linear subsystem of $|K_E + L_E|$ (where $\mathcal{L}_{|E} = \mathcal{O}_E(L_E)$). So, in order to bound $\overline{d}_i$, we can not apply always Clifford's Lemma. In fact we have

$$\begin{array}{ll} \overline{d}_i \geq 2\overline{r}_i - 2 & \text{if } \ \overline{r}_i \leq h \\ \overline{d}_i = \overline{r}_i + h - 1 & \text{if } \ \overline{r}_i \geq h+1 \end{array}$$

by Clifford's Lemma and Riemann-Roch on E.

Note that $\overline{r}_{\ell_2} = \text{rank}\mathcal{G} = \text{rank}\mathcal{E} - \text{rank}\mathcal{F} = g - h$. Hence, if $g - h \geq h + 1$ (i.e., if $g \geq 2h + 1$) we get

$$\forall i \quad 1 \leq i \leq \ell_2 \quad \overline{d}_i \geq \frac{g-1}{g-h-1}(\overline{r}_i - 1) \tag{4.10}$$

just looking in the $(\overline{r}, \overline{d})$-plane the bounds given above. The line $\overline{d} = \frac{g-1}{g-h-1}(\overline{r} - 1)$ passes through (1,0) and $(g-h, g-1)$.

Note that in this case we can define $\overline{N}_{\ell_2+1} = \overline{N}_{\ell_2}$ or $\overline{N}_{\ell_2+1} = \sigma^* K_{S/B}$. Both possibilities give $\overline{e}_i = 2g - 2$. It follows from (4.10) that

$$\begin{array}{rll} \overline{d}_i + \overline{d}_{i+1} & \geq 2\frac{g-1}{g-h-1}\overline{r}_i - \frac{g-1}{g-h-1} & \text{for } i \leq \ell_2 - 1 \\ \overline{d}_{\ell_2} + \overline{d}_{\ell_2+1} = 2\overline{d}_{\ell_2} & \geq 2\frac{g-1}{g-h-1}\overline{r}_{\ell_2} - 2\frac{g-1}{g-h-1} & \end{array}$$

Hence (4.9) reads

$$K_{S/B}^2 \geq 2\sum_{i=1}^{\ell_2}(\overline{d}_i+\overline{d}_{i+1})(\overline{\mu}_i-\overline{\mu}_{i+1}) \geq \sum_{i=1}^{\ell_2}\left(4\tfrac{g-1}{g-h-1}\overline{r}_i - 2\tfrac{g-1}{g-h-1}\right)(\overline{\mu}_i-\overline{\mu}_{i+1})$$

$$-2\tfrac{g-1}{g-h-1}\overline{\mu}_{\ell_2} = 4\tfrac{g-1}{g-h-1}\chi_2 - 2\tfrac{g-1}{g-h-1}(\overline{\mu}_1+\overline{\mu}_{\ell_2})$$

Using the indices $\{i_1,i_2\}=\{1,\ell_2\}$ in Proposition 1.11 (see Remark 1.12) we get

$$K_{S/B}^2 \geq 2\left[(\overline{d}_1+\overline{d}_{\ell_2})(\overline{\mu}_1-\overline{\mu}_{\ell_2})+(\overline{d}_{\ell_2}+\overline{d}_{\ell_2+1})(\overline{\mu}_{\ell_2})\right] \geq 2\overline{d}_{\ell_2}(\overline{\mu}_1+\overline{\mu}_{\ell_2}) = 2(g-1)(\overline{\mu}_1+\overline{\mu}_{\ell_2})$$

which produces, eliminating $(\overline{\mu}_1+\overline{\mu}_{\ell_2})$

$$K_{S/B}^2 \geq 4\frac{g-1}{g-h}\chi_2$$

■

Summing up we can give the main result

Theorem 4.11 *Let $f: S \longrightarrow B$ be a genus g, relatively minimal, non isotrivial fibration. Assume f is a double cover fibration of a fibration τ of genus $h \geq 1$ (if $g \geq 4h+2$ just assume the general fibre F is a double cover of a smooth curve E of genus h).*

(i) If $g \geq 4h+1$ then

$$\lambda(f) \geq 4 + \frac{4(h-1)(g-4h-1)}{(g-h)(g-4h-1)+2(g-1)h^2} \geq 4$$

(ii) If $g \geq 2h+1$ and we set $\delta = 2(\lambda(\tau)-4)+\frac{2(g-2h+1)}{h^2}$ then

$$\lambda(f) \geq 4\frac{(g-1)}{(g-h)}\frac{(\delta+4)}{(\delta+4\frac{(g-1)}{(g-h)})}.$$

PROOF: If $g \geq 4h+2$, the double cover of F onto curves of genus h is unique by Lemma 4.7. Hence we have that $f: S \longrightarrow B$ is a double cover of a genus h fibration $\tau: V \longrightarrow B$ by Remark 4.6.

(i) From Proposition 4.10 (i), looking independently to the cases $\frac{\mu_1}{h} \geq \mu_{\ell_1}$ and $\mu_{\ell_1} \geq \frac{\mu_1}{h}$ we always get

$$K^2_{S/B} - 4\chi_f \geq \frac{2(g-4h-1)}{h}\mu_1 \geq \frac{2(g-4h-1)}{h^2}\chi_1$$

since $g-4h-1 \geq 0$, $\mu_1 = \mu_1(\mathcal{F}) \geq \frac{\deg \mathcal{F}}{\mathrm{rk}\mathcal{F}} = \frac{\chi_1}{h}$ ($\mathcal{F}_1$ is the maximal unstabilizing subsheaf of $\mathcal{F}$).

Using $\chi_1 + \chi_2 = \chi_f$ we also have from Proposition 4.10 (ii) and (iii)

$$\begin{aligned} K^2_{S/B} &\geq 8\tfrac{g(g-1)}{g^2+g-1}\chi_1 \\ K^2_{S/B} &\geq 4\tfrac{g-1}{g-h}\chi_f - 4\tfrac{g-1}{g-h}\chi_1 \end{aligned}$$

Consider in the (x,y)-plane the three inequalities

$$\begin{aligned} y &\geq 4\chi_f + \tfrac{2(g-4h-1)}{h^2}x = f_1(x) \\ y &\geq 8\tfrac{g(g-1)}{g^2+g-1}x = f_2(x) \\ y &\geq 4\tfrac{g-1}{g-h}\chi_f - 4\tfrac{g-1}{g-h}x = f_3(x) \end{aligned}$$

Independently of x we have that $y \geq f_1(a)$ where a verifies $f_1(a) = f_3(a)$ and so

$$a = \frac{4(h-1)h^2}{4(g-1)h^2 + 2(g-4h-1)(g-h)}\chi_f$$

Hence

$$K^2_{S/B} \geq \left(4 + \frac{4(h-1)(g-4h-1)}{(g-h)(g-4h-1) + 2(g-1)h^2}\right)\chi_f$$

(ii) From the proof of Proposition 4.10 (i) we obtain:

$$K^2_{S/B} - 4\chi_f \geq 2(K^2_{V/B} - 4\chi_1) + K_{V/B}R = 2(\lambda(\tau) - 4)\chi_1 + K_{V/B}R$$

$$K_{V/B}R \geq \frac{2(g-2h+1)}{h}\mu_1 \geq \frac{2(g-2h+1)}{h^2}\chi_1$$

and from (iii):

$$K^2_{S/B} \geq 4\frac{g-1}{g-h} = 4\frac{g-1}{g-h}\chi_f - 4\frac{g-1}{g-h}\chi_1$$

and hence the conclusion follows by dividing by χ_f and eliminating the quotient χ_1/χ_f. ■

Remark 4.12 As a particular case of Theorem 4.11 (i), we obtain the bound $\lambda(f) \geq 4$ for bielliptic fibrations of genus $g \geq 5$ of Theorem 4.3. Note that in fact under the assumption $g \geq 4h+1$ we only can get $\lambda(f) = 4$ if $h = 1$ (so bielliptic fibrations) or $g = 4h+1$. Observe also that from Theorem 4.11 (ii) we obtain a natural influence of $\lambda(\tau)$ in $\lambda(f)$.

With a bit more care we can get more information about Problem 3 (see introduction) in the case of double covers:

Theorem 4.13 *Let $f : S \longrightarrow B$ be a relatively minimal, non locally trivial double cover fibration of $\tau : V \longrightarrow B$. Let F and E be general fibres of f and τ respectively and let $g = g(F)$, $h = g(E)$. Assume $h \geq 1$. Then*

$$\lambda(f) \geq 4$$

provided one of the following conditions hold

(i) $\lambda(\tau) \geq 4$

(ii)$g \geq 2h+11$ and F is not tetragonal.

PROOF: (i) It follows immediately from (see proof of Proposition 4.10 (i)):

$$K^2_{S/B} - 4\chi_f \geq 2(K^2_{V/B} - 4\chi_1) + K_{V/B}R$$

and the nefness of $K_{V/B}$.

(ii) Since $g \geq 2h+11$ then F is not hyperelliptic (cf. [27] V Corollary 2). We can assume F is not bielliptic since $2{\cdot}1+11 = 13 \geq 6$ and hence Theorem 4.3 applies. We can also assume F is not trigonal otherwise $\lambda(f) \geq \frac{14(g-1)}{3g+1} \geq 4$ if $g \geq 9$, using [65] Main Theorem.

Consider the Harder-Narasimhan filtration of $\mathcal{E} = f_*\omega_{S/B}$

$$0 = \mathcal{E}_0 \subseteq \mathcal{E}_1 \subseteq \ldots \subseteq \mathcal{E}_\ell = \mathcal{E}$$

with slopes $\gamma_1 > \ldots > \gamma_\ell \geq 0$. The linear systems $|R_i|$ induced by each piece on F have degree d_i and projective dimension at least $r_i - 1$ ($r_i = \mathrm{rk}\mathcal{E}_i$). Note that if $|R_i|$ induces a map φ_i we have

If $\deg\varphi_i = 1$	$d_i \geq 3r_i - 4$	(if $d_i \leq g-1$), $d_i \geq \frac{3r_i+g-4}{2}$ (otherwise)
If $\deg\varphi_i = 2$	$d_i \geq 2r_i + 2$	(since F is not hyperelliptic nor bielliptic)
If $\deg\varphi_i = 3$	$d_i \geq 3r_i$	(since F is not trigonal)
If $\deg\varphi_i \geq 4$	$d_i \geq 4(r_i - 1)$	

Observe that, since $R_i \leq R_{i+1}$, the map φ_i factorizes through φ_{i+1} and then the degree of φ_{i+1} must divide the degree of φ_i. Note also that $r_{i+1} \geq r_i + 1$ and $d_{i+1} \geq d_i$. Then we can prove that $d_i + d_{i+1} \geq 4r_i + 1$ with a few exceptions. Indeed $|R_i|$ does not define any map only if $i = 1$ $(r_1, d_1) = (1, 0)$. Then $d_2 \geq 5 = 4r_1 + 1$ except if $d_2 = 2, 3, 4$. All these possibilities imply $r_2 = 2$ according to the previous inequalities and hence F would be hyperelliptic, trigonal or tetragonal, all of these being impossible by hypothesis. From now on we assume $r_i \geq 2$.

If $\deg\varphi_i \geq 2$ then $d_i \geq 2r_i$ and hence $d_{i+1} + d_i \geq 2d_i + 1 \geq 4r_i + 1$ if $d_i < d_{i+1}$; if $d_i = d_{i+1}$ then $\varphi_i = \varphi_{i+1}$ and hence $d_i + d_{i+1} \geq 4r_i + 2$.

If $\deg\varphi_i = 1$ then also $\deg\varphi_{i+1} = 1$. If $d_i, d_{i+1} \leq g - 1$ then $d_i + d_{i+1} \geq 3r_i - 4 + 3r_{i+1} - 4 \geq 6r_i - 5 \geq 4r_i + 1$ since $r_i \geq 3$ (φ_i is birational). If $d_i \leq g - 1$, $d_{i+1} \geq g$ then $d_i + d_{i+1} \geq 2d_i + 1 \geq 6r_i - 7 \geq 4r_i + 1$ except if $r_i = 3$. But then $d_i + d_{i+1} \geq (3 \cdot 3 - 4) + g = g + 5 \geq 13 = 4r_i + 1$ since $g \geq 11$ by hypothesis.

Finally assume $d_i, d_{i+1} \geq g$, being φ_i and φ_{i+1} birational maps. Then

$$d_i + d_{i+1} \geq \frac{3r_i + g - 4}{2} + \frac{3r_{i+1} + g - 4}{2} \geq 3r_i + g - 4 + \frac{3}{2} \geq 4r_i + 1$$

if $r_i \leq g - 3$ (the case $r_i = g - 3$ needs a bit more care).

Assume $r_i = g - 2$. If $r_{i+1} = g$ then $d_{i+1} = 2g - 2$ and we are done. If $r_{i+1} = g - 1$ then the only case to check is $d_i = 2g - 5$, $d_{i+1} = 2g - 3$. Note that then $h^0(F, K_F - R_i) = 1$ since F is not hyperelliptic nor trigonal. By Riemann-Roch

$$r_i = h^0(F, R_i) = 1 + d_i + 1 - g = g - 3$$

a contradiction.

Assume $r_i = g-1$. Then $d_i = 2g-3$, $(r_{i+1}, d_{i+1}) = (r_\ell, d_\ell) = (g, 2g-2)$ and $d_i + d_{i+1} = 4g-5 = 4r_i - 1$.

For $r_i = r_\ell = g$ we have $d_\ell + d_{\ell+1} = 2d_\ell = 4g-4 = 4r_\ell - 4$.

So we conclude from Xiao's inequality (Proposition 1.11, Remark 1.12)

$$\begin{aligned} K^2_{S/B} &\geq \sum_{i=1}^{\ell}(d_i + d_{i+1})(\gamma_i - \gamma_{i+1}) \geq \\ &\geq \sum_{i=1}^{\ell}(4r_i+1)(\gamma_i - \gamma_{i+1}) - 2(\gamma_{\ell-1} - \gamma_\ell) - 5\gamma_\ell \\ &= 4\chi_f + \gamma_1 - 2\gamma_{\ell-1} - 3\gamma_\ell \end{aligned} \tag{4.11}$$

if $r_{\ell-1} = g-1$, $d_{\ell-1} = 2g-3$; otherwise

$$K^2_{S/B} \geq \sum_{i=1}^{\ell}(4r_i+1)(\gamma_i - \gamma_{i+1}) - 5\mu_\ell = 4\chi_f + \gamma_1 - 5\gamma_\ell$$

Let us consider first the general case. If $\gamma_1 \geq 5\gamma_\ell$ we are done. Assume $\gamma_1 < 5\gamma_\ell$. If $\mathcal{F} = \tau_*\omega_{V/B}$, $\mathcal{G} = \tau_*(\omega_{V/B} \otimes \mathcal{L})$ and considering the notations as in Proposition 4.8 we have that $\gamma_\ell = \min\{\mu_{\ell_1}, \overline{\mu}_{\ell_2}\} \leq \mu_{\ell_1}$, $\gamma_1 = \max\{\mu_1, \overline{\mu}_1\} \geq \mu_1$ by Corollary 4.9. Hence we have $\mu_1 \leq \gamma_1 < 5\gamma_\ell \leq 5\mu_{\ell_1}$. Then we get from Proposition 4.10 (i)

$$\begin{aligned} K^2_{S/B} &\geq 4\chi_f - 4(\mu_1 + \mu_{\ell_1}) + 2(g-2h+1)\max\{\tfrac{\mu_1}{h}, \mu_{\ell_1}\} \geq \\ &\geq 4\chi_f - 24\mu_{\ell_1} + 2(g-2h+1)\max\{\tfrac{\mu_1}{h}, \mu_{\ell_1}\}. \end{aligned}$$

If $h \geq 5$ or $\mu_{\ell_1} \geq \frac{\mu_1}{h}$ we have $\max\{\frac{\mu_1}{h}, \mu_{\ell_1}\} = \mu_{\ell_1}$ and hence

$$K^2_{S/B} \geq 4\chi_f + 2(g-2h-11)\mu_{\ell_1} \geq 4\chi_f$$

when $g \geq 2h+11$.

If $h = 2, 3, 4$ and $\mu_{\ell_1} < \frac{\mu_1}{h}$ then

$$K^2_{S/B} \geq 4\chi_f - 24\mu_{\ell_1} + 2(g-2h+1)\frac{\mu_1}{h} > 4\chi_f + \frac{2(g-2h-11)}{h}\mu_1 \geq 4\chi_f$$

when $g \geq 2h + 11$.

Consider finally the special case $r_{\ell-1} = g - 1$, $d_{\ell-2} = 2g - 3$. Remember that the Cartier divisor $R_{\ell-1}$ on F is induced by a decomposition

$$(N_{\ell-1} + \mu_{\ell-1}F) + Z_{\ell-1} = H_{\ell-1} + Z_{\ell-1} = \sigma^* K_{S/B}$$

so in this case $Z_{\ell-1}$ is a section of f, such that $Z_{\ell-1}F = K_F - R_{\ell-1}$. The base point free linear system $|H_{\ell-1}|$ on $\widetilde{S}$ is induced by the piece $\mathcal{E}_{\ell-1}$ of the Harder-Narasimhan decomposition of $\mathcal{E}$, which has rank $r_{\ell-1} = g - 1$. According to Proposition 4.8 $\mathcal{E}_{\ell-1} = \mathcal{F}_{\psi(\ell-1)} \oplus \mathcal{G}_{\varphi(\ell-1)}$. Since $\mathrm{rk}\mathcal{E}_{\ell-1} = \mathrm{rk}\mathcal{E} - 1$ we only have two possibilities: either $\mathcal{F}_{\psi(\ell-1)} = \mathcal{F}_{\ell_1}$, $\mathcal{G}_{\varphi(\ell-1)} = \mathcal{G}_{\ell_2-1}$ and $\overline{r}_{\ell_2-1} = g - h - 1$ or $\mathcal{F}_{\psi(\ell-1)} = \mathcal{F}_{\ell_1-1}$, $r_{\ell_1-1} = h - 1$, $\mathcal{G}_{\varphi(\ell-1)} = \mathcal{G}_{\ell_2}$.

We claim that the second possibility can not occur. Indeed consider the double cover $\pi : F \longrightarrow E$. We have that

$$H^0(F, \omega_F) \cong H^0(E, \omega_E) \oplus H^0(E, \omega_E \otimes \mathcal{L}_{|E})$$

More concretely, if D is the ramification divisor of π in F and $t \in H^0(F, \mathcal{O}_F(D))$ vanishes along D we have that, given $s \in H^0(F, \omega_F)$, $s = t \cdot \pi^*(s_1) + \pi^*(s_2)$ where $s_1 \in H^0(E, \omega_E)$, $s_2 \in H^0(E, \omega_E \otimes \mathcal{L}_{|E})$.

We have $V \subseteq H^0(F, \omega_F)$ a codimension one subspace which produces after taking out the base point the linear series $|R_{\ell-1}|$ on F. The second possibility asserts that $V = V_1 \oplus V_2$ in the above decomposition, where $V_2 = H^0(E, \omega_E \otimes \mathcal{L}_{|E})$ and $V_1 \subseteq H^0(E, \omega_E)$ is a codimension one subspace.

But then, since V_2 has clearly no base point, sections of type $\pi^*(s_2)$, $s_2 \in V_2$, can not have a base point in F, a contradiction.

So we have the following decompositions

$$\begin{aligned} \mathcal{E}_\ell &= \mathcal{F}_{\ell_1} \oplus \mathcal{G}_{\ell_2} \\ \mathcal{E}_{\ell-1} &= \mathcal{F}_{\ell_1} \oplus \mathcal{G}_{\ell_2-1} \qquad \overline{r}_{\ell_2-1} = g - h - 1 \end{aligned}$$

If $\mathcal{E}_{\ell-2} = \mathcal{F}_j \oplus \mathcal{G}_k$ we have several possibilities according to Proposition 4.8:

If $j = \ell_1$, $k = \ell_2 - 2$ then $\gamma_{\ell-1} = \mu(\mathcal{E}_{\ell-1}/\mathcal{E}_{\ell-2}) = \mu(\mathcal{G}_{\ell_2-1}/\mathcal{G}_{\ell_2-2}) = \overline{\mu}_{\ell_2-1}$ and $\mu_{\ell_1} > \gamma_{\ell-1}$.

If $j = \ell_1 - 1$, $k = \ell_2 - 1$ then $\gamma_{\ell-1} = \mu(\mathcal{F}_{\ell_1}/\mathcal{F}_{\ell_1-1}) = \mu_{\ell_1}$.

If $j = \ell_1 - 1$, $k = \ell_2 - 2$ then $\gamma_{\ell-1} = \mu_{\ell_1} = \overline{\mu}_{\ell_2-1}$.

In any case we get $\gamma_{\ell-1} \leq \mu_{\ell_1}$. Since always happens that $\gamma_1 \geq \mu_1$ and $\gamma_\ell \leq \mu_{\ell_1}$, (4.11) reads

$$K^2_{S/B} \geq 4\chi_f + \gamma_1 - 2\gamma_{\ell-1} - 3\gamma_\ell \geq 4\chi_f + \mu_1 - 5\mu_{\ell_1}$$

If $\mu_1 \geq 5\mu_{\ell_1}$ we are done. If $\mu_1 < 5\mu_{\ell_1}$ we can repeat the argument of the general case. ∎

Remark 4.14 Observe that both bounds found in Theorem 4.11 are very close to $\lambda_{\mathrm{exp}} = 4\frac{g-1}{g-h} = 4 + 4\frac{h-1}{g-h}$ when $h << g$. Hence, fixing h and increasing g, that bound is assimptotically sharp since we have examples with slope equal to λ_{exp} (see Example 4.29).

4.3 The slope of non-Albanese fibrations

We are interested in understanding the behaviour of the slope of a fibration depending on the irregularity.

Let $f : S \longrightarrow B$ be a fibred surface. If F is a general fibre and $i : F \hookrightarrow S$ is the natural inclusion we can deduce from the natural diagram

$$F[r][d]^i JF[d]^{i_*} S[r]^{alb_S}[d]^f Alb(S)[d]^{f_*} B[r]JB$$

that $b \leq q(S) \leq b + g$. If $q(S) = b + g$ the fibration is locally trivial (cf. [12]). On the other hand, from the diagram and considering the universal property of Albanese varieties it follows easily that $q(S) = b$ if and only if $alb_S(S) = alb_B(B)$. Hence, if $b \geq 1$, equality $q(S) = b$ is equivalent to say that $B \cong alb_S(S)$ and to say that S is of Albanese dimension 1. In this case we say that the fibration is an Albanese fibration. We will say that f is a non-Albanese fibration if $q(S) > b$ (equivalently, if $b = 0$, $q(S) \geq 1$ or S is of Albanese general type).

As for the upper bound of the irregularity it is known that when f is not locally trivial $q < \frac{5g+1}{6}$ (cf. [92]; in [64] slightly more accurate bounds are given). If $b = 0$ Xiao proves that $q \leq \frac{1}{2}(g+1)$ and conjectures that in general $q - b \leq \frac{1}{2}(g+1)$ (cf. [93]). This conjecture is known to be false (see [75] for a counterexample) but the counterexamples suggest that only the constant term should be slightly increased. We can state

Conjecture (modified Xiao's conjecture on the irregularity of nonlinear pencils). There exists a constant c such that if f is not locally trivial then

$$q(S) \leq \frac{1}{2}g + c.$$

There are two known results on the influence of $q(S)$ on the slope. In [92] Xiao proves that $\lambda(f) \geq 4$ whenever $q(S) - b \geq 1$. In [64] Konno obtains

$$\lambda(f) \geq \frac{4g(g-1)}{(2g-1)(g-(q-b))}$$

Note that this last inequality is an increasing function of $q-b$ but that we need $q - b \geq \frac{1}{2}(g+1)$ to have $\lambda(f) \geq 4$. According to Xiao's conjecture this range is doubtful to happen. Also note that, as a function on g, this bound tends to be 2 as g grows.

It is clear from these results that the natural variable to deal with is $q(S)-b$.

We divide the study in two parts according to whether f is a double cover fibration or not. When f is a double cover fibration we get a lower bound when $g \geq 4h+1$ (notations as in §4.2) which is at least four and depends on the contribution of $q(V)$ on $q(S)$.

In the general case (when f is not a double cover) we give as a lower bound an increasing function $\alpha(q-b)$ such that $\alpha(2) = 4$.

4.3.1 The double cover case

We follow the constructions and notations of §4.2.

Lemma 4.15 *Let $\mathcal{F} = \tau_* \omega_{V/B}, \mathcal{G} = \tau_*(\omega_{V/B} \otimes \mathcal{L})$ as in §4.2. Let $s = q(S) - b, s_1 = q(V) - b$ and $s_2 = s - s_1$. Then, if according to Fujita's decomposition we set $\mathcal{E} = \mathcal{H} \oplus \mathcal{O}_B^{\oplus s}$ we have*

$$\mathcal{F} = \mathcal{H}_1 \oplus \mathcal{O}_B^{\oplus s_1}$$

$$\mathcal{G} = \mathcal{H}_2 \oplus \mathcal{O}_B^{\oplus s_2}$$
$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$$

PROOF: The decomposition $\mathcal{F} = \mathcal{H}_1 \oplus \mathcal{O}_B^{\oplus s_1}$ comes out from Fujita's decomposition applied to the fibration τ. Note that according to Theorem 3.1 (v) we have

$$H^0(B, \mathcal{H}^*) = H^0(B, \mathcal{H}_1^*) = 0.$$

Hence we have

$$s = h^0(B, \mathcal{E}^*) = h^0(B, \mathcal{F}^*) + h^0(B, \mathcal{G}^*) = s_1 + h^0(B, \mathcal{G}^*)$$

and so there is an epimorphism $\mathcal{G} \longrightarrow \mathcal{O}_B^{\oplus s_2}$. By composing with the projection $\mathcal{E} \longrightarrow \mathcal{G}$ and using the splitting property given in Theorem 3.1 (v) we obtain that in fact

$$\mathcal{G} = \mathcal{H}_2 \oplus \mathcal{O}_B^{\oplus s_2}$$

From this follows that $h^0(B, \mathcal{H}_2^*) = 0$. Finally note that we have

$$\mathcal{H} \oplus \mathcal{O}_B^{\oplus s} = \mathcal{E} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{O}_B^{\oplus s}.$$

Since

$$Hom_B(\mathcal{H}, \mathcal{O}_B) = H^0(B, \mathcal{H}^*) = 0$$
$$Hom_B(\mathcal{H}_1, \mathcal{O}_B) = H^0(B, \mathcal{H}_1^*) = 0$$
$$Hom_B(\mathcal{H}_2, \mathcal{O}_B) = H^0(B, \mathcal{H}_2^*) = 0$$

we have that in fact

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$$

■

Theorem 4.16 *Let $f : S \longrightarrow B$ be a relatively minimal double cover fibration of $\tau : V \longrightarrow B$. Let E, F be the fibres of τ and f respectively. Let $h = g(E)$, $g = g(F)$ and s, s_1, s_2 as in lemma 4.15. Assume f is not locally trivial and that $g \geq 4h + 1$. Then* *(i) If* $g \geq 2h + s_2 + 1$

$$\lambda(f) \geq 4 + 4\frac{(h-1)(g-4h-1)}{(g-s_2-h)(g-4h-1)+2h(h-s_1)(g-s_2-1)}$$

(ii) If $g \leq 2h + s_2$

$$\lambda(f) \geq 4 + 8\frac{(g-3)(g-4h-1)}{(g-4h-1)+4(g-1)(h-s_1)h}$$

PROOF:

(i) The same proof as in Theorem 4.11 works with the substitution of g by $g - s_2$ and noting that $\mu_1 = \mu(\mathcal{F}_1) \geq \frac{\chi_1}{h-s_1}$ in Proposition 4.10.

(ii) Again the proof of Theorem 4.11 works with the following modifications:

In Proposition 4.10 (i) use $\mu_1 = \mu(\mathcal{F}_1) \geq \frac{\chi_1}{h-s_1}$ to get $K^2_{S/B} \geq 4\chi_f + \frac{2(g-4h-1)}{h(h-s_1)}\chi_1$.

In Proposition 4.10 (iii) we must only consider the possibility $\overline{d}_i \geq 2\overline{r}_i - 2$ in (4.10), since $g - h - s_2 \leq h$. Then for every i, $1 \leq i \leq \ell_2 - 1$, $\overline{d}_i + \overline{d}_{i+1} \geq 4\overline{r}_i - 2$; $\overline{d}_{\ell_2} + \overline{d}_{\ell_2+1} = 4\overline{r}_{\ell_2} - 4$.

Then we obtain

$$K^2_{S/B} \geq 8\frac{g-1}{g+1}\chi_2.$$

Both inequalities give

$$\lambda(f) \geq 4 + 8\frac{(g-3)(g-4h-1)}{(g-4h-1)+4(g-1)(h-s_1)h}$$

■

Corollary 4.17 *With the above hypotheses, if* $q(S) = q(V) = b + h$ *and* $g \geq 2h + 1$ *then*

$$\lambda(f) \geq 4\frac{g-1}{g-h}$$

PROOF: We have $s_2 = 0$, $s_1 = s = q(V) - b = h$ and Theorem 4.16 (i) gives the required bound. ■

Remark 4.18 Condition $s_2 = 0$ happens in most cases; indeed, observe that by definition $s_2 = q(S) - q(V) = h^1(V, \mathcal{L}^{-1})$. If the ramification locus of π has some property of positivity then $h^1(V, \mathcal{L}^{-1}) = 0$ by Ramanujam or Kawamata-Viehweg vanishing theorems.

Since $g \geq 2h+1$ generically holds ($g \geq 2h-1$ holds for any double cover and equality holds only in the étale case) then Theorem 4.17 under the assumption $s_2 = 0$ reads

$$\lambda(f) \geq 4 + 4\frac{(h-1)(g-4h-1)}{(g-h)(g-4h-1) + 2h(h-s)(g-1)}$$

As pointed out in Corollary 4.17, this bound is sharp as far as $h = s$. Although $h \neq s$ we can give an example were this bound is almost sharp. Indeed in Example 4.32 we found a double cover fibration with $s_2 = 0, h = 3, g = 4m + 5 \quad (m$ arbitrarily large) and slope arbitrarily near to

$$\widetilde{\lambda} = 4 + \frac{8}{g-3}$$

Observe that with this data the previous bound reads

$$\lambda(f) \geq 4 + \frac{8(g-13)}{(g-3)(g-13) + 6(g-1)} \sim 4 + \frac{8}{g+3}.$$

Finally note that, for a fixed h, our bounds tend to be 4 when g grows. This is exactly the real behaviour: see Example 4.29 and Example 4.34 with $\alpha = 2$.

4.3.2 The non-double cover case

Theorem 4.19 *Let $f : S \longrightarrow B$ be a relatively minimal fibration which is not a double cover fibration. Assume $g = g(F) \geq 5$ and that f is not locally trivial. Let $s = q(S) - b \geq 1$.*

Then

(i) $\lambda(f) \geq 4$

(ii) If $s \geq 2$ and $g \geq \frac{3}{2}s + 2$ then

$$\lambda(f) \geq \frac{8g(g-1)(4g-3s-10)}{8g(g-1)(g-s-2)+3(s-2)(2g-1)} \quad \text{if } F \text{ is not trigonal}$$

$$\lambda(f) \geq \frac{4g(g-1)(4g-3s-10)}{4g(g-1)(g-s-2)+(g-4)(2g-1)} \quad \text{if } F \text{ is trigonal}$$

(iii) if $g < \frac{3}{2}s + 2$ then

$$\lambda(f) \geq \frac{4g(g-1)(2g-7)}{\frac{4}{3}g(g-1)(g-3)+(g-4)(2g-1)}$$

Proof:

(i) This is due to Xiao ([92]). We give here an alternative and very short proof. If $q(S) > b$ then for every $n >> 0$ there exists a $n-$torsion element $\mathcal{L} \in \mathrm{Pic}^0(S)$, such that for $1 \leq i \leq n-1 \quad \mathcal{L}^{\otimes i} \neq \mathcal{O}_F$. The étale cover $\widetilde{S} \longrightarrow S$ induced by $\mathcal{L}$ induces a fibration $\widetilde{f} : \widetilde{S} \longrightarrow B$ such that $\lambda(\widetilde{f}) = \lambda(f)$, and with genus of the fibre $\widetilde{g} \sim ng$. Since this is true for $n >> 0$ and the general bound (4.0.1) gives

$$\lambda(f) \geq 4 - \frac{4}{g}$$

we get $\lambda(f) \geq 4$ by a limit process.

(ii), (iii) Consider Fujita's decomposition $\mathcal{E} = f_*\omega_{S/B} = \mathcal{A} \oplus \mathcal{Z}$ with $\mathcal{Z} = \mathcal{O}_B^{\oplus s}$. Consider the Harder-Narasimhan filtration of $\mathcal{A}$:

$$0 = \mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \ldots \subseteq \mathcal{A}_\ell = \mathcal{A}$$

As in §1.2 we produce nef $\mathbb{Q}$-divisors N_i, and effective divisors Z_i in a suitable blow-up of $S \quad \sigma : \widetilde{S} \longrightarrow S$ such that

$$N_i + \mu_i F + Z_i \equiv N_j + \mu_j F + Z_j \equiv \sigma^* K_{S/B}$$

where $\{\mu_i\}$ are the Harder-Narasimhan slopes of $\mathcal{A}$. Note that we can define $N_{\ell+1} = \sigma^* K_{S/B}$, $Z_{\ell+1} = 0$, $\mu_{\ell+1} = 0$. Observe also that, if $r_i = \mathrm{rk}\mathcal{A}_i$, $\sum\limits_{i=1}^{\ell} r_i(\mu_i - \mu_{i+1}) = \deg\mathcal{A} = \deg\mathcal{E} = \chi_f$.

Each N_i induces on F a base point free linear system of degree d_i and (projective) dimension greater or equal than $r_i - 1$. Note that $N_i + \mu_i F = H_i$ is induced by a map $\varphi_i : S \longrightarrow \mathbb{P}_B(\mathcal{A}_i)$ which restricted to fibres induces the above linear system. By hypothesis φ_i is never a double cover onto the image and so the induced map ψ_i on F is not a double cover. Hence we have

$$\begin{aligned} d_i &\geq 3(r_i - 1) \qquad \text{if } \deg\psi_i \geq 3 \\ d_i &\geq 3r_i - 4 \qquad \text{if } \deg\psi_i = 1 \text{ and } d_i \leq g-1 \\ d_i &\geq \tfrac{3r_i+g-4}{2} \qquad \text{if } \deg\psi_i = 1 \text{ and } d_i \geq g \end{aligned}$$

the latest two inequalities being "Clifford plus" Lemma ([11], [77]). Considering the above inequalities in the (r, d)-plane, we have the following two possibilities (note that the lines $d = 3r-4$ and $d = \frac{3r+g-4}{2}$ meet exactly at the point $(r = \frac{1}{3}(g+4), d = g)$) depending on rank$\mathcal{A} = g - s$.

Case 1.- $g - s \geq \frac{1}{3}(g+4)$

In this case note that for every $1 \leq i \leq \ell$, $d_i \geq \frac{2g-\frac{3}{2}s-5}{g-s-2}r - \frac{g-4}{g-s-2}$ (this border line joining the point (2,3) and the point $(g-h, 2g-\frac{3}{2}h-2)$) except if $(r_1, d_1) = (1, 0)$. Note that $g - s - 2 > 0$ since $g \geq \frac{3}{2}s + 2$.

Note also that by definition we have $d_{\ell+1} = 2g-2$. So for $1 \leq i \leq \ell$ we get (since $r_{i+1} \geq r_i + 1$)

$$d_i + d_{i+1} \geq \frac{4g-3s-10}{g-s-2} r_i - \frac{3(s-2)}{2(g-s-2)} =: Ar_i + B$$

except if $(r_1, d_1) = (1, 0)$ and $(r_2, d_2) = (2, 3)$. In this exceptional case we get

$$d_1 + d_2 - Ar_1 - B = 3 - A - B = -\frac{g - \frac{3}{2}s - 1}{g-s-2}$$

If this happens F is trigonal since has a linear system of degree 3 and dimension 1.

Applying Xiao's formula we get, in the general case,

$$\begin{aligned} K^2_{S/B} &\geq \sum_{i=1}^{\ell}(d_i + d_{i+1})(\mu_i - \mu_{i+1}) \geq \sum_{i=1}^{\ell} Ar_i(\mu_i - \mu_{i+1}) + \sum_{i=1}^{\ell} B(\mu_i - \mu_{i+1}) = \\ &= A\chi_f + B\mu_1 = \tfrac{4g-3s-10}{g-s-2}\chi_f - \tfrac{3(s-2)}{2(g-s-2)}\mu_1 \end{aligned}$$

Applying again Konno's bound (cf. [64] p. 680):

$$K_{S/B}^2 \geq \frac{4g(g-1)}{2g-1}\mu_1$$

we can eliminate μ_1 and get

$$K_{S/B}^2 \geq \frac{8g(g-1)(4g-3s-10)}{8g(g-1)(g-s-2)+3(s-2)(2g-1)}\chi_f$$

Note that this bound is a strictly increasing function of s and that $K_{S/B}^2 \geq 4\chi_f$ if $s \geq 2$.

In the exceptional case (when F is trigonal) we get

$$K_{S/B}^2 \geq A\chi_f + B\mu_1 - \frac{g-\frac{3}{2}s-1}{g-s-2}(\mu_1-\mu_2) \geq A\chi_f + \left(B - \frac{g-\frac{3}{2}s-1}{g-s-2}\right)\mu_1$$

The same argument using $K_{S/B}^2 \geq \frac{4g(g-1)}{2g-1}\mu_1$ yields

$$K_{S/B}^2 \geq \frac{4g(g-1)(4g-3s-10)}{4g(g-1)(g-s-2)+(g-4)(2g-1)}\chi_f$$

which is also a strictly increasing function of s. In this case we need $s \geq 4$ to get $K_{S/B}^2 \geq 4\chi_f$.

Case 2.- $g-s \leq \frac{1}{3}(g+4)$

Let $\overline{s} = \left[\frac{2}{3}g - \frac{4}{3}\right]$. Under our hypotheses $s \geq \overline{s}$, so we can take $\overline{\mathcal{A}} = \mathcal{A} \oplus \mathcal{O}_B^{\oplus(s-\overline{s})}$ instead of $\mathcal{A}$. Hence we get according to whether we are in the general or in the special case

$$\begin{aligned} K_{S/B}^2 &\geq \tfrac{8g(g-1)(4g-3\overline{s}-10)}{8g(g-1)(g-\overline{s}-2)+3(\overline{s}-2)(2g-1)}\chi_f &\geq \tfrac{8g(g-1)(2g-7)}{\frac{8}{3}g(g-1)(g-3)+(2g-9)(2g-1)}\chi_f \\ K_{S/B}^2 &\geq \tfrac{4g(g-1)(4g-3\overline{s}-10)}{4g(g-1)(g-\overline{s}-2)+(g-4)(2g-1)}\chi_f &\geq \tfrac{4g(g-1)(2g-7)}{\frac{4}{3}g(g-1)(g-3)+(g-4)(2g-1)}\chi_f \end{aligned}$$

since both expressions are increasing functions of s and $\overline{s} \geq \frac{2}{3}g - 1$. Note that the second bound is slightly smaller than the first one. ■

Remark 4.20 In the case (iii) of the Theorem we could consider that for $1 \leq i \leq \ell$, $d_i \geq 3r_i - 4$ and hence $d_i + d_{i+1} \geq 6r_i - 5$ for $1 \leq i \leq \ell - 1$. But for $i = \ell$ we would have $d_\ell + d_{\ell+1} \geq 2d_\ell + 1 \geq 6r_\ell - 7$ which produces

$$K^2_{S/B} \geq 6\chi_f - (5\mu_1 + 2\mu_\ell)$$

Hence using Xiao's inequality with indexes $\{1, \ell\}$

$$K^2_{S/B} \geq (d_1 + d_\ell)(\mu_1 - \mu_\ell) + (d_\ell + d_{\ell+1})\mu_\ell \geq d_\ell(\mu_1 + \mu_\ell) \geq (3g - 3s - 4)(\mu_1 + \mu_\ell)$$

we get

$$K^2_{S/B} \geq 6\frac{3g - 3s - 4}{3g - 3s + 1}\chi_f$$

which depends on s and is better than (iii) for some special values of (g, s) but is a *decreasing* function of s.

Observe that according to Xiao's Conjecture on the irregularity quoted above, the range $g < \frac{3}{2}(q(S) - b) + 2)$ (case (iii) of the theorem) is doubtful to occur.

Remark 4.21 In the above theorem we worked with $\mathcal{Z} = \mathcal{O}_B^{\oplus(q(S)-b)}$ and $s = \mathrm{rank}\mathcal{Z}$. In most parts of the proof we only use that $\deg\mathcal{Z} = 0$. Hence, we get the same bounds in (ii) if we define s to be the rank of the degree zero part in Fujita's decomposition of $\mathcal{E} = f_*\omega_{S/B}$ $(s \geq q(S) - b)$. Note that then the argument of Theorem 4.19 (iii) does not work since we do not know whether $\mathcal{Z}$ can be cut in pieces of the length we need. In fact, if Fujita's Conjecture holds (see §3) $\mathcal{Z}$ would became trivial after an étale base change. Since the slope is invariant under étale base changes we would have that Theorem 4.19 (iii) also would hold for this new definition of s.

In any case the bound of the previous Remark holds for the general definition of s.

Finally if $s = 1$ (the only case not covered by (ii) and (iii)) we also get $\lambda(f) \geq 4$ with this new definition of s. Indeed, if $\mathcal{L}$ is a rank one, degree zero summand of $\mathcal{E}$, then by Theorem 3.4 it is torsion and hence it becomes trivial after étale base change. Using again invariance of the slope we can assume $\mathcal{L}$ to be $\mathcal{O}_B$ and apply Theorem 4.19 (i).

Remark 4.22 Remember that if F is trigonal we have (cf. [65])

$$\lambda(f) \geq \frac{14(g-1)}{3g+1}$$

which is better that Theorem 4.19 (ii) (special case) for $g \gg s = q - b$ and that gives $\lambda(f) \geq 4$ if $g \geq 9$.

Remark 4.23 As a function on g (fixing s) the bounds of Theorem 4.19 tend to be 4 when g grows (compare [64] Lemma 2.7 where this limit is 2). We have some more information of the assimptotic behaviour:

Theorem 4.24 *Let $f : S \longrightarrow B$ be a relatively minimal, non locally trivial fibration. Let F be a fibre of f, $g = g(F)$ and $q = q(S)$. Assume f is not a double cover fibration and that $s = q - b \geq 1$ (i.e., f is a non-Albanese fibration). Let $\mathcal{C} = \{\pi_i : S \longrightarrow C_i$ fibrations, $c_i = g(C_i) \geq 2, \pi_i \neq f\}_{i \in I}$. Assume $\mathcal{C} \neq \emptyset$ and let $c = max\{c_i | i \in I\}$. Then*

(i) $\lambda(f) \geq 4 + \frac{c-1}{g-c}$

(ii) If, moreover, $\dim alb(S) = 1$ *(then necessarily $b = 0$) we have*

$$\lambda(f) \geq 4 + \frac{q-1}{g-q}$$

Proof: Remember that if f is a non-Albanese fibration then either $\dim alb(S) = 2$ or $b = 0$ (provided $q(S) \neq 0$).

Let $\pi : S \longrightarrow C$ be the fibration with maximal base genus $c \geq 2$ (if $\dim alb(S) = 1$, then $c = q$ and $\pi = alb$).

Since in any case $f^*Pic^0(B)$ does not contain $\pi^*Pic^0(C)$ we can choose for $n \gg 0$, a n-torsion element $\mathcal{L} \in Pic^0(C)$ such that $\pi^*\mathcal{L}^{\otimes i} \notin f^*Pic^0(B)$ for $1 \leq i \leq n-1$. Consider the base change

$$\widetilde{S}[r]^{\widetilde{\pi}}[d]^{\widetilde{\alpha}}\widetilde{C}[d]^{\alpha}S[r]^{\pi}[d]^{f}CB$$

and let $\widetilde{f} = f \circ \widetilde{\alpha}$. Since $\mathcal{L}^{\otimes i}_{|F} \neq \mathcal{O}_F$ for $1 \leq i \leq n-1$, $\widetilde{f}$ has connected fibres and so $\widetilde{f}$ is again a fibration over B. Let $\widetilde{F}$ be the fibre of $\widetilde{f}$. Then if $\widetilde{g} = g(\widetilde{F})$,

$$\widetilde{g} - 1 = n(g-1).$$

Moreover we have

$$q(\widetilde{S}) = h^1(\widetilde{S}, \mathcal{O}_{\widetilde{S}}) = h^1(S, \mathcal{O}_S) + \sum_{i=1}^{n-1} h^1(S, (\pi^* \mathcal{L}^{-i}))$$

From the exact sequence

$$0 \longrightarrow H^1(B, \mathcal{L}^{-i}) \longrightarrow H^1(S, \pi^* \mathcal{L}^{-i}) \longrightarrow H^0(B, (R^1 \pi_* \mathcal{O}_S) \otimes \mathcal{L}^{-i}) \longrightarrow 0$$

and using that $h^0(B, (R^1\pi_*\mathcal{O}_S) \otimes \mathcal{L}^{-i}) = 0$ except for a finite number of sheaves $\mathcal{L}^{-i} \in Pic^0(C)$ (which can be avoided with the election of $\mathcal{L}$ (see Theorem 3.4 and Proposition 3.3)), we get

$$\widetilde{s} = q(\widetilde{S}) - b = q(S) - b + (n-1)(c-1) = s + (n-1)(c-1)$$

since $h^1(B, \mathcal{L}^{-i}) = c - 1$ by Riemann-Roch. In particular, $\widetilde{s} \geq 2$ if $n \geq 2$.

Note that if F is trigonal then $\widetilde{F}$ is not if $n \gg 0$ (see Lemma 5.12). On the other hand

$$\lim_{n \to \infty} \frac{\widetilde{g}}{\widetilde{s}} = \frac{g-1}{c-1} \geq 2$$

since the map $\pi_{|F} : F \longrightarrow C$ is at least of degree two (if it were of degree 1 clearly $F \cong C$ and $S = B \times C$). Hence if $n \gg 0$ the case $\widetilde{g} < \frac{3}{2}\widetilde{s} + 2$ can not occur.

So if $n \gg 0$ we are under the hypotheses of Theorem 4.19 (ii) (non trigonal case). Using that the slope is invariant under étale changes of S (cf. [92]) we get

$$\lambda(f) = \lambda(\widetilde{f}) \geq \frac{8\widetilde{g}(\widetilde{g}-1)(4\widetilde{g} - 3\widetilde{s} - 10)}{8\widetilde{g}(\widetilde{g}-1)(\widetilde{g} - \widetilde{s} - 2) + 3(\widetilde{s} - 2)(2\widetilde{g} - 1)}$$

for $\widetilde{g} = n(g-1) + 1$, $\widetilde{s} = s + (n-1)(c-1)$ and $n \in \mathbb{N}$ arbitrarily large. So we can take limit as n grows and get

$$\lambda(f) \geq 4 + \frac{c-1}{g-c}$$

In case $\dim alb(S) = 1$ then clearly $b = 0$ and $c = q$. ∎

Corollary 4.25 *Let $f : S \longrightarrow B$ be as in Theorem 4.19. Assume $\lambda(f) < 4 + \frac{1}{g-2}$. Then S has no other fibration onto a curve of genus greater or equal than two.*

Corollary 4.26 *Let S be a minimal surface with $q(S) \geq 2$ and $F \subseteq S$ an irreducible curve of geometric genus g. Assume $h^0(S, \mathcal{O}_S(F)) \geq 2$ and let $f : \widetilde{S} \longrightarrow \mathbb{P}^1$ be a relatively minimal fibration with fibre F. If F is not a double cover and $\lambda(f) < 4 + \frac{q-1}{g-q}$ then S is of Albanese general type.*

4.4 Fibred surfaces with low slope

As pointed out in (4.0.7), fibred surfaces with slope $\lambda(f) < 4$ seem to have very special properties, as follows basically from the work of Xiao in [92]. We recall

Conjecture. Let $f : S \longrightarrow B$ be a non-locally trivial, relatively minimal fibred surface. If $\lambda(f) < 4$ then $\mathcal{E} = f_*\omega_{S/B}$ is ample and (if $g >> 0$) F is hyperelliptic.

As for this conjecture, as a consequence of previous results in this chapter and Chapter 3, we can contribute with

Theorem 4.27 *Let $f : S \longrightarrow B$ be a non locally trivial, relatively minimal fibred surface. Let F be a general fibre, $g = g(F)$ and $b = g(B)$. Assume $\lambda(f) < 4$. Then*

(i) $q(S) = b$

(ii) $\mathcal{E} = f_*\omega_{S/B}$ is ample provided one of the following conditions holds

(a) F is non hyperelliptic

(b) $b \leq 1$

(c) $g \leq 3$

Proof: (i) Cf. [92] and §4.2, §4.3.

(ii) (a) If F is non hyperelliptic, according to [92] p.463, it only remains open the case of a degree zero, rank one quotient $\mathcal{L}$ of $\mathcal{E}$. By Theorem 3.4 such $\mathcal{L}$ is torsion and then, after an étale base change

$$\widetilde{S}[r][d]^{\widetilde{f}} S[d]^{f} \widetilde{b}[r]^{\sigma} B$$

we have $\widetilde{f}_*\omega_{\widetilde{S}/\widetilde{B}} = \sigma^* f_*\omega_{S/B} \longrightarrow \sigma^*\mathcal{L} = \mathcal{O}_{\widetilde{B}}$ and hence $\lambda(\widetilde{f}) \geq 4$ since $q(\widetilde{S}) \geq g(\widetilde{B}) + 1$. As the slope is invariant under étale base changes we have $\lambda(f) = \lambda(\widetilde{f}) \geq 4$.

(b) The same argument works as any stable degree zero locally free sheaf on an elliptic curve has rank one ([4]).

(c) If $g = 2$ then $\mathcal{E} = \mathcal{A} \oplus \mathcal{L}$ or $\mathcal{E} = \mathcal{A}$ with $\mathcal{A}$ ample and $\mathcal{L}$ torsion of rank one, so we are done. If $g = 3$ the only non trivial case is $\mathcal{E} = \mathcal{A} \oplus \mathcal{F}$ with $\mathcal{A}$

an ample line bundle and $\mathcal{F}$ a étale, degree zero, rank two vector bundle. But then by [64] Lemma 2.7

$$K_{S/B}^2 \geq \frac{4g(g-1)}{2g-1}\chi_f = \frac{24}{5}\chi_f$$

a contradiction. ∎

In [92] Theorem 3.3, Xiao proves that a non Albanese fibration with $\lambda = 4$ verifies that $\mathcal{E} = f_*\omega_{S/B} = \mathcal{F} \oplus \mathcal{O}_B$ with $\mathcal{F}$ semistable. We can give the following refinement

Proposition 4.28 *Let $f : S \longrightarrow B$ be a relatively minimal non locally trivial fibration. If $\lambda(f) = 4$ then $\mathcal{E} = f_*\omega_{S/B}$ has at most one degree zero, rank one quotient $\mathcal{L}$*

Moreover, in this case $\mathcal{E} = \mathcal{A} \oplus \mathcal{L}$ with $\mathcal{A}$ semistable.

PROOF: As in the previous theorem $\mathcal{L}$ is torsion and hence trivial after an étale base change. But then

$$\widetilde{f}_*\omega_{\widetilde{S}/\widetilde{B}} = \widetilde{\mathcal{A}} \oplus \mathcal{O}_{\widetilde{B}}, \quad \widetilde{\mathcal{A}} = \sigma^*\mathcal{A}$$

.

Hence, applying Xiao's result quoted above, $\widetilde{\mathcal{A}}$ is semistable. Then $\mathcal{A}$ is also semistable by [70] Proposition 3.2. ∎

4.5 Examples

We construct here several families of examples of fibred surfaces that we needed in the previous sections to check the sharpness (at least assimptotically) of the bounds found. The first one is an example of Xiao.

Example 4.29 (cf. [92] Example 4.3; we refer there for details). Let B, A be smooth curves of genus b and h respectively. Let $\delta_1 \in \mathrm{Div}(B), \delta_2 \in \mathrm{Div}(A)$ divisors of big enough degree $2x$ and $2y$ respectively. Consider the two projections $\pi_1 : B \times A \longrightarrow B$, $\pi_2 : B \times A \longrightarrow A$ and a smooth divisor $R \in |\pi_1^*(\delta_1) + \pi_2^*(\delta_2)|$. Note that R is even and so we can consider the double cover S ramified along R. S has an induced fibration $f : S \longrightarrow B$ with general fibre F which is a double cover of A and the following invariants

$$\begin{aligned} g &= g(F) = 2h + y - 1 \\ q &= q(S) = b + h \\ \lambda(f) &= 4\frac{g-1}{g-h} \end{aligned}$$

In particular, if $h = 0$ we get hyperelliptic fibrations with slope $4\frac{g-1}{g}$; if $h = 1$ we get bielliptic fibrations with slope exactly 4 and g arbitrarily large.

Example 4.30 Our starting point is the previous example. Let A, B be to smooth curves of genus a and b respectively, and let Z be the double cover of $B \times A$ as in 4.29. Z has natural fibrations over B and A; let E and L be the respective general fibres. A standard use of formulas for double covers (cf. [7] p.182) gives

$$\begin{aligned} g(L) &= 2b + x - 1 \\ h = g(E) &= 2a + y - 1 \\ K_{Z/B} &\sim \rho^*(K_{B\times A/B} + \frac{1}{2}R) \sim xE + (2a - 2 + y)L \\ \chi\mathcal{O}_Z &= 2(a-1)(b-1) + (a-1)x + (b-1)y + xy \end{aligned}$$

Let now n, m be big enough integers and consider $\delta = nE + mL$ and $\Delta \in |2\delta|$ a smooth divisor. Let $\pi : S \longrightarrow Z$ be the double cover ramified along Δ and let f be the natural fibration of S over B. Let F be the general fibre of f and $g = g(F)$. Note that F is a double cover of E. Again a standard computation shows

$$
\begin{aligned}
&g = 2h + 2m - 1 \\
&q = q(S) = q(Z) = a + b \\
&K_{S/B} \sim \pi^*(K_{Z/B} + \delta) = (x+n)F + (2a - 2 + y + m)\pi^*(L) \\
&\chi\mathcal{O}_S = 4(a-1)(b-1) + 2(a-1)x + 2(b-1)y + 2xy + (2a-2+y)n + \\
&\quad +(2b - 2 + x)m + 2nm
\end{aligned}
$$

and hence

$$
\lambda(f) = \frac{8(x+n)(2a-2+y+m)}{2(a-1)x + 2xy + (2a-2+y)n + (x+2n)m}
$$

Observe that in this case we have

$$
\lambda_{\text{exp}} = 4\frac{g-1}{g-h} = \frac{8(2a-2+y+m)}{2a-2+y-2m}
$$

which certainly coincides with the limit of $\lambda(f)$ as n grows as proven in §4.2.

A simple computation shows that

$$
\lambda(f) \geq \lambda_{\text{exp}} \Longleftrightarrow m \geq y
$$

Note that our construction allows us to take $m \geq y$ or $m < y$ as needed. Hence we can construct families of double cover fibrations such that $\lambda(f) < \lambda_{\text{exp}}$. Nevertheless observe that for this examples $g < 4h$.

Example 4.31 This example deals again with the construction of successive double covers but the starting point is not so natural. Let G, H be two smooth elliptic curves with a fixed group structure. Let $\epsilon \in \text{Div}^0(G)$ such that $2\epsilon = 0$ and consider the involution σ of $G \times H$ such that $\sigma(e, h) = (e + \epsilon, -h)$. Clearly

σ has no fixed points and hence $X = G \times H/ < \sigma >$ is a smooth surface. Note that $G^{'} = G/ < \sigma >$ is an elliptic curve and that $H/ < \sigma >= \mathbb{P}^1$. The double cover $G \longrightarrow G^{'}$ is determined by a two torsion element $\eta \in \mathrm{Pic}^0(G^{'})$ and the double cover $H \longrightarrow \mathbb{P}^1$ ramifies at 4 points. By construction of X we have a natural double cover

$$\gamma : X \longrightarrow G^{'} \times \mathbb{P}^1$$

determined by $\alpha = \pi^*_{G'}(\eta) + \pi^*_{\mathbb{P}^1}(2p)$. Hence we have

$$\begin{aligned} &K_X = \gamma^*(K_{G' \times \mathbb{P}^1} + \alpha) = g_1^*(\eta) \\ &\chi\mathcal{O}_X = 0 \end{aligned}$$

where $g_1 = \pi_{G'} \circ \gamma, \quad g_2 = \pi_{\mathbb{P}^1} \circ \gamma.$

Let $t \in \mathbb{P}^1$ be a ramification point of $H \longrightarrow \mathbb{P}^1$. By construction $g_2^*(t)$ is 2-divisible. Let $\bar{G} = \frac{1}{2}g_2^*(t)$, let G_1 be a general fibre of g_1 and $d \geq 1$ (note that if G is the general fibre of g_2, $2\bar{G} \sim G$). If $\delta = dG_1 + \bar{G}$, then 2δ moves without base points and hence $|2\delta|$ has smooth members. We can consider then the associated double cover $\rho : Z \longrightarrow X$ which has two natural fibrations l_1 and l_2 onto $G^{'}$ and $\mathbb{P}^1$ respectively. Let L and E be the respective general fibres. By construction we have

$$\begin{aligned} &LE = 4 \\ &h = g(E) = 2g(G) + 2d - 1 = 2d + 1 \\ &g(L) = 2 \\ &K_Z = \rho^*(K_X + \delta) = l^*(\eta) + dL + \frac{1}{2}E \\ &\chi\mathcal{O}_Z = d \end{aligned}$$

Finally let m, n big enough and take a smooth member $R \in |2mL + 2nE|$. Consider the associated double cover $\pi : S \longrightarrow Z$ and let $f : S \longrightarrow \mathbb{P}^1$ be the induced fibration. Let F be its general fibre. We have

$$g = g(F) = 2h + 4m - 1 = 4d + 4m + 1$$

$$\chi\mathcal{O}_S = 2d + 2nd + m + 4mn$$
$$K_S^2 = 8(d+m)(2n+1)$$

and hence

$$\lambda(f) = \frac{8(d+m)(2n+1) + 8(4d+4m)}{6d + 2nd + 5m + 4mn}$$

Note that after a base change of f not ramified on the images of singular fibres, we can get new fibrations with the same slope and with the base curve of arbitrarily large genus.

In this case we have

$$\lambda_{\text{exp}} = 4\frac{g-1}{g-h} = \frac{16d + 16m}{2d + 4m}$$

As n grows, the slope of f tends to be the expected one. Again we can immediately check that

$$\lambda(f) \geq \lambda\text{exp} \Longleftrightarrow 5m \geq d$$

Observe that m can be fixed and d arbitrarily increased. Indeed, note that E is bielliptic and that $\eta : E \longrightarrow G$ ramifies in $2h - 2 = 2d$ points. Let $p_1, p_2 \in E$ two of such ramification points. Then $|2p_1 + 2p_2| = |\eta^*(q_1 + q_2)|$ is a g_4^1 in E which moves without base points. We can choose then $n >> 0$ and a smooth member $R \in |l^*(q_1 + q_2) + 2nE| \quad (l^*(q_1 + q_2) \sim 4L)$ and construct S with $m = 2$ and d arbitrary. Nevertheless observe that in order to get $\lambda(f) < \lambda_{\text{exp}}$ we have $g \sim 2h$.

Finally observe that S has another fibration $g : S \longrightarrow G'$. A similar computation shows that in this case $\lambda(g) \geq \bar{\lambda}_{\text{exp}}$ always holds.

Example 4.32 Let A be an abelian surface with a base point free linear system $|C|$, $C^2 = 4$ (an abelian surface with a polarization of type (1,2), for instance). Then g(C)=3. Take C_1, C_2 two smooth and transversal members and let $\sigma : \widetilde{A} \longrightarrow A$ be the blow-up at the 4 base points. We have then a fibration $\tau : \widetilde{A} \longrightarrow \mathbb{P}^1$ with general fibre $\widetilde{C}$ a curve of genus 3. Let $E =$

$E_1+E_2+E_3+E_4$ be the σ-exceptional reduced and irreducible divisor. Note that $\tau(E)=\mathbb{P}^1$. Let $n>>m>>0$ and $\delta=n\widetilde{C}+mE$; then $|2\delta|$ has no base point (if $m>>0$, $|2\delta_{|C_t}|$ has no base point for every $t\in\mathbb{P}^1$; so we can apply Lemma 1.2 or 1.3).

We can take then a smooth member $R\in|2\delta|$ and consider the associated double cover $\pi: S\longrightarrow\widetilde{A}$. Let $f: S\longrightarrow\mathbb{P}^1$ be the induced fibration, and let F be a general fibre. Then F is a double cover of $\widetilde{C}$ and we have (note that $K_{\widetilde{A}}=\sigma^*K_A+E=E$)

$$\begin{aligned}
&h=g(\widetilde{C})=3\\
&g=g(F)=4m+5\\
&K_S^2=2(K_{\widetilde{A}}+\delta)^2=8(m+1)(2n-m-1)\\
&\chi\mathcal{O}_S=2\chi\mathcal{O}_{\widetilde{A}}+\frac{1}{2}\delta K_{\widetilde{A}}+\frac{1}{2}\delta^2=2m(2n-m)+2(n-m)
\end{aligned}$$

Moreover, observe that δ is nef and big since $\delta^2>0$ and $|2\delta|$ moves without base points. Then we can apply Kawamata-Viehweg vanishing theorem and get that $h^1(\widetilde{A},\mathcal{O}_{\widetilde{A}}(-\delta))=0$; hence

$$q(S)=q(\widetilde{A})=2$$

Finally we obtain

$$\lambda(f)=\frac{8(m+1)(2n-m-1)+32(m+1)}{2m(2n-m)+2(n-m)+8(m+1)}$$

Fixing m and making n as big as needed we obtain fibrations with $g=4m+5, h=3$ and the slope arbitrarily near to

$$\frac{16(m+1)}{4m+2}=4+\frac{8}{g-3}.$$

Example 4.33

Let Y, B be a smooth surface and a smooth curve respectively. Denote π_Y, π_B the two natural projections of $Y \times B$. Let $C, E \in Div(Y)$ and $\gamma, \eta \in Div(B)$ such that there exist smooth divisors $L \in |2(\pi_Y^*(C) + \pi_B^*(\gamma))|$ and $V \in |2(\pi_Y^*(E) + \pi_B^*(\eta))|$. Let $\tau : Z \longrightarrow Y \times B$ the double cover branched on L and let $S = \tau^* V$. We denote $\pi = \tau_{|S}$, $\Phi = \pi_{B|V}$, $f = \Phi \circ \pi$. Assume $L_{|V}$ is a smooth divisor. Then S is smooth and $f : S \longrightarrow B$ is a fibration. If $m = \deg(\eta)$, $n = \deg(\gamma)$, $g = g(F)$ (F fibre of f) then

$$\lambda(f) = 6\frac{4(g-1)(n+m) + 2m(K_Y + C + E)^2}{(3n+6m)(g-1) + 12\chi\mathcal{O}_Y + 6m(g(C)-1) + 3nEC}$$

Indeed note that $\pi : S \longrightarrow V$ is ramified on $L_{|V}$ and that $\pi_{|F} : F \longrightarrow E$ is a double cover verifying $g - 1 = E^2 + K_Y E + EC$.

We have $K_V = (K_{Y\times B} + V)_{|V}$, $K_{Y\times B} = \pi_Y^*(K_Y) + \pi_B^*(K_B)$. From standard formulas of smooth double covers (cf. [7] p.182) we have

$$\begin{aligned} \chi\mathcal{O}_S &= 2\chi\mathcal{O}_V + \tfrac{1}{4}LK_V + \tfrac{1}{8}L^2 \\ K_S^2 &= 2K_V^2 + 2LK_V + \tfrac{1}{4}L^2 \end{aligned}$$

Then the computation follows by Riemann-Roch on $Y \times B$.

Let Y be a $K3$ surface with a genus 2 hyperplane section E. Take $E = C$ and $B = \mathbb{P}^1$, $n = m = 1$. We get then a double cover fibration with

$$\lambda(f) = 4$$

$$g = 5$$

$$h = 2$$

Hence, the slope 4 can be achieved by double cover fibrations with $h > 1$ (note that F can be chosen non-bielliptic).

Example 4.34

A standard example of a fibration $f : S \longrightarrow B$ is obtained considering a smooth divisor $S \subseteq Y \times B = Z$ where Y is a smooth surface, B is a smooth curve and f is the restriction on S of the natural projection π_B : $Y \times B \longrightarrow B$. We consider $\pi_Y : Y \times B \longrightarrow Y$ the other projection and two divisors $F \in Div(Y)$, $\eta \in Div(B)$ such that there exists a smooth divisor $S \in |\pi_Y^*(F)+\pi_B^*(\eta)|$. We will call product-type fibration this sort of fibrations. We will do the computations under the assumption that $\deg(\eta) = n > 0$, F is ample and $S \in Div(Z)$ is ample; so by the Kodaira vanishing Theorem we obtain

$$h^1(Z, K_Z + S) = h^2(Z, K_Z + S) = 0$$

Thus by the adjoint sequence

$$0 \longrightarrow \mathcal{O}_Z(K_Z) \longrightarrow \mathcal{O}_Z(K_Z + S) \longrightarrow \mathcal{O}_S(K_S) \longrightarrow 0$$

we have:

$$h^0(S, K_S) = h^0(Z, K_Z + S) - h^0(Z, K_Z) + h^1(Z, K_Z),$$

and

$$h^1(S, K_S) = h^2(Z, K_Z)$$

On the other hand since $Z = Y \times B$ we have:

$$h^2(Z, K_Z) = q(Y) + b \qquad h^0(Z, K_Z) = p_g(Y)b \qquad h^1(Z, K_Z) = bq(Y) + p_g(Y)$$

and

$$H^0(Z, K_Z + S) = H^0(Y, K_Y + F) \otimes H^0(B, K_B + \eta)$$

Moreover by our assumption

$$h^0(B, K_B + \eta) = b - 1 + n$$

and

$$h^0(Y, K_Y + F) = \chi\mathcal{O}_Y + \frac{1}{2}(K_Y F + F^2)$$

In particular we have $q(S) = b + q(Y)$ so note that $s = q(S) - b = q(Y)$. Also

$$p_g(S) = \left(\chi(\mathcal{O}_Y) + \frac{1}{2}(K_Y F + F^2)\right)(b + n - 1) - p_Y b + (p_g(Y) + q_Y b),$$

that is:

$$\chi(\mathcal{O}_S) = n\chi(\mathcal{O}_Y) + (n-1)(g-1) + bg - b$$

Since

$$K_S^2 = (K_Z + S)^2 S = n(K_Y + F)^2 + 2(2b - 2 + b)(K_Y + F)F$$

and

$$\chi = n\chi(\mathcal{O}_Y) + n(g-1)$$

we get

$$\lambda(f) = \frac{6g - 6 + K_Y^2 + K_Y F}{\chi(\mathcal{O}_Y) + g - 1}.$$

Let Y be a ruled surface over a curve C of genus h, and let e be the invariant of Y in the sense of [43] V.2. Take a smooth divisor $F \in |\alpha C_0 + \beta D|$, where $D = \mathbb{P}^1$ is a ruling and C_0 is a section with $C_0^2 = -e$. Observe that $s = q(Y) - g(C) = h$. A simple computation shows that if β grows the induced fibration f has a slope arbitrarily near to

$$\widetilde{\lambda} = 6 - \frac{2}{\alpha - 1} < 6.$$

For $\alpha = 2$ we get $\widetilde{\lambda} = 4$; for $\alpha = 3$ we obtain $\widetilde{\lambda} = 5$. Note that in general, when $\alpha = 3$, F is not a double cover.

Chapter 5

The slope of fibred threefolds

In this chapter we consider fibrations $f : T \longrightarrow B$ from a projective threefold T onto a smooth curve B. We consider always that T is normal, $\mathbb{Q}$-factorial with at most canonical singularities and that f is relatively minimal, i.e. the restriction of K_T to any fibre F of f is nef. As in the case of fibred surfaces we are interested on the cases where F is of general type. Given any fibration $g : \widetilde{T} \longrightarrow B$ from a smooth threefold $\widetilde{T}$ and with fibres of general type we can always get its relatively minimal associated fibration by divisorial contractions and flips (cf. [54], [72]).

As in the case of surfaces our aim is to relate the numerical invariants of T, B and F. First of all note that under our assumptions K_T (and hence $K_{T/B} = K_T - f^*K_B$) is a Weil, $\mathbb{Q}$-Cartier divisor. We can consider its associated divisorial sheaves ω_T and $\omega_{T/B}$, the canonical sheaf of T and the relative canonical sheaf of f respectively. Now $\mathcal{E} = f_*\omega_{T/B}$ is a locally free sheaf on B of rank $p_g(F)$. We have then the well defined numerical invariants (note that the first one may be a rational number):

$$K^3_{T/B} = K^3_T - 6K^2_F(b-1)$$

$$\Delta_f = \deg \mathcal{E}$$

$$\chi_f = (-1)^{\dim T}(\chi\mathcal{O}_T - \chi\mathcal{O}_B\chi\mathcal{O}_F) = \chi\mathcal{O}_F\chi\mathcal{O}_B - \chi\mathcal{O}_T.$$

From Chapter 3 we know that $\mathcal{E}$ is a nef vector bundle and hence that $\Delta_f \geq 0$. From the nefness of direct images of multiples of the relative dualizing sheaf it follows that $K_{T/B}$ is also nef (see [73], Theorem 1.4) and so that $K^3_{T/B} \geq 0$. Moreover, if $K^3_{T/B} = 0$ then f is isotrivial (cf. [73] corollary 1.5). When $\Delta_f = 0$ we can also deduce the isotriviality of f under some additional hypotheses (see Lemma 5.3 below).

As for the irregularity we have in the same way of fibred surfaces

$$b \leq q(T) \leq b + q(F)$$

and that, when $b \geq 1$, $b = q(T)$ if and only if the Albanese dimension of T is one and f is the Albanese fibration of T. Now is not more true that the upper equality $q(T) = b + q(F)$ implies the isotriviality of f.

Some other new phenomena appear. In the case of fibred surfaces equality $\Delta_f = \chi_f$ holds. This is not true in the case of fibred threefolds where in general $\Delta_f \neq \chi_f$ (in fact what happens is that $\Delta_f \geq \chi_f$). Observe that from the geographical point of view χ_f is more interesting than Δ_f.

With this language we can state the only known general result on the geography of fibred threefolds over curves, due to Ohno (cf. [73]).

Theorem 5.1 *Let $f : T \longrightarrow B$ be a relatively minimal fibration of surfaces of general type over a smooth curve of genus b.*

(i) If $p_g(F) \geq 3$ and $|K_F|$ is not composed of a pencil, then

$$K^3_{T/B} \geq (4 - \frac{8}{p_g(F)})\Delta_f \geq (4 - \frac{8}{p_g(F)})\chi_f$$

(ii) If $|K_F|$ is composed of a pencil and F is not a surface with $K^2_F = 1$, $p_g(F) = 2, q(F) = 0$ then

$$K^3_{T/B} \geq (4 - \frac{4}{p_g(F)})\Delta_f \geq (4 - \frac{4}{p_g(F)})\chi_f$$

(iii) If $K^2_F = 1, p_g(F) = 2$ and $q(F) = 0$, then

$$K^3_{T/B} \geq \Delta_f \geq \chi_f$$

(iv) If $p_g(F) = 1$, then

$$K^3_{T/B} \geq (K^2_F)\Delta_f \geq (K^2_F)\chi_f$$

(v) If $p_g(F) = 0$, then

$$K^3_{T/B} \geq \frac{2}{3}l(2) \quad \text{if} \quad K^2_F \geq 2$$
$$K^3_{T/B} \geq \frac{4}{3}l(2) \quad \text{if} \quad K^2_F = 1$$

where $l(2)$ is the second correction term in the plurigenera formula of Reid-Fletcher for T.

We can define two different slopes for fibred threefolds with fibres of general type

$$\lambda_1(f) = \frac{K^3_{T/B}}{\Delta_f}$$
$$\lambda_2(f) = \frac{K^3_{T/B}}{\chi_f}$$

whenever $\Delta_f, \chi_f \neq 0$; the results of Ohno have an easy translation in terms of these. But there is a new problem: it is not known whether $\chi_f \geq 0$. In fact in [73] this question is avoided. If $\chi_f < 0$, Theorem 5.1 still holds, since always $K^3_{T/B} \geq 0$ holds, but it gives no information. In §5.1 we prove that $\chi_f \geq 0$ holds in most cases and give some examples of $\chi_f < 0$. We also study in this section the behaviour of both slopes under certain natural operations.

We consider then fibrations with $\Delta_f \neq 0, \chi_f > 0$ and study the two slopes defined above. We obtain a behaviour of the slope quite analogous to the slope of fibred surfaces. In fact, and surprisingly enough, the behaviour is better in some cases.

First of all (cf. Theorem 5.11) we obtain that both slopes verify a considerably better bound than Ohno's one as far as $p_g >> 0$ and F is not fibred by hyperelliptic, trigonal or tetragonal curves: Ohno's bound is assimptotically 4 whereas our bound tends to be 9. Moreover we obtain alternative bounds in the case F is fibred by hyperelliptic, trigonal or tetragonal curves. Observe that the behaviour of the slopes correspond completely to that hoped for fibred surfaces (see Problem 1 in the introduction to Chapter 4): the non-existence of certain special pencils on F make the slope bigger.

Then we consider the influence of the irregularity on the slope. It turns out that $\lambda_2(f)$ is sensible to $q(T) - b$. We prove that $\lambda_2(f) \geq 9$ whenever $q(T) - b > 0$ and F is not special in the above sense (see Theorem 5.14). Moreover we can give alternative bounds in the special cases and a structure result for those non Albanese fibrations with $\lambda_2 < 9$.

Finally we study fibrations with very low slope ($\lambda_2 < 4$). These are known to exist (cf. [73] p.664); in [73], Ohno gives a classification of them in seven (possible) families (see theorem 5.19 below). We prove that the same phenomena that appears in fibred surfaces hold: if $\lambda_2 < 4$ then $q(T) = b$ (i.e. $q(T) = b = 0$ or f is an Albanese fibration), $\mathcal{E} = f_*\omega_{T/B}$ is ample if $b \leq 1$ (in fact for all b provided Fujita's conjecture holds) and finally the canonical map of F is not birational in most cases. In fact we prove that F is fibred by hyperelliptic curves if $p_g(F) \geq 8$ (of genus 2 if $p_g(F) \geq 15$) and by hyperelliptic or trigonal curves otherwise, except for one residual case which is doubtful to exist (see Theorem 5.20 and Remark 5.21).

5.1 Preliminary results

Definition 5.2 Let $f : T \longrightarrow B$ be a fibration of a normal, $\mathbb{Q}$-factorial, projective threefold with only canonical singularities onto a smooth curve. Let F be the general fibre of f and $b = g(B)$. We define

$$\Delta_f = \deg f_* w_{T/B}$$
$$\chi_f = \chi\mathcal{O}_B \chi\mathcal{O}_F - \chi\mathcal{O}_T$$

Lemma 5.3

(i) $\Delta_f = \chi_f + \deg R^1 f_*\omega_{T/B} \geq \chi_f$

(ii) $\Delta_f \geq 0$. *If* $\Delta_f = 0$ *and* $|K_F|$ *is birational, then* f *is isotrivial.*

(iii) If $\beta : \widetilde{T} \longrightarrow T$ *is a nonsingular model of* T *and* $\widetilde{f} = f \circ \beta$*, then* $\chi_f = \chi_{\widetilde{f}}$, $\Delta_f = \Delta_{\widetilde{f}}$.

PROOF:

(i) Follows from [73] Lemma 2.4 and 2.5. We sketch briefly a proof. The equality $\chi_f = \deg f_*\omega_{T/B} - \deg R^1 f_*\omega_{T/B}$ follows by Leray spectral sequence and duality on T (observe that canonical singularities are Cohen-Macaulay, cf. [78]). The inequality $\Delta_f \geq \chi_f$ follows from the nefness of $R^1 f_*\omega_{T/B}$ (see Chapter 3).

(ii) $\Delta_f \geq 0$ follows from the nefness of $\mathcal{E}$ ([30] and chapter 3). If $\Delta_f = 0$ and $|K_F|$ is birational we can apply [57] I (see also [71], 7.64).

(iii) Canonical singularities are rational (cf. [25]) and hence $R^i\beta_*\mathcal{O}_{\widetilde{T}} = 0$ for $i \geq 1$ (cf. [56], p.50). Hence $\chi\mathcal{O}_{\widetilde{T}} = \chi\mathcal{O}_T$. The same holds for general fibres F and $\widetilde{F}$ of f and $\widetilde{f}$ respectively, so $\chi_f = \chi_{\widetilde{f}}$.

By Grauert-Riemenschneider's vanishing we have $R^i\beta_*\omega_{\widetilde{T}} = 0$ for $i \geq 1$. Hence using the spectral sequence $E_2^{p,q} = R^p f_*(R^q\beta_*\omega_{\widetilde{T}}) \Rightarrow R^{p+q}\widetilde{f}_*\omega_{\widetilde{T}}$ we obtain that for every $i \geq 0$ $R^i f_*\omega_T = R^i\widetilde{f}_*\omega_{\widetilde{T}}$ holds

■

Definition 5.4 With the above notations if f is relatively minimal and the general fibre F of f is of general type we define

$$\lambda_1(f) = \frac{K^3_{T/B}}{\Delta_f} \quad \text{if} \quad \Delta_f \neq 0$$

$$\lambda_2(f) = \frac{K^3_{T/B}}{\chi_f} \quad \text{if} \quad \chi_f > 0$$

Remark 5.5 In the case of fibrations of surfaces over curves we actually have $\Delta_f = \chi\mathcal{O}_S - \chi\mathcal{O}_B\chi\mathcal{O}_F = \chi_f \geq 0$. Also we had that vanishing holds only in the locally trivial case. Then when f is not locally trivial we defined $\lambda(f) = K^2_{S/B}\big/\Delta_f = K^2_{S/B}\big/\chi_f$. Here we have two different possibilities for the slope of f : $K^3_{T/B}\big/\Delta_f$ or $K^3_{T/B}\big/\chi_f$. As we will see in §5.2, the natural methods provide lower bounds for $\lambda_1(f)$ (hence also for $\lambda_2(f)$: see Lemma 5.6 below). Note that from the *geographical* point of view the most interesting one is $\lambda_2(f)$. But for this election we do not know whether $\chi_f \geq 0$. The aim of this section is to show that this actually happens for general fibrations.

Lemma 5.6 *Assume $\chi_f > 0$. Then*

(i) $\lambda_2(f) \geq \lambda_1(f)$.

(ii) If $\sigma : \widetilde{B} \longrightarrow B$ is a finite map whose branch locus does not contain the images of singular fibres of f and $\widetilde{f} : \widetilde{T} = T \underset{B}{\times} \widetilde{B} \longrightarrow \widetilde{B}$ is the induced fibration, then $\lambda_i(f) = \lambda_i(\widetilde{f})$, $i = 1, 2$.

(iii) If $\widetilde{T} \xrightarrow{\alpha} T$ is an étale map such that $\widetilde{f} = f \circ \alpha$ has connected fibres, then $\lambda_2(f) = \lambda_2(\widetilde{f})$ (but in general $\lambda_1(f) \neq \lambda_1(\widetilde{f})$).

PROOF: (i) is obvious.

(ii) If σ does not ramify over the images of singular fibres of f then $\widetilde{T} = T \underset{B}{\times} \widetilde{B}$ is again a normal, $\mathbb{Q}$-factorial relatively minimal threefold over $\widetilde{B}$ with only canonical singularities (cf. [71], 4.10). Clearly $K^3_{\widetilde{T}/\widetilde{B}} = (\deg\sigma)K^3_{T/B}$ and $\deg R^i\widetilde{f}_*\omega_{\widetilde{T}/\widetilde{B}} = (\deg\sigma)R^if_*\omega_{T/B}$ by flat base change. Then $n\Delta_f = \Delta_{\widetilde{f}}$, $n\chi_f = \chi_{\widetilde{f}}$ and we are done.

(iii) From the standard theory of cyclic coverings (cf. [89]) we get that $\alpha_*\omega_{\widetilde{T}} = \oplus_{i=0}^{n-1}(\omega_T \otimes \mathcal{L}^{\otimes i})$. Since α is finite we have

$$R^j\widetilde{f}_*w_{\widetilde{T}/B} = \bigoplus_{i=0}^{n-1} R^jf_*(w_{T/B} \otimes \mathcal{L}^{\otimes i})$$

and

$$K^3_{\widetilde{T}/B} = n(K_{T/B} + (n-1)L)^3 = nK^3_{T/B}\,.$$

Hence

$$\begin{aligned}\chi_{\widetilde{f}} &= \deg\, \widetilde{f}_*w_{\widetilde{T}/B} - \deg\, R^1\widetilde{f}_*w_{\widetilde{T}/B} = \\ &= \textstyle\sum_{i=0}^{n-1}(\deg\, f_*(w_{T/B} \otimes \mathcal{L}^{\otimes i}) - \deg\, R^1f_*(w_{T/B} \otimes \mathcal{L}^{\otimes i})) = n\chi_f.\end{aligned}$$

Indeed, to prove this last equality observe first that by relative duality $\deg R^2f_*\omega_{T/B} = \deg f_*\mathcal{O}_T = \deg\mathcal{O}_B$. On the other hand for $1 \leq i \leq n-1$, $h^2(F, \omega_F \otimes \mathcal{L}^{\otimes i}_{|F}) = h^0(F, \mathcal{L}^{\otimes i}_{|F}) = 0$ since $\mathcal{L}^{\otimes i}_{|F}$ is non trivial. Then $\mathrm{rank}(R^2f_*(\omega_{T/B} \otimes \mathcal{L}^{\otimes i})) = 0$; since $R^2f_*(\omega_{T/B} \otimes \mathcal{L}^{\otimes i})$ is a subsheaf of $R^2\widetilde{f}_*\omega_{\widetilde{T}/B}$ (which is locally free, see Chapter 3) it is torsion free and hence $R^2f_*(\omega_{T/B} \otimes$

$\mathcal{L}^{\otimes i}) = 0$. Finally note that $\chi(T, \omega_{T/B} \otimes \mathcal{L}^{\otimes i}) = \chi(T, \omega_T)$ ($\mathcal{L} \in Pic^0(T)$). Then from Leray's spectral sequence and Riemann-Roch on B we get

$$\begin{aligned} \chi(\omega_{T/B}) &= \chi(T, \omega_{T/B} \otimes \mathcal{L}^{\otimes i}) = \sum_{j=0}^{2} (-1)^j \chi(B, R^j f_*(\omega_{T/B} \otimes \mathcal{L}^{\otimes i})) = \\ &= \left[\sum_{j=0}^{2} (-1)^j \mathrm{deg} R^j f_*(\omega_{T/B} \otimes \mathcal{L}^{\otimes i}) \right] + \chi(F, \omega_F \otimes \mathcal{L}^{\otimes i}_{|F}) \chi \mathcal{O}_B. \end{aligned}$$

Since again $\chi(F, \omega_F \otimes \mathcal{L}^{\otimes i}_{|F}) = \chi(F, \omega_F)$ we obtain that

$$\sum_{j=0}^{2} (-1)^j \mathrm{deg} R^j f_*(\omega_{T/B} \otimes \mathcal{L}^{\otimes i}) = \mathrm{deg} f_*(\omega_{T/B}) - \mathrm{deg} R^1 f_*(\omega_{T/B}) = \chi_f.$$

Then $\lambda_2(f) = \lambda_2(\widetilde{f})$.

Note that in general deg $f_*(w_{T/B} \otimes \mathcal{L}^{\otimes i}) \neq$ deg $f_* w_{T/B}$ and hence λ_1 need not be invariant. $\blacksquare$

The question now is whether $\chi_f \geq 0$ holds. This is not always true (see Remark 5.8). We give criteria for its non-negativity depending on the Albanese dimension of T. By Remark 5.3 we have that $\chi_f = \chi_{\widetilde{f}}$ where $\widetilde{f} = f \circ \beta$, $\beta : \widetilde{T} \longrightarrow T$ being a desingularization. Hence we can assume T is smooth.

First of all, consider $t \in B$, such that F_t is smooth, and the Albanese maps

$$F_t[r]^{\mathrm{alb}_{F_t}}[d]^{i_t}\mathrm{Alb}(F_t)[d]_{(i_t)_*}T[r]^{\mathrm{alb}_T}[d]^f\mathrm{Alb}(T)[d]_{f_*}B[r]^{\mathrm{alb}_B}\mathrm{Alb}(B)$$

and let $\Sigma = \mathrm{alb}_T(T)$. We set $a = \mathrm{albdim}(T) = \dim \Sigma$. Let $S \longrightarrow \Sigma$ be a minimal desingularization of Σ and $\pi : \widetilde{T} \longrightarrow S$ the induced map on a birational model of T.

Note that by rigidity, Im $(i_t)_* = A$ is an abelian variety independent of t, of dimension $q(T) - b$.

Consider also the induced map $Pic^0 T \xrightarrow{(i_t)^*} Pic^0(F_t)$ which image is $\widehat{A} \hookrightarrow Pic^0(F_t)$. We say that f is *special* if for general $t \in B$ $\widehat{A} \hookrightarrow (h_t)^*(Pic^0 C_t) \subseteq Pic^0(F_t)$, for some $h_t : F_t \longrightarrow C_t$ a fibration over a curve of genus $g(C_t) \geq 2$. Otherwise we say that f is *general*.

Theorem 5.7 *Let $f : T \longrightarrow B$ be a fibration of a normal, $\mathbb{Q}$-factorial, projective threefold with only canonical singularities onto a smooth curve of genus b. Let F be a general fibre of f. Let $a = \dim alb(T)$.*

Then $\chi_f = \chi\mathcal{O}_B\chi\mathcal{O}_F - \chi\mathcal{O}_T \geq 0$ provided one of the following conditions holds

(i) $b \leq 1$ *and* $\chi\mathcal{O}_T \leq 0$.

(ii) $a = 2$ *and* $h^{\circ}(S, \pi_* w_{\widetilde{T}/S}) \neq 0$.

(iii) $a = 3$, *f is special and has no non-reduced fibres.*

(iv) $a = 3$ *and f is general.*

Remark 5.8 Part (i) of the theorem is trivial since when $b \leq 1$ $\chi_f \geq -\chi\mathcal{O}_T$. We only want to remark that condition $\chi\mathcal{O}_T \leq 0$ holds in most cases. Indeed, if T is smooth and K_T ample then $\chi\mathcal{O}_T = \frac{1}{24}c_1c_2 < 0$ by Miyaoka-Yau inequality. Also, if T is minimal and Gorenstein $\chi\mathcal{O}_T \leq 0$ holds (cf. [70]). Finally, if $a = 3$ then $\chi\mathcal{O}_T \leq 0$ by a consequence of generic vanishing results (see [35]). Observe that if $a = 0$ then necessarily $q(T) = b = 0$ and hence this possibility is included in (i).

Extra conditions included for the cases $a = 2, 3$ are very mild. This is clear for the case $a = 3$. In the case $a = 2$ observe that if E is a general curve on S and H is its pullback on $\widetilde{T}$, then $h^0(S, \pi_*\omega_{\widetilde{T}/S}) = h^0(E, \pi_*\omega_{H/E})$ (see proof of part (iii) of the theorem). Then, remember that by Proposition 3.1.(v) $\pi_*\omega_{H/E}$ is nef and contains an ample vector bundle, so condition $h^0(E, \pi_*\omega_{H/E}) \neq 0$ is not very restrictive.

We want to stress that in the statement of the theorem some hypotheses are needed since $\chi_f \geq 0$ *is not always true.* Indeed, we can construct counterexamples as follows. Let (C_i, τ_i) (i=1,2,3) be smooth curves with an involution. Let $D_i = C_i/\tau_i$ and $g_i = g(C_i) \geq 2$, $b_i = g(D_i)$. Consider $X = C_1 \times C_2 \times C_3$ and $\tau : X \longrightarrow X$ the involution acting on C_i as τ_i. Consider $T = X/\tau$. Then T is a threefold of general type with a finite number of (canonical) singularities, endowed with a fibration $f : T \longrightarrow D_1 =: B$ with general fibre $F \cong C_2 \times C_3$ (hence it is isotrivial). Since $H^0(T, \Lambda^i\Omega^1_T) \cong H^0(X, \Lambda^i\Omega^1_X)^\tau$ (see [29]) a standard computation shows that

$$\begin{aligned}
\chi\mathcal{O}_B &= 1 - b_1\\
\chi\mathcal{O}_F &= (1-g_1)(1-g_2)\\
h^1\mathcal{O}_T &= q(T) = b_1 + b_2 + b_3\\
h^2\mathcal{O}_T &= b_1b_2 + b_1b_3 + b_2b_3 + (g_1-b_1)(g_2-b_2) + (g_1-b_1)(g_3-b_3)+\\
&\quad +(g_2-b_2)(g_3-b_3)\\
h^3\mathcal{O}_T &= b_1b_2b_3 + (g_1-b_1)(g_2-b_2)b_3 + (g_1-b_1)(g_3-b_3)b_2+\\
&\quad +(g_2-b_2)(g_3-b_3)b_1.
\end{aligned}$$

If we take $b_1 = b_2 = b_3 = 0$ (C_i must be hyperelliptic then) we obtain $a = 0$ and $\chi_f < 0$. Any base change of this fibration to a curve of positive genus produces a new fibration with $q(\widetilde{T}) = \widetilde{b} \geq 1$ (hence with $a = 1$) and $\chi_{\widetilde{f}} < 0$.

If we take $b_1 = b_2 = 1$, $b_3 = 0$ we obtain again $\chi_f < 0$ and $q(T) > b$ (so $a \geq 2$).

Finally we must say that we do not have any reasonable criteria for the nonnegativity of χ_f when $a = 1$, which corresponds to the case when $q(T) = b$, $f_* = \mathrm{Id}$ and alb_T factors through f (f is, then, an Albanese fibration). Nevertheless this is precisely the case in which we are not interested in the main theorem of the chapter (cf. Theorem 5.14).

Proof: (ii) We can assume T smooth. We can also assume, by the same arguments, that $\pi : T \longrightarrow S$ has ramification locus contained in a normal crossings divisor and that is flat using a *flattening* base change ([71], 1.10).

We have now a factorization of f

$$T[dd]_f[dr]^\pi S[dl]^g B$$

where g need not to be a relatively minimal fibration. Let $C_t = g^{-1}(t)$ be a general fibre ($g(C_t) \geq 1$) and $\pi_t : F_t \longrightarrow C_t$ the induced fibration. Let G be a general fibre of π_t. Note that we have $(\pi_t)_*\omega_{F_t/C_t} = \pi_*\omega_{T/S} \otimes \mathcal{O}_{C_t}$ and hence

$(R^1\pi_t)_*\mathcal{O}_{F_t} = (R^1\pi_*\mathcal{O}_T)\otimes\mathcal{O}_{C_t}$ by relative duality on C_t (see proof of Theorem 3.4).

Take a n-torsion element $\mathcal{L} \in Pic^0(S)$ verifying that for $1 \le i \le n-1$ $\mathcal{L}^{\otimes i} \notin g^*(Pic^0(B))$ (this is possible since S is of Albanese general type by construction) and such that $h^\circ(C_t, R^1(\pi_t)_*\mathcal{O}_{F_t}\otimes\mathcal{L}_{|C_t}) = 0$ (this is also possible since $\{\widetilde{\mathcal{L}} \in Pic^0(C_t) \mid h^\circ(C_t, (R^1\pi_t)_*\mathcal{O}_{F_t}\otimes\widetilde{\mathcal{L}}) \ne 0\}$ is a finite set (see Proposition 3.3) and the image of $Pic^0(S) \longrightarrow Pic^0(C_t)$ is a subtorus of positive dimension otherwise $q(S) = b$, a contradiction).

Let $\mathcal{M} = \pi^*\mathcal{L} \in Pic^0(T)$. Since π has a normal crossings ramification locus we have that $R^1\pi_*\mathcal{O}_T$ is locally free (cf. [59], Theorem 2.6 and Chapter 3) and hence $g_*(R^1\pi_*\mathcal{O}_T\otimes\mathcal{L})$ is torsion free (hence it is locally free since B is a smooth curve). Then

$$g_*(R^1\pi_*\mathcal{O}_T\otimes\mathcal{L}) = 0$$

since rk $g_*(R^1\pi_*\mathcal{O}_T\otimes\mathcal{L}) = h^\circ(C_t, R^1(\pi_t)_*\mathcal{O}_{F_t}\otimes\mathcal{L}_{|C_t}) = 0$ by election of $\mathcal{L}$.

Using the spectral sequence $E_2^{p,q} = R^p g_*(R^q\pi_*\mathcal{F}) \Rightarrow R^{p+q}f_*\mathcal{F}$ and that $R^2\pi_* = R^2 g_* = 0$ by dimension of fibres we have

$$R^2 f_*(\mathcal{M}) = R^1 g_*(R^1\pi_*(\mathcal{M})) = R^1 g_*(R^1\pi_*\mathcal{O}_T\otimes\mathcal{L}). \tag{5.1}$$

We observe that $R^2 f_*(\mathcal{M})$ is locally free. Indeed $\mathcal{M} = \pi^*\mathcal{L}$; since $\mathcal{L}$ is torsion and $\mathcal{L}_{|C}^{\otimes i} \ne \mathcal{O}_C$ for $1 \le i \le n-1$ we can consider the induced étale base change of π:

$$\widehat{T}[r][d]_{\widehat{\pi}}T[d]^{\pi}\widehat{S}[r][dr]_{\widehat{g}}S[d]^{g}B$$

and get

$$R^j\widehat{f}_*\omega_{\widehat{T}/B} = \bigoplus_{i=0}^{n-1} R^j f_*(\omega_{T/B}\otimes\mathcal{M}^{\otimes i}) \tag{5.2}$$

$$R^j\widehat{f}_*\mathcal{O}_{\widehat{T}} = \bigoplus_{i=0}^{n-1} R^j f_*(\mathcal{M}^{\otimes i}).$$

Hence $R^2 f_*(\mathcal{M})$ is locally free, being a subsheaf of $R^2\widehat{f}_*\mathcal{O}_{\widehat{T}}$ (which is locally free by Proposition 3.1).

Finally remember that for fibrations of surfaces over curves we have (see Chapter 4; we do not need for this that the fibration be relatively minimal):

$$\deg\,(\pi_t)_*\omega_{F_t/C_t} = \chi\mathcal{O}_F - \chi\mathcal{O}_C\chi\mathcal{O}_G; \quad \deg\, g_*\omega_{S/B} = \chi\mathcal{O}_S - \chi\mathcal{O}_C\chi\mathcal{O}_B.$$

Now we can compute

$$\begin{array}{lll}
\chi\mathcal{O}_T & = \chi(\pi_*\mathcal{O}_T) - \chi(R^1\pi_*\mathcal{O}_T) = & \\
& = \chi\mathcal{O}_S - \chi(R^1\pi_*\mathcal{O}_T \otimes \mathcal{L}) & \text{since} \quad \mathcal{L} \in Pic^0(S) \\
& = \chi\mathcal{O}_S + \chi(R^1 g_*(R^1\pi_*\mathcal{O}_T \otimes \mathcal{L})) & \text{by Leray} \\
& = \chi\mathcal{O}_S + \chi(R^2 f_*(\mathcal{M})) & \text{by (5.1)} \\
& = \chi\mathcal{O}_S + \chi(f_*(\omega_{T/B} \otimes \mathcal{M}^{-1})^*) & \text{by relative duality} \\
& = \chi\mathcal{O}_S - \deg(f_*(\omega_{T/B} \otimes \mathcal{M}^{-1})) + h^0(F, \omega_F \otimes \mathcal{M}^{-1}_{|F})\chi\mathcal{O}_B & \text{by R.R. on } B
\end{array}$$

By election of $\mathcal{L}$, Serre duality on B and relative duality on B and C we obtain

$$h^0(F, \omega_F \otimes \mathcal{M}^{-1}_{|F}) = h^0(C, \pi_*\omega_{F/C} \otimes \omega_C \otimes \mathcal{L}^{-1}_{|C}) = \chi(\pi_*\omega_{F/C} \otimes \omega_C \otimes \mathcal{L}^{-1}_{|C}) \quad (5.3)$$

$$\begin{aligned}
&= -\chi(R^1\pi_*\mathcal{O}_F \otimes \mathcal{L}) = -\chi(R^1\pi_*\mathcal{O}_F) = (\chi\mathcal{O}_F - \chi\mathcal{O}_C\chi\mathcal{O}_G) - g(G)\chi\mathcal{O}_C \\
&= \chi\mathcal{O}_F - \chi\mathcal{O}_C.
\end{aligned}$$

Hence:

$$\chi_f = \chi\mathcal{O}_F\chi\mathcal{O}_B - \chi\mathcal{O}_T = \deg(f_*(\omega_{T/B} \otimes \mathcal{M}^{-1})) - \deg(g_*\omega_{S/B}) \qquad (5.4)$$

Now we use the hypothesis: $\pi_*\omega_{T/S}$ has a section and hence we have an injection

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \pi_*\omega_{T/S}$$

which gives

$$0 \longrightarrow \omega_{S/B} \otimes \mathcal{L}^{-1} \longrightarrow \pi_*\omega_{T/S} \otimes \omega_{S/B} \otimes \mathcal{L}^{-1} = \pi_*(\omega_{T/B} \otimes \mathcal{M}^{-1})$$

and so

$$0 \longrightarrow g_*(\omega_{S/B} \otimes \mathcal{L}^{-1}) \xrightarrow{\tau} g_*(\pi_*(\omega_{T/B} \otimes \mathcal{M}^{-1})) = f_*(\omega_{T/B} \otimes \mathcal{M}^{-1}) \qquad (5.5)$$

Note that $\deg(g_*(\omega_{S/B} \otimes \mathcal{L}^{-1})) = \deg g_*\omega_{S/B}$. Indeed, by election of $\mathcal{L}$ we have that $h^1(C, \omega_C \otimes \mathcal{L}^{-1}_{|C}) = 0$; on the other hand $R^1 g_*(\omega_{S/B} \otimes \mathcal{L}^{-1})$ is locally free, being a subsheaf of $R^1\widetilde{g}_*\omega_{\widetilde{S}/B}$ for the étale cover $\widetilde{S} \longrightarrow S$ induced by $\mathcal{L}$. Hence $R^1 g_*(\omega_{S/B} \otimes \mathcal{L}^{-1}) = 0$. Then use Leray spectral sequence and Riemann-Roch on B and S as in the proof of Lemma 5.6 (iii).

In order to finish the proof it suffices to check that $f_*(\omega_{T/B} \otimes \mathcal{M}^{-1})$ is nef, since from (5.4) and (5.5) we have that $\chi_f = \deg \operatorname{Coker}(\tau)$ (which will be positive if $f_*(\omega_{T/B} \otimes \mathcal{M}^{-1})$ is nef). By (5.2) we have that $f_*(\omega_{T/B} \otimes \mathcal{M}^{-1}) = f_*(\omega_{T/B} \otimes \mathcal{M}^{\otimes(n-1)})$ is nef since it is a quotient of a nef vector bundle.

(iii) Assume that for general $t \in B$ we have a fibration $h_t : F_t \longrightarrow C_t$. Let $\overset{\circ}{B} \subseteq B$ be a non-empty open set such that $f^\circ : \overset{\circ}{T} \longrightarrow \overset{\circ}{B}$ is smooth and for every $t \in \overset{\circ}{B}$ there exists such a h_t. We can consider now the fibration of abelian varieties $\psi : \mathrm{Alb}_{\overset{\circ}{T}/\overset{\circ}{B}} \longrightarrow \overset{\circ}{B}$. For every $t \in \overset{\circ}{B}$ we have an abelian subvariety $K_t = \ker(\mathrm{Alb}F_t \longrightarrow \mathrm{Alb}C_t) \hookrightarrow \mathrm{Alb}F_t = \psi^{-1}(t)$. Then we can apply §1.4 and get, up to base change, a relative abelian subvariety $K \hookrightarrow \mathrm{Alb}_{\overset{\circ}{T}/\overset{\circ}{B}}$ over $\overset{\circ}{B}$. Let $J = \mathrm{Alb}_{\overset{\circ}{T}/\overset{\circ}{B}}\big/K$. Consider the natural map, up to base change, $\varphi : \overset{\circ}{T} \longrightarrow \mathrm{Alb}_{\overset{\circ}{T}/\overset{\circ}{B}} \longrightarrow J$ over $\overset{\circ}{B}$. For general $t \in \overset{\circ}{B}$, $\varphi_t : F_t \longrightarrow J_t$ has as image C_t by construction. Let $\overset{\circ}{S} = \varphi(\overset{\circ}{T})$ and complete the map to get

$$\widetilde{T}[d][dr]^{\pi} T[d]_f \overline{T}[l]_{\bar{\sigma}}[r][d]_{\bar{f}} S[dl] B \overline{B}[l]^{\sigma}$$

Note that we are in the same situation for $\bar{f}$ as in (ii). We have even more since by construction the hypothesis $h^0(\pi_*\omega_{T/S}) > 0$ holds; indeed, let E be a general curve on S and let H be its pullback on T. We have as in proof of Theorem 3.4 that

$$\pi_*\omega_{H/E} = (\pi_*\omega_{T/S}) \otimes \mathcal{O}_E.$$

If E is ample enough, we have

$$h^0(S, \pi_*\omega_{T/S} \otimes \mathcal{O}_S(-E)) = h^1(S, \pi_*\omega_{T/S} \otimes \mathcal{O}_S(-E)) = 0.$$

Hence $h^0(S, \pi_*\omega_{T/S}) = h^0(E, \pi_*\omega_{H/E})$. In the case of fibred surfaces

$h^0(E, \pi_* \omega_{H/E}) \geq q(H) - g(E)$ according to Fujita's decomposition (see Chapter 3). Finally note that, by [16], $q(H) - g(E) \geq q(T) - q(S) \geq 1$.

So we can apply the same argument than in (ii) and get $\chi_{\bar{f}} \geq 0$.

We apply now the hypothesis: f has no non-reduced fibres. In fact, after some blow-ups in T and $\overline{T}$, that do not modify χ_f and $\chi_{\bar{f}}$, we can assume f to be semistable, and $\overline{T}$ smooth. Then we have

$$\overline{T}[r]^{\alpha}[dr]_{\bar{f}} T \underset{B}{\times} \overline{B}[d]^{f'}[r] T[d]^{f} \overline{B}[r]_{\sigma} B$$

where α is induced by $\overline{f}$ and $\overline{\sigma}$. Since f is semistable we can apply base change theorem ([71], 4.10) and get

$$\overline{f}_* \omega_{\overline{T}/\overline{B}} = f'_* \omega_{T \underset{B}{\times} \overline{B}/\overline{B}} = \sigma^*(f_* \omega_{T/B}) \, .$$

In fact we also have the same equality for $R^1 f_*$: take $\mathcal{H}$ a very ample line bundle in T and let H be a general smooth member of its associated linear system. We have in a natural way

$$0 \longrightarrow f_* \omega_{T/B} \longrightarrow f_*(\omega_{T/B} \otimes \mathcal{H}) \longrightarrow f_* \omega_{H/B} \longrightarrow R^1 f_* \omega_{T/B} \longrightarrow 0$$

since $R^1 f_*(\omega_{T/B} \otimes \mathcal{H}) = 0$ (by Kodaira vanishing $h^1(F, \omega_F \otimes \mathcal{H}_{|F}) = 0$ and $R^1 f_*(\omega_{T/B} \otimes \mathcal{H})$ is locally free by the trick of a cyclic cover used in (ii)).

Note that all of them are locally free. Hence we have that after taking σ^* we still have a long exact sequence. Considering the analogous exact sequence for $\overline{f}$ and the natural maps we get

$$0[r] \sigma^*(f_* \omega_{T/B})[r] \sigma^*(f_* \omega_{T/B} \otimes \mathcal{H})[r] \sigma^*(f_* \omega_{H/B})[r] \sigma^*(R^1 f_* \omega_{T/B})[r] 0 0[r] \bar{f}_*(\omega_{\overline{T}/\overline{B}})[r][u]^{\cong} \bar{f}_*(\omega_{\overline{T}/\overline{B}} \otimes$$

where γ is naturally induced and exhaustive. Since $R^1 \bar{f}_* \omega_{\overline{T}/\overline{B}}$ and $\sigma^*(R^1 f_* \omega_{T/B})$ are both locally free sheaves of the same rank over $\overline{B}$, γ is an isomorphism.

So we have

$$0 \leq \chi_{\bar{f}} = \deg \bar{f}_* \omega_{\overline{T}/\overline{B}} - \deg\, R^1 \bar{f}_* \omega_{\overline{T}/\overline{B}} = n(\deg f_* \omega_{T/B} - \deg R^1 f_* \omega_{T/B}) = n \chi_f \, .$$

(iv) Since T is of Albanese general type, then so is F_t for $t \in B$ general. We can apply then Theorem 1.34 to get that $\{\mathcal{L} \in Pic^0(F_t) |\, h^1(F_t, \mathcal{L}) \neq 0\}$ is the union of subtori $h_i^*(Pic^0(C_i))$ for fibrations $h_i : F_t \longrightarrow C_i$ with $g(C_i) \geq 2$ and a finite number of (torsion) points.

Under our assumptions we can take a n-torsion element $\mathcal{L} \in Pic^0(T)$ such that for $1 \leq i \leq n-1$, $h^1(F_t, \mathcal{L}^{\otimes i}_{|F_t}) = 0$.

Hence as in Lemma 5.6 (iii) if we consider the étale cover $\sigma : \widetilde{T} \longrightarrow T$ associated to $\mathcal{L}$, and $\widetilde{f} = f \circ \sigma$ we have that $f_*(\omega_{T/B} \otimes \mathcal{L})$ is a quotient of $\widetilde{f}_*\omega_{\widetilde{T}/B}$ hence it is nef.

Since $h^1(F_t, \mathcal{L}^{-1}_{|F_t}) = 0$ we have $R^1 f_*(\omega_{T/B} \otimes \mathcal{L}) = 0$ (as above it is locally free) and hence

$$\begin{aligned}\chi_f &= \deg f_*\omega_{T/B} - \deg R^1 f_*\omega_{T/B} = \\ &= \deg f_*(\omega_{T/B} \otimes \mathcal{L}) - \deg R^1 f_*(\omega_{T/B} \otimes L) = \deg f_*(\omega_{T/B} \otimes \mathcal{L}) \geq 0\end{aligned}$$

■

5.2 Slopes of fibred threefolds

We give here a lower bound for $\lambda_1(f)$ (and hence for $\lambda_2(f)$ provided it is well defined) in the case of a relatively minimal fibred threefold with fibres of general type. The bounds we obtain are considerably better than Ohno's ones ([73] Main Theorem 1; see Theorem 5.1) as far as $p_g(F) \gg 0$.

First we need some results on linear systems on surfaces of general type.

Lemma 5.9 *Let F be a minimal surface of general type such that $p_g(F) \geq 3$ and let $\tau : \widetilde{F} \longrightarrow F$ be a birational morphism. Let $0 \leq P \leq Q \leq \tau^* K_F$ be two nef and effective divisors, such that the complete linear systems $|P|$ and $|Q|$ are base point free. Let $r \leq s$ be the ranks of $|P|$ and $|Q|$ respectively. Let Σ be the image of $\widetilde{F}$ through the map φ induced by $|P|$. Then*

(i) If φ is a generically finite map then we have

- $P(\tau^* K_F) \geq P^2 \geq 2r - 4 + 2q(\Sigma)$ *if φ is a double cover of a geometrically ruled surface Σ.*
- $P(\tau^* K_F) \geq P^2 \geq 3r - 7$ *otherwise*

(ii) If $|P|$ is composed with a pencil of generic fibre D of (geometric) genus g, $\widehat{D} = \tau_* D$, and $|Q|$ induces a generically finite map then we have

- $QP \geq 2(r-1)$
- $QP \geq 3(r-1)$ *except if D is hyperelliptic and $|Q|_{|D} = g^1_2$.*
- $QP \geq 4(r-1)$ *except if D is hyperelliptic or trigonal and $|Q|_{|D} = g^1_2$ or g^1_3.*
- $QP \geq 5(r-1)$ *except if D is hyperelliptic, trigonal or tetragonal and $|Q|_{|D} = g^1_2$, $2g^1_2$, g^1_3 or g^1_4, or D is of genus 2 or 3.*
- $P(\tau^* K_F) \geq (2g-2)(r-1)$ *or* $(2g-2)r$, *according to whether the pencil is rational or not, if the pencil $|\widehat{D}|$ in F has no base point.*
- $P(\tau^* K_F) \geq (2p_a(\widehat{D}) - 2 - \widehat{D}^2)(r-1)$ *if the pencil $|\widehat{D}|$ has some base point.*

(iii) If $|K_F|$ is composed with a pencil of generic fibre D as in (ii) then we have

- $P(\tau^* K_F) \geq (2g-2)(r-1)$ *or* $(2g-2)r$, *according to whether the pencil is rational or not, if the pencil $|\widehat{D}|$ in F has no base point.*
- $P(\tau^* K_F) \geq \max\{\sqrt{2(g-1)\left(1 - \frac{1}{p_g(F)}\right)(p_g(F)-1)\widehat{D}^2},$ $(2p_a(\widehat{D}) - 2 - \widehat{D}^2)(r-1)\}$ *otherwise.*

Proof: (i) Since P is nef and $P \leq \tau^* K_F$ we obviously have $P(\tau^* K_F) \geq P^2$. It is a well known fact that deg $\Sigma \geq r - 2 + q(\Sigma)$ if Σ is geometrically ruled and that deg $\Sigma \geq 2r - 4$ otherwise (see [10]).

Let $a = \deg \varphi$. If $a \geq 3$ then $P^2 \geq 3\deg\Sigma \geq 3(r-2) > 3r - 7$. If $a = 2$ and Σ is not geometrically ruled $P^2 \geq 2(2r-4) = 4r - 8 \geq 3r - 7$. If Σ is geometrically ruled $P^2 \geq 2\deg\Sigma \geq 2r - 4 + 2q(\Sigma)$.

If $a = 1$, let $C \in |P|$ be a smooth curve ($|P|$ has no base point). Then $2P_{|C} \leq (\tau^* K_F + P)_{|C} \leq (K_{\widetilde{F}} + P)_{|C} = K_C$. So $\deg P_{|C} \leq g(C) - 1$. We can then apply "Clifford plus" lemma (cf. [11]) and get $P^2 = \deg P_{|C} \geq 3h^0(C, P_{|C}) - 4 \geq 3h^0(\widetilde{F}, P) - 7 = 3r - 7$.

(ii) Let $\varphi_P(F) = C \subseteq \mathbb{P}^{r-1}$. The map $F \longrightarrow C$ may not have connected fibres; consider the Stein factorization of φ_P, $F \longrightarrow \widetilde{C} \longrightarrow C$. Note that then we have $P \equiv \alpha D$ where D is an irreducible smooth curve such that $D^2 = 0$ and $\alpha = \alpha_1\alpha_2$ where $\alpha_1 = \deg(\widetilde{C} \longrightarrow C)$ and $\alpha_2 \geq r-1$ (and equality holds only when C is rational). The pencil $|P|$ is said to be rational if $\widetilde{C} = \mathbb{P}^1$ and irrational otherwise. Note that in general also $\alpha \geq r-1$ and $\alpha \geq r$ if the pencil is irrational.

Since $P \leq Q$, the map φ_P factors through φ_Q. Let $\Sigma = \varphi_Q(F)$ and consider the induced map $\psi : \Sigma \longrightarrow C$. By construction clearly $\varphi_Q(D) \subseteq \psi^{-1}(t)$ for $t \in C$ (note that $\psi^{-1}(t)$ does not need to be connected).

We have that $QP = \alpha_2(\alpha_1 QD) \geq (r-1)(\alpha_1 QD)$. Let a be the degree of $\varphi_{Q|D}$, $\overline{D} = \varphi_Q(D)$ and $d = \deg\overline{D}$. Note that a divides $\deg\varphi_Q$ although we will not use it. We have then $\alpha_1 QD = \alpha_1 ad$. Note that $ad \geq 2$ (otherwise $\widetilde{F}$ would be covered by rational curves) and hence $QP \geq 2(r-1)$. If $QP < 3(r-1)$ then $\alpha_1 = 1$, $a = 2$, $d = 1$ (if $a = 1$, $d = 2$ again $\widetilde{F}$ is covered by rational curves). Hence D is hyperelliptic and $|Q|_{|D} = g_2^1$.

If $QP < 4(r-1)$ then $\alpha_1 ad \leq 3$. If $\alpha_1 ad = 2$ the previous argument holds. If $\alpha_1 ad = 3$ then $\alpha_1 = 1$, $a = 3$, $d = 1$ (if $a = 1$, $d = 3$, $\widetilde{F}$ would be covered by elliptic or rational curves, a contradiction since F is of general type). Then D is trigonal and $|Q|_{|D} = g_3^1$.

If $QP < 5(r-1)$ then $\alpha_1 ad \leq 4$ and we only must study the case $\alpha_1 ad = 4$. Four possibilities may occur. Either $\alpha_1 = 2$, $a = 2$, $d = 1$ (then D is hyperelliptic and $|Q|_{|D} = g_2^1$) or $\alpha_1 = 1$, $a = 4$, $d = 1$ (then D is tetragonal and $|Q|_D = g_4^1$) or $\alpha_1 = 1$, $a = 2$, $d = 2$ (then D is again hyperelliptic and $|Q|_D = 2g_2^1$) or $\alpha_1 = a = 1, d = 4$ (then D has at most genus 3; in particular D is also hyperelliptic or trigonal).

Finally note that $D\tau^*K_F = 2g-2$ if $|\widehat{D}|$ has no base point and $D\tau^*K_F = \widehat{D}K_F = 2p_a(\widehat{D}) - 2 - \widehat{D}^2$ (by adjunction formula on F) otherwise. Hence the result follows from $P(\tau^*K_F) = \alpha D(\tau^*K_F)$ and the previous bound of α.

(iii) The first result is analogous to (ii) having in mind that if $P \leq \tau^*K_F$ then $|P|$ is composed with a pencil of the same genus than $|K_F|$ and with the same general fibre.

Part of the second statement follows as in (ii). For the rest consider that from Hodge Index theorem $(K_F\widehat{D})^2 \geq K_F^2\widehat{D}^2$ and that when $|K_F|$ is composed of a pencil of genus zero then $K_F^2 \geq 2(g-1)\left(1 - \frac{1}{p_g(F)}\right)(p_g(F)-1)$

([64], Lemma 3.3). ■

Definition 5.10 We say that a linear pencil $|Q|$ on F is of type (r, g, p), $r \geq 2$, $g \geq 2$, $p = 0, 1$ if $|Q|$ is a complete linear system of r-gonal (but not s-gonal for $s < r$) curves of (geometric) genus g, rational if $p = 0$, irrational if $p = 1$. If $|Q|$ is a rational pencil, we call $\widehat{D}$ a generic member and $\delta = K_F\widehat{D} = 2p_a(\widehat{D}) - 2 - \widehat{D}^2$. If $|Q|$ is base point free, then clearly $\delta = 2g - 2$.

Now we can state the main result of this section:

Theorem 5.11 *Let T be a normal, $\mathbb{Q}$-factorial, projective threefold with only canonical singularities and let $f : T \longrightarrow B$ be a relatively minimal fibration onto a smooth curve of genus b. Let F be the general fibre.*

Assume F is of general type, $p_g(F) \geq 3$ and that $\chi_f = \chi\mathcal{O}_F\chi\mathcal{O}_B - \chi\mathcal{O}_T > 0$. Then

(i) If $|K_F|$ is not composed with a pencil and $|K_F|$ has no hyperelliptic, trigonal or tetragonal subpencil, then

$$\lambda_2(f) \geq \lambda_1(f) \geq 9\left(1 - \frac{17}{3p_g(F) + 10}\right)$$

(ii) If $|K_F|$ is composed with a pencil with generic fibre $\widehat{D}$ of (geometric) genus g, then

$$\begin{array}{ll} \lambda_2(f) \geq \lambda_1(f) \geq 4g - 4 & \textit{if the pencil is irrational} \\ \lambda_2(f) \geq \lambda_1(f) \geq \frac{p_g(F)}{p_g(F)+1}(4p_a(\widehat{D}) - 4 - 2\widehat{D}^2) & \textit{if the pencil is rational} \end{array}$$

(iii) If $|K_F|$ is not composed, has no hyperelliptic subpencils, and has subpencils $\{|Q_i|\}_{i \in I}$ of type (r_i, g_i, p_i) with $r_i = 3$, or 4 then

$$\lambda_2(f) \geq \lambda_1(f) \geq \min_{i \in I}\{\lambda_{r_i}^{p_i}(Q_i)\}$$

where

$$\lambda_3^1(Q) = 9 - \tfrac{9}{4g-7} - \varepsilon_3^1(p_g(F), g)$$

$$\lambda_3^0(Q) = \begin{cases} 9 - \frac{9}{\delta-3} - \varepsilon_3^0(p_g(F), \delta) & \textit{if} \quad \delta \geq 7, \\ 6\left(1 - \frac{17}{3p_g(F)+10}\right) & \textit{otherwise} \end{cases}$$

$$\lambda_4^1(Q) = 9 - \frac{3}{4g-9} - \varepsilon_4^1(p_g(F), g)$$

$$\lambda_4^0(Q) = \begin{cases} 9 - \frac{3}{\delta-5} - \varepsilon_4^0(p_g(F), \delta) & \textit{if} \quad \delta \geq 7, \\ 8\left(1 - \frac{17}{3p_g(F)+10}\right) & \textit{otherwise} \end{cases}$$

(iv) If $|K_F|$ has hyperelliptic subpencils $\{|Q_i|\}_{i\in I}$ of type $(2, g_i, p_i)$ then

$$\lambda_2(f) \geq \lambda_1(f) \geq \min_{i\in I}\{\lambda^{p_i}_{r_i}(Q_i)\}$$

where

$$\lambda^1_2(Q) = \begin{cases} 6 - \frac{2}{2g-3} - \varepsilon^1_2(p_g(F), g) & \text{if} \quad g \geq 3, \\ 4\left(1 - \frac{1}{p_g(F)-1}\right) & \text{if} \quad g = 2 \end{cases}$$

$$\lambda^0_2(Q) = \begin{cases} 6 - \frac{4}{\delta-2} - \varepsilon^0_2(p_g(F), \delta) & \text{if} \quad \delta \geq 4, \\ 4\left(1 - \frac{9}{2p_g(F)+5}\right) & \text{otherwise} \end{cases}$$

and where $\varepsilon^p_r \sim o(\frac{1}{p_g(F)})$ are the following positive functions

$$\varepsilon^1_4 = \frac{(68g-159)(36g-84)}{(4g-9)^2(3p_g(F)-7)+(68g-159)(4g-9)} \qquad \varepsilon^0_4 = \frac{(17\delta-91)(9\delta-48)}{(\delta-5)^2(3p_g(F)-7)+(\delta-5)(17\delta-91)}$$

$$\varepsilon^1_3 = \frac{36(g-2)(68g-137)}{(4g-7)^2(3p_g(F)-7)+(68g-137)(4g-7)} \qquad \varepsilon^0_3 = \frac{(9\delta-36)(17\delta-69)}{(\delta-3)^2(3p_g(F)-7)+(\delta-3)(17\delta-69)}$$

$$\varepsilon^1_2 = \frac{4(3g-5)(9g-17)}{(2g-3)^2(p_g(F)-2)+(9g-17)(2g-3)} \qquad \varepsilon^0_2 = \frac{(9\delta-24)(6\delta-16)}{(\delta-2)^2(2p_g(F)-4)+(\delta-2)(9\delta-24)}$$

Remark 5.12 The statement of Theorem 5.11 looks considerably easier if we consider the bounds for $p_g(F) >> 0$ (in which case the functions ε tend to be zero). This behaviour will play a special role in next section. We also observe that even if $\chi_f < 0$ then the bounds in the theorem hold for $\lambda_1(f)$ as far as $\Delta_f \neq 0$.

PROOF: We consider $\mathcal{E} = f_*\omega_{T/B}$ and its Harder-Narasimhan filtration

$$0 = \mathcal{E}_0 \subseteq \mathcal{E}_1 \subseteq \ldots \subseteq \mathcal{E}_\ell = \mathcal{E}$$

with slopes $\mu_1 > \mu_2 > \ldots > \mu_\ell \geq 0$ and ranks $1 \leq r_1 < r_2 < \ldots < r_\ell = p_g(F)$. As in §1.2, each piece induces a Cartier divisor P_i on $\widetilde{F}$ such that the linear system $|P_i|$ has projective dimension at least $r_i - 1$. We define as usual $\mu_{\ell+1} = 0$, $P_{\ell+1} = P_\ell$. We remark that following §1.2 we could define, if necessary, $P_{\ell+1} = \tau^*K_F \geq P_\ell$ ($|P_\ell|$ is the moving part of $|\tau^*K_F|$), although this possibility will only be used in very special computations and will be specifically pointed out. We have the inequalities given in Proposition 1.11 and Remark 1.12. Remember that we have $\Delta_f = \deg \mathcal{E} = \sum_{i=1}^{\ell} r_i(\mu_i - \mu_{i+1})$.

Consider first the case $|K_F|$ composed. Using Lemma 5.9 (iii) and Remark 1.12 we get, if the pencil is irrational

$$\begin{aligned} K^3_{T/B} &\geq \sum_{i=1}^{\ell}(P_i + P_{i+1})(\tau^* K_F)(\mu_i - \mu_{i+1}) \geq \\ &\geq \sum_{i=1}^{\ell-1}((4g-4)r_i + (2g-2))(\mu_i - \mu_{i+1}) + (4g-4)r_\ell\mu_\ell = \\ &= (4g-4)\Delta_f + (2g-2)(\mu_1 - \mu_\ell) \geq (4g-4)\Delta_f \end{aligned}$$

using that $r_{i+1} \geq r_i + 1$. If the pencil is rational and $\widehat{D}$ is a generic member of its moving part then we have

$$\begin{aligned} K^3_{T/B} &\geq \sum_{i=1}^{\ell-1}((4p_a(\widehat{D}) - 4 - 2\widehat{D}^2)r_i - (2p_a(\widehat{D}) - 2 - \widehat{D}^2))(\mu_i - \mu_{i+1}) + \\ &\qquad +(4p_a(\widehat{D}) - 4 - 2\widehat{D}^2)(r_\ell - 1)\mu_\ell = \\ &= (4p_a(\widehat{D}) - 4 - 2\widehat{D}^2)\Delta_f - (2p_a(\widehat{D}) - 2 - \widehat{D}^2)(\mu_1 + \mu_\ell) \end{aligned}$$

By Remark 1.12 using the indexes $\{1, \ell\}$

$$\begin{aligned} K^3_{T/B} &\geq (P_1 + P_\ell)(\tau^* K_F)(\mu_1 - \mu_\ell) + 2P_\ell(\tau^* K_F)\mu_\ell \geq \\ &\geq P_\ell(\tau^* K_F)(\mu_1 + \mu_\ell) \geq \\ &\geq (2p_a(\widehat{D}) - 2 - \widehat{D}^2)p_g(F)(\mu_1 + \mu_\ell) \end{aligned}$$

And hence eliminating $(\mu_1 + \mu_\ell)$ from the above inequalities

$$\left(1 + \frac{1}{p_g(F)}\right) K^3_{T/B} \geq (4p_a(\widehat{D}) - 4 - 2\widehat{D}^2)\Delta_f$$

which proves (ii).

From now on we assume $|K_F|$ not composed. Let

$$m = \min\{k \mid |P_k| \quad \text{induces a generically finite map}\} \leq \ell$$

By Remark 1.12 we have

$$K_{T/B}^3 \geq \sum_{i=1}^{m-1} (P_i + P_{i+1}) P_m (\mu_i - \mu_{i+1}) + \sum_{i=m}^{n} (P_i^2 + P_i P_{i+1} + P_{i+1}^2)(\mu_i - \mu_{i+1})$$

Note that for $i \geq m$ we have $P_{i+1}^2 \geq P_i P_{i+1} \geq P_i^2$ and that if $P_i P_{i+1} = P_i^2$ then $P_i = P_{i+1}$. Indeed, we have $P_{i+1} = P_i + D_i$, with $D_i \geq 0$. Hence $P_{i+1}^2 = P_{i+1}(P_i + D_i) \geq P_{i+1} P_i = (P_i + D_i) P_i \geq P_i^2$ since P_i and P_{i+1} are nef and D_i effective. If $P_i P_{i+1} = P_i^2$ we would have $P_i D_i = 0$. Since $|P_i|$ is base point free and is not composed, Hodge Index Theorem applies and hence either $D_i^2 < 0$ (which is impossible since then $P_{i+1}^2 = P_i^2 + 2P_i D_i + D_i^2 < P_i^2$) or $D_i = 0$. So we get $P_i = P_{i+1}$.

Assume $|K_F|$ has no hyperelliptic subpencil (in particular, the maps induced by the linear systems $|P_i|$ are never double covers of geometrically ruled surfaces). Then

$$\begin{array}{lll} \text{for} & m \leq i \leq \ell - 1 & P_i^2 + P_i P_{i+1} + P_{i+1}^2 \geq 9r_i - 17 \\ \text{for} & i = \ell & P_\ell^2 + P_\ell P_{\ell+1} + P_{\ell+1}^2 \geq 9r_\ell - 21 \end{array} \tag{5.6}$$

Indeed, we call φ_i the map induced by $|P_i|$ and $a_i = \deg\varphi_i$. Note that $r_i \geq 3$. First consider the case $m \leq i \leq \ell - 1$. By Lemma 5.9, if φ_i and φ_{i+1} are not double covers of geometrically ruled surfaces we have $P_i^2 \geq 3r_i - 7$ and $P_{i+1}^2 \geq 3r_{i+1} - 7 \geq 3r_i - 4$; if $P_i \neq P_{i+1}$ then $P_i P_{i+1} > P_i^2 \geq 3r_i - 7$ and we are done. If $P_i = P_{i+1}$ then $(P_i^2 + P_i P_{i+1} + P_{i+1}^2) = 3P_{i+1}^2 \geq 9r_{i+1} - 21 \geq 9r_i - 12$.

If $i = \ell$ the result follows immediately from the previous considerations.

Similarly, if $|K_F|$ admits hyperelliptic subpencils

$$\begin{array}{lll} \text{for} & m \leq i \leq \ell - 1 & P_i^2 + P_i P_{i+1} + P_{i+1}^2 \geq 6r_i - 9 \\ \text{for} & i = \ell & P_\ell^2 + P_\ell P_{\ell+1} + P_{\ell+1}^2 \geq 6r_\ell - 12 \end{array} \tag{5.7}$$

Indeed, $9r_i - 17 \geq 6r_i - 9$ (since $r_i \geq 3$) so we only have to check the case $a_i = 2$ and $\varphi_i(\widetilde{F})$ a geometrically ruled surface. Since φ_i factorizes through φ_{i+1}, $a_{i+1} = 1$ or 2. In any case $P_{i+1}^2 \geq 2r_{i+1} - 4 \geq 2r_i - 2$. If $P_i \neq P_{i+1}$ we have $P_i P_{i+1} > P_i^2 \geq 2r_i - 4$ and we are done. If $P_i = P_{i+1}$ then $P_i^2 + P_{i+1} P_i + P_{i+1}^2 = 3P_{i+1}^2 \geq 6r_i - 6$. For $i = \ell$ the assertion is clear.

Observe that since $P_1 \leq \ldots \leq P_{m-1}$, all the maps φ_i induced by $|P_i|$ are composed of the same pencil (with the only exception of $r_1 = 1$, $P_1 = 0$ for which we have no defined map φ_1). Indeed if $i < j \leq m-1$ the map φ_i factors through the map φ_j. Since $\varphi_i(\widetilde{F})$ and $\varphi_j(\widetilde{F})$ are curves, both maps have, up to Stein factorization, the same fibre.

Let P_{m-1} (and hence P_i for $i \leq m-1$) be of type (r, g, p). Now

$$\text{If} \quad i = m-1 \quad \text{then} \quad (P_{m-1} + P_m)P_m \geq \begin{cases} 10r_{m-1} - 10 & \text{if} \quad r \geq 5 \\ 8r_{m-1} - 8 & \text{if} \quad r = 4 \\ 6r_{m-1} - 6 & \text{if} \quad r = 3 \\ 4r_{m-1} - 4 & \text{if} \quad r = 2 \end{cases} \tag{5.8}$$

For this just note that $(P_{m-1} + P_m)P_m \geq 2P_{m-1}P_m$ since $P_{m-1} \leq P_m$ and P_m is nef. Then apply Lemma 5.9. Note that even if $r_{m-1} = r_1 = 1$ (hence $P_1 = 0$) $(P_{m-1} + P_m)P_m \geq 10r_{m-1} - 10$ holds.

Finally, we have

$$\text{If} \quad 1 \leq i \leq m-2 \quad \text{then} \quad (P_i + P_{i+1})P_m \geq \begin{cases} 10r_i - 5 & \text{if} \quad r \geq 5 \\ 8r_i - 4 & \text{if} \quad r = 4 \\ 6r_i - 3 & \text{if} \quad r = 3 \\ 4r_i - 2 & \text{if} \quad r = 2 \end{cases} \tag{5.9}$$

which follows immediately from Lemma 5.9, even if $P_i = P_1 = 0$ ($r_1 = 1$).

If $\Delta_f = \deg \mathcal{E} = \sum_{i=1}^{\ell} r_i(\mu_i - \mu_{i+1})$, call $\Delta_1 = \sum_{i=1}^{m-1} r_i(\mu_i - \mu_{i+1})$ and $\Delta_2 = \Delta_f - \Delta_1$.

Let us proof first (i) and (iii); we can assume then that $|K_F|$ has no hyperelliptic subpencil. We get, using (5.6),(5.8) and (5.9)

If $\quad r \geq 5$ (5.10)

$$K^3_{T/B} \geq 10\Delta_1 + 9\Delta_2 - 5\mu_1 - 5\mu_{m-1} - 7\mu_m - 4\mu_\ell \geq 9\Delta_f - 17\mu_1 - 4\mu_\ell$$

If $\quad r = 4$

$$K^3_{T/B} \geq 8\Delta_1 + 9\Delta_2 - 4\mu_1 - 4\mu_{m-1} - 9\mu_m - 4\mu_\ell \geq 8\Delta_1 + 9\Delta_2 - 17\mu_1 - 4\mu_\ell$$

If $\quad r = 3$

$$K^3_{T/B} \geq 6\Delta_1 + 9\Delta_2 - 3\mu_1 - 3\mu_{m-1} - 11\mu_m - 4\mu_\ell \geq 6\Delta_1 + 9\Delta_2 - 17\mu_1 - 4\mu_\ell$$

Note that the bound for $r \geq 5$ also holds for $m = 2, \quad P_{m-1} = P_1 = 0$ $(r_1 = 1)$, or $\quad m = 1$. Using now Remark 1.12 and Lemma 5.9 (i) we have

$$K^3_{T/B} \geq P^2_\ell(\mu_1 + 2\mu_\ell) \geq (3p_g(F) - 7)(\mu_1 + 2\mu_\ell)$$

and so (we use $-17\mu_1 - 4\mu_\ell \geq -17(\mu_1 + 2\mu_\ell)$; note that μ_ℓ may be zero):

$$\text{If} \quad r \geq 5 \quad \left(1 + \frac{17}{3p_g(F) - 7}\right) K^3_{T/B} \geq 9\Delta_f \tag{5.11}$$

$$\text{If} \quad r = 4 \quad \left(1 + \frac{17}{3p_g(F) - 7}\right) K^3_{T/B} \geq 8\Delta_1 + 9\Delta_2$$

$$\text{If} \quad r = 3 \quad \left(1 + \frac{17}{3p_g(F) - 7}\right) K^3_{T/B} \geq 6\Delta_1 + 9\Delta_2$$

The first inequality proves (i) and holds also when $m = 2, \quad P_{m-1} = P_1 = 0 \quad (r_1 = 1)$, or $\quad m = 1$.

In order to prove (iii) we can assume from now on that $r = 3$ or 4, otherwise we have (i) which is stronger than (iii).

We can also assume $m \geq 2$ and, as pointed out, that $|P_{m-1}|$ is composed with a pencil.

We divide the argument according to whether the pencil $|P_{m-1}|$ is irrational or not.

If the pencil is rational we use the same notation than in Lemma 5.9 and Definition 5.10 and set $\widehat{D}$ for the (possible singular) general element of the linear system $|\tau_* P_i| = |Q_i|$ in F (with possible base points).

Then using Lemma 5.9 and according to whether the pencil is irrational or not we have

$$\begin{aligned}
&\text{for} \quad i \leq m-2 \\
&\quad (P_i + P_{i+1})(\tau^* K_F) \geq (4g-4)r_i + (2g-2) \quad (\text{except if } P_i = P_1 = 0,\ r_1 = 1) \\
&\quad (P_i + P_{i+1})(\tau^* K_F) \geq (4p_a(\widehat{D}) - 4 - 2\widehat{D}^2)r_i - (2p_a(\widehat{D}) - 2 - \widehat{D}^2) \\
&\text{for} \quad i = m-1 \\
&\quad (P_{m-1} + P_m)(\tau^* K_F) \geq 2P_{m-1}(\tau^* K_F) \geq (4g-4)r_{m-1} \\
&\quad (P_{m-1} + P_m)(\tau^* K_F) \geq 2P_{m-1}(\tau^* K_F) \geq (4p_a(\widehat{D}) - 4 - 2\widehat{D}^2)(r_{m-1} - 1) \\
&\text{for} \quad m \leq i \leq \ell - 1
\end{aligned} \tag{5.12}$$

$$(P_i + P_{i+1})(\tau^* K_F) \geq P_i^2 + P_{i+1}^2$$

for $\quad i = \ell$

$$(P_\ell + P_{\ell+1}^2)(\tau^* K_F) \geq 2P_\ell^2$$

Using Remark 1.12 we know

$$K_{T/B}^3 \geq \sum_{i=1}^{\ell} (P_i + P_{i+1})(\tau^* K_F)(\mu_i - \mu_{i+1})$$

and so we can conclude

If $\quad r = 3, 4, \ p = 1$ (5.13)

$$K_{T/B}^3 \geq (4g-4)\Delta_1 + 6\Delta_2 - 11\mu_m - 3\mu_\ell \geq (4g-4)\Delta_1 + 6\Delta_2 - 11\mu_1 - 3\mu_\ell$$

Finally,

If $\quad r = 3, 4, \ p = 0$ (5.14)

$$\begin{aligned} K_{T/B}^3 &\geq (4p_a(\widehat{D}) - 4 - 2\widehat{D}^2)\Delta_1 + 6\Delta_2 - (2p_a(\widehat{D}) - 2 - \widehat{D}^2)(\mu_1 - \mu_{m-1}) - \\ &-(4p_a(\widehat{D}) - 4 - 2\widehat{D}^2)(\mu_{m-1} - \mu_m) - 11\mu_m - 3\mu_\ell \geq \\ &\geq (2p_a(\widehat{D}) - 2 - \widehat{D}^2)\Delta_1 + 6\Delta_2 - 11\mu_1 - 3\mu_\ell \end{aligned}$$

since $\Delta_1 \geq (\mu_1 - \mu_{m-1}) + 2(\mu_{m-1} - \mu_m)$ (this is immediate if $m - 1 \geq 2$; if $m = 2$, then $r_1 \geq 2$ and $\Delta_1 = r_1(\mu_1 - \mu_2) \geq 2(\mu_1 - \mu_m)$).

Note also that these formulas include the possibility $r_1 = 1$, $P_1 = 0$.

Using now that $K_{T/B}^3 \geq (3p_g(F) - 7)(\mu_1 + 2\mu_\ell)$ we get

$$\left(1 + \frac{11}{3p_g(F) - 7}\right) K_{T/B}^3 \geq \begin{cases} (4g-4)\Delta_1 + 6\Delta_2 & \text{if } r = 3, 4, \ p = 1 \\ (2p_a(\widehat{D}) - 2 - \widehat{D}^2)\Delta_1 + 6\Delta_2 & \text{if } r = 3, 4, \ p = 0 \end{cases}$$

Considering simultaneously this last inequality together with (5.11) and using that $\Delta_2 = \Delta_f - \Delta_1$ we get (iii). All the arguments work similarly so we only give details of one of them. Assume P_{m-1} is a tetragonal irrational pencil. Then last inequality together with (5.11) give

$$\left(1+\frac{17}{3p_g(F)-7}\right)K^3_{T/B} \geq 8\Delta_1 + 9\Delta_2 = 8\Delta_f + \Delta_2$$

$$\left(1+\frac{11}{3p_g(F)-7}\right)K^3_{T/B} \geq (4g-4)\Delta_1 + 6\Delta_2 = (4g-4)\Delta_f - (4g-10)\Delta_2$$

Observe that since $|P_{m-1}|$ is not an hyperelliptic pencil, then $g \geq 3$ and so we can get a lower bound for Δ_2 from the second inequality and we obtain

$$\left[1+\frac{17}{3p_g(F)-7}+\frac{1}{4g-10}\left(1+\frac{11}{3p_g(F)-7}\right)\right]K^3_{T/B} \geq \left[8+\frac{4g-4}{4g-10}\right]\Delta_f$$

which gives $\lambda^1_4(P_{m-1})$. As for the computation of $\lambda^0_4(P_{m-1})$ or $\lambda^0_3(P_{m-1})$ we only must be careful when $\delta \leq 6$ since then the corresponding second inequality does not give a lower bound for Δ_2. If this case happens, then just deduce from (5.11)

$$\left(1+\frac{17}{3p_g(F)-7}\right)K^3_{T/B} \geq 8\Delta_f \qquad \text{if the pencil is tetragonal}$$

$$\left(1+\frac{17}{3p_g(F)-7}\right)K^3_{T/B} \geq 6\Delta_f \quad \text{if the pencil is trigonal}$$

which give the special values of λ^0_r in (iii).

Of course we must consider all the possibilities for $|P_{m-1}|$ being trigonal or tetragonal subpencils of $|K_F|$ and so we must consider the minimum of all such lower bounds.

Finally we must prove (iv). Assume $|K_F|$ has hyperelliptic subpencils. Then it may happens that for some $i \geq m \quad \varphi_i$ is of degree two onto a ruled surface. Also may happen that $r = 2$. Altogether, Remark 1.12, Lemma 5.9 and inequalities (5.10), (5.11), (5.13) and (5.14) read

$$K^3_{T/B} \geq P^2_\ell(\mu_1 + 2\mu_\ell) \geq (2p_g(F)-4)(\mu_1+2\mu_\ell)$$

If $\quad r \geq 3 \quad \text{or} \quad m = 1 \quad \text{or} \quad m = 2 \quad P_{m-1} = P_1 = 0$
$K_{T/B}^3 \geq 6\Delta_1 + 6\Delta_2 - 3\mu_1 - 3\mu_{m-1} - 3\mu_m - 3\mu_\ell \geq 6\Delta_f - 9\mu_1 - 3\mu_\ell$

and so

$$\text{If} \quad r \geq 3 \quad \left(1 + \frac{9}{2p_g(F) - 4}\right) K_{T/B}^3 \geq 6\Delta_f \tag{5.15}$$

If $\quad r = 2$
$K_{T/B}^3 \geq 4\Delta_1 + 6\Delta_2 - 2\mu_1 - 2\mu_{m-1} - 5\mu_m - 3\mu_\ell \geq 4\Delta_1 + 6\Delta_2 - 9\mu_1 - 3\mu_\ell$

and so

$$\text{If} \quad r = 2 \quad \left(1 + \frac{9}{2p_g(F) - 4}\right) K_{T/B}^3 \geq 4\Delta_1 + 6\Delta_2 \tag{5.16}$$

If $r = 2$ $p = 0$

$$K_{T/B}^3 \geq (2p_a(\widehat{D}) - 2 - \widehat{D}^2)\Delta_1 + 4\Delta_2 - 6\mu_1 - 2\mu_\ell$$

If $\quad r = 2$ $p = 1$
$K_{T/B}^3 \geq (4g - 4)\Delta_1 + 4\Delta_2 - 2\mu_m - 2\mu_\ell \geq (4g - 4)\Delta_1 + 4\Delta_2 - 2\mu_1 - 2\mu_\ell$

This last inequality needs an extra explanation for the coefficient of μ_m. If $r = 2$, we have an hyperelliptic pencil on $\widetilde{F}$. Let D be a general irreducible member. For $i \geq m$, $|P_i|_{|D|}$ is a base point free linear subsystem of $|K_D|$ by adjunction and hence maps D onto $\mathbb{P}^1$. Hence, $\Sigma_i = \varphi_i(\widetilde{F})$ is always a ruled surface. Moreover, φ_i is of degree at least two. Since φ_{m-1} factors through Σ_i for $i \geq m$, then $q(\Sigma_i) \geq 1$ (the pencil is irrational). Hence for $i \geq m$:

$$P_i(\tau^* K_F) \geq P_i^2 \geq (\deg \varphi_i)(r_i - 2 + q(\Sigma_i)) \geq 2r_i - 2\,.$$

From here we get

$$\left(1+\frac{2}{2p_g(F)-4}\right)K^3_{T/B} \geq (4g-4)\Delta_1 + 4\Delta_2 \qquad \text{if } r=2,\ p=1 (5.17)$$

$$\left(1+\frac{6}{2p_g(F)-4}\right)K^3_{T/B} \geq (2p_a(\widehat{D})-2-\widehat{D}^2)\Delta_1 + 4\Delta_2 \qquad \text{if } r=2,\ p=0$$

If $r=2$ (i.e., $|P_{m-1}|$ is one of the hyperelliptic subpencils of $|K_F|$) then we can proceed as in (iii) using (5.16) and (5.17) and inequalities in (iv) follow. Here the exceptional bounds appear in the cases rational and irrational. When $p=0, \quad \delta \leq 4$ (5.16) gives

$$\left(1+\frac{9}{2p_g(F)-4}\right)K^3_{T/B} \geq 4\Delta_f$$

and so

$$\lambda_1(f) \geq 4\left(1-\frac{9}{2p_g(F)+5}\right)$$

When $p=1, \quad g=2$ we have

$$\lambda_1(f) \geq 4\left(1-\frac{1}{p_g(F)-1}\right)$$

If $r \geq 3$ or $|P_{m-1}|$ is not composed with a pencil the situation can only be better; indeed, in this case inequality (5.15) holds, which is better than inequality (5.16) and hence it is better than any inequalities coming from (5.16) as (iv) are. ■

5.3 The slope of non Albanese fibred threefolds

As in Chapter 4 we say that $f: T \longrightarrow B$ is a non-Albanese fibration if $q(T) > b$. We want to analyze the influence of this fact in the slope as in Xiao's result for surfaces (cf. [92], Corollary 2.1) and in the results of §4.3. Our main tool will be Theorem 5.11 together with an argument of étale covers.

First of all we need to control when étale covers of curves are d-gonal.

Lemma 5.12 *Let D be a smooth curve and $\mathcal{L} \in Pic^0(D)$ a n-torsion element.*

Let $\alpha : \widetilde{D} \longrightarrow D$ the associated étale cover of degree n.

Assume $\widetilde{D}$ has a unique (base point free) g^1_d; then

(i) D has a $g^1_{\bar{d}}$.

(ii) $n|d$.

Proof: (i) Let φ be the automorphism of order n of $\widetilde{D}$ such that $D \cong \widetilde{D}_{/<\varphi>}$. Since $\widetilde{D}$ has a unique g^1_d, φ commutes with the map $\ell : \widetilde{D} \longrightarrow \mathbb{P}^1$ and hence there is an order $\bar{n}$ automorphism $\bar{\varphi}$ of $\mathbb{P}^1$ such that

$$\widetilde{D}[d]_\ell[r]^\varphi \widetilde{D}[d]^\ell \mathbb{P}^1[r]^{\bar{\varphi}} \mathbb{P}^1$$

commutes. Clearly $\overline{n}|n$ and $\frac{n}{\overline{n}}|d$. Hence we have an induced degree $\overline{d} = \frac{d\overline{n}}{n}$ map $D = \widetilde{D}_{/<\varphi>} \longrightarrow \mathbb{P}^1_{/<\bar{\varphi}>} = \mathbb{P}^1$. Composing with a map $\mathbb{P}^1 \longrightarrow \mathbb{P}^1$ of degree $\frac{n}{\overline{n}}$ we have the desired $g^1_{\bar{d}}$ on D.

(ii) Let ψ be the automorphism of $\mathbb{P}^1$ given by $\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$ with $\alpha^n = 1$, $\alpha^i \neq 1$ if $i < n$. Consider the diagonal action of $\mathbb{Z}_n$ on $\mathbb{P}^1 \times \widetilde{D}$ by ψ and φ respectively. Put $S = (\mathbb{P}^1 \times \widetilde{D})_{/\mathbb{Z}_n}$. We have a natural fibration

$$\pi : S \longrightarrow \mathbb{P}^1_{/\mathbb{Z}_n} = \mathbb{P}^1$$

such that $\pi^{-1}(t) \cong \widetilde{D}$ for general $t \in B$ and $\pi^{-1}(\overline{(1:0)}) = n(\widetilde{D}_{/<\varphi>}) = nD$. Moreover S is smooth along $\pi^{-1}(\overline{(1:0)})$ (cf. [82] pp. 65, 66), so for our purposes can be considered smooth.

Then π verifies the hypotheses of Theorem 1.30 (ii) and then one has

$$\widehat{S}[d]_\sigma[dr]S[d]_\pi[r]^\Phi R[dl]^\tau \mathbb{P}^1$$

where R is a ruled surface, Φ is of degree d and σ solves Φ. Notice that $\widehat{S}$ still has a fibre of multiplicity n over $\overline{(1:0)}$. Since R is ruled, τ has a section so $\pi \circ \sigma$ has a d-section Z. Then $d = \widetilde{D}Z = nDZ$ is a multiple of n. ∎

Lemma 5.13 *Let F be a surface of general type and $\mathcal{L} \in Pic^0(F)$ a n-torsion element. Let $\alpha : \widetilde{F} \longrightarrow F$ be the associated étale cover of degree n. Then, if n is prime and big enough* *(i)* $\widetilde{F}$ *has no rational pencil of d-gonal curves* $(d = 2, 3)$.

(ii) If $\widetilde{F}$ *has an irrational pencil of d-gonal curves* $(d = 2, 3)$ *of genus* g*, then so has* F *and there exists a base change*

$$\widetilde{F}[d]_{\widetilde{h}}[r]^{\alpha}F[d]^{h}\widetilde{C}[r]C$$

such that $\mathcal{L} = h^*(\mathcal{M}) \in h^*(Pic^0(C))$ *and* $\widetilde{C} \longrightarrow C$ *is induced by* $\mathcal{M}$.

(iii) If $\widetilde{F}$ *has a pencil of tetragonal curves then, either so has* F *and there exists a base change diagram as in (ii) (and necessarily the pencil is irrational), or the pencil* $\{\widetilde{D}\}$ *is of bielliptic curves and* F *has a pencil* $\{D\}$ *of bielliptic (hence tetragonal) or hyperelliptic curves such that* $\alpha^*(D) = \widetilde{D}$ *(and* $2g(\widetilde{D}) - 2 = n(2g(D) - 2)$*) where*

$$@C = 0pt@R = 1truecm\widetilde{F}[dr]_{\widetilde{h}} \xrightarrow{\alpha} F[dl]^{h}C$$

PROOF: Assume first $\widetilde{F}$ has a base point free pencil $\widetilde{h} : \widetilde{F} \longrightarrow \widetilde{C}$. Let $\widetilde{D}$ be a general fibre and let $D = (\alpha(\widetilde{D}))_{\text{red}}$. Clearly D is smooth since $\widetilde{D}$ moves algebraically.

If $\mathcal{L}_{|D} \neq \mathcal{O}_D$, then since n is prime $\mathcal{L}_{|D}^{\otimes i} \neq \mathcal{O}_D$ for $1 \leq i \leq n-1$ and hence $\alpha^*(D)$ is a connected smooth étale cover of D containing $\widetilde{D}$ and so $\alpha^*(D) = \widetilde{D}$.

If $D^2 > 0$, by [69], $\mathcal{L}_{|D} \neq \mathcal{O}_D$ and by the previous argument $\alpha^*(D) = \widetilde{D}$, $0 = \widetilde{D}^2 = nD^2 > 0$, a contradiction. So necessarily we have $D^2 = 0$.

If $\mathcal{L}_{|D} = \mathcal{O}_D$ then $\alpha^*(D) = \widetilde{D}_1 + \ldots + \widetilde{D}_n$ $(\widetilde{D}_1 = \widetilde{D})$, $\widetilde{D}_i\widetilde{D}_j = 0$, $\widetilde{D}_i \neq \widetilde{D}_j$ if $i \neq j$, and hence we have a factorization

$$\widetilde{F}[r]^{\alpha}[d]_{\widetilde{h}}F[d]^{h}\widetilde{C}[r]^{\beta}C$$

such that $\mathcal{L} \in h^*Pic^0(C)$ and β is an étale cover. In particular $g(C) \geq 1$.

Now we want to explore the possibility $\widetilde{D} = \alpha^*(D)$. Since n is big enough then so is $g(\widetilde{D})$ and hence, if $\widetilde{D}$ has a g^1_d $(d = 2, 3, 4)$ it is unique except if $d = 4$ and $\widetilde{D}$ is bielliptic. So Lemma 5.12 (ii) says that $\widetilde{D}$ has no g^1_d $(d = 2, 3, 4)$ as far as n does not divide d except when $\widetilde{D}$ is bielliptic.

In this case, let σ be the (unique, if $g(\widetilde{D}) \geq 6$) bielliptic involution of $\widetilde{D}$. Let φ be the rank n automorphism of $\widetilde{D}$ induced by α. Since σ is unique we must have that $\varphi \circ \sigma \circ \varphi^{-1} = \sigma$ and hence there exists an automorphism $\bar{\varphi}$ of the elliptic base curve E such that the following diagram commutes

$$\widetilde{D}[r]^{\varphi}[d]_{2:1}\widetilde{D}[d]^{2:1}E = \widetilde{D}_{/<\sigma>}[r]^{\bar{\varphi}}E$$

Hence there is an induced degree two map $D = \widetilde{D}_{/<\varphi>} \longrightarrow E_{/<\bar{\varphi}>} = E'$, where E' is $\mathbb{P}^1$ or an elliptic curve according $\bar{\varphi}$ has fixed points or not. Then D is hyperelliptic or bielliptic (hence tetragonal). $\blacksquare$

Then we can consider which is the influence of the irregularity of T in the slope of f as in §4.3. We have an exceptional good behaviour:

Theorem 5.14 *Let $f : T \longrightarrow B$ be a relatively minimal fibration of a normal, $\mathbb{Q}-$factorial, projective threefold with only canonical singularities onto a smooth curve of genus b. Let F be a general fibre. Assume F is of general type, $p_g(F) \geq 3$, and that $\chi_f = \chi\mathcal{O}_F\chi\mathcal{O}_B - \chi\mathcal{O}_T > 0$.*

Then, if $q(T) > b$ we have

(i) $\lambda_2(f) \geq 4$.

(ii) If F has no irrational pencil of d-gonal curves ($d = 2, 3, 4$) then $\lambda_2(f) \geq 9$.

(iii) If F has an irrational tetragonal pencil of (minimal) genus g and no irrational pencil of trigonal curves nor pencil of hyperelliptic curves then $\lambda_2(f) \geq 9 - \frac{3}{4g-9}$.

(iv) If f has an irrational tetragonal pencil of (minimal) genus g_1, an irrational trigonal pencil of (minimal) genus g_2 and no hyperelliptic pencil, then

$$\lambda_2(f) \geq \min\left\{9 - \frac{3}{4g_1 - 9}, 9 - \frac{9}{4g_2 - 7}\right\}.$$

(v) If F has an irrational hyperelliptic pencil of (minimal) genus g, then $\lambda_2(f) \geq 6 - \frac{2}{2g-3}$.

(vi) If F has a rational hyperelliptic pencil and none irrational, then $\lambda_2(f) \geq 6$.

(vii) If $\lambda_2(f) < 9$ then either F has a rational pencil of hyperelliptic curves or there exists, perhaps up to base change, a factorization of f

$$T@-- >[r]^h[d]^f S[dl]^g B$$

where S is a smooth surface fibred over B by curves C_t of genus $g(C_t) \geq 1$ and h is everywhere defined at the general fibre F_t of f, such that

- *for $t \in B$ general the image of $((i_t)^* : Pic^0(T) \longrightarrow Pic^0(F_t))$ lies in $h_t^*(Pic^0(C_t)) \subseteq Pic^0(F_t)$.*
- *for $s \in S$ general $D_s = h^{-1}(s)$ is hyperelliptic, trigonal or tetragonal (necessarily hyperelliptic or of genus 3 if $\lambda_2(f) < 8$).*

PROOF: Note that (i) follows from (ii), (iii), (iv), (v) and (vi). Since $q(T) > b$ then for every $n \gg 0$ there exists a n-torsion element $\mathcal{L} \in Pic^0 T \setminus f^*(Pic^0(B))$ such that for $1 \leq i \leq n-1$, $\mathcal{L}_{|F}^{\otimes i} \neq \mathcal{O}_F$. Then we can construct the associated étale cover $\alpha : \widetilde{T} \longrightarrow T$ as in Lemma 5.6 (iii) and get $\widetilde{f} = f \circ \alpha$ such that $\lambda_2(f) = \lambda_2(\widetilde{f})$.

If $\widetilde{F}$ is the fibre of $\widetilde{f}$, we have an étale cover $\alpha_1 : \widetilde{F} \longrightarrow F$ and hence $p_g(\widetilde{F}) \geq \chi\mathcal{O}_{\widetilde{F}} = n\chi\mathcal{O}_F \geq n$ (note that $q(\widetilde{F}) \geq q(F) \geq 1$ since $q(T) > b$). Since we can do this process for n as big as needed, we can take in the bounds of Theorem 5.11 limit when $p_g(F)$ goes to infinity.

Assume that $|K_F|$ is composed. We have $p_g(\widetilde{F}) \geq n$ and either the genus of the fibre of the pencil or the genus of the base curve increases, except if $q(F) = 1$ and the pencil is elliptic. When $|K_F|$ is composed the pencil can only be rational or elliptic and the genus of the fibre is at most 5 provided $p_g \gg 0$ ([11], [91]). So if n is big enough and the pencil is rational $|K_{\widetilde{F}}|$ can not be composed. Since $\lambda_2(f) = \lambda_2(\widetilde{f})$ we can assume $|K_F|$ not composed with a rational pencil.

Finally, if the pencil is elliptic and $q(F) = 1$, note that we can apply Theorem 5.11 (ii) and get that $\lambda_2(f) \geq 12$ if $g \geq 4$, $\lambda_2(f) \geq 8$ if $g = 3$, $\lambda_2(f) \geq 4$ if $g = 2$. So we have that (ii), (iii), (iv), (v) and (vi) hold. From now on we assume $|K_F|$ is not composed.

When F has a fibration $h : F \longrightarrow C$ we have an induced map $\widetilde{h} = h \circ \alpha_1 : F \longrightarrow C$. This map may not have connected fibres and hence factorizes through an étale cover $\widetilde{C} \longrightarrow C$. We have two possibilities.

It may exist an unbounded sequence $\{n_i\}_{i\in\mathbb{N}} \subseteq \mathbb{N}$ such that for every i $\widetilde{h}_{n_i}$ has connected fibres over C (hence it is a fibration) or for every $n \geq n_0$, $\widetilde{h}_n$ factorizes through a non trivial étale cover $\widetilde{C}_n \longrightarrow C$.

In any case we have that, if g_n is the genus of the fibration $\widetilde{F}_n \longrightarrow \widetilde{C}_n$, $g \leq g_n \leq n(g-1)+1$, the border cases being the two extreme possibilities.

If $C = \mathbb{P}^1$ (rational pencil) then $\widetilde{C}_n = C$ for all n and the sequence $\{\delta_n\}$ is unbounded. If F has a pencil of tetragonal curves which are bielliptic, then by Lemma 5.13 (iii) again may happen that $\{g_n\}$ is unbounded. Otherwise by using Lemma 5.13, we have that $g_n = g$ holds for all n.

If the sequence $\{g_n\}$ is bounded, since $\lim_{n\to\infty} p_g(\widetilde{F}_n) = \infty$, we can consider Theorem 5.11 (iii), (iv) and get the bounds of (iii), (iv), (v) and (vi) (note that $g_n = g$ is the worst case).

Finally assume $\{g_n\}$ (or δ_n) is unbounded. We must take limit in the bounds of Theorem 5.11 when g (or δ) and $p_g(F)$ simultaneously (and linearly) grow. In all the cases, the limit is 9.

If F has no irrational d-gonal pencil ($d = 2, 3, 4$) so has not $\widetilde{F}$ by Lemma 5.13. If F has a rational $d-$gonal pencil ($d = 2, 3, 4$) we know yet that $\lambda_2(f) \geq 9$. So we can assume that if F verifies the hypotheses of Theorem 5.11 (i) then so does $\widetilde{F}$ and so we get $\lambda_2(f) \geq 9$ in the limit process. This proves (ii).

In order to prove (vii) note that in the previous arguments we always have $\lambda_2(f) \geq 9$ except when there exists $h_t : F_t \longrightarrow C_t$ with hyperelliptic, trigonal or tetragonal fibres (such that $g(C_t) \geq 1$ when non-hyperelliptic) and for every $\mathcal{L} \in Pic^0(T)$ the étale cover $\widetilde{F_t} \longrightarrow F_t$ given by $\mathcal{L}_{|F_t}$ factorizes through an étale cover of C_t. This says that Im $((i_t)^* Pic^0 T \longrightarrow Pic^0(F_t))$ lies in the subtorus $h_t^* Pic^0 C_t$.

In order to glue all the maps h_t we can proceed as in Theorem 5.7 (iii). ■

Corollary 5.15 *With the same notations as in theorem 5.14, if $q(T) > b$ then*

(i) If $\lambda_2(f) < 9$ then F is fibred by hyperelliptic, trigonal or tetragonal curves.

(ii) If $\lambda_2(f) < 8$ then F is fibred by genus 3 or hyperelliptic curves.

(iii) If $\lambda_2(f) < \frac{16}{3}$ then F is fibred by genus 2 curves.

Corollary 5.16 *With the same notations as in theorem 5.14, if $\mathcal{E} = f_* \omega_{T/B}$ has a quotient of rank one and degree zero, then the same conclusions as in theorem 5.14 hold.*

In particular, if $b = 0, 1$, F is not fibred by d-gonal curves ($d = 2, 3, 4$) and $\lambda_2(f) < 9$ then $\mathcal{E}$ is ample.

PROOF: The same argument of Theorem 4.27 holds. ■

Remark 5.17 Although in Theorem 5.11 we only must take care of subpencils of $|K_F|$, in the proof of Theorem 5.14 we must take care of subpencils of $|K_{\widetilde{F}}|$ for any étale cover $\widetilde{F} \longrightarrow F$, hence corresponding to *arbitrary* pencils in F. Hence the hypotheses that appear in the statement of Theorem 5.14 can not be restricted to subpencils of $|K_F|$.

Example 5.18 In the case of fibred surfaces is not easy to find examples with low slope to check the sharpness of the bounds. So it is natural that in the case of fibred threefolds natural examples does not lie near the bounds. We only can give a family of examples of fibred threefolds with F fibred by genus two curves and with slope arbitrarily near to 6. For this consider a ruled surface R onto a smooth curve C of genus m, and let B be a smooth curve of genus b. Let $Y = R \times B$ and consider a suitable double cover $T \longrightarrow Y$. If the ramification locus is suitable chosen, the general fibre F of the induced fibration $f : T \longrightarrow B$ has a genus two fibration. A standard computation shows that $\lambda_2(f)$ is arbitrarily near to 6 provided $m \geq 1$ (in fact equal to 6 if $m = 1$). Observe that by construction $q(T) - b \geq m$ and so f is a non Albanese fibration. So we can conclude that the bound 6 has certainly some meaning for fibrations with general fibre fibred by hyperelliptic curves. The same construction produces examples with arbitrary $g \geq 3$ but then $\lambda_2(f)$ is far from 6.

5.4 Fibred threefolds with low slope

In [73], the following possibilities for fibred threefolds with fibre of general type and $\lambda_2(f) < 4$ are listed.

Theorem 5.19 (Ohno, [73] Main Theorem 2).*Let $f : T \longrightarrow B$ be a relatively minimal fibred threefold as in Theorem 5.11. Assume F is of general type. If $K_{T/B}^3 < 4(\chi\mathcal{O}_B\chi\mathcal{O}_F - \chi\mathcal{O}_T)$ then F has one of the following properties:*

(i) F carries a linear pencil of curves of genus two.

(ii) $K_F^2 \leq 2p_g(F) - 1$

(iii) $K_F^2 = 2p_g(F)$, $p_g(F) \geq 3$, $q(F) \leq 2$ and $|K_F|$ is not composed ($q(F) = 2$ only if $p_g(F) = 3$).

(iv) $|K_F|$ is not composed and

- $K_F^2 = 8$, $p_g(F) = 3$, $q(F) \leq 1$ *or*
- $K_F^2 = 9$, $p_g(F) = 4$, $q(F) \leq 1$ *or*
- $K_F^2 = 7$, $p_g(F) = 3$, $q(F) \leq 2$

(v) $K_F^2 = 4$ or 5, $p_g(F) = 2$ and the movable part of $|K_F|$ is a linear pencil of curves of genus three with only one base point.

(vi) $K_F^2 = 2$ or 3 and $p_g(F) = 1$

(vii) $p_g(F) = 0$

Moreover Ohno gives an example of fibration of type (i).

Following the analogous conjectured result for fibred surfaces (see Chapter 4, introduction), fibred threefolds with low $\lambda_2(f)$ should be very special from the point of view of the canonical map. In the case of fibred surfaces, the fibration should be hyperelliptic provided $g \gg 0$. The analogy in the case of threefolds is clear: the canonical map $\varphi_{|K_F|}$ should not be *general*. If $p_g(F) \leq 2$ the canonical map is clearly very special. We can prove that if $p_g(F) \geq 8$, F is fibred by hyperelliptic curves (in fact of genus 2 is $p_g(F) \geq 15$) and hence $\varphi_{|K_F|}$ has at least degree two. In the remaining cases $3 \leq p_g(F) \leq 7$ we also prove that the canonical map of F has degree 3 up to some exceptions.

The positivity properties of fibred surfaces with low slope (see Theorem 4.27) hold similarly here.

Theorem 5.20 *Let $f : T \longrightarrow B$ be a relatively minimal fibration of a normal, $\mathbb{Q}-$factorial projective threefold T with only canonical singularities onto a smooth curve B of genus b. Assume that the general fibre F is of general type, $p_g(F) \geq 3$ and $\chi_f = \chi\mathcal{O}_F\chi\mathcal{O}_B - \chi\mathcal{O}_T > 0$.*

Then if $\lambda_2(f) < 4$ we have

(i) $q(T) = b$

(ii) $\mathcal{E} = f_*\omega_{T/B}$ *has no invertible rank zero quotient sheaf (in particular, $\mathcal{E}$ is ample provided $b \leq 1$ or Fujita's conjecture holds).*

(iii) If $p_g(F) \geq 15$ F has a rational pencil of curves of genus 2.

(iv) If $p_g(F) \leq 14$ then one of the following holds

(a) F has a rational pencil of hyperelliptic curves

(b) F has a rational pencil of trigonal curves, $q(F) = 0$ and

. *either the canonical map of F is of degree 3 and either $p_g(F) = 3$, $\quad 3 \leq K_F^2 \leq 8$ or $p_g(F) = 4, 5 \quad 3p_g(F) - 6 \leq K_F^2 \leq 9$*

. *or the canonical map of F is birational and*

$$p_g(F) = 4 \quad 5 \leq K_F^2 \leq 9$$
$$5 \leq p_g(F) \leq 7 \quad 3p_g(F) - 7 \leq K_F^2 \leq 2p_g(F)$$

(c) F is canonical, $p_g(F)=4$, $q(F)=0$, $K_F^2=5$.

Remark 5.21 It is doubtful that the cases of fibre F canonical in (iv)(b) and in (iv)(c) occur. In Chapter 2 we proved that then $K_{T/B}^3 \geq 4\chi_f$, provided T is Gorenstein, so any example should necessarily have T non Gorenstein.

Proof:

The first two statements follow from Theorem 5.14 and Corollary 5.16. Following the list of Ohno in Theorem 5.19, if $\lambda_2(f)<4$ and $p_g(F)\geq 3$, $|K_F|$ is not composed.

We follow the notations of the proof of Theorem 5.11.

Assume F has a rational hyperelliptic pencil. We must proof that the pencil is of genus 2 provided $p_g(F)\geq 15$. Call $\delta = K_F\widehat{D}$. If the (geometric) genus of $\widehat{D}$ is not 2 then we observe that $\delta \geq 4$ (if $\widehat{D}^2=0$ then $\delta = 2g-2$; if $\widehat{D}^2>0$ Hodge index theorem gives $\delta^2 \geq K_F^2 \geq 2p_g(F)-4 \geq 26$). Formula (5.14) reads for $r=2, p=0$

$$K_{T/B}^3 \geq 2\delta\Delta_1 + 4\Delta_2 - \delta(\mu_1-\mu_{m-1}) - 2\delta(\mu_{m-1}-\mu_m) - 6\mu_m - 2\mu_\ell$$
$$\geq 2\delta\Delta_1 + 4\Delta_2 - 2\delta\mu_1$$

and using that $K_{T/B}^3 \geq (2p_g(F)-4)(\mu_1+2\mu_\ell)$ we get

$$(1+\frac{2\delta}{2p_g(F)-4})K_{T/B}^3 \geq 2\delta\Delta_1 + 4\Delta_2$$

which together with (5.16) gives that $K_{T/B}^3 \geq 4\Delta_f$ provided $p_g(F)\geq 15$.

Assume F has no rational hyperelliptic pencil. According to Remark 1.12 we must check when the coefficient of $(\mu_i-\mu_{i+1})$ is greater or equal than $4r_i$.

Take i such that $m \leq i \leq \ell-1$. If $a_i=2$, then $P_i^2 \geq 2r_i-2$ if the image is ruled (since F has no hyperelliptic rational pencil, $\deg\varphi_i(F) \geq r_i-1$) or $P_i^2 \geq 4r_i-8$ otherwise. In any case $P_i^2 \geq 2r_i-2$ ($r_i\geq 3$ since $|P_i|$ is not composed), and hence $P_i^2+P_iP_{i+1}+P_{i+1}^2 \geq 3P_i^2 \geq 6r_i-6 \geq 4r_i$.

If $a_i=3$ then $a_{i+1}=1$ or 3 and hence $P_{i+1}^2 \geq 3r_{i+1}-7 \geq 3r_i-4$. If $P_i \neq P_{i+1}$ then $P_i^2+P_iP_{i+1}+P_{i+1}^2 \geq 2P_i^2+1+P_{i+1}^2 \geq 9r_i-15 \geq 4r_i$. If $P_i=P_{i+1}$, $P_i^2+P_iP_{i+1}+P_{i+1}^2 = 3P_{i+1}^2 \geq 9r_i-12 \geq 4r_i$.

If $a_i \geq 4$, $P_i^2 + P_iP_{i+1} + P_{i+1}^2 \geq 3P_i^2 \geq 12r_i - 24 \geq 4r_i$. Finally if $a_i = 1$ then $a_{i+1} = 1$ and hence, by the same argument as in (5.6), $P_i^2 + P_iP_{i+1} + P_{i+1}^2 \geq 9r_i - 17 \geq 4r_i$ (since $r_i \geq 4$ if φ_i is birational).

Let $i = \ell$. As pointed out at the beginning of proof of Theorem 5.11, we can set $P_{\ell+1} = \tau^* K_F$. Hence, if $P_\ell = P_{\ell+1} = \tau^* K_F$ we have $P_\ell^2 + P_\ell P_{\ell+1} + P_{\ell+1}^2 = 3P_\ell^2$ but if $P_{\ell+1} \neq P_\ell$ we have $P_\ell^2 + P_\ell P_{\ell+1} + P_{\ell+1}^2 \geq K_F^2 + 2P_\ell^2 + 1 \geq 3P_\ell^2 + 2$. Having this in mind we obtain that $P_\ell^2 + P_\ell P_{\ell+1} + P_{\ell+1}^2 \geq 4r_\ell$ except when $r_\ell = p_g(F) = 4$, $P_\ell^2 = K_F^2 = 3p_g - 7 = 5$ and F is canonical or $r_\ell = p_g(F) = 3$, $P_\ell^2 = 3p_g - 6 = 3$, $K_F^2 = 3p_g(F) - 6$ or $3p_g(F) - 5$ and the canonical map is of degree three. In both cases we have necessarily $q(F) = 0$ (see Chapter 2 for the canonical case and [96] for the degree 3 case).

Take i such that $1 \leq i \leq m - 1$. If $r_i = 1$ (then $i = 1$ and $P_1 = 0$) we have $(P_1 + P_2)P_m \geq 4r_1 = 4$ except when $r_2 = 2$ and P_m induces a g_3^1 in the fibre of the rational pencil $|P_2|$. Assume $r_i \geq 2$. If $a_m = 2$, then $P_iP_m \geq 2r_i$; for this we must look at the proof of Lemma 5.9 (ii). Assume $2r_i - 1 \geq P_iP_m \geq \alpha_2(\alpha_1 ad) \geq (\alpha_1 ad)(r_i - 1)$; we have that $|P_m|_{|D} = g_2^1$, hence $a = 2$ and necessarily $\alpha_1 = d = 1$; if $\alpha_2 = r_i - 1$ the pencil would be rational (since $\alpha_1 = 1$) which is impossible by our assumptions; hence $\alpha_2 \geq r_i$ which is again impossible. Then $(P_i + P_{i+1})P_m \geq 2P_iP_m \geq 4r_i$. If $a_m \geq 3$ then by Lemma 5.9 $P_iP_m \geq 3(r_i - 1) \geq 2r_i$ except if $r_i = 2$, $a_m = 3$. In this exceptional case, if $P_{i+1} \neq P_m$ then $(P_i + P_{i+1})P_m \geq (3r_i - 3) + (3r_{i+1} - 3) \geq 6r_i - 3 = 9 > 8 = 4r_i$; if $P_{i+1} = P_m$ then $(P_i + P_m)P_m \geq 8 = 4r_i$ except if $4 \geq P_m^2 \geq 3r_m - 6$, i.e., $r_1 = 2$, $r_2 = 3$, $m = 2$, $a_m = 3$ (which again produces a rational trigonal pencil in F). Finally if $a_m = 1$ $P_iP_m \geq 4r_i - 4$ (Lemma 5.9) and hence $P_iP_m \geq 2r_i$, $(P_i + P_{i+1})P_m \geq 4r_i$.

So we can conclude than either $p_g(F) = 4$, $q(F) = 0$, $K_F^2 = 5$ or F has a rational trigonal pencil.

Note that in the discussion above, when F has a rational trigonal pencil, $|P_m|$ induces a degree 3 map. Hence the canonical map of F can only be of degree 1 or 3. In any case $K_F^2 \geq 3p_g(F) - 7$. Hence, applying Theorem 5.19 we have $3p_g(F) - 7 \leq K_F^2 \leq 2p_g(F)$ (if $p_g(F) \geq 5$) and hence $p_g(F) \leq 7$, $K_F^2 \leq 2p_g(F) \leq 14$.

Finally we must prove that $q(F) = 0$. If $q(F) = 1$, then $K_F^2 \geq 3p_g(F) + 7q(F) - 7 = 3p_g(F)$ (cf. [62]) which is impossible. Assume $q(F) \geq 2$. If $|K_F|$ is birational we have $3p_g(F) - 4 \leq K_F^2 \leq 2p_g(F) - 1$ (by Chapter 2, if $q(F) \geq 2$ we have $K_F^2 \geq 3p_g(F) + q(F) - 7$ but if equality holds then

$q(F) \geq 3$) which is impossible. If $|K_F|$ induces a map of degree 3 we have $3p_g(F)-3 \leq K_F^2 \leq 2p_g(F)$ (cf. [96] and Theorem 5.19) or $p_g(F)=3$, $q(F)=2$, $K_F^2=7$; so in any case we get $p_g(F)=3$, $K_F^2 \geq 6$. Following the above discussion the only possibilities for the Harder-Narasimhan filtration of $\mathcal{E}$ are $r_1=2$, $r_2=3$ or $r_1=1$, $r_2=2$, $r_3=3$. The first one gives

$$K_{T/B}^3 \geq (P_1+P_2)P_2(\mu_1-\mu_2)+3P_2^2\mu_2 \geq 9(\mu_1-\mu_2)+18\mu_2 \geq 8(\mu_1-\mu_2)+12\mu_2 = 4\Delta_f$$

The last one gives

$$\begin{aligned} K_{T/B}^3 &\geq (P_1+P_2)P_3(\mu_1-\mu_2)+(P_2+P_3)P_3(\mu_2-\mu_3)+3P_3^2\mu_3 \\ &\geq 3(\mu_1-\mu_2)+9(\mu_2-\mu_3)+18\mu_3 \geq 4(\mu_1-\mu_2)+8(\mu_2-\mu_3)+12\mu_3 = 4\Delta_f \end{aligned}$$

if $\mu_2-\mu_3 \geq \mu_1-\mu_2$; otherwise consider

$$\begin{aligned} K_{T/B}^3 &\geq (P_1+P_3)P_3(\mu_1-\mu_3)+3P_3^2\mu_3 \\ &\geq 6(\mu_1-\mu_3)+18\mu_3 \geq 4(\mu_1-\mu_2)+8(\mu_2-\mu_3)+12\mu_3 = 4\Delta_f. \end{aligned}$$

So we have necessarily $q(F)=0$.

As for the restrictions for $(p_g(F), K_F^2)$ when the canonical map is of degree 3, we refer to [76], [96], [61].

Finally we note that in Chapter 2 we prove that if T is Gorenstein and $|K_F|$ is birational, then $\lambda_2(f) \geq 4$. Then the last exceptional case and those verifying $3 \leq p_g(F) \leq 7$ and $|K_F|$ birational only can happen when T is not Gorenstein. ■

Remark 5.22 We recall that in §1.3, Corollary 1.21 we also get lower bounds for the slope $\lambda_2(f)$ whenever $\mathcal{E}$ is semistable or the canonical image of F is contained in few quadrics (hence $p_g(F)$ is necessarily very low), but the bound contains a negative term which can only be assumed to be zero if T is Gorenstein.

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