

## CHAPTER 1

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# Introduction

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*“The beginning of knowledge is  
the discovery of something we do not understand.”*

FRANK HERBERT (1920-1986)

In this introductory chapter we give a motivation for the thesis. After a very concise introduction to the field of research, we define the objective and scope of this work. In the last section, the structure of this thesis and a summary of the contents and contributions is specified.

### 1.1 Motivation

Discontinuous events characterise the behaviour of a large number of dynamical systems of relevance in applied science and engineering. Examples can be found in the occurrence of impacts in mechanical systems, stick-slip motion in oscillators with friction, switchings in electronic circuits and hybrid dynamics in control systems. They are also present in models from economy and biology. Dynamical systems with discontinuous events fall into a wide group of systems that are often referred to as discontinuous or nonsmooth dynamical systems.

In general, those physical systems can operate in different modes, and the transition from one mode to another is often much shorter in time than the time scale of the dynamics of the individual modes. In order to study

this class of dynamical systems from a macroscopic level it may be very advantageous and useful to model the transitions as being instantaneous (discrete transitions). For instance, diode elements in electric circuits are often modeled as being ideal elements which are either blocking or conducting without voltage drop, and mathematical models of mechanical systems subjected to dry friction are considered to switch between a slip mode and a (pure) stick mode. Consequently, some difficulties are raised due to the features of such systems, such as the use of multi-valued functions.

Despite of the growing evidence of their widespread use, there is no effective systematic theory of such systems. While the last decades have witnessed an explosive development in the theory of smooth dynamical systems, many fundamental problems remain open for nonsmooth dynamical systems. These include theoretical issues (such as well-posedness, stability and numerical algorithms), and more importantly, practical issues of relevance to applications. For example, in industry, virtual prototyping is a crucial step in the design of most mechanical, electrical and electronic systems because of money savings and improved reliability. At the same time many of these physical systems are difficult to model and to simulate numerically because, as mentioned, they involve complex dynamical phenomena like nonsmooth characteristics (diodes, MOS transistors), impacts, friction, switching devices, etc. However, much of the available software for these systems does not incorporate appropriate routines for the treatment of nonsmoothness.

All around the world, researchers and industrialists are now urging the development of appropriate novel analytical and numerical tools for nonsmooth systems. These should support the derivation and development of models of systems characterised by nonsmooth dynamics at the required level of complexity. More precisely, these tools should not be too simplistic in order to capture the qualitative behaviour of the real physical systems of interest. But, at the same time, they should be simple enough to allow a reliable analysis, numerical simulations and control design of the systems.

Nonsmooth dynamical systems are also known to exhibit sudden losses of structural stability under parameter variations. For example, dry-friction oscillators have been shown to exhibit a sudden jump from a regular periodic solution to deterministic chaos (see figure 1.1). Chattering and sliding phenomena are also observed which correspond to unwanted high-frequency switching behaviour in the systems of interest. But now, after an ongoing research effort we know that, because of their discontinuous nature, nonsmooth dynamical systems can exhibit an entirely novel class of

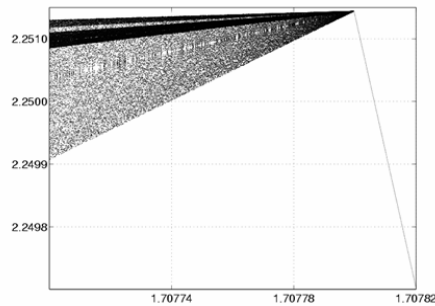


Figure 1.1: Sudden transition from a regular periodic solution to chaos.

bifurcations. These are termed *Discontinuous – induced bifurcations* (DIBs) or *C – bifurcations* to distinguish them from the bifurcations also occurring in smooth systems. Such bifurcations include phenomena such as *grazing* (near-tangencies of the system trajectories with the discontinuity sets in phase space) or *border – collision* (interaction between the equilibrium of a switched flow or map with one or more phase space boundaries). Despite the successes of nonsmooth bifurcation theory in explaining the complex behaviour observed, for instance, in power electronic converters or mechanical systems with impact and friction, many open problems still remain.

## 1.2 Nonsmooth dynamical systems

### 1.2.1 Introduction

There are many concrete problems in applied science and engineering where nonsmooth phenomena play an important role. Examples can be found in mechanical systems subjected to unilateral constraints [22, 88, 101, 122, 136], Coulomb friction [11, 12, 98] or impacts [12, 22, 65, 66, 104]. They are also present in electrical networks if nonsmooth characteristics are used to represent switches and in control theory they frequently appear when discontinuous controls are involved.

From a mathematical viewpoint, problems of this kind are not easy to handle, since the resulting models are dynamical systems whose right-hand sides are not continuous or not differentiable. Moreover, in many cases the solutions have to observe additional restrictions that frequently

appear in the form of inequality constraints. This means that an extension of the differential equations concept is needed to describe discontinuous behaviour.

Several works have been devoted to introducing the mathematical basis on nonsmooth dynamical systems. There are, at least, three different popular approaches to deal with such systems: the *differential inclusion* [1, 2, 9, 64] approach, the *complementarity formalism* [146, 27, 80] approach and the *hybrid system* [22] approach. The different approaches have shown to be good for different purposes, although all of them have some drawbacks which make them more suitable for some tasks than for others.

In what follows we will introduce some terminology related to nonsmooth dynamical systems.

### 1.2.2 Classification of nonsmooth dynamical systems

Nonsmooth dynamical systems can be described by sets of piecewise-smooth (PWS) ordinary differential equations (ODEs). These are smooth in regions  $G_i$  of phase space with smoothness being lost across the boundaries  $\Sigma_{i,j}$  between adjacent regions. We will refer to such boundaries as *nonsmoothness sets* or *switching manifolds*. Specifically, we have

$$\dot{x} = F(x, \mu); \tag{1.1}$$

where  $F : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^n$  is a PWS function,  $\mu \in \mathbb{R}^m$  is a parameter vector and  $x \in \mathbb{R}^n$  the state vector. In each of the phase space regions  $G_i$ , the system dynamics is described by a different functional form,  $f_i$ , of the system vector field. Following [98] we divide the nonsmooth dynamical systems of interest into three different categories, depending on the discontinuity type of their orbits and vector fields (see figure 1.2):

- Systems exposed to discontinuities (or jumps) in the state or *impacting systems*. Examples are impacting systems with velocity reversals or vibro-impacting machines [22]. Such systems are more commonly formulated as a *hybrid system* with discontinuous jumps (such as a restitution law) described by auxiliary maps.
- Systems with discontinuous vector fields, i.e.  $f_i(x_0, \mu) \neq f_j(x_0, \mu)$ ,  $\forall x_0 \in \Sigma_{ij}$ . These systems are also called *Filippov systems* [64]. Examples are systems with visco-elastic supports and dry friction [71, 125]. Models of power electronic voltage converters can be also considered [44, 67].

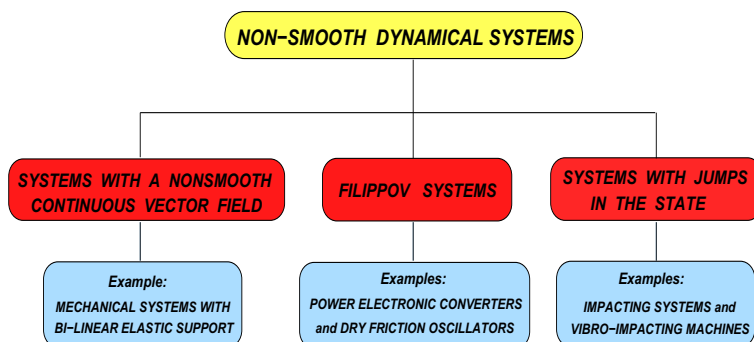


Figure 1.2: Classification of non-smooth dynamical systems.

- Nonsmooth systems whose orbits and vector fields are everywhere continuous, or piecewise smooth continuous systems. Such systems have continuously-differentiable orbits but discontinuities in the first or higher derivatives of  $f$ , i.e.  $f_i(x_0, \mu) = f_j(x_0, \mu)$ , but  $\exists n \in \mathbb{N} / \frac{\partial^n f_i}{\partial x^n}(x_0, \mu) \neq \frac{\partial^n f_j}{\partial x^n}(x_0, \mu), \forall x_0 \in \Sigma_{ij}$ . Examples include mechanical systems with bi-linear elastic support [131].

### 1.2.3 Literature Survey of discontinuity-induced bifurcations

Recently, it has been shown that nonsmooth systems can exhibit complex dynamics whose occurrence cannot be explained using bifurcation analysis tools developed for smooth dynamical systems. For example, one of the most striking feature of this class of systems is that they often exhibit sudden transitions from periodic attractors to chaos in the absence of any period-doubling or other bifurcation cascade usually observed in their smooth counterparts. In fact, a new class of bifurcations, or *nonsmooth bifurcations*, needed to be introduced to explain these phenomena. As further detailed below, so-called border collision and grazing bifurcations were shown to be the main cause for such unexpected transitions which were left unexplained for a relatively long time in the nonlinear dynamics literature.

Apart from standard bifurcations (flip, fold, Hopf, etc.), systems from all three of the classes presented in the previous section can undergo topological changes to their phase portraits that are unique to nonsmooth systems. For example, as a parameter is varied, an equilibrium point in one of

the open domains can approach a boundary. Many possibilities can emerge from such a nonsmooth bifurcation point. For example, Hopf-like bifurcations can occur, generating limit cycles from the boundary equilibrium, but with a linear rather than square-root growth in amplitude [64, 94, 96]. There are also nonsmooth analogues of folds [98]. Limit cycles of discontinuous systems can also exhibit other novel nonsmooth transitions as elucidated in the pioneering work in the Russian literature, such as that of Feigin [59, 60, 61, 62, 63]. In Feigin’s work, all scenarios involving invariant sets undergoing a non-structurally stable interaction with a discontinuity set were given the collective name of *C-bifurcations* (*C* stands for the Russian word for “sewing”). As remarked in [52], however, such definition does not necessarily imply a bifurcation in the strict mathematical sense of transition to a topologically non-equivalent phase portrait (see, e.g. [96]), because the existence and stability of invariant sets can be unaffected by such an interaction. So, it might be more correct to refer to such discontinuity-set driven events as *nonsmooth transitions* rather than bifurcations in the classical sense. Due to this fact the name that is currently used to term these nonsmooth transitions is *discontinuous-induced bifurcations* (DIBs).

Bifurcations of fixed points in PWS discrete-time maps were more recently studied in the West by Yorke and collaborators [114, 115, 116, 150]. Interactions of periodic points with the discontinuity set in continuous piecewise-linear (PWL) maps were termed *border-collision bifurcations*. It was later shown in [45] that border-collision bifurcations may be interpreted in terms of Feigin’s theory. Thus, they lead to a number of bifurcation scenarios (saddle-node like cases, period-doublings, transcritical-like transitions), which can be classified in  $n$ -dimensional cases by applying a set of appropriate conditions on the eigenvalues of the linear part of a PWS map on either side of the discontinuity.

Another important class of DIBs are the so-called *grazing bifurcations*, when a limit cycle undergoes either a tangential interaction with a discontinuity set, or passes through the intersection between two discontinuity sets. To analyse and classify the dynamics that can follow from such limit cycle transitions, an established technique is to derive PWS maps as ‘normal forms’ for the bifurcation. Iterates of the derived maps then give the existence of nearby invariant sets. The derivation of such maps is based on the concept of the *zero time discontinuity mapping* (ZDM) introduced by Nordmark [109]. These normal forms were derived for all three classes of nonsmooth dynamical systems mentioned above [49, 50, 69, 51, 107].

Interestingly, none of the cases, other than a boundary-intersection crossing in a Filippov system [50], does lead to a locally piecewise-linear map of the kind studied by Feigin or Yorke. Instead, depending on the local properties of the vector fields across the discontinuity, tangential grazing in the absence of sliding leads to a ZDM with an  $O(k + 1/2)$  term, for some non-negative integer  $k$  [49, 69]. In contrast, boundary intersection in the absence of sliding leads to maps with a discontinuity of one order less than that of the vector field. There are more possibilities of grazing bifurcations in vector fields in which sliding occurs. In general these all lead to maps with jumps in derivatives of higher than linear order, except for the so-called *grazing-sliding* case, which yields a piecewise-linear normal form map. However, this map is not of the form used by Feigin and Yorke, because it is non-invertible on one side of the discontinuity. Such non-invertibility has significant implications regarding possible bifurcation scenarios that can be observed.

So far, the investigation of DIBs has focused on one-parameter transitions. Here, we should comment on the notion of the codimension of DIBs. Assuming the weak definition as non-generic interaction with a discontinuity set, an utilitarian definition of the codimension of a bifurcation completely analogous to the one used for smooth system can be proposed (see e.g. [96]). Namely, the codimension of a bifurcation is defined as the difference between the dimension of the parameter space and the corresponding set of parameters for which the bifurcation occurs.

## 1.3 Objective and Scope of the Thesis

### 1.3.1 General Objective

The main aim of this thesis is the analysis of the structural stability under parameter variations (bifurcation analysis) in nonsmooth or discontinuous dynamical systems. In particular, we focus on the study of different mechanical systems which involve impacts and dry-friction. It will be shown that these systems can exhibit unexpected transitions due to the presence of discontinuity boundaries partitioning the state space into different system functional forms. These transitions, termed discontinuity-induced bifurcations (DIBs), are unique to nonsmooth and discontinuous dynamical systems and cannot be observed in their smooth counterparts.

In addition the so-called complementarity formalism have been used as mathematical framework in order to model some switched electronics

systems in view of control and numerical applications. Moreover, a software has been developed for the simulation of the mentioned nonsmooth and discontinuous dynamical systems under the EU Project SICONOS in order to tackle the necessity of dealing with hit crossing, impacts, sliding and chatter in a robust and easily applicable way.

### 1.3.2 Specific Objectives

In order to develop the general objective of this work the following complementarity objectives will be considered :

- Modeling of some power electronics systems using the complementarity formalism with a view to control and numerical applications.
- Simulation of some nonsmooth dynamical systems following different strategies (using event-driven methods, time-stepping methods or smoothing the system) and study of the differences between the obtained results.
- Continuation of periodic orbits in nonsmooth dynamical systems.
- Study and classification of discontinuity-induced bifurcations in different nonsmooth dynamical systems. We have considered mechanical systems with impacts and dry friction.
- Analysis of bifurcations given by sliding or border-collision, paying special attention to codimension 2 bifurcations.
- Bifurcation analysis using different types of friction in mechanical systems with dry friction.
- Calculations of basins of attraction.
- Participation in the development of the SICONOS Platform as an expert user.

### 1.3.3 Scope

The main results of this thesis are concerned with the analysis of the structural stability under parameter variations (bifurcation analysis) in nonsmooth and discontinuous dynamical systems. We focus our attention on the study of mechanical systems as cam-follower systems, dry friction



oscillators and two-block stick-slip systems. For each system we have analysed the complex behaviour that characterises these systems.

It is known that the complementarity formalism is a good framework for studying nonsmooth systems with a large amount of nonsmooth interactions. As a first step for studying large electrical networks with diodes and switchings, we have modeled some basic dc-dc power converters. For systems with a single diode, an analytical condition for the presence of generalized discontinuous conduction modes (GDCM), characterized by a reduction of the dimension of the effective dynamics, has been proved.

This thesis has been developed under the framework of the European project SICONOS (IST-2001-37172). One of the SICONOS objectives has been the development of a platform for modeling and simulating nonsmooth dynamical systems (NSDS). Some algorithms for the analysis of NSDS (basins of attraction, bifurcations,...) have been implemented using this numerical software. The author developed this task during stages at the University of Bristol and the INRIA institute. Some benchmarks to illustrate the platform abilities have been implemented.

## 1.4 Outline of the Thesis and Contributions

This thesis studies nonsmooth dynamical systems with particular applications to power electronics systems and mechanical systems with impact and dry friction. From a mathematical point of view, it contributes to the study of these systems with emphasis on the numerical computing. Although closed form solutions can be available, finally, one must resort to numerical methods to compute periodic orbits, bifurcations, invariant manifolds and basins of attraction. From an engineering point of view, this thesis both contributes to answering some questions about the behaviour observed in experimental works, and to generating new questions to be answered by the engineering community. This thesis is structured in eight chapters and the contents of each other is explained in the following paragraphs.

Chapter 2 contains some background of the theory for modeling using the complementarity formalism. This theory gives us the necessary tools to model some basic dc-dc power converters as complementarity systems. It is shown that, for each position of the switches, the dynamics is given by a linear complementarity problem to which standard techniques can be applied. For systems with a single diode, an analytical condition for the presence of generalized discontinuous conduction modes (GDCM), charac-

terized by a reduction of the dimension of the effective dynamics, can be proved. This result is used to identify the GDCM for the switch configurations of each dc-dc power converter. Simulation results, showing a variety of behaviors, such as persistent or re-entering GDCM, are presented. A Parallel Resonant Converter is also modeled as a complementarity system and some conditions for GDCM are shown. Finally, some open problems referred mainly to control are stated. An important part of this chapter has already been published in the *IEEE Transactions on Circuits and Systems* [C.Batlle, E.Fossas, I.Merillas and A.Miralles, 2005].

The analysis of a cam-follower system is studied in Chapter 3. This kind of systems can be considered as driven impact oscillators. Therefore, several nonsmooth phenomena as first detachment, transitions from complete to uncomplete chattering and nonsmooth bifurcations of periodic orbits are presented. In order to study these complex behaviours we have stated analytical explanations as necessary conditions for periodic orbits with a single impact. We will also explain the suddenly jump of a period orbit to chaos, which is termed as corner-impact bifurcation. Finally, coexistence of attractors is shown using domains of attraction calculated with a standard cell mapping method. The main part of this chapter was partially done during a stage at University of Naples Federico II and has been submitted to *International Journal of Bifurcation and Chaos* [I.Merillas, G.Osorio, P.T. Piironen, M.di Bernardo, E.Fossas].

Sliding bifurcations in a dry friction oscillator are dealt with in Chapter 4. A dry friction introduced by Popp has been considered. This kind of system can be studied using the Filippov theory. Therefore, bifurcations due to the interaction between the trajectory and the discontinuity surface can be found. The four possible scenarios of sliding bifurcations are shown and the calculation of their normal-form mappings is also detailed. Finally, a nonsmooth codimension 2 bifurcation is presented. A large part of this chapter has been submitted to *International Journal of Bifurcation and Chaos* [I.Merillas, U. Galvanetto, C.Batlle].

Chapter 5 is concerned with the study of two-block stick-slip system. The two-block stick-slip system is another example where nonsmooth transition can be obtained. This example has direct applications to geology and geophysics. Different models have been considered in order to approach to the behaviour of earthquakes and fault dynamics. Basically these models are Filippov systems with two discontinuity surfaces. Each model have been simulated and we show different nonsmooth behaviours using an event map. This chapter has already been published in different proceedings of

congress.

In Chapter 6 the Siconos software, dedicated to simulation of non-smooth dynamical systems (NSDS), is presented. After motivating the development of this tool, we give a overview of the SICONOS software and the way NSDS are modeled and simulated within the platform. Routines for analysis (stability, bifurcations, invariant manifolds, ...) of NSDS implemented in the platform, during my stages at the University of Bristol and INRIA, are explained in detail. To conclude, several representative samples are shown in order to illustrate the SICONOS platform abilities.

Finally, in Chapter 7 the conclusions of the thesis are presented and several open problems are outlined for further research.

## 1. Introduction

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