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DOCTORAL THESIS

A Contribution to Consensus Modeling in Decision-Making by means of Linguistic Assessments

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A la tieta

Declaration of Authorship

I, Jordi Montserrat-Adell, do hereby declare that the thesis entitled *"A Contribution to Consensus Modeling in Decision-Making by means of Linguistic Assessments"*, submitted to Universitat Politècnica de Catalunya (UPC-BarcelonaTech) for the award of the degree of **Doctor of Philosophy in Applied Mathematics**, is an original and authentic piece of work that satisfies the university rules with respect to plagiarism and collusion.

I further confirm that the work embodied in this thesis is the record of an independent research work done by myself between 2015 and 2017 under the supervision and guidance of Prof. Dr. Mónica Sánchez and Prof. Dr. Núria Agell. I attest that the forenamed dissertation has been written by myself and that I have fully referenced and acknowledged all material incorporated as secondary resources.

I also certify that this work has not been previously accepted in substance for any degree and is not being concurrently submitted in candidature for any degree in any other university or institute of higher learning.

Barcelona, 13th June, 2018.

Abstract

Doctor of Philosophy in Applied Mathematics

A Contribution to Consensus Modeling in Decision-Making by means of Linguistic Assessments

by Jordi MONTSERRAT-ADELL

Decision-making is an active field of research. Specifically, in recent times, a lot of contributions have been presented on decision-making under linguistic assessments. To tackle this kind of processes, hesitant fuzzy linguistic term sets have been introduced to grasp the uncertainty inherent in human reasoning when expressing preferences. This thesis introduces an extension of the set of hesitant fuzzy linguistic term sets to capture differences between non-compatible assessments. Based on this extension, a distance between linguistic assessments is defined to quantify differences between several opinions. This distance is used in turn to present a representative opinion from a group in a decision-making process. In addition, different consensus measures are introduced to determine the level of agreement or disagreement within a decision-making group and are used to define a decision maker's profile to keep track of their dissension with respect to the group as well as their level of hesitancy. Furthermore, with the aim of allowing decision makers to choose the linguistic terms that they feel more comfortable with, the concept of free double hierarchy hesitant fuzzy linguistic term set is developed in this thesis. Finally, a new approach of the TOPSIS methodology for processes in which the assessments are given by means of free double hierarchy hesitant fuzzy information is presented to rank alternatives under these circumstances.

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Les primers paraules d'agraïment d'aquesta tesi són per la Mónica i la Núria. Els vull agrair totes les hores que han dedicat i la paciència que han tingut. D'elles n'he après molt més que matemàtiques, però m'agradaria destacar que, per sobre de tot, he après que en equip es treballa millor.

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Jenn has been my traveling companion during these three years. She has made stressing situations more bearable. I want to thank her for sharing the tough times as well as the good ones with me.

I want to show my gratitude to Prof. Zeshui Xu for his hospitality and kindness during my visit at Sichuan University. I could not have a better host. I also appreciate the opportunity of collaborating with him and all what I learned from him. I want to give thanks to Xunjie for collaborating together during my visit and for taking the time to revise my work. I also want to thank all the office 703 mates for welcoming me and for taking care of me during my stay, with special thanks to Yue, Peijia, and Lala. 谢谢你们！

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List of Abbreviations

DHHFLE	**D**ouble **H**ierarchy **H**esitant **F**uzzy **L**inguistic **E**lement
DHHFLTS	**D**ouble **H**ierarchy **H**esitant **F**uzzy **L**inguistic **T**erm **S**et
DHLT	**D**ouble **H**ierarchy **L**inguistic **T**erm
DHLTS	**D**ouble **H**ierarchy **L**inguistic **T**erm **S**et
DM	**D**ecision **M**aker
FDH	**F**ree **D**ouble **H**ierarchy
FDHHFLE	**F**ree **D**ouble **H**ierarchy **H**esitant **F**uzzy **L**inguistic **E**lement
FDHHFLTS	**F**ree **D**ouble **H**ierarchy **H**esitant **F**uzzy **L**inguistic **T**erm **S**et
FDHLT	**F**ree **D**ouble **H**ierarchy **L**inguistic **T**erm
FDHLTS	**F**ree **D**ouble **H**ierarchy **L**inguistic **T**erm **S**et
GDM	**G**roup **D**ecision-**M**aking
HFE	**H**esitant **F**uzzy **E**lement
HFLD	**H**esitant **F**uzzy **L**inguistic **D**escription
HFLE	**H**esitant **F**uzzy **L**inguistic **E**lements
HFLTS	**H**esitant **F**uzzy **L**inguistic **T**erm **S**et
HFS	**H**esitant **F**uzzy **S**et
LTS	**L**inguistic **T**erm **S**et
MADM	**M**ultiple-**A**ttribute **D**ecision-**M**aking
MAGDM	**M**ultiple-**A**ttribute **G**roup **D**ecision-**M**aking
MCDM	**M**ultiple-**C**riteria **D**ecision-**M**aking
QR	**Q**ualitative **R**easoning
TOPSIS	**T**echnique for **O**rder of **P**reference by **S**imilarity to **I**deal **S**olution

"Where all think alike, no one thinks very much."

Walter Lippmann

Chapter 1

Introduction

1.1 Motivation

Making decisions is a daily action. Should I stay or should I go? Where should I go on vacation? Which car model should I buy? What is the best candidate for a job position? Yet, to make a decision is not only part of people's daily life, it is also part of business. It is quite common that company managers have to make relevant decisions and support them under complex and uncertain problems. Because of this everyday nature and its usefulness for some important research fields, decision-making has been an increasingly interesting topic in the recent years, specially when it comes to Multiple-Criteria Decision-Making (MCDM).

MCDM refers to structuring and solving decision problems in the presence of multiple criteria. For instance, to decide where to go on vacation, these criteria could be, for example, the price, weather, cultural interest and amusement of each destination. Instead, to choose one car model to buy, the criteria would be the reviews in different car magazines of the models that we are considering.

As the previous examples show, MCDM methodologies can be used for different kind of decision-making problems. On the one hand, we can deal with Multiple-Attribute Decision-Making (MADM) situations in which the different criteria correspond to several features of the alternative or candidate to be assessed. Focusing on the job candidate example, the owner of a clothing store, trying to choose the best shop assistant would be an MADM process. In this case, one single person, the owner, has to make the decision based on several aspects such as previous experience, languages, references and so on.

On the other hand, MCDM tools can be used as well to approach situations in which the different criteria correspond to the opinion of a set of experts or Decision Makers (DMs). Under these circumstances, we talk about Group Decision-Making (GDM) situations. Back to the job candidate example, a case in which a consulting firm wants to hire the best job candidate basing the decision on the global opinion about the candidate from several members of a human resources recruitment committee would be a GDM process.

Finally, these two kind of situations can be combined together when each member of the group has to consider different attributes before giving their assessment.

In this case we talk about Multi-Attribute Group Decision-Making (MAGDM) problems. This would be the case, for instance, of a recruitment committee in which each member of the committee evaluate separately different aspects of the candidate's profile instead of giving a global opinion.

MCDM has been an active discipline of research for the last 50 years. Since 1970s several methodologies have been studied and presented to try to improve how to deal with these widespread situations [16, 38]. A valuable branch of these contributions is focused on linguistic MCDM problems, in which a qualitative opinion is preferred rather than a quantitative opinion. Computing with words was introduced in 1999 and was widely accepted given that qualitative descriptions are much more accepted than quantitative ones in some non-technological fields, and even in some technological ones, given that it is closer to usual language [79]. Because of this acceptance, the development of structures using linguistic assessments has speeded up a lot in recent times.

To this aim, fuzzy sets are a useful tool to mathematically deal with qualitative descriptions [77]. In this regard, Likert scales such as *totally disagree, disagree, neither disagree nor agree, agree, totally agree* can be understood as fuzzy linguistic term sets. This kind of structures have been very used in the literature to ease the study of linguistic MCDM situations.

Nevertheless, indecision is quite common in human reasoning and DMs could not be very sure about which linguistic label from the scale to choose. In front of this, a possible solution is to allow the DMs to hesitate and choose more than one label. This fact led to the introduction of absolute order-of-magnitude qualitative models [64] and, later, Hesitant Fuzzy Linguistic Terms Sets (HFLTSs) in 2012 to better capture such uncertainty [53].

The motivation of this tesis is to present, based on the initial studies on the fields of Qualitative Reasoning (QR) [63] and HFLTSs [53], a contribution to MCDM by means of hesitant fuzzy linguistic assessments, with a special focus on the study of consensus modeling, which includes from finding a consensus opinion to calculating consensus measures for GDM situations.

Consensus opinions can be understood as central opinions of the group, which are useful in any of the previous examples. In the store shop example, it can be used to determine the overall suitability of each candidate, while in the consulting firm example, it can be used to aggregate the opinion of the different members of the committee. In addition, consensus measures are appropriate to determine the level of agreement or disagreement between the different criteria, either features or DMs.

1.2   State of the Art

The research line in which this thesis is framed is the study of new mathematical structures for multi-criteria decision aiding under uncertainty. To deal with this kind of processes, Zadeh introduced in 1965 the concept of fuzzy sets [77]. Ever since

then, several extensions such as the intuitionistic fuzzy sets [5, 6] or the Hesitant Fuzzy Sets (HFSs) [28, 62, 72] have been presented.

However, given that in some areas people prefer to use a qualitative reasoning better than a quantitave one, this thesis focuses on decision-making processes in which opinions are given by means of linguistic assessments. To this end, Zadeh also introduced the concepts of linguistic variable [78] and computing with words [79]. From then on, several contributions have been developed on that field, some of them focusing on different linguistic representation models [27, 39, 65] and some others dealing with decision systems that are able to compute with linguistic variables [24, 40, 42].

Furthermore, different studies have shown that, in general, humans do not use purely quantitative models when expressing preferences and interests but are more comfortable using global or abstract forms based on qualitative or linguistic information [4, 24, 65]. Similarly, in MAGDM environments, the design of systems that try to aid decision-making is considered appropriate to allow the description or evaluation of alternatives to be made by non-numerical values. It is also capable to reflect the available knowledge, which is, in general, imprecise and involving uncertainty [14, 25, 29, 61]. In the literature, this uncertainty has been modeled with intervals or fuzzy values through a linguistic approach [52, 53, 58].

With regards to the set of linguistic labels used, two main groups of approaches for linguistic modeling in a considered fuzzy environment can be defined. The models in the first group are based on a totally ordered set of linguistic labels [73]. On the contrary, the models in the second group involve different levels of accuracy or multi-granularity and, therefore, they do not rely on a fully ordered set of linguistic labels [10, 22, 24, 49].

In addition, it should be noted that the different methods of MAGDM include a stage of aggregation or information fusion. While some of them use aggregation operators [60, 73], others are based on reference point methods [2], and others involve a consensus process to obtain a compromise solution [4, 7, 41, 49].

Once this general overview on MAGDM has been presented, let us confine ourselves to the most recent contributions in the literature with respect to the most closely related concepts to the contents of this thesis. Therefore, the following paragraphs focus with much more detail on the state of the art of the specific research line of this thesis: absolute order-of-magnitude qualitative models and HFLTSs.

Following previous studies framed in the order-of-magnitude qualitative reasoning [65], Prats *et al.* constructed the extended set of qualitative labels L over a well-ordered set and they proved its lattice structure [50]. The qualitative descriptions of a given set are also defined as L-fuzzy sets. The underlying idea of L-fuzzy sets is analogous to the concept of subset or the concept of fuzzy set [17]. In the same way that any function $f : \Lambda \rightarrow \{0,1\}$ defines an ordinary subset of a set Λ, whose characteristic function is f, and any function $f : \Lambda \rightarrow [0,1]$ defines a fuzzy set on Λ, whose membership function is f, an L-fuzzy set is defined by a membership function $Q : \Lambda \rightarrow L$. The elements of an L-fuzzy set are assigned to elements of a

lattice rather than degrees of membership. In this case, the lattice is the extended set of qualitative labels L over a well-ordered set.

In addition, in the case where the well-ordered set is finite, a suitable distance between L-fuzzy sets is introduced based on the properties of the lattice L, as well as the concept of the information contained in a qualitative label, leading to a formal definition of the entropy of an L-fuzzy set as a Lebesgue integral. In the discrete case, this integral becomes a weighted average of the information of the labels, corresponding to the Shannon entropy in information theory.

Prats *et al.* provided a new general representation of linguistic descriptions by unifying ordinal and fuzzy perspectives [51]. It proposes fuzzy-qualitative labels as a generalization of the concept of qualitative labels over a well-ordered set. Fuzzy-qualitative descriptions are defined to model the assessments of a group of experts when evaluating different alternatives by using linguistic descriptions. A remarkable theorem that characterizes finite fuzzy partitions using fuzzy-qualitative labels, the cores and supports of which are qualitative labels, is established. This theorem provides a mathematical justification for commonly-used fuzzy partitions of real intervals via trapezoidal (or triangular) fuzzy sets.

A mathematical framework and new methodologies for group decision-making under multi-granular and multi-attribute linguistic assessments have been thoroughly investigated [2, 56, 57]. On the one hand, a new approach is presented based upon qualitative reasoning techniques for representing and synthesizing the information given by a group of evaluators [2]. To represent non-trivial domain knowledge, the alternatives to be ranked are characterized by a set of features, which are evaluated by each member of the group through linguistic labels corresponding to ordinal values. Different levels of precision are considered to draw the distinctions required by evaluators' reasoning processes. The method used for ranking alternatives is based on comparing distances to an optimal reference point or gold standard.

On the other hand, a degree of consensus and distances between linguistic assessments for multi-criteria group decision-making are presented [56, 57]. Distances in the space of qualitative assessments are defined from the geodesic distance in graph theory and the Minkowski distance. The degree of consensus is introduced through the concept of entropy of a qualitatively-described system. Optimal assessments in terms of both proximity to all the expert opinions in the group and the degree of consensus are used to compare opinions and to define a methodology to rank multi-attribute alternatives. This new approach is able to manage situations where the assessments given by experts involve different levels of precision.

Subsequently, Rodríguez *et al.* introduced in 2012 the concept of Hesitant Fuzzy Linguistic Term Set (HFLTS) [53] with the aim of combining HFSs and QR. Due to the parallelism between the extended set of qualitative labels L over a well-ordered set (in the case where the well-ordered set is finite), and the set of Hesitant Fuzzy Linguistic Term Sets (HFLTSs) over a well-ordered set, some of the results obtained on the first field [2, 50, 57] can also be analyzed in the HFLTSs framework.

HFLTSs were later redefined by Liao *et al.* in a mathematical form [35]. From then on, several aspects of the HFLTSs have been studied such as hesitant fuzzy

linguistic information aggregation techniques [19, 68], hesitant fuzzy linguistic measure methods [20, 32, 33, 35], hesitant fuzzy linguistic operational laws [18], hesitant fuzzy linguistic preference relations [35, 36, 80, 81] and hesitant fuzzy linguistic decision making methods [20, 32].

Amongst all the related topics, this thesis focuses on the study of consensus of a GDM situation by means of HFLTSs. Several contributions have been presented in this field. While the aim of some of them is to present a consensus reaching algorithm [11], some others focus on introducing consensus measures to quantify the level of agreement in this kind of scenarios. Within this second kind of approaches, two main group can be defined based on a key difference. On the one hand, some results seek to determine the level of agreement of the whole group [55, 69, 70]. On the other hand, the aim of the second kind of approaches is to gauge the agreement of a certain DM of the group with respect to the whole group [12, 71]. In Chapter 4, a further study on the differences between these approaches is developed.

Notwithstanding the utility of HFLTSs, in some situations, this kind of linguistic expressions are not able to depict the complexity inherent in human reasoning when evaluating with linguistic assessments with enough details. Some authors have studied how to define linguistic expressions more complex than single linguistic terms as reviewed by Rodríguez *et al.* [54]. In order to provide a more precise tool, Gou *et al.* presented the concept of Double Hierarchy Hesitant Fuzzy Linguistic Term Sets (DHHFLTSs) [21]. This structure allows each decision maker to choose one term from a first hierarchy Linguistic Term Set (LTS) and later choose another term from a second hierarchy LTS gaining more accuracy on the linguistic assessment. Therefore, linguistic expression such as *very bad* or *almost perfect* may arise as possible linguistic assessments to be used by the DMs. A deeper explanation on DHHFLTS is given in Chapter 5.

Finally, in the last years, in the frame of multi-criteria decision aiding methods based on preference modeling, several approaches to enhance existing methods have been introduced. As an example, Afsordegan *et al.* present a modified Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) method for multi-criteria group decision-making with qualitative linguistic labels [1] and Gou *et al.* introduce a new approach of the Multiple Objective Optimization on the basis of Ratio Analysis plus Full Multiplicative Form (MULTIMOORA) method for GDM processes in which the assessments are made by means of DHHFLTSs [21].

1.3 Objectives of the Thesis

From a theoretical point of view, the main goal of this Ph.D. thesis is to define and study new mathematical structures, metrics, and criteria aggregation methods. In addition, these concepts are used throughout this thesis to introduce new consensus models and ranking methodologies that take into account some aspects that were disregarded by the already existing ones. All these contributions are based on fuzzy and qualitative reasoning for group decision-making and social interaction.

To this aim, this Ph.D. thesis is oriented towards the following specific objectives:

O1 To define new metrics between HFLTSs that better describe the concept of distance between DMs, users' or experts' opinions when those are expressed by means of linguistic assessments. These new metrics must capture details that were disregarded by the existing ones.

O2 To extend the set of HFLTSs into a larger set with negative elements that offsets the main drawbacks of the existing structure of HFLTSs. This extension must enable us to discriminate between pairs of non-overlapping assessments by calculating the gap between them.

O3 To redefine the new metrics in the extended set of HFLTSs in order to analyze and measure differences among linguistic assessments for group or collaborative decision-making.

O4 To use the introduced metrics to present a method for determining a central opinion in a GDM situation as a representative assessment of the group. This method must consider the aforementioned gap between non overlapping assessments and must be computed in a reasonable time.

O5 To define consensus measures for GDM environments to quantify the level of agreement within the group. These measures must improve the properties of the already existing ones.

O6 To present a DM's profile that keeps track of their previous performances in other GDM situations. This profile must consider both the level of hesitancy of the DM when assessing and the level of discrepancy between their opinion and the group's opinion.

O7 To introduce a new mathematical structure based on the HFLTSs that enable the DMs of a GDM process to choose their own linguistic expressions. These expressions do not necessarily have to be the same for all the DMs involved. This structure will give more freedom to the DMs by letting them choose the linguistic expressions that they feel more comfortable with.

O8 To develop a ranking method based on GDM processes in which assessments are given by means of the aforesaid structure. This method must deal with the fact that each DM can be using different linguistic expressions.

1.4 Contributions of the Thesis

The contributions of this Ph.D. thesis can be summarized as follows:

C1 The first contribution of this thesis is the introduction of a new distance measure within the set HFLTSs based on the previous work of Agell *et al.* [3], in which a distance measure was already defined. The drawbacks of the already defined distances have been pointed out and offset with the introduction of the new distance. The main novelty of the introduced distance with respect to the already existing ones is the fact that it is based on a concept called concordance of HFLTSs instead of the usual intersection of HFLTSs. This concordance

provides more details than just the empty set for non-overlapping assessments. This contribution covers the objective **O1**.

The results of this study have been published in the following article:

- Jordi Montserrat-Adell, Núria Agell, Mónica Sánchez, Francesc Prats, Francisco Javier Ruiz. Modeling group assessments by means of hesitant fuzzy linguistic term sets. *Journal of Applied Logic*, 27 (2017) 40–50. DOI: 10.1016/j.jal.2016.11.005. Impact Factor: 0.838. Quartile: Q1 (JCR Category: Logic).

C2 The second contribution is the extension of the set of HFLTSs to a larger set. Besides the usual HFLTSs that are now called positive HFLTSs, this new set also contains negative HFLTSs, which represent gaps between assessments, and zero HFLTSs, which represents consecutiveness of assessments. This extended model enables the generalization of the distance from the objective **O1** to the whole new set. In addition, this extended structure is used to provide a group representative assessment based on the idea of central opinion of the group by minimizing the addition of distances to each DM's opinion. This contribution covers the objectives **O2**, **O3**, and **O4**.

Preliminary works on this topic have been presented in the following international conferences:

- Jordi Montserrat-Adell, Núria Agell, Mónica Sánchez, Francisco Javier Ruiz. Extended set of hesitant fuzzy linguistic term sets. In *XVIII Congreso Español sobre Tecnologías y Lógica Fuzzy (ESTYLF)*, Donostia, Spain, 2016.

- Jordi Montserrat-Adell, Núria Agell, Mónica Sánchez, Francisco Javier Ruiz. From qualitative absolute order-of-magnitude to the extended set of hesitant fuzzy linguistic term sets. In *29th International Workshop on Qualitative Reasoning (QR)*, New York, USA, 2016.

The results of this study have been published in the proceedings of the following international conference:

- Jordi Montserrat-Adell, Núria Agell, Mónica Sánchez, Francisco Javier Ruiz. A representative in group decision by means of the extended set of hesitant fuzzy linguistic term sets. In proceedings of the *13th Modeling Decisions for Artificial Intelligence (MDAI)*, Sant Julià de Lòria, Andorra, 2016. Lecture Notes in Artificial Intelligence, Springer International Publishing (2016) 56–67. DOI: 10.1007/978-3-319-45656-0_5.

C3 The third contribution is the study of consensus (agreement or discrepancy) in a GDM process. To this end, different degrees of consensus are defined. These degrees, which quantifies the level of agreement in GDM processes, are approached in two different ways. On the one hand, a collective degree of consensus is presented to determine the overall level of agreement of the whole group. On the other hand, an individual degree of consensus is also introduced to measure the level of agreement a specific DM with respect to the rest of the group. Furthermore, this individual degree of consensus, together with a

measure of the level of hesitancy of each DM, are used to present a precision-dissension profile for each expert. This profile is useful to summarize the main characteristics of the assessments of a DM: hesitant or not and discrepant or not. This contribution covers the objectives **O5** and **O6**.

Preliminary works on this topic have been presented in the following international conference:

- Jordi Montserrat-Adell, Núria Agell, Mónica Sánchez, Francisco Javier Ruiz. A consensus degree for hesitant fuzzy linguistic decision-making. In *26th IEEE International Conference on Fuzzy Systems (FUZZ-IEEE)*, Naples, Italy, 2017.

The results of this study have been published in the following article:

- Jordi Montserrat-Adell, Núria Agell, Mónica Sánchez, Francisco Javier Ruiz. Consensus, dissension and precision in group decision making by means of an algebraic extension of hesitant fuzzy linguistic term sets. *Information Fusion*, 42 (Supplement C) (2018) 1–11. DOI: 10.1016/ j.inffus.2017.09.004. Impact Factor: 5.667. Quartile: Q1 (JCR Categories: Computer Sciences & Artificial Intelligence and Computer Sciences, Theory & Methods).

C4 The last contribution is the introduction of a new methodology that let the DMs choose their own linguistic terms. Gou *et al.* introduced a second hierarchy of linguistic terms to let the DMs be more precise on their assessments [21]. Based on this work, this thesis proposes an extensions of this model in which the second hierarchy can be different for each linguistic term and for each DM. The aim of this methodology is to let the DMs feel more comfortable by using the linguistic expressions that they prefer. Finally, this model is used to present a new approach of the TOPSIS method suitable to rank alternatives in this kind of situations. This contribution covers the objectives **O7** and **O8**.

The results of this study have been submitted to the following journal and are currently under review:

- Jordi Montserrat-Adell, Zeshui Xu, Xunjie Gou, Núria Agell. Free double hierarchy hesitant fuzzy linguistic term sets: An application on ranking alternatives in GDM. *Information Fusion*. Currently under review. Impact Factor: 5.667. Quartile: Q1 (JCR Categories: Computer Sciences & Artificial Intelligence and Computer Sciences, Theory & Methods).

Table 1.1 summarizes the main contributions of this PhD thesis.

From a different point of view, this thesis has also contributed to establish research collaborations between UPC-BarcelonaTech, ESADE Business School and Sichuan University. This partnership arose due to the stage that I did in Chengdu to collaborate with Professor Zeshui Xu. Chapter 5 of this thesis is the result of this collaboration.

In addition, not only research collaborations with Sichuan University have been established, but also with University of Granada, where I am planning to go on stage

TABLE 1.1: Summary of the main contributions of this thesis.

Obj.	Cont.	Conferences	Journal	Chapter
O1	**C1**		*Journal of Applied Logic* [47] (Q$_1$)	2
O2 **O3** **O4**	**C2**	ESTYLF 2016[a] QR 2016[a] MDAI 2016	*Lecture Notes on Artificial Intelligence* [45]	3
O5 **O6**	**C3**	FUZZ-IEEE 2017[a]	*Information Fusion* [46] (Q$_1$)	4
O7 **O8**	**C4**		*Information Fusion*[b] (Q$_1$)	5

[a] Preliminary works.
[b] Currently under review.

to work with the Soft Computing and Intelligent Information Systems (SCI^{2}S) research group, leaded by Professor Francisco Herrera to explore the lines of future research opened by this thesis.

Finally, the results of this thesis are framed on and have contributed to the development of the INVITE Research Project (TIN2016-80049-C2-1-R and TIN2016-80049-C2-2-R (AEI/FEDER, UE)), funded by the Spanish Ministry of Science and Information Technology.

1.5   Outline of the Thesis

The rest of the present document is structured into the following chapters that correspond to the articles that present the previously summarized contributions:

- Chapter 2 is the article *Modeling group assessments by means of hesitant fuzzy linguistic term sets*, which presents the results from the contribution **C1**.

- Chapter 3 is the article *A representative in group decision by means of the extended set of hesitant fuzzy linguistic term sets*, which presents the results from the contribution **C2**.

- Chapter 4 is the article *A consensus degree for hesitant fuzzy linguistic decision-making*, which presents the results from the contribution **C3**.

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- Chapter 5 is the article *Free double hierarchy hesitant fuzzy linguistic term sets: An application on ranking alternatives in GDM*, which presents the results from the contribution **C4**.

- Chapter 6 presents the main conclusions of the thesis and some lines of future research.

The link between each contribution of the thesis and the chapter that it corresponds to is also summarized in Table 1.1.

Chapter 2

Modeling Group Assessments by means of Hesitant Fuzzy Linguistic Term Sets

2.1   Introduction

Different approaches have been developed in the fuzzy set literature involving linguistic modeling to handle the imprecision and uncertainty inherent in human preference reasoning [14, 25, 29, 49, 61]. In addition, several extensions of classic fuzzy sets theory have been established to include different levels of precision or multi-granularity in linguistic modeling [10, 22, 53]. Hesitant Fuzzy Linguistic Term Sets (HFLTSs) were introduced to capture the human way of reasoning involving different levels of precision. To this end, a set of linguistic expressions is defined based on the concept of hesitancy [53].

L-fuzzy sets are considered as a generalization of the classic fuzzy sets with range values of membership functions in a lattice L [17]. Classic fuzzy sets can be considered as a special case of the L-fuzzy sets with L = [0, 1]. The relation between L-fuzzy sets and other extensions of fuzzy sets, such as intuitionistic fuzzy sets and interval-valued fuzzy sets, has been analyzed in several studies [10, 66].

In this chapter, we define a lattice structure on the set of HFLTSs over a set of linguistic terms, $\mathcal{H}_{\mathcal{S}}$, based on the literature related to absolute order-of-magnitude spaces with different levels of precision or multi-granularity [15, 50, 65]. This allows us to consider hesitant fuzzy linguistic descriptions (HFLDs) as L-fuzzy sets based on this lattice. The set $\mathcal{F}_{\mathcal{H}}$ of all the $\mathcal{H}_{\mathcal{S}}$-fuzzy sets is also introduced.

In group assessment processes where decision makers (DMs) are assessing different alternatives by means of hesitant fuzzy linguistic term sets, the assessments provided by each DM are modeled as a HFLD. To study differences between HFLDs representing the assessments of each DM of a group, we present two distances in $\mathcal{H}_{\mathcal{S}}$ between HFLTSs, and their associated distances in $\mathcal{F}_{\mathcal{H}}$ between HFLDs.

Taking into consideration the different perspectives of the DMs in the decision-making group, we present a HFLD that characterizes the group via an aggregation of

linguistic preferences. In addition, a centroid of the group is presented for each distance in $\mathcal{F}_\mathcal{H}$, as the HFLD that minimizes the addition of distances to the HFLDs of all the DMs in the group. Distances between HFLDs are used to measure differences between the DMs.

The rest of this chapter is organized as follows: first, Section 2.2 presents the lattice of hesitant fuzzy linguistic term sets. The concept of hesitant fuzzy linguistic description is introduced in Section 2.3. In Section 2.4, two distances between HFLDs are defined by means of two distances between HFLTSs. A new approach for group preference modeling based on an aggregation of HFLDs and the distances between them is presented in Section 2.5. Finally, Section 2.6 contains the main conclusions and lines of future research.

2.2 The Lattice of Hesitant Fuzzy Linguistic Term Sets

In this section, we briefly review some basic concepts related to HFLTSs [2, 50, 53, 56]. This enables us to provide the set of HFLTSs with a lattice structure, to define a partial order and a compatibility relation in this set.

From here on, let S be a finite totally ordered set of linguistic terms, $\mathcal{S} = \{a_1, \ldots, a_n\}$, with $a_1 < \ldots < a_n$.

Definition 2.1. ([53]) A *hesitant fuzzy linguistic term set (HFLTS)* over $\mathcal{S}$ is a subset of consecutive linguistic terms of $\mathcal{S}$, i.e. $\{x \in \mathcal{S} \mid a_i \leq x \leq a_j\}$, for some $i, j \in \{1, \ldots, n\}$ with $i \leq j$.

The HFLTS $\mathcal{S}$ is called the *full HFLTS*. and it is also denoted by the symbol ?. Moreover, the empty set $\{\} = \varnothing$ is also considered as a HFLTS and it is called the *empty HFLTS*.

From now on, the non-empty HFLTS $H = \{x \in \mathcal{S} \mid a_i \leq x \leq a_j\}$ is also denoted by $[a_i, a_j]$. If $i = j$, $[a_i, a_i]$ is the singleton $\{a_i\}$. The set of all HFLTSs over S is denoted by $\mathcal{H}_\mathcal{S}$:

$$\mathcal{H}_\mathcal{S} = \{[a_i, a_j] \mid i, j \in \{1, \ldots, n\}, i \leq j\} \cup \{\varnothing\}$$

A simple calculation proves that the cardinality of $\mathcal{H}_\mathcal{S}$ is $|\mathcal{H}_\mathcal{S}| = 1 + \frac{n(n+1)}{2}$.

The union and complement [53] are not closed operations on the set $\mathcal{H}_\mathcal{S}$. Indeed, the union of two non-empty HFLTSs $[a_i, a_j]$ and $[a_{i'}, a_{j'}]$ is a HFLTS if and only if $[a_i, a_j] \cap [a_{i'}, a_{j'}] \neq \varnothing$ or $i = j' + 1$ or $i' = j + 1$. On the other hand, the complement of a non-empty HFLTS $[a_i, a_j]$ is a HFLTS if and only if $i = 1$ or $j = n$. The intersection of HFLTSs is a closed binary operation on the set $\mathcal{H}_\mathcal{S}$.

The *connected union,* $\sqcup$, of HFLTSs [50] is a closed binary operation on the set $\mathcal{H}_\mathcal{S}$, which is defined as follows:

Definition 2.2. The *connected union of two HFLTSs* is the least element of $\mathcal{H}_\mathcal{S}$, based on the subset inclusion relation $\subseteq$, that contains both HFLTSs.

As proven in the general case of order-of-magnitude spaces over a well-ordered (finite or infinite) set [50], the binary operations intersection and connected union provide a lattice structure to the set $\mathcal{H}_\mathcal{S}$ of HFLTSs.

Proposition 2.1. $(\mathcal{H}_\mathcal{S}, \sqcup, \cap)$ is a lattice.

Proof. The two operations $\sqcup$ and $\cap$ are clearly idempotent and commutative. The intersection is associative, and $(H_1 \sqcup H_2) \sqcup H_3 = H_1 \sqcup (H_2 \sqcup H_3)$ because $H_1 \sqcup (H_2 \sqcup H_3)$ is the least element that contains $H_1 \sqcup H_2$ and H_3. The two absorption laws can be checked in a similar straightforward manner. □

The lattice $(\mathcal{H}_\mathcal{S}, \sqcup, \cap)$ is not distributive. A counterexample in which the property $H_1 \cap (H_2 \sqcup H_3) = (H_1 \cap H_2) \sqcup (H_1 \cap H_3)$ does not hold is given in the case where $\mathcal{S}$ has at least three linguistic terms, considering $a_1, a_2, a_3 \in \mathcal{S}$ such that $a_1 < a_2 < a_3$ and the following three HFLTSs: $H_2 = \{a_1\}, H_1 = \{a_2\}, H_3 = \{a_3\}$.

The partial order $\leq$ in the lattice is given by: $H_1 \leq H_2 \Leftrightarrow H_1 \sqcup H_2 = H_1 \Leftrightarrow H_1 \cap H_2 = H_2 \Leftrightarrow H_1 \supseteq H_2$. Therefore, this order is the inverse subset inclusion relation and we call it *to be less or equally precise than*.

Definition 2.3. For any non-empty HFLTS $[a_i, a_j]$ and $[a_{i'}, a_{j'}]$, we say that $[a_i, a_j]$ is *less or equally precise than* $[a_{i'}, a_{j'}]$ if and only if $[a_i, a_j] \supseteq [a_{i'}, a_{j'}]$, i.e., $i \leq i'$ and $j' \leq j$.

Then, the least element in the lattice $\mathcal{H}_\mathcal{S}$ is $0_{\mathcal{H}_\mathcal{S}} = ? = \mathcal{S}$ because it is the least precise HFLTS, and the greatest element is $1_{\mathcal{H}_\mathcal{S}} = \emptyset$ because $H \supseteq \emptyset$ for all $H \in \mathcal{H}_\mathcal{S}$.

In Figure 2.1 the diagram of the lattice $(\mathcal{H}_\mathcal{S}, \sqcup, \cap)$ is depicted.
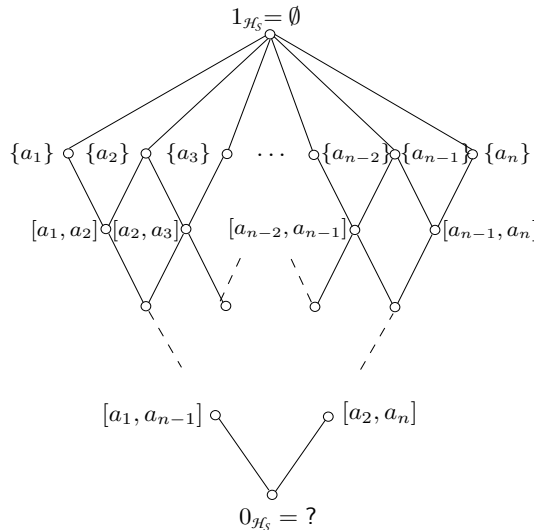
FIGURE 2.1: Diagram of the lattice $(\mathcal{H}_\mathcal{S}, \sqcup, \cap)$ [50].

The relation *to be compatible* between non-empty HFLTSs is defined, inspired by the concept of qualitative equality in absolute order-of-magnitude qualitative spaces [65] as follows:

Definition 2.4. For any non-empty HFLTS $[a_i, a_j]$ and $[a_{i'}, a_{j'}]$, we say that $[a_i, a_j]$ and $[a_{i'}, a_{j'}]$ are *compatible* if and only if $[a_i, a_j] \cap [a_{i'}, a_{j'}] \neq \varnothing$, i.e., $j \geq i'$ or $j' \geq i$.

Let us consider a simple example to illustrate the above definitions.

Example 2.1. Given the set of linguistic terms: $\mathcal{S} = \{a_1, a_2, a_3, a_4\}$ with: $a_1 = $ *slightly good*, $a_2 = $ *moderately good*, $a_3 = $ *very good*, $a_4 = $ *extremely good*, and the following HFLTSs:

$\{a_3\}$ = *very good,*

$[a_1, a_3]$ = *not extremely good,*

$? = [a_1, a_4]$ = *unknown, and*

$\{a_1\}$ = *slightly good,*

identifying $a_i = \{a_i\}, \forall i = \{1, 2, 3, 4\}$.

The relation to be *less or equally precise than* among the first three HFLTSs gives: $? \supseteq [a_1, a_3] \supseteq \{a_3\}$. However, $\{a_1\}$ are $\{a_3\}$ are not comparable by this relation. In addition, $\{a_3\}$, $[a_1, a_3]$ and $?$ are *compatible* since their pairwise intersections are non-empty, while $\{a_1\}$ and $\{a_3\}$ are not compatible.

Two distances between HFLTSs will be introduced in Section 2.4 based on the properties of the lattice $(\mathcal{H}_{\mathcal{S}}, \sqcup, \cap)$.

2.3   Hesitant Fuzzy Linguistic Descriptions

The concept of an *L-fuzzy set on a non-empty set* Λ was introduced by Goguen [17] as a function $f : \Lambda \to L$, where L is a lattice. This concept is applied to the case of the lattice $(\mathcal{H}_{\mathcal{S}}, \sqcup, \cap)$ of HFLTSs over a finite totally ordered set of linguistic terms $\mathcal{S}$ in the following definitions.

Definition 2.5. An $\mathcal{H}_{\mathcal{S}}$*-fuzzy set on* Λ is a function $F_H : \Lambda \to \mathcal{H}_{\mathcal{S}}$.

Note that any $f : \Lambda \to \{0, 1\}$ defines an ordinary set or crisp set on Λ, that is, a subset of Λ, whose characteristic function is f. If $f : \Lambda \to [0, 1]$, then f defines a fuzzy set on Λ, where for each $\lambda \in \Lambda$, $f(\lambda)$ is the degree of membership of λ. We can therefore consider an $\mathcal{H}_{\mathcal{S}}$-fuzzy set as a function $F_H : \Lambda \to \mathcal{H}_{\mathcal{S}}$ that assigns to each element of Λ a HFLTS from $\mathcal{H}_{\mathcal{S}}$ instead of a degree of membership.

Definition 2.6. The *set* $\mathcal{F}_{\mathcal{H}}$ *of* $\mathcal{H}_{\mathcal{S}}$*-fuzzy sets on* Λ is:

$$\mathcal{F}_{\mathcal{H}} = (\mathcal{H}_{\mathcal{S}})^{\Lambda} = \{F_H \mid F_H : \Lambda \to \mathcal{H}_{\mathcal{S}}\}.$$

Definition 2.7. A *Hesitant fuzzy linguistic description (HFLD) of the set* Λ *by* $\mathcal{H}_{\mathcal{S}}$ is an $\mathcal{H}_{\mathcal{S}}$-fuzzy set F_H on Λ such that for all $\lambda \in \Lambda$, $F_H(\lambda)$ is a non-empty HFLTS, i.e., $F_H(\lambda) \in \mathcal{H}_{\mathcal{S}} - \{\varnothing\}$.

From now on, the set Λ will represent a set of alternatives, and a HFLD will be used to model a DM's assessment of the alternatives in Λ. Note that missing values (such as DK/NA/REF) will be considered as $?$.

Example 2.2. Following Example 2.1, and given the same set $\mathcal{S}$ of linguistic terms, let us consider $\Lambda = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$, then

$$
\begin{array}{rccc}
F_H : & \Lambda & \longrightarrow & \mathcal{H}_{\mathcal{S}} \\
& \lambda_1 & \mapsto & \{a_3\} \\
& \lambda_2 & \mapsto & [a_1, a_3] \\
& \lambda_3 & \mapsto & ? \\
& \lambda_4 & \mapsto & \{a_1\}
\end{array}
$$

is a HFLD of the set Λ.

2.4 Distances between Hesitant Fuzzy Linguistic Descriptions

In order to define a first distance between HFLDs, that measures differences in the assessments of DMs, we previously consider the following distance between non-empty HFLTSs:

Definition 2.8. Given $H_1, H_2 \in \mathcal{H}_{\mathcal{S}} - \{\varnothing\}$, the *distance D_1 between H_1 and H_2* is defined as:
$$D_1(H_1, H_2) = card(H_1 \sqcup H_2) - card(H_1 \cap H_2)$$

As proven in the case of order-of-magnitude spaces over a finite well-ordered set [50], D_1 fulfills the distance requirements. This distance between non-empty HFLTSs induces a distance between HFLDs as follows:

Definition 2.9. Let us consider F_H^1 and F_H^2 two HFLDs of a finite set $\Lambda = \{\lambda_1, \ldots, \lambda_r\}$ by means of $\mathcal{H}_{\mathcal{S}}$, with $F_H^1(\lambda_i) = H_i^1$ and $F_H^2(\lambda_i) = H_i^2$, for all $i \in \{1, \ldots, r\}$. Then, the *distance $D_1^{\mathcal{F}}$ between these two HFLDs* is defined as:

$$D_1^{\mathcal{F}}(F_H^1, F_H^2) = \sum_{t=1}^{r} D_1(H_t^1, H_t^2) \tag{2.I}$$

Expression 2.I provides a distance in the set $(\mathcal{H}_{\mathcal{S}} - \{\varnothing\})^{\Lambda}$, i.e., a distance between HFLDs. In fact, each HFLD F_H of the set Λ by $\mathcal{H}_{\mathcal{S}}$ can be identified with the r-dimensional vector $(H_1, \ldots, H_r) \in (\mathcal{H}_{\mathcal{S}} - \{\varnothing\})^r = (\mathcal{H}_{\mathcal{S}} - \{\varnothing\}) \times \cdots \times (\mathcal{H}_{\mathcal{S}} - \{\varnothing\})$ whose components are $H_i = F_H(\lambda_i)$, for all $i \in \{1, \ldots, r\}$. Therefore the set $(\mathcal{H}_{\mathcal{S}} - \{\varnothing\})^{\Lambda}$ can be identified with the Cartesian product $(\mathcal{H}_{\mathcal{S}} - \{\varnothing\})^r$, and the Cartesian product of metric spaces is a metric space using the product distance and the city-block norm, which in this case results in Formula 2.I.

Remark 2.1. The maximum value for D_1 between two HFLTSs from $\mathcal{H}_{\mathcal{S}} - \{\varnothing\}$, where $\mathcal{S} = \{a_1, \ldots, a_n\}$, is n. This case is given, for instance, when $H_1 = \{a_1\}$

and $H_2 = \{a_n\}$, among others. Consequently, the maximum value for $D_1^{\mathcal{F}}$ between two HFLDs of the set $\Lambda = \{\lambda_1, \ldots, \lambda_r\}$ is $r \cdot n$.

Let us consider a simple example to illustrate the computation of this distance between HFLDs.

Example 2.3. Let us consider $\mathcal{S} = \{a_1, a_2, a_3, a_4\}$ as in Examples 2.1 and 2.2, and F_H^1 and F_H^2 two HFLDs of the set $\Lambda = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ by $\mathcal{H}_{\mathcal{S}}$ given in Table 2.1.

TABLE 2.1: F_H^1 and F_H^2 from Example 2.3.

	F_H^1	F_H^2
λ_1	$\{a_3\}$	$\{a_1\}$
λ_2	$[a_1, a_3]$	$\{a_4\}$
λ_3	?	$[a_2, a_4]$
λ_4	$\{a_1\}$	$\{a_4\}$

Therefore:

$$D_1^{\mathcal{F}}(F_H^1, F_H^2) = \sum_{t=1}^{4} (card(H_t^1 \sqcup H_t^2) - card(H_t^1 \cap H_t^2)) =$$

$$(3 - 0) + (4 - 0) + (4 - 3) + (4 - 0) = 12.$$

In this case, the distance between two HFLDs ranges from 0 to 16, which gives us a reference to frame the obtained result.

To capture differences among pairs of HFLTSs that are at the same distance D_1, we introduce the following measure of agreement that takes into consideration the gap between a pair of HFLTSs:

Definition 2.10. Given $H_1, H_2 \in \mathcal{H}_{\mathcal{S}} - \{\varnothing\}$, the *concordance between H_1 and H_2* is defined as:

$$\mathcal{C}(H_1, H_2) = \begin{cases} card(H_1 \cap H_2) & \text{if} \quad H_1 \cap H_2 \neq \varnothing \\ -card((H_1 \sqcup H_2) \cap \overline{H_1} \cap \overline{H_2}) & \text{if} \quad H_1 \cap H_2 = \varnothing \end{cases}$$

where $\overline{H} = \{x \in \mathcal{S} \mid x \notin H\}$ is the complement of H with respect to $\mathcal{S}$.

It is straightforward to see that if $H_1 = [a_i, a_j]$ and $H_2 = [a_{j+k}, a_l]$, with $k > 0$, then $\mathcal{C}(H_1, H_2) = -(k-1)$. Moreover, notice that the concordance between two HFLTSs is positive if and only if the two HFLTSs are compatible. In addition, the aim of the concordance is to consider how much in common two HFLTSs have or how big is the gap between them in case that they have nothing in common. According to the concordance, we present a new distance between non-empty HFLTSs as:

Definition 2.11. Given $H_1, H_2 \in \mathcal{H}_{\mathcal{S}} - \{\varnothing\}$, the *distance D_2 between H_1 and H_2* is defined as:

$$D_2(H_1, H_2) = card(H_1 \sqcup H_2) - \mathcal{C}(H_1, H_2)$$

In order to prove that D_2 is a distance, we will see that it is equivalent to the geodesic distance in the graph $\mathcal{H}_\mathcal{S} - \{\emptyset\}$, based on measuring the length of the shortest path between two elements of the graph [23]. In $\mathcal{H}_\mathcal{S} - \{\emptyset\}$, the shortest path between two HFLTSs can always be reached passing through the connected union of both of them. In Figure 2.2, we can see, as an example, the shortest path between $\{a_1\}$ and $[a_2, a_3]$ working with $\mathcal{S} = \{a_1, a_2, a_3, a_4\}$. In this case, the length of the shortest path is 3.
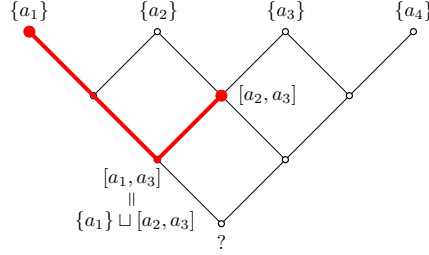
FIGURE 2.2: Shortest path between $\{a_1\}$ and $[a_2, a_3]$.

Lemma 2.1. D_2 can be equivalently expressed as:

$$D_2(H_1, H_2) = 2 \cdot card(H_1 \sqcup H_2) - card(H_1) - card(H_2)$$

Proof. We see that $2 \cdot \mathrm{card}(H_1 \sqcup H_2) - \mathrm{card}(H_1) - \mathrm{card}(H_2) = \mathrm{card}(H_1 \sqcup H_2) - \mathrm{card}(H_1 \cap H_2) + \mathrm{card}((H_1 \sqcup H_2) \cap \overline{H_1} \cap \overline{H_2})$. Indeed, if $H_1 \cap H_2 \neq \emptyset$, both parts are equal to $\mathrm{card}(H_1 \sqcup H_2) - \mathrm{card}(H_1 \cap H_2)$, while if $H_1 \cap H_2 = \emptyset$, then both parts are $\mathrm{card}(H_1 \sqcup H_2) + \mathrm{card}((H_1 \sqcup H_2) \cap \overline{H_1} \cap \overline{H_2})$. $\square$

Proposition 2.2. D_2 is equivalent to the geodesic distance in the graph $\mathcal{H}_\mathcal{S} - \{\emptyset\}$.

Proof. By Lemma 2.1, $D_2(H_1, H_2) = 2 \cdot \mathrm{card}(H_1 \sqcup H_2) - \mathrm{card}(H_1) - \mathrm{card}(H_2) = (\mathrm{card}(H_1 \sqcup H_2) - \mathrm{card}(H_1)) + (\mathrm{card}(H_1 \sqcup H_2) - \mathrm{card}(H_2)) = \ell(H_1, H_1 \sqcup H_2) + \ell(H_2, H_1 \sqcup H_2) = \ell(H_1, H_2)$, where $\ell(H, H')$ is the length of the shortest path from H to H'. $\square$

Once we have proved that D_2 is a distance between HFLTSs, we can use it to define an associated distance between HFLDs, analogously to what we did for D_1:

Definition 2.12. Let us consider F_H^1 and F_H^2 two HFLDs of a set $\Lambda = \{\lambda_1, \ldots, \lambda_r\}$ by means of $\mathcal{H}_\mathcal{S}$, with $F_H^1(\lambda_i) = H_i^1$ and $F_H^2(\lambda_i) = H_i^2$, for all $i \in \{1, \ldots, r\}$. Then, the *distance $D_2^\mathcal{F}$ between these two HFLDs* is defined as:

$$D_2^\mathcal{F}(F_H^1, F_H^2) = \sum_{t=1}^r D_2(H_t^1, H_t^2)$$

Remark 2.2. The maximum value for D_2 between two HFLTSs from $\mathcal{H}_\mathcal{S} - \{\emptyset\}$, where $\mathcal{S} = \{a_1, \ldots, a_n\}$, is $2n - 2$. This case is given only when $H_1 = \{a_1\}$ and $H_2 = \{a_n\}$. Consequently, the maximum value for $D_2^\mathcal{F}$ between two HFLDs of the set $\Lambda = \{\lambda_1, \ldots, \lambda_r\}$ is $r \cdot (2n - 2)$.

In order to illustrate this new distance, let us see the following example:

Example 2.4. Let F_H^1 and F_H^2 be the two HFLDs from Example 2.3 of the set Λ by $\mathcal{H}_S$ given in Table 2.1. Therefore:

$$D_2^{\mathcal{F}}(F_H^1, F_H^2) = \sum_{t=1}^{4} (card(H_t^1 \sqcup H_t^2) - \mathcal{C}(H_t^1, H_t^2)) =$$

$$(3 - (-1)) + (4 - 0) + (4 - 3) + (4 - (-2)) = 15.$$

In this case, the distance between two HFLDs ranges from 0 to 24, which gives us a reference to frame the obtained result.

The two distances that have been proposed can be compared as follows:

Proposition 2.3. Given two non-empty HFLTSs, H_1 and H_2, from $\mathcal{H}_S - \{\varnothing\}$,

$$D_1(H_1, H_2) \leq D_2(H_1, H_2).$$

Proof. It is enough to rewrite $D_2(H_1, H_2)$ as:

$$D_2(H_1, H_2) = card(H_1 \sqcup H_2) - card(H_1 \cap H_2) + card((H_1 \sqcup H_2) \cap \overline{H_1} \cap \overline{H_2}) =$$

$$D_1(H_1, H_2) + card((H_1 \sqcup H_2) \cap \overline{H_1} \cap \overline{H_2}) \geq D_1(H_1, H_2).$$

$\square$

Proposition 2.3 can be generalized to the distance between HFLDs as follows:

Proposition 2.4. Given two HFLDs, F_H^1 and F_H^2, of a set $\Lambda = \{\lambda_1, \ldots, \lambda_r\}$,

$$D_1^{\mathcal{F}}(F_H^1, F_H^2) \leq D_2^{\mathcal{F}}(F_H^1, F_H^2).$$

Proof. Taking into account Definitions 2.9 and 2.12, then, by Proposition 2.3, the proof becomes trivial. $\square$

To illustrate these two propositions, Table 2.2 summarizes the results from Examples 2.3 and 2.4.

TABLE 2.2: Distances between F_H^1 and F_H^2 from Example 2.3.

	F_H^1	F_H^2	$D_1(H_t^1, H_t^2)$	$D_2(H_t^1, H_t^2)$
λ_1	$\{a_3\}$	$\{a_1\}$	3	4
λ_2	$[a_1, a_3]$	$\{a_4\}$	4	4
λ_3	?	$[a_2, a_4]$	1	1
λ_4	$\{a_1\}$	$\{a_4\}$	4	6
			$D_1^{\mathcal{F}}(F_H^1, F_H^2) = 12$	$D_2^{\mathcal{F}}(F_H^1, F_H^2) = 15$

2.5 Modeling Group Assessments

In this section, we analyze from two different perspectives how to summarize the assessments given by a group of DMs that are assessing a set of alternatives by means of HFLTSs. To this end, the lattice structure of $\mathcal{H}_{\mathcal{S}}$-fuzzy sets and the distances defined in Section 2.4 are considered to aggregate the DMs' assessments of alternatives.

We consider two possible representatives to summarize the group's assessments: Firstly, the connected union in $\mathcal{H}_{\mathcal{S}}$-fuzzy sets and secondly, the HFLD of the set of alternatives Λ that minimizes the addition of distances to the assessments of all the DMs in the group, with respect to the two distances presented in Section 2.4, $D_1^{\mathcal{F}}$ and $D_2^{\mathcal{F}}$.

The connected union among $\mathcal{H}_{\mathcal{S}}$-fuzzy sets can be considered as a reasonable way to model the group assessment, because it provides a HFLD compatible with all the HFLDs in the group for all the alternatives. Notice that the intersection among $\mathcal{H}_{\mathcal{S}}$-fuzzy sets cannot be used to model the group assessments because some of its values may result in the null HFLTS. If so, the intersection would not be a HFLD.

Definition 2.13. Let Λ be a set of alternatives and G a group of k DMs. Let $F_H^1, \ldots, F_H^k$ be the HFLDs of Λ provided by the DMs in G. The *HFLD of Λ associated to the connected union of the assessments in group G* is defined as:

$$
\begin{aligned}
F_H^G : \quad \Lambda \quad &\longrightarrow \quad \mathcal{H}_{\mathcal{S}} - \{\varnothing\} \\
\lambda \quad &\mapsto \quad F_H^G(\lambda) = F_H^1(\lambda) \sqcup \ldots \sqcup F_H^k(\lambda)
\end{aligned}
$$

However, this way of representing the group's assessment tends very fast to ? in most of cases, because it is very sensitive to outliers. In addition, it does not consider the precision that DMs in the group use. For this reason, to solve these drawbacks, a representative of the group of DMs as a *centroid of the group* is defined by means of the concept of a distance as follows:

Definition 2.14. Let Λ be a set of r alternatives, G a group of k DMs and $F_H^1, \ldots, F_H^k$ the HFLDs of Λ provided by the DMs in G, then, for any distance $D^{\mathcal{F}}$ in $\mathcal{F}_{\mathcal{H}}$, a *centroid of the group with respect to $D^{\mathcal{F}}$* is:

$$
F_H^C = arg \min_{F_H^X \in (\mathcal{H}_{\mathcal{S}} - \{\varnothing\})^r} \sum_{t=1}^k D^{\mathcal{F}}(F_H^X, F_H^t),
$$

identifying each HFLD F_H with the vector $(H_1, \ldots, H_r) \in (\mathcal{H}_{\mathcal{S}} - \{\varnothing\})^r$, where $F_H(\lambda_i) = H_i$, for all $i = 1, \ldots, r$.

In the specific case of the two distances presented in Section 2.4, $D_1^{\mathcal{F}}$ and $D_2^{\mathcal{F}}$, the corresponding centroids will be denoted as F_{H1}^C and F_{H2}^C respectively.

Note that, for a given distance, more than one HFLD can produce the minimum value for the sum of distances in the above definition. Thus, a group of DMs can have more than one centroid with respect to the same distance. In addition, neither the HFLD of the connected union nor those of the centroids of the group with respect

to any distance have necessarily to coincide with any HFLD provided by a DM in the group.

Example 2.5. Following Examples 2.1, 2.2, 2.3 and 2.4, where $\mathcal{S} = \{a_1, a_2, a_3, a_4\}$ and $\Lambda = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$, let us consider a group G of five DMs. The HFLDs of the set Λ by $\mathcal{H}_\mathcal{S}$ corresponding to the DMs in G are given in Table 2.3 (columns from 1 to 5). Column 6 shows the HFLD associated to the connected union, F_H^G, columns 7 and 8 show the two centroids of the group, $F_{H\,1}^{C_1}$ and $F_{H\,1}^{C_2}$, according to $D_1^\mathcal{F}$, and in the last column we find the unique centroid of the group, F_{H2}^C, with respect to $D_2^\mathcal{F}$. An exhaustive search has been conducted to obtain the centroids of the group $F_{H\,1}^{C_1}$, $F_{H\,1}^{C_2}$ and F_{H2}^C.

TABLE 2.3: All HFLDs of Λ from Example 2.5.

	F_H^1	F_H^2	F_H^3	F_H^4	F_H^5	F_H^G	$F_{H\,1}^{C_1}$	$F_{H\,1}^{C_2}$	$F_{H\,2}^{C_1}$
λ_1	$\{a_3\}$	$\{a_1\}$	$[a_1, a_2]$	$[a_1, a_3]$	$\{a_2\}$	$[\boldsymbol{a_1, a_3}]$	$[\boldsymbol{a_1, a_2}]$	$[\boldsymbol{a_1, a_2}]$	$[\boldsymbol{a_1, a_2}]$
λ_2	$[a_1, a_3]$	$\{a_4\}$	$\{a_1\}$	$\{a_4\}$	$\{a_1\}$	?	$\{\boldsymbol{a_1}\}$	$\{\boldsymbol{a_1}\}$	$[\boldsymbol{a_1, a_3}]$
λ_3	?	$[a_2, a_4]$	$\{a_3\}$	$\{a_1\}$	$[a_3, a_4]$	?	$[\boldsymbol{a_3, a_4}]$	$[\boldsymbol{a_2, a_4}]$	$[\boldsymbol{a_2, a_4}]$
λ_4	$\{a_1\}$	$\{a_4\}$	$[a_3, a_4]$	$\{a_3\}$	$[a_3, a_4]$	?	$[\boldsymbol{a_3, a_4}]$	$[\boldsymbol{a_3, a_4}]$	$[\boldsymbol{a_3, a_4}]$

Note that, as it can be seen in Table 2.3, the considered group of DMs has two centroids according to $D_1^\mathcal{F}$ that just differ in their values corresponding to λ_3: $[a_3, a_4]$ and $[a_2, a_4]$. However, since $[a_2, a_4] \supseteq [a_3, a_4]$, one can choose $F_{H\,1}^{C_1}$ as the most precise centroid representing the group with respect to $D_1^\mathcal{F}$.

Figure 2.3 depicts, for each element in Λ, a graphical representation of the HFLTSs given by the DMs in G, together with the HFLTSs corresponding to the HFLD associated to the connected union, F_{HG}, and to the three centroids of the group, $F_{H\,1}^{C_1}$, $F_{H\,1}^{C_2}$ and F_{H2}^C.
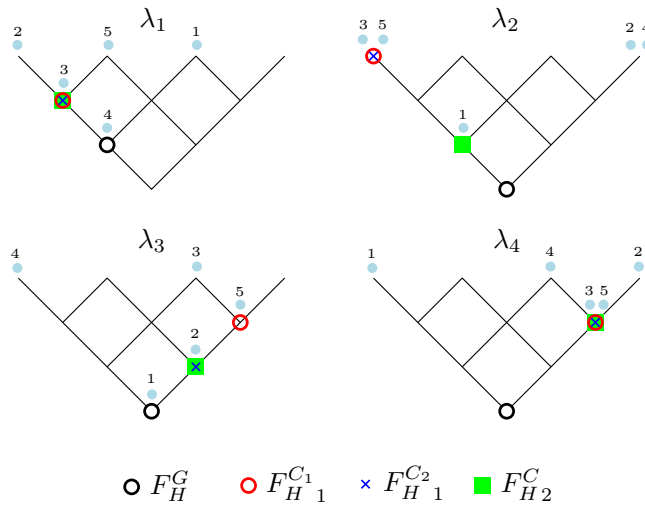
FIGURE 2.3: Graphical representation of Example 2.5.

Finally, Table 2.4 presents the distance matrices, with respect to $D_1^{\mathcal{F}}$ and $D_2^{\mathcal{F}}$ respectively, computed for each pair of HFLDs in the group G expanded with F_H^G and the corresponding centroids for each distance: $F_{H\,1}^{C_1}$ and $F_{H\,1}^{C_2}$ in the first case, and F_{H2}^C in the second case.

TABLE 2.4: Distance matrices from Example 2.5.

(A) Distance $D_1^{\mathcal{F}}$.

$D_1^{\mathcal{F}}$	F_H^1	F_H^2	F_H^3	F_H^4	F_H^5	F_H^G	$F_{H\,1}^{C_1}$	$F_{H\,1}^{C_2}$
F_H^1	0	12	12	12	10	6	11	10
F_H^2		0	8	8	8	10	7	6
F_H^3			0	9	2	9	1	2
F_H^4				0	11	9	10	10
F_H^5					0	9	1	2
F_H^G						0	8	7
$F_{H\,1}^{C_1}$							0	1
$F_{H\,1}^{C_2}$								0

(B) Distance $D_2^{\mathcal{F}}$.

$D_2^{\mathcal{F}}$	F_H^1	F_H^2	F_H^3	F_H^4	F_H^5	F_H^G	F_{H2}^C
F_H^1	0	15	12	13	11	6	9
F_H^2		0	10	8	10	9	6
F_H^3			0	12	2	9	4
F_H^4				0	14	9	10
F_H^5					0	9	4
F_H^G						0	5
F_{H2}^C							0

We can observe similar results by analyzing the values of the distances provided in Tables 2.4a and 2.4b. Assessments corresponding to the descriptions given by DM 3 and DM 5 are the closest ones with respect to both distances. In the same way, the most distant pairs of assessments correspond to the pairs: DM 1 and DM 2, DM 1 and DM 3 and DM 1 and DM 4 with respect to $D_1^{\mathcal{F}}$, Whilst according to $D_2^{\mathcal{F}}$, the tie is broken and the most distant ones are DM 1 and DM 2. We can also observe that assessments provided by DM 3 and DM 5 are the closest ones to the centroids of the group in both cases. Finally, the assessment corresponding to the descriptions given by DM 1 is the closest one to the assessment associated to the connected union with respect to both distances. It is also one of the two most distant assessments from the centroids of the group, together with the assessment given by DM 4.

Note that these distances matrices quantifying the similarity in between pairwise linguistic expressions could be used in other pattern recognition contexts, such as clustering, classification or ranking [8, 30]. In addition, the use of HFLTSs will allow the definition of fuzzy outputs able to capture the inherent complexity underlying in end-users' opinions

2.6 Conclusions

This chapter proposes a theoretical framework to model group assessments on the basis of HFLTSs. To this aim, the concept of distance between DMs in group decision-making when DMs' assessments are expressed using HFLTSs is studied. This concept allows similarities and differences among DMs' opinions to be analyzed.

From a well-ordered set $\mathcal{S}$ of linguistic terms, the set of hesitant fuzzy linguistic term sets $\mathcal{H}_{\mathcal{S}}$ has been provided with two closed aggregation operations, connected

union and intersection, which are suitable to be used on reasoning and comparisons. In addition, the two operations provide $\mathcal{H}_\mathcal{S}$ with a lattice structure. The hesitant fuzzy linguistic descriptions of a set Λ are defined as $\mathcal{H}_\mathcal{S}$-fuzzy sets.

Two distances between HFLDs have been proposed. The first distance, D_1, is built directly from connected union and intersection. The second distance, D_2, coincides with D_1 in the case that there is a non-empty intersection between the considered pair of HFLTSs and, intuitively, corresponds to adding the gap between them to D_1 if their intersection is empty.

Finally, the concept of centroid of a decision-making group is introduced by minimizing the addition of distances to the assessments of all the DMs in the group. The two proposed distances are used to do a further study of the corresponding centroids, which can be used as representatives of the opinions of the group of DMs. Moreover, the distances between each DM and the centroid can be considered as a measure of agreement within the group. Lastly, most dissident DMs in the group can be easily identified by means of distances to the centroid.

The proposed structure based on distances and centroids is not only limited to decision making scenarios. It provides a general model suitable for comparing opinions between end-users in general when expressed in terms of ordered linguistic terms.

Future research is oriented towards three main directions. First, the design of an algorithm for the computation of the proposed centroids of a decision-making group. Second, based on the proposed centroids, a study will be addressed to analyze risk measurement and validity assurance of the actions derived from a decision outcome. This analysis will be oriented towards the improvement of consensus reaching processes by focussing in the dissident DMs. Finally, a real case study will be conducted in the marketing research area to analyze customers preferences in a retailing context.

Chapter 3

A Representative in Group Decision by means of the Extended Set of Hesitant Fuzzy Linguistic Term Sets

3.1 Introduction

Different approaches involving linguistic assessments have been introduced in the fuzzy set literature to deal with the impreciseness and uncertainty connate with human preference reasoning [14, 25, 29, 49, 61]. Additionally, different extensions of fuzzy sets have been presented to give more realistic assessments when uncertainty increases [10, 22, 53]. In particular, Hesitant Fuzzy Sets were introduced by Torra to capture this kind of uncertainty and hesitancy [62]. Following this idea, Hesitant Fuzzy Linguistic Term Sets (HFLTSs) were introduced to deal with situations in which linguistic assessments involving different levels of precision are used [53]. In addition, a lattice structure is provided to the set of HFLTSs [47].

In this chapter, we present an extension of the set of HFLTSs, $\overline{\mathcal{H}_{\mathcal{S}}}$, based on an equivalence relation on the usual set of HFLTSs. This enables us to establish differences between non-compatible HFLTSs. An order relation and two closed operation over this set are also introduced to define a new lattice structure in $\overline{\mathcal{H}_{\mathcal{S}}}$.

In order to describe group decision situations in which Decision Makers (DMs) are evaluating different alternatives, Hesitant Fuzzy Linguistic Descriptions (HFLDs) were presented [47]. A distance between HFLTSs is defined based on the lattice of $\overline{\mathcal{H}_{\mathcal{S}}}$. This allows us to present a distance between HFLDs that we can use to quantify differences among assessments of different DMs. Taking into consideration this distance, a group representative is suggested to describe the whole group assessment. Due to this representative is the HFLD that minimizes distances with the assessments of all the DMs, it is called the centroid of the group.

The rest of this chapter is organized as follows: first, Section 3.2 presents a brief review of HFLTSs and its lattice structure. The lattice of the extended set of HFLTSs is introduced in Section 3.3. In Section 3.4, the distances between HFLTSs and HFLDs are defined and the centroid of the group is presented in Section 3.5. Lastly, Section 3.6 contains the main conclusions and lines of future research.

3.2 The Lattice of Hesitant Fuzzy Linguistic Term Sets

In this section we present a brief review of some concepts about HFLTSs already presented in the literature that are used throughout this chapter [47, 53].

From here on, let $\mathcal{S}$ denote a finite total ordered set of linguistic terms, $\mathcal{S} = \{a_1, \ldots, a_n\}$ with $a_1 < \cdots < a_n$.

Definition 3.1. ([53]) A *hesitant fuzzy linguistic term set (HFLTS)* over $\mathcal{S}$ is a subset of consecutive linguistic terms of S, i.e. $\{x \in \mathcal{S} \,|\, a_i \leq x \leq a_j\}$, for some $i, j \in \{1, \ldots, n\}$ with $i \leq j$.

The HFLTS $\mathcal{S}$ is called the *full HFLTS*. Moreover, the empty set $\{\} = \varnothing$ is also considered as a HFLTS and it is called the *empty HFLTS*.

For the rest of this chapter, the non-empty HFLTS, $H = \{x \in \mathcal{S} \mid a_i \leq x \leq a_j\}$, is denoted by $[a_i, a_j]$. Note that, if $j = i$, the HFLTS $[a_i, a_i]$ is expressed as the singleton $\{a_i\}$.

The set of all the possible HFLTSs over $\mathcal{S}$ is denoted by $\mathcal{H}_{\mathcal{S}}$, being $\mathcal{H}_{\mathcal{S}}^* = \mathcal{H}_{\mathcal{S}} - \{\varnothing\}$ the set of all the non-empty HFLTSs. This set is provided with a lattice structure with the two following operations: on the one hand, the *connected union of two HFLTSs,* $\sqcup$, which is defined as the least element of $\mathcal{H}_{\mathcal{S}}$, based on the subset inclusion relation $\subseteq$, that contains both HFLTSs, and on the other hand, the intersection of HFLTSs, $\cap$, which is defined as the usual intersection of sets [47]. The reason of including the empty HFLTS in $\mathcal{H}_{\mathcal{S}}$ is to make the intersection of HFLTSs a closed operation in $\mathcal{H}_{\mathcal{S}}$.

For the sake of comprehensiveness, let us introduce the following example that is used throughout all this chapter to depict all the concepts defined.

Example 3.1. Given the set of linguistic terms $\mathcal{S} = \{a_1, a_2, a_3, a_4, a_5\}$, being $a_1 = $ *very bad*, $a_2 = $ *bad*, $a_3 = $ *regular*, $a_4 = $ *good*, $a_5 = $ *very good*, possible linguistic assessments and their corresponding HFLTSs by means of $\mathcal{S}$ would be:

TABLE 3.1: Assessments and their corresponding HFLTSs by means of $\mathcal{S}$ from Example 3.1.

Assessments	HFLTSs
$A = $ *"between bad and regular"*	$H_A = [a_2, a_3]$
$B = $ *"bad"*	$H_B = \{a_2\}$
$C = $ *"above regular"*	$H_C = [a_4, a_5]$
$D = $ *"below regular"*	$H_D = [a_1, a_2]$
$E = $ *"not very good"*	$H_E = [a_1, a_4]$

3.3 The Extended Lattice of Hesitant Fuzzy Linguistic Term Sets

With the aim of describing differences between couples of HFLTSs with empty intersections, an extension of the intersection of HFLTSs is presented in this section, resulting their intersection if it is not empty or a new element that we will call *negative HFLTS* related to the rift, or gap, between them if their intersection is empty. In order to present said extension of the intersection between HFLTSs, we first need to introduce the mathematical structure that allows us to define it as a closed operation. To this end, we define the extended set of HFLTSs in an analogous way to how integer numbers are defined based on an equivalence relation on the natural numbers. To do so, we first present some needed concepts:

Definition 3.2. Given two non-empty HFLTSs, $H_1, H_2 \in \mathcal{H}_{\mathcal{S}}^*$, we define:

(a) The *gap between H_1 and H_2* as:

$$gap(H_1, H_2) = (H_1 \sqcup H_2) \cap \overline{H_1} \cap \overline{H_2},$$

where $\overline{H} = \{x \in \mathcal{S} \mid x \notin H\}$ is the complement of H with respect to $\mathcal{S}$.

(b) H_1 and H_2 are *consecutive* if and only if $H_1 \cap H_2 = \varnothing$ and $gap(H_1, H_2) = \varnothing$.

Proposition 3.1. Given two non-empty HFLTSs, $H_1, H_2 \in \mathcal{H}_{\mathcal{S}}^*$, the following properties are met:

1. $gap(H_1, H_2) = gap(H_2, H_1)$.

2. If $H_1 \subseteq H_2$, $gap(H_1, H_2) = \varnothing$.

3. If $H_1 \cap H_2 \neq \varnothing$, $gap(H_1, H_2) = \varnothing$.

4. If $H_1 \cap H_2 = \varnothing$, $gap(H_1, H_2) \neq \varnothing$ or H_1 and H_2 are consecutive.

5. If H_1 and H_2 are consecutive, there exist $j \in \{2, \ldots, n-1\}$, $i \in \{1, \ldots, j\}$ and $k \in \{j+1, \ldots, n\}$, such that $H_1 = [a_i, a_j]$ and $H_2 = [a_{j+1}, a_k]$ or $H_2 = [a_i, a_j]$ and $H_2 = [a_{j+1}, a_k]$.

Proof. The proof is straightforward. □

Note that neither $[a_1, a_j]$ nor $[a_i, a_n]$ can ever be the result of the *gap* between two HFLTSs for any i and for any j.

Notation. Given two consecutive HFLTSs, $H_1 = [a_i, a_j]$ and $H_2 = [a_{j+1}, a_k]$, then a_j and a_{j+1} are called the linguistic terms that provide the consecutiveness of H_1 and H_2.

Example 3.2. Following Example 3.1, $gap(H_B, H_C) = \{a_3\}$, while the HFLTSs H_A and H_C are consecutive and their consecutiveness is given by $\{a_3\}$ and $\{a_4\}$.

Definition 3.3. Given two pairs of non-empty HFLTSs, (H_1, H_2) and (H_3, H_4), the *equivalence relation* $\sim$, is defined as:

$$(H_1, H_2) \sim (H_3, H_4) \iff \begin{cases} H_1 \cap H_2 = H_3 \cap H_4 \neq \varnothing \\ \vee \\ gap(H_1, H_2) = gap(H_3, H_4) \neq \varnothing \\ \vee \\ \text{both pairs are consecutive and} \\ \text{their consecutiveness is provided} \\ \text{by the same linguistic terms} \end{cases}$$

It can be easily seen that $\sim$ relates couples of non-empty HFLTSs with the same intersection if they are compatible, with consecutiveness provided by the same linguistic terms if they are consecutive and with the same gap between them in the case that they are neither compatible nor consecutive.

Example 3.3. Following Example 3.1, the pairs of HFLTSs (H_A, H_B) and (H_A, H_D) are related according to $\sim$ given that they have the same intersection, $\{a_2\}$. Additionally, $(H_C, H_B) \sim (H_C, H_D)$ since they have the same gap between them, $\{a_3\}$.

Applying this equivalence relation over the set of all the pairs of non-empty HFLTSs, we get the quotient set $(\mathcal{H}_{\mathcal{S}}^*)^2 / \sim$, whose equivalence classes can be labeled as:

- $[a_i, a_j]$ for the class of all pairs of compatible non-empty HFLTSs with intersection $[a_i, a_j]$, for all $i, j = 1, \ldots, n$ with $i \leq j$.

- $-[a_i, a_j]$ for the class of all pairs of incompatible non-empty HFLTSs whose gap is $[a_i, a_j]$, for all $i, j = 2, \ldots, n-1$ with $i \leq j$.

- α_i for the class of all pairs of consecutive non-empty HFLTSs whose consecutiveness is provided by $\{a_i\}$ and $\{a_{i+1}\}$, for all $i = 1, \ldots, n-1$.

For completeness and symmetry reasons, $(\mathcal{H}_{\mathcal{S}}^*)^2 / \sim$ is represented as shown in Figure 3.1 and stated in the next definition.

Example 3.4. Subsequent to this labeling, and following Example 3.1, the pair (H_C, H_B) belongs to the class $-\{a_3\}$ and so does the pair (H_C, H_D). The pair (H_C, H_A) belongs to the class α_3 and the pair (H_C, H_E) belongs to the class $\{a_4\}$.

Definition 3.4. Given a set of ordered linguistic term sets $\mathcal{S} = \{a_1, \ldots, a_n\}$, the *extended set of HFLTSs, $\overline{\mathcal{H}_{\mathcal{S}}}$*, is defined as:

$$\overline{\mathcal{H}_{\mathcal{S}}} = (-\mathcal{H}_{\mathcal{S}}^*) \cup \mathcal{A} \cup \mathcal{H}_{\mathcal{S}}^*,$$

where $-\mathcal{H}_{\mathcal{S}}^* = \{-H \mid H \in \mathcal{H}_{\mathcal{S}}^*\}$ and $\mathcal{A} = \{\alpha_0, \ldots, \alpha_n\}$.

In addition, by analogy with real numbers $-\mathcal{H}_{\mathcal{S}}^*$ is called the *set of negative HFLTSs,* $\mathcal{A}$ is called the *set of zero HFLTSs,* and, from now on, $\mathcal{H}_{\mathcal{S}}^*$ is called the *set positive HFLTSs.*

Note that HFLTSs can be characterized by couples of zero HFLTSs. This leads us to introduce a new notation for HFLTSs:
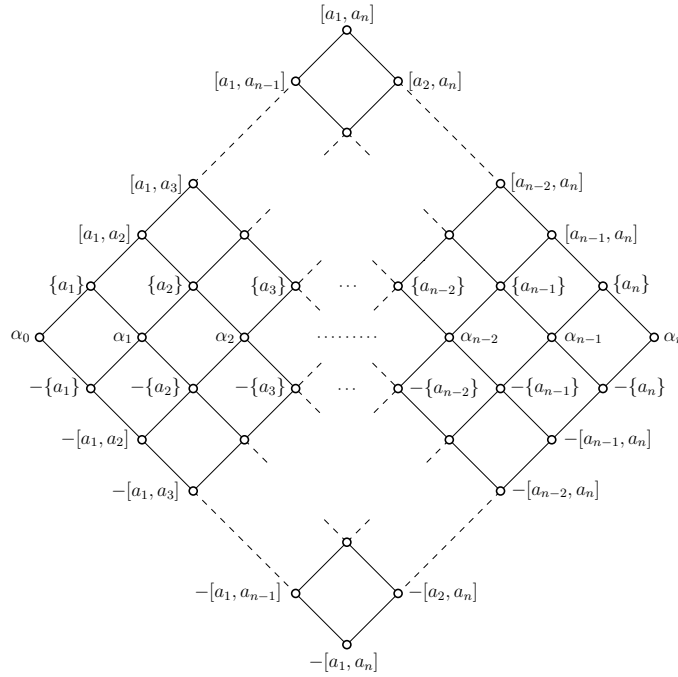
FIGURE 3.1: Graph of the extended set of HFLTSs.

Notation. Given a HFLTS, $H \in \overline{\mathcal{H}_{\mathcal{S}}}$, it can be expressed as $H = \langle \alpha_i, \alpha_j \rangle$, where the first zero HFLTS identifies the bottom left to top right diagonal and the second one identifies the top left to bottom right diagonal. Thus, $\langle \alpha_i, \alpha_j \rangle$ corresponds with $[a_{i+1}, a_j]$ if $i < j$, with $-[a_{i+1}, a_j]$ if $i > j$ and α_i if $i = j$.

This notation is used in the following definition that we present in order to latter introduce an order relation within $\overline{\mathcal{H}_{\mathcal{S}}}$.

Definition 3.5. Given $H \in \overline{\mathcal{H}_{\mathcal{S}}}$ described by $\langle \alpha_i, \alpha_j \rangle$ the *coverage of H* is defined as:

$$cov(H) = \{ \langle \alpha_{i'}, \alpha_{j'} \rangle \in \overline{\mathcal{H}_{\mathcal{S}}} \mid i' \geq i \wedge j' \leq j \}.$$

Example 3.5. The coverage of H_A from Example 3.1 can be seen in Figure 3.2.
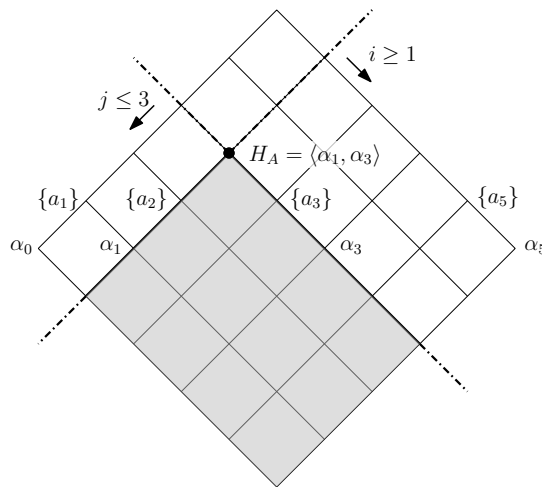
FIGURE 3.2: Coverage of H_A from Example 3.1.

The concept of coverage of a HFLTS enables us to define the *extended inclusion relation* between elements of $\overline{\mathcal{H}_\mathcal{S}}$.

Definition 3.6. The *extended inclusion relation in* $\overline{\mathcal{H}_\mathcal{S}}$, $\preccurlyeq$, is defined as:

$$\forall H_1, H_2 \in \overline{\mathcal{H}_\mathcal{S}}, \quad H_1 \preccurlyeq H_2 \iff H_1 \in cov(H_2).$$

Note that, restricting to only the positive HFLTSs, the extended inclusion relation coincides with the usual subset inclusion relation. According to this relation in $\overline{\mathcal{H}_\mathcal{S}}$, we can define the *extended connected union* and the *extended intersection* as closed operations within the set $\overline{\mathcal{H}_\mathcal{S}}$ as follows:

Definition 3.7. Given $H_1, H_2 \in \overline{\mathcal{H}_\mathcal{S}}$, the *extended connected union of H_1 and H_2*, $H_1 \sqcup H_2$, is defined as the least element that contains H_1 and H_2, according to the extended inclusion relation.

Definition 3.8. Given $H_1, H_2 \in \overline{\mathcal{H}_\mathcal{S}}$, the *extended intersection of H_1 and H_2*, $H_1 \sqcap H_2$, is defined as the largest element being contained in H_1 and H_2, according to the extended inclusion relation.

It is straightforward to see that the extended connected union of two positive HFLTSs coincides with the connected union presented by Montserrat-Adell *et al.* [47]. This justifies the use of the same symbol. About the extended intersection of two positive HFLTSs, it results the usual intersection of sets if they overlap and the *gap* between them if they do not overlap. Notice that the empty HFLTS is not needed to make the extended intersection a closed operation in $\overline{\mathcal{H}_\mathcal{S}}$.

Proposition 3.2. Given two non-empty HFLTSs, $H_1, H_2 \in \mathcal{H}_\mathcal{S}^*$, if $H_1 \preccurlyeq H_2$, then $H_1 \sqcup H_2 = H_2$ and $H_1 \sqcap H_2 = H_1$.

Proof. The proof is straightforward. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Example 3.6. Figure 3.3 provides an example with the extended connected union and the extended intersection of H_B and H_C and of H_A and H_E from Example 3.1: $H_B \sqcup H_C = [a_2, a_5]$, $H_B \sqcap H_C = -\{a_3\}$, $H_A \sqcup H_E = H_E$ and $H_A \sqcap H_E = H_A$.

(A) $H_B \sqcup H_C$ and $H_B \sqcap H_D$.

(B) $H_A \sqcup H_E$ and $H_A \sqcap H_E$.
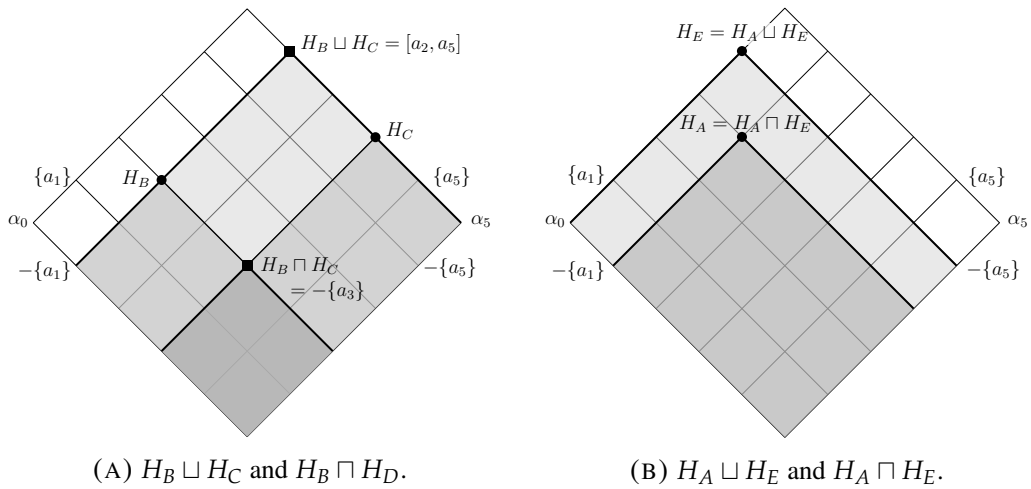
FIGURE 3.3: $\sqcup$ and $\sqcap$ of HFLTSs.

Proposition 3.3. $\left(\overline{\mathcal{H}_{\mathcal{S}}}, \sqcup, \sqcap \right)$ is a distributive lattice.

Proof. According to their respective definitions, both operations, $\sqcup$ and $\sqcap$, are trivially commutative and idempotent.

The associative property of $\sqcup$ is met since $(H_1 \sqcup H_2) \sqcup H_3 = H_1 \sqcup (H_2 \sqcup H_3)$ given that both parts equal the least element that contains H_1, H_2 and H_3. About the associativeness of $\sqcap$, $(H_1 \sqcap H_2) \sqcap H_3 = H_1 \sqcap (H_2 \sqcap H_3)$ given that in both cases it results the largest element contained in H_1, H_2 and H_3.

Finally, the absorption laws are satisfied given that: on the one hand $H_1 \sqcup (H_1 \sqcap H_2) = H_1$ given that $H_1 \sqcap H_2 \preccurlyeq H_1$ and on the other hand $H_1 \sqcap (H_1 \sqcup H_2) = H_1$ given that $H_1 \preccurlyeq H_1 \sqcup H_2$.

Furthermore, the lattice $\left(\overline{\mathcal{H}_{\mathcal{S}}}, \sqcup, \sqcap \right)$ is distributive given that none of its sublattices are isomorphic to the diamond lattice, M_3, or the pentagon lattice, N_5. $\qquad\square$

3.4 A Distance between Hesitant Fuzzy Linguistic Term Sets

In order to define a distance between HFLTSs, we introduce a generalization of the concept of cardinal of a positive HFLTS to all the elements of the extended set of HFLTSs.

Definition 3.9. Given $H \in \overline{\mathcal{H}_{\mathcal{S}}}$, the *width of H* is defined as:

$$\mathcal{W}(H) = \begin{cases} card(H) & if \quad H \in \mathcal{H}_{\mathcal{S}}^*, \\ 0 & if \quad H \in \mathcal{A}, \\ -card(-H) & if \quad H \in (-\mathcal{H}_{\mathcal{S}}^*). \end{cases}$$

Note that the width of a HFLTS could be related as well with the height on the graph of $\overline{\mathcal{H}_{\mathcal{S}}}$, associating the zero HFLTSs with height 0, the positive HFLTSs with positive heights and the negative HFLTSs with negative values of heights as shown in Figure 3.4.

Proposition 3.4. $D(H_1, H_2) = \mathcal{W}(H_1 \sqcup H_2) - \mathcal{W}(H_1 \sqcap H_2)$ provides a distance in the lattice $\left(\overline{\mathcal{H}_{\mathcal{S}}}, \sqcup, \sqcap \right)$.

Proof. $D(H_1, H_2)$ defines a distance given that it is equivalent to the geodesic distance in the graph $\overline{\mathcal{H}_{\mathcal{S}}}$. The geodesic distance between H_1 and H_2 is the length of the shortest path to go from H_1 to H_2. Due to the fact that $H_1 \sqcap H_2 \preccurlyeq H_1 \sqcup H_2$, $\mathcal{W}(H_1 \sqcup H_2) - \mathcal{W}(H_1 \sqcap H_2)$ is the length of the minimum path between $H_1 \sqcup H_2$ and $H_1 \sqcap H_2$. Thus, we have to check that the length of the shortest path between $H_1 \sqcup H_2$ and $H_1 \sqcap H_2$ coincides with the length of the shortest path between H_1 and H_2.

If one of them belong to the coverage of the other one, let us suppose that $H_1 \preccurlyeq H_2$, then $H_1 \sqcup H_2 = H_2$ and $H_1 \sqcap H_2 = H_1$ and the foregoing assertion becomes

obvious. If not, H_1, $H_1 \sqcup H_2$, H_2 and $H_1 \sqcap H_2$ define a parallelogram on the graph. Two consecutive sides of this parallelogram define the shortest path between $H_1 \sqcup H_2$ and $H_1 \sqcap H_2$ while two other consecutive sides of the same parallelogram define the shortest path between H_1 and H_2. Thus, the assertion becomes true as well. $\qquad \square$

Proposition 3.5. Given two HFLTSs, $H_1, H_2 \in \overline{\mathcal{H}_S}$, then $D(H_1, H_2) \leq 2n$. If, in addition, $H_1, H_2 \in \mathcal{H}_S^*$, then $D(H_1, H_2) \leq 2n - 2$.

Proof. If $H_1, H_2 \in \overline{\mathcal{H}_S}$, then, the most distant pair is α_0 and α_n. Then,

$$\mathcal{W}(\alpha_0 \sqcup \alpha_n) - \mathcal{W}(\alpha_0 \sqcap \alpha_n) = \mathcal{W}([a_1, a_n]) - \mathcal{W}(-[a_1, a_n]) =$$

$$n - (-n) = 2n.$$

If $H_1, H_2 \in \mathcal{H}_S^*$, then, the most distant pair is $\{a_1\}$ and $\{a_n\}$. Then,

$$\mathcal{W}(\{a_1\} \sqcup \{a_n\}) - \mathcal{W}(\{a_1\} \sqcap \{a_n\}) = \mathcal{W}([a_1, a_n]) - \mathcal{W}(-[a_2, a_{n-1}]) =$$

$$n - (-(n-2)) = 2n - 2.$$

$$\square$$

Notice that for positive HFLTSs, $D(H_1, H_2)$ coincides with the distance $D_2(H_1, H_2)$ introduced by Montserrat-Adell *et al.* [47]. Additionally, in this case, the distance presented can also be calculated as $D([a_i, a_j], [a_{i'}, a_{j'}]) = |i - i'| + |j - j'|$.

Example 3.7. Figure 3.4 shows the width of the extended connected union and the extended intersection of H_B and H_C from Example 3.1. According to these results, $D(H_B, H_C) = \mathcal{W}(H_B \sqcup H_C) - \mathcal{W}(H_B \sqcap H_C) = 4 - (-1) = 5$.

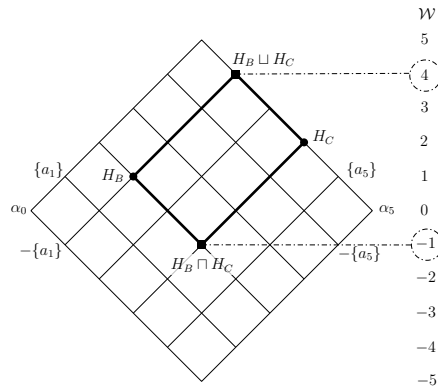FIGURE 3.4: Distance between H_B and H_C from Example 3.1.

3.5   A Representative of a Group Assessment

The aim of this section is to model the assessments given by a group of Decision Makers (DMs) that are evaluating a set of alternatives $\Lambda = \{\lambda_1, \ldots, \lambda_r\}$ by means of positive HFLTSs over $\mathcal{S} = \{a_1, \ldots, a_n\}$. To do so, we use the definition of

Hesitant Fuzzy Linguistic Description (HFLD) introduced by Montserrat-Adell *et al.* [47].

Definition 3.10. A *Hesitant fuzzy linguistic description of the set* Λ *by* $\mathcal{H}_{\mathcal{S}} - \{\varnothing\}$ is a function F_H on Λ such that for all $\lambda \in \Lambda$, $F_H(\lambda)$ is a non-empty HFLTS, i.e., $F_H(\lambda) \in \mathcal{H}_{\mathcal{S}} - \{\varnothing\}$.

According to this definition, we can extend the distance between HFLTSs presented in Section 3.4 to a distance between HFLDs as follows:

Definition 3.11. Let us consider F_H^1 and F_H^2 two HFLDs of a set $\Lambda = \{\lambda_1, \dots, \lambda_r\}$ by means of $\mathcal{H}_{\mathcal{S}}$, with $F_H^1(\lambda_i) = H_i^1$ and $F_H^2(\lambda_i) = H_i^2$, for all $i \in \{1, \dots, r\}$. Then, the *distance* $D^{\mathcal{F}}$ *between these two HFLDs* is defined as:

$$D^{\mathcal{F}}(F_H^1, F_H^2) = \sum_{t=1}^{r} D(H_t^1, H_t^2).$$

Thus, given a set ok k DMs, we have k different HFLDs of the set of alternatives Λ. In order to summarize this k different assessments, we propose a HFLD that serves as a group representative.

Definition 3.12. Let Λ be a set of r alternatives, G a group of k DMs and $F_H^1, \dots, F_H^k$ the HFLDs of Λ provided by the DMs in G, then, the *centroid of the group* is:

$$F_H^C = arg \min_{F_H^x \in (\mathcal{H}_{\mathcal{S}}^*)^r} \sum_{t=1}^{k} D^{\mathcal{F}}(F_H^x, F_H^t),$$

identifying each HFLD F_H with the vector $(H_1, \dots, H_r) \in (\mathcal{H}_{\mathcal{S}}^*)^r$, where $F_H(\lambda_i) = H_i$, for all $i = 1, \dots, r$.

Note that the HFLD of the centroid of the group does not have to coincide with any of the HFLDs given by the DMs. In addition, there can be more than one HFLDs minimizing the addition of distances to the assessments given by the DMs, so the centroid of the group is not necessarily unique. Consequently, we proceed with a further study of the possible unicity of the centroid of the group.

Proposition 3.6. For a specific alternative λ, let $F_H^1(\lambda), \dots, F_H^k(\lambda)$ be the HFLTSs given as assessments of λ by a group of k DMs. Then, if $F_H^p(\lambda) = [a_{i_p}, a_{j_p}], \forall p \in \{1, \dots, k\}$, the set of all the HFLTSs associated to the centroid of the group for λ is:

$$\{[a_i, a_j] \in \mathcal{H}_{\mathcal{S}}^* \mid i \in med(i_1, \dots, i_k), j \in med(j_1, \dots, j_k)\},$$

where $med(\ )$ contains the median of the values sorted from smallest to largest if k is odd or any integer number between the two central values sorted in the same order if k is even.

Proof. It is straightforward to check that the distance D between HFLTSs is equivalent to the Manhattan distance, also known as taxicab distance, because the graph of $\overline{\mathcal{H}_{\mathcal{S}}}$ can be seen as a grid. Thus, finding the HFLTSs that corresponds to the centroid

of the group is reduced to finding the HFLTSs in the grid that minimizes the addition of distances to the other HFLTSs given by the DMs.

The advantage of the taxicab metric is that it works with two independent components, in this case, initial linguistic term and ending linguistic term. Therefore, we can solve the problem for each component separately. For each component, we have a list of natural numbers and we want to find the one minimizing distances. It is well known that the median is the number satisfying a minimum addition of distances to all the points, generalizing the median to all the numbers between the two central ones if there is an even amount of numbers.

Thus, all the HFLTSs satisfying a minimum addition of distances are:

$$\{[a_i, a_j] \in \overline{\mathcal{H}_{\mathcal{S}}} \mid i \in med(i_1, \ldots, i_k), j \in med(j_1, \ldots, j_k)\}.$$

Finally, we have to check that the HFLTSs associated to the centroid are positive HFLTSs for the F_H^C to be a HFLD. If $F_H^p(\lambda) = [a_{i_p}, a_{j_p}] \in \mathcal{H}_{\mathcal{S}}^*, \forall p \in \{1, \ldots, k\}$, that means $i_p \leq j_p, \forall p \in \{1, \ldots, k\}$. Therefore, if k is odd, the median of $i_1, \ldots, i_k$ is less than or equal to the median of $j_1, \ldots, j_k$, and if k is even, the minimum value of $med(i_1, \ldots, i_k)$ is less than or equal than the maximum value of $med(j_1, \ldots, j_k)$. Accordingly, there is always at least one HFLTS associated to the centroid which is a positive HFLTS. Thus,

$$\{[a_i, a_j] \in \mathcal{H}_{\mathcal{S}}^* \mid i \in med(i_1, \ldots, i_k), j \in med(j_1, \ldots, j_k)\}.$$

$\square$

Example 3.8. Let us assume that H_A, H_B, H_C, H_D, H_E from Example 3.1 are the assessments given by 5 DMs about the same alternative. In such case, $med(2, 2, 4, 1, 1) = 2$ and $med(3, 2, 5, 2, 4) = 3$, and, therefore, the central assessment for this alternative is $[a_2, a_3]$.

Corollary 3.1. For a group of k DMs, if k is odd, the centroid of the group is unique.

Proof. If k is odd, both medians are from a set with an odd amount of numbers, so both medians are unique. Therefore, the corresponding HFLTS minimizing the addition of distances is also unique. $\square$

Corollary 3.2. For each alternative in Λ, the set of all the HFLTSs corresponding to any centroid of the group is a connected set in the graph of $\overline{\mathcal{H}_{\mathcal{S}}}$.

Proof. If k is odd, by Corollary 3.1, the proof results obvious. If k is even, by the definition of $med(\ )$, the set of possible results is also connected. $\square$

Example 3.9. Let G be a group of 5 DMs assessing a set of alternatives $\Lambda = \{\lambda_1, \ldots, \lambda_4\}$ by means of HFLTSs over the set $\mathcal{S} = \{a_1, a_2, a_3, a_4, a_5\}$ from Example 3.1, and let $F_H^1, F_H^2, F_H^3, F_H^4, F_H^5$ the HFLDs describing their corresponding assessments shown in Table 3.2 together with the HFLD corresponding to the centroid of the group.

TABLE 3.2: HFLDs from Example 3.9.

	F_H^1	F_H^2	F_H^3	F_H^4	F_H^5	F_H^C
λ_1	$[a_2, a_3]$	$\{a_2\}$	$[a_4, a_5]$	$[a_1, a_2]$	$[a_1, a_4]$	$[\boldsymbol{a_2, a_3}]$
λ_2	$[a_1, a_2]$	$\{a_1\}$	$[a_2, a_3]$	$[a_1, a_2]$	$\{a_2\}$	$[\boldsymbol{a_1, a_2}]$
λ_3	$[a_3, a_5]$	$\{a_3\}$	$\{a_4\}$	$[a_1, a_4]$	$[a_2, a_4]$	$[\boldsymbol{a_3, a_4}]$
λ_4	$[a_4, a_5]$	$\{a_5\}$	$\{a_5\}$	$\{a_5\}$	$[a_1, a_2]$	$\{\boldsymbol{a_5}\}$

As the last alternative shows, the centroid of the group is not sensible to outliers, due to the fact that is based on the calculation of two medians.

3.6 Conclusions and future research

This chapter presents an extension of the set of Hesitant Fuzzy Linguistic Term Sets by introducing the concepts of negative and zero HFLTSs to capture differences between pair of non-compatible HFLTSs. This extension enables the introduction of a new operation studying the intersection and the gap between HFLTSs at the same time. This operation is used to define a distance between HFLTSs that allows us to analyze differences between the assessments given by a group of decision makers. Based on the study of these differences, a centroid of the group has been proposed.

Future research is focused in two main directions. First, the study of the consensus level of the total group assessments to analyze the agreement or disagreement within the group. And secondly, a real case study will be performed in the marketing research area to examine consensus and heterogeneities in consumers' preferences.

Chapter 4

Consensus, Dissension and Precision in Group Decision Making by means of an Algebraic Extension of Hesitant Fuzzy Linguistic Term Sets

4.1 Introduction

Several studies have shown that, in general, people do not use purely quantitative models when expressing preferences and interests and are more comfortable using global or abstract forms, which can be understood as models based on qualitative or linguistic information [4, 24, 65]. Analogously, in Group Decision-Making (GDM) environments, the design of systems to facilitate decision-making processes is considered suitable for describing alternatives to be made in terms of non-numerical values and reflect the uncertainty inherent in human reasoning [25, 29, 34, 48, 75]. In the literature, this impreciseness has been modeled with intervals or fuzzy values through a linguistic approach [52, 53, 58].

Rodríguez *et al.* introduced the Hesitant Fuzzy Linguistic Term Sets (HFLTSs) over a well-ordered set of linguistic labels to deal with decision-making situations through hesitant fuzzy linguistic assessments [53]. In this way, one can express not only the uncertainty but also the hesitancy inherent in human reasoning. There are several contributions in the literature that have studied HFLTSs, their properties, aggregation functions, preference relations, distances and so on [12, 21, 33, 47, 67]. These approaches have contributed either from a theoretical point of view or by proposing different applications. An algebraic extension of the set of HFLTSs is presented by Montserrat-Adell *et al.* to take into account the gap between non-overlapping assessments[45].

In recent times, consensus in GDM problems through HFLTSs has been studied by several approaches [11, 12, 55, 69, 70, 71]. While some of them focus on the aim of quantifying the level of agreement, some others focus on the consensus reaching process. The problem is set, for all of them, with a group of experts or Decision Makers (DMs) evaluating a set of several alternatives by means of HFLTSs. Despite this, some differences emerge among the approaches that try to quantify the

consensus level. A first key difference between them is that, while some approaches study, for each alternative, the consensus of an expert with respect to the rest of the group [12, 71], others study the consensus of the whole group on each alternative [55, 69, 70]. Both types of consensus approaches might be useful under different kinds of situations: while approaches of the first type can be used to evaluate the relation of each expert with respect to the group, approaches of the second type can be used to evaluate the available alternatives. For instance, when in a GDM process the most dissenting decision makers are asked to reconsider their opinion, a measure of the first kind should be used. On the contrary, when everyone is asked to reconsider his or her assessment on the most controversial alternative, a second type measure should be used instead. In this chapter, we propose a new measure of consensus that can be adapted to the measurement of both individual and collective consensus.

The second main difference among approaches lies in whether the definition of the measure of consensus is based on the concept of distance or on the concept of similarity. On the one hand, the consensus level presented by Dong *et al.* [12] is a distance-based measure. According to the distance that they use, if two opinions do not overlap, the consensus level is always zero, regardless how far apart the opinions are. This null value for the measure is because the distance used does not take into consideration the gap between HFLTSs in the cases in which the intersection is the empty set. In this chapter we define more accurate agreement measures, based on the distance presented by Montserrat-Adell that does take into consideration this gap [47]. On the other hand, the rest of the measures [55, 69, 70, 71] are not distance-based but similarity-based. The concept of similarity between HFLTSs is presented by Rodríguez and Martínez [55], and later used by Wu and Xu, based on the comparison, between two experts, of their preferences for a given alternative over another one [69] and extended as a comparison, between two experts, of their assessment of a specific alternative. In any case, this similarity concept neither takes into consideration how distant non-overlapping assessments are nor the level of hesitancy used by the experts when assessing an alternative [70]. The measures presented in this chapter solve these issues by considering both the hesitancy of the assessments and the gap between them if they do not overlap.

Selecting or prioritizing suitable experts or DMs is a frequent problem in GDM applications in real situations [9, 31]. This chapter introduces the concepts of preciseness and dissent of an expert assessing a set of alternatives. These concepts allow the definition of an expert's profile, which keeps track of how experts have made his/her previous assessments with respect to how precise or how dissenting they are. These profiles characterize the up-to-date behavior of experts in GDM processes and can be useful for the task of selecting the appropriate experts to form part of future committees or decision groups.

The rest of this chapter is structured as follows: first, Section 4.2 presents a summary of the basic concepts in the literature that are used throughout the chapter. A new degree of consensus for the whole group on each alternative is introduced in Section 4.3 with a further comparison study with other similar measures. Section 4.4 defines a different degree of consensus for an expert with respect to the group and it is also compared with the similar existing measures. A precision-dissension profile

is presented in Section 4.5 to keep track of the assessments of a DM within several groups. Finally, Section 4.6 presents the main conclusion and lines of future research.

4.2 Theoretical framework

The aim of this section is to provide a summary of basic concepts related to HFLTSs that appear throughout this chapter. In particular, a special focus on the distance between HFLTSs that is used in this work is required.

From this point onwards, let $\mathcal{S}$ denote a finite total ordered set of linguistic terms, $\mathcal{S} = \{a_1, \ldots, a_n\}$ with $a_1 < \cdots < a_n$.

Definition 4.1. ([53]) A *hesitant fuzzy linguistic term set (HFLTS)* over $\mathcal{S}$ is a subset of consecutive linguistic terms of $\mathcal{S}$, i.e., $\{x \in \mathcal{S} \mid a_i \leq x \leq a_j\}$, for some $i, j \in \{1, \ldots, n\}$ with $i \leq j$.

Following the concept of uncertain linguistic term introduced by Xu [74], in this chapter we will denote HFLTSs by linguistic intervals. Thus, for the rest of this chapter, the HFLTS defined as $\{x \in \mathcal{S} \mid a_i \leq x \leq a_j\}$ is denoted as $[a_i, a_j]$ or, if $j = i$, $\{a_i\}$. In addition, $\mathcal{H}_{\mathcal{S}}$ represents the set of all the possible HFLTSs over $\mathcal{S}$ including the empty HFLTS, $\varnothing$.

In order to define a suitable distance between two HFLTSs that takes into consideration not just the intersection of them, but also the gap between them if they do not intersect, an algebraic extension of the set $\mathcal{H}_{\mathcal{S}}^* = \mathcal{H}_{\mathcal{S}} - \{\varnothing\}$ is presented by Montserrat-Adell *et al.* [45] as $\overline{\mathcal{H}_{\mathcal{S}}}$ different than the extension presented by Wang [67] that includes HFLTS with non-consecutive linguistic terms from $\mathcal{S}$. This algebraic extension includes the concepts of the *negative HFLTSs*, $-\mathcal{H}_{\mathcal{S}}^* = \{-H | H \in \mathcal{H}_{\mathcal{S}}^*\}$, the *zero HFLTSs*, $\mathcal{A} = \{\alpha_0, \ldots, \alpha_n\}$ and the *positive HFLTSs*, $\mathcal{H}_{\mathcal{S}}^*$. The graph of this set is presented in Figure 4.1.
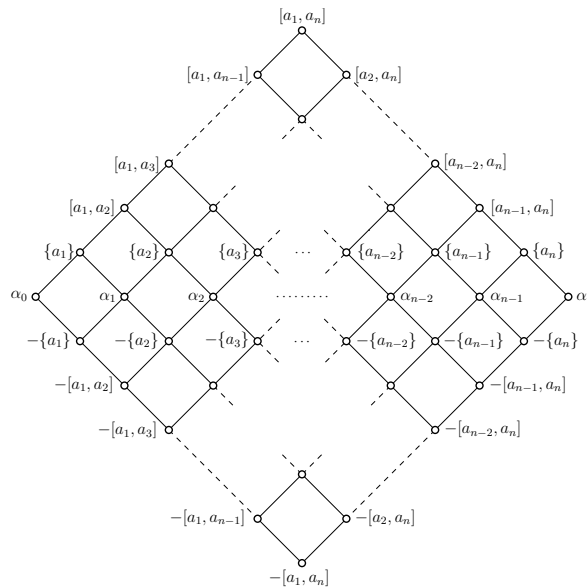

FIGURE 4.1: Graph of the extended set of HFLTSs, $\overline{\mathcal{H}_{\mathcal{S}}}$.

In the frame of $\overline{\mathcal{H}_\mathcal{S}}$, an *extended inclusion relation* is introduced based on the graph of $\overline{\mathcal{H}_\mathcal{S}}$ (Figure 4.1) and the usual inclusion relation between HFLTSs. Figure 4.2 shows, as an example, all the elements of $\overline{\mathcal{H}_\mathcal{S}}$ included in $[a_1, a_2]$ according to the extended inclusion relation. Additionally, this extended inclusion relation is used to extend the connected union and the intersection of HFLTSs to an operation between elements of $\overline{\mathcal{H}_\mathcal{S}}$.

Definition 4.2. ([45]) Given $H_1, H_2 \in \overline{\mathcal{H}_\mathcal{S}}$, then:

a) The *extended connected union of H_1 and H_2*, $H_1 \sqcup H_2$, is defined as the least element that contains H_1 and H_2, according to the extended inclusion relation.

b) The *extended intersection of H_1 and H_2*, $H_1 \sqcap H_2$, is defined as the largest element being contained in H_1 and H_2, according to the extended inclusion relation.

As an example, Figure 4.3 shows the extended connected union and the extended intersection of $[a_1, a_2]$ and $\{a_4\}$.
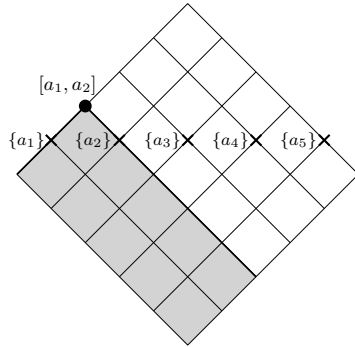
FIGURE 4.2: Elements of $\overline{\mathcal{H}_\mathcal{S}}$ included in $[a_1, a_2]$.
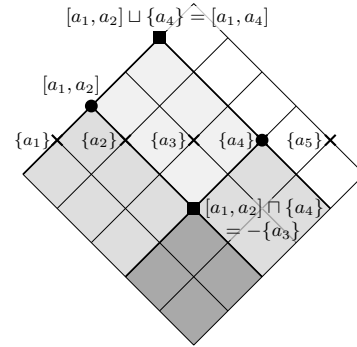
FIGURE 4.3: $\sqcup$ and $\sqcap$ of $[a_1, a_2]$ and $\{a_4\}$.

The negative and zero HFLTSs appear only as a result of the extended intersection of two elements H_1 and H_2 from $\mathcal{H}_\mathcal{S}^*$. If $H_1 \sqcap H_2 = -[a_i, a_j]$ with $i \leq j$, then there is a gap of $[a_i, a_j]$ between them. Whilst, if $H_1 \sqcap H_2 = \alpha_i$, then H_1 and H_2 are consecutive, with one of them ending at a_i and the other one starting at a_{i+1}.

Finally, given $H \in \overline{\mathcal{H}_\mathcal{S}}$, the *width* of H, $\mathcal{W}(H)$, is defined as the cardinal of H if $H \in \mathcal{H}_\mathcal{S}^*$, $-card(-H)$ if H is a negative HFLTS or 0 if H is a zero HFLTS. All these concepts are used to introduce the following distance between HFLTSs:

Definition 4.3. ([45]) Let $H_1, H_2 \in \overline{\mathcal{H}_\mathcal{S}}$, then $D(H_1, H_2) := \mathcal{W}(H_1 \sqcup H_2) - \mathcal{W}(H_1 \sqcap H_2)$ provides a distance in $\overline{\mathcal{H}_\mathcal{S}}$.

Remark 4.1. Notice that since the $\mathcal{W}$ operator is based on the concept of cardinal, it works under the assumption of a uniformly distributed set of linguistic terms $\mathcal{S}$. If this is not the case, the cardinal operator should be replaced in the definition of *width* by a measure μ on $\mathcal{H}_\mathcal{S}$, such that $\mu(H)$ represents the *size* of the semantic content of H, for all $H \in \mathcal{H}_\mathcal{S}$.

The distance provided by Definition 4.3 has three main advantages with respect to other measures between HFLTSs existing in the literature [33]: first of all, this new

measure takes explicitly into consideration the gap between two non-overlapping HFLTSs; secondly, it is simply computed even between HFLTSs with different cardinalities and, finally, this measure satisfies the triangle inequality and, therefore, it is a distance. From here on, all computations of distances between HFLTSs appearing in this chapter are done based on this definition. For this reason, and for the sake of comprehensiveness, let us present the following example to illustrate all the foregoing concepts:

Example 4.1. Let $a_1 = $ *very bad*, $a_2 = $ *bad*, $a_3 = $ *regular*, $a_4 = $ *good* and $a_5 = $ *very good* be 5 linguistic labels defining the set $\mathcal{S} = \{a_1, a_2, a_3, a_4, a_5\}$. Then, three possible assessments by means of $\mathcal{S}$ are $A = $ *"below regular"*, $B = $ *"very good"* and $C = $ *"neither very good nor very bad"* and their corresponding HFLTS by means of $\mathcal{S}$ are $H_A = [a_1, a_2]$, $H_B = \{a_5\}$ and $H_C = [a_2, a_4]$ respectively. The extended connected union and extended intersection of all the possible pairs among H_A, H_B and H_C are shown in Figure 4.4.
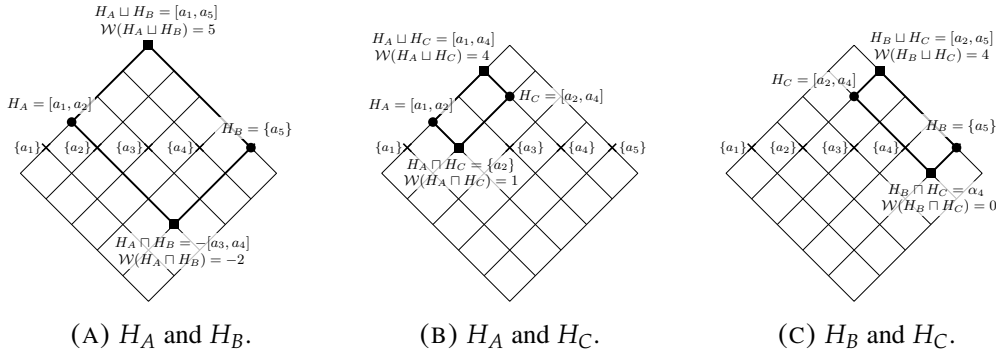
FIGURE 4.4: Extended connected unions and extended intersections from Example 4.1.

According to these results, $D(H_A, H_B) = 5 - (-2) = 7$, $D(H_A, H_C) = 4 - 1 = 3$ and $D(H_B, H_C) = 4 - 0 = 4$.

Remark 4.2. In order to ease future computations, it is important to note that, as proved by Montserrat-Adell *et al.* , the presented distance is equivalent to the taxicab metric in the graph of $\overline{\mathcal{H}_\mathcal{S}}$ [45]. Therefore, if $H_1 = [a_{i_1}, a_{j_1}]$ and $H_2 = [a_{i_2}, a_{j_2}]$, then $D(H_1, H_2)$ can be calculated as $|i_1 - i_2| + |j_1 - j_2|$. This fact can be easily seen in the previous example and in Figure 4.1.

The next step in any GDM situation is to assess not just one single alternative, but a set of them. With the aim of dealing with this kind of situations, Montserrat-Adell *et al.* developed the concept of *Hesitant Fuzzy Linguistic Description (HFLD)* of a set of alternatives $\Lambda = \{\lambda_1, \ldots, \lambda_r\}$ as a function F_H on Λ such that for all $\lambda \in \Lambda$, $F_H(\lambda) \in \mathcal{H}_\mathcal{S}^*$ [47]. For the rest of this chapter, each DM or expert is modeled by a HFLD.

Following this definition, the distance D between HFLTSs is extended to a distance, $D^\mathcal{F}$, between HFLDs as the addition of the distances between the corresponding HFLTSs for each alternative in Λ. Formally,

Definition 4.4. ([45]) Let F_H^1 and F_H^2 be two HFLDs of a set $\Lambda = \{\lambda_1, \ldots, \lambda_r\}$ by means of $\mathcal{H}_\mathcal{S}$, with $F_H^1(\lambda_i) = H_i^1$ and $F_H^2(\lambda_i) = H_i^2$, for all $i \in \{1, \ldots, r\}$. Then,

the *distance* $D^{\mathcal{F}}$ between F_H^1 and F_H^2 is defined as:

$$D^{\mathcal{F}}(F_H^1, F_H^2) = \sum_{t=1}^r D(H_t^1, H_t^2).$$

Finally, the distance $D^{\mathcal{F}}$ is used to propose a central opinion (or centroid) of a group of DMs about a set of alternatives Λ as the HFLD that minimizes the addition of distances to the opinion of each expert.

Definition 4.5. ([45]) Let Λ be a set of r alternatives, G a group of k DMs and $F_H^1, \ldots, F_H^k$ the HFLDs of Λ provided by the DMs in G. Then, a *centroid of the group* is:

$$F_H^C = arg \min_{F_H^x \in (\mathcal{H}_S^*)^r} \sum_{i=1}^k D^{\mathcal{F}}(F_H^x, F_H^i).$$

Notice that this centroid does not have to be unique and this might lead us to some issues when working with the centroid. To fix this problem, let us consider the following remark.

Remark 4.3. For an easier calculation of the centroid, let us note that it is already proved [45] that, for each specific alternative $\lambda \in \Lambda$, if $F_H^p(\lambda) = [a_{i_p}, a_{j_p}]$ for $p \in \{1, \ldots, k\}$, then the set of all the HFLTSs associated to the centroid of the group for λ is:

$$\{[a_i, a_j] \in \mathcal{H}_S^* \mid i \in \mathcal{M}(i_1, \ldots, i_k), j \in \mathcal{M}(j_1, \ldots, j_k)\},$$

where $\mathcal{M}(\ )$ is the set that contains just the median of the index values if k is odd or any integer number between the two central index values sorted from smallest to largest if k is even. Therefore, if k is odd, the centroid is unique, while if k is even, the controid might be not unique. Henceforth, to avoid possible problems with a non-unique centroid, when there are more than one possible centroid of the group, the one with a highest cardinality, which can be understood as the most hesitant one, is considered as $F_H^C(\lambda)$. Thus, $F_H^C(\lambda) = [a_{i^*}, a_{j^*}]$, where $i^* = \min(\mathcal{M}(i_1, \ldots, i_k))$ and $j^* = \max(\mathcal{M}(j_1, \ldots, j_k))$.

Example 4.2. Let G be a group of 5 DMs assessing a set of alternatives $\Lambda = \{\lambda_1, \ldots, \lambda_4\}$ by means of HFLTSs over the set $\mathcal{S} = \{a_1, \ldots, a_5\}$ from Example 4.1, and let $F_H^1, F_H^2, F_H^3, F_H^4, F_H^5$ be the HFLDs modeling their corresponding assessments shown in the Table 4.1. Then, the centroid of the group, F_H^C, can be easily computed by median calculations as stated in Remark 4.3 providing the results shown in the same table.

TABLE 4.1: Centroid of the group G for Λ from Example 4.2.

	F_H^1	F_H^2	F_H^3	F_H^4	F_H^5	F_H^C
λ_1	$[a_1, a_2]$	$\{a_2\}$	$[a_1, a_5]$	$[a_4, a_5]$	$[a_2, a_3]$	$[a_2, a_3]$
λ_2	$[a_2, a_4]$	$\{a_3\}$	$[a_1, a_5]$	$[a_3, a_4]$	$[a_2, a_3]$	$[a_2, a_4]$
λ_3	$[a_4, a_5]$	$\{a_5\}$	$[a_4, a_5]$	$[a_1, a_2]$	$[a_4, a_5]$	$[a_4, a_5]$
λ_4	$\{a_3\}$	$\{a_3\}$	$[a_2, a_3]$	$[a_3, a_4]$	$\{a_3\}$	$\{a_3\}$

Note that, contrary to some other common aggregation operators such as the union, the centroid of the group is robust with respect to extreme hesitancies in one expert. When aggregating with the union, a big hesitancy in the opinion of one of the experts implies a big hesitancy in the central opinion. That is not the case with the centroid from Definition 4.5. This can be seen, for instance, in alternative λ_1, where the assessment of one of the experts is $[a_1, a_5]$, but the centroid is $[a_2, a_3]$. That it to say that a large hesitancy of a DM does not necessarily imply a lack of precision of the centroid.

Note that, since in this example there are 5 DMs, which is an odd number, the centroid of the group obtained from Definition 4.5 is unique.

4.3 Collective consensus

In this section, a new degree of consensus of the whole group on a specific alternative or a set of alternatives is introduced based on the distance proposed by Montserrat-Adell *et al.* [45]. This new measure seeks to quantify the level of agreement within a group of DMs on a specific alternative or a set of alternatives. A further study on the properties of the introduced measure and a comparison with the similar existing measures in the literature are also presented in this section. Finally, an example is provided to illustrate the commented properties.

4.3.1 A collective degree of consensus

The idea of this new degree of consensus arises with the need of finding a measure that depends neither on the number of DMs assessing the alternatives nor on the number of linguistic labels used in $\mathcal{S}$. Thus, the degree of consensus presented in this section is a normalization of the addition of distances between the centroid of the group and each of the HFLDs given by the DMs. In order to define this normalization, the first step is to study the maximum value that this addition of distances can take.

Lemma 4.1. Let F_H^1, F_H^2 be two HFLDs of the set of alternatives $\Lambda = \{\lambda_1, \ldots, \lambda_r\}$ by means of $\mathcal{S} = \{a_1, \ldots, a_n\}$. Then,

$$D^{\mathcal{F}}(F_H^1, F_H^2) \leq r \cdot (2n - 2).$$

Proof. According to Definition 4.3, the most distant HFLTSs are $H_1 = \{a_1\}$ and $H_2 = \{a_n\}$. In this case, $H_1 \sqcup H_2 = [a_1, a_n]$ and $H_1 \sqcap H_2 = -[a_2, a_{n-1}]$. Thus, $D(H_1, H_2) = \mathcal{W}([a_1, a_n]) - \mathcal{W}(-[a_2, a_{n-1}]) = n - (-(n-2)) = 2n - 2$. Consequently, the most distant HFLDs are those that for all the alternatives, the corresponding two HFLTSs used by each HFLD are the most distant ones. In such case,

$$D^{\mathcal{F}}(F_H^1, F_H^2) = \sum_{i=1}^{r}(2n - 2) = r \cdot (2n - 2).$$

$\square$

Therefore, Lemma 4.1 can be used to find an upper bound for the addition of distances between the centroid of a group and each of the DMs of the assessing group.

Proposition 4.1. Let $F_H^1, \ldots, F_H^k$ be the HFLDs of a group of k DMs of the set of alternatives $\Lambda = \{\lambda_1, \ldots, \lambda_r\}$ by means of $\mathcal{S} = \{a_1, \ldots, a_n\}$, and let F_H^C be the centroid of the group. Then,

$$\sum_{i=1}^{k} D^{\mathcal{F}}(F_H^C, F_H^i) \leq k \cdot r \cdot (n-1).$$

Proof. If k is even, the worst-case scenario is given when, for each of the alternatives $k/2$ of the DMs have assessed it with $\{a_1\}$, and the other $k/2$ of the DMs have assessed it with $\{a_n\}$. In such case, calculating the corresponding medians, we get that any HFLD could be considered as the centroid of the group given that all of them give the same addition of distances, but, according to Remark 4.3, $\mathcal{F}_H^C(\lambda_i) = [a_1, a_n]$ for $i = 1, \ldots, r$, then:

$$\sum_{i=1}^{k} D^{\mathcal{F}}(F_H^C, F_H^i) = \frac{k}{2} \cdot r \cdot (n-1) + \frac{k}{2} \cdot r \cdot (n-1) = k \cdot r \cdot (n-1).$$

If k is odd, the worst-case scenario is met when, for each of the alternatives, $\lfloor k/2 \rfloor$ of the DMs have assessed it with $\{a_1\}$ and $\lfloor k/2 \rfloor$ of the DMs have assessed it with $\{a_n\}$, regardless what is the last HFLTS. If so, based on the median calculations, the centroid of the group is equal, for each alternative, to this last HFLTS, and the addition of distances is equal to $(k-1) \cdot r \cdot (n-1)$. Choosing, for example, the last HFLTS to be $\{a_1\}$ for all the alternatives, then:

$$\sum_{i=1}^{k} D^{\mathcal{F}}(F_H^C, F_H^i) = \left(\left\lfloor \frac{k}{2} \right\rfloor + 1 \right) \cdot 0 + \left\lfloor \frac{k}{2} \right\rfloor \cdot r \cdot (2n-2)$$

$$= (k-1) \cdot r \cdot (n-1) \leq k \cdot r \cdot (n-1).$$

$\square$

Corollary 4.1. Under the same conditions as in Property 4.1, in the particular case where $r = 1$, just one single alternative to be assessed, the upper bound results to be $k \cdot (n-1)$.

The upper bounds provided in Proposition 4.1 and Corollary 4.1 for the total addition of distances between the centroid of the group and all the HFLD of the group enables us to proceed with the normalization that leads us to the definition of a measure of agreement within the group, in a similar way to other measures [12, 70], as follows:

Definition 4.6. Let $F_H^1, \ldots, F_H^k$ be the HFLDs given by a group G of k DMs about the set of alternatives $\Lambda = \{\lambda_1, \ldots, \lambda_r\}$ by means of $\mathcal{S} = \{a_1, \ldots, a_n\}$ and let

F_H^C the centroid of the group, being $H_j^i = F_H^i(\lambda_j)$ for $i \in \{1, \ldots, k, C\}$. Then, the *degree of consensus of G on* λ_j is defined as:

$$\delta_{\lambda_j}(G) = 1 - \frac{\sum\limits_{i=1}^{k} D(H_j^C, H_j^i)}{k \cdot (n-1)}.$$

Analogously, the *degree of consensus of G on* Λ is defined as:

$$\delta_{\Lambda}(G) = 1 - \frac{\sum\limits_{i=1}^{k} D^{\mathcal{F}}(F_H^C, F_H^i)}{k \cdot r \cdot (n-1)}.$$

Note that, by Proposition 4.1, $0 \leq \delta_{\Lambda}(G) \leq 1$. The closer to 0 $\delta_{\Lambda}(G)$ is, the closer to its maximum value the addition of distances is, which implies a lot of disagreement. On the contrary, the closer to 1 $\delta_{\Lambda}(G)$ is, the smaller the addition of distances is, and that means a high level of agreement. The same reasoning is valid for the degree of consensus of one specific alternative.

Notice also that, the upper bound given by Proposition 4.1 can be reached only when k is even. Thus, if k is odd, the degree of consensus cannot be zero. This fact is coherent given that situations with maximum disagreement arise when half of the experts assess an alternative with the worst linguistic label and the other half do it with the best linguistic label. Obviously, this situation is only possible with an even number of opinions.

Property 4.1. Let G be a group of k DMs assessing a set of alternatives $\Lambda = \{\lambda_1, \ldots, \lambda_r\}$ by means of $\mathcal{S} = \{a_1, \ldots, a_n\}$. Then,

$$\delta_{\Lambda}(G) = \frac{\sum\limits_{j=1}^{r} \delta_{\lambda_j}(G)}{r}.$$

Proof. Let $F_H^1, \ldots, F_H^k$ be the HFLDs given by the DMs and F_H^C the centroid of the group, being $H_j^i = F_H^i(\lambda_j)$ for $i \in \{1, \ldots, k, C\}$. Then,

$$\frac{\sum\limits_{j=1}^{r} \delta_{\lambda_j}(G)}{r} = \frac{\sum\limits_{j=1}^{r} 1 - \frac{\sum\limits_{i=1}^{k} D(H_j^C, H_j^i)}{k \cdot (n-1)}}{r} = \frac{r - \frac{\sum\limits_{i=1}^{k} \sum\limits_{j=1}^{r} D(H_j^C, H_j^i)}{k \cdot (n-1)}}{r}$$

$$= 1 - \frac{\sum\limits_{i=1}^{k} D^{\mathcal{F}}(F_H^C, F_H^i)}{k \cdot r \cdot (n-1)} = \delta_{\Lambda}(G).$$

$\square$

This property states the consistency between the degree of consensus on each alternative and on the whole set of alternatives.

4.3.2 Comparison with existing measures

This section presents a comparison of the degree of consensus defined in Section 4.3.1 with similar existing measures. Out of all the agreement measures for GDM by means of HFLTSs summarized in the Introduction, the ones defined as a degree of consensus on the alternatives are those presented by Rodríguez and Martínez [55] and by Wu and Xu [69, 70]. When calculating the agreement on an alternative λ_j, there is a main difference between these two measures: the first and third degrees are defined based on the preference of said alternative over another alternative $\lambda_k, \forall k \neq j$, while the second one is based just on the assessment of λ_j, regardless the assessment of the rest of alternatives. This leads us to the automatic conclusion that the most similar measure to the one presented in this chapter is the second one. For this reason, we proceed with a further study to compare the results provided by both measures.

To begin with, let us summarize the measure presented by Wu and Xu [70]. They defined the consensus level within all the DMs for an alternative as the average of all the similarity degrees between any pair of DMs about this alternative. This similarity degree is based on what they call the mean (or expected value) of a HFLTS, which is just the center of the HFLTS in the case of a set $\mathcal{S}$ with uniform and symmetric linguistic labels. Translated to the notation used in this chapter, in which $H^i = [a_{x_i}, a_{y_i}]$ for $i \in \{1, \dots, k\}$ are the assessments given by a group of k DMs about an alternative λ by means of $\mathcal{S} = \{a_1, \dots, a_n\}$, the consensus level within all the DMs for λ defined by Wu and Xu can be calculated as

$$ca_\lambda = \frac{\sum\limits_{i=1}^{k} \sum\limits_{j>i}^{k} \left(1 - \frac{\left| \frac{x_j+y_j}{2} - \frac{x_i+y_i}{2} \right|}{(n-1)} \right)}{\binom{k}{2}}. \tag{4.I}$$

Remark 4.4. The fact that this measure ignores the width of the HFLTSs and, in the case with uniform and symmetric linguistic labels, is based just on the mean of the HFLTSs, implies that the hesitancy of each expert is not taken into consideration. Therefore, the similarity degree of two experts assessing an alternative with HFLTSs with the same expected value but with different levels of hesitancy would be 1, the maximum.

On the other hand, the degree of consensus presented in Section 4.3.1 can be rewritten in a similar way as shown in the following lemma:

Lemma 4.2. Let $H^1, \dots, H^k$ be the assessments of a group G of k DMs about an alternative λ, and let H^C be the centroid of G for λ, where $H^i = [a_{x_i}, a_{y_i}]$ for $i \in \{1, \dots, k, C\}$. Then,

$$\delta_\lambda(G) = 1 - \frac{\sum\limits_{i=1}^{k} |x_i - x_C| + |y_i - y_C|}{k \cdot (n-1)}.$$

Proof. The proof is straightforward by Definition 4.6 and Remark 4.3. □

In order to compare the two consensus measures, we first need the following definition:

Definition 4.7. Let $H^1, \ldots, H^k$ be a collection of HFLTSs over $\mathcal{S}$, where $H^i = [a_{x_i}, a_{y_i}]$ for $i \in \{1, \ldots, k\}$. Then,

(a) H^i is *lower* than H^j, $H_i \preccurlyeq H_j$, if $x_i \leq x_j$ and $y_i \leq y_j$.

(b) $H^1, \ldots, H^k$ are *sorted* if $H^1 \preccurlyeq H^2 \preccurlyeq \ldots \preccurlyeq H^k$.

(c) $H^1, \ldots, H^k$ are *sortable* if there exists a permutation of them which is sorted.

Property 4.2. Let $H^1, \ldots, H^k$ be the assessments of a group G of k DMs about an alternative λ. Then,

$$\delta_\lambda(G) \leq ca_\lambda$$

and the equality is met when $H^1, \ldots, H^k$ are sortable and the $k-2$ central opinions are the same.

Proof. For this proof, let us assume $H^i = [a_{x_i}, a_{y_i}]$ for $i \in \{1, \ldots, k, C\}$. Thus, beginning with Equation 4.I,

$$ca_\lambda = \frac{\sum\limits_{i=1}^{k} \sum\limits_{j>i}^{k} \left(1 - \frac{\left| \frac{x_j + y_j}{2} - \frac{x_i + y_i}{2} \right|}{(n-1)} \right)}{\binom{k}{2}}$$

$$= \frac{\binom{k}{2} - \frac{1}{2 \cdot (n-1)} \cdot \sum\limits_{i=1}^{k} \sum\limits_{j>i}^{k} \left(|x_j + y_j - x_i - y_i| \right)}{\binom{k}{2}}$$

$$= 1 - \frac{\sum\limits_{i=1}^{k} \sum\limits_{j>i}^{k} |x_j + y_j - x_i - y_i|}{\frac{k \cdot (k-1)}{2} \cdot 2 \cdot (n-1)}$$

$$\geq 1 - \frac{\sum\limits_{i=1}^{k} \sum\limits_{j>i}^{k} |x_j - x_i| + |y_j - y_i|}{k \cdot (k-1) \cdot (n-1)}$$

$$= 1 - \frac{\sum\limits_{i=1}^{k} \sum\limits_{j>i}^{k} |x_j - x_C - x_i + x_C| + |y_j - y_C - y_i + y_C|}{k \cdot (k-1) \cdot (n-1)}$$

$$\geq 1 - \frac{\sum\limits_{i=1}^{k} \sum\limits_{j>i}^{k} |x_j - x_C| + |x_i - x_C| + |y_j - y_C| + |y_i - y_C|}{k \cdot (k-1) \cdot (n-1)}$$

$$= 1 - \frac{\sum\limits_{i=1}^{k} (k-1) \cdot |x_i - x_C| + (k-1) \cdot |y_i - y_C|}{k \cdot (k-1) \cdot (n-1)}$$

$$= 1 - \frac{(k-1) \left(\sum\limits_{i=1}^{k} |x_i - x_C| + |y_i - y_C| \right)}{k \cdot (k-1) \cdot (n-1)}$$

$$= 1 - \frac{\sum\limits_{i=1}^{k} |x_i - x_C| + |y_i - y_C|}{k \cdot (n-1)} = \delta_\lambda(G).$$

Additionally, for the first inequality to be an equality $(x_j - x_i)$ and $(y_j - y_i)$ have to have the same sign for any $j > i$, which means that $H^1, \ldots, H^k$ have to be sorted. Since the order of the DMs is not important, it is enough for $H^1, \ldots, H^k$ to be sortable. On the other hand, for the equality to be met in the second inequality, $(x_i - x_C)$ and $(x_j - x_C)$ have to have opposite signs or be zero for any $j > i$, and analogously for $(y_i - y_C)$ and $(y_j - y_C)$. Given that, because of the previous condition, we can assume $H^1, \ldots, H^k$ to be sorted, this happens only if $x_2 = \ldots = x_{k-1} = x_C$ and $y_2 = \ldots = y_{k-1} = y_C$. $\qquad\square$

The reason why $\delta_\lambda(G) \leq ca_\lambda$ is explained by the fact that ca_λ does not take into account the hesitancy of the experts and, therefore, for some alternatives the consensus level is higher that what it would be expected.

Additionally, if these degrees of consensus are applied to to end-users of a product instead of a set of experts, then the number of DMs might be very large, and the time complexity of calculating the consensus level for an alternative becomes a crucial point. Given that the degree of consensus presented by Rodríguez and Martínez [55] and by Wu and Xu [69] compute the similarity between each pair of DMs about the preference of the studied alternative over all the other ones one by one, its time complexity is $O(rk^2)$, where k is the number of DMs within the group and r is the number of alternatives to be assessed. The consensus level introduced by Wu and Xu [70] studies the similarity between each pair of DMs on a specific alternative, without comparing it with the rest of alternatives. Thus, its time complexity is $O(k^2)$. On the contrary, the degree of consensus presented in Section 4.3.1 only makes one comparison with the central opinion. Therefore, its time complexity is $O(1)$ once the centroid of the group for the studied alternative is computed. Since this centroid, as staten before, is based on the median calculation, which is known to be done in linear time, the time complexity of $\delta_{\lambda_i}(G)$ is $O(k)$.

Table 4.2 summarizes the main characteristics of the different collective degrees of consensus using HFLTSs.

4.3.3 An illustrative example on collective consensus

For an easier understanding of the introduced degree of consensus, in this subsection we present a clarifying example to illustrate its computation. The same example

TABLE 4.2: Comparison of the presented collective degrees of consensus.

	Rodríguez and Martínez [55]	Wu and Xu [69]	Wu and Xu [70]	Montserrat-Adell *et al.*
Year	2015	2016	2016	2017
Groupal consensus	✓	✓	✓	✓
Individual consensus				
Distance-based				✓
Similarity-based	✓	✓	✓	
Preference similarity	✓	✓		
Alternative similarity			✓	
Pairwise comparison	✓	✓	✓	
Central opinion comparison				✓
Considers gap				✓
Considers hesitancy				✓
Time complexity[a,b]	$O(rk^2)$	$O(rk^2)$	$O(k^2)$	$O(1) + \mathcal{T}_C$

[a] $\mathcal{T}_C$ stands for the time complexity of calculating the central opinion.
[b] For the overall degree of consensus of a set of r alternatives, all times are multiplied by r.

is also used to point out its properties commented in Section 4.3.2 with respect to similar existing measures.

Example 4.3. Following Example 4.2, where G is a group of 5 DMs assessing a set of alternatives $\Lambda = \{\lambda_1, \ldots, \lambda_4\}$ by means of HFLTSs over the set $\mathcal{S} = \{a_1, \ldots, a_5\}$, with the assessments provided in Table 4.1, we can now proceed to compute the degree of consensus on each of the alternatives in Λ as shown in Table 4.3, where D_j^i stands for $D(H_j^C, H_j^i)$, as well as the degree of consensus for the whole set Λ.

TABLE 4.3: Degree of consensus on each alternative and on the set Λ from Example 4.3.

	D_j^1	D_j^2	D_j^3	D_j^4	D_j^5	$\sum_{i=1}^{5} D_j^i$	$\delta_{\lambda_j}(G)$
λ_1	2	1	3	4	0	10	0.5
λ_2	0	2	2	1	1	6	0.7
λ_3	0	1	0	6	0	7	0.65
λ_4	0	0	1	1	0	2	0.9
Λ	2	4	6	12	1	25	0.6875

In order to illustrate the properties presented in Section 4.3.2, we can now use the methodology introduced by Wu and Xu [70] to calculate their degree of consensus on each alternative λ_j, ca_j, for $j = 1, \ldots 4$. To this end, the first step is to calculate the similarity matrix for each alternative, showing, in a scale from 0 to 1, the agreement between each pair of experts on the corresponding alternative. These similarity

coefficients are calculated as one minus the difference between the middle points of the corresponding HFLTSs over $n - 1$, where n is the cardinality of $\mathcal{S}$ ($n = 5$ in this example). These similarity matrices are shown in Figure 4.5.

$$
\begin{pmatrix}
- & 0.875 & 0.625 & 0.25 & 0.75 \\
 & - & 0.75 & 0.375 & 0.875 \\
 & & - & 0.625 & 0.875 \\
 & & & - & 0.5 \\
 & & & & -
\end{pmatrix}
$$

(A) λ_1

$$
\begin{pmatrix}
- & 1 & 1 & 0.875 & 0.875 \\
 & - & 1 & 0.875 & 0.875 \\
 & & - & 0.875 & 0.875 \\
 & & & - & 0.75 \\
 & & & & -
\end{pmatrix}
$$

(B) λ_2

$$
\begin{pmatrix}
- & 0.875 & 1 & 0.25 & 1 \\
 & - & 0.875 & 0.125 & 0.875 \\
 & & - & 0.25 & 1 \\
 & & & - & 0.25 \\
 & & & & -
\end{pmatrix}
$$

(C) λ_3

$$
\begin{pmatrix}
- & 1 & 0.875 & 0.875 & 1 \\
 & - & 0.875 & 0.875 & 1 \\
 & & - & 0.75 & 0.875 \\
 & & & - & 0.875 \\
 & & & & -
\end{pmatrix}
$$

(D) λ_4

FIGURE 4.5: Similarity matrices for each alternative from Example 4.3.

Once these matrices are calculated, the next step to get ca_j is, for each alternative, to compute the average of the similarity between each pair of experts. Table 4.4 presents a comparison of the results of $\delta_{\lambda_j}(G)$ and ca_j on each alternative.

TABLE 4.4: δ_{λ_j} and ca_j for the alternatives in Λ from Example 4.3.

	$\delta_{\lambda_j}(G)$	ca_j
λ_1	0.5	0.65
λ_2	0.7	0.9
λ_3	0.65	0.65
λ_4	0.9	0.9

As staten in Property 4.2, $\delta_{\lambda_j}(G) \leq ca_j$ for all the alternatives, being the equality met in alternatives λ_3 and λ_4. In Table 4.1, it can be seen that, for λ_3, the assessments are sortable as $F_H^4(\lambda_3) \preccurlyeq F_H^1(\lambda_3) = F_H^3(\lambda_3) = F_H^5(\lambda_3) \preccurlyeq F_H^2(\lambda_3)$, while for λ_4, they are sortable as $F_H^3(\lambda_4) \preccurlyeq F_H^1(\lambda_4) = F_H^2(\lambda_4) = F_H^5(\lambda_4) \preccurlyeq F_H^4(\lambda_4)$. On the contrary, the assessments for alternatives λ_1 and λ_2 are not sortable, for instance $F_H^3(\lambda_1)$ and $F_H^5(\lambda_1)$ or $F_H^1(\lambda_2)$ and $F_H^3(\lambda_2)$, and, therefore, $\delta_{\lambda_j}(G) < ca_j$.

Additionally, alternatives λ_2 and λ_4 are a clear example for Remark 4.4. Again, in Table 4.1, it can be seen that the HFLTSs used by the experts to assess the two alternatives have the same mean, but the level of hesitancy of the answers is different in the two cases. Given that there is much more hesitancy on λ_2, it seems intuitive that the degree of consensus on this alternative is lower than the one on λ_4, where there is much more coincidence of opinions. Table 4.4 shows that $ca_2 = ca_4$ given that this measure does not take into account the hesitancy of the experts while $\delta_{\lambda_2}(G) < \delta_{\lambda_4}(G)$.

This leads us to the conclusion that, under the HFLTSs-based GDM framework, $\delta_{\lambda_j}(G)$ provides a measure of the consensus of a group of experts on a set of alternatives closer to common-sense reasoning.

4.4 Individual consensus

This section studies the idea of consensus within a group of DMs as the agreement of an expert with respect to the group instead of the agreement of the whole group on an alternative as in Section 4.3. To this end, a convenient degree of consensus is defined for each expert. Even though there are some other measures already defined in the literature, the convenience of a new measure is explained by the fact that the previous ones present some issues like not considering the hesitancy of the assessments or not considering the gap between non-overlapping assessments. Additionally, this degree is compared with similar already existing measures and also exemplified to point out its properties.

4.4.1 An individual degree of consensus

As in Definition 4.6, this new measure is thought to be on a scale from 0 to 1 independently from the number of linguistic labels used in $\mathcal{S}$ and the number of DMs in the group. The degree of consensus presented in this section is a normalization of the distance between the opinion of the expert and the centroid of the group as follows:

Definition 4.8. Let G be a group of DMs, $\epsilon_1, \ldots, \epsilon_k$, assessing a set of alternatives $\Lambda = \{\lambda_1, \ldots, \lambda_r\}$ by means of HFLTSs over $\mathcal{S} = \{a_1, \ldots, a_n\}$, and let F_H^i and F_H^C be the HFLDs of ϵ_i for $i = 1, \ldots, k$ and the centroid of the group respectively, with $H_j^i = F_H^i(\lambda_j)$ for $i \in \{1, \ldots, k, C\}$. Then, the *degree of consensus of ϵ_i with respect to G on λ_j* is defined as:

$$\delta_{\lambda_j}^G(\epsilon_i) = 1 - \frac{D(H_j^C, H_j^i)}{2n - 2}.$$

Analogously, the *degree of consensus of ϵ_i with respect to G on Λ* is defined as:

$$\delta_\Lambda^G(\epsilon_i) = 1 - \frac{D^{\mathcal{F}}(F_H^C, F_H^i)}{r \cdot (2n - 2)}.$$

By Lemma 4.1, the upper bound for the distance between two HFLTSs is $2n - 2$ and the one for the distance between two HFLDs is $r \cdot (2n - 2)$. Thus, it can be easily seen that both $\delta_{\lambda_j}^G(\epsilon_i)$ and $\delta_\Lambda^G(\epsilon_i)$ range between 0 and 1. The closer to 1 these coefficients are, the more similar the opinion of ϵ_i is to the centroid, while the closer to 0 the more dissidence there is.

Note that this degree of consensus is 1 only when the opinion of the expert coincides with the centroid of the group and it is 0 if and only if the opinion of the expert is $\{a_1\}$ and the centroid is $\{a_n\}$ or vice versa.

Property 4.3. Let G be a group of DMs, $\epsilon_1, \ldots, \epsilon_k$, assessing a set of alternatives $\Lambda = \{\lambda_1, \ldots, \lambda_r\}$ by means of $\mathcal{S} = \{a_1, \ldots, a_n\}$. Then, for $i = 1, \ldots, k$,

$$\delta_\Lambda^G(\epsilon_i) = \frac{\sum\limits_{j=1}^{r} \delta_{\lambda_j}^G(\epsilon_i)}{r}.$$

Proof. Let $F_H^1, \ldots, F_H^k$ be the HFLDs given by the DMs and F_H^C the centroid of the group, being $H_j^i = F_H^i(\lambda_j)$ for $i \in \{1, \ldots, k, C\}$. Then,

$$\frac{\sum\limits_{j=1}^{r} \delta_{\lambda_j}^G(\epsilon_i)}{r} = \frac{\sum\limits_{j=1}^{r} 1 - \frac{D(H_j^C, H_j^i)}{(2n-2)}}{r} = \frac{r - \frac{\sum\limits_{j=1}^{r} D(H_j^C, H_j^i)}{(2n-2)}}{r}$$

$$= 1 - \frac{D^{\mathcal{F}}(F_H^C, F_H^i)}{r \cdot (2n-2)} = \delta_\Lambda^G(\epsilon_i).$$

$\square$

In the same way than Property 4.1, this property provides consistency to the definition of the degree of consensus of an expert with respect to a group on an alternative and on a set of alternatives.

4.4.2 Comparison with existing measures

As staten before, the degree of consensus for experts introduced in Section 4.4.1 is similar to some of the measures presented in the literature. The aim of this section is to compare the degree of consensus defined in Section 4.4.1 with the most similar existing ones.

From the agreement measures by GDM by means of HFLTSs presented in the Introduction, those defined as degrees of consensus for an expert are the ones introduced by Dong *et al.* [12] and by Wu and Xu [71].

On the one hand, Dong *et al.* defined the consensus level of ϵ_i on an alternative based on the intersection and the union of the opinion of ϵ_i and a central opinion as:

$$CL_i = \frac{\text{card}(H^i \cap H^C)}{\text{card}(H^i \cup H^C)},$$

being H^i the opinion of ϵ_i and H^C the central opinion. The main issue with this consensus level is that, in the case of an empty intersection between the opinion of the expert and the central opinion, the result is always 0, without taking into consideration how far H^i is from H^C. The reason that explains this is the fact that CL_i is

based on a distance between HFLTSs, that contrarily to the one from Definition 4.3, does not take into account the gap between two HFLTSs with null intersection.

Because of this reason, we have considered more interestingly to proceed with a further study to compare the results provided by the consensus measure for experts introduced by Wu and Xu [71] with the one given by the degree of consensus for experts presented in this chapter.

In order to carry on this comparison, we first need to introduce the consensus level proposed by Wu and Xu It is based on the same idea of similarity than ca_j in Equation 4.I, but in this case, between the opinion of the expert and a central opinion. In this case, we use the centroid from Definition 4.5 as central opinion. Therefore, if $F_H^i(\lambda) = [a_{x_i}, a_{y_i}]$ is the opinion of expert ϵ_i on λ and $F_H^C(\lambda) = [a_{x_C}, a_{y_C}]$ is the centroid of the group on λ, then the degree of consensus presented by Wu and Xu is defined as:

$$SM_\lambda^i = 1 - \frac{\left| \frac{x_i + y_i}{2} - \frac{x_C + y_C}{2} \right|}{n - 1}, \tag{4.II}$$

where n is the cardinal of $\mathcal{S}$. Additionally, they defined the overall consensus level for expert ϵ_i on the set of alternatives $\Lambda = \{\lambda_1, \ldots, \lambda_r\}$, SM_i, as the average of $SM_1^i, \ldots, SM_r^i$.

On the other hand, the following lemma rewrites the degree of consensus from Section 4.4.1 in a similar way.

Lemma 4.3. Let G be a group of DMs, $\epsilon_1, \ldots, \epsilon_k$, whose assessments about alternative λ are $H^i = [a_{x_i}, a_{y_i}]$ for $i = 1, \ldots, k$, and let $H^C = [a_{x_C}, a_{y_C}]$ be the centroid of the group for λ. Then,

$$\delta_\lambda^G(\epsilon_i) = 1 - \frac{|x_i - x_C| + |y_i - y_C|}{(2n - 2)}.$$

Proof. The proof is straightforward from Definition 4.8 and Remark 4.2. □

With the foregoing lemma, we can proceed to compare the two measures.

Property 4.4. Let G be a group of DMs, $\epsilon_1, \ldots, \epsilon_k$, whose assessments about alternative λ are $H^1, \ldots, H^k$ respectively. Then,

$$\delta_\lambda^G(\epsilon_i) \leq SM_\lambda^i$$

and the equality is met when H^i and H^C are sortable, being H^C the centroid of group G for λ.

Proof. For this proof, let us assume $H^i = [a_{x_i}, a_{y_i}]$ for $i \in \{1, \ldots, k\}$. Thus, beginning with Equation 4.II,

$$SM_\lambda^i = 1 - \frac{\left| \frac{x_i + y_i}{2} - \frac{x_C + y_C}{2} \right|}{n - 1} = 1 - \frac{\frac{1}{2}|x_i + y_i - x_C - y_C|}{n - 1}$$

$$= 1 - \frac{|x_i - x_C + y_i - y_C|}{2n - 2} \geq 1 - \frac{|x_i - x_C| + |y_i - y_C|}{2n - 2} = \delta_\lambda^G(\epsilon_i).$$

In addition, for the inequality to be an equality, $x_i - x_C$ and $y_i - y_C$ must have the same sign or at least one of them has to be 0, which is equivalent to $x_i \leq x_C$ and $y_i \leq y_C$, i.e. $H^i \preccurlyeq H^C$, or $x_i \geq x_C$ and $y_i \geq y_C$, i.e. $H^C \preccurlyeq H^i$. Therefore, H^i and H^C have to be sortable. $\qquad\square$

Corollary 4.2. Let G be a group of k DMs assessing a set of alternatives Λ. Then, for any expert ϵ_i, $i \in \{1, \dots, k\}$, $\delta_\Lambda^G(\epsilon_i) \leq SM_i$. In addition, the equality is met when, for any alternative $\lambda_j \in \Lambda$, $F_H^i(\lambda_j)$ and $F_H^C(\lambda_j)$ are sortable, being F_H^i and F_H^C the HFLDs of ϵ_i and the centroid of the group respectively.

Proof. The proof is straightforward from Properties 4.3 and 4.4 and the definition of SM_i. $\qquad\square$

In an analogous way to Property 4.2 in Section 4.3, this property and its corollary show that the degree of consensus for experts introduced in Section 4.4.1 can capture differences among situations in which the measure presented by Wu and Xu cannot.

Lastly, referring to the time complexity, measures presented in by Dong *et al.* [12] and by Wu and Xu [71] have the same time complexity than the one presented in Section 4.4.1, which is a constant time plus the time of computing the central opinion for λ_j. Using the centroid from Definition 4.5, which is computed in linear time as commented in the previous section, the time complexity for $\delta_{\lambda_j}^G(\epsilon_i)$ is $O(k)$ where k is the number of DMs within the group.

Table 4.5 summarizes the main characteristics of the presented individual consensus measures.

4.4.3   An illustrative example on individual consensus

For the seek of clarifying the calculation of the degree of consensus for each expert, let us present an example. In the same example, the foregoing properties can also be checked.

Example 4.4. Following Example 4.2, where G is a group of 5 DMs assessing a set of alternatives $\Lambda = \{\lambda_1, \dots, \lambda_4\}$ by means of HFLTSs over the set $\mathcal{S} = \{a_1, \dots, a_5\}$, with the assessments provided in Table 4.1, we can now use the presented methodology to compute the degree of consensus for each expert. For instance,

$$\delta_{\lambda_1}^G(\epsilon_1) = 1 - \frac{D(H_1^C, H_1^1)}{2n - 2} = 1 - \frac{D([a_2, a_3], [a_1, a_2])}{2n - 2} = 1 - \frac{2}{8} = 0.75.$$

Following the same steps for all the experts and alternatives, we get the results shown in Table 4.6.

TABLE 4.5: Comparison of the presented individual degrees of consensus.

	Dong *et al.* [12]	Wu and Xu [71]	Montserrat-Adell *et al.*
Year	2015	2016	2017
Groupal consensus			
Individual consensus	✓	✓	✓
Distance-based	✓		✓
Similarity-based		✓	
Preference similarity			
Alternative similarity		✓	
Pairwise comparison			
Central opinion comparison	✓	✓	✓
Considers gap			✓
Considers hesitancy	✓		✓
Time complexity[a,b]	$O(1) + \mathcal{T}_C$	$O(1) + \mathcal{T}_C$	$O(1) + \mathcal{T}_C$

[a] $\mathcal{T}_C$ stands for the time complexity of calculating the central opinion.

[b] For the overall degree of consensus of a set of r alternatives, all times are multiplied by r.

TABLE 4.6: Degrees of consensus $\delta^G_{\lambda_j}(\epsilon_i)$ and $\delta^G_\Lambda(\epsilon_i)$ from Example 4.4.

$\delta^G_{\lambda_j}(\epsilon_i)$	ϵ_1	ϵ_2	ϵ_3	ϵ_4	ϵ_5
λ_1	0.75	0.875	0.625	0.5	1
λ_2	1	0.75	0.75	0.875	0.875
λ_3	1	0.875	1	0.25	1
λ_4	1	1	0.875	0.875	1
Λ	0.9375	0.875	0.8125	0.625	0.96875

Analogously, we can calculate the consensus level presented by Wu and Xu [71] following Equation 4.II, as for instance,

$$SM_1^1 = 1 - \frac{\left| \frac{1+2}{2} - \frac{2+3}{2} \right|}{n-1} = 1 - \frac{|-1|}{4} = 0.75.$$

In the same way, we can compute all the consensus levels as shown in Table 4.7.

Property 4.4 can be easily checked by comparing results from Tables 4.6 and 4.7. It is clear that $\delta^G_{\lambda_j}(\epsilon_i) = SM_j^i$ except for expert ϵ_2 on alternative λ_2 and expert ϵ_3 on alternatives λ_1 and λ_2, where $\delta^G_{\lambda_j}(\epsilon_i) < SM_j^i$. In this three cases, the opinion of the expert is not sortable with the centroid of the group, while in any other case, it is.

Notice also that, in the cases where the two consensus measures are different, the one presented by Wu and Xu [71] is greater given the fact that it only cares about the center of the HFLTS without taking into consideration either the hesitancy of the DMs or the existing gaps between opinions. For this reason, for instance,

TABLE 4.7: Consensus levels SM_j^i and SM_i from Example 4.4.

SM_j^i	ϵ_1	ϵ_2	ϵ_3	ϵ_4	ϵ_5
λ_1	0.75	0.875	0.875	0.5	1
λ_2	1	1	1	0.875	0.875
λ_3	1	0.875	1	0.25	1
λ_4	1	1	0.875	0.875	1
Λ	0.9375	0.9375	0.9375	0.625	0.96875

$SM_2^2 = SM_2^3 = 1$, even if the opinions of experts ϵ_2 and ϵ_3 are not the same than the centroid of the group for alternative λ_2. This leads us to a situation in which, experts ϵ_1, ϵ_2 and ϵ_3 share the same overall consensus level, $SM_1 = SM_2 = SM_3$, but, comparing F_H^1, F_H^2 and F_H^3 with respect to F_H^C, it seems quite intuitive that their coincidence with the central opinion should not be the same. By contrast, in Table 4.6 we can see that this problem is fixed given that $\delta_\Lambda^G(\epsilon_3) < \delta_\Lambda^G(\epsilon_2) < \delta_\Lambda^G(\epsilon_1)$.

4.5 A precision-dissension profile

Sometimes, when choosing DMs to assess a set of alternatives, a more precise expert is preferable to a more hesitant one. Sometimes a more dissenting expert is interesting to open a door to innovation, or sometimes it is just the other way around. The aim of this section is to present an expert's profile that keeps track of how experts have done their previous assessments to know how precise or how dissenting they are.

This profile might be useful to whoever has to choose among several decision makers to be part of a GDM situation because he or she can know beforehand the main characteristics of each expert's assessments. For instance, if we want to have a committee where common decisions are easily taken, we will choose un certain decision makers whose opinions are always close to the average opinion, which means a low precision and a low dissension. On the contrary, if we prefer a committee where polarized opinions are strongly defended, we should choose determined decision makers whose opinions tend to be far away from the central opinion, which means a high precision as well as a high dissension.

To this end, we present two numerical descriptors that characterize the assessment of a decision maker. Firstly, similarly to the notion of determinacy presented by Ma *et al.* [37], we introduce the concept of preciseness of an expert assessing a set of alternatives as a discrete version of determinacy. Both the preciseness and the determinacy seek to quantify the certainty of an expert but, while the determinacy is based on areas calculated as fuzzy integrals, the preciseness is based on the number of linguistic labels from $\mathcal{S}$ that the experts uses.

Definition 4.9. Let ϵ_i be a DM assessing a set of alternatives $\Lambda = \{\lambda_1, \ldots, \lambda_r\}$ by means of HFLTSs over $\mathcal{S} = \{a_1, \ldots, a_n\}$, and let F_H^i be his HFLD about Λ, being

$H_j^i = F_H^i(\lambda_j)$. Then, the *preciseness of ϵ_i on* Λ is defined as:

$$\pi_\Lambda(\epsilon_i) = \frac{\sum_{j=1}^{r} \frac{n-\text{card}(H_j^i)}{n-1}}{r}.$$

Note that, given that $\text{card}(H_j^i)$ is between 1 and n for any $j \in \{1,\ldots,r\}$, $\pi_\Lambda(\epsilon_i)$ ranges from 0 to 1, being 0 when $\text{card}(H_j^i) = n$ for any j and being 1 when $\text{card}(H_j^i) = 1$ for any j. Thus, the closer to 1 $\pi_\Lambda(\epsilon_i)$ is, the more precise ϵ_i has been with his assessments. Whilst, if $\pi_\Lambda(\epsilon_i)$ is close to 0, it means that there is more hesitancy in the assessments of ϵ_i about Λ.

Secondly, we also introduce the concept of dissent of an expert with respect to a group as follows:

Definition 4.10. Let $\epsilon_1,\ldots,\epsilon_k$ be a group G of DMs assessing a set of alternatives $\Lambda = \{\lambda_1,\ldots,\lambda_r\}$ by means of HFLTSs over $\mathcal{S} = \{a_1,\ldots,a_n\}$, and let F_H^i be the HFLD of ϵ_i for $i = 1,\ldots,k$ and F_H^C the centroid of the group. Then, the *dissent of ϵ_i on Λ with respect to G* is defined as:

$$\sigma_\Lambda^G(\epsilon_i) = 1 - \delta_\Lambda^G(\epsilon_i).$$

Notice that, again, $\sigma_\Lambda^G(\epsilon_i)$ moves between 0 and 1 for any $i \in \{1,\ldots,k\}$. The smaller $\sigma_\Lambda^G(\epsilon_i)$ is, the closer the opinion of the expert ϵ_i and the central opinion are, being exactly 0 if $F_H^i = F_H^C$.

With these two measures, a profile for each expert assessing a set of alternatives can be defined as:

Definition 4.11. Let $\epsilon_1,\ldots,\epsilon_k$ be a group G of DMs assessing a set of alternatives $\Lambda = \{\lambda_1,\ldots,\lambda_r\}$ by means of HFLTSs over $\mathcal{S} = \{a_1,\ldots,a_n\}$. Then, the *precision-dissension profile of ϵ_i on Λ with respect to G* is defined as:

$$\phi_\Lambda^G(\epsilon_i) = (\pi_\Lambda(\epsilon_i), \sigma_\Lambda^G(\epsilon_i)).$$

For the seek of a better understanding, let us present the following example illustrating the previous concepts.

Example 4.5. Following Example 4.2, with the assessments about the set of alternatives Λ shown in Table 4.1, the preciseness and the dissent of each expert can be calculated, as, for instance,

$$\pi_\Lambda(\epsilon_1) = \frac{\frac{5-2}{4} + \frac{5-3}{4} + \frac{5-2}{4} + \frac{5-1}{4}}{4} = 0.75$$

and

$$\sigma_\Lambda^G(\epsilon_1) = 1 - 0.9375 = 0.0625,$$

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given that $\delta_\Lambda^G(\epsilon_1)$ was already calculated in Example 4.4. Thus,

$$\phi_\Lambda^G(\epsilon_1) = (0.75, 0.0625).$$

Repeating this process for all the experts, we get the results shown in Table 4.8.

TABLE 4.8: Preciseness and dissent of each expert on Λ from Example 4.5.

	ϵ_1	ϵ_2	ϵ_3	ϵ_4	ϵ_5
$\pi_\Lambda(\epsilon_i)$	0.75	1	0.375	0.75	0.8125
$\sigma_\Lambda^G(\epsilon_i)$	0.0625	0.125	0.1875	0.375	0.03125
$\phi_\Lambda^G(\epsilon_i)$	$(0.75, 0.0625)$	$(1, 0.125)$	$(0.375, 0.1875)$	$(0.75, 0.375)$	$(0.8125, 0.03125)$

It can be seen that ϵ_2 has a preciseness of 1 given that he has assessed all the alternatives with just one linguistic label without hesitation. In contrast, ϵ_3 has a very low preciseness due to a a big hesitancy on his assessments. For instance, he has assessed two alternatives with all the possible linguistic labels.

On the other hand, ϵ_4 has the highest dissent of the whole group. This fact can be corroborated by having a look at Figure 4.6, which is a graphical representation of the assessments provided in Table 4.1, where it is clear than F_H^4 is the most distant assessment to the central opinion in almost all the alternatives. On the contrary, F_H^1 and F_H^5 are equal to the central opinion in almost all the alternatives, and that is why ϵ_1 and ϵ_5 have the lowest dissent of the group.
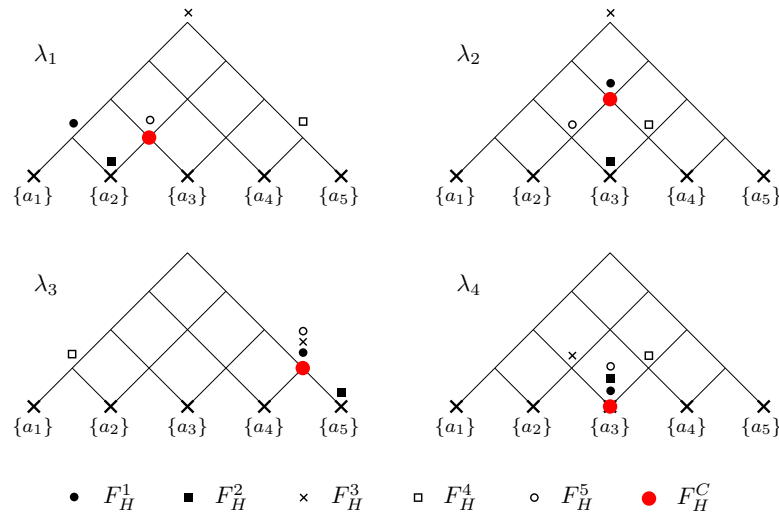
FIGURE 4.6: HFLDs from Example 4.2.

Finally, if an expert has assessed more than one set of alternatives within several groups, the information of each different situation can be combined as follows:

Definition 4.12. Let ϵ be a DM that has assessed the sets of alternatives $\Lambda_1, \ldots, \Lambda_m$ within the groups $G_1, \ldots, G_m$ respectively. Then:

(a) The *preciseness of ϵ* is defined as $\pi^m(\epsilon) = \dfrac{\sum\limits_{l=1}^{m} \pi_{\Lambda_l}(\epsilon)}{m}$.

(b) The *dissent of ϵ* is defined as $\sigma^m(\epsilon) = \dfrac{\sum\limits_{l=1}^{m} \sigma_{\Lambda_l}^{G_l}(\epsilon)}{m}$.

(c) The *precision-dissension profile of ϵ* is defined as $\Phi^m(\epsilon) = (m, \pi^m(\epsilon), \sigma^m(\epsilon))$.

With $\Phi^m(\epsilon)$ one can know the characteristics of the assessments of expert ϵ regarding precision and dissension after evaluating m different sets of alternatives within their respective groups.

4.6 Conclusions and future work

Based on the weak points of existing consensus measures for GDM by means of HFLTSs, two consensus measures are defined in this chapter in order to capture differences among situations in which the previous measures are not able to make a difference.

On the one hand, a consensus level is defined for the whole group on a specific alternative as a normalization of the addition of distances from a central opinion to the opinion of each expert of the group, and an analogous definition is given for a set of several alternatives instead of just one of them. On the other hand, the consensus level is defined for each expert with respect to the rest of the group based on the distance between his/her opinion and the central opinion for both one specific alternative and a set of alternatives.

Additionally, a study is carried out to compare the presented measures with the similar existing ones and concludes that the measures presented in this chapter are more accurate in situations in which existing measures consider the level of agreement to be the same but where common sense suggests they should be different. Moreover, the comparison study also shows that the collective degree of consensus presented in this chapter has a lower time complexity than the existing measures.

Lastly, a profile of an expert is presented to keep track of the precision and dissension in his/her assessments with a view to using this information for future experts selection processes.

Future work will focus on two main directions. From a theoretical point of view, a dynamical study will be carried out on both the consensus-reaching process and the precision-dissension profile of DMs in several GDM processes. In particular, the proposed consensus measures will be used to measure polarization in this kind of scenarios. From a practical point of view, all the introduced concepts are already being implemented in a real case example framed in the city tourism management field.

Acknowledgements

Chapter 5

Free Double Hierarchy Hesitant Fuzzy Linguistic Term Sets: An Application on the TOPSIS Methodology

5.1 Introduction

Fuzzy sets were introduced by Zadeh to deal with uncertain decision-making processes [77]. Several extensions have been presented since then such as the Intuitionistic Fuzzy Sets [5, 6] or the Hesitant Fuzzy Sets (HFSs) [28, 62, 72]. However, in some areas, people prefer to use a qualitative reasoning better than a quantitative reasoning. To this end, Zadeh also introduced the concept of linguistic variable [78]. From then on, several studies have been developed on that field [24, 39, 40, 42, 43, 65].

With the aim of combining HFSs and qualitative reasoning, Rodríguez *et al.* introduced the concept of Hesitant Fuzzy Linguistic Term Set (HFLTS) [53] that was later redefined in a mathematical form by Liao *et al.* [35]. So far, several contributions presented in the literature have studied several aspects of the HFLTSs such as hesitant fuzzy linguistic information aggregation techniques [19, 68], hesitant fuzzy linguistic measure methods [20, 33, 35], hesitant fuzzy linguistic operational laws [18], hesitant fuzzy linguistic preference relations [35, 36, 81] and hesitant fuzzy linguistic decision-making methods [20, 45, 46].

Nonetheless, in some situations, HFLTSs are not able to depict with enough details the complexity inherent in human reasoning when evaluating with linguistic assessments. Some authors have studied how to define linguistic expressions more complex than single linguistic terms as reviewed by Rodríguez *et al.* [54]. In order to provide a more precise tool, Gou *et al.* presented the concept of Double Hierarchy Hesitant Fuzzy Linguistic Term Sets (DHHFLTSs) [21]. This structure allows each decision maker to choose one term from a first hierarchy Linguistic Term Set (LTS) and later choose another term from a second hierarchy LTS gaining more accuracy on the linguistic assessment.

DHHFLTSs are a very useful tool to deal with qualitative assessments, yet they present some shortcomings given that the second hierarchy LTS has to be the same for every single term of the first hierarchy LTS. This leads us to three main issues:

Firstly, misleading or meaningless linguistic expressions may appear as a result of using a fixed second hierarchy LTS. For instance, while "*extremely*" has a strong positive meaning on "*good*", it does not have the same positive meaning when it is applied to "*regular*". In addition, while a term like "*close to*" makes sense when applied to "*perfect*", it should not be applied to a linguistic term like "*normal*". This is due to the fact that "*close to normal*" can be understood in two different meanings (worse than average or better than average).

Secondly, not all linguistic terms need the same range of precision for their corresponding second hierarchy LTSs. As an example, linguistic terms such as "*bad*" or "*good*", in general, accept a much wider variety of precision than terms such as "*null*" or "*perfect*".

Lastly, all decision makers are forced to use the same second hierarchy LTS. It is known that the decision makers have their own preferences about which linguistic expressions to use. For instance, for the linguistic term "*perfect*", one decision maker could prefer to use the second hierarchy LTS {"*not far from*", "*almost*", "*completely*"}, and another one could feel more comfortable by using {"*close to*", "*totally*"}.

In this chapter, we present a new structure that overcomes these three issues called Free Double Hierarchy Hesitant Fuzzy Linguistic Term Sets (FDHHFLTSs), whose elements are called Free Double Hierarchy Hesitant Fuzzy Linguistic Elements (FD-HHFLEs). Based on the introduced structure, each decision maker involved in Group Decision-Making (GDM) situation is allowed to choose the second hierarchy LTS that he or she thinks that suits it better, with as many terms as desired.

Furthermore, an order and a distance between FDHHFLEs are also presented in this chapter in order to compare and quantify distances between linguistic assessments provided by the decision makers by means of the aforementioned structure. These order and distance are used to introduce a free double hierarchy approach based on the well-known multi-criteria decision-making TOPSIS ranking method, enabling us to rank alternatives that have been assessed by means of free double hierarchy hesitant fuzzy linguistic information.

The rest of this chapter is structured as follows: First, Section 5.2 summarizes basic concepts already introduced in the literature that will be used throughout the work. The new free double hierarchy structure is introduced in Section 5.3. Section 5.4 introduces an order and a distance for FDHHFLEs. A free double hierarchy approach based on the TOPSIS method is presented in Section 5.5 as well as a simulated example to illustrate the presented approach. Finally, Section 5.6 summarizes the main conclusions and points out the directions of future research.

5.2 Preliminaries

This section presents an overview of some concepts already introduced in the literature regarding HFLTSs and DHHFLTSs that will be used throughout the chapter to present the new contributions.

Note that, since this chapter of the thesis is a collaboration with Sichuan University, and it is oriented towards the lines of research opened by Gou *et al.* [21], the terminology of this chapter slightly differs from the rest of the thesis. Specifically, the expression hesitant fuzzy linguistic element refers to the assessment of a single feature of a certain alternative, which in the previous chapters was called hesitant fuzzy linguistic term set, while the expression hesitant fuzzy linguistic term set refers to the assessment of all the features of a certain alternative, which was called hesitant fuzzy linguistic description in the previous chapters.

5.2.1 Hesitant Fuzzy Linguistic Term Sets

HFSs were introduced by Torra [62] as a function that returns a subset of $[0, 1]$ as possible membership degrees. Later, Xia and Xu [72] expressed the concept of HFS in a mathematical way as $A = \{< x, h_A(x) >| x \in X\}$ with $h_A(X) \subset [0, 1]$ denoting the possible membership degrees of the element $x \in X$ to the set A. Moreover, $h = h_A(x)$ is called a Hesitant Fuzzy Element (HFE) and Θ denotes the set of all HFEs.

Rodríguez *et al.* extended the HFSs to define the concept of HFLTS as an ordered finite subset of consecutive linguistic terms of a given LTS [53]. An extension of this definition was presented in a mathematical way by Liao *et al.* [35] as follows:

Definition 5.1 ([35]). Let $X = \{x_1, x_2, \ldots, x_n\}$ be a fixed set and $\mathcal{S} = \{s_t \mid t = -\tau, \ldots, -1, 0, 1, \ldots, \tau\}$ be a LTS. Then, a *Hesitant Fuzzy Linguistic Term Set (HFLTS) on X*, $H_{\mathcal{S}}$, can be expressed in a mathematical form as:

$$H_{\mathcal{S}} = \{< x_i, h_{\mathcal{S}}(x_i) >| x_i \in X\},$$

where $h_{\mathcal{S}}(x_i)$ is a subset of some linguistic terms in $\mathcal{S}$ and can be expressed as:

$$h_{\mathcal{S}}(x_i) = \{s_{\phi_l}(x_i) \mid s_{\phi_l}(x_i) \in \mathcal{S}; \, l = 1, \ldots, L_i; \, \phi_l \in \{-\tau, \ldots, -1, 0, 1, \ldots, \tau\}\}$$

being L_i the number of linguistic terms in $h_{\mathcal{S}}(x_i)$ and $s_{\phi_l}(x_i)$, for $l = 1, \ldots, L_i$, the consecutive terms of $\mathcal{S}$ in $h_{\mathcal{S}}(x_i)$. Analogous to the HFSs and HFEs, $h_{\mathcal{S}}(x_i)$ denotes the possible membership degrees of the linguistic variable x_i to $\mathcal{S}$ and it is called a *Hesitant Fuzzy Linguistic Element (HFLE)* and Φ denotes the set of all possible HFLEs.

Some contributions have presented approaches on how to extend the discrete form of $\mathcal{S}$ to a continuous form [19, 27, 76], such as the introduction of the following transformation function between the continuous HFLE and HFE:

Definition 5.2 ([19]). Let $\mathcal{S} = \{s_t \mid t = -\tau, \ldots, -1, 0, 1, \ldots, \tau\}$ be a LTS, $h_{\mathcal{S}} = \{s_{\phi_l} \mid s_{\phi_l} \in \mathcal{S}; l = 1, \ldots, L; \phi_l \in [-\tau, \tau]\}$ be a continuous HFLE containing L linguistic terms in it and $h_{\gamma} = \{\gamma_l \mid \gamma_l \in [0, 1]; l = 1, \ldots, L\}$ be a HFE. Then:

a) The *transformation functions* g and g^{-1} between each subscript ϕ_l of the linguistic term s_{ϕ_l} and the membership degree γ_l that expresses the equivalent information are, respectively:

- $$g: \quad [-\tau, \tau] \longrightarrow [0, 1]$$
$$\phi_l \longmapsto g(\phi_l) = \frac{\phi_l + \tau}{2\tau} = \gamma_l,$$

- $$g^{-1}: \quad [0, 1] \longrightarrow [-\tau, \tau]$$
$$\gamma_l \longmapsto g^{-1}(\gamma_l) = (2\gamma_l - 1)\tau = \phi_l.$$

b) The *transformation functions* G and G^{-1} between the continuous HFLE $h_{\mathcal{S}}$ and the HFE h_{γ} are, respectively:

- $$G: \quad \Phi_{\mathcal{C}} \longrightarrow \Theta$$
$$h_{\mathcal{S}} \longmapsto G(h_{\mathcal{S}}) = \{\gamma_l \mid \gamma_l = g(\phi_l)\} = h_{\gamma},$$

- $$G^{-1}: \quad \Theta \longrightarrow \Phi_{\mathcal{C}}$$
$$h_{\gamma} \longmapsto G^{-1}(h_{\gamma}) = \{s_{\phi_l} \mid \phi_l = g^{-1}(\gamma_l)\} = h_{\mathcal{S}},$$

being $\Phi_{\mathcal{C}}$ the set of all possible continuous HFLEs.

These functions allow us to translate descriptions from a qualitative context into the equivalent ones from a quantitative context.

5.2.2 Double Hierarchy Hesitant Fuzzy Linguistic Term Sets

In GDM problems, it is common that linguistic labels such as "*good*" or "*low*" are not suitable enough to describe the opinion of the decision maker. To this aim, Gou *et al.* presented a double hierarchy approach for HFLTSs in which more accurate evaluations like "*just right good*" or "*a little low*" can be provided [21]. Linguistic hierarchy has been classical concept in the literature of computing with words [13, 26]. Yet, the approach considered in this chapter follows the direction of Gou *et al.* [21].

Definition 5.3 ([21]). Let $\mathcal{S} = \{s_t \mid t = -\tau, \ldots, -1, 0, 1, \ldots, \tau\}$ and $\mathcal{O} = \{o_k \mid k = -\zeta, \ldots, -1, 0, 1, \ldots, \zeta\}$ be the first and second hierarchy LTSs, respectively, being fully independent. A *Double Hierarchy Linguistic Term Set (DHLTS)*, $\mathcal{S}_{\mathcal{O}}$, can be expressed in a mathematical way as:

$$\mathcal{S}_{\mathcal{O}} = \{s_{t<o_k>} \mid t = -\tau, \ldots, -1, 0, 1, \ldots, \tau; k = -\zeta, \ldots, -1, 0, 1, \ldots, \zeta\}.$$

Each $s_{t<o_k>}$ is called a *Double Hierarchy Linguistic Term (DHLT)*, where o_k expresses the second hierarchy linguistic term when the first hierarchy linguistic term is s_t.

The concept of DHLTS is used on hesitant fuzzy linguistic information to incorporate the second hierarchy to the idea of HFLTS and HFLE as follows:

Definition 5.4 ([21]). Let $\mathcal{S}_{\mathcal{O}} = \{s_{t<o_k>} \mid t = -\tau, \ldots, -1, 0, 1, \ldots, \tau; k = -\zeta, \ldots, -1, 0, 1, \ldots, \zeta\}$ be a DHLTS, then a *Double Hierarchy Hesitant Fuzzy Linguistic Term Set (DHHFLTS) on X*, $H_{\mathcal{S}_{\mathcal{O}}}$, can be expressed in a mathematical form as:

$$H_{\mathcal{S}_{\mathcal{O}}} = \{< x_i, h_{\mathcal{S}_{\mathcal{O}}}(x_i) >| x_i \in X\}$$

where $h_{\mathcal{S}_{\mathcal{O}}}(x_i)$ is a set of some values in $\mathcal{S}_{\mathcal{O}}$, denoted as:

$$h_{\mathcal{S}_{\mathcal{O}}}(x_i) = \{s_{\phi_l<o_{\varphi_l}>}(x_i) \mid s_{\phi_l<o_{\varphi_l}>} \in \mathcal{S}_{\mathcal{O}}; l = 1, \ldots, L_i;$$
$$\phi_l \in \{-\tau, \ldots, -1, 0, 1, \ldots, \tau\}; \varphi_l \in \{-\zeta, \ldots, -1, 0, 1, \ldots, \zeta\}\}$$

being L_i the number of DHLTs in $h_{\mathcal{S}_{\mathcal{O}}}(x_i)$ and $s_{\phi_l<o_{\varphi_l}>}(x_i)$, for $i = 1, \ldots, L_i$, the consecutive terms of $\mathcal{S}_{\mathcal{O}}$ in $h_{\mathcal{S}_{\mathcal{O}}}(x_i)$. Analogous to the case of HFLTSs and HFLEs, $h_{\mathcal{S}_{\mathcal{O}}}(x_i)$ denotes the possible membership degrees of the linguistic variable x_i to $\mathcal{S}_{\mathcal{O}}$ and it is called a *Double Hierarchy Hesitant Fuzzy Linguistic Element (DHHFLE)*, and $\Phi \times \Psi$ denotes the set of all possible DHHFLEs.

To clarify the foregoing definition, let us present the following example:

Example 5.1. Let $\mathcal{S} = \{s_{-2} = \text{"null"}, s_{-1} = \text{"bad"}, s_0 = \text{"regular"}, s_1 = \text{"good"}, s_2 = \text{"perfect"}\}$ be the first hierarchy LTS with $\tau = 2$ and let $\mathcal{O} = \{o^1_{-3} = \text{"hardly"}, o^1_{-2} = \text{"slightly"}, o^1_{-1} = \text{"pretty"}, o^1_0 = \text{"simply"}, o^1_1 = \text{"very"}, o^1_2 = \text{"unusually"}, o^1_3 = \text{"extremely"}\}$ with $\zeta = 3$. Then, "*hardly good*" is a DHLT from $\mathcal{S}_{\mathcal{O}}$ and "*between pretty good and very good*" is a possible linguistic assessment that corresponds to the DHHFLE $\{s_{1<o_{-1}>}, s_{1<o_0>}, s_{1<o_1>}\}$. Figure 5.1 shows the second hierarchy for the linguistic term $s_1 = \text{"good"}$ from the first hierarchy.
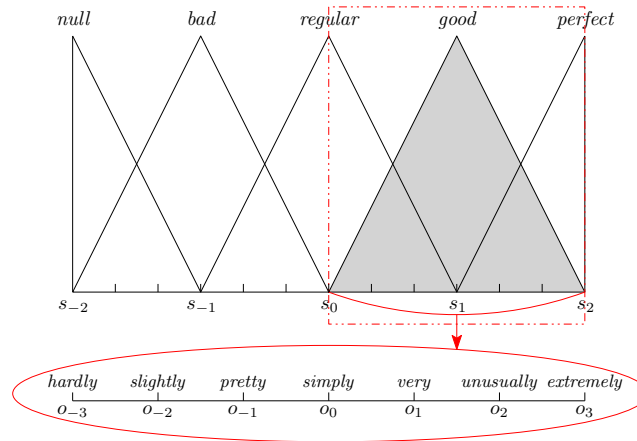

FIGURE 5.1: DHLTS $\mathcal{S}_{\mathcal{O}}$ from Example 5.1.

Lastly, a way to extend DHHFLEs to the continuous DHHFLEs was also presented by Gou *et al.* [21] just based solely on the continuation of the second hierarchy while the first hierarchy remains discrete and the corresponding functions to transform the continuous DHHFLEs into HFEs and vice versa were also defined. Despite

this, in Section 5.3.1 we present a more accurate definition of those functions taking into consideration some cases that were disregarded by the original definition.

5.3   Free Double Hierarchy

In some situations, the second hierarchy cannot be the same for all the linguistic terms in the first hierarchy. Additionally, we often want a larger second hierarchy LTS for some first hierarchy linguistic terms than for others. In order to present a more suitable approach for these kinds of situations, in this section, the concept of DHLTS is extended to a new definition that allows the second hierarchy terms to be different for each of the first hierarchy terms. So, the introduced methodology can capture in a better way the linguistic assessments given by the decision makers when assessing alternatives. Furthermore, this structure is applied to depict hesitant fuzzy linguistic information and to present an extension of the DHHFLTSs. Finally, the last part of this section presents the corresponding transformation functions that allow us to extend the aforementioned concepts from the discrete form to the continuous one in an analogous way to the double hierarchy case. To this end, we start this section by developing the preliminaries on DHHFLTSs a bit more.

5.3.1   Developed Preliminaries

For the sake of the consistency of this chapter, in this section we present a more accurate definition of the transformation functions for DHHFLTSs from the discrete version to the continuous one.

Definition 5.5. Let $\mathcal{S}_{\mathcal{O}} = \{s_{t<o_k>} \mid t = -\tau, \ldots, -1, 0, 1, \ldots, \tau; k = -\zeta, \ldots, -1, 0, 1, \ldots, \zeta\}$ be a DHLTS, $h_{\mathcal{S}_{\mathcal{O}}} = \{s_{\phi_l<o_{\varphi_l}>} \mid s_{\phi_l<o_{\varphi_l}>} \in \mathcal{S}_{\mathcal{O}}; l = 1, \ldots, L; \phi_l \in \{-\tau, \ldots, \tau\}; \varphi_l \in [-\zeta, \zeta]\}$ be a continuous DHHFLE containing L linguistic terms in it and $h_{\gamma} = \{\gamma_l \mid \gamma_l \in [0,1]; l = 1, \ldots, L\}$ be a HFE. Then:

a) The *transformation functions* f and f^{-1} between each pair of subscripts (ϕ_l, φ_l) of the linguistic term $s_{\phi_l<o_{\varphi_l}>}$ and the membership degree γ_l that expresses the equivalent information are, respectively:

$$
\begin{aligned}
&\bullet\ f: \quad \{-\tau, \ldots, \tau\} \times [-\zeta, \zeta] \longrightarrow \qquad\quad [0,1] \\
&\qquad\qquad (\phi_l, \varphi_l) \qquad\quad \longmapsto \qquad f(\phi_l, \varphi_l) = \frac{\varphi_l + (\tau + \phi_l)\zeta}{2\zeta\tau} = \gamma_l,
\end{aligned}
$$

$$
\begin{aligned}
&\bullet\ f^{-1}: \qquad\quad [0,1] \qquad\quad \longrightarrow \{-\tau, \ldots, \tau\} \times [-\zeta, \zeta] \\
&\qquad\qquad\quad \gamma_l \qquad\qquad \longmapsto \qquad f^{-1}(\gamma_l) = (\phi_l, \varphi_l) =
\end{aligned}
$$

$$
= \begin{cases}
\begin{bmatrix}
(2\tau\gamma_l - \tau - 1, \zeta) & \text{(only if } \gamma_l \neq 0) \\
\vee \\
(2\tau\gamma_l - \tau, 0) \\
\vee \\
(2\tau\gamma_l - \tau + 1, -\zeta) & \text{(only if } \gamma_l \neq 1)
\end{bmatrix} & \text{if } (2\tau\gamma_l - \tau) \in \mathbb{Z}, \\[2em]
\begin{bmatrix}
(\lfloor 2\tau\gamma_l - \tau \rfloor, (2\tau\gamma_l - \tau - \lfloor 2\tau\gamma_l - \tau \rfloor)\zeta) \\
\vee \\
(\lceil 2\tau\gamma_l - \tau \rceil, (2\tau\gamma_l - \tau - \lceil 2\tau\gamma_l - \tau \rceil)\zeta)
\end{bmatrix} & \text{if } (2\tau\gamma_l - \tau) \notin \mathbb{Z}.
\end{cases}
$$

b) The *transformation functions* F and F^{-1} between the continuous DHHFLE $h_{\mathcal{S}_{\mathcal{O}}}$ and the HFE h_γ are, respectively:

- $F: \quad \Phi \times \Psi_{\mathcal{C}} \longrightarrow \quad \Theta$
$\quad\quad\quad h_{\mathcal{S}_{\mathcal{O}}} \quad \longmapsto \quad F(h_{\mathcal{S}_{\mathcal{O}}}) \ = \ \{\gamma_l \mid \gamma_l = f(\phi_l, \varphi_l)\} = h_\gamma,$

- $F^{-1}: \quad \Theta \quad \longrightarrow \Phi \times \Psi_{\mathcal{C}}$
$\quad\quad\quad h_\gamma \quad \longmapsto \quad F^{-1}(h_\gamma) \ = \ \{s_{\phi_l < o_{\varphi_l}>} \mid (\phi_l, \varphi_l) = f^{-1}(\gamma_l)\} = h_{\mathcal{S}_{\mathcal{O}}},$

being $\Phi \times \Psi_{\mathcal{C}}$ the set of all possible continuous DHHFLEs.

Remark 5.1. Note that the function f is not a bijection given that it is not injective. As an example, for $\phi_l = -\tau + 1, \ldots, \tau - 1$, $f(\phi_l - 1, \zeta) = f(\phi_l, 0) = f(\phi_l + 1, -\zeta) = \dfrac{\tau + \phi_l}{2\tau}$, as shown in Figure 5.2a for $\phi_l = -1$. Therefore, f^{-1} is not uniquely defined and it leads to different results for the same value of $\gamma_l \in [0, 1]$. Figure 5.2b shows another example in which we can see that $s_{0 < o_{1.5}>}$ and $s_{1 < o_{-1.5}>}$ share the same image for the function f, which is $\frac{15}{25}$. In addition, we can see that, among others, a hypothetical $s_{0.5 < o_0>}$ would also have the same image. However, it is not considered given that, as stated before, the extension from the discrete version to the continuous one is made based solely on the second hierarchy.

(A) $f^{-1}\left(\frac{1}{4}\right).$
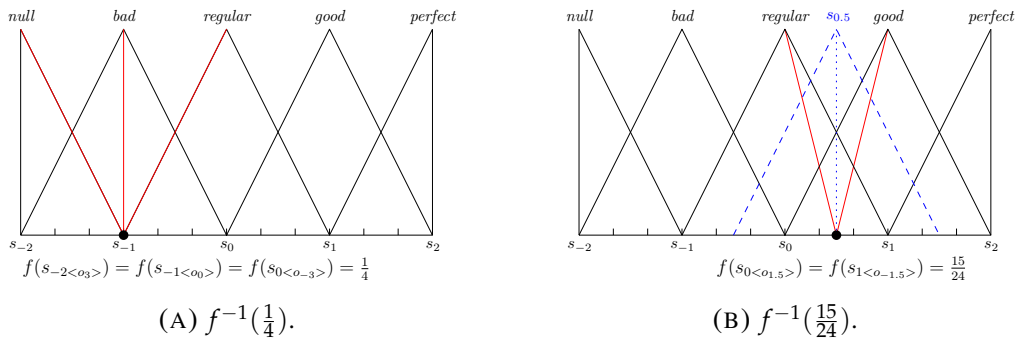
(B) $f^{-1}\left(\frac{15}{24}\right).$

FIGURE 5.2: DHLTs with the same value of the transformation function f.

5.3.2 Free Double Hierarchy Linguistic Term Sets

DHLTSs are a useful tool to describe, in a mathematical way, possible linguistic assessments provided by the decision makers in a group decision-making problem. Yet

they present some shortcomings such as the fact that the second hierarchy scale is the same for all the linguistic terms of the first hierarchy. However, whilst "*extremely*" applies well for some linguistic terms such as "*good*" (resulting in "*extremely good*"), it does not apply that well for some other linguistic terms, leading to confused, or even meaningless, linguistic terms such as "*extremely regular*". Same thing happens with "*almost perfect*", which is a clear and common linguistic expression, and "*almost regular*", which is not clear. In order to fix this issue, we propose the following extension of the DHLTSs:

Definition 5.6. Let $\mathcal{S} = \{s_t \mid t = -\tau, \ldots, -1, 0, 1, \ldots, \tau\}$ and $\mathcal{O}^t = \{o_k^t \mid k = -\zeta_t, \ldots, -1, 0, 1, \ldots, \zeta_t\}$ for all $t \in \{-\tau, \ldots, -1, 0, 1, \ldots, \tau\}$ be the first and second hierarchies of LTSs respectively. Then, a *Free Double Hierarchy Linguistic Term Set (FDHLTS)*, $\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}$, can be expressed in a mathematical form as:

$$\mathcal{S}_{\mathcal{O}}^{\mathcal{F}} = \{s_{t<o_k^t>} \mid t = -\tau, \ldots, -1, 0, 1, \ldots, \tau; k = -\zeta_t, \ldots, -1, 0, 1, \ldots, \zeta_t\}.$$

Each $s_{t<o_k^t>}$ is called a *Free Double Hierarchy Linguistic Term (FDHLT)*, where o_k^t expresses the second hierarchy linguistic term when the first hierarchy linguistic term is s_t.

Remark 5.2. For symmetry reasons, as it can be seen in Figure 5.3, the FDHLTs contained in $\mathcal{O}_-^{-\tau} = \{s_{-\tau<o_{-\zeta_{-\tau}}^{-\tau}>}, s_{-\tau<o_{-\zeta_{-\tau}+1}^{-\tau}>}, \ldots, s_{-\tau<o_{-1}^{-\tau}>}\}$ as well as in $\mathcal{O}_+^{\tau} = \{s_{\tau<o_1^{\tau}>}, s_{\tau<o_2^{\tau}>}, \ldots, s_{\tau<o_{\zeta_\tau}^{\tau}>}\}$ should be dismissed. Therefore, from now on, $\mathcal{O}^{-\tau}$, $\mathcal{O}^{\tau}$ and $\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}$ are used throughout this chapter, without loss of generality, for $(\mathcal{O}^{-\tau} \setminus \mathcal{O}_-^{-\tau})$, $(\mathcal{O}^{\tau} \setminus \mathcal{O}_+^{\tau})$ and $\mathcal{S}_{\mathcal{O}}^{\mathcal{F}} \setminus (\mathcal{O}_-^{-\tau} \cup \mathcal{O}_+^{\tau})$ respectively to simplify the notation.

Notice that, according to Definition 5.6, in a FDHLTS, the granularity of the second hierarchy can be different for each linguistic term of the first hierarchy. This fact fixes another shortcoming presented by the DHLTSs.

Even though it has been shown that the granularity of a linguistic term set must be smaller than 9 because of limitation of human ability [44, 59], in this case, since we are considering two different hierarchies, we understand that each hierarchy takes part of a different linguistic term set. In addition, given that each decision maker will be asked to choose, for each linguistic term of the first scale, the granularity that he or she prefers for the second hierarchy, the number of total linguistic expressions could be as low as the cardinality of the first linguistic scale.

With the aim of simplifying the introduction of the concepts presented throughout this work, we define the following order relation between FDHLTs:

Definition 5.7. Let $s_{t_1<o_{k_1}^{t_1}>}$ and $s_{t_2<o_{k_2}^{t_2}>}$ be two FDHLTs of $\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}$, then we define:

$$s_{t_1<o_{k_1}^{t_1}>} \preccurlyeq s_{t_2<o_{k_2}^{t_2}>} \iff (t_1 < t_2) \vee ((t_1 = t_2) \wedge (k_1 \leq k_2)),$$

with the equality satisfied only when $t_1 = t_2$ and $k_1 = k_2$.

For a better comprehension of the FDHLTSs, in the following, let us illustrate the foregoing definition with an example:

Example 5.2. Let $\mathcal{S} = \{s_{-2} = \text{``null''}, s_{-1} = \text{``bad''}, s_0 = \text{``regular''}, s_1 = \text{``good''}, s_2 = \text{``perfect''}\}$ be the first hierarchy LTS with $\tau = 2$ and let

$$\mathcal{O}^{-2} = \{o_0^{-2} = \text{``completely''}, o_1^{-2} = \text{``almost''}, o_2^{-2} = \text{``close to''}\},$$
$$\mathcal{O}^{-1} = \{o_{-3}^{-1} = \text{``extremely''}, o_{-2}^{-1} = \text{``unusually''}, o_{-1}^{-1} = \text{``very''}, o_0^{-1} = \text{``simply''},$$
$$o_1^{-1} = \text{``pretty''}, o_2^{-1} = \text{``slightly''}, o_3^{-1} = \text{``hardly''}\},$$
$$\mathcal{O}^0 = \{o_{-2}^0 = \text{``very low''}, o_{-1}^0 = \text{``low''}, o_0^0 = \text{``medium''}, o_1^0 = \text{``high''},$$
$$o_2^0 = \text{``very high''}\},$$
$$\mathcal{O}^1 = \{o_{-3}^1 = \text{``hardly''}, o_{-2}^1 = \text{``slightly''}, o_{-1}^1 = \text{``pretty''}, o_0^1 = \text{``simply''},$$
$$o_1^1 = \text{``very''}, o_2^1 = \text{``unusually''}, o_3^1 = \text{``extremely''}\},$$
$$\mathcal{O}^2 = \{o_{-2}^2 = \text{``close to''}, o_{-1}^2 = \text{``almost''}, o_0^2 = \text{``completely''}\}$$

be the respective second hierarchy LTSs for each s_t, for $t = -2, \ldots, 2$ (with $\zeta_{-2} = \zeta_0 = \zeta_2 = 2$ and $\zeta_{-1} = \zeta_1 = 3$), defining the associated FDHLTS $\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}$ represented in Figure 5.3. Thus, the possible linguistic assessments such as "*slightly bad*" or "*almost perfect*" can be expressed by means of FDHLTs of $\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}$ as $s_{-1<o_2^{-1}>}$ and $s_{2<o_{-1}^2>}$.
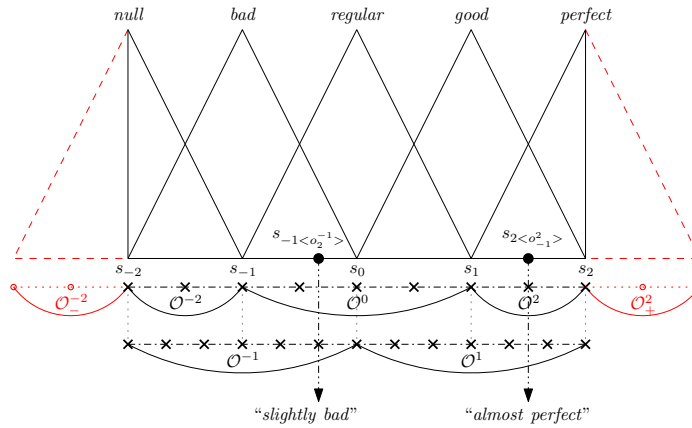
FIGURE 5.3: FDHLTS $\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}$ from Example 5.2.

Example 5.2 brings to light the utility of the FDHLTs as they can properly describe linguistic assessments in a more precise way.

5.3.3 Free Double Hierarchy Hesitant Fuzzy Linguistic Term Sets

Even though FDHLTs are useful to describe in a mathematical way a possible linguistic assessment from a decision maker, they cannot capture the vacillation inherent in human reasoning. In an analogous way to how HFLTSs are defined from ordinary LTSs, a new structure can be defined from the FDHLTSs to capture such indecision:

Definition 5.8. Let $\mathcal{S}_{\mathcal{O}}^{\mathcal{F}} = \{s_{t<o_k^t>} \mid t = -\tau, \ldots, -1, 0, 1, \ldots, \tau; k = -\zeta_t, \ldots, -1, 0, 1, \ldots, \zeta_t\}$ be a FDHLTS, then, a *Free Double Hierarchy Hesitant Fuzzy Linguistic*

Term Set (FDHHFLTS) on X, $H_{\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}}$, can be expressed in a mathematical form as:

$$H_{\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}} = \{< x_i, h_{\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}}(x_i) >| \, x_i \in X\},$$

where $h_{\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}}$ is a set of some consecutive linguistic terms of $\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}$ denoted as:

$$h_{\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}}(x_i) = \{s_{\phi_l < o_{\varphi_l}^{\phi_l}>}(x_i) \mid s_{\phi_l < o_{\varphi_l}^{\phi_l}>} \in \mathcal{S}_{\mathcal{O}}^{\mathcal{F}}; l = 1, \ldots, L_i;$$
$$\phi_l \in \{-\tau, \ldots, -1, 0, 1, \ldots, \tau\}; \varphi_l \in \{-\zeta_{\phi_l}, \ldots, -1, 0, 1, \ldots, \zeta_{\phi_l}\}\}$$

with L_i being the number of FDHLTs in $h_{\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}}(x_i)$, and $s_{\phi_l < o_{\varphi_l}^{\phi_l}>}(x_i)$, for $l = 1, \ldots, L_i$, the consecutive terms of $\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}$ in $h_{\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}}(x_i)$.

Additionally, $h_{\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}}(x_i)$ denotes the possible membership degrees of the linguistic variable x_i to $\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}$ and, for convenience, it is called a *Free Double Hierarchy Hesitant Fuzzy Linguistic Element (FDHHFLE),* and $\Phi \otimes \Psi$ denotes the set of all possible FDHHFLEs.

Remark 5.3. Note that, since the terms in $h_{\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}}(x_i)$ have to be continuous, any FD-HHFLE will always contain all the terms between two FDHLTs. Therefore, if all the FDHLTs included in $h_{\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}}(x_i)$ are from the same linguistic term from the first hierarchy, s_t, then we write one by one all the FDHLTs. Otherwise, for simplicity, $h_{\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}}(x_i)$ can be characterized by one single element per each linguistic term from the first hierarchy that, to a greater or lesser extent, take part of $h_{\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}}(x_i)$. Thus, if for a linguistic term of the first hierarchy, s_t, all the possible second hierarchy terms are included in $h_{\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}}(x_i)$, we just write s_t without specifying the second hierarchy. On the contrary, for the linguistic terms of the first hierarchy s_t whose possible second hierarchy terms are not all included in $h_{\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}}(x_i)$, we just write the lower or upper bound of the FDHLTs taking part of the FDHHFLE.

For instance, on the one hand, let $h_{\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}}(x_1)$ be the FDHHFLE including all the FDHLTs from $s_{1 < o_{-1}^1>}$ to $s_{1 < o_2^1>}$, then $h_{\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}}(x_1)$ is written as $\{s_{1 < o_{-1}^1>}, s_{1 < o_0^1>}, s_{1 < o_1^1>}, s_{1 < o_2^1>}\}$. On the other hand, let $h_{\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}}(x_2)$ be the FDHHFLE including all the FDHLTs from $s_{0 < o_{-1}^0>}$ to $s_{2 < o_1^2>}$, then $h_{\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}}(x_2)$ is written as $\{s_{0 < o_{-1}^0>}, s_1, s_{2 < o_1^2>}\}$.

For GDM problems, it is convenient to consider the following context-free grammar that generates the suitable language for assessing the different alternatives.

Definition 5.9. Let $\mathcal{S}_{\mathcal{O}}^{\mathcal{F}} = \{s_{t < o_k^t>} \mid t = -\tau, \ldots, -1, 0, 1, \ldots, \tau; k = -\zeta_t, \ldots, -1, 0, 1, \ldots, \zeta_t\}$ be a FDHLTS, then the *Free Double Hierarchy Hesitant context-free*

grammar, $\Gamma_H = \{\ddot{V}_N, \ddot{V}_T, \ddot{I}, \ddot{P}\}$, can be defined as follows:

$\ddot{V}_N = \{\langle\textit{FDH primary term}\rangle, \langle\textit{FDH composite term}\rangle,$
$\qquad \langle\textit{1H-term }[-\tau]\rangle, \langle\textit{1H-term }[-\tau+1]\rangle, \ldots, \langle\textit{1H-term }[\tau]\rangle,$
$\qquad \langle[-\tau]\textit{ 2H-term}\rangle, \langle[-\tau+1]\textit{ 2H-term}\rangle, \ldots, \langle[\tau]\textit{ 2H-term}\rangle,$
$\qquad \langle\textit{unary relation}\rangle, \langle\textit{binary relation}\rangle, \langle\textit{conjunction}\rangle\},$

$\ddot{V}_T = \{\textit{at least}, \textit{at most}, \textit{between}, \textit{and}, s_{-\tau}, s_{-\tau+1} \ldots, s_\tau,$
$\qquad o_0^{-\tau}, o_1^{-\tau}, \ldots, o_{\zeta_{-\tau}}^{-\tau}, o_{-\zeta_{-\tau+1}}^{-\tau+1}, o_{-\zeta_{-\tau+1}+1}^{-\tau+1}, \ldots, o_{\zeta_{-\tau+1}}^{-\tau+1}, \ldots,$
$\qquad o_{-\zeta_{\tau-1}}^{\tau-1}, o_{-\zeta_{\tau-1}+1}^{\tau-1}, \ldots, o_{\zeta_{\tau-1}}^{\tau-1}, o_{-\zeta_\tau}^{\tau}, o_{-\zeta_\tau+1}^{\tau}, \ldots, o_0^{\tau}\},$

$\ddot{I} = \{\langle\textit{FDH primary term}\rangle, \langle\textit{FDH composite term}\rangle\} \in \ddot{V}_N,$

$\ddot{P} = \{\langle\textit{FDH composite term}\rangle ::= \langle\textit{unary relation}\rangle\langle\textit{FDH primary term}\rangle|$
$\qquad \langle\textit{binary relation}\rangle\langle\textit{FDH primary term}\rangle\langle\textit{conjunction}\rangle\langle\textit{FDH primary term}\rangle;$
$\qquad \langle\textit{FDH primary term}\rangle ::= \langle[-\tau]\textit{ 2H-term}\rangle\langle\textit{1H-term }[-\tau]\rangle|$
$\qquad\quad \langle[-\tau+1]\textit{ 2H-term}\rangle\langle\textit{1H-term }[-\tau+1]\rangle, |\ldots|,$
$\qquad\quad \langle[\tau]\textit{ 2H-term}\rangle\langle\textit{1H-term }[\tau]\rangle;$
$\qquad \langle\textit{1H-term }[-\tau]\rangle ::= s_{-\tau};$
$\qquad \langle[-\tau]\textit{ 2H-term}\rangle ::= o_0^{-\tau}|o_1^{-\tau}|\ldots|o_{\zeta_{-\tau}}^{-\tau};$
$\qquad \langle\textit{1H-term }[-\tau+1]\rangle ::= s_{-\tau+1};$
$\qquad \langle[-\tau+1]\textit{ 2H-term}\rangle ::= o_{-\zeta_{-\tau+1}}^{-\tau+1}|o_{-\zeta_{-\tau+1}+1}^{-\tau+1}|\ldots|o_{\zeta_{-\tau+1}}^{-\tau+1};$
$\qquad \vdots$
$\qquad \langle\textit{1H-term }[\tau-1]\rangle ::= s_{\tau-1};$
$\qquad \langle[\tau-1]\textit{ 2H-term}\rangle ::= o_{-\zeta_{\tau-1}}^{\tau-1}|o_{-\zeta_{\tau-1}+1}^{\tau-1}|\ldots|o_{\zeta_{\tau-1}}^{\tau-1};$
$\qquad \langle\textit{1H-term }[\tau]\rangle ::= s_\tau;$
$\qquad \langle[\tau]\textit{ 2H-term}\rangle ::= o_{-\zeta_\tau}^{\tau}|o_{-\zeta_\tau+1}^{\tau}|\ldots|o_0^{\tau};$
$\qquad \langle\textit{unary relation}\rangle ::= \textit{at least} \mid \textit{at most};$
$\qquad \langle\textit{binary relation}\rangle ::= \textit{between};$
$\qquad \langle\textit{conjunction}\rangle ::= \textit{and}\}.$

The language generated by the context-free grammar Γ_H, $L(\Gamma_H)$, defines the set of expressions that can be used by the decision makers to provide their assessments that are later translated into FDHHFLEs by means of the following transformation function:

Definition 5.10. Let $\mathcal{S}_{\mathcal{O}}^{\mathcal{F}} = \{s_{t<o_k^t>} \mid t = -\tau, \ldots, -1, 0, 1, \ldots, \tau; k = -\zeta_t, \ldots, -1, 0, 1, \ldots, \zeta_t\}$ be a FDHLTS and $L(\Gamma_H)$ be the language generated by Γ_H, then the *transformation function*

$$E_{\Gamma_H} : L(\Gamma_H) \longrightarrow \Phi \otimes \Psi$$

can be defined as:

- $E_{\Gamma_H}(o_k^t \, s_t) = \{s_{t<o_k^t>}\};$

- $E_{\Gamma_H}(\text{at least } o_k^t \, s_t) = \begin{cases} \{s_{t<o_k^t>}, s_{t+1}, s_{t+2}, \ldots, s_\tau\} & \text{if } t < \tau, \\ \{s_{t<o_k^t>}, s_{t<o_{k+1}^t>}, \ldots, s_{t<o_0^t>}\} & \text{if } t = \tau; \end{cases}$

- $E_{\Gamma_H}(\text{at most } o_k^t \, s_t) = \begin{cases} \{s_{-\tau}, s_{-\tau+1}, \ldots, s_{t-1}, s_{t<o_k^t>}\} & \text{if } t > -\tau, \\ \{s_{t<o_0^t>}, s_{t<o_1^t>}, \ldots, s_{-t<o_k^t>}\} & \text{if } t = -\tau; \end{cases}$

- $E_{\Gamma_H}(\text{between } o_{k_1}^{t_1} \, s_{t_1} \text{ and } o_{k_2}^{t_2} \, s_{t_2}) = E_{\Gamma_H}(\text{between } o_{k_2}^{t_2} \, s_{t_2} \text{ and } o_{k_1}^{t_1} \, s_{t_1}) =$

$$= \begin{cases} \{s_{t_1<o_{k_1}^{t_1}>}, s_{t_1+1}, \ldots, s_{t_2-1}, s_{t_2<o_{k_2}^{t_2}>}\} & \text{if} \quad t_1 < t_2, \\ \{s_{t_2<o_{k_2}^{t_2}>}, s_{t_2+1}, \ldots, s_{t_1-1}, s_{t_1<o_{k_1}^{t_1}>}\} & \text{if} \quad t_2 < t_1, \\ \{s_{t_1<o_{k_1}^{t_1}>}, s_{t_1<o_{k_1+1}^{t_1}>}, \ldots, s_{t_1<o_{k_2}^{t_1}>}\} & \text{if} \quad (t_1 = t_2) \wedge (k_1 < k_2), \\ \{s_{t_1<o_{k_2}^{t_1}>}, s_{t_1<o_{k_2+1}^{t_1}>}, \ldots, s_{t_1<o_{k_1}^{t_1}>}\} & \text{if} \quad (t_1 = t_2) \wedge (k_2 < k_1), \\ \{s_{t_1<o_{k_1}^{t_1}>}\} & \text{if} \quad (t_1 = t_2) \wedge (k_1 = k_2). \end{cases}$$

Finally, the concept of envelope can be generalized for FDHHFLEs as follows:

Definition 5.11. Let $h_{\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}}$ be a FDHHFLE by means of the FDHFLTS $\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}$, then the *envelope of* $h_{\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}}$, $env(h_{\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}})$, is defined as a double hierarchy linguistic interval whose limits are the lower and upper bounds of $h_{\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}}$, $h_{\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}}^-$ and $h_{\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}}^+$ respectively. Thus,

$$env(h_{\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}}) = [h_{\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}}^-, h_{\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}}^+].$$

Example 5.3. Considering the FDHLTS, $\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}$, from Example 5.2, some of the possible linguistic assessments are "*at least very good*", "*between slightly bad and pretty good*", "*between very good and extremely good*" and "*almost null*". By Definitions 5.9, 5.10, and 5.11, the associated FDHHFLEs and their respective envelopes can be obtained as follows:

Linguistic assessment	*FDHHFLE*	*envelope*
"*at least very good*"	$h_1 = \{s_{1<o_1^1>}, s_2\}$	$[s_{1<o_1^1>}, s_{2<o_0^2>}]$
"*between slightly bad and pretty good*"	$h_2 = \{s_{-1<o_2^{-1}>}, s_0, s_{1<o_{-1}^1>}\}$	$[s_{-1<o_2^{-1}>}, s_{1<o_{-1}^1>}]$
"*between very good and extremely good*"	$h_3 = \{s_{1<o_1^1>}, s_{1<o_2^1>}, s_{1<o_3^1>}\}$	$[s_{1<o_1^1>}, s_{1<o_3^1>}]$
"*almost null*"	$h_4 = \{s_{-2<o_1^{-2}>}\}$	$[s_{-2<o_1^{-2}>}, s_{-2<o_1^{-2}>}]$

Furthermore, we can define some operations between FDHHFLEs that are used in Section 5.4.2 to present a distance between FDHHFLEs.

Definition 5.12. Let $\mathcal{S}_{\mathcal{O}}^{\mathcal{F}} = \{s_{t<o_k^t>} \mid t = -\tau, \ldots, -1, 0, 1, \ldots, \tau; k = -\zeta_t, \ldots, -1,$
$0, 1, \ldots, \zeta_t\}$ be a FDHLTS, and h_1 and h_2 be two FDHHFLEs by means of $\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}$ whose
envelopes are $[s_{t_1^-<o_{k_1^-}^{t_1^-}>}, s_{t_1^+<o_{k_1^+}^{t_1^+}>}]$ and $[s_{t_2^-<o_{k_2^-}^{t_2^-}>}, s_{t_2^+<o_{k_2^+}^{t_2^+}>}]$ respectively. Then:

- The *connected union of h_1 and h_2*, $h_1 \sqcup h_2$, is defined as:

$$h_1 \sqcup h_2 = \{s_{t<o_k^t>} \in \mathcal{S}_{\mathcal{O}}^{\mathcal{F}} \mid$$
$$\min\{s_{t_1^-<o_{k_1^-}^{t_1^-}>}, s_{t_2^-<o_{k_2^-}^{t_2^-}>}\} \preccurlyeq s_{t<o_k^t>} \preccurlyeq \max\{s_{t_1^+<o_{k_1^+}^{t_1^+}>}, s_{t_2^+<o_{k_2^+}^{t_2^+}>}\}\}.$$

- The *intersection of h_1 and h_2*, $h_1 \cap h_2$, is defined as:

$$h_1 \cap h_2 = \{s_{t<o_k^t>} \in \mathcal{S}_{\mathcal{O}}^{\mathcal{F}} \mid$$
$$\max\{s_{t_1^-<o_{k_1^-}^{t_1^-}>}, s_{t_2^-<o_{k_2^-}^{t_2^-}>}\} \preccurlyeq s_{t<o_k^t>} \preccurlyeq \min\{s_{t_1^+<o_{k_1^+}^{t_1^+}>}, s_{t_2^+<o_{k_2^+}^{t_2^+}>}\}\}.$$

- The *gap between h_1 and h_2*, $h_1 \lozenge h_2$, is defined as:

$$h_1 \lozenge h_2 = \{s_{t<o_k^t>} \in \mathcal{S}_{\mathcal{O}}^{\mathcal{F}} \mid$$
$$\min\{s_{t_1^+<o_{k_1^+}^{t_1^+}>}, s_{t_2^+<o_{k_2^+}^{t_2^+}>}\} \prec s_{t<o_k^t>} \prec \max\{s_{t_1^-<o_{k_1^-}^{t_1^-}>}, s_{t_2^-<o_{k_2^-}^{t_2^-}>}\}\}.$$

Notice that, extending $\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}$ with a hypothetical empty FDHHFLE, these three operations between FDHHFLEs are closed operations. Instead, the ordinary union of sets has not been considered because of the fact that it is not a closed operation given that its elements do not have to be the continuous FDHLTs.

Remark 5.4. For two given FDHHFLEs, h_1 and h_2, if $h_1 \cap h_2 \neq \varnothing$, then $h_1 \lozenge h_2 = \varnothing$ and vice versa. On the contrary, the reciprocals are not true. That is to say that the intersection and the gap between two FDHHFLEs can be both empty at the same time but they cannot be non-empty simultaneously.

Example 5.4. Following Example 5.3, we can obtain the connected union, intersection and gap of each pair of FDHHFLEs. As an example:

(h_i, h_j)	$h_i \sqcup h_j$	$h_i \cap h_j$	$h_i \lozenge h_j$
(h_1, h_2)	$\{s_{-1<o_2^{-1}>}, s_0, s_1, s_2\}$	$\varnothing$	$\{s_{1<o_0^1>}\}$
(h_1, h_3)	$\{s_{1<o_1^1>}, s_2\}$	$\{s_{1<o_1^1>}, s_{1<o_2^1>}, s_{1<o_3^1>}\}$	$\varnothing$

5.3.4 From the Discrete Version to the Continuous One

As it has been done with HFLEs and DHHFLEs in previous approaches, FDHHFLEs can also be extended from their discrete definition to a continuous one. There are

several ways in which this extension can be done, but in this chapter we focus on a continuation based on the LTS corresponding to the second hierarchy similar to what has been done by Gou *et al.* for DHHFLEs [21]. Thus, the general form of a continuous FDHHFLE is

$$h_{\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}}(x_i) = \{s_{\phi_l<o_{\varphi_l}^{\phi_l}>}(x_i) \mid s_{\phi_l<o_{\varphi_l}^{\phi_l}>} \in \mathcal{S}_{\mathcal{O}}^{\mathcal{F}}; l = 1, \ldots, L_i;$$
$$\phi_l \in \{-\tau, \ldots, -1, 0, 1, \ldots, \tau\}; \varphi_l \in [-\zeta_{\phi_l}, \zeta_{\phi_l}]\},$$

with the corresponding extension of the continuous FDHLTS as:

$$\mathcal{S}_{\mathcal{O}}^{\mathcal{F}} = \{s_{t<o_k^t>} \mid t = -\tau, \ldots, -1, 0, 1, \ldots, \tau; k \in = [-\zeta_t, \zeta_t]\}.$$

Note that the same graphical representation presented in Section 5.3.3 is useful for the continuous FDHHFLEs as well.

Thereupon, the corresponding transformation function between the continuous FDHHFLEs and HFEs can be defined as follows:

Definition 5.13. Let $\mathcal{S}_{\mathcal{O}}^{\mathcal{F}} = \{s_{t<o_k^t>} \mid t = -\tau, \ldots, -1, 0, 1, \ldots, \tau; k = -\zeta_t, \ldots, -1, 0, 1, \ldots, \zeta_t\}$ be a FDHLTS, $h_{\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}} = \{s_{\phi_l<o_{\varphi_l}^{\phi_l}>} \mid s_{\phi_l<o_{\varphi_l}^{\phi_l}>} \in \mathcal{S}_{\mathcal{O}}^{\mathcal{F}}; l = 1, \ldots, L; \phi_l \in \{-\tau, \ldots, \tau\}; \varphi_l \in [-\zeta_{\phi_l}, \zeta_{\phi_l}]\}$ be a continuous FDHHFLE containing L linguistic terms in it and $h_\gamma = \{\gamma_l \mid \gamma_l \in [0, 1]; l = 1, \ldots, L\}$ be a HFE. Let $\zeta = \max\{\zeta_0, \zeta_1, \ldots, \zeta_k\}$, then:

a) The *transformation functions* $f_{\mathcal{F}}$ and $f_{\mathcal{F}}^{-1}$ between each pair of subscripts (ϕ_l, φ_l) of the linguistic term $s_{\phi_l<o_{\varphi_l}^{\phi_l}>}$ and the membership degree γ_l that expresses the equivalent information are, respectively:

$$\cdot\; f_{\mathcal{F}}: \quad \{-\tau, \ldots, \tau\} \times [-\zeta, \zeta] \longrightarrow \qquad [0,1]$$
$$(\phi_l, \varphi_l) \longmapsto \quad f_{\mathcal{F}}(\phi_l, \varphi_l) \;=\; \gamma_l \;=$$

$$= \begin{cases} \dfrac{(\tau + \phi_l)}{2\tau} + \dfrac{\varphi_l}{2\zeta_{\phi_l}\tau} & \text{if } \varphi_l \in [-\zeta_{\phi_l}, \zeta_{\phi_l}], \\ \nexists & \text{if } \varphi_l \notin [-\zeta_{\phi_l}, \zeta_{\phi_l}], \end{cases}$$

$$\cdot\; f_{\mathcal{F}}^{-1}: \qquad [0,1] \qquad \longrightarrow \{-\tau, \ldots, \tau\} \times [-\zeta, \zeta]$$
$$\gamma_l \qquad \longmapsto \qquad f_{\mathcal{F}}^{-1}(\gamma_l) \;=\; (\phi_l, \varphi_l) \;=$$

$$= \begin{cases} \begin{bmatrix} (2\tau\gamma_l - \tau - 1, \zeta_{2\tau\gamma_l - \tau - 1}) \; (\text{only if } \gamma_l \neq 0) \\ \vee \\ (2\tau\gamma_l - \tau, 0) \\ \vee \\ (2\tau\gamma_l - \tau + 1, -\zeta_{2\tau\gamma_l - \tau + 1}) \, (\text{only if } \gamma_l \neq 1) \end{bmatrix} & \text{if } (2\tau\gamma_l - \tau) \in \mathbb{Z}, \\[1em] \begin{bmatrix} (\lfloor 2\tau\gamma_l - \tau \rfloor, (2\tau\gamma_l - \tau - \lfloor 2\tau\gamma_l - \tau \rfloor)\zeta_{\lfloor 2\tau\gamma_l - \tau \rfloor}) \\ \vee \\ (\lceil 2\tau\gamma_l - \tau \rceil, (2\tau\gamma_l - \tau - \lceil 2\tau\gamma_l - \tau \rceil)\zeta_{\lceil 2\tau\gamma_l - \tau \rceil}) \end{bmatrix} & \text{if } (2\tau\gamma_l - \tau) \notin \mathbb{Z}. \end{cases}$$

b) The *transformation functions* $F_{\mathcal{F}}$ and $F_{\mathcal{F}}^{-1}$ between the continuous FDHHFLE $h_{\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}}$ and the HFE h_{γ} are, respectively:

- $F_{\mathcal{F}} : \quad \Phi \otimes \Psi_{\mathcal{C}} \longrightarrow \qquad \Theta$
  $$h_{\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}} \longmapsto F_{\mathcal{F}}(h_{\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}}) = \{\gamma_l \mid \gamma_l = f_{\mathcal{F}}(\phi_l, \varphi_l)\} = h_{\gamma},$$

- $F_{\mathcal{F}}^{-1} : \quad \Theta \qquad \longrightarrow \quad \Phi \otimes \Psi_{\mathcal{C}}$
  $$h_{\gamma} \longmapsto F_{\mathcal{F}}^{-1}(h_{\gamma}) = \{s_{\phi_l < o_{\varphi_l}^{\phi_l}>} \mid (\phi_l, \varphi_l) = f_{\mathcal{F}}^{-1}(\gamma_l)\} = h_{\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}},$$

being $\Phi \otimes \Psi_{\mathcal{C}}$ the set of all possible continuous FDHHFLEs.

As stated in Remark 5.1, once again this transformation function is not a bijection, so $f_{\mathcal{F}}^{-1}$ is not uniquely defined. Figure 5.4a shows an example of two continuous FDHLTs sharing the same image according to the function $f_{\mathcal{F}}$. Thus, the expected value for DHHFLEs introduced by Gou *et al.* [21], that is based on the transformation function, presents some issues because of this reason. As an example, the linguistic assessments such as "*at least very good*" and "*between very good and extremely good*" from Example 5.3 would have the same expected value.

5.4 Order and Distance among Free Double Hierarchy Hesitant Fuzzy Linguistic Elements

In this section, we present two mathematical relations within the set of FDHHFLEs. On the one hand, an order is defined to allow linguistic assessments by means of FD-HHFLEs to be sorted. On the other hand, to capture the differences among opinions, a distance is presented.

5.4.1 An Order among Free Double Hierarchy Hesitant Fuzzy Linguistic Elements

The presented order is based on the idea of expected value of a linguistic assessment. As stated at the end of Section 5.3, the transformation function and the expected value introduced by Gou *et al.*present some issues [21]. Hence, in order to define a new definition of expected value that gets rid of these issues, we propose a new transformation function between FDHLTSs and the continuous interval [0,1], similar to a cumulative area function.

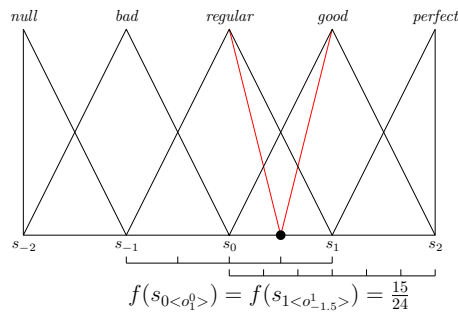**Definition 5.14.** Let $\mathcal{S}_{\mathcal{O}}^{\mathcal{F}} = \{s_{t < o_k^t>} \mid t = -\tau, \ldots, -1, 0, 1, \ldots, \tau; k = -\zeta_t, \ldots, -1, 0, 1, \ldots, \zeta_t\}$ be a FDHLTS, then, the *cumulative function, $\mathcal{A}$,* maps the continuous

FDHLTSs onto $[0, 1]$ as follows:

$$\mathcal{A}: \quad \mathcal{S}_{\mathcal{O}}^{\mathcal{F}} \quad \longrightarrow \quad [0, 1]$$

$$s_{t<o_k^t>} \quad \longmapsto \quad \mathcal{A}(s_{t<o_k^t>}) = \begin{cases} (t + \tau) \cdot \dfrac{1}{2\tau} + \dfrac{k/(2\tau\zeta_t)}{2} & \text{if } \zeta_t \neq 0, \\ (t + \tau) \cdot \dfrac{1}{2\tau} & \text{if } \zeta_t = 0. \end{cases}$$

Note that the function $\mathcal{A}$ is indeed a bijection, so both the images and the preimages of the function are uniquely defined. As shown in Figures 5.4b and 5.4c, the two FDHLTs that in Figure 5.4a have the same image for the function $f_{\mathcal{F}}$, they have different images for the function $\mathcal{A}$.

(A) Possible FDHLTs for $f_{\mathcal{F}}^{-1}(\frac{15}{24})$.
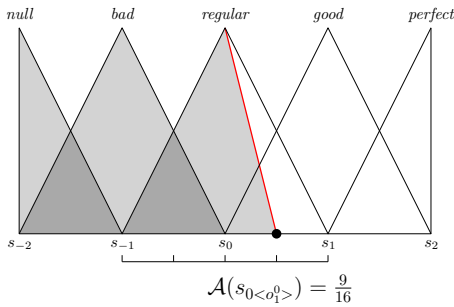
(B) Cumulative area up to $s_{0<o_1^0>}$.
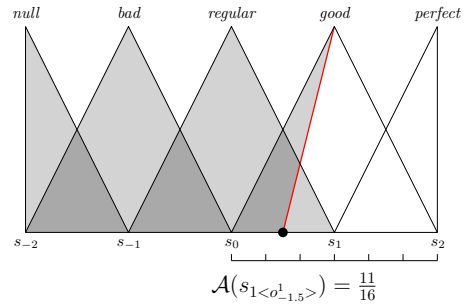
(C) Cumulative area up to $s_{1<o_{-1.5}^1>}$.

FIGURE 5.4: FDHLTs with the same value of $f_{\mathcal{F}}$ but different values of $\mathcal{A}$.

The function $\mathcal{A}$ can also be used to introduce a new measure of hesitancy of a FDHHFLE as follows:

Definition 5.15. Let $\mathcal{S}_{\mathcal{O}}^{\mathcal{F}} = \{s_{t<o_k^t>} \mid t = -\tau, \ldots, -1, 0, 1, \ldots, \tau; k = -\zeta_t, \ldots, -1, 0, 1, \ldots, \zeta_t\}$ be a FDHLTS and $h_{\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}}$ be a continuous FDHHFLE by means of $\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}$ whose envelope is $[s_{t^-<o_{k^-}^{t^-}>}, s_{t^+<o_{k^+}^{t^+}>}]$, then, the *area of* $h_{\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}}$ is defined as:

$$Area(h_{\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}}) = \mathcal{A}(s_{t^+<o_{k^+}^{t^+}>}) - \mathcal{A}(s_{t^-<o_{k^-}^{t^-}>}).$$

If only the discrete FDHHFLEs are used, which is the common case in any GDM problem, then a continuity correction factor has to be applied, and then:

$$
Area(h_{\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}}) = \begin{cases}
\delta[\mathcal{A}(s_{t<o_{k+\frac{1}{2}}^t>}), \frac{1/2}{2\tau}] - & \\
\quad \delta[\mathcal{A}(s_{t<o_{k-\frac{1}{2}}^t>}), 0] & \text{if} \qquad t = -\tau, \\
& \\
\delta[\mathcal{A}(s_{t<o_{k+\frac{1}{2}}^t>}), \frac{t+\tau+1/2}{2\tau}] - & \\
\quad \delta[\mathcal{A}(s_{t<o_{k-\frac{1}{2}}^t>}), \frac{t+\tau-1/2}{2\tau}] & \text{if} \quad t = -\tau+1, \dots, \tau-1, \\
& \\
\delta[\mathcal{A}(s_{t<o_{k+\frac{1}{2}}^t>}), 1] - & \\
\quad \delta[\mathcal{A}(s_{t<o_{k-\frac{1}{2}}^t>}), \frac{2\tau-1/2}{2\tau}] & \text{if} \qquad t = \tau,
\end{cases}
$$

where $\delta[a, b]$ takes as value a, if it exists, or b if a does not exist.

Additionally, for convenience of future uses of the function *Area*, we extend this definition to a hypothetical empty FDHHFLE, $\emptyset$, as $Area(\emptyset) = 0$.

For an easier understanding of this continuity correction factor, Figure 5.5 shows a graphical representation of the areas of the FDHHFLEs from Example 5.3. Notice that the FDHHFLEs corresponding to the assessments "*at least very good*" and "*between very good and extremely good*" (Figures 5.5a and 5.5c respectively) have different areas. Thus, whatever it is defined based on the area will show different results for this two assessments.
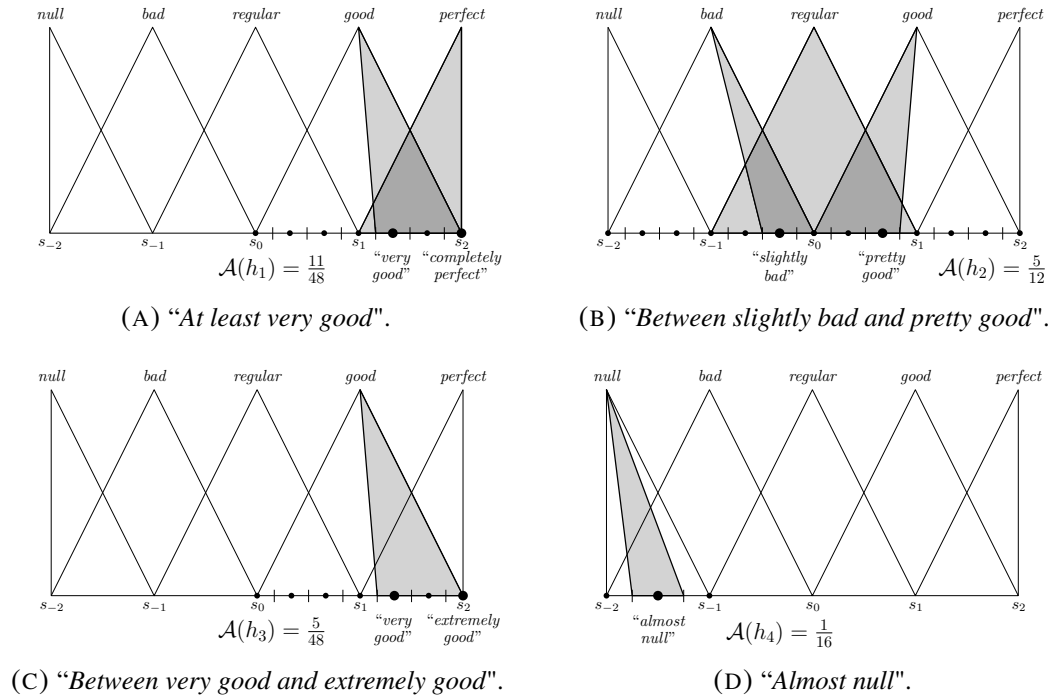

(A) "*At least very good*".

(B) "*Between slightly bad and pretty good*".

(C) "*Between very good and extremely good*".

(D) "*Almost null*".

FIGURE 5.5: Graphical representation of the FDHHFLEs from Example 5.3.

Hence, based on the definition of the *area* of a FDHHFLE, we can now propose a new definition for the expected value of a FDHHFLE that fixes the issue presented by the one proposed by Gou *et al.* [21].

Definition 5.16. Let $\mathcal{S}_{\mathcal{O}}^{\mathcal{F}} = \{s_{t<o_k^t>} \mid t = -\tau, \ldots, -1, 0, 1, \ldots, \tau; k = -\zeta_t, \ldots, -1, 0, 1, \ldots, \zeta_t\}$ be a FDHLTS and $h_{\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}}$ be a continuous FDHHFLE by means of $\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}$ whose envelope is $[s_{t^-<o_{k^-}^{t^-}>}, s_{t^+<o_{k^+}^{t^+}>}]$, then, the *expected value of* $h_{\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}}$ is defined as:

$$E(h_{\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}}) = \frac{\mathcal{A}(s_{t^+<o_{k^+}^{t^+}>}) + \mathcal{A}(s_{t^-<o_{k^-}^{t^-}>})}{2}.$$

If only the discrete FDHHFLEs are used, then the continuity correction factor has to be applied again, resulting in:

$$E(h_{\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}}) = \begin{cases} \frac{1}{2}\left(\delta[\mathcal{A}(s_{t<o_{k+\frac{1}{2}}^t>}), \frac{1/2}{2\tau}] + \right. & \\ \left. \qquad \delta[\mathcal{A}(s_{t<o_{k-\frac{1}{2}}^t>}), 0] \right) & \text{if} \qquad t = -\tau, \\[2em] \frac{1}{2}\left(\delta[\mathcal{A}(s_{t<o_{k+\frac{1}{2}}^t>}), \frac{t+\tau+1/2}{2\tau}] + \right. & \\ \left. \qquad \delta[\mathcal{A}(s_{t<o_{k-\frac{1}{2}}^t>}), \frac{t+\tau-1/2}{2\tau}] \right) & \text{if} \quad t = -\tau+1, \ldots, \tau-1, \\[2em] \frac{1}{2}\left(\delta[\mathcal{A}(s_{t<o_{k+\frac{1}{2}}^t>}), 1] + \right. & \\ \left. \qquad \delta[\mathcal{A}(s_{t<o_{k-\frac{1}{2}}^t>}), \frac{2\tau-1/2}{2\tau}] \right) & \text{if} \qquad t = \tau. \end{cases}$$

Finally, the expected value of a FDHHFLE can be used to introduce an order within the set $\Phi \otimes \Psi$ as follows:

Definition 5.17. Let $h_{\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}}^1$ and $h_{\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}}^2$ be two FDHHFLEs of $\Phi \otimes \Psi$, then we define:

$$h_{\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}}^1 \circledotimes h_{\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}}^2 \iff \begin{cases} E(h_{\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}}^1) < E(h_{\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}}^2) \\ \vee \\ E(h_{\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}}^1) = E(h_{\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}}^2) \quad \wedge \quad Area(h_{\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}}^1) \geq Area(h_{\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}}^2), \end{cases}$$

with the equality satisfied only when

$$E(h_{\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}}^1) = E(h_{\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}}^2) \quad \wedge \quad Area(h_{\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}}^1) = Area(h_{\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}}^2).$$

Let us present the following example to clarify the foregoing concepts:

Example 5.5. Following Example 5.3, the area and the expected value of each assessment can be calculated by Definitions 5.15 and 5.16. For instance,

$$Area(h_1) = 1 - \frac{37}{48} = \frac{11}{48} \quad \text{and} \quad E(h_1) = \frac{1 + 37/48}{2} = \frac{85}{96}.$$

For the rest of assessments, the calculations can be proceeded in an analogous way leading to the following results:

Linguistic assessment	FDHHFLE	Area	Expected value
"at least very good"	$h_1 = \{s_{1<o_1^1>}, s_2\}$	11/48	85/96
"between slightly bad and pretty good"	$h_2 = \{s_{-1<o_2^{-1}>}, s_0, s_{1<o_{-1}^1>}\}$	5/12	25/48
"between very good and extremely good"	$h_3 = \{s_{1<o_1^1>}, s_{1<o_2^1>}, s_{1<o_3^1>}\}$	5/48	79/96
"almost null"	$h_4 = \{s_{-2<o_1^{-2}>}\}$	1/16	1/16

Therefore, the four assessments can be sorted as $h_4 \otimes h_2 \otimes h_3 \otimes h_1$.

Remark 5.5. The best possible FDHHFLE according to the order from Definition 5.17 is $\{s_{\tau<o_0^\tau>}\}$. Note that $E(\{s_{\tau<o_0^\tau>}\})$ is not exactly 1 because of the continuity correction factor. In fact, the more elements there are in the second hierarchy LTS for s_τ, the closer to 1 $E(\{s_{\tau<o_0^\tau>}\})$ is. This makes sense because the more you know about a certain topic the more precise you can be, so you can choose a larger second hierarchy LTS and the alternatives assessed with $\{s_{\tau<o_0^\tau>}\}$ are closer to *perfection*. On the contrary, if someone is not very comfortable assessing a specific topic, he or she can choose a less precise second hierarchy LTS and, therefore, the alternatives assessed with $\{s_{\tau<o_0^\tau>}\}$ are not necessarily that close to *perfection*, so $E(\{s_{\tau<o_0^\tau>}\})$ is farther away from 1. Analogously, the same thing happens with the worst possible FDHHFLE, $\{s_{\tau<o_0^\tau>}\}$, whose expected value is not exactly 0.

5.4.2 A Distance between Free Double Hierarchy Hesitant Fuzzy Linguistic Elements

Based on the distance between HFLTSs introduced by Montserrat-Adell *et al.* [47], the following distance between FDHHFLEs is proposed:

Proposition 5.1. Let $\mathcal{S}_\mathcal{O}^\mathcal{F}$ be a FDHLTS, and $\Phi \otimes \Psi$ be the set of all possible FDHFHLEs by means of $\mathcal{S}_\mathcal{O}^\mathcal{F}$. Then,

$$D(h_1, h_2) = Area(h_1 \sqcup h_2) - Area(h_1 \cap h_2) + Area(h_1 \between h_2)$$

defines a distance in $\Phi \otimes \Psi$, where h_1 and h_2 are two FDHHFLEs.

Proof. In order to ease the reading of this chapter, the proof of this property is provided in Appendix 5.A at the end of the chapter. $\square$

This distance is based on the classical distance provided by the difference between the union and the intersection of two elements with an essential change. This variation is the fact that, in case that the two considered elements have an empty intersection, then the gap between them is also taken into consideration to compute the distance between them. This gap is used as a measure of how far from coinciding these opinions are.

Remark 5.6. Note that the two most distant FDHHFLEs are $\{s_{-\tau<o_0^{-\tau}>}\}$ and $\{s_{\tau<o_0^{\tau}>}\}$. In this case, $d(\{s_{-\tau<o_0^{-\tau}>}\}, \{s_{\tau<o_0^{\tau}>}\}) = 1 + (1 - \epsilon_{-\tau} - \epsilon_{\tau})$, being $\epsilon_{-\tau}$ and ϵ_{τ} the areas of $\{s_{-\tau<o_0^{-\tau}>}\}$ and $\{s_{\tau<o_0^{\tau}>}\}$, which depend on $\zeta_{-\tau}$ and ζ_{τ} respectively. In fact, as stated in Remark 5.5, the larger $\zeta_{-\tau}$ and ζ_{τ} are, the smaller $\epsilon_{-\tau}$ and ϵ_{τ} will be. Consequently, the supremum value for this distance is 2.

Let us illustrate the foregoing distance with an example as follows:

Example 5.6. Following Examples 5.3 and 5.4, we can graphically represent the connected unions, intersections and gaps of the FDHHFLEs h_1 and h_2 and between h_1 and h_3 as in Figure 5.6.

(A) $h_1 \sqcup h_2$ and $h_1 \between h_2$.　　　　(B) $h_1 \sqcup h_3$ and $h_1 \cap h_3$.
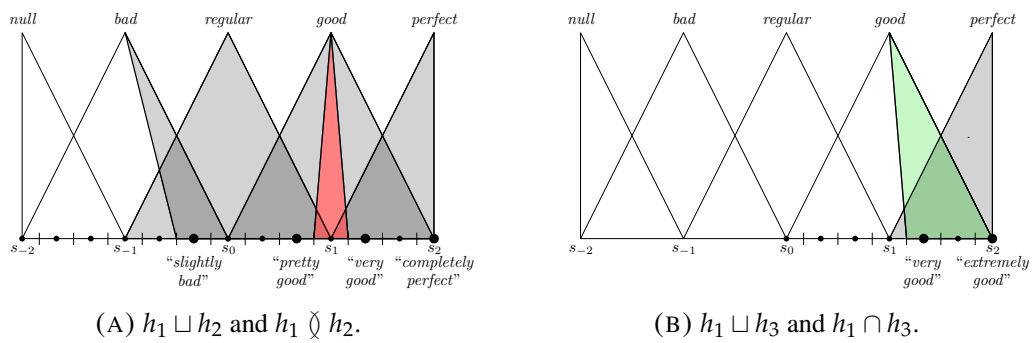
FIGURE 5.6: Connected unions (all shaded regions), intersections (green) and gaps (red) from Example 5.6.

Now, we just need to calculate the respective areas in order to get the distance value:

(h_i, h_j)	$Area(h_i \sqcup h_j)$	$Area(h_i \cap h_j)$	$Area(h_i \between h_j)$	$D(h_i, h_j)$
(h_1, h_2)	$11/16$	0	$1/24$	$35/48$
(h_1, h_3)	$11/48$	$5/48$	0	$1/8$

After introducing an order and a distance between FDHHFLEs, we can now proceed to present a free double hierarchy approach based on the Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) method.

5.5 Free Double Hierarchy Hesitant Fuzzy Linguistic TOPSIS Approach

In this section, we present a free double hierarchy approach based on the well-known TOPSIS method to rank alternatives in a GDM problem. Among all the

multiple-criteria decision-making ranking methods, we have chosen to use the TOP-SIS method because it is known to be useful for those problems in which the evaluations of the alternatives are given in different units and magnitudes. In this case, each DM can use a different second hierarchy LTS, which could lead us to a similar situation than working with different magnitudes. To illustrate the usefulness of this approach, a simulated example on tourism management in Barcelona is also presented.

5.5.1 Free Double Hierarchy Hesitant Fuzzy Linguistic TOPSIS Methodology

A GDM problem with free double hierarchy hesitant fuzzy linguistic information can be described as follows: Let $A = \{A_1, A_2, \ldots, A_m\}$ be a set of alternatives that have to be assessed by a set of experts $E = \{E_1, E_2, \ldots, E_n\}$ by means of FDHHFLEs. Let $S = \{s_t \mid t = -\tau, \ldots, \tau\}$ be a LTS used as a common first hierarchy LTS, and let each decision maker choose the second hierarchy LTS that he or she prefers to use for each linguistic term in S. Let $\Gamma_H^1, \Gamma_H^2, \ldots, \Gamma_H^n$ be the context-free grammars generated by the the second hierarchy LTSs chosen by each decision maker. Then, each expert, E_i, assesses all the alternatives in A by means of linguistic expressions from the context-free grammar Γ_H^i. The aim of the Free Double Hierarchy Hesitant Fuzzy Linguistic TOPSIS (FDHHFL-TOPSIS) approach that we will develop below is to rank the alternatives taking into account the opinions of all the decision makers.

To this end, the following steps have to be followed:

1. We express as FDHHFLEs all the linguistic assessments given by the experts to evaluate the alternatives and create the following decision-making matrix:

$$M = \begin{pmatrix} h_{11} & h_{12} & \ldots & h_{1n} \\ h_{21} & h_{22} & \ldots & h_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ h_{m1} & h_{m2} & \ldots & h_{mn} \end{pmatrix},$$

where h_{ij} stands for the FDHHFLE that corresponds to the linguistic assessment used by the expert E_j to assess the alternative A_i.

2. For each decision maker, E_j, we find his or her best and worst assessments according the order provided in Definition 5.17:

$$h_j^+ \quad \text{and} \quad h_j^-.$$

3. For each alternative A_i, we calculate the distance between the FDHHFLE used by E_j to evauate it and h_j^+ and h_j^-, using the distance from Proposition 5.1:

$$D_{ij}^+ = D(h_{ij}, h_i^+) \quad \text{and} \quad D_{ij}^- = D(h_{ij}, h_i^-).$$

4. For each alternative, A_i, we add the distances between each expert's assessment of A_i and his or her best assessment and repeat the same with the distances to the worst assessment:

$$D_i^+ = \sum_{j=1}^{n} D_{ij}^+ \quad \text{and} \quad D_i^- = \sum_{j=1}^{n} D_{ij}^-.$$

5. For each alternative, A_i, we find its similarity to an ideal solution as:

$$S_i = \frac{D_i^-}{D_i^+ + D_i^-},$$

which is obviously a value between 0 and 1. It takes the value 1 only when $D_i^+ = 0$, which means that the alternative A_i gets the best assessment from all the experts, and it takes the value 0 only when $D_i^+ = 0$, which means that A_i gets every expert's worst assessment.

6. The alternatives in A can be sorted according to their value of S_i, being the alternative with the largest S_i the highest ranked one.

This approach can be applied to any GDM problem with the aim of ranking alternatives as long as the assessments of the decision makers are provided as free double hierarchy hesitant fuzzy linguistic information. The usefulness of this structure is that it allows the experts to be more precise using their own words, given that they can use the second hierarchy that they prefer.

5.5.2   A Simulated Example on Tourism Management in Barcelona

In this section, we apply the FDHHFL-TOPSIS method presented in 5.5.1 into a simulated practical GDM problem involving tourist attractions in the city of Barcelona.

In recent years, tourism has increased a lot in Barcelona and it has become a trending topic of discussion due to the skepticism of some residents about how beneficial this increase of tourism is for Barcelona inhabitants and for the city itself. Because of this, a new phenomenon called *tourismphobia* has arisen in some sectors of the city.

Frequently, what is good for tourist does not coincide with what is good for residents and this non-coincidence leads to a debate on what to prioritize. This confrontation between locals and tourists' interests turns specially intense in areas surrounding some of the most visited tourist attractions in the city.

Hence, tourism management policies carried out by the different entities that take care of the tourist attractions are extremely important to find a balance between the great number of tourists and their needs, such as restaurants, guided tours, souvenirs shops or going out places among others, and the daily life of residents and their serenity, specially at night. For this reason, the city council is interested into knowing which of the current policies are giving better results.

In order to evaluate these policies, a member of the city council, a spokesperson of neighborhood associations, a manager of the tourist agencies labor union and a representative of the tertiary sector (E_1, E_2, E_3 and E_4 respectively) are asked to assess the management of five of the most famous attractions of the city: Sagrada Familia, Camp Nou, Ciutat Vella neighborhood, Park Güell and Tibidabo (A_1, A_2, A_3, A_4 and A_5 respectively). The aim is to aggregate the opinion of the four experts to determine which tourist attractions are carrying out better policies.

To provide these assessments, the experts are asked to use a common scale $\mathcal{S} = \{s_{-2} = \text{"null"}, s_{-1} = \text{"bad"}, s_0 = \text{"regular"}, s_1 = \text{"good"}, s_2 = \text{"perfect"}\}$ as a first hierarchy LTS, but they are allowed to choose a second hierarchy LTS in their own words, to express themselves as they feel more comfortable, for each of the labels of the first hierarchy. Thus, for instance, while the city council member (E_1) chooses to evaluate with the same second hierarchy than in Example 5.2, the neighborhood associations spokesperson (E_2) prefers to not use a second hierarchy and assess the alternatives just with the linguistic labels in $\mathcal{S}$. Table 5.1 shows the second hierarchy LTSs chosen by each expert for each linguistic term in $\mathcal{S}$.

TABLE 5.1: Second hierarchy LTSs used by each expert.

	E_1	E_2	E_3	E_4
s_{-2}	$o_0^{-2} = $ "completely" $o_1^{-2} = $ "almost" $o_2^{-2} = $ "close to"	$o_0^{-2} = $ " "	$o_0^{-2} = $ "totally" $o_1^{-2} = $ "roughly"	$o_0^{-2} = $ "totally" $o_1^{-2} = $ "roughly"
s_{-1}	$o_{-3}^{-1} = $ "extremely" $o_{-2}^{-1} = $ "unusually" $o_{-1}^{-1} = $ "very" $o_0^{-1} = $ "simply" $o_1^{-1} = $ "pretty" $o_2^{-1} = $ "slightly" $o_3^{-1} = $ "hardly"	$o_0^{-1} = $ " "	$o_{-3}^{-1} = $ "extremely" $o_{-2}^{-1} = $ "unusually" $o_{-1}^{-1} = $ "very" $o_0^{-1} = $ "simply" $o_1^{-1} = $ "pretty" $o_2^{-1} = $ "slightly" $o_3^{-1} = $ "hardly"	$o_{-1}^{-1} = $ "very" $o_0^{-1} = $ "simply" $o_1^{-1} = $ "slightly"
s_0	$o_{-2}^0 = $ "very low" $o_{-1}^0 = $ "low" $o_0^0 = $ "medium" $o_1^0 = $ "high" $o_2^0 = $ "very high"	$o_0^0 = $ " "	$o_0^0 = $ " "	$o_{-1}^0 = $ "lower" $o_0^0 = $ "middle" $o_1^0 = $ "upper"
s_1	$o_{-3}^1 = $ "hardly" $o_{-2}^1 = $ "slightly" $o_{-1}^1 = $ "pretty" $o_0^1 = $ "simply" $o_1^1 = $ "very" $o_2^1 = $ "unusually" $o_3^1 = $ "extremely"	$o_0^1 = $ " "	$o_{-3}^1 = $ "hardly" $o_{-2}^1 = $ "slightly" $o_{-1}^1 = $ "pretty" $o_0^1 = $ "simply" $o_1^1 = $ "very" $o_2^1 = $ "unusually" $o_3^1 = $ "extremely"	$o_{-1}^1 = $ "slightly" $o_0^1 = $ "simply" $o_1^1 = $ "very"
s_2	$o_{-2}^2 = $ "close to" $o_{-1}^2 = $ "almost" $o_0^2 = $ "completely"	$o_0^2 = $ " "	$o_{-1}^2 = $ "roughly" $o_0^2 = $ "totally"	$o_{-1}^2 = $ "roughly" $o_0^2 = $ "totally"

Later, each expert is given a list of criteria that he or she has to take into account when evaluating the different alternatives, such as price, accessibility, noise or influx among others. For this reason, since several criteria have to be considered for the final assessment, then a hesitant result is allowed.

According to all the aforementioned arguments, the assessments gathered from the five experts are given as free double hierarchy hesitant fuzzy linguistic information and, therefore, the FDHHFL-TOPSIS method can be used to rank the alternatives according to the results achieved by the corresponding policies carried out in

each of the studied tourist attractions. Table 5.2 shows all the linguistic assessments provided by the each expert on each of the alternatives.

TABLE 5.2: Assessments given by the experts using a free double hierarchy.

	E_1	E_2	E_3	E_4
A_1	*"between very bad and slightly bad"*	*"at most regular"*	*"between regular and pretty good"*	*"between slightly good and roughly perfect"*
A_2	*"between high regular" and slightly good"*	*"at least good"*	*"hardly good"*	*"between simply bad" and upper regular"*
A_3	*"between close to null and very bad"*	*"null"*	*"between regular and slightly good"*	*"between lower regular and simply good"*
A_4	*"between pretty good" and extremely good"*	*"good"*	*"at least extremely good"*	*"at least very good'*
A_5	*"at least very good"*	*"perfect"*	*"regular"*	*"between roughly null" and slightly bad"*

Now we have to follow the steps described in Section 5.5.1 one by one to get the final ranking:

Step 1: To translate the linguistic assessments into FDHHFLEs to get the decision-making matrix M. Table 5.3 shows the resulting matrix M.

TABLE 5.3: FDHHFLEs corresponding to the assessments given by the experts.

	E_1	E_2	E_3	E_4
A_1	$[s_{-1<o_{-1}^{-1}>}, s_{-1<o_2^{-1}>}]$	$[s_{-2<o_0^{-2}>}, s_{0<o_0^0>}]$	$[s_{0<o_0^0>}, s_{1<o_{-1}^1>}]$	$[s_{1<o_{-1}^1>}, s_{2<o_{-1}^2>}]$
A_2	$[s_{0<o_1^0>}, s_{1<o_{-2}^1>}]$	$[s_{1<o_0^1>}, s_{2<o_0^2>}]$	$\{s_{1<o_{-3}^1>}\}$	$[s_{-1<o_0^{-1}>}, s_{0<o_1^0>}]$
A_3	$[s_{-2<o_2^{-2}>}, s_{-1<o_{-1}^{-1}>}]$	$\{s_{-2<o_0^{-2}>}\}$	$[s_{0<o_0^0>}, s_{1<o_{-2}^1>}]$	$[s_{0<o_{-1}^0>}, s_{1<o_0^1>}]$
A_4	$[s_{1<o_{-1}^1>}, s_{1<o_3^1>}]$	$\{s_{1<o_0^1>}\}$	$[s_{1<o_3^1>}, s_{2<o_0^2>}]$	$[s_{1<o_1^1>}, s_{2<o_0^2>}]$
A_5	$[s_{1<o_1^1>}, s_{2<o_0^2>}]$	$\{s_{2<o_0^2>}\}$	$\{s_{0<o_0^0>}\}$	$[s_{-2<o_1^{-2}>}, s_{-1<o_1^{-1}>}]$

Step 2: To find the best and worst assessments from each expert based on the order from Definition 5.7. Table 5.4 shows h_j^+ and h_j^- for each expert E_j.

TABLE 5.4: Best and worst assessments from each expert.

	E_1	E_2	E_3	E_4
h_j^+	$[s_{1<o_1^1>}, s_{2<o_0^2>}]$	$\{s_{2<o_0^2>}\}$	$[s_{1<o_3^1>}, s_{2<o_0^2>}]$	$[s_{1<o_1^1>}, s_{2<o_0^2>}]$
h_j^-	$[s_{-2<o_2^{-2}>}, s_{-1<o_{-1}^{-1}>}]$	$\{s_{-2<o_0^{-2}>}\}$	$\{s_{0<o_0^0>}\}$	$[s_{-2<o_1^{-2}>}, s_{-1<o_1^{-1}>}]$

Step 3: To calculate, for each alternative, the distance between the assessment of an expert and his or her best assessment and repeat the same process with the worst assessment, using the distance from Proposition 5.1. Table 5.5a shows the corresponding distances to the best assessment while Table 5.5b shows the distances to the worst assessment.

TABLE 5.5: Distances to best and worst assessments.

(A) Distances to best assessment.

A_i	D_{i1}^+	D_{i2}^+	D_{i3}^+	D_{i4}^+	D_i^+
A_1	1.2292	1.25	0.75	0.25	3.4792
A_2	0.5521	0.25	0.5833	1	2.3854
A_3	1.4479	1.75	0.7917	0.625	4.6146
A_4	0.2083	0.375	0	0	0.5833
A_5	0	0	0.8542	1.375	2.2292

(B) Distances to worst assessment.

A_i	D_{i1}^-	D_{i2}^-	D_{i3}^-	D_{i4}^+	D_i^-
A_1	0.2188	0.5	0.1042	1.125	1.9479
A_2	0.8958	1.5	0.2708	0.375	3.0417
A_3	0	0	0.0625	0.75	0.8125
A_4	1.2396	1.375	0.8542	1.375	4.8438
A_5	1.4479	1.75	0	0	3.1979

Step 4: To add, for each alternative, the distances from Step 3, for both the best and worst assessments respectively. Last column of Table 5.5a shows D_i^+ for each alternative A_i and so does Table 5.5b with D_i^-.

Step 5: To calculate the similarity degree of each alternative to an ideal solution. Table 5.6 shows these values for all of the alternatives.

TABLE 5.6: Similarities to ideal solution.

A_i	A_1	A_2	A_3	A_4	A_5
S_i	0.3589	0.5605	0.1497	0.8925	0.5893

Step 6: To rank the alternatives based on their similarity degrees from Step 5. The ranking results are as follows:

$$A_3 \lhd A_1 \lhd A_2 \lhd A_5 \lhd A_4.$$

Therefore, after applying the FDHHFL-TOPSIS method, we can conclude that the management of Park Güell is the one working better while Sagrada Familia and specially the neighborhood of Ciutat Vella, should work harder in order to find solutions to the relationship between tourists and residents.

Comparison with Different Linguistic TOPSIS Methods

To evaluate the effectiveness of the FDHHFL-TOPSIS methodology, we now solve the same GDM problem using the HFL-TOPSIS and the DHHFL-TOPSIS methodologies to compare the obtained results.

A) *Hesitant Fuzzy Linguistic TOPSIS*

In this case, each expert is asked to provide their assessments using a HFLE by means of $\mathcal{S} = \{s_{-2} = \text{``null''}, s_{-1} = \text{``bad''}, s_0 = \text{``regular''}, s_1 = \text{``good''}, s_2 = \text{``perfect''}\}$ without using any second hierarchy. Under these circumstances, the assessments provided by the experts are as shown in Table 5.7.

Then, following Step 1, these assessment have to be rewritten as HFLEs; by Step 2, the best and worst assessments have to be found; and by Steps 3 and 4, the distances between each alternative and the best and worst assessments have to be computed.

TABLE 5.7: Assessments given by the experts using a single hierarchy.

	E_1	E_2	E_3	E_4
A_1	*"bad"*	*"at most regular"*	*"between regular and good"*	*"between good and perfect"*
A_2	*"between regular" and good"*	*"at least good"*	*"good"*	*"between bad" and regular"*
A_3	*"between null and bad"*	*"null"*	*"between regular and good"*	*"between regular and good"*
A_4	*"good"*	*"good"*	*"at least good"*	*"at least good'*
A_5	*"at least good"*	*"perfect"*	*"regular"*	*"between null" and bad"*

In this case, since the assessments are given as HFLEs, we use the distance between HFLEs introduced by Montserrat-Adell *et al.* [47], in which the distance used for the FDHHFL-TOPSIS method is inspired. In addition, according to Step 5, the similarity degrees of each alternative to an ideal solution are calculated. All of these steps are shown in Table 5.8.

TABLE 5.8: Similarity degrees of each alternative to an ideal solution.

	E_1	E_2	E_3	E_4	D_i^+	D_i^-	S_i
A_1	$\{s_{-1}\}$	$[s_{-2},s_0]$	$[s_0,s_1]$	$[s_1,s_2]$	11	10	0.4762
A_2	$[s_0,s_1]$	$[s_1,s_2]$	$\{s_1\}$	$[s_{-1},s_0]$	8	15	0.6522
A_3	$[s_{-2},s_{-1}]$	$\{s_{-2}\}$	$[s_0,s_1]$	$[s_0,s_1]$	14	5	0.2631
A_4	$\{s_1\}$	$\{s_1\}$	$[s_1,s_2]$	$[s_1,s_2]$	2	20	0.9091
A_5	$[s_1,s_2]$	$\{s_2\}$	$\{s_0\}$	$[s_{-2},s_{-1}]$	9	14	0.6087
h_j^+	$[s_1,s_2]$	$\{s_2\}$	$[s_1,s_2]$	$[s_1,s_2]$			
h_j^-	$[s_{-2},s_{-1}]$	$\{s_{-2}\}$	$\{s_0\}$	$[s_{-2},s_{-1}]$			

Finally, following Step 6, the five alternatives can be ranked according to their similarity degree to an ideal solution as follows:

$$A_3 \lhd A_1 \lhd A_5 \lhd A_2 \lhd A_4.$$

As it can be seen, the alternatives A_2 and A_5 have reversed their ranking with respect to the FDHHFL-TOPSIS results. This is due to the fact that the free double hierarchy allows the experts to be more precise when assessing the alternatives and to feel more comfortable with the linguistic scale that they have to use. For these reasons, a lot of details that are disregarded using a single hierarchy can be captured using FDHHFL assessments.

If we carefully analyze the original assessments (using FDHHFL information) of alternatives A_2 and A_5, we realize that the ones for A_2 use lower labels of the second hierarchy. In particular E_1 uses o^1_{-2} and E_3 uses o^1_{-3}. On the contrary, among the assessments for A_5 no one has used any negative label of the second hierarchy. This distinction is not captured by using a single hierarchy, and that is the reason why the two alternatives reverse their orders with respect to the case in which a free double hierarchy is used.

B) *Double Hierarchy Hesitant Fuzzy Linguistic TOPSIS*

In this case, each expert is asked to provide their assessments using a DHHFLE by means of $\mathcal{S} = \{s_{-2} = \text{``null''}, s_{-1} = \text{``bad''}, s_0 = \text{``regular''}, s_1 = \text{``good''}, s_2 = \text{``perfect''}\}$ and the same fixed second hierarchy for all the experts and for all the linguistic terms. In this case, the second hierarchy is $\mathcal{O} = \{o^1_{-1} = \text{``slightly''}, o^1_0 = \text{``simply''}, o^1_1 = \text{``very''}\}$, sorted in the reverse order for the negative term of $\mathcal{S}$. Under these circumstances, the assessments provided by the experts are as shown in Table 5.9.

TABLE 5.9: Linguistic assessments given by the experts using a free double hierarchy.

	E_1	E_2	E_3	E_4
A_1	*"between simply bad and slightly bad"*	*"at most very regular"*	*"between slightly regular and simply good"*	*"between slightly good and slightly perfect"*
A_2	*"between very regular" and slightly good"*	*"at least slightly good"*	*"slighty good"*	*"between simply bad" and very regular"*
A_3	*"between slightly null and simply bad"*	*"at most slightly null"*	*"between slightly regular and slightly good"*	*"between slightly regular and simply good"*
A_4	*"between simply good" and very good"*	*"between slightly good and very good"*	*"at least very good"*	*"at least very good'*
A_5	*"at least simply good"*	*"at least slightly perfect"*	*"between slightly regular" and very regular"*	*"between slightly null" and slightly bad"*

Now, following Step 1, these assessments have to be rewritten as DHHFLEs; by Step 2, the best and worst assessments have to be found. These results are shown in Table 5.10.

TABLE 5.10: DHHFLEs corresponding to the assessments given by the experts.

	E_1	E_2	E_3	E_4
A_1	$[s_{-1<o_0^{-1}>}, s_{-1<o_1^{-1}>}]$	$[s_{-2<o_0^{-2}>}, s_{0<o_1^{0}>}]$	$[s_{0<o_{-1}^{0}>}, s_{1<o_0^{1}>}]$	$[s_{1<o_{-1}^{1}>}, s_{2<o_{-1}^{2}>}]$
A_2	$[s_{0<o_1^{0}>}, s_{1<o_{-1}^{1}>}]$	$[s_{1<o_{-1}^{1}>}, s_{2<o_0^{2}>}]$	$\{s_{1<o_{-1}^{1}>}\}$	$[s_{-1<o_0^{-1}>}, s_{0<o_1^{0}>}]$
A_3	$[s_{-2<o_1^{-2}>}, s_{-1<o_0^{-1}>}]$	$[s_{-2<o_0^{-2}>}, s_{-2<o_1^{-2}>}]$	$[s_{0<o_{-1}^{0}>}, s_{1<o_{-1}^{1}>}]$	$[s_{0<o_{-1}^{0}>}, s_{1<o_0^{1}>}]$
A_4	$[s_{1<o_0^{1}>}, s_{1<o_1^{1}>}]$	$[s_{1<o_{-1}^{1}>}, s_{1<o_1^{1}>}]$	$[s_{1<o_1^{1}>}, s_{2<o_0^{2}>}]$	$[s_{1<o_1^{1}>}, s_{2<o_0^{2}>}]$
A_5	$[s_{1<o_0^{1}>}, s_{2<o_0^{2}>}]$	$[s_{2<o_{-1}^{2}>}, s_{2<o_0^{2}>}]$	$[s_{0<o_{-1}^{0}>}, s_{0<o_1^{0}>}]$	$[s_{-2<o_1^{-2}>}, s_{-1<o_1^{-1}>}]$
h_j^+	$[s_{1<o_0^{1}>}, s_{2<o_0^{2}>}]$	$[s_{2<o_{-1}^{2}>}, s_{2<o_0^{2}>}]$	$[s_{1<o_1^{1}>}, s_{2<o_0^{2}>}]$	$[s_{1<o_1^{1}>}, s_{2<o_0^{2}>}]$
h_j^-	$[s_{-2<o_1^{-2}>}, s_{-1<o_0^{-1}>}]$	$[s_{-2<o_0^{-2}>}, s_{-2<o_1^{-2}>}]$	$[s_{0<o_{-1}^{0}>}, s_{0<o_1^{0}>}]$	$[s_{-2<o_1^{-2}>}, s_{-1<o_1^{-1}>}]$

For this case, given that the DHHFLEs ca be understood as a special case of FD-HHFLEs, we can use the same distance presented in Proposition 5.1. Hence, we can proceed with Steps 3, 4 & 5 to calculate the similarities degrees of each alternative to an ideal solution. These results are shown in Table 5.11.

Lastly, following Step 6, the five alternatives can be ranked according to their similarity degree to an ideal solution as follows:

$$A_3 \lhd A_1 \lhd A_2 = A_5 \lhd A_4.$$

TABLE 5.11: Similarity degrees using the DHHFL-TOPSIS.

	D_i^+	D_i^-	S_i
A_1	3.25	2	0.381
A_2	2.1875	3.0625	0.5833
A_3	4.4375	0.8125	0.1548
A_4	0.5	4.75	0.9048
A_5	2.1875	3.0625	0.5833

In this case, we can see that the similarity degrees of the alternatives A_2 and A_5 are the same, so the method is not able to sort them as one being higher ranked than the other one.

To summarize the results, Table 5.12 shows the similarity degrees obtained by the different methodologies.

TABLE 5.12: Comparison of the different similarity degrees.

	A_1	A_2	A_3	A_4	A_5
HFL-TOPSIS	0.4762	0.6522	0.2631	0.9091	0.6087
DHHFL-TOPSIS	0.381	0.5833	0.1548	0.9048	0.5833
FDHHFL-TOPSIS	0.3589	0.5605	0.1497	0.8925	0.5893

We can see that, when a single hierarchy is used, the alternative A_2 is better ranked than the alternative A_5 because of the aforementioned reasons. When the DMs are allowed to use a double hierarchy, they can be more precise and capture some of the details disregarded by the single hierarchy. Yet, in this case, the two alternatives are equally ranked. Instead, by using a free double hierarchy in which the DMs are asked to choose the linguistic labels that they prefer for the second hierarchy to be more precise or less precise depending on their knowledge the level of accuracy of the results is higher. In this case, we can finally conclude that the alternative A_5 is higher ranked than the alternative A_2.

5.6 Conclusions and future work

Based on the weak points that HFLTSs and DHHFLTSs have in the GDM problems, a new structure is presented in this chapter to capture linguistic assessments with more details. This structure enables the decision makers to be more accurate when evaluating an alternative by means of linguistic terms.

On the one hand, Free Double Hierarchy Linguistic Term Sets are introduced as a double hierarchy LTS in which the second hierarchy LTS can be different for each term of the first hierarchy LTS. Thus, each decision maker can choose the second hierarchy LTS that better suits each linguistic term of the first hierarchy according to his/her criterion.

On the other hand, Free Double Hierarchy Hesitant Fuzzy Linguistic Elements and Free Double Hierarchy Hesitant Fuzzy Linguistic Term Sets are defined as a useful tool to depict the hesitancy inherent in human reasoning.

Lastly, an order and a distance between FDHHFLEs are defined to enable us to present a free double hierarchy approach based on the TOPSIS method, called the FDHHFL-TOPSIS. This method is useful to sort alternatives in a GDM situation when the decision makers provide their assessments by means of free double hierarchy linguistic information in order to be more precise.

Future research is focused on two main directions: on the one hand, other methods to aggregate free double hierarchy hesitant fuzzy linguistic information will be studied as well as new measures within the set of FDHHFLTSs such as other distance definitions, similarity measures or preference relations.

On the other hand, the structure of FDHHFLTSs will also be applied on the field of recommender systems among end-users that express their opinions by means of this kind of linguistic information.

5.A   Proof of Proposition 5.1

In order to prove Proposition 5.1, let us first present two useful lemmas in order to simplify the proof:

Lemma 5.1. Given two FDHHFLEs, h_1 and h_2, $D_2(h_1, h_2)$ can be equivalently expressed as:

$$D(h_1, h_2) = 2 \cdot Area(h_1 \sqcup h_2) - Area(h_1) - Area(h_2).$$

Proof. We must see that $2 \cdot Area(h_1 \sqcup h_2) - Area(h_1) - Area(h_2) = Area(h_1 \sqcup h_2) - Area(h_1 \cap h_2) + Area(h_1 \between h_2)$. If $h_1 \cap h_2 \neq \varnothing$, both parts are equal to $Area(h_1 \sqcup h_2) - Area(h_1 \cap h_2)$, while if $h_1 \cap h_2 = \varnothing$, then both parts are $Area(h_1 \sqcup h_2) + Area(h_1 \between h_2)$. □

Lemma 5.2. Given three FDHHFLEs, h_1, h_2 and h_3,

$$Area(h_1 \sqcup h_3) \leq Area(h_1 \sqcup h_2) + Area(h_2 \sqcup h_3) - Area(h_2).$$

Proof. To start with, let us express the value of the function *Area* of the connected union of any pair of FDHHFLEs as:

$$Area(h \sqcup h') = Area(h) + Area(h') - Area(h \cap h') + Area(h \between h'). \quad (5.I)$$

Using this expression with $h = h_1 \sqcup h_3$ and $h' = h_2$, we get:

$$Area((h_1 \sqcup h_3) \sqcup h_2) = Area(h_1 \sqcup h_3) + Area(h_2) - $$
$$Area((h_1 \sqcup h_3) \cap h_2) + Area((h_1 \sqcup h_3) \between h_2),$$

which, after rearranged, becomes:

$$Area(h_1 \sqcup h_3) = Area(h_1 \sqcup h_2 \sqcup h_3) - Area(h_2) +$$
$$Area((h_1 \sqcup h_3) \cap h_2) - Area((h_1 \sqcup h_3) \between h_2). \qquad (5.\text{II})$$

Recalculating $Area(h_1 \sqcup h_2 \sqcup h_3)$ using again Equation 5.I, we get that $Area(h_1 \sqcup h_2 \sqcup h_3) = Area((h_1 \sqcup h_2) \sqcup (h_2 \sqcup h_3)) = Area(h_1 \sqcup h_2) + Area(h_2 \sqcup h_3) - Area((h_1 \sqcup h_2) \cap (h_2 \sqcup h_3)) + Area(\varnothing) = Area(h_1 \sqcup h_2) + Area(h_2 \sqcup h_3) - Area((h_1 \sqcup h_2) \cap (h_2 \sqcup h_3))$, and replacing in 5.II:

$$
\begin{aligned}
Area(h_1 \sqcup h_3) &= Area(h_1 \sqcup h_2) + Area(h_2 \sqcup h_3) - \\
&\quad Area((h_1 \sqcup h_2) \cap (h_2 \sqcup h_3)) - Area(h_2) + \\
&\quad Area((h_1 \sqcup h_3) \cap h_2) - Area((h_1 \sqcup h_3) \between h_2) \\
&= Area(h_1 \sqcup h_2) + Area(h_2 \sqcup h_3) - Area(h_2) + \\
&\quad Area((h_1 \sqcup h_3) \cap h_2) - Area((h_1 \sqcup h_2) \cap (h_2 \sqcup h_3)) - \\
&\quad Area((h_1 \sqcup h_3) \between h_2) \\
&= Area(h_1 \sqcup h_2) + Area(h_2 \sqcup h_3) - Area(h_2) + \Delta.
\end{aligned}
$$

At this point, we just need to study the sign of the term $\Delta = Area((h_1 \sqcup h_3) \cap h_2) - Area((h_1 \sqcup h_2) \cap (h_2 \sqcup h_3)) - Area((h_1 \sqcup h_3) \between h_2)$.

- If $(h_1 \sqcup h_3) \cap h_2 \neq \varnothing$, then:

$$
\left.
\begin{aligned}
Area((h_1 \sqcup h_3) \cap h_2) &\leq Area(h_2) \\
Area((h_1 \sqcup h_2) \cap (h_2 \sqcup h_3)) &\geq Area(h_2) \\
Area((h_1 \sqcup h_3) \between h_2) &= 0
\end{aligned}
\right\} \implies \Delta \leq 0.
$$

- If $(h_1 \sqcup h_3) \cap h_2 = \varnothing$, then:

$$
\left.
\begin{aligned}
Area((h_1 \sqcup h_3) \cap h_2) &= 0 \\
Area((h_1 \sqcup h_2) \cap (h_2 \sqcup h_3)) &\geq Area(h_2) \\
Area((h_1 \sqcup h_3) \between h_2) &\geq 0
\end{aligned}
\right\} \implies \Delta \leq 0.
$$

Thus, in any case, $\Delta \leq 0$, and, therefore,

$$Area(h_1 \sqcup h_3) \leq Area(h_1 \sqcup h_2) + Area(h_2 \sqcup h_3) - Area(h_2).$$

$\square$

With these two lemmas, we can now proceed with our aim of proving Proposition 5.1. For an easier reading, let us recall it:

Proposition 5.1. Let $\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}$ be a FDHLTS, and $\Phi \otimes \Psi$ be the set of all possible FD-HFHLEs by means of $\mathcal{S}_{\mathcal{O}}^{\mathcal{F}}$. Then,

$$D(h_1, h_2) = Area(h_1 \sqcup h_2) - Area(h_1 \cap h_2) + Area(h_1 \between h_2)$$

defines a distance in $\Phi \otimes \Psi$, where h_1 and h_2 are two FDHHFLEs.

Proof. We will check the conditions that D must satisfy to be a distance using the equivalent definition, presented in Lemma 5.1, $D(h_1, h_2) = 2 \cdot Area(h_1 \sqcup h_2) - Area(h_1) - Area(h_2)$.

- Non-negativity: $D(h_1, h_2) = 2 \cdot Area(h_1 \sqcup h_2) - Area(h_1) - Area(h_2) \geq 0$ given that $h_1 \subset h_1 \sqcup h_2$ and $h_2 \subset h_1 \cap h_2$.

- Identity of indiscernibles: If $D(h_1, h_2) = 0$, then $2 \cdot Area(h_1 \sqcup h_2) = Area(h_1) + Area(h_2)$, and for the same reason above, $h_1 = h_1 \sqcup h_2$ and $h_2 = h_1 \sqcup h_2$, which implies $h_1 = h_2$. On the other hand, $D(h_1, h_1) = 2 \cdot Area(h_1 \sqcup h_1) - Area(h_1) - Area(h_1) = 0$ given that $h_1 \sqcup h_1 = h_1$. Thus, $D(h_1, h_2) = 0 \Leftrightarrow h_1 = h_2$.

- Symmetry: $D(h_1, h_2) = D(h_2, h_1)$ given the symmetry of the connected union, the intersection and the gap operators.

- Triangular inequality: $D(h_1, h_2) + D(h_2, h_3) = 2 \cdot Area(h_1 \sqcup h_2) - Area(h_1) - Area(h_2) + 2 \cdot Area(h_2 \sqcup h_3) - Area(h_2) - Area(h_3) = 2 \cdot (Area(h_1 \sqcup h_2) + Area(h_2 \sqcup h_3) - Area(h_2)) - Area(h_1) - Area(h_3) \geq 2 \cdot Area(h_1 \sqcup h_3) - Area(h_1) - Area(h_3) = D(h_1, h_3)$ by Lemma 5.2.

$\square$

Acknowledgements

Chapter 6

Conclusion

6.1 Conclusions

The main contributions of this thesis are framed on the research line of the development of mathematical structures to deal with multiple-criteria decision problems under linguistic assessments. Yet, this is a large discipline of research including a lot of different topics. This thesis is oriented towards the study of situations in which the decision makers are allowed to hesitate when giving their assessments. Under these circumstances, the assessments provided by the decision makers of a GDM group are known as hesitant fuzzy linguistic assessments.

To deal with this situations, Rodríguez *et al.* introduced in 2012 the Hesitant Fuzzy Linguistic Term Sets (HFLTSs) [53]. The most remarkable results on this field obtained in this thesis are summarized in this chapter.

Chapter 2 proposes a theoretical framework to model group assessments on the basis of HFLTSs. To this end, the concept of distance between DMs in GDM by means of hesitant fuzzy linguistic information is a key notion. This conception allows similarities and differences between DM's assessments to be analyzed.

From a well-ordered set $\mathcal{S}$ of linguistic terms, the set of hesitant fuzzy linguistic term sets $\mathcal{H}_{\mathcal{S}}$ has been provided with two closed aggregation operations, connected union and intersection. These two operations provide $\mathcal{H}_{\mathcal{S}}$ with a lattice structure and are suitable to be used on reasoning and comparisons. The hesitant fuzzy linguistic descriptions of a set Λ are defined as $\mathcal{H}_{\mathcal{S}}$-fuzzy sets to describe the opinion of a DM about a set of alternatives.

Therefore, the first main contribution of this thesis is the introduction of a new distance measure within the set HFLTSs. This distance is based on previous work done by Agell *et al.*, in which a distance is already defined [3]. Yet, the existing distance presents some drawbacks such as returning equal values of distances between pairs of assessments that, according to human common sense, they should not be at the same distance. This issue is due to the fact that it is computed just based on the difference between the cardinalities of the connected union and the intersection. Hence, it is not taking into consideration that when two elements have an empty intersection, they might be very close or very far away from each other.

To overcome this point, not only the intersection between two HFLTSs has to be taken into consideration, but also the gap between them in case that these assessments do not overlap. To this end, the concept of concordance between two HFLTSs is defined. This concordance returns the cardinality of the intersection between them in case it is not empty and the cardinality of the gap between them in case that the intersection is empty.

The introduced distance is based on the difference between the cardinality of the connected union and the concordance in such a way that it is able to distinguish between couples of non overlapping assessments.

In addition, Chapter 2 also presents the concept of the centroid of a decision-making group by minimizing the addition of distances to the assessments of all the DMs in the group. This centroid can be understood as a central opinion of the group and the distances from each DM to the centroid can be thought as a measure of disagreement of the DM with respect to the group.

The proposed structure based on distances and centroids is not only limited to decision making scenarios. It also provides a general model suitable for comparing opinions between end-users when they express their preferences in terms of ordered linguistic terms.

The results from Chapter 2 have been published in the *Journal of Applied Logic* (Impact Factor: 0.838 [Q1 (JCR Category: Logic)]).

Following this research line, Chapter 3 presents the second main contribution of this thesis, which is an extension of the set of HFLTSs with new elements that are used to determine the gaps between non-overlapping HFLTSs. This extension is made by defining an equivalence relation in an analogous way to how negative numbers are defined from positive numbers. This leads us to the introduction of positive, negative and zero HFLTSs. The positive HFLTSs are the original ones, the negatives HFLTSs are the ones used to describe a gap and the zero HFLTSs are those used to describe consecutiveness of two HFLTSs, i.e., empty intersection and empty gap between them.

Additionally, all results from Chapter 2 are redefined in terms of this new set, including, among others, the order relation, operations, lattice structure and HFLDs. Special attention is paid to the distance between HFLTSs and the centroid of a GDM group. With the basis of the presented extended model, the computation of the distance and the centroid is simplified. In particular, the centroid is proved to be found as the median of a set of values, which can be calculated in linear time.

Preliminary works on the results from Chapter 3 have been presented in *XVIII Congreso Español sobre Tecnologías y Lógica Fuzzy (ESTYLF)* and *29th International Workshop on Qualitative Reasoning (QR)*. The final results have been presented in *13th Modeling Decisions for Artificial Intelligence (MDAI)* and they have been published in *Lecture Notes in Artificial Intelligence*.

Chapter 4 focuses on the study of the consensus of a GDM situation. On this field, the third main contribution of this thesis is the introduction of two new consensus measures for GDM processes by means of HFLTSs based on the weak points of the

already existing similar measures. These two degrees of consensus are introduced to capture differences among situations in which the previous measures are not able to make a difference.

On the one hand, a degree of consensus is defined for the whole group on a specific alternative. This degree of consensus is computed as a normalization of the addition of distances from the central opinion to the opinion of each DM of the group, being the central opinion modeled by the centroid introduced in the previous chapters. In addition, an analogous definition is given not just for one single alternative but for the whole set of alternatives.

On the other hand, the other degree of consensus is defined for each DM with respect to the whole group, based on the distance between his or her opinion and the central opinion. Once again, this degree of consensus is presented for one specific alternative and for a whole set of alternatives. This degree can be understood as a measure of the dissent of the DM with respect to the rest of the group.

Furthermore, for both degrees of consensus a comparison study with similar existing measures is carried out. As a conclusion, the measures presented in Chapter 4 are more accurate in situations in which existing measures consider the level of agreement to be the same but where common sense suggests they should be different. This is because the new measures take into account two important things that the already existing measures do not take. Firstly, since the presented measures are based on the distance from the previous chapters, the gap between non-overlapping assessments is considered. Secondly, the level of hesitancy in the assessments provided by the DMs is also disregarded by most of the existing measures, while, with the proposed ones, it is not.

On top of that, the comparison study also shows that the new measures have a lower complexity time than the existing measures, which is explained, among other factors, because of the low complexity time of finding the centroid.

Finally, a profile of a DM is introduced to keep track of his or her performances in GDM processes by means of hesitant fuzzy linguistic information. This profile is useful for situations in which the DMs that take part of a decision-making group have to be chosen with certain characteristics such as, for instance, not hesitating at all when assessing alternatives or having very different opinions from the majority of people. To this end, the profile records his or her level of precision and dissent with respect to the whole group in previous GDM processes as well as the total number of groups that he has been part of.

Preliminary works on the results from Chapter 4 have been presented in *26th IEEE International Conference on Fuzzy Systems (FUZZ-IEEE)* and the final results have been published in the journal *Information Fusion* (Impact Factor: 5.667 [Q1 (JCR Categories: Computer Sciences & Artificial Intelligence and Computer Sciences, Theory & Methods)]).

Chapter 5 follows the line of research opened by Gou *et al.* with the introduction of the Double Hierarchy Hesitant Fuzzy Linguistic Term Sets (DHHFLTSs) [21]. DHHFLTSs include a second hierarchy LTS to create more accurate linguistic expressions such as *almost perfect*. Yet, this second hierarchy LTS has to be the same

for all the linguistic terms in the first LTS and for all the DMs in the group. Hence, the last main contribution of this thesis is the introduction of a new structure called Free Double Hierarchy Hesitant Fuzzy Linguistic Term Sets (FDHHFLTSs) that offsets the drawbacks presented by the DHHFLTSs. The aim of this new structure is to capture linguistic assessments with more detail.

FDHHFLTSs are introduced as a double hierarchy LTS in which each DM can choose the second hierarchy LTS that better suits each linguistic term of the first hierarchy according to his or her criterion.

In addition, the distance from previous chapters is extended to this new model to present a free double hierarchy ranking method based on the TOPSIS method, called the FDHHFL-TOPSIS. This method enables us to rank alternatives in GDM processes in which the DMs provide their assessments by means of free double hierarchy linguistic information in order to be more precise.

In this chapter, we apply the presented FDHHFL-TOPSIS method into a simulated practical GDM problem involving tourist attractions in the city of Barcelona. This problem illustrates the usefulness of the presented methodology.

The results from Chapter 5 have been submitted to the journal *Information Fusion* (Impact Factor: 5.667 [Q1 (JCR Categories: Computer Sciences & Artificial Intelligence and Computer Sciences, Theory & Methods)]) and are currently under review.

6.2 Future work

The work presented in this thesis is a contribution in the exploration of the GDM problems under linguistic assessments. This is an interesting and relevant field of research with a big variety of topics to explore. Several directions of future work regarding the concepts introduced by this thesis have been identified while working on it. These directions are grouped in two main different perspectives: theoretical research and applied research.

From a theoretical point of view, there are several research lines. Firstly, an interesting direction of research is to extend all the concepts presented un Chapters 2, 3, and 4 to the structure of FDHHFLTSs introduced in Chapter 5. This theoretical extension would lead us to the appearance of positive, negative, and zero FDHHFLTSs to capture, respectively, intersections, gaps, and consecutiveness between opinions, as well as two degrees of consensus for GDM situations with free double hierarchy hesitant fuzzy linguistic information and decision maker's profile to keep track of his or her previous performances in this kind of situations.

Secondly, another interesting line of future research is the introduction of a consensus reaching process based on the degrees of consensus proposed in Chapter 4 extended to the structure of FDHHFLTSs. This consensus reaching process is an iterative process in which, in each round of the process, both the collective and the individual degrees of consensus are computed. The individual degrees of consensus enables us to identify the most dissident DMs to ask them to reconsider their opinion

in order to increase the collective degree of consensus until we reach a certain consensus threshold. This process will permit each DM to utilize linguistic terms that reflect more adequately their level of uncertainty and to be dynamically aware of their agreement in each round. It will also has the ability to reach consensus automatically with no need for either a moderator or a final interaction among DMs.

In addition, it is quite common that some DMs are more demanding than other DMs. Because of this, there might be two DMs with the same opinion about an alternative but giving different assessments to it. Hence, a different direction of future research is to use the DM's profile to introduce a strictness measure for each DM. With this strictness measure, the assessments provided by each DMs could be rescaled depending on how harsh he or she is.

Lastly, we have to take into consideration that it is likely that DMs are much more reticent to change their mind from *very good* to *almost perfect* than from *above average* to *slightly good*. Thus this changes could be treated different to compute the value of the distance between two opinions. To fix this problem, another possible line of future research is to give weights to the edges of the graph of the lattice of the extended set of HFLTSs.

From an applied point of view, an attractive research line is to interpret all the proposed methodologies in terms of a large groups of decision makers such as the set of all the end-users of a certain service. In this case, these methodologies could be used in the field of recommender systems given that the end-users with similar opinions could be identified by means of the proposed distance and the proposed degrees of consensus.

In this direction, some ongoing work is going on on applying the introduced techniques into a real case application framed in the INVITE research project (TIN2016-80049-C2-1-R and TIN2016-80049-C2-2-R (AEI/FEDER, UE)), funded by the Spanish Ministry of Science and Information Technology. We are currently working on an application on hotels rating that can be used as a pilot test for future uses of this methodology. This application is based on real data of the reviews of hotels in Rome obtained from a review website. In the future, we would also like to apply it to tourism real data obtained from Barcelona City Hall.

Said application associates an interval rating (hesitant term) together with a measure of consensus to the hotel ratings derived from a group of reviewers. Specifically, it gives recommender systems the ability to extend reviewer opinions from ratings to hesitant fuzzy linguistic term sets by combining the opinion of ratings and written reviews. From each set of extended reviewer opinions it considers the centroid to be the global opinion of each hotel. In this way, group consensus can be measured for each hotel and used to differentiate hotels having the same ratings.

The contributions of this application are threefold. First, it introduces hesitancy in the assessment of each reviewer by means of sentiment analysis. Second the centroid allows us to fuse the information introduced in the text and the reviewer?s rating. Third, the consensus measure allows us to better understand previous ratings allowing reviewers of recommenders systems to immediately identify which of the hotels will have more variability in their reviews. From a general perspective, the

ability to distinguish between items having the same ratings could be beneficial to intelligent personal assistants. Rather than offering a list of the top items based on ratings, an intelligent personal assistant may suggest a single alternative to the user. This scenario would be more reflective of a conversation between friends. Furthermore, some experiments will be run to test the applicability of the methodology in real recommendation scenarios.

In addition, in the frame of the INVITE research project, we are currently developing a method which uses behavioral attributes to segment visitors and tourists when visiting an attraction. Our degrees of consensus will be used in a hierarchical agglomerative clustering algorithm to work with ordinal variables and select the best segmentation. The method will be implemented in an application to a particular tourist attraction in Barcelona (Park Güell) considering a period over the course of two years and surveys from 2937 visitors.

As seen, the modeling of consensus in decision-making processes by means of linguistic assessments opens a wide range of options to be considered for future research, including from the most theoretical ones to its application on real case situations.

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