Appendix A

Fundamentals of Contact Analysis

A.1 Introduction

In this Appendix the basic principles of bi-dimensional contact analysis are described. At the beginning, the basic equilibrium equations derived from the Continuum Mechanics are applied for two contacting bodies. The necessary conditions to be fulfilled for the contacting bodies are derived. Finally, the equation of perfect friction is written to emphasize the principles of Coulomb's dry friction, already seen in Chapter 2.

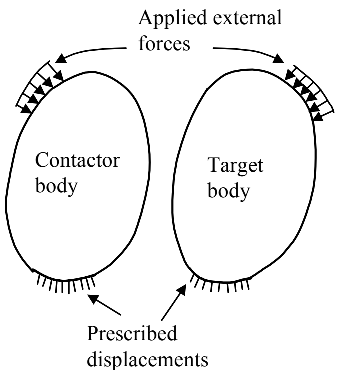
A.2 Bi-dimensional Contact

A particularly difficult nonlinear behavior to analyze is contact between two or more bodies. Contact problems range from frictionless contact in small displacements to contact with friction in general large strain inelastic conditions. Although the formulation of the contact conditions is the same in all these cases, the solution of the nonlinear problems can in some analyses be much more difficult than in other cases. The nonlinearity of the analysis problem is influenced not only by the geometric and material nonlinearities considered so far but also by the contact conditions [48].

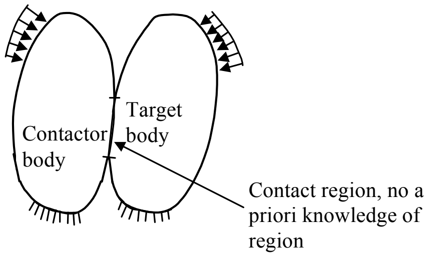
Fig. A.1 shows schematically the problem considered. This figure represents two generic bodies which are arbitrarily denoted as *contactor* and *target*.

The basic conditions of contact along the contact surfaces are such that no material overlap can occur; as a result, the contact forces that are developed act along the region of contact upon the target and the contactor. These forces are equal and opposite. The normal tractions can only exert compressive action, and the tangential tractions satisfy a law of frictional resistance.

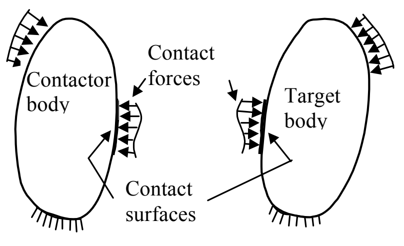
Let S^{IJ} and S^{JI} denote the surfaces associated with the contactor and target bodies, respectively. Also, let $\mathbf{n}$ be the unit outward normal to S^{IJ} and let $\mathbf{s}$ be a vector such that



(a) Condition prior to contact



(b) Condition at contact



(c) Forces acting on contactor and target bodies

Figure A.1 Schematic representation of the bi-dimensional contact problem

$\mathbf{n}$, $\mathbf{s}$ form an orthogonal right-hand basis (see Fig. A.2). The contact tractions ${}^t\mathbf{f}^{IJ}$ acting on S^{IJ} can be decomposed into normal and tangential components corresponding to $\mathbf{n}$ and $\mathbf{s}$ on S^{JI},

$$ {}^t\mathbf{f}^{IJ} = \gamma\mathbf{n} + \tau\mathbf{s} \tag{A.1} $$

where γ and τ are the normal and tangential traction components (for brevity of notation a superscript will not be used). Hence,

$$ \gamma = {}^t\mathbf{f}^{IJ} \cdot \mathbf{n}; \qquad \tau = {}^t\mathbf{f}^{IJ} \cdot \mathbf{s} \tag{A.2} $$

To define the actual values of $\mathbf{n}$, $\mathbf{s}$ that are going to be used in contact calculations, consider a generic point X on S^{IJ} and let $Y^*(X,t)$ be the point on S^{JI} satisfying

$$ \left|\overrightarrow{X - Y^*}(X,t)\right|_2 = \min_{Y \in S^{JI}} \left\{ \left|\overrightarrow{X - Y}\right|_2 \right\} \tag{A.3} $$

where $\overrightarrow{X - Y^*}(X,t)$ and $\overrightarrow{X - Y}$ are the distances from X to S^{JI} at points Y^* and Y, respectively.

The (signed) distance from X to S^{JI} is given by

$$ g(X,t) = (\overrightarrow{X - Y^*}) \cdot \mathbf{n}^* \tag{A.4} $$

where $\mathbf{n}^*$ is the unit 'normal vector' at $Y^*(X,t)$ (see Fig. A.2) and $\mathbf{n}^*$, $\mathbf{s}^*$ are used in (A.1) corresponding to the point X. The function g is the gap function for the contact surface pair.

With these definitions, the conditions for normal contact can now be written as

$$ g \geq 0; \qquad \gamma \geq 0; \qquad g\gamma = 0 \tag{A.5} $$

where the last equation expresses the fact that if $g > 0$, then $\gamma = 0$, and viceversa. Fig. A.3a illustrates the conditions given by (A.5) for normal contact.

To include frictional conditions, assume that Coulomb's law of friction holds pointwise on the contact surface and that μ is the coefficient of friction. This assumption means of course that frictional effects are included in a very simplified manner.

Let α be defined as a dimensionless variable given by

$$ \alpha = \frac{\tau}{\mu\gamma} \tag{A.6} $$

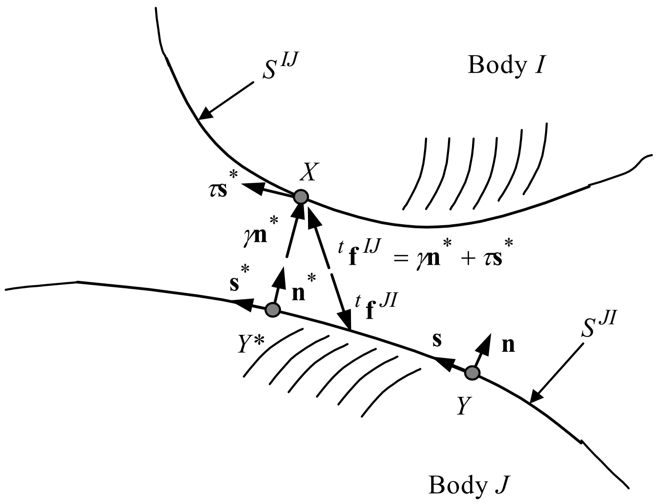


Figure A.2 Bi-dimensional contact analysis definitions

where $\mu\gamma$ is the 'frictional resistance', and the magnitude of the relative tangential velocity

$$\dot{u}(X,t) = \left[\dot{\mathbf{u}}^J \mid_{Y^*(X,t)} - \dot{\mathbf{u}}^I \mid_{(X,t)}\right] \cdot \mathbf{s}^* \tag{A.7}$$

corresponding to the unit tangential vector $\mathbf{s}^*$ at $Y^*(X,t)$. Hence, $\dot{u}(X,t)\mathbf{s}^*$ is the tangential velocity at time t of the material point at Y^* relative to the material point X^*. With these definitions Coulomb's law of friction states

$$\begin{aligned} |\alpha| &\leq 1 \\ |\alpha| &< 1 \text{ implies } \dot{u} = 0 \\ |\alpha| &= 1 \text{ implies } \operatorname{sgn}(\dot{u}) = \operatorname{sgn}(\tau) \end{aligned} \tag{A.8}$$

Fig. A.3b illustrates these interface conditions.

A.3 Perfect Friction

As said in Sect. 2.2, the law of perfect friction states that the force of friction is proportional to the load and is independent of the apparent area of contact and the other state variables. Combined to the impenetrability condition the criterion of perfect friction takes the form

$$F = N \tan\phi + C \qquad \text{contact} \tag{A.9}$$

$$F = F_{\max} = N \tan\phi_0 + C = \mu N + C \qquad (\phi_0 > \phi) \qquad \text{slip} \tag{A.10}$$

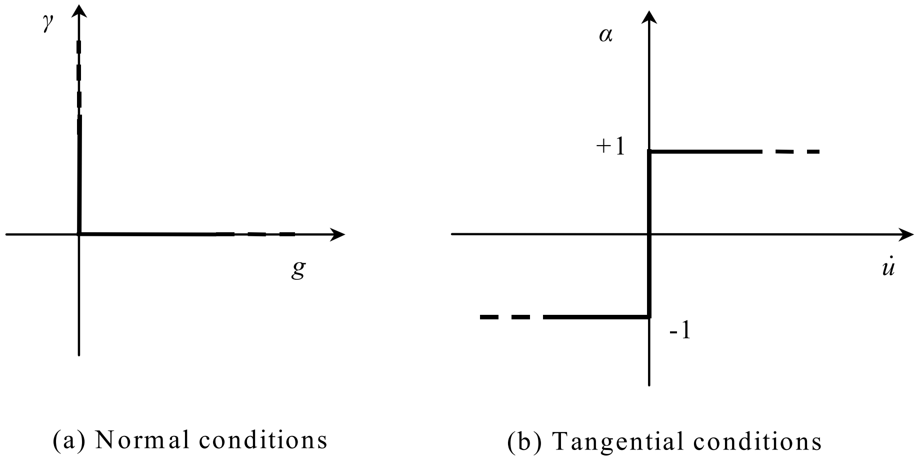


Figure A.3 Interface conditions for the contact analysis

where $F = |\mathbf{F}| = \sqrt{F_1^2 + F_2^2}$ denotes the euclidean norm of $\mathbf{F}$, which is the tangential component of the contact force, $\mu = \tan\phi_0$ is the coefficient of friction (ϕ_0 is the angle of rugosity) and C is a constant characterizing adhesion [68]. In geometric terms the criterion assumes the shape of a truncated cone in the normal-tangential axes attached to the contact point (see Fig. A.4). It is the analogue of the Drucker-Prager criterion [69] in plasticity and constitutes the determining ingredient of Coulomb's law of friction.

For $\mu = 0$ the cone degenerates into a cylinder which corresponds to a force of friction independent of the load. It is the analogue of the deviatoric energy criterion of Von Mises [57].

Now, considering the bi-dimensional case, and assuming $C = 0$, the truncated cone of Fig. A.4 becomes a planar cone (see Fig. A.5). The directions of the axes are now along F_1 ($|\mathbf{F}| = F_1$) and N. This new cone will be set on an inclined plane to get the relationship between the maximum friction force, $F_{\max}$ and the normal force N for the case of bidimensional contact [60].

Fig. A.5 shows the weight $\mathbf{W}$ in equilibrium with the resulting force $\mathbf{R}$ of the complex and unknown stress distribution exerted from the underlying wedge upon the material surface in the contact zone. Thus, $\mathbf{R} - \mathbf{W} = \mathbf{0}$ and the forces act along the same line. The projection of the force $\mathbf{R}$ on the normal surface is $N = W\cos\phi$ and the tangential component is $F = W\sin\phi$. Experience show that a limitation of the shear component exists when ϕ is increasing. A maximum value of ϕ, the *angle of static friction* ϕ_0, limits the range of equilibrium. The condition of static friction may be expressed by the inequality $\phi < \phi_0$ or $F < F_0 = W\sin\phi_0 = N\tan\phi_0$. The *coefficient of static friction* $\mu = \tan\phi_0$ depends on the smoothness and cleanness of the contacting surfaces and on the materials in contact. The numerical values for absolutely dry contact are determined experimentally often in connection

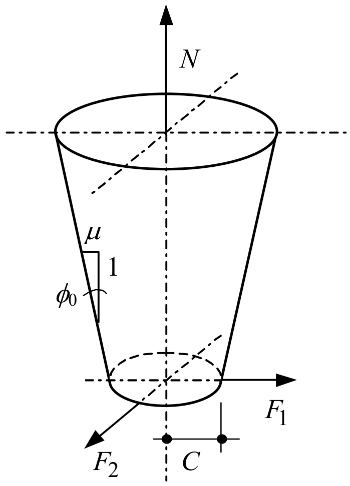


Figure A.4 Perfect friction represented by the Coulomb's cone

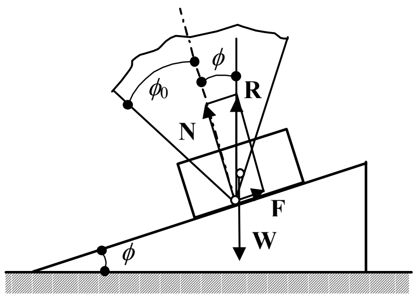


Figure A.5 A rigid body at rest in contact with an inclined plane: the cone of static friction with abrasive testing (a field called *tribology*) and are available in tabulated form. For reasons of safety, μ is set equal to the generally smaller value of the coefficient of friction μ in sliding contact. In such a case Coulomb's law of dry friction is given by

$$F = F_{\max} = \mu N \tag{A.11}$$

which is equal to Eq. (A.10). This expression holds with constant coefficient for a limited range of 'slow' relative velocities [1, 2, 58, 70]. If it is assumed that μ is known, the condition of static friction is easily proved by drawing a cone of half aperture ϕ_0, with its axis in the N-direction and tip at the point of application of $\mathbf{R}$ in the zone of contact. Equilibrium is safe as long as $\mathbf{R}$ points into the interior of this *cone of static friction*. Some average values of μ are the following: for steel in contact with steel, 0.2; for steel contacting bronze, 0.3; for metal in contact with glass, 0.5 and so on [35].