CHAPTER 2

Modeling switched power converters using the complementarity formalism

"One geometry cannot be more true than another; it can only be more convenient."

J.H.Poincaré (1854-1912)

This chapter is concerned with the modeling of power converters using the complementarity formalism. First, a concise introduction to the theory of complementarity systems is given. Then, we model some basic dc-dc power converters with a single diode (buck, boost, buck-boost and Čuk) as linear cone complementarity systems. After fixing the position of the switches, the dynamics is given by a linear complementarity system which incorporates, in a natural way, the description of generalized discontinuous conduction modes (GDCM), characterized by a reduction of the dimension of the effective dynamics. Analytical state-space conditions for the presence of a GDCM have been stated in each example and simulation results, showing a variety of behaviours, such as persistent or re-entering GDCM, are also presented. The modeling, analysis and simulation of a parallel resonant converter (PRC) which has four diodes illustrate the convenience of the complementarity formalism to simulate electrical systems with a large number of ideal diodes. Finally, we present the simulation of a boost converter with a sliding mode control, even though a general control theory for complementarity systems is not still developed.

2.1 Introduction

Power electronics devices are widely used in different real-life applications from industrial, commercial and aerospace environments. The needs to convert electrical energy at high efficiency (power conversion problems) in practical situations have been an important point in the development of the field of power electronics. Moreover, an intensive development in many aspects of technology, including power devices, control methods, circuit design, passive components, etc., have been made in the last four decades and power converters have taken advantage of all that. However, like in many areas of engineering, power electronics is mainly motivated by practical applications, and this fact allows that a particular circuit topology or system implementation has found widespread applications long before it is thoroughly analyzed and most of its subtleties uncovered. For instance, the widespread application of a simple switching converter may date back to more than three decades ago. However, good analytical models allowing a better understanding and systematic circuit design were only developed in late 1970's, and in-depth analytical and modeling work is still being actively pursued today. Furthermore, nonlinear phenomena, despite being commonly found in power electronic circuits, have only received formal treatments in very recent years.

Power converters fall into a group of systems that are often referred to as *nonsmooth dynamical systems* (NSDS) or *piecewise smooth* (PWS) systems due to the physical behaviour of devices used as switches and diodes. In recent years there have been a lot of analysis and research into NSDS and there are, at least, three popular different approaches to deal with NSDS, the *complementarity system* approach [24, 82], the *differential inclusion* approach [9] and the *hybrid system* approach [90]. The different approaches have shown to be good for different purposes, depending of what the goal with the analysis have been. For instance, it is well known that the complementarity formalism is a suitable framework for simulating mechanical systems with a large number of constraints [106], but however, the hybrid system approach has shown to be particularly successful when trying to understand and explain qualitative dynamical changes caused by the interaction of invariant sets, such as limit cycles, and discontinuity surfaces [92]. In the present chapter we will use the first approach, the complementarity system approach, to model and study some characteristics of the dynamical behaviour of switched power converters.

Dynamic networks with diodes can be recast as linear complementarity systems (LCS), [29]. LCS are obtained as follows. Take a standard linear system, select a number of input/output pairs (u_i, y_i) and impose for each of these pairs that at each time t both $u_i(t)$ and $y_i(t)$ must be nonnegative, and at least one of them should be zero (nonnegativity+orthogonality):

$$u_i(t) \geq 0, \quad y_i(t) \geq 0, \quad u_i(t)y_i(t) = 0, \quad \forall i, \quad \forall t.$$

These are called the "complementarity conditions" (CC), and are denoted collectively as

$$0 \leq u \perp y \geq 0.$$

The pairs (u_i, y_i) are called "complementary variables". CC are well-known in mathematical programming [19], although not usually in combination with differential equations. In the context of electrical circuits, imposing complementarity conditions simply means that some ports are terminated by ideal diodes, with the current i_D and (minus) the voltage $-v_D$ as complementary variables.

Associated to each complementary pair (u_i, y_i) there are two general situations allowed by the CC: either $u_i = 0$ and $y_i \geq 0$ or $y_i = 0$ and $u_i \geq 0$. In electrical engineering terminology, diodes may be blocking or conducting. If there are p diodes, one has 2^p of these binary choices and the system can be in any of 2^p so-called "modes". For power converters one has, in addition to (ideal) diodes, some (ideal) switches which are arbitrarily closed or open by a control law. Ideal switches do not dissipate or store power, and hence the product of current and voltage for any of them is zero, $i_S v_S = 0$. This resembles part of a CC; however one does not have, in general, a positiveness condition in this case (although some physical realizations of the switch may impose some kind of partial positiveness; see [30]). To achieve this, a generalization of linear complementarity systems known as linear cone complementarity systems (LCCS) can be considered [29].

Since some of the complementary variables are linked to state variables by static relations, the evolution of the later can bring some of the former to zero; they cannot decrease any further without violating the CC and this may force a nonsmooth change in some of the other complementary variables so that the vector field for the state variables takes the correct sign. Therefore, the study of existence and uniqueness of solution trajectories given an initial condition (well-posedness) of complementarity systems is particularly relevant in order to check the validity of the mathematical model and set up simulation algorithms for such systems.

Existence and uniqueness of solutions in linear electrical networks consisting of (linear) resistors, capacitors, inductors, capacitors, gyrators, transformers (RLCGT), ideal diodes and current and/or voltage sources have been proved under some technical assumptions as passivity in [28]. These assumptions are satisfied for some basic dc-dc power converters with single diode as buck, boost, buck-boost and Čuk. Using these techniques analytical results can be obtained and the state space conditions for the above jump to take place can be given in detail. If a complementary variable is nonzero, the complementarity condition forces its conjugate to remain equal to zero for a while, and hence one or more state variables (or combination of them) may be kept to a constant value for a time. This situation, in the case when the variable is a current, is known as a "discontinuous conduction mode" (DCM) in the power converter literature [58]. Since this can happen to a wider class of variables, other than currents, we call this situation "generalized discontinuous conduction mode" (GDCM). The GDCM lasts until a switch state change takes place, or until the companion complementary variable returns to zero due to the dynamics of the other state variables. DCMs have been extensively studied in the literature in connection with control algorithms, *i.e.* switching policies, such as PWM (see [126] and references therein), where averaged methods or small signal frequency domain descriptions are generally used, or in connection to bifurcation theory and chaos [121]. However, the aim in this chapter will be to obtain, for a given switch position, an exact state-space condition which indicates the appearance of a GDCM.

Nevertheless, considering linear electrical systems with diodes in parallel some assumptions imposed in [28] to assure existence and uniqueness of solutions are not hold. Indeed, we will present a Parallel Resonant Converter (PRC) where solutions of the complementarity variables are not unique but the state-space solution is unique. Therefore, results given in [28] must be extended in order to cover a wider class of electrical systems.

Another open problem is concerned with control theory for complementarity systems. Design of robust and efficient controllers for complementarity systems is now being investigated but still there are no general results. In this chapter we have simulated a boost converter using a sliding mode control. Such control works pretty well but can not be extended to other electrical systems.

Along this chapter we have used numerical algorithms for simulating complementarity systems (see Appendix A for further details). Firstly, we have considered a backward Euler squeme in order to discretize our dynamical system and as a result we obtain a Linear Complementarity Problem (LCP) for each time step. In order to solve such LCP we have used a Lemke algorithm although another algorithms can be also used as Murty's algorithm, Latin algorithm, Non Smooth Newton algorithm, etc.

This chapter is organized as follows. Section 2.2 presents the general LCS formulation. In Section 2.3 the specific forms for the buck, boost, buck-boost and Čuk are displayed. Section 2.4 starts with a general result which then is applied to the converters just mentioned in order to obtain specific conditions of GDCM for each of them. Furthermore, simulations for the boost and Čuk converter which corroborate the theoretical predictions and display some interesting phenomena are also shown. In Section 2.6 a PRC is modeled and analysed. We will show that existence and uniqueness of solutions for linear electrical systems must be extended for the cases of systems with diodes in parallel. A boost converter with a sliding mode control is implemented and simulated in Section 2.7. Finally, we summarize our results in Section 2.8 and discuss some open questions.

2.2 Modeling using complementarity formalism

In what follows we will introduce some terminology and background material related to the modeling using the complementarity formalism (for further details, the general theory for complementarity systems is covered in [27, 80]).

2.2.1 Complementarity systems

Inequalities have played an important role in many research fields including mathematical programming and economics. It is surprising to see that inequalities have received little attention in system theory except for the matrix inequalities which appear in most of multivariable linear control theory. One reason might be that combining inequalities and differential equations means giving up the smoothness properties that form the basis of much of the theory of dynamical systems. However, in many situations it seems reasonable to study dynamics together with inequalities. Thus, we will introduce the linear complementarity problem (LCP) that is a very important tool in mathematical programming and in problems using inequalities.

The name "linear complementarity problem" has underwent several changes until 1965, and the current name was proposed by Cottle. Then, it was later used in a paper by Cottle, Habetler and Lemke (1970). After that paper, many results and algorithms have been obtained in this field (see more details in [19, 36]). Although diverse instances of the linear complementarity problem can be traced to publications as far back as 1940, concentrated study of the LCP began in the mid 1960's.

Basically, the LCP consists of finding a vector in a finite-dimensional real vector space that satisfies a certain system of inequalities. Specifically, given a vector $q \in \mathbb{R}^k$ and a matrix $M \in \mathbb{R}^{k\times k}$, the LCP is to find a vector $u \in \mathbb{R}^k$ such that

$$u \geq 0 \tag{2.1}$$

$$y = q + Mu \geq 0 \tag{2.2}$$

$$u^T y = 0 \tag{2.3}$$

or to show that no such vector u exists.We denote the above LCP by the pair (q, M).

The conditions $u \geq 0, \quad y \geq 0, \quad u^T y = 0$ are called "complementarity conditions" (CC) and are denoted collectively as $0 \leq u \perp y \geq 0$. The pairs (u, y) are called "complementary variables".

Matrix classes play a strong role in the theory of the LCP. In fact, positive definite matrices allow us to give the following existence result for the LCP [36]:

Theorem 1. If $M \in \mathbb{R}^{k\times k}$ is positive definite, then the LCP(q, M) has an unique solution for all $q \in \mathbb{R}^k$.

In general, the LCP with a positive semi-definite matrix can have multiple solutions. For instance, the LCP with:

$$q = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \; M = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

has solutions

$$u^{(1)} = (1, 0), \quad u^{(2)} = (0, 1), \quad u^{(3)} = (\frac{1}{2}, \frac{1}{2})$$

In fact, there is a strong relation between the LCP and the class of matrices M such that the LCP(q, M) has an unique solution for all vectors q. For this purpose, we introduce the class of **P**-matrices:

Definition 1. A matrix $M \in \mathbb{R}^{k\times k}$ is said to be a **P**-matrix if all its principal minors are positive. The class of such matrices is denoted **P**.

Theorem 2. A matrix $M \in \mathbb{R}^{k \times k}$ is a **P**-matrix if and only if the LCP(q, M) has a unique solution for all vectors $q \in \mathbb{R}^k$.

Linear electronic networks, specially the power converters considered in this chapter, can be seen as the dynamical extensions of LCPs and fall into a class called complementarity systems. A complementarity system is a combination of a standard dynamical system and complementarity conditions. In a mechanical context such combinations of differential equations and complementarity conditions had already been used by Lötstedt [101]. Van der Schaft and Schumacher were among the first that formulated the equations of complementarity systems (or complementary slackness systems) in a general setting [146]. In the most general form complementarity systems are described by the differential and algebraic equations

$$F(\dot{x}(t), x(t)) = 0 \tag{2.4}$$

$$y(t) = g(x(t)) \in \mathbb{R}^k \tag{2.5}$$

$$u(t) = h(x(t)) \in \mathbb{R}^k \tag{2.6}$$

together with the complementarity conditions $0 \leq u(t) \perp y(t) \geq 0$. In this formulation t denotes the time variable, $x(t)$ the state variable and $u(t)$ and $y(t)$ the complementarity variables at time t.

A special complementarity system occurs when (2.4), (2.5) and (2.6) are replaced by an "input/state/output system" of the form

$$\dot{x}(t) = f(x(t), u(t)) \tag{2.7}$$

$$y(t) = g(x(t), u(t)) \tag{2.8}$$

These systems are called "semi-explicit" complementarity systems. Moreover, if the input/state/output system is taken to be linear we obtain a *linear complementarity system* (LCS):

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Ew(t), & (2.9)\\ y(t) &= Cx(t) + Du(t) + Fw(t), & (2.10) \end{aligned}$$

where A, B, C, D, E and F are (constant) matrices and vector of suitable dimensions.

If one allows for positiveness to be relaxed to various degrees for the corresponding pairs, then one gets a cone complementarity system (CCS). There are three possible cases of CCS depending on the degree of relaxation: $u \perp y \geq 0$, $0 \leq u \perp y$ and $u \perp y$. If the system is linear we have then a linear cone complementarity system (LCCS).

In order to establish existence and uniqueness of solutions for complementarity systems, specifically for electrical circuits, it is shown in [28] that passivity of the dynamical system plays an important role. Indeed, existence and uniqueness of solutions is proven for passive networks with ideal diodes and external inputs under some assumptions. In the following subsection we will describe the aforementioned results.

2.2.2 Passivity in Linear Systems

In this part we outline the implications of the notion of passivity in the context of LCS. As it is well-known from circuit theory, the matrix quadruple (A, B, C, D) is not arbitrary but has a certain structure. Indeed, the circuit should satisfy a passivity property when there is no external input present.

Definition 1 . [7]A linear system $\Sigma(A, B, C, D)$ given by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{2.11}$$

is called *passive*, or *dissipative* with respect to the supply rate $u^T y$, if there exists a nonnegative function $V : \mathbb{R}^n \to \mathbb{R}_+$ such that for all $t_0 \leq t_1$ and all trajectories (x, u, y) of the system the following inequality holds:

$$V(x(t_0)) + \int_{t_0}^{t_1} u^T(t)y(t)dt \geq V(x(t_1)) \tag{2.12}$$

If it exists, the function V is called a *storage function.*

The following proposition is one of the classical results of systems and control theory.

Proposition 1 . *[7] Consider a system $\Sigma(A, B, C, D)$ in which (A, B, C) is a minimal representation. The following statements are equivalent.*

- $\Sigma(A, B, C, D)$ *is passive.*
- *The transfer matrix $G(s) := D + C(sI - A)^{-1}B$ is positive real, i.e., $x^*[G(\lambda) + G^*(\lambda)]x \geq 0$ for all complex vector x and all $\lambda \in \mathbb{C}$ such that $Re(\lambda) > 0$ and λ is not an eigenvalue of A.*

- *The matrix inequalities*

$$\begin{pmatrix} A^T K + KA & KB - C^T \\ B^T K - C & -(D + D^T) \end{pmatrix} \geq 0 \tag{2.13}$$

and $K = K^T \geq 0$ *have a solution* K.

Moreover, in case $\Sigma(A, B, C, D)$ *is passive, all solutions* K *to the linear matrix inequalities (2.13) are positive definite and* K *is a solution to (2.13) if and only if* $V(x) = \frac{1}{2}x^T K x$ *defines a storage function of the system* $\Sigma(A, B, C, D)$.

2.3 Power converters as complementarity systems

We present next the models of several dc-dc converters in this formalism. The basic schemes for the boost, buck, buck-boost and Čuk converters appear in Figures 2.1, 2.2, 2.3 and 2.4, respectively.

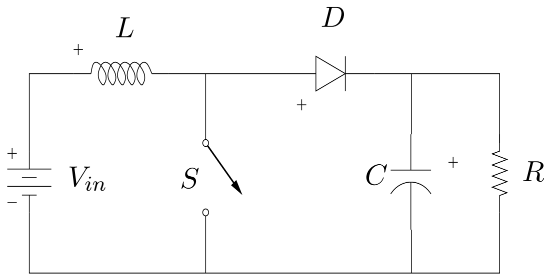

Figure 2.1: The boost converter

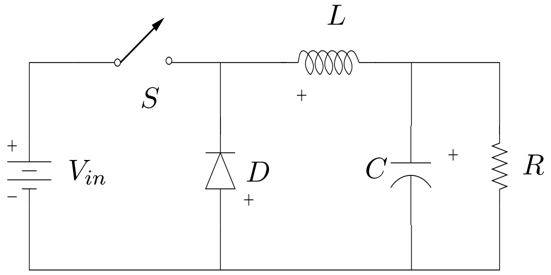

Figure 2.2: The buck converter

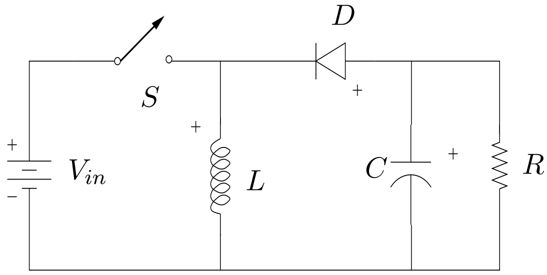

Figure 2.3: The buck-boost converter

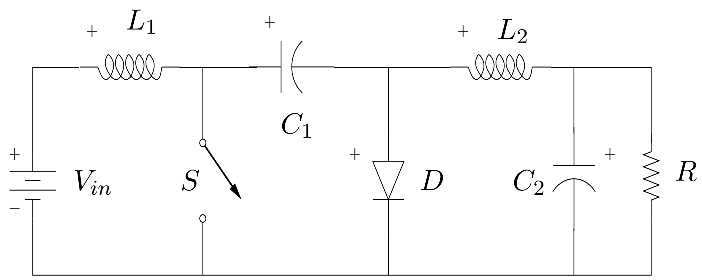

Figure 2.4: The Čuk converter

To model these systems we will introduce port variables (v_D, i_D) and (v_S, i_S) for the diode and the switch. We will consider two-quadrant switches, such that $i_S \geq 0$ but v_S is unrestricted. Hence the model for an arbitrary switch configuration will be a LCSS. Once the switch configuration is specified (S open *i.e.* $i_S = 0$ or S closed *i.e.* $v_S = 0$), the switch variables can be eliminated and one is left with an standard LCS with the diode variables.

2.3.1 Boost converter

We use as state variables the current in the inductance and the voltage on the capacitor, i.e $x_1 = i_L$ and $x_2 = v_C$. The state space equations are then

$$\dot{x}_1 = -\frac{1}{L}x_2 - \frac{1}{L}v_D + \frac{1}{L}V_{in} \tag{2.14}$$

$$\dot{x}_2 = \frac{1}{C}x_1 - \frac{1}{RC}x_2 - \frac{1}{C}i_S \tag{2.15}$$

We can formulate this as a CCS with

$$\begin{array}{lcl} u_1 = -v_D & \rightarrow & y_1 = i_D = x_1 - u_2 \\ u_2 = i_S & \rightarrow & y_2 = v_S = x_2 - u_1 \end{array}$$

In matrix notation

$$\begin{cases} \dot{x} = Ax + Bu + EV_{in} \\ y = Cx + Du + FV_{in} \\ 0 \leq y_1 \perp u_1 \geq 0 \\ y_2 \perp u_2 \geq 0 \end{cases} \tag{2.16}$$

where

$$A = \begin{pmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{pmatrix} \qquad B = \begin{pmatrix} \frac{1}{L} & 0 \\ 0 & -\frac{1}{C} \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \qquad E = \begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix} \qquad F = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

2.3.2 Buck converter

With the same variables as for the boost converter, we get

$$\dot{x}_1 = -\frac{1}{L}x_2 - \frac{1}{L}v_D \tag{2.17}$$

$$\dot{x}_2 = \frac{1}{C}x_1 - \frac{1}{RC}x_2 \tag{2.18}$$

$$\begin{array}{lcl} u_1 = -v_D & \rightarrow & y_1 = i_D = x_1 - u_2 \\ u_2 = i_S & \rightarrow & y_2 = v_S = V_{in} - u_1 \end{array}$$

Then

$$\begin{cases} \dot{x} = Ax + Bu + EV_{in} \\ y = Cx + Du + FV_{in} \\ 0 \leq y_1 \perp u_1 \geq 0 \\ y_2 \perp u_2 \geq 0 \end{cases} \tag{2.19}$$

with

$$A = \begin{pmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{pmatrix} \qquad B = \begin{pmatrix} \frac{1}{L} & 0 \\ 0 & 0 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \qquad E = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad F = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

2.3.3 Buck-boost converter

$$\dot{x}_1 = \frac{1}{L}V_{in} - \frac{1}{L}v_S \tag{2.20}$$

$$\dot{x}_2 = -\frac{1}{RC}x_2 - \frac{1}{C}i_D \tag{2.21}$$

$$\begin{array}{lcl} u_1 = v_S & \rightarrow & y_1 = i_S = x_1 - u_2 \\ u_2 = i_D & \rightarrow & y_2 = -v_D = -x_2 - u_1 + V_{in} \end{array}$$

$$\begin{cases} \dot{x} = Ax + Bu + EV_{in} \\ y = Cx + Du + FV_{in} \\ 0 \leq y_1 \perp u_1 \\ 0 \leq y_2 \perp u_2 \geq 0 \end{cases} \tag{2.22}$$

$$A = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{RC} \end{pmatrix} \qquad B = \begin{pmatrix} -\frac{1}{L} & 0 \\ 0 & -\frac{1}{C} \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \qquad E = \begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix} \qquad F = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

2.3.4 Čuk converter

We take as state variables $x_1 = i_{L1}$, $x_2 = i_{L2}$, $x_3 = v_{C1}$ and $x_4 = v_{C2}$. Then

$$\dot{x}_1 = \frac{1}{L_1}V_{in} - \frac{1}{L_1}x_3 - \frac{1}{L_1}v_D \tag{2.23}$$

$$\dot{x}_2 = \frac{1}{L_2}v_D - \frac{1}{L_2}x_4 \tag{2.24}$$

$$\dot{x}_3 = \frac{1}{C_1}x_1 - \frac{1}{C_1}i_S \tag{2.25}$$

$$\dot{x}_4 = \frac{1}{C_2}x_2 - \frac{1}{C_2R}x_4 \tag{2.26}$$

$$\begin{array}{lcl} u_1 = -v_D & \rightarrow & y_1 = i_D = x_1 - x_2 - u_2 \\ u_2 = i_S & \rightarrow & y_2 = v_S = x_3 - u_1 \end{array}$$

$$\begin{cases} \dot{x} = Ax + Bu + EV_{in} \\ y = Cx + Du + FV_{in} \\ 0 \leq y_1 \perp u_1 \geq 0 \\ y_2 \perp u_2 \geq 0 \end{cases} \tag{2.27}$$

$$A = \begin{pmatrix} 0 & 0 & -\frac{1}{L_1} & 0 \\ 0 & 0 & 0 & -\frac{1}{L_2} \\ \frac{1}{C_1} & 0 & 0 & 0 \\ 0 & \frac{1}{C_2} & 0 & -\frac{1}{RC_2} \end{pmatrix} \qquad B = \begin{pmatrix} \frac{1}{L_1} & 0 \\ -\frac{1}{L_2} & 0 \\ 0 & -\frac{1}{C_1} \\ 0 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad D = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \qquad E = \begin{pmatrix} \frac{1}{L_1} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$F = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

2.4 Generalized discontinuous conduction modes for systems with a single diode

Our approach to computing conditions for the presence of a GDCM follows ideas presented in [145]. It is based on assuming right-analyticity of solutions in state-space. This means that any quantity can be computed on $(t_0, t_0 + \epsilon)$, for some $\epsilon > 0$, if the quantity and its derivatives are known in t_0. To be more precise, let us consider a couple of complementary variables, u and y, and let $\mathcal{U} \equiv (u^{(0)}, u^{(1)}, u^{(2)}, \ldots)$, $\mathcal{Y} = (y^{(0)}, y^{(1)}, y^{(2)}, \ldots)$ denote the values of u and y and their successive right-time derivatives at $t = t_0$. To ensure $0 \leq u \perp y \geq 0$ on $[t_0, t_0 + \epsilon)$, one of the following must be true

$$\mathcal{U} \succeq 0 \quad \text{and} \quad \mathcal{Y} = 0, \tag{2.28}$$

or

$$\mathcal{Y} \succeq 0 \quad \text{and} \quad \mathcal{U} = 0, \tag{2.29}$$

where $\succeq$ means lexicographic nonnegativity [36], *i.e.* all the terms are zero or the first nonzero term is positive. If the terms of the sequence are the Taylor coefficients of an analytic function, lexicographic nonnegativity ensures nonnegativity of the function in an open interval. We will also consider finite sequences $\mathcal{U}_k \equiv (u^{(0)}, u^{(1)}, \ldots, u^{(k)})$, $\mathcal{Y}_k = (y^{(0)}, y^{(1)}, \ldots, y^{(k)})$. A pair of sequences, finite or not, satisfying either (2.28) or (2.29), will be called *valid*.

Consider a dynamical system of the form

$$\begin{aligned} \dot{x} &= f(x,z) + au, && (2.30)\\ \dot{z} &= g(x,z,u), && (2.31)\\ y &= \beta x + \alpha, && (2.32) \end{aligned}$$

with $x \in \mathbb{R}$, $z \in \mathbb{R}^{n-1}$, $a \neq 0$, $\beta \neq 0$, and where $u \in \mathbb{R}$ and $y \in \mathbb{R}$ are complementary variables, $0 \leq u \perp y \geq 0$. This is a relative degree $\rho = 1$ system, and from (2.30), (2.31) and (2.32) one can compute the equations linking the values of u, y and their successive time derivatives at $t = t_0$. Using the notation explained above, the first of these are

$$\begin{aligned} y^{(0)} &= \beta x_0 + \alpha \equiv \beta\gamma_0, & (2.33)\\ y^{(1)} &= \beta f(x_0, z_0) + a\beta u^{(0)} \equiv \beta\gamma_1 + a\beta u^{(0)}, & (2.34)\\ y^{(2)} &= \beta\partial_x f(x_0, z_0)(f(x_0, z_0) + au^{(0)}) \\ &+ \beta\partial_z f(x_0, z_0) g(x, z, u^{(0)}) + a\beta u^{(1)} \\ &\equiv \beta\gamma_2 + a\beta u^{(1)}, & (2.35) \end{aligned}$$

and, in general,

$$y^{(k)} = \beta\gamma_k + a\beta u^{(k-1)}, \tag{2.36}$$

where γ_k depends on x_0, z_0 and the time derivatives of u at $t = t_0$ up to order $k - 2$.

As explained in [145], a method for constructing smooth solutions starting from x_0, z_0 can be obtained by solving a series of what are called there dynamical complementarity problems (DCP). A DCP(k) consists in finding valid sequences $\mathcal{U}_k$ and $\mathcal{Y}_k$ satisfying relations (2.33), (2.34),... up to (2.36). In general, DCP(k) may have many solutions (for instance, $u^{(0)}$ does not appear in (2.33), and hence it is free, apart from being non-negative, for DCP(0)); however, since the conditions of DCP(k) are a subset of those of DCP($k+1$), the solutions of DCP($k+1$) must be chosen among those of DCP(k). This is called the *nesting property* of the DCP.

We say that the system is in mode J_k if *all* the solutions of DCP(k) satisfy $\mathcal{Y}_k \succ 0$, and that the system is in mode I_k if *all* the solutions of DCP(k) verify $\mathcal{U}_k \succ 0$; otherwise, the system is said to be in mode K_k[1]. Due to the nesting property, if the system is in mode J_k (I_k) it will be in mode J_l (resp. I_l) for any $l > k$. We will assume that

1. $y^{(0)} \geq 0$,
2. $a\beta > 0$.

These conditions ensure in our case existence and uniqueness of smooth solutions starting from x_0, z_0 (theorems 3.1 and 3.2 of [145]). Then the DCPs are as follows.

[1]We use the term *mode* in a sense different from that of [145]; ours is adapted to the fact that only a pair of complementary variables are present.

DCP(0): Find a valid pair $(u^{(0)})$, $(y^{(0)})$ such that (2.33) holds. Two situations are possible:

Case 1: $y^{(0)} > 0$. This forces $u^{(0)} = 0$. The system is in mode J_0.

Case 2: $y^{(0)} = 0$. In this case $u^{(0)} \geq 0$ is still free. The system is in mode K_0.

DCP(1): Find a valid pair $(u^{(0)}, u^{(1)})$, $(y^{(0)}, y^{(1)})$ such that (2.33) and (2.34) hold. Since the conditions of DCP(0) are a subset of these, we start with the solutions obtained there.

Case 1: Since $y^{(0)} > 0$ we must have $u^{(1)} = 0$ for the pair to be valid. One also gets $y^{(1)} = \beta\gamma_1$. Notice that the sign of $y^{(1)}$ does not matter since already $y^{(0)} > 0$. The system is in mode J_1.

Case 2: In this case (2.34) becomes a 1-dimensional LCP for $y^{(1)}$, $u^{(0)}$, which always has solution due to $a\beta > 0$. Three subcases are possible.

Case 2.1: $\beta\gamma_1 > 0$. The only solution to the LCP is $u^{(0)} = 0$, $y^{(1)} = \beta\gamma_1 > 0$; one must choose $u^{(1)} = 0$ and the system is in mode J_1.

Case 2.2: $\gamma_1 = 0$. Now we have $u^{(0)} = 0$, $y^{(1)} = 0$; any $u^{(1)} \geq 0$ is valid and the system is in mode K_1.

Case 2.3: $\beta\gamma_1 < 0$. The solution is $u^{(0)} = -\frac{\gamma_1}{a} = -\frac{\beta\gamma_1}{a\beta} > 0$, $y^{(1)} = 0$; any $u^{(1)} \in \mathbb{R}$ is valid and the system is in mode I_1.

DCP(2): Find a valid pair $(u^{(0)}, u^{(1)}, u^{(2)})$, $(y^{(0)}, y^{(1)}, y^{(2)})$ such that (2.33), (2.34) and (2.35) hold. As is the transition to DCP(0) to DCP(1), solutions coming from modes I_1 or J_1 will yield solutions in I_2 and J_2, respectively, so the only case worth studying is 2.2, for which now (2.35) is an LCP for $y^{(2)}$, $u^{(1)}$. Again this always has solution due to $a\beta > 0$, and three situations can be encountered:

Case 2.2.1: $\beta\gamma_2 > 0$. The solution is $u^{(1)} = 0$, $y^{(2)} = \beta\gamma_2 > 0$; one must choose $u^{(2)} = 0$ and the system is in mode J_2.

Case 2.2.2: $\gamma_2 = 0$. Now $u^{(1)} = 0$, $y^{(2)} = 0$; any $u^{(2)} \geq 0$ is valid and the system is in mode K_2.

Case 2.2.3: $\beta\gamma_2 < 0$. The solution is $u^{(1)} = -\frac{\gamma_2}{a} = -\frac{\beta\gamma_2}{a\beta} > 0$, $y^{(2)} = 0$; any $u^{(2)} \in \mathbb{R}$ is valid and the system is in mode I_2.

Successive DCPs can be solved, and we assume that after a finite number of steps the system ends up in a J or I mode. The general case can be summarized as follows:

Proposition 1. Under the conditions of the preceding discussion,

- if the above procedure enters a J mode for the first time when solving DCP(k), then $u^{(l)} = 0$ for all $l \in \mathbb{N}$, $y^{(l)} = 0$ for $l = 0, \ldots, k-1$, and $y^{(k)} = \beta\gamma_k > 0$.
- if the above procedure enters a I mode for the first time when solving DCP(k), then $y^{(l)} = 0$ for all $l \in \mathbb{N}$, $u^{(l)} = 0$ for $l = 0, \ldots, k-2$, and $u^{(k-1)} = -\frac{\gamma_k}{a} > 0$.

Proposition 2. Assume that $y(t) > 0$, and hence $u(t) = 0$, for $t \in (t_0 - \tilde{\epsilon}, t_0)$, for some $\tilde{\epsilon} > 0$. Then, if for some $k \geq 1$,

$$\gamma_0 = 0, \text{ and } \gamma_l = 0, \forall l = 1, \ldots, k-1, \text{ but } \beta\gamma_k < 0 \tag{2.37}$$

one has that

- the $(k-1)$th time derivative of u has a jump at t_0, going from 0 to $-\gamma_k/a > 0$.
- there exists $\epsilon > 0$ such that $y(t) = 0$, $u(t) > 0$ for $t \in (t_0, t_0 + \epsilon)$, and hence we have a GDCM.

We call (2.37) the kth-order GDCM condition (GDCMC).

Proof. Obvious from Proposition 1 and the discussion leading to it, and the fact that all the derivatives of u are zero for $t \in (t_0 - \tilde{\epsilon}, t_0)$.

As a corollary, we have

Proposition 3. Assume $y(t) > 0$ for $t \in (t_0 - \tilde{\epsilon}, t_0)$, for some $\tilde{\epsilon} > 0$, and that $y(t_0) = 0$ and $\beta\gamma_1 < 0$. Then u has a discontinuity at $t = t_0$, from 0 to $-\gamma_1/a > 0$, and there exists $\epsilon > 0$ such that $y(t) = 0$ and $u(t) > 0$ for $t \in (t_0, t_0 + \epsilon)$.

The situation presented in Proposition 3 is the one normally encountered both in simulation and in experiment; higher order GDCM conditions are difficult to meet, since they require several state space quantities to be zero simultaneously.

2.5 Application to the power converters with single diode.

In this part we will obtain the first order GDCMC for the two positions of the switch of the power converters presented in the previous Section.

2.5.1 Boost converter

The equations are

$$
\begin{aligned}
L\dot{x}_1 &= -x_2 - v_D + V_{in},\\
C\dot{x}_2 &= x_1 - \frac{1}{R}x_2 - i_S,\\
i_D &= x_1 - i_S,\\
v_S &= v_D + x_2.
\end{aligned}
$$

$\boxed{S \text{ closed } (v_S = 0)}$

$$
\begin{aligned}
L\dot{x}_1 &= V_{in},\\
C\dot{x}_2 &= i_D - \frac{1}{R}x_2,\\
x_2 &= -v_D.
\end{aligned}
$$

We apply Proposition 3 with $x = x_2$, $z = x_1$, $u_D = i_D$, $w_D = -v_D$, $f(x,z) = -x_2/(RC)$, $\alpha = 0$, $\beta = 1$, $a = 1$. The DCMC is given by

$$
x_2(t_0) = 0, \quad \gamma = -\frac{1}{RC}x_2(t_0) < 0
$$

which is clearly impossible.

$\boxed{S \text{ open } (i_S = 0)}$

$$
\begin{aligned}
L\dot{x}_1 &= -x_2 - v_D + V_{in},\\
C\dot{x}_2 &= x_1 - \frac{1}{R}x_2,\\
x_1 &= i_D.
\end{aligned}
$$

We apply Proposition 3 with $x = x_1$, $z = x_2$, $u_D = -v_D$, $w_D = i_D$, $f(x,z) = \frac{1}{L}(V_{in} - x_2)$, $\alpha = 0$, $\beta = 1$, $a = 1/L$. The DCMC is given by

$$
x_1(t_0) = 0, \quad \gamma = \frac{1}{L}(V_{in} - x_2(t_0)) < 0,
$$

This implies $x_2(t_0) > V_{in}$, which is the normal form of operation for the boost converter. Notice that in this case f depends on z. However, since $x_1(t) = 0$ for $t \in (t_0, t_0 + \nu)$, we have that $x_2(t)$ remains above V_{in}, so $u_D(t) = x_2(t) - V_{in} > 0$ for all $t \in (t_0, t_0 + \nu)$. Hence, at $t = t_0 + \nu$ we still have $x = 0$, $w_D = 0$ and $u_D > 0$ and nothing changes thereafter.

2.5.2 Buck converter

We have

$$\begin{aligned} L\dot{x}_1 &= -x_2 - v_D, \\ C\dot{x}_2 &= x_1 - \frac{1}{R}x_2, \\ i_D &= x_1 - i_S, \\ v_S &= V_{in} + v_D. \end{aligned}$$

$\boxed{S \text{ closed } (v_S = 0)}$

$$\begin{aligned} L\dot{x}_1 &= -x_2 + V_{in}, \\ C\dot{x}_2 &= x_1 - \frac{1}{R}x_2, \\ x_1 &= i_D + i_S. \end{aligned}$$

The system evolves without any jump because the vector field in this configuration does not depend on the diode variables. No DCM exists.

$\boxed{S \text{ open } (i_S = 0)}$

$$\begin{aligned} L\dot{x}_1 &= -x_2 - v_D, \\ C\dot{x}_2 &= x_1 - \frac{1}{R}x_2, \\ x_1 &= i_D. \end{aligned}$$

We apply Proposition 3 with $x = x_1$, $z = x_2$, $u_D = -v_D$, $w_D = i_D$, $f(x, z) = -\frac{1}{L}x_2$, $\alpha = 0$, $\beta = 1$, $a = 1/L$. The DCMC is given by

$$x_1(t_0) = 0, \quad \gamma = -\frac{1}{L}x_2(t_0)) < 0,$$

This implies $x_2(t_0) > 0$, which is the normal form of operation for the buck converter. The same considerations exposed for the boost converter apply here.

2.5.3 Buck-boost converter

We have

$$\begin{aligned} L\dot{x}_1 &= V_{in} - v_S, \\ C\dot{x}_2 &= -\frac{1}{R}x_2 - i_D, \\ i_S &= x_1 - i_D, \\ v_D &= x_2 + v_S - V_{in}. \end{aligned}$$

$\boxed{S \text{ closed } (v_S = 0)}$

$$\begin{aligned} L\dot{x}_1 &= V_{in}, \\ C\dot{x}_2 &= -\frac{1}{R}x_2 - i_D, \\ x_2 &= v_D + V_{in}. \end{aligned}$$

We apply Proposition 3 with $x = x_2$, $z = x_1$, $u_D = i_D$, $w_D = -v_D$, $f(x,z) = -\frac{1}{RC}x_2$, $\alpha = V_{in}$, $\beta = -1$, $a = -1/C$. The DCMC is given by

$$x_2(t_0) = V_{in}, \quad \gamma = -\frac{1}{RC}x_2(t_0)) > 0,$$

which is impossible. No DCM exists.

$\boxed{S \text{ open } (i_S = 0)}$

$$\begin{aligned} L\dot{x}_1 &= x_2 - v_D, \\ C\dot{x}_2 &= -x_1 - \frac{1}{R}x_2, \\ x_1 &= i_D. \end{aligned}$$

We apply Proposition 3 with $x = x_1$, $z = x_2$, $u_D = -v_D$, $w_D = i_D$, $f(x,z) = \frac{1}{L}x_2$, $\alpha = 0$, $\beta = 1$, $a = 1/L$. The DCMC is given by

$$x_1(t_0) = 0, \quad \gamma = \frac{1}{L}x_2(t_0)) < 0,$$

This implies $x_2(t_0) < 0$.

2.5.4 Čuk converter

We have

$$\begin{aligned} L_1\dot{x}_1 &= V_{in} - x_3 - v_D, \\ L_2\dot{x}_2 &= -x_4 + v_D, \\ C_1\dot{x}_3 &= x_1 - i_S, \\ C_2\dot{x}_4 &= x_2 - \frac{1}{R}x_4, \\ i_D &= x_1 - x_2 - i_S, \\ v_S &= x_3 + v_D. \end{aligned}$$

$\boxed{S \text{ closed } (v_S = 0)}$

$$
\begin{aligned}
L_1\dot{x}_1 &= V_{in}, \\
L_2\dot{x}_2 &= -x_3 - x_4, \\
C_1\dot{x}_3 &= x_2 + i_D, \\
C_2\dot{x}_4 &= x_2 - \frac{1}{R}x_4, \\
x_3 &= -v_D.
\end{aligned}
$$

We apply Proposition 3 with $x = x_3$, $z = (x_1, x_2, x_4)$, $u_D = i_D$, $w_D = -v_D$, $f(x,z) = \frac{1}{C_1}x_2$, $\alpha = 0$, $\beta = 1$, $a = 1/C_1$. The DCMC is given by

$$
x_3(t_0) = 0, \quad \gamma = \frac{1}{C_1}x_2(t_0) < 0.
$$

This implies $x_2(t_0) < 0$.

$\boxed{S \text{ open } (i_S = 0)}$

$$
\begin{aligned}
L_1\dot{x}_1 &= -x_3 - v_D + V_{in}, \\
L_2\dot{x}_2 &= -x_4 + v_D, \\
C_1\dot{x}_3 &= x_1, \\
C_2\dot{x}_4 &= x_2 - \frac{1}{R}x_4, \\
i_D &= x_1 - x_2.
\end{aligned}
$$

Notice that, if $x = x_1 - x_2$, then

$$
\dot{x} = \frac{V_{in}}{L_1} - \frac{1}{L_1}x_3 + \frac{1}{L_2}x_4 - \left(\frac{1}{L_1} + \frac{1}{L_2}\right)v_D.
$$

Thus, we can apply Proposition 3 with $x = x_1 - x_2$, $z = (x_1 + x_2, x_3, x_4)$, $u_D = -v_D$, $w_D = i_D$, $f(x,z) = V_{in}/L_1 - x_3/L_1 + x_4/L_2$, $\alpha = 0$, $\beta = 1$, $a = 1/L_1 + 1/L_2$.

The DCMC is given by

$$
x_1(t_0) = x_2(t_0), \quad \gamma = \frac{V_{in}}{L_1} - \frac{x_3(t_0)}{L_1} + \frac{x_4(t_0)}{L_2} < 0,
$$

2.5.5 Simulations

As shown in [27], a backwards Euler integration method yields a stable algorithm when complementarity conditions are present. For a LCS of the following form

$$\begin{aligned} \dot{x} &= Ax + Bu + E, \\ y &= Cx + Du + F, \end{aligned}$$

the backwards Euler scheme is

$$\begin{aligned} x_{k+1} &= x_k + h(Ax_{k+1} + Bu_{k+1} + E) \\ y_{k+1} &= Cx_{k+1} + Du_{k+1} + F, \end{aligned}$$

where h is the time step size.

At each step, one must solve the LCP

$$\begin{aligned} y_{k+1} &= (hC(\,I - hA)^{-1}B + D)u_{k+1} \\ &+ \; C(\,I - hA)^{-1}x_k \\ &+ \; hC(\,I - hA)^{-1}E + F \qquad (2.38) \\ 0 \le y_{k+1} &\perp \; u_{k+1} \ge 0, \qquad (2.39) \end{aligned}$$

for given x_k, and then compute

$$x_{k+1} = (\,I - hA)^{-1}\,(x_k + h(Bu_{k+1} + E))$$

from the obtained value of u_{k+1}. In general, an LCP must be solved using specialized algorithms (such as Lemke's [19]). However, for a single pair of complementary variables, an explicit computation can be done. First of all, (2.38) can be written as $y_{k+1} = Mu_{k+1} + q_k$. Assume $M > 0$ (which is the case for any of the switch configurations of our converters). Then the LCP is solved as follows:

- if $q_k > 0$, then $u_{k+1} = 0$ and $y_{k+1} = q_k$,
- if $q_k < 0$, then $u_{k+1} = -q_k/M$ and $y_{k+1} = 0$,
- if $q_k = 0$, then $u_{k+1} = 0$ and $y_{k+1} = 0$.

We have implemented this algorithm in `Matlab` to check the GDCMC for the boost and the Čuk converter. We have used the system parameter values (in SI units) $L = 750 \cdot 10^{-6}$, $C = 220 \cdot 10^{-6}$, $R = 10$ and $V_{in} = 24$ for the boost converter and $L_1 = 750 \cdot 10^{-6}$, $L_2 = 800 \cdot 10^{-6}$, $C_1 = 220 \cdot 10^{-6}$, $C_2 = 130 \cdot 10^{-6}$, $R = 10$ and $V_{in} = 24$ for the Čuk converter. The initial conditions are $x(0) = (2, 1)$ for the boost converter and $x(0) = (2, 1, 1, 1)$ for both simulations of the Čuk, while the fixed integration step has been chosen as $h = 10^{-6}$.

Simulation results are displayed in figures 2.5, 2.6 and 2.7. For the boost with open switch, a GDCM is reached and left in finite time, and then the system decays to an equilibrium point. For the Čuk converter with switch closed, the GDCM is entered multiple times, while for open switch the system does not leave the GDCM, although a re-entrant behaviour can be obtained if the other initial conditions are used. All this corroborates the theoretical predictions.

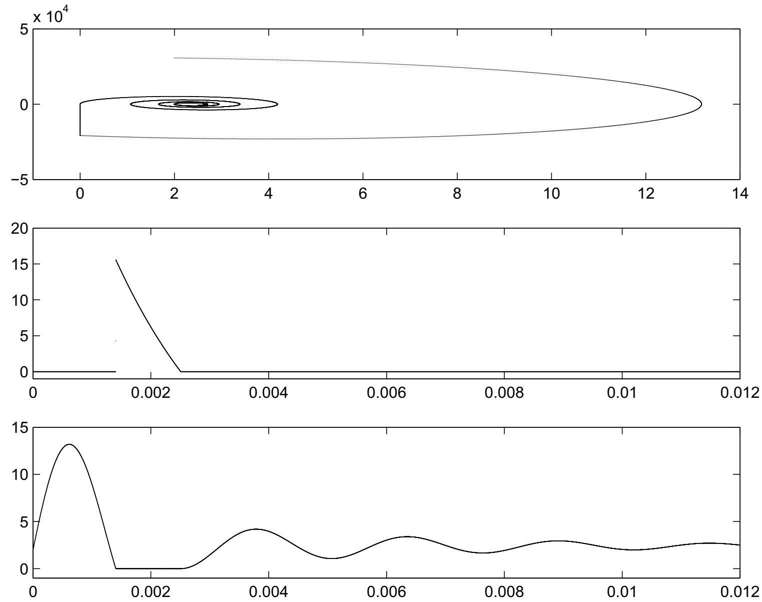

Figure 2.5: GDCM for the boost converter with switch open. Upper: x_1 on the horizontal axis and Γ on the vertical one. Middle: u as a function of time. Lower: y as a function of time. The GDCM has a finite duration and it is not re-entrant for the parameters and initial conditions chosen.

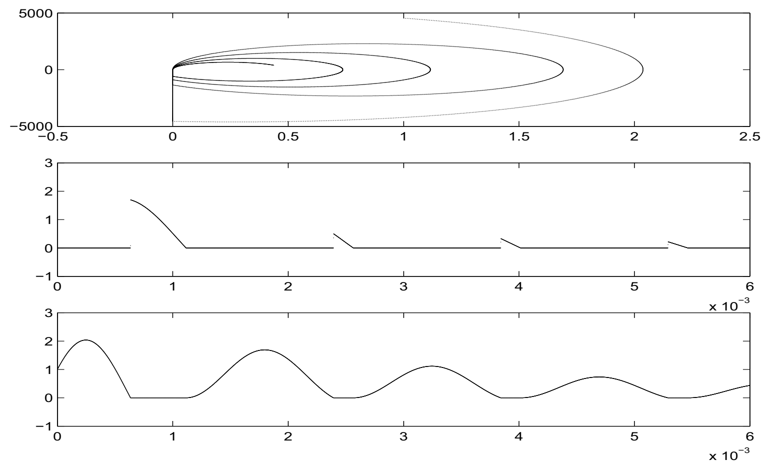

Figure 2.6: GDCM for the Cuk converter with switch closed. Upper: x_3 on the horizontal axis and Γ on the vertical one. Middle: u as a function of time. Lower: y as a function of time. The GDCM has a finite duration, but it is re-entrant for the parameters and initial conditions chosen.

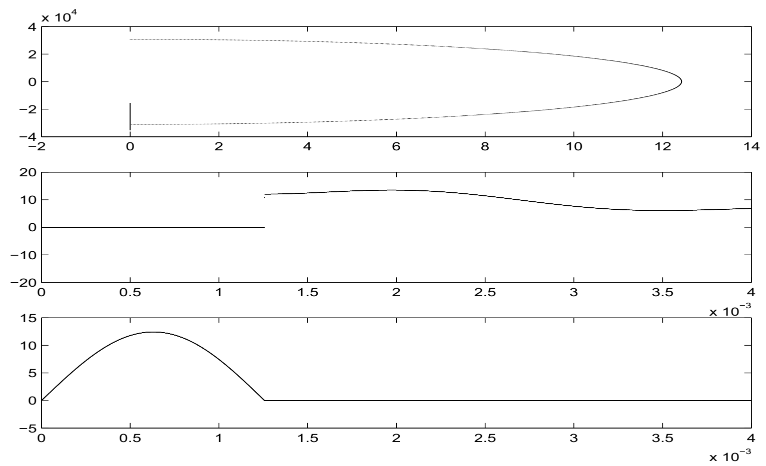

Figure 2.7: GDCM for the Cuk converter with switch open. Upper: $x_1 - x_2$ on the horizontal axis and Γ on the vertical one. Middle: u as a function of time. Lower: y as a function of time. For the parameters and initial conditions used, the GDCM lasts indefinitely.

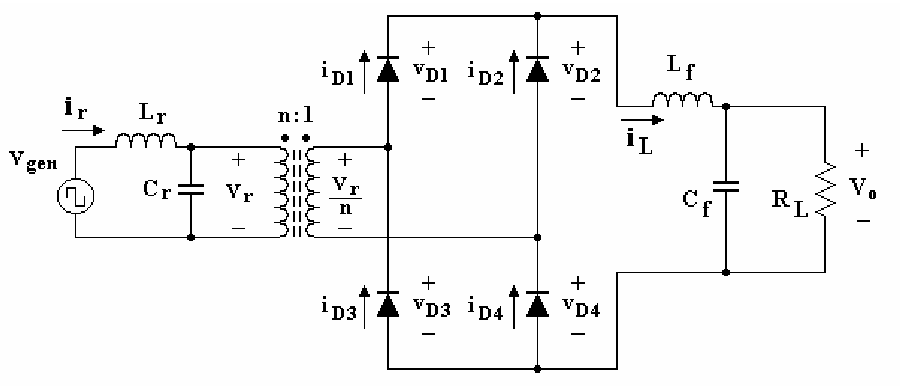

Figure 2.8: A Power Resonant Convert diagram

2.6 Parallel Resonant Converter as LCS

According to the general form of an LCS given by eq. 2.9 and 2.10, we can model the parallel resonant converter (PRC) presented in Figure 2.8 as an LCS. The complementarity description is obtained as follows:

We take as state variables $x_1 = i_r, x_2 = v_r, x_3 = i_L$ and $x_4 = v_0$, and $u_1 = i_{D1}, u_2 = i_{D3}, u_3 = v_{D2}, u_4 = v_{D4}, y_1 = v_{D1}, y_2 = v_{D3}, y_3 = i_{D2}$ and $y_4 = i_{D4}$ as complementarity variables. Then, in matrix notation,

$$\left\{ \begin{array}{c} \dot{x}(t) = Ax(t) + Bu(t) + ESign(\sin(wt)), \\ y(t) = Cx(t) + Du(t) + FSign(\sin(wt)), \\ 0 \leq y \perp u \geq 0 \end{array} \right. \tag{2.40}$$

with

$$A = \begin{pmatrix} 0 & -\frac{1}{L_r} & 0 & 0 \\ \frac{1}{C_r} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{L_f} \\ 0 & 0 & \frac{1}{C_f} & -\frac{1}{R_L C_f} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\frac{1}{nC_r} & \frac{1}{nC_r} & 0 & 0 \\ 0 & 0 & \frac{1}{L_f} & \frac{1}{L_f} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & -\frac{1}{n} & 0 & 0 \\ 0 & \frac{1}{n} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, D = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$E = \begin{pmatrix} \frac{1}{L_r} \\ 0 \\ 0 \\ 0 \end{pmatrix}, F = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Following the definition of passivity in linear systems given in section 2.2.2 it can be proved that the PRC that we are considering is passive. Indeed, an example of storage function for that system is $V(x) = \frac{1}{2}x^T Kx$, where K is given by:

$$K = \begin{pmatrix} L_r & 0 & 0 & 0 \\ 0 & C_r & 0 & 0 \\ 0 & 0 & L_f & 0 \\ 0 & 0 & 0 & C_f \end{pmatrix}$$

At this point, in order to assure the existence and uniqueness of solutions the following assumption is used in [28]:

Assumption 1 . *The system $\Sigma(A, B, C, D)$ is passive with the storage function $x \to \frac{1}{2}x^T Kx$ where K is definite positive and $col(B, D + D^T)$ has full column rank.*

However, this assumption is not satisfied by our example because $col(B, D+D^T)$ has not full column rank. Nevertheless in the following subsections we will prove the existence and uniqueness of solutions for this particular example.

2.6.1 Initial and Local Well-Posedness

In this section, we are interested in existence and uniqueness of initial solutions for the PRC. Before analysing our example, we recall the so-called rational complementarity problem.

Problem RCP: Let A, B, C, D, E, F be matrices of appropriate dimensions. Define rational matrix functions $T(s)$ and $G(s)$ by $T(s) = C(sI - A)^{-1}x_0 + [F + C(sI - A)^{-1}E]w(s)$ and $G(s) = C(sI - A)^{-1}B + D$.

For given x_0, find strictly proper rational functions $y(s)$ and $u(s)$ such that the equality

$$y(s) = T(s)x_0 + G(s)u(s) \tag{2.41}$$

holds, and there exists and $s_0 \in \mathbb{R}$ such that for all $s \geq s_0$ we have

$$y(s) \geq 0, \quad u(s) \geq 0, \quad u(s)^T y(s) = 0. \tag{2.42}$$

The solution of the RCP gives a solution for the LCS. In our example, the matrix functions are given by:

$$G(s) = \begin{pmatrix} \frac{L_r s}{(s^2C_rL_r+1)} & -\frac{L_r s}{(s^2C_rL_r+1)} & 1 & 0 \\ -\frac{L_r s}{(s^2C_rL_r+1)} & \frac{L_r s}{(s^2C_rL_r+1)} & 0 & 1 \\ -1 & 0 & \frac{(sR_lC_f+1)}{(s^2L_fR_lC_f+sL_f+R_l)} & \frac{(sR_lC_f+1)}{(s^2L_fR_lC_f+sL_f+R_l)} \\ 0 & -1 & \frac{(sR_lC_f+1)}{(s^2L_fR_lC_f+sL_f+R_l)} & \frac{(sR_lC_f+1)}{(s^2L_fR_lC_f+sL_f+R_l)} \end{pmatrix}$$

$$T(s) = \begin{pmatrix} -\frac{L_r(V_{gen} + sx_1 + C_rs^2x_2)}{(s^2C_rL_r+1)} \\ \frac{L_r(Vgen + sx_1 + C_rs^2x_2)}{(s^2C_rL_r+1)} \\ \frac{L_fsx_3(sR_lC_f+1) + R_lC_fsx_4}{(s^2L_fR_lC_f + sL_f + R_l)} \\ \frac{L_fsx_3(sR_lC_f+1) + R_lC_fsx_4}{(s^2L_fR_lC_f + sL_f + R_l)} \end{pmatrix}$$

It is easy to see that the G(s) matrix is not P-matrix. Then, uniqueness is not assured. Moreover, the RCP can be solved by considering the following assumption:

$$G(s) = \begin{pmatrix} a(s) & -a(s) & 1 & 0 \\ -a(s) & a(s) & 0 & 1 \\ -1 & 0 & b(s) & b(s) \\ 0 & -1 & b(s) & b(s) \end{pmatrix}, \quad T(s) = \begin{pmatrix} -c(s) \\ c(s) \\ d(s) \\ d(s) \end{pmatrix}$$

with $a(s), b(s) \geq 0, \forall s \geq s_0$. The functions $b(s), c(s)$ can take any value. Then, with this assumption we have analysed case by case all the possible solutions.

- **Case 1:** $u = (0,0,0,0)$. This case has solution if and only if $c(s) = 0$ and $d(s) \geq 0$. In this case the solution is $u = (0,0,0,0)$, $y = (0,0,d(s),d(s))$.

- **Case 2:** $y_1 = u_2 = u_3 = u_4 = 0$. This case has solution if and only if $c(s), d(s) \geq 0$ and $d(s) - \frac{c(s)}{a(s)} \geq 0$. The solution is given by $u = (\frac{c(s)}{a(s)}, 0, 0, 0)$,$y = (0, 0, d(s) - \frac{c(s)}{a(s)}, d(s))$.

- **Case 3:** $u_1 = y_2 = u_3 = u_4 = 0$. This case has solution if and only if $c(s) \leq 0$, $d(s) \geq 0$ and $d(s) + \frac{c(s)}{a(s)} \geq 0$. The solution is given by $u = (0, \frac{-c(s)}{a(s)}, 0, 0)$,$y = (0, 0, d(s), d(s) + \frac{c(s)}{a(s)})$.

- **Case 4.** $u_1 = u_2 = y_3 = u_4 = 0$. This case has solution if and only if $c(s) \geq 0$, $d(s) \leq 0$ and $c(s) + \frac{d(s)}{b(s)} \leq 0$. In this case the solution is $u = (0, 0, \frac{-d(s)}{b(s)}, 0)$,$y = (-(c(s) + \frac{d(s)}{b(s)}), -c(s), 0, 0)$.

- **Case 5.** $u_1 = u_2 = u_3 = y_4 = 0$. This case has solution if and only if $c(s), d(s) \leq 0$ and $c(s) - \frac{d(s)}{b(s)} \geq 0$. In this case the solution is $u = (0, 0, 0, -\frac{d(s)}{b(s)})$, $y = (-c(s), c(s) - \frac{d(s)}{b(s)}, 0, 0)$.

- **Case 6.** $y_1 = y_2 = u_3 = u_4 = 0$. This case has multiple solutions. If $c(s), d(s) \geq 0$ and $d(s) - \frac{c(s)}{a(s)} \geq 0$ then $u = (l, l - \frac{c(s)}{a(s)}, 0, 0)$ and $y = (0, 0, d - l, d + \frac{c(s)}{a(s)} - l)$ with $\frac{c(s)}{a(s)} \leq l \leq d$. If $c(s) \leq 0$, $d(s) \geq 0$ and $d(s) + \frac{c(s)}{a(s)} \geq 0$ then $u = (l, l - \frac{c(s)}{a(s)}, 0, 0)$ and $y = (0, 0, d - l, d + \frac{c(s)}{a(s)} - l)$ with $0 \leq l \leq d + \frac{c(s)}{a(s)}$.

- **Case 7.** $y_1 = u_2 = y_3 = u_4 = 0$. It has solution if and only if $c(s) \geq 0$ and moreover one of this conditions is hold:

    1. $d(s) \geq 0$ and $d(s) - \frac{c(s)}{a(s)} \leq 0$.
    2. $d(s) \leq 0$ and $c(s) + \frac{d(s)}{b(s)} \geq 0$.

  Then, the solution is given by $u = (\frac{b(s)c(s)+d(s)}{1+a(s)b(s)}, 0, \frac{c(s)-a(s)d(s)}{1+a(s)b(s)}, 0)$, and $y = (0, \frac{c(s)-a(s)d(s)}{1+a(s)b(s)}, 0, \frac{b(s)c(s)+d(s)}{1+a(s)b(s)})$.

- **Case 8.** $y_1 = u_2 = u_3 = y_4 = 0$. This case has solution if and only if $c(s) = 0$ and $d(s) \leq 0$. In this case the solution is $u = (0, 0, 0, \frac{-d(s)}{b(s)})$, $y = (0, \frac{-d(s)}{b(s)}, 0, 0)$.

- **Case 9.** $u_1 = y_2 = y_3 = u_4 = 0$. This case has solution if and only if $c(s) = 0$ and $d(s) \leq 0$. In this case the solution is $u = (0, 0, \frac{-d(s)}{b(s)}, 0)$, $y = (\frac{-d(s)}{b(s)}, 0, 0, 0)$.

- **Case 10.** $u_1 = y_2 = u_3 = y_4 = 0$. It has solution if and only if $c(s) \leq 0$ and moreover one of this conditions is hold:

  1. $d(s) \geq 0$ and $d(s) + \frac{c(s)}{a(s)} \leq 0$.
  2. $d(s) \leq 0$ and $c(s) - \frac{d(s)}{b(s)} \leq 0$.

  Then, the solution is given by $u = (0, \frac{d(s)-b(s)c(s)}{1+a(s)b(s)}, 0, \frac{-(c(s)+a(s)d(s))}{1+a(s)b(s)})$, and $y = (\frac{-(c(s)+a(s)d(s))}{1+a(s)b(s)}, 0, \frac{d(s)-b(s)c(s)}{1+a(s)b(s)}, 0)$.

- **Case 11.** $u_1 = u_2 = y_3 = y_4 = 0$. This case has multiple solutions. If $c(s) \geq 0$, $d(s) \leq 0$ and $c(s) + \frac{d(s)}{b(s)} \leq 0$ then $u = (0, 0, l, -(\frac{d(s)}{b(s)} + l))$ and $y = (l - c(s), (c(s) - \frac{d(s)}{b(s)}) - l), 0, 0$ with $c(s) \leq l \leq -\frac{d(s)}{b(s)}$. If $c(s), d(s) \leq 0$ and $c(s) - \frac{d(s)}{b(s)} \geq 0$ then $u = (0, 0, l, -(\frac{d(s)}{b(s)} + l))$ and $y = (l - c(s), (c(s) - \frac{d(s)}{b(s)}) - l), 0, 0$ with $0 \leq l \leq c(s) - \frac{d(s)}{b(s)}$.

- **Case 12.** $u_1 = y_2 = y_3 = y_4 = 0$. This case has solution if and only if $c(s), d(s) \leq 0$ and $c(s) - \frac{d(s)}{b(s)} \geq 0$. The solution is given by $u = (0, 0, c(s) - \frac{d(s)}{b(s)}, -c(s))$,$y = (-\frac{d(s)}{b(s)}, 0, 0, 0)$.

- **Case 13.** $y_1 = u_2 = y_3 = y_4 = 0$. This case has solution if and only if $c(s) \geq 0$, $d(s) \leq 0$ and $c(s) + \frac{d(s)}{b(s)} \leq 0$. In this case the solution is $u = (0, 0, c(s), -(c(s) + \frac{-d(s)}{b(s)}))$,$y = (0, -\frac{d(s)}{b(s)}), 0, 0)$.

- **Case 14.** $y_1 = y_2 = u_3 = y_4 = 0$. This case has solution if and only if $c(s) \leq 0$, $d(s) \geq 0$ and $d(s) + \frac{c(s)}{a(s)} \geq 0$. The solution is given by $u = (d(s) + \frac{c(s)}{a(s)}, d(s), 0, 0)$,$y = (0, 0, -\frac{c(s)}{a(s)}, 0)$.

- **Case 15.** $y_1 = y_2 = y_3 = u_4 = 0$. This case has solution if and only if $c(s), d(s) \geq 0$ and $d(s) - \frac{c(s)}{a(s)} \geq 0$. The solution is given by $u = (d(s), d(s) - \frac{c(s)}{a(s)}, 0, 0)$,$y = (0, 0, 0, \frac{c(s)}{a(s)})$.

- **Case 16.** $y_1 = y_2 = y_3 = y_4 = 0$. This case has solution if and only if $c(s) = 0$ and $d(s) \geq 0$. In this case the solution is $u = (d(s), d(s), 0, 0)$, $y = (0, 0, 0, 0)$.

In the following cases:

- If $c(s), d(s) \geq 0$ and $d(s) - \frac{c(s)}{a(s)} \geq 0$,

- If $c(s) \leq 0$, $d(s) \geq 0$ and $d(s) + \frac{c(s)}{a(s)} \geq 0$,
- If $c(s) \geq 0$, $d(s) \leq 0$ and $c(s) + \frac{d(s)}{b(s)} \leq 0$,
- If $c(s), d(s) \leq 0$ and $c(s) - \frac{d(s)}{b(s)} \geq 0$.

there are multiple solutions. However, the solutions of the LCP in such regions give the same solution for the state space. This fact is due to the fact that the diodes are in parallel. This result shows that an extended theory is needed in order to cover such cases.

2.7 Simulation of a boost converter as LCS with SMC

In this section we will simulate a boost converter as a linear complementarity system with a sliding mode control (SMC). Although there is no a general control theory for complementarity systems we have achieved to control this power converter doing some manipulations with the variables and using a SMC (see appendix B for a detailed description of this control technique).

Let us consider a 2-dimensional, non linear (actually bi-linear) dc-dc converter used for stepping-up voltages. Step-up or boost converters are used in battery powered devices, where the electronic circuit requires a higher operating voltage than the battery can supply, e.g. mobile phones, notebooks, camera flashes, ...The diagram of this converter have been shown in Figure 2.1.

Power converters are variable structure systems because of the abrupt topological changes that the circuit, commanded by a discontinuous control action, undergoes. They constitute a natural field of application of nonlinear control techniques. Several authors have applied sliding control techniques to the regulation problem in basic dc-to-dc power converters. Power converters contain switches and diodes; then, they can also be modeled in the frame of complementary systems. Therefore, we will joint both things in order to simulate a boost converter.

2.7.1 Dynamical equation

For modeling this system port variables (v_D, i_D) and (v_S, i_S) for the diode and the switch will be considered. Constrains for the diode are as usual $0 \leq -v_D \perp i_S \geq 0$ while for the switch $i_S \geq 0$ but v_S is unrestricted, which corresponds to a two-quadrant switch. Once the switch configuration is specified (switch open i.e. $i_S = 0$ or closed i.e. $v_S = 0$), the switch variables can be eliminated and one is left with a standard LCS with the diode variables.

By considering as state variables the current in the inductance and the voltage on the capacitor, i.e., i_L and v_C, it follows that this circuit can be modeled by the time invariant state space equations:

$$L\frac{di_L}{dt} = V_{in} - v_C - v_D \tag{2.43}$$
$$C\frac{dv_C}{dt} = i_L - i_S - \frac{v_C}{R_L}, \tag{2.44}$$

where V_{in} is the initial voltage that satisfies the condition: $V_{in} > 0$. In addition we have the port equations,

$$i_S = i_L - i_D \tag{2.45}$$
$$v_D = v_S - v_C, \tag{2.46}$$

and the complementarity conditions,

$$0 \leq -v_D \perp i_D \geq 0 \qquad v_S \perp i_S \geq 0$$

In order to make the control variable explicit take $i_S = u \cdot \hat{i}_S$ and $(1-u) \cdot \hat{v}_S$. Then, the LCS model becomes

$$L\frac{di_L}{dt} = V_{in} - v_C - v_D \tag{2.47}$$
$$C\frac{dv_C}{dt} = i_L - u\hat{i}_S - \frac{v_C}{R_L} \tag{2.48}$$
$$u\hat{i}_S = i_L - i_D \tag{2.49}$$
$$v_D = (1-u)\hat{v}_S - v_C, \tag{2.50}$$

and the only remaining complementarity condition is

$$0 \leq -v_D \perp i_D \geq 0.$$

Finally, diode variables can be removed from state equations, resulting in

$$L\frac{di_L}{dt} = V_{in} - (1-u)\hat{v}_S \tag{2.51}$$
$$C\frac{dv_C}{dt} = i_L - u\hat{i}_S - \frac{v_C}{R_L}, \tag{2.52}$$

Let us consider a generic sliding surface $\sigma(i_L, v_C) := k_1 L i_L + k_2 C v_C + k_3$. Note that the transversality condition is $k_1 \hat{v}_S - k_2 \hat{i}_S \neq 0$ and the equivalent control

$$u_{eq} = \frac{k_1 \left(V_{in} - \hat{v}_S\right) + k_2 \left(i_L - \frac{v_C}{R_L}\right)}{k_2 \hat{i}_S - k_1 \hat{v}_S}$$

Parameters k_1, k_2 and k_3 have to be chosen so that the ideal sliding dynamics be stable.

Then, the control action is defined as:

$$u = \begin{cases} 0 & \text{if } \sigma \cdot (k_1 \hat{v}_S - k_2 \hat{i}_S) > 0 \\ 1 & \text{if } \sigma \cdot (k_1 \hat{v}_S - k_2 \hat{i}_S) < 0 \end{cases} \tag{2.53}$$

provided that $0 < u_{eq} < 1$.

While the system is in continuous conduction mode

1. $u = 0$ implies $i_S = 0$ and $i_D = i_L \neq 0$, hence $v_D = 0$ and $\hat{v}_S = v_C$.
2. $u = 1$ implies $v_S = 0$ and $-v_D = v_C \neq 0$, hence $i_D = 0$ and $\hat{i}_S = i_L$.

Taking all of this into account the switching logic can be rewritten as

- if $u = 0$ and $\sigma \cdot (k_1 v_C) < 0$, then switch to $u = 1$.
- if $u = 1$ and $\sigma \cdot (k_2 i_L) < 0$, then switch to $u = 0$.

However if the system is in discontinuous conduction mode

1. $u = 0$ implies $i_S = 0$ and $i_D = i_L = 0$, hence $v_D = V_{in} - v_c$ and $\hat{v}_S = V_{in}$.
2. $u = 1$ implies $v_S = 0$ and $-v_D = v_C = 0$, hence $i_D = 0$ and $\hat{i}_S = i_L$.

Taking all of this into account the switching logic can be rewritten as

- if $u = 0$, $i_L = 0$ and $\sigma \cdot k_1 < 0$, then switch to $u = 1$.
- if $u = 1$, $v_C = 0$ and $\sigma \cdot (k_2 i_L) < 0$, then switch to $u = 0$.

Remark: : A natural but naive candidate for sliding surface is $v_C = v^*$. It is qualified as naive because it is well known that these converters are non-minimum phase with respect to the output voltage. Thus, an indirect control has to be considered. For the election of the sliding surface it has to be taken into account the ideal sliding dynamics: the transient performance as well as the steady-state. The former will depend on the slope of the sliding surface, for the later the sliding surface has to pass through the desired equilibrium point. This design depends on system parameters and results in a non robust controlled system. In order to avoid this drawback, the voltage error integral is used as an additional variable and the sliding surface is completed with an integral term. Since the aim of this report is limited to an introduction to the subject, we assume the system parameters are well known; particularly V_{in} and R_L.

2.7.2 Simulation

As we have mentioned before, for the numerical approximation of the solutions of switched electrical networks the Backward Euler time-stepping scheme is frequently used. For LCS the method consists of discretizing the system description by applying the well known backward Euler integration routine and imposing the complementarity conditions at every time step. This comes down to the computation of u^h_{k+1}, y^h_{k+1}, and x^h_{k+1} given x^h_k through the solving a linear complementarity problem. In general a linear complementarity problem may have multiple solutions or have no solutions at all. In that case we can assure the unique solvability of the problem because we have a P-matrix for each LCS.

For the boost converter analysis, the state variables (the current in the inductance and the voltage on the capacitor, i.e., i_L and v_C) and the sliding surface are the basic results we want to export out.

For simulations we use these fixed parameters:

Design parameter $L = 750 \cdot 10^{-6}$ H., $C = 220 \cdot 10^{-6}$ F., $R_L = 10\ \Omega$, $V_{in} = 24$ v., $v^* = 60$ v., $k_1 = 1$, $k_2 = 0$, $k_3 = -15$, $\Delta = 0.35$.

Initial conditions $i^0_L = 0$ A., $v^0_c = 24$ v.

Simulation results for state variables and the sliding surface are shown. An hysteresis of width Δ has been added to the sliding surface equation.

2.8 Conclusions

The basic power converters can be formulated as linear cone complementarity systems (LCCS). For a given switch configuration, the resulting system can be cast strictly in the linear complementarity system (LCS) formalism.

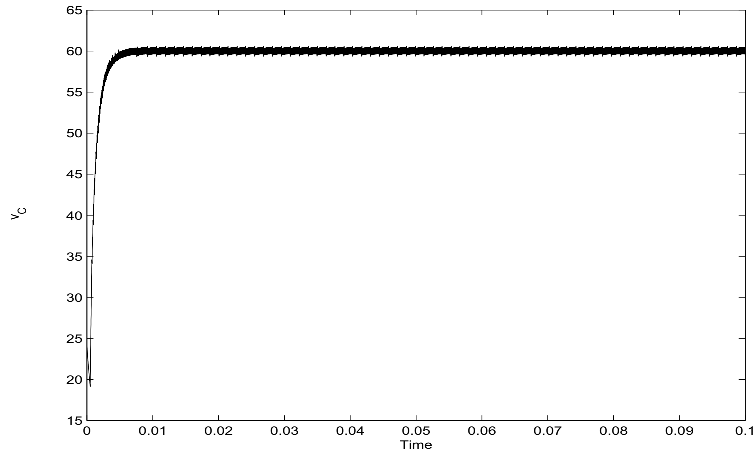

Figure 2.9: Voltage on the capacitor

We have presented a simple, analytical test to look for generalized discontinuous conduction modes in power converters with a single diode. This test can be verified at several orders; the higher the order, the smoother the change in u. We have applied the test to the Čuk converter, and found the conditions under which generalized discontinuous conduction modes can appear. We have performed simulations and checked the theoretical prediction. In particular, we have found a variety of behaviors, such as a re-entrant GDCM. Although we have centered our exposition on switched power converters, the results apply as well to any other LCS with a single complementary pair.

Using the formalism of [145] as has been applied here, systems with several diodes can be treated. It is straightforward to extend the study to the case of decoupled diodes, *i.e.* to the case when βa, now a matrix, is diagonal; for the nondiagonal case, a case-by-case study of the LCP problems appearing in the successive DCPs will be needed to obtain analytical results, although numerical algorithms can always be used.

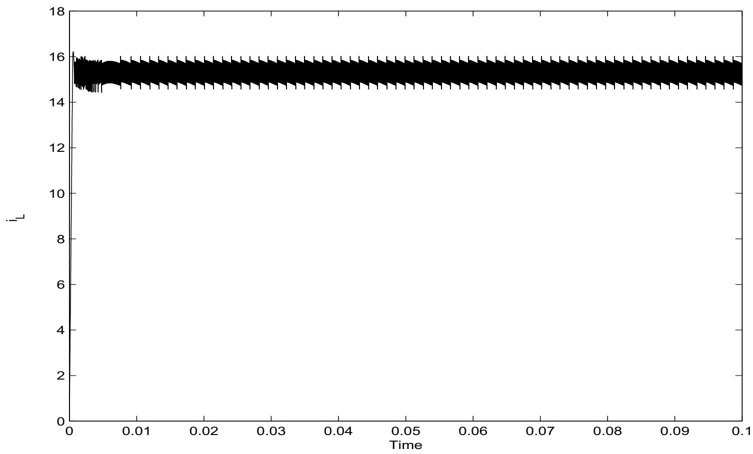

Figure 2.10: Current in the inductance

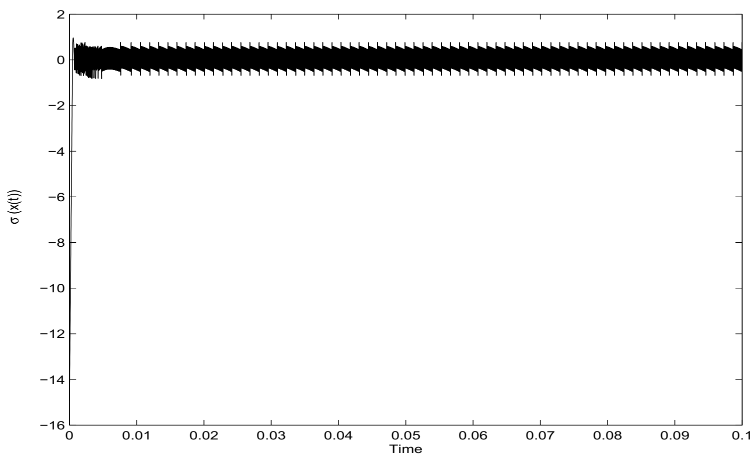

Figure 2.11: Sliding surface

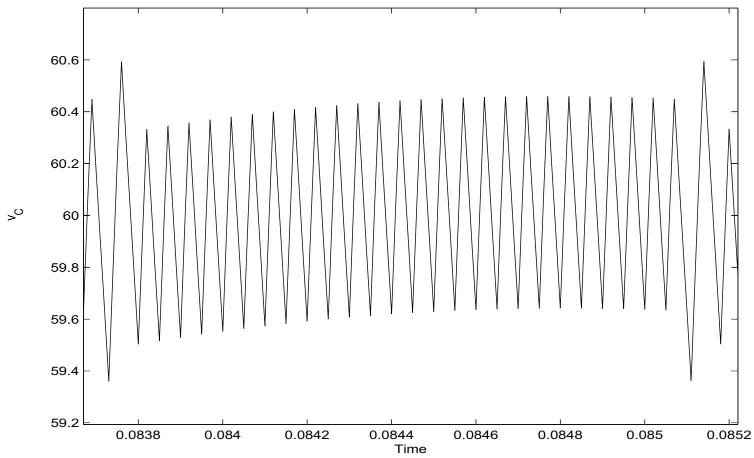

Figure 2.12: Zoom of the voltage on the capacitor

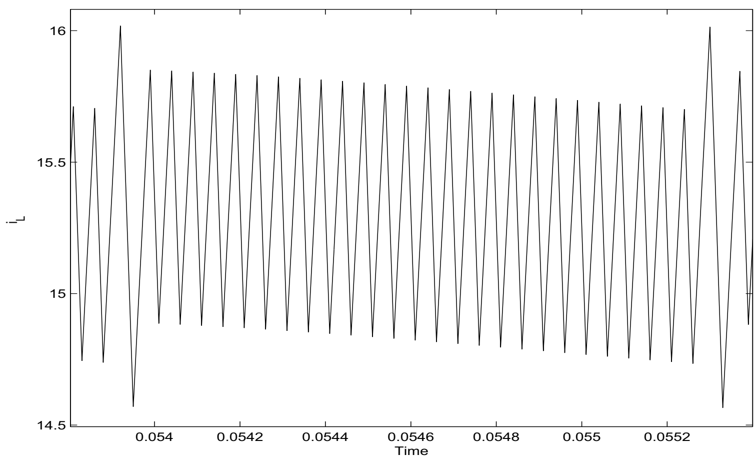

Figure 2.13: Zoom of the current in the inductance

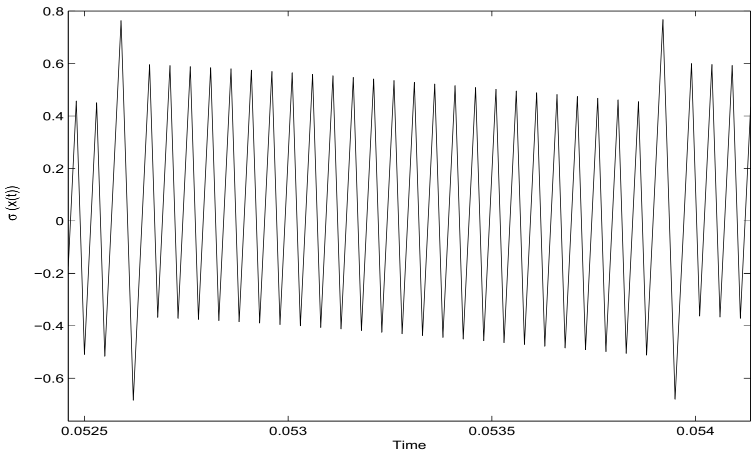

Figure 2.14: Zoom of the sliding surface