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Thesis Advisor
Anna Lladó

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Jordi Moragas Vilarnau

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Anna Lladó

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UNIVERSITAT POLITÈCNICA DE CATALUNYA

Jordi Moragas Vilarnau
Departament de Matemàtica Aplicada IV
Universitat Politècnica de Catalunya
Edifici C3, Jordi Girona 1-3
E-08034, Barcelona
<jmoragas@ma4.upc.edu>

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Abstract

This work is a contribution to the study of various problems that arise from two strongly connected areas of the Graph Theory: graph labelings and graph decompositions.

Most graph labelings trace their origins to the ones presented in 1967 by Rosa. One of these labelings, widely known as the graceful labeling, originated as a means of attacking the conjecture of Ringel, which states that the complete graph K_{2m+1} can be decomposed into m copies of a given tree of size m. Here, we study related labelings that give some approaches to Ringel's conjecture, as well as to another conjecture by Graham and Häggkvist that, in a weak form, asks for the decomposition of a complete bipartite graph by a given tree of appropriate size.

Our main contributions in this topic are the proof of the latter conjecture for double sized complete bipartite graphs being decomposed by trees with large growth and prime number of edges, and the proof of the fact that every tree is a large subtree of two trees for which both conjectures hold respectively. These results are mainly based on a novel application of the so-called polynomial method by Alon.

Another kind of labelings, the magic labelings, are also treated. Motivated by the notion of magic squares in Number Theory, in these type of labelings we want to assign integers to the parts of a graph (vertices, edges, or vertices and edges) in such a way that the sums of the labels assigned to certain substructures of the graph remain constant. We develop techniques based on partitions of certain sets of integers with some additive conditions to construct cycle-magic labelings, a new brand introduced in this work that extends the classical magic labelings.

Magic labelings do not provide any graph decomposition, but the techniques that we use to obtain them are the core of another decomposition problem, the ascending subgraph decomposition (ASD).

In 1987, was conjectured by Alavi, Boals. Chartrand, Erdős and Oellerman that every graph has an ASD. Here, we study ASD of bipartite graphs, a class of graphs for which the conjecture has not been shown to hold. We give a necessary and a sufficient condition on the one sided degree sequence of a bipartite graph in order that it admits an ASD by star forests. Here the techniques are based on the existence of edge-colorings in bipartite multigraphs.

Motivated by the ASD conjecture we also deal with the sumset partition problem, which asks for a partition of $[n]$ in such a way that the sum of the elements of each part is equal to a prescribed value. We give a best possible condition for the modular version of the sumset partition problem that allows us to prove the best known results in the integer case for n a prime. The proof is again based on the polynomial method.

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1

Introduction

In this chapter we first introduce the basic notation and terminology that will be eventually used in the forthcoming chapters. In Section 1.2 we present the general framework and the main motivations of this work. Finally, in Section 1.3, the polynomial method of Alon is described. We will use this general technique to develop different key parts of our work and this is the reason why we introduce it here.

1.1 Basic structures and definitions

We denote by $\mathbb{Z}$ the set of integer numbers. For any two integers $n < m$ we denote by $[n, m]$ the interval of all integers $n \leq x \leq m$. The set of the first n positive integers $[1, n]$ can be denoted simply by $[n]$. The ring $\mathbb{Z}/n\mathbb{Z}$ of integers modulo n is denoted by $\mathbb{Z}_n$. For a real number x we write $\lfloor x \rfloor$ for the greatest integer $\leq x$ and $\lceil x \rceil$ for the least integer $\geq x$. Given a finite set X, we denote its cardinality by $|X|$. A collection $\mathcal{P}$ of k non-empty subsets of X, $\mathcal{P} = \{X_1, \ldots, X_k\}$ is a *partition* of X if $\cup_{i=1}^{k} X_i = X$ and $X_i \cap X_j = \emptyset$ for all $1 \leq i < j \leq k$. If we want to emphasize $|\mathcal{P}|$, we say that $\mathcal{P}$ is a *k-partition* of X. If all the elements of $\mathcal{P}$ have the same cardinality, then $\mathcal{P}$ is said to be a *k-equipartition* of X. For a finite subset X of $\mathbb{Z}$ or $\mathbb{Z}_n$ we write, $\sum X = \sum_{x \in X} x$. The symmetric group of all permutations of a set of cardinality k will be denoted by $Sym(k)$ and for a particular $\sigma \in Sym(k)$ we write $sgn(\sigma)$ to denote its sign.

1.1.1 Graphs

The basic objects that we will deal with are simple finite graphs. Notation in Graph Theory is not uniform in the literature. We use the basic terminology for these objects following the notation found in the textbook by Diestel [15]. We recall some basic notations here for the commodity of the reader. A *graph* $G = (V, E)$ is a pair of sets satisfying $E \subseteq [V]^2$, where $[V]^2$ is the set of all 2-element subsets of V. Thus we will assume that a graph has no loops or multiple edges, otherwise we will use the term *multigraph*. The sets $V = V(G)$ and $E = E(G)$ are respectively the vertices and edges of G. The *order* of the graph is $|V|$ and the *size* is $|E|$. If $u, v \in V$ define an edge $e = \{u, v\} \in E$ of G, we denote this edge simply by $e = uv$ and say that u and v are adjacent. Given a vertex $u \in V$, the edges incident with u are precisely $e \in E$ such that $u \in e$. The *complement* $\overline{G}$ of G is the graph with $V(\overline{G}) = V(G)$ and $uv \in E(\overline{G}) \Longleftrightarrow uv \notin E(G)$.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. We call G_1 and G_2 *isomorphic*, and write $G_1 \simeq G_2$, if there exists a bijection $\phi : V_1 \longrightarrow V_2$ with

$$uv \in E_1 \Longleftrightarrow \phi(u)\phi(v) \in E_2 \text{ for all } u, v \in V_1;$$

ϕ is called an *isomorphism*.

G' is a *subgraph* of G if $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G) \cap [V(G')]^2$; if $E(G') = E(G) \cap [V(G')]^2$ we say that G' is an *induced subgraph* of G. Given a set of vertices $V' \subseteq V$ of a graph $G = (V, E)$, we say that the graph $G' = (V', E')$ is the *graph induced by* V' if

$$E' = \{e \in E : \exists u, v \in V' \text{ with } e = uv\}.$$

Similarly, given a set of edges $E' \subseteq E$, the graph $G' = (V', E')$ is the *graph induced by* E' if

$$V' = \{u \in V : \exists e \in E' \text{ with } u \in e\}.$$

If $A \subseteq V$ is any set of vertices of $G = (V, E)$, we denote by $G - A$ the graph obtained from G by deleting all the vertices in A and their incident edges. If $F \subseteq E$ is any set of edges, we write by $G - F$ the graph $(V, E \setminus F)$. We will use the following notation that describes a special deletion scheme. If $A \subseteq V$ is a set of vertices of G, $G \setminus A$ is the set $(V(G) \setminus A) \cup E(G)$, that is, we do not remove the edges incident to the vertices in A as in the usual vertex deletion process. In general, the remaining object is not a graph but

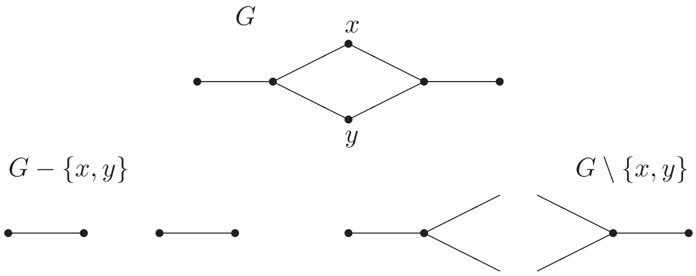

Figure 1.1 Vertex deletion processes.

we will still talk about its set of vertices and edges. Fig. 1.1 shows the two vertex removing schemes considered.

Given a vertex $u \in V$ of a graph $G = (V, E)$, the neighborhood of u are the vertices $v \in V$ such that there exists an edge joining them. We denote this set by $N_G(u)$ or briefly by $N(u)$. The cardinality of $N(u)$ is the *degree* of the vertex u that we usually denote by $d_G(u)$ or $d(u)$ if there is no risk of confusion. A vertex of degree 1 is called an *end vertex* or a *leaf*. The number

$$\delta(G) = \min\{d(u) : \ u \in V\}$$

is the *minimum degree* of G and

$$\Delta(G) = \max\{d(u) : \ u \in V\}$$

is the *maximum degree* of G. If all the vertices have the same degree d, then we say that G is *d-regular* or simply *regular*.

A graph is said to be the *complete graph* on n vertices, and is denoted by K_n, if its order is n and has all the possible adjacencies between the vertices. A graph G is *bipartite* if it is possible to partition the set $V(G)$ into two sets A and B, called the *partite sets* of G, such that no edge of G has both endpoints in the same partite set. A *t-partite* graph is similarly defined. We will often denote a bipartite graph with partite sets A and B by $G(A, B)$. If a bipartite graph $G(A, B)$ has all the possible adjacencies between the vertices of the two partite sets, then it is called *complete bipartite* and denoted by $K_{n,m}$ if $|A| = n$ and $|B| = m$. The graph $K_{1,n}$ is called a *star* with n spokes and denoted by S_n. Some examples are displayed in Fig. 1.2.

A *path* P_h is a graph with vertex set $V = \{v_0, v_1, \ldots, v_h\}$, with $v_i \neq v_j$ if $i \neq j$, and edge set $E = \{v_0v_1, v_1v_2, \ldots, v_{h-1}v_h\}$. The *length of the path* is

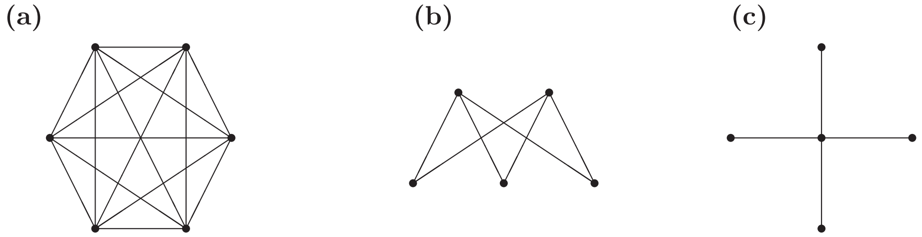

Figure 1.2 **(a)** The complete graph K_6. **(b)** The complete bipartite graph $K_{2,3}$. **(c)** The star $S_4 = K_{1,4}$.

its number of edges h. A *cycle* C_h is a closed path ($v_0 = v_h$) with h different vertices and h denotes the *length of the cycle*. A cycle of length n with an extra vertex connected to each of the vertices of the cycle is called a *wheel* and denoted by W_n.

We say that a graph G is *connected* if for each pair of vertices there exists a path joining them. A maximal connected subgraph of G is called a *component* of G. G is called k-*connected* if $|V(G)| > k$ and $G - X$ is connected for every set $X \subset V(G)$ with $|X| < k$. The *distance* between any pair of vertices u and v of a connected graph G is defined as the minimum length of a path joining them and denoted by $d_G(u, v)$ or simply by $d(u, v)$. The *eccentricity* of a vertex $u \in V(G)$ is the maximum distance between u and any other vertex of $V(G)$, and the *diameter* of G is the maximum eccentricity of the vertices of G.

One of the most important class of graphs that we will deal with is the class of trees. An acyclic graph is called a *forest*, and a connected forest is a *tree*. Recall that a connected graph with n vertices is a tree if and only if it has $n-1$ edges and also that trees are bipartite. These facts will be used later on. The *base tree* of a tree is obtained by deleting all leaves and their incident edges; see Fig. 1.3 for an example. A *caterpillar* is a tree whose base tree is a path. Similarly, a *lobster* is a tree whose base tree is a caterpillar. A *matching* is a forest in which each component is K_2.

We finish this section with a key definition for our study. A *decomposition* of a graph G is a partition $\mathcal{P}$ of its set of edges. The graph induced by each part of $\mathcal{P}$ is called a *factor*. When each factor of the decomposition is isomorphic to a graph H, we say that H decomposes G and write $H|G$. An H-*decomposition* of K_n is also known as an H-*design* of order n. Fig. 1.4 shows a graph that can be decomposed by K_3.

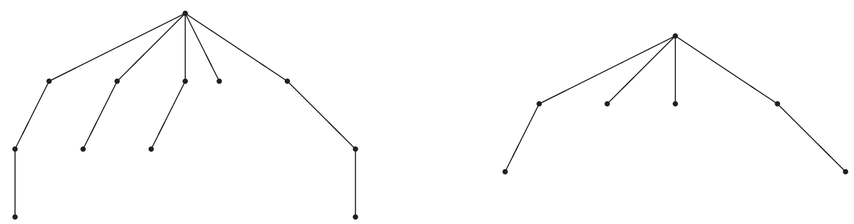

Figure 1.3 A tree and its base tree.

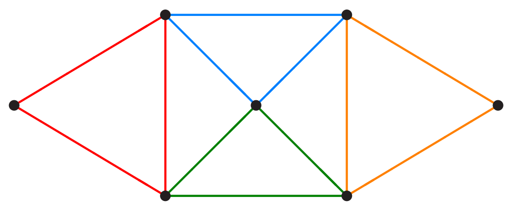

Figure 1.4 $K_3|G$.

Graph decompositions will be our main study in Chapters 3 and 5. If we admit that some edges can overlap, then we are talking about graph coverings. A *covering* of G is a family of subgraphs $H_1, \dots, H_k$ such that each edge of $E(G)$ belongs to at least one of the subgraphs. In this case, it is said that G admits an $(H_1, \dots, H_k)$-covering. If every H_i is isomorphic to a given graph H, we say that G has an H-covering. Graph coverings will be used to define an important graph labeling object of our study in Chapter 2. Finally, a *packing* of G is a family of subgraphs $H_1, \dots, H_k$ such that each edge of $E(G)$ belongs to at most one of the subgraphs, and $E(H_i) \cap E(H_j) = \emptyset$, $1 \leq i < j \leq k$.

1.2 Framework and motivations

This work is devoted to the study of various problems that arise from two main subjects in Graph Theory: graph labelings and graph decompositions. We will see in the development of the thesis the connection between these subjects but, roughly speaking, appropriate graph labelings provide graph decompositions.

A graph labeling is the assignment of integers to the edges or vertices, or both, subject to certain conditions. Most graph labelings trace their origins to the labelings presented by Rosa in his 1967 paper [47]. Rosa identified four types of successively weaker labelings, which he called α-, β-, σ- and ρ-valuations. β-valuations were later renamed *graceful* by Golomb [26] and the name has been popular since then. A *β-valuation* of a graph G with q edges is an injection f from the vertices of G to the set $\{0, 1, \ldots, q\}$ such that, when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct. Rosa introduced β-valuations as well as the other mentioned labelings as tools for decomposing the complete graph into isomorphic subgraphs. In particular, β-valuations originated as a means of attacking the conjecture of Ringel [46], which says that the complete graph K_{2m+1} can be decomposed into $2m+1$ subgraphs each isomorphic to a given tree with m edges. It is proved that a graph with m edges having a graceful labeling decomposes K_{2m+1}.

There are a lot of variants of these graceful-type labelings, for instance, the k-graceful labelings, the cordial labelings or the Hamming-graceful labelings. The survey of Gallian [24], which contains more than 1000 references on the subject, is a good guideline for all them. A related labeling is the also well-known harmonious labeling which naturally arises in the study of Graham and Sloane [27] on modular versions of additive basis problems stemming from error-correcting codes.

We study similar labelings as a tool to decompose graphs in smaller pieces and also as interesting problems on their own. This is the case of the magic-type labelings, motivated by the notion of magic squares in number theory. In these kind of labelings we want to assign integers, under some conditions, to the parts of a graph (vertices, edges, or vertices and edges) in such a way that the sums of the labels assigned to certain substructures of the graph remain constant. A good example is the *edge-magic total labeling* defined in 1970 by Kotzig and Rosa [37], which given a graph $G(V, E)$ asks for a bijection $f : V \cup E \to \{1, 2, \ldots, |V| + |E|\}$ such that for all edges $xy \in E$, $f(x) + f(y) + f(xy)$ is constant. The textbook on magic graphs by Wallis [49] is a good reference for this and other closely related magic labelings.

On the other hand, graph decompositions, known for its applications in combinatorial design theory, have been studied since the mid nineteenth century. There are a lot of decomposition problems but, among all of them, stand out the decompositions of complete or complete bipartite graphs by a given tree and the ascending subgraph decomposition of a graph. Both decompositions will be treated here.

The origin of the decomposition of a complete graph by a given tree is the already mentioned Ringel's conjecture, which gives rise to the conjecture of Graham and Häggkvist (see, e.g., [30]) that, in a weak form, says that every tree with m edges decomposes the complete bipartite graph $K_{m,m}$. Many partial results are known on both conjectures that motivate our work, mainly the ones that state that the addition of a certain number of vertices and edges to the tree results on a tree for which one of the conjectures hold [33, 35, 40]. The attempt to decompose larger complete graphs [30, 40] has also been a starting point for us.

The philosophy of the ascending subgraph decomposition is quite different from the above one. Introduced in 1987 by Alavi, Boals, Chartrand, Erdös and Oellerman [2], asks for a decomposition of a graph G of size $\binom{n+1}{2}$ into n subgraphs $H_1, \ldots, H_n$ such that H_i has i edges and is isomorphic to a subgraph of H_{i+1}, $i = 1, \ldots, n-1$. In the same paper, they conjectured that every graph with the stated size has such a decomposition and that a star forest with each component having size between n and $2n-2$ has an ascending subgraph decomposition with each H_i being a star. Nowadays, the first conjecture is still open and the second one was proved in 1994 by Ma, Zhou and Zhou [44]. The first conjecture gives us the motivation to study the problem for the class of bipartite graphs, a class for which the conjecture has not been shown to hold yet. But is in the proof of the second conjecture where a related problem arises: the sumset partition problem.

The sumset partition problem asks for a partition of a certain set of positive integers, in such a way that the sum of the elements inside each part is equal to a prescribed value. To characterize the structure of the sequence of prescribed values when the set to partition is $[n]$, has been a hard topic up to now. Some sufficient conditions can be found in the literature, [7, 21, 44, 12], each one being more general than the preceding. One of the reasons for us to study this problem, as well as the interest by itself, is the connection with the ascending subgraph decomposition problem for bipartite graphs and the study of a generalization of magic labelings.

We next summarize the contents of the forthcoming chapters, which contain the bulk of the work.

Chapter 2 deals with the magic-type labelings described above. We present a generalization of the concept introduced by Gutiérrez and Lladó [28], in which the labeling is required to be constant on each member of a covering of the target graph (the classical magic valuations correspond to the decomposition of a graph by its edges). In this chapter we show the connection between the partition of certain sets of integers and this general kind of magic labelings. Finally, we focuss on the special case where the members of the covering are cycles.

In Chapter 3 we study the conjectures by Ringel and by Graham and Häggkvist stated above. The first part of the chapter is motivated by a paper of Lladó and López [40] where it is shown that a tree with m edges and it growth ratio at least $\sqrt{2}$ decomposes $K_{2m,2m}$. Our approach to the problem uses an extension of the bigraceful labeling (see e.g. [40]) that leads us to the $\mathcal{G}$-bigraceful labeling. We can improve the last bound for trees with a prime number of edges. To obtain appropriate labelings, we use the polynomial method of Alon [3, 4, 5], an algebraic method based on finding nonzero valuations of a multivariate polynomial over a field that will be described in the following section. In the last part of the chapter, we consider an arbitrary tree T and show that it can be embedded into two trees with n and n' edges, bounded with respect to the size of T, such that decompose K_{2n+1} and $K_{n',n'}$ respectively. These last results are based on the well-known Kneser's theorem (see, e.g., [45]) from Additive Theory.

Chapter 4 is completely devoted to the sumset partition problem. We obtain general sufficient conditions for sequences of prescribed sums, and we also completely characterize the sequences of length at most 4. The obtained results have direct consequences on Chapter 5. At the end, we consider a modular version of the problem and show that for p an odd prime, $\mathbb{Z}_p$, except one element in some cases, can be partitioned in such a way that each part adds up to any prescribed sum, being the sequence of prescribed sums of length at most $\frac{p-1}{2}$. The result is obtained from the application of the polynomial method.

Finally, in Chapter 5, we first show the strong connection between the sumset partition problem and the ascending subgraph decompositions. We use the results obtained in Chapter 4 to obtain ascending decompositions of bipartite graphs in which each subgraph is a star. We also construct ascending subgraph decompositions for bipartite graphs with one partite set of cardinality at most 4. We finish the chapter by finding ascending subgraph decompositions of bipartite graphs in which each factor is a forest of stars. The techniques of this part are based on the construction of adequate bipartite multigraphs that admit a proper edge-coloring subject to certain conditions.

1.3 Polynomial method

Among the variety of tools presented in this thesis, which range from Additive Theory to purely combinatorial ones, the polynomial method of Alon stands out. We next present the main result of Alon, and a pair of facts, that will be used many times in the sequel.

The use of polynomial methods in combinatorics can be traced back to the applications in combinatorial geometry in the late 1970's, see the survey of Blokhuis [9] on this kind of applications. A particular kind of these methods was elaborated by Alon and his collaborators in a series of applications, first in the works of Alon and Tarsi on the list chromatic number of a graph [6] in the late 1980's and then in Combinatorial Number Theory, in the solution of the Erdös-Heilbron conjecture on distinct sums by Alon, Nathanson and Rusza [5]. These applications led Alon to establish a quite powerful tool which he named *Combinatorial Nullstellensatz* [3] for its connection with the Nullstellensatz by Hilbert, but it is usually referred to as the *polynomial method* of Alon (see also the survey by Kàrolyi on this topic [32]).

The basic idea of one part of the method is based in the simple observation that a nonzero polynomial of degree n can not vanish in a set of more than n points. For multivariate polynomials an analogous less trivial result holds which is in the basis of many of the applications. Its statement is summarized in the following theorem.

Theorem 1.1 (Alon, [3]) *Let F be an arbitrary field, and let $f(x_1, \ldots, x_n)$ be a polynomial in $F[x_1, \ldots, x_n]$. Suppose that the degree $deg(f)$ of f is $\sum_{i=1}^{n} t_i$, where each t_i is a nonnegative integer, and suppose that the coefficient of $\prod_{i=1}^{n} x^{t_i}$ in f is nonzero. Then, if $S_1, \ldots, S_n$ are subsets of F with $|S_i| > t_i$, there are $s_1 \in S_1, s_2 \in S_2, \ldots, s_n \in S_n$ so that*

$$f(s_1, \ldots, s_n) \neq 0.$$

□

In our applications we usually build polynomials which contain Vandermonde polynomials as factors. We denote by $V(x_1, \ldots, x_n) = \prod_{1 \le i < j \le n}(x_i - x_j)$ the *Vandermonde polynomial* on the variables $x_1, \ldots, x_n$ over some field F. The polynomial takes nonzero value in a point $(a_1, \ldots, a_n) \in F^n$ if and only if the coordinates are pairwise distinct. Recall that the expansion of the polynomial has the form

$$V(x_1,\dots,x_n)=\sum_{\sigma\in Sym(n)} sgn(\sigma)x_{\sigma(1)}^{n-1}x_{\sigma(2)}^{n-2}\cdots x_{\sigma(n)}^{0}. \tag{1.1}$$

It can be shown that, in the homogeneous polynomial $V(x_1,\dots,x_n)^2 = \prod_{1\le i<j\le n}(x_i-x_j)^2$, the monomial in which the exponents of each variable are balanced has coefficient $\pm n!$, that is,

$$[x_1^{n-1}\cdots x_n^{n-1}](V(x_1,\dots,x_n))^2=(-1)^{\binom{n}{2}}n!, \tag{1.2}$$

see, e.g., Alon [4]. We shall use this fact later on.

2

$\boldsymbol{H}$-magic labelings

This chapter is devoted to the study of a generalization of magic labelings. We first present the generalization introduced by Gutiérrez and Lladó [28] of the classical concept of magic graph. In this generalization, the labeling is required to be constant on each member of a covering of the target graph, while the classical magic valuations correspond to the covering of the graph by its edges. In Section 2.2, we summarize some of the results obtained in [28] to describe the general framework for a complete understanding of the problem. Then, in Section 2.3, we show the connection between the partition of certain sets of integers and this general kind of magic labelings. In the last section, we focuss on the special case where the members of the covering are cycles. The results of this section appear in [42].

2.1 Introduction

Let $G = (V, E)$ be a finite simple graph. An (edge)covering of G is a family of subgraphs $H_1, \ldots, H_k$ such that each edge of E belongs to at least one of the subgraphs H_i, $1 \leq i \leq k$. In this case, it is said that G admits an $(H_1, \ldots, H_k)$-(edge)covering. If every H_i is isomorphic to a given graph H, we say that G has an H-covering.

Suppose that $G = (V, E)$ admits an H-covering. A bijective function

$$f : V \cup E \to \{1, 2, \ldots, |V| + |E|\},$$

is an H-*magic* labeling of G whenever, for every subgraph $H' = (V', E')$ of G isomorphic to H,

$$f(H') \stackrel{\text{def}}{=} \sum_{v \in V'} f(v) + \sum_{e \in E'} f(e)$$

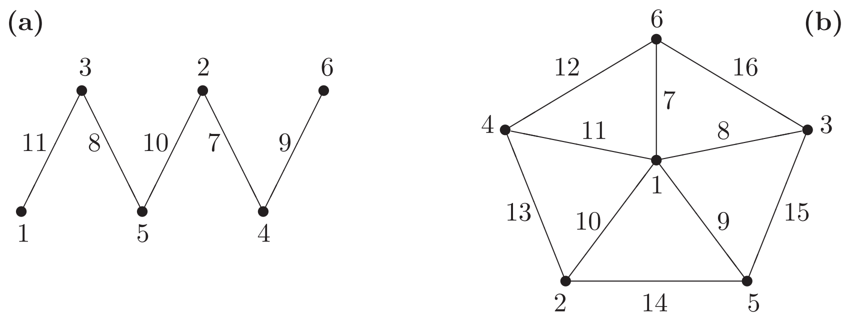

Figure 2.1 **(a)** P_3-supermagic labeling of P_6. **(b)** C_3-supermagic labeling of W_5.

is constant. In this case we say that the graph G is H-magic. If the restriction of f on the vertices takes the first $|V|$ possible values, $f(V) = \{1, \ldots, |V|\}$, then G is said to be H-*supermagic*. The constant value that every copy of H takes under the labeling f is denoted by $m(f)$ in the magic case and by $s(f)$ in the supermagic case. Fig. 2.1 shows an example of a P_3-supermagic labeling with $s(f) = 28$ and a C_3-supermagic labeling with $s(f) = 41$.

The notion of H-magic graphs was first introduced by Gutiérrez and Lladó [28] as an extension of the magic valuation given by Rosa [47] in 1967 (see also [37]), which corresponds to the case $H = K_2$. Supermagic labelings have been considered in [16] (see also [1]). There are other closely related notions of magic labelings in the literature; see the survey of Gallian [24] and the references therein, and the textbook on Magic Graphs by Wallis [49]

When $H = K_2$, we say that a K_2-magic or a K_2-supermagic graph is simply *magic* or *supermagic*. Some authors in this case use the terminology *total-magic* or *super total-magic* labeling, in order to stress the fact that both vertices and edges are labeled.

2.2 Star and path-magic graphs

In this section we summarize some of the results obtained by Gutiérrez and Lladó [28] concerning H-magic graphs for H a star $K_{1,h}$ or a path P_h.

2.2.1 Star-magic coverings

It is clear that, for any pair of positive integers $n \geq h$, the star $K_{1,n}$ can be covered by a family of $\binom{n}{h}$ stars $K_{1,h}$. The very first result obtained in [28] is that the star $K_{1,n}$ is $K_{1,h}$-supermagic for any $1 \leq h \leq n$.

The authors next study the H-(super)magic behavior of complete graphs and complete bipartite graphs when H is a star. It is well-known that the complete graph K_n is not magic for any order larger than six [13, 36, 38]. It is also known [37] that complete bipartite graphs of any order are magic. Using local arguments, they conclude that if G is a d-regular graph, then G is not $K_{1,h}$-magic for any $1 < h < d$. As a direct consequence of the last fact, they obtain that the complete graph K_n is not $K_{1,h}$-magic for any $1 < h < n-1$ and the complete bipartite graph $K_{n,n}$ is not $K_{1,h}$-magic for any $1 < h < n$.

Using a classical result about the existence of magic squares, they show the extremal case for complete bipartite graphs.

Theorem 2.1 (**[28]**) *The complete bipartite graph $K_{n,n}$ is $K_{1,n}$-magic for $n \geq 1$.* □

The extremal case for complete bipartite graphs with respect to the star-supermagic property is also proved. For that, they use a result dealing with 2-partitions of sets of consecutive integers.

Theorem 2.2 (**[28]**) *For each integer $n > 1$, the complete bipartite graph $K_{n,n}$ is not $K_{1,n}$-supermagic.* □

They next study the same question for general complete bipartite graphs $K_{r,s}$ when $1 < r < s$. In [16] it is proved that the only complete bipartite graphs that are supermagic are the stars. Next theorem says that in fact there is no integer $1 < h < s$ for which $K_{r,s}$ admits a $K_{1,h}$-supermagic labeling. It also states that $K_{r,s}$ is $K_{1,s}$-supermagic, which is an extension of the result given in [16].

Theorem 2.3 (**[28]**) *For any pair of integers $1 < r < s$, the complete bipartite graph $K_{r,s}$ is $K_{1,h}$-supermagic if and only if $h = s$.* □

2.2.2 Path-magic coverings

The second part of [28] studies the H-(super)magic behavior of paths, complete graphs and cycles when H is a path.

The first result on this problem concerns the path-supermagic behavior of paths and states that the path P_n is P_h-supermagic for every $2 \leq h \leq n$.

It has already been mentioned that the complete graph K_n is not magic for any integer $n > 6$. They prove that if G is a P_h-magic graph with $h > 2$, then G is C_h-free, implying that complete graphs are not path-magic for any path of length larger than 2.

All cycles are magic, see [25]. It is also known that only the odd cycles are supermagic [16]. By adding an additional divisibility condition they get path-supermagic labelings of cycles as shown in the next theorem.

Theorem 2.4 ([28]) *Let n and h be positive integers with $2 \leq h < n$. If $\gcd(n, h(h-1)) = 1$ then the cycle C_n is P_h-supermagic.* □

2.3 Equipartitions with given sums

In this section we describe additional results from [28] that motivate our partition method to obtain cycle-magic labelings of the target graph that will be shown in the next section. The authors prove that for every graph H, verifying some weak conditions, there are infinite families of connected and non-connected H-magic graphs using results about set equipartitions. We extend here these results about set equipartitions and we also introduce the concept of well-distributed equipartition, that will be very useful to show the supermagic behavior of certain classes of graphs.

We first need some preliminary notation.

Let $\mathcal{P} = \{X_1, \ldots, X_k\}$ be a partition of a set X of integers. The set of subset sums of $\mathcal{P}$ is denoted by $\sum \mathcal{P} = \{\sum X_1, \ldots, \sum X_k\}$. If all elements of $\mathcal{P}$ have the same cardinality, then $\mathcal{P}$ is said to be a k-equipartition of X.

We shall describe a partition $\mathcal{P} = \{X_1, \ldots, X_k\}$ of a set $X = \{x_1, x_2, \ldots, x_n\}$ by giving a k-coloring on the elements of X in such a way that X_i contains all the elements with color i, $1 \leq i \leq k$. For example, the coloring $(1, 2, 1, 2, 2, 1)$ means that $X_1 = \{x_1, x_3, x_6\}$ and $X_2 = \{x_2, x_4, x_5\}$. When some pattern of colors $(c_1, c_2, \ldots, c_r)$ is repeated t times we write $(c_1, c_2, \ldots, c_r)^t$. For instance, the coloring $(1, 2, 1, 2, 2, 1)$ is denoted by $(1, 2)^2(2, 1)$.

We say that a k-equipartition $\mathcal{P} = \{X_1, \ldots, X_k\}$ of a set of integers $X = \{x_1 < x_2 < \cdots < x_{hk}\}$ is *well-distributed* if for each $0 \leq j < h$, the elements $x_l \in X$, with $l \in [jk+1, (j+1)k]$, belong to distinct parts of $\mathcal{P}$. In other words, the coloring which gives the partition is bijective on each of the h disjoint blocks of length k of consecutive elements in X.

For instance, $\mathcal{P}_1 = \{\{1,4,5\},\{2,3,6\}\}$ and $\mathcal{P}_2 = \{\{1,3,5\},\{2,4,6\}\}$, are well-distributed 2-equipartitions of $X = [1,6]$ (the associated colorings are $(1,2,2,1,1,2)$ and $(1,2,1,2,1,2)$ respectively) while $\mathcal{P}_3 = \{\{1,2,3\},\{4,5,6\}\}$ is not.

We will use the next two lemmas for k-equipartitions. It can be easily checked that the proofs given in [28] provide in fact well-distributed equipartitions.

Lemma 2.5 ([28]) *Let h and k be two positive integers. For each integer $0 \leq t \leq \lfloor h/2 \rfloor$, there exists a well-distributed k-equipartition $\mathcal{P}$ of $[1, hk]$ such that $\sum \mathcal{P}$ is an arithmetic progression of difference $d = h - 2t$.* □

Lemma 2.6 ([28]) *Let h and k be two positive integers. If h or k are not both even, there exists a well-distributed k-equipartition $\mathcal{P}$ of $[1, hk]$ such that $\sum \mathcal{P}$ is a set of consecutive integers.* □

Lemmas 2.5 and 2.6 allow us to obtain infinite families of H-magic graphs for a given graph H under some weak conditions.

Lemma 2.5 has a simple application for the construction of an infinite family of H-magic non-connected graphs as it is shown in the following result.

Theorem 2.7 ([28]) *Let H be a graph with $|V(H)| + |E(H)|$ even. Then the disjoint union $G = kH$ of k copies of H is H-magic.* □

As an application of Lemma 2.6, they obtain the following result that provides infinite families of connected H-magic graphs. It is based on the following graph operation. Let G and H be two graphs and $e \in E(H)$ a distinguished edge in H. We denote by $G * eH$ the graph obtained from G by gluing a copy of H to each edge of G by the distinguished edge $e \in E(H)$.

Theorem 2.8 ([28]) *Let H be a 2-connected graph and let G be an H-free supermagic graph. Let k be the size of G and $h = |V(H)| + |E(H)|$. Assume that h and k are not both even. Then, for each edge $e \in E(H)$, the graph $G * eH$ is H-magic.* □

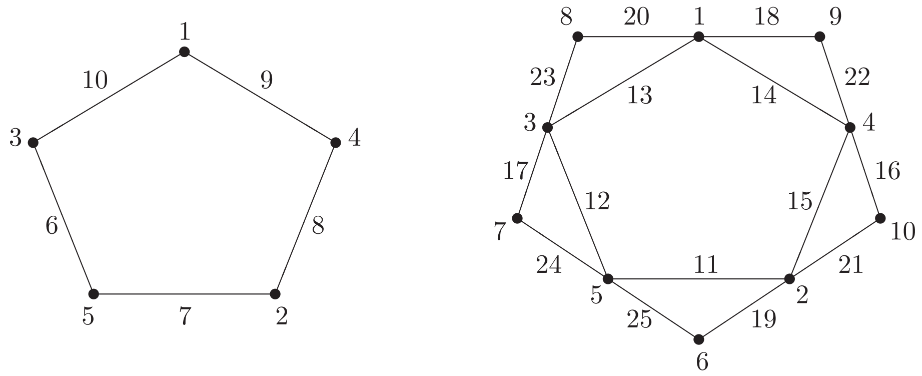

Figure 2.2 A supermagic labeling of C_5 and the C_3-magic labeling of $C_5 * eC_3$.

Fig. 2.2 depicts a labeling obtained from the proof of Theorem 2.8 given in [28] for $G = C_5$ and $H = C_3$.

We finish this section with a key lemma that will be very useful to obtain the results of the next section. It provides well-distributed equipartitions where all the parts have the same sum.

Lemma 2.9 *Let $h \geq 3$ be an odd integer. If either*

(1) *k is odd and $X = [1, hk]$, or*

(2) *k is even and $X = [1, hk+1] \setminus \{k/2+1\}$,*

then there is a well-distributed k-equipartition $\mathcal{P}$ of X such that $|\sum \mathcal{P}| = 1$.

Proof.

(1) By Lemma 2.6 there exists a well-distributed k-equipartition $\mathcal{P}' = \{Y_1, \ldots, Y_k\}$ of the interval $Y = [1, (h-1)k]$ such that

$$\sum \mathcal{P}' = \{\sum Y_1 + (i-1) : \ 1 \leq i \leq k\}.$$

Consider the partition $\mathcal{P} = \{X_1, \ldots, X_k\}$ of $[1, hk]$, where

$$X_i = Y_i \cup \{(1-i) + hk\}, \ 1 \leq i \leq k.$$

It is clear that $\mathcal{P}$ is a k-equipartition of $[1, hk]$.

As $\mathcal{P}'$ is a well-distributed k-equipartition of $[1,(h-1)k]$ and there is one element of each part in $[(h-1)k+1, hk]$, $\mathcal{P}$ is also well-distributed.

In addition, for any $1 \leq i \leq k$ we have,

$$\sum X_i = \sum Y_1 + (i-1) + (1-i) + hk = \sum Y_1 + hk,$$

which is independent of i and therefore $|\sum \mathcal{P}| = 1$.

(2) Let k be an even positive integer and $X = [1, hk+1] \setminus \{k/2+1\}$.

Set $A = [1, k+1] \setminus \{k/2+1\}$ and $B = [k+2, hk+1]$. Clearly, $|A| = k$, $|B| = (h-1)k$ and $X = A \cup B$.

Consider now the partition $\mathcal{P} = \{X_1, \dots, X_k\}$ given by the following k-coloring of $A \cup B$.

Color the k elements of A by

$$(k/2, k/2-1, \dots, 1)(k, k-1, \dots, k/2+1).$$

Now color the $(h-1)k$ elements of B by

$$(k/2+1, 1, k/2+2, 2, \dots, k, k/2)(k, k-1, \dots, 1)^{\frac{h-3}{2}+1}(1, 2, \dots, k)^{\frac{h-3}{2}}.$$

It is clear by the coloring that $\mathcal{P}$ is well-distributed. Moreover, for $1 \leq i \leq k/2$, we have,

$$\sum X_i - \sum X_1 = (k/2+1-i-k/2) + (k+1+2i-k-3) +$$
$$+ \left(\frac{h-3}{2} + 1\right)(1-i) + \frac{h-3}{2}(i-1) = 0.$$

A similar computation shows that $\sum X_i - \sum X_1$ takes the same value when $k/2 < i \leq k$, so that $|\sum \mathcal{P}| = 1$.

□

Remark 2.10 *Note that the statements of the three above lemmas can be extended to any integer translation $a + X = \{a + x_1, \dots, a + x_n\}$ of the set $X = \{x_1, \dots, x_n\}$.* □

2.4 Cycle-magic graphs

In this section we study H-magic labelings when H is a cycle C_r. In this case we speak of *cycle-magic* labelings and *cycle-magic* graphs. A related notion of *face-magic* labelings of a planar graph G asks for a total labeling such that the sum over the vertices and edges of each face of a planar embedding of G is constant; see, for instance, Bacca [8]. When G has a planar embedding in which all faces have the same number r of edges, a C_r-magic labeling of G is also a face magic labeling of the graph.

The section continues with the following new results. We prove that the wheel W_n with $n \geq 5$ odd is C_3-magic and that the cartesian product of a C_4-free supermagic graph with K_2 is C_4-magic. In particular the odd prisms and books are C_4-supermagic. We also show that the windmill $W(r,k)$ is C_r-magic, thus providing a family of C_r-magic graphs for each $r \geq 3$. Finally, it is also shown that subdivided wheels and uniform Θ-graphs are cycle-magic. All these results rely on the application of the partition lemmas described in the previous section.

2.4.1 C_3 and C_4-magic graphs

Let $W_n = C_n + \{v\}$ denote the wheel with a rim of order n. Clearly W_n admits a covering by triangles. As an another application of Lemma 2.6, we next show that any odd wheel is a C_3-supermagic graph.

Theorem 2.11 *The wheel W_n for $n \geq 5$ odd, is C_3-supermagic.*

Proof. Denote by $v_1, v_2, \ldots, v_n$ the vertices in the n-cycle of the wheel W_n and by v its central vertex. For $1 \leq i \leq n$ let $N_i = \{v_i, v_i v\}$.

Define a total labeling f of W_n on $[1, 3n+1]$ as follows. Set $f(v) = 1$, $f(v_n v_1) = 2n+2$ and for $1 \leq i < n$, $f(v_i v_{i+1}) = 3n + 2 - i$. Therefore, $f(E(C_n)) = [2n+2, 3n+1]$.

We have to define f on $N = \cup_{i=1}^{n} N_i$ in such a way that $f(N) = [2, 2n+1]$.

Since n is odd, by Lemma 2.6 there is a well-distributed n-equipartition $\mathcal{P} = \{X_1, \ldots, X_n\}$ of the set $X = 1 + [1, 2n]$, such that $\sum X_i = \sum X_1 + (i-1)$. Let $X_i = \{x_{i,1} < x_{i,2}\}$. Since $\mathcal{P}$ is well-distributed, we have $1 < x_{i,1} \leq n+1$ and $n+1 < x_{i,2} \leq 1+2n$.

Let α be the permutation of $[1, n]$ given by

$$\alpha(i) = \begin{cases} i/2, & i \text{ even;} \\ (n+i)/2, & i \text{ odd.} \end{cases}$$

Since n is odd, α is a permutation of $[1, n]$. Moreover $\alpha(i) + \alpha(i+1) = i + (n+1)/2$ for $1 \leq i \leq n-1$ and $\alpha(n) + \alpha(1) = (3n+1)/2$.

Define f on each N_i by the bijection from N_i to $X_{\alpha(i)}$ given by

$$f(v_i) = x_{\alpha(i),1} \text{ and } f(vv_i) = x_{\alpha(i),2}.$$

Note that $1 < f(v_i) \leq n+1$ and $n+1 < f(vv_i) \leq 2n+1$, so that $f(V(N)) = [2, n+1]$ and $f(E(N)) = [n+2, 2n+1]$. Hence, if f is C_3-magic, then it is C_3-supermagic.

Let us show that $\sum f(H)$ is constant in every triangle H of W_n. Now we prove that f take the same sum in every subgraph H of W_n isomorphic to C_3. Since $n \geq 5$, each triangle H has vertex set either $\{v, v_i, v_{i+1}\}$ for some $1 \leq i < n$, or $\{v, v_n, v_1\}$. Therefore,

$$\begin{aligned} \sum f(H) &= \sum f(N_i) + \sum f(N_{i+1}) + f(v_i v_{i+1}) + f(v) \\ &= 2\sum X_1 + \alpha(i) + \alpha(i+1) - 2 + (3n+2-i) + 1 \\ &= 2\sum X_1 + i + (n+1)/2 + (3n+1) - i \\ &= 2\sum X_1 + (7n+3)/2 \\ &= \sum f(N_n) + \sum f(N_1) + f(v_n v_1) + f(v), \end{aligned}$$

which is independent of i as claimed. This completes the proof. $\square$

Fig. 2.3 shows an example of the C_3-supermagic labeling defined in the above proof.

Remark 2.12 *Fig. 2.4 shows two quite different C_3-magic labelings of the wheels W_4 and W_6. We do not know if the wheel W_{2r} with $r > 3$ is a C_3-magic graph.* $\square$

Another application of Lemma 2.6 provides a large family of C_4-supermagic graphs. The *cartesian product* of two graphs G_1 and G_2 is the graph $G = G_1 \times G_2$ with

$$V(G) = V(G_1) \times V(G_2)$$

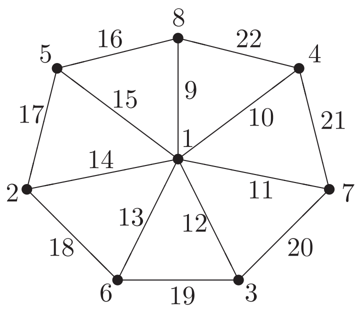

Figure 2.3 C_3-supermagic labeling of W_7

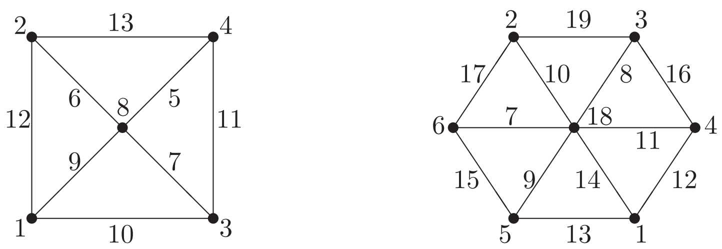

Figure 2.4 C_3-magic labelings of W_4 and W_6.

and

$$(x_1, y_1)(x_2, y_2) \in E(G) \iff \begin{cases} x_1 = x_2 \text{ and } y_1 y_2 \in E(G_2) \text{ or} \\ y_1 = y_2 \text{ and } x_1 x_2 \in E(G_1). \end{cases}$$

Clearly, for any graph G, the cartesian product $G \times K_2$ can be covered by 4-cycles.

Theorem 2.13 *Let G be a C_4-free supermagic graph of odd size. Then, the graph $G \times K_2$ is C_4-supermagic.*

Proof. Let n and m be, respectively, the order and size of $G = (V, E)$. We have to show a C_4-supermagic total labeling of $G \times K_2$ with the integers in $[1, 3n + 2m]$.

For each vertex $x \in V(G)$ denote by $x_0, x_1 \in V(G \times K_2)$ the corresponding vertices in the two copies of G and $x_0 x_1 \in E(G \times K_2)$ the edge joining them. Denote by $A_x = \{x_0, x_1, x_0 x_1\}$ and by $A = \cup_{x \in V} A_x \subset V(G \times K_2) \cup E(G \times K_2)$. We have $|A| = 3n$. Now, for each edge $xy \in E(G)$, denote by $B_{xy} = \{x_0y_0, x_1y_1\}$ the corresponding edges in the two copies of G and $B = \cup_{xy\in E(G)} B_{xy}$. We have $|B| = 2m$. Clearly, $\{A, B\}$ is a partition of the set $V(G \times K_2) \cup E(G \times K_2)$.

By Lemma 2.6 there is a well-distributed n-equipartition $\mathcal{P}_1 = \{X_1, \ldots, X_n\}$ of the set $[1, 3n]$, such that $\sum X_i = a + i$ for some integer a.

Since m is odd, Lemma 2.6 also ensures a well-distributed m-equipartition $\mathcal{P}_2 = \{Y_1, \ldots, Y_m\}$ of $3n + [1, 2m]$ such that $\sum Y_i = b + i$ for some integer b.

Let f be a supermagic labeling of G with supermagic sum $s(f)$. Define a total labeling f' of $G \times K_2$ as follows. For $x \in V(G)$ define f' on A_x by any bijection from A_x to $X_{f(x)}$ (the bijection depends on f). Similarly, for $xy \in E(G)$ define f' on B_{xy} by any bijection from B_{xy} to $Y_{f(xy)-n}$ (again the bijection depends on f). Then, the map f' is a bijection from $V(G\times K_2)\cup E(G\times K_2)$ to $[1, 3n+2m]$. In addition, as $\mathcal{P}_1$ is well-distributed in $[1, 3n]$, we can choose f' verifying $f'(V(G \times K_2)) = [1, 2n]$.

Now, let H be a subgraph of $G \times K_2$ isomorphic to a 4-cycle. Since G is C_4-free, every 4-cycle H of $G \times K_2$ has the form,

$$V(H) \cup E(H) = A_x \cup A_y \cup B_{xy},$$

where $x, y \in V(G)$ and $xy \in E(G)$. Then, the sum of the elements in any 4-cycle H of $G \times K_2$ is

$$\begin{aligned} f'(H) &= f'(A_x) + f'(A_y) + f'(B_{xy}) \\ &= 2a + f(x) + f(y) + b + f(xy) - n, \\ &= 2a + b + s(f) - n. \end{aligned}$$

independent of H. □

As an application of Theorem 2.13 we have the next corollary.

Corollary 2.14 *The following two families of graphs are C_4-supermagic for n odd.*

(1) *The prims, $C_n \times K_2$.*

(2) *The books, $K_{1,n} \times K_2$.* □

2.4.2 C_r-magic graphs

Here, we give a family of C_r-supermagic graphs for any integer $r \geq 3$. Let C_r be a cycle of length $r \geq 3$. Consider the graph $W(r,k)$ obtained by identifying one vertex in each of the $k \geq 2$ disjoint copies of the cycle C_r. The resulting graphs are called *windmills*, and $W(3,k)$ is also known as the *friendship graph*. Note that windmills clearly admit a C_r-covering. We next show that they are C_r-supermagic graphs.

A remark to keep in mind is that, for the proofs of the last part of this chapter, we shall use the notation introduced in Chapter 1 for $G \setminus A$ where $A \subseteq V(G)$. Recall that this is defined by the deletion of the vertices in A but not the incident edges.

Theorem 2.15 *For any two integers $r \geq 3$ and $k \geq 2$, the windmill $W(r,k)$ is C_r-supermagic.*

Proof. Let $G_1, \ldots, G_k$ be the r-cycles of $W(r,k)$ and let v their only common vertex. Denote by $G^* = W(r,k) \setminus \{v\}$ and its set of vertices and edges by V^* and E^* respectively. Therefore, we have $|V^*| = (r-1)k$ and $|E^*| = rk$.

We want to define a C_r-supermagic total labeling f of $W(r,k)$ with the integers from $[1,(2r-1)k+1]$ such that $f(V(W(r,k))) = f(V^* \cup \{v\}) = [1,(r-1)k]+1$.

Suppose first that k is odd. Let $f(v) = 1$.

By Lemma 2.9 (1) there is a k-equipartition $\mathcal{P} = \{X_1, \ldots, X_k\}$ of the set $1+[1,(2r-1)k]$ such that $|\sum \mathcal{P}| = 1$. Furthermore, as it is well-distributed, in each set X_i there are $r-1$ elements less or equal than $1+(r-1)k$.

Define f on each $G_i^* = G_i \setminus \{v\}$, $1 \leq i \leq k$, by any bijection from G_i^* to X_i such that $f(V_i^*) \subset [1,(r-1)k]+1$, where V_i^* is the set of vertices of G_i^*.

Suppose now that k is even. In this case, let $f(v) = k/2+1$.

By Lemma 2.9 (2) there is a k-equipartition $\mathcal{P} = \{X_1, \ldots, X_k\}$ of the set $[1,(2r-1)k+1] \setminus \{k/2+1\}$ such that $|\sum \mathcal{P}| = 1$. Furthermore, there are $r-1$ elements less or equal than $1+k(r-1)$ in each set X_i.

In this case, define also $f(G_i^*)$ by any bijection from G_i^* to X_i such that $f(V_i^*) \subset [1,(r-1)k+1] \setminus \{k/2+1\}$.

In both cases, for each $1 \leq i \leq k$,

$$f(G_i) = \sum X_i + f(v).$$

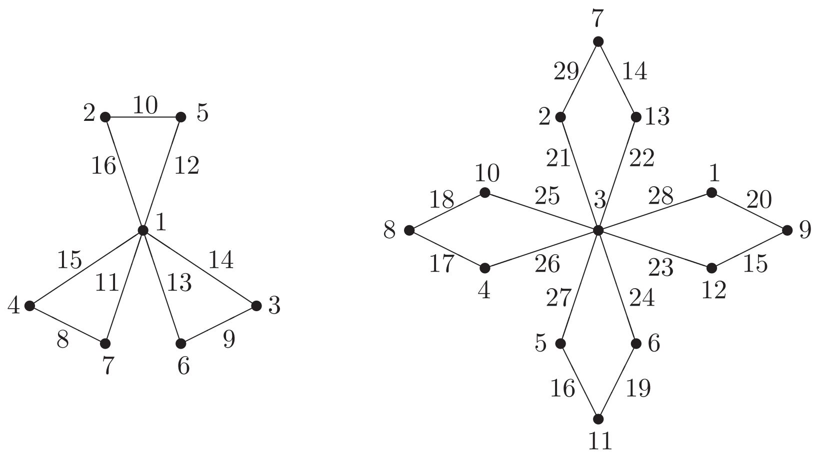

Figure 2.5 C_k-supermagic labelings of $W(k,k)$, for $k = 3, 4$.

Hence f is a C_r-supermagic labeling of the windmill $W(r,k)$. □

See Fig. 2.5 for examples of cycle-supermagic labelings of windmills for different parities of the cycles.

Next, we consider a family of graphs obtained by subdivisions of a wheel. Given the wheel W_n, we denote by $v_1, v_2, \ldots, v_n$ the vertices in the n-cycle and by v its central vertex, as in the proof of Theorem 2.11. The *subdivided wheel* $W_n(r,k)$ is the graph obtained from the wheel W_n by replacing each radial edge vv_i, $1 \le i \le n$ by a vv_i-path of size $r \ge 1$, and every external edge v_iv_{i+1} by a v_iv_{i+1}-path of size $k \ge 1$. It is clear that, $|V(W_n(r,k))| = n(r+k)+1$ and $|E(W_n(r,k))| = n(r+k)$.

Fig. 2.6 shows a subdivided wheel from W_6.

Theorem 2.16 *Let r and k be two positive integers. The subdivided wheel $W_n(r,k)$ is C_{2r+k}-magic for any odd $n \neq \frac{2r}{k}+1$. Furthermore, $W_n(r,1)$ is C_{2r+1}-supermagic.*

Proof. Let $n \ge 3$ be an odd integer.

Denote by v the central vertex of the subdivided wheel $W_n(r,k)$ and by $v_1, v_2, \ldots, v_n$ the remaining vertices of degree > 2. For $1 \le i \le n$ let P_i be the vv_i-path of length $r \ge 1$.

Let $P_i^* = P_i \setminus \{v\}$, $1 \le i \le n$ and $P^* = \cup_{i=1}^n P_i^*$.

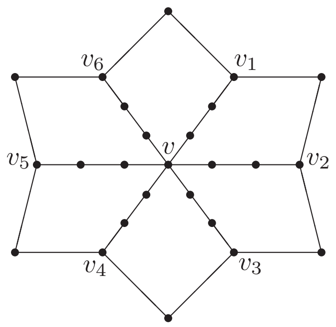

Figure 2.6 The subdivided wheel $W_6(3,2)$.

Suppose first $k=1$.

In this case, we want a C_{2r+1}-magic labeling f on $W_n(r,1)$ with the integers in $[1, 1+2nr+n]$ such that $f(V)=[1, nr+1]$. Let $f(v)=1$ and $f(v_nv_1)=2nr+2$, and label the remaining edges of the external cycle of $W_n(r,1)$ by $f(v_iv_{i+1})=2nr+2+n-i$, $1\le i<n$.

The only elements left to label are the ones in P^*, with $|P^*|=2nr$. Since $n\ge 3$ is odd, Lemma 2.6 ensures the existence of a well-distributed n-equipartition $\mathcal{P}_1=\{X_1,\ldots,X_n\}$ of the set $1+[1,2nr]$ such that $\sum X_i = a+i, 1\le i\le n$ for some constant a. Moreover, as $\mathcal{P}_1$ is well-distributed, each X_i has r elements in $1+[1,nr+1]$. Now define f on each P_i^* by a bijection with $X_{\alpha(i)}$ that assigns the first r values of $[1,nr]$ to the vertices, where α is the following permutation of $[1,n]$.

$$\alpha(i)=\begin{cases} i/2, & i \text{ even;}\\ (n+i)/2, & i \text{ odd.}\end{cases}$$

Note that, as n is odd, $\alpha(n)+\alpha(1)=n+(n+1)/2$ and, for $1\le i<n$, we have $\alpha(i)+\alpha(i+1)=i+(n+1)/2$.

Therefore, f is clearly a bijection from $W_n(r,1)$ to $[1, n(2r+1)+1]$, and $f(V)=[1,nr+1]$.

Now, since $n\ne 2r+1$, for every subgraph H of $W_n(r,1)$ isomorphic to C_{2r+1}, we have either

$$\begin{aligned} V(H)\cup E(H) &= \{v\}\cup P_n^*\cup\{v_nv_1\}\cup P_1^* \text{ or}\\ V(H)\cup E(H) &= \{v\}\cup P_i^*\cup\{v_iv_{i+1}\}\cup P_{i+1}^*, \text{ for some } 1\le i<n.\end{aligned}$$

Then, for each $1 \leq i < n$, we have

$$\begin{aligned} \sum f(H) &= \sum f(P_i^*) + \sum f(P_{i+1}^*) + f(v_i v_{i+1}) + f(v) \\ &= 2a + \alpha(i) + \alpha(i+1) + (n(2r+1) + 2 - i) + 1 \\ &= 2a + 2nr + \frac{3n+7}{2}, \end{aligned}$$

which is independent of i. A similar computation shows that $\sum f(P_n^*) + \sum f(P_1^*) + f(v_n v_1) + f(v)$ has the same value. Hence f is a C_{2r+1}-supermagic labeling of $W_n(r,1)$.

Suppose now $k > 1$.

In this case, for each $1 \leq i \leq n$, let Q_i be the $v_i v_{i+1}$-path of length $k \geq 1$. Denote by $Q_i^* = Q_i \setminus \{v_i, v_{i+1}\}$ and $Q^* = \cup_i Q_i^*$.

By Lemma 2.6 there is a well-distributed n-equipartition $\mathcal{P}_2 = \{Y_1, \ldots, Y_n\}$, of the set $2nr + [1, n(2k-1)]$ such that, for $1 \leq i \leq n$, $\sum Y_i = b + i$ for some constant b.

Define a total labeling f of $W_n(r,k)$ on $[1, n(2k-1)]$ as follows. Set $f(v) = 2n(r+k) - n + 1$. Define f on P_i^* by any bijection from P_i^* to $X_{\alpha(i)}$, where $\mathcal{P}_1 = \{X_1, \ldots, X_k\}$ and α are defined as in the above case. Define f on Q_i^* by any bijection to Y_{n+1-i}.

Since $n \neq \frac{2r}{k} + 1$, every subgraph H of $W_n(r,k)$ isomorphic to C_{2r+k} verifies either

$$\begin{aligned} V(H) \cup E(H) &= \{v\} \cup P_n^* \cup Q_n^* \cup P_1^* \text{ or} \\ V(H) \cup E(H) &= \{v\} \cup P_i^* \cup Q_i^* \cup P_{i+1}^*, \text{ for some } 1 \leq i < n. \end{aligned}$$

Then, for each $1 \leq i < n$ we have,

$$\begin{aligned} \sum f(H) &= \sum f(P_i^*) + \sum f(P_{i+1}^*) + f(Q_i^*) + f(v) \\ &= 2a + \alpha(i) + \alpha(i+1) + b + (n+1-i) + 2n(r+k) - n + 1 \\ &= 2a + b + 2n(r+k) + \frac{n+5}{2}. \end{aligned}$$

It is also immediate to check that the labels of the remaining $(2r+k)$-cycle have also the same sum. □

Fig. 2.7 shows examples of cycle-supermagic labelings defined in the above proof.

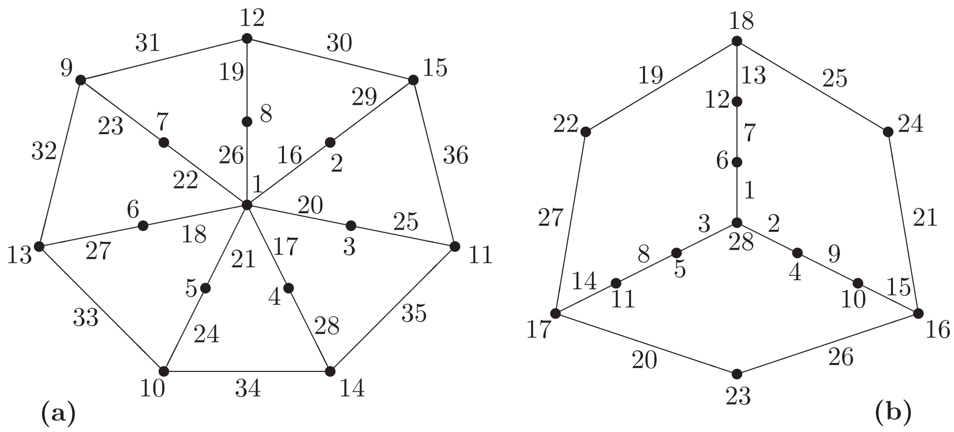

Figure 2.7 **(a)** C_5-supermagic labeling of $W_7(2,1)$. **(b)** C_8-magic labeling of $W_3(3,2)$.

We finish by giving another family of cycle-supermagic graphs. Recall that, for a sequence $k_1, \ldots, k_n$ of positive integers, the graph $\Theta(k_1, \ldots, k_n)$ consists of n internally disjoint paths of orders $k_1 + 1, \ldots, k_n + 1$ joined by two end vertices u and v. When all the paths have the same size p, this graph, denoted by $\Theta_n(p)$, admits a C_{2p}-covering. We next show that such a graph is cycle-supermagic.

Theorem 2.17 *The graph $\Theta_n(p)$ is C_{2p}-supermagic for $n, p \geq 2$.*

Proof. Let u and v be the common end vertices of the paths $P_1, \ldots, P_n$ in $\Theta_n(p) = (V, E)$. Denote by $P_i^* = P_i \setminus \{u, v\}$, $1 \leq i \leq n$.

We want to define a total labeling f of $\Theta_n(p)$ with integers from the interval $[1, (2 + (p-1)n) + np]$ such that $f(V) = [1, 2 + (p-1)n]$.

If n is odd, Lemma 2.9 (1) provides an n-equipartition $\mathcal{P} = \{X_1, \ldots, X_n\}$ of the set $2 + [1, (2p-1)n]$ such that $\sum X_1 = \cdots = \sum X_n = a$, for some constant a, and, as it is well-distributed, in each X_i there are $p-1$ integers less or equal than $2 + (p-1)n$.

Define a total labeling f on $\Theta_n(p)$ as follows. Set $f(u) = 1$, $f(v) = 2$ and f on P_i^* by a bijection from P_i^* to X_i such that the $p-1$ numbers in each X_i that are less or equal than $2 + (p-1)n$ are used for the $p-1$ vertices in each P_i^*.

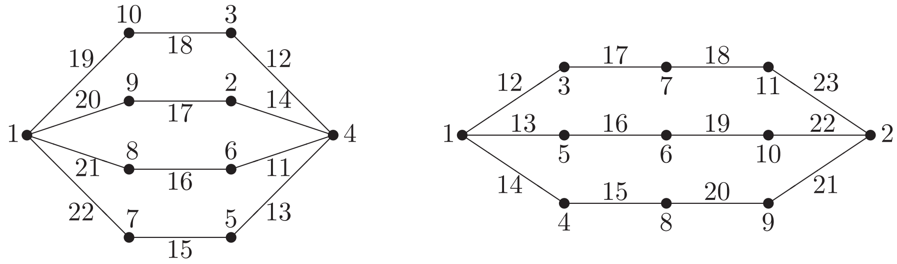

Figure 2.8 C_6-supermagic labeling of $\Theta_4(3)$ and C_8-supermagic labeling of $\Theta_3(4)$.

Every subgraph H of $\Theta_n(p)$ isomorphic to C_{2p} is of the form

$$V(H) \cup E(H) = \{u\} \cup P_i^* \cup \{v\} \cup P_j^*,$$

for $1 \le i < j \le n$.

It is easy to check that $\sum f(H) = 2a + 3$.

Assume now that n is even. By Lemma 2.9 (2) there exists an n-equipartition $\mathcal{P} = \{X_1, \ldots, X_n\}$ of the set $[1, (2p-1)n + 2] \setminus \{1, n/2 + 2\}$ such that $\sum X_1 = \cdots = \sum X_n = a$, for some constant a. Moreover, since $\mathcal{P}$ is well-distributed, in each X_i there are $p-1$ numbers less or equal than $2+(p-1)n$.

Now, we proceed as before but setting $f(v) = n/2 + 2$. It is immediate to check that we indeed get a C_{2p}-supermagic labeling of $\Theta_n(p)$. $\square$

In Fig. 2.8 two supermagic labelings of $\Theta_n(p)$ for different parities of n and p are displayed.

3

Decomposing by trees

A conjecture of Graham and Häggkvist states that every tree with m edges decomposes every $2m$-regular graph and every bipartite m-regular graph. In this chapter we present two results that can be seen as approximations to the complete solution of the conjecture.

Let T be a tree with a prime number p of edges. In Section 3.3 we show that if the growth ratio of T at some vertex v_0 satisfies $\rho(T, v_0) \geq \phi^{1/2}$, where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio, then T decomposes $K_{2p,2p}$. We also prove that if T has at least $p/3$ leaves then it decomposes $K_{2p,2p}$. This improves previous results by Häggkvist [30] and by Lladó and López [40]. The results follow from an application of Alon's Combinatorial Nullstellensatz [3] to obtain bigraceful labelings and they are collected in [11].

In Section 3.4 we consider a tree T of size m (not necessarily prime) and we show that there exists an integer n with $n \leq \lceil (3m-1)/2 \rceil$ and a tree T_1 with n edges such that decomposes K_{2n+1} and contains T. We also show that there exists an integer n' with $n' \leq 2m-1$ and a tree T_2 with n' edges such that T_2 decomposes $K_{n',n'}$ and contains T. In the latter case, we can improve the bound if there exists a prime p such that $\lceil 3m/2 \rceil \leq p < 2m-1$. The results of this part can be found in [41].

3.1 Introduction

A decomposition of a graph G is a partition $\mathcal{P}$ of its set of edges. When the graph induced by each part of $\mathcal{P}$ is isomorphic to a graph H, we say that H decomposes G and write $H|G$.

A famous conjecture of Ringel from 1963 states that every tree with m edges decomposes the complete graph K_{2m+1} [46]. In spite of the hundreds of papers that have appeared in the literature on the subject (see the dynamic survey of Gallian [24]), Ringel's conjecture is still wide open. Graham and Häggkvist proposed the following generalization of Ringel's conjecture; see, *e.g.*, [30]:

Conjecture 3.1 (Graham and Häggkvist) *Every tree with m edges decomposes every $2m$-regular graph and every bipartite m-regular graph.* □

Conjecture 3.1 in particular asserts that every tree with m edges decomposes the complete bipartite graph $K_{m,m}$. In the sequel we will refer to this particularization of Conjecture 3.1.

Some partial results are known on Conjecture 3.1 that motivate our study. Concerning the bipartite case, Häggkvist [30] showed, among other results, that any tree with m edges and at least $(m+1)/2$ leaves decomposes $K_{2m,2m}$. The authors of [40] showed that some families of trees, like trees whose base tree is a caterpillar, d-ary trees with d odd or trees of diameter at most five, decompose $K_{m,m}$ where m is the number of edges of the tree. In the same paper the authors also showed that a tree with m edges and growth ratio $\rho(T, v_0) \geq \sqrt{2} \approx 1.414\ldots$ at some vertex v_0 decomposes $K_{2m,2m}$. The *growth ratio* of T at vertex v_0 is defined as

$$\rho(T, v_0) = \min\{\frac{|V_{i+1}|}{|V_i|},\ i = 0, 1, \ldots, h-1\},$$

where V_i denote the set of vertices at distance i from v_0 and h denotes the eccentricity of v_0. In Section 3.3 we show the following improvement of this result.

Theorem 3.2 *Let T be a tree with a prime number p of edges. If the growth ratio of T at some vertex v_0 satisfies $\rho(T, v_0) \geq \phi^{1/2} \approx 1.272\ldots$, where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio, then T decomposes $K_{2p,2p}$.* □

We also prove a similar but independent result. The base tree of a tree T is obtained by removing all its end vertices. Let $T = T^{(0)}$ and define $T^{(i)}$ as the base tree of $T^{(i-1)}$ for $i \geq 1$. The *base growth ratio* of T is defined as

$$\rho_b(T) = \min\{\frac{|L_{i-1}|}{|L_i|},\ i = 1, \ldots, h'\},$$

where L_i is the set of leaves of $T^{(i)}$ for $0 \leq i < h'$, $|L_{h'}| = 1$, and h' is the minimum positive integer k such that $T^{(k)}$ is a tree with at most one leaf.

Theorem 3.3 *Let T be a tree with a prime number p of edges. If the base growth ratio of T satisfies $\rho_b(T) \geq \phi$, where $\phi = \frac{1+\sqrt{5}}{2}$, then T decomposes $K_{2p,2p}$.* □

The following examples show that Theorems 3.2 and 3.3 are independent. Let T be a binary tree with eccentricity h such that $p = 2^{h+1} - 1$ is a prime and let T' be the tree obtained from T by adding a new leaf to a vertex in the last level. Then T' has size p and we clearly have $\rho_b(T) = 2$ so that T' decomposes $K_{2p,2p}$ by Theorem 3.3. However, one can easily check that $\rho(T', v) \leq 1$ for each vertex v so that T' does not satisfy the hypothesis of Theorem 3.2. On the other hand, let T be a tree with p edges, p a prime, and a vertex v with eccentricity h such that the levels from v satisfy $|V_2| = c|V_1|$, $|V_i| \geq c|V_{i-1}|$, $2 < i \leq h$, and with all leaves in V_h, where $\phi^{1/2} < c < \phi$. Then $\rho_b(T) = \rho(T, v) = c$ so that T satisfies the hypothesis of Theorem 3.2 but not the ones of Theorem 3.3.

Theorems 3.2 and 3.3 will be derived from the stronger statements of Theorems 3.11 and 3.15 respectively, which provide the same conclusion depending essentially on the cardinality of the last two levels of growth in each case.

We also improve the result of Häggkvist above mentioned on the decomposition by trees with large number of leaves in the case of trees with a prime number of edges.

Theorem 3.4 *Let p be a prime and let T be a tree with p edges. If T has at least $p/3$ leaves then it decomposes $K_{2p,2p}$.* □

In one of the early papers on the subject, Kotzig [35] showed that the substitution of an edge by a sufficiently large path in an arbitrary tree results in a tree T for which Ringel's conjecture holds. Thus every tree is homeomorphic to a tree for which the conjecture holds. On the other hand Kézdy [33] showed that the addition of an unspecified number of leaves to a vertex of a tree results in a tree with n edges that decomposes K_{2n+1}. An analogous result for the decomposition of $K_{n,n}$ was proved in [40]. Therefore, every tree contains the base tree of some tree for which both conjectures hold. However, neither result gives a quantitative estimate of the number of additional vertices that will suffice to make a tree decompose the appropriate complete graph.

In Section 3.4 we consider an approximation to both conjectures and prove that every tree is a large subtree of two trees for which the conjectures hold respectively. Specifically we prove:

Theorem 3.5 *Let T be a tree with m edges.*

(1) *For every odd $n \geq 2m-1$, there exists a tree T' with n edges that contains T and T' decomposes $K_{n,n}$.*

(2) *For every prime $p \geq \lceil 3m/2 \rceil$, there exists a tree T' with p edges that contains T and T' decomposes $K_{p,p}$.* $\square$

Theorem 3.6 *Let T be a tree with m edges. For every $n \geq \lceil (3m-1)/2 \rceil$, there exists a tree T' with n edges that decomposes K_{2n+1} and contains T.* $\square$

The results of this chapter follow by an application of a general technique to obtain cyclic decompositions of complete bipartite graphs, which will be described in the following section.

3.2 Labelings and cyclic decompositions

The classical approach to the decomposition problem of graphs uses labeling techniques that aim to find *cyclic decompositions*. A tree T with m edges cyclically decomposes K_{2m+1} if there is an injection $\phi : V(T) \to [0, 2m]$ such that, for each edge xy of T, the $2m+1$ pairs

$$\{\phi(x) + k \pmod{2m+1},\ \phi(y) + k \pmod{2m+1}\},$$

for $k = 0, 1, \ldots, 2m$, are pairwise disjoint. Thus the translations

$$\phi(V(T)) + k \pmod{2m+1},$$

for $k = 0, 1, \ldots, 2m$, give $2m+1$ edge-disjoint copies of T in K_{2m+1}. Similarly, T cyclically decomposes $K_{m,m}$ if there is a map $\phi : V(T) \to [0, m-1]$ that is injective on each partite set of T such that the translations

$$\phi(V(T)) + k \pmod{m}$$

produce m edge-disjoint copies of T in $K_{m,m}$.

In this section we recall the well-known graph labelings that make possible the cyclic decompositions and we introduce new generalizations of them that will allow us to prove the main results of this chapter.

One of the ingredients of the proofs of this chapter is the polynomial method of Alon [3]. We will use Theorem 1.1 described in Chapter 1 to obtain the graph labelings used for the decompositions. The application of the polynomial method to other related graph labeling problems can be seen in [31], [33] and [34].

3.2.1 Bigraceful labelings

An appropriate bipartite labeling, the bigraceful labeling, was first introduced by Ringel and Lladó, see, e.g., [40]. A *bigraceful* labeling of a tree T with m edges and partite sets A and B is a map f of V on the integers $\{0, 1, \dots, m-1\}$ such that the restriction of f to each partite set is injective and the induced edge values, $f_E(uv) = f(u) - f(v)v$ for an edge $uv \in E(T)$ and $u \in A$, are pairwise distinct and must lie in $\{0, 1, \dots, m-1\}$. These authors conjectured that all trees are bigraceful. Since a tree T that admits a bigraceful labeling cyclically decomposes $K_{m,m}$, this conjecture would imply Conjecture 3.1 for the complete bipartite graph.

Here we consider the following modification of the bigraceful labeling introduced above, which take values in an arbitrary abelian group (see [11]).

Let $H = H(A, B)$ be a bipartite graph with partite sets A and B and let $(\mathcal{G}, +)$ be an abelian group. A map $f : A \cup B \to \mathcal{G}$ is $\mathcal{G}$-*bigraceful* if

(i) the restrictions of f to each partite set are injective maps, and

(ii) the induced values of f over the edges of H are pairwise distinct, where for an edge $e = uv$, the induced value of f on e is $f_E(uv) = f(u) + f(v)$.

We will say that a $\mathbb{Z}_m$-bigraceful map of a bipartite graph H with m edges is a *modular bigraceful* labeling. Note that if H admits a bigraceful labeling f then it admits a modular bigraceful labeling f': if A and B are the two partite sets of H just define $f'(x) = f(x) \pmod{m}$ if $x \in A$ and $f'(x) = -f(x) \pmod{m}$ if $x \in B$.

Note also that in the definition of the bigraceful labeling, the two partite sets of the bipartite graph play an asymmetric role, since the induced edge values are defined as the difference between labels of vertices in A minus the labels of the vertices in B and must be positive or zero, implying that the labels in A should be greater than the ones in B. We avoid this asymmetry in condition (ii) above since the labels are in a group. In order to perform the cyclic decompositions from a $\mathcal{G}$-bigraceful map f we can use the auxiliary labeling f_1 defined as f in A and as $-f$ in B. Fig. 3.1 shows an example.

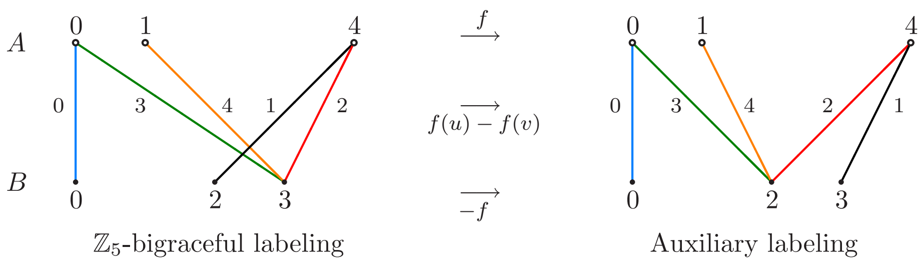

Figure 3.1 Modular bigraceful labeling of a tree and the corresponding auxiliary labeling

With this remark in mind, it is shown in [40, Lemma 1.1] that, if a bipartite graph H admits a $\mathcal{G}$-bigraceful map on a group $\mathcal{G}$ of order m, then the complete bipartite graph $K_{m,m}$ contains m edge-disjoint copies of H. In particular, if H has m edges then H decomposes $K_{m,m}$. Nevertheless, instead of trying to construct a $\mathbb{Z}_m$-bigraceful labeling for an arbitrary tree of size m, and thus decompose $K_{m,m}$, we will work with larger groups in order to decompose $K_{2m,2m}$, a somewhat easier task for which our tools provide positive results.

Fig. 3.2 shows the cyclic decomposition of $K_{5,5}$ by the tree of Fig. 3.1. The auxiliary labeling obtained from the modular bigraceful map gives five different slopes for the edges of the tree and therefore, when performing the translations of the tree, there will be no edge overlaps.

Remark 3.7 *In* [41] *a slight modification of the definition of the $\mathcal{G}$-bigraceful labelings is considered. This modification consists in defining the induced edge values by $f_E(uv) = f(v) - f(u)$, with $u \in A$ and $v \in B$. It is clear that both definitions are equivalent since we can switch from one to the other simply by changing the labels of one partite set by their inverses in $\mathcal{G}$. Therefore, all the decomposition properties also hold with this modification.*

Throughout this chapter, and for the sake of simplicity, we shall use the original additive definition from [11]. □

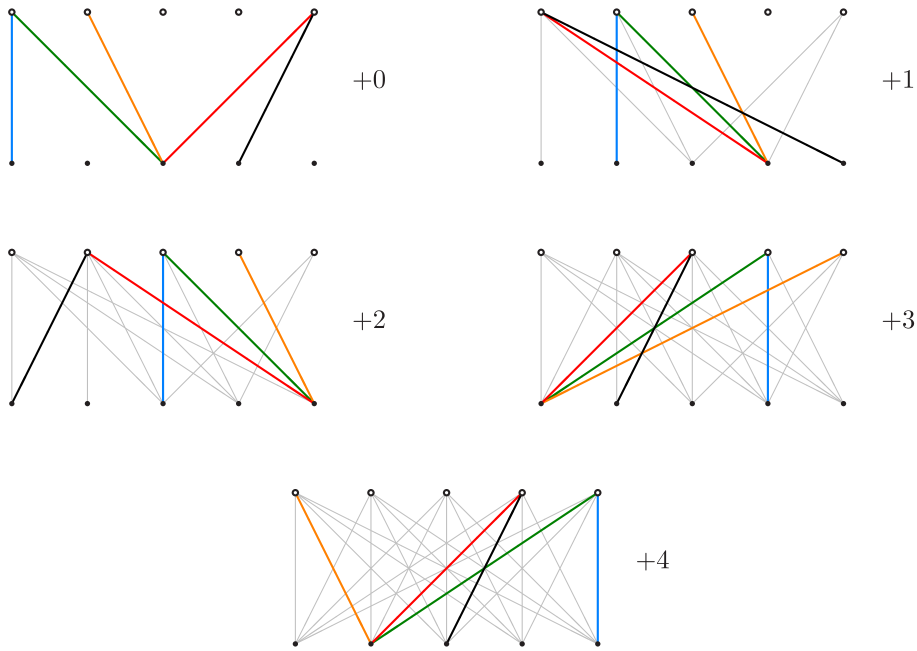

Figure 3.2 Cyclic decomposition of $K_{5,5}$.

3.2.2 ρ-valuations

A *ρ-valuation* of a graph H on m edges is an injection $\rho : V(H) \to \mathbb{Z}_{2m+1}$ such that the induced edge labels $\rho_E(uv) := \rho(u) - \rho(v)$, for $uv \in E(H)$, satisfy

$$\rho_E(e) \neq \pm\rho_E(f) \pmod{2m+1},$$

for all distinct pairs of edges $e, f \in E(H)$. Rosa [47] proved that a graph H with m edges cyclically decomposes K_{2m+1} if and only if it admits a ρ-valuation.

For our present purposes we define a relaxation in the definition of a ρ-valuation. Given a graph H with m edges and given $n \geq m$, a *ρ_n-valuation* is an injection $\rho_n : V(H) \to \mathbb{Z}_{2n+1}$ such that the induced edge labels defined as above (but now taking the differences modulo $2n+1$) are pairwise distinct (see [41]).

3.3 Decomposing $K_{2p,2p}$

In this section we prove Theorems 3.2, 3.3 and 3.4.

3.3.1 The basic lemmas

The main results are based on the following two lemmas. They are obtained through an application of the polynomial method.

Lemma 3.8 *Let p be a prime and let T be a tree with m edges and partite sets A and B. Let $A_0 \subset A$ be a set of end vertices of T. If*

$$p - m \geq |A \setminus A_0|,$$

then every $\mathbb{Z}_p$-bigraceful map of $T - A_0$ can be extended to a $\mathbb{Z}_p$-bigraceful map of T.

Proof. Let f' be a $\mathbb{Z}_p$-bigraceful map of $T' = T - A_0$. Set $r = |A_0|$ and $r' = |A \setminus A_0|$.

Let $x_1, \ldots, x_r$ be the vertices of A_0. Denote by $g_B(x_i)$ the vertex in B adjacent to x_i, $i = 1, \ldots, r$. Consider the following polynomials in $\mathbb{Z}_p[z_1, \ldots, z_r]$:

$$\begin{aligned} P_1 &= V(z_1, \ldots, z_r), \\ P_2 &= V(z_1 - f'(g_B(x_1)), \ldots, z_r - f'(g_B(x_r))), \\ P_3 &= \prod_{i=1}^{r} \prod_{a \in A \setminus A_0} (z_i - f'(g_B(x_i)) - f'(a)). \end{aligned}$$

Let $P = P_1 P_2 P_3$. Note that

$$P = (V(z_1, \ldots, z_r))^2 z_1^{r'} \cdots z_r^{r'} + \text{ terms of lower degree.}$$

By (1.2), the polynomial P has the monomial of maximum degree

$$z_1^{r+r'-1} \cdots z_r^{r+r'-1},$$

with coefficient $\pm r!$, and since $r < p$, it is nonzero modulo p.

Let $C \subset \mathbb{Z}_p$ be the set of edge values of f' on T'. Since $p - m \geq r'$ and $|A| = r + r'$, we have

$$p - |C| = (|A| + |B| - 1) + (p - m) - (r' + |B| - 1) = r + (p - m) > r + r' - 1.$$

By Theorem 1.1, there exists $\mathbf{a} = (a_1, \ldots, a_r)$ such that $a_i \in \mathbb{Z}_p \backslash C$, $1 \le i \le r$, with $P(\mathbf{a}) \neq 0$.

Define f on A_0 by $f(x_i) = a_i - f'(g_B(x_i))$, $1 \le i \le r$. In this way, $a_1, \ldots, a_r$ are precisely the edge values of f on the edges connecting B with A_0, which are different from the edge values of f' on T' (since $a_i \notin C$). Since $P_1(\mathbf{a}) \neq 0$ these edge values a_i are pairwise distinct. Since $P_2(\mathbf{a}) \neq 0$, f is injective on A_0. Finally, since $P_3(\mathbf{a}) \neq 0$, the values $g_B(x_i) + a_i$ do not belong to $f'(A \setminus A_0)$ and f is injective on the whole set A. Thus f is a $\mathbb{Z}_p$-bigraceful map of T. $\square$

Next lemma shows the way to decompose $K_{2p,2p}$.

Lemma 3.9 *Let T be a tree with a prime number p of edges and partite sets A and B. Let $T_0 = T - B_0 - A_0$ where $B_0 \subset B$ is a set of end vertices of T and $A_0 \subset A$ is a set of end vertices of $T - B_0$. If T_0 admits a $\mathbb{Z}_p$-bigraceful map then T decomposes $K_{2p,2p}$.*

Proof. Consider the graph G with vertex set $\mathbb{Z}_p \times \mathbb{Z}_4$ and vertex (α, β) is adjacent to $(\alpha + i, \beta + 1)$ for each $i \in \mathbb{Z}_p$. This graph G is isomorphic to $K_{2p,2p}$. We consider G as an edge-colored graph, the edge $(\alpha, \beta)(\alpha + i, \beta + 1)$ being colored $i \in \mathbb{Z}_p$.

Let f_0 be a $\mathbb{Z}_p$-bigraceful map of T_0. Consider the map $f_0' : V(T_0) \to \mathbb{Z}_p \times \mathbb{Z}_4$ defined as $f_0'(x) = (f_0(x), 1)$ for $x \in A \setminus A_0$ and $f_0'(y) = (f_0(y), 2)$ for $y \in B \setminus B_0$. Thus f_0' is an embedding of T_0 in G such that the colors of the edges, which are the edge-values of f_0, are pairwise distinct.

We will extend f_0' to an embedding f' of T in G in such a way that the colors of the edges will be pairwise distinct. The argument follows the same lines as the proof of Lemma 3.8.

Let C_0 be the edge values of f_0 on T_0. Set $r = |A_0|$, $r' = |A \setminus A_0|$, $s = |B_0|$ and $s' = |B \setminus B_0|$.

Let $x_1, \ldots, x_r$ be the vertices in A_0. Consider the polynomials

$$\begin{aligned} P_1^A &= V(z_1, \ldots, z_r), \\ P_2^A &= V(z_1 - f_0(g_B(x_1)), \ldots, z_r - f_0(g_B(x_r))), \end{aligned}$$

where $g_B(x_i)$ is the vertex in $B \setminus B_0$ adjacent to x_i, $1 \le i \le r$. By (1.2), the polynomial

$$P^A = P_1^A P_2^A = (V(z_1, \ldots, z_r))^2 + \text{ terms of lower degree}$$

has a monomial of maximum degree

$$z_1^{r-1}\cdots z_r^{r-1}$$

with coefficient $\pm r! \not\equiv 0 \pmod p$, as $r < p$. Since

$$p - |C_0| = (r + r' + s + s' - 1) - (r' + s' - 1) > r - 1,$$

by Theorem 1.1 there is $\mathbf{a} = (a_1, \ldots, a_r)$ with $a_i \in \mathbb{Z}_p \setminus C_0$ with $P^A(\mathbf{a}) \neq 0$.

Define an extension $f_1(x)$ of f_0 to $T_1 = T_0 + A_0$ by $f_1(v) = f_0(v)$ if $v \in T_0$ and $f_1(x_i) = a_i - f_0(g_B(x_i))$ if $x_i \in A_0$.

Define $f'(x_i) = (f_1(x_i), 3) = (a_i - f_0(g_B(x_i)), 3)$, $1 \leq i \leq r$. Since $P_2^A(\mathbf{a}) \neq 0$, the values of f' on A_0 are pairwise distinct, and since $P_1^A(\mathbf{a}) \neq 0$ the a_i's are pairwise distinct (and different from the edge values of f_0 since $a_i \notin C_0$).

Similarly, let $\{y_1, \ldots, y_s\} = B_0$ and now let $g_A(y_i)$ denote the vertex in A adjacent to y_i, $1 \leq i \leq s$. The polynomial $P^B = P_1^B P_2^B$, where

$$\begin{aligned} P_1^B &= V(z_1, \ldots, z_s), \\ P_2^B &= V(z_1 - f_1(g_A(y_1)), \ldots, z_s - f_1(g_A(y_s))), \end{aligned}$$

has a monomial of maximum degree

$$z_1^{s-1}\cdots z_s^{s-1}$$

with coefficient $\pm s! \not\equiv 0 \pmod p$, as $s < p$. Let $C_1 = C_0 \cup \{a_1, \ldots, a_r\}$. Since

$$p - |C_1| = (r + r' + s + s' - 1) - (r + r' + s' - 1) > s - 1,$$

again by Theorem 1.1 there is $\mathbf{b} = (b_1, \ldots, b_s)$ with $b_i \in \mathbb{Z}_p \setminus C_1$ with $P^B(\mathbf{b}) \neq 0$. Define $f'(y_i) = (b_i - f_1(g_A(y_i)), 0)$, $1 \leq i \leq s$. Since $P_2^A(\mathbf{b}) \neq 0$, the values of f' on B_0 are pairwise distinct, and since $P_1^B(\mathbf{b}) \neq 0$ the b_i's are pairwise distinct (and they do not belong to C_1).

Thus we have extended f_0' to an embedding f' of the whole tree T in G in such a way that the colors of the edges are pairwise distinct.

Each translation $(\alpha, \beta) \mapsto (\alpha, \beta) + (i, j)$ with fixed (i, j), preserves the edge colors. Hence, the translations by the vectors $(i, 0)$, $1 \leq i \leq p$ and the vectors $(0, j)$, $0 \leq j \leq 3$, give $4p$ edge-disjoint copies of T covering all the edges of G exactly once, so that T decomposes $G \simeq K_{2p,2p}$. $\square$

3.3.2 Trees with large growth ratio

In this section we shall prove Theorem 3.2. The result will be derived from the more general Theorem 3.11 below.

In what follows we use the following notation. Let T be a tree with partite sets A and B, and let $v_0 \in A$ be a fixed vertex of T with eccentricity h. Denote by V_i, $0 \le i \le h$, the set of vertices of T at distance i from v_0. We also define, for $0 \le r \le h$, $A_r^+ = A \cap \bigcup_{j \le r} V_j$ and $B_r^+ = B \cap \bigcup_{j \le r} V_j$. Note that, since $v_0 \in A$, we have $A_r^+ = \bigcup_{j \le r,\ j \text{ even}} V_j$ and $B_r^+ = \bigcup_{j \le r,\ j \text{ odd}} V_j$.

Lemma 3.10 *Let p be a prime and let T be a tree with m edges. Let v_0 be a vertex of the tree with eccentricity $h \ge 2$. If*

$$p - m \ge \max\{|A_{h-2}^+|, |B_{h-2}^+|\}$$

then T admits a $\mathbb{Z}_p$-bigraceful map.

Proof. For $k = 1, 2, \ldots, h$, let $T_k = T - (V_{k+1} \cup \cdots \cup V_h)$ denote the subtree of T induced by the first k levels of T. Suppose that T_{k-1} admits a $\mathbb{Z}_p$-bigraceful labeling f'.

If m' is the size of T_k then $p - m' \ge p - m \ge \max\{|A_{h-2}^+|, |B_{h-2}^+|\} \ge \max\{|A_{k-2}^+|, |B_{k-2}^+|\}$. Hence T_k satisfies the hypothesis of Lemma 3.8 (with $A_0 = V_k$) and there is a $\mathbb{Z}_p$-bigraceful labeling of T_k. Since T_1 is a star, which clearly admits a $\mathbb{Z}_p$-bigraceful labeling, the result follows by an iterated application of Lemma 3.8. □

Theorem 3.11 *Let p be a prime and let T be a tree with p edges. Let v_0 be a vertex of the tree with eccentricity $h \ge 4$. If*

$$|V_h| + |V_{h-1}| \ge \max\{|A_{h-4}^+|, |B_{h-4}^+|\}$$

then T decomposes $K_{2p,2p}$.

Proof. Let $T' = T - V_h - V_{h-1}$. Since $p - |E(T')| = |V_h| + |V_{h-1}| \ge \max\{|A_{h-4}^+|, |B_{h-4}^+|\}$, Lemma 3.10 implies that T' admits a $(\mathbb{Z}_p)$-bigraceful labeling and then by Lemma 3.9, T decomposes $K_{2p,2p}$. □

The hypothesis of Theorem 3.11 are illustrated in Fig. 3.3.

To prove that a tree with growth ratio $\sqrt{\frac{1+\sqrt{5}}{2}}$ and p edges decomposes $K_{2p,2p}$ we will need the following technical lemma.

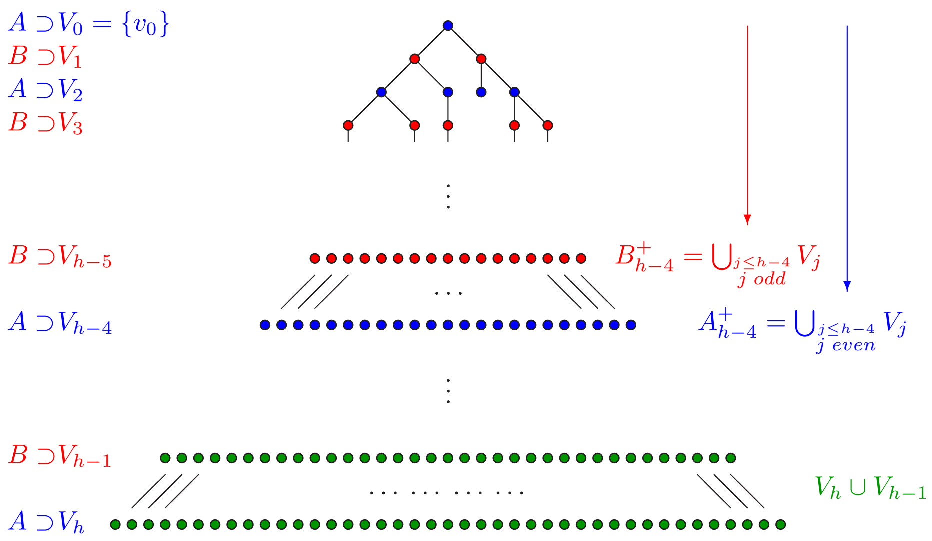

Figure 3.3 Hypothesis of Theorem 3.11

Lemma 3.12 *Let T be a tree with growth ratio $\alpha \ge 1$ at some vertex v_0 with eccentricity $h \ge 4$. Then*

$$\max\{|A^+_{h-4}|, |B^+_{h-4}|\} < \frac{c}{\alpha^4},$$

where $c = |A|$ if h is even and $c = |B|$ if h is odd.

Proof. Suppose first that h is even. We have

$$\begin{aligned} |A^+_{h-4}| - |B^+_{h-4}| &= \sum_{\substack{j \le h-4 \\ j\ even}} |V_j| - \sum_{\substack{j \le h-4 \\ j\ odd}} |V_j| \\ &= |V_0| + (|V_2| - |V_1|) + \cdots + (|V_{h-4}| - |V_{h-5}|) \ge 0 \end{aligned}$$

since $|V_{i+1}| \ge |V_i|$ for each $i = 0, 1, \ldots, h-1$. Thus $\max\{|A^+_{h-4}|, |B^+_{h-4}|\} = |A^+_{h-4}|$.

On the other hand,

$$\begin{aligned}|A^+_{h-4}| = \sum_{\substack{j\le h-4\\ j\ even}} |V_j| &\le \frac{1}{\alpha}\sum_{\substack{j\le h-3\\ j\ odd}} |V_j| < \frac{1}{\alpha^2}\sum_{\substack{j\le h-2\\ j\ even}} |V_j| \\ &\le \frac{1}{\alpha^3}\sum_{\substack{j\le h-1\\ j\ odd}} |V_j| < \frac{1}{\alpha^4}\sum_{\substack{j\le h\\ j\ even}} |V_j| = \frac{1}{\alpha^4}|A|,\end{aligned}$$

which proves the inequality.

The case h odd can be similarly seen by exchanging the roles of A^+_{h-4} and B^+_{h-4}. □

Proof of Theorem 3.2. Since trees with diameter at most 7 decompose $K_{2p,2p}$ [40, Corollary 3.2], we can assume that the eccentricity of v_0 is $h \ge 4$.

Suppose that h is even. By Lemma 3.12,

$$\begin{aligned}\max\{|A^+_{h-4}|, |B^+_{h-4}|\} &< \frac{1}{\alpha^4}|A| \\ &\le \frac{|V_h|}{\alpha^4}\left(1 + \frac{1}{\alpha^2} + \frac{1}{\alpha^4} + \cdots + \frac{1}{\alpha^h}\right) \\ &= \frac{|V_h|}{\alpha^4}\left(\frac{1 - 1/\alpha^{h+2}}{1 - 1/\alpha^2}\right).\end{aligned}$$

Therefore, if $\frac{1-1/\alpha^{h+2}}{\alpha^4(1-1/\alpha^2)} \le 1$ then we are on the hypothesis of Theorem 3.11. Last inequality holds if $\alpha^4(1 - \frac{1}{\alpha^2}) = \alpha^4 - \alpha^2 \ge 1$, and this is true for all $\alpha \ge \sqrt{\frac{1+\sqrt{5}}{2}}$.

A similar reasoning for h odd gives the same conclusion. □

3.3.3 Trees with large base growth ratio

We can study the decomposition of $K_{2p,2p}$ from a similar, but not equivalent, point of view that leads to the proof of Theorems 3.3 and 3.4. These two theorems will be derived from the more general Theorem 3.15 below. We first state the following direct consequences of Lemma 3.8.

Lemma 3.13 *Let p be a prime and let T be a tree with m edges. Let f' be a $\mathbb{Z}_p$-bigraceful map of the base tree T' of T. If $p - m \ge \max\{|A'|, |B'|\}$, where A' and B' are the two partite sets of T' then f' can be extended to a $\mathbb{Z}_p$-bigraceful map of T.*

Proof. Let $A \supset A'$ and $B \supset B'$ be the partite sets of T. Let $A_0 = A \setminus A'$ and $B_0 = B \setminus B'$.

Since $p - (m - |B_0|) \geq p - m \geq |A'| = |A \setminus A_0|$, it follows from Lemma 3.8 that f' can be extended to a $\mathbb{Z}_p$-bigraceful labeling f_1 of $T' + A_0$. Similarly, since $p - m \geq |B'| = |B \setminus B_0|$, Lemma 3.8 implies again that f_1 can be extended to a $\mathbb{Z}_p$-bigraceful labeling of the whole tree T. $\square$

Corollary 3.14 *Let p be a prime and let T be a tree with m edges. If $p - m \geq \max\{|A'|, |B'|\}$, where A' and B' are the two partite sets of the base tree T' of T, then T admits a $\mathbb{Z}_p$-bigraceful map.*

Proof. Note that the condition $p - m \geq \max\{|A'|, |B'|\}$ is also satisfied by T': if m' is the number of edges of T' then $p - m' > p - m \geq \max\{|A'|, |B'|\} > \max\{|A''|, |B''|\}$, where A'' and B'' are the two partite sets of the base tree T'' of T'. In particular, this condition is also satisfied by each tree in the sequence $T, T', T'', \ldots, T^{(i)}, \ldots, T^{(h')}$, where $T^{(i)}$ is the base tree of $T^{(i-1)}$, $1 \leq i \leq h'$. Since $T^{(h')}$ consists eventually of an edge or a single vertex, which trivially admits a $\mathbb{Z}_p$-bigraceful labeling, the iterated application of Lemma 3.13 gives the result. $\square$

Theorem 3.15 *Let p be a prime and let T be a tree with p edges. Let T' be the base tree of T and let T'' be the base tree of T'. If*

$$|E(T) \setminus E(T')| \geq \max\{|A''|, |B''|\},$$

where A'' and B'' are the partite sets of T'', then T decomposes $K_{2p,2p}$.

Proof. Let $m' = |E(T')|$. By Corollary 3.14 and the condition $p - m' = |E(T) \setminus E(T')| \geq \max\{|A''|, |B''|\}$, the base tree T' of T admits a $\mathbb{Z}_p$-bigraceful labeling. Then, the result follows by Lemma 3.9. $\square$

Fig. 3.4 depicts an example of two trees that show that the statements of Theorems 3.11 and 3.15 are independent. The tree T_1 satisfies the hypothesis of Theorem 3.15 but not the ones of Theorem 3.11, whereas T_2 fulfills the requirements of Theorem 3.11 but not the ones in Theorem 3.15.

Proof of Theorem 3.3. Recall that we denote by L_i the set of leaves of $T^{(i)}$, $i = 0, 1, \ldots, h' - 1$, $|L_{h'}| = 1$, where $T^{(i)}$ is the base tree of $T^{(i-1)}$ for $i \geq 1$ and h' is the minimum positive integer k such that $T^{(k)}$ is a tree with

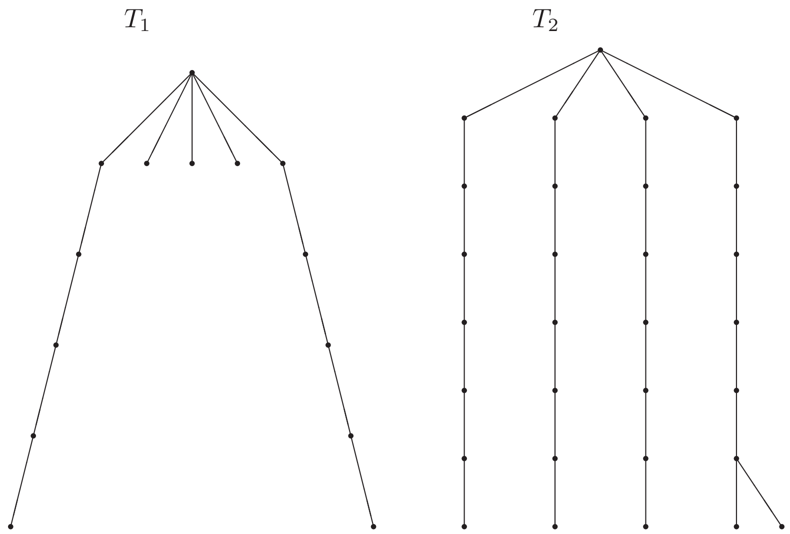

Figure 3.4 Example of trees illustrating the independence of Theorems 3.11 and 3.15.

at most one leaf. We have

$$\begin{aligned}\max\{|A''|, |B''|\} &\leq \sum_{i=2}^{h'} |L_i| \leq \frac{1}{\alpha^2} \sum_{i=0}^{h'} |L_i| \\ &\leq \frac{1}{\alpha^2}|L_0| \left(1 + \frac{1}{\alpha} + \cdots + \frac{1}{\alpha^{h'}}\right) \\ &= \frac{1}{\alpha^2}|L_0| \frac{1 - 1/\alpha^{h'+1}}{1 - 1/\alpha},\end{aligned}$$

where $|L_0| = |E(T) \setminus E(T')|$. Hence, if $\frac{1}{\alpha^2}\frac{1-1/\alpha^{h'+1}}{1-1/\alpha} \leq 1$ we are on the hypothesis of the Theorem 3.15. Since $0 < 1 - 1/\alpha^{h'-1} \leq 1$, it suffices that $\alpha^2(1 - 1/\alpha) \geq 1$. This last inequality holds for $\alpha \geq \phi$. □

Proof of Theorem 3.4. We may assume that T', the base tree of T, is not a caterpillar since otherwise we know that T decomposes $K_{p,p}$ [40].

Let $m \geq p/3$ be the number of leaves of T and let T'' the base tree of T' and A'', B'' its partite sets with $|A''| \geq |B''|$. Note that the number m'' of leaves of T'' satisfies $m'' \geq |A''| - |B''| + 1$ and that the number of leaves of

T' satisfies $m' \geq m''$. Hence,

$$p + 1 = m + m' + |A''| + |B''| \geq p/3 + 2|A''| + 1,$$

which implies $|A''| \leq p/3 \leq m$. Therefore T satisfies the hypothesis of Theorem 3.15 and thus it decomposes $K_{2p,2p}$. □

3.4 Large subtrees

To prove Theorems 3.5 and 3.6 we shall show that a tree T with m edges can be embedded in a tree of the stated size that admits either a modular bigraceful labeling or a ρ-valuation.

We also use a well-known theorem by Kneser. The *stabilizer* $H(C)$ of a subset C in an abelian group $\mathcal{G}$ is defined by

$$H(C) = \{g \in \mathcal{G} : C + g = C\}.$$

In other words, $H(C)$ is the largest subgroup of $\mathcal{G}$ that has the property $H(C) + C = C$. If $\mathcal{G}$ is finite, then $|H(C)|$ divides both $|\mathcal{G}|$ and $|C|$.

Theorem 3.16 (Kneser; see, *e.g.*, [45]) *If A and B are finite non-empty subsets of an abelian group satisfying $|A + B| \leq |A| + |B| - 1$, and H is the stabilizer of $A + B$, then*

$$|A + B| = |A + H| + |B + H| - |H|.$$

□

The next lemma, which is based on Kneser's theorem, will be used later to prove the existence of appropriate labelings.

Lemma 3.17 *Let r be a positive integer and let X_1, X_2, Y be non-empty subsets of $\mathbb{Z}_r$ with $|X_1| \geq |X_2|$ and $|Y| > 1$. If the following condition holds*

$$r - |X_1| - |X_2| = |Y| - 1, \tag{3.1}$$

then $|X_1 + Y| > |X_2|$.

Proof. If $|X_1 + Y| \leq |X_2|$, then we must have $|X_1 + Y| = |X_2| = |X_1| < |X_1| + |Y| - 1$. By Theorem 3.16,

$$|X_1 + Y| = |X_1 + H| + |Y + H| - |H|,$$

where H is the stabilizer of $X_1 + Y$. From this relation and $|X_1 + Y| = |X_1|$ we deduce that $|Y + H| = |H|$ and therefore $|Y| \leq |H|$.

Now, since $|H|$ divides the left hand-side of (3.1), $|H|$ must also divide $|Y| - 1$. Finally, $|Y| > 1$ implies that $|H| \leq |Y| - 1$, contradicting $|Y| \leq |H|$. □

3.4.1 Proof of Theorem 3.5

We first show that a tree that admits a $\mathbb{Z}_n$-bigraceful map can be embedded in a tree with n edges that decomposes $K_{n,n}$.

Lemma 3.18 *Every tree T that admits a $\mathbb{Z}_n$-bigraceful map with n odd is a subtree of a tree T' with n edges that admits a modular bigraceful labeling.*

Proof. Let m be the number of edges of T. Let f be a $\mathbb{Z}_n$-bigraceful map of T. Clearly $n \geq m$. We define a sequence of trees $T_m, T_{m+1}, \ldots, T_n$, with $T_m = T$ and $T_n = T'$, by adding one leaf at each step and extend f on T' as a modular bigraceful map.

Suppose we have defined T_i and a $\mathbb{Z}_n$-bigraceful map f on T_i for some i such that $m \leq i < n$. Let A_i and B_i be the two partite sets of T_i with $|A_i| \geq |B_i|$. Let

$$\begin{aligned} A'_i &= -f(A_i), \\ B'_i &= f(B_i), \\ C_i &= \{f(x) + f(y) : xy \in E(T_i)\}, \\ D_i &= \mathbb{Z}_n \setminus C_i. \end{aligned}$$

Since T_i is a tree, we have the following relation among these sets:

$$|A'_i| + |B'_i| = n - |D_i| + 1. \tag{3.2}$$

It suffices to prove that $|D_i + A'_i| > |B'_i|$. In this case there exists $d \in D_i$ and some $a \in f(A_i)$ such that $d - a \in \mathbb{Z}_n \setminus B'_i$. Define $T_{i+1} = T_i + e_{i+1}$, where e_{i+1} joins the vertex in A_i labeled a to a new vertex v_{i+1} and $f(v_{i+1}) = d - a$; this gives the extension of f to T_{i+1}.

Since $|D_i| = n - |C_i| = n - i \geq 1$, either $|D_i| = 1$ or $|D_i| > 1$. In the former case ($i = n-1$), since n, which equals $|A'_{n-1}| + |B'_{n-1}|$, is odd, $|D_{n-1} + A'_{n-1}| = |A'_{n-1}| > |B'_{n-1}|$. In the latter case we apply Lemma 3.17 with $r = n$, $X_1 = A'_i$, $X_2 = B'_i$ and $Y = D_i$; the condition (3.1) of the lemma holds by (3.2). □

In view of Lemma 3.18, and using the cyclic decomposition from [40], to prove the statement (1) of Theorem 3.5 it suffices to show that every tree T with m edges admits a $\mathbb{Z}_n$-bigraceful labeling for every odd $n \geq 2m - 1$. The next lemma shows that this is indeed the case.

Lemma 3.19 *Every tree T with m edges admits a $\mathbb{Z}_n$-bigraceful map for every n such that $n \geq m + \max\{|A|, |B|\} - 1$, where A and B are the partite sets of T.*

Proof. The proof is by induction on m, the result being obvious for $m = 1$. Let u be a leaf of T with neighbor v. Suppose first that $u \in A$. Let $T' = T - u$, choose an integer n such that $n \geq m + \max\{|A|, |B|\} - 1$, and let f be a $\mathbb{Z}_n$-bigraceful map on T'. Let

$$\begin{aligned} C &= \{f(x) + f(y) : xy \in E(T')\}, \\ D &= \mathbb{Z}_n \setminus C. \end{aligned}$$

Since $|D - f(v)| = |D| = n - m + 1 \geq \max\{|A|, |B|\} \geq |A|$, there exists $d \in D$ such that $d - f(v) \notin f(A \setminus \{u\})$. Extending f to T by defining $f(u) = d - f(v)$ produces a $\mathbb{Z}_n$-bigraceful labeling of T. If $u \in B$, we can perform an analogous reasoning. □

Statement (2) of Theorem 3.5 may give a better upper bound for the minimum n for which we can ensure that there is a tree T' with n edges containing a given tree T with the property that T' decomposes $K_{n,n}$. We use the following simple lemma.

Lemma 3.20 *A tree T with partite sets A and B such that $|A| \geq |B|$ has at least $|A| - |B| + 1$ leaves in A.*

Proof. Let $A' \subset A$ be the set of non-leaves in A, and let $T' = T - (A \setminus A')$. Then $|A'| + |B| - 1 = |E(T')| = \sum_{x \in A'} d(x) \geq 2|A'|$. Hence $|A'| \leq |B| - 1$, and T has at least $|A| - |A'| \geq |A| - |B| + 1$ leaves in A. □

Lemma 3.21 *Let T be a tree with m edges. If p is a prime such that $p \geq \lceil 3m/2 \rceil$, then there is a $\mathbb{Z}_p$-bigraceful map of T.*

Proof. Let A and B be the partite sets of T with $|A| \geq |B|$. By Lemma 3.20 there is a set $A_0 \subset A$ of leaves such that $|A'| = |A \setminus A_0| = |B|$. Let $T' = T - A_0$. Since $|B| \leq \lceil m/2 \rceil$ and $p \geq m + |B|$, it follows from Lemma 3.19 that there is a $\mathbb{Z}_p$-bigraceful map f' of T'. If $A_0 = \emptyset$ we are done. Otherwise, let C' denote the set of edge values of f'. Thus C' is a subset of $\mathbb{Z}_p$ of cardinality $2|A'| - 1$.

Let $A_0 = \{a_1, \ldots, a_k\}$. Let $b_{\sigma(i)}$ be the vertex in B adjacent to a_i, for $1 \leq i \leq k$. Consider the polynomial $P \in \mathbb{Z}_p[z_1, \ldots, z_k]$ defined as

$$P = \prod_{1\leq i<j\leq k} (z_i - z_j) \prod_{1\leq i<j\leq k} (z_i - b'_{\sigma(i)} - (z_j - b'_{\sigma(j)})) \prod_{1\leq i\leq k} \prod_{a\in A'} (z_i - b'_{\sigma(i)} - a'),$$

where $b'_{\sigma(i)} = f'(b_{\sigma(i)})$ and $a' = f'(a)$. We can write

$$P = \prod_{1\leq i<j\leq k} (z_i - z_j)^2 \prod_{1\leq i\leq k} z_i^{|A'|} + \text{ terms of lower degree.}$$

It is known that the coefficient of the monomial $\prod_{i=1}^k z_i^{k-1}$ in the expansion of $\prod_{1\leq i<j\leq k}(z_i - z_j)^2$ is $(-1)^{\binom{k}{2}} k!$ (see, *e.g.*, [4]), which is nonzero modulo p. Therefore P has a monomial

$$z_1^{k+|A'|-1} \cdots z_k^{k+|A'|-1}$$

of maximum degree with nonzero coefficient. Let $D = \mathbb{Z}_p \setminus C'$. Note that $|D| = p - |C'| \geq \lceil 3(2|A'| + k - 1)/2 \rceil - 2|A'| + 1 \geq |A'| + k$. By Theorem 1.1, there exist $d_1, \ldots, d_k \in D$ such that $P(d_1, \ldots, d_k) \neq 0$. Extend f' on T' to f on T by defining $f(a_i) = d_i - f'(b_{\sigma(i)})$. Since

$$\prod_{1\leq i\leq k} \prod_{a\in A'} (d_i - b'_{\sigma(i)} - a') \neq 0,$$

the values of f on A_0 are different from the ones on A'; since

$$\prod_{1\leq i<j\leq k} (d_i - b'_{\sigma(i)} - (d_j - b'_{\sigma(j)})) \neq 0,$$

these values are pairwise distinct. Finally, since

$$\prod_{1\leq i<j\leq k} (d_i - d_j) \neq 0,$$

the edge values $d_1, \ldots, d_k$ of the edges incident to $a_1, \ldots, a_k$ are distinct and, since $d_i \in \mathbb{Z}_p \setminus C'$, they are also different from the ones taken by f on T'. Thus f is a $\mathbb{Z}_p$-bigraceful map of T. □

Theorem 3.5 (2) follows from Lemmas 3.21 and 3.18, and using the cyclic decomposition from [40].

3.4.2 Proof of Theorem 3.6

Following the ideas of the proof of Theorem 3.5, we give an upper bound for the number of edges that have to be added to an arbitrary tree T to obtain a tree that admits a ρ-valuation in terms of the size of T.

Lemma 3.22 *Every tree T with m edges has a ρ_n-valuation for every $n \geq \lceil (3m-1)/2 \rceil$.*

Proof. Let $T_1, T_2, \ldots, T_m$ be trees such that $T_m = T$, T_1 has one edge v_0v_1, and T_{i+1} is obtained from T_i by adding a leaf v_{i+1} adjacent to some $u \in V(T_i)$. Define a ρ_n-valuation of T inductively as follows.

Define $f(v_0) = x_0 \in \mathbb{Z}_{2n+1}$, $f(v_1) = x_1 \in \mathbb{Z}_{2n+1}$ arbitrarily, with $x_0 \neq x_1$. Suppose f is defined on T_i for $1 \leq i < m$, and let

$$\begin{aligned} V_i &= f(V(T_i)), \\ C_i &= \{\pm(f(x) - f(y)) : xy \in E(T_i)\} \cup \{0\}, \\ D_i &= \mathbb{Z}_{2n+1} \setminus C_i. \end{aligned}$$

Since $|D_i + f(u)| = |D_i| = 2n+1-2i-1 \geq m+1 > |V_i|$, there exists $d \in D_i$ such that $d + f(u) \in \mathbb{Z}_{2n+1} \setminus V_i$. Thus we can define $f(v_{i+1}) = d + f(u)$. At the end, we have a ρ_n-valuation of T. □

Lemma 3.23 *Every tree T of size m that admits a ρ_n-valuation for $n \geq m$ can be embedded into a tree T' of size n that admits a ρ-valuation.*

Proof. If $n = m$ we are done. Otherwise, let f be the ρ_n-valuation of T. We define a sequence of trees $T_m, T_{m+1}, \ldots, T_n$ with $T_m = T$ and $T_n = T'$, by adding one leaf at each step and extend f to T' as a ρ-valuation.

Suppose we have defined T_i and a ρ_n-valuation f on T_i for some i such that $m \le i < n$. Let

$$\begin{aligned} V_i &= f(V(T_i)), \\ C_i &= \{\pm(f(x) - f(y)) : xy \in E(T_i)\} \cup \{0\}, \\ D_i &= \mathbb{Z}_{2n+1} \setminus C_i. \end{aligned}$$

Since T_i is a tree, we have the following relation:

$$2|V_i| - 1 = 2n + 1 - |D_i|. \tag{3.3}$$

Since $|D_i| = 2n + 1 - |C_i| = 2n - 2i \ge 2$ we can apply Lemma 3.17 with $r = 2n + 1$, $X_1 = X_2 = V_i$, and $Y = D_i$ to obtain $|D_i + V_i| > |V_i|$. By (3.3), condition (3.1) of Lemma 3.17 holds. Therefore there exists $d \in D_i$ and some $a \in V_i$ such that $d + a \in \mathbb{Z}_{2n+1} \setminus V_i$. Let $T_{i+1} = T_i + e_{i+1}$ where e_{i+1} joins the vertex in V_i labeled with a to a new vertex v_{i+1}. By defining $f(v_{i+1}) = d + a$ we extend f to a ρ_n-valuation of T_{i+1}. By iterating this procedure we eventually get a ρ-valuation of a tree T' that contains T as a subtree. □

Theorem 3.6 is a direct consequence of Lemmas 3.22 and 3.23, and the fact that a graph with m edges cyclically decomposes K_{2m+1} if and only if it admits a ρ-valuation (Rosa, [47]).

Another related result is given by Van Bussel [10, Theorem 1]; it implies that every tree with m edges has a ρ_n-valuation, with $n = 2m - diam(T)$. Since a random tree has diameter of order $\sqrt{m}$, this lower bound is in general worse than the one obtained in Theorem 3.6 (see also Lemma 3.22).

4

Sumset partition problem

A sequence $m_1 \geq m_2 \geq \cdots \geq m_k$ of k positive integers is *n-realizable* if there is a partition $X_1, \ldots, X_k$ of the set $[n]$ such that the sum of the elements in X_i is m_i for each $i = 1, 2, \ldots, k$. In Section 4.2 we study the n-realizable sequences by adopting a different viewpoint from the one that has been previously used in the literature. We consider the n-realizability of a sequence in terms of the length k of the sequence or the values of its distinct elements. We characterize all n-realizable sequences with $k \leq 4$ and, for $k \geq 5$, we prove that $n \geq 4k-1$, $m_k > 4k-1$ and $\sum_{i=1}^{k} m_i = \binom{n+1}{2}$ are sufficient conditions for a sequence to be n-realizable. We also obtain characterizations for n-realizable sequences whose elements are almost all below n using some results on complete sets. These characterizations complement the one used in connection with the ascending subgraph decomposition conjecture of Alavi *et al.* [2]. Finally, in Section 4.3, we consider the modular version of the problem and prove that all sequences in $\mathbb{Z}_p$ of length $k \leq (p-1)/2$ are realizable for any prime $p \geq 3$. The bound on k is best possible. The main results of this chapter are summarized in [43].

4.1 Introduction

The sumset partition problem consists, given a sequence $m_1, m_2, \ldots, m_k$ of k positive integers and a set X of positive integers, in finding a partition of X into k subsets $X_1, X_2, \ldots, X_k$ such that the sum of the elements in X_i is m_i, for each $i = 1, 2, \ldots, k$.

For the sequences we will follow the notation of [12]. A sequence of positive integers $m_1, m_2, \ldots, m_k$ that is ordered in a *non-increasing* way, that is, $m_1 \geq m_2 \geq \cdots \geq m_k$, will be denoted by $\langle m_1, m_2, \ldots, m_k \rangle$. If the ordering is not specified, we will simply denote a sequence by $(m_1, m_2, \ldots, m_k)$. The parameter k is the *length* of the sequence. A sequence $\langle m_1, \ldots, m_k \rangle$ may be also denoted as $\langle u_1^{\alpha_1}, \ldots, u_r^{\alpha_r} \rangle$, where $u_i < u_{i+1}$, $1 \leq i \leq r-1$ and $\sum_{i=1}^{r} \alpha_i = k$. We also assume that for all i, $\alpha_i \geq 1$. Note that in the latter form, the sequence is depicted in increasing way. For example $\langle 7, 7, 7, 4, 2, 2 \rangle$ can be also denoted by $\langle 2^2, 4, 7^3 \rangle$. From here on we will use the two notations indistinctively if there is no risk of confusion.

As said at the beginning of the chapter, a sequence $(m_1, m_2, \ldots, m_k)$ is said to be n-realizable if the set $X = [n]$ can be partitioned into k mutually disjoint subsets $X_1, \ldots, X_k$ such that $\sum X_i = m_i$ for each $1 \leq i \leq k$. In this context, a sequence $(m_1, m_2, \ldots, m_k)$ that verifies $\sum_{i=1}^{k} m_i = \binom{n+1}{2} (= \sum[n])$ is said to be *n-feasible*. Obviously, if a sequence is n-realizable, then it is n-feasible. Here we want to characterize the n-feasible sequences that are n-realizable.

The study of n-realizable sequences was motivated by the ascending subgraph decomposition problem posed by Alavi, Boals, Chartrand, Erdös and Oellerman in [2], which asks for a decomposition of a given graph G of size $\binom{n+1}{2}$ by subgraphs $H_1, \ldots, H_n$ where H_i has size i and is a subgraph of H_{i+1} for each $i = 1, \ldots, n-1$. These authors conjectured that a forest of stars of size $\binom{n+1}{2}$ with each component having at least n edges admits an ascending subgraph decomposition by stars. This is equivalent to the fact that every n-feasible sequence $\langle m_1, m_2, \ldots, m_k \rangle$ with $m_k \geq n$ is n-realizable, a result proved by Ma, Zhou and Zhou [44]. We will deal with the ascending subgraph decomposition problem of bipartite graphs in Chapter 5. Other instances of the sumset partition problem have been also considered in the literature, some of them related to graph decomposition problems; see for instance [7, 18, 21, 22, 23, 42]. In particular, the following n-feasible sequences have been shown to be n-realizable.

(1) $\langle m, m, \ldots, m, l \rangle$, where $m \geq n$ [7];

(2) $\langle m+1, m+1, \ldots, m+1, m, m, \ldots, m \rangle$, where $m \geq n$ [21];

(3) $\langle m+k-2, m+k-3, \ldots, m+1, m, l \rangle$, where $m \geq n$ [21];

(4) $\langle m_1, m_2, \ldots, m_k \rangle$, where $m_k \geq n$ [44];

(5) $\langle m_1, m_2, \ldots, m_k \rangle$, where $m_{k-1} \geq n$ [12].

The condition $m_k \geq n$ is far from being necessary for a sequence to be n-realizable. Chen, Fu, Wang and Zhou [12] showed that $m_{k-1} \geq n$ (sequence

(5) above) is a weaker sufficient condition, which is somewhat best possible in view of the fact that a sequence with $m_{k-1} = m_k = 1$ is never n-realizable. However the characterization of n-realizable sequences is still a wide open problem. In our work we adopt a different viewpoint and consider the n-realizability of a sequence in terms of the length k of the sequence or the values of its distinct elements.

4.2 Sequences in $\mathbb{Z}$

Here we consider the classic treatment of the problem where the set to be partitioned is $[n] \subset \mathbb{Z}$. The partition of $\mathbb{Z}_n$ into subsets with prescribed sums shall be considered in Section 4.3. We first introduce the notion of forbidden sequences, which are used to discard an n-feasible sequence when exactly it contains a forbidden subsequence. Then, we will use some results on complete sets of integers, sets whose subset sums realize all possible values (see, e.g., [48]), to obtain new characterizations for n-realizable sequences.

4.2.1 Forbidden subsequences

We say that a sequence $\langle m_1, m_2, \ldots, m_k \rangle$ is simply *realizable* if there exist pairwise disjoint subsets $X_1, \ldots, X_k$ of $[m_1]$ such that $\sum X_i = m_i$ for each $i = 1, 2, \ldots, k$. We say that a sequence is *forbidden* if it is not realizable. Note that if an n-feasible sequence contains a forbidden sequence as a subsequence then it is obviously not n-realizable. A remark to keep in mind is that an n-feasible sequence not containing any forbidden sequence, does not necessarily have to be n-realizable, as shown in the following example.

Example 4.1 *Consider the sequence $\langle 13, 12, 11, 9, 4, 3, 2, 1 \rangle$. It is 10-feasible and clearly does not have any forbidden subsequence since all the elements of the sequence are pairwise distinct. We can easily check that the sequence is not 10-realizable. The only possibility for the last 6 elements is $X_8 = \{1\}$, $X_7 = \{2\}$, $X_6 = \{3\}$, $X_5 = \{4\}$, $X_4 = \{9\}$ and $X_3 = \{5, 6\}$. Therefore, there is only left the set $\{7, 8, 10\}$ to obtain $\langle 13, 12 \rangle$, which is impossible.* $\square$

Let $\mathcal{F}_k$ be the set of minimal forbidden sequences of length k, that is, every sequence in $\mathcal{F}_k$ does not contain any proper forbidden subsequence. For example,

$$\begin{aligned}\mathcal{F}_2 &= \{\langle 1,1\rangle, \langle 2,2\rangle\};\\ \mathcal{F}_3 &= \{\langle 3,3,1\rangle, \langle 3,3,2\rangle, \langle 3,3,3\rangle, \langle 4,4,1\rangle, \langle 4,4,3\rangle, \langle 4,4,4\rangle\}.\end{aligned}$$

Realizable sequences of small length can be characterized only in terms of minimal forbidden sequences.

For small values of n a computer search shows that, for $n \leq 7$, the only n-feasible sequences of length three which are not n-realizable are the ones that contain a subsequence in $\mathcal{F}_2$. Similarly, for $n \in \{4,5,6,7\}$, an n-feasible sequence $\langle m_1, m_2, m_3, m_4\rangle$ is n-realizable if and only if it does not contain a subsequence in $\mathcal{F}_2 \cup \mathcal{F}_3$, and is not any of the sequences in

$$\mathcal{S} = \{\langle 6,6,2,1\rangle, \langle 8,7,3,3\rangle, \langle 8,8,3,2\rangle, \langle 10,10,4,4\rangle, \langle 14,8,3,3\rangle\},$$

where the first sequence in $\mathcal{S}$ is 5-feasible and belongs to $\mathcal{F}_4$, the next two sequences are 6-feasible and the last two sequences are 7-feasible. By computer search one may verify that, for $n = 8$, the only sequences of length four which are n-feasible and not n-realizable contain a forbidden sequence of length at most three. The algorithms which have been used to perform the computer search giving these results for $n \leq 8$ and $k \leq 4$ are detailed in the Appendix A.

Theorem 4.2 *An n-feasible sequence $\langle m_1, m_2, m_3\rangle$ is n-realizable if and only if it does not contain the subsequences in $\mathcal{F}_2$.*

Proof. The proof is by induction on n. By the above remarks we may assume $n > 7$.

Let $m = \langle m_1, m_2, m_3\rangle$ be an n-feasible sequence not containing the subsequences $\langle 1,1\rangle$ nor $\langle 2,2\rangle$ with $n > 7$. Since the length of the sequence is 3, the greatest element of the sequence, m_1, should be at least $n+1$. If not, we would have $n(n+1)/2 = m_1 + m_2 + m_3 \leq 3n$, which implies $n \leq 5$.

Then, we can define $m' = (m_1 - n, m_2, m_3)$, which is a $(n-1)$-feasible sequence. If m' is $(n-1)$-realizable, then by adding n to the set with sum $m_1 - n$ we get a partition of $[1,n]$ that fits with the given sequence. Otherwise, the sequence m' has a subsequence in $\mathcal{F}_2$ implying that our original sequence is $\langle n+1, m_2, 1\rangle$ with $1 \leq m_2 \leq n+1$ or $\langle n+2, m_2, 2\rangle$ with $2 \leq m_2 \leq n+2$. In either case, $n(n+1)/2 = m_1 + m_2 + m_3 \leq 2 + 2(n+2)$ implying $n \leq 7$. □

Theorem 4.3 *Let $n > 7$. An n-feasible sequence $\langle m_1, m_2, m_3, m_4\rangle$ is n-realizable if and only if it does not contain a subsequence in $\mathcal{F} = \mathcal{F}_2 \cup \mathcal{F}_3$.*

Proof. The proof is by induction on n. By the above remarks we know that the result is true for $n = 8$. Let $m = \langle m_1, m_2, m_3, m_4\rangle$ be an n-feasible sequence that does not contain a subsequence in $\mathcal{F}$ and $n > 8$. Since the length of the sequence is 4, the greatest element of the sequence, m_1, should be at least $n+1$. If not, we would have $n(n+1)/2 = m_1 + m_2 + m_3 + m_4 \leq 4n$, which implies $n \leq 7$. Consider the $(n-1)$-feasible sequence $m' = (m_1 - n, m_2, m_3, m_4)$. If m' is $(n-1)$-realizable then by adding n to the part with sum $m_1 - n$ we get a partition of $[1, n]$ with the desired sums. Otherwise, by the induction hypothesis, it contains a subsequence in $\mathcal{F}$, so that $m_1 - n \in \{1, 2, 3, 4\}$.

If $m_1 - n = 1$, then the original sequence is $m = \langle n+1, m_2, m_3, 1\rangle$. Therefore, $n(n+1)/2 = m_1 + m_2 + m_3 + m_4 \leq 1 + 3(n+1)$ implying $n \leq 7$.

If $m_1 - n = 2$, then $m = \langle n+2, m_2, m_3, 2\rangle$ or $m = \langle n+2, m_2, 2, 1\rangle$ or $m = \langle n+2, m_2, 3, 3\rangle$. In either case, $n(n+1)/2 \leq 2 + 3(n+2)$ obtaining that $n \leq 7$.

If $m_1 - n = 3$, then $m = \langle n+3, m_2, 4, 4\rangle$ or others with lower sum. In either case, $n(n+1)/2 \leq 8 + 2(n+3)$ and $n \leq 7$.

If $m_1 - n = 4$ then $m = \langle n+4, m_2, 4, 4\rangle$ or others with lower sum. In either case, $n(n+1)/2 \leq 8 + 2(n+4)$ and $n \leq 7$. $\square$

For $k \geq 5$ we have the following sufficient condition:

Theorem 4.4 *All n-feasible sequences $\langle m_1, \ldots, m_k\rangle$ with $n \geq 4k-1$ and $m_k > 4k-1$ are n-realizable for $k \geq 5$.*

Proof. The proof is by induction on n. If $n = 4k-1$, is the result of Ma, Zhou and Zhou [44].

Suppose $n > 4k-1$ and let $\langle m_1, m_2, \ldots, m_k\rangle$ be an n-feasible sequence with $m_k > 4k-1$.

Since the length of the sequence is k, the greatest element of the sequence, m_1, must be at least $n+1$, otherwise we would have $n(n+1)/2 = m_1 + \cdots + m_k \leq kn$, which implies $n \leq 2k-1$.

Consider the sequence $(m_1 - n, m_2, \ldots, m_k)$, which is $(n-1)$-feasible. If $m_1 - n > 4k-1$ then, by the induction hypothesis, we can realize this sequence in $[n-1]$. That is, there is a partition $X_1, \dots, X_k$ of $[n-1]$ such that

$$\sum(X_1) = m_1 - n, \sum(X_2) = m_2, \dots, \sum(X_k) = m_k.$$

Then, by taking $X_1' = X_1 \cup \{n\}$ and $X_2' = X_2, \dots, X_k' = X_k$, we obtain an n-realization of our original sequence.

Suppose now that $m_1 - n \leq 4k - 1$. Therefore, $m_1 \leq 4k + n - 1$. Since the original sequence is n-feasible, we know that

$$n(n+1)/2 = m_1 + \cdots + m_k \leq km_1 \leq k(4k + n - 1),$$

implying $n^2 + (1 - 2k)n + (2k - 8k^2) \leq 0$. This last inequality is satisfied when

$$n \leq \frac{2k - 1 + \sqrt{(1-2k)^2 - 8k + 32k^2}}{2} = 4k - 1.$$

But we were in the case $n - 1 \geq 4k - 1$, contradicting the last inequality. □

Theorem 4.4 shows that, for n sufficiently large, the condition on the smaller term of an n-realizable sequence which ensures its n-realizability, can be expressed as a function of its length k instead as a function on n itself, as it is in the previously known results. We believe that essentially the same lower bound on n should suffice to ensure n-realizability as long as the sequence does not contain forbidden subsequences. The conjectured value of the lower bound for n is given at the end of this section after collecting some additional evidence. This also raises the question of characterizing the family of minimal forbidden sequences of a given length. We also formulate a conjecture at the end of this section which states that such sequences must be somewhat dense.

We now continue by characterizing realizable sequences. Note that sequences with no repeated elements are not forbidden by definition. Therefore non-realizable sequences contain repetitions. Clearly, if a sequence $\langle u_1^{\alpha_1}, \dots, u_r^{\alpha_r} \rangle$ of length k is realizable, we must have $u_r \geq k$. The following lemma states a non-trivial lower bound on u_r when the sequence $\langle u_1^{\alpha_1}, \dots, u_r^{\alpha_r} \rangle$ is realizable. For an element u_i from a sequence $\langle u_1^{\alpha_1}, \dots, u_r^{\alpha_r} \rangle$, we denote by $\rho_i = \lfloor \frac{u_i + 1}{2} \rfloor$, which is the number of representations of u_i as a sum of two nonnegative integers,

$$\rho_i = |\{\{a, b\} : \; 0 \leq a < b \leq u_i \text{ and } a + b = u_i\}| \, .$$

Lemma 4.5 *Suppose that $m = \langle u_1^{\alpha_1}, \ldots, u_r^{\alpha_r} \rangle$ is realizable. Then*

$$\rho_r - 1 \geq \sum_{i=1}^{r} (\alpha_i - 1) + \sum_{i=1}^{r'} \alpha_i,$$

where r' is the number of u_i's less than ρ_r. Moreover, if the equality holds then all sets in a realization of m have cardinality at most two.

Proof. Let $\mathcal{X} = \{X_i^j : 1 \leq i \leq r, 1 \leq j \leq \alpha_i \text{ and } \sum(X_i^j) = u_i\}$ be a realization of m. Then,

$$\begin{aligned}
\rho_r - 1 &\geq \sum_{\substack{1 \leq i \leq r \\ 1 \leq j \leq \alpha_i}} \left| X_i^j \cap [1, \rho_r - 1] \right| \\
&\geq \sum_{i=1}^{r'} (|X_i^1| + \cdots + |X_i^{\alpha_i}|) + \sum_{i=r'+1}^{r} (|X_i^1| - 1 + \cdots + |X_i^{\alpha_i}| - 1) \\
&\geq \sum_{i=1}^{r'} (2\alpha_i - 1) + \sum_{i=r'+1}^{r} (\alpha_i - 1) \\
&= \sum_{i=1}^{r} (\alpha_i - 1) + \sum_{1}^{r'} \alpha_i.
\end{aligned}$$

The second inequality can be seen in the following way. If $i \in [1, r']$, then the corresponding u_i is less than ρ_r and therefore the sets X_i^j, for $1 \leq j \leq \alpha_i$, have all its elements in the interval $[1, \rho_r - 1]$. If $i \in [r' + 1, r]$, one element of each set may be outside of the interval. Finally, the third inequality is obtained since the cardinality of the sets $X_i^1, \ldots, X_i^{\alpha_i}$, for all $1 \leq i \leq r$, is always at least two, except one of them that may contain only a single element. The equality is attained if all the sets have cardinality at most two. □

Given a sequence $m = \langle u_1^{\alpha_1}, \ldots, u_r^{\alpha_r} \rangle$, we introduce the edge-colored graph $G = G_{u_1,\ldots,u_r}$. The set of vertices is $[0, u_r]$, and the set E of edges is defined as $E = E_1 \cup \cdots \cup E_r$ where for each $1 \leq i \leq r$, $E_i = \{\{l, u_i - l\} : 0 \leq l \leq \lfloor \frac{u_i+1}{2} \rfloor\}$ and we color the edges of E_i with u_i. Each realization of m by sets of cardinality at most two corresponds to a matching in $G_{u_1,\ldots,u_r}$ except that two or more edges can meet at 0. In Fig. 4.1 two examples of these graphs are depicted.

Using these graphs, we can obtain sufficient conditions for the realizability of an arbitrary sequence.

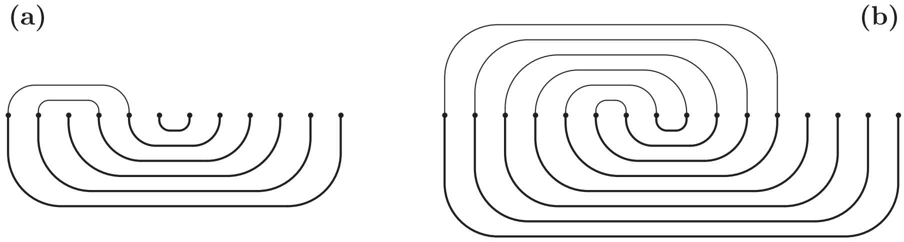

Figure 4.1 **(a)** The graph $G_{4,11}$. **(b)** The graph $G_{11,15}$.

Theorem 4.6 *A sequence $m = \langle u_1^{\alpha_1}, \dots, u_r^{\alpha_r} \rangle$ is realizable if the following conditions hold:*

$$\begin{aligned} \alpha_1 &\leq \rho_1, \\ \alpha_t + \sum_{i=1}^{t-1} 2\alpha_i &\leq \rho_t + (t-1), \qquad t = 2, \dots, r. \end{aligned}$$

Moreover, in this case there exists a realization with r sets of cardinality one and the remaining sets of cardinality two.

Proof. Consider the subsequences $m_t = \langle u_1^{\alpha_1}, \dots, u_t^{\alpha_t} \rangle$. For $t = 1$, if $\alpha_1 \leq \rho_1$ it is clear that m_1 is realizable by $\alpha_1 - 1$ sets of cardinality two and the set $\{u_1\}$ of cardinality one.

Let $t \geq 2$, and suppose that m_{t-1} is realizable by sets of cardinality two and the sets $\{u_1\}, \dots, \{u_{t-1}\}$. This realization of m_{t-1} induces a matching in the graph $G_{u_1,\dots,u_t}$. Now, we want to obtain α_t sets that add up to u_t using free elements, that is, vertices that are not in any edge of the matching except 0. We pick up the edge $\{0, u_t\}$. The number of remaining edges colored u_t is $\rho_t - 1 = \left\lfloor \frac{u_t+1}{2} \right\rfloor - 1$. In the worst case, we can assume that every edge of the matching touches two different u_t-edges, except the $t-1$ edges of the matching incident with 0, which only touch one u_t-edge. Then, the number $\rho_t - 1$ of remaining u_t-edges minus the number of u_t-edges touching an edge of the matching is at least

$$\rho_t - 1 - \sum_{i=1}^{t-1} (2\alpha_i - 1) = \rho_t - 1 - \sum_{i=1}^{t-1} 2\alpha_i + t - 1.$$

From the t-th restriction we have that $\alpha_t - 1$ is precisely less or equal than this quantity and therefore, we can select $\alpha_t - 1$ free u_t-edges to complete the realization of m_t. □

The converse of Theorem 4.6 is not true in general. For instance, take the sequence $\langle u_1^{\alpha_1}, \ldots, u_5^{\alpha_5} \rangle = \langle 1, 5, 6, 7, 9^2 \rangle$, for which $10 = \alpha_5 + \sum_{i=1}^{4} 2\alpha_i > \lfloor \frac{u_5+1}{2} \rfloor + 4 = 9$. But the sequence is realizable with $X_1^1 = \{1\}$, $X_2^1 = \{5\}$, $X_3^1 = \{6\}$, $X_4^1 = \{7\}$, $X_5^1 = \{9\}$, $X_5^2 = \{2, 3, 4\}$. Next theorem characterizes completely the sequences with large gaps showing that, for these sequences, the converse of Theorem 4.6 is indeed true.

Theorem 4.7 *A sequence $m = \langle u_1^{\alpha_1}, \ldots, u_r^{\alpha_r} \rangle$ with $u_{i+1} > 2u_i$ for each $1 \le i \le r-1$, is realizable if and only if the the restrictions of the statement of Theorem 4.6 hold. In this case, the sequence can be realized by sets of cardinality at most two.*

Proof. If the sequence is realizable, the subsequences $m_t = \langle u_1^{\alpha_1}, \ldots, u_t^{\alpha_t} \rangle$ are also realizable for each $1 \le t \le r$. We can apply Lemma 4.5 for each m_t, implying

$$\rho_t - 1 \ge \sum_{i=1}^{t} (\alpha_i - 1) + \sum_{i=1}^{t-1} \alpha_i, \qquad 1 \le t \le r,$$

where the second sum adds up to $t-1$ since for all t, $u_t > 2u_{t-1}$. Therefore,

$$\rho_t + (t-1) \ge 2\alpha_1 + \cdots + 2\alpha_{t-1} + \alpha_t \qquad 1 \le t \le r.$$

$\square$

In order to show the complexity in the characterization of realizable sequences $\langle u_1^{\alpha_1}, \ldots, u_r^{\alpha_r} \rangle$ when $u_{i+1} \le 2u_i$ for some $1 \le i \le r-1$, we present the following case that gives a tighter sufficient condition for a very specific sequence.

Theorem 4.8 *The sequence $m = \langle u^{\alpha}, v^{\beta} \rangle$ with u, v odd and $u < v < 2u$ is realizable by sets of cardinality at most two if*

(1) $\alpha = 0$ *and*
$$\beta \le \frac{v+1}{2};$$

(2) $kA + 1 \le \alpha \le (k+1)A$ *for some* $0 \le k \le N_1 - 1$, *and*
$$\alpha + \beta \le \frac{v+1}{2} - k;$$

(3) $(N_1+k)A-k+1 \leq \alpha \leq (N_1+k+1)A-k-1$ *for some* $0 \leq k \leq N_2-1$, *and*
$$\alpha+\beta \leq \frac{v+1}{2} - N_1 - k;$$

where $A = \left\lceil \frac{u+1}{v-u} \right\rceil$, $N_1 = \frac{v-p+1}{2}$, $N_2 = \frac{p-1-u}{2}$ *and* $p = \left\lceil \frac{u+1}{v-u} \right\rceil (v-u)$.

Proof. We start the proof by analyzing the basic properties of the components of the graph $G_{u,v}$.

Each one of the components is a path that starts at a vertex x of the interval $[u+1, v]$ and, by alternating v-edges and u-edges, finishes at another point of the same interval. Let $P(x)$ denote the path that starts at vertex $x \in [u+1, v]$. We have that

$$P(x) = \{x_1 = x, x_2 = v - x_1, x_3 = u - x_2, x_4 = v - x_3, \dots, l(x)\}.$$

Since the path finishes always with a v-edge, we know that the last vertex of the path is $l(x) = Q(x)(v-u) + (v-x)$, where $Q(x)$ is the minimum integer such that $l(x) \in [u+1, v]$. Therefore, $Q(x) = \left\lceil \frac{u+x+1-v}{v-u} \right\rceil$ and

$$l(x) = \left\lceil \frac{u+x+1-v}{v-u} \right\rceil (v-u) + (v-x), \qquad x \in [u+1, v]. \tag{4.1}$$

To see the behavior of the function $l(x)$ we should focuss on the quotient $Q(x)$. It takes its maximum value at the point $x = v$, that is, $Q(v) = \left\lceil \frac{u+1}{v-u} \right\rceil$, and takes its minimum value at the point $x = u+1$, $Q(u+1) = \left\lceil \frac{2u-v+2}{v-u} \right\rceil$. The difference between the numerators of $Q(v)$ and $Q(u+1)$ is $v-u-1$, and this means that $Q(v) = Q(u+1)+1$. From this observation, the fact that l is an involution and equation (4.1), we can partition the interval of vertices $[u+1, v]$ in two parts.

$$[u+1, v] = \underbrace{[u+1, l(u+1)]}_{\Gamma_2} \cup \underbrace{[l(v), v]}_{\Gamma_1},$$

such that for each vertex $x \in \Gamma_i$, $l(x)$ is the symmetric of x inside Γ_i, $i = 1, 2$. If we define $p = l(v) = \left\lceil \frac{u+1}{v-u} \right\rceil (v-u)$, we have that $\Gamma_1 = [p, v]$ and $\Gamma_2 = [u+1, p-1]$.

The components of the graph $G_{u,v}$ can be partitioned in two classes C_1 and C_2, being C_i the components that have the end vertices in Γ_i, $i = 1, 2$. The number of components that are in each class is $N_1 = |C_1| = \frac{v-p+1}{2}$ and $N_2 = |C_2| = \frac{p-u-1}{2}$.

Consider now the total number of u-edges belonging to a component $P(x)$, $x \in [u+1, v]$, which is precisely $Q(x)$. Since $Q(x)$ is constant inside each interval Γ_i, $i = 1, 2$, we have that the number of u-edges for a component in C_1 is $A = Q(v) = \left\lceil \frac{u+1}{v-u} \right\rceil$, and the number of u-edges for a component in C_2 is $A - 1$.

Once we have defined all the parameters about the components of $G_{u,v}$, we can continue by characterizing the exponents (α, β) for which the sequence $\langle u^\alpha, v^\beta \rangle$ is realizable.

By choosing all the v-edges in each of the components we get the maximal exponent $(0, \beta)$ with $\beta = \frac{v+1}{2}$. Therefore, all the exponents described by restriction (1) will be also realizable.

We start with the components in C_1 and concretely with $P(v)$. Replace the second v-edge by the first u-edge to obtain $(\alpha, \beta) = (1, \frac{v+1}{2} - 1)$. We proceed by exchanging v-edges by u-edges one by one along the path until $(\alpha, \beta) = (A, \frac{v+1}{2} - A)$. This gives the following restriction: if $1 \leq \alpha \leq A$, then $\alpha + \beta \leq \frac{v+1}{2}$, which corresponds to restriction (2) with $k = 0$. We next proceed with the rest of the components of C_1, and doing, one by one, the same procedure of interchanging v-edges by u-edges. With these components we must be careful because we have to remove the first v-edge to be able to add the first u-edge, since they do not meet at 0 like it was with the component $P(v)$. Therefore, we obtain the following set of restrictions:

$$\begin{gathered}
\text{If } A+1 \leq \alpha \leq 2A, \text{ then } \alpha + \beta \leq \frac{v+1}{2} - 1, \\
\text{if } 2A+1 \leq \alpha \leq 3A, \text{ then } \alpha + \beta \leq \frac{v+1}{2} - 2, \\
\vdots \\
\text{if } (N_1 - 1)A + 1 \leq \alpha \leq N_1 A, \text{ then } \alpha + \beta \leq \frac{v+1}{2} - (N_1 - 1).
\end{gathered}$$

All these restrictions, together with the one for $P(v)$, are precisely the ones in (2).

At this point, we continue the edge interchanging with the components of the class C_2. Now, we should take in account that the number of u-edges in each component is $A - 1$. One can easily see that the resulting restrictions are the ones in (3). $\square$

Note that the restrictions of Theorem 4.8 are more accurate than the general restrictions of Theorem 4.6. For the sequence $\langle u^{\alpha}, v^{\beta}\rangle = \langle 5^2, 7^2\rangle$, the conditions of Theorem 4.6 do not hold since $6 = \beta + 2\alpha > \left\lfloor \frac{v+1}{2} \right\rfloor + (2-1) = 5$. Dealing with the hypothesis of Theorem 4.8 we have that $A = 3$, $N_1 = 1$ and $N_2 = 0$ and therefore we have to look at restriction (2) since for $k = 0(\leq N_1 - 1)$, $1 = 0A + 1 \leq \alpha = 2 \leq (0+1)A = 3$. In this case, we can conclude that the sequence is definitively realizable as $4 = \alpha + \beta \leq \frac{v+1}{2} = 4$. In fact, $X_1^1 = \{5\}$, $X_1^2 = \{2,3\}$, $X_2^1 = \{7\}$, and $X_2^2 = \{1,6\}$ is a realization.

A corollary of Theorem 4.7 is that the largest element u_r in a forbidden sequence $\langle u_1^{\alpha_1}, \ldots, u_r^{\alpha_r}\rangle$ of $\mathcal{F}_k$, where the distinct elements have the growth established in the statement of the theorem, verifies $u_r < 4k$. This bound appears also in Theorem 4.4.

Corollary 4.9 *Let $m = \langle u_1^{\alpha_1}, \ldots, u_r^{\alpha_r}\rangle \in \mathcal{F}_k$ with $u_{i+1} > 2u_i$, $1 \leq i \leq r-1$. Then $u_r < 4k$.*

Proof. Since the sequence m is not realizable, by Theorem 4.7, one or more of the following restrictions must fail:

$$2\alpha_1 + \cdots + 2\alpha_{i-1} + \alpha_i \leq \rho_i + (i-1)$$

for $1 \leq i \leq r$.

We consider the sequence $m' = \langle u_1^{\alpha_1}, \ldots, u_{r-1}^{\alpha_{r-1}}\rangle$, which is realizable. Now, again from Theorem 4.7, the above restrictions should be attained for $1 \leq i \leq r-1$. Therefore, only the last restriction, for $i = r$, must be false. Then,

$$\rho_r + (r-1) < 2\alpha_1 + \cdots + 2\alpha_{r-1} + \alpha_r < 2(\alpha_1 + \cdots + \alpha_r) = 2k,$$

implying $\frac{u_r+1}{2} - \frac{1}{2} < 2k - r + 1 < 2k$. Finally, we get $u_r < 4k$. □

These results suggest the following problem. Let $f(k)$ be the largest element in a sequence of $\mathcal{F}_k$. We have,

Proposition 4.10 *For $k \geq 3$, $f(k) \geq 4k - 9$.*

Proof. For $k \geq 3$, define $a = 2k - 5$. Take the sequence $m = \langle a^{\frac{a+1}{2}}, 2a + 1^2\rangle$ of length k. It is easy to check that the inequality of Lemma 4.5 does not hold for m and thus, m is not realizable. We will see that all its subsequences of length $k-1$, and therefore all its proper subsequences, are realizable.

Case 1, $m' = \langle a^{\frac{a+1}{2}-1}, 2a + 1^2\rangle$.

Take $\mathcal{X} = \{\{a, a+1\}, \{2a+1\}, \{1, a-1\}, \{2, a-2\}, \{3, a-3\}, \ldots, \{\frac{a-1}{2}, \frac{a+1}{2}\}\}$. Clearly, the first two sets add up to $2a+1$ and the last $\frac{a+1}{2} - 1$ add up to a.

Case 2, $m' = \langle a^{\frac{a+1}{2}}, 2a+1\rangle$.

Take $\mathcal{X} = \{\{2a+1\}, \{a\}, \{1, a-1\}, \{2, a-2\}, \{3, a-3\}, \ldots, \{\frac{a-1}{2}, \frac{a+1}{2}\}\}$. In this case, the first set adds up to $2a+1$ and the remaining $\frac{a+1}{2}$ sets add up to a.

So clearly $m \in \mathcal{F}_k$ and its maximum value is $2a+1 = 2(2k-5)+1 = 4k-9$, implying that for all k, $f(k) \geq 4k-9$. □

We conjecture that the above lower bound is close to the true value.

Conjecture 4.11 *For all $k \geq 2$, $f(k) < 4k$.* □

A conjecture that implies Conjecture 4.11, as we will next see, is the following one.

Conjecture 4.12 *Let $\langle m_1, \ldots, m_k\rangle$ be a realizable sequence. Then there exists a realization $X_1, \ldots, X_k$ with $|\cup_{i=1}^{k} X_i| \leq 2k$.* □

It is easy to show that if Conjecture 4.12 is true, then $f(k) < 4k$. Let $m = \langle u_1^{\alpha_1}, \ldots, u_r^{\alpha_r}\rangle \in \mathcal{F}_k$. As the sequence $\langle u_1^{\alpha_1}, \ldots, u_{r-1}^{\alpha_{r-1}}\rangle$ is realizable, by Conjecture 4.12 there exists a realization

$$\mathcal{X} = \{X_i^j : \ 1 \leq i \leq r-1, \ 1 \leq j \leq \alpha_i \text{ and } \sum(X_i^j) = u_i\}$$

with $\sum_{i,j} |X_i^j| \leq 2\alpha_1 + \cdots + 2\alpha_{r-1}$. Using the same reasoning as in the proof of Theorem 4.6, if

$$\rho_r - \sum_{i,j} |X_i^j| \geq \alpha_r,$$

the sequence m would be realizable. Therefore,

$$\frac{u_r}{2} \leq \left\lfloor \frac{u_r+1}{2} \right\rfloor < \sum_{i,j} |X_i^j| + \alpha_r \leq 2\alpha_1 + \cdots + 2\alpha_{r-1} + \alpha_r < 2k,$$

implying that $u_r < 4k$.

At this point, we pose the following key conjecture that shows the close connection between the realizable sequences and the n-realizable sequences.

Conjecture 4.13 *Let $m = \langle m_1, \ldots, m_k\rangle$ be an n-feasible sequence. If m is realizable and $n \geq 4k$ then m is n-realizable.* □

In other words, the above conjecture states that, for $n \geq 4k$, a sequence of length k is n-realizable if and only if it does not contain minimal forbidden sequences.

4.2.2 Using complete sets

This section is devoted to analyze the n-realizability of sequences with small values. The main tool is the use of so-called complete sets of integers, a notion related to a vast area in Additive Number Theory.

Let $S(X) = \{\sum Y, Y \subseteq X\}$ denote the set of all subset sums of X. We say that a set X of positive integers is *complete* if

$$S(X) = \{0, 1, 2, \ldots, \sum X\},$$

that is, the subset sums cover all the interval from 0 to $\sum X$. It is clear that

$$S([n]) = \{0, 1, 2, \ldots, n(n+1)/2\},$$

that is, $[n]$ is a complete set.

There are several results on complete sets in $\mathbb{Z}_n$ and in the integers, see for example the book of Tao and Vu [48].

We will start with the following key observation:

Proposition 4.14 *Let X be a complete set. Then $X \cup \{a\}$ is complete if and only if $a \leq \sum X + 1$.*

Proof. Since X is complete, the set $S(X \cup \{a\})$ is clearly the union of two intervals, $I_1 = \{0, 1, \ldots, \sum X\}$ and $I_2 = \{a, a+1, \ldots, a + \sum X\}$. Therefore, $I_1 \cup I_2$ is another interval, and thus $X \cup \{a\}$ is complete, if and only if $a \leq \sum X + 1$. □

Using the above proposition, we are able to state the following result.

Theorem 4.15 *Let be $X = \{1, 2, \ldots, m\} \cup \{a_1\} \cup \{a_2\} \cup \cdots \cup \{a_k\}$ with $m + 1 < a_1 < \cdots < a_k$ and let $l = \frac{m(m+1)}{2}$. If*

$$a_1 \leq l + 1, \tag{4.2}$$

and

$$a_i - a_{i-1} \leq l + 1, \; 1 < i \leq k, \tag{4.3}$$

then X is complete.

Proof. From condition (4.2) and Proposition 4.14, the set $\{1, 2, \ldots, m\} \cup \{a_1\}$ is complete. Then, from condition (4.3) and again by Proposition 4.14, we can add successively the elements $a_2, a_3, \ldots, a_k$ and the resulting set will be still complete. $\square$

First, we are going to analyze how is the construction suggested by Theorem 4.15.

Given n, we want to remove the maximum number of elements of $[n]$ in such a way that the remaining set is still complete. The naive approximation is the following one:

Take the minimum $m = m_1$ such that $\frac{(m_1+1)m_1}{2} \geq n - 1$ and remove all the elements from $m_1 + 1$ to $n - 1$. The remaining set will be

$$X_1 = \{1, 2, \ldots, m_1\} \cup \{n\}.$$

It is easy to check that

$$m_1 = \left\lceil \frac{-1 + \sqrt{8n - 7}}{2} \right\rceil,$$

and that the number of removed elements from $[n]$ is

$$h_{n,1} = n - m_1 - 1 = \left\lfloor n - \frac{1 + \sqrt{8n - 7}}{2} \right\rfloor.$$

But Theorem 4.15 gives us a better construction:

We can reduce substantially this $m = m_1$ if we do not remove all the numbers from $m + 1$ to $n - 1$, but almost all. The construction is the following.

$$X_2 = \{1, 2, \ldots, m_2\} \cup \{l + 1\} \cup \{2l + 2\} \cup \cdots \cup \{sl + s\} \cup \{n\},$$

where $l = \frac{(m_2+1)m_2}{2}$, s is the maximum value such that $sl+s \leq n$ ($s = \left\lfloor \frac{n}{l+1} \right\rfloor$), and m_2 will be taken as the point that maximizes the number of holes given by

$$h_{n,2}(m) = (l - m) + \underbrace{l + \cdots + l}_{s-1} + (n - 1 - sl - s). \tag{4.4}$$

If we put the values of l and s inside (4.4), we obtain

$$h_{n,2}(m) = n - m - 1 - \left\lfloor \frac{n}{\frac{m(m+1)}{2} + 1} \right\rfloor. \tag{4.5}$$

n	m_1	$h_{n,1}$	m_2	$h_{n,2}$	$h_{n,2} - h_{n,1}$
20	6	13	3	14	1
50	10	39	5	41	2
100	14	85	6	89	4
1000	45	954	15	976	22
10000	141	9858	33	9949	91
100000	447	99552	73	99889	337
1000000	1414	998585	158	999762	1177

Table 4.1 Comparing constructions.

Finally, in order to work numerically with $h_{n,2}$, we will take the approximation given by

$$h_{n,2}(m) \simeq \frac{-m^3 + (n-2)m^2 + (n-3)m - 2}{m^2 + m + 2}. \tag{4.6}$$

The desired m_2 will be the point that maximizes (4.6).

In Table 4.1 there are computed the parameters m_1 and m_2 for each construction and the corresponding number of holes for different values of n. For $n = 20, 50, 100$ the values for m_2 and $h_{n,2}$ are exact. For the rest, the approximation by (4.6) is used.

For example, the sets constructed for $n = 100$ are

$$X_1 = \{1, \ldots, 14\} \cup \{100\},\ h_{100,1} = 85,$$

and

$$X_2 = \{1, 2, 3, 4, 5, 6\} \cup \{22\} \cup \{44\} \cup \{66\} \cup \{88\} \cup \{100\},\ h_{100,2} = 89.$$

This analysis motivates the interesting subject of describing the structure of complete sets over the nonnegative integers. The natural question that arises is: what is the maximum number of elements that can be removed from $[n]$, except n, and how should they be eliminated so that the resulting set is still complete. However, it is not the objective of this work to study this issue in depth.

Next, we present the following corollary of Theorem 4.15 as a means to say something about the sequences.

Corollary 4.16 *Let $H \subset [n]$ with $n \notin H$. If $\min H \geq \frac{3+\sqrt{8|H|+1}}{2}$, then $X = [n] \setminus H$ is complete.*

Proof. If we define $m + 1 = \min H$, the set X can be described in the following way.

$$[n] \setminus H = \{1, 2, \ldots, m\} \cup \{a_1\} \cup \cdots \cup \{a_k = n\},$$

with $m + 1 < a_1 < \cdots < a_k$.

From the condition $\min H \geq \frac{3+\sqrt{8|H|+1}}{2}$, we can deduce that $|H| \leq \frac{m(m-1)}{2}$. Now, for a pair of consecutive numbers a_{i-1} and a_i, we have

$$a_i - a_{i-1} \leq |H| + 1 \leq \frac{m(m-1)}{2} + 1 \leq \frac{m(m+1)}{2} + 1,$$

satisfying the condition (4.3) of Theorem 4.15, and

$$a_1 \leq |H| + m + 1 \leq \frac{m(m-1)}{2} + m + 1 = \frac{m(m+1)}{2} + 1,$$

satisfying also the condition (4.2) of Theorem 4.15. □

Example 4.17 *For $n = 100$, and $|H| = 50$, the bound given by the corollary is $\min H \geq (3 + \sqrt{8 \cdot 50 + 1})/2 \simeq 11.52$, so we can remove up to 50 numbers from 12 to 99 and the remaining set is still complete.* □

This corollary of Theorem 4.15 leads us to the following consequence about sequences.

Theorem 4.18 *Let $S = \langle m_1, \ldots, m_k \rangle$ be a non-increasing n-feasible sequence. If*

$$n > m_3 > \cdots > m_k \geq \frac{3 + \sqrt{8k - 15}}{2},$$

then S is n-realizable.

Proof. Let $X_i = \{m_i\}$, $3 \leq i \leq k$. Let $H = \{m_3, \ldots, m_k\}$, which has cardinality $k - 2$. Since

$$\min H = m_k \geq \frac{3 + \sqrt{8k - 15}}{2} = \frac{3 + \sqrt{8|H| + 1}}{2},$$

by Corollary 4.16, $[n] \setminus \{m_3, \ldots, m_k\}$ is complete. Therefore, there exists a subset $X_2 \subseteq [n] \setminus \{m_3, \ldots, m_k\}$ such that $\sum X_2 = m_2$. Finally, with the remaining elements, define the set $X_1 = [n] \setminus (X_2 \cup \{m_3, \ldots, m_k\})$, for which

$$\sum X_1 = n(n+1)/2 - (m_2 + m_3 + \cdots + m_k) = m_1.$$

□

Using the following result of Lev, we can eliminate the bound on m_k by adding a new condition on n.

Theorem 4.19 (Lev, [39]) *Let $A \subseteq [n]$ be a set of $a \geq 1$ integers, and assume that $a \geq \frac{2n+4}{3}$. Then*

$$[2n - 2a + 1, \sum A - (2n - 2a + 1)] \subseteq S(A).$$

□

Theorem 4.20 *Let $m = \langle m_1, \ldots, m_k \rangle$ be a non-increasing n-feasible sequence. If $n \geq 3k - 2$ and $n > m_3 > \cdots > m_k$ then m is n-realizable.*

Proof. Since $n > m_3 > \cdots > m_k$ we can take the sets $X_i = \{m_i\}$ for each $3 \leq i \leq k$. Assume that $m_2 > n$, otherwise we can take $X_2 = \{m_2\}$ and the remaining elements add up to m_1 (as m is n-feasible) and we are done.

Let $A = [n] \setminus \{m_3, \ldots, m_k\}$ be the set of remaining elements of the interval. Then, since the sequence is n-feasible, we have $\sum A = m_1 + m_2$.

Now, $|A| = n - k + 2$ and, as $n \geq 3k - 2$, we have $|A| \geq \frac{2n+4}{3}$. So we can apply Theorem 4.19 obtaining that

$$\begin{aligned}
& [2n - 2|A| + 1, m_1 + m_2 - (2n - 2|A| + 1)] \subseteq S(A) \\
\Longrightarrow\ & [2n - 2(n - k + 2) + 1, m_1 + m_2 - (2n - 2(n - k + 2) + 1)] \subseteq S(A) \\
\Longrightarrow\ & [2k - 3, m_1 + m_2 - (2k - 3)] \subseteq S(A) \\
\Longrightarrow\ & [n, m_1 + m_2 - n] \subseteq S(A).
\end{aligned}$$

Since $m_1 \geq m_2 > n$, clearly both m_1 and m_2 belong to the interval and we can obtain, for instance, $X_2 \subseteq A$ with $\sum X_2 = m_2$. The remaining elements $X_1 = [n] \setminus (\{m_3, \ldots, m_k\} \cup X_2)$ will add up to m_1. □

Theorems 4.18 and 4.20 complement the result of Chen, Fu, Wang and Zhou [12] in the sense that in the latter almost all elements of the sequence must be larger than n, while here we require that almost all should be below n.

4.3 Modular sumset partition problem

We conclude the chapter by considering the modular version of the problem. A sequence $(m_1, \dots, m_k)$ of elements in $\mathbb{Z}_n$ is *realizable modulo* n if there is a family $X_1, \dots, X_k$ of pairwise disjoint sets of $\mathbb{Z}_n$ such that $\sum X_i = m_i$ for each $i = 1, 2, \dots, k$. Note that, for n odd, the sequence $(m, m, \dots, m)$ of length $(n+1)/2$ is clearly not realizable for every $m \neq 0$. Theorem 4.21 shows that sequences in $\mathbb{Z}_p$ of length at most $(p-1)/2$ for any prime $p \geq 3$ are always realizable. Its proof uses the polynomial method of Alon described in Chapter 1. The application of the Combinatorial Nullsetellensatz to similar problems can be seen in [4] and [14].

Theorem 4.21 *Let $p \geq 3$ be a prime number and $k = (p-1)/2$. For each sequence $(m_1, \dots, m_k)$ of elements in $\mathbb{Z}_p$ with $\sum_{i=1}^k m_i = M$, there is a partition $X_1, \dots, X_k$ of $\mathbb{Z}_p \setminus \{-M\}$ with $|X_i| = 2$ and $\sum X_i = m_i$, $i = 1, \dots, k$.*

Proof. Consider the following polynomial in $\mathbb{Z}_p[x_1, \dots, x_k, y_1, \dots, y_k]$:

$$f = V(x_1, \dots, x_k, y_1, \dots, y_k) \prod_{i=1}^{k} \left(1 - (x_i + y_i - m_i)^{p-1}\right).$$

The polynomial f takes a nonzero value if and only if there are pairwise distinct elements $(a_1, \dots, a_k, b_1, \dots, b_k)$ such that $a_i + b_i = m_i$, $1 \leq i \leq k$. If this is the case, the sets $X_i = \{a_i, b_i\}$ fulfill the conclusion of the statement. Indeed, the X_i's have cardinality two and they are pairwise disjoint, so they form a partition of $\mathbb{Z}_p \setminus \{\gamma\}$ for some γ. Since $\gamma + \sum_{i=1}^k m_i = \sum_{x \in \mathbb{Z}_p} x = 0$, we have $\gamma = -\sum_{i=1}^k m_i = -M$.

Consider the monomial

$$x_1^{p-1} \cdots x_k^{p-1} y_1^{p-2} \cdots y_k^{p-2} \tag{4.7}$$

in the expansion of f, which is a monomial of maximum degree. Let us show that this monomial has a nonzero coefficient, so that, by the Theorem 1.1, f takes a nonzero value in $(\mathbb{Z}_p)^{2k}$.

We identify every monomial $x_1^{\alpha_1} \cdots x_k^{\alpha_k} y_1^{\beta_1} \cdots y_k^{\beta_k}$ by its exponent sequence $(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_k)$.

The monomial (4.7) arises in the expansion of f whenever we multiply one monomial with exponent sequence $(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_k)$ in the expansion of $V(x_1, \dots, x_k, y_1, \dots, y_k)$ by a monomial with exponent sequence

$(\alpha'_1, \ldots, \alpha'_k; \beta'_1, \ldots, \beta'_k)$ in the expansion of $\prod_{i=1}^{k} \left(1 - (x_i + y_i - m_i)^{p-1}\right)$ verifying

$$\begin{array}{rclr} \alpha_i + \alpha'_i & = & p-1, & i = 1, \ldots, k; \\ \beta_i + \beta'_i & = & p-2, & i = 1, \ldots, k. \end{array}$$

Since $\alpha'_i + \beta'_i = p - 1$, the above relations lead to

$$\alpha_i + \beta_i = p - 2, \qquad i = 1, \ldots, k. \tag{4.8}$$

The last relation implies that these are precisely the only monomials from the Vandermonde polynomial that contribute to monomial (4.7). Moreover, this monomial arise in the decomposition by multiplying each monomial from the Vandermonde polynomial

$$(\alpha_1, \ldots, \alpha_k; \beta_1, \ldots, \beta_k) \quad \text{with } \alpha_i + \beta_i = p-2, \;\; 1 \leq i \leq k,$$

by the following monomial from the remaining factor of f:

$$(-1)\binom{p-1}{\alpha_1} x_1^{p-1-\alpha_1} y_1^{p-2-\beta_1} \cdots \binom{p-1}{\alpha_k} x_k^{p-1-\alpha_k} y_k^{p-2-\beta_k}. \tag{4.9}$$

We know that the expansion of the Vandermonde polynomial is

$$V(x_1, \ldots, x_k, y_1, \ldots, y_k) =$$

$$= \sum_{\tau \in Sym(2k)} sgn(\tau) x_1^{\tau(0)} x_2^{\tau(1)} \cdots x_k^{\tau(k-1)} y_1^{\tau(k)} y_2^{\tau(k+1)} \cdots y_k^{\tau(2k-1)} \tag{4.10}$$

(see Chapter 1, (1.1)). Observe that for a given τ we have $\alpha_i = \tau(i-1)$ and $\beta_i = \tau(k+i-1)$ for each $1 \leq i \leq k$. Now, from relation (4.8), we have that the only permutations τ in which we are interested are precisely the ones that satisfy $\tau(i) + \tau(k+i) = p - 2$ for each $0 \leq i \leq k-1$. We can obtain such permutations by considering every permutation σ of $[0, k-1]$ and, for each $i \in [0, k-1]$, one of the two choices:

$$\begin{array}{rcl} \tau(i) = \sigma(i) & \text{and} & \tau(k+i) = p-2-\sigma(i), \;\text{ or} \\ \tau(i) = p-2-\sigma(i) & \text{and} & \tau(k+i) = \sigma(i). \end{array}$$

With exactly these permutations, we obtain the desired exponent sequences from the Vandermonde polynomial. Each one of these exponent sequences combined with the corresponding one from (4.9) will verify all the above relations. Thus, we have $2^k k!$ such pairs of monomials.

Observe that given a permutation σ of $[0, k-1]$, one particular exponent $i \in [0, k-1]$ is initially in position $\alpha_{\sigma^{-1}(i)+1}$, and then we have the two choices for this position:

$$\alpha_{\sigma^{-1}(i)+1} \in \{i, p-2-i\}.$$

A pair of monomials corresponding to the permutation σ and all the choices being $\alpha_{\sigma^{-1}(i)+1} = i$ for all $1 \leq i \leq k$, has coefficient

$$C_\sigma = \rho(k) \prod_{i=0}^{k-1} \binom{p-1}{\alpha_{\sigma^{-1}(i)+1}} = \rho(k) \prod_{i=0}^{k-1} \binom{p-1}{i}, \tag{4.11}$$

where $\rho(k) = (-1)^{\lfloor \frac{k}{2} \rfloor + 1}$ is the sign of τ from (4.10) with an extra -1 from (4.9). Note that τ, when all the choices are the first option, is a permutation by σ of the first k elements and the same permutation, but inverted, of the last k elements. Therefore, $(-1)^{\lfloor \frac{k}{2} \rfloor}$ is the number of transpositions that should be applied to τ in order to obtain the same permutation σ of the last k elements, implying $sgn(\tau) = (sgn(\sigma))^2(-1)^{\lfloor \frac{k}{2} \rfloor} = (-1)^{\lfloor \frac{k}{2} \rfloor}$.

Observe that the coefficient C_σ does not depend on the permutation σ.

A pair corresponding to the permutation σ and a particular choice for every exponent $i \in [0, k-1]$, which is initially in position $\alpha_{\sigma^{-1}(i)+1}$, has coefficient

$$C_{\sigma;q_1,\ldots,q_s} = (-1)^s \rho(k) \prod_{i=0}^{k-1} \binom{p-1}{i} \left(\frac{p-1-q_1}{q_1+1}\right) \cdots \left(\frac{p-1-q_s}{q_s+1}\right),$$

where $q_l \in [0, k-1]$, $1 \leq l \leq s$, are the exponents for which the choice has been $\alpha_{\sigma^{-1}(q_l)+1} = p-2-q_l$. We next show that this expression arises from (4.11) and the relation $\binom{n}{i+1} = \binom{n}{i}\frac{n-i}{i+1}$ for $n > i$. Indeed, when in the pair $(\alpha_{\sigma^{-1}(i)+1}, \beta_{\sigma^{-1}(i)+1})$ we choose the second option ($\alpha_{\sigma^{-1}(i)+1} = p-2-i$), we are changing the corresponding coefficient $\binom{p-1}{i}$ of the first option by

$$\binom{p-1}{p-2-i} = \binom{p-1}{i+1} = \binom{p-1}{i}\frac{p-1-i}{i+1}.$$

The factor $(-1)^s$ shows the fact that when we choose the second option for one particular exponent, the sign of the permutation τ changes by -1.

Again, the coefficients $C_{\sigma;q_1,\dots,q_s}$ do not depend on the permutation σ, and thus all the $k!$ permutations with the same choices will contribute the same to the coefficient. If we denote $j_l = q_l + 1$, $1 \le l \le s$, the contribution of these $k!$ permutations to the coefficient is

$$C_{j_1,\dots,j_s} = k!(-1)^s \rho(k) \prod_{i=0}^{k-1} \binom{p-1}{i} \binom{\frac{p-j_1}{j_1}} \binom{\frac{p-j_2}{j_2}} \cdots \binom{\frac{p-j_s}{j_s}}.$$

Finally we have to consider all the possible choices and therefore the coefficient of the monomial (4.7) is

$$\begin{aligned} C &= k!\rho(k) \prod_{i=0}^{k-1} \binom{p-1}{i} + \sum_{s=1}^{k} \sum_{1\le j_1<j_2<\cdots<j_s\le k} C_{j_1,\dots,j_s} \\ &= k!\rho(k) \prod_{i=0}^{k-1} \binom{p-1}{i} \left(1 + \sum_{s=1}^{k} (-1)^s \lambda(s,k)\right), \end{aligned}$$

where

$$\lambda(s,k) = \sum_{1\le j_1<j_2<\cdots<j_s\le k} \binom{\frac{p-j_1}{j_1}} \binom{\frac{p-j_2}{j_2}} \cdots \binom{\frac{p-j_s}{j_s}}.$$

The goal now is to prove that C is nonzero modulo p. We have that

$$\begin{aligned} \lambda(s,k) &\equiv \sum_{1\le j_1<j_2<\cdots<j_s\le k} (-j_1)(-j_2)\cdots(-j_s) \frac{1}{j_1 j_2 \cdots j_s} \\ &\equiv \sum_{1\le j_1<j_2<\cdots<j_s\le k} (-1)^s \\ &\equiv (-1)^s \binom{k}{s} \pmod{p}. \end{aligned}$$

Then, the coefficient C of the monomial (4.7) can be expressed modulo p as

$$\begin{aligned}
C &\equiv k!\rho(k)\prod_{i=0}^{k-1}\binom{p-1}{i}\left(1+\sum_{s=1}^{k}(-1)^s\lambda(s,k)\right)\\
&\equiv k!\rho(k)\prod_{i=0}^{k-1}\binom{p-1}{i}\left(1+\sum_{s=1}^{k}(-1)^{2s}\binom{k}{s}\right)\\
&\equiv k!\rho(k)\prod_{i=0}^{k-1}\binom{p-1}{i}\sum_{s=0}^{k}\binom{k}{s}\\
&\equiv k!\rho(k)\prod_{i=0}^{k-1}\binom{p-1}{i}2^k,
\end{aligned}$$

which is nonzero since $\prod_{i=0}^{k-1}\binom{p-1}{i} \equiv \pm 1 \pmod p$, $k! \not\equiv 0 \pmod p$ and 2^k is obviously nonzero modulo p. This completes the proof. $\square$

Theorem 4.21 also applies for shorter sequences as we can see in the following corollary.

Corollary 4.22 *Let p be an odd prime. For each sequence $(m_1,\dots,m_r)$ of elements in $\mathbb{Z}_p$ with $r \le (p-1)/2$ and $\sum_{i=1}^r m_i = M$, there is a partition $X_1,\dots,X_r$ of $\mathbb{Z}_p \setminus \{-M\}$ with $|X_i| > 1$ and $\sum X_i = m_i$, $i = 1,\dots,r$.*

Proof. If $r = k$ then it is Theorem 4.21. Assume that $r < k$, and consider the following sequence of length k

$$m'_1 = m_1,\dots,m'_r = m_r, m'_{r+1} = 0,\dots,m'_k = 0.$$

Now, we apply Theorem 4.21 that gives a partition $X'_1,\dots,X'_k$ of $\mathbb{Z}_p \setminus \{-M\}$ such that $\sum X'_i = m'_i$ and $|X'_i| > 1$, $1 \le i \le k$. Finally, the following sets

$$X_1 = X'_1 \cup \left(\cup_{i=r+1}^k X'_i\right) \text{ and } X_2 = X'_2,\dots,X_r = X'_r$$

are a partition of $\mathbb{Z}_p \setminus \{-M\}$ with $|X_i| > 1$ and the desired sumset. $\square$

If a sequence $(m_1,\dots,m_r)$ is *feasible modulo* p, that is, if $\sum_{i=1}^r m_i = 0$, we can extend the result to length $(p+1)/2$.

Corollary 4.23 *Let p be an odd prime and $k = (p-1)/2$. For each sequence $(m_1,\dots,m_r)$ of elements in $\mathbb{Z}_p$ with $r \le k+1$ and $\sum_{i=1}^r m_i = 0$, there is a partition $X_1,\dots,X_r$ of $\mathbb{Z}_p$ with $\sum X_i = m_i$, $i = 1,\dots,r$ and*

(1) *if* $r \leq k$, $|X_i| > 1$, $1 \leq i \leq r$;

(2) *if* $r = k+1$, *we can choose* m_i, $1 \leq i \leq r$, *for which* $|X_i| = 1$. *The remaining sets will have cardinality two.*

Proof. If $r \leq k = (p-1)/2$, we can apply Corollary 4.22 and obtain a partition of $\mathbb{Z}_p \setminus \{0\}$. Therefore we can add 0 to any set without changing its sum to obtain the total partition.

If $r = k+1$, select the desired m_i, $1 \leq i \leq r$, and take the truncated sequence $(m_1, \ldots, m_{i-1}, m_{i+1}, \ldots, m_r)$ of k elements. Let $J = \{1, \ldots, i-1, i+1, \ldots, r\}$. Note that $\sum_{j \in J} m_j = -m_i$. So we can apply Theorem 4.21 to this sequence and obtain a partition $X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_r$ of $\mathbb{Z}_p \setminus \{m_i\}$ with $|X_j| = 2$ and $\sum X_j = m_j$, $j \in J$. Finally define $X_i = \{m_i\}$. The collection $X_1, \ldots, X_r$ is a partition of $\mathbb{Z}_p$ and $\sum X_j = m_j$, $1 \leq j \leq r$. □

We finish this chapter by providing an alternative simple proof of the result of Chen, Fu, Wang and Zhou for $n = p$ prime.

Theorem 4.24 ([12]) *Let* $p \geq 3$ *be a prime and let* $m = \langle m_1, \ldots, m_k \rangle$ *be a* p*-feasible sequence. If* $m_{k-1} \geq p$ *then* m *is* p*-realizable.*

Proof. We can assume that $m_k < p$, otherwise we are done by the result of Ma, Zhou and Zhou [44].

Suppose now that $m_{k-1} = p$. Consider the sequence $\langle m_1, \ldots, m_{k-2}, m_k \rangle$, which is $(p-1)$-feasible and $m_{k-2} \geq p$. By the inductive hypothesis, we can obtain a partition $X_1, \ldots, X_{k-2}, X_k$ of $[p-1]$ such that $\sum X_i = m_i$, $i = 1, \ldots, k-2, k$. Adding the set $X_{k-1} = p$ to the collection, we obtain a partition of $[p]$ with $\sum X_i = m_i$, $1 \leq i \leq k$. Therefore, we can also assume that $m_{k-1} \geq p+1$.

Now, $p(p+1)/2 > m_1 + \cdots + m_{k-1} \geq (k-1)(p+1)$ implies $k \leq (p+1)/2$. Denote by m_i' the representative modulo p of m_i in $[1, p]$, $1 \leq i \leq k$. Observe that $m_k' = m_k$. Let $t = (p+1)/2 - k$ (can be zero) and consider the sequence

$$(m_1', \ldots, m_k', \underbrace{p, \ldots, p}_{t})$$

of length $(p+1)/2$. Since $\sum_{i=1}^k m_i' + tp \equiv p(p+1)/2 + tp \equiv 0 \pmod{p}$, by Corollary 4.23 (2) there is a partition

$$X_1', \ldots, X_k', Y_1, \ldots, Y_t$$

of $[1, p]$ with $\sum X'_i \equiv m'_i \pmod{p}$ for $i = 1, \ldots, k$, $\sum Y_j \equiv 0 \pmod{p}$ for $j = 1, \ldots, t$, $|X'_k| = 1$ and $|X'_1| = \cdots = |X'_{k-1}| = |Y_1| = \cdots = |Y_t| = 2$. Therefore, we have that $X'_k = \{m_k\}$, $\sum X'_i \in \{m'_i, m'_i + p\}$ for $i = 1, \ldots, k-1$, and $\sum Y_j = p$ for $j = 1, \ldots, t$. Moreover,

$$\{X'_1, X'_2, \ldots, X'_{k-1}, X'_k = \{m_k\}, Y_1, \ldots, Y_t\}$$

is a partition of $[1, p]$, implying

$$\sum_{i=1}^{k}\left(\sum X'_i\right) + \sum_{j=1}^{t}\left(\sum Y_j\right) = p(p+1)/2 = m_1 + \cdots + m_k. \tag{4.12}$$

Since $m_{k-1} > p$ we have $\sum X'_i \leq m_i$ for each $i = 1, \ldots, k$, $m_i - \sum X'_i$ is a multiple of p for $i = 1, \ldots, k$, and it follows from (4.12) that

$$\sum_{j=1}^{t}\left(\sum Y_j\right) = \sum_{i=1}^{k}\left(m_i - \sum X'_i\right).$$

Hence, by joining the Y_j's to the X'_i's appropriately, we can obtain a partition $X_1, \ldots, X_k$ of $[1, p]$ with $\sum X_i = m_i$ for each $1 \leq i \leq k$. $\square$

5

Ascending subgraph decompositions of bipartite graphs

Let G be a graph of size $\binom{n+1}{2}$ for some integer $n \geq 1$. G is said to have an ascending subgraph decomposition if can be decomposed into n subgraphs $H_1, \ldots, H_n$ such that H_i has i edges and is isomorphic to a subgraph of H_{i+1}, $i = 1, \ldots, n-1$. In this chapter we deal with ascending subgraph decompositions of bipartite graphs, considering two different approaches.

In Section 5.2 we obtain sufficient conditions for a bipartite graph to have an ascending subgraph decomposition into stars, mainly based on the results about the sumset partition problem from Chapter 4. The connection between these two problems is shown in Section 5.1. In the same vein, in Section 5.3, we obtain an ascending subgraph decomposition for a bipartite graph $G(A, B)$ when $|A| \leq 4$ using the results for short sequences also from Chapter 4.

The second approach consists in finding ascending subgraph decompositions for a bipartite graph in which each factor of the decomposition is a forest of stars. In Section 5.4 we show that every bipartite graph G with $\binom{n+1}{2}$ edges such that the degree sequence $(d_1, \ldots, d_k)$ with $d_i > d_{i+1}$ for each $1 \leq i \leq k-1$, of one of the partite sets satisfies $d_1 \geq (k-1)(n-k+1)$ and $d_i \geq n-i+2$ for each $2 \leq i \leq k$, admits an ascending subgraph decomposition into star forests. We also give a necessary condition on the degree sequence of G to have an ascending subgraph decomposition into star forests that is not far from the above sufficient one. Our results are based on the existence of certain matrices that we call ascending, and the construction of edge-colorings for some bipartite graphs with parallel edges.

5.1 Introduction

Let G be a graph of positive size q, and let n be the positive integer with $\binom{n+1}{2} \leq q \leq \binom{n+2}{2}$. Then G is said to have an ascending subgraph decomposition, which we will denote by ASD, if G can be decomposed into n subgraphs $H_1, \ldots, H_n$ without isolated vertices such that H_i is isomorphic to a proper subgraph of H_{i+1} for $i = 1, \ldots, n-1$. Furthermore, if every subgraph H_i is a star (matching, path, star forest, ...), then we say that G admits an ascending star (matching, path, star forest, ...) subgraph decomposition or simply a star (matching, path, star forest, ...) ASD. In Fig. 5.1 an ascending subgraph decomposition of a graph G of size $\binom{4+1}{2} = 10$ is shown.

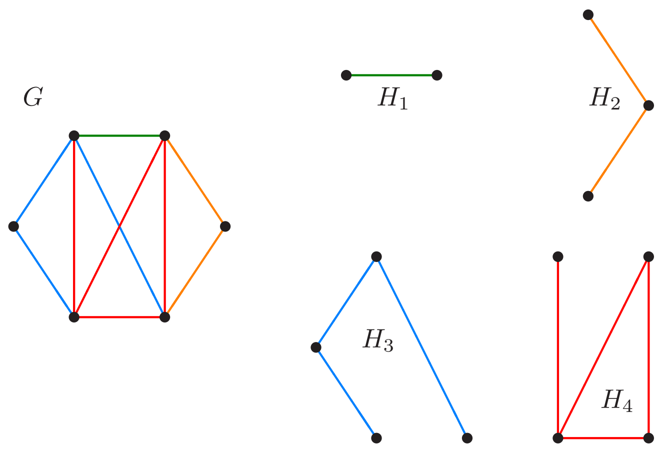

Figure 5.1 ASD of G

In 1987 Alavi, Boals, Chartrand, Erdös and Oellerman proposed two conjectures:

Conjecture 5.1 (Alavi et al., [2]) *A graph of positive size has an ASD.* □

Conjecture 5.2 (Alavi et al., [2]) *A star forest of size* $\binom{n+1}{2}$ *with each component having size between* n *and* $2n-2$ *inclusively has a star ASD.* □

In the same paper they reduced the verification of the first conjecture to the following equivalent version:

Conjecture 5.3 (Alavi et al., [2]) *Every graph of size $\binom{n+1}{2}$ for $n \geq 1$, has an ASD.* $\square$

Conjecture 5.2 was proved by Ma, Zhou and Zhou in [44], and it is equivalent to the n-realizability of sequence (4) in page 54. The condition was later weakened to the effect that the smallest component of the star forest can have size below n; this was obtained by Chen, Fu, Wang and Zhou in [12] and it is equivalent to the n-realizability of sequence (5) in page 54. In order to obtain their proofs, they used the connection between the ASD problem and the sumset partition problem treated in Chapter 4.

Let us show the connection between these two problems. Consider a graph G with $\binom{n+1}{2}$ edges and N vertices. If G admits a star ASD, by orienting the edges of each star of the decomposition towards the leaves we get an orientation of the edges of G. Let $m_i = d^+(v_i)$, $1 \leq i \leq N$, where $d^+(v_i)$ is the out-degree of v_i in this orientation for some ordering of the vertices of G. Then the sequence $(m_1, m_2, \ldots, m_N)$ is n-realizable (the sizes of the stars in the star ASD provide a realization). Conversely, if G admits an orientation such that the sequence $(d^+(v_1), d^+(v_2), \ldots, d^+(v_N))$ is n-realizable then G clearly admits a star ASD.

Conjecture 5.1 and its equivalent Conjecture 5.3 have turned out to be much more difficult. We can find three main directions to deal with them in the literature.

The first one concerns the number of vertices of the graph. Note that a graph G with $\binom{n+1}{2}$ edges has at least $n+1$ vertices, which corresponds to the complete graph. Faudree, Gyárfás and Schelp proved in [18] that the complete graph K_{n+1} has a star and a path ASD. They also showed that a graph with $n+2$ vertices has a star ASD.

The second direction is related with the maximum degree $\Delta(G)$. In 1990, Fu [19] proved that a graph G of size $\binom{n+1}{2}$ and $\Delta(G) \leq \frac{n-1}{2}$ has a matching ASD. Moreover, in the same paper showed that if $\Delta(G) \leq \frac{n+1}{2}$ then G has an ASD. These results extend pervious partial results concerning the maximum degree of G that can be found in [2, 18].

Finally, some authors have studied the conjecture for certain classes of graphs. It has already been commented that [12] and [44] are devoted to star ASD of star forests. Faudree and Gould [17] proved that a forest with $\binom{n+1}{2}$ edges has an ASD with each member a star forest. In 2002 Fu and Hu obtained that regular graphs have an ASD [23]. The same authors showed that complete multipartite graphs also admit an ASD [22], which extends the result given by Fu [20] for complete bipartite graphs.

In the definition of an ASD of a graph, we require that each subgraph of the decomposition must be isomorphic to a proper subgraph of a greater factor. A closely related packing problem is considered in [29] by loosening this requirement. The authors conjectured that the complete graph K_{2n+1} can be decomposed into n trees of sizes $1, 2, \ldots, n$. Observe that this conjecture is also related with Ringel's conjecture, treated in Chapter 3, which asks for the decomposition of K_{2n+1} into $2n+1$ copies of a given tree of size n.

Our work is focused on the existence of ASD of bipartite graphs. In particular, we know from the above results that a bipartite graph G with $\binom{n+1}{2}$ edges has an ASD if:

(1) G is regular;

(2) G is complete bipartite;

(3) $\Delta(G) \leq \frac{n+1}{2}$.

Therefore, our main objective is to obtain sufficient conditions for a non-complete, not necessarily regular, bipartite graph G with $\Delta(G) > \frac{n+1}{2}$ to have an ASD. Moreover, in view of the equivalence between Conjectures 5.1 and 5.3, we will always consider graphs of size $\binom{n+1}{2}$.

For this chapter we use the definition and the notation as well as the known results of the sumset partition problem described in Chapter 4.

5.2 Star ASD

There is a strong connection between the sumset partition problem and the star ASD of a bipartite graph. Let $G(A, B)$ be a bipartite graph of size $\binom{n+1}{2}$ and $A = \{a_1, \ldots, a_k\}$. We denote by $d_A = \langle d_1, d_2, \ldots, d_k \rangle$ the *degree sequence* of the vertices in A, which are ordered in such a way that d_A is a non-increasing sequence. By the definition, d_A is an n-feasible sequence, and it is clear that d_A is n-realizable if and only if G admits a star ASD with every star centered at the vertices of A. Therefore, we can translate every result obtained on the sumset partition problem about n-realizable sequences to the current problem. We have directly, from the result of Chen, Fu, Wang and Zhou [12], that if $d_{k-1} \geq n$ then $G(A, B)$ admits a star ASD. Moreover, from Theorem 4.4 we have the following corollary.

Corollary 5.4 *Let $G(A,B)$ be a bipartite graph of size $\binom{n+1}{2}$ and degree sequence $d_A = \langle d_1, d_2, \ldots, d_k \rangle$. If $n \geq 4k-1$ and $d_k \geq 4k$ then G admits a star ASD.* □

Using results on complete sets of positive integers, we obtained Theorems 4.18 and 4.20, which have the following direct implications on the star ASD problem for bipartite graphs.

Corollary 5.5 *Let $G(A,B)$ be a bipartite graph of size $\binom{n+1}{2}$ and degree sequence $d_A = \langle d_1, d_2, \ldots, d_k \rangle$. If*

$$n > d_3 > \cdots > d_k \geq \frac{3+\sqrt{8k-15}}{2},$$

then G admits a star ASD. □

Corollary 5.6 *Let $G(A,B)$ be a bipartite graph of size $\binom{n+1}{2}$ and degree sequence $d_A = \langle d_1, d_2, \ldots, d_k \rangle$. If $n \geq 3k-2$ and $n > m_3 > \cdots > m_k$ then G admits a star ASD.* □

5.3 Small partite set

We can obtain ASD of bipartite graphs $G(A,B)$ when $|A| \leq 4$ using the results for short sequences from Chapter 4. In particular, we will use Theorems 4.2 and 4.3 to obtain that every bipartite graph $G(A,B)$ with $\binom{n+1}{2}$ edges and $|A| \leq 4$ has an ASD if $n \geq 11$. For the proof of this result, we will denote by ASD_m a partial ASD of a graph with $\binom{n+1}{2}$ edges and $n \geq m$, consisting on the m first subgraphs of the total ASD($= ASD_n$).

Proposition 5.7 *Let $G(A,B)$ be a bipartite graph with $M = \binom{n+1}{2}$ edges. Then G admits an ASD if*

(1) $|A| = 1$ *and* $n \geq 1$;

(2) $|A| = 2$ *and* $n \geq 2$;

(3) $|A| = 3$ *and* $n \geq 3$;

(4) $|A| = 4$ *and* $n \geq 11$.

Proof.

(1) If $|A| = 1$, then for every $n \geq 1$ G is a star with M edges, which admits a star ASD.

(2) If $|A| = 2$, every n-feasible degree sequence $d_A = \langle d_1, d_2 \rangle$ for $n \geq 2$ is n-realizable and G admits a star ASD.

(3) If $|A| = 3$ we consider the degree sequence $d_A = \langle d_1, d_2, d_3 \rangle$ of the vertices $\{a_1, a_2, a_3\}$ of A. From Theorem 4.2, d_A is n-realizable and thus G admits a star ASD unless $\langle d_2, d_3 \rangle \in \mathcal{F}_2 = \{\langle 1, 1 \rangle, \langle 2, 2 \rangle\}$. Therefore, for $n \geq 3$, we have three cases:

- $\langle d_1, d_2, d_3 \rangle = \langle 2, 2, 2 \rangle$

  This case, which corresponds to $n = 3$, clearly admits an ASD.

- $\langle d_2, d_3 \rangle = \langle 1, 1 \rangle$

  Form the vertex a_2 we can obtain a star S_1. There are $M' = \binom{n+1}{2} - 2$ edges incident with a_1 and, since $n \geq 3$, we have that

  $$2 + \cdots + (n-1) + (n-1) = M'.$$

  Take stars $S_2, \ldots, S_{n-1}$ centered at a_1, which together with S_1 form an ASD_{n-1} of G. The last graph in the ASD_n of G is the star induced by the $(n-1)$ remaining edges from a_1 an the edge incident with a_3.

- $\langle d_2, d_3 \rangle = \langle 2, 2 \rangle$

  From the vertices a_2 and a_3 we can obtain an ASD_2 consisting of stars S_1 and S_2, and remains a single edge e. There are $M' = \binom{n+1}{2} - 4$ edges incident with a_1 and, since $n \geq 4$ (the case $n = 3$ corresponds to $\langle d_1, d_2, d_3 \rangle = \langle 2, 2, 2 \rangle$), we have that

  $$3 + \cdots + (n-1) + (n-1) = M'.$$

  Take stars $S_3, \ldots, S_{n-1}$ centered at a_1, which together with the stars S_1 and S_2 form an ASD_{n-1} of G. The star induced by the $(n-1)$ remaining edges of a_1 and the edge e is the last factor of G.

(4) If $|A| = 4$ and $n \geq 11$ we consider the degree sequence $d_A = \langle d_1, d_2, d_3, d_4 \rangle$ of the vertices $\{a_1, a_2, a_3, a_4\}$ of A. From Theorem 4.3, d_A is n-realizable and thus G admits a star ASD unless $\langle d_1, d_2, d_3, d_4 \rangle$ has a subsequence in

$$\begin{aligned} \mathcal{F}_2 \cup \mathcal{F}_3 \quad = \quad & \{\langle 1, 1 \rangle, \langle 2, 2 \rangle, \langle 3, 3, 1 \rangle, \langle 3, 3, 2 \rangle, \\ & \langle 3, 3, 3 \rangle, \langle 4, 4, 1 \rangle, \langle 4, 4, 3 \rangle, \langle 4, 4, 4 \rangle\}. \end{aligned}$$

We need now to check all these cases one by one:

- $\langle d_3, d_4\rangle = \langle 1, 1\rangle$

  If $n \geq 7$ we can choose $(n-2)+(n-1)$ edges incident with a_1 and not touching the two remaining edges incident with a_3 and a_4. Since the sequence $(d_1-(n-2)-(n-1), d_2)$ is $(n-2)$-feasible we can take a star ASD_{n-2} of the remaining edges incident with a_1 and a_2. The last two graphs in the ASD_n of G are the star induced by the $(n-2)$ chosen edges and the edge incident with a_3, and the star induced by the $(n-1)$ chosen edges and the edge incident with a_4.

- $\langle d_3, d_4\rangle = \langle 2, 2\rangle$

  If $n \geq 11$ we can choose $(n-3)+(n-2)+(n-2)$ edges incident witha_1 and not touching the four edges incident with a_3 and a_4. Since the sequence $(d_1-(n-3)-(n-2)-(n-2), d_2)$ is $(n-3)$-feasible we can take a star ASD_{n-3} of the remaining edges incident with a_1 and a_2. The last three graphs in the ASD_n of G are the star induced by the $(n-3)$ chosen edges and one edge incident with a_3, the star induced by the first $(n-2)$ chosen edges and the other edge incident with a_3, and the star induced by the last $(n-2)$ chosen edges and the two edges incident with a_4.

- $\langle d_2, d_3, d_4\rangle = \langle 2, 2, 1\rangle$

  From the vertices a_3 and a_4 we can obtain an ASD_2 consisting of the stars S_1 and S_2. There are $M' = \binom{n+1}{2} - 5$ edges incident with a_1. If $n \geq 5$, we have that

  $$3 + \cdots + (n-2) + (n-2) + (n-1) = M',$$

  and we can take $(n-2)+(n-1)$ edges incident with a_1 and not touching the two edges incident with a_2. From the remaining edges of a_1 take stars $S_3, \ldots, S_{n-2}$, which together with the stars S_1 and S_2 form an ASD_{n-2} of G. Finally, the last two factors of G will be the star induced by the $(n-2)$ chosen edges and one of the edges incident with a_2, and the star induced by the $(n-1)$ chosen edges and the other edge incident with a_2.

- $\langle d_2, d_3, d_4\rangle = \langle 3, 3, 1\rangle$

  From the vertices a_2 and a_3 we can obtain an ASD_3 consisting of the stars S_1, S_2 and S_3. There are $M' = \binom{n+1}{2} - 7$ edges incident with a_1.

If $n \geq 5$, we have that

$$4 + \cdots + (n-1) + (n-1) = M',$$

and we can take stars $S_4, \ldots, S_{n-1}$ centered at a_1, which together with the stars S_1, S_2 and S_3 form an ASD_{n-1} of G. The star induced by the $(n-1)$ remaining edges of a_1 and the edge incident with a_4 is the last factor of G.

- $\langle d_2, d_3, d_4 \rangle = \langle 3, 3, 2 \rangle$

  From the vertices a_2 and a_3 we can obtain an ASD_3 consisting of the stars S_1, S_2 and S_3. There are $M' = \binom{n+1}{2} - 8$ edges incident with a_1. If $n \geq 6$, we have that

$$4 + \cdots + (n-2) + (n-2) + (n-1) = M',$$

  and we can take $(n-2) + (n-1)$ edges incident with a_1 and not touching the two edges incident with a_4. From the remaining edges of a_1 take stars $S_4, \ldots, S_{n-2}$, which together with the stars S_1, S_2 and S_3 form an ASD_{n-2} of G. Finally, the last two factors of G will be the star induced by the $(n-2)$ chosen edges and one of the edges incident with a_4, and the star induced by the $(n-1)$ chosen edges and the other edge incident with a_4.

- $\langle d_2, d_3, d_4 \rangle = \langle 3, 3, 3 \rangle$

  From the vertices a_2 and a_3 we can obtain an ASD_3 consisting of the stars S_1, S_2 and S_3. There are $M' = \binom{n+1}{2} - 9$ edges incident with a_1. If $n \geq 6$, we have that

$$4 + \cdots + (n-2) + (n-2) + (n-2) = M',$$

  and we can take $(n-2) + (n-2)$ edges incident with a_1 and not touching the three edges incident with a_4. From the remaining edges of a_1 take stars $S_4, \ldots, S_{n-2}$, which together with the stars S_1, S_2 and S_3 form an ASD_{n-2} of G. Finally, the last two factors of G will be the star induced by the first $(n-2)$ chosen edges and one of the edges incident with a_4, and the star induced by the last $(n-2)$ chosen edges and the two remaining edges incident with a_4.

- $\langle d_2, d_3, d_4 \rangle = \langle 4, 4, 1 \rangle$

  From the vertices a_2 and a_3 we can obtain stars S_1, S_3 and S_4. There are $M' = \binom{n+1}{2} - 9$ edges incident with a_1. If $n \geq 6$, we have that

  $$2 + 5 + \cdots + (n-1) + (n-1) = M'.$$

  Take stars $S_2, S_5, \ldots, S_{n-1}$, which together with the stars S_1, S_3 and S_4 form an ASD_{n-1} of G. The star induced by the $(n-1)$ remaining edges of a_1 and the edge incident with a_4 is the last factor of G.

- $\langle d_2, d_3, d_4 \rangle = \langle 4, 4, 3 \rangle$

  From the vertices a_2, a_3 and a_4 we can obtain stars S_1, S_2, S_3 and S_4, and remains a single edge e. There are $M' = \binom{n+1}{2} - 11$ edges incident with a_1. If $n \geq 6$, we have that

  $$5 + \cdots + (n-1) + (n-1) = M'.$$

  Take stars $S_5, S_6, \ldots, S_{n-1}$, which together with the stars S_1, S_2, S_3 and S_4 form an ASD_{n-1} of G. The star induced by the $(n-1)$ remaining edges of a_1 and the edge e is the last factor of G.

- $\langle d_2, d_3, d_4 \rangle = \langle 4, 4, 4 \rangle$

  From the vertices a_2 a_3 and a_4 we can obtain an ASD_4 consisting of the stars S_1, S_2, S_3 and S_4, and remain two edges e and f. There are $M' = \binom{n+1}{2} - 12$ edges incident with a_1. If $n \geq 7$, we have that

  $$5 + \cdots + (n-2) + (n-2) + (n-1) = M',$$

  and we can take $(n-2) + (n-1)$ edges incident with a_1 and not touching the two edges e and f. From the remaining edges of a_1 take stars $S_5, S_6, \ldots, S_{n-2}$, which together with the stars S_1, S_2, S_3 and S_4 form an ASD_{n-2} of G. Finally, the last two graphs in the ASD_n of G will be the star induced by the $(n-2)$ chosen edges and the edge e, and the star induced by the $(n-1)$ chosen edges and the edge f.

$\square$

We can observe from the above proof, that in almost all of the cases (including when de degree sequence d_A is n-realizable) we are decomposing the graph with stars. In some cases we need star forests, and for cases $\langle 1, 1 \rangle$ and $\langle 2, 2 \rangle$ for length 3, and $\langle 3, 3, 1 \rangle$, $\langle 4, 4, 1 \rangle$ and $\langle 4, 4, 3 \rangle$ for length 4, the last factor may be a star and an edge hanging from a spoke. Should be pointed out that for these cases the proof can be slightly modified to obtain star forest decompositions. Another observation is that, for $|A| = 4$, the bound $n \geq 11$ is given by the single case $\langle d_3, d_4 \rangle = \langle 2, 2 \rangle$. Hence, if the degree sequence does not contain it, $n > 7$ is enough.

5.4 Star forest ASD

In order to weaken the sufficient conditions for a bipartite graph to obtain an ASD, here we consider star forest decompositions instead of star decompositions. In all this section we will consider that a star forest of $G(A, B)$ has the centers of all the stars in the partite set A.

5.4.1 Reduction lemma

Given a bipartite graph $G(A, B)$ with degree sequence $d_A = \langle d_1, \ldots, d_k \rangle$, we define the *reduced graph* $G_{d_A}(A, R)$ as the bipartite graph with same partite set A and $R = \{r_1, \ldots, r_{d_1}\}$ such that every vertex a_i is adjacent to the d_i first vertices $r_1, \ldots, r_{d_i}$ of R, $1 \leq i \leq k$. In Fig. 5.2 a bipartite graph and its reduced graph are shown.

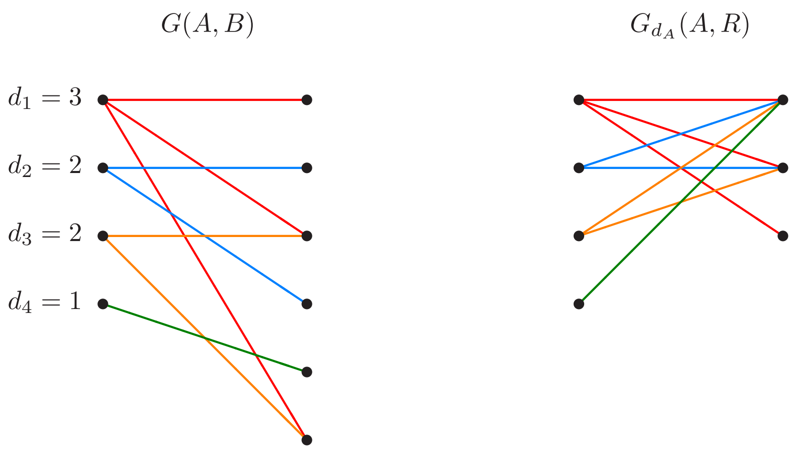

Figure 5.2 The bipartite graph $G(A, B)$ and its reduced graph $G_{d_A}(A, R)$.

Since in the reduced graph we are collapsing all the edges into the first vertices, it seems plausible that if the reduced graph admits a star forest ASD then the original graph must admit also a star forest ASD. Lemma 5.9 below shows that this is indeed true. In order to prove it, we first need the following notions.

A *proper edge-coloring* of a graph (or multigraph) is and assignment of colors to its edges in such a way that two edges that are incident with the same vertex should have different colors. From here on, we will consider a multigraph as a graph with parallel edges but without loops.

Let $G(A, B)$ be a bipartite multigraph and let $\mathcal{L} = \{L_a : a \in A\}$ be a family of lists of colors. We say that G can be properly edge-colored with $\mathcal{L}$ if the graph admits a proper edge-coloring and, for every $a \in A$, the edges incident with a are colored with a color from the list L_a.

To prove Lemma 5.9 we use the following result.

Theorem 5.8 (Häggkvist, [30]) *Let $H(V, W)$ be a bipartite multigraph. If H admits a proper edge-coloring such that each vertex $v \in V$ is incident with edges colored $\{1, 2, \ldots, d(v)\}$, then H can be properly edge-colored for an arbitrary assignment of lists $\{L(v) : v \in V\}$ such that $|L(v)| = d(v)$ for each $v \in V$.* □

Lemma 5.9 (Reduction lemma) *Let $G(A, B)$ be a bipartite graph with partite set $A = \{a_1, \ldots, a_k\}$ and degree sequence $d_A = \langle d_1, \ldots, d_k\rangle$. If the reduced graph $G_{d_A}(A, R)$ has a decomposition*

$$F'_1, \ldots, F'_t,$$

where F'_i is a star forest for each $i = 1, \ldots, t$, then G has a decomposition

$$F_1, \ldots, F_t$$

where each F_i is a star forest and $d_{F_i}(a_i) = d_{F'_i}(a_i)$ for $i = 1, \ldots, t$.

Proof. Let C be the $k \times t$ matrix whose entry c_{ij} is the number of edges incident to a_i in the star forest F'_j of the decomposition of $G_{d_A}(A, R)$.

Consider the bipartite graph $H(A, U)$, $U = \{u_1, \ldots, u_t\}$, where a_i is joined with u_j with c_{ij} parallel edges. Now, for each pair (i, j), color the c_{ij} parallel edges of H with the neighbors of a_i in the forest F'_j. Note that in this way we get a proper edge-coloring of H: two edges incident with a vertex a_i receive different colors since the original bipartite graph has no multiple edges, and two edges incident to a vertex u_j receive different colors since F'_j is a star forest.

Consider the original graph $G(A,B)$ with partite sets $A=\{a_1,\dots,a_k\}$ and $B=\{b_1,\dots,b_q\}$. By the definition of the bipartite multigraph H, each vertex a_i in A is incident with edges colored $1,2,\dots,d_i$. By Theorem 5.8, there is a proper edge-coloring of H in which the edges incident to vertex a_i in A receive the colors from the list $L(a_i)$ of neighbors of a_i in the original graph G. Now construct F_s for $s=1,\dots,t$ by letting the edge a_ib_j, $1\le i\le k$ and $1\le j\le q$, be in F_s, whenever the edge a_iu_s is colored b_j in the latter edge-coloring of H. Hence, F_s has the same number of edges than F'_s and the degree of a_i in F_s is $d_{F_s}(a_i)=c_{is}$, the same as in F'_s. Moreover, since the coloring is proper, F'_s is a star forest. This concludes the proof. □

Example 5.10 *To illustrate how is the decomposition of a graph from the decomposition of the reduced graph we consider the following example. Let $G(A,B)$ be the bipartite graph depicted in Fig. 5.3, which has degree sequence $d_A=\langle 5,3,1,1\rangle$.*

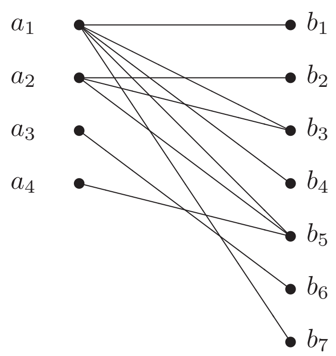

Figure 5.3 The graph $G(A,B)$.

For this graph we can obtain a star forest decomposition F'_1,F'_2,F'_3,F'_4 of its reduced graph as shown in the first part of Fig. 5.4. From this decomposition we have that the $k\times t$ matrix C defined in the above proof is

$$C=\begin{pmatrix}0&2&0&3\\1&0&2&0\\0&0&1&0\\0&0&0&1\end{pmatrix}.$$

The bipartite multigraph $H(A,U)$ obtained from matrix C is shown in Fig. 5.5, where it is properly edge-colored. In the same figure we can see a proper edge-coloring guaranteed by Theorem 5.8 in which each vertex from A is incident with colors from its neighbors in $G(A,B)$.

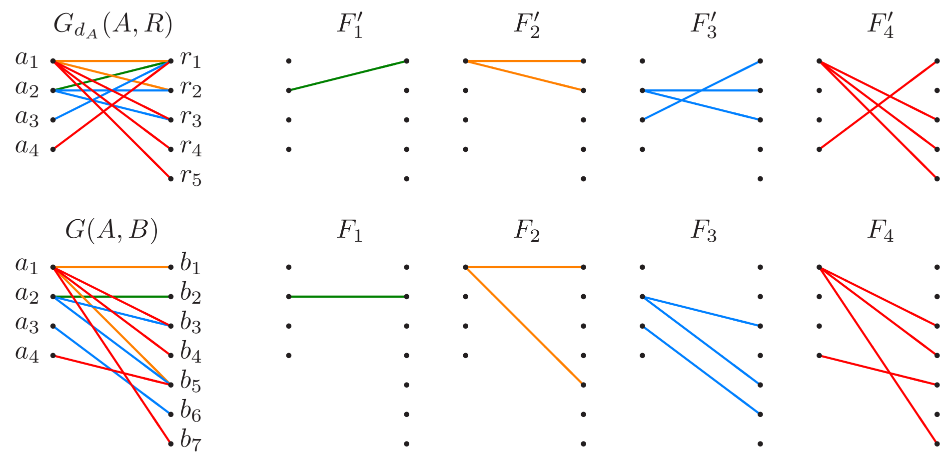

Figure 5.4 The reduced graph $G_{d_A}(A,R)$ with its star forest decomposition and the translation to the graph $G(A,B)$.

Finally, we return to Fig. 5.4 to view the translation of the star forest decomposition of the reduced graph to the original graph via the edge-coloring of the bipartite multigraph $H(A,U)$. □

We will obtain later on sufficient conditions for the decomposition of reduced graphs but first we present a necessary condition for every bipartite graph to admit a star forest ASD.

As said in the beginning of this section, for a bipartite graph $G(A,B)$, we only consider star forest decompositions with the stars centered at the vertices of A. We say that a degree sequence $d = \langle d_1, \ldots, d_k \rangle$ with $\sum_{i=1}^{k} d_i = \binom{n+1}{2}$ is *strongly decomposable* if every bipartite graph $G(A,B)$ with $d_A = d$ admits a star forest ASD with the centers of the stars in A.

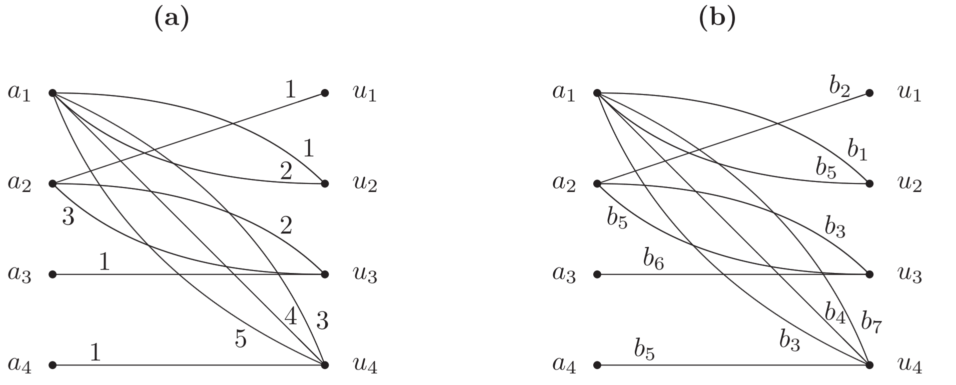

Figure 5.5 **(a)** The multigraph $H(A, U)$ obtained from matrix C. **(b)** $H(A, U)$ properly edge-colored with the vertices of B.

Lemma 5.11 *If the sequence $\langle d_1, \ldots, d_k \rangle$ is strongly decomposable then*

$$\sum_{i=1}^{t} d_i \geq \sum_{i=1}^{t} (n - i + 1) \tag{5.1}$$

for each $t = 1, \ldots, k$.

Proof. Consider the bipartite graph $G = G(A, B)$ with partite sets $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_{d_1}\}$ such that $a_i \in A$ is adjacent to the first d_i elements $\{b_j : 1 \leq j \leq d_i\}$, $i = 1, \ldots, k$. Let $F_1, \ldots, F_n$ be a star forest ASD of G. Since F_n has n leaves in B we clearly have $|B| = d_1 \geq n$.

Suppose that $d_1 + \cdots + d_{t-1} \geq n + (n-1) + \cdots + (n-t+2)$ for some $t \geq 2$. If $d_t \geq n - t + 1$ then the inequality extends to $d_1 + \cdots + d_{t-1} + d_t \geq n + (n-1) + \cdots + (n-t+2) + (n-t+1)$. Suppose that

$$n - t - d_t \geq 0. \tag{5.2}$$

We will compute the minimum number of edges incident with the vertices $a_1, \ldots, a_t$ and thus give a bound for $d_1 + \cdots + d_t$.

F_n has n edges, and at least $(n - d_t)$ edges are adjacent to the vertices $\{b_{d_t+1}, \ldots, b_{d_1}\}$ and therefore adjacent to the vertices $\{a_1, \ldots, a_{t-1}\}$. Likewise, F_{n-i} has at least $(n - i + 1 - d_t)$ edges adjacent to the vertices

$\{a_1, \ldots, a_{t-1}\}$, $i = 1, \ldots, t$. From (5.2) we know that all these quantities are positive. Moreover, every vertex a_i, with $1 \leq i \leq t$, has d_t edges incident with the vertices $b_1, \ldots, b_{d_t}$ still not counted. Hence,

$$\begin{aligned}\sum_{i=1}^{t} d_i &\geq td_t + (n - d_t) + (n - 1 - d_t) + \cdots + (n - t + 1 - d_t) \\ &= n + (n-1) + \cdots + (n - t + 1).\end{aligned}$$

□

5.4.2 Ascending matrices

Given two k-dimensional vectors $c = (c_1, \ldots, c_k)$ and $c' = (c'_1, \ldots, c'_k)$, we say that $c \preceq c'$ if there is a permutation $\sigma \in Sym(k)$ such that $c_i \leq c'_{\sigma(i)}$ for $i = 1, 2, \ldots, k$. In other words, after reordering the components of each vector in non-increasing order, the i-th component of c is not larger than the i-th component of c'. This definition is motivated by the following remark.

Remark 5.12 *Let F, F' be two forests of stars with centers $x_1, \ldots, x_k$ and $x'_1, \ldots, x'_k$ respectively. Then F is isomorphic to a subgraph of F' if and only if $(d_F(x_1), \ldots, d_F(x_k)) \preceq (d_{F'}(x'_1), \ldots, d_{F'}(x'_k))$.* □

Given a sequence $d = \langle d_1, \ldots, d_k \rangle$ of positive integers with $\sum_{i=1}^{k} d_i = \binom{n+1}{2}$, we say that a $k \times n$ matrix C with nonnegative integer entries is a *d-ascending matrix* if it satisfies the following three properties:

(A1) $\sum_{j=1}^{n} c_{ij} = d_i$, $i = 1, \ldots, k$,

(A2) $\sum_{i=1}^{k} c_{ij} = n - j + 1$, $j = 1, \ldots, n$,

(A3) $c^j \succeq c^{j+1}$, $j = 1, \ldots, n-1$, where c^j denotes the j-th column of C.

Example 5.13 *The following 4×7 matrix is d-ascending for the sequence $d = \langle 10, 8, 5, 5 \rangle$, which is 7-feasible.*

$$C = \begin{pmatrix} 3 & 2 & 2 & 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

□

We next show that the existence of these matrices give, in fact, star forest ASD of reduced graphs and therefore, by Lemma 5.9, of every bipartite graph with the same degree sequence on a partite set. To show this we will need the following well-known theorems by König and Hall, which can be easily seen that also hold for bipartite graphs with parallel edges. The *edge-chromatic number* $\chi'(G)$ of a graph (or multigraph) G is the minimum number of colors needed to obtain a proper edge-coloring of G.

Theorem 5.14 (König, see, e.g., [15]) *If G is a bipartite graph then*

$$\chi'(G) = \Delta(G).$$

$\square$

A *matching of* A in a bipartite graph $G(A, B)$ is a matching of G such that each vertex of A is incident to an edge of the matching. For a subset X of the vertices of a graph, we denote by $N(X)$ to all the neighbors of the vertices of X, that is,

$$N(X) = \cup_{v \in X} N(v).$$

Theorem 5.15 (Hall, see, e.g., [15]) *A bipartite graph $G(A, B)$ contains a matching of A if and only if*

$$|N(X)| \geq |X| \qquad (\textit{marriage condition})$$

for all $X \subseteq A$. $\square$

Before proving the main result, we will show the following technical lemma by using Theorems 5.14 and 5.15.

Lemma 5.16 *Let $H(A, U)$ be a bipartite multigraph. If*

$$\delta(A) \geq \Delta(U),$$

where $\delta(A) = \min_{a \in A} d(a)$ and $\Delta(U) = \max_{u \in U} d(u)$, then there is a proper edge-coloring of H such that each vertex $a \in A$ is incident to edges colored with $\{1, 2, \ldots, d(a)\}$.

Proof. Given a $X \subseteq A$ we denote by $e(X, N(X))$ the set of edges that join the vertices of X with their neighbors.

We have that

$$|N(X)|\Delta(U) \geq e(X, N(X)) \geq |X|\delta(A) \geq |X|\Delta(U),$$

which implies $|N(X)| \geq |X|$ and the marriage condition holds. By Theorem 5.15 there is a matching M from A to U in H. Let $A_{\Delta(A)} \subset A$ be the set of vertices with degree $\Delta(A)$ in A. If $\Delta(A) > \Delta(U)$, then color the edges in M incident to vertices in $A_{\Delta(A)}$ with $\Delta(A)$ and remove these edges from H. The resulting multigraph H' still satisfies $\delta_{H'}(A) \geq \Delta_{H'}(U)$ but now $\Delta_{H'}(A) = \Delta_H(A) - 1$. By iterating this process we eventually reach a multigraph H_0 with $m_0 = \Delta_{H_0}(A) = \delta_{H_0}(A) = \Delta_H(U)$, which can be properly edge-colored, by using Theorem 5.14, with m_0 colors. At the end, it is clear that the edges incident with each vertex a in A are colored with $\{1, 2, \ldots, d_H(a)\}$. □

Recall that a non-increasing degree sequence $(d_1, \ldots, d_k)$ is usually denoted by $\langle d_1, \ldots, d_k \rangle$. We introduce here a similar notation for *strictly decreasing* sequences. If $d_i > d_{i+1}$ for each $i = 1, \ldots, k-1$ we denote the sequence by

$$\langle d_1, \ldots, d_k \rangle_S.$$

With the stated previous results we are now able to prove the main decomposition theorem.

Theorem 5.17 *Let $G(A, B)$ be a bipartite graph with $\binom{n+1}{2}$ edges and degree sequence $d_A = \langle d_1, \ldots, d_k \rangle_S$. Suppose that there is a d_A-ascending matrix C such that $c_{ij} \geq 1$ for each pair (i, j) with $i + j \leq k$. If $d_i \geq n - i + 1$, $i = 1, \ldots, k$, then G admits a star forest ASD.*

Proof. Let $H(A, U)$ be the bipartite multigraph with $U = \{u_1, \ldots, u_n\}$ and c_{ij} parallel edges joining $a_i \in A$ with $u_j \in U$.

Suppose that H admits a proper edge-coloring in which each vertex a_i in A is incident with the colors $\{1, 2, \ldots, d_i\}$, $1 \leq i \leq k$. This coloring can be used to obtain a family of subgraphs

$$F_1, \ldots, F_n$$

of the reduced graph $G_{d_A}(A, R)$ by letting F_s, $s = 1, \ldots, n$, consist of the edges $a_i r_j$, $1 \leq i \leq k$ and $1 \leq j \leq d_1$, such that $a_i u_{n-s+1}$ is colored j in the edge-colored multigraph $H(A, U)$. Thus F_s is a star forest and has degree sequence $d_A(F_s) = (c_{1,n-s+1}, \ldots, c_{k,n-s+1})$ in $G_{d_A}(A, R)$. By the column sum property (A2) of the matrix C, the star forest F_s has $\sum_{i=1}^{k} c_{i,n-s+1} = s$ edges and, by the ascending column property (A3) and Remark 5.12, it is isomorphic to a subgraph of F_{s+1}. Finally, by the row sum property (A1), $d_{F_1}(a_i) + \cdots + d_{F_n}(a_i) = d_i$ for each $i = 1, \ldots, k$, therefore the collection $F_1, \ldots, F_n$ of star forests forms an ASD of $G_{d_A}(A, R)$.

It follows from Lemma 5.9 that the given graph $G(A, B)$ admits an analogous star forest decomposition.

Therefore, we only need to prove that $H(A, U)$ admits such a coloring.

For each $s = 1, \ldots, k-1$ denote by M_s the matching formed by the s edges $a_i u_{s-i+1}$, $1 \le i \le s$, in H (such matchings exist by the condition $c_{ij} \ge 1$ for each pair (i, j) with $i + j \le k$).

Consider first the bipartite multigraph

$$H'(A, U) = H - (M_1 \cup \cdots \cup M_{k-1})$$

consisting of the remaining edges, and let $d'_A = \langle d'_1, \ldots, d'_k \rangle$ be the degree sequence of A in H'. Since $d_i \ge n - i + 1$ and from each vertex a_i we have removed $k-i$ incident edges, $1 \le i \le k-1$, we have $d'_i = d_i - (k-i) \ge n-k+1$ for each $i = 1, \ldots, k-1$, and $d'_k = d_k \ge n - k + 1$. On the other hand, each vertex u_i, $i = 1, \ldots, n$, has $d_H(u_i) = n - i + 1$ by the property (A2). Hence, the vertices u_i with $i = k, \ldots, n$, have degree in H (and H') at most $n-k+1$. Moreover, each one of the remaining vertices u_i ($i = 1, \ldots, k-1$) belongs to $k - i$ colored matchings, so that its degree is $d_{H'}(u_i) = n - k + 1$. Therefore, for H', $\delta(A) = n-k+1 \ge \Delta(U)$ and by Lemma 5.16 there is an edge-coloring such that each vertex a_i is incident with edges colored $\{1, \ldots, d'_i\}$, $1 \le i \le k$.

Now, we will successively add the removed matchings $M_{k-1}, M_{k-2}, \ldots, M_1$ to H' in the following way.

Let $t = \min\{d'_i : 1 \le i \le k\}$ and $r = \max\{d'_i : 1 \le i \le k\}$. Since the sequence d_A is strictly decreasing we have that $t = d'_k$.

If $t = r$, we can add all the matchings $M_{k-1}, \ldots, M_1$ to H' by assigning the color $r + k - s$ to the edges of M_s, and we have obtained a proper edge-coloring of H such that each vertex a_i in A is incident with edges colored $\{1, 2, \ldots, d_i\}$, $1 \le i \le k$.

Suppose that $t < r$. Since H' is proper edge-colored, the edges colored with j define a matching Q_j of H' for each $1 \le j \le r$. Consider the set of edges

$$Q = Q_{t+1} \cup \cdots \cup Q_r.$$

Now, we increase one unity the color of each edge in Q and then we add the matching M_{k-1} to H' by assigning to its edges the free color $t+1$. In this way, each vertex a_i is incident with edges colored $\{1,\ldots,d'_i+1\}$ for $i=1,\ldots,k-1$, and its degree has also increased one unity. The vertex a_k is incident with edges colored $\{1,\ldots,t\}=\{1,\ldots,d'_k=d_k\}$.

At this point we can repeat the same procedure and successively add the matchings $M_{k-2},\ldots,M_1$. At the end, we obtain a proper edge-coloring of H such that each vertex a_i in A is incident with edges colored $\{1,2,\ldots,d_i\}$, $1\le i\le k$. □

Note that the condition $d_i\ge n-i+1$, $1\le i\le k$, from the hypothesis of Theorem 5.17, is not far from the necessary condition (5.1) from Lemma 5.11.

Now, our main goal is to prove the existence of an adequate ascending matrix for a given degree sequence. We actually believe that such matrices always exists.

Conjecture 5.18 *For every n-feasible sequence $d=\langle d_1,\ldots,d_k\rangle$ with $d_i\ge n-i+1$, $1\le i\le k$, there exists a $k\times n$ ascending matrix $C=(c_{ij})$ with $c_{ij}\ge 1$ for each pair (i,j) with $i+j\le k$.* □

We have obtained two approximations to Conjecture 5.18, which in turn, by Theorem 5.17, will provide star forest ASD for some degree sequences. The first result is the following.

Theorem 5.19 *Let $G(A,B)$ be a bipartite graph with $\binom{n+1}{2}$ edges and degree sequence $d_A=\langle d_1,\ldots,d_k\rangle_S$. If*

(1) $d_1\ge(k-1)(n-k+1)$,

(2) $d_i\ge n-i+2$, $i=2,\ldots,k$;

then there exists a star forest ASD of G.

Proof. Define a matrix C' in the following way. Let the first row be

$$(\underbrace{n-k+1,\ldots,n-k+1}_{k-1},n-k+1,n-k,n-k-1,\ldots,2,1),$$

and the i-th row be $(\underbrace{1,\ldots,1}_{k-i+1},0,\ldots,0)$, for $i=2,\ldots,k$.

From this construction we have that C' has the ascending column property (A3).

It is clear that the sum of the elements of the column j is $(n-k+1)+(k-j) = n-j+1$ for $j = 1,\ldots,k-1$, and for $j = k,\ldots,n$, the only nonzero elements are the $n-j+1$ elements of the first row. Therefore the matrix C' has the column sum property (A2).

Let $d'_2,\ldots,d'_k$ be the row sums of C'. From condition (2):

$$d_i - d'_i \geq n-i+2-(k-i+1) = n-k+1.$$

Let $d'_1 = (k-1)(n-k+1)$. From condition (1):

$$d_1 - d'_1 \geq 0.$$

Consider the sequence

$$S = (d_1 - d'_1, d_2 - d'_2, \ldots, d_k - d'_k).$$

Then,

$$\begin{aligned}\sum_{i=1}^{k}(d_i - d'_i) &= \binom{n+1}{2} - (k-1)(n-k+1) - \sum_{i=2}^{k} k-i+1 \\ &= \frac{n^2+3n+k^2-3k-2nk+2}{2} \\ &= \binom{n-k+1}{2}.\end{aligned}$$

Therefore, S is $(n-k+1)$-feasible. Since all the elements of the sequence are above $n-k+1$ with the possible exception of $d_1 - d'_1$, by the result of Chen et al. (sequence (5) in page 54), there is a partition $X_1,\ldots,X_k$ of the set $[n-k+1]$ such that $\sum(X_i) = d_i - d'_i$, $1 \leq i \leq k$. The elements of this set appear precisely in the last positions of the first row, which have zeros below them. Observe that if $d_1 - d'_1 = 0$, we can consider the same sequence and set $X_1 = \emptyset$.

We define the matrix $C = (c_{ij})$ to be the same matrix as C' but applying the following switchings to the last $n-k+1$ columns. For every $x \in [n-k+1]$, which is in the set X_i for some $1 \leq i \leq k$, switch it with the corresponding zero in the same column and row i. In this way, we are not altering the ascending column property (A3) and the column sum property (A2) since we are permuting elements in the same column; and finally, for each row $i = 1, \ldots, k$,

$$\sum_{j=1}^{n} c_{ij} = d'_i + \sum X_i = d_i.$$

Therefore, the matrix C has the row sum property (A1) and it is clear that $c_{ij} \geq 1$ for each pair (i, j) with $i + j \leq k$. Thus we are in the hypothesis of Theorem 5.17 and G admits a star forest ASD. □

Example 5.20 *Let $d_A = \langle 16, 8, 7, 5 \rangle_S$, for which $k = 4$ and $n = 8$. The constructed matrices from the proof of Theorem 5.19 are the following.*

$$C' = \left(\begin{array}{ccc|ccccc} 5 & 5 & 5 & 5 & 4 & 3 & 2 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \qquad C = \left(\begin{array}{ccc|ccccc} 5 & 5 & 5 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 5 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 3 & 2 & 0 \\ 1 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \end{array}\right)$$

□

The second result is:

Theorem 5.21 *Let $G(A, B)$ be a bipartite graph with $\binom{n+1}{2}$ edges and degree sequence $d_A = \langle d_1, \ldots, d_k \rangle_S$. If*

(1) *$d_i \geq n - i + 1$, $1 \leq i \leq k$, and*

(2) *the $(n-k)$-feasible sequence $(d_1 - n, d_2 - (n-1), \ldots, d_k - (n-k+1))$ is $(n-k)$-realizable,*

then there exists a star forest ASD of G.

Proof. Let ϵ_i denote the n dimensional vector with entries 1 till the i-th coordinate and entries 0 for the rest:

$$\epsilon_i = (\underbrace{1, 1, \ldots, 1}_{i}, \underbrace{0, \ldots, 0}_{n-i})$$

Construct an $n \times n$ matrix C' with row i the vector ϵ_{n-i+1}.

From condition (1), the sum of each of the first k rows of C' is at most the value d_i, so the first k rows have to be completed and the last $n-k$ deleted. To do this, we consider the sequence

$$d' = (d_1 - n, d_2 - (n-1), \dots, d_k - (n-k+1)),$$

which is clearly $(n-k)$-feasible. In the case that for some i, $d_i = n - i + 1$, the row is completed and we consider the sequence d' of length $k-1$ by removing the 0.

We know from condition (2) that d' is $(n-k)$-realizable, so there is a partition $\mathcal{P} = \{X_1, \dots, X_k\}$ of the set $[n-k]$ such that $\sum X_i = d_i - (n-i+1)$. Now, we construct the ascending matrix $C = (c_{ij})$ from C' in the following way:

For every set $X_i = \{\alpha_1, \dots, \alpha_t\}$, we remove the rows of C' with vectors $\epsilon_{\alpha_1}, \dots, \epsilon_{\alpha_t}$ and add all them to the row i. Since $\mathcal{P}$ is a partition of the set $[n-k]$, we are deleting the $n-k$ last rows of C', thus C is a $k \times n$ matrix. Moreover,

$$\sum_{j=1}^{n} c_{ij} = n - i + 1 + \sum (X_i) = di, \qquad 1 \leq i \leq k,$$

and the matrix C has the row sum property (A1). The column sum property (A2) is obvious since every column j of C' has $n-j+1$ unities. Finally it is clear that every time that we add a row ϵ_{α_j} to another row, we do not break the ascending column property (A3).

Hence, $C = (c_{ij})$ is a $k \times n$ d_A-ascending matrix and clearly $c_{ij} \geq 1$ for each pair (i,j) with $i + j \leq k$. Thus we are in the hypothesis of Theorem 5.17 and G admits a star forest ASD. $\square$

Example 5.22 *Let $d_A = \langle 19, 17, 11, 8 \rangle_S$, for which $k = 4$ and $n = 10$. The starting matrix C' is*

$$C' = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The sequence $d' = (19-10, 17-9, 11-8, 8-7) = (9, 8, 3, 1)$ *is* 6*-realizable. Consider the following realization:*

$$X_1 = \{5+4\}, \qquad X_2 = \{6+2\}, \qquad X_3 = \{3\}, \qquad X_4 = \{1\}.$$

Thus, once deleted the last 6 *rows and added properly to the first* k *rows, the matrix* C' *is transformed to the ascending matrix*

$$C = \begin{pmatrix} 3 & 3 & 3 & 3 & 2 & 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

$\square$

A

Non-realizable sequences

In Chapter 4, Section 4.2, we need to do some verifications. Specifically, we want to check the n-feasible but not n-realizable sequences of length $k = 3$, $n = 4, 5, 6, 7$ and the same sequences of length $k = 4$, $n = 4, 5, 6, 7, 8$. In order to do these calculations that would be tedious by hand, we present a simple algorithm that given n and k computes the n-feasible sequences of length k that are not n-realizable. We call it $\textsc{NonReal}(n, k)$. In order to describe this algorithm, we will need the following technical function Φ that adds one element in each sequence of a sequence list. It is defined as

$$\Phi(\emptyset, m) = \emptyset,$$

$$\begin{aligned}&\Phi(\{(a_1^1, \ldots, a_{l_1}^1), \ldots, (a_1^s, \ldots, a_{l_s}^s)\}, m) = \\ &= \{(a_1^1, \ldots, a_{l_1}^1, m), \ldots, (a_1^s, \ldots, a_{l_s}^s, m)\},\end{aligned}$$

where m is an arbitrary integer and a_j^i are also arbitrary integers indexed by the sequence in which belong (the total number of sequences is $s \geq 1$) and the position in each sequence (the length of each sequence is $l_i \geq 0$). A sequence of length zero is denoted by $()$.

$\textsc{NonReal}(n, k)$ is split into two main procedures. The first procedure is a well-known recursive algorithm that gives all the partitions of the set $[n]$ into k non-empty parts. The entries of the procedure are n and k and it returns a set of sequences of length n, each one representing one specific k-partition, in such a way that the element $a_j \in \{1, \ldots, k\}$ of a returned sequence $(a_1, \ldots, a_n)$ says that the element j lies in part a_j. The procedure is detailed in Algorithm 1. To obtain all the different n-realizable sequences of length k we only have to sum all the parts of each k-partition and discard the repeated sequences.

Algorithm 1 Partitions of $[n]$ in k parts

procedure SetPartitions(n, k)
  var i, out
  if $n = 1$ and $k \neq 1$ **then** RETURN$(\emptyset)$ **end if**
  if $n = 1$ and $k = 1$ **then** RETURN$(\{(1)\})$ **end if**
  if $n \neq 1$ **then**
    $out := \Phi(\text{SetPartitions}(n-1, k-1), k)$
    for $i = 1$ to k **do**
      $out := out \cup \Phi(\text{SetPartitions}(n-1, k), i)$
    end for
    RETURN(out)
  end if
end procedure

The second procedure is another well-known recursive algorithm that gives all the integer partitions of a positive integer m, that is, all the possible ways to add up to m with nonzero summands. Algorithm 2 takes an integer m and returns the set of all different sequences such that the sum of the elements of each sequence is m. To obtain the partitions of the integer m, the procedure should be called as Partitions(m, m).

Algorithm 2 Integer partitions of m

procedure Partitions(m, l)
  var i, out
  if $m = 0$ **then** RETURN$(\{()\})$ **end if**
  if $m \neq 0$ **then**
    $out := \emptyset$
    for $i = 1$ to $\min(m, l)$ **do**
      $out := out \cup \Phi(\text{Partitions}(m-i, i), i)$
    end for
    RETURN(out)
  end if
end procedure

Since we want the n-feasible sequences of length k, we should only keep the sequences of length k from Partitions$\left(\frac{n(n+1)}{2}, \frac{n(n+1)}{2}\right)$.

The final algorithm $\textsc{NonReal}(n, k)$ that returns all the n-feasible sequences of length k that are not n-realizable can be done by simply comparing the n-feasible sequences given by the first procedure with the n-realizable sequences given by the second one.

For Theorem 4.2, we run the algorithm $\textsc{NonReal}(n, k)$ for $k = 3$ and $n = 4, 5, 6, 7$ with the following results.

$$\begin{aligned}
\textsc{NonReal}(4,3) &= \{\langle 6,2,2\rangle, \langle 8,1,1\rangle\}\\
\textsc{NonReal}(5,3) &= \{\langle 11,2,2\rangle, \langle 13,1,1\rangle\}\\
\textsc{NonReal}(6,3) &= \{\langle 17,2,2\rangle, \langle 19,1,1\rangle\}\\
\textsc{NonReal}(7,3) &= \{\langle 24,2,2\rangle, \langle 26,1,1\rangle\}
\end{aligned}$$

For Theorem 4.3, we run the algorithm for $k = 4$ and $n = 8$ with the following result.

$$\begin{aligned}
\textsc{NonReal}(8,4) = \{&\langle 16,16,2,2\rangle, \langle 17,15,2,2\rangle, \langle 17,17,1,1\rangle, \langle 18,14,2,2\rangle,\\
&\langle 18,16,1,1\rangle, \langle 19,13,2,2\rangle, \langle 19,15,1,1\rangle, \langle 20,12,2,2\rangle,\\
&\langle 20,14,1,1\rangle, \langle 21,11,2,2\rangle, \langle 21,13,1,1\rangle, \langle 22,10,2,2\rangle,\\
&\langle 22,12,1,1\rangle, \langle 23,9,2,2\rangle, \langle 23,11,1,1\rangle, \langle 24,4,4,4\rangle,\\
&\langle 24,8,2,2\rangle, \langle 24,10,1,1\rangle, \langle 25,4,4,3\rangle, \langle 25,7,2,2\rangle,\\
&\langle 25,9,1,1\rangle, \langle 26,6,2,2\rangle, \langle 26,8,1,1\rangle, \langle 27,3,3,3\rangle,\\
&\langle 27,4,4,1\rangle, \langle 27,5,2,2\rangle, \langle 27,7,1,1\rangle, \langle 28,3,3,2\rangle,\\
&\langle 28,4,2,2\rangle, \langle 28,6,1,1\rangle, \langle 29,3,2,2\rangle, \langle 29,3,3,1\rangle,\\
&\langle 29,5,1,1\rangle, \langle 30,2,2,2\rangle, \langle 30,4,1,1\rangle, \langle 31,2,2,1\rangle,\\
&\langle 31,3,1,1\rangle, \langle 32,2,1,1\rangle, \langle 33,1,1,1\rangle\}
\end{aligned}$$

Finally, to obtain the set $\mathcal{S}$ in page 56, we run the algorithm for $k = 4$ and $n = 4, 5, 6, 7$.

$$\begin{aligned}
\textsc{NonReal}(4,4) = \{&\langle 3,3,2,2\rangle, \langle 3,3,3,1\rangle, \langle 4,2,2,2\rangle, \langle 4,4,1,1\rangle,\\
&\langle 5,2,2,1\rangle, \langle 5,3,1,1\rangle, \langle 6,2,1,1\rangle, \langle 7,1,1,1\rangle\}
\end{aligned}$$

$$\begin{aligned}
\textsc{NonReal}(5,4) \quad = \quad & \{\langle 4,4,4,3\rangle, \langle 6,3,3,3\rangle, \langle 6,4,4,1\rangle, \langle 6,5,2,2\rangle,\\
& \langle 6,6,2,1\rangle^*, \langle 7,3,3,2\rangle, \langle 7,4,2,2\rangle, \langle 7,6,1,1\rangle,\\
& \langle 8,3,2,2\rangle, \langle 8,3,3,1\rangle, \langle 8,5,1,1\rangle, \langle 9,2,2,2\rangle,\\
& \langle 9,4,1,1\rangle, \langle 10,2,2,1\rangle, \langle 10,3,1,1\rangle, \langle 11,2,1,1\rangle,\\
& \langle 12,1,1,1\rangle\}
\end{aligned}$$

$$\begin{aligned}
\textsc{NonReal}(6,4) \quad = \quad & \{\langle 8,7,3,3\rangle^*, \langle 8,8,3,2\rangle^*, \langle 9,4,4,4\rangle, \langle 9,8,2,2\rangle,\\
& \langle 10,4,4,3\rangle, \langle 10,7,2,2\rangle, \langle 10,9,1,1\rangle, \langle 11,6,2,2\rangle,\\
& \langle 11,8,1,1\rangle, \langle 12,3,3,3\rangle, \langle 12,4,4,1\rangle, \langle 12,5,2,2\rangle,\\
& \langle 12,7,1,1\rangle, \langle 13,3,3,2\rangle, \langle 13,4,2,2\rangle, \langle 13,6,1,1\rangle,\\
& \langle 14,3,2,2\rangle, \langle 14,3,3,1\rangle, \langle 14,5,1,1\rangle, \langle 15,2,2,2\rangle,\\
& \langle 15,4,1,1\rangle, \langle 16,2,2,1\rangle, \langle 16,3,1,1\rangle, \langle 17,2,1,1\rangle,\\
& \langle 18,1,1,1\rangle\}
\end{aligned}$$

$$\begin{aligned}
\textsc{NonReal}(7,4) \quad = \quad & \{\langle 10,10,4,4\rangle^*, \langle 12,12,2,2\rangle, \langle 13,11,2,2\rangle, \langle 13,13,1,1\rangle,\\
& \langle 14,8,3,3\rangle^*, \langle 14,10,2,2\rangle, \langle 14,12,1,1\rangle, \langle 15,9,2,2\rangle,\\
& \langle 15,11,1,1\rangle, \langle 16,4,4,4\rangle, \langle 16,8,2,2\rangle, \langle 16,10,1,1\rangle,\\
& \langle 17,4,4,3\rangle, \langle 17,7,2,2\rangle, \langle 17,9,1,1\rangle, \langle 18,6,2,2\rangle,\\
& \langle 18,8,1,1\rangle, \langle 19,3,3,3\rangle, \langle 19,4,4,1\rangle, \langle 19,5,2,2\rangle,\\
& \langle 19,7,1,1\rangle, \langle 20,3,3,2\rangle, \langle 20,4,2,2\rangle, \langle 20,6,1,1\rangle,\\
& \langle 21,3,2,2\rangle, \langle 21,3,3,1\rangle, \langle 21,5,1,1\rangle, \langle 22,2,2,2\rangle,\\
& \langle 22,4,1,1\rangle, \langle 23,2,2,1\rangle, \langle 23,3,1,1\rangle, \langle 24,2,1,1\rangle,\\
& \langle 25,1,1,1\rangle\}
\end{aligned}$$

The sequences marked with * are precisely the only ones not containing any subsequence from $\mathcal{F}_2 \cup \mathcal{F}_3$.

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