On Normal Forms and Splitting of Separatrices in Reversible Systems

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Memòria presentada per a aspirar al grau de
Doctor en Matemàtica Aplicada

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Barcelona, setembre del 2003

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Introduction

It is difficult, in the Theory of Dynamical Systems, to draw a boundary line between conservative laws and symmetries because often their effects on the dynamics are very similar. This is the case of the Hamiltonian and *reversible* systems. Briefly, a system of ordinary differential equations

$$\dot{x} = F(x,t), \qquad x \in \mathbb{R}^n, \; t \in \mathbb{R} \tag{1}$$

is called *time-reversible* if there exists a phase space involution $\mathfrak{G}$ (that is, $\mathfrak{G}^2 = \text{id}$, $\mathfrak{G} \neq \text{id}$) such that system (1) remains invariant under the action of $\mathfrak{G}$ and a reversion in time's arrow $t \longmapsto -t$. From an analytical point of view, this is equivalent to say that $\mathfrak{G}$ conjugates F with its opposite field $-F$. If the involution $\mathfrak{G}$ is linear we will refer to them as (time) *linearly* reversible systems. In the case of diffeomorphims, $\mathfrak{G}$ conjugates the corresponding diffeo with its inverse. From a dynamical point of view, one of the main consequences is that if $x = \varphi(t)$ is a solution of (1) then so is the same solution evolving backwards in time and applying the phase space involution $\mathfrak{G}$, that is $x = \mathfrak{G}(\varphi(-t))$. For instance, consider the motion of an ideal pendulum (with no loss of energy due to friction) and film its evolution. If we, later, played this film with a projector, we could not distinguish from the images if we were playing the film forwards or backwards. This is only a simple example of a time-reversible system but, in fact, this is not a coincidence. Reversibility and time-reversibility are present in a lot of branches of Mathematics, Physics, Astronomy, Biology, Chemistry, ...

For years, *reversibility* (from now on, we will use the terms *reversible* and *reversibility* to indicate *time-reversible* and *time-reversibility*) was considered a property of some Hamiltonian systems, which allowed a reduction of the number of significant variables and a simplification, in general, of the model.

It was not until the end of the 1970's and, specially during the decade of the 80's, that reversibility received again the interest of scientist as a feature independent of the Hamiltonian character (see, for instance [29, 3, 4, 51]). An example of the richness of this kind of systems is the fact that they can exhibit simultaneously conservative-like and dissipative-like behaviors. Indeed, the reversible systems can have Kolmogorov tori which are invariant under the phase flow and the involution $\mathfrak{G}$ (see [44, 3, 51, 52] and satisfy some other results concerning a reversible version of Lyapunov center theorem [29] and Lyapunov stability [38, 39]. However, Nekhoroshev-like results concerning exponential stability (*effective* stability) cannot be carried out to reversible systems. These results depend strongly on some geometrical properties which are satisfied by the Hamiltonian system but are not by the reversible ones. For more information about this topic see, for instance, [7] and [9, §4.2.5]. For a discussion concerning also the reversible case, see [39]. Moreover, outside the manifold Fix $\mathfrak{G}$ of fixed points by the involution $\mathfrak{G}$, reversible systems can exhibit attractors and repellers (in fact, if they have an attractor at a point x_0 then the symmetric point $\mathfrak{G}\, x_0$ is a repeller, and *vice versa*), which is a typical dissipative feature. Although they are not very common, there exist in the literature examples of reversible systems that are not Hamiltonian. For instance:

Example 1: It is due to Sevryuk [51, page 144]. Consider the following system in $\mathbb{C}^2$:

$$\left\{ \begin{array}{rcl} \dot{z}_1 &=& \mathrm{i}\, z_1\, (\alpha_1 + z_2 \bar{z}_2) \\ \dot{z}_2 &=& \mathrm{i}\, \alpha_2\, z_2 \end{array} \right. \tag{2}$$

where $\alpha_1, \alpha_2 \in \mathbb{R}$. This system is reversible with respect to the involution $(z_1, z_2) \longmapsto (\bar{z}_1, \bar{z}_2)$. However, there does not exist any symplectic structure with respect to which system (2) is Hamiltonian.

Example 2: (Politi *et al.* [47]) Under some conditions, an externally injected class B laser can be described by a 3-dimensional system of differential equations of the form

$$\left\{ \begin{array}{rcl} \dot{x} &=& zx + y + c_1 \\ \dot{y} &=& zy - x \\ \dot{z} &=& c_2 - x^2 - y^2 \end{array} \right. \tag{3}$$

where c_1, c_2 are parameters. This is system is reversible with respect to the linear involution $\mathfrak{G} : (x, y, z) \longmapsto (-x, y, -z)$ and presents conservative-like structures and typical dissipative ones. A related simplified 2-dimensional model of system (3) is

$$\left\{ \begin{array}{rcl} \dot{x} &=& f_1 y + f_2 y^2 + f_3 x^2 \\ \dot{y} &=& g_1 x + g_2 xy \end{array} \right. \tag{4}$$

where f_j, g_j, $j = 1, 2$ are real parameters. This system is reversible with respect to the involution $\mathfrak{G} : (x, y) \longmapsto (-x, y)$. If we take values of f_1, $f_2 \neq 0$, g_1 and g_2 satisfying

$$f_1 g_1 < 0 \qquad \text{and} \qquad f_1\,(f_1 g_2 - f_2 g_1)\, f_2 < 0,$$

then system (4) has two symmetric fixed points (lying on Fix $\mathfrak{G} = \{x = 0\}$): the origin that is an elliptic point and another equilibrium point which is hyperbolic. Around them, Hamiltonian-like structure is observed. But, moreover, it has two other fixed points, lying outside $\{x = 0\}$ that are symmetrical, in the sense that one is the image by $\mathfrak{G}$ of the other one, being one of them an attractor and the other one, a repeller.

Example 3: This third example is due to Champneys [17] and comes from a model for an optical 0 with grating which generalizes the classical Massive Thirring model. Roughly speaking, the partial differential equations arising from this problem can be led into a second-order complex ordinary differential equation, which is Hamiltonian, with Hamilton function given by

$$H = D \left|U'\right|^2 + \Omega \left|U\right|^2 + |U|^4 + \frac{1}{2}\left(U^2 + (\bar{U})^2\right).$$

Here $u(x,t) = \mathrm{e}^{-\mathrm{i}\Omega t} U(x)$ represents the solitary wave we are looking for, D is the effective diffraction coefficient and $\bar{U}$ denotes the conjugate of U. To model the effects of nonlinear dispersion (or diffraction) Champneys introduces an extra term depending on a coefficient β, giving rise to an equation

$$DU'' + \mathrm{i}U' + \Omega U + U\left|U\right|^2 + \bar{U} = \mathrm{i}\beta \left(U \left|U\right|^2\right)'. \tag{5}$$

This equation can be viewed as a four-dimensional dynamical system in the real variables $\Re U$, $\Im U$, $\Re U'$ and $\Im U'$ and, in particular, it is reversible with respect to the involutions

$$\mathfrak{G} : (U, U') \longmapsto (\bar{U}, -\bar{U}'), \qquad \mathfrak{G}' : (U, U') \longmapsto (-\bar{U}, \bar{U}').$$

It can be seen that, for $\beta = 0$ this Hamiltonian system has a saddle-center equilibrium point at the origin with a homoclinic connection, provided one takes $D > 0$ and $|\Omega| < 1$. Using a numerical approach, Champneys proves that for non-zero values of β equation (5) is not any more Hamiltonian (but still reversible).

This memory deals with reversible systems. More precisely, we have focused our attention on two topics intimately related to the person of Henri Poincaré (next year it will be celebrated the 150th anniversary of his birth). This topics, which Poincaré introduced in his thesis (1890), are the *Normal Form Theory* and the phenomenon of the *Splitting of Separatrices.* The first one has become one of the most useful tools to study the dynamics of a nonlinear system around an equilibrium. The second one is related to the transversal intersection of invariant manifolds arising from the perturbation of a system having an homoclinic connection.

The aim of this work is to contribute to the study of this two topics, looking for similar results to the ones obtained in the Hamiltonian context and establishing some connections between both type of systems.

Let us start with *Normal Forms.* Since they were introduced by Poincaré, they have been one of the most used tool in Dynamical Systems Theory. The literature devoted to them is very extensive as well as the number of authors involved (for instance, see [14, 20] for a general overview on this topic). Very briefly, the main problem of the Normal Form Theory is, given a system

$$\dot{z} = F(z) = \Lambda\, z + O(z^2), \tag{6}$$

with an equilibrium point (say the origin), to seek for a change of variables $z = \Phi(\zeta) = \zeta + O(\zeta^2)$ (usually called a *normalizing transformation*) leading system (6) into its *simplest* form, say

$$\dot{\zeta} = N(\zeta) = \Lambda\, \zeta + O(\zeta^2). \tag{7}$$

This is called *normal form* and is just formed by the so-called *resonant terms* (see the introduction of Chapter 1), terms closely related to the characteristic exponents of system (6) at the origin (Poincaré-Dulac Theorem).

Our first idea was to study the convergence of the normal form for an analytic reversible system in the vicinity of a *saddle-center* or a *saddle-focus* equilibrium point. This is the case if their characteristic exponents are $\{\pm\lambda,\ \pm\alpha\,\mathrm{i}\}$ or $\{\pm\lambda \pm \alpha\,\mathrm{i}\}$, $\lambda, \alpha > 0$, respectively. The proof of this convergence becomes the reversible version of a celebrated result due to Moser [43], which generalized the well-known Lyapunov's Theorem to the case of a 2-degrees of freedom Hamiltonian system. Devaney [29] proved an analogous result of Moser's Theorem for reversible systems, but using a topological argument. Later, other proofs were given due to Moser and Webster [45] and Sevryuk [51]. Our aim was to give a new proof of this theorem of Moser based on a completely constructive scheme (and, therefore, implementable on a computer). However, at that moment, we became aware of a paper of DeLatte [22], inspired on ideas of Moser. The thesis of that paper can be summarized as follows. Take, for instance, a general 2-dimensional analytic system (6) with a hyperbolic equilibrium point at the origin (that is, with characteristic exponents $\{\pm\lambda\}$, $\lambda > 0$). To normalize it we must seek for an analytic transformation $z = \Phi(\zeta)$ leading (6) into a normal form (7). This means that $N = \Phi^* F = (D\Phi)^{-1} F(\Phi)$ or, in other words, that the equation

$$D\Phi\, N = F\,(\Phi)$$

is satisfied. For example, if we are dealing with a Hamiltonian system, the vector field N will be the one of the *Birkhoff Normal Form* (BNF in short) [42]. The idea of Moser and DeLatte is to consider another vector field B, formed also by resonant-type terms, in such a way that the equality

$$D\Phi\, N + B = F\,(\Phi) \tag{8}$$

holds. DeLatte proved that these vector fields $N(\zeta)$, $B(\zeta)$ and the transformation $z = \Phi(\zeta)$ converged (analytically) in some cases. And moreover, in those cases, if the system was Hamiltonian then B had to vanish and we met therefore the usual BNF.

This result led us to the possibility of extending this method to the problem we wanted to solve. That is, if there exists an analytic transformation $z = \Phi(\zeta)$, leading a *general* (non-necessarily reversible) system (6) into a form satisfying (8), in a vicinity of saddle-center or a saddle-focus equilibrium point. If such a transformation exists we say that it leads system (6) into pseudo-normal form (ΨNF in short). Since we know that in the Hamiltonian case this problem is already solved by the BNF theory, it is natural to look for a vector field N, in the expression (8), of the same type as the one provided by the BNF. Moreover, if we expect ΨNF to become a generalization of the BNF, they will have to coincide when we deal with a Hamiltonian system. Consequently, B will have to vanish.

Chapter 1 is devoted to the proof of the convergence of this ΨNF, as claimed in Theorem 1.2. The methodology employed to do it has consisted essentially on handling formal power series in several variables, performance of *constructive* recurrent schemes and formal solution of vectorial homological equations. The convergence is linear with respect to the order in the spatial variable $\zeta \in \mathbb{C}^4$. To prove such convergence it has not been used the celebrated *majorants method*, but another one similar to one used in [25].

Later, we applied Theorem 1.2 to the particular cases of a Hamiltonian system and a reversible system (with the reversibility non-necessarily linear). The result obtained (as it was expected) is that in both cases B vanishes and, therefore, we have a local equivalence between Hamiltonian and reversible systems around saddle-center or saddle-focus equilibrium points. In particular, several consequences can be derived for planar systems around hyperbolic and elliptic equilibrium points (see Theorem 1.3), specially in the case that B does not vanish. It is remarkable that this ΨNF-method permits to measure, in some sense, the *Hamiltoniability* (and, therefore, its local *integrability*) by means of an analytic vector field B. This $B(\zeta)$ has a particular form (see (1.7), for instance) which depends on two (one in the planar case) analytic scalar-valued functions b_1, b_2. Consequently, one can apply methods coming from the Theory of Analytic functions to estimate the number of zeros of these functions b_1, b_2. In fact, the idea of using these ΨNF (suggested by Moser) is closely related to the *Translated curve* approach suggested by Moser in 1962 to find invariant curves and first used by Rüssmann (1982).

Chapter 2 is devoted to the study of the ΨNF in the planar case. Like it was done by Moser in [42], we prove some properties about the ΨNF, the transformations preserving such a form and their effect on the coefficients of N and B. A possible interesting application of these results is to the case of an analytic planar system having a linear center at the origin, namely,

$$\left\{ \begin{array}{rcrcl} \dot{x} & = & -y & + & \widehat{f}(x,y) \\ \dot{y} & = & x & + & \widehat{g}(x,y) \end{array} \right. , \tag{9}$$

with $\widehat{f}$ and $\widehat{g}$ analytic functions starting with terms of order at least 2 in x, y. It is known (see, for instance, [51, page 144]) that system (9) is Hamiltonian if and only if it is reversible. Even more, this is also equivalent to say that (9) has a *center* at the origin and that there exists a convergent transformation leading it into BNF. A possible way to ensure this convergence is, using Bruno's approach, to check that the formal normal form N obtained satisfies the so-called *Condition A* (see the introduction of Chapter 1). Unfortunately, this is a very strong restriction since it must be checked at any order (in the spatial variables). Consequently, given an analytic planar system with a *linear center* at the origin, that is of the type (9) it is not possible, as a rule, to determine whether the BNF is convergent or not. In other words, if the origin is a center or a focus (see, for example, [14, §1.11,§4]).

Interesting results concerning the convergence of Poincaré-Dulac normal forms (that is, non necessarily Hamiltonian) for general analytic systems have been obtained in presence of symmetries (see [19, 20, 15] and references therein).

Coming back to system (9), we can consider the particular case where $\widehat{f}$ and $\widehat{g}$ are polynomials,

$$\left\{ \begin{array}{rcrcl} \dot{x} & = & -y & + & P(x,y) \\ \dot{y} & = & x & + & Q(x,y) \end{array} \right. . \tag{10}$$

The problem can be then formulated as looking for the conditions one has to impose on the coefficients of P and Q ensuring that the origin (a linear center) is in fact a center (and, therefore, not a focus). This apparently simple question is known as the *center-focus problem* and is a famous question intimately related to a local version of 16th Hilbert's Problem. In fact, it still remains open and it has been solved in just a few cases: when P and Q are quadratic polynomials, if they are cubic polynomials without terms of order 2, if they form a Liénard-type system, ... As an example, this problem with (full) cubic polynomials P and Q has not been solved yet. For more information on this topic see, for instance, [6, 53, 54, 50].

One of the usual tools to approach this problem is based on the computation of the *Lyapunov-constants* which are related to the coefficients of a formal first integral of the system (see Lemma 2.2) and are polynomials in the coefficients of P and Q. It is known that system (10) has a center at the origin if and only if all these Lyapunov constants vanish. An interesting problem is to find which is the minimal number of Lyapunov constants one has to take equal to zero in order to ensure that the equilibrium point is a center.

Applying the ΨNF method to system (10) it follows that B depends on an analytic scalar-valued function b. In Section §3, Chapter 2, it is proved that the coefficients of this function b (in its turn also polynomials in the coefficients of the system) satisfy some similar properties to the ones satisfied by the Lyapunov constants. This fact, lead us to approach the center-focus problem for some known cases (essentially, when P and Q have degree 2 or 3) using a ΨNF-scheme. The idea is to consider generic polynomials P and Q, perform ΨNF up to a given order in the spatial variables and study the polynomial functions forming the coefficients of b. For this purpose we built our own algebraic manipulator (an example of the outputs obtained are given at the end of Chapter 2) and studied the use of *Gröbner Bases* to factorize polynomials in several variables. The problems we found are given in the same Chapter. It is worth mentioning that this project was a joint work with A. Guillamon and F. Planas, supported by an UPC-grant. Our aim is to continue with this problem in a close future.

Coming back to the problem of the ΨNF, it is quite natural to wonder if this method is also convergent in some other situations. Since the second topic we are interested in is the Splitting of Separatrices for reversible systems, it seems reasonable to consider the problem of the convergence of the normal form around a hyperbolic periodic orbit and around a whiskered torus. It is reasonable to expect that in both cases the ΨNF can be also convergent. Chapter 3 is devoted to the first case. There, it is considered a general analytic periodic perturbation of an integrable Hamiltonian system (whose corresponding BNF is convergent). Although the idea is essentially the same as the one used for the saddle-center and saddle-focus equilibria, the perturbative approach employed here introduced changes homological equation (see formula (3.31) in Chapter 3) as well as in its solution. We expect this formal perturbative approach to be very close to the one appearing in the quasi-periodic problem, which arises when considering a quasi-periodic perturbation of an integrable system around a whiskered torus. The main result of Chapter 3 is given in Theorem 3.1. Like in Chapter 1, the same consequences concerning ΨNF and BNF can be derived for Hamiltonian and Reversible systems. The methodology applied consists on handling formal expansions in Taylor-Fourier series, solving vectorial equations by means of recurrent procedures and proving their (analytic) convergence following a similar argument to the one in Chapter 1.

As mentioned in Chapter 5, one of our future works will be devoted to the proof of the convergence of the ΨNF in a vicinity of a whiskered torus. Although the formal solution can be analogous to the periodic situation, the problems of the small denominators will require a quadratic order scheme to ensure the convergence.

Finally, we arrive to the last part of this memory, which concerns the topic of the *Splitting of Separatrices in reversible systems* and to which Chapter 4 is devoted. This is an important phenomenon which seems to be one of the main causes of the stochastic behavior in the Hamiltonian systems. When Poincaré introduced in his thesis a perturbative method to measure the size of this splitting, he was already aware that its predicted size was exponentially small in the perturbative parameter. However, it was not until the end of the 80s and during the 90s that effective measurements and asymptotic formulas of such exponentially small splitting were given. In 1997, A. Delshams and T.M. Seara wrote a paper [24] that pretended to expound a general method to deal with this problem when one perturbed with a rapidly oscillatory forcing a $1\frac{1}{2}$-degrees of freedom Hamiltonian system having a homoclinic connection. In that paper they obtained an asymptotic formula for the area of the lobe remaining between the invariant curves and the angle at the first homoclinic point. The size of this splitting was exponentially small in the perturbative parameter and was given in first order by the Melnikov function.

Thus, it seem to us that it could be interesting to reproduce this result in the case of a 2-dimensional linearly reversible system subjected to a fast periodic perturbation that preserved the same reversibility. There are essentially two the basic results that support the method presented in [24]: the convergence of the BNF around the hyperbolic periodic orbit and the transversal intersection on the invariant curves. In our reversible case, the convergence of the BNF around a hyperbolic periodic orbit had been given in Chapter 3. Like in the Hamiltonian problem, the size of this domain of convergence is independent of the perturbative parameters. On the other hand, the transversal intersection between the invariant curves of the associated Poincaré map (with respect to time) is ensured by the linear reversibility of the system. The final result is given in Theorem 4.1 (Chapter 4) and is completely analogous to the Hamiltonian one given at [24] for the angle between the invariant curves at the first homoclinic intersection. Due to the reversibility of our system, this first homoclinic point lies always at the symmetry line, the set of points which are fixed by the linear spatial involution. We want to stress the fact that in a reversible context the area of the lobes between these curves it is not an invariant and, therefore, its measurement has no interesting meaning.

The methodology employed in this chapter contains the use of the normal forms (linear normal form, non-linear BNF and an extension of this normal form along an invariant manifold, based on results from Da Silva, Ozorio, Douady and Vieira [21, 46]), parameterization of invariant manifolds, extension of solutions and construction of flow-box coordinates to measure the splitting size.

Then, a suggestive question arises: provided we were able to prove the convergence of the BNF for quasi-periodic reversible systems around a whiskered torus, could we extend the results given at [25] concerning the measurement of the splitting of the invariant whiskers for a suitable Hamiltonian system, to the case of a linearly reversible system ? This problem is still open. We are planning to study it following similar ideas to those used in Chapters 3 and 4.

To end, let us stress the following significant fact: in 1890 Poincaré introduced the Normal Form Theory and a perturbative method to measure the Splitting of Separatrices; nowadays, 113 years later, they continue been basic tools in the study and understanding of the dynamics in nonlinear dynamical systems.

Chapter 1

ΨNF near saddle-center or saddle-focus equilibria.

§1 Introduction and main results

Since normal forms were introduced by Poincaré they have become a very useful tool to study the local qualitative behavior of dynamical systems around equilibria. Consequently, the literature devoted to this topic has been very extensive as the amount of authors involved (Poincaré, Dulac, Siegel, Birkhoff, Lyapunov, Sternberg, Arnold, Moser, Bibikov, Bruno and many others; for a general background see for instance [2, 14, 20] and references therein). In a few words, given a system

$$\dot{X} = F(X) = \Lambda X + O(X^2), \tag{1.1}$$

around an equilibrium, say the origin $X = 0$, a general normal form procedure consists on looking for a (formal power series close to the identity) transformation $X = \Phi(\chi)$ in such a way that the new system $\dot{\chi} = \Phi^* F(\chi)$ takes its *simplest* form. This is called *normal form* and contains only the so-named *resonant terms*, monomials whose powers are intimately related to the characteristic exponents of system (1.1) at the origin.

More precisely, if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ is the vector formed by the characteristic exponents of system (1.1) at the origin, i.e. the eigenvalues of the $m \times m$-matrix Λ, then λ is called *resonant* if there exist $p_1, p_2, \ldots, p_m \in \{0, 1, 2, 3, \ldots\}$, satisfying $|p| := p_1 + p_2 + \cdots + p_m \geq 2$, such that

$$\lambda_s = p_1\lambda_1 + p_2\lambda_2 + \cdots + p_m\lambda_m = \langle p, \lambda \rangle \tag{1.2}$$

for some $s \in \{1, 2, \ldots, m\}$. If $z = (z_1, z_2, \ldots, z_m) \in \mathbb{C}^m$ and $p = (p_1, p_2, \ldots, p_m) \in (\mathbb{N} \cup \{0\})^m$, we say that

$$z^p e_s = z_1^{p_1} z_2^{p_2} \cdots z_m^{p_m} e_s$$

with $\underset{s}{e}^\top = (0, \ldots, 0, \overset{(s)}{1}, 0, \ldots, 0)$, is a *resonant monomial* if p and s satisfy (1.2). Thus, an analytic system

$$\dot{\chi} = N(\chi), \tag{1.3}$$

with $\chi = (\zeta_1, \zeta_2, \ldots, \zeta_m)$ and $N(\chi) = (n_1(\chi), n_2(\chi), \ldots, n_m(\chi))$, is said to be in *normal form* if all the terms in $N(\chi)$ are resonant.

In this work, we will focus our attention on analytic vector fields and will be specially concerned with the convergence of the *normalizing transformation* Φ.

There are two well-known cases where a polynomial normal form is achieved. The convergence of its normalizing transformation depends only on the location of the vector of characteristic exponents $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ on the complex plane and on some arithmetical properties. Namely,

(*i*) when λ belongs to the Poincaré domain, that is, the convex hull of the set $\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ does not contain the origin;

(*ii*) when λ belongs to the complementary of this domain, the so-called Siegel's domain, and satisfies a Diophantine condition.

In the first case, the Theorem of Poincaré–Dulac ensures the convergence of a normalizing transformation conjugating the original system to a system having only resonant terms. Since in this situation there is just a finite number of resonant monomials, the normal form is a polynomial. In the second case, the Diophantine condition permits to bound the small divisors appearing in the normalizing transformation (see, for instance, [2, Chapter 5, §24]) and its convergence is also derived (Siegel's Theorem). The original system is conjugated to its linear part, again a polynomial.

However, resonant normal forms with an infinite number of terms do arise in some important families of dynamical systems, like the Hamiltonian or the reversible ones. In such contexts the characteristic exponents come in pairs $\{\pm\lambda\}$ and, therefore, they always belong to the Siegel's domain. In these cases, convergence results depend not only on the location of the characteristic exponents and their arithmetical properties but also on the kind of formal normal form they exhibit. In 1971, Bruno (see [11, Chapter II, §3, §4]) provided sufficient and, in some particular sense, necessary conditions ensuring this convergence. He denominated them conditions A and ω. The so-called condition ω depends on arithmetic properties of the vector of characteristic exponents $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ and can be checked explicitly. On the contrary, condition A imposes a strong restriction on the normal form forcing it (up to all order !) to depend only on one or two scalar functions (see [11, pages 173–175]). These conditions read as follows.

Condition A: There exist formal power series $a(\chi)$, $b(\chi)$ such that in equation (1.3)

$$N(\chi) = \begin{pmatrix} \lambda_1\zeta_1 \\ \lambda_2\zeta_2 \\ \vdots \\ \lambda_m\zeta_m \end{pmatrix} a(\chi) + \begin{pmatrix} \overline{\lambda_1}\zeta_1 \\ \overline{\lambda_2}\zeta_2 \\ \vdots \\ \overline{\lambda_m}\zeta_m \end{pmatrix} b(\chi),$$

where $\overline{\lambda_j}$ denotes the conjugate of λ_j.

Condition ω: Set

$$\omega_k = \min |\lambda_s - \langle p, \lambda\rangle|$$

for $\lambda_s - \langle p, \lambda\rangle \neq 0$ and $2 \leq |p| < 2^k$. Then

$$\sum_{k\geq 1} 2^{-k} \ln \omega_k$$

converges.

Typically, for any vector of characteristic exponents $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$, ω_k tends to zero as k goes to infinity. Since $\lambda_s - \langle p, \lambda \rangle$ appear as denominators in the normalizing transformation Φ this is the so-called *small divisors* phenomenon. If $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$ satisfies condition ω then these small divisors can be bounded. Notice that this condition is imposed only on the non-resonant terms of the system. If condition ω is satisfied it follows the existence of $c, \nu > 0$ such that

$$|\lambda_s - \langle p, \lambda \rangle| > c \exp(-\nu|p|)$$

(see [11, page 140]). With respect to these two conditions, in [11, Theorem 4, page 186], Bruno asserts

Theorem 1.1 ([Bruno]) *Given a system* (1.1)*, if its characteristic exponents* $(\lambda_1, \lambda_2, \dots, \lambda_m)$ *satisfy condition* ω *and its (formal) normal form satisfies condition* A*, then there exists a convergent analytic transformation* $X = \Phi(\chi)$ *transforming system* (1.1) *into normal form.*

In some sense conversely, if a normal form (1.3) can be obtained from an analytic system, such that neither of the two conditions ω and A is satisfied, then there exists a system (1.1) having (1.3) as its normal form and such that any transformation leading it into normal form is divergent [11, page 186].

In order to show how difficult is to check condition A, it is particularly interesting to consider the case of a planar analytic vector field with the origin being a linear center equilibrium point, that is, with purely imaginary characteristic exponents. Although a formal normalizing transformation can be built for such system, its convergence can be ensured *a priori* just in case one knows that the origin is a center. Indeed, this fact forces the corresponding formal normal form to satisfy condition A. On the other hand, if the origin is a focus, we have as a rule divergence (for more details, see [14, pages 121–122]). In the case this planar vector field is polynomial, one can look for the minimal number of conditions on their coefficients ensuring the origin to be a center. This is a famous question intimately related to a local version of the 16th Hilbert's problem called the *center-focus problem*, which still remains open.

There are very few cases where the fulfillment of condition A follows from the nature of the original system. Some of them arise in Hamiltonian systems, where the normal form is called the *Birkhoff normal form* (BNF in short). We recall that a system (1.1) is *(locally) Hamiltonian* if there exists a function $H \in \mathcal{C}^r(U)$, where $r = 2, \dots, \infty, \omega$ and U is a neighborhood of the origin, and a 2-form $\tilde{\omega} \in \Omega^2(U)$ such that $\tilde{\omega}(F, \cdot) = dH$. Note that for a Hamiltonian system condition A admits an equivalent reformulation in terms of the Hamiltonian function (the so-called condition H, [12, page 225]).

Thus, let us consider a Hamiltonian system and assume it has no small divisors. This implies that condition ω is trivially satisfied. However, this means that we can only deal with Hamiltonian systems of one or two degrees of freedom, since any Hamiltonian system having more than 2 degrees of freedom presents always small divisors. Indeed, for a 1-degree of freedom system, BNF becomes $\dot{\chi} = N(\chi)$, with $\chi = (\xi, \eta) \in \mathbb{C}$,

$$N = \begin{pmatrix} \xi a(\xi\eta) \\ -\eta a(\xi\eta) \end{pmatrix} \tag{1.4}$$

and characteristic exponents $\pm\lambda$, where $\lambda = a(0) \in \mathbb{R}$. The assumption $\lambda \neq 0$ makes this normal form to satisfy condition A.

Consider now a 2-degrees of freedom system and denote by $\{\pm\lambda_1, \pm\lambda_2\}$ its characteristic exponents at the origin. It is not difficult to check that if $\lambda_1/\lambda_2 \notin \mathbb{R}$ then its BNF $\dot{\chi} = N(\chi)$ satisfies condition A. This means that the origin has to be

(*i*) a *saddle-focus*, if $\{\pm\lambda_1, \pm\lambda_2\} = \{\pm\lambda \pm \mathrm{i}\alpha\}$ with $\lambda, \alpha \in \mathbb{R} \setminus \{0\}$, or

(*ii*) a *saddle-center*, if $\{\pm\lambda_1, \pm\lambda_2\} = \{\pm\lambda, \pm\mathrm{i}\alpha\}$ and $\lambda, \alpha \in \mathbb{R} \setminus \{0\}$.

In these cases N can be written as

$$(1.5) \qquad (a) \quad N = \begin{pmatrix} \xi a_1(\xi\eta, \mu\nu) \\ -\eta a_1(\xi\eta, \mu\nu) \\ \mu a_2(\xi\eta, \mu\nu) \\ -\nu a_2(\xi\eta, \mu\nu) \end{pmatrix} \quad \text{or} \quad (b) \quad N = \begin{pmatrix} \xi a_1(\xi\eta, \mu^2+\nu^2) \\ -\eta a_1(\xi\eta, \mu^2+\nu^2) \\ \nu a_2(\xi\eta, \mu^2+\nu^2) \\ -\mu a_2(\xi\eta, \mu^2+\nu^2) \end{pmatrix},$$

respectively, where $a_j(0,0) = \lambda_j$, $j = 1,2$ and $\chi = (\xi, \eta, \mu, \nu) \in \mathbb{C}^4$.

It was Lyapunov [36] in 1907, who provided a first result in this direction. Namely, he proved that given a real analytic Hamiltonian system with characteristic exponents at the origin

$$\{\pm\lambda_1(\text{pure imaginary}), \pm\lambda_2, \ldots, \pm\lambda_n\}$$

satisfying that

$$\lambda_s \neq m\lambda_1$$

for any $m \in \mathbb{N}$ and $s \in \{2, 3, \ldots, n\}$, there exists always a one-parameter analytic family of periodic solutions in a neighborhood of this equilibrium point (for a detailed proof, see for instance [55]). In other words, he proved the existence of a convergent normalizing transformation leading this system into BNF with respect to the variable associated to the characteristic exponent λ_1.

Later, in 1958, Moser [43] extended this result to the case of the equilibrium having characteristic exponents

$$\{\pm\lambda_1, \pm\lambda_2, \ldots, \pm\lambda_n\}$$

verifying that

(*i*) λ_1, λ_2 are independent over $\mathbb{R}$;

(*ii*) $\lambda_s \neq m_1\lambda_1 + m_2\lambda_2$ for any m_1, $m_2 \in \mathbb{N}$, and $s \in \{3, 4, \ldots, n\}$.

As it has mentioned above, this corresponds to the origin being a *saddle-focus* or a *saddle-center* equilibrium point. Moser proved the existence of an analytic convergent transformation leading the original system into BNF (with respect to the variables associated to λ_1, λ_2).

Recently, a new proof of this theorem has been provided by Giorgilli [30] putting special emphasis on the Hamiltonian character of the system (a characteristic which does not appear in Moser's proof).

At this point, it seems natural to wonder about the convergence of a normalizing transformation Φ in the case of a general system. The analogy with the Hamiltonian case suggests to consider 2-dimensional and 4-dimensional systems with characteristic exponents at the equilibrium point being (i) $\pm\lambda$, $\lambda \in \mathbb{C} \setminus \{0\}$ and (ii) $\{\pm\lambda_1, \pm\lambda_2\}$, respectively, since they have no small divisors and condition ω is clearly satisfied. Thus, the problem becomes to investigate *how far* (and in which way) are these systems from the fulfillment of condition A (and, therefore, of having a convergent normalizing transformation Φ). The case (i) was studied in [26]. The aim of the present work is to deal with case (ii), a general analytic system (1.1) with a saddle-focus or a saddle-center equilibrium point at the origin.

Our intention is to compare such system with a Hamiltonian one, where BNF is convergent, and to build a kind of convergent extended BNF. We will ask it to have BNF as a particular situation and we expect to obtain some interesting information even in case condition A is not satisfied.

Let us be more precise. As it has been said, it is well-known in the saddle-focus or saddle-center Hamiltonian cases the existence of a convergent transformation $X = \Phi(\chi)$ leading system (1.1) into BNF, that is, the transformed system being of the form

$$\dot{\chi} = (\Phi^* F)(\chi) = N(\chi), \tag{1.6}$$

where N is of the first or second type in (1.5) respectively. Notice that equation (1.6) is equivalent to say that

$$D\Phi \; N = F \circ \Phi.$$

Our approach, which comes from ideas of Moser and DeLatte [22], consists on looking for a *remainder term* of the form

$$(a) \quad \widehat{B} = \begin{pmatrix} \xi\widehat{b}_1(\xi\eta, \mu\nu) \\ \eta\widehat{b}_1(\xi\eta, \mu\nu) \\ \mu\widehat{b}_2(\xi\eta, \mu\nu) \\ \nu\widehat{b}_2(\xi\eta, \mu\nu) \end{pmatrix} \quad \text{or} \quad (b) \quad \widehat{B} = \begin{pmatrix} \xi\widehat{b}_1(\xi\eta, \mu^2+\nu^2) \\ \eta\widehat{b}_1(\xi\eta, \mu^2+\nu^2) \\ \mu\widehat{b}_2(\xi\eta, \mu^2+\nu^2) \\ \nu\widehat{b}_2(\xi\eta, \mu^2+\nu^2) \end{pmatrix}, \tag{1.7}$$

depending if we are considering the saddle-focus or saddle-center case, respectively, satisfying $\widehat{b}_1(0,0) = \widehat{b}_2(0,0) = 0$ and such that the equality

$$D\Phi \; N + \widehat{B} = F \circ \Phi \tag{1.8}$$

holds. Hence forward $\widehat{G}$ will denote vector fields constituted by formal powers series starting with terms of order at least 2. Notice that (1.8) is equivalent to saying that the new system is of the form

$$\dot{\chi} = N(\chi) + (D\Phi(\chi))^{-1}\,\widehat{B}(\chi)$$

which is not, as a rule, a normal form. Thus, we will say that $X = \Phi(\chi)$ transforms system (1.1) into *pseudo-normal form* (ΨNF in short).

The interest, we think, of this construction lies in the following facts: first, it constitutes an extension of the BNF and, therefore, in the contexts where BNF *converges* they must coincide; second, this procedure is *convergent* in some situations where BNF does not apply and, thus, it translates the problem of the existence of a convergent normalizing transformation to the one of determining if some analytic scalar-valued functions $\widehat{b}_1$ and $\widehat{b}_2$ vanish. Moreover, even in the case that these functions do not vanish, some interesting dynamical consequences can be derived from this pseudo-normal form.

Theorem 1.2 (Main Theorem) *Given a system*

$$\dot{X} = F(X) = \Lambda X + \widehat{F}(X), \tag{1.9}$$

analytic around the origin (an equilibrium) and with characteristic exponents $\{\pm\lambda_1, \pm\lambda_2\}$ *equal to*

- $\{\pm\lambda \pm \mathrm{i}\alpha\}$ *with* $\lambda, \alpha \in \mathbb{R} \setminus \{0\}$ *(saddle-focus case), or*
- $\{\pm\lambda, \pm\mathrm{i}\alpha\}$ *with* $\lambda, \alpha \in \mathbb{R} \setminus \{0\}$ *(saddle-center case),*

there exist an analytic transformation $X = \Phi(\chi) = \chi + \widehat{\Phi}(\chi)$ *and convergent analytic vector fields* N, *as in* (1.5), *and* $\widehat{B}$ *as in* (1.7) *in such a way that the equality*

$$D\Phi\ N + \widehat{B} = F \circ \Phi$$

holds.

Section §2 is devoted to the proof of this theorem.

Remark 1 *The proof of this theorem is constructive. It is based on a recurrent scheme which provides the coefficients of* Φ, N *and* $\widehat{B}$ *order by order. Moreover, a condition for determining the radius of convergence of these vector fields is provided in equation* (1.75).

Remark 2 *As it is usual in Normal Form Theory, computations will be carried out complexifying the variables. It is not difficult to check that the corresponding* ΨNF *vector fields in the* real *case are of the form*

$$N(\zeta) = \begin{pmatrix} \xi a_1(\xi\eta, \mu^2+\nu^2) \\ -\eta a_1(\xi\eta, \mu^2+\nu^2) \\ \nu a_2(\xi\eta, \mu^2+\nu^2) \\ -\mu a_2(\xi\eta, \mu^2+\nu^2) \end{pmatrix} \qquad \widehat{B}(\zeta) = \begin{pmatrix} \xi\widehat{b}_1(\xi\eta, \mu^2+\nu^2) \\ \eta\widehat{b}_1(\xi\eta, \mu^2+\nu^2) \\ \mu\widehat{b}_2(\xi\eta, \mu^2+\nu^2) \\ \nu\widehat{b}_2(\xi\eta, \mu^2+\nu^2) \end{pmatrix},$$

in the saddle-center case and

$$N(\zeta) = \begin{pmatrix} \xi a_1(\xi\eta, \mu\nu) \\ -\eta a_1(\xi\eta, \mu\nu) \\ \mu a_2(\xi\eta, \mu\nu) \\ -\nu a_2(\xi\eta, \mu\nu) \end{pmatrix} \qquad \widehat{B}(\zeta) = \begin{pmatrix} \xi\widehat{b}_1(\xi\eta, \mu\nu) \\ \eta\widehat{b}_1(\xi\eta, \mu\nu) \\ \mu\widehat{b}_2(\xi\eta, \mu\nu) \\ \nu\widehat{b}_2(\xi\eta, \mu\nu) \end{pmatrix},$$

in the saddle-focus one, where $(\xi, \eta, \mu, \nu) \in \mathbb{R}^4$ *and the functions* a_ℓ *and* b_ℓ, $\ell = 1, 2$, *are real.*

A first consequence of Theorem 1.2 is that, if the initial system is *Hamiltonian* then the ΨNF becomes BNF. This is the thesis of the following proposition whose proof has been deferred to Section §3.

Proposition H1 *System* (1.9) *is Hamiltonian in a neighborhood of the origin if and only if* $\widehat{B}$ *vanishes (and, therefore,* ΨNF *becomes* BNF*).*

In the case that system (1.9) is a 2-degrees of freedom Hamiltonian, this proposition provides a new proof for the celebrated Moser's-Lyapunov theorem

Corollary H2 [Lyapunov, Moser] *For an analytic Hamiltonian system around a saddle-focus or a saddle-center equilibrium,* BNF *is convergent.*

Some other consequences can be derived from a partial reading of Theorem 1.2. Namely, a linear center can be seen as a particular subsystem of the general saddle-center case. Indeed, if we write explicitly system (1.9),

$$(1.10) \qquad \begin{cases} \dot{x} &= \lambda x + \widehat{f}_1(x,y,q,p) \\ \dot{y} &= -\lambda y + \widehat{f}_2(x,y,q,p) \\ \dot{q} &= \alpha p + \widehat{f}_3(x,y,q,p) \\ \dot{p} &= -\alpha q + \widehat{f}_4(x,y,q,p) \end{cases}$$

for $\widehat{f}_1(0,0,q,p) = \widehat{f}_2(0,0,q,p) = 0$ and fix $x = y = 0$, we obtain the following planar system

$$(1.11) \qquad \begin{cases} \dot{q} &= \alpha p + \widehat{f}_3(q,p) \\ \dot{p} &= -\alpha q + \widehat{f}_4(q,p) \end{cases}$$

Here $\widehat{f}_j(q,p)$, $j = 3,4$ denote $\widehat{f}_j(0,0,q,p)$ This is the framework where the previously cited *center-focus problem* takes place. In this case Theorem 1.2 provides the existence of a transformation $(q,p) = \Phi(\mu,\nu)$ and vector fields $N(\mu,\nu)$ and $\widehat{B}(\mu,\nu)$, of the form

$$(1.12) \qquad N = \begin{pmatrix} \nu a(\mu^2+\nu^2) \\ -\mu a(\mu^2+\nu^2) \end{pmatrix}, \qquad \widehat{B} = \begin{pmatrix} \mu\widehat{b}(\mu^2+\nu^2) \\ \nu\widehat{b}(\mu^2+\nu^2) \end{pmatrix},$$

analytic in a neighborhood of the origin, with $a(0) = \alpha$, $b(0) = 0$, and satisfying $D\Phi\ N + \widehat{B} = F_c \circ \Phi$, where $F_c(p,q) = (\alpha p + \widehat{f}_3(q,p), -\alpha q + \widehat{f}_4(q,p))$. The following corollary is a reformulation of Proposition H1.

Corollary H3 *Assume $\widehat{f}_3$, $\widehat{f}_4$ analytic at the origin. Then, the following statements are equivalent.*

- (i) *System* (1.11) *is (locally) Hamiltonian.*
- (ii) *The origin is a center.*
- (ii) *The function $\widehat{b}(\mu^2+\nu^2)$ in* (1.12) *provided by Theorem 1.2 vanishes identically.*

On the other hand, assuming $\widehat{f}_3 \equiv \widehat{f}_4 \equiv 0$ in system (1.10) (that is, the origin is a center in the (q,p)-variables), taking polar coordinates, scaling time if necessary and fixing an invariant cycle, we have a system of the form

$$(1.13) \qquad \begin{cases} \dot{x} &= \lambda x + \widehat{g}_1(x,y,\theta) \\ \dot{y} &= -\lambda y + \widehat{g}_2(x,y,\theta) \\ \dot{\theta} &= 1 \end{cases},$$

where $\gamma = \{x = y = 0\}$ is now a *hyperbolic periodic orbit* (of characteristic exponents $\pm\lambda$, $\lambda > 0$) and $\widehat{g}_1$, $\widehat{g}_2$ are analytic functions of x, y and θ. For such a system we have from Proposition H1 the following result.

Corollary H4 [Moser [42]] *Assume* (1.13) *is an analytic Hamiltonian system. Then, there exists a convergent transformation leading system* (1.13) *into* ΨNF *in a neighborhood of* γ *and this* ΨNF *coincides with the* BNF.

Remark 3 *The original result due to Moser is also valid assuming only* $\widehat{g}_1$ *and* $\widehat{g}_2$ *to be* $\mathcal{C}^1$ *with respect to the angular variable* θ. *With a similar scheme to the one presented in this paper,* Corollary H4 *can also be proved under these weaker assumptions.*

Up to this point, the results already presented follow from a suitable reading of Theorem 1.2 in a Hamiltonian framework. However, this is not the unique context where they can be applied. Namely, these results have a counterpart in the well known setting of the *reversible systems.*

We say that a system $\dot{X} = F(X)$ is $\mathfrak{G}$(time-)*reversible* (or simply, $\mathfrak{G}$-*reversible*) if it is invariant under $X \longmapsto \mathfrak{G}(X)$ and a reversion in the direction of time $t \mapsto -t$, with $\mathfrak{G}$ being an involutory diffeomorphism, that is, $\mathfrak{G}^2 = \mathrm{id}$ and $\mathfrak{G} \neq \mathrm{id}$. From this definition, it turns out that F satisfies

$$\mathfrak{G}^* F = -F, \tag{1.14}$$

where $\mathfrak{G}^* F = (D\mathfrak{G})^{-1} F(\mathfrak{G})$. The diffeomorphism $\mathfrak{G}$ is commonly called a *reversing* involution of this system and is, in general, non linear. In this work we are dealing with analytic systems, so we will consider analytic involutions $\mathfrak{G}$. A set S which is invariant under the action of $\mathfrak{G}$ (that is, $\mathfrak{G}(S) \subseteq S$) is called $\mathfrak{G}$-*symmetric* or, simply, *symmetric* if there is no problem of misunderstanding. Since we are dealing with systems in a neighborhood of an equilibrium point or a periodic orbit, from now on we will assume always that these elements are symmetric with respect to the corresponding involution $\mathfrak{G}$.

Important examples of reversible systems are provided by the BNF (1.5). For instance, the BNF around a saddle-center equilibrium point

$$\left\{ \begin{array}{rcl} \dot{\xi} & = & \xi a_1(\xi\eta, \mu^2 + \nu^2) \\ \dot{\eta} & = & -\eta a_1(\xi\eta, \mu^2 + \nu^2) \\ \dot{\mu} & = & \nu a_2(\xi\eta, \mu^2 + \nu^2) \\ \dot{\nu} & = & -\mu a_2(\xi\eta, \mu^2 + \nu^2) \end{array} \right.$$

is $\mathfrak{R}$-reversible, $\mathfrak{R}$ being the linear involution $(\xi, \eta, \mu, \nu) \longmapsto (\eta, \xi, \mu, -\nu)$. Analogously, the BNF around a saddle-focus equilibrium point is reversible with respect to the linear involution $(\xi, \eta, \mu, \nu) \mapsto (\eta, \xi, \nu, \mu)$.

Proposition R1 *System* (1.9) *is reversible in a neighborhood of the origin if and only if* $\widehat{B}$ *vanishes (and, therefore,* ΨNF *becomes* BNF*).*

Remark 4 *The Reversible Lyapunov Theorem was proven by Devaney* [29] *in both the smooth and the analytic case, using a geometrical approach. An alternative proof for this theorem is due to Vanderbauwhede* [57] *(see also* [51] *and* [37]*, for an extension to families of analytic reversible vector fields).*

The proof of this proposition is deferred to Section §3. Notice that, in particular, it implies that locally Hamiltonian and locally reversible is the same around this equilibrium point. Like in the Hamiltonian case, we have

Corollary R2 *Corollaries* H3 *and* H4 *also hold substituting Hamiltonian by reversible.*

From these results, it seems natural to look for a summarizing statement connecting both contexts, the Hamiltonian and the reversible. Indeed, we can summarize the previous statements in the following theorem.

Theorem 1.3 *Let us consider an analytic system*

$$\dot{X} = F(X) \tag{1.15}$$

and assume that one of the following three situations holds (corresponding to dimensions 2, 3 *and* 4, *respectively),*

- *(i)* $X = (q,p) \in \mathbb{R}^2$ *and the origin is a linear center equilibrium point (like in system* (1.11)*).*
- *(ii)* $X = (x,y,\theta) \in \mathbb{R}^2 \times \mathbb{T}$ *and* $\gamma = \{x = y = 0\}$ *is a hyperbolic periodic orbit (like in system* (1.13)*).*
- *(iii)* $X = (x,y,q,p) \in \mathbb{R}^4$ *and the origin is a saddle-center or saddle-focus equilibrium point (like in system* (1.10)*).*

Then, in a neighborhood of the corresponding critical element, the following statements are equivalent

- *(i) System* (1.15) *is Hamiltonian (with respect to some suitable* 2*-form* ω*).*
- *(ii) System* (1.15) *is reversible (with respect to some suitable reversing involution* $\mathfrak{G}$*).*
- *(iii) The analytic vector field* $\widehat{B}$ *(as in* (1.7)*) provided by Theorem* 1.2 *vanishes.*

Remark 5 *This local duality around critical elements between Hamiltonian and reversible systems is quite common. As an example, see for instance* [40]*, where it is proved this equivalence in the case of a non-semisimple* $1:1$ *resonance, which occurs when two pairs of purely imaginary eigenvalues of the linearized system collide. Nevertheless, there exist also counter examples of such equivalence. For instance, see the one given at* [49]*, where it is given a class of area preserving mappings, with linear part the identity, which are not reversible.*

Beyond the consequences provided by Theorem 1.2 in the Hamiltonian or reversible frameworks, this ΨNF-approach can be useful to find out isolated periodic orbits in other situations.

For instance, in [26] it is shown that for the center-focus problem (case (i) in Theorem 1.3) each zero of the analytic function b, defined in (1.12), gives rise to a limit cycle of system (1.11) close to the origin.

Now, consider system (1.15) with the origin being a saddle-center equilibrium point (case (iii) in Theorem 1.3). Let N and $\widehat{B}$, as in (1.5b), (1.7b), be the analytic vector fields provided by Theorem 1.2. Assume this system (1.15) is not locally Hamiltonian (neither reversible, therefore). Equivalently, functions $\widehat{b}_1$, $\widehat{b}_2$ in equation (1.7b) do not vanish simultaneously. Then the transformed system becomes of the form

$$\dot{\chi} = N(\chi) + (D\Phi(\chi))^{-1}\,\widehat{B}(\chi)$$

or, more precisely,

$$
(1.16)\qquad \begin{pmatrix} \dot\xi \\ \dot\eta \\ \dot\mu \\ \dot\nu \end{pmatrix} = \begin{pmatrix} \xi a_1(\xi\eta, \mu^2+\nu^2) \\ -\eta a_1(\xi\eta, \mu^2+\nu^2) \\ \nu a_2(\xi\eta, \mu^2+\nu^2) \\ -\mu a_2(\xi\eta, \mu^2+\nu^2) \end{pmatrix} + (D\Phi(\chi))^{-1} \begin{pmatrix} \xi \widehat{b}_1(\xi\eta, \mu^2+\nu^2) \\ \eta \widehat{b}_1(\xi\eta, \mu^2+\nu^2) \\ \mu \widehat{b}_2(\xi\eta, \mu^2+\nu^2) \\ \nu \widehat{b}_2(\xi\eta, \mu^2+\nu^2) \end{pmatrix}.
$$

Assume that $\widehat{b}_2$ does not vanish identically but there exists, at least, a value $I_* > 0$ satisfying $\widehat{b}_2(0, I_*) = 0$. If we take initial conditions $\xi^0 = \eta^0 = 0$ in (1.16) it follows that $\xi(t) = \eta(t) = 0$ $\forall t$ and, therefore, $\mu^2 + \nu^2 = I_*$ becomes a limit cycle of the restricted system

$$
(1.17)\qquad \left\{ \begin{array}{rcl} \dot\mu &=& \nu a_2(0, I_*) \\ \dot\nu &=& -\mu a_2(0, I_*) \end{array} \right.
$$

where, for small enough values of I_*, we have $a_2(0, I_*) = \alpha + O(I_*) \neq 0$. That is,

$$
\Gamma_* = \{\mu^2 + \nu^2 = I_*\}
$$

is a hyperbolic periodic orbit of system (1.17) with period $2\pi/a_2(0, I_*)$ and characteristic exponent $a_1(0, I_*) = \lambda + O(I_*)$. Consequently,

$$
\Gamma = \Phi(\Gamma_*) = \{\Phi(0, 0, \mu, \nu) : \mu^2 + \nu^2 = I_*\}
$$

is a hyperbolic periodic orbit of system (1.15). It is also straightforward to parameterize the corresponding (local) stable and unstable invariant manifolds of Γ. Namely, there exists $\delta > 0$, given by the radius of convergence of the ΨNF, such that

$$
(1.18)\qquad \begin{array}{rcl} W^{\mathrm{s}}_{\mathrm{loc}}(\Gamma) &=& \left\{\Phi(0, \eta^0 e^{-t a_1(0,I_*)}, \mu, \nu) : \left|\eta^0\right| < \delta,\ \mu^2 + \nu^2 = I_*\right\}, \\ W^{\mathrm{u}}_{\mathrm{loc}}(\Gamma) &=& \left\{\Phi(\xi^0 e^{t a_1(0,I_*)}, 0, \mu, \nu) : \left|\eta^0\right| < \delta,\ \mu^2 + \nu^2 = I_*\right\}. \end{array}
$$

We finish this introduction summarizing this result.

Corollary 1.1 *Consider system* (1.15) *where the origin is a saddle-center equilibrium point (case* (*iii*) *in Theorem* 1.3*) and let* N *and* $\widehat{B}$*, as in* (1.5*b*) , (1.7*b*)*, be the analytic vector fields provided by Theorem 1.2. Assume that the (analytic) function* $I \mapsto \widehat{b}_2(0, I)$*, defined in a neighborhood of the origin, does not vanish identically (so system* (1.15) *is neither Hamiltonian nor reversible). Thus, every positive zero of* $\widehat{b}_2(0, *)$ *gives rise to a hyperbolic periodic orbit of system* (1.15)*. Moreover, parameterizations for the (local) stable and unstable invariant manifolds associated to this periodic orbit are given by* (1.18)*.*

§2 Proof of the Main Theorem

§2.1 The formal solution: a first approach

It is worth noting that both cases, the origin being a saddle-focus or being a saddle-center, can be treated formally with the same argument. Moreover, we will deal first with the case of a *complex* ΨNF and will derive subsequently the case of a *real* ΨNF. Indeed, let us assume that we have complexified the original variables in such a way that the new (complex) matrix Λ is diagonal. Under this common approach, we will refer often to $\{\pm\lambda_1,\ \pm\lambda_2\}$ as the characteristic exponents of the origin, meaning $\{\pm\lambda \pm \mathrm{i}\alpha\}$ in the first case and $\{\pm\lambda,\ \pm\mathrm{i}\alpha\}$ in the second one, respectively, always with $\lambda, \alpha > 0$. Moreover, it is not difficult to check that with such unified notation the vector fields N and $\widehat{B}$ take the same form (1.5a) and (1.7a), respectively, in both cases. This will be their formal aspect along this proof if nothing against is explicitly said.

The sketch of the proof follows the standard pattern: first, we will look for a formal solution of equation

$$D\Phi\ N + \widehat{B} = F \circ \Phi \tag{1.19}$$

by means of a recurrent scheme, that will consist on two steps, an initial approach and a final refinement. Later on, it will be introduced a norm which will allow us to establish the convergence of the functions involved.

Thus, let us start with the first part. We recall that $\widehat{G}$ denotes that G is formed by formal power series beginning with terms of order at least 2. Now, since the linear part of $F(X) = \Lambda X + \widehat{F}(X)$ (or shorter, $F = \Lambda + \widehat{F}$) is in normal form, we have that the linear part of N is just Λ (notice that Λ represents also the complex matrix of eigenvalues $\pm\lambda_1,\ \pm\lambda_2$) . Writing $\Phi = \mathrm{id} + \widehat{\Phi}$ and $N = \Lambda + \widehat{N}$, equation (1.19) becomes

$$D\widehat{\Phi}\ N - \Lambda\widehat{\Phi} = \widehat{F} \circ \Phi - \widehat{N} - \widehat{B}. \tag{1.20}$$

Assume that we already know $\widehat{\Phi}$, $\widehat{N}$ and $\widehat{B}$ up to some order K and let us see which difficulties involves the computation of the terms of order $K+1$ of $\widehat{\Phi}$. From equation (1.20) we realize that we only have to consider the terms up to order $K+1$ of equation

$$D\widehat{\Phi}\ N - \Lambda\widehat{\Phi} = \widehat{H}. \tag{1.21}$$

where $\widehat{H} = \widehat{F} \circ \Phi$ only contains terms up to order K of $\widehat{\Phi}$. The terms in $\widehat{N}$ and $\widehat{B}$ of order $K+1$ will be determined later. By direct computation, writing

$$\widehat{\Phi} = (\widehat{\phi}^{(1)}, \widehat{\phi}^{(2)}, \widehat{\phi}^{(3)}, \widehat{\phi}^{(4)}), \qquad \widehat{H} = (\widehat{h}^{(1)}, \widehat{h}^{(2)}, \widehat{h}^{(3)}, \widehat{h}^{(4)}),$$

with

$$\begin{aligned} \widehat{\phi}^{(i)}(\xi,\eta,\mu,\nu) &= \sum \phi^{(i)}_{jk\ell m}\ \xi^j \eta^k \mu^\ell \nu^m, \\ \widehat{h}^{(i)}(\xi,\eta,\mu,\nu) &= \sum h^{(i)}_{jk\ell m}\ \xi^j \eta^k \mu^\ell \nu^m, \end{aligned}$$

for $i = 1, \dots, 4$, and using that

$$N = \begin{pmatrix} \xi a_1(\xi\eta, \mu\nu) \\ -\eta a_1(\xi\eta, \mu\nu) \\ \mu a_2(\xi\eta, \mu\nu) \\ -\nu a_2(\xi\eta, \mu\nu) \end{pmatrix} = \begin{pmatrix} \xi\lambda_1 + \cdots \\ -\eta\lambda_1 + \cdots \\ \mu\lambda_2 + \cdots \\ -\nu\lambda_2 + \cdots \end{pmatrix}, \tag{1.22}$$

the terms up to order $K+1$ of equation (1.21) come from the following system,

$$\begin{aligned} \left(\xi\widehat{\phi}^{(1)}_\xi - \eta\widehat{\phi}^{(1)}_\eta\right) a_1(\xi\eta,\mu\nu) + \left(\mu\widehat{\phi}^{(1)}_\mu - \nu\widehat{\phi}^{(1)}_\nu\right) a_2(\xi\eta,\mu\nu) - \lambda_1\widehat{\phi}^{(1)} &= \widehat{h}^{(1)}, \\ \left(\xi\widehat{\phi}^{(2)}_\xi - \eta\widehat{\phi}^{(2)}_\eta\right) a_1(\xi\eta,\mu\nu) + \left(\mu\widehat{\phi}^{(2)}_\mu - \nu\widehat{\phi}^{(2)}_\nu\right) a_2(\xi\eta,\mu\nu) + \lambda_1\widehat{\phi}^{(2)} &= \widehat{h}^{(2)}, \\ \left(\xi\widehat{\phi}^{(3)}_\xi - \eta\widehat{\phi}^{(3)}_\eta\right) a_1(\xi\eta,\mu\nu) + \left(\mu\widehat{\phi}^{(3)}_\mu - \nu\widehat{\phi}^{(3)}_\nu\right) a_2(\xi\eta,\mu\nu) - \lambda_2\widehat{\phi}^{(3)} &= \widehat{h}^{(3)}, \\ \left(\xi\widehat{\phi}^{(4)}_\xi - \eta\widehat{\phi}^{(4)}_\eta\right) a_1(\xi\eta,\mu\nu) + \left(\mu\widehat{\phi}^{(2)}_\mu - \nu\widehat{\phi}^{(4)}_\nu\right) a_2(\xi\eta,\mu\nu) + \lambda_2\widehat{\phi}^{(4)} &= \widehat{h}^{(4)}, \end{aligned}$$

where $\widehat{\phi}_\xi$ represents $\frac{\partial\widehat{\phi}}{\partial\xi}$, etc. Hence, since $a_s(\xi\eta,\mu\nu) = \lambda_s + \cdots$, the terms of order $K+1$ of $\widehat{\Phi}$ have to satisfy

$$
\begin{aligned}
\phi^{(1)}_{jk\ell m} &= \frac{h^{(1)}_{jk\ell m}}{\lambda_1(j-k-1)+\lambda_2(\ell-m)}, &&\text{if } j\neq k+1 \text{ or } \ell\neq m,\\
\phi^{(2)}_{jk\ell m} &= \frac{h^{(2)}_{jk\ell m}}{\lambda_1(j-k+1)+\lambda_2(\ell-m)}, &&\text{if } k\neq j+1 \text{ or } \ell\neq m,\\
\phi^{(3)}_{jk\ell m} &= \frac{h^{(3)}_{jk\ell m}}{\lambda_1(j-k)+\lambda_2(\ell-m-1)}, &&\text{if } j\neq k \text{ or } \ell\neq m+1,\\
\phi^{(4)}_{jk\ell m} &= \frac{h^{(4)}_{jk\ell m}}{\lambda_1(j-k)+\lambda_2(\ell-m+1)}, &&\text{if } j\neq k \text{ or } m\neq \ell+1.
\end{aligned}
\tag{1.23}
$$

It is clear from these equations that terms of the form

$$
\begin{pmatrix}
\xi\sum\phi^{(1)}_{k+1,kmm}\ (\xi\eta)^k(\mu\nu)^m\\[2ex]
\eta\sum\phi^{(2)}_{j,j+1,\ell\ell}\ (\xi\eta)^j\mu\nu)^\ell\\[2ex]
\mu\sum\phi^{(3)}_{kk,m+1,m}\ (\xi\eta)^k(\mu\nu)^m\\[2ex]
\nu\sum\phi^{(4)}_{jj,\ell,\ell+1}\ (\xi\eta)^j(\mu\nu)^\ell
\end{pmatrix}
\tag{1.24}
$$

cannot be determined and remain in principle arbitrary. In terms of simply linear algebra this amounts to say that the transformation Φ is completely determined once it has been fixed its projection on a suitable vectorial subspace, called *resonant subspace.*

§2.2 Definition of the projections

The type of coefficients appearing in expression (1.24) and the remarks above motivate the following definition.

Definition 1 *Given a formal series* $h(\xi,\eta,\mu,\nu) = \sum h_{jk\ell m}\ \xi^j\eta^k\mu^\ell\nu^m$, *we define the projections*

$$
\begin{aligned}
P_1h &:= \xi\sum_{k\geq 0,\ m\geq 1} h_{k+1,kmm}\ (\xi\eta)^k(\mu\nu)^m,\\
P_2h &:= \eta\sum_{j\geq 0,\ \ell\geq 1} h_{j,j+1,\ell\ell}\ (\xi\eta)^j(\mu\nu)^\ell,\\
P_3h &:= \mu\sum_{k\geq 1,\ m\geq 0} h_{kk,m+1,m}\ (\xi\eta)^k(\mu\nu)^m,\\
P_4h &:= \nu\sum_{j\geq 1,\ \ell\geq 0} h_{jj\ell,\ell+1}\ (\xi\eta)^j(\mu\nu)^\ell.
\end{aligned}
$$

Analogously, if $H = (h^{(1)},h^{(2)},h^{(3)},h^{(4)})$ *is a (formal) vector field we define*

$$\mathcal{P}H := (P_1h^{(1)},P_2h^{(2)},P_3h^{(3)},P_4h^{(4)})$$

and $\mathcal{R}H := H - \mathcal{P}H$.

As it has been noticed before, $\mathcal{P}\widehat{\Phi}$ corresponds to the terms which remain arbitrary from the solution of equation (1.21). Moreover, vector fields N and $\widehat{B}$ are invariant under the action of $\mathcal{P}$. This property will be used in the solution of equation (1.20). In this sense, we have the following lemma, whose proof is omitted since it consists on straightforward computations.

Lemma 1.1 *Given* $N = \Lambda + \widehat{N}$ *of the form* (1.22)*, the operator* $\mathcal{L}_N$ *defined as*

$$\mathcal{L}_N\Psi := D\Psi\ N - \Lambda\Psi \tag{1.25}$$

satisfies the following properties.

(*i*) $\mathcal{L}_N\Psi$ *is linear with respect to* Ψ *and* N*, that is*

$$\mathcal{L}_N\left(\Psi + \Psi'\right) = \mathcal{L}_N\Psi + \mathcal{L}_N\Psi' \qquad\qquad \mathcal{L}_{N+N'}\Psi = \mathcal{L}_N\Psi + \mathcal{L}_{N'}\Psi.$$

(*ii*) $\mathcal{L}_N$ *preserves order, that is,* $\mathcal{L}_N\Psi$ *and* Ψ *start with terms in* (ξ,η,μ,ν) *of the same order.*

(*iii*) *The projections* $\mathcal{P}$ *and* $\mathcal{R}$ *commute with* $\mathcal{L}_N$*, that is,*

$$\mathcal{P}\left(\mathcal{L}_N\Psi\right) = \mathcal{L}_N\left(\mathcal{P}\Psi\right), \qquad\qquad \mathcal{R}\left(\mathcal{L}_N\Psi\right) = \mathcal{L}_N\left(\mathcal{R}\Psi\right).$$

§2.3 The recurrent scheme

Let us come back to the solution of equation (1.20). Having in mind the definition of the operator $\mathcal{L}_N$ it can be written as

$$\mathcal{L}_N\widehat{\Phi} = \widehat{F}\circ\Phi - \widehat{N} - \widehat{B}, \tag{1.26}$$

which is of type (1.21) provided we take $\widehat{H} = \widehat{F}\circ\Phi - \widehat{N} - \widehat{B}$. In a first approach to this kind of equations we have shown that they could be solved recurrently for those terms in $\Phi = \mathrm{id} + \widehat{\Phi}$ of type $\mathcal{R}\widehat{\Phi}$, remaining those of the form $\mathcal{P}\widehat{\Phi}$ arbitrary. This fact suggests the idea of splitting the transformation we are looking for, Φ, into $\mathrm{id} + \mathcal{P}\widehat{\Phi} + \mathcal{R}\widehat{\Phi}$, to determine $\mathcal{R}\widehat{\Phi}$ from equation (1.26) and to choose a suitable value for $\mathcal{P}\widehat{\Phi}$.

Remark 6 *In Normal Form theory it is standard to set* $\mathcal{P}\widehat{\Phi} = 0$ *in order to simplify the computations. However, it could be useful to take advantage of this freedom in some concrete situations.*

Applying $\mathcal{R}$ onto equation (1.26),

$$\mathcal{R}\left(\mathcal{L}_N\widehat{\Phi}\right) = \mathcal{R}\left(\widehat{F}(\Phi)\right) - \mathcal{R}\widehat{N} - \mathcal{R}\widehat{B},$$

using Lemma 1.1 and taking into account that $\mathcal{R}\widehat{N} = \mathcal{R}\widehat{B} = 0$ if $\widehat{N}$ and $\widehat{B}$ are assumed to be of the form (1.5*a*) and (1.7*a*), respectively, we obtain the equation

$$\mathcal{L}_N\left(\mathcal{R}\widehat{\Phi}\right) = \mathcal{R}\left(\widehat{F}(\Phi)\right). \tag{1.27}$$

On the other hand, applying now $\mathcal{P}$ onto (1.26), taking again into account Lemma 1.1, the fact that $\mathcal{P}\widehat{N} = \widehat{N}$, $\mathcal{P}\widehat{B} = \widehat{B}$ and choosing $\mathcal{P}\widehat{\Phi} \equiv 0$, it follows that

$$\widehat{N} + \widehat{B} = \mathcal{P}\left(\widehat{F}(\Phi)\right). \tag{1.28}$$

A usual way to deal with such kind of equations is to consider it as a fixed point problem. Thus, we can set

$$\mathcal{P}\widehat{\Phi} \equiv 0,$$

take initial values

$$\Phi^{(1)} = \mathrm{id}, \quad N^{(1)} = \Lambda, \quad \widehat{B}^{(1)} = 0 \tag{1.29}$$

and obtain, recurrently,

$$\begin{array}{rcl} \Phi^{(K+1)} & = & \mathrm{id} + \mathcal{R}\widehat{\Phi}^{(K+1)} \\ N^{(K+1)} & = & \Lambda + \widehat{N}^{(K+1)} \\ \widehat{B}^{(K+1)} & & \end{array} \tag{1.30}$$

from equations

$$\mathcal{L}_{N^{(K)}}\left(\mathcal{R}\widehat{\Phi}^{(K+1)}\right) = \mathcal{R}\left(\widehat{F}\left(\Phi^{(K)}\right)\right) \tag{1.31}$$

$$\widehat{N}^{(K+1)} + \widehat{B}^{(K+1)} = \mathcal{P}\left(\widehat{F}\left(\Phi^{(K)}\right)\right). \tag{1.32}$$

We will see now how these two equations can be solved formally.

§2.3.1 Solution of a $\mathcal{L}_N(\mathcal{R}\widehat{\Psi}) = \mathcal{R}\widehat{H}$-type equation

Assuming that we know the coefficients of N and $\mathcal{R}\widehat{H}$ up to a given order K, the coefficients of $\mathcal{R}\widehat{\Psi}$ of the same order will be determined from

$$\mathcal{L}_N\left(\mathcal{R}\widehat{\Psi}\right) = \mathcal{R}\widehat{H}. \tag{1.33}$$

Indeed, writing

$$\mathcal{R}\widehat{\Psi} = \begin{pmatrix} \widehat{\psi}_1(\xi,\eta,\mu,\nu) \\ \widehat{\psi}_2(\xi,\eta,\mu,\nu) \\ \widehat{\psi}_3(\xi,\eta,\mu,\nu) \\ \widehat{\psi}_4(\xi,\eta,\mu,\nu) \end{pmatrix}, \qquad \mathcal{R}\widehat{H} = \begin{pmatrix} \widehat{h}_1(\xi,\eta,\mu,\nu) \\ \widehat{h}_2(\xi,\eta,\mu,\nu) \\ \widehat{h}_3(\xi,\eta,\mu,\nu) \\ \widehat{h}_4(\xi,\eta,\mu,\nu) \end{pmatrix}, \tag{1.34}$$

where

$$\begin{array}{rcl} \widehat{\psi}_w(\xi,\eta,\mu,\nu) & = & \displaystyle\sum_{j+k+\ell+m\geq 2} \psi^{(w)}_{jk\ell m}\,\xi^j\eta^k\mu^\ell\nu^m \\ \widehat{h}_w(\xi,\eta,\mu,\nu) & = & \displaystyle\sum_{j+k+\ell+m\geq 2} \widehat{h}^{(w)}_{jk\ell m}\,\xi^j\eta^k\mu^\ell\nu^m \end{array}$$

for $w = 1, \ldots, 4$, and taking into account that

$$N(\xi,\eta,\mu,\nu) = \begin{pmatrix} \xi a_1(\xi\eta,\mu\nu) \\ -\eta a_1(\xi\eta,\mu\nu) \\ \mu a_2(\xi\eta,\mu\nu) \\ -\nu a_2(\xi\eta,\mu\nu) \end{pmatrix}$$

with $a_i(\xi\eta,\mu\nu) = \lambda_i + \widehat{a}_i(\xi\eta,\mu\nu)$, it follows that the left-hand side of (1.33) is equivalent to

$$D\left(\mathcal{R}\widehat{\Psi}\right) N - \Lambda\,\mathcal{R}\Psi =$$

$$\begin{pmatrix} \widehat{\psi}_{1,\xi} & \widehat{\psi}_{1,\eta} & \widehat{\psi}_{1,\mu} & \widehat{\psi}_{1,\nu} \\ \widehat{\psi}_{2,\xi} & \widehat{\psi}_{2,\eta} & \widehat{\psi}_{2,\mu} & \widehat{\psi}_{2,\nu} \\ \widehat{\psi}_{3,\xi} & \widehat{\psi}_{3,\eta} & \widehat{\psi}_{3,\mu} & \widehat{\psi}_{3,\nu} \\ \widehat{\psi}_{4,\xi} & \widehat{\psi}_{4,\eta} & \widehat{\psi}_{4,\mu} & \widehat{\psi}_{4,\nu} \end{pmatrix} \begin{pmatrix} \xi a_1 \\ -\eta a_1 \\ \mu a_2 \\ -\nu a_2 \end{pmatrix} - \begin{pmatrix} \lambda_1 \widehat{\psi}_1 \\ -\lambda_1 \widehat{\psi}_2 \\ \lambda_2 \widehat{\psi}_3 \\ -\lambda_2 \widehat{\psi}_4 \end{pmatrix} =$$

$$\begin{pmatrix} \left(\xi\widehat{\psi}_{1,\xi} - \eta\widehat{\psi}_{1,\eta}\right) a_1 + \left(\mu\widehat{\psi}_{1,\mu} - \nu\widehat{\psi}_{1,\nu}\right) a_2 - \lambda_1\widehat{\psi}_1 \\ \left(\xi\widehat{\psi}_{2,\xi} - \eta\widehat{\psi}_{2,\eta}\right) a_1 + \left(\mu\widehat{\psi}_{2,\mu} - \nu\widehat{\psi}_{2,\nu}\right) a_2 + \lambda_1\widehat{\psi}_2 \\ \left(\xi\widehat{\psi}_{3,\xi} - \eta\widehat{\psi}_{3,\eta}\right) a_1 + \left(\mu\widehat{\psi}_{3,\mu} - \nu\widehat{\psi}_{3,\nu}\right) a_2 - \lambda_2\widehat{\psi}_3 \\ \left(\xi\widehat{\psi}_{4,\xi} - \eta\widehat{\psi}_{4,\eta}\right) a_1 + \left(\mu\widehat{\psi}_{4,\mu} - \nu\widehat{\psi}_{4,\nu}\right) a_2 + \lambda_2\widehat{\psi}_4 \end{pmatrix} =$$

$$\begin{pmatrix} ((j-k-1)\lambda_1 + (\ell-m)\lambda_2) + \left(\xi\widehat{\psi}_{1,\xi} - \eta\widehat{\psi}_{1,\eta}\right)\widehat{a}_1 + \left(\mu\widehat{\psi}_{1,\mu} - \nu\widehat{\psi}_{1,\nu}\right)\widehat{a}_2 \\ ((j-k+1)\lambda_1 + (\ell-m)\lambda_2) + \left(\xi\widehat{\psi}_{2,\xi} - \eta\widehat{\psi}_{2,\eta}\right)\widehat{a}_1 + \left(\mu\widehat{\psi}_{2,\mu} - \nu\widehat{\psi}_{2,\nu}\right)\widehat{a}_2 \\ ((j-k)\lambda_1 + (\ell-m-1)\lambda_2) + \left(\xi\widehat{\psi}_{3,\xi} - \eta\widehat{\psi}_{3,\eta}\right)\widehat{a}_1 + \left(\mu\widehat{\psi}_{3,\mu} - \nu\widehat{\psi}_{3,\nu}\right)\widehat{a}_2 \\ ((j-k)\lambda_1 + (\ell-m+1)\lambda_2) + \left(\xi\widehat{\psi}_{4,\xi} - \eta\widehat{\psi}_{4,\eta}\right)\widehat{a}_1 + \left(\mu\widehat{\psi}_{4,\mu} - \nu\widehat{\psi}_{4,\nu}\right)\widehat{a}_2 \end{pmatrix}.$$

We can refer to this vector field, in short, as

$$\left(L_N^{(1)}\widehat{\psi}_1,\, L_N^{(2)}\widehat{\psi}_2,\, L_N^{(3)}\widehat{\psi}_3,\, L_N^{(4)}\widehat{\psi}_4\right)$$

and write its components, in formal power series expansions, as

$$L_N^{(w)}\widehat{\psi}_w(\xi,\eta,\mu,\nu) = \sum_{j+k+\ell+m\geq 2} \tilde{g}^{(w)}_{jk\ell m}(\xi\eta,\mu\nu)\,\psi^{(w)}_{jk\ell m}\,\xi^j\eta^k\mu^\ell\nu^m, \tag{1.35}$$

being

$$\tilde{g}^{(w)}_{jk\ell m}(\xi\eta,\mu\nu) := \gamma^{(w)}_{jk\ell m}(\lambda) + (j-k)\,\widehat{a}_1(\xi\eta,\mu\nu) + (\ell-m)\,\widehat{a}_2(\xi\eta,\mu\nu),$$

with

$$\gamma^{(w)}_{jk\ell m}(\lambda) := \begin{cases} (j-k-1)\lambda_1 + (\ell-m)\lambda_2 & \text{if } w=1 \\ (j-k+1)\lambda_1 + (\ell-m)\lambda_2 & \text{if } w=2 \\ (j-k)\lambda_1 + (\ell-m-1)\lambda_2 & \text{if } w=3 \\ (j-k)\lambda_1 + (\ell-m+1)\lambda_2 & \text{if } w=4. \end{cases}$$

Notice, from equation (1.35), that $\mathcal{L}_N$ acts on $\mathcal{R}\widehat{\Psi}$ *multiplying* each coefficient $\psi_{jk\ell m}$ by a function of the products $\xi\eta$ and $\mu\nu$. To take advantage of this feature we will express our formal series expansions in a more convenient way which will highlight those terms of the form $(\xi\eta)^p$ and $(\mu\nu)^q$. A similar idea was suggested in [25]. In our case it works as follows. For any component $\widehat{\psi}_w$ of $\mathcal{R}\widehat{\Psi}$ we have

$$\widehat{\psi}_w(\xi,\eta,\mu,\nu) = \sum_{j+k+\ell+m\geq 2} \psi^{(w)}_{jk\ell m}\, \xi^j \eta^k \mu^\ell \nu^m = \sum_{j+k+\ell+m\geq 2} \psi^{(w)}_{jk\ell m}\, \xi^{j-k}\, (\xi\eta)^k\, \mu^{\ell-m}\, (\mu\nu)^m\,. \tag{1.36}$$

Defining $p = j-k$, $q = \ell - m$ and taking into account that $j+k+\ell+m \geq 2$, $p+k \geq 0$ and $q+m \geq 0$, this expansion is equivalent to

$$\sum_{p,q\in\mathbb{Z}} \psi^{(w)}_{pq}(\xi\eta,\mu\nu)\, \xi^p \mu^q \tag{1.37}$$

where

$$\psi^{(w)}_{pq}(\xi\eta,\mu\nu) = \sum_{(k,m)\in Q_{pq}} \psi^{(w)}_{p+k,k,q+m,m}\, (\xi\eta)^k\, (\mu\nu)^m \tag{1.38}$$

and

$$Q_{pq} := \left\{ (k,m) \in (\mathbb{N}\cup\{0\})^2 : \begin{array}{ccc} k & \geq & \max\{0,-p\} \\ m & \geq & \max\{0,-q\} \end{array},\ k+m \geq 1 - \frac{p+q}{2} \right\}.$$

In the same way, for $\mathcal{R}\widehat{H}$ we get

$$\widehat{h}_w(\xi,\eta,\mu,\nu) = \sum_{p,q\in\mathbb{Z}} h^{(w)}_{pq}(\xi\eta,\mu\nu)\, \xi^p \mu^q$$

where

$$h^{(w)}_{pq}(\xi,\eta,\mu,\nu) = \sum_{(k,m)\in Q_{pq}} h^{(w)}_{p+k,k,q+m,m}\, (\xi\eta)^k\, (\mu\nu)^m\,.$$

With this notation formula (1.35) becomes

$$\sum_{p,q\in\mathbb{Z}} g^{(w)}_{pq}(\xi\eta,\mu\nu)\, \psi^{(w)}_{pq}(\xi\eta,\mu\nu)\, \xi^p \mu^q$$

where now

$$g^{(w)}_{pq}(\xi\eta,\mu\nu) := \Gamma^{(w)}_{pq}(\lambda) + p\,\widehat{a}_1(\xi\eta,\mu\nu) + q\,\widehat{a}_2(\xi\eta,\mu\nu)$$

being

$$\Gamma^{(w)}_{pq}(\lambda) := \begin{cases} (p-1)\lambda_1 + q\lambda_2 & \text{if } w=1 \\ (p+1)\lambda_1 + q\lambda_2 & \text{if } w=2 \\ p\lambda_1 + (q-1)\lambda_2 & \text{if } w=3 \\ p\lambda_1 + (q+1)\lambda_2 & \text{if } w=4. \end{cases} \tag{1.39}$$

Thus, equality

$$\mathcal{L}_N(\mathcal{R}\widehat{\Psi}) = \mathcal{R}\widehat{H}$$

gives rise to the equations

$$L_N^{(w)}\widehat{\psi}_w(\xi,\eta,\mu,\nu) = \widehat{h}_w(\xi,\eta,\mu,\nu)$$

or, in formal series expansions,

$$\sum_{p,q\in\mathbb{Z}} g_{pq}^{(w)}(\xi\eta,\mu\nu)\,\psi_{pq}^{(w)}(\xi\eta,\mu\nu)\,\xi^p\mu^q = \sum_{p,q\in\mathbb{Z}} h_{pq}^{(w)}(\xi\eta,\mu\nu)\,\xi^p\mu^q,$$

whose formal solution is given by

$$\widehat{\psi}_w(\xi,\eta,\mu,\nu) = \sum_{p,q\in\mathbb{Z}} \psi_{pq}^{(w)}(\xi\eta,\mu\nu)\,\xi^p\mu^q \tag{1.40}$$

with the functions $\psi_{pq}^{(w)}(\xi\eta,\mu\nu)$ coming from

$$\psi_{pq}^{(w)}(\xi\eta,\mu\nu) = \frac{h_{pq}^{(w)}(\xi\eta,\mu\nu)}{g_{pq}^{(w)}(\xi\eta,\mu\nu)} = \frac{h_{pq}^{(w)}(\xi\eta,\mu\nu)}{\Gamma_{pq}^{(w)}(\lambda) + p\,\widehat{a}_1(\xi\eta,\mu\nu) + q\,\widehat{a}_2(\xi\eta,\mu\nu)}, \tag{1.41}$$

for $w = 1,2,\ldots,4$ and $p,q\in\mathbb{Z}$. With this notation coefficients with $p=\pm1$ and $q=0$ or $p=0$ and $q=\pm1$ are those belonging to the projection $\mathcal{P}\widehat{\Psi}$.

§2.3.2 Solution of a $\widehat{N}+\widehat{B}=\mathcal{P}\widehat{H}$-type equation

As it has been done for equations of type $\mathcal{L}_N(\mathcal{R}\widehat{\Psi}) = \mathcal{R}\widehat{H}$ we are going to prove that equation $\widehat{N}+\widehat{B}=\mathcal{P}\widehat{H}$ determines uniquely the coefficients of $\widehat{N}$ and $\widehat{B}$ provided they are of type (1.5a) and (1.7a), respectively, and that $\widehat{H}$ is known. Thus, writing

$$\begin{aligned} \widehat{N} &= (\xi\widehat{a}_1, -\eta\widehat{a}_1, \mu\widehat{a}_2, -\nu\widehat{a}_2) \\ \widehat{B} &= \left(\xi\widehat{b}_1, \eta\widehat{b}_1, \mu\widehat{b}_2, \nu\widehat{b}_2\right), \\ \mathcal{P}\widehat{H} &= \left(\xi\widehat{h}_1, \eta\widehat{h}_2, \mu\widehat{h}_3, \nu\widehat{h}_4\right), \end{aligned} \tag{1.42}$$

where $\widehat{a}_i$, $\widehat{b}_i$ and $\widehat{h}_w$ are functions of $\xi\eta$ and $\mu\nu$, for $i=1,2$ and $w=1,2,\ldots,4$, the solution of this equation is given explicitly by

$$\begin{aligned} \widehat{a}_1 &= \tfrac{1}{2}\left(\widehat{h}_1 - \widehat{h}_2\right), & \widehat{b}_1 &= \tfrac{1}{2}\left(\widehat{h}_1 + \widehat{h}_2\right), \\ \widehat{a}_2 &= \tfrac{1}{2}\left(\widehat{h}_3 - \widehat{h}_4\right), & \widehat{b}_2 &= \tfrac{1}{2}\left(\widehat{h}_3 + \widehat{h}_4\right). \end{aligned} \tag{1.43}$$

Notice that the form of the functions $\widehat{a}_i$, $\widehat{b}_i$ and $\widehat{h}_w$ implies that $\mathcal{P}\widehat{H}$, $\widehat{N}$ and $\widehat{B}$ only contain terms of odd order.

§2.4 The recurrent scheme: an improvement

One of the features of this procedure is that it provides a constructive (and, therefore, implementable on a computer) way to determine $\widehat{\Phi}$, N and $\widehat{B}$. To do it we need to define (and allocate memory for them) data vectors representing these vector fields. Unfortunately, the scheme above implies to handle (and to recompute) the *complete* vectors storing $\widehat{\Phi}$, N and $\widehat{B}$, at any step of the process. This makes it slow and not much efficient. In this sense it is easy to refine it by paying attention on the order of the solutions of equations (1.31)–(1.32).

Before going on with this refinement, let us introduce some notation. We will denote $G = \mathcal{O}_{[K]}$ if G is a homogeneous polynomial in the spatial variables ξ, η, μ, ν of order exactly K. Besides, we will write $G = \mathcal{O}_K$ if G contains only terms of order greater or equal than K in these variables and $G = \mathcal{O}_{\leq K}$ if all the terms in G are of order less or equal than K. Thus, we have

Lemma 1.2 *At any step* $K \geq 1$ *of the process* (1.29)–(1.32) *the following estimates hold*

$$\begin{array}{lcll} \mathcal{R}\widehat{\Phi}^{(K+1)} & - & \mathcal{R}\widehat{\Phi}^{(K)} & = \mathcal{O}_{K+1}, \\ \widehat{N}^{(K+1)} & - & \widehat{N}^{(K)} & = \mathcal{O}_{K+1}, \\ \widehat{B}^{(K+1)} & - & \widehat{B}^{(K)} & = \mathcal{O}_{K+1}. \end{array}$$

Proof. We proceed inductively.

- For $K = 1$, from the initial values (1.29)

  $$\Phi^{(1)} = \mathrm{id}, \quad N^{(1)} = \Lambda, \quad \widehat{B}^{(1)} = 0$$

  one has that $\mathcal{R}\widehat{\Phi}^{(1)} = \widehat{N}^{(1)} = \widehat{B}^{(1)} = 0$. In this case formula (1.29) reads

  $$\mathcal{L}_{N^{(1)}}\left(\mathcal{R}\widehat{\Phi}^{(2)}\right) = \mathcal{R}\left(\widehat{F}\left(\Phi^{(1)}\right)\right). \tag{1.44}$$

  Its right-hand side becomes

  $$\mathcal{R}\left(\widehat{F}\left(\Phi^{(1)}\right)\right) = \mathcal{R}\left(\widehat{F}\left(\mathrm{id}\right)\right) = \mathcal{R}\widehat{F}$$

  and its left-hand side

  $$\mathcal{L}_{N^{(1)}}\left(\mathcal{R}\widehat{\Phi}^{(2)}\right) = D\left(\mathcal{R}\widehat{\Phi}^{(2)}\right)\Lambda - \Lambda\,\mathcal{R}\widehat{\Phi}^{(2)} = \left[\Lambda, \mathcal{R}\widehat{\Phi}^{(2)}\right],$$

  where $[G,H] = (DH)\,G - (DG)\,H$ stands for the Lie bracket of the vector fields G and H. We recall also that, abusing of the notation, we denote with the same symbol Λ the matrix Λ and the vector field $\Lambda\,\mathrm{id}$. Thus formula (1.44) becomes

  $$\left[\Lambda, \mathcal{R}\widehat{\Phi}^{(2)}\right] = \mathcal{R}\widehat{F}.$$

  Using that $\mathcal{R}\widehat{F} = \mathcal{O}_2$ and that the Lie bracket preserves the order it follows that $\mathcal{R}\widehat{\Phi}^{(2)} = \mathcal{O}_2$ and, consequently, $\mathcal{R}\widehat{\Phi}^{(2)} - \mathcal{R}\widehat{\Phi}^{(1)} = \mathcal{O}_2$.
  With respect to $\widehat{N}$ and $\widehat{B}$, we have now that

  $$\widehat{N}^{(2)} + \widehat{B}^{(2)} = \mathcal{P}\left(\widehat{F}\left(\Phi^{(1)}\right)\right) = \mathcal{P}\left(\widehat{F}\left(\mathrm{id}\right)\right) = \mathcal{P}\widehat{F}.$$

From formulas (1.43) and having in mind that $\mathcal{P}\widehat{F} = \mathcal{O}_3$ it turns out that $\widehat{N}^{(2)}$, $\widehat{B}^{(2)} = \mathcal{O}_3$ so, in particular,

$$\widehat{N}^{(2)} - \widehat{N}^{(1)} = \mathcal{O}_2, \qquad \widehat{B}^{(2)} - \widehat{B}^{(1)} = \mathcal{O}_2.$$

- Assume now, as induction hypothesis, that

$$\begin{array}{lcllcl} \mathcal{R}\widehat{\Phi}^{(K)} & - & \mathcal{R}\widehat{\Phi}^{(K-1)} & = & \mathcal{O}_K, \\ \widehat{N}^{(K)} & - & \widehat{N}^{(K-1)} & = & \mathcal{O}_K, \\ \widehat{B}^{(K)} & - & \widehat{B}^{(K-1)} & = & \mathcal{O}_K. \end{array} \tag{1.45}$$

hold for any arbitrary step $K-1$. To avoid a cumbersome notation, let us denote

$$\begin{aligned} \mathcal{R}\Delta\widehat{\Phi}^{(K)} &:= \mathcal{R}\widehat{\Phi}^{(K+1)} - \mathcal{R}\widehat{\Phi}^{(K)} \\ \Delta\widehat{N}^{(K)} &:= \widehat{N}^{(K+1)} - \widehat{N}^{(K)} \\ \Delta\widehat{B}^{(K)} &:= \widehat{B}^{(K+1)} - \widehat{B}^{(K)}. \end{aligned}$$

With such notation, hypothesis (1.45) becomes just

$$\mathcal{R}\Delta\widehat{\Phi}^{(K-1)} = \mathcal{O}_K, \quad \Delta\widehat{N}^{(K-1)} = \mathcal{O}_K, \quad \Delta\widehat{B}^{(K-1)} = \mathcal{O}_K$$

and we want to prove that

$$\mathcal{R}\Delta\widehat{\Phi}^{(K)} = \mathcal{O}_{K+1}, \quad \Delta\widehat{N}^{(K)} = \mathcal{O}_{K+1}, \quad \Delta\widehat{B}^{(K)} = \mathcal{O}_{K+1}.$$

Subtracting equation (1.31) for two consecutive steps $K-1$ and K we get

$$\begin{aligned} \mathcal{L}_{N^{(K)}}\left(\mathcal{R}\widehat{\Phi}^{(K+1)}\right) - \mathcal{L}_{N^{(K-1)}}\left(\mathcal{R}\widehat{\Phi}^{(K)}\right) = \\ \mathcal{R}\left(\widehat{F}\left(\Phi^{(K)}\right)\right) - \mathcal{R}\left(\widehat{F}\left(\Phi^{(K-1)}\right)\right). \end{aligned} \tag{1.46}$$

Using the linearity of the operator $\mathcal{L}$ (see Lemma 1.1) we have

$$\begin{aligned} \mathcal{L}_{N^{(K)}}\left(\mathcal{R}\widehat{\Phi}^{(K+1)}\right) &= \mathcal{L}_{N^{(K)}}\left(\mathcal{R}\widehat{\Phi}^{(K)}\right) + \mathcal{L}_{N^{(K)}}\left(\mathcal{R}\Delta\widehat{\Phi}^{(K)}\right) = \\ & \mathcal{L}_{N^{(K-1)}}\left(\mathcal{R}\widehat{\Phi}^{(K)}\right) + \mathcal{L}_{\Delta\widehat{N}^{(K-1)}}\left(\mathcal{R}\widehat{\Phi}^{(K)}\right) + \mathcal{L}_{N^{(K)}}\left(\mathcal{R}\Delta\widehat{\Phi}^{(K)}\right), \end{aligned}$$

so the left-hand side of equation (1.46) becomes

$$\begin{aligned} &\mathcal{L}_{N^{(K)}}\left(\mathcal{R}\widehat{\Phi}^{(K+1)}\right) - \mathcal{L}_{N^{(K-1)}}\left(\mathcal{R}\widehat{\Phi}^{(K)}\right) = \\ &\mathcal{L}_{\Delta\widehat{N}^{(K-1)}}\left(\mathcal{R}\widehat{\Phi}^{(K)}\right) + \mathcal{L}_{N^{(K)}}\left(\mathcal{R}\Delta\widehat{\Phi}^{(K)}\right) = \\ &D\left(\mathcal{R}\widehat{\Phi}^{(K)}\right)\Delta\widehat{N}^{(K-1)} + \mathcal{L}_{N^{(K)}}\left(\mathcal{R}\Delta\widehat{\Phi}^{(K)}\right) = \\ &\mathcal{L}_{N^{(K)}}\left(\mathcal{R}\Delta\widehat{\Phi}^{(K)}\right) + \mathcal{O}_{K+1}, \end{aligned}$$

where we have taken into account that $\Delta\widehat{N}^{(K-1)}$ has null linear part. With respect to the right-hand side of (1.46), expanding it in Taylor series, it turns out that

$$\begin{aligned}
&\widehat{F}\left(\Phi^{(K)}\right) - \widehat{F}\left(\Phi^{(K-1)}\right) = \\
&D\widehat{F}\left(\Phi^{(K-1)}\right)\mathcal{R}\Delta\widehat{\Phi}^{(K-1)} + \\
&\sum_{j\geq 2}\frac{1}{j!}D^j\widehat{F}\left(\Phi^{(K-1)}\right)\left(\mathcal{R}\Delta\widehat{\Phi}^{(K-1)}\right)^j.
\end{aligned} \tag{1.47}$$

Since $D\widehat{F}(\Phi^{(K-1)}) = \mathcal{O}_1$ and $\mathcal{R}\Delta\widehat{\Phi}^{(K-1)} = \mathcal{O}_K$, by induction hypothesis, it turns out that

$$\mathcal{R}\left(\widehat{F}(\Phi^{(K)})\right) - \mathcal{R}\left(\widehat{F}(\Phi^{(K-1)})\right) = \mathcal{O}_{K+1}.$$

Consequently, equation (1.46) becomes of type

$$\mathcal{L}_{N^{(K)}}\left(\mathcal{R}\Delta\widehat{\Phi}^{(K)}\right) = \mathcal{O}_{K+1}.$$

Since $\mathcal{L}$ preserves the order (see Lemma 1.1) it follows that $\mathcal{R}\Delta\widehat{\Phi}^{(K)} = \mathcal{O}_{K+1}$.

Concerning $\Delta\widehat{N}^{(K)}$ and $\Delta\widehat{B}^{(K)}$ we proceed in the same way. Subtracting formula (1.32) for $K-1$ and K one obtains that

$$\Delta\widehat{N}^{(K)} + \Delta\widehat{B}^{(K)} = \mathcal{P}\left(\widehat{F}\left(\Phi^{(K)}\right) - \widehat{F}\left(\Phi^{(K)}\right)\right).$$

From (1.47) we know that

$$\mathcal{P}\left(\widehat{F}\left(\Phi^{(K)}\right) - \widehat{F}\left(\Phi^{(K)}\right)\right) = \mathcal{O}_{K+1}$$

and therefore, using (1.43), it follows that $\Delta\widehat{N}^{(K)}$, $\Delta\widehat{B}^{(K)} = \mathcal{O}_{K+1}$, which concludes the proof.

□

An important consequence of this lemma is the reduction of the computational effort of the recurrent scheme: in the K-th step of our recurrent scheme the coefficients of order less or equal than K computed from the previous iteration will remain invariant. Therefore, from now onwards we will consider

$$\widehat{\Phi}^{(K+1)} = \mathcal{O}_{\leq K+1}, \qquad \widehat{N}^{(K+1)} = \mathcal{O}_{\leq K+1}, \qquad \widehat{B}^{(K+1)} = \mathcal{O}_{\leq K+1},$$

obtained from the equations (1.31)–(1.31) taken only up to order $K+1$

$$\left\{\mathcal{L}_{N^{(K)}}\left(\mathcal{R}\widehat{\Phi}^{(K+1)}\right)\right\}_{\leq K+1} = \left\{\mathcal{R}\left(\widehat{F}\left(\Phi^{(K)}\right)\right)\right\}_{\leq K+1} \tag{1.48}$$

$$\widehat{N}^{(K+1)} + \widehat{B}^{(K+1)} = \left\{\mathcal{P}\left(\widehat{F}(\Phi^{(K)})\right)\right\}_{\leq K+1}. \tag{1.49}$$

This implies in particular that

$$\Delta\widehat{\Phi}^{(K)} = \mathcal{O}_{[K+1]}, \qquad \Delta\widehat{N}^{(K)} = \mathcal{O}_{[K+1]}, \qquad \Delta\widehat{B}^{(K)} = \mathcal{O}_{[K+1]}.$$

From a computational point of view, at any step K of this recurrent scheme it would be just necessary to compute these incremental terms. Besides, notice that since $\widehat{N}^{(K)}$ and $\widehat{B}^{(K)}$ contain only terms of odd order, it follows that

$$\widehat{N}^{(2J)} - \widehat{N}^{(2J-1)} = \widehat{B}^{(2J)} - \widehat{B}^{(2J-1)} = 0 \tag{1.50}$$

or, equivalently,

$$\Delta\widehat{N}^{(2J-1)} = \Delta\widehat{B}^{(2J-1)} = 0,$$

for any $J \geq 2$.

§2.5 Convergence of the recurrent scheme

§2.5.1 Definition of the norm, estimates and technical lemmas

The domains we consider are those of type

$$\overline{\mathcal{D}_\sigma} = \{z = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n \;:\; |z_j| \leq \sigma \quad j = 1, 2, \dots, n\},$$

where $r > 0$ and $|\cdot|$ denotes the standard modulo. By an analytic function $f(z)$ on $\overline{\mathcal{D}_\sigma}$ we mean a function with Taylor expansion

$$f(z) = \sum_{\alpha \in (\mathbb{N}\cup\{0\})^n} f_\alpha z^\alpha \tag{1.51}$$

(absolutely) convergent for any $z \in \overline{\mathcal{D}_\sigma}$. We use the standard multi-index notation: if

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in (\mathbb{N}\cup\{0\})^n \quad \text{and} \quad z = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n$$

one sets

$$\begin{aligned} |\alpha| &= \alpha_1 + \alpha_2 + \cdots + \alpha_n \\ \alpha! &= \alpha_1!\,\alpha_2!\cdots\alpha_n! \\ z^\alpha &= z_1^{\alpha_1} z_2^{\alpha_2}\cdots z_n^{\alpha_n} \\ D^\alpha &= \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1}\partial z_2^{\alpha_2}\cdots\partial z_n^{\alpha_n}} \end{aligned}$$

and $1 = (1, 1, \dots, 1)$, $0 = (0, 0, \dots, 0)$. Moreover, in $(\mathbb{N}\cup\{0\})^n$ we consider the following partial ordering:

$$\alpha \geq \beta \quad \text{whenever} \quad \alpha_j \geq \beta_j \;\; \text{for } j = 1, 2, \dots, n.$$

Given a function f analytic on $\overline{\mathcal{D}_\sigma}$ we consider the following norms: the *supremum norm*

$$\|f\|_{\infty,\sigma} = \sup_{z\in\overline{\mathcal{D}_\sigma}} |f(z)|$$

and the 1-*norm*

$$\|f\|_{1,\sigma} = \sum_{|\alpha|\geq 0} |f_\alpha|\,\sigma^{|\alpha|}. \tag{1.52}$$

For a vector field $F = (f_1, f_2, \ldots, f_n) : \overline{\mathcal{D}_\sigma} \subseteq \mathbb{C}^n \longmapsto \mathbb{C}^n$ we define

$$\|F\|_{\infty,\sigma} = \sup_{i=1,\ldots,n} \|f_i\|_{\infty,\sigma}\,, \qquad \|F\|_{1,\sigma} = \frac{1}{n}\sum_{i=1,\ldots,n} \|f_i\|_{1,\sigma} \tag{1.53}$$

and analogously if $F : \overline{\mathcal{D}_\sigma} \subseteq \mathbb{C}^n \longmapsto \mathbb{M}_{n,n}(\mathbb{C}^n)$. The next lemma list some properties of these norms. We omit its proof since it is standard.

Lemma 1.3 *Let f be an analytic function on $\overline{\mathcal{D}_{\sigma_1}}$ satisfying that $f(0) = 0$ and assume $0 < \sigma_2 \leq \sigma_1$. Then, the following properties hold:*

(i) $\|f\|_{\infty,\sigma_2} \leq \|f\|_{1,\sigma_2}$.

(ii) Let $\Phi = (\phi_1, \phi_2, \ldots, \phi_n) : \overline{\mathcal{D}_{\sigma_2}} \subseteq \mathbb{C}^n \longmapsto \mathbb{C}^n$ be analytic on $\overline{\mathcal{D}_{\sigma_2}}$ and satisfying that $\|\Phi\|_{\infty,\sigma_2} \leq \sigma_1$ (that is, $\Phi\left(\overline{\mathcal{D}_{\sigma_2}}\right) \subseteq \overline{\mathcal{D}_{\sigma_1}}$). Then we have

$$\|f \circ \Phi\|_{1,\sigma_2} \leq \|f\|_{1,\sigma_1}\,.$$

If $F = (f_1, \ldots, f_n)$ is an analytic vector field on $\overline{\mathcal{D}_{\sigma_1}}$ the same estimate holds for $\|F \circ \Phi\|_{1,\sigma_2}$.

(iii) Let g be an analytic function on $\overline{\mathcal{D}_\sigma}$ satisfying that $|g(z)| \geq C\ \forall z \in \overline{\mathcal{D}_\sigma}$. Then, one has that

$$\left\|\frac{1}{g}\right\|_{1,\sigma} \leq \frac{1}{C}.$$

(iv) If $G_{[K]} = \mathcal{O}_{[K]}$ and $H_{[L]} = \mathcal{O}_{[L]}$ are homogeneous polynomials of orders K and L, respectively, with $K \neq L$, then

$$\left\|G_{[K]} + H_{[L]}\right\|_{1,\sigma_2} = \left\|G_{[K]}\right\|_{1,\sigma_2} + \left\|H_{[L]}\right\|_{1,\sigma_2}.$$

From this point up to the end of this section we will prove some technical results which will be used during the proof of the convergence of the recurrent scheme introduced in Sections §2.3 and §2.4. In particular, next lemma provides a lower bound for $|q_1\lambda_1 + q_2\lambda_2|$ which works in both cases, when the equilibrium point is a *saddle-center* or a *saddle-focus* (whose characteristic exponents are given by $\{\pm\lambda,\ \mathrm{i}\alpha\}$ and $\{\pm\lambda \pm \mathrm{i}\alpha\}$, respectively).

Lemma 1.4 *Let us define*

$$\omega_\infty = \omega_\infty(\Lambda) := \min\{\lambda,\ \alpha\} \tag{1.54}$$

where we assume $\lambda, \alpha > 0$. Then, we have that

$$|q_1\lambda_1 + q_2\lambda_2| \geq \left(\sqrt{q_1^2 + q_2^2}\right)\omega_\infty$$

for any $q_1, q_2 \in \mathbb{Z}$.

Proof. We proceed separately. Thus,

(*a*) *Saddle-center case*: as it has been mentioned above, we have $\lambda_1 = \lambda$ and $\lambda_2 = \mathrm{i}\alpha$ so

$$|q_1\lambda_1 + q_2\lambda_2| = \sqrt{q_1^2\lambda^2 + q_2^2\alpha^2} \geq \left(\sqrt{q_1^2 + q_2^2}\right) \min\{\lambda,\, \alpha\} = \left(\sqrt{q_1^2 + q_2^2}\right) \omega_\infty.$$

(*b*) *Saddle-focus case*: now we have $\lambda_1 = \lambda + \mathrm{i}\alpha$ and $\lambda_2 = \lambda - \mathrm{i}\alpha$. Then,

$$|q_1\lambda_1 + q_2\lambda_2| = |(q_1 + q_2)\lambda + (q_1 - q_2)\mathrm{i}\alpha| = \sqrt{(q_1 + q_2)^2\lambda^2 + (q_1 - q_2)^2\alpha^2}.$$

If $q_1 q_2 > 0$, using that $|q_1| + |q_2| \geq \sqrt{q_1^2 + q_2^2}$, one obtains that

$$\sqrt{(q_1 + q_2)^2\lambda^2 + (q_1 - q_2)^2\alpha^2} \geq \sqrt{(q_1 + q_2)^2\lambda^2} = (|q_1| + |q_2|)\,\lambda \geq \left(\sqrt{q_1^2 + q_2^2}\right) \omega_\infty.$$

On the other hand, if $q_1 q_2 < 0$ then

$$\sqrt{(q_1 + q_2)^2\lambda^2 + (q_1 - q_2)^2\alpha^2} \geq \sqrt{(q_1 - q_2)^2\alpha^2} = (|q_1| + |q_2|)\,\alpha \geq \left(\sqrt{q_1^2 + q_2^2}\right) \omega_\infty,$$

which proves the lemma.

□

Remark 7 *In fact,* ω_∞ *constitutes a lower bound for the values* ω_k *introduced by Bruno in* condition ω *(see Section* §1*). Moreover, notice that, in the* saddle-center *case, one has that*

$$\rho(\Lambda^{-1}) = \omega_\infty^{-1}$$

where $\rho(M)$ *is the* spectral radius *of the matrix* M*, defined as the maximum of the modulus of their eigenvalues.*

Now, we present a basic result which provides estimates for the vector fields $\mathcal{R}\widehat{\Psi}$, $\widehat{N}$ and $\widehat{B}$ that are solution of the equations

$$\begin{aligned} \widehat{N} + \widehat{B} &= \mathcal{P}\widehat{H} \\ \mathcal{L}_N\left(\mathcal{R}\widehat{\Psi}\right) &= \mathcal{R}\widehat{H} \end{aligned} \tag{1.55}$$

and whose formal approach has been derived in Sections §2.3.2 and §2.3.1, respectively.

Proposition 1.1 *Let us consider a vector field* $\widehat{H}$ *analytic on* $\overline{\mathcal{D}_\sigma}$ *and let* $\mathcal{R}\widehat{\Psi}$ *and* $\widehat{N}$, $\widehat{B}$ *(of the form* (1.5*a*) and (1.7*a*)*, respectively) be the solutions of equations* (1.55)*, (formally) derived in Sections* §2.3.2 *and* §2.3.1*. Then, the following estimates hold.*

(*i*) *First, we have*

$$\left\|\widehat{N}\right\|_{1,\sigma}, \; \left\|\widehat{B}\right\|_{1,\sigma} \leq \left\|\widehat{H}\right\|_{1,\sigma}.$$

(*ii*) *Moreover,*

$$\left\|\mathcal{R}\widehat{\Psi}\right\|_{1,\sigma} \leq \frac{\left\|\widehat{H}\right\|_{1,\sigma}}{\omega_\infty\left(1-\frac{4}{\sigma\,\omega_\infty}\left\|\widehat{H}\right\|_{1,\sigma}\right)}$$

provided we assume that the bound

$$\left\|\widehat{H}\right\|_{1,\sigma} < \frac{\sigma\omega_\infty}{4} \tag{1.56}$$

is satisfied.

Proof.

(*i*) From equation (1.42) and formulas (1.43) it follows straightforwardly that $\|\widehat{N}\|_{1,\sigma}$ and $\|\widehat{B}\|_{1,\sigma}$ are both bounded by $\|\mathcal{P}\widehat{H}\|_{1,\sigma}$ and by $\|\widehat{H}\|_{1,\sigma}$.

(*ii*) In Section §2.3.1 we dealt with equation

$$\mathcal{L}_N\left(\mathcal{R}\widehat{\Psi}\right) = \mathcal{R}\widehat{H}$$

where N had the form (1.5a) and we wrote $\mathcal{R}\widehat{H} = (\widehat{h}_1, \widehat{h}_2, \ldots, \widehat{h}_4)$ where

$$\widehat{h}_w(\xi,\eta,\mu,\nu) = \sum_{p,q\in\mathbb{Z}} h^{(w)}_{pq}(\xi\eta,\mu\nu)\,\xi^p\mu^q \tag{1.57}$$

for $w = 1, 2, \ldots, 4$, with

$$h^{(w)}_{pq}(\xi\eta,\mu\nu) = \sum_{(k,m)\in Q_{pq}} h^{(w)}_{p+k,k,q+m,m}\,(\xi\eta)^k\,(\mu\nu)^m \tag{1.58}$$

and

$$Q_{pq} := \left\{(k,m)\in(\mathbb{N}\cup\{0\})^2 : \begin{array}{rcl} k & \geq & \max\{0,-p\} \\ m & \geq & \max\{0,-q\}\end{array}, \; k+m \geq 1-\frac{p+q}{2}\right\}.$$

We note that, by definition of the 1-norm (1.52), we have that

$$\begin{aligned}\left\|\widehat{h}_w\right\|_{1,\sigma} &= \sum_{j+k+\ell+m\geq 2} |h_{jk\ell m}|\,\sigma^{j+k+\ell+m} = \\ &\sum_{p,q\in\mathbb{Z}}\sum_{(k,m)\in Q_{pq}}\left|h^{(w)}_{p+k,k,q+m,m}\right|\,\sigma^{p+q+2(k+m)} = \\ &\sum_{p,q\in\mathbb{Z}}\left\|h^{(w)}_{pq}(\xi\eta,\mu\nu)\,\xi^p\mu^q\right\|_{1,\sigma}.\end{aligned} \tag{1.59}$$

From that section we also know that its solution $\mathcal{R}\widehat{\Psi} = (\widehat{\psi}_1, \widehat{\psi}_2, \ldots, \widehat{\psi}_4)$ is given, in terms of formal power series, by

$$\widehat{\psi}_w(\xi,\eta,\mu,\nu) = \sum_{p,q\in\mathbb{Z}} \psi^{(w)}_{pq}(\xi\eta,\mu\nu)\,\xi^p\mu^q \tag{1.60}$$

where $\psi^{(w)}_{pq}(\xi\eta,\mu\nu)$ are obtained from

$$\psi^{(w)}_{pq}(\xi\eta,\mu\nu) = \frac{h^{(w)}_{pq}(\xi\eta,\mu\nu)}{\Gamma^{(w)}_{pq}(\lambda) + p\,\widehat{a}_1(\xi\eta,\mu\nu) + q\,\widehat{a}_2(\xi\eta,\mu\nu)}, \tag{1.61}$$

for $w = 1,2,\ldots,4$, $p,q\in\mathbb{Z}$ and the coefficients $\Gamma^{(w)}_{pq}(\lambda)$ as defined in (1.39). Notice that the functions $\psi^{(w)}_{pq}$ in (1.61) are rational functions of $\xi\eta$, $\mu\nu$. Therefore, equation (1.60) is not an standard representation in power series, that is, formula (1.58) does not apply to $\psi^{(w)}_{pq}$.

To estimate the 1-norm of $\mathcal{R}\widehat{\Psi}$ on $\overline{\mathcal{D}_\sigma}$ we have to bound their components. Indeed, using Lemma 1.3(i), we can write

$$\begin{aligned}
&\left\|\widehat{\psi}_w\right\|_{1,\sigma} \le \sum_{p,q\in\mathbb{Z}} \left\|\psi^{(w)}_{pq}(\xi\eta,\mu\nu)\,\xi^p\mu^q\right\|_{1,\sigma} = \\
&\sum_{p,q\in\mathbb{Z}} \left\| \frac{h^{(w)}_{pq}(\xi\eta,\mu\nu)\xi^p\mu^q}{\Gamma^{(w)}_{pq}(\lambda) + p\,\widehat{a}_1(\xi\eta,\mu\nu) + q\,\widehat{a}_2(\xi\eta,\mu\nu)} \right\|_{1,\sigma} \le \\
&\sum_{p,q\in\mathbb{Z}} \left\|h^{(w)}_{pq}(\xi\eta,\mu\nu)\xi^p\mu^q\right\|_{1,\sigma} \left\| \frac{1}{\Gamma^{(w)}_{pq}(\lambda) + p\,\widehat{a}_1(\xi\eta,\mu\nu) + q\,\widehat{a}_2(\xi\eta,\mu\nu)} \right\|_{1,\sigma}
\end{aligned} \tag{1.62}$$

The next lemma gives an upper bound for the second norm appearing in this formula (1.62).

Lemma 1.5 *Consider* $\Gamma^{(w)}_{pq}(\lambda)$ *as defined in* (1.39) *and* $\widehat{a}_1(\xi\eta,\mu\nu)$, $\widehat{a}_2(\xi\eta,\mu\nu)$ *coming from* (1.5a). *Then, for any* $p,q\in\mathbb{Z}$ *and* $(\xi,\eta,\mu,\nu)\in\overline{\mathcal{D}_\sigma}$, *we have that*

$$\left|\Gamma^{(w)}_{pq}(\lambda) + p\,\widehat{a}_1(\xi\eta,\mu\nu) + q\,\widehat{a}_2(\xi\eta,\mu\nu)\right| \ge \omega_\infty \left(1 - \frac{4}{\sigma\omega_\infty}\left\|\widehat{H}\right\|_{1,\sigma}\right)$$

provided estimate (1.56) *is satisfied.*

Proof. (lemma) We will distinguish two cases:

(a) If $|p|+|q|\ge 1$ it follows that

$$\left|\Gamma^{(w)}_{pq}(\lambda) + p\,\widehat{a}_1(\xi\eta,\mu\nu) + q\,\widehat{a}_2(\xi\eta,\mu\nu)\right| \ge \left|\,\left|\Gamma^{(w)}_{pq}(\lambda)\right| - \|p\,\widehat{a}_1 + q\,\widehat{a}_2\|_{\infty,\sigma}\right| \tag{1.63}$$

From the definition of $\Gamma^{(w)}_{pq}$ in (1.39) and applying Lemma 1.4 it turns out that

$$\left|\Gamma^{(w)}_{pq}(\lambda)\right| \ge M_{pq}\,\omega_\infty,$$

where we define

$$M_{pq} := \min\left\{\sqrt{(|p|-1)^2+q^2}, \sqrt{p^2+(|q|-1)^2}\right\}.$$

We recall that the terms $h_{pq}^{(w)}(\xi\eta,\mu\nu)$ with $|p|=1$ and $q=0$ or $p=0$ and $|q|=1$ vanish since they belong to the projection $\mathcal{P}\widehat{H}$ so, in particular, this implies that

$$M_{pq} \geq 1. \tag{1.64}$$

Moreover, it is clear that

$$|p|,\, |q| \leq 2M_{pq}. \tag{1.65}$$

Coming back to equation (1.63) we have that

$$\begin{aligned} &\left|\left|\Gamma_{pq}^{(w)}(\lambda)\right| - \|p\,\widehat{a}_1 + q\,\widehat{a}_2\|_{\infty,\sigma}\right| \geq \\ &M_{pq}\,\omega_\infty \left|1 - \frac{1}{M_{pq}\,\omega_\infty}\|p\,\widehat{a}_1 + q\,\widehat{a}_2\|_{\infty,\sigma}\right| \geq \\ &\omega_\infty\left(1 - \frac{4}{\sigma\,\omega_\infty}\left\|\widehat{H}\right\|_{1,\sigma}\right), \end{aligned}$$

where it has been taken into account the assumption (1.56) and, by (1.65), (1.43) and Lemma 1.3(i), that

$$\begin{aligned} &\frac{1}{M_{pq}\,\omega_\infty}\|p\,\widehat{a}_1 + q\,\widehat{a}_2\|_{\infty,\sigma} \leq \\ &\frac{1}{\omega_\infty}\left(\frac{|p|}{M_{pq}}\|\widehat{a}_1\|_{\infty,\sigma} + \frac{|q|}{M_{pq}}\|\widehat{a}_2\|_{\infty,\sigma}\right) \leq \frac{2}{\omega_\infty}\left(\|\widehat{a}_1\|_{\infty,\sigma} + \|\widehat{a}_2\|_{\infty,\sigma}\right) = \\ &\frac{2}{\sigma\,\omega_\infty}\left(\|\sigma\,\widehat{a}_1\|_{\infty,\sigma} + \|\sigma\,\widehat{a}_2\|_{\infty,\sigma}\right) \leq \frac{4}{\sigma\,\omega_\infty}\left\|\widehat{H}\right\|_{\infty,\sigma} \leq \frac{4}{\sigma\,\omega_\infty}\left\|\widehat{H}\right\|_{1,\sigma}. \end{aligned}$$

(b) If $p=q=0$ one has that

$$\left|\Gamma_{pq}^{(w)}(\lambda) + p\,\widehat{a}_1(\xi\eta,\mu\nu) + q\,\widehat{a}_2(\xi\eta,\mu\nu)\right| = \left|\Gamma_{00}^{(w)}(\lambda)\right| \geq \omega_\infty, \tag{1.66}$$

and, in particular, assuming again (1.56),

$$\left|\Gamma_{00}^{(w)}(\lambda)\right| \geq \omega_\infty \geq \omega_\infty\left(1 - \frac{4}{\sigma\,\omega_\infty}\left\|\widehat{H}\right\|_{1,\sigma}\right).$$

This concludes the proof of this lemma.

♣

Since we are assuming that (1.56) holds, we can apply this lemma together with lemma 1.3(iii) and, therefore, it follows that

$$\left\|\frac{1}{\Gamma_{pq}^{(w)}(\lambda) + p\,\widehat{a}_1(\xi\eta,\mu\nu) + q\,\widehat{a}_2(\xi\eta,\mu\nu)}\right\|_{1,\sigma} \leq \frac{1}{\omega_\infty\left(1 - \frac{4}{\sigma\,\omega_\infty}\left\|\widehat{H}\right\|_{1,\sigma}\right)}.$$

Thus, estimate (1.62) jointly with (1.59) gives

$$\left\|\widehat{\psi}_w\right\|_{1,\sigma} \leq \frac{1}{\omega_\infty\left(1-\frac{4}{\sigma\,\omega_\infty}\left\|\widehat{H}\right\|_{1,\sigma}\right)} \sum_{p,q\in\mathbb{Z}} \left\|h_{pq}^{(w)}(\xi\eta,\mu\nu)\xi^p\mu^q\right\|_{1,\sigma} = \frac{\left\|\widehat{h}_w\right\|_{1,\sigma}}{\omega_\infty\left(1-\frac{4}{\sigma\,\omega_\infty}\left\|\widehat{H}\right\|_{1,\sigma}\right)},$$

for $w=1,2,\ldots,4$. Finally, using (1.53), it turns out that

$$\left\|\mathcal{R}\widehat{\Psi}\right\|_{1,\sigma} \leq \frac{\left\|\widehat{H}\right\|_{1,\sigma}}{\omega_\infty\left(1-\frac{4}{\sigma\,\omega_\infty}\left\|\widehat{H}\right\|_{1,\sigma}\right)}.$$

□

§2.5.2 Proof of the convergence

To ease the reading of this proof, let us recall briefly the problem we are dealing with. Let consider a system

$$\dot{X} = F(X) = \Lambda + \widehat{F}(X) \tag{1.67}$$

where F is analytic on a domain $\overline{\mathcal{D}_R}$ and having at $X=0$ a saddle-focus or saddle-center equilibrium point with characteristic exponents $\{\pm\lambda_1,\pm\lambda_2\}$ equal to $\{\pm\lambda\pm\mathrm{i}\alpha\}$ and $\{\pm\lambda,\,\pm\mathrm{i}\alpha\}$, respectively. As it has been seen at the beginning of Section §2.1, we can assume the matrix Λ to be written in (complex) diagonal form. This allows us to deal with both cases using a unified approach. We also recall that, again in Section §2.1, we introduced the notation Λ to denote both the matrix Λ and the vector field $\Lambda\mathrm{id}$. We will only use explicitly the second expression in cases of possible misunderstanding.

Our aim is the following: we are looking for an analytic transformation $X=\Phi(\chi)=\chi+\widehat{\Phi}(\chi)$ and analytic vector fields N and $\widehat{B}$ (that we can assume to be of the form (1.5a) and (1.7a), respectively) such that the equality

$$D\Phi\,N + \widehat{B} = F\left(\Phi\right) \tag{1.68}$$

is satisfied. We say in that case that Φ leads system (1.67) into ΨNF. To get such transformation and vector fields we have developed in Sections §2.3 and §2.4 the following *recurrent scheme* to whose convergence proof is devoted this section. Setting the following condition on $\widehat{\Phi}$,

$$\mathcal{P}\widehat{\Phi} \equiv 0, \tag{1.69}$$

we take initial values

$$\Phi^{(1)} = \mathrm{id}, \quad N^{(1)} = \Lambda, \quad \widehat{B}^{(1)} = 0 \tag{1.70}$$

and obtain, recurrently,

$$\begin{aligned} \Phi^{(K+1)} &= \mathrm{id} + \mathcal{R}\widehat{\Phi}^{(K+1)} \\ N^{(K+1)} &= \Lambda + \widehat{N}^{(K+1)} \\ \widehat{B}^{(K+1)} & \end{aligned} \tag{1.71}$$

with

$$\widehat{\Phi}^{(K+1)} = \mathcal{O}_{\leq K+1}, \qquad \widehat{N}^{(K+1)} = \mathcal{O}_{\leq K+1}, \qquad \widehat{B}^{(K+1)} = \mathcal{O}_{\leq K+1},$$

from equations

$$\left\{\mathcal{L}_{N^{(K)}}\left(\mathcal{R}\widehat{\Phi}^{(K+1)}\right)\right\}_{\leq K+1} = \left\{\mathcal{R}\left(\widehat{F}\left(\Phi^{(K)}\right)\right)\right\}_{\leq K+1} \tag{1.72}$$

$$\widehat{N}^{(K+1)} + \widehat{B}^{(K+1)} = \left\{\mathcal{P}\left(\widehat{F}\left(\Phi^{(K)}\right)\right)\right\}_{\leq K+1}. \tag{1.73}$$

Once set our problem, let us start with the proof. Let us consider a positive constant $0 < \gamma < 1$ (in order to simplify the estimates, we can assume $\gamma \geq 1/2$, which is not restrictive). As it is commonly done in Normal Form Theory, we can scale our system by means of a change $X = \alpha Z$, where $\alpha > 0$ is a constant to determine. Thus we have a new system

$$\dot{Z} = F_\alpha(Z) := \Lambda + \alpha^{-1}\widehat{F}(\alpha Z), \tag{1.74}$$

with F_α analytic on $\overline{\mathcal{D}_r}$, where $r := \alpha^{-1}R$. Let us consider a positive constant $0 < \gamma < 1$. In order to simplify the estimates, we can assume $\gamma \geq 1/2$, which is not restrictive. Then, since $\widehat{F}_\alpha$ starts with terms of order at least 2, we can choose α big enough (so r small enough) in such a way that the following estimate holds

$$\left\|\widehat{F}_\alpha\right\|_{1,r} \leq \left(\frac{(1-\gamma)\omega_\infty}{8}\right) r. \tag{1.75}$$

Calling again Z and F_α as X and F, respectively, we can assume our system (1.67) to be analytic on $\overline{\mathcal{D}_r}$ and satisfying (1.75). We are going to prove that the limit vector fields Φ, N and $\widehat{B}$ obtained from this recurrent scheme satisfy (1.68) and are analytic on $\overline{\mathcal{D}_{\gamma r}}$ (and therefore, reversing the scaling, on $\overline{\mathcal{D}_{\gamma R}}$).

For ease of reading we will itemize the proof in several parts: the first one will provide some estimates on the approximations provided by the recurrent scheme; in the second one, their convergence will be derived.

(*i*) Consider system (1.67) having F analytic on a domain $\overline{\mathcal{D}_r}$ and satisfying the assumption (1.75). Apply onto it the recurrent scheme (1.69)–(1.73) and consider the sequences

$$\left\{\left\|\Phi^{(K)}\right\|_{1,s}\right\}_K, \quad \left\{\left\|N^{(K)}\right\|_{1,s}\right\}_K, \quad \left\{\left\|\widehat{B}^{(K)}\right\|_{1,s}\right\}_K, \tag{1.76}$$

defined for $K \geq 1$ and being $s = \gamma r$. Then, the following properties are satisfied.

(a) They increase monotonically, that is,

$$\begin{aligned}\left\|\Phi^{(K+1)}\right\|_{1,s} &\geq \left\|\Phi^{(K)}\right\|_{1,s}\\ \left\|N^{(K+1)}\right\|_{1,s} &\geq \left\|N^{(K)}\right\|_{1,s}\\ \left\|\widehat{B}^{(K+1)}\right\|_{1,s} &\geq \left\|\widehat{B}^{(K)}\right\|_{1,s}.\end{aligned}$$

(b) All these sequences are uniformly upper-bounded. Precisely, for all $K \geq 1$ we have that

$$\left\|\Phi^{(K)}\right\|_{1,s} \leq r \tag{1.77}$$

and that

$$\left\|N^{(K)}\right\|_{1,s}, \ \left\|\widehat{B}^{(K)}\right\|_{1,s} \leq \|F\|_{1,r}. \tag{1.78}$$

Let us prove these assertions.

(a) From Lemma 1.2 we have that

$$\begin{aligned}\Phi^{(K+1)} &= \Phi^{(K)} + \mathcal{R}\Delta\widehat{\Phi}^{(K)},\\ N^{(K+1)} &= N^{(K)} + \Delta N^{(K)},\\ \widehat{B}^{(K+1)} &= \widehat{B}^{(K)} + \Delta\widehat{B}^{(K)},\end{aligned}$$

where $\mathcal{R}\Delta\widehat{\Phi}^{(K)}$, $\Delta N^{(K)}$ and $\Delta\widehat{B}^{(K)}$ are all three $\mathcal{O}_{[K+1]}$, except in the case of odd K where one has that

$$\Delta N^{(2J-1)} = \Delta\widehat{B}^{(2J-1)} = 0.$$

Therefore, taking into account Lemma 1.3(iv), it turns out that

$$\begin{aligned}\left\|\Phi^{(K+1)}\right\|_{1,s} = \left\|\Phi^{(K)} + \mathcal{R}\Delta\widehat{\Phi}^{(K)}\right\|_{1,s} =\\ \left\|\Phi^{(K)}\right\|_{1,s} + \left\|\mathcal{R}\Delta\Phi^{(K)}\right\|_{1,s} \geq \left\|\Phi^{(K)}\right\|_{1,s}.\end{aligned}$$

In the same way it can be proved for $\|N^{(K+1)}\|_{1,s}$ and $\left\|\widehat{B}^{(K+1)}\right\|_{1,s}$.

(b) To see it we proceed inductively. Thus, for $K = 1$ equation (1.72) becomes

$$\left\{\mathcal{L}_{N^{(1)}}\left(\mathcal{R}\widehat{\Phi}^{(2)}\right)\right\}_{\leq 2} = \left\{\mathcal{R}\left(\widehat{F}(\Phi^{(1)})\right)\right\}_{\leq 2}.$$

Having in mind that $N^{(1)} = \Lambda$ (so $\widehat{N}^{(1)} = 0$), $\Phi^{(1)} = \mathrm{id}$ and the definition (1.25) of the operator $\mathcal{L}$, this equation is equivalent to

$$D\left(\mathcal{R}\widehat{\Phi}^{(2)}\right)\Lambda - \Lambda\,\mathcal{R}\widehat{\Phi}^{(2)} = \mathcal{R}F_{[2]}$$

and to

$$\left[\Lambda, \mathcal{R}\widehat{\Phi}^{(2)}\right] = F_{[2]},$$

where $[H,G] = (DG)H - (DH)G$ stands for the Lie bracket of the vector fields H and G. Now, from Proposition 1.1(ii), taking into account that $\widehat{a}_1^{(1)} = \widehat{a}_2^{(1)} = 0$ (the functions appearing in $\widehat{N}^{(1)}$) and using estimate (1.66) it follows that

$$\left\|\mathcal{R}\widehat{\Phi}^{(2)}\right\|_{1,s} \leq \frac{\|F_{[2]}\|_{1,s}}{\omega_\infty}$$

and, in particular,

$$\left\|\mathcal{R}\widehat{\Phi}^{(2)}\right\|_{1,s} \leq \frac{\left\|\widehat{F}\right\|_{1,r}}{\omega_\infty}. \tag{1.79}$$

Thus, applying Lemma 1.3(iv), the assumption $\mathcal{P}\widehat{\Phi} = 0$ and the estimate (1.75), one obtains that

$$\left\|\Phi^{(2)}\right\|_{1,s} \leq s + \frac{\left\|\widehat{F}\right\|_{1,r}}{\omega_\infty} \leq \gamma r + \frac{1-\gamma}{8} r \leq \gamma r + (1-\gamma) r = r.$$

Concerning vector fields $N^{(2)}$ and $\widehat{B}^{(2)}$ we have that

$$N^{(2)} = N^{(1)} = \Lambda, \qquad \widehat{B}^{(2)} = \widehat{B}^{(1)} = 0$$

and, therefore, estimate (1.78) is trivially satisfied.
Thus, by induction hypothesis, assume that the following bounds

$$\begin{aligned} \left\|\Phi^{(K)}\right\|_{1,s} &\leq r \\ \left\|N^{(K)}\right\|_{1,s}, \left\|\widehat{B}^{(K)}\right\|_{1,s} &\leq \|F\|_{1,r} \end{aligned}$$

hold. We are going to show that they are also true for $K+1$. We start dealing with equation (1.72), namely,

$$\left\{\mathcal{L}_{N^{(K)}}\left(\mathcal{R}\widehat{\Phi}^{(K+1)}\right)\right\}_{\leq K+1} = \left\{\mathcal{R}\left(\widehat{F}\left(\Phi^{(K)}\right)\right)\right\}_{\leq K+1}.$$

This equation is of type $\mathcal{L}_N(\mathcal{R}\widehat{\Psi}) = \mathcal{R}\widehat{H}$ provided we take

$$N = N^{(K)}, \qquad \mathcal{R}\widehat{\Psi} = \mathcal{R}\widehat{\Phi}^{(K+1)}, \qquad \widehat{H} = \widehat{F}\left(\Phi^{(K)}\right)$$

and consider just terms up to order $K+1$. Setting $\sigma = s$ and taking into account estimate (1.75), the induction hypothesis and Lemma 1.3(i, ii) it follows that

$$\left\|\widehat{H}\right\|_{1,s} = \left\|\left(\widehat{F}(\Phi^{(K)})\right)\right\|_{1,s} \leq \left\|\widehat{F}\right\|_{1,r} \leq \left(\frac{(1-\gamma)\omega_\infty}{8}\right) r.$$

Using that $1/2 \le \gamma < 1$ and that $s = \gamma r$, this estimate reads

$$\left\|\widehat{H}\right\|_{1,s} \le \left(\frac{(1-\gamma)\omega_\infty}{8}\right) r \le \frac{\gamma}{8}\,\omega_\infty r = \frac{s\,\omega_\infty}{8} < \frac{s\,\omega_\infty}{4},$$

which is assumption (1.56). Applying Proposition 1.1(*ii*) and that

$$1 - \frac{4}{\gamma r\,\omega_\infty}\left\|\widehat{F}\left(\Phi^{(K)}\right)\right\|_{1,\gamma r} = 1 - \frac{4}{\gamma r\,\omega_\infty}\left\|\widehat{H}\right\|_{1,s} \ge 1 - \left(\frac{4}{s\,\omega_\infty}\right)\left(\frac{s\,\omega_\infty}{8}\right) = 1 - \frac{1}{2} = \frac{1}{2}$$

we obtain

$$\left\|\mathcal{R}\widehat{\Phi}^{(K+1)}\right\|_{1,\gamma r} \le \frac{\left\|\widehat{F}\left(\Phi^{(K)}\right)\right\|_{1,\gamma r}}{\omega_\infty\left(1 - \frac{4}{\gamma r\,\omega_\infty}\left\|\widehat{F}\left(\Phi^{(K)}\right)\right\|_{1,\gamma r}\right)} \le \frac{\frac{(1-\gamma)\omega_\infty}{8}\,r}{\omega_\infty/2} = \frac{(1-\gamma)r}{4}.$$

Finally, from Lemma 1.3(*iv*) one obtains that

$$\left\|\Phi^{(K+1)}\right\|_{1,s} = \left\|\Phi^{(K+1)}\right\|_{1,\gamma r} = \|\mathrm{id}\|_{1,\gamma r} + \left\|\mathcal{R}\widehat{\Phi}^{(K+1)}\right\|_{1,\gamma r} \le \gamma r + \frac{(1-\gamma)r}{4} \le \gamma r + (1-\gamma)r = r.$$

Concerning $N^{(K+1)}$ and $\widehat{B}^{(K+1)}$, having in mind the induction hypothesis $\left\|\Phi^{(K)}\right\|_{1,s} \le r$, equation (1.73) and section §2.3.2, one obtains that

$$\left\|\widehat{N}^{(K+1)}\right\|_{1,s} \le \left\|\widehat{F}\right\|_{1,r}$$

and

$$\left\|\widehat{B}^{(K+1)}\right\|_{1,s} \le \left\|\widehat{F}\right\|_{1,r} \le \|F\|_{1,r}\,.$$

Since $N^{(K+1)} = \Lambda + \widehat{N}^{(K+1)}$ and $F = \Lambda + \widehat{F}$ it turns out that

$$\left\|N^{(K+1)}\right\|_{1,s} \le \|F\|_{1,r}$$

which concludes the proof of (b).

(*ii*) At (*i*) it has been proved that the sequences

$$\left\{\left\|\Phi^{(K)}\right\|_{1,s}\right\}_K, \quad \left\{\left\|N^{(K)}\right\|_{1,s}\right\}_K, \quad \left\{\left\|\widehat{B}^{(K)}\right\|_{1,s}\right\}_K,$$

increase monotonically and are uniformly upper-bounded. Applying onto them the Ascoli-Arzelà theorem it follows that they admit convergent subsequences

$$\left\{\left\|\Phi^{(K_J)}\right\|_{1,s}\right\}_J, \quad \left\{\left\|N^{(K_J)}\right\|_{1,s}\right\}_J, \quad \left\{\left\|\widehat{B}^{(K_J)}\right\|_{1,s}\right\}_J.$$

Therefore, if we define a vector field Φ given by

$$\Phi(\chi) := \lim_{J\to\infty} \Phi^{(K_J)}(\chi)$$

for any $\chi \in \overline{\mathcal{D}_s}$, it follows that the limit

$$\|\Phi\|_{1,s} = \lim_{J\to\infty} \left\|\Phi^{(K_J)}\right\|_{1,s}$$

exists and is finite. From Weierstrass theorem it follows that Φ is an analytic vector field on $\overline{\mathcal{D}_s} = \overline{\mathcal{D}_{\gamma r}}$. Moreover, since the recurrent scheme (1.69)–(1.73) and Lemma 1.2, provide vector fields $\Phi^{(K+1)}$ of the form

$$\Phi^{(K+1)} = \Phi^{(K)} + \mathcal{R}\Delta\widehat{\Phi}^{(K)}$$

where $\mathcal{R}\Delta\widehat{\Phi}^{(K)} = \mathcal{O}_{[K+1]}$, it can be derived that the subsequence $\left\{\left\|\Phi^{(K_J)}\right\|_{1,s}\right\}_J$ is, in fact, the complete sequence $\left\{\left\|\Phi^{(K)}\right\|_{1,s}\right\}_K$. In a similar way one obtains N and $\widehat{B}$, analytic vector fields on $\overline{\mathcal{D}_{\gamma r}}$ defined as

$$N := \lim_K N^{(K)}, \qquad \widehat{B} := \lim_K \widehat{B}^{(K)}.$$

Together with Φ, they satisfy that

$$D\Phi\, N + \widehat{B} = F\,(\Phi)$$

and therefore, they lead system (1.67) into ΨNF. This concludes the proof of the Main Theorem.

§3 Proof of Propositions H1 and R1

§3.1 Proof of Proposition H1

It is clear that if $\widehat{B} \equiv 0$ then ΨNF is just BNF so, let us consider the converse situation. To fix ideas, let us deal with a 4-dimensional Hamiltonian system with the origin being a *saddle-center* equilibrium point. The *saddle-focus* case can be done in a similar way. Assume moreover that the center variables have been complexified (becoming complex conjugated). Applying Moser's Theorem [43], we know the existence of an analytic convergent transformation Ψ, close to the identity, leading it into BNF,

$$\left\{\begin{array}{rcl} \dot\xi &=& \xi\, a_1(\xi\eta,\mu\nu) \\ \dot\eta &=& -\eta\, a_1(\xi\eta,\mu\nu) \\ \dot\mu &=& \mu\, a_2(\xi\eta,\mu\nu) \\ \dot\nu &=& -\nu\, a_2(\xi\eta,\mu\nu) \end{array}\right. \tag{1.80}$$

with $a_1(\xi\eta,\mu\nu) = \lambda + \cdots$ and $a_2(\xi\eta,\mu\nu) = \mathrm{i}\alpha + \cdots$. It is clear that $\tilde h_1(\xi\eta) = \xi\eta\, a_1(\xi\eta, 0) = \lambda\xi\eta + \cdots$ and $\tilde h_2(\mu\nu) = \mu\nu\, a_2(0,\mu\nu) = \mathrm{i}\alpha\mu\nu + \cdots$ are independent *first integrals* of system (1.80) and, therefore,

$$\begin{array}{rcl} h_1 &=& \tilde h_1 \circ \Psi^{-1} = \lambda xy + \cdots \\ h_2 &=& \tilde h_2 \circ \Psi^{-1} = \mathrm{i}\alpha uv + \cdots \end{array}$$

are independent first integrals of the original one. Let Φ be the convergent analytic transformation leading the initial system into ΨNF, that is, such that the new system is of the form

$$\dot{\chi} = N(\chi) + \left(D\Phi(\chi)\right)^{-1} \widehat{B}(\chi) \tag{1.81}$$

where $\chi = (\xi, \eta, \mu, \nu)$ denotes now the ΨNF-variables. Since Φ starts with the identity and h_1, h_2 are independent first integrals of the original system, it follows that $\breve{h}_1 = h_1 \circ \Phi$ and $\breve{h}_2 = h_2 \circ \Phi$ are first integrals of (1.81) and, moreover, they begin with $\lambda\xi\eta + \cdots$ and $\mathrm{i}\alpha\mu\nu + \cdots$, respectively. Indeed, they satisfy

$$D\breve{h}_j \left(N + (D\Phi)^{-1}\widehat{B}\right) \equiv 0 \tag{1.82}$$

for $j = 1, 2$. Assume now that $\widehat{B} \neq 0$ so its minimal order terms are

$$\begin{pmatrix} \xi b_{rs}^{(1)} (\xi\eta)^r (\mu\nu)^s + \cdots \\ \eta b_{rs}^{(1)} (\xi\eta)^r (\mu\nu)^s + \cdots \\ \mu b_{r's'}^{(2)} (\xi\eta)^{r'} (\mu\nu)^{s'} + \cdots \\ \nu b_{r's'}^{(2)} (\xi\eta)^{r'} (\mu\nu)^{s'} + \cdots \end{pmatrix}$$

with $b_{rs}^{(1)} \neq 0$ or $b_{r's'}^{(2)} \neq 0$ (and $r+s$ not necessarily equal to $r'+s'$). Using that $\breve{h}_1 = \lambda\xi\eta + \cdots$ and $(D\Phi)^{-1} = \mathrm{I} - (D\widehat{\Phi}) + \cdots$, the term of type $(\xi\eta)^{\ell}(\mu\nu)^m$ of minimal order corresponding to the left-hand side of equation (1.82), for $j = 1$, is given by

$$(\lambda\eta + \cdots \quad \lambda\xi + \cdots \quad 0 + \cdots \quad 0 + \cdots) \begin{pmatrix} \xi b_{rs}^{(1)} (\xi\eta)^r (\mu\nu)^s + \cdots \\ \eta b_{rs}^{(1)} (\xi\eta)^r (\mu\nu)^s + \cdots \\ * \\ * \end{pmatrix} =$$

$$2\lambda b_{rs}^{(1)} (\xi\eta)^{r+1} (\mu\nu)^s + \cdots$$

Since $\lambda \neq 0$ it implies that $b_{rs}^{(1)} = 0$. Applying the same argument to equation (1.82) with $j = 2$, and using that $\alpha \neq 0$, it follows that $b_{r's'}^{(2)} = 0$, which contradicts the assumption of $\widehat{B} \neq 0$. Consequently, $\widehat{B}$ vanishes.

§3.2 Proof of Proposition R1

The problem of the convergence of the ΨNF (and BNF) around an equilibrium is certainly a local problem. In the reversible setting, this implies that both the linearized system and the reversing involution can be taken in suitable way. Namely,

Lemma 1.6 *Let us consider a system*

$$\dot{X} = F(X)$$

analytic around the origin, a saddle-center or a saddle-focus equilibrium, and assume it is reversible with respect to an (in principle, non linear) involutory diffeomorphism $\mathfrak{G}$. *Suppose* *that the origin is a fixed point of $\mathfrak{S}$. Then there exists an analytic change of variables $X \longmapsto Z$, defined in a neighborhood of the origin, such that in the new coordinates the linearized system becomes*

$$\dot{Z} = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & \mathrm{i}\alpha & 0 \\ 0 & 0 & 0 & -\mathrm{i}\alpha \end{pmatrix} Z$$

or

$$\dot{Z} = \begin{pmatrix} \lambda + \mathrm{i}\alpha & 0 & 0 & 0 \\ 0 & \lambda - \mathrm{i}\alpha & 0 & 0 \\ 0 & 0 & -(\lambda + \mathrm{i}\alpha) & 0 \\ 0 & 0 & 0 & -(\lambda - \mathrm{i}\alpha) \end{pmatrix} Z,$$

depending if we are in the saddle-center or saddle-focus case, respectively, and assuming $\lambda, \alpha > 0$. Moreover, in these coordinates and for both cases, the symmetry $\mathfrak{S}$ can be taken of the form

$$Z \longmapsto \mathfrak{R} Z,$$

where $\mathfrak{R}$ is given by the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{1.83}$$

Proof. The proof of this lemma is essentially contained in [51]. For the *saddle-center* case, it is proved there the existence of a coordinate system, with center at the origin, in whose variables, say $Z = (u_1, u_2, z, \overline{z})$, the linearized system is given by

$$\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{z} \\ \dot{\overline{z}} \end{pmatrix} = \begin{pmatrix} 0 & \lambda & 0 & 0 \\ \lambda & 0 & 0 & 0 \\ 0 & 0 & \mathrm{i}\alpha & 0 \\ 0 & 0 & 0 & -\mathrm{i}\alpha \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ z \\ \overline{z} \end{pmatrix}$$

and the linear part of the involution $\mathfrak{S}$ has the form

$$\begin{pmatrix} u_1 \\ u_2 \\ z \\ \overline{z} \end{pmatrix} \longmapsto \begin{pmatrix} u_1 \\ -u_2 \\ \overline{z} \\ z \end{pmatrix},$$

with $z \in \mathbb{C}$ and $u_1, u_2 \in \mathbb{R}$. In that proof, it has been used that the linearization of an involution around one of its fixed points is also an involution. Performing the linear transformation

$$(v_1, v_2, z, \overline{z}) = \left(\frac{u_1 + u_2}{2}, \frac{u_1 - u_2}{2}, z, \overline{z} \right)$$

we reach the claimed result about the form of the linear parts of F and $\mathfrak{S}$. Now, by Bochner's theorem (see [8, 41]) there exists an analytic change of variables, defined in a neighborhood of the fixed point, the origin, which conjugates the symmetry $\mathfrak{S}$ to its linear part

$$(v_1, v_2, z, \overline{z}) \longmapsto (v_2, v_1, \overline{z}, z).$$

With respect to the *saddle-focus* case, it works in a similar way. In this situation the first change of variables takes X to

$$Z = (z_1, z_2, z_3, z_4) = (z_1, \overline{z_1}, z_3, \overline{z_3}),$$

the linearization of the system $\dot{X} = F(X)$ to

$$\begin{pmatrix} \dot{z_1} \\ \dot{z_2} \\ \dot{z_3} \\ \dot{z_4} \end{pmatrix} = \begin{pmatrix} \lambda + \mathrm{i}\alpha & 0 & 0 & 0 \\ 0 & \lambda - \mathrm{i}\alpha & 0 & 0 \\ 0 & 0 & -(\lambda + \mathrm{i}\alpha) & 0 \\ 0 & 0 & 0 & -(\lambda - \mathrm{i}\alpha) \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}$$

and the linearization of $\mathfrak{G}$ to $Z \longmapsto \overline{Z}$, which is the involution given by the matrix (1.83). The local analytic conjugacy with $\mathfrak{G}$ is again provided by Bochner's theorem.

□

Therefore, it is not restrictive to assume that our system is written, in a neighborhood of the origin, in the form

$$\dot{X} = \Lambda + \widehat{F}(X),$$

with

$$\Lambda = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & \mathrm{i}\alpha & 0 \\ 0 & 0 & 0 & -\mathrm{i}\alpha \end{pmatrix} X$$

in the saddle-center case, or

$$\Lambda = \begin{pmatrix} \lambda + \mathrm{i}\alpha & 0 & 0 & 0 \\ 0 & \lambda - \mathrm{i}\alpha & 0 & 0 \\ 0 & 0 & -(\lambda + \mathrm{i}\alpha) & 0 \\ 0 & 0 & 0 & -(\lambda - \mathrm{i}\alpha) \end{pmatrix} X,$$

in the case of a saddle-focus, respectively. Moreover, we can assume it to be reversible with respect to the linear involution

$$\mathfrak{R} : (x_1, x_2, x_3, x_4) \longmapsto (x_2, x_1, x_4, x_3).$$

Thus, the reversibility condition (1.14) reads

$$\mathfrak{R}F(\mathfrak{R}X) = -F(X). \tag{1.84}$$

Once we have set the linear framework, we present a property which characterizes those transformations that preserve a given linear reversibility.

Lemma 1.7 *Let Ψ be a diffeomorphism satisfying*

$$\mathfrak{R}\Psi(\mathfrak{R}\chi) = \Psi(\chi). \tag{1.85}$$

Then the transformation $X = \Psi(\chi)$ preserves the $\mathfrak{R}$-reversibility, that is, the new system

$$\dot{\chi} = G(\chi) := (\Psi^* F)(\chi)$$

is also $\mathfrak{R}$-reversible.

Proof. To see that $\dot{\chi} = G(\chi)$ is $\mathfrak{R}$-reversible we have to check that $\mathfrak{R}G(\mathfrak{R}\chi) = -G(\chi)$. Differentiating both sides of equation (1.85) we get

$$\mathfrak{R}\, D\Psi\,(\mathfrak{R}\chi)\,\mathfrak{R} = D\Psi(\chi).$$

Using this property and equations (1.84), (1.85) it follows that

$$\begin{aligned}
&\mathfrak{R}G(\mathfrak{R}\chi) = \mathfrak{R}\left((D\Psi(\mathfrak{R}\chi))^{-1}\; F\left(\Psi(\mathfrak{R}\chi)\right)\right) = \\
&\quad \mathfrak{R}\left((D\Psi(\mathfrak{R}\chi))^{-1}\; F\left(\mathfrak{R}\Psi(\chi)\right)\right) = -\mathfrak{R}\left((D\Psi(\mathfrak{R}\chi))^{-1}\; \mathfrak{R}F\left(\Psi(\chi)\right)\right) = \\
&\quad -\mathfrak{R}\left(\left(\mathfrak{R}\,(D\Psi(\chi))^{-1}\,\mathfrak{R}\right)\left(\mathfrak{R}F\left(\Psi(\chi)\right)\right)\right) = -\left(D\Psi(\chi)\right)^{-1}\; F\left(\Psi(\chi)\right) = \\
&\quad -\left(\Psi^{*}F\right)(\chi) = -G(\chi),
\end{aligned}$$

which concludes the proof of this lemma.

□

The proof of Proposition R1 is based on the following two points:

- Applying Theorem 1.2, there exist an analytic transformation $X = \Phi(\chi)$ and analytic vector fields $N(\chi)$, $\widehat{B}(\chi)$ leading the original system into ΨNF, provided the origin is a saddle-center or saddle-focus equilibrium point. That is, satisfying the equality

$$D\Phi\, N + \widehat{B} = F \circ \Phi.$$

- We will prove that the vector fields obtained from the recurrent scheme satisfy: (a) the transformation $X = \Phi(\chi)$ verifies relation (1.85), so it preserves $\mathfrak{R}$-reversibility; (b) N and $\widehat{B}$ are $\mathfrak{R}$-reversible. This last property will imply that $\widehat{B}$ has to vanish and, therefore, ΨNF will become BNF.

Lemma 1.8 *Let us consider an $\mathfrak{R}$-reversible system*

$$\dot{X} = F(X) = \Lambda + \widehat{F}(X)$$

with $\mathfrak{R} : (x_1, x_2, x_3, x_4) \longmapsto (x_2, x_1, x_4, x_3)$, analytic on a neighborhood of the origin, that we assume to be a saddle-center or a saddle-focus equilibrium point. Let us take $\Phi^{(K)}$, $N^{(K)}$ and $\widehat{B}^{(K)}$, the vector fields provided by the ΨNF-recurrent scheme: set $\mathcal{P}\widehat{\Phi} = 0$, take initial values

$$\Phi^{(1)} = id, \quad N^{(1)} = \Lambda, \quad \widehat{B}^{(1)} = 0$$

and obtain, recurrently,

$$\begin{aligned}
\Phi^{(K+1)} &= id + \mathcal{R}\widehat{\Phi}^{(K+1)} \\
N^{(K+1)} &= \Lambda + \widehat{N}^{(K+1)} \\
\widehat{B}^{(K+1)} &
\end{aligned}$$

with

$$\widehat{\Phi}^{(K+1)} = \mathcal{O}_{\leq K+1}, \qquad \widehat{N}^{(K+1)} = \mathcal{O}_{\leq K+1}, \qquad \widehat{B}^{(K+1)} = \mathcal{O}_{\leq K+1},$$

from equations

$$\left\{\mathcal{L}_{N^{(K)}}\left(\mathcal{R}\widehat{\Phi}^{(K+1)}\right)\right\}_{\leq K+1} = \left\{\mathcal{R}\left(\widehat{F}\left(\Phi^{(K)}\right)\right)\right\}_{\leq K+1} \tag{1.86}$$

$$\widehat{N}^{(K+1)} + \widehat{B}^{(K+1)} = \left\{\mathcal{P}\left(\widehat{F}\left(\Phi^{(K)}\right)\right)\right\}_{\leq K+1}. \tag{1.87}$$

Then, the following assertions hold,

(*i*) *For any* $K \geq 1$, *the vector field* $\Phi^{(K)}$ *satisfies* (1.85) *and the vector fields* $N^{(K)}$ *and* $\widehat{B}^{(K)}$ *are* $\mathfrak{R}$*-reversible, that is,*

$$\mathfrak{R}\Phi^{(K)}(\mathfrak{R}\chi) = \Phi^{(K)}(\chi),$$

and

$$\mathfrak{R}N^{(K)}(\mathfrak{R}\chi) = -N^{(K)}(\chi), \qquad \mathfrak{R}\widehat{B}^{(K)}(\mathfrak{R}\chi) = -\widehat{B}^{(K)}(\chi).$$

(*ii*) *The vector fields* Φ, N *and* $\widehat{B}$ *provided by Theorem 1.2 and defined as*

$$\Phi = \lim_{K\to\infty} \Phi^{(K)}, \qquad N = \lim_{K\to\infty} N^{(K)}, \qquad \widehat{B} = \lim_{K\to\infty} \widehat{B}^{(K)},$$

verify that

- *The change of variables* $X = \Phi(\chi)$ *satisfies relation* (1.85) *and, therefore, it preserves the* $\mathfrak{R}$*-reversibility;*
- N *and* $\widehat{B}$ *are* $\mathfrak{R}$*-reversible.*

(*iii*) *Since* $\widehat{B}$ *is* $\mathfrak{R}$*-reversible it turns out that* $\widehat{B}$ *vanishes.*

Remark 8 *As it happened in the general case,* Φ *is also convergent if we fix* $\mathcal{P}\widehat{\Phi}$ *equal to any analytic function, convergent in the same domain as* $\mathcal{R}\widehat{\Phi}$ *and verifying* (1.85).

Proof. It is based in some statements that we list and prove separately. Namely,

(*a*) *If a vector field* H *is* $\mathfrak{R}$*-reversible then its projections* $\mathcal{P}H$ *and* $\mathcal{R}H$ *are also* $\mathfrak{R}$*-reversible.*

Having in mind the definition of the projections $\mathcal{P}$ and $\mathcal{R}$ we can write

$$H(\chi) = \mathcal{P}H(\chi) + \mathcal{R}H(\chi) = \begin{pmatrix} \xi h_1(\xi\eta, \mu\nu) \\ \eta h_2(\xi\eta, \mu\nu) \\ \mu h_3(\xi\eta, \mu\nu) \\ \nu h_4(\xi\eta, \mu\nu) \end{pmatrix} + \begin{pmatrix} \tilde{h}_1(\xi, \eta, \mu, \nu) \\ \tilde{h}_2(\xi, \eta, \mu, \nu) \\ \tilde{h}_3(\xi, \eta, \mu, \nu) \\ \tilde{h}_4(\xi, \eta, \mu, \nu) \end{pmatrix}.$$

Since H is $\mathfrak{R}$-reversible we have that $\mathfrak{R}\,H(\mathfrak{R}\chi) + H(\chi) = 0$. With respect the first term it turns out that

$$
\begin{aligned}
\mathfrak{R}\, H(\mathfrak{R}\chi) \quad &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \eta h_1(\xi\eta, \mu\nu) \\ \xi h_2(\xi\eta, \mu\nu) \\ \nu h_3(\xi\eta, \mu\nu) \\ \mu h_4(\xi\eta, \mu\nu) \end{pmatrix} + \\
& \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \tilde{h}_1(\eta, \xi, \nu, \mu) \\ \tilde{h}_2(\eta, \xi, \nu, \mu) \\ \tilde{h}_3(\eta, \xi, \nu, \mu) \\ \tilde{h}_4(\eta, \xi, \nu, \mu) \end{pmatrix} = \\
& \begin{pmatrix} \xi h_2(\xi\eta, \mu\nu) \\ \eta h_1(\xi\eta, \mu\nu) \\ \mu h_4(\xi\eta, \mu\nu) \\ \nu h_3(\xi\eta, \mu\nu) \end{pmatrix} + \begin{pmatrix} \tilde{h}_2(\eta, \xi, \nu, \mu) \\ \tilde{h}_1(\eta, \xi, \nu, \mu) \\ \tilde{h}_4(\eta, \xi, \nu, \mu) \\ \tilde{h}_3(\eta, \xi, \nu, \mu) \end{pmatrix}
\end{aligned}
$$

Therefore, from

$$
\begin{aligned}
\mathfrak{R}\, H(\mathfrak{R}\chi) + H(\chi) = &\begin{pmatrix} \xi\left(h_2(\xi\eta, \mu\nu) + h_1(\xi\eta, \mu\nu)\right) \\ \eta\left(h_1(\xi\eta, \mu\nu) + h_2(\xi\eta, \mu\nu)\right) \\ \mu\left(h_4(\xi\eta, \mu\nu) + h_3(\xi\eta, \mu\nu)\right) \\ \nu\left(h_3(\xi\eta, \mu\nu) + h_4(\xi\eta, \mu\nu)\right) \end{pmatrix} + \\
&\begin{pmatrix} \tilde{h}_2(\eta, \xi, \nu, \mu) + \tilde{h}_1(\xi, \eta, \mu, \nu) \\ \tilde{h}_1(\eta, \xi, \nu, \mu) + \tilde{h}_2(\xi, \eta, \mu, \nu) \\ \tilde{h}_4(\eta, \xi, \nu, \mu) + \tilde{h}_3(\xi, \eta, \mu, \nu) \\ \tilde{h}_3(\eta, \xi, \nu, \mu) + \tilde{h}_4(\xi, \eta, \mu, \nu) \end{pmatrix} = 0
\end{aligned}
$$

and using that

$$
\mathcal{P} \begin{pmatrix} \tilde{h}_2(\eta, \xi, \nu, \mu) \\ \tilde{h}_1(\eta, \xi, \nu, \mu) \\ \tilde{h}_4(\eta, \xi, \nu, \mu) \\ \tilde{h}_3(\eta, \xi, \nu, \mu) \end{pmatrix} = \mathcal{P} \begin{pmatrix} \tilde{h}_1(\xi, \eta, \mu, \nu) \\ \tilde{h}_2(\xi, \eta, \mu, \nu) \\ \tilde{h}_3(\xi, \eta, \mu, \nu) \\ \tilde{h}_4(\xi, \eta, \mu, \nu) \end{pmatrix} = 0
$$

it follows that

$$
\begin{aligned}
h_2(\xi\eta, \mu\nu) \;+\; h_1(\xi\eta, \mu\nu) &= 0 \\
h_4(\xi\eta, \mu\nu) \;+\; h_3(\xi\eta, \mu\nu) &= 0
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{h}_2(\xi, \eta, \mu, \nu) \;+\; \tilde{h}_1(\eta, \xi, \nu, \mu) &= 0 \\
\tilde{h}_4(\xi, \eta, \mu, \nu) \;+\; \tilde{h}_3(\eta, \xi, \nu, \mu) &= 0.
\end{aligned}
$$

That is,

$$
\mathcal{P}H(\chi) = \begin{pmatrix} \xi h_1(\xi\eta, \mu\nu) \\ -\eta h_1(\xi\eta, \mu\nu) \\ \mu h_3(\xi\eta, \mu\nu) \\ -\nu h_3(\xi\eta, \mu\nu) \end{pmatrix}, \qquad \mathcal{R}H(\chi) = \begin{pmatrix} \tilde{h}_1(\xi, \eta, \mu, \nu) \\ -\tilde{h}_1(\eta, \xi, \nu, \mu) \\ \tilde{h}_3(\xi, \eta, \mu, \nu) \\ -\tilde{h}_3(\eta, \xi, \nu, \mu) \end{pmatrix}.
$$

From this expression, it is straightforward to check that

$$
\mathfrak{R}\,(\mathcal{P}H)\,(\mathfrak{R}\chi) = -\mathcal{P}H(\chi), \qquad \mathfrak{R}\,(\mathcal{R}H)\,(\mathfrak{R}\chi) = -\mathcal{R}H(\chi).
$$

(*b*) *Let* $\mathcal{R}\widehat{\Psi}$ *be the solution of an equation of type*

$$\mathcal{L}_N\left(\mathcal{R}\widehat{\Psi}\right) = \mathcal{R}\widehat{H}.$$

Then, if $\mathcal{R}\widehat{H}$ *is* $\mathfrak{R}$*-reversible it follows that*

$$\mathfrak{R}\left(\mathcal{R}\widehat{\Phi}\right)(\mathfrak{R}\chi) = \mathcal{R}\widehat{\Psi}(\chi),$$

this is, the transformation $X = \chi + \mathcal{R}\widehat{\Psi}(\chi)$ *preserves the* $\mathfrak{R}$*-reversibility.*

To see it, let us consider

$$\mathcal{R}\widehat{\Psi}(\chi) = \begin{pmatrix} \widehat{\psi}_1(\xi,\eta,\mu,\nu) \\ \widehat{\psi}_2(\xi,\mu,\mu,\nu) \\ \widehat{\psi}_3(\xi,\mu,\mu,\nu) \\ \widehat{\psi}_4(\xi,\mu,\mu,\nu) \end{pmatrix}, \qquad \mathcal{R}\widehat{H}(\chi) = \begin{pmatrix} \widehat{h}_1(\xi,\eta,\mu,\nu) \\ \widehat{h}_2(\xi,\eta,\mu,\nu) \\ \widehat{h}_3(\xi,\eta,\mu,\nu) \\ \widehat{h}_4(\xi,\eta,\mu,\nu) \end{pmatrix}$$

and write them in the form

$$\begin{aligned} \widehat{\psi}_w(\xi,\eta,\mu,\nu) &= \sum_{j+k+\ell+m\geq 2} \psi^{(w)}_{jk\ell m}\,\xi^j\eta^k\mu^\ell\nu^m \\ \widehat{h}_w(\xi,\eta,\mu,\nu) &= \sum_{j+k+\ell+m\geq 2} h^{(w)}_{jk\ell m}\,\xi^j\eta^k\mu^\ell\nu^m \end{aligned}$$

for $w = 1,2,\ldots,4$. Since $\mathcal{R}\widehat{H}$ is $\mathfrak{R}$-reversible one has that

(1.88) $$\widehat{h}_2(\xi,\eta,\mu,\nu) = -\widehat{h}_1(\xi,\eta,\mu,\nu), \qquad \widehat{h}_4(\xi,\eta,\mu,\nu) = -\widehat{h}_3(\xi,\eta,\mu,\nu),$$

which is equivalent to

$$h^{(2)}_{jk\ell m} = -h^{(1)}_{kjm\ell}, \qquad h^{(4)}_{jk\ell m} = -h^{(3)}_{kjm\ell}.$$

In a similar way it follows that to prove that $\mathfrak{R}(\mathcal{R}\widehat{\Psi})(\mathfrak{R}\chi) = \mathcal{R}\widehat{\Psi}(\chi)$ it is enough to check that

$$\widehat{\psi}_2(\xi,\eta,\mu,\nu) = \widehat{\psi}_1(\xi,\eta,\mu,\nu), \qquad \widehat{\psi}_4(\xi,\eta,\mu,\nu) = \widehat{\psi}_3(\xi,\eta,\mu,\nu)$$

or, as before, that

(1.89) $$\psi^{(2)}_{jk\ell m} = \psi^{(1)}_{kjm\ell}, \qquad \psi^{(4)}_{jk\ell m} = \psi^{(3)}_{kjm\ell}.$$

We will see that the first condition in (1.89) holds and, therefore, that $h^{(2)}_{jk\ell m} = h^{(1)}_{kjm\ell}$ is also satisfied. With respect to the second condition in (1.89), and consequently

$$\widehat{\psi}_4(\xi,\eta,\mu,\nu) = \widehat{\psi}_3(\xi,\eta,\mu,\nu),$$

it works in the same way. Thus, we recall, from Section §2.3.1, that equation

$$\mathcal{L}_N(\mathcal{R}\widehat{\Psi}) = \mathcal{R}\widehat{H}$$

can be written, in a vectorial form, as

$$\left(L_N^{(1)}\widehat{\psi}_1,\, L_N^{(2)}\widehat{\psi}_2,\, L_N^{(3)}\widehat{\psi}_3,\, L_N^{(4)}\widehat{\psi}_4\right) = \left(\widehat{h}_1, \widehat{h}_2, \widehat{h}_3, \widehat{h}_4\right) \tag{1.90}$$

where

$$L_N^{(w)}\widehat{\psi}_w(\xi,\eta,\mu,\nu) = \sum_{j+k+\ell+m\geq 2} \tilde{g}^{(w)}_{jk\ell m}(\xi\eta,\mu\nu)\,\psi^{(w)}_{jk\ell m}\,\xi^j\eta^k\mu^\ell\nu^m,$$

being

$$\tilde{g}^{(w)}_{jk\ell m}(\xi\eta,\mu\nu) := \gamma^{(w)}_{jk\ell m}(\lambda) + (j-k)\,\widehat{a}_1(\xi\eta,\mu\nu) + (\ell-m)\,\widehat{a}_2(\xi\eta,\mu\nu),$$

with

$$\gamma^{(w)}_{jk\ell m}(\lambda) = \begin{cases} (j-k-1)\lambda_1 + (\ell-m)\lambda_2 & \text{if } w=1 \\ (j-k+1)\lambda_1 + (\ell-m)\lambda_2 & \text{if } w=2 \\ (j-k)\lambda_1 + (\ell-m-1)\lambda_2 & \text{if } w=3 \\ (j-k)\lambda_1 + (\ell-m+1)\lambda_2 & \text{if } w=4. \end{cases}$$

It is also derived from the same section that, provided

$$N = (\xi\, a_1(\xi\eta,\mu\nu), -\eta\, a_1(\xi\eta,\mu\nu), \mu\, a_2(\xi\eta,\mu\nu), -\nu\, a_2(\xi\eta,\mu\nu))$$

and $\mathcal{R}\widehat{H}$ are known, the formal solution $\widehat{\psi}_w(\xi,\eta,\mu,\nu)$ of equations (1.90) is uniquely determined. Thus, writing (1.90) for $w=1$ and applying the involution $\mathfrak{R}$, one obtains

$$L_N^{(1)}\widehat{\psi}_1(\eta,\xi,\nu,\mu) = \widehat{h}_1(\eta,\xi,\mu,\nu). \tag{1.91}$$

From the left-hand side of (1.91) it follows

$$\begin{aligned} L_N^{(1)}\widehat{\psi}_1(\eta,\xi,\nu,\mu) &= \sum_{j+k+\ell+m\geq 2} \tilde{g}^{(1)}_{jk\ell m}(\eta\xi,\nu\mu)\,\psi^{(1)}_{jk\ell m}\,\eta^j\xi^k\nu^\ell\mu^m = \\ &\sum_{k+j+m+\ell\geq 2} \tilde{g}^{(1)}_{kjm\ell}(\xi\eta,\mu\nu)\,\psi^{(1)}_{kjm\ell}\,\xi^j\eta^k\mu^\ell\nu^m = \\ &-\sum_{j+k+\ell+m\geq 2} \tilde{g}^{(2)}_{jk\ell m}(\xi\eta,\mu\nu)\,\psi^{(1)}_{kjm\ell}\,\xi^j\eta^k\mu^\ell\nu^m, \end{aligned}$$

where it has been used that

$$\begin{aligned} \gamma^{(1)}_{kjm\ell}(\lambda) &= (k-j-1)\lambda_1 + (m-\ell)\lambda_2 = \\ &-((j-k+1)\lambda_1 + (\ell-m)\lambda_2) = -\gamma^{(2)}_{jk\ell m}(\lambda) \end{aligned}$$

and

$$\begin{aligned} \tilde{g}^{(1)}_{kjm\ell}(\xi\eta,\mu\nu) &= \gamma^{(1)}_{kjm\ell}(\lambda) + (k-j)\,\widehat{a}_1(\xi\eta,\mu\nu) + (m-\ell)\,\widehat{a}_2(\xi\eta,\mu\nu) = \\ &-\gamma^{(2)}_{jk\ell m}(\lambda) - (j-k)\,\widehat{a}_1(\xi\eta,\mu\nu) - (\ell-m)\,\widehat{a}_2(\xi\eta,\mu\nu) = -\tilde{g}^{(2)}_{jk\ell m}(\xi\eta,\mu\nu). \end{aligned}$$

Concerning the right-hand side of (1.91), having in mind (1.88), we have that

$$\widehat{h}_1(\eta,\xi,\mu,\nu) = -\widehat{h}_2(\xi,\eta,\mu,\nu)$$

so, finally, equation (1.91) becomes equivalent to the equation

$$\sum_{j+k+\ell+m\geq 2} \tilde{g}^{(2)}_{jk\ell m}(\xi\eta,\mu\nu)\,\psi^{(1)}_{kjm\ell}\,\xi^j\eta^k\mu^\ell\nu^m = \widehat{h}_2(\xi,\eta,\mu,\nu). \tag{1.92}$$

Since the unique solution of

$$L_N^{(2)}\widehat{\psi}(\xi,\eta,\mu,\nu) = \sum_{j+k+\ell+m\geq 2} \tilde{g}^{(2)}_{jk\ell m}(\xi\eta,\mu\nu)\,\psi_{jk\ell m}\,\xi^j\eta^k\mu^\ell\nu^m = \widehat{h}_2(\xi,\eta,\mu,\nu)$$

is given by

$$\widehat{\psi}(\xi,\eta,\mu,\nu) = \sum_{j+k+\ell+m\geq 2} \psi^{(2)}_{jk\ell m}\,\xi^j\eta^k\mu^\ell\nu^m$$

it follows, comparing with (1.92), that

$$\psi^{(1)}_{kjm\ell} = \psi^{(2)}_{jk\ell m}$$

for $j+k+\ell+m \geq 2$.

(*c*) *If H is $\mathfrak{R}$-reversible then $H_{\leq K}$ (constituted by its terms of order less or equal than K) is also $\mathfrak{R}$-reversible, for any $K \geq 1$.*

This assertion comes directly from the fact that the equality $\mathfrak{R}H(\mathfrak{R}\chi) = -H(\chi)$ must be satisfied at any order.

We are now in conditions of proving the assertions (*i*), (*ii*) and (*iii*).

(*i*) First, note that from its form, $\widehat{N}^{(K)}$ satisfies that

$$\mathfrak{R}\widehat{N}^{(K)}(\mathfrak{R}\chi) = -\widehat{N}^{(K)}(\chi)$$

and, therefore, so does $N^{(K)}$,

$$\mathfrak{R}N^{(K)}(\mathfrak{R}\chi) = -N^{(K)}(\chi)\,.$$

Thus, $N^{(K)}$ is $\mathfrak{R}$-reversible for any $K \geq 1$. We are going to prove that $\Phi^{(K)}$ verifies condition (1.85) and $\widehat{B}^{(K)}$ is $\mathfrak{R}$-reversible using an inductive argument.

For $K = 2$ (the case $K = 1$ is trivial) we have that

$$\left\{\mathcal{L}_{N^{(1)}}\left(\mathcal{R}\widehat{\Phi}^{(2)}\right)\right\}_{\leq 2} = \left\{\mathcal{R}\left(\widehat{F}\left(\Phi^{(1)}\right)\right)\right\}_{\leq 2}$$

or, simplifying,

$$\mathcal{L}_\Lambda\left(\mathcal{R}\widehat{\Phi}^{(2)}\right) = F_{[2]}.$$

Applying properties (*c*) and (*b*) above one obtains that $\mathcal{R}\widehat{\Phi}^{(2)}$ preserves $\mathfrak{R}$-reversibility. On the other hand, $\widehat{B}^{(2)} = 0$ so it is trivially a $\mathfrak{R}$-reversible vector field.

Assume, by *induction hypotheses* that, for a given $K \geq 1$,

- $\Phi^{(K)} = \mathrm{id} + \mathcal{R}\widehat{\Phi}^{(K)}$ satisfies (1.85) (so it preserves $\mathfrak{R}$-reversibility),
- $\widehat{B}^{(K)}$ is a $\mathfrak{R}$-reversible vector field.

Using these induction hypotheses and applying properties (a), (c) and (b) on equation (1.86) it follows that $\mathcal{R}\widehat{\Phi}^{(K+1)}$ preserves $\mathfrak{R}$-reversibility. Consequently, so does $\Phi^{(K+1)} = \mathrm{id} + \mathcal{R}\widehat{\Phi}^{(K+1)}$. Besides, from equation (1.87) we have

$$\widehat{B}^{(K+1)} = \left\{\mathcal{P}\left(\widehat{F}(\Phi^{(K)})\right)\right\}_{\leq K+1} - \widehat{N}^{(K+1)}.$$

By induction hypothesis $\Phi^{(K)} = \mathrm{id} + \mathcal{R}\widehat{\Phi}^{(K)}$ preserves $\mathfrak{R}$-reversibility so, taking into account (a), (c) and the fact that $\widehat{N}^{(K+1)}$ is $\mathfrak{R}$-reversible, it turns out that $\widehat{B}^{(K+1)}$ is also $\mathfrak{R}$-reversible.

(ii) We have to prove now that the limit transformation $X = \Phi(\chi)$, whose convergence comes from the proof of Theorem 1.2, preserves $\mathfrak{R}$-reversibility. Using the result above, we have that

$$\mathfrak{R}\,\Phi^{(K)}(\mathfrak{R}\chi) = \Phi^{(K)}(\chi)$$

holds for any $K \geq 1$. Letting K tend to infinity it follows that

$$\mathfrak{R}\,\Phi(\mathfrak{R}\chi) = \Phi(\chi),$$

so Φ preserves $\mathfrak{R}$-reversibility. With respect to $\widehat{N}$, its $\mathfrak{R}$-reversibility comes straightforwardly from its form and its convergence. Concerning $\widehat{B}$ a similar argument to the one used for Φ applies. Thus $\widehat{B}$ is $\mathfrak{R}$-reversible and consequently we have that

$$\mathfrak{R}\,\widehat{B}(\mathfrak{R}\chi) = -\widehat{B}(\chi). \tag{1.93}$$

(iii) In particular, formula (1.93) implies that

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \eta\widehat{b}_1(\xi\eta,\mu\nu) \\ \xi\widehat{b}_1(\xi\eta,\mu\nu) \\ \nu\widehat{b}_2(\xi\eta,\mu\nu) \\ \mu\widehat{b}_2(\xi\eta,\mu\nu) \end{pmatrix} = \begin{pmatrix} \xi\widehat{b}_1(\xi\eta,\mu\nu) \\ \eta\widehat{b}_1(\xi\eta,\mu\nu) \\ \mu\widehat{b}_2(\xi\eta,\mu\nu) \\ \nu\widehat{b}_2(\xi\eta,\mu\nu) \end{pmatrix} = -\begin{pmatrix} \xi\widehat{b}_1(\xi\eta,\mu\nu) \\ \eta\widehat{b}_1(\xi\eta,\mu\nu) \\ \mu\widehat{b}_2(\xi\eta,\mu\nu) \\ \nu\widehat{b}_2(\xi\eta,\mu\nu) \end{pmatrix}$$

and, therefore,

$$\widehat{b}_1(\xi\eta,\mu\nu) = -\widehat{b}_1(\xi\eta,\mu\nu) \qquad \text{and} \qquad \widehat{b}_2(\xi\eta,\mu\nu) = -\widehat{b}_2(\xi\eta,\mu\nu),$$

so $\widehat{b}_1(\xi\eta,\mu\nu) = \widehat{b}_2(\xi\eta,\mu\nu) = 0$ and the lemma is proved.

□

From this lemma the proof of Proposition R1 follows straightforwardly. The transformation Φ preserves $\mathfrak{R}$-reversibility, the vector field N is $\mathfrak{R}$-reversible and $\widehat{B} = 0$ so, in fact, the ΨNF is nothing else but the BNF.

Chapter 2

ΨNF for a planar system

§1 Integrability and ΨNF

In Chapter 1 it was proved the convergence of the ΨNF in the case of an analytic system close to a saddle-center or a saddle-focus equilibria. Moreover, it was noticed (see Theorem 1.3) the equivalence between being Hamiltonian (and reversible) and the fact that ΨNF was just BNF. A particular case of such situation is a planar analytic system in a neighborhood of a hyperbolic equilibrium point. The next result expresses such equivalence in terms of local integrability.

Theorem 2.1 (Criterium of integrability) *Let us consider a planar system*

$$\dot{z} = F(z) = \Lambda z + \widehat{F}(z), \qquad z \in \mathbb{C}^2 \tag{2.1}$$

where Λ *is a diagonal matrix* $\{\lambda, -\lambda\}$ *with* $\lambda \neq 0$*, analytic in some domain around the origin. From Theorem* 1.2 *we know the existence of an analytic transformation*

$$z = \Phi(\zeta) = \zeta + \widehat{\Phi}(\zeta)$$

and analytic vector fields N, $\widehat{B}$,

$$N(\xi,\eta) = \begin{pmatrix} \xi a(\xi\eta) \\ -\eta a(\xi\eta) \end{pmatrix}, \qquad \widehat{B}(\xi,\eta) = \begin{pmatrix} \xi b(\xi\eta) \\ \eta b(\xi\eta) \end{pmatrix}, \tag{2.2}$$

where $a(\xi\eta) = \lambda + \cdots$, *leading system* (2.1) *into* ΨNF, *that is, satisfying*

$$D\Phi\, N + \widehat{B} = F(\Phi). \tag{2.3}$$

Then, we have that system (2.1) *has a first integral* $h(z)$ *if and only if* $b \equiv 0$. *Moreover, if* $h(z) = h(x,y)$ *is a first integral of system* (2.1) *then it has the form* $h = \tilde{h} \circ \Phi^{-1}$, *where* $\tilde{h}(\zeta) = \tilde{h}(\xi\eta)$ *depends only on the product* $\xi\eta$.

Remark 9 *We recall that an non-constant scalar function is called a* first integral *of system* (2.1) *if it is constant on its solutions.*

Proof. If $b \equiv 0$ then ΨNF becomes BNF and, therefore, any function of the form $\tilde{h}(\xi\eta)$ is a first integral of the transformed system $\dot{\zeta} = N(\zeta)$. Moreover, this is the only type of first integrals it admits. From this first integral it is straightforward to obtain a first integral for the initial system by applying the inverse change of variables. This prove the theorem in one sense.

To prove it in the other one, we will follow some ideas given by Siegel and Moser in [55, §30]. Concretely, we will assume that system (2.1) has a first integral h and that $b(\xi\eta)$ does not vanish identically. This will lead us to a contradiction so , therefore, $b(\xi\eta)$ will have to be identically zero. Thus, let us consider $h(z)$ a first integral of (2.1) and assume that

$$b(\xi\eta) = b_\ell(\xi\eta)^\ell + \cdots \tag{2.4}$$

with $b_\ell \neq 0$, for some $m \geq 1$. In this proof $\cdots$ denotes *higher order terms.*

Performing the change of variables $z = \Phi(\zeta)$ system (2.1) becomes

$$\dot{\zeta} = N(\zeta) + (D\Phi(\zeta))^{-1}\,\widehat{B}(\zeta) \tag{2.5}$$

and has $\tilde{h}(\zeta) = (h \circ \Phi)(\zeta)$ as a first integral, that we can expand in power series as

$$\tilde{h}(\xi, \eta) = \tilde{h}_M(\xi, \eta) + \tilde{h}_{M+1}(\xi, \eta) + \cdots$$

$\tilde{h}_J(\xi, \eta)$ being homogeneous polynomials of order J in ξ, η. We can assume also that $\tilde{h}_M(\xi, \eta)$ does not vanish identically. Since $\tilde{h}$ is a first integral of (2.5) it derives that the equality

$$D\tilde{h}\,\left(N + (D\Phi)^{-1}\,\widehat{B}\right) = 0 \tag{2.6}$$

holds at any order in the variable $\zeta = (\xi, \eta)$. Since Φ starts with the identity and $\widehat{B}$ is $\mathcal{O}_3$ in ξ, η it follows that the homogeneous polynomial of minimal order appearing in the left-hand side of (2.6) comes from $D\tilde{h}\,N$

$$\left(\frac{\partial}{\partial\xi}\tilde{h}_M(\xi, \eta) \quad \frac{\partial}{\partial\eta}\tilde{h}_M(\xi, \eta) \right) \cdot \left(\begin{array}{c} \lambda\xi + \cdots \\ -\lambda\eta - \cdots \end{array} \right), \tag{2.7}$$

which corresponds to the lowest order terms in $D\tilde{h}\,N$. Writing

$$\tilde{h}_M(\xi, \eta) = \sum_{j+k=M} h_{jk}^{(M)} \xi^j \eta^k,$$

and equating (2.7) to the right-hand side of (2.6) it follows that

$$\lambda \sum_{j+k=M} (j-k) h_{jk}^{(M)} \xi^j \eta^k = 0.$$

Consequently, only coefficients $h_{jk}^{(M)}$ with $j \neq k$, can be different from zero, so then $\tilde{h}(\xi, \eta)$ starts with a term of th type $h_m(\xi\eta)^m$, where $m = M/2$ and $h_m \neq 0$. In a similar way, we seek in equation (2.6) for those terms of type $(\xi\eta)^s$ having minimal order in ζ. Notice first that $D\tilde{h}(\xi, \eta)\,N(\xi, \eta)$ does not contribute to this kind of terms. Namely, writing

$$\tilde{h}(\xi, \eta) = \cdots + c_\ell(\xi\eta)^\ell + \cdots + d_{jk}\xi^j\eta^k,$$

where $j \neq k$, then it follows that

$$
\begin{aligned}
&D\tilde{h}(\xi,\eta)\ N(\xi,\eta)\\
&= \Big(\cdots + \ell c_\ell (\xi\eta)^{\ell-1}\eta + \cdots + j d_{jk}\xi^{j-1}\eta^k \\
&\quad + \cdots + \ell c_\ell(\xi\eta)^{\ell-1}\xi + \cdots + k d_{jk}\xi^j\eta^{k-1} + \cdots\Big)\begin{pmatrix} \xi A(\xi\eta) \\ -\eta A(\xi\eta)\end{pmatrix}\\
&= \cdots + c_\ell(\ell-\ell)(\xi\eta)^\ell + \cdots + d_{jk}(j-k)\xi^j\eta^k + \cdots\\
&= d_{jk}(j-k)\xi^j\eta^k + \cdots
\end{aligned}
$$

so, all the terms of type $(\xi\eta)^s$ vanish and only remain those of type $\xi^j\eta^k$. Therefore, these terms must be provided by $D\tilde{h}\,(D\Phi)^{-1}\widehat{B}$. Since Φ starts with the identity it turns out that $(D\Phi)^{-1}\widehat{B} = \widehat{B} + \cdots$. Then, taking into account (2.4) and that $\tilde{h}(\xi\eta) = h_m(\xi\eta)^m + \cdots$ it follows that the terms of type $(\xi\eta)^s$ on the left-hand side of equality (2.6) come from

$$
\begin{aligned}
&\begin{pmatrix} m h_m(\xi\eta)^{m-1}\eta + \cdots & m h_m(\xi\eta)^{m-1}\xi + \cdots \end{pmatrix}\cdot\begin{pmatrix} \xi b_\ell(\xi\eta)^\ell + \cdots \\ \eta b_\ell(\xi\eta)^\ell + \cdots\end{pmatrix}\\
&= 2m h_m b_\ell (\xi\eta)^{m+\ell} + \cdots\ ,
\end{aligned}
$$

Equating to 0 we obtain that $m h_m b_\ell = 0$, which is a contradiction since m, h_m and b_ℓ do not vanish. Consequently, it follows that $b(\xi\eta)$ must vanish.

□

§2 The group of transformations preserving the ΨNF

From the last theorem it is clear that the (analytic) scalar function $b(I)$ contains all the information about the local integrability of system (2.1) around the origin.

Let us assume for a while that system (2.1) is locally Hamiltonian. In that case Moser [42] proved the convergence of the pass to BNF and the existence of a set of analytic invariants associated to (2.1). Moreover, in that paper, Moser also proved that the transformations preserving the form of the BNF had a group-like structure which he called *group of self-transformations* of the normal form. Later, this property was generalized by Bruno [11, Chapter I] to the Poincaré-Dulac normal form, proving that any transformation close to the identity containing only resonant-type terms preserved the same kind of normal form. Conversely, he showed that any transformation preserving the *form* of the normal form had to be necessarily of that type (that is, formed only by resonant terms).

The aim of this section is to extend these results to the ΨNF case, showing that similar properties hold. This is given in the following proposition, which characterizes the set of analytic transformations preserving the ΨNF. Like in the BNF case, this set presents a group-like structure.

Proposition 2.1 *Let us consider system* (2.1) *and assume that the hypotheses of theorem* (2.1) *hold. Then we following assertions are satisfied.*

(*i*) *If we perform an analytic transformation*

$$\zeta = \Psi(\chi) = \chi + \widehat{\Psi}(\chi)$$

given by

(2.8)
$$\left\{\begin{array}{lcl}\xi &=& \tilde{\xi}\,\psi(\tilde{\xi}\tilde{\eta})\\ \eta &=& \tilde{\eta}\,\psi(\tilde{\xi}\tilde{\eta})\end{array}\right.$$

then it follows that the form of the ΨNF *is preserved, that is, the equality*

(2.9)
$$D\,(\Phi\circ\Psi)\;N^{*}+\widehat{B}^{*}=F\,(\Phi\circ\Psi)\,,$$

holds, where

$$N^{*}=\Psi^{*}N,\qquad\qquad \widehat{B}^{*}=\widehat{B}\circ\Psi.$$

Moreover, we have that

$$\mathcal{P}\,(\Phi\circ\Psi)=(\mathcal{P}\Phi)\circ\Psi,$$

$\mathcal{P}$ *being the projection defined in the precedent Chapter.*

(*ii*) *Conversely, if*

(2.10)
$$\zeta=\Psi(\chi)=\chi+\widehat{\Psi}(\chi)$$

is an analytic transformation preserving the form *of the* ΨNF (2.5)*, that is, verifying equality* (2.9)*, then* (2.10) *must be of the form* (2.8)*.*

Proof.

(*i*) Since $z=\Phi(\zeta)$ leads system (2.1) into ΨNF it derives that equation (2.3) holds, that is

$$D\Phi(\zeta)\,N(\zeta)+\widehat{B}(\zeta)=F\,(\Phi(\zeta))$$

or, in other words, that the new system takes the form

(2.11)
$$\dot{\zeta}=(\Phi^{*}F)\,(\zeta)=(D\Phi(\zeta))^{-1}\,F\,(\Phi(\zeta))=N(\zeta)+(D\Phi(\zeta))^{-1}\,\widehat{B}(\zeta).$$

Performing a transformation $\zeta=\Psi(\chi)$ system (2.11) becomes

(2.12)
$$\begin{array}{l}\dot{\chi}=(D\Psi(\chi))^{-1}\,N\,(\Psi(\chi))+(D\Psi(\chi))^{-1}\,((D\Phi)\,(\Psi(\chi)))^{-1}\,\widehat{b}\,(\Psi(\chi))=\\ \qquad(\Psi^{*}N)\,(\chi)+(D\,(\Phi\circ\Psi))^{-1}\,(\chi)\widehat{B}\,(\Psi(\chi))\,.\end{array}$$

To prove that this transformation preserves the *form* of the ΨNF it is enough to check that

$$N^{*}(\chi)=(\Psi^{*}N)\,(\chi),\qquad\qquad \widehat{B}^{*}(\chi)=\widehat{B}\,(\Psi(\chi))$$

are of type

$$\left(\begin{array}{c}\tilde{\xi}\,a^{*}(\tilde{\xi}\tilde{\eta})\\ -\tilde{\eta}\,a^{*}(\tilde{\xi}\tilde{\eta})\end{array}\right),\qquad\qquad \left(\begin{array}{c}\tilde{\xi}\,b^{*}(\tilde{\xi}\tilde{\eta})\\ \tilde{\eta}\,b^{*}(\tilde{\xi}\tilde{\eta})\end{array}\right),$$

respectively. With respect to N^{*} we have

$$J(\tilde{\xi}\tilde{\eta}):=\det D\Psi(\tilde{\xi},\tilde{\eta})=\left|\begin{array}{cc}\psi+\tilde{\xi}\tilde{\eta}\psi' & \tilde{\xi}^{2}\psi'\\ \tilde{\eta}^{2}\psi' & \psi+\tilde{\xi}\tilde{\eta}\psi'\end{array}\right|=\psi^{2}+\left(\tilde{\xi}\tilde{\eta}\right)\,(\psi^{2})',$$

where $\psi = \psi(\tilde{\xi}\tilde{\eta})$ and $'$ stands for $\frac{d}{du}\psi(u)$. So, writing

$$N(\Psi(\chi)) = \begin{pmatrix} \tilde{\xi}\,\psi(\tilde{\xi}\tilde{\eta})\,a(\tilde{\xi}\tilde{\eta}\,\psi^2(\tilde{\xi}\tilde{\eta})) \\ -\tilde{\eta}\,\psi(\tilde{\xi}\tilde{\eta})\,a(\tilde{\xi}\tilde{\eta}\,\psi^2(\tilde{\xi}\tilde{\eta})) \end{pmatrix} =: \begin{pmatrix} \tilde{\xi}\psi\,g(\tilde{\xi}\tilde{\eta}) \\ -\tilde{\eta}\psi\,g(\tilde{\xi}\tilde{\eta}) \end{pmatrix},$$

it follows that

$$\begin{aligned}
&(\Psi^* N)(\chi) = (D\Psi(\chi))^{-1}\,N(\Psi(\chi)) = \\
&\frac{1}{J(\tilde{\xi}\tilde{\eta})}\begin{pmatrix} \psi + \tilde{\xi}\tilde{\eta}\,\psi' & -\tilde{\xi}^2\,\psi' \\ -\tilde{\eta}^2\,\psi' & \psi + \tilde{\xi}\tilde{\eta}\,\psi' \end{pmatrix}\begin{pmatrix} \tilde{\xi}\,\psi\,g(\tilde{\xi}\tilde{\eta}) \\ -\tilde{\eta}\,\psi\,g(\tilde{\xi}\tilde{\eta}) \end{pmatrix} = \\
&\frac{1}{J(\tilde{\xi}\tilde{\eta})}\begin{pmatrix} \tilde{\xi}\,\psi^2\,g + \tilde{\xi}^2\tilde{\eta}\,\psi\psi'\,g + \tilde{\xi}^2\tilde{\eta}\,\psi'\psi\,g \\ -\tilde{\xi}\tilde{\eta}^2\,\psi\psi'\,g - \tilde{\eta}\,\psi^2\,g - \tilde{\xi}\tilde{\eta}^2\,\psi'\psi\,g \end{pmatrix} = \\
&\frac{1}{J(\tilde{\xi}\tilde{\eta})}\begin{pmatrix} \tilde{\xi}\,J(\tilde{\xi}\tilde{\eta})\,g(\tilde{\xi}\tilde{\eta}) \\ -\tilde{\eta}\,J(\tilde{\xi}\tilde{\eta})\,g(\tilde{\xi}\tilde{\eta}) \end{pmatrix} = \begin{pmatrix} \tilde{\xi}\,g(\tilde{\xi}\tilde{\eta}) \\ -\tilde{\eta}\,g(\tilde{\xi}\tilde{\eta}) \end{pmatrix} =: \begin{pmatrix} \tilde{\xi}\,a^*(\tilde{\xi}\tilde{\eta}) \\ -\tilde{\eta}\,a^*(\tilde{\xi}\tilde{\eta}) \end{pmatrix},
\end{aligned}$$

which is the claimed form. In particular, this means that

$$a^*(\tilde{\xi}\tilde{\eta}) = a\left(\tilde{\xi}\tilde{\eta}\,\psi^2(\tilde{\xi}\tilde{\eta})\right).$$

It is straightforward to check that $\widehat{B}^*$ is of the same form as $\widehat{B}$. Namely,

$$\widehat{B}^*(\chi) = \begin{pmatrix} \tilde{\xi}\,\psi(\tilde{\xi}\tilde{\eta})\,b\left(\tilde{\xi}\tilde{\eta}\,\psi^2(\tilde{\xi}\tilde{\eta})\right) \\ \tilde{\eta}\,\psi(\tilde{\xi}\tilde{\eta})\,b\left(\tilde{\xi}\tilde{\eta}\,\psi^2(\tilde{\xi}\tilde{\eta})\right) \end{pmatrix} =: \begin{pmatrix} \tilde{\xi}\,b^*(\tilde{\xi}\tilde{\eta}) \\ \tilde{\eta}\,b^*(\tilde{\xi}\tilde{\eta}) \end{pmatrix}.$$

Let us now consider $\Phi = (\phi_1, \phi_2)$ with

$$\phi_\ell(\xi,\eta) = \sum_{j+k\geq 1} \phi_{jk}^{(\ell)}\,\xi^j\eta^k,$$

where $\ell = 1,2$. Since $\mathcal{P}\Phi = (P_1\phi_1, P_2\phi_2)$ to prove that $\mathcal{P}(\Phi\circ\Psi) = (\mathcal{P}\Phi)\circ\Psi$ is equivalent to verify that

$$P_1\,(\phi_1 \circ \Psi) = (P_1\phi_1)\circ\Psi,$$

for $\ell = 1,2$. We will do it for $\ell = 1$. The second case works analogously. Thus,

$$\begin{aligned}
P_1(\phi_1\circ\Psi)(\tilde{\xi},\tilde{\eta}) &= P_1\left(\sum_{j+k\geq 1} \phi_{jk}^{(1)}\,\tilde{\xi}^j\tilde{\eta}^k\left(\psi_1(\tilde{\xi}\tilde{\eta})\right)^j\left(\psi(\tilde{\xi}\tilde{\eta})\right)^k\right) \\
&= \sum_{k\geq 0}\phi_{k+1,k}^{(1)}\left(\tilde{\xi}\,\psi(\tilde{\xi}\tilde{\eta})\right)^{k+1}\left(\tilde{\eta}\,\psi(\tilde{\xi}\tilde{\eta})\right)^k = ((P_1\phi_1)\circ\Psi)\,(\tilde{\xi},\tilde{\eta}).
\end{aligned}$$

(*ii*) We know that our system (2.1) can be led into ΨNF by means of the transformation $z = \Phi(\zeta)$. This implies that equality (2.3) is satisfied. If we assume that the change of variables $\zeta = \Psi(\chi)$ preserves the ΨNF it is derived that equation (2.9) is satisfied.

Moreover, the relation between both expressions is given by (2.12), which leads to the following equalities

$$N^* = \Psi^* N, \qquad \widehat{B}^* = \widehat{B} \circ \Psi. \tag{2.13}$$

If we write $\Psi(\tilde{\xi},\tilde{\eta}) = (\psi_1(\tilde{\xi},\tilde{\eta}), \psi_2(\tilde{\xi},\tilde{\eta}))$ then the first component in the last equality of (2.13) reads

$$b^*(\tilde{\xi}\tilde{\eta}) = \psi_1(\tilde{\xi},\tilde{\eta})\, b\left(\psi_1(\tilde{\xi},\tilde{\eta})\,\psi_2(\tilde{\xi},\tilde{\eta})\right).$$

Expanding the right-hand side of this expression we obtain

$$\psi_1(\tilde{\xi},\tilde{\eta})\left(b_1\,\psi_1(\tilde{\xi},\tilde{\eta})\,\psi_2(\tilde{\xi},\tilde{\eta}) + b_2\left(\psi_1(\tilde{\xi},\tilde{\eta})\,\psi_2(\tilde{\xi},\tilde{\eta})\right)^2 + \cdots\right),$$

so it is clear that if we want it to be of type $b^*(\tilde{\xi}\tilde{\eta})$ no term of type $\tilde{\xi}^j\tilde{\eta}^k$ with $j \neq k$ can appear in the expansion of ψ_1 and ψ_2. Consequently, the transformation $\zeta = \Psi(\chi)$ satisfies that $\mathcal{R}\Psi = 0$ or, in other words, that it is of the form

$$\left\{\begin{array}{lcl} \xi & = & \tilde{\xi}\,\psi_1(\tilde{\xi}\tilde{\eta}) \\ \eta & = & \tilde{\eta}\,\psi_2(\tilde{\xi}\tilde{\eta}) \end{array}\right.$$

Using again that $\widehat{B}^* = \widehat{B} \circ \Psi$ and that

$$\widehat{B}^* = \begin{pmatrix} \tilde{\xi}\, b^*(\tilde{\xi}\tilde{\eta}) \\ \tilde{\eta}\, b^*(\tilde{\xi}\tilde{\eta}) \end{pmatrix}$$

it derives straightforwardly that $\psi_1(\tilde{\xi}\tilde{\eta}) = \psi_2(\tilde{\xi}\tilde{\eta})$ and the assertion is proved.

□

Let us consider system (2.1) and assume again that the hypotheses of theorem (2.1) hold. Then, the ΨNF vector field $\widehat{B}$ is of the form

$$\widehat{B}(\xi,\eta) = \begin{pmatrix} \xi b(\xi\eta) \\ \eta b(\xi\eta) \end{pmatrix}, \tag{2.14}$$

where

$$b(\xi\eta) = \sum_{m\geq 1} b_m\ (\xi\eta)^m. \tag{2.15}$$

From the construction of the ΨNF it is clear that the b_m's can be seen as analytic functions depending on the precedent constants $b_1, b_2, \ldots, b_{m-1}$ and with coefficients depending also 0 on the coefficients of the initial system (2.1), that is,

$$b_m = b_m(b_1,\ldots,b_{m-1},F) = \sigma_{m,m}(F) + \sum_{\ell=1}^{m-1} \sigma_{m,\ell}(F)\, b_\ell. \tag{2.16}$$

In fact, since the term b_m has been computed at the $2m+1$-th step of the recurrent process, it follows that the coefficients $\sigma_{m,\ell}$ are polynomials in the coefficients of the system. Associated to the $\{b_m\}_{m\geq 1}$, let us consider the sequence $\{\beta_m\}_{m\geq 1}$, where β_m is a polynomial in the coefficients of the system (2.1) defined as

$$\beta_m = \beta_m(F) := \sigma_{m,m}. \tag{2.17}$$

In other words, if define $\mathcal{J}_j$ as the ideal generated by the *functions* $\{b_1, b_2, \ldots, b_j\}$, we have that

$$\beta_m \equiv \sigma_{m,m}(F) \qquad \text{mod } \mathcal{J}_{m-1}.$$

Assume now that we have an analytic transformation $\zeta = \Psi(\chi)$ of the form (2.8), which preserves the ΨNF, and consider $\widehat{B}^*$ the corresponding ΨNF-vector field. Let us assume that $\widehat{B}^*$ can be written as

$$\widehat{B}^* = \begin{pmatrix} \tilde{\xi}\, b^*(\tilde{\xi}\tilde{\eta}) \\ \tilde{\eta}\, b^*(\tilde{\xi}\tilde{\eta}) \end{pmatrix}$$

with

$$b^*(\tilde{\xi}\tilde{\eta}) = \sum_{m\geq 1} b^*_m \left(\tilde{\xi}\tilde{\eta}\right)^m.$$

As before, we can assume b^*_m's to be of the form

$$b^*_m = b^*_m(b^*_1, \ldots, b^*_{m-1}, F) = \sigma^*_{m,m}(F) + \sum_{\ell=1}^{m-1} \sigma^*_{m,\ell}(F)\, b^*_\ell$$

and define, analogously, for any $m \geq 1$,

$$\beta^*_m = \beta^*_m(F) := \sigma^*_{m,m}.$$

Moreover, defining $\mathcal{J}^*_m$ as the ideal generated by the *functions* $\{b^*_1, b^*_2, \ldots, b^*_m\}$, one obtains also that

$$\beta^*_m \equiv \sigma^*_{m,m}(F) \qquad \text{mod } \mathcal{J}^*_{m-1}.$$

Then we have the following result.

Proposition 2.2 *The sequences* $\{\beta_m\}_{m\geq 1}$ *and* $\{\beta^*_m\}_{m\geq 1}$ *defined above satisfy, for any* $m \geq 1$, *that*

$$\beta^*_m = \beta_m,$$

as a function on the coefficients of the system (2.1).

Proof. From Proposition 2.1 it follows that $\widehat{B}^*$ and $\widehat{B}$ verify that $\widehat{B}^* = \widehat{B} \circ \Psi$. Therefore,

$$\widehat{B}^* = \begin{pmatrix} \tilde{\xi}\, b^*(\tilde{\xi}\tilde{\eta}) \\ \tilde{\eta}\, b^*(\tilde{\xi}\tilde{\eta}) \end{pmatrix} = \begin{pmatrix} \tilde{\xi}\, \psi(\tilde{\xi}\tilde{\eta})\, b\left(\tilde{\xi}\tilde{\eta}\, \psi(\tilde{\xi}\tilde{\eta})^2\right) \\ \tilde{\eta}\, \psi(\tilde{\xi}\tilde{\eta})\, b\left(\tilde{\xi}\tilde{\eta}\, \psi(\tilde{\xi}\tilde{\eta})^2\right) \end{pmatrix}, \tag{2.18}$$

Since the transformation Ψ is of the form (2.8) and starts with the identity we have that

$$\psi(\tilde{\xi}\tilde{\eta}) = 1 + \psi_1(\tilde{\xi}\tilde{\eta}) + \psi_2(\tilde{\xi}\tilde{\eta})^2 + \cdots. \tag{2.19}$$

Thus, using (2.18) and the first step of the recurrent scheme, it is easily derived that

$$b_1 = \sigma_{1,1}(F) = \sigma^*_{1,1}(F) = b^*_1$$

and, therefore, $\beta_1^* = \beta_1$. So, let us assume by induction hypothesis that

$$\beta_s^* = \beta_s, \qquad s = 1, 2, \ldots, m-1,$$

or, equivalently $\sigma_{s,s}^*(F) = \sigma_{s,s}(F)$ for $s = 1, \ldots, m-1$. Again, from (2.18) it follows that

$$b^*(\tilde{\xi}\tilde{\eta}) = b\left(\tilde{\xi}\tilde{\eta}\,\psi^2(\tilde{\xi}\tilde{\eta})\right) = \sum_{q\geq 1} b_q \left(\tilde{\xi}\tilde{\eta}\right)^q \left(\psi(\tilde{\xi}\tilde{\eta})\right)^{2q}. \tag{2.20}$$

To determine the coefficient of $b^*(\tilde{\xi}\tilde{\eta})$ accompanying $(\tilde{\xi}\tilde{\eta})^m$ we will use the following result, called *J. C. P. Miller formula*

Lemma 2.1 ([32, page 48]) *Let us consider the following formal power series*

$$B_a := 1 + \binom{a}{1}x + \binom{a}{2}x^2 + \binom{a}{3}x^3 + \cdots,$$

commonly called Binomial series, *and* $P := p_1x + p_2x^2 + p_3x^3 + \cdots$. *Then, the formal power series corresponding to the composition* $B_a \circ P$ *is given by*

$$c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots$$

where the coefficients c_n *can be computed recurrently from the formula*

$$c_n = \frac{1}{n}\sum_{k=0}^{n-1}\left(a(n-k) - k\right)\, c_k\, p_{n-k} = \frac{1}{n}\sum_{k=1}^{n}\left((a+1)k - n\right)\, c_{n-k}\, p_k,$$

for $n = 1, 2, \ldots$.

Thus, using this result, formula (2.20) and the expression (2.19) for ψ, it follows that

$$\begin{aligned}
b^*(\tilde{\xi}\tilde{\eta}) &= \sum_{q\geq 1} b_q \left(\tilde{\xi}\tilde{\eta}\right)^q \left(\psi(\tilde{\xi}\tilde{\eta})\right)^{2q} = \\
&\sum_{q\geq 1} b_q \left(\tilde{\xi}\tilde{\eta}\right)^q \left(1 + \widehat{\psi}(\tilde{\xi}\tilde{\eta})\right)^{2q} = \sum_{q\geq 1} b_q \left(\tilde{\xi}\tilde{\eta}\right)^q \left(B_{2q}\circ\widehat{\psi}\right)(\tilde{\xi}\tilde{\eta}) = \\
&\sum_{q\geq 1} b_q \left(\tilde{\xi}\tilde{\eta}\right)^q \left(1 + c_1^{(2q)}(\tilde{\xi}\tilde{\eta}) + c_2^{(2q)}(\tilde{\xi}\tilde{\eta})^2 + c_3^{(2q)}(\tilde{\xi}\tilde{\eta})^3 + \cdots\right)
\end{aligned}$$

where the coefficients $c_n^{(2q)}$ are polynomial functions depending on $b_1, b_2, \ldots, b_{m-1}$ and the coefficients of system (2.1) and can be computed from

$$c_n^{(2q)} = \frac{1}{n}\sum_{k=1}^{n}\left((2q+1)k - n\right)\, c_{n-k}^{(2q)}\,\psi_k$$

for $n = 1, 2, \ldots$ and provided we define $c_0^{(2q)} := 1$. Therefore, the coefficient b_m^* accompanying $(\tilde{\xi}\tilde{\eta})^m$ in $b^*(\tilde{\xi}\tilde{\eta})$ is given by

$$b_m^* = \sum_{q=1}^{m} b_q\, c_{m-q}^{(2q)} = \sum_{q=1}^{m-1} b_q\, c_{m-q}^{(2q)} + b_m = \sum_{q=1}^{m-1} b_q\, c_{m-q}^{(2q)} + \left(\sigma_{m,m}(F) + \sum_{\ell=1}^{m-1}\sigma_{m,\ell}(F)\, b_\ell\right),$$

where we it has been used expression (2.16). Since

$$b_m^* = \sigma_{m,m}^*(F) + \sum_{\ell=1}^{m-1} \sigma_{m,\ell}^*(F)\, b_\ell$$

and using the induction hypothesis, it follows that $\beta_m = \sigma_{m,m}$ is equal to $\sigma_{m,m}^*(F)$ and, therefore, $\beta_m = \beta_m^*$.

□

§3 An approach to the center-focus problem using ΨNF

§3.1 The problem of a center

Let us consider in this section the case of real polynomial planar systems of the form

$$\left\{ \begin{array}{rcl} \dot{x} & = & P(x,y) \\ \dot{y} & = & Q(x,y) \end{array} \right. , \tag{2.21}$$

where the origin is an equilibrium point. Following up Darboux works (1870's), Poincaré stated the problem of giving conditions on the polynomials P and Q to ensure the existence of a first integral of system (2.21). The work of Poincaré was also continued by Painlevé. A proof of the interest that this kind of problems had aroused was the fact that this subject figured on Hilbert's list of problems (1900). When Hilbert formulated his 16th problem he divided it in two parts, one of them concerning real algebraic curves and the other wondering about the maximum number of limit cycles that could appear in a polynomial system (2.21) as a function of its degree. As it will be seen later, this question is intimately related to the problem of discerning whether a system (2.21) having a linear center at the origin (that is, with pure imaginary characteristic exponents $\pm i\alpha$) is in fact a center or a focus.

This second problem is known in the literature as the *center-focus problem*. Roughly speaking, it consists on looking for the conditions on the coefficients of polynomials P and Q in (2.21) ensuring that the origin is a center. A general answer to this question is still an open problem and up to now it has been only possible to give satisfactory answers in partial situations (for quadratic polynomials P and Q we point up the works of Bautin, Li and Zoladek; systems with non linear homogeneous part of degree three were studied by Zoladek, Sibirskii, ... and a wide list of authors; for more details and references about this problem see, for instance, [1, 50]).

In these setting we have the following theorem which connects the existence of a center at the equilibrium point with the local integrability of system (2.21). To simplify the exposition, we will assume during this section that system (2.21) has an equilibrium point at the origin with pure imaginary characteristic exponents, that we can consider to be $\pm i\alpha$, $\alpha \neq 0$. In a short way, we will say that our system has a *linear center* at the origin.

Theorem 2.2 (Poincaré, 1885) *The origin of a polynomial system* (2.21)*, having a linear center at the origin is a center if and only if it has a first integral which is analytic in a neighborhood of the origin.*

In 1892, Lyapunov generalized Poincaré's result to the case of an analytic system (2.21).

Theorem 2.3 (Lyapunov, 1892) *Given a* (2.21) *with analytic functions P and Q and having a linear center at the origin, we have that the origin is a center if and only if there exists an analytic first integral on some domain around the origin.*

In [53], Shi Songling stated the following lemma which allowed to compute formally this first integral using computer algebra calculations.

Lemma 2.2 (Shi Songling, 1981) *Consider system* (2.21) *with P and Q polynomials of degree m, having a linear center at the origin. Then, if F is a formal first integral (that is, without determining its convergence but just formally) of this system, it follows that there exist $V_1, V_2, \ldots, V_i, \ldots \in \mathbb{Q}[p_{20}, \ldots, q_{0m}]$, where $p_{20}, \ldots, q_{0m}$ are the coefficients of P and Q, such that*

$$DF = \sum_{i=1}^{\infty} V_i \left(x^2 + y^2\right)^{i+1} = 0.$$

The polynomials V_i in the coefficients of P and Q are called *Lyapunov constants.* Nevertheless, V_1 is uniquely determined from the system this is not true, as a rule, for the rest. The uniqueness comes in the following sense,

Theorem 2.4 (Shi Sonling [54]) *Let $\mathcal{A}$ be the ring $\mathbb{Q}[p_{20}, \ldots, q_{0m}]$, where $\{p_{20}, \ldots, q_{0m}\}$ are the coefficients of P and Q. Given a set of Poincaré-Lyapunov constants $V_1, V_2, \ldots, V_i$ let $\mathcal{J}_{k-1} = (V_1, V_2, \ldots, V_{k-1})$ be the ideal of $\mathcal{A}$ generated by $V_1, V_2, \ldots, V_{k-1}$.*

Then, if $\{V_1', V_2', \ldots, V_i'\}$ is another set of Lyapunov constants, one has that

$$V_k \equiv V_k' \mod (\mathcal{J}_{k-1}).$$

Therefore, having in mind Theorem 2.2, the origin is a center if and only if all the $V_k's$ 0. If we define

$$\mathcal{J} = (V_1, V_2, V_3, \ldots, V_k, \ldots),$$

the ideal generated by all the Lyapunov constants, we have, applying Hilbert's basis Theorem, that $\mathcal{J}$ is finitely generated so there exist $W_1, W_2, \ldots, W_q \in \mathcal{J}$ such that $\mathcal{J} = (W_1, W_2, \ldots, W_q)$. In other words, systems of type (2.21) for whose all the Lyapunov constants vanish (so having a center at the origin) can be characterized by a finite number of equations. These equations involve rational combinations of these Lyapunov constants.

§3.2 ΨNF and the center-focus problem

It is known that the convergence of the analytic (Birkhoff) normal form for planar analytic systems cannot be *a priori* determined in the case of a system with a linear 0 at the equilibrium point. Even more, in the general case Bruno and Walcher [15] have proved that this is equivalent to the existence of a non-trivial local one-parameter group of symmetries of the given system. In Chapter 1 we proved the equivalence between the convergence of the BNF and the fact that the vector field $\widehat{B}$ (2.14) appearing in its ΨNF vanished. In the particular case of a planar system this means that the coefficients b_j in (2.15) must be zero. Thus, for an analytic planar system (2.21) around a linear center, the following assertions are equivalent

(*i*) the origin is a center,

(*ii*) system (2.21) is (locally) Hamiltonian, (*iii*) system (2.21) is (locally) reversible,

(*iv*) BNF is convergent in a neighborhood of the origin,

(*v*) the coefficients b_j in (2.15) vanish,

(*vi*) the corresponding Lyapunov constants of system (2.21) are zero.

The last two sentences suggest a possible relation between the set of coefficients $\{b_j\}_j$ and the Lyapunov constants of (2.21). This impression is also supported on the similarity between the definition of the sequence $\{\beta_m\}_m$ in (2.17) and the way that the Lyapunov constants are defined for a general system. This led us to think that the ideal generated for both sets of constants could be the same and was the motivation of a joint research project with F. Planas and A. Guillamon that we developed from 1997 until 1999 (project UPC-9712). The aim of that work was to apply the ΨNF-method to the case of a polynomial system of the plane having a linear center at the origin. We built our own algebraic manipulator to carry out the computation of the ΨNF of system (2.21) and provided explicit formulas for the coefficients a_j and b_j as a polynomials on the coefficients of P and Q. We considered *Gröbner basis* in order to recognize whether two polynomials in some variables belonged to the same ideal but, unfortunately, this method presents to big difficulties: (a) the problem of the decomposition of polynomial in some variables in terms of *simpler* polynomials is still not satisfactory solved; (b) Second, the commonly used *Buchberger's algorithm* to compute a Gröbner basis from a set of polynomials has an upper bound for the degree of the polynomials of the basis of the order of

$$(nd)^{(n+1)\,2^{s+1}}$$

(Giusti, Möller and Mora), where n is the number of variables (that is, the number of essential coefficients of P and Q), d is the maximal degree of P and Q and s is the dimension of the ideal (which is bounded by n). In the simplest case, when $d = 2$, it is enough to consider $n = 5$ coefficients and the previous bound is $10^{6\cdot 2^{s+1}} \geq 10^{24}$!!. We will finish this section showing some features of the ΨNF-procedure in the polynomial case (2.21). At the end of this Chapter, there have been included two examples of 0 of our algebraic manipulator.

Thus, let us consider a planar system

$$\left\{\begin{array}{rclclcl} \dot{x} &=& P(x,y) &=& -y &+& \widehat{P}(x,y) \\ \dot{y} &=& Q(x,y) &=& x &+& \widehat{Q}(x,y) \end{array}\right.$$

$\widehat{P}$ and $\widehat{Q}$ being polynomials of degree m. Complexifying the variables and scaling, if necessary, this system can be written as

$$\left\{\begin{array}{rcl} \dot{z}_1 &=& f(z_1,z_2) \\ \dot{z}_2 &=& \overline{f(z_1,z_2)} = \bar{f}(z_2,z_1) \end{array}\right. \tag{2.22}$$

where $z_2 = \overline{z_1}$,

$$f(u,v) = \sum_{j,k} f_{jk}\, u^j v^k \qquad \bar{f}(u,v) = \sum_{j,k} \overline{f_{jk}}\, u^j v^k$$

and $f(z_1,z_2) = \mathrm{i}\, z_1 + \widehat{f}(z_1,z_2)$. The second equation in (2.22) is just the complex conjugate of the first one. Thus, any transformation $z = \Phi(\zeta) = \zeta + \widehat{\Phi}(\zeta)$, with $z = (z_1,z_2)$ and $\zeta = (\xi,\eta)$, of the form

$$\Phi(\xi,\eta) = \begin{pmatrix} \phi(\xi,\eta) \\ \overline{\phi}(\eta,\xi) \end{pmatrix} \tag{2.23}$$

preserves this *real* structure. It is not difficult to check that the recurrent scheme presented in Chapter 1 to compute the ΨNF preserves the structure (2.23) at any order. Moreover we have the following result.

Lemma 2.3 *Let us consider a system* (2.22) *and let* $z = \Phi(\zeta) = \zeta + \widehat{\Phi}(\zeta)$ *be an analytic transformation of type* (2.23) *leading it into* ΨNF, *that is verifying that*

$$D\Phi\, N + \widehat{B} = F\,(\Phi) \tag{2.24}$$

where $F(z_1,z_2) = (f(z_1,z_2), \bar{f}(z_2,z_1))$, *and* N, $\widehat{B}$ *are analytic vector fields of the form*

$$N(\xi,\eta) = \begin{pmatrix} \xi a(\xi\eta) \\ -\eta a(\xi\eta) \end{pmatrix}, \qquad \widehat{B}(\xi,\eta) = \begin{pmatrix} \xi b(\xi\eta) \\ \eta b(\xi\eta) \end{pmatrix}, \tag{2.25}$$

where $a(\xi\eta) = \mathrm{i} + \cdots$. *Then we have that*

$$a(\xi\eta) \text{ is pure imaginary} \iff b(\xi\eta) \text{ is real.}$$

Proof. From (2.25) it follows that equation (2.24) is equivalent to

$$\begin{aligned} \Gamma_1(\xi,\eta)\,a(\xi\eta) \;&+\; \xi\,b(\xi\eta) \;=\; f\left(\phi(\xi,\eta),\overline{\phi}(\eta,\xi)\right) \\ \Gamma_2(\xi,\eta)\,a(\xi\eta) \;&+\; \eta\,b(\xi\eta) \;=\; \bar{f}\left(\overline{\phi}(\eta,\xi),\phi(\xi,\eta)\right) \end{aligned}$$

where we define

$$\Gamma_j(\xi,\eta) := \xi\,\frac{\partial\phi_j}{\partial\xi} - \eta\,\frac{\partial\phi_j}{\partial\eta},$$

for $\phi_1 = \phi$ and $\phi_2 = \overline{\phi}$. This operator $\Gamma_j(\xi,\eta)$ satisfies the following symmetry relation,

$$\Gamma_2(\eta,\xi) = -\bar{\Gamma}_1(\xi,\eta).$$

Actually, if we write

$$\phi(\xi,\eta) = \sum_{j+k\geq 1} \phi_{jk}\,\xi^j\eta^k$$

then it follows that

$$\overline{\phi}(\eta,\xi) = \sum_{j+k\geq 1} \overline{\phi_{kj}}\,\xi^j\eta^k.$$

Thus, since

$$\Gamma_1(\xi,\eta) = \sum_{j+k\geq 1} (j-k)\,\phi_{jk}\,\xi^j\eta^k$$

it derives that

$$\overline{\Gamma_1}(\eta,\xi) = \sum_{j+k\geq 1} \overline{(k-j)\,\phi_{kj}}\,\xi^j\eta^k =$$

$$\sum_{j+k\geq 1} (k-j)\,\overline{\phi_{kj}}\,\xi^j\eta^k = \eta\frac{\partial\overline{\phi}}{\partial\eta}(\xi,\eta) - \xi\frac{\partial\overline{\phi}}{\partial\xi}(\xi,\eta) = -\Gamma_2(\xi,\eta).$$

Then, having in mind that

$$\overline{f\left(\phi(\xi,\eta),\overline{\phi}(\eta,\xi)\right)} = \bar{f}\left(\overline{\phi}(\eta,\xi),\phi(\xi,\eta)\right)$$

it turns out that

$$\begin{aligned}\Gamma_2(\xi,\eta)\,a(\xi\eta)+\eta\,b(\xi\eta) &= \overline{\Gamma_1(\xi,\eta)\,a(\xi\eta)+\xi\,b(\xi\eta)} = \\ \overline{\Gamma_1}(\eta,\xi)\,\bar{a}(\xi\eta)+\eta\,\bar{b}(\xi\eta) &= -\Gamma_2(\xi,\eta)\,\bar{a}(\xi\eta)+\eta\,\bar{b}(\xi\eta)\end{aligned}$$

and so

$$\Gamma_2(\xi,\eta)\left(a(\xi\eta)+\bar{a}(\xi\eta)\right) = \eta\left(\bar{b}(\xi\eta)-b(\xi\eta)\right)$$

or, equivalently,

$$\Gamma_2(\xi,\eta)\,\Re a(\xi\eta) = -\mathrm{i}\,\eta\,\Im b(\xi\eta). \tag{2.26}$$

Since $\Gamma_2(\xi,\eta) = -\eta+\widehat{\Gamma_2}(\xi,\eta)\neq 0$, where $\widehat{\ }$ means terms of order at least 2 in ξ, η, it follows the claimed result.

□

In the Qualitative Theory of planar systems, the function $a(u)$ plays an important rôle in the case that the linear center becomes a center. It is related to the periods of closed orbits around this point. More precisely, if the origin is a center ΨNF becomes BNF and the *normalized* system is

$$\begin{cases}\dot{\xi} &= \xi\,a(\xi\eta)\\ \dot{\eta} &= -\eta\,a(\xi\eta)\end{cases}$$

Taking polar coordinates ($\xi = r\cos\theta$, $\eta = r\sin\theta$) this is equivalent to

$$\begin{cases}\dot{r} &= 0\\ \dot{\theta} &= -\mathrm{i}\,a(r^2)\end{cases}. \tag{2.27}$$

If we denote by $r(\theta,\rho)$ the solution of system (2.27) satisfying that $r(0,\rho)=\rho$ at time $t=0$, we get that

$$\begin{aligned}r(t) &= \rho\\ \theta(t) &= -\mathrm{i}\,a(\rho^2)t, \qquad t\geq 0.\end{aligned}$$

In this context it is known that $a(u)$ is pure imaginary, $a(u)=\mathrm{i}\,(1+a_1u+a_2u^2+\cdots)$ so the following expansion can be derived for the (real) angular variable $\theta(t)$

$$\theta(t) = \left(1+a_1\rho^2+a_2\rho^4+\cdots+a_j\rho^{2j}+\cdots\right)t.$$

Defining the *period function* $T(\rho)$ as the time the solution $r(\theta,\rho)$ needs to reach the Poincaré section $\theta=0 \pmod{2\pi}$ we have

$$2\pi = \theta\left(T(\rho)\right) = \left(1+a_1\rho^2+a_2\rho^4+\cdots\right)T(\rho)$$

and, consequently,

$$T(\rho) = 2\phi\left(1+a_1\rho^2+a_2\rho^4+\cdots\right)^{-1}. \tag{2.28}$$

A power series expansion of $T(\rho)$ in (2.28) is easily obtained using that $(1+\delta)^{-1} = 1-\delta+\delta^2-\delta^3+\cdots$ provided δ small. Since ΨNF just generalizes BNF it is reasonable to expect that the function $a(\xi\eta)$ provided by the ΨNF-procedure is also pure imaginary. And therefore, applying the previous lemma, that $b(\xi\eta)$ is real.

ΨNF up to order 9 in x, y and order 8 in the parameters

$$\dot{x} = f(x, y)$$

$$\dot{y} = g(x, y)$$

with

$$f(x, y) = \{\ -1\}\, y + \{\ +2a_1\ \ +1a_4\}\, xy + O(10)$$

$$g(x, y) = \{\ +1\}\, x + \{\ +1a_1\}\, x^2 + \{\ -1a_1\}\, y^2 + O(10)$$

Performing the transformation

$$\begin{cases} x &= \{\ +1\}\, z_1 + \{\ +1\}\, z_2 + O(10) \\ y &= \{\ -1i\}\, z_1 + \{\ +1i\}\, z_2 + O(10) \end{cases}$$

the initial system can be written in the form

$$\dot{z_1} = f^*(z_1, z_2)$$

$$\dot{z_2} = g^*(z_1, z_2)$$

with

$$f^*(z_1, z_2) = \{\ +1i\}\, z_1 + \{\ -0.5ia_4\}\, z_1^2 + \{\ +2ia_1\ \ +0.5ia_4\}\, z_2^2 + O(10)$$

$$g^*(z_1, z_2) = \{\ -1i\}\, z_2 + \{\ -2ia_1\ \ -0.5ia_4\}\, z_1^2 + \{\ +0.5ia_4\}\, z_2^2 + O(10)$$

Change of variables:

$$z_1 = \phi^{(1)}(\xi, \eta)$$
$$z_2 = \phi^{(2)}(\xi, \eta)$$

with fixed projections

$$P_1\phi^{(1)}(\xi, \eta) = O(10)$$

$$P_{-1}\phi^{(2)}(\xi, \eta) = O(10)$$

Figure 2.1: Output of our algebraic manipulator (possible center) (1)

Results:

$A(\xi\eta) = \{\ \ +1i\} + O(10)$

$b(\xi\eta) = O(10)$

Checking:

$$||(D\Phi)N + B - F(\Phi)||_\infty$$

$\nabla\phi^{(1)}(\xi,\eta)N(\xi,\eta) - f^{(1)}(\Phi(\xi,\eta)) = O(10)$

Error in the first component: 0.000000e+00

$\nabla\phi^{(2)}(\xi,\eta)N(\xi,\eta) + \eta b(\xi\eta) - f^{(2)}(\Phi(\xi,\eta)) = O(10)$

Error in the second component: 0.000000e+00

Figure 2.2: Output of our algebraic manipulator (possible center) (2)

ΨNF up to order 11 in x, y and order 13 in the parameters

$$\dot{x} = f(x,y)$$

$$\dot{y} = g(x,y)$$

with

$$f(x,y) = \{\ -1\}\, y + \{\ -1a_2\}\, x^2 + \{\ +2a_1\ \ +1a_4\}\, xy + \{\ +1a_5\}\, y^2 + O(12)$$

$$g(x,y) = \{\ +1\}\, x + \{\ +1a_1\}\, x^2 + \{\ +2a_2\ \ +1a_3\}\, xy + \{\ -1a_1\}\, y^2 + O(12)$$

Performing the transformation

$$\begin{cases} x = \{\ +1\}\, z_1 + \{\ +1\}\, z_2 + O(12) \\ y = \{\ -1i\}\, z_1 + \{\ +1i\}\, z_2 + O(12) \end{cases}$$

the initial system can be written in the form

$$\dot{z_1} = f^*(z_1, z_2)$$

$$\dot{z_2} = g^*(z_1, z_2)$$

with

$$f^*(z_1,z_2) = \{\ +1i\}\, z_1 + \{\ +0.5a_2\ \ +0.5a_3\ \ -0.5ia_4\ \ -0.5a_5\}\, z_1^2 + \{\ -1a_2\ \ +1a_5\}\, z_1 z_2 + \{\ +2ia_1\ \ -1.5a_2\ \ -0.5a_3\ \ +0.5ia_4\ \ -0.5a_5\}\, z_2^2 + O(12)$$

$$g^*(z_1,z_2) = \{\ -1i\}\, z_2 + \{\ -2ia_1\ \ -1.5a_2\ \ -0.5a_3\ \ -0.5ia_4\ \ -0.5a_5\}\, z_1^2 + \{\ -1a_2\ \ +1a_5\}\, z_1 z_2 + \{\ +0.5a_2\ \ +0.5a_3\ \ +0.5ia_4\ \ -0.5a_5\}\, z_2^2 + O(12)$$

Change of variables:

$$z_1 = \phi^{(1)}(\xi, \eta)$$

$$z_2 = \phi^{(2)}(\xi, \eta)$$

with fixed projections

$$P_1 \phi^{(1)}(\xi,\eta) = \{\ +1\}\, \xi + \{\ +1a_6\}\, \xi^2 \eta + \{\ +1a_7\}\, \xi^3 \eta^2 + \{\ +1a_8\}\, \xi^4 \eta^3 + O(12)$$

$$P_{-1} \phi^{(2)}(\xi,\eta) = \{\ +1\}\, \eta + \{\ +1a_6\}\, \xi \eta^2 + \{\ +1a_7\}\, \xi^2 \eta^3 + \{\ +1a_8\}\, \xi^3 \eta^4 + O(12)$$

Results:

$$b(\xi\eta) = \{\ -0.5a_2a_4\ \ +0.5a_4a_5\}\, (\xi\eta)^1 + \{\ -1.5a_2a_4a_6\ \ +1.5a_4a_5a_6\ \ +1.66667a_1a_2^2a_3\ \ +0.333333a_1a_2a_3^2\ \ -2.22222a_1^2a_2a_4\ \ -1a_2^3a_4\ \ -1a_2^2a_3a_4\ \ +0.111111a_2a_3^2a_4\ \ -1.44444a_1a_2a_4^2\ \ -0.222222a_2a_4^3\ \ -3.33333a_1a_2a_3a_5\ \ -0.333333a_1a_3^2a_5\ \ +2.22222a_1^2a_4a_5\ \ +1.33333a_2^2a_4a_5\ \ +1.55556a_2a_3a_4a_5\ \ -0.111111a_3^2a_4a_5\ \ +1.44444a_1a_4^2a_5\ \ +0.222222a_4^3a_5\ \ +1.66667a_1a_3a_5^2\ \ -1.88889a_2a_4a_5^2\ \ -0.555556a_3a_4a_5^2\ \ +1.55556a_4a_5^3\}\, (\xi\eta)^2 + \cdots\cdots\cdots\cdots$$

Figure 2.3: Output of our algebraic manipulator (no center) (1)

$$\cdots\{\ -1.5a_2a_4a_7\ +1.5a_4a_5a_7\ +8.33333a_1a_2^2a_3a_6\ +1.66667a_1a_2a_3^2a_6\ -11.1111a_1^2a_2a_4a_6$$
$$-5a_2^3a_4a_6\ -5a_2^2a_3a_4a_6\ +0.555556a_2a_3^2a_4a_6\ -7.22222a_1a_2a_4^2a_6\ -1.11111a_2a_4^3a_6\ -16.6667a_1a_2a_3a_5a_6$$
$$-1.66667a_1a_3^2a_5a_6\ +11.1111a_1^2a_4a_5a_6\ +6.66667a_2^2a_4a_5a_6\ +7.77778a_2a_3a_4a_5a_6\ -0.555556a_3^2a_4a_5a_6$$
$$+7.22222a_1a_4^2a_5a_6\ +1.11111a_4^3a_5a_6\ +8.33333a_1a_3a_5^2a_6\ -9.44444a_2a_4a_5^2a_6\ -2.77778a_3a_4a_5^2a_6$$
$$+7.77778a_4a_5^3a_6\ +20a_1^3a_2^2a_3\ +18.75a_1a_2^4a_3\ +3a_1^3a_2a_3^2\ +18.125a_1a_2^3a_3^2\ +3.91667a_1a_2^2a_3^3$$
$$+0.208333a_1a_2a_3^4\ -11.1605a_1^4a_2a_4\ -23.6667a_1^2a_2^3a_4\ -3.625a_2^5a_4\ -9.07407a_1^2a_2^2a_3a_4\ -6.08333a_2^4a_3a_4$$
$$+2.96605a_1^2a_2a_3^2a_4\ +0.725694a_2^3a_3^2a_4\ +2.07176a_2^2a_3^3a_4\ +0.213349a_2a_3^4a_4\ -14.1605a_1^3a_2a_4^2$$
$$-20.9444a_1a_2^3a_4^2\ -15.9815a_1a_2^2a_3a_4^2\ -0.0169753a_1a_2a_3^2a_4^2\ -5.82407a_1^2a_2a_4^3\ -4.51736a_2^3a_4^3$$
$$-3.8588a_2^2a_3a_4^3\ -0.281636a_2a_3^2a_4^3\ -0.725309a_1a_2a_4^4\ +0.00501543a_2a_4^5\ -40a_1^3a_2a_3a_5\ -45a_1a_2^3a_3a_5$$
$$-3a_1^3a_3^2a_5\ -39.875a_1a_2^2a_3^2a_5\ -6.33333a_1a_2a_3^3a_5\ -0.208333a_1a_3^4a_5\ +11.1605a_1^4a_4a_5\ +57.5185a_1^2a_2^2a_4a_5$$
$$+7.54167a_2^4a_4a_5\ +12.9877a_1^2a_2a_3a_4a_5\ +11.125a_2^3a_3a_4a_5\ -2.96605a_1^2a_3^2a_4a_5\ -0.417824a_2^2a_3^2a_4a_5$$
$$-2.74383a_2a_3^3a_4a_5\ -0.213349a_3^4a_4a_5\ +14.1605a_1^3a_4^2a_5\ +51.5926a_1a_2^2a_4^2a_5\ +27.8827a_1a_2a_3a_4^2a_5$$
$$+0.0169753a_1a_3^2a_4^2a_5\ +5.82407a_1^2a_4^3a_5\ +9.98727a_2^2a_4^3a_5\ +6.61728a_2a_3a_4^3a_5\ +0.281636a_3^2a_4^3a_5$$
$$+0.725309a_1a_4^4a_5\ -0.00501543a_4^5a_5\ +20a_1^3a_3a_5^2\ +42.5a_1a_2^2a_3a_5^2\ +28.375a_1a_2a_3^2a_5^2\ +2.41667a_1a_3^3a_5^2$$
$$-66.358a_1^2a_2a_4a_5^2\ -12.6944a_2^3a_4a_5^2\ -3.91358a_1^2a_3a_4a_5^2\ -13.6898a_2^2a_3a_4a_5^2\ +0.334491a_2a_3^2a_4a_5^2$$
$$+0.672068a_3^3a_4a_5^2\ -54.5123a_1a_2a_4^2a_5^2\ -11.9012a_1a_3a_4^2a_5^2\ -7.9564a_2a_4^3a_5^2\ -2.75849a_3a_4^3a_5^2$$
$$-25a_1a_2a_3a_5^3\ -6.625a_1a_3^2a_5^3\ +32.5062a_1^2a_4a_5^3\ +15.6019a_2^2a_4a_5^3\ +13.1775a_2a_3a_4a_5^3\ -0.642361a_3^2a_4a_5^3$$
$$+23.8642a_1a_4^2a_5^3\ +2.4865a_4^3a_5^3\ +8.75a_1a_3a_5^4\ -16.3225a_2a_4a_5^4\ -4.52932a_3a_4a_5^4\ +9.49846a_4a_5^5\}\,(\xi\eta)^3+$$
$$\cdots$$
$$\cdots\cdots\cdots$$
$$\cdots -11457.9a_1a_2^3a_3^3a_5^3\ -387.219a_1^3a_3^4a_5^3\ -2639.2a_1a_2^2a_3^4a_5^3\ -231.808a_1a_2a_3^5a_5^3\ -5.56018a_1a_3^6a_5^3$$
$$+5191.65a_1^6a_4a_5^3\ +60773.9a_1^4a_2^2a_4a_5^3\ +44262.7a_1^2a_2^4a_4a_5^3\ +1440.13a_2^6a_4a_5^3\ +6842.64a_1^4a_2a_3a_4a_5^3$$
$$+18821.8a_1^2a_2^3a_3a_4a_5^3\ +2177.04a_2^5a_3a_4a_5^3\ -3434.98a_1^4a_3^2a_4a_5^3\ -13516.5a_1^2a_2^2a_3^2a_4a_5^3\ -389.588a_2^4a_3^2a_4a_5^3$$
$$-5884a_1^2a_2a_3^3a_4a_5^3\ -1830.27a_2^3a_3^3a_4a_5^3\ -445.952a_1^2a_3^4a_4a_5^3\ -870.883a_2^2a_3^4a_4a_5^3\ -134.816a_2a_3^5a_4a_5^3$$
$$-6.30878a_3^6a_4a_5^3\ +11531.8a_1^5a_4^2a_5^3\ +92835.9a_1^3a_2^2a_4^2a_5^3\ +35436.7a_1a_2^4a_4^2a_5^3\ +36140.8a_1^3a_2a_3a_4^2a_5^3$$
$$+37645.2a_1a_2^3a_3a_4^2a_5^3\ +378.43a_1^3a_3^2a_4^2a_5^3\ +9349.93a_1a_2^2a_3^2a_4^2a_5^3\ -255.603a_1a_2a_3^3a_4^2a_5^3\ -106.308a_1a_3^4a_4^2a_5^3$$
$$+8873.19a_1^4a_4^3a_5^3\ +45477.3a_1^2a_2^2a_4^3a_5^3\ +4770.58a_2^4a_4^3a_5^3\ +22075.4a_1^2a_2a_3a_4^3a_5^3\ +6354.26a_2^3a_3a_4^3a_5^3$$
$$+1277.71a_1^2a_3^2a_4^3a_5^3\ +2548.06a_2^2a_3^2a_4^3a_5^3\ +276.922a_2a_3^3a_4^3a_5^3\ +2.93504a_3^4a_4^3a_5^3\ +3078.06a_1^3a_4^4a_5^3$$
$$+8375.88a_1a_2^2a_4^4a_5^3\ +4702.46a_1a_2a_3a_4^4a_5^3\ +413.907a_1a_3^2a_4^4a_5^3\ +486.948a_1^2a_4^5a_5^3\ +506.33a_2^2a_4^5a_5^3$$
$$+314.666a_2a_3a_4^5a_5^3\ +38.6062a_3^2a_4^5a_5^3\ +29.2916a_1a_4^6a_5^3\ +0.286527a_4^7a_5^3\ +4740.44a_1^5a_3a_5^4\ +29243.1a_1^3a_2^2a_3a_5^4$$
$$+9638.44a_1a_2^4a_3a_5^4\ +15994.3a_1^3a_2a_3^2a_5^4\ +13251.2a_1a_2^3a_3^2a_5^4\ +1350.67a_1^3a_3^3a_5^4\ +5514.76a_1a_2^2a_3^3a_5^4$$
$$+763.826a_1a_2a_3^4a_5^4\ +29.9299a_1a_3^5a_5^4\ -37468.7a_1^4a_2a_4a_5^4\ -44095.9a_1^2a_2^3a_4a_5^4\ -1807.26a_2^5a_4a_5^4$$
$$-1754.79a_1^4a_3a_4a_5^4\ -14846.2a_1^2a_2^2a_3a_4a_5^4\ -2552.08a_2^4a_3a_4a_5^4\ +6573.21a_1^2a_2a_3^2a_4a_5^4\ -116.365a_2^3a_3^2a_4a_5^4$$
$$+1091.56a_1^2a_3^3a_4a_5^4\ +854.689a_2^2a_3^3a_4a_5^4\ +257.204a_2a_3^4a_4a_5^4\ +18.8122a_3^5a_4a_5^4\ -55063.9a_1^3a_2a_4^2a_5^4$$
$$-32768.6a_1a_2^3a_4^2a_5^4\ -9519.99a_1^3a_3a_4^2a_5^4\ -25485.4a_1a_2^2a_3a_4^2a_5^4\ -3386.03a_1a_2a_3^2a_4^2a_5^4\ +55.6956a_1a_3^3a_4^2a_5^4$$
$$-25629.9a_1^2a_2a_4^3a_5^4\ -4091.91a_2^3a_4^3a_5^4\ -5773.97a_1^2a_3a_4^3a_5^4\ -3966.08a_2^2a_3a_4^3a_5^4\ -981.605a_2a_3^2a_4^3a_5^4$$
$$-46.1242a_3^3a_4^3a_5^4\ -4320.5a_1a_2a_4^4a_5^4\ -1207.26a_1a_3a_4^4a_5^4\ -238.172a_2a_4^5a_5^4\ -75.674a_3a_4^5a_5^4\ -13774.5a_1^3a_2a_3a_5^5$$
$$-6888.29a_1a_2^3a_3a_5^5\ -3250.25a_1^3a_3^2a_5^5\ -6828.25a_1a_2^2a_3^2a_5^5\ -1637.84a_1a_2a_3^3a_5^5\ -104.796a_1a_3^4a_5^5$$
$$+10758.7a_1^4a_4a_5^5\ +33929.5a_1^2a_2^2a_4a_5^5\ +1969.81a_2^4a_4a_5^5\ +7527.71a_1^2a_2a_3a_4a_5^5\ +2369.53a_2^3a_3a_4a_5^5$$
$$-1791.26a_1^2a_3^2a_4a_5^5\ +226.501a_2^2a_3^2a_4a_5^5\ -293.482a_2a_3^3a_4a_5^5\ -37.9326a_3^4a_4a_5^5\ +15350.8a_1^3a_4^2a_5^5$$
$$+24038.6a_1a_2^2a_4^2a_5^5\ +11644.9a_1a_2a_3a_4^2a_5^5\ +341.294a_1a_3^2a_4^2a_5^5\ +6676.38a_1^2a_4^3a_5^5\ +2860.88a_2^2a_4^3a_5^5$$
$$+1690.27a_2a_3a_4^3a_5^5\ +145.977a_3^2a_4^3a_5^5\ +976.262a_1a_4^4a_5^5\ +43.4479a_4^5a_5^5\ +3355a_1^3a_3a_5^6\ +4204.65a_1a_2^2a_3a_5^6$$
$$+2543.91a_1a_2a_3^2a_5^6\ +212.54a_1a_3^3a_5^6\ -18850a_1^2a_2a_4a_5^6\ -1993.39a_2^3a_4a_5^6\ -1740.63a_1^2a_3a_4a_5^6$$
$$-1697.78a_2^2a_3a_4a_5^6\ +29.3031a_2a_3^2a_4a_5^6\ +28.2903a_3^3a_4a_5^6\ -13032.9a_1a_2a_4^2a_5^6\ -2665.01a_1a_3a_4^2a_5^6$$
$$-1455.47a_2a_4^3a_5^6\ -408.731a_3a_4^3a_5^6\ -2009.67a_1a_2a_3a_5^7\ -483.922a_1a_3^2a_5^7\ +5463.53a_1^2a_4a_5^7\ +1997.64a_2^2a_4a_5^7$$
$$+949.805a_2a_3a_4a_5^7\ -121.17a_3^2a_4a_5^7\ +3727.57a_1a_4^2a_5^7\ +347.463a_4^3a_5^7\ +543.238a_1a_3a_5^8\ -1673.26a_2a_4a_5^8$$
$$-265.598a_3a_4a_5^8\ +688.652a_4a_5^9\}\,(\xi\eta)^5+O(12)$$

Figure 2.4: Output of our algebraic manipulator (no center) (2)

Chapter 3

ΨNF around a hyperbolic periodic orbit

§1 Introduction and main results

This chapter is devoted to the convergence of the pass to ΨNF for an analytic system in a neighborhood of a hyperbolic periodic orbit. The tools used to build such a change of variables and the corresponding ΨNF-vector fields N and $\widehat{B}$ are essentially the same introduced in Chapter 1 and the argument employed to prove their convergence is also similar. However, there are some differences with respect to the treatment given there that have lead us to include here a quite detailed study. Namely, in this case we will follow a perturbative approach, starting from a integrable system (for what BNF is convergent) and looking for a convergent ΨNF-scheme when a periodic analytic perturbation is added. This means that the homological equation to solve is a bit more complicated than the one introduced in Chapter 1 but presenting, however the same important features, the projections $\mathcal{P}$, $\mathcal{R}$ and the functional operator $\mathcal{L}_N$. Apart from this, the notation, properties and initial formal solutions presented in this chapter can be trivially extended to the *quasi-periodic* case by assuming $\theta = (\theta_1, \theta_2, \ldots, \theta_n) \in \mathbb{T}^n$, $\omega(\varepsilon) = (\omega_1(\varepsilon), \omega_2(\varepsilon), \ldots, \omega_n(\varepsilon)) \in \mathbb{R}^n$ and taking $\mathbb{Z}^n$ instead of $\mathbb{Z}$. The main difference arises when one is solving (formally) for the coefficients of a given order $K+1$ of the the transformation Φ: in the quasi-periodic case we will have *small divisors* (that is denominators close to zero in formulas similar to the ones given in (3.24) that will difficult the convergence of the procedure; like in the Hamiltonian case, to control them, we will need to assume that the *frequency vector* $\omega(\varepsilon)$ satisfies some Diophantine condition. One of our future working plans is to study the convergence of this ΨNF in the case of an analytic system depending quasi-periodically on time.

Coming back to the target of this chapter, let us consider an analytic system

$$\dot{z} = F_\circ(z) = \Lambda_\circ z + \widehat{F}_\circ(z), \tag{3.1}$$

where

$$\Lambda_\circ = \begin{pmatrix} \lambda_\circ & 0 \\ 0 & -\lambda_\circ \end{pmatrix}$$

being $z \in \mathbb{C}$ and $\lambda_\circ > 0$. Henceforth we will denote by $\widehat{G}$ to indicate that the vector field $\widehat{G}$ starts with terms of order at least 2 with respect to the spatial variable z. Assume that system (3.1) can be led into BNF, that is, there exists an analytic transformation

$$z = \Phi_\circ(\zeta) = \zeta + \widehat{\Phi}_\circ(\zeta) \tag{3.2}$$

with $\zeta = (\xi, \eta)$, taking it into the form $\dot{\zeta} = N_\circ(\zeta)$ where

$$N_\circ(\zeta) = \begin{pmatrix} \xi a_\circ(\xi\eta) \\ -\eta a_\circ(\xi\eta) \end{pmatrix}$$

and $a_\circ(\xi\eta) = \lambda_\circ + \widehat{a}_\circ(\xi\eta)$. Since $N_\circ = \Phi_\circ^* F_\circ = (D\Phi_\circ)^{-1} F_\circ(\Phi_\circ)$, this is equivalent to say that the equality

$$D\Phi_\circ\, N_\circ = F_\circ\,(\Phi_\circ) \tag{3.3}$$

holds at any point ζ in the domain of convergence. We are interested on the study of the ΨNF of an integrable system (3.1) when we modify it with an small periodic perturbation, that is, when we consider a system

$$\left\{ \begin{array}{rcl} \dot{z} & = & F(z, \theta, \varepsilon, \mu) := F_\circ(z) + F_\mu(z, \theta, \varepsilon), \\ \dot{\theta} & = & \omega(\varepsilon) \end{array} \right. , \tag{3.4}$$

where F_μ is assumed to be analytic with respect the spatial variable z, 2π-periodic and analytic in θ, with $\omega(\varepsilon) \in \mathbb{R}$ and μ, $\varepsilon > 0$ independent small parameters. Moreover, we will suppose F_μ to be of the type

$$F_\mu(z, \theta, \varepsilon) = \mu\varepsilon^q F_1(z, \theta, \varepsilon),$$

where $q \geq 0$ so, in fact, this system can be seen as a perturbation of system (3.1). Since (3.1) has a hyperbolic equilibrium point at the origin, with eigenvalues $\pm\lambda_\circ$, it follows that

$$\gamma_\circ := \{(0, 0, \theta)\}_{\theta \in \mathbb{T}}$$

is a hyperbolic periodic orbit of the system

$$\left\{ \begin{array}{rcl} \dot{z} & = & F_\circ(z), \\ \dot{\theta} & = & \omega(\varepsilon) \end{array} \right.$$

with characteristic exponents $\pm\lambda_\circ$.

Remark 10 *Concerning $\omega(\varepsilon)$, two cases can be essentially considered: if $\omega(\varepsilon) = 1$ it corresponds to a periodic perturbation; if $\omega(\varepsilon) = 1/\varepsilon$ it can be seen as a rapidly forcing of the original system (which appears when one is dealing with the problem of the splitting of separatrices) or, taking θ as the new time, as a nearly integrable system with slow dynamics.*

On the other hand, the previous result can be also proved if the dependence of F_μ on θ is $\mathcal{C}^1$ instead of analytic.

It is well known that, for μ small enough, there is still a hyperbolic periodic orbit γ_μ of system (3.4) which remains close to $\gamma_\circ$ (for an explicit construction of this γ_μ see, for instance, [23, page 439]). Moreover, using Floquet Theory, we know the existence of a change of variables leading the linear part of F to constant coefficients. Therefore, in order to simplify the computations, let us assume that F_μ is given in such a way that F_μ vanishes at $z=0$ (so γ_μ becomes $\gamma_\circ$) and that its linear part does not depend on θ. Moreover, we will assume also that this linear part is in written in diagonal form, that is,

$$F_\mu(z,\theta,\varepsilon) = \Lambda_\mu(\varepsilon)z + \widehat{F}_\mu(z,\theta,\varepsilon),$$

with

$$\Lambda_\mu = \begin{pmatrix} \lambda_\mu(\varepsilon) & 0 \\ 0 & -\lambda_\mu(\varepsilon) \end{pmatrix}, \tag{3.5}$$

and $\lambda_\mu(\varepsilon) \in \mathbb{R}$. Denoting $X = (z,\theta) = (x,y,\theta)$ and $\chi = (\zeta,\theta) = (\xi,\eta,\theta)$, our aim is to seek for an analytic transformation

$$X = \Phi^{\ltimes}(\chi,\varepsilon,\mu)$$

of the form

$$\begin{pmatrix} z \\ \theta \end{pmatrix} = \begin{pmatrix} \Phi(z,\theta,\varepsilon,\mu) \\ \theta \end{pmatrix} = \begin{pmatrix} \zeta \\ \theta \end{pmatrix} + \begin{pmatrix} \widehat{\Phi}(z,\theta,\varepsilon,\mu) \\ 0 \end{pmatrix}, \tag{3.6}$$

and analytic vector fields

$$N(\zeta,\varepsilon,\mu) = \begin{pmatrix} \xi a(\xi\eta,\varepsilon,\mu) \\ -\eta a(\xi\eta,\varepsilon,\mu) \\ \omega(\varepsilon) \end{pmatrix}, \qquad \widehat{B}(\zeta,\varepsilon,\mu) = \begin{pmatrix} \xi\widehat{b}(\xi\eta,\varepsilon,\mu) \\ \eta\widehat{b}(\xi\eta,\varepsilon,\mu) \\ 0 \end{pmatrix} \tag{3.7}$$

leading system (3.4) into ΨNF, that is, satisfying the equality

$$D\Phi^{\ltimes} N + \widehat{B} = (F,\omega(\varepsilon)) \circ \Phi^{\ltimes}. \tag{3.8}$$

Thus, the main result concerning the existence and convergence of such a change of variables and vector fields can be established as follows.

Theorem 3.1 (Main Theorem) *Let us consider a system*

$$\begin{cases} \dot{z} &= F(z,\theta,\varepsilon,\mu) := F_\circ(z) + F_\mu(z,\theta,\varepsilon), \\ \dot{\theta} &= \omega(\varepsilon) \end{cases}, \tag{3.9}$$

where F is defined on a domain $\overline{\mathcal{D}_{r_\circ}} \times \mathcal{T}_\rho$ (see Section §2.7.1), analytic in z and 2π-periodic and analytic in θ. Moreover, F_μ is of the form $\mu\varepsilon^q F_1(z,\theta,\varepsilon)$, $\omega(\varepsilon) > 0$ and μ, $\varepsilon > 0$ are small parameters. Let us assume that for the unperturbed system *($\mu = 0$)*

$$\dot{z} = F_\circ(z) = \Lambda_\circ z + \widehat{F}_\circ(z), \tag{3.10}$$

where

$$\Lambda_\circ = \begin{pmatrix} \lambda_\circ & 0 \\ 0 & -\lambda_\circ \end{pmatrix}$$

and $\lambda_\circ > 0$, we have the existence of a transformation $z = \Phi_\circ(\zeta) = \zeta + \widehat{\Phi}_\circ(\zeta)$ and a vector field $N_\circ(\zeta) = (\xi a_\circ(\xi\eta), \eta a_\circ(\xi\eta))$, with $a_\circ(\xi\eta) = \lambda_\circ + \widehat{a}_\circ(\xi\eta)$, both analytic on $\overline{\mathcal{D}_{\frac{3}{4}r_\circ}}$ and leading system (3.10) *into* ΨNF*, that is, satisfying that*

$$D\Phi_\circ N_\circ = F_\circ(\Phi_\circ).$$

Then, for μ and ε small enough, there exist a transformation $X = \Phi^{\ltimes}(\chi, \varepsilon, \mu)$ of the form (3.6) *and vector fields N, $\widehat{B}$ as in* (3.7)*, defined for $(\zeta, \theta) \in \overline{\mathcal{D}_{\frac{r_\circ}{2}}} \times \mathcal{T}_\rho$, analytic in ζ and 2π-periodic and analytic in θ, leading system* (3.9) *into* ΨNF*, that is, satisfying*

$$D\Phi^{\ltimes}\, N + \widehat{B} = (F, \omega(\varepsilon)) \circ \Phi^{\ltimes}.$$

Moreover, we have that for $\mu \to 0$, $\varepsilon \to 0^+$

$$\Phi^{\ltimes} = (\Phi_\circ, \theta) + O(\mu\varepsilon^q), \qquad N^{\ltimes} = (N_\circ, \omega(\varepsilon)) + O(\mu\varepsilon^q) \quad \text{and} \quad \widehat{B} = O(\mu\varepsilon^q),$$

where $O(\mu\varepsilon^q)$ denotes a generic function of the form $\mu\varepsilon^q H$, with H bounded.

Similar arguments to the ones used in Chapter 1 yield the following result.

Corollary 3.1 *If system* (3.9) *is Hamiltonian or reversible then* ΨNF *becomes* BNF.

§2 Proof of the Main Theorem

§2.1 Some notation

Before starting with a formal approach to our problem, let us introduce some notation in order to simplify the exposition. Thus, we will use the same name for a vector field $F(X, \varepsilon, \mu)$: $\mathbb{R}^2 \times \mathbb{T} \to \mathbb{R}^2$ as for a vector field

$$\begin{pmatrix} F(X,\varepsilon,\mu) \\ 0 \end{pmatrix}.$$

On the contrary, we will add the symbol $^{\ltimes}$ if the vector field contains non-vanishing angular component. For instance, applying it onto system (3.4), this can be written as

$$\begin{pmatrix} \dot z \\ \dot\theta \end{pmatrix} = \begin{pmatrix} F_\circ(z) + F_\mu(z,\theta,\varepsilon) \\ \omega(\varepsilon) \end{pmatrix} = \begin{pmatrix} 0 \\ \omega(\varepsilon) \end{pmatrix} + \begin{pmatrix} F_\circ(z) \\ 0 \end{pmatrix} + \begin{pmatrix} F_\mu(z,\theta,\varepsilon) \\ 0 \end{pmatrix}$$

or, equivalently, as

$$\dot X = F^{\ltimes}(X,\varepsilon,\mu) := F_\circ^{\ltimes}(z,\varepsilon) + F_\mu(X,\varepsilon) = \Omega(\varepsilon) + F_\circ(z) + F_\mu(X,\varepsilon), \tag{3.11}$$

where we define

$$\Omega(\varepsilon) := \begin{pmatrix} 0 \\ \omega(\varepsilon) \end{pmatrix}.$$

As mentioned above, we are considering $F_\circ(z) = \Lambda_\circ z + \widehat{F}_\circ(z)$ so, in a similar way, we can express

$$F_\mu(z,\theta,\varepsilon) = \Lambda_\mu(\varepsilon) z + \widehat{F}_\mu(z,\theta,\varepsilon)$$

and assume the matrix $\Lambda_\mu(\varepsilon)$ to be written in the diagonal form (3.5). Thus, system (3.11) admits the following equivalent expressions

$$\begin{aligned} \dot X &= F^{\ltimes}(X,\varepsilon,\mu) = \\ &F_\circ^{\ltimes}(z) + F_\mu(X,\varepsilon,\mu) = \\ &\Big(\Omega(\varepsilon) + \Lambda_\circ z + \widehat{F}_\circ(z)\Big) + \Big(\Lambda_\mu(\varepsilon) z + \widehat{F}_\mu(X,\varepsilon,\mu)\Big) = \\ &\Omega(\varepsilon) + (\Lambda_\circ + \Lambda_\mu(\varepsilon))\, z + \Big(\widehat{F}_\circ(z) + \widehat{F}_\mu(X,\varepsilon,\mu)\Big) = \\ &\Omega(\varepsilon) + \Lambda(\varepsilon) z + \widehat{F}(X,\varepsilon,\mu), \end{aligned} \tag{3.12}$$

where we have defined

$$\Lambda(\varepsilon) := \Lambda_\circ + \Lambda_\mu(\varepsilon), \qquad \widehat{F}(X,\varepsilon,\mu) := \begin{pmatrix} \widehat{F}_\circ(z) + \widehat{F}_\mu(z,\theta,\varepsilon) \\ 0 \end{pmatrix}.$$

Concerning the vector fields constituting its (in principle, formal) ΨNF we write them in the following equivalent forms

$$\begin{aligned} X = \Phi^{\ltimes}(\chi,\varepsilon,\mu) = &\\ &\Phi_\circ^{\ltimes}(\chi,\varepsilon) + \widehat{\Phi}_\mu(\chi,\varepsilon) = \chi + \widehat{\Phi}_\circ(\zeta) + \widehat{\Phi}_\mu(\chi,\varepsilon) = \\ &\begin{pmatrix} \zeta \\ \theta \end{pmatrix} + \begin{pmatrix} \widehat{\Phi}_\circ(\zeta) \\ 0 \end{pmatrix} + \begin{pmatrix} \widehat{\Phi}_\mu(\zeta,\theta,\varepsilon) \\ 0 \end{pmatrix} = \\ &\begin{pmatrix} \zeta \\ \theta \end{pmatrix} + \begin{pmatrix} \widehat{\Phi}(\zeta,\theta,\varepsilon,\mu) \\ 0 \end{pmatrix} = \chi + \widehat{\Phi}(\chi,\varepsilon,\mu), \end{aligned} \tag{3.13}$$

with respect to the change of variables and

$$N^{\ltimes}(\zeta,\varepsilon,\mu) := \begin{pmatrix} \xi\, a(\xi\eta,\varepsilon,\mu) \\ -\eta\, a(\xi\eta,\varepsilon,\mu) \\ \omega(\varepsilon) \end{pmatrix}, \qquad \widehat{B}(\zeta,\varepsilon,\mu) := \begin{pmatrix} \xi\, \widehat{b}(\xi\eta,\varepsilon,\mu) \\ \eta\, \widehat{b}(\xi\eta,\varepsilon,\mu) \\ 0 \end{pmatrix},$$

with respect to the ΨNF vector fields. Moreover, we can write $N^{\ltimes}$ as

$$\begin{aligned} N^{\ltimes}(\zeta,\varepsilon,\mu) = N_\circ^{\ltimes}(\zeta,\varepsilon) + N_\mu(\zeta,\varepsilon) = &\\ &\Omega(\varepsilon) + N_\circ(\zeta) + N_\mu(\zeta,\varepsilon) = \\ &\Omega(\varepsilon) + (\Lambda_\circ + \Lambda_\mu(\varepsilon))\,\zeta + \left(\widehat{N}_\circ(\zeta) + \widehat{N}_\mu(\zeta,\varepsilon)\right) = \\ &\Omega(\varepsilon) + \Lambda(\varepsilon)\zeta + \widehat{N}(\zeta,\varepsilon,\mu). \end{aligned} \tag{3.14}$$

From equation (3.14), it is also derived that

$$\begin{aligned} a(\xi\eta,\varepsilon,\mu) = a_\circ(\xi\eta) + a_\mu(\xi\eta,\varepsilon) = &\\ &(\lambda_\circ + \widehat{a}_\circ(\xi\eta)) + (\lambda_\mu(\varepsilon) + \widehat{a}_\mu(\xi\eta,\varepsilon)) = \\ &(\lambda_\circ + \lambda_\mu(\varepsilon)) + \widehat{a}_\circ(\xi\eta) + \widehat{a}_\mu(\xi\eta,\varepsilon) = \lambda(\varepsilon) + \widehat{a}(\xi\eta,\varepsilon,\mu). \end{aligned}$$

Once setting all this notation, the ΨNF equation (3.8) becomes

$$D\Phi^{\ltimes}\, N^{\ltimes} + \widehat{B} = F^{\ltimes}\left(\Phi^{\ltimes}\right). \tag{3.15}$$

§2.2 Formal solution: starting up

Lemma 3.1 *Assuming that equation* (3.3),

$$D\Phi_\circ\, N_\circ = F_\circ\left(\Phi_\circ\right)$$

holds, it follows that the equality (3.8)

$$D\Phi^{\ltimes}\, N^{\ltimes} + \widehat{B} = F^{\ltimes}\left(\Phi^{\ltimes}\right)$$

is equivalent to

$$\left(D\widehat{\Phi}_\circ N_\mu - \Lambda_\mu \widehat{\Phi}_\circ\right) + \left(D\widehat{\Phi}_\mu N^\ltimes - \Lambda \widehat{\Phi}_\mu\right) + \widehat{N}_\mu + \widehat{B} = \widehat{G_F}\left(\widehat{\Phi}_\mu\right) \tag{3.16}$$

where

$$\begin{aligned}\widehat{G_F}\left(\widehat{\Phi}_\mu\right) = \widehat{G_F}\left(\Phi_\circ^\ltimes, \widehat{\Phi}_\mu\right) &:= \widehat{F}\left(\Phi^\ltimes\right) - \widehat{F}_\circ\left(\Phi_\circ^\ltimes\right) = \\ &\widehat{F}_\mu\left(\Phi_\circ^\ltimes\right) + \left(\widehat{F}\left(\Phi^\ltimes\right) - \widehat{F}\left(\Phi_\circ^\ltimes\right)\right) = \\ &\widehat{F}_\mu\left(\Phi_\circ^\ltimes\right) + \left(\widehat{F}\left(\Phi_\circ^\ltimes + \widehat{\Phi}_\mu\right) - \widehat{F}\left(\Phi_\circ^\ltimes\right)\right) = \\ &\widehat{F}_\mu\left(\Phi_\circ^\ltimes\right) + \sum_{m\geq 1} \frac{1}{m!} D^m \widehat{F}\left(\Phi_\circ^\ltimes\right)\left(\widehat{\Phi}_\mu\right)^m .\end{aligned} \tag{3.17}$$

Proof. With respect to the left-hand side of equation (3.8), using (3.13) and (3.14), one has that

$$\begin{aligned}D\Phi^\ltimes N^\ltimes + \widehat{B} = D\left(\Phi_\circ^\ltimes + \widehat{\Phi}_\mu\right)\left(N_\circ^\ltimes + \widehat{N}_\mu\right) + \widehat{B} = \\ D\Phi_\circ^\ltimes N_\circ^\ltimes + D\Phi_\circ^\ltimes N_\mu + D\widehat{\Phi}_\mu N^\ltimes + \widehat{B}.\end{aligned} \tag{3.18}$$

Concerning its right-hand side, it follows that

$$\begin{aligned}F^\ltimes\left(\Phi^\ltimes\right) &= F_\circ^\ltimes\left(\Phi^\ltimes\right) + F_\mu\left(\Phi^\ltimes\right) = \\ &F_\circ^\ltimes\left(\Phi_\circ^\ltimes\right) + \left(F_\circ^\ltimes\left(\Phi^\ltimes\right) - F_\circ^\ltimes\left(\Phi_\circ^\ltimes\right)\right) + F_\mu\left(\Phi^\ltimes\right) = \\ &F_\circ^\ltimes\left(\Phi_\circ^\ltimes\right) + \left(F_\circ^\ltimes\left(\Phi^\ltimes\right) - F_\circ^\ltimes\left(\Phi_\circ^\ltimes\right)\right) + F_\mu\left(\Phi_\circ^\ltimes\right) + \left(F_\mu\left(\Phi^\ltimes\right) - F_\mu\left(\Phi_\circ^\ltimes\right)\right).\end{aligned} \tag{3.19}$$

Dealing with each part separately we obtain

(*a*) first,

$$\begin{aligned}&F_\circ^\ltimes\left(\Phi^\ltimes\right) - F_\circ^\ltimes\left(\Phi_\circ^\ltimes\right) = \\ &\quad\Omega(\varepsilon) + \Lambda_\circ \Phi + \widehat{F}_\circ\left(\Phi^\ltimes\right) - \Omega(\varepsilon) - \Lambda_\circ \Phi_\circ - \widehat{F}_\circ\left(\Phi_\circ^\ltimes\right) = \\ &\quad\Lambda_\circ\left(\Phi - \Phi_\circ\right) + \widehat{F}_\circ\left(\Phi^\ltimes\right) - \widehat{F}_\circ\left(\Phi_\circ^\ltimes\right) = \Lambda_\circ \widehat{\Phi}_\mu + \left(\widehat{F}_\circ\left(\Phi^\ltimes\right) - \widehat{F}_\circ\left(\Phi_\circ^\ltimes\right)\right).\end{aligned}$$

(*b*) Moreover, it follows that

$$\begin{aligned}&F_\mu\left(\Phi^\ltimes\right) - F_\mu\left(\Phi_\circ^\ltimes\right) = \\ &\quad\Lambda_\mu \Phi + \widehat{F}_\mu\left(\Phi^\ltimes\right) - \Lambda_\mu \Phi_\circ - \widehat{F}_\mu\left(\Phi_\circ^\ltimes\right) = \\ &\quad\Lambda_\mu \widehat{\Phi}_\mu + \left(\widehat{F}_\mu\left(\Phi^\ltimes\right) - \widehat{F}_\mu\left(\Phi_\circ^\ltimes\right)\right).\end{aligned}$$

(*c*) Finally, $F_\mu\left(\Phi_\circ^\ltimes\right) = \Lambda_\mu \Phi_\circ + \widehat{F}_\mu\left(\Phi_\circ^\ltimes\right)$.

Applying formulas (*a*)–(*c*), equation (3.19) becomes

$$\begin{aligned}F^\ltimes\left(\Phi^\ltimes\right) &= F_\circ^\ltimes\left(\Phi_\circ^\ltimes\right) + \left(\Lambda_\circ \widehat{\Phi}_\mu + \Lambda_\mu \widehat{\Phi}_\mu + \Lambda_\mu \Phi_\circ\right) + \\ &\left(\left(\widehat{F}_\circ\left(\Phi^\ltimes\right) - \widehat{F}_\circ\left(\Phi_\circ^\ltimes\right)\right) + \widehat{F}_\mu\left(\Phi_\circ^\ltimes\right) + \left(\widehat{F}_\mu\left(\Phi^\ltimes\right) - \widehat{F}_\mu\left(\Phi_\circ^\ltimes\right)\right)\right) = \\ &F_\circ^\ltimes\left(\Phi_\circ^\ltimes\right) + \Lambda_\mu \Phi_\circ + \Lambda \widehat{\Phi}_\mu + \widehat{G_F}\left(\widehat{\Phi}_\mu\right),\end{aligned} \tag{3.20}$$

where

$$\widehat{G_F}\left(\widehat{\Phi}_\mu\right) = \widehat{G_F}\left(\Phi_\circ^\ltimes, \widehat{\Phi}_\mu\right) := \widehat{F}_\mu\left(\Phi_\circ^\ltimes\right) + \left(\widehat{F}\left(\Phi^\ltimes\right) - \widehat{F}\left(\Phi_\circ^\ltimes\right)\right) = \widehat{F}_\mu\left(\Phi_\circ^\ltimes\right) + \sum_{m\geq 1}\frac{1}{m!}D^m\widehat{F}\left(\Phi_\circ^\ltimes\right)\left(\widehat{\Phi}_\mu\right)^m.$$

Therefore, from equations (3.18) and (3.20), it follows that equation (3.8) is equivalent to

$$D\Phi_\circ^\ltimes N_\circ^\ltimes + D\Phi_\circ^\ltimes N_\mu + D\widehat{\Phi}_\mu N^\ltimes + \widehat{B} = F_\circ^\ltimes\left(\Phi_\circ^\ltimes\right) + \Lambda_\mu\Phi_\circ + \Lambda\widehat{\Phi}_\mu + \widehat{G_F}\left(\widehat{\Phi}_\mu\right). \tag{3.21}$$

Moreover, we have that

$$D\Phi_\circ^\ltimes N_\circ^\ltimes = \begin{pmatrix} D\Phi_\circ & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} N_\circ \\ \omega(\varepsilon) \end{pmatrix} = \begin{pmatrix} D\Phi_\circ N_\circ \\ \omega(\varepsilon) \end{pmatrix},$$

with

$$F_\circ^\ltimes\left(\Phi_\circ^\ltimes\right) = \Omega(\varepsilon) + F_\circ\left(\Phi_\circ\right) = \begin{pmatrix} F_\circ\left(\Phi_\circ\right) \\ \omega(\varepsilon) \end{pmatrix},$$

where it has been used that $F_\circ^\ltimes(X,\varepsilon) = \Omega(\varepsilon) + F_\circ(z)$. Since equation (3.3) is satisfied, it follows that

$$D\Phi_\circ^\ltimes N_\circ^\ltimes = F_\circ^\ltimes\left(\Phi_\circ^\ltimes\right).$$

Applying it onto (3.21) one has the equation

$$D\Phi_\circ^\ltimes N_\mu + D\widehat{\Phi}_\mu N^\ltimes + \widehat{B} = \Lambda_\mu\Phi_\circ + \Lambda\widehat{\Phi}_\mu + \widehat{G_F}\left(\widehat{\Phi}_\mu\right),$$

with $\widehat{G_F}$ defined in (3.17) or, reordering terms,

$$\left(D\Phi_\circ^\ltimes N_\mu - \Lambda_\mu\Phi_\circ\right) + \left(D\widehat{\Phi}_\mu N^\ltimes - \Lambda\widehat{\Phi}_\mu\right) + \widehat{B} = \widehat{G_F}\left(\Phi_\circ^\ltimes, \widehat{\Phi}_\mu\right). \tag{3.22}$$

Finally, since $\Phi_\circ^\ltimes(\chi) = \chi + \widehat{\Phi}_\circ(\zeta)$ it follows that

$$\begin{aligned} D\Phi_\circ^\ltimes N_\mu - \Lambda_\mu\Phi_\circ &= N_\mu + D\widehat{\Phi}_\circ N_\mu - \Lambda_\mu\Phi_\circ = \\ &\left(\Lambda_\mu\zeta + \widehat{N}_\mu\right) + D\widehat{\Phi}_\circ N_\mu - \Lambda_\mu\zeta - \Lambda_\mu\widehat{\Phi}_\circ = \\ &\widehat{N}_\mu + D\widehat{\Phi}_\circ N_\mu - \Lambda_\mu\widehat{\Phi}_\circ, \end{aligned}$$

and, therefore, equation (3.22) gives

$$\left(D\widehat{\Phi}_\circ N_\mu - \Lambda_\mu\widehat{\Phi}_\circ\right) + \left(D\widehat{\Phi}_\mu N^\ltimes - \Lambda\widehat{\Phi}_\mu\right) + \widehat{N}_\mu + \widehat{B} = \widehat{G_F}\left(\widehat{\Phi}_\mu\right)$$

as the lemma claims. □

The standard pattern to solve this kind of problems consists essentially of two steps:

(*i*) *A formal solution*: let us consider a vector field $G^{\ltimes}(\chi,\varepsilon,\mu)$ and write it in the form

$$\begin{pmatrix} g_1(\xi,\eta,\theta,\varepsilon,\mu) \\ g_2(\xi,\eta,\theta,\varepsilon,\mu) \\ g_3(\theta,\varepsilon) \end{pmatrix}$$

where $g_3(\theta,\varepsilon)$ takes the values θ, $\omega(\varepsilon)$ or 0. We recall that in the third case we denote G instead of $G^{\ltimes}$. Assume that the functions $g_\ell(\chi,\varepsilon,\mu)$, $\ell=1,2$, can be represented in formal Taylor-Fourier series expansion, that is,

$$g_\ell(\xi,\eta,\theta,\varepsilon,\mu)=\sum_{\substack{j+k\geq 2\\ s\in\mathbb{Z}}} g^{(\ell)}_{jks}(\varepsilon,\mu)\,\xi^j\eta^k \mathrm{e}^{\mathrm{i}s\theta}.$$

Moreover, let us denote $G=\mathcal{O}_{[K]}$ if G is a homogeneous polynomial with respect to the spatial variables ξ,η of order exactly K. Besides, we will write $G=\mathcal{O}_K$ if G contains only terms of order greater or equal than K with respect to these variables and $G=\mathcal{O}_{\leq K}$ if all the terms in G are of order less or equal than K in ξ,η. Henceforth, when we refer to the *order* of a vector field (or a function) we will always mean with respect to the spatial variables (and for any spatial order fixed, for any order with respect to the angular variable).

The aim of this step of the procedure is to provide a recurrent scheme solving our equation (3.16), for $\widehat{\Phi}_\mu$, N_μ and $\widehat{B}$, order by order. In general this method does not always work and needs some kind of *triangular* structure in the equation to solve.

(*ii*) *The convergence of the recurrent scheme*: in this step one has to prove that the formal Taylor-Fourier expansions obtained for $\widehat{\Phi}_\mu$, N_μ and $\widehat{B}$ are absolutely convergent in a suitable domain. Therefore they will represent analytic vector fields on that region.

Along this section, and their corresponding subsections, we will deal with the first step, the formal solution. The second step, related to its convergence, is deferred to Section §2.7. Thus, the development of this section will be divided in different parts

- Before presenting a recurrent scheme solving equation (3.16), we will consider a restricted situation and will try to solve it for some terms. From that partial solution we will learn which are the problems we will meet in the complete scheme. This will be done in Section §2.3.
- The considerations extracted from the previous item will lead us to the definition of some suitable projections in Section §2.4. These projections will allow us to solve formally equation (3.16) by means of a first recurrent scheme. Section §2.5 will be devoted to it.
- Finally, in Section §2.6 we will improve that sooner scheme, paying special attention on its applicability on a computer.

§2.3 A first approach

Take again equation (3.16)

$$\left(D\widehat{\Phi}_\circ N_\mu-\Lambda_\mu\widehat{\Phi}_\circ\right)+\left(D\widehat{\Phi}_\mu N^{\ltimes}-\Lambda\widehat{\Phi}_\mu\right)+\widehat{N}_\mu+\widehat{B}=\widehat{G_F}\left(\Phi^{\ltimes}_\circ,\widehat{\Phi}_\mu\right)$$

and assume, for a while, that we have already computed the terms of N_μ, $\widehat{B}$ of order less or equal than $K+1$ and of $\widehat{\Phi}_\mu$ up to order K. Since $\Phi_\circ$ and $\widehat{N}_\circ$ are known by hypothesis we also know $N^{\ltimes}$ up to order $K+1$ (included). Therefore, we wonder about the following questions: can be obtained, from these hypotheses, the terms of $\widehat{\Phi}_\mu$ of order exactly $K+1$ satisfying equation (3.16) ? Even in the case this can be done, do they present numerical problems, like small divisors ?

Notice that, from the definition of $\widehat{G_F}$ (3.17), its terms of order exactly $K+1$ are completely determined from those of $\Phi_\circ$ and $\widehat{\Phi}_\mu$ of order K or less. A similar argument works for $(D\widehat{\Phi}_\circ N_\mu - \Lambda_\mu \widehat{\Phi}_\circ)$ and $\widehat{N}_\mu + \widehat{B}$ with regard to $\Phi_\circ$, N_μ and $\widehat{B}$. So, in principle, the terms of $\widehat{\Phi}_\mu$ of order $K+1$ satisfying (3.16) can be determined from

$$\left\{ D\widehat{\Phi}_\mu N^{\ltimes} - \Lambda \widehat{\Phi}_\mu \right\}_{[K+1]} = \widehat{H}_{[K+1]}$$

where $\{G\}_{[M]}$ and $G_{[M]}$ indicate the terms of order exactly M of the vector field G and $\widehat{H}_{[K+1]}$ is known. Moreover,

$$D\widehat{\Phi}_\mu N^{\ltimes} - \Lambda \widehat{\Phi}_\mu = D\widehat{\Phi}_\mu \left(\Omega(\varepsilon) + \Lambda\zeta + \widehat{N} \right) - \Lambda \widehat{\Phi}_\mu =$$
$$D\widehat{\Phi}_\mu \left(\Omega(\varepsilon) + \Lambda\zeta \right) - \Lambda \widehat{\Phi}_\mu + D\widehat{\Phi}_\mu \widehat{N} = \left[\Omega(\varepsilon) + \Lambda\zeta, \widehat{\Phi}_\mu \right] + D\widehat{\Phi}_\mu \widehat{N},$$

where $[G_1, G_2] = (DG_2)\, G_1 - (DG_1)\, G_2$ is the *Lie bracket of the vector fields* G_1 *and* G_2. Since $\widehat{N}$ starts with terms of order, at least, 3 (that is, $\widehat{N} = \mathcal{O}_3$), it follows that the first equation where the terms of $\widehat{\Phi}_\mu$ of order $K+1$ appear is of the type

$$\left[\Omega(\varepsilon) + \Lambda\zeta, \widehat{\Phi}_\mu \right] = \widehat{H}_{[K+1]} \tag{3.23}$$

where $\widehat{H}_{[K+1]}$ now contains also those terms of order $K+1$ coming from $D\widehat{\Phi}_\mu \widehat{N}$ (which depend on the terms of $\widehat{N}$ and $\widehat{\Phi}_\mu$ of order less or equal than K). Then, let us assume

$$\widehat{\Phi}_\mu(\xi,\eta,\theta,\varepsilon) = \begin{pmatrix} \widehat{\phi}_1(\xi,\eta,\theta,\varepsilon) \\ \widehat{\phi}_2(\xi,\eta,\theta,\varepsilon) \\ 0 \end{pmatrix}, \qquad \widehat{H}(\xi,\eta,\theta,\varepsilon) = \begin{pmatrix} \widehat{h}_1(\xi,\eta,\theta,\varepsilon) \\ \widehat{h}_2(\xi,\eta,\theta,\varepsilon) \\ 0 \end{pmatrix}$$

with

$$\begin{aligned} \widehat{\phi}_\ell(\xi,\eta,\theta,\varepsilon) &= \sum_{\substack{j+k\geq 2 \\ s\in\mathbb{Z}}} \phi^{(\ell)}_{jks}(\varepsilon)\, \xi^j \eta^k \mathrm{e}^{\mathrm{i}s\theta} \\ \widehat{h}_\ell(\xi,\eta,\theta,\varepsilon) &= \sum_{\substack{j+k\geq 2 \\ s\in\mathbb{Z}}} h^{(\ell)}_{jks}(\varepsilon)\, \xi^j \eta^k \mathrm{e}^{\mathrm{i}s\theta} \end{aligned}$$

for $\ell = 1, 2$, and recall that

$$\Lambda = \Lambda(\varepsilon) = \begin{pmatrix} \lambda(\varepsilon) & 0 \\ 0 & -\lambda(\varepsilon) \end{pmatrix}.$$

Indeed, an equivalent expression for equation (3.23) in terms of formal Taylor-Fourier series is given by

$$\begin{pmatrix} \frac{\partial\widehat{\phi}_1}{\partial\xi} & \frac{\partial\widehat{\phi}_1}{\partial\eta} & \frac{\partial\widehat{\phi}_1}{\partial\theta} \\ \frac{\partial\widehat{\phi}_2}{\partial\xi} & \frac{\partial\widehat{\phi}_2}{\partial\eta} & \frac{\partial\widehat{\phi}_2}{\partial\theta} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda(\varepsilon)\xi \\ -\lambda(\varepsilon)\eta \\ \omega(\varepsilon) \end{pmatrix} - \begin{pmatrix} \lambda(\varepsilon)\widehat{\phi}_1 \\ -\lambda(\varepsilon)\widehat{\phi}_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \widehat{h}_1(\xi,\eta,\theta) \\ \widehat{h}_2(\xi,\eta,\theta) \\ 0 \end{pmatrix},$$

and also by

$$\begin{aligned} \lambda(\varepsilon)\left(\frac{\partial\widehat{\phi}_1(\varepsilon)}{\partial\xi}\,\xi - \frac{\partial\widehat{\phi}_1}{\partial\eta}\,\eta - \widehat{\phi}_1\right) + \frac{\partial\widehat{\phi}_1}{\partial\theta}\,\omega(\varepsilon) &= \widehat{h}_1 \\ \lambda(\varepsilon)\left(\frac{\partial\widehat{\phi}_2(\varepsilon)}{\partial\xi}\,\xi - \frac{\partial\widehat{\phi}_2}{\partial\eta}\,\eta + \widehat{\phi}_2\right) + \frac{\partial\widehat{\phi}_2}{\partial\theta}\,\omega(\varepsilon) &= \widehat{h}_2. \end{aligned}$$

Equating the previous equations, we obtain that the coefficients of $\widehat{\Phi}_\mu$ accompanying those terms of order exactly $K+1$ are given by the following formulas

$$\begin{aligned} \phi^{(1)}_{jks} &= \frac{h^{(1)}_{jks}}{\lambda(j-k-1)+\mathrm{i}s\omega(\varepsilon)} && \text{if } j \neq k+1 \text{ or } s \neq 0 \\ \phi^{(2)}_{jks} &= \frac{h^{(2)}_{jks}}{\lambda(j-k+1)+\mathrm{i}s\omega(\varepsilon)} && \text{if } k \neq j+1 \text{ or } s \neq 0 \end{aligned} \tag{3.24}$$

for $j+k=K+1$. From these expressions two important consequences can be derived. Namely,

(*i*) Not all the terms of $\widehat{\Phi}_\mu$ of order $K+1$ can be computed. Precisely, the terms of $\widehat{\Phi}_\mu$ which *cannot* be obtained from these formulas are those of type

$$\begin{pmatrix} \xi\sum\phi^{(1)}_{k+1,k,0}(\varepsilon)\,(\xi\eta)^k \\ \eta\sum\phi^{(2)}_{j,j+1,0}(\varepsilon)\,(\xi\eta)^j \\ 0 \end{pmatrix} \tag{3.25}$$

That is, they correspond to those coefficients satisfying that $\lambda(j-k-1)+\mathrm{i}s\omega(\varepsilon)$ or $\lambda(j-k+1)+\mathrm{i}s\omega(\varepsilon)$ vanishes. In other words, they are *resonant terms* of our system.

(*ii*) Even in those cases when we can compute $\phi^{(\ell)}_{jks}(\varepsilon)$ (and still missing how to compute the terms of $\widehat{N}_\mu$ and $\widehat{B}$) this does not ensure a probable convergence of such a (still poor) scheme. It is well known that the smallness of the denominators appearing in (3.24) can lead to the divergence of the series involved. This phenomenon is the so-called *small divisors* problem. In our case, like it happened in the *saddle-center or saddle-focus cases* (see Chapter 1), there are no small divisors. In particular, this fact will have as consequence the no reduction in the domain of convergence of the ΨNF (and, therefore, of the BNF) with respect to the angular variable θ.

§2.4 Definition of the projections

From the previous explanation, it seems natural to distinguish in $\widehat{\Phi}_\mu$ between those terms which can be obtained from equations (3.24) and those terms (3.25) which remain arbitrary. This fact motivates the following definition.

Definition 2 *Given a formal power series*

$$g(\xi,\eta,\theta,\varepsilon)=\sum_{\substack{j+k\geq 1\\ s\in\mathbb{Z}}} g_{jks}(\varepsilon)\,\xi^j\eta^k\mathrm{e}^{\mathrm{i}s\theta},$$

we define the projections

$$\begin{aligned}(P_1 g)(\xi,\eta,\varepsilon) &:= \xi\sum_{k\geq 0} g_{k+1,k,0}(\varepsilon)\ (\xi\eta)^k\,,\\ (P_2 g)(\xi,\eta,\varepsilon) &:= \eta\sum_{j\geq 0} g_{j,j+1,0}(\varepsilon)\ (\xi\eta)^j\,.\end{aligned}$$

Analogously, for a vector field $G(\xi,\eta,\theta,\varepsilon)=(g_1(\xi,\eta,\theta,\varepsilon),g_2(\xi,\eta,\theta,\varepsilon),g_3(\theta,\varepsilon))$, *where* $g_3(\theta,\varepsilon)$ *can take the values* θ, $\omega(\varepsilon)$ *or* 0, *we define*

$$(\mathcal{P}G)(\xi,\eta,\varepsilon):=\begin{pmatrix}(P_1\,g_1)(\xi,\eta,\varepsilon)\\ (P_2\,g_2)(\xi,\eta,\varepsilon)\\ g_3(\theta,\varepsilon)\end{pmatrix}$$

Moreover, we define also the projection $\mathcal{R}$ *as*

$$\mathcal{R}G:=G-\mathcal{P}G$$

or, in components

$$(\mathcal{R}G)(\xi,\eta,\varepsilon):=\begin{pmatrix}(R_1\,g_1)(\xi,\eta,\varepsilon)\\ (R_2\,g_2)(\xi,\eta,\varepsilon)\\ (R_3\,g_3)(\xi,\eta,\varepsilon)\end{pmatrix}=\begin{pmatrix}g_1(\xi,\eta,\varepsilon)-(P_1\,g_1)(\xi,\eta,\varepsilon)\\ g_2(\xi,\eta,\varepsilon)-(P_2\,g_2)(\xi,\eta,\varepsilon)\\ 0\end{pmatrix}.$$

Remark 11 *From this definition, it follows that the terms in* $\widehat{\Phi}_\mu$ *which can not be computed from equation (3.24), that is, those of type (3.25), all belong to the projection* $\mathcal{P}\widehat{\Phi}_\mu$.

Lemma 3.2 *Let us assume* $\widehat{\Psi}$, M *being vector fields of the form*

$$\widehat{\Psi}(\chi,\varepsilon)=\begin{pmatrix}\widehat{\psi}_1(\xi,\eta,\theta,\varepsilon)\\ \widehat{\psi}_2(\xi,\eta,\theta,\varepsilon)\\ 0\end{pmatrix}\tag{3.26}$$

and

$$M(\zeta,\varepsilon)=\begin{pmatrix}\xi u(\xi\eta,\varepsilon)\\ -\eta u(\xi\eta,\varepsilon)\\ \tilde{u}(\varepsilon)\end{pmatrix}\tag{3.27}$$

where $\tilde{u}(\varepsilon)$ take the values $\omega(\varepsilon)$ or 0. In other words, M takes the form

$$M(\zeta) = \Omega(\varepsilon) + A\zeta + \widehat{M}(\zeta, \varepsilon) \qquad \text{or} \qquad M(\zeta) = A\zeta + \widehat{M}(\zeta, \varepsilon),$$

respectively. Then, we define the functional operator $\mathcal{L}$ as

$$\mathcal{L}_M \widehat{\Psi} := D\widehat{\Psi}\, M - A\widehat{\Psi}. \tag{3.28}$$

Moreover, let us consider vector fields $\widehat{\Psi}$, $\widehat{\Psi}_1$ and $\widehat{\Psi}_2$ of the type (3.26), vector fields M_1, M_2 of type (3.27) and $\widehat{B}$ given by

$$\widehat{B}(\zeta, \varepsilon) = \begin{pmatrix} \xi \widehat{b}(\xi\eta, \varepsilon) \\ \eta \widehat{b}(\xi\eta, \varepsilon) \\ 0 \end{pmatrix}$$

Then, the following properties are satisfied.

(*i*) *The operator $\mathcal{L}$ is linear with respect to both vector fields, that is,*

$$\begin{aligned} \mathcal{L}_{M_1+M_2} \widehat{\Psi} &= \mathcal{L}_{M_1} \widehat{\Psi} + \mathcal{L}_{M_2} \widehat{\Psi} \\ \mathcal{L}_M \left(\widehat{\Psi}_1 + \widehat{\Psi}_2 \right) &= \mathcal{L}_M \widehat{\Psi}_1 + \mathcal{L}_M \widehat{\Psi}_2. \end{aligned}$$

(*ii*) *$\mathcal{L}$ preserves the order with respect to the spatial variables, that is, $\mathcal{L}_M \widehat{\Psi}$ and $\widehat{\Psi}$ start with terms of the same order with respect to $\zeta = (\xi, \eta)$.*

(*iii*) *We have that*

$$\mathcal{P}M = M, \qquad \mathcal{P}\widehat{B} = \widehat{B},$$

or, equivalently,

$$\mathcal{R}M \equiv 0, \qquad \mathcal{R}\widehat{B} \equiv 0.$$

(*iv*) *The projections $\mathcal{P}$ and $\mathcal{R}$ commute with $\mathcal{L}$, that is,*

$$\mathcal{P}\left(\mathcal{L}_M \widehat{\Psi} \right) = \mathcal{L}_M \left(\mathcal{P}\widehat{\Psi} \right), \qquad \mathcal{R}\left(\mathcal{L}_M \widehat{\Psi} \right) = \mathcal{L}_M \left(\mathcal{R}\widehat{\Psi} \right).$$

We omit the proof of this lemma since it consists on straightforward computations.

§2.5 A recurrent scheme

Once the projections $\mathcal{P}$, $\mathcal{R}$ and the linear operator $\mathcal{L}$ have been defined and some of their properties introduced, we come back to the solution of equation (3.16),

$$\left(D\widehat{\Phi}_\circ N_\mu - \Lambda_\mu \widehat{\Phi}_\circ \right) + \left(D\widehat{\Phi}_\mu N^\ltimes - \Lambda \widehat{\Phi}_\mu \right) + \widehat{N}_\mu + \widehat{B} = \widehat{G_F}\left(\widehat{\Phi}_\mu \right)$$

From formulas (3.24) and (3.25) it is clear that we cannot determine from equation (3.16) the terms of $\mathcal{P}\widehat{\Phi}_\mu$. This fact suggests to split it in two complementary parts

$$\widehat{\Phi}_\mu = \mathcal{P}\widehat{\Phi}_\mu + \mathcal{R}\widehat{\Phi}_\mu,$$

fix a value for $\mathcal{P}\widehat{\Phi}_\mu$ and solve this equation for $\mathcal{R}\widehat{\Phi}_\mu$. This is an standard procedure in Normal Form Theory (see, for instance [42, 11, 12]). To simplify computations it is usual to consider

$$\mathcal{P}\widehat{\Phi}_\mu \equiv 0, \tag{3.29}$$

but it could be useful in some situations to take advantage of this freedom. Notice that this is also the case of vector field $\Phi_\circ$ in (3.2) leading the unperturbed system (3.1) into BNF. As before, we will assume $\Phi_\circ$ to satisfy that

$$\mathcal{P}\widehat{\Phi}_\circ \equiv 0. \tag{3.30}$$

Moreover, having in mind the definition of the operator $\mathcal{L}$ in (3.28) equation (3.16) can be rewritten as

$$\mathcal{L}_{N_\mu}\widehat{\Phi}_\circ + \mathcal{L}_{N^\ltimes}\widehat{\Phi}_\mu + \widehat{N}_\mu + \widehat{B} = \widehat{G_F}\left(\widehat{\Phi}_\mu\right). \tag{3.31}$$

Applying now the projection $\mathcal{R}$ onto both sides of this equation, one obtains

$$\mathcal{R}\left(\mathcal{L}_{N_\mu}\widehat{\Phi}_\circ + \mathcal{L}_{N^\ltimes}\widehat{\Phi}_\mu + \widehat{N}_\mu + \widehat{B}\right) = \mathcal{R}\left(\widehat{G_F}\left(\widehat{\Phi}_\mu\right)\right). \tag{3.32}$$

Using lemma 3.2 there follow the linearity of $\mathcal{R}$, that $\mathcal{R}\widehat{N}_\mu = \mathcal{R}\widehat{B} = 0$ and that

$$\mathcal{R}\left(\mathcal{L}_{N_\mu}\widehat{\Phi}_\circ\right) = \mathcal{L}_{N_\mu}\left(\mathcal{R}\widehat{\Phi}_\circ\right), \qquad \mathcal{R}\left(\mathcal{L}_{N^\ltimes}\widehat{\Phi}_\mu\right) = \mathcal{L}_{N^\ltimes}\left(\mathcal{R}\widehat{\Phi}_\mu\right).$$

Thus, equation (3.31) is equivalent to

$$\mathcal{L}_{N_\mu}\left(\mathcal{R}\widehat{\Phi}_\circ\right) + \mathcal{L}_{N^\ltimes}\left(\mathcal{R}\widehat{\Phi}_\mu\right) = \mathcal{R}\left(\widehat{G_F}\left(\widehat{\Phi}_\mu\right)\right). \tag{3.33}$$

In a similar way, we can apply $\mathcal{P}$ onto (3.31) and obtain

$$\mathcal{P}\left(\mathcal{L}_{N_\mu}\widehat{\Phi}_\circ + \mathcal{L}_{N^\ltimes}\widehat{\Phi}_\mu + \widehat{N}_\mu + \widehat{B}\right) = \mathcal{P}\left(\widehat{G_F}\left(\widehat{\Phi}_\mu\right)\right).$$

As before, from lemma 3.2 it is derived the linearity of $\mathcal{P}$, the equalities $\mathcal{P}\widehat{N}_\mu = \widehat{N}_\mu$, $\mathcal{P}\widehat{B} = \widehat{B}$ and

$$\begin{array}{rcl}\mathcal{P}\left(\mathcal{L}_{N_\mu}\widehat{\Phi}_\circ\right) & = & \mathcal{L}_{N_\mu}\left(\mathcal{P}\widehat{\Phi}_\circ\right) = 0\\ \mathcal{P}\left(\mathcal{L}_{N^\ltimes}\widehat{\Phi}_\mu\right) & = & \mathcal{L}_{N^\ltimes}\left(\mathcal{P}\widehat{\Phi}_\mu\right) = 0,\end{array}$$

where assumptions (3.30) and (3.29) have been taken into account. Consequently, it turns out the following equation

$$\widehat{N}_\mu + \widehat{B} = \mathcal{P}\left(\widehat{G_F}\left(\widehat{\Phi}_\mu\right)\right). \tag{3.34}$$

From these formulas (3.33) and (3.34) we can establish a *first recurrent scheme*. Namely, assuming that the vector fields $\Phi_\circ$ (such that $\mathcal{P}\widehat{\Phi}_\circ = 0$), $N_\circ$ and $F_\circ$ are known, we take initial values

$$\widehat{\Phi}_\mu^{(1)} = 0, \qquad {N_\mu}^{(1)} = \Lambda_\mu(\varepsilon)\zeta, \qquad \widehat{B}^{(1)} = 0. \tag{3.35}$$

Then, taking $\mathcal{P}\widehat{\Phi}_\mu{}^{(K)} = 0$, for any $K \geq 1$, and using that $N^\ltimes = N_\circ^\ltimes + N_\mu$, we look for

$$\begin{aligned} \widehat{\Phi}_\mu^{(K+1)} &= \mathcal{R}\widehat{\Phi}_\mu{}^{(K+1)} \\ N_\mu{}^{(K+1)} &= \Lambda_\mu(\varepsilon)\zeta + \widehat{N}_\mu^{(K+1)} \\ \widehat{B}^{(K+1)}, & \end{aligned} \tag{3.36}$$

obtained from the recurrent equations

$$\mathcal{L}_{N_\mu{}^{(K)}}\left(\mathcal{R}\widehat{\Phi}_\circ\right) + \mathcal{L}_{N_\circ^\ltimes + N_\mu{}^{(K)}}\left(\mathcal{R}\widehat{\Phi}_\mu{}^{(K+1)}\right) = \mathcal{R}\left(\widehat{G_F}\left(\widehat{\Phi}_\mu^{(K)}\right)\right) \tag{3.37}$$

$$\widehat{N}_\mu^{(K+1)} + \widehat{B}^{(K+1)} = \mathcal{P}\left(\widehat{G_F}\left(\widehat{\Phi}_\mu^{(K)}\right)\right). \tag{3.38}$$

The following subsections will be devoted to the formal solution of equations of type (3.33) and (3.34). Precisely,

(*i*) with respect to (3.33), we will seek for a vector field $\mathcal{R}\widehat{\Psi}$ solution of an equation of type

$$\mathcal{L}_{N^\ltimes}\left(\mathcal{R}\widehat{\Psi}\right) = \mathcal{R}\widehat{H},$$

where we assume $N^\ltimes$ and $\widehat{H}$ known. In some sense this is natural if we have in mind our intention of applying a triangular scheme: assuming that we know $N^\ltimes$ (or, equivalently, N_μ) and $\widehat{\Phi}_\mu$ up to a given order K, we wonder about which equation must be satisfied by the terms of $\mathcal{R}\widehat{\Phi}_\mu$ of order less or equal than $K+1$.

(*ii*) Concerning equation (3.34), this can be seen as looking for vector fields $\widehat{N}_\mu$, $\widehat{B}$ solving an equation of the type

$$\widehat{N}_\mu + \widehat{B} = \mathcal{P}\widehat{H},$$

with $\widehat{H}$ also known.

§2.5.1 Solution of a $\mathcal{L}_{N^\ltimes}(\mathcal{R}\widehat{\Psi}) = \mathcal{R}\widehat{H}$-type equation

We will follow the same notation introduced in [25] and also used in [28]. The idea is to rearrange the series expansions of the vector fields to facilitate the formal solution of

$$\mathcal{L}_{N^\ltimes}\left(\mathcal{R}\widehat{\Psi}\right) = \mathcal{R}\widehat{H}. \tag{3.39}$$

Thus, let us consider a function $\widehat{g}(\zeta, \theta, \varepsilon)$ given in Taylor-Fourier series expansion,

$$\widehat{g}(\zeta, \theta, \varepsilon) = \widehat{g}(\xi, \eta, \theta, \varepsilon) = \sum_{\substack{j+k\geq 2 \\ s\in\mathbb{Z}}} g_{jks}(\varepsilon)\, \xi^j \eta^k \mathrm{e}^{\mathrm{i}s\theta},$$

and rewrite it as

$$\sum_{\substack{j+k\geq 2 \\ s\in\mathbb{Z}}} g_{jks}(\varepsilon)\ (\xi\eta)^k\, \xi^{j-k} \mathrm{e}^{\mathrm{i}s\theta}.$$

Defining $\ell = j - k$, since $j \geq 0$, $k \geq 0$ and $j + k \geq 2$, the new limits are given by $\ell + k \geq 0$, $k \geq 0$ and $\ell + 2k \geq 2$. Therefore,

$$\widehat{g}(\xi,\eta,\theta,\varepsilon) = \sum_{\ell,s\in\mathbb{Z}} \sum_{k\geq\max\{0,1-\ell/2,-\ell\}} g_{\ell+k,ks}(\varepsilon)\ (\xi\eta)^k\, \xi^\ell \mathrm{e}^{\mathrm{i}s\theta} =$$

$$\sum_{\ell,s\in\mathbb{Z}} \left(\sum_{k\geq\max\{0,1-\ell/2,-\ell\}} g_{\ell+k,ks}(\varepsilon)\ (\xi\eta)^k \right) \xi^\ell \mathrm{e}^{\mathrm{i}s\theta} = \sum_{\ell,s\in\mathbb{Z}} g_{\ell s}(\xi\eta,\varepsilon)\, \xi^\ell \mathrm{e}^{\mathrm{i}s\theta},$$

where we have defined

$$g_{\ell s}(\xi\eta,\varepsilon) := \sum_{k\geq\max\{0,1-\ell/2,-\ell\}} g_{\ell+k,ks}(\varepsilon)\, (\xi\eta)^k\,. \tag{3.40}$$

Once introduced this notation, let us come back to equation (3.39) and write

$$\mathcal{R}\widehat{\Psi}(\chi,\varepsilon) = \begin{pmatrix} \widehat{\psi}_1(\xi,\eta,\theta,\varepsilon) \\ \widehat{\psi}_2(\xi,\eta,\theta,\varepsilon) \\ 0 \end{pmatrix}, \qquad \mathcal{R}\widehat{H}(\chi,\varepsilon) = \begin{pmatrix} \widehat{h}_1(\xi,\eta,\theta,\varepsilon) \\ \widehat{h}_2(\xi,\eta,\theta,\varepsilon) \\ 0 \end{pmatrix}$$

and take

$$N^{\ltimes}(\zeta,\varepsilon) = \begin{pmatrix} \xi a(\xi\eta,\varepsilon) \\ -\eta a(\xi\eta,\varepsilon) \\ \omega(\varepsilon) \end{pmatrix} = \Omega(\varepsilon) + \Lambda(\varepsilon)\zeta + \widehat{N}(\zeta,\varepsilon),$$

with $a(\xi\eta,\varepsilon) = \lambda(\varepsilon) + \widehat{a}(\xi\eta,\varepsilon)$. In principle, assume that we express their components in series expansion of the form

$$\widehat{\psi}_w(\xi,\eta,\theta,\varepsilon) = \sum_{\substack{j+k\geq 2\\ s\in\mathbb{Z}}} \psi^{(w)}_{jks}(\varepsilon)\, \xi^j\eta^k \mathrm{e}^{\mathrm{i}s\theta}, \qquad \widehat{h}_w(\xi,\eta,\theta,\varepsilon) = \sum_{\substack{j+k\geq 2\\ s\in\mathbb{Z}}} h^{(w)}_{jks}(\varepsilon)\, \xi^j\eta^k \mathrm{e}^{\mathrm{i}s\theta}, \tag{3.41}$$

for $w = 1,2$. Using the definition of $\mathcal{L}$ (3.28), the left-hand side of equation (3.39) is given by

$$\mathcal{L}_{N^{\ltimes}}\left(\mathcal{R}\widehat{\Psi}\right) = D\left(\mathcal{R}\widehat{\Psi}\right) N^{\ltimes} - \Lambda(\varepsilon)\mathcal{R}\widehat{\Psi} =$$

$$\begin{pmatrix} \frac{\partial\widehat{\psi}_1}{\partial\xi} & \frac{\partial\widehat{\psi}_1}{\partial\eta} & \frac{\partial\widehat{\psi}_1}{\partial\theta} \\ \frac{\partial\widehat{\psi}_2}{\partial\xi} & \frac{\partial\widehat{\psi}_2}{\partial\eta} & \frac{\partial\widehat{\psi}_2}{\partial\theta} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi a(\xi\eta,\varepsilon) \\ -\eta a(\xi\eta,\varepsilon) \\ \omega(\varepsilon) \end{pmatrix} - \begin{pmatrix} \lambda(\varepsilon)\widehat{\psi}_1 \\ -\lambda(\varepsilon)\widehat{\psi}_2 \\ 0 \end{pmatrix} =$$

$$\begin{pmatrix} \left(\frac{\partial\widehat{\psi}_1}{\partial\xi}\,\xi - \frac{\partial\widehat{\psi}_1}{\partial\eta}\,\eta\right) a(\xi\eta,\varepsilon) + \frac{\partial\widehat{\psi}_1}{\partial\theta}\,\omega(\varepsilon) - \lambda(\varepsilon)\widehat{\psi}_1 \\ \left(\frac{\partial\widehat{\psi}_2}{\partial\xi}\,\xi - \frac{\partial\widehat{\psi}_2}{\partial\eta}\,\eta\right) a(\xi\eta,\varepsilon) + \frac{\partial\widehat{\psi}_2}{\partial\theta}\,\omega(\varepsilon) + \lambda(\varepsilon)\widehat{\psi}_2 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \lambda(\varepsilon)\left(\frac{\partial\widehat{\psi}_1}{\partial\xi}\,\xi - \frac{\partial\widehat{\psi}_1}{\partial\eta}\,\eta - \widehat{\psi}_1\right) + \frac{\partial\widehat{\psi}_1}{\partial\theta}\,\omega(\varepsilon) + \left(\frac{\partial\widehat{\psi}_1}{\partial\xi}\,\xi - \frac{\partial\widehat{\psi}_1}{\partial\eta}\,\eta\right)\widehat{a}(\xi\eta,\varepsilon) \\ \lambda(\varepsilon)\left(\frac{\partial\widehat{\psi}_2}{\partial\xi}\,\xi - \frac{\partial\widehat{\psi}_2}{\partial\eta}\,\eta + \widehat{\psi}_2\right) + \frac{\partial\widehat{\psi}_2}{\partial\theta}\,\omega(\varepsilon) + \left(\frac{\partial\widehat{\psi}_2}{\partial\xi}\,\xi - \frac{\partial\widehat{\psi}_2}{\partial\eta}\,\eta\right)\widehat{a}(\xi\eta,\varepsilon) \\ 0 \end{pmatrix} =:$$

$$\begin{pmatrix} \left(L^{(1)}_{N^{\ltimes}}\right)(\xi,\eta,\theta,\varepsilon) \\ \left(L^{(2)}_{N^{\ltimes}}\right)(\xi,\eta,\theta,\varepsilon) \\ 0 \end{pmatrix}.$$

It is not difficult to check that these functions admit the following expression in Taylor-Fourier series

$$\left(L^{(w)}_{N^{\ltimes}}\right)(\xi,\eta,\theta,\varepsilon) = \sum_{\substack{j+k\geq 2\\ s\in\mathbb{Z}}} \tilde{g}^{(w)}_{jks}(\xi\eta,\varepsilon)\,\xi^j\eta^k \mathrm{e}^{\mathrm{i}s\theta}, \tag{3.42}$$

where, for $w=1,2$, one has

$$\tilde{g}^{(w)}_{jks}(\xi\eta,\varepsilon) := \gamma^{(w)}_{jks}(\xi\eta,\varepsilon) + (j-k)\,\widehat{a}(\xi\eta,\varepsilon) \tag{3.43}$$

and

$$\gamma^{(w)}_{jks}(\lambda,\omega,\varepsilon) := \begin{cases} \lambda(\varepsilon)\,(j-k-1) + \mathrm{i}s\omega(\varepsilon) & \text{if } \ w=1 \\ \lambda(\varepsilon)\,(j-k+1) + \mathrm{i}s\omega(\varepsilon) & \text{if } \ w=2 \end{cases} \tag{3.44}$$

Applying onto expansions (3.41) the rearrangement introduced at the beginning of this section, the following equivalent expression are derived

$$\left(L^{(w)}_{N^{\ltimes}}\widehat{\psi}_w\right)(\xi,\eta,\theta,\varepsilon) = \sum_{\ell,s\in\mathbb{Z}} g^{(w)}_{\ell s}(\xi\eta,\varepsilon)\psi^{(w)}_{\ell s}(\xi\eta,\varepsilon)\,\xi^{\ell}\mathrm{e}^{\mathrm{i}s\theta}, \tag{3.45}$$

where, using (3.43) and (3.44), we have

$$\begin{aligned} \psi^{(w)}_{\ell s}(\xi\eta,\varepsilon) &= \sum_{k\geq \max\{0,1-\ell/2,-\ell\}} \psi^{(w)}_{\ell+k,ks}(\varepsilon)\ (\xi\eta)^k \\ g^{(w)}_{\ell s}(\xi\eta,\varepsilon) &= \Gamma^{(w)}_{\ell s}(\lambda,\omega,\varepsilon) + \ell\,\widehat{a}(\xi\eta,\varepsilon), \end{aligned} \tag{3.46}$$

with

$$\Gamma^{(w)}_{\ell s}(\lambda,\omega,\varepsilon) := \begin{cases} \lambda(\varepsilon)\,(\ell-1) + \mathrm{i}s\omega(\varepsilon) & \text{if } \ w=1 \\ \lambda(\varepsilon)\,(\ell+1) + \mathrm{i}s\omega(\varepsilon) & \text{if } \ w=2 \end{cases} \tag{3.47}$$

Therefore, equation (3.39) becomes

$$\left(L^{(w)}_{N^{\ltimes}}\widehat{\psi}_w\right)(\xi,\eta,\theta,\varepsilon) = \widehat{h}_w(\xi,\eta,\theta,\varepsilon),$$

for $w=1,2$. Writing $\widehat{h}_w$ in the form

$$\widehat{h}_w(\xi,\eta,\theta,\varepsilon) = \sum_{\ell,s\in\mathbb{Z}} h^{(w)}_{\ell s}(\xi\eta,\varepsilon)\,\xi^{\ell}\mathrm{e}^{\mathrm{i}s\theta},$$

it turns out that the solution $\mathcal{R}\widehat{\Psi} = (\widehat{\psi}_1,\widehat{\psi}_2,0)$ of (3.39) is given by

$$\widehat{\psi}_w(\xi,\eta,\theta,\varepsilon) = \sum_{\ell,s\in\mathbb{Z}} \psi^{(w)}_{\ell s}(\xi\eta,\varepsilon)\,\xi^{\ell}\mathrm{e}^{\mathrm{i}s\theta}, \tag{3.48}$$

where the functions $\psi^{(w)}_{\ell s}(\xi\eta,\varepsilon)$ are obtained from

$$\psi^{(w)}_{\ell s}(\xi\eta,\varepsilon) = \frac{h^{(w)}_{\ell s}(\xi\eta,\varepsilon)}{g^{(w)}_{\ell s}(\xi\eta,\varepsilon)},$$

or, equivalently, from

$$\psi_{\ell s}^{(w)}(\xi\eta,\varepsilon)=\frac{h_{\ell s}^{(w)}(\xi\eta,\varepsilon)}{\Gamma_{\ell s}^{(w)}(\lambda,\omega,\varepsilon)+\ell\,\widehat{a}(\xi\eta,\varepsilon)}, \tag{3.49}$$

with $\Gamma_{\ell s}^{(w)}$ defined in (3.47) and $w=1,2$. Notice that $\psi_{\ell s}^{(w)}(\xi\eta,\varepsilon)$ is a rational function in the variable $\xi\eta$ so (3.48) does not constitute an standard representation in Taylor (formal power) series expansion. Consequently, formula (3.40) does not apply for $\psi_{\ell s}^{(w)}$. Moreover, since $\widehat{\psi}_w(\xi\eta,\theta,\varepsilon)$, $w=1,2$, are the components of the vector field $\mathcal{R}\widehat{\Psi}$, it follows that

$$P_1\widehat{\psi}_1=P_2\widehat{\psi}_2=0,$$

where P_1,P_2 are the two first components of the projection $\mathcal{P}$ (see Definition 2). With the new notation (3.39)–(3.40), this is the same as saying that

$$\psi_{10}(\xi\eta,\varepsilon)=\psi_{-10}(\xi\eta,\varepsilon)=0.$$

For $\widehat{h}_1$ and $\widehat{h}_2$ the same property holds.

§2.5.2 Formal solution of a $\widehat{N}_\mu+\widehat{B}=\mathcal{P}\widehat{H}$-type equation

Let us consider

$$\widehat{N}_\mu(\xi,\eta,\varepsilon)=\begin{pmatrix}\xi\widehat{a}_\mu(\xi\eta,\varepsilon)\\ -\eta\widehat{a}_\mu(\xi\eta,\varepsilon)\\ 0\end{pmatrix},\qquad \widehat{B}(\xi,\eta,\varepsilon)=\begin{pmatrix}\xi\widehat{b}(\xi\eta,\varepsilon)\\ \eta\widehat{b}(\xi\eta,\varepsilon)\\ 0\end{pmatrix}$$

and write in this case

$$\mathcal{P}\widehat{H}(\xi,\eta,\varepsilon)=\begin{pmatrix}\xi\widehat{h}_1(\xi\eta,\varepsilon)\\ -\eta\widehat{h}_2(\xi\eta,\varepsilon)\\ 0\end{pmatrix}.$$

Then, equation

$$\widehat{N}_\mu+\widehat{B}=\mathcal{P}\widehat{H}$$

becomes

$$\begin{aligned}\xi\left(\widehat{a}_\mu(\xi\eta,\varepsilon)+\widehat{b}(\xi\eta,\varepsilon)\right)&=\xi\widehat{h}(\xi\eta,\varepsilon)\\ \eta\left(-\widehat{a}_\mu(\xi\eta,\varepsilon)+\widehat{b}(\xi\eta,\varepsilon)\right)&=\eta\widehat{h}(\xi\eta,\varepsilon),\end{aligned}$$

whose solution follows straightforwardly,

$$\widehat{a}_\mu(\xi\eta,\varepsilon)=\frac{1}{2}\left(\widehat{h}_1(\xi\eta,\varepsilon)-\widehat{h}_2(\xi\eta,\varepsilon)\right),\qquad \widehat{b}(\xi\eta,\varepsilon)=\frac{1}{2}\left(\widehat{h}_1(\xi\eta,\varepsilon)+\widehat{h}_2(\xi\eta,\varepsilon)\right). \tag{3.50}$$

Notice that equations (3.50) determine uniquely the coefficients of $\widehat{a}_\mu(\xi\eta,\varepsilon)$ and $\widehat{b}(\xi\eta,\varepsilon)$ as a function of those of $\widehat{h}_1$ and $\widehat{h}_2$.

§2.6 Improving the recurrent scheme

From the precedent sections it is clear that the recurrent scheme presented in this work is implementable on a computer. Moreover, as it was done in Chapter 1 (and in [28]), it is not difficult to modify the scheme (3.35)– (3.38) in order to save computations and space of memory. This improvement is based on the following result.

Lemma 3.3 *At any step* $K \geq 1$, *the vector fields* $\mathcal{R}\widehat{\Phi}_\mu{}^{(K+1)}$, $N_\mu{}^{(K+1)}$ *and* $\widehat{B}^{(K+1)}$, *obtained from equations* (3.37)–(3.38) *satisfy the following properties,*

$$\begin{array}{lcll} \mathcal{R}\widehat{\Phi}_\mu{}^{(K+1)} & - & \mathcal{R}\widehat{\Phi}_\mu{}^{(K)} & = \mathcal{O}_{K+1} \\ N_\mu{}^{(K+1)} & - & N_\mu{}^{(K)} & = \mathcal{O}_{K+1} \\ \widehat{B}^{(K+1)} & - & \widehat{B}^{(K)} & = \mathcal{O}_{K+1}. \end{array}$$

Proof. We will prove it inductively. Thus,

(*i*) for $K = 1$ equation (3.37) reads

$$\mathcal{L}_{N_\mu{}^{(1)}}\left(\mathcal{R}\widehat{\Phi}_\circ\right) + \mathcal{L}_{N_\circ + N_\mu{}^{(1)}}\left(\mathcal{R}\widehat{\Phi}_\mu{}^{(2)}\right) = \mathcal{R}\left(\widehat{G_F}\left(\widehat{\Phi}_\mu^{(1)}\right)\right). \tag{3.51}$$

Since $\widehat{\Phi}_\mu^{(1)} = 0$, from definition (3.17) it follows that

$$\widehat{G_F}(0) = \widehat{F}_\mu\left(\Phi_\circ^\ltimes\right).$$

Moreover, we have that

$$\Phi_\circ^\ltimes = \Omega(\varepsilon) + \Phi_\circ, \qquad N_\mu{}^{(1)} = \Lambda_\mu(\varepsilon)\zeta.$$

Applying all together onto (3.51) one obtains

$$\mathcal{L}_{\Lambda_\mu(\varepsilon)\zeta}\left(\mathcal{R}\widehat{\Phi}_\circ\right) + \mathcal{L}_{N_\circ + \Lambda_\mu(\varepsilon)\zeta}\left(\mathcal{R}\widehat{\Phi}_\mu{}^{(2)}\right) = \mathcal{R}\left(\widehat{F}_\mu\left(\Phi_\circ^\ltimes\right)\right)$$

or, equivalently,

$$\left[\Lambda(\varepsilon)\zeta, \mathcal{R}\widehat{\Phi}_\circ\right] + \mathcal{L}_{N_\circ + \Lambda_\mu(\varepsilon)\zeta}\left(\mathcal{R}\widehat{\Phi}_\mu{}^{(2)}\right) = \mathcal{R}\left(\widehat{F}_\mu\left(\Phi_\circ^\ltimes\right)\right),$$

where we recall that $[H_1, H_2]$ stands for the Lie bracket of the vector fields H_1 and H_2. This equation can also be written as

$$\mathcal{L}_{N_\circ + \Lambda_\mu(\varepsilon)\zeta}\left(\mathcal{R}\widehat{\Phi}_\mu{}^{(2)}\right) = \mathcal{R}\left(\widehat{F}_\mu\left(\Phi_\circ^\ltimes\right)\right) - \left[\Lambda(\varepsilon)\zeta, \mathcal{R}\widehat{\Phi}_\circ\right]. \tag{3.52}$$

Since $\widehat{F}_\mu$ has order at least 2 and $\left[\Lambda(\varepsilon)\zeta, \mathcal{R}\widehat{\Phi}_\circ\right]$ has the same order as $\mathcal{R}\widehat{\Phi}_\circ$ (which is $\mathcal{O}_2$), we get that the right-hand side of (3.52) is $\mathcal{O}_2$. Consequently,

$$\mathcal{L}_{N_\circ + \Lambda_\mu(\varepsilon)\zeta}\left(\mathcal{R}\widehat{\Phi}_\mu{}^{(2)}\right) = \mathcal{O}_2.$$

From Lemma 3.26(*ii*), we know that $\mathcal{L}$ preserves order, so it turns out that

$$\mathcal{R}\widehat{\Phi}_\mu{}^{(2)} = \mathcal{O}_2.$$

Having in mind that $\mathcal{R}\widehat{\Phi}_\mu{}^{(1)} = 0$ it follows, finally, that

$$\mathcal{R}\widehat{\Phi}_\mu{}^{(2)} - \mathcal{R}\widehat{\Phi}_\mu{}^{(1)} = \mathcal{R}\widehat{\Phi}_\mu{}^{(2)} = \mathcal{O}_2.$$

With respect to $\widehat{N}_\mu$ and $\widehat{B}$, equation (3.38) becomes

$$\widehat{N}_\mu^{(2)} + \widehat{B}^{(2)} = \mathcal{P}\left(\widehat{G_F}\left(\widehat{\Phi}_\mu^{(1)}\right)\right) = \mathcal{P}\left(\widehat{F}_\mu\left(\Phi_\circ^\ltimes\right)\right), \tag{3.53}$$

which is of type $\widehat{N}_\mu + \widehat{B} = \mathcal{P}\widehat{H}$. From its solvability (see section §2.5.2), it is derived that $\widehat{N}_\mu$ and $\widehat{B}$ are of the same order as the vector field $\mathcal{P}\widehat{H}$. Thus, applied to (3.53) and using that $\mathcal{P}(\widehat{F}_\mu(\Phi_\circ^\ltimes)) = \mathcal{O}_3$, it follows that $\widehat{N}_\mu^{(2)}$ and $\widehat{B}^{(2)}$ are both $\mathcal{O}_3$. In particular, since $\widehat{N}^{(1)} = \widehat{B}^{(1)} = 0$, we have that

$$\widehat{N}_\mu^{(2)} - \widehat{N}_\mu^{(1)} = \mathcal{O}_2, \qquad \widehat{B}^{(2)} - \widehat{B}^{(1)} = \mathcal{O}_2,$$

and also that

$$N_\mu{}^{(2)} - N_\mu{}^{(1)} = \mathcal{O}_2.$$

This proves our lemma for the case $K = 1$.

(*ii*) Let us assume now that the following estimates are satisfied for a given $K \geq 1$,

$$\begin{array}{lclcl} \mathcal{R}\widehat{\Phi}_\mu{}^{(K)} & - & \mathcal{R}\widehat{\Phi}_\mu{}^{(K-1)} & = & \mathcal{O}_K \\ N_\mu{}^{(K)} & - & N_\mu{}^{(K-1)} & = & \mathcal{O}_K \\ \widehat{B}^{(K)} & - & \widehat{B}^{(K-1)} & = & \mathcal{O}_K. \end{array}$$

We will prove first that $\mathcal{R}\widehat{\Phi}_\mu{}^{(K+1)} - \mathcal{R}\widehat{\Phi}_\mu{}^{(K)} = \mathcal{O}_{K+1}$. To do it, consider equation (3.37) for two consecutive values of K. Precisely,

$$\begin{array}{rcl} \mathcal{L}_{N_\mu{}^{(K)}}\left(\mathcal{R}\widehat{\Phi}_\circ\right) + \mathcal{L}_{N_\circ^\ltimes + N_\mu{}^{(K)}}\left(\mathcal{R}\widehat{\Phi}_\mu{}^{(K+1)}\right) & = & \mathcal{R}\left(\widehat{G_F}\left(\widehat{\Phi}_\mu^{(K)}\right)\right) \\ \mathcal{L}_{N_\mu{}^{(K-1)}}\left(\mathcal{R}\widehat{\Phi}_\circ\right) + \mathcal{L}_{N_\circ^\ltimes + N_\mu{}^{(K-1)}}\left(\mathcal{R}\widehat{\Phi}_\mu{}^{(K)}\right) & = & \mathcal{R}\left(\widehat{G_F}\left(\widehat{\Phi}_\mu^{(K-1)}\right)\right). \end{array} \tag{3.54}$$

Subtracting them, its left-hand side becomes

$$\mathcal{L}_{N_\mu{}^{(K)} - N_\mu{}^{(K-1)}}\left(\mathcal{R}\widehat{\Phi}_\circ\right) + \mathcal{L}_{N_\circ + N_\mu{}^{(K)}}\left(\mathcal{R}\widehat{\Phi}_\mu{}^{(K+1)}\right) - \mathcal{L}_{N_\circ + N_\mu{}^{(K-1)}}\left(\mathcal{R}\widehat{\Phi}_\mu{}^{(K)}\right), \tag{3.55}$$

where it has been taken into account the linearity of $\mathcal{L}$. We deal with each part of this equation separately.

- Concerning the first term, since the linear term of $N_\mu{}^{(K)} - N_\mu{}^{(K-1)}$ vanishes, it follows that

$$\mathcal{L}_{N_\mu{}^{(K)} - N_\mu{}^{(K-1)}}\left(\mathcal{R}\widehat{\Phi}_\circ\right) = D\left(\mathcal{R}\widehat{\Phi}_\circ\right)\left(N_\mu{}^{(K)} - N_\mu{}^{(K-1)}\right) = \mathcal{O}_{K+1}, \tag{3.56}$$

  where it has been taken into account that $D(\mathcal{R}\widehat{\Phi}_\circ) = \mathcal{O}_2$ and the induction hypothesis $N_\mu{}^{(K)} - N_\mu{}^{(K-1)} = \mathcal{O}_K$.

– Regarding to

$$\mathcal{L}_{N_\circ+N_\mu{}^{(K)}}\left(\mathcal{R}\widehat{\Phi}_\mu{}^{(K+1)}\right)-\mathcal{L}_{N_\circ+N_\mu{}^{(K-1)}}\left(\mathcal{R}\widehat{\Phi}_\mu{}^{(K)}\right), \tag{3.57}$$

writing $N_\mu{}^{(K)}$ as $N_\mu{}^{(K-1)}+(N_\mu{}^{(K)}-N_\mu{}^{(K-1)})$ and using again the linearity of $\mathcal{L}$, it turns out that

$$\begin{aligned}\mathcal{L}_{N_\circ+N_\mu{}^{(K)}}&\left(\mathcal{R}\widehat{\Phi}_\mu{}^{(K+1)}\right)=\\ &\mathcal{L}_{N_\circ+N_\mu{}^{(K-1)}+(N_\mu{}^{(K)}-N_\mu{}^{(K-1)})}\left(\mathcal{R}\widehat{\Phi}_\mu{}^{(K+1)}\right)=\\ &\mathcal{L}_{N_\circ+N_\mu{}^{(K-1)}}\left(\mathcal{R}\widehat{\Phi}_\mu{}^{(K+1)}\right)+\mathcal{L}_{N_\mu{}^{(K)}-N_\mu{}^{(K-1)}}\left(\mathcal{R}\widehat{\Phi}_\mu{}^{(K+1)}\right).\end{aligned}$$

Since $N_\mu{}^{(K)}-N_\mu{}^{(K-1)}=\mathcal{O}_K$, this equation can be written as

$$\mathcal{L}_{N_\circ+N_\mu{}^{(K)}}\left(\mathcal{R}\widehat{\Phi}_\mu{}^{(K+1)}\right)=\mathcal{L}_{N_\circ+N_\mu{}^{(K-1)}}\left(\mathcal{R}\widehat{\Phi}_\mu{}^{(K+1)}\right)+\mathcal{O}_{K+1}$$

and therefore, the expression (3.57) becomes

$$\mathcal{L}_{N_\circ+N_\mu{}^{(K-1)}}\left(\mathcal{R}\widehat{\Phi}_\mu{}^{(K+1)}-\mathcal{R}\widehat{\Phi}_\mu{}^{(K)}\right)+\mathcal{O}_{K+1}. \tag{3.58}$$

Consequently, from this estimate and using (3.56), expression (3.55) is given by

$$\mathcal{L}_{N_\circ+N_\mu{}^{(K-1)}}\left(\mathcal{R}\widehat{\Phi}_\mu{}^{(K+1)}-\mathcal{R}\widehat{\Phi}_\mu{}^{(K)}\right)+\mathcal{O}_{K+1}. \tag{3.59}$$

Let us consider now the subtraction of both formulas in (3.54) and take its right-hand side part, that is,

$$\mathcal{R}\left(\widehat{G_F}\left(\widehat{\Phi}_\mu^{(K)}\right)\right)-\mathcal{R}\left(\widehat{G_F}\left(\widehat{\Phi}_\mu^{(K-1)}\right)\right). \tag{3.60}$$

Since

$$\widehat{G_F}\left(\widehat{\Phi}_\mu\right)=\widehat{F}_\mu\left(\Phi_\circ^\ltimes\right)+\sum_{m\geq 1}\frac{1}{m!}D^m\widehat{F}\left(\Phi_\circ^\ltimes\right)\left(\widehat{\Phi}_\mu\right)^m,$$

one obtains that

$$\begin{aligned}\mathcal{R}\left(\widehat{G_F}\left(\widehat{\Phi}_\mu^{(K)}\right)\right)&-\mathcal{R}\left(\widehat{G_F}\left(\widehat{\Phi}_\mu^{(K-1)}\right)\right)=\\ &\sum_{m\geq 1}\frac{1}{m!}D^m\widehat{F}\left(\Phi_\circ^\ltimes\right)\left(\left(\widehat{\Phi}_\mu^{(K)}\right)^m-\left(\widehat{\Phi}_\mu^{(K-1)}\right)^m\right),\end{aligned}$$

whose lowest order term is achieved for $m=1$, namely, for

$$D\widehat{F}\left(\Phi_\circ^\ltimes\right)\left(\widehat{\Phi}_\mu^{(K)}-\widehat{\Phi}_\mu^{(K-1)}\right). \tag{3.61}$$

Using that $D\widehat{F}(\Phi_\circ^\ltimes)=\mathcal{O}_1$ and that, by induction hypothesis, $\widehat{\Phi}_\mu^{(K)}-\widehat{\Phi}_\mu^{(K-1)}=\mathcal{O}_K$, it follows that (3.61) is $\mathcal{O}_{K+1}$ and, consequently, so is (3.60). Joining this estimate and the bound (3.59), one obtains from subtracting the two equations in (3.54) that

$$\begin{aligned}\mathcal{L}_{N_\circ+N_\mu{}^{(K-1)}}&\left(\mathcal{R}\widehat{\Phi}_\mu{}^{(K+1)}-\mathcal{R}\widehat{\Phi}_\mu{}^{(K)}\right)=\\ &\mathcal{R}\left(\widehat{G_F}\left(\widehat{\Phi}_\mu^{(K)}\right)\right)-\mathcal{R}\left(\widehat{G_F}\left(\widehat{\Phi}_\mu^{(K-1)}\right)\right)+\mathcal{O}_{K+1}=\mathcal{O}_{K+1}.\end{aligned}$$

Having in mind again that $\mathcal{L}$ preserves the order in the spatial variables, it follows finally that

$$\mathcal{R}\widehat{\Phi}_\mu{}^{(K+1)} - \mathcal{R}\widehat{\Phi}_\mu{}^{(K)} = \mathcal{O}_{K+1},$$

as the lemma claims.

We are proving now that

$$\widehat{N}_\mu^{(K+1)} - \widehat{N}_\mu^{(K)} = \mathcal{O}_{K+1}, \qquad \widehat{B}^{(K+1)} - \widehat{B}^{(K)} = \mathcal{O}_{K+1}.$$

Indeed, following the same argument as above, we can consider the equation (3.38) and write it for two consecutive values of K,

$$\begin{array}{ccccc} \widehat{N}_\mu^{(K+1)} & + & \widehat{B}^{(K+1)} & = & \mathcal{P}\left(\widehat{G_F}\left(\widehat{\Phi}_\mu^{(K)}\right)\right) \\ \widehat{N}_\mu^{(K)} & + & \widehat{B}^{(K)} & = & \mathcal{P}\left(\widehat{G_F}\left(\widehat{\Phi}_\mu^{(K-1)}\right)\right). \end{array}$$

Subtracting them, it follows

$$(3.62) \qquad \left(\widehat{N}_\mu^{(K+1)} - \widehat{N}_\mu^{(K)}\right) + \left(\widehat{B}^{(K+1)} - \widehat{B}^{(K)}\right) = \\ \mathcal{P}\left(\widehat{G_F}\left(\widehat{\Phi}_\mu^{(K)}\right)\right) - \mathcal{P}\left(\widehat{G_F}\left(\widehat{\Phi}_\mu^{(K-1)}\right)\right).$$

Using the same argument as in (3.60), we obtain that

$$\mathcal{P}\left(\widehat{G_F}\left(\widehat{\Phi}_\mu^{(K)}\right)\right) - \mathcal{P}\left(\widehat{G_F}\left(\widehat{\Phi}_\mu^{(K-1)}\right)\right) = \mathcal{O}_{K+1}.$$

Since equation (3.62) is of type $\widehat{N}_\mu + \widehat{B} = \mathcal{P}\widehat{H}$, one has that

$$\widehat{N}_\mu^{(K+1)} - \widehat{N}_\mu^{(K)} \qquad \text{and} \qquad \widehat{B}^{(K+1)} - \widehat{B}^{(K)}$$

have the same order as $\mathcal{P}\left(\widehat{G_F}\left(\widehat{\Phi}_\mu^{(K)}\right)\right) - \mathcal{P}\left(\widehat{G_F}\left(\widehat{\Phi}_\mu^{(K-1)}\right)\right)$. Therefore,

$$\begin{array}{ccccc} \widehat{N}_\mu^{(K+1)} & - & \widehat{N}_\mu^{(K)} & = & \mathcal{O}_{K+1} \\ \widehat{B}^{(K+1)} & - & \widehat{B}^{(K)} & = & \mathcal{O}_{K+1}, \end{array}$$

analogously,

$$\begin{array}{ccccc} N_\mu{}^{(K+1)} & - & N_\mu{}^{(K)} & = & \mathcal{O}_{K+1} \\ \widehat{B}^{(K+1)} & - & \widehat{B}^{(K)} & = & \mathcal{O}_{K+1}, \end{array}$$

and the lemma is finally proved.

□

The main consequence of this lemma is that it is enough to consider the vector fields

$$\widehat{\Phi}_\mu^{(K)}, \quad N_\mu{}^{(K)}, \quad \widehat{B}^{(K)}$$

provided by the recurrent scheme (3.35)– (3.38) just up to order K, that is

$$\widehat{\Phi}_\mu^{(K)},\ N_\mu{}^{(K)},\ \widehat{B}^{(K)} \quad \text{are all} \ \ \mathcal{O}_{\leq K}.$$

Therefore, our *final recurrent scheme* is rewritten in the following form: assuming that the vector fields $\Phi_\circ$ (such that $\mathcal{P}\widehat{\Phi}_\circ = 0$), $N_\circ$ and $F_\circ$ are known, take again initial values

$$\widehat{\Phi}_\mu^{(1)} = 0, \qquad {N_\mu}^{(1)} = \Lambda_\mu(\varepsilon)\zeta, \qquad \widehat{B}^{(1)} = 0. \tag{3.63}$$

Taking $\mathcal{P}{\widehat{\Phi}_\mu}^{(K)} = 0$, for any $K \geq 1$, and using that $N^{\ltimes} = N_\circ^{\ltimes} + N_\mu$, we obtain

$$\begin{array}{rcl} \widehat{\Phi}_\mu^{(K+1)} & = & \mathcal{R}{\widehat{\Phi}_\mu}^{(K+1)} \\ {N_\mu}^{(K+1)} & = & \Lambda_\mu(\varepsilon)\zeta + \widehat{N}_\mu^{(K+1)} \\ \widehat{B}^{(K+1)}, & & \end{array} \tag{3.64}$$

satisfying that

$$\widehat{\Phi}_\mu^{(K+1)},\ {N_\mu}^{(K+1)},\ \widehat{B}^{(K+1)} \quad \text{are all } \mathcal{O}_{\leq K+1},$$

from the recurrent equations

$$\left\{\mathcal{L}_{{N_\mu}^{(K)}}\left(\mathcal{R}\widehat{\Phi}_\circ\right)\right\}_{\leq K+1} + \left\{\mathcal{L}_{N_\circ^{\ltimes}+{N_\mu}^{(K)}}\left(\mathcal{R}{\widehat{\Phi}_\mu}^{(K+1)}\right)\right\}_{\leq K+1} = \left\{\mathcal{R}\left(\widehat{G_F}\left(\widehat{\Phi}_\mu^{(K)}\right)\right)\right\}_{\leq K+1} \tag{3.65}$$

$$\widehat{N}_\mu^{(K+1)} + \widehat{B}^{(K+1)} = \left\{\mathcal{P}\left(\widehat{G_F}\left(\widehat{\Phi}_\mu^{(K)}\right)\right)\right\}_{\leq K+1}. \tag{3.66}$$

Remark 12 *Since $\mathcal{P}(\widehat{G_F}(\widehat{\Phi}_\mu^{(K)}))$ only contains terms of odd order, it is straightforward to check that*

$$\widehat{N}_\mu^{(2J+2)} = \widehat{N}_\mu^{(2J+1)}, \qquad \widehat{B}^{(2J+2)} = \widehat{B}^{(2J+1)},$$

for any $J \geq 0$.

§2.7 Convergence of the recurrent scheme

§2.7.1 Definition of the norms

Given positive numbers σ and ρ, we consider the following type of domains

$$\begin{array}{rcl} \overline{\mathcal{D}_\sigma} & = & \left\{z = (z_1, z_2) \in \mathbb{C}^2 \,:\, |z_\ell| \leq \sigma,\ \ell = 1, 2\right\} \\ \mathcal{T}_\rho & = & \left\{\theta \in \mathbb{C} :\ \Re\theta \in \mathbb{T},\ |\Im\theta| \leq \rho\right\}. \end{array}$$

We deal with functions $g(z,\theta)$ being 2π-periodic with respect to θ, so having an expansion in Fourier series of the form

$$g(z,\theta) = \sum_{s\in\mathbb{Z}} g_s(z)\, \mathrm{e}^{\mathrm{i}s\theta}$$

where we recall that its Fourier coefficients can be obtained from

$$g_s(z) = \frac{1}{2\pi}\int_{\mathbb{T}} g(z,\theta)\mathrm{e}^{-\mathrm{i}s\theta}\, d\theta.$$

Moreover, we will assume that these functions $g_s(z)$ satisfy $g_s(0) = 0$ and admit an expansion in formal (Taylor) power series around the origin,

$$g_s(z) = \sum_{j+k\geq 1} g_{jks}\, z_1^j z_2^k.$$

Thus, by a 2π-periodic in θ function $g(z,\theta)$ analytic on $\overline{\mathcal{D}_\sigma} \times \mathcal{T}_\rho$ we mean a function with Taylor-Fourier expansion

$$g(z,\theta) = \sum_{s\in\mathbb{Z}} g_s(z)\, \mathrm{e}^{\mathrm{i}s\theta} = \sum_{\substack{j+k\geq 1\\ s\in\mathbb{Z}}} g_{jks}\, z_1^j z_2^k\, \mathrm{e}^{\mathrm{i}s\theta}, \tag{3.67}$$

absolutely convergent for any $(z,\theta) \in \overline{\mathcal{D}_\sigma} \times \mathcal{T}_\rho$. In particular, this implies that, for any $s \in \mathbb{Z}$, the functions $g_s(z)$ are also analytic on $\overline{\mathcal{D}_\sigma}$.

We are going to introduce the norms we will use along this convergence proof. First of all, and like it was done in Chapter 1, we consider the following norms for functions of the type $g(z)$: the *supremum norm*

$$\|g\|_{\infty,\sigma} = \sup_{z\in\overline{\mathcal{D}_\sigma}} |g(z)|$$

and the 1-*norm*

$$\|g\|_{1,\sigma} = \sum_{j+k\geq 1} g_{jk}\, \sigma^{j+k}.$$

For a vector field $G = (g_1, g_2, \ldots, g_m) : \overline{\mathcal{D}_\sigma} \subseteq \mathbb{C}^2 \longmapsto \mathbb{C}^m$ we define

$$\|G\|_{\infty,\sigma} = \sup_{i=1,\ldots,m} \|g_i\|_{\infty,\sigma}, \qquad \|G\|_{1,\sigma} = \frac{1}{m} \sum_{i=1,\ldots,m} \|g_i\|_{1,\sigma} \tag{3.68}$$

and analogously if $G : \overline{\mathcal{D}_\sigma} \subseteq \mathbb{C}^2 \longmapsto \mathbb{M}_{m,m}(\mathbb{C}^m)$. These norms satisfy the following properties, whose proof is standard.

Lemma 3.4 *Let g be an analytic function on $\overline{\mathcal{D}_\sigma}$ satisfying that $g(0) = 0$. Then, the following properties hold:*

(i) $\|g\|_{\infty,\sigma} \leq \|g\|_{1,\sigma}$.

(ii) Let $g_{[K]} = \mathcal{O}_{[K]}$ and $h_{[L]} = \mathcal{O}_{[L]}$ be homogeneous polynomials of orders K and L, respectively, with $K \neq L$, that is

$$g_{[K]}(z) = \sum_{j+k=K} g_{jk}\, z_1^j z_2^k, \qquad h_{[L]}(z) = \sum_{j+k=L} h_{jk}\, z_1^j z_2^k.$$

Then, we have that

$$\left\|g_{[K]} + h_{[L]}\right\|_{1,\sigma} = \left\|g_{[K]}\right\|_{1,\sigma} + \left\|h_{[L]}\right\|_{1,\sigma}.$$

Now, for a function $g(z,\theta)$ of the form (3.67), we consider the norms

$$\|g\|_{\infty,\sigma,\rho} = \sum_{s\in\mathbb{Z}} \|g_s(z)\|_{\infty,\sigma}\, \mathrm{e}^{|s|\rho} \tag{3.69}$$

$$\|g\|_{1,\sigma,\rho} = \sum_{s\in\mathbb{Z}} \|g_s(z)\|_{1,\sigma}\, \mathrm{e}^{|s|\rho}. \tag{3.70}$$

As above, we can extend these definitions to vector fields. Indeed, for $G = (g_1, g_2, \dots, g_m) : \overline{\mathcal{D}_\sigma} \times \mathcal{T}_\rho \subseteq \mathbb{C}^3 \longmapsto \mathbb{C}^m$ we define

$$\|G\|_{\infty,\sigma,\rho} = \sup_{i=1,\dots,m} \|g_i\|_{\infty,\sigma,\rho}\,, \qquad \|G\|_{1,\sigma,\rho} = \frac{1}{m} \sum_{i=1,\dots,m} \|g_i\|_{1,\sigma,\rho} \tag{3.71}$$

and, in a similar way, for $G : \overline{\mathcal{D}_\sigma} \times \mathcal{T}_\rho \subseteq \mathbb{C}^3 \longmapsto \mathbb{M}_{m,m}(\mathbb{C}^m)$. The following standard lemma relates norms and composition. Precisely,

Lemma 3.5 *(i) Let $G(\zeta,\theta)$ be a vector field analytic on $\overline{\mathcal{D}_{\sigma_1}} \times \mathcal{T}_\rho$ and let us consider $\Phi(\zeta,\theta) = (\phi_1(\zeta,\theta), \phi_2(\zeta,\theta), \theta)$, analytic on $\overline{\mathcal{D}_{\sigma_2}} \times \mathcal{T}_\rho$, $0 < \sigma_2 \le \sigma_1$, satisfying that*

$$\|\phi_\ell\|_{1,\sigma_2,\rho} \le \sigma_1$$

for $\ell = 1, 2$. Then, the following bound holds,

$$\|G \circ \Phi\|_{1,\sigma_2,\rho} \le \|G\|_{1,\sigma_1,\rho}\,.$$

(ii) If g is analytic on $\overline{\mathcal{D}_\sigma}$ and satisfies that $|g(\zeta)| \ge C$ $\forall \zeta \in \overline{\mathcal{D}_\sigma}$, then it follows that

$$\left\|\frac{1}{g}\right\|_{1,\sigma} \le \frac{1}{C}.$$

Proof. (i) is standard. With respect to (ii): since $g(\zeta)$ is analytic on $\overline{\mathcal{D}_\sigma}$ and verifies that $|g(\zeta)| \ge C$ for any $\zeta \in \overline{\mathcal{D}_\sigma}$, it follows that $h(\zeta) = 1/g(\zeta)$ is also analytic on the same domain and satisfies that $|h(\zeta)| \le 1/C$ $\forall \zeta \in \overline{\mathcal{D}_\sigma}$. Consequently, $\|h\|_{\infty,\sigma} \le 1/C$. Then, applying (i), we have that

$$\|h\|_{1,\sigma} = \|\zeta \circ h(\zeta)\|_{1,\sigma} \le \|\zeta\|_{1,1/C} = \frac{1}{C}.$$

□

§2.7.2 Some technical lemmas and estimates

The aim of this section is to provide estimates for the vector fields solution of equations of the type $\widehat{N}_\mu + \widehat{B} = \mathcal{P}\widehat{H}$ and $\mathcal{L}_{N^\kappa}(\mathcal{R}\widehat{\Psi}) = \mathcal{R}\widehat{H}$ and whose formal approach has been obtained in the precedent sections. These estimates will be presented in Proposition 3.1, while next lemma gives a lower bound for the denominators appearing in the solution of equation $\mathcal{L}_{N^\kappa}(\mathcal{R}\widehat{\Psi}) = \mathcal{R}\widehat{H}$.

Lemma 3.6 *Let us consider*

$$\Gamma_{\ell s}^{(w)}(\lambda, \omega, \varepsilon) + \ell\, \widehat{a}(\xi\eta, \varepsilon),$$

where $\ell, s \in \mathbb{Z}$, $(\xi, \eta) \in \overline{\mathcal{D}_\sigma}$, $\Gamma_{\ell s}^{(w)}$ *as defined in* (3.47), *that is,*

$$\Gamma_{\ell s}^{(w)}(\lambda, \omega, \varepsilon) = \begin{cases} \lambda(\varepsilon)\,(\ell - 1) + \mathrm{i} s\omega(\varepsilon) & \textit{if} \;\; w = 1 \\ \lambda(\varepsilon)\,(\ell + 1) + \mathrm{i} s\omega(\varepsilon) & \textit{if} \;\; w = 2 \end{cases},$$

with $s \neq 0$ *or* $\ell \neq \pm 1$, *and* $\widehat{a}(\xi\eta, \varepsilon)$ *coming from*

$$\widehat{N}(\xi, \eta, \varepsilon) = \begin{pmatrix} \xi \widehat{a}(\xi\eta, \varepsilon) \\ -\eta \widehat{a}(\xi\eta, \varepsilon) \\ 0 \end{pmatrix} = \begin{pmatrix} \xi\,(\widehat{a}_\circ(\xi\eta) + \widehat{a}_\mu(\xi\eta, \varepsilon)) \\ -\eta\,(\widehat{a}_\circ(\xi\eta) + \widehat{a}_\mu(\xi\eta, \varepsilon)) \\ 0 \end{pmatrix}$$

Moreover, let us assume that

$$0 < \frac{\lambda_\circ}{2} \leq \lambda(\varepsilon) \leq \frac{3}{2}\lambda_\circ. \tag{3.72}$$

Then, the following bound is satisfied

$$\left|\Gamma_{\ell s}^{(w)}(\lambda, \omega, \varepsilon) + \ell\, \widehat{a}(\xi\eta, \varepsilon)\right| \geq \begin{cases} \lambda_\circ/4 & \textit{if} \;\; s = 0 \textit{ (and, therefore, } \ell \neq \pm 1) \\ |\omega(\varepsilon)| & \textit{if} \;\; s \neq 0 \end{cases}$$

provided that

$$\|\widehat{a}_\circ(\xi\eta)\|_{1,\sigma} \;,\; \|\widehat{a}_\mu(\xi\eta, \varepsilon)\|_{1,\sigma} \;\leq \lambda_\circ/16 \tag{3.73}$$

hold.

Proof. We will distinguish between the cases $s \neq 0$ and $s = 0$. Thus,

(*i*) *Case* $s \neq 0$. Since $\widehat{a}(\xi\eta, \varepsilon)$ is a real function, $\omega(\varepsilon) \in \mathbb{R}^n$ and using the definition of $\Gamma_{\ell s}^{(w)}(\lambda, \omega, \varepsilon)$ it follows that

$$\left|\Gamma_{\ell s}^{(w)}(\lambda, \omega, \varepsilon) + \ell\, \widehat{a}(\xi\eta, \varepsilon)\right| \geq |s\omega(\varepsilon)| \geq |\omega(\varepsilon)|\,,$$

for any $(\xi, \eta) \in \overline{\mathcal{D}_\sigma}$.

(*ii*) *Case* $s = 0$. From

$$\left|\Gamma_{\ell 0}^{(w)}(\lambda, \omega, \varepsilon) + \ell\, \widehat{a}(\xi\eta, \varepsilon)\right| = \begin{cases} |\lambda(\varepsilon)(\ell - 1) + \ell\, \widehat{a}(\xi\eta, \varepsilon)| & \text{if} \;\; w = 1 \\ |\lambda(\varepsilon)(\ell + 1) + \ell\, \widehat{a}(\xi\eta, \varepsilon)| & \text{if} \;\; w = 2 \end{cases},$$

it follows that

$$\left|\Gamma_{\ell 0}^{(w)}(\lambda, \omega, \varepsilon) + \ell\, \widehat{a}(\xi\eta, \varepsilon)\right| \geq \begin{cases} |\lambda(\varepsilon)|\ell - 1| - |\ell|\; |\widehat{a}(\xi\eta, \varepsilon)|| & \text{if} \;\; w = 1 \\ |\lambda(\varepsilon)|\ell + 1| - |\ell|\; |\widehat{a}(\xi\eta, \varepsilon)|| & \text{if} \;\; w = 2 \end{cases}.$$

Recall that $s = 0$ implies that $\ell \neq \pm 1$. Let us consider the case $w = 1$. Taking into account lemma 3.4(i), it turns out that

$$0 \leq |\widehat{a}(\xi\eta,\varepsilon)| \leq \|\widehat{a}(\xi\eta,\varepsilon)\|_{\infty,\sigma} \leq \|\widehat{a}(\xi\eta,\varepsilon)\|_{1,\sigma}\,.$$

Using also assumption (3.72) and hypothesis (3.73) one obtains

$$\begin{aligned}
&|\lambda(\varepsilon)|\ell-1| - |\ell|\,|\widehat{a}(\xi\eta,\varepsilon)|| \geq \left|\frac{\lambda_\circ}{2}|\ell-1| - |\ell|\,\|\widehat{a}(\xi\eta,\varepsilon)\|_{1,\sigma}\right| = \\
&\quad |\ell-1|\left|\frac{\lambda_\circ}{2} - \frac{|\ell|}{|\ell-1|}\,\|\widehat{a}(\xi\eta,\varepsilon)\|_{1,\sigma}\right| \geq |\ell-1|\left|\frac{\lambda_\circ}{2} - 2\,\|\widehat{a}(\xi\eta,\varepsilon)\|_{1,\sigma}\right| \geq \\
&\quad |\ell-1|\left|\frac{\lambda_\circ}{2} - 2\left(\|\widehat{a}_\circ(\xi\eta)\|_{1,\sigma} + \|\widehat{a}_\mu(\xi\eta,\varepsilon)\|_{1,\sigma}\right)\right| \geq |\ell-1|\left|\frac{\lambda_\circ}{2} - 2\left(\frac{\lambda_\circ}{16} + \frac{\lambda_\circ}{16}\right)\right| \geq \\
&\quad |\ell-1|\left|\frac{\lambda_\circ}{2} - \frac{\lambda_\circ}{4}\right| = |\ell-1|\,\frac{\lambda_\circ}{4} \geq \frac{\lambda_\circ}{4}.
\end{aligned}$$

In a similar way, it can be proved for $w = 2$ that

$$|\lambda(\varepsilon)|\ell+1| - |\ell|\,|\widehat{a}(\xi\eta,\varepsilon)|| \geq \frac{\lambda_\circ}{4}$$

and therefore,

$$\left|\Gamma^{(w)}_{\ell s}(\lambda,\omega,\varepsilon) + \ell\,\widehat{a}(\xi\eta,\varepsilon)\right| \geq \frac{\lambda_\circ}{4}.$$

□

Proposition 3.1 *Let us consider vector fields $\widehat{H}$ and $\widehat{N}_\circ$ analytic on $\overline{\mathcal{D}_\sigma} \times \mathcal{T}_\rho$ and $\widehat{N}_\mu$, $\widehat{B}$ analytic on $\overline{\mathcal{D}_\sigma}$, respectively. Let $\widehat{B}$ and $\mathcal{R}\widehat{\Psi}$ be the solutions of the equations*

$$\begin{aligned}
\widehat{N}_\mu + \widehat{B} &= \mathcal{P}\widehat{H} \\
\mathcal{L}_{N^\ltimes}\left(\mathcal{R}\widehat{\Psi}\right) &= \mathcal{R}\widehat{H},
\end{aligned}$$

which have been formally obtained in Sections §2.5.1 and §2.5.2. Moreover, assume that

$$N^\ltimes(\zeta,\varepsilon) = \Omega(\varepsilon) + \Lambda(\varepsilon)\zeta + \widehat{N}_\circ(\zeta) + \widehat{N}_\mu(\zeta,\varepsilon) =$$

$$\tag{3.74} \begin{pmatrix} 0 \\ 0 \\ \omega(\varepsilon) \end{pmatrix} + \begin{pmatrix} \begin{pmatrix} \lambda(\varepsilon) & 0 \\ 0 & -\lambda(\varepsilon) \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix} \\ 0 \end{pmatrix} + \begin{pmatrix} \xi\widehat{a}_\circ(\xi\eta) \\ -\eta\widehat{a}_\circ(\xi\eta) \\ 0 \end{pmatrix} + \begin{pmatrix} \xi\widehat{a}_\mu(\xi\eta,\varepsilon) \\ -\eta\widehat{a}_\mu(\xi\eta,\varepsilon) \\ 0 \end{pmatrix}.$$

Then, the following assertions hold,

(i) *Concerning $\widehat{N}_\mu$ and $\widehat{B}$ we have that*

$$\left\|\widehat{N}_\mu\right\|_{1,\sigma}\,,\ \left\|\widehat{B}\right\|_{1,\sigma} \leq \left\|\widehat{H}\right\|_{1,\sigma,\rho}.$$

In particular,

$$\|\widehat{a}_\mu(\xi\eta,\varepsilon)\|_{1,\sigma} \leq \frac{1}{\sigma}\left\|\widehat{H}\right\|_{1,\sigma,\rho}.$$

(*ii*) *The bound*

$$\left\|\mathcal{R}\widehat{\Psi}\right\|_{1,\sigma,\rho} \leq \frac{5}{\lambda_\circ}\left\|\widehat{H}\right\|_{1,\sigma,\rho}$$

holds, provided that the estimates

$$\frac{1}{\sigma}\left\|\widehat{H}\right\|_{1,\sigma,\rho}\ ,\ \left\|\widehat{a}_\circ(\xi\eta)\right\|_{1,\sigma}\ \leq \frac{\lambda_\circ}{16} \tag{3.75}$$

are satisfied.

Proof.

(*i*) From the formal solution of equation $\widehat{N}_\mu + \widehat{B} = \mathcal{P}\widehat{H}$, derived in formula (3.50) it is clear that $\left\|\widehat{N}_\mu\right\|_{1,\sigma}$ and $\left\|\widehat{B}\right\|_{1,\sigma}$ are both bounded by $\left\|\mathcal{P}\widehat{H}\right\|_{1,\sigma,\rho}$ and, consequently, by $\left\|\widehat{H}\right\|_{1,\sigma,\rho}$. Concerning the second part, writing

$$\mathcal{P}\widehat{H}(\xi,\eta,\varepsilon) = \begin{pmatrix} \xi\widehat{h}_1(\xi\eta,\varepsilon) \\ -\eta\widehat{h}_2(\xi\eta,\varepsilon) \\ 0 \end{pmatrix},$$

using again formula (3.50) for $\widehat{a}_\mu(\xi\eta,\varepsilon)$,

$$\widehat{a}_\mu(\xi\eta,\varepsilon) = \frac{1}{2}\left(\widehat{h}_1(\xi\eta,\varepsilon) - \widehat{h}_2(\xi\eta,\varepsilon)\right),$$

and definition (3.68) it turns out that

$$\begin{aligned}&\left\|\widehat{a}_\mu(\xi\eta,\varepsilon)\right\|_{1,\sigma} \leq \\ &\frac{1}{2\sigma}\left(\left\|\sigma\widehat{h}_1(\xi\eta,\varepsilon)\right\|_{1,\sigma} + \left\|\sigma\widehat{h}_2(\xi\eta,\varepsilon)\right\|_{1,\sigma}\right) \leq \\ &\frac{1}{\sigma}\left\|\mathcal{P}\widehat{H}\right\|_{1,\sigma} \leq \frac{1}{\sigma}\left\|\widehat{H}\right\|_{1,\sigma,\rho}.\end{aligned}$$

(*ii*) Let us recall that, in Section §2.5.1 we dealt with vector fields

$$\mathcal{R}\widehat{\Psi}(\chi,\varepsilon) = \begin{pmatrix} \widehat{\psi}_1(\xi,\eta,\theta,\varepsilon) \\ \widehat{\psi}_2(\xi,\eta,\theta,\varepsilon) \\ 0 \end{pmatrix}, \qquad \mathcal{R}\widehat{H}(\chi,\varepsilon) = \begin{pmatrix} \widehat{h}_1(\xi,\eta,\theta,\varepsilon) \\ \widehat{h}_2(\xi,\eta,\theta,\varepsilon) \\ 0 \end{pmatrix}$$

and

$$N^{\ltimes}(\zeta,\varepsilon) = \begin{pmatrix} \xi a(\xi\eta,\varepsilon) \\ -\eta a(\xi\eta,\varepsilon) \\ \omega(\varepsilon) \end{pmatrix},$$

where, from (3.74), we know that

$$a(\xi\eta,\varepsilon) = \lambda(\varepsilon) + \widehat{a}(\xi\eta,\varepsilon) = (\lambda_\circ + \lambda_\mu(\varepsilon)) + \widehat{a}_\circ(\xi\eta,\varepsilon) + \widehat{a}_\mu(\xi\eta,\varepsilon). \tag{3.76}$$

Moreover, we assumed for their components the following (rearranged) series expansions

$$\begin{aligned}\widehat{h}_w(\xi,\eta,\theta,\varepsilon) &= \sum_{\ell,s\in\mathbb{Z}} h^{(w)}_{\ell s}(\xi\eta,\varepsilon)\,\xi^\ell \mathrm{e}^{\mathrm{i}s\theta}\\ \widehat{\psi}_w(\xi,\eta,\theta,\varepsilon) &= \sum_{\ell,s\in\mathbb{Z}} \psi^{(w)}_{\ell s}(\xi\eta,\varepsilon)\,\xi^\ell \mathrm{e}^{\mathrm{i}s\theta}\end{aligned}$$

for $w=1,2$, where

$$\begin{aligned}h^{(w)}_{\ell s}(\xi\eta,\varepsilon) &= \sum_{k\geq\max\{0,1-\ell/2,-\ell\}} h^{(w)}_{\ell+k,ks}(\varepsilon)\,(\xi\eta)^k\,,\\ \psi^{(w)}_{\ell s}(\xi\eta,\varepsilon) &= \sum_{k\geq\max\{0,1-\ell/2,-\ell\}} \psi^{(w)}_{\ell+k,ks}(\varepsilon)\,(\xi\eta)^k\,,\end{aligned}$$

respectively. We recall also, that the formal solution of equation $\mathcal{L}_{N^\ltimes}(\mathcal{R}\widehat{\Psi})=\mathcal{R}\widehat{H}$ was given by the formula (3.49)

$$\psi^{(w)}_{\ell s}(\xi\eta,\varepsilon)=\frac{h^{(w)}_{\ell s}(\xi\eta,\varepsilon)}{\Gamma^{(w)}_{\ell s}(\lambda,\omega,\varepsilon)+\ell\,\widehat{a}(\xi\eta,\varepsilon)},$$

with $\Gamma^{(w)}_{\ell s}$ defined in (3.47). To estimate the norm $\left\|\mathcal{R}\widehat{\Psi}\right\|_{1,\sigma,\rho}$ we must deal first with their components. Therefore, one can write

$$\begin{aligned}&\left\|\widehat{\psi}_w(\xi,\eta,\theta,\varepsilon)\right\|_{1,\sigma,\rho}\leq\sum_{\ell,s\in\mathbb{Z}}\left\|\psi^{(w)}_{\ell s}(\xi\eta,\varepsilon)\,\xi^\ell\right\|_{1,\sigma}\mathrm{e}^{|s|\rho}\leq\\ &\sum_{\ell,s\in\mathbb{Z}}\left\|h^{(w)}_{\ell s}(\xi\eta,\varepsilon)\,\xi^\ell\right\|_{1,\sigma}\left\|\frac{1}{\Gamma^{(w)}_{\ell s}(\lambda,\omega,\varepsilon)+\ell\,\widehat{a}(\xi\eta,\varepsilon)}\right\|_{1,\sigma}\mathrm{e}^{|s|\rho}=\\ &\sum_{\ell\in\mathbb{Z}}\left\|h^{(w)}_{\ell 0}(\xi\eta,\varepsilon)\,\xi^\ell\right\|_{1,\sigma}\left\|\frac{1}{\Gamma^{(w)}_{\ell 0}(\lambda,\omega,\varepsilon)+\ell\,\widehat{a}(\xi\eta,\varepsilon)}\right\|_{1,\sigma}+\\ &\sum_{\substack{\ell\in\mathbb{Z}\\ s\in\mathbb{Z}\setminus\{0\}}}\left\|h^{(w)}_{\ell s}(\xi\eta,\varepsilon)\,\xi^\ell\right\|_{1,\sigma}\left\|\frac{1}{\Gamma^{(w)}_{\ell s}(\lambda,\omega,\varepsilon)+\ell\,\widehat{a}(\xi\eta,\varepsilon)}\right\|_{1,\sigma}\mathrm{e}^{|s|\rho},\end{aligned}\tag{3.77}$$

where the initial series has been divided in two parts: a first one corresponding to those terms having $s=0$ and a second one with the rest. We proceed to bound each sum separately.

(*a*) With respect to the sum for $s=0$,

$$\sum_{\ell\in\mathbb{Z}}\left\|h^{(w)}_{\ell 0}(\xi\eta,\varepsilon)\,\xi^\ell\right\|_{1,\sigma}\left\|\frac{1}{\Gamma^{(w)}_{\ell 0}(\lambda,\omega,\varepsilon)+\ell\,\widehat{a}(\xi\eta,\varepsilon)}\right\|_{1,\sigma},$$

we have from (*i*) and hypothesis (3.75) that

$$\begin{aligned}\|\widehat{a}_\circ(\xi\eta)\|_{1,\sigma} &\leq \frac{\lambda_\circ}{16}\\ \|\widehat{a}_\mu(\xi\eta,\varepsilon)\|_{1,\sigma} &\leq \frac{1}{\sigma}\left\|\widehat{H}\right\|_{1,\sigma,\rho}\leq\frac{\lambda_\circ}{16}.\end{aligned}$$

Therefore, we can apply Lemma 3.6 to obtain

$$\left|\Gamma_{\ell 0}^{(w)}(\lambda,\omega,\varepsilon)+\ell\,\widehat{a}(\xi\eta,\varepsilon)\right|\geq\frac{\lambda_\circ}{4}\qquad\forall(\xi,\eta)\in\overline{\mathcal{D}_\sigma}$$

and use Lemma 3.5(ii) to get that

$$\left\|\frac{1}{\Gamma_{\ell 0}^{(w)}(\lambda,\omega,\varepsilon)+\ell\,\widehat{a}(\xi\eta,\varepsilon)}\right\|_{1,\sigma}\leq\frac{4}{\lambda_\circ}.$$

Thus, it follows that

$$\sum_{\ell\in\mathbb{Z}}\left\|h_{\ell 0}^{(w)}(\xi\eta,\varepsilon)\,\xi^\ell\right\|_{1,\sigma}\left\|\frac{1}{\Gamma_{\ell 0}^{(w)}(\lambda,\omega,\varepsilon)+\ell\,\widehat{a}(\xi\eta,\varepsilon)}\right\|_{1,\sigma}\leq$$
$$\frac{4}{\lambda_\circ}\sum_{\ell\in\mathbb{Z}}\left\|h_{\ell 0}^{(w)}(\xi\eta,\varepsilon)\,\xi^\ell\right\|_{1,\sigma}\leq\frac{4}{\lambda_\circ}\left\|\widehat{H}\right\|_{1,\sigma,\rho}.$$

(b) Following the same idea as before, using

$$\|\widehat{a}(\xi\eta,\varepsilon)\|_{1,\sigma}\leq\|\widehat{a}_\circ(\xi\eta)\|_{1,\sigma}+\|\widehat{a}_\mu(\xi\eta,\varepsilon)\|_{1,\sigma}\leq\frac{\lambda_\circ}{16}+\frac{\lambda_\circ}{16}=\lambda_\circ/8,$$

Lemma 3.6 for $s\neq 0$, Lemma 3.5(ii) and having in mind that $\omega(\varepsilon)=1/\varepsilon$, the following expression is derived,

$$\sum_{\substack{\ell\in\mathbb{Z}\\ s\in\mathbb{Z}\setminus\{0\}}}\left\|h_{\ell s}^{(w)}(\xi\eta,\varepsilon)\,\xi^\ell\right\|_{1,\sigma}\left\|\frac{1}{\Gamma_{\ell s}^{(w)}(\lambda,\omega,\varepsilon)+\ell\,\widehat{a}(\xi\eta,\varepsilon)}\right\|_{1,\sigma}\mathrm{e}^{|s|\rho}\leq$$
$$\varepsilon\sum_{\substack{\ell\in\mathbb{Z}\\ s\in\mathbb{Z}\setminus\{0\}}}\left\|h_{\ell s}^{(w)}(\xi\eta,\varepsilon)\,\xi^\ell\right\|_{1,\sigma}\mathrm{e}^{|s|\rho}\leq\varepsilon\left\|\widehat{H}\right\|_{1,\sigma,\rho}.$$

Joining together the estimates obtained in (a) and (b) and assuming ε small, equation (3.77) becomes

$$\left\|\widehat{\psi}_w(\xi,\eta,\theta,\varepsilon)\right\|_{1,\sigma,\rho}\leq\left(\frac{4}{\lambda_\circ}+\varepsilon\right)\left\|\widehat{H}\right\|_{1,\sigma,\rho}\leq\frac{5}{\lambda_\circ}\left\|\widehat{H}\right\|_{1,\sigma,\rho}.$$

Consequently, from definition (3.71), we get

$$\left\|\mathcal{R}\widehat{\Psi}\right\|_{1,\sigma,\rho}\leq\frac{5}{\lambda_\circ}\left\|\widehat{H}\right\|_{1,\sigma,\rho}.$$

□

§2.8 Proof of the convergence

In this section we are going to prove the convergence of the ΨNF-recurrent scheme introduced in formulas (3.63)– (3.66). Precisely, assume that the transformation $z = \Phi_\circ(\zeta) = \zeta + \mathcal{R}\widehat{\Phi}_\circ(\zeta)$ leads system (3.1) into BNF, that is, verifies

$$D\Phi_\circ N_\circ = F_\circ(\Phi_\circ)\,.$$

Take initial values

$$\widehat{\Phi}_\mu^{(1)} = 0, \qquad N_\mu{}^{(1)} = \Lambda_\mu(\varepsilon)\zeta, \qquad \widehat{B}^{(1)} = 0, \tag{3.78}$$

fix $\mathcal{P}\widehat{\Phi}_\mu{}^{(K)} = 0$, for any $K \geq 1$, and write $N^\ltimes = N_\circ^\ltimes + N_\mu$. Then, we obtain recurrently

$$\begin{aligned} \widehat{\Phi}_\mu^{(K+1)} &= \mathcal{R}\widehat{\Phi}_\mu{}^{(K+1)} \\ N_\mu{}^{(K+1)} &= \Lambda_\mu(\varepsilon)\zeta + \widehat{N}_\mu^{(K+1)} \\ \widehat{B}^{(K+1)}, & \end{aligned} \tag{3.79}$$

satisfying that $\widehat{\Phi}_\mu^{(K+1)}$, $N_\mu{}^{(K+1)}$ and $\widehat{B}^{(K+1)}$ are all $\mathcal{O}_{\leq K+1}$, from equations

$$\begin{aligned} \left\{\mathcal{L}_{N_\mu{}^{(K)}}\left(\mathcal{R}\widehat{\Phi}_\circ\right)\right\}_{\leq K+1} &+ \left\{\mathcal{L}_{N_\circ^\ltimes + N_\mu{}^{(K)}}\left(\mathcal{R}\widehat{\Phi}_\mu{}^{(K+1)}\right)\right\}_{\leq K+1} = \\ & \left\{\mathcal{R}\left(\widehat{G_F}\left(\widehat{\Phi}_\mu^{(K)}\right)\right)\right\}_{\leq K+1} \end{aligned} \tag{3.80}$$

$$\widehat{N}_\mu^{(K+1)} + \widehat{B}^{(K+1)} = \left\{\mathcal{P}\left(\widehat{G_F}\left(\widehat{\Phi}_\mu^{(K)}\right)\right)\right\}_{\leq K+1}. \tag{3.81}$$

Using exactly the same argument as in the *saddle-center and saddle-focus case* (see Chapter 1, Section §2.5.2), the following result can be proved.

Proposition 3.2 *Let us assume that system* (3.4) *is analytic on* $\overline{\mathcal{D}_{r_\circ}} \times \mathcal{T}_\rho$ *and satisfies the properties introduced in Section* §1. *Let us consider the sequences*

$$\begin{aligned} &\left\{\left\|\Phi_\circ^\ltimes + \mathcal{R}\widehat{\Phi}_\mu{}^{(K+1)}\right\|_{1,r_\circ/2,\rho}\right\}_K \\ &\left\{\left\|N_\circ^\ltimes + N_\mu{}^{(K+1)}\right\|_{1,r_\circ/2}\right\}_K \\ &\left\{\left\|\widehat{B}^{(K+1)}\right\|_{1,r_\circ/2}\right\}_K. \end{aligned} \tag{3.82}$$

Then, the following assertions hold:

(*i*) *The sequences defined in* (3.82) *increase monotonically and are upper-bounded, that is, satisfy that*

$$\begin{aligned} \left\|\Phi_\circ^\ltimes + \mathcal{R}\widehat{\Phi}_\mu{}^{(K+1)}\right\|_{1,r_\circ/2,\rho} &\geq \left\|\Phi_\circ^\ltimes + \mathcal{R}\widehat{\Phi}_\mu{}^{(K)}\right\|_{1,r_\circ/2,\rho} \\ \left\|N_\circ^\ltimes + N_\mu{}^{(K+1)}\right\|_{1,r_\circ/2} &\geq \left\|N_\circ^\ltimes + N_\mu{}^{(K)}\right\|_{1,r_\circ/2} \\ \left\|\widehat{B}^{(K+1)}\right\|_{1,r_\circ/2} &\geq \left\|\widehat{B}^{(K)}\right\|_{1,r_\circ/2}, \end{aligned}$$

and

$$(3.83)\qquad \begin{array}{rcl} \left\| \Phi_\circ^\ltimes + \mathcal{R}\widehat{\Phi}_\mu^{(K+1)} \right\|_{1,r_\circ/2,\rho} & \leq & r_\circ, \\ \left\| N_\circ^\ltimes + N_\mu^{(K+1)} \right\|_{1,r_\circ/2}, \ \left\| \widehat{B}^{(K+1)} \right\|_{1,r_\circ/2} & \leq & \mathbf{C} r_\circ. \end{array}$$

(*ii*) *Using* (*i*) *and taking into account Lemma* 3.3 *it follows that the vector fields* $\Phi^\ltimes$, $N^\ltimes$ *and* $\widehat{B}$, *defined as*

$$\Phi^\ltimes := \lim_{K\mapsto\infty} \left(\Phi_\circ^\ltimes + \mathcal{R}\widehat{\Phi}_\mu^{(K)} \right)$$

and

$$N^\ltimes := \lim_{K\mapsto\infty} \left(N_\circ^\ltimes + N_\mu^{(K)} \right), \qquad \widehat{B} := \lim_{K\mapsto\infty} \widehat{B}^{(K+1)},$$

are analytic on $\overline{\mathcal{D}_{r_\circ/2}} \times \mathcal{T}_\rho$, $\overline{\mathcal{D}_{r_\circ/2}}$ *and again* $\overline{\mathcal{D}_{r_\circ/2}}$, *respectively. Moreover they lead system* (3.4) *into* ΨNF, *that is, satisfying the equality*

$$D\Phi^\ltimes N^\ltimes + \widehat{B} = F\left(\Phi^\ltimes\right)$$

Proof. Because of the similarity with the proof given in Chapter 1, Section §2.5.2 and in order to bore as less as possible the reader, we will present here only the proof (with details) for the estimates (3.83). The rest of the results can be obtained following exactly the same arguments employed in Section §2.5.2. With respect to the proof of (3.83) we will distinguish two parts: the first one will provide necessary estimates on the unperturbed system and the perturbation; in the second one the proof of estimates (3.83) following an inductive argument will be carried out. Indeed,

(i) **Estimates of the system**

Lemma 3.7 *Let us consider the unperturbed system* (3.4),

$$\dot{z} = F_\mu(z) = \Lambda_\circ z + \widehat{F}_\circ(z),$$

with $\Lambda_\circ$ *a diagonal matrix* $\{\pm\lambda_\circ\}$, $\lambda_\circ > 0$. *Since* $\widehat{F}_\circ$ *starts with terms of order at least* 2 *in* z, *like is was done in Chapter* 1, *Section* §2.5.2, *we can scale our system in such a way that in the new domain of definition,* $\overline{\mathcal{D}_{r_\circ}}$ *the following bound holds,*

$$(3.84)\qquad \left\| \widehat{F}_\circ \right\|_{1,r_\circ} \leq \frac{c_\circ\,\lambda_\circ}{16}\, r_\circ,$$

for a constant $0 < c_\circ \leq 1/2$. *Moreover, we know the existence of analytic transformation* $z = \Phi_\circ(\zeta) = \zeta + \widehat{\Phi}_\circ(\zeta)$ *and vector field* $N_\circ(\zeta) = N_\circ(\xi,\eta) = (\xi a_\circ(\xi\eta), -\eta a_\circ(\xi\eta))$, *where* $a_\circ(\xi\eta) = \lambda_\circ + \widehat{a}_\circ(\xi\eta)$, *leading system* (3.4) *into* BNF, *that is, verifying that* $D\Phi_\circ N_\circ = F_\circ(\Phi_\circ)$. *Then, the following properties are satisfied,*

(*a*) $\Phi_\circ$ *is analytic on* $\overline{\mathcal{D}_{\frac{3}{4}r_\circ}}$, *verifies that* $\mathcal{P}\widehat{\Phi}_\circ = 0$ *and that*

$$\|\Phi_\circ\|_{1,\frac{3}{4}r_\circ} \leq \frac{13}{16} r_\circ, \qquad \left\| D\left(\mathcal{R}\widehat{\Phi}_\circ\right)\right\|_{1,\frac{r_\circ}{2}} \leq \frac{1}{4}.$$

(b) $N_\circ$ *is analytic on* $\overline{\mathcal{D}_{\frac{3}{4}r_\circ}}$ *and we have also that*

$$\|\widehat{a}_\circ(\xi\eta)\|_{1,\frac{3}{4}r_\circ} \leq \frac{c_\circ\,\lambda_\circ}{12} \leq \frac{\lambda_\circ}{24}.$$

Proof. (Lemma) These results come from the convergence proof of Chapter 1, Section §2.5.2. In that section it was proved that, if $\overline{\mathcal{D}_{r_\circ}}$ was the domain of analyticity of the initial system, then the corresponding iterates (leading to $\Phi_\circ$) satisfied that

$$\left\|\mathcal{R}\widehat{\Phi}_\circ^{(K+1)}\right\|_{1,\gamma r_\circ} \leq \left(\frac{1-\gamma}{4}\right) r_\circ$$

for any $K \geq 1$ and $1/2 \leq \gamma < 1$. Choosing $\gamma = 3/4$ it follows that

$$\left\|\mathcal{R}\widehat{\Phi}_\circ^{(K+1)}\right\|_{1,\frac{3}{4}r_\circ} \leq \frac{r_\circ}{16},$$

for any $K \geq 1$. In particular, it also holds for the limit $\mathcal{R}\Phi_\circ = \lim_K \mathcal{R}\widehat{\Phi}_\circ^{(K+1)}$, so

$$\left\|\mathcal{R}\widehat{\Phi}_\circ\right\|_{1,\frac{3}{4}r_\circ} \leq \frac{r_\circ}{16}$$

and, consequently,

$$\|\Phi_\circ\|_{1,\frac{3}{4}r_\circ} \leq \|\mathrm{id}\|_{1,\frac{3}{4}r_\circ} + \left\|\mathcal{R}\widehat{\Phi}_\circ\right\|_{1,\frac{3}{4}r_\circ} \leq \frac{3}{4}r_\circ + \frac{r_\circ}{16} = \frac{13}{16}r_\circ.$$

Applying now Cauchy estimates one obtains that

$$\left\|D\left(\mathcal{R}\widehat{\Phi}_\circ\right)\right\|_{1,\frac{r_\circ}{2}} \leq \frac{1}{r_\circ/4}\left\|\mathcal{R}\widehat{\Phi}_\circ\right\|_{1,\frac{3}{4}r_\circ} \leq \frac{4}{r_\circ}\cdot\frac{r_\circ}{16} = \frac{1}{4}.$$

Concerning $N_\circ$, we have from the same section that

$$\widehat{N}_\circ = \mathcal{P}\left(F_\circ\left(\Phi_\circ\right)\right),$$

and, therefore, using Lemma 3.5 and estimate (3.84),

$$\left\|\widehat{N}_\circ\right\|_{1,\frac{3}{4}r_\circ} \leq \|\mathcal{P}\left(F_\circ\left(\Phi_\circ\right)\right)\|_{1,\frac{3}{4}r_\circ} \leq \left\|\widehat{F}_\circ\right\|_{1,r_\circ} \leq \frac{c_\circ\,\lambda_\circ}{16}\,r_\circ.$$

In particular,

$$\|\widehat{a}_\circ(\xi\eta)\|_{1,\frac{3}{4}r_\circ} = \frac{1}{\frac{3}{4}r_\circ}\left\|\left(\frac{3}{4}r_\circ\right)\widehat{a}_\circ(\xi\eta)\right\|_{1,\frac{3}{4}r_\circ} =$$

$$\frac{1}{\frac{3}{4}r_\circ}\|\xi\widehat{a}_\circ(\xi\eta)\|_{1,\frac{3}{4}r_\circ} \leq \frac{4}{3r_\circ}\left\|\widehat{N}_\circ\right\|_{1,\frac{3}{4}r_\circ} \leq \frac{4}{3r_\circ}\cdot\frac{c_\circ\,\lambda_\circ}{16}\,r_\circ = \frac{c_\circ\,\lambda_\circ}{12} \leq \frac{\lambda_\circ}{24}$$

if one takes into account that $0 < c_\circ \leq 1/2$. ♣

With respect to the perturbation we have the following result, whose proof can be derived 0.

Lemma 3.8 *Since the perturbation vector field*

$$F_\mu(z,\theta,\varepsilon) = \Lambda_\mu(\varepsilon)z + \widehat{F}_\mu(z,\theta,\varepsilon),$$

where

$$\Lambda_\mu = \begin{pmatrix} \lambda_\mu(\varepsilon) & 0 \\ 0 & -\lambda_\mu(\varepsilon) \end{pmatrix},$$

and $\lambda_\mu(\varepsilon) \in \mathbb{R}$, *is assumed to be of the type*

$$F_\mu(z,\theta,\varepsilon) = \mu\varepsilon^q F_1(z,\theta,\varepsilon),$$

then it follows that

$$\|F_\mu\|_{1,r_\circ,\rho} \leq c_\mu \mu\varepsilon^q \lambda_\circ, \qquad |\lambda_\mu(\varepsilon)| \leq c_\lambda \mu\varepsilon^q \lambda_\circ,$$

for suitable positive constants c_μ *and* c_λ.

(ii) **Estimates for the recurrent scheme**

To prove estimates (3.83) it is enough to check that

$$\text{(3.85)} \qquad \left\|\mathcal{R}\widehat{\Phi}_\mu{}^{(K)}\right\|_{1,\frac{r_\circ}{2},\rho}, \ \left\|\widehat{N}_\mu^{(K)}\right\|_{1,\frac{r_\circ}{2}} \ \text{and} \ \left\|\widehat{B}^{(K)}\right\|_{1,\frac{r_\circ}{2}} \quad \text{are } O(\mu\varepsilon^q), \quad K \geq 1$$

for small enough values of μ and ε and $\mathcal{R}\widehat{\Phi}_\mu{}^{(K)}$, $\widehat{N}_\mu^{(K)}$ and $\widehat{B}^{(K)}$ being the vector fields provided by the recurrent scheme (3.78)–(3.81). Indeed, if inequalities (3.85) are satisfied, having in mind (i), we can choose μ and ε small enough such that

$$\left\|\mathcal{R}\widehat{\Phi}_\mu\right\|_{1,\frac{r_\circ}{2},\rho} \leq \frac{7}{16} r_\circ$$

and, therefore,

$$\|\Phi\|_{1,\frac{r_\circ}{2},\rho} \leq \|\Phi_\circ\|_{1,\frac{r_\circ}{2}} + \left\|\mathcal{R}\widehat{\Phi}_\mu\right\|_{1,\frac{r_\circ}{2},\rho} \leq \frac{9}{16} r_\circ + \frac{7}{16} r_\circ = r_\circ.$$

For $\|N\|_{1,\frac{r_\circ}{2}}$ a similar result can be obtained.

Notice that all the estimates presented here are referred to the spatial components and there is not any comment concerning the angular variable. This abuse of notation is simply due to the fact that with respect to this angular component we just deal with the identity.

Thus, let us prove estimates (3.85) inductively. From (3.78) it is clear that they hold for the case $K = 1$. So, let us assume by induction hypothesis, that $\left\|\mathcal{R}\widehat{\Phi}_\mu{}^{(K)}\right\|_{1,\frac{r_\circ}{2},\rho}$, $\left\|\widehat{N}_\mu^{(K)}\right\|_{1,\frac{r_\circ}{2}}$ and $\left\|\widehat{B}^{(K)}\right\|_{1,\frac{r_\circ}{2}}$ are order $O(\mu\varepsilon^q)$.

Lemma 3.9 *Assuming the induction hypothesis* (3.85) *holds, the following estimates are satisfied:*

(*a*)

$$\left\| \widehat{G_F}\left(\mathcal{R}\widehat{\Phi}_\mu{}^{(K)}\right)\right\|_{1,\frac{r_\circ}{2},\rho} = O(\mu\varepsilon^q).$$

(*b*) *We have that*

$$\left\|\widehat{N}_\mu^{(K+1)}\right\|_{1,\frac{r_\circ}{2}} \text{ and } \left\|\widehat{B}^{(K+1)}\right\|_{1,\frac{r_\circ}{2}} \text{ are } O(\mu\varepsilon^q)$$

In particular, $\left\|\widehat{a}_\mu^{(K+1)}\right\|_{1,\frac{r_\circ}{2}}$ *is also* $O(\mu\varepsilon^q)$.

(*c*) $\left\|\mathcal{L}_{N_\mu{}^{(K)}}\left(\mathcal{R}\widehat{\Phi}_\circ\right)\right\|_{1,\frac{r_\circ}{2},\rho}$ *is* $O(\mu\varepsilon^q)$ *and, consequently*

$$\left\|\mathcal{R}\left(\widehat{G_F}\left(\mathcal{R}\widehat{\Phi}_\mu{}^{(K)}\right)\right) - \mathcal{L}_{N_\mu{}^{(K)}}\left(\mathcal{R}\widehat{\Phi}_\circ\right)\right\|_{1,\frac{r_\circ}{2},\rho} = O(\mu\varepsilon^q).$$

(*d*) *Finally, for* μ *and* $\varepsilon > 0$ *small enough, it follows that*

$$\left\|\mathcal{R}\widehat{\Phi}_\mu{}^{(K+1)}\right\|_{1,\frac{r_\circ}{2},\rho} = O(\mu\varepsilon^q),$$

which concludes the of (3.85).

Proof. (Lemma)

(*a*) From the definition of $\widehat{G_F}$ in (3.17) and using Lemma 3.5 it follows that

$$\begin{aligned}
\left\|\widehat{G_F}\left(\mathcal{R}\widehat{\Phi}_\mu{}^{(K)}\right)\right\|_{1,\frac{r_\circ}{2},\rho} &= \left\|\widehat{F}_\mu\left(\Phi_\circ^\ltimes\right) - \left(\widehat{F}\left(\Phi^\ltimes\right) - \widehat{F}\left(\Phi_\circ^\ltimes\right)\right)\right\|_{1,\frac{r_\circ}{2},\rho} \le \\
&\left\|\widehat{F}_\mu\left(\Phi_\circ^\ltimes\right)\right\|_{1,\frac{r_\circ}{2},\rho} + \left\|\widehat{F}\left(\Phi^\ltimes\right) - \widehat{F}\left(\Phi_\circ^\ltimes\right)\right\|_{1,\frac{r_\circ}{2},\rho} \le \\
&c_\mu\mu\varepsilon^q + \left\|\widehat{F}\left(\Phi^\ltimes\right) - \widehat{F}\left(\Phi_\circ^\ltimes\right)\right\|_{1,\frac{r_\circ}{2},\rho}.
\end{aligned} \tag{3.86}$$

Applying Taylor formula we have

$$\left\|\widehat{F}\left(\Phi^\ltimes\right) - \widehat{F}\left(\Phi_\circ^\ltimes\right)\right\|_{1,\frac{r_\circ}{2},\rho} \le \left\|D\widehat{F}\left(\Phi_\circ^\ltimes + \beta\widehat{\Phi}_\mu\right)\right\|_{1,\frac{r_\circ}{2},\rho}\left\|\widehat{\Phi}_\mu\right\|_{1,\frac{r_\circ}{2},\rho},$$

where $0 < \beta < 1$. Since

$$\left\|\Phi_\circ^\ltimes + \beta\widehat{\Phi}_\mu\right\|_{1,\frac{r_\circ}{2},\rho} \le \|\Phi_\circ\|_{1,\frac{3}{4}r_\circ} + \left\|\widehat{\Phi}_\mu\right\|_{1,\frac{r_\circ}{2},\rho} \le \frac{13}{16}r_\circ + c_\Phi\mu\varepsilon^q \le \frac{7}{8}r_\circ$$

for μ, ε small enough, it follows that

$$\begin{aligned}
&\left\|D\widehat{F}\left(\Phi_\circ^\ltimes + \beta\widehat{\Phi}_\mu\right)\right\|_{1,\frac{r_\circ}{2},\rho} \le \left\|D\widehat{F}\right\|_{1,\frac{7}{8}r_\circ,\rho} \le \frac{8}{r_\circ}\left\|\widehat{F}\right\|_{1,r_\circ,\rho} \le \\
&\quad \frac{8}{r_\circ}\left(\left\|\widehat{F}_\circ\right\|_{1,r_\circ} + \left\|\widehat{F}_\mu\right\|_{1,r_\circ,\rho}\right) \le \frac{8}{r_\circ}\left(\frac{c_\circ\lambda_\circ}{16}r_\circ + c_\mu\mu\varepsilon^q\right) \le \frac{8}{r_\circ}\left(2\frac{c_\circ\lambda_\circ}{16}r_\circ\right) = c_\circ\lambda_\circ,
\end{aligned}$$

where it has been used Cauchy estimates and Lemmas 3.7 and 3.8. Using that $\left\|\widehat{\Phi}_\mu\right\|_{1,\frac{r_\circ}{2},\rho} \leq c\,\mu\varepsilon^q$ it turns out that

$$\left\|D\widehat{F}\left(\Phi_\circ^\ltimes + \beta\widehat{\Phi}_\mu\right)\right\|_{1,\frac{r_\circ}{2},\rho}\left\|\widehat{\Phi}_\mu\right\|_{1,\frac{r_\circ}{2},\rho} = O(\mu\varepsilon^q)$$

and, finally, $\left\|\widehat{G_F}\left(\mathcal{R}\widehat{\Phi}_\mu{}^{(K)}\right)\right\|_{1,\frac{r_\circ}{2},\rho} = O(\mu\varepsilon^q)$.

(*b*) It follows straightforwardly from the fact that

$$\left\|\widehat{N}_\mu^{(K+1)}\right\|_{1,\frac{r_\circ}{2}}, \left\|\widehat{B}^{(K+1)}\right\|_{1,\frac{r_\circ}{2}} \leq \left\|\mathcal{P}\left(\widehat{G_F}\left(\mathcal{R}\widehat{\Phi}_\mu{}^{(K)}\right)\right)\right\|_{1,\frac{r_\circ}{2},\rho}.$$

(*c*) We just need to prove that

$$\left\|\mathcal{L}_{N_\mu{}^{(K)}}\left(\mathcal{R}\widehat{\Phi}_\circ\right)\right\|_{1,\frac{r_\circ}{2},\rho} = O(\mu\varepsilon^q)$$

since the claimed result is then obtained taking into account bound (*a*). Thus, using the definition of $\mathcal{L}$, it follows

$$\left\|\mathcal{L}_{N_\mu{}^{(K)}}\left(\mathcal{R}\widehat{\Phi}_\circ\right)\right\|_{1,\frac{r_\circ}{2},\rho} \leq \left\|D\left(\mathcal{R}\widehat{\Phi}_\circ\right)\right\|_{1,\frac{r_\circ}{2}}\left\|N_\mu{}^{(K)}\right\|_{1,\frac{r_\circ}{2}} + \|\Lambda_\mu(\varepsilon)\|_{1,\frac{r_\circ}{2}}\left\|\mathcal{R}\widehat{\Phi}_\circ\right\|_{1,\frac{r_\circ}{2}}.$$

Then, having in mind that $\left\|N_\mu{}^{(K)}\right\|_{1,\frac{r_\circ}{2}}$ and $\|\Lambda_\mu(\varepsilon)\|_{1,\frac{r_\circ}{2}}$ are both $O(\mu\varepsilon^q)$ and using Lemma 3.7(*a*), it turns out that $\left\|\mathcal{L}_{N_\mu{}^{(K)}}\left(\mathcal{R}\widehat{\Phi}_\circ\right)\right\|_{1,\frac{r_\circ}{2},\rho}$ is $O(\mu\varepsilon^q)$.

(*d*) Writing formula (3.80) in the form

$$\left\{\mathcal{L}_{N_\circ^\ltimes + N_\mu{}^{(K)}}\left(\mathcal{R}\widehat{\Phi}_\mu{}^{(K+1)}\right)\right\}_{\leq K+1} = \left\{\mathcal{R}\left(\widehat{G_F}\left(\widehat{\Phi}_\mu^{(K)}\right)\right) - \mathcal{L}_{N_\mu{}^{(K)}}\left(\mathcal{R}\widehat{\Phi}_\circ\right)\right\}_{\leq K+1},$$

we can consider this equation as one of the type

$$\mathcal{L}_{N_\circ^\ltimes + N_\mu{}^{(K)}}\left(\mathcal{R}\widehat{\Phi}_\mu{}^{(K+1)}\right) = \mathcal{R}\widehat{H}$$

provided we take

$$\mathcal{R}\widehat{H} = \left\{\mathcal{R}\left(\widehat{G_F}\left(\widehat{\Phi}_\mu^{(K)}\right)\right) - \mathcal{L}_{N_\mu{}^{(K)}}\left(\mathcal{R}\widehat{\Phi}_\circ\right)\right\}_{\leq K+1}.$$

Moreover, from Lemma 3.7(*b*) one has that

$$\|\widehat{a}_\circ(\xi\eta)\|_{1,\frac{r_\circ}{2}} \leq \|\widehat{a}_\circ(\xi\eta)\|_{1,\frac{3}{4}r_\circ} \leq \frac{\lambda_\circ}{24} < \frac{\lambda_\circ}{16}$$

and from estimate (*c*) it follows that

$$\left\|\mathcal{R}\widehat{H}\right\|_{1,\frac{r_\circ}{2},\rho} = O(\mu\varepsilon^q) \leq \frac{\lambda_\circ}{16}\left(\frac{r_\circ}{2}\right),$$

so we can apply Proposition 3.1 and obtain that $\left\|\mathcal{R}\widehat{\Phi}_\mu{}^{(K+1)}\right\|_{1,\frac{r_\circ}{2},\rho} = O(\mu\varepsilon^q)$.

♣

□

Chapter 4

Splitting of separatrices in 2-dimensional periodic reversible systems

§1 Introduction and main result

As it has been mentioned at the introduction of this thesis, the phenomenon of the splitting of separatrices (that is, the transversal intersection of invariant manifolds coming from a homoclinic connection) seems to be one of the main causes of the stochastic behavior in Hamiltonian systems. To measure the size of this splitting, Poincaré (1890) introduced a method based on a perturbative approach. Nevertheless the results he obtained were not completely rigorous, he was already aware that the size of this splitting predicted by his method was exponentially small in the perturbative parameter ε. This method, rediscovered 70 years later, is known as Melnikov or Poincaré-Melnikov method.

It was at the end of the 80's, that this problem received again the interest of the scientific community (see, for instance, at [24] and references therein), providing effective computations of this exponentially small splitting size in several situations and depending on the type and size of the perturbative forcing.

Among these papers, there is one [24], written by Delshams and Seara, where a quite general outline is given of the problem of measuring the size of splitting of separatrices for a Hamiltonian system with one and a half degrees of freedom. In that paper the authors validate Melnikov's method to measure this splitting size, which appears to be exponentially small in the perturbative parameter and given, in first order, by the Melnikov function.

The aim of this chapter is to extend the results given at [24] for reversible systems. Precisely, we will consider an integrable system, with the origin being a saddle equilibrium point and having an homoclinic connection (for some more details, see hypothesis (h_1)). In principle, we will consider this integrable system to be Hamiltonian but it could be also possible to take it reversible, provided their local equivalence around this kind of equilibrium point (see Chapter 1). Moreover, we will assume this system to be (time) linearly reversible, that is, invariant under the action of a linear spatial involution and a reversion in the time variable $t \longmapsto -t$. We start this study with this simpler case since it is the first natural step before consider a more general situation. Moreover, linear reversibilities are very common in dynamical systems and are, in fact, of easy detection. As usual, we will perturb this system with an small perturbation, with a rapidly periodic dependence on time and preserving the same reversibility as the initial system. Consequently, under the action of this perturbation, the invariant manifolds associated to the new equilibrium (a periodic orbit close to the origin) do not, as a rule, coincide and give rise to infinite number of intersections. The problem of the Splitting of Separatrices consists, essentially, in measuring the *distance* between both invariant manifolds. This can be estimate in 0 ways, namely, from measuring the distance, the angle at the first intersection or, in the Hamiltonian case, the are of the first lobe contained between them. In our reversible setting, the purpose of this chapter is to validate the exponentially expressions provided by the Melnikov function for the angle between the invariant curves, provided that the area of the lobes is not an invariant. We will follow closely the paper [24], not only in many of the techniques employed but even the notation.

It is worth noting that the reversibility plays an important rôle in this problem. Namely, it ensures the transversal intersection between the (symmetric) invariant curves, the fact that this first homoclinic point lies on the symmetry manifold (formed by the points which are fixed for the spatial involution) and provides suitable *reversible* parameterizations of the invariant curves. As it will be notice later, this *symmetry* will be also present in the Melnikov and the *splitting* functions.

Let us start in a more precise way. Thus, let us consider the Hamiltonian system

$$\left\{\begin{array}{lcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & f(x_1) \end{array}\right. \tag{4.1}$$

with Hamilton function given by

$$h^{(0)}(x) = h^{(0)}(x_1, x_2) = \frac{x_2^2}{2} + V(x_1)$$

and $f(x_1) = -V'(x_1)$. This system is (time)-reversible with respect to the linear spatial involution

$$\mathfrak{R} : (x_1, x_2) \longmapsto (x_1, -x_2).$$

Let us assume that system (4.1) satisfies the following hypotheses:

(h_1) The origin is a saddle equilibrium point with characteristic exponents $\pm\lambda_0$, with $\lambda_0 > 0$. Moreover, there exists a homoclinic solution $x^{(0)}(t) = ({x_1}^{(0)}(t), {x_2}^{(0)}(t))$ verifying that $x^{(0)}(t) \longrightarrow (0,0)$ for $t \longrightarrow \pm\infty$. This solution is commonly called a *separatrix*. Due to the reversibility of (4.1) it follows that the homoclinic solution $x^{(0)}(t)$ satisfies that $x^{(0)}(-t) = \mathfrak{R}x^{(0)}(t)$. Let us assume that it has been parameterized in such a way that for $t = 0$ it lies on the symmetry line, that is,

$$x^{(0)}(0) \in \text{Fix } \mathfrak{R} = \{x = (x_1, x_2) : \ \mathfrak{R}x = x\}.$$

(h_2) The function $f(x_1)$ is real entire and ${x_2}^{(0)}(t) = \dot{x}_1^{(0)}(t)$ is analytic on a strip $|\Im u| < a$, with a pole of order r at the points $u = \pm a\,\mathrm{i}$ as the unique singularity at the lines $|\Im u| = a$.

Let us now consider the following perturbation of system (4.1)

$$\left\{\begin{array}{lcl} \dot{x}_1 & = & x_2 + \mu\varepsilon^p g_1(x, t/\varepsilon) \\ \dot{x}_2 & = & f(x_1) + \mu\varepsilon^p g_2(x, t/\varepsilon) \end{array}\right. \tag{4.2}$$

or, in a more compact way,

$$\dot{x} = J\nabla h^{(0)}(x) + \mu\varepsilon^p G(x, t/\varepsilon),$$

with

$$\nabla h^{(0)}(x) = \begin{pmatrix} \partial_1 h^{(0)}(x) \\ \partial_2 h^{(0)}(x) \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad G(x, t/\varepsilon) = \begin{pmatrix} g_1(x, t/\varepsilon) \\ g_2(x, t/\varepsilon) \end{pmatrix},$$

where ∂_j denotes derivative with respect x_j. Concerning the perturbation, let us suppose that it satisfies the following hypothesis:

(h_3) The vector field $G(x, \theta) = (g_1(x, \theta), g_2(x, \theta))$ is analytic in x, θ and 2π-periodic in θ, and we assume it to be $\mathfrak{R}$-reversible, that is, satisfying

$$\mathfrak{R}\, G(\mathfrak{R}\, x, -t/\varepsilon) = -G(x, t/\varepsilon).$$

Moreover, as it was done in [24], we introduce the assumption

$$\int_0^{2\pi} G(x, \theta) d\theta = 0,$$

that is, G has zero mean. Like there, this is not essential but it allows, by means of two steps of averaging, to increase the order of the perturbation and, consequently, to get better final estimates. See [24] for more details.

We also assume that it takes the following forms with respect to the spatial variable x:

- If f is 2π-periodic, we will consider g_1, g_2 to be trigonometric polynomials in x_1 and polynomials in x_2. Moreover, the case $G(x, \theta) = (0, x_1 g(\theta))$ is also allowed.
- If f is not 2π-periodic, we will consider G only depending on θ, that is, of the form $(g_1(\theta), g_2(\theta))$ and verifying the symmetry conditions

$$g_1(-\theta) = -g_1(\theta), \qquad g_2(-\theta) = g_2(\theta),$$

induced by the $\mathfrak{R}$-reversibility.

Like it was done in [24], from this hypothesis (h_3) we have that $G = (g_1, g_2)$ can be expressed as sums of monomials in x. Since $x^{(0)}(u)$ has poles of order r at the complex points $u = \pm a\,\mathrm{i}$, so have $g_1(x^{(0)}(u))$ and $g_2(x^{(0)}(u))$. Let us take ℓ_1, ℓ_2 the greatest of the orders of these poles for $g_1(x^{(0)}(u))$ and $g_2(x^{(0)}(u))$, respectively, and define $\ell = \max\{\ell_1, \ell_2\}$. In some sense, ℓ can be considered as the maximal order of the perturbation on the separatrix. We will call ℓ the *order of the perturbation on the separatrix.* For a detailed list of remarks and comments concerning these hypotheses we refer the reader to [24].

Remark 13 *In* [24] *an analogous definition for this* order of the perturbation *is introduced. However, since that paper deals with Hamiltonian systems, the perturbation is given in terms of the Hamiltonian function*

$$h(x, t/\varepsilon) = h^0(x) + \mu\varepsilon^p h^1(x, t/\varepsilon).$$

Moreover, the homoclinic solution $x^0(u)$ *is assumed to have a pole of order* $r \geq 1$ *at the complex points* $u = \pm a\,\mathrm{i}$ *and, therefore,* ℓ_{DS} *is defined as the greatest order of the pole* $h^1(x^0(u), \theta)$ *has at* $u = \pm a\,\mathrm{i}$. *From an straightforward computation it follows that* $\partial_1 h^1(x^0(u), \theta)$ *and* $\partial_2 h^1(x^0(u), \theta)$ *have poles of order* $\ell_{DS} + 1 - r$ *and* $\ell_{DS} - r$*, respectively. In our notation, this means that the perturbation vector field is given by*

$$G(x, t/\varepsilon) = \begin{pmatrix} \partial_2 h^1(x, t/\varepsilon) \\ -\partial_1 h^1(x, t/\varepsilon) \end{pmatrix},$$

and, consequently, $\ell_1 = \ell_{DS} - r$, $\ell_2 = \ell_{DS} + 1 - r$. *Thus,* $\ell = \max\{\ell_1, \ell_2\} = \ell_{DS} + 1 - r$. *In the Hamiltonian case it is not difficult to observe that* $\ell_{DS} \geq r - 1$ *(see* [24, Remark *R*3]*) and, therefore,* $\ell \geq 0$.

Since system (4.2) is $2\pi\varepsilon$-periodic we can consider its corresponding (time) Poincaré map $\mathcal{P}$, defined as the map which sends a given point x_p to its image by the flow after a time $2\pi\varepsilon$,

$$\mathcal{P}(x_p) := x(x_p, 2\pi\varepsilon).$$

Since (4.2) is $\mathfrak{R}$-reversible it is known (see, for instance [51, 48]) that its associated (time) Poincaré map $\mathcal{P}$ is also reversible, with respect to the same reversing involution $\mathfrak{R}$. We recall that, in the case of maps, this means that it satisfies $\mathfrak{R} \circ \mathcal{P} \circ \mathfrak{R} = \mathcal{P}^{-1}$.

Let us come back to our system and make some comments about the dynamics before and after considering the perturbation. For the case $\mu = 0$, since system (4.1) is autonomous, we have that both phase portraits coincide. In fact, it is foliated by the level curves of the Hamiltonian $h^{(0)}$. Assuming, as usual, that $V(0) = 0$ it follows that the homoclinic orbit $x^{(0)}$ is contained in the level curve $h^{(0)}(x) = 0$. Because of the $\mathfrak{R}$-reversibility, the orbits are invariant under the action of the involution $\mathfrak{R}$ and, in particular, the equilibrium point $(0, 0)$ is also fixed by $\mathfrak{R}$.

For $\mu \neq 0$, the dynamics of system (4.2) becomes more complicated and, consequently, also the phase portrait of $\mathcal{P}$. It presents a hyperbolic point x_*, close to $(0, 0)$, whose stable and unstable invariant manifolds do not, in general, coincide. However, since our perturbation is $\mathfrak{R}$-reversible, some important consequences can be derived:

(*a*) The hyperbolic point x_* belongs to the symmetry line Fix $\mathfrak{R}$, that is, the set of points that remain invariant when we apply $\mathfrak{R}$ onto them.

(*b*) It is known that if x_0 is a hyperbolic point of an $\mathfrak{R}$-reversible mapping L with stable and unstable invariant manifolds $W^s(x_0)$ and $W^u(x_0)$, respectively, then the following equalities are satisfied

$$\mathfrak{R}\{W^s(x_0)\} = W^u(\mathfrak{R}\,x_0), \qquad \mathfrak{R}\{W^u(x_0)\} = W^s(\mathfrak{R}x_0).$$

In the particular case that $x_0 \in$ Fix $\mathfrak{R}$, that is $\mathfrak{R}\,x_0 = x_0$, these equalities imply that

$$\mathfrak{R}\{W^s(x_0)\} = W^u(x_0), \qquad \mathfrak{R}\{W^u(x_0)\} = W^s(x_0)$$

and therefore, they always intersect transversally each other at a homoclinic point in Fix $\mathfrak{R}$. In our case, since $x_* \in$ Fix $\mathfrak{R}$ it follows that their corresponding invariant manifolds intersect transversally at the symmetry line.

Remark 14 *Although these properties have been presented for a linear involution $\mathfrak{R}$, they also hold for general $\mathfrak{G}$-reversible system or map, with $\mathfrak{G}$ a non linear involutory diffeomorphism (see, for instance,* [51, 48]*).*

The main result of this Chapter is to give an asymptotic formula for the value of the angle between the invariant curves associated to x_* at the first intersection. As in [24], this formula validates the exponentially small terms provided by the *Melnikov function*

$$M(s,\varepsilon) = \int_{-\infty}^{+\infty} \mathcal{L}_G h^{(0)}(x^{(0)}(t+s,t/\varepsilon))\,dt = \int_{-\infty}^{+\infty} \left(\nabla h^{(0)}(x^{(0)}(t-s))\right)^{\top} G(x^{(0)}(t-s),t/\varepsilon)\,dt, \tag{4.3}$$

denoting $\mathcal{L}_H f$ the Lie derivative of the function f with respect to the vector field H. The $\mathfrak{R}$-reversibility of systems (4.1) and (4.2) leads this Melnikov function to satisfy the following properties, whose proof will be given in Lemma 4.2:

(i) $M(s,\varepsilon)$ is $2\pi\varepsilon$-periodic.

(ii) $M(s,\varepsilon)$ is an odd function, that is $M(-s,\varepsilon) = -M(s,\varepsilon)$. Consequently, $M(0,\varepsilon) = 0$ and it has zero mean.

(iii) From the previous properties it follows that the expansion of the Melnikov function in Fourier series has no zero terms, that is,

$$M(s,\varepsilon) = \sum_{k\neq 0} M_k(\varepsilon)\,\mathrm{e}^{\mathrm{i}ks/\varepsilon}.$$

Thus, our main result reads as follows.

Theorem 4.1 (Main Theorem) *Assume that systems* (4.1) *and* (4.2) *satisfy hypotheses* (h_1)–(h_3) *and that* $\gamma := p - \ell \geq -1$. *Then, for* $\varepsilon \to 0^+$, $\mu \to 0$, *the following formula holds*

$$\sin\alpha = \mu\varepsilon^p \frac{M'(0,\varepsilon)}{\left\|\left(x^{(0)}(0)\right)'\right\|^2} + O\left(\mu^2\varepsilon^{2\gamma-1},\mu\varepsilon^p\right)\mathrm{e}^{-a/\varepsilon},$$

where M is the Melnikov function (4.3) *and α is the angle between the invariant curves at the symmetric homoclinic point $x^h = (x_1^h, 0)$.*

Our proof will follow almost exactly the argument of [24]. Even more, to ease a parallel reading we have preserved in many cases the same notation. However, in spite of this similarity, there are some important points where the treatment will be different:

- The Normal Form Theorem: we need a procedure ensuring the convergence of the Birkhoff Normal Form for periodic reversible systems in a neighborhood of a symmetric periodic orbit (symmetric means here, invariant under the action of the reversing involution). This result is given at Chapter 3.

- Using the results from Da Silva, Ozorio and Douady [21] and Ozorio and Vieira [46], the Birkhoff Normal Form obtained for our reversible system will be extended along (for instance) the stable invariant manifold. Precisely, we will extend it beyond Fix $\mathfrak{R}$ and will contain a *rectangle* $\mathcal{R}$ of length and width independent of μ and ε. It will be in all this region evolving along the stable manifold where the flow-box coordinates will be defined.

- Applying an slightly modified version of the Extension Theorem given at [24], we will extend the parameterization of the unstable invariant manifold until it reaches this region $\mathcal{R}$.

- From the $\mathfrak{R}$-reversibility of the system, the transversal intersection (at Fix $\mathfrak{R}$) between the invariant manifolds is derived. The computations here will follow quite closely those in [24].

Moreover, to ease the reading, next Section explains the argument and the results used in the proof of this Main Theorem, while their proofs have been deferred to the following ones.

Before starting with the proof, let us notice that there are some well known integrable Hamiltonian systems verifying hypotheses $(h_1), (h_2)$ which are also linearly reversible. Among them, we list the following classical examples:

1. The *pendulum*, given by the equations

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -\sin x_1 \end{array} \right.$$

is reversible with respect to the spatial linear involutions

$$\mathfrak{R}_1 : (x_1, x_2) \longmapsto (-x_1, x_2) \qquad \text{and} \qquad \mathfrak{R}_2 : (x_1, x_2) \longmapsto (x_1, -x_2).$$

It has homoclinic orbits $\Gamma_\pm = \left\{ \left(x_1^{(0)}(t), \pm x_2^{(0)}(t) \right) \right\}$, where $x_1^{(0)}(t) = 2\arctan(\sinh t)$, $x_2^{(0)}(t) = \dot{x}_1^{(0)}(t)$ has a pole of order $r = 1$ at $u = \pm \pi \mathrm{i}/2$.

2. The *Duffing* equation,

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & x_1 - x_1^3 \end{array} \right.$$

is reversible with respect to the involutions $\mathfrak{R}_1$ and $\mathfrak{R}_2$ introduced above. Moreover, it has two homoclinic orbits defined by $\Gamma_\pm = \left\{ \left(\pm x_1^{(0)}(t), x_2^{(0)}(t) \right) \right\}$, where $x_1^{(0)}(t) = \sqrt{2}/\cosh t$, $x_2^{(0)}(t) = \dot{x}_1^{(0)}(t)$ has a pole of order $r = 2$ at $u = \pm \pi \mathrm{i}/2$.

3. The *cubic potential* system

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & x_1 - x_1^2 \end{array} \right.$$

which is $\mathfrak{R}_2$-reversible and has the homoclinic orbit $\Gamma_\pm = \left\{ \left(x_1^{(0)}(t), x_2^{(0)}(t) \right) \right\}$, with $x_1^{(0)}(t) = (\sqrt{3}/2)(\cosh(t/2))^{-2}$, $x_2^{(0)}(t) = \dot{x}_1^{(0)}(t)$ having a pole of order $r = 3$ at $u = \pm \pi \mathrm{i}$.

§2 Proof of the Main Theorem

Before dealing with the proof of the theorem, let us introduce some notation concerning reversible systems. Along this work, if there is no problem of misunderstanding, we will use the same letter $\mathfrak{R}$ for the reversing involution as for the associated matrix. That is, for instance, we will denote by $\mathfrak{R}$ both the involution $\mathfrak{R} : (x_1, x_2) \longmapsto (x_1, -x_2)$ and the linear matrix

$$\mathfrak{R} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Abusing of the language, we are writing in the same way $\mathfrak{R}(x)$ and $\mathfrak{R}x$. Moreover, from the definition of $\mathfrak{R}$, it follows that the manifold Fix $\mathfrak{R}$ is just the symmetry line $\{x_2 = 0\}$.

Now, we list some basic properties of reversible systems that we will use during this chapter. Its proof is omitted since it consists on straightforward computations.

Lemma 4.1 *Let $\Psi(x,t)$ be a diffeomorphism satisfying that $\widetilde{\mathfrak{R}}\Psi(\widetilde{\mathfrak{R}}x, -t) = \Psi(x,t)$, with $\widetilde{\mathfrak{R}}$ a linear involution. Let F be an $\widetilde{\mathfrak{R}}$-reversible vector field, that is, satisfying $\widetilde{\mathfrak{R}}\,F(\widetilde{\mathfrak{R}}\,x, -t) = -F(x,t)$. Then, the following assertions hold:*

- (i) *The transformation $y = \Psi(x,t)$ preserves the $\widetilde{\mathfrak{R}}$-reversibility, that is, $\dot{y} = (\Psi^* F)(y,t)$ is also $\widetilde{\mathfrak{R}}$-reversible. If a diffeomorphism Ψ satisfies $\widetilde{\mathfrak{R}}\,\Psi(\widetilde{\mathfrak{R}}\,x, -t) = \Psi(x,t)$ we will call it* $\widetilde{\mathfrak{R}}$-symmetric.
- (ii) *If Ψ is $\widetilde{\mathfrak{R}}$-symmetric then Ψ^{-1} is also $\widetilde{\mathfrak{R}}$-symmetric.*
- (iii) *$D\Psi$ is $\widetilde{\mathfrak{R}}$-symmetric and DF is $\widetilde{\mathfrak{R}}$-reversible, where D denotes the differential with respect to x.*
- (iv) *The Lie bracket $[\Psi, F]$ (with respect to the spatial variable x) is $\widetilde{\mathfrak{R}}$-reversible, that is,*

$$\widetilde{\mathfrak{R}}\,[\Psi, F]\,(\widetilde{\mathfrak{R}}\,x, -t) = -\,[\Psi, F]\,(x,t).$$

- (v) *With respect to the time variable t it follows that $\partial_t \Psi(x,t)$ is $\widetilde{\mathfrak{R}}$-reversible. In a similar way, one has that $\int F(x,t)\,dt$ is $\widetilde{\mathfrak{R}}$-symmetric.*

Theorem 4.2 (Normal Form Theorem) *The following properties are satisfied for the $\mathfrak{R}$-reversible system* (4.2)*:*

- (i) *There exists a hyperbolic $2\pi\varepsilon$-periodic orbit $\gamma_p(t/\varepsilon)$ close to the origin, with*

$$\bar{\gamma}_p(\theta) = \mu\varepsilon^{p+1} P(0,\theta) + O(\mu\varepsilon^{p+2}),$$

where $P = (P_1, P_2)$ satisfies that

$$\partial_\theta P(x,\theta) = G(x,\theta) \qquad \text{and} \qquad \int_0^{2\pi} P(x,\theta)\,d\theta = 0. \tag{4.4}$$

Moreover, $\bar{\gamma}_p$ is $\mathfrak{R}$-symmetric, that is

$$\mathfrak{R}\bar{\gamma}_p(-\theta) = \bar{\gamma}_p(\theta). \tag{4.5}$$

(ii) There exists a change of variables

$$x = \Psi(\zeta, \theta), \qquad \theta = \theta, \tag{4.6}$$

with

$$\Psi(\zeta,\theta) = \Psi(\zeta,\theta,\mu,\varepsilon) = \Psi^{(0)}(\zeta) + \mu\varepsilon^{p+1}\Psi^{(1)}(\zeta,\theta),$$

where $\zeta = (\zeta_1, \zeta_2)$ *and* $\theta = t/\varepsilon$, *leading system* (4.2) *into the normal form*

$$\left\{ \begin{array}{rcl} \dot\zeta_1 &=& F'\left(\left(\zeta_1^2 - \zeta_2^2\right)/2, \mu, \varepsilon\right)\, \zeta_2 \\ \dot\zeta_2 &=& F'\left(\left(\zeta_1^2 - \zeta_2^2\right)/2, \mu, \varepsilon\right)\, \zeta_1, \end{array} \right. \tag{4.7}$$

where $F'(I)$ *denotes derivative with respect to* I. *This transformation* (4.6) *satisfies the following properties:*

(a) $\Psi(\zeta,\theta)$ *is* 2π*-periodic and analytic in* θ.

(b) *With respect to* ζ, $\Psi(\zeta,\theta)$ *is analytic on a region* $\overline{\mathcal{D}_*}$ *which contains the ball* $\{\zeta = (\zeta_1,\zeta_2) \in \mathbb{C}^2 : |\zeta_j| \le R_*\}$ *and a strip of width* R_* *along the stable invariant manifold of system* (4.7), *crossing the symmetry line* $\{\zeta_2 = 0\}$ *up to a point* p *whose distance to this line is much bigger than* $2\pi\varepsilon$. *The value* R_* *is a positive number independent of* μ, ε.

(c) *the transformation* $x = \Psi(\zeta,\theta)$ *is* $\mathfrak{R}$*-symmetric, that is, it verifies* $\mathfrak{R}\Psi(\mathfrak{R}\zeta, -\theta) = \Psi(\zeta,\theta)$. *In particular, this implies that it preserves* $\mathfrak{R}$*-reversibility.*

(d) *The function* F *verifies that*

$$F(I,\mu,\varepsilon) = \lambda I + O(I^2), \quad F'(I,\mu,\varepsilon) = \lambda + O(I), \quad \lambda = \lambda_0 + O(\mu\varepsilon^{p+2}).$$

Moreover, the change of variables $z = \Psi^{(0)}(\zeta)$ *is* $\mathfrak{R}$*-symmetric and leads system* (4.1) *into its normal form*

$$\left\{ \begin{array}{rcl} \dot\zeta_1 &=& (F^{(0)})'\left((\zeta_1^2 - \zeta_2^2)/2\right)\, \zeta_2 \\ \dot\zeta_2 &=& (F^{(0)})'\left((\zeta_1^2 - \zeta_2^2)/2\right)\, \zeta_1, \end{array} \right. \tag{4.8}$$

where $F^{(0)}(I) = \lambda_0 I + O(I^2)$.

The main consequences of this theorem can be summarized as follows

(i) It determines a region of convergence for the normal form which is independent of μ, ε. More precisely, this domain of convergence can be considered as $\overline{D_{R_*}} = \overline{B_{R_*}} \cup \overline{V_{R_*}}$ where

$$\overline{B_{R_*}} = \left\{\zeta = (\zeta_1,\zeta_2) \in \mathbb{C}^2 \ : \ \|\zeta\| \le R_*\right\} \tag{4.9}$$

and

$$\overline{V_{R_*}} = \left\{\zeta = (\zeta_1,\zeta_2) \in \mathbb{C}^2 \ : \ \|\zeta\| \le R_{*h} \ , \ d\left(\zeta, \{\zeta_1 = \zeta_2\}\right) \le R_*\right\}, \tag{4.10}$$

where $\|\zeta\|^2 = |\zeta_1|^2 + |\zeta_2|^2$.

(*ii*) The periodic orbit γ_p and the normalizing transformation Ψ are $O(\mu\varepsilon^{p+1})$-close to those for the unperturbed system, 0 and $\Psi^{(0)}$, respectively.

(*iii*) The system in normal form for the full system (4.2), given by (4.7) is $O(\mu\varepsilon^{p+2})$-close to the corresponding one (4.8) for the unperturbed system. Besides, their characteristic exponents λ and λ_0 are also $O(\mu\varepsilon^{p+2})$-close. This extra ε is due to hypothesis that $G = (g_1, g_2)$ has zero mean.

We will use the fact that system (4.7) can be explicitly solved, to obtain good parameterizations for the invariant manifolds associated to system (4.2). As it standard, the idea will be to construct them for the normalized ζ-variables and to transport them to the x-variables using the change of variables (4.6). The following corollary (analogously to the one given at [24]) details this construction.

Corollary 4.1 (Local invariant manifolds) *There exist parameterizations, $x^s(t,s)$ and $x^u(t,s)$, of the stable and unstable invariant manifolds associated to system (4.2), defined in the regions*

$$\begin{aligned} D^s &:= \{(t,s) \in \mathbb{R} \times \mathbb{C} \,:\, t + \Re s \geq -T_1\}, \\ D^u &:= \{(t,s) \in \mathbb{R} \times \mathbb{C} \,:\, t + \Re s \leq -T_0\}, \end{aligned}$$

where T_1, T_0 are positive constants independent of μ, ε. In these domains, the parameterizations $x^u(t,s)$ and $x^s(t,s)$ satisfy the following properties:

(*i*) *Fixed s, the function $x^u(t,s)$ is a solution of system (4.2). With respect to s, we have that $s \longmapsto x^u(t,s)$ is a real analytic function. The same properties hold for $x^s(t,s)$. Moreover, due to the $\mathfrak{R}$-reversibility of system (4.2), they can be chosen to satisfy the relations*

$$\mathfrak{R}\, x^u(-t,-s) = x^s(t,s), \qquad \mathfrak{R}\, x^s(-t,-s) = x^u(t,s).$$

(*ii*) *They verify that*

$$x^s(t + 2\pi\varepsilon, s) = x^s(t, s + 2\pi\varepsilon), \qquad x^u(t + 2\pi\varepsilon, s) = x^u(t, s + 2\pi\varepsilon).$$

Therefore, the local stable and unstable invariant curves of the Poincaré map associated to system (4.2) are just given by

$$C^s_{\mathrm{loc}} = \{x^s(2\pi n\varepsilon, s)\}_{n\in\mathbb{N}}, \qquad C^u_{\mathrm{loc}} = \{x^u(2\pi n\varepsilon, s)\}_{n\in\mathbb{N}},$$

respectively, in their domains of convergence D^s and D^u.

(*iii*) *The parameterizations $x^u(t,s)$, $x^s(t,s)$ coincide with the homoclinic solution $x^{(0)}(t+s)$ for $\mu = 0$. Precisely, one has that*

$$\begin{aligned} x^u(t,s) &= x^{(0)}(t+s) + \mu\varepsilon^{p+1} P\left(x^{(0)}(t+s), t/\varepsilon\right) + O(\mu\varepsilon^{p+2}), \qquad t + \Re s \leq -T_0 \\ x^s(t,s) &= x^{(0)}(t+s) + \mu\varepsilon^{p+1} P\left(x^{(0)}(t+s), t/\varepsilon\right) + O(\mu\varepsilon^{p+2}), \qquad t + \Re s \geq -T_1, \end{aligned}$$

where P is given at (4.4).

(*iv*) *For values* $(t,s) \in D^u$ *we have the following asymptotic formula*

$$x^u(t,s) = \gamma_p(t/\varepsilon) + x^{(0)}(t+s) + O(\mu\varepsilon^{p+1}e^{\lambda(t+s)}) + O(\mu\varepsilon^{p+2}e^{t+s}), \tag{4.11}$$

and for $(t,s) \in D^s$,

$$x^s(t,s) = \gamma_p(t/\varepsilon) + x^{(0)}(t+s) + O(\mu\varepsilon^{p+1}e^{-\lambda(t+s)}) + O(\mu\varepsilon^{p+2}e^{-(t+s)}). \tag{4.12}$$

Remark 15 *It is important to stress that the parameterizations* $x^s(t,s)$ *and* $x^u(t,s)$ *of the stable and unstable manifolds, respectively, are not uniquely determined. In fact, as it was also noticed in* [24], *any transformation* $s = S + \phi(S)$ *with* ϕ $2\pi\varepsilon$*-periodic and of size* $O(\mu\varepsilon^{p+1})$, *provides new parameterizations* $\widetilde{x}^s(t,S)$ *and* $\widetilde{x}^u(t,S)$ *of the invariant manifolds. This change produces a shift in their domain of analyticity and preserves the same properties. We will use this freedom later, when introducing a suitable splitting function.*

The domain where we are considering the parameterization of the unstable local manifold $W^u_{\text{loc}}(\gamma_p)$ is essentially the image by Ψ of an closed ball around the equilibrium point $\zeta = 0$. Therefore, it becomes an closed set around the periodic orbit γ_p in the original variables (x,θ), that we can consider as $\{(x,\theta) \in \mathbb{C} \times \mathbb{R} \,:\, \|x - \gamma_p(\theta)\| \le r_*\}$, with r_* independent of μ, ε. On the other hand, from the Normal Form Theorem, it follows that the domain where the parameterization of the stable manifold $W^s(\gamma_p)$ has been defined, is a region evolving along the unperturbed separatrix $x^{(0)}(u)$. This region $\mathcal{U}$ has an strip of width independent of μ, ε, reaches and crosses the symmetry line $\{x_2 = 0\}$ and contains, beyond this line, a *rectangle* $\mathcal{R}$ of length much bigger than $2\pi\varepsilon$.

One important consequence of Theorem 4.2 and, precisely, from its normal form system (4.7), is that it can be solved explicitly. In particular, defining new variables (S,E) as

$$S = \frac{-\log\left(\frac{\zeta_1-\zeta_2}{2}\right)}{F'\left(\frac{\zeta_1^2-\zeta_2^2}{2}\right)}, \qquad E = F\left(\frac{\zeta_1^2-\zeta_2^2}{2}\right) \tag{4.13}$$

and composing it with the change Ψ provided by Theorem 4.2, we will have on the region $\mathcal{U} \setminus W^u_{\text{loc}}(\gamma_p)$ a flow-box coordinates system. Precisely,

Corollary 4.2 (Flow-Box Theorem) *There exists a change of variables*

$$(S,E,\theta) = (\mathcal{S}(x,\theta), \mathcal{E}(x,\theta), \theta)\,, \qquad (S,E,\theta) \in \mathcal{V}, \tag{4.14}$$

analytic in x, θ *and* 2π*-periodic in* θ, *defined on* $\mathcal{U} \setminus W^u_{\text{loc}}(\gamma_p)$, *leading system* (4.2) *into the flow-box form*

$$\dot{S} = 1, \qquad \dot{E} = 0.$$

Moreover, the following properties are satisfied:

(*i*) $\mathcal{S}(x,\theta)$ *and* $\mathcal{E}(x,\theta)$ *are* $O(\mu\varepsilon^{p+1})$*-close to* $\mathcal{S}^{(0)}(x)$ *and* $\mathcal{E}^{(0)}(x)$, *respectively, the corresponding change* $(S^{(0)}, E^{(0)}) = (\mathcal{S}^{(0)}(x), \mathcal{E}^{(0)}(x) = h^{(0)}(x))$ *for the unperturbed system* (4.1).

(*ii*) *Denoting by* $(x,\theta) = (\chi(S,E,\theta),\theta)$ *the inverse change of* (4.14), *we have*

$$\chi(S,E,\theta) = \chi^{(0)}(S,E) + O(\mu\varepsilon^{p+1}),$$

being $x = \chi^{(0)}(S,E)$ *the inverse change of* $(S,E) = (\mathcal{S}^{(0)}(x), \mathcal{E}^{(0)}(x))$.

(*iii*) *Along the stable manifold* $x^s(t,s)$ *we have*

$$\mathcal{S}\left(x^s(t,s), t/\varepsilon\right) = t + s, \qquad \mathcal{E}\left(x^s(t,s), t/\varepsilon\right) = 0. \tag{4.15}$$

Up to this moment we have proved the existence of *flow-box* coordinates on the domain $\mathcal{U} \setminus W^u_{\rm loc}(\gamma_p)$, defined along the stable manifold $W^s(\gamma_p)$ and, in particular, on the rectangle $\mathcal{R}$ defined above. The idea now is to extend the parameterization $x^u(t,s)$ of the unstable local invariant manifold $W^u_{\rm loc}(\gamma_p)$, defined on $\{(x,\theta) \in \mathbb{C}\times\mathbb{R} \ : \ \|x - \gamma_p(\theta)\| \le r_*\}$, up to the symmetry line. In this way, we would have defined on the rectangle $\mathcal{R}$, both parameterizations and a flow-box coordinates system: a good place to measure the splitting between both manifolds. Unfortunately, the parameterization $x^u(t,s)$ is $O(\mu\varepsilon^{p+1})$-close to the unperturbed separatrix $x^{(0)}$ and this has, on its turn, singularities in the complex field placed at $u = \pm a\,\mathrm{i}$. This fact can difficult the control of the growth of this parameterization and a bit more technical extension is needed. This result is provided by the following Extension Theorem, which is an slightly adapted version of the one given by Delshams and Seara at [24]. Like there, it is proved for general solutions of system (4.2).

Theorem 4.3 (Extension Theorem) *Let* $x^{(0)}(t+s)$ *be the separatrix of the unperturbed system* (4.1) *and* $x(t,s)$ *a family of solutions of system* (4.2), *satisfying the initial condition*

$$x(t_0,s) - x^{(0)}(t_0+s) - \mu\varepsilon^{p+1} P\left(x^{(0)}(t_0+s), t_0/\varepsilon\right) = O(\mu\varepsilon^{p+2}), \tag{4.16}$$

where P *has been defined at* (4.4), $s \in \mathbb{C}$, $|\Im s| \le a - \varepsilon$ *and* $t_0 + \Re s = -T$. *Then, if* $\gamma = p - \ell \ge -1$ *it follows that* $x(t,s)$ *is defined on the domain*

$$\mathcal{D}^u_\varepsilon := \left\{(t,s) \in \mathbb{R}\times\mathbb{C} \ : \ |\Im s| \le a - \varepsilon, \quad -T \le t + \Re s \le 0\right\}.$$

Moreover, in such domain, $x(t,s)$ *satisfies that*

$$x(t,s) - x^{(0)}(t+s) = O(\mu\varepsilon^{\gamma+1}). \tag{4.17}$$

Remark 16 *The main feature of this Extension Theorem is that it provides an extension of the family of solutions* $x(t,s)$ *in a complex strip (up to a distance* ε *of* $|\Im u| = a$*). The final estimate* (4.17) *shows the* price *one has to* pay *in the distance to the unperturbed homoclinic. With respect to the extension on the reals, there is no loss of accuracy. Indeed, if* T *is fixed and* $s \in \mathbb{R}$, *the* real *extended solution satisfies*

$$x(t,s) - x^{(0)}(t+s) = \mu\varepsilon^{p+1} P\left(x^{(0)}(t+s), t/\varepsilon\right) + O(\mu\varepsilon^{p+2}),$$

which is of the same order as the initial condition (4.16).

From Corollary 4.1 it follows that the parameterization $x^u(t,s)$ of the unstable local invariant manifold $W^u_{\text{loc}}(\gamma_p)$ satisfies the condition (4.16) for $t_0 = -T_0 - \Re s$. Applying to $x^u(t,s)$ the Extension Theorem, we obtain that it can be extended up to reach, for $t + \Re s = 0$ the symmetry line $\{x_2 = 0\}$. Thus, since $-T_0 \le -T_1$ we have for $-T_1 \le t + \Re s \le 0$ that both parameterizations $x^s(t,s)$ and $x^u(t,s)$ stay at the rectangle $\mathcal{R}$, a region where the flow-box coordinates are also defined. This implies that we can evaluate over $x^u(t,s)$ the flow-box functions, giving rise to the following functions

$$\mathcal{S}^u(s) := \mathcal{S}\left(x^u(t,s), t/\varepsilon\right) - t, \qquad \mathcal{E}^u(s) := \mathcal{E}\left(x^u(t,s), t/\varepsilon\right), \tag{4.18}$$

for $-T_1 \le t + \Re s \le 0$ and $|\Im s| \le a - \varepsilon$. From Corollary 4.2 it follows that $\mathcal{S}(x,t/\varepsilon) - t$ and $\mathcal{E}(x,t/\varepsilon)$ are local first integrals of system (4.2) and, consequently, $\mathcal{S}^u$ and $\mathcal{E}^u$ do not depend on t. Moreover, from the properties (i) and (ii) of Corollary 4.1, it is derived their analyticity on $|\Im s| \le a - \varepsilon$ and $2\pi\varepsilon$-periodicity in s. Next proposition asserts that $\mathcal{E}^u(s)$ can be well-approximated by the Melnikov function.

Proposition 4.1 *The functions $\mathcal{S}^u(s)$ and $\mathcal{E}^u(s)$ defined above, satisfy the following bounds, provided $\gamma = p - (\ell - r) \ge 0$:*

(i) *For $s \in \mathbb{C}$ such that $|\Im s| \le a - \varepsilon$, we have that*

$$\mathcal{E}^u(s) = \mu\varepsilon^p M(s,\varepsilon) + O\left(\mu\varepsilon^p \mu\varepsilon^{\gamma - r - \ell}, \mu\varepsilon^{p+1}\right).$$

(ii) *For $s \in \mathbb{R}$ and defining $\mathcal{E}^u_0 = \dfrac{1}{2\pi\varepsilon}\displaystyle\int_0^{2\pi\varepsilon} \mathcal{E}^u(s)\, ds$, it follows that*

$$\mathcal{E}^u(s) - \mathcal{E}^u_0 = \mu\varepsilon^p M(s,\varepsilon) + O\left(\mu\varepsilon^p \mu\varepsilon^{\gamma - r - \ell}, \mu\varepsilon^{p+1}\right) \mathrm{e}^{-a/\varepsilon}.$$

(iii) *For $s \in \mathbb{R}$, $S = \mathcal{S}^u(s)$ is real analytic and invertible, and its inverse $s = s^u(S)$ satisfies that $s^u(S) - S$ is $O(\mu\varepsilon^{p+1})$ and $2\pi\varepsilon$-periodic in S.*

Some properties and symmetries of system (4.2) have a counterpart in its corresponding Melnikov function. In particular, we have the following result, where two of them are listed.

Lemma 4.2 *The Melnikov function associated to the $\mathfrak{R}$-reversible system* (4.2),

$$M(s,\varepsilon) = \int_{-\infty}^{+\infty} \mathcal{L}_G h^{(0)}(x^{(0)}(t+s, t/\varepsilon))\, dt = \int_{-\infty}^{+\infty} \left(\nabla h^{(0)}(x^{(0)}(t-s))\right)^\top G(x^{(0)}(t-s), t/\varepsilon)\, dt$$

is odd and $2\pi\varepsilon$-periodic in s, that is,

$$M(-s,\varepsilon) = -M(s,\varepsilon), \qquad M(s + 2\pi\varepsilon, \varepsilon) = M(s,\varepsilon).$$

In particular, since M is odd, it follows that $M(0,\varepsilon) = 0$.

Proof. We start proving that it is odd. First, we will prove the following assertions:

(*a*) *The matrices* $\mathfrak{R}$ *and* J *1, that is,* $\mathfrak{R}J = -J\mathfrak{R}$, *where*

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and $\mathfrak{R} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ *is the matrix associated to the linear reversing involution* $\mathfrak{R}$: $(x_1, x_2) \longmapsto (x_1, -x_2)$.

(*b*) *We have also that* $\left(\nabla h^{(0)}(\mathfrak{R}x)\right)^\top = \left(\nabla h^{(0)}(x)\right)^\top \mathfrak{R}$.

Indeed, (*a*) is straightforward to check. Concerning (*b*), since our system $\dot{x} = J\nabla h^{(0)}(x)$ is $\mathfrak{R}$-reversible, it turn out that $\mathfrak{R}J\nabla h^{(0)}(\mathfrak{R}x) = -J\nabla h^{(0)}(x)$ and, in particular, that

$$\nabla h^{(0)}(\mathfrak{R}x) = -J^{-1}\mathfrak{R}J\nabla h^{(0)}(x).$$

Therefore, using that $\mathfrak{R}J = -J\mathfrak{R}$, we get

$$\nabla h^{(0)}(\mathfrak{R}x) = -J^{-1}\mathfrak{R}J\nabla h^{(0)}(x) = J^{-1}J\mathfrak{R}\nabla h^{(0)}(x) = \mathfrak{R}\nabla h^{(0)}(x)$$

and, consequently, $\left(\nabla h^{(0)}(\mathfrak{R}x)\right)^\top = \left(\nabla h^{(0)}(x)\right)^\top \mathfrak{R}$.

♣

Having in mind these assertions, we have

$$M(-s,\varepsilon) = \int_{-\infty}^{+\infty} \mathcal{L}_G h^{(0)}(x^{(0)}(t-s))\,dt = \int_{-\infty}^{+\infty} \left(\nabla h^{(0)}(x^{(0)}(t-s))\right)^\top G(x^{(0)}(t-s), t/\varepsilon)\,dt.$$

Changing $t \mapsto -t$ and using that $x^{(0)}(-t-s) = \mathfrak{R}x^{(0)}(t+s)$ (since it is a solution of the $\mathfrak{R}$-reversible system (4.1)), the latter expression is equivalent to

$$\int_{-\infty}^{+\infty} \left(\nabla h^{(0)}(\mathfrak{R}x^{(0)}(t+s))\right)^\top G(\mathfrak{R}x^{(0)}(t+s), -t/\varepsilon)\,dt. \tag{4.19}$$

Since G is $\mathfrak{R}$-reversible it derives that

$$G(\mathfrak{R}x^{(0)}(t+s), -t/\varepsilon) = -\mathfrak{R}\,G(x^{(0)}(t+s), t/\varepsilon).$$

Moreover, from (*b*), we have that

$$\left(\nabla h^{(0)}(\mathfrak{R}x^{(0)}(t+s))\right)^\top = \left(\nabla h^{(0)}(x^{(0)}(t+s))\right)^\top \mathfrak{R}.$$

Hence, applying these properties onto expression (4.19) this becomes

$$-\int_{-\infty}^{+\infty} \left(\nabla h^{(0)}(x^{(0)}(t+s))\right)^\top \mathfrak{R}^2 G(x^{(0)}(t+s), t/\varepsilon)\,dt = -M(s,\varepsilon).$$

The $2\pi\varepsilon$-periodicity in s of $M(s,\varepsilon)$ comes directly from performing the change $t \mapsto t + 2\pi\varepsilon$ on the integral defining M and using that $G(x,\theta)$ is 2π-periodic in θ.

□

Coming back to the flow-box variables (S, E), we recall that for the stable parameterization $x^s(t,s)$ they admit the following simple expression

$$(S,E) = (t+s,0)$$

while, for the unstable one $x^u(t,s)$, they satisfied

$$(S,E) = (t+\mathcal{S}^u(s), \mathcal{E}^u(s)),$$

as defined in (4.18). This means, in particular, that the stable and unstable curves of the Poincaré map associated to system (4.2) are given by

$$C^s = \{(S,E) = (2\pi n\varepsilon + s, 0)\}_{n\in\mathbb{N}}$$

and

$$C^u = \{(S,E) = (2\pi n\varepsilon + \mathcal{S}^u(s), \mathcal{E}^u(s))\}_{n\in\mathbb{N}},$$

respectively. Notice that in (S, E)-coordinates, C^s has the equation $E = 0$. Thus, it is natural to introduce the so-called *splitting function* ψ, defined implicitly by

$$\psi(2\pi n\varepsilon + \mathcal{S}^u(s)) = \mathcal{E}^u(s)$$

or, simply,

$$\psi(\mathcal{S}^u(s)) = \mathcal{E}^u(s), \tag{4.20}$$

if we take into account the $2\pi\varepsilon$-periodicity of the functions $\mathcal{S}^u(s)-s$ and $\mathcal{E}^u(s)$. This function will allow us to study the evolution of the unstable curve C^u with respect to the stable C^s (given by $E = 0$). To do it, it will be useful to get an equivalent explicit definition for ψ. The idea is to rewrite formula (4.20) in terms of the S-variable. Indeed, from Proposition 4.1(iii), we know that $S = \mathcal{S}^u(s)$ is invertible for any $s \in \mathbb{R}$ and satisfies that its inverse $s = s^u(S) = S + O(\mu\varepsilon^{p+1})$ is analytic and $2\pi\varepsilon$-periodic in S. Therefore, ψ admits the following explicit definition

$$\psi(S) = \mathcal{E}^u(s^u(S)). \tag{4.21}$$

This implies that we need to rewrite our parameterizations $x^s(t,s)$ and $x^u(t,s)$ in terms of the new parameter S. From Remark 15, we know that any change of the form $s = S+\phi(S)$, with ϕ analytic and $2\pi\varepsilon$-periodic in S and of size $O(\mu\varepsilon^{p+1})$, provides new parameterizations for the stable and unstable invariant manifolds. Since in our case, these properties are satisfied by the change $s = s^u(S) = S + O(\mu\varepsilon^{p+1})$ (see Proposition 4.1(iii)), we can consider the new parameterizations

$$\widetilde{x}^u(t,S) := x^u(t, s^u(S))$$

for the unstable invariant manifold and, using that $\mathcal{S}(x^s(t,s), t/\varepsilon) = t+s$,

$$\widetilde{x}^s(t,S) := x^s(t,S)$$

for the stable one. Therefore, having in mind that $\mathcal{E}(x^s(t,S),t/\varepsilon) = \mathcal{E}(\widetilde{x}^s(t,s),t/\varepsilon) = 0$, the splitting function ψ admits the following equivalent expressions

$$
\begin{aligned}
\psi(S) = \mathcal{E}(x^u(t,s^u(S)),t/\varepsilon) = & \\
& \mathcal{E}(x^u(t,s^u(S)),t/\varepsilon) - \mathcal{E}(x^s(t,S),t/\varepsilon) = \\
& \mathcal{E}(\widetilde{x}^u(t,S),t/\varepsilon) - \mathcal{E}(\widetilde{x}^s(t,S),t/\varepsilon) = \\
& \mathcal{E}(\widetilde{x}^u(t,S),t/\varepsilon).
\end{aligned} \tag{4.22}
$$

Let us prove that ψ measures the splitting. It is done in the following proposition. The proof of Theorem 4.1 follows readily from it.

Proposition 4.2 *The splitting function ψ is analytic and $2\pi\varepsilon$-periodic. Moreover it satisfies the following properties:*

(*i*) *There exists a homoclinic point $x^h = x^u(0,0) = x^s(0,0)$ belonging to the symmetry line Fix $\mathfrak{R}$, with $\mathcal{S}^u(0) = 0$. Therefore, $\psi(h_n) = 0$ for $h_n = 2\pi n\varepsilon$, $n \in \mathbb{N}$ and, moreover, we have that*

$$\psi'(h_n) = \frac{\partial \widetilde{x}^u}{\partial S}(0,h_n) \wedge \frac{\partial x^s}{\partial S}(0,h_n) = \left\| \frac{\partial \widetilde{x}^u}{\partial S}(0,h_n) \right\| \left\| \frac{\partial x^s}{\partial S}(0,h_n) \right\| \sin \alpha_n,$$

where $u \wedge v$ denotes the exterior product of the vectors u and v, $\alpha_n = \alpha(0,h_n)$ is the angle between $x^u(0,2\pi n\varepsilon) = \widetilde{x}^u(0,h_n)$ and $x^s(0,h_n)$. Furthermore, the value of $\psi'(h_n)$ is independent of n.

(*ii*) *For $S \in \mathbb{R}$, the function ψ satisfies that*

$$\psi(S) = \mu\varepsilon^p M(S,\varepsilon) + O(\mu^2\varepsilon^{2(p-\ell)}, \mu\varepsilon^{p+1})\, \mathrm{e}^{-a/\varepsilon}.$$

§3 Proofs of Theorems, Corollaries and Propositions used in the proof of the Main Theorem

§3.1 Proof of the Normal Form Theorem

This section is devoted to the proof of the Normal Form Theorem 4.2 and it has been divided in several parts:

(*i*) Before starting with the normalization, we perform onto system (4.2) two steps of *averaging*. Since the perturbation vector field G is assumed to have zero mean, this will allow us to increase the order of the perturbation. This is not basic, but it is useful if we are interested on sharp lower bounds for the parameter p, the order of the perturbation.

(*ii*) Here we start the standard process of normalizing a system dealing, in principle, with the *linear normal form*: we will prove the existence of a symmetric hyperbolic periodic orbit, close to the origin, for the perturbed system; we will consider a reference system around this periodic orbit and will apply reversible Floquet Theory.

(*iii*) Having in mind the results presented in Chapter 3, we will obtain a *non-linear Birkhoff normal form* (BNF in short) for our system. Since BNF is reversible with respect to the involution $\mathfrak{R}' : (x_1, x_2) \longmapsto (x_2, x_1)$, before applying this theorem we will perform a linear transformation Ω leading our $\mathfrak{R}$-reversible system to another equivalent system reversible with respect to $\mathfrak{R}'$.

(*iv*) Following the ideas given by Da Silva, Ozorio and Douady [21] and Ozorio and Vieira [46], we will *extend the normalizing transformation* provided by the Theorem above to a region, which contains a ball around the origin and evolves along the separatrix of the unperturbed system up to a suitable point.

(*v*) Finally, we will perform the change of variables Ω^{-1} in order to have the normal form system $\mathfrak{R}$-reversible.

Let us deal with the details. First of all, to simplify the notation, we perform the change of time $\theta = t/\varepsilon$ in system (4.2), which becomes

$$\left\{ \begin{array}{lcl} x_1' &=& \varepsilon x_2 + \mu\varepsilon^{p+1} g_1(x,\theta) \\ x_2' &=& \varepsilon f(x_1) + \mu\varepsilon^{p+1} g_2(x,\theta) \end{array} \right. \tag{4.23}$$

provided we denote $' = d/d\theta$.

§3.1.1 Averaging the initial system

Lemma 4.3 *There exists a change of variables*

$$x = \Pi(\bar{x},\theta,\mu,\varepsilon) = \bar{x} + \mu\varepsilon^{p+1} P(\bar{x},\theta) + O(\mu\varepsilon^{p+2}), \tag{4.24}$$

with Π *satisfying*

$$\mathfrak{R}\,\Pi(\mathfrak{R}\,\bar{x}, -\theta,\mu,\varepsilon) = \Pi(\bar{x},\theta,\mu,\varepsilon), \tag{4.25}$$

(that is, $\mathfrak{R}$-symmetric) analytic in $\bar{x}$, 2π-periodic and analytic in θ, leading system

$$x' = \varepsilon J\nabla h^{(0)}(x) + \mu\varepsilon^{p+1}G(x,\theta,\mu,\varepsilon) \tag{4.26}$$

into system

$$\bar{x}' = \varepsilon J\nabla h^{(0)}(\bar{x}) + \mu\varepsilon^{p+3}\widetilde{G}(\bar{x},\theta,\mu,\varepsilon) \tag{4.27}$$

where $\widetilde{G}$ is analytic in $\bar{x}$, 2π-periodic and analytic in θ.

Proof. The idea is to apply two steps of *averaging* onto system (4.26). Precisely, performing a change of the form (4.24),

$$\begin{aligned} x = \Pi(\bar{x},\theta,\mu,\varepsilon) = \bar{x} + \mu\varepsilon^{p+1}P(\bar{x},\theta) + O(\mu\varepsilon^{p+2}) = \\ \bar{x} + \mu\varepsilon^{p+1}P(\bar{x},\theta) + \mu\varepsilon^{p+2}\widetilde{P}(\bar{x},\theta) + O(\mu\varepsilon^{p+3}) \end{aligned}$$

onto (4.26) it follows

$$\begin{aligned} &\varepsilon J\nabla h^{(0)}\left(\bar{x} + \mu\varepsilon^{p+1}P + \mu\varepsilon^{p+2}\widetilde{P}\right) + \mu\varepsilon^{p+1}G\left(\bar{x} + \mu\varepsilon^{p+1}P + \mu\varepsilon^{p+2}\widetilde{P},\theta\right) = \\ &\bar{x}' + \mu\varepsilon^{p+1}\left(\partial_1 P\,\bar{x}_1' + \partial_2\widetilde{P}\,\bar{x}_2' + \partial_\theta P\right) + \mu\varepsilon^{p+2}\left(\partial_1\widetilde{P}\,\bar{x}_1' + \partial_2\widetilde{P}\bar{x}_2' + \partial_\theta\widetilde{P}\right), \end{aligned} \tag{4.28}$$

where ∂_1, ∂_2 and ∂_θ denote differential with respect to the variables $\bar{x}_1$, $\bar{x}_2$ and θ, respectively. After some standard computations, the left-hand side of (4.28) can be expressed as

$$\begin{aligned} &\varepsilon J\nabla h^{(0)}(\bar{x}) + \mu\varepsilon^{p+1}G(\bar{x},\theta) + \mu\varepsilon^{p+2}D\left(J\nabla h^{(0)}\right)(\bar{x})\,P(\bar{x},\theta) + \\ &\mu^2\varepsilon^{2(p+1)}DG(\bar{x},\theta)\,P(\bar{x},\theta) + O(\mu\varepsilon^{p+3}). \end{aligned}$$

In a similar way, the right-hand side of equation (4.28) becomes

$$\begin{aligned} &\bar{x}' + \mu\varepsilon^{p+1}\partial_\theta P(\bar{x},\theta) + \mu\varepsilon^{p+2}\left(\partial_1 P(\bar{x},\theta)\,\bar{x}_2' + \partial_2\widetilde{P}(\bar{x},\theta)\,f(\bar{x}_1) + \partial_\theta\widetilde{P}(\bar{x},\theta)\right) + \\ &\mu^2\varepsilon^{2(p+1)}\left(\partial_1 P(\bar{x},\theta)\,g_1(\bar{x},\theta) + \partial_2 P(\bar{x},\theta)\,g_2(\bar{x},\theta)\right) + O(\mu\varepsilon^{p+3}). \end{aligned} \tag{4.29}$$

Joining expressions (4.29) and (4.29) together, it follows

$$\begin{aligned} \bar{x}' = &\varepsilon J\nabla h^{(0)}(\bar{x}) + \mu\varepsilon^{p+1}\left(G(\bar{x},\theta) - \partial_\theta P(\bar{x},\theta)\right) + \\ &\mu\varepsilon^{p+2}\left(D\left(J\nabla h^{(0)}\right)(\bar{x})\,P(\bar{x},\theta) - \partial_1 P(\bar{x},\theta)\,\bar{x}_2 - \partial_2 P(\bar{x},\theta)\,f(\bar{x}_1) - \partial_\theta\widetilde{P}(\bar{x},\theta)\right) + \\ &\mu^2\varepsilon^{2(p+1)}\left(DG(\bar{x},\theta)\,P(\bar{x},\theta) - \partial_1 P(\bar{x},\theta)\,g_1(\bar{x},\theta) - \partial_2 P(\bar{x},\theta)\,g_2(\bar{x},\theta)\right), \end{aligned}$$

which is equivalent to

$$\begin{aligned} \bar{x}' = &\varepsilon J\nabla h^{(0)}(\bar{x}) + \mu\varepsilon^{p+1}\left(G(\bar{x},\theta) - \partial_\theta P(\bar{x},\theta)\right) + \\ &\mu\varepsilon^{p+2}\left(\left[P(\bar{x},\theta), J\nabla h^{(0)}(\bar{x})\right] - \partial_\theta\widetilde{P}(\bar{x},\theta)\right) + \mu^2\varepsilon^{2(p+1)}\left[P(\bar{x},\theta), G(\bar{x},\theta)\right] + O(\mu\varepsilon^{p+3}), \end{aligned}$$

where $[L, M]$ denotes the Lie bracket (with respect to the spatial variables) of the vector fields L and M.

To eliminate those terms of order $\mu\varepsilon^{p+1}$ and $\mu\varepsilon^{p+2}$ we ask P, $\widetilde{P}$ to satisfy the equations

$$\left\{ \begin{array}{rcl} \partial_\theta P(\bar{x},\theta) & = & G(\bar{x},\theta) \qquad \text{with } \int_0^{2\pi} P(\bar{x},\theta)\,d\theta = 0 \\ \partial_\theta \widetilde{P}(\bar{x},\theta) & = & \left[P(\bar{x},\theta), J\nabla h^{(0)}(\bar{x})\right] \end{array} \right. . \tag{4.30}$$

Once this is done, the previous equation reads

$$\bar{x}' = \varepsilon J\nabla h^{(0)}(\bar{x}) + \mu^2\varepsilon^{2(p+1)}\left[P(\bar{x},\theta), G(\bar{x},\theta)\right] + O(\mu\varepsilon^{p+3})$$

so, in order to prove the lemma, we need to check that $\mu^2\varepsilon^{2(p+1)}[P(\bar{x},\theta), G(\bar{x},\theta)]$ is, in fact, $O(\mu\varepsilon^{p+3})$. With regard to $\ell = \max\{\ell_1,\ell_2\}$ we can distinguish two cases:

(*a*) If $\ell \geq 1$: using $\gamma \geq p - \ell \geq 0$ it follows that $p \geq \ell \geq 1$, so then $O(\mu^2\varepsilon^{2(p+1)}) = O(\mu\varepsilon^{p+3})$ and $\mu^2\varepsilon^{2(p+1)}[P(\bar{x},\theta), G(\bar{x},\theta)] = O(\mu\varepsilon^{p+3})$.

(*b*) If $\ell = 0$: by definition of ℓ, this implies that $\ell_1 = \ell_2 = 0$. We are then in the case where $G(x,\theta) = (g_1(\theta), g_2(\theta))$. From equations (4.30) it is derived that $P(\bar{x},\theta) = (\psi_1(\theta), \psi_2(\theta))$ and, therefore, $[\widehat{\Phi}^{(1)}, G]$ vanishes. Consequently, $\mu^2\varepsilon^{2(p+1)}[P(\bar{x},\theta), G(\bar{x},\theta)] = O(\mu\varepsilon^{p+3})$ also in this case.

We have to check that change of variables (4.24), with P, $\widetilde{P}$ defined from (4.30) are $\mathfrak{R}$-symmetric, but this follows straightforwardly from applying Lemma 4.1.

□

Remark 17 *It could seem useful to continue with this averaging process. However, notice that the resulting vector field $\widetilde{G}$ does not have zero mean, so a new averaging step would change the integrable system and, therefore, their separatrices.*

§3.1.2 The linear normal form

Recall that the unperturbed system $x' = \varepsilon J\nabla h^0(x)$ has a hyperbolic fixed point at the origin. Then, it is well known that for $\mu\varepsilon^{p+2}$ small enough the full system (4.27)

$$\bar{x}' = \varepsilon J\nabla h^{(0)}(\bar{x}) + \mu\varepsilon^{p+3}\widetilde{G}(\bar{x},\theta,\mu,\varepsilon)$$

has a 2π-periodic hyperbolic orbit close to the origin. Precisely,

Lemma 4.4 *System* (4.27) *has a hyperbolic* 2π*-periodic orbit* $\bar{\gamma}_p(\theta) = (\bar{\gamma}_1(\theta), \bar{\gamma}_2(\theta))$ *which is* $O(\mu\varepsilon^{p+2})$. *Moreover, it verifies that* $\mathfrak{R}\,\bar{\gamma}_p(-\theta) = \bar{\gamma}_p(\theta)$, *so* $\{\bar{\gamma}_p(\theta)\}_{\theta\in\mathbb{T}}$ *is an* $\mathfrak{R}$*-symmetric set.*

Proof. As it is standard, we consider the Poincaré map associated to the 2π-periodic system (4.27), having in our case a hyperbolic fixed point at the origin. If $\mu\varepsilon^{p+2}$ small, it is known that the Poincaré map of the full system has another hyperbolic fixed point which is $O(\mu\varepsilon^{p+2})$. Since the Poincaré map of a $\mathfrak{R}$-reversible system is also reversible with respect to the same 0 $\mathfrak{R}$ (see, for instance [48]) it follows that this new fixed point is also a fixed point of $\mathfrak{R}$. Coming back to our system (4.27) it is derived the existence of a hyperbolic periodic orbit $\bar{\gamma}_p(\theta)$ invariant under the action of $\mathfrak{R}$ and a reversion of time's arrow.

□

From this lemma, it follows that the initial full system (4.2) has an $\mathfrak{R}$-symmetric $2\pi\varepsilon$-periodic orbit $\gamma_p(t/\varepsilon)$, near the origin, given by

$$\gamma_p(\theta) = (\gamma_1(\theta), \gamma_2(\theta)) = \mu\varepsilon^{p+1}P(0,\theta) + O(\mu\varepsilon^{p+2}),$$

where $\partial_\theta P(x,\theta) = G(x,\theta)$. Following the usual approach, to ease the study of the local behavior around this $\mathfrak{R}$-symmetric periodic orbit $\bar{\gamma}_p$, we send it to the origin of coordinates. It is done by means of the change of variables

$$y = \Upsilon(\bar{x},\theta) := \bar{x} - \bar{\gamma}_p(\theta).$$

Notice that this transformation preserves the $\mathfrak{R}$-reversibility. Indeed,

$$\mathfrak{R}\,\Upsilon(\mathfrak{R}\,\bar{x},-\theta) = \mathfrak{R}\,(\mathfrak{R}\,\bar{x} - \bar{\gamma}_p(-\theta)) = \mathfrak{R}^2\bar{x} - \mathfrak{R}\,\bar{\gamma}_p(-\theta) = \bar{x} - \bar{\gamma}_p(-\theta) = \Upsilon(\bar{x},\theta), \tag{4.31}$$

where it has been taking into account that $\mathfrak{R}$ is an involution and that $\bar{\gamma}_p$ is $\mathfrak{R}$-symmetric. Performing such a change of variables, system (4.27) becomes

$$y' = \varepsilon J\nabla h^{(0)}(y) + \mu\varepsilon^{p+3}\breve{G}(y,\theta,\mu,\varepsilon) \tag{4.32}$$

which begins with

$$\begin{aligned} y' &= \varepsilon\begin{pmatrix} y_2 \\ f(y_1) \end{pmatrix} + \mu\varepsilon^{p+3}\breve{G}(y,\theta,\mu,\varepsilon) = \\ &\quad \varepsilon\begin{pmatrix} y_2 \\ \lambda_0 y_1 + O(y_1^2) \end{pmatrix} + \mu\varepsilon^{p+3}\breve{G}(y,\theta,\mu,\varepsilon) = \varepsilon\begin{pmatrix} y_2 \\ \lambda_0 y_1 \end{pmatrix} + O\left(y_1^2, \mu\varepsilon^{p+3}\right). \end{aligned} \tag{4.33}$$

In Chapter1 it was showed that BNF was reversible with respect to the linear involution

$$\mathfrak{R}' : (x_1, x_2) \longmapsto (x_2, x_1).$$

Since we are dealing with a *similar* (that is, equivalent by means of an invertible transformation) involution $\mathfrak{R}$ our first step will consist on performing a linear change of variables transforming system (4.32) in a system $\mathfrak{R}'$-reversible. The linear normal form will come from applying Reversible Floquet Theory (see [51], for instance).

Lemma 4.5 *The following assertions are satisfied:*

(*i*) *The linear change of variables $y = \Omega\, z$, given by*

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

transforms system (4.32) *into system*

$$z' = \varepsilon J\nabla\kappa^{(0)}(z) + \mu\varepsilon^{p+3}\bar{G}(z,\theta,\mu,\varepsilon) \tag{4.34}$$

where

$$\kappa^{(0)}(z) := \Omega^{-1}h^{(0)}(\Omega\, z)\Omega^{-1}, \qquad \bar{G}(z,\theta,\mu,\varepsilon) = \Omega^{-1}\breve{G}(\Omega\, z,\theta,\mu,\varepsilon).$$

System (4.34) *is now $\mathfrak{R}'$-reversible and begins with*

$$\begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+\lambda_0^2 & -(1-\lambda_0^2) \\ 1-\lambda_0^2 & -(1+\lambda_0^2) \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + O\left(z^2\right).$$

(*ii*) *There exists a change of variables*

$$z = A(\theta,\mu,\varepsilon)Z,$$

linear in Z, 2π*-periodic and analytic in* θ, *preserving the* $\mathfrak{R}'$*-reversibility*

$$\mathfrak{R}'\,A(-\theta,\mu,\varepsilon)\,\mathfrak{R}' = A(\theta,\mu,\varepsilon), \tag{4.35}$$

with $A(\theta,\mu,\varepsilon) = A_0 + O(\mu\varepsilon^{p+2})$, *being*

$$A_0 = c_f \begin{pmatrix} 1+\lambda_0 & 1-\lambda_0 \\ 1-\lambda_0 & 1+\lambda_0 \end{pmatrix} \tag{4.36}$$

$c_f \neq 0$ *a constant, leading system* (4.34) *into* BNF *up to order* 2*:*

$$Z' = \varepsilon J\nabla H^{(0)}(Z) + \mu\varepsilon^{p+3}\breve{G}(Z,\theta,\mu,\varepsilon) = \varepsilon\Lambda Z + O\left(Z^2\right), \tag{4.37}$$

where

$$\Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$$

and $\lambda = \lambda(\mu,\varepsilon) = \lambda_0 + O(\mu\varepsilon^{p+2})$. *In particular, the change* $z = A_0 Z$ *leads the unperturbed system* $z' = \varepsilon J\nabla\kappa^{(0)}(z)$ *into* BNF *up to order* 2*:*

$$Z' = \varepsilon\Lambda_0 Z + O\left(Z^2\right),$$

being

$$\Lambda_0 = \begin{pmatrix} \lambda_0 & 0 \\ 0 & -\lambda_0 \end{pmatrix}.$$

Proof.

(*i*) First, recall that our system (4.32) is $\mathfrak{R}$-reversible so this implies that

$$\begin{aligned} h^{(0)}(\mathfrak{R}y) &= h^{(0)}(y) \;\Rightarrow\; \mathfrak{R}\nabla h^{(0)}(\mathfrak{R}y) = \nabla h^{(0)}(y), \\ \breve{G}(y,\theta,\mu,\varepsilon) &= -\mathfrak{R}\,\breve{G}(\mathfrak{R}y,-\theta,\mu,\varepsilon). \end{aligned} \tag{4.38}$$

Now, let us consider the linear transformation $y = \Omega z$, where

$$\Omega = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \tag{4.39}$$

is a rotation of angle $\pi/4$. This change of variables satisfies that

$$\mathfrak{R}' = \Omega^{-1}\mathfrak{R}\,\Omega, \qquad \Omega J = J\Omega \Leftrightarrow J\Omega^{-1} = \Omega^{-1}J, \qquad \det\Omega = 1. \tag{4.40}$$

Applying the transformation $y = \Omega z$ onto system (4.32) and using properties (4.40) the new system becomes

$$z' = \varepsilon J\Omega^{-1}\nabla h^{(0)}(\Omega\, z) + \mu\varepsilon^{p+3}\bar{G}(z,\theta,\mu,\varepsilon), \tag{4.41}$$

where

$$\bar{G}(z,\theta,\mu,\varepsilon) := \Omega^{-1}\breve{G}(\Omega z,\theta,\mu,\varepsilon). \tag{4.42}$$

We need now to verify that the new system (4.41) is $\mathfrak{R}'$-reversible. To see it is enough to check that the following identities are satisfied (*a*) $\mathfrak{R}'\left(J\Omega^{-1}\nabla h^{(0)}(\Omega\,\mathfrak{R}'\,z)\right) = -J\Omega^{-1}\nabla h^{(0)}(\Omega z)$.

(*b*) $\mathfrak{R}'\,\bar{G}(\mathfrak{R}'\,z, -\theta, \mu, \varepsilon) = -\bar{G}(z, \theta, \mu, \varepsilon)$.

Indeed,

(*a*) Using properties (4.38) and (4.40) and the fact that $\mathfrak{R}'\,J\mathfrak{R}' = -J$ it follows that

$$\mathfrak{R}'\left(J\Omega^{-1}\nabla h^{(0)}(\Omega\,\mathfrak{R}'\,z)\right) = \mathfrak{R}'\left(J\Omega^{-1}\nabla h^{(0)}(\mathfrak{R}\,\Omega\,z)\right) = \\ \mathfrak{R}'\left(J\Omega^{-1}\,\mathfrak{R}\nabla h^{(0)}(\Omega\,z)\right) = \left(\mathfrak{R}'\,J\,\mathfrak{R}'\right)\Omega^{-1}\nabla h^{(0)}(\Omega\,z) = -J\,\Omega^{-1}\nabla h^{(0)}(\Omega z).$$

(*b*) Analogously, using the same properties and the fact that $\mathfrak{R}^2 = \mathrm{id}$, it turns out that

$$\mathfrak{R}'\,\bar{G}(\mathfrak{R}'\,z, -\theta) = \mathfrak{R}'\,\bar{G}(\mathfrak{R}'\,z, -\theta, \mu, \varepsilon) = \mathfrak{R}'\,\Omega^{-1}\breve{G}(\Omega\,\mathfrak{R}'\,z, -\theta) = \\ \mathfrak{R}'\,\Omega^{-1}\breve{G}(\Omega\,\mathfrak{R}'\,\Omega^{-1}y, -\theta) = \mathfrak{R}'\,\Omega^{-1}\breve{G}(\mathfrak{R}\,y, -\theta) = \mathfrak{R}'\,\Omega^{-1}\left(-\mathfrak{R}\,\breve{G}(y,\theta)\right) = \\ -\mathfrak{R}'\,\Omega^{-1}\mathfrak{R}\,\breve{G}(\Omega\,z, \theta) = -\Omega^{-1}\mathfrak{R}^2\breve{G}(\Omega\,z, \theta) = -\Omega^{-1}\breve{G}(\Omega\,z, \theta) = -\bar{G}(z, \theta).$$

(*ii*) It follows directly from applying reversible Floquet Theory to system (4.34), where this change of variables can be chosen to preserve the $\mathfrak{R}'$-reversibility. In the case of the *unperturbed* transformation $z = A_0 Z$ this implies that $\mathfrak{R}'\,A_0\mathfrak{R}' = A_0$.

□

§3.1.3 The non-linear normal form

Applying now the results presented in Chapter 3, we have that system (4.37) can be led into (non linear) BNF in a neighborhood of $Z = 0$ by means of a convergent transformation. Moreover, this transformation preserves the $\mathfrak{R}'$-reversibility and its radius of convergence does not depend on μ, ε. Precisely, we have the following result.

Proposition 4.3 (BNF-Theorem) *There exists a change of variables*

$$Z = \widetilde{\Phi}(\xi, \theta, \mu, \varepsilon) = \widetilde{\Phi}^{(0)}(\xi) + \mu\varepsilon^{p+2}\widetilde{\Phi}^{(1)}(\xi, \theta, \mu, \varepsilon)$$

analytic for $\xi \in \overline{\mathcal{D}_{\widetilde{R}_0}}$, *with* $\widetilde{R}_0 > 0$ *and*

$$\overline{\mathcal{D}_{\widetilde{R}_0}} = \left\{\zeta = (\xi_1, \xi_2) \in \mathbb{C}^2 : \quad |\xi_j| \le \widetilde{R}_0, \quad j = 1, 2\right\}, \tag{4.43}$$

2π-periodic and analytic in θ, satisfying

$$\mathfrak{R}'\,\Phi(\mathfrak{R}'\,\xi, -\theta) = \Phi(\xi, \theta), \tag{4.44}$$

(that is, preserving $\mathfrak{R}'$-reversibility), leading system (4.37) *into* BNF

$$\dot{\xi} = N(\xi, \mu, \varepsilon) = N^{(0)}(\xi) + \mu\varepsilon^{p+2}N^{(1)}(\xi, \mu, \varepsilon), \tag{4.45}$$

where

$$N(\xi,\mu,\varepsilon) = \begin{pmatrix} \xi_1\, a(\xi_1\xi_2,\mu,\varepsilon) \\ -\xi_2\, a(\xi_1\xi_2,\mu,\varepsilon) \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + O(\xi^2).$$

It is important to stress the fact that the radius of convergence of $\widetilde{\Phi}$, that is $\widetilde{R}_0$, does not depend on μ, ε. Moreover, the transformation $Z = \widetilde{\Phi}^{(0)}$ leads the unperturbed system $Z' = \varepsilon J \nabla H^{(0)}(Z)$ into its BNF

$$\dot{\xi} = N^{(0)}(\xi) = \begin{pmatrix} \xi_1\, a^{(0)}(\xi_1\xi_2,\mu,\varepsilon) \\ -\xi_2\, a^{(0)}(\xi_1\xi_2,\mu,\varepsilon) \end{pmatrix} = \begin{pmatrix} \lambda_0 & 0 \\ 0 & -\lambda_0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + O(\xi^2)$$

and is also convergent for $\zeta \in \overline{\mathcal{D}_{\widetilde{R}_0}}$. Defining $F^{(0)}(I) = a^{(0)}(I)$, $F^{(1)}(I,\mu,\varepsilon) = a^{(1)}(I,\mu,\varepsilon)$ and $F(I,\mu,\varepsilon) = a(I,\mu,\varepsilon)$, we can rewrite equation (4.45) *as*

$$\dot{\xi}_1 = F'(\xi_1\xi_2,\mu,\varepsilon)\xi_1, \qquad \dot{\xi}_2 = -F'(\xi_1\xi_2,\mu,\varepsilon)\xi_2, \tag{4.46}$$

where $F'(I,\mu,\varepsilon)$ means derivative with respect to I. Moreover, it follows that

$$F(I,\mu,\varepsilon) = F^{(0)}(I) + \mu\varepsilon^{p+2} F^{(1)}(I,\mu,\varepsilon)$$

with $F(I,\mu,\varepsilon) = \lambda I + O(I^2)$ and $\lambda = \lambda_0 + O(\mu\varepsilon^{p+2})$.

Remark 18 *As it was noticed in Chapter 1, this result implies that, in a neighborhood of a symmetric hyperbolic periodic orbit, the Hamiltonian and reversible behaviors coincide. In particular, locally around $Z = 0$, our system is Hamiltonian with Hamilton function $\varepsilon\widetilde{H}$ where*

$$\widetilde{H}(\xi,\theta,\mu,\varepsilon) = F(\xi_1\xi_2,\mu,\varepsilon) = F^{(0)}(\xi_1\xi_2) + \mu\varepsilon^{p+2} F^{(1)}(\xi_1\xi_2,\mu,\varepsilon).$$

§3.1.4 The extended normal form

From the results of Da Silva, Ozorio and Douady [21] and Ozorio and Vieira [46], we have

Proposition 4.4 (Extended BNF**))** *There exists a transformation $Z = \Phi(\zeta,\theta,\mu,\varepsilon)$, with*

$$\Phi(\zeta,\theta,\mu,\varepsilon) = \Phi^{(0)}(\zeta) + \mu\varepsilon^{p+2}\Phi^{(1)}(\zeta,\theta,\mu,\varepsilon)$$

analytic in ζ, 2π-periodic and analytic in θ, preserving $\mathfrak{R}'$-reversibility, leading system (4.37) *into* BNF *system* (4.46).

This transformation is defined on a domain $\overline{D^*}$ which evolves along the stable invariant manifold $\mathcal{W}^s$ associated to system (4.46) *and crosses the symmetry line $\{\zeta_1 = \zeta_2\}$ up to a point p which is at a distance bigger than ε from this line. This region $\overline{D^*}$ contains an strip of width $w = O(1)$, containing $\mathcal{W}^s$, which is independent of μ, ε. Moreover, the origin belongs to $\overline{D^*}$ and in the regions where both Φ and $\widetilde{\Phi}$ (provided by the* BNF*-Theorem) are defined, they coincide.*

The transformation $Z = \Phi^{(0)}(\zeta)$ corresponds to the extended normalizing transformation of the unperturbed system.

Proof.

The proof follows essentially from the papers [21, 46]. The difference with respect to the treatment given there is that they perform an infinite construction for the extended normalizing transformation Φ. Such a scheme allows them to reach (with an exponentially decay of the strip width) the homoclinic point and apply it for numerical purposes. In our case this is easier since we are just interested on crossing (an a bit more) the symmetry line. As a consequence, our extended normal form will be formed by a finite number of functions.

From Proposition 4.3 we know the existence of an analytic transformation

$$Z = \widetilde{\Phi}(\zeta,\theta,\mu,\varepsilon) = \widetilde{\Phi}^{(0)}(\zeta) + \mu\varepsilon^{p+2}\widetilde{\Phi}^{(1)}(\zeta,\theta,\mu,\varepsilon) \tag{4.47}$$

leading system (4.37) into the BNF

$$\left\{ \begin{array}{rcl} \dot{\zeta}_1 & = & F'(\zeta_1\zeta_2,\mu,\varepsilon)\,\zeta_1 \\ \dot{\zeta}_2 & = & -F'(\zeta_1\zeta_2,\mu,\varepsilon)\,\zeta_2 \end{array} \right. \tag{4.48}$$

This transformation is convergent in a domain $\overline{\mathcal{D}_{\widetilde{R}_0}}$ given by (4.43). System (4.48) can be solved directly and, therefore, the following explicit expression can be derived for its associated map *flow at time t*

$$\nu_t(\zeta^0) = \nu_t(\zeta_1^0,\zeta_2^0) = \left(\zeta_1^0 \mathrm{e}^{t\,F'(\zeta_1^0\zeta_2^0,\mu,\varepsilon)}, \zeta_2^0 \mathrm{e}^{-t\,F'(\zeta_1^0\zeta_2^0,\mu,\varepsilon)}\right)$$

with $\nu_0(\zeta^0) = \zeta^0$. This map ν_t is $O(\mu\varepsilon^{p+2})$-close to the map $\nu_t^{(0)}$, the map *flow at time t* corresponding to unperturbed BNF. It is known (see, for instance [48]) that the map flow at time t associated to a time-reversible system is symmetric with respect to the same spatial involution. In our case, this means that the map $\nu_t : \zeta^0 \longmapsto \nu_t(\zeta^0)$ preserves the $\mathfrak{R}'$-reversibility, where we recall that $\mathfrak{R}' : (\zeta_1,\zeta_2) \longmapsto (\zeta_2,\zeta_1)$.

Let ψ_t be flow at time t associated to the initial system (4.37). Since this system was $\mathfrak{R}'$-reversible it turns out that $\psi_t : z \longmapsto \psi_t(z)$ is a $\mathfrak{R}'$-symmetric transformation. Then, we define the following vector fields

$$\begin{array}{rcll} \Phi^{(0)} & := & \widetilde{\Phi} & \\ \Phi^{(m)} & = & \psi_{-m} \circ \widetilde{\Phi} \circ \nu_m & \qquad m \geq 1 \end{array} \tag{4.49}$$

where $\widetilde{\Phi}$ is the normalizing transformation (4.47) provided by Proposition 4.3. Since $\widetilde{\Phi}$, ν_t and ψ_t preserve $\mathfrak{R}'$-reversibility, it follows that $\Phi^{(m)}$ is also $\mathfrak{R}'$-reversible preserving. From definition (4.49) is clear that

$$\Phi^{(m)} = \psi_{-1} \circ \Phi^{(m-1)} \circ \nu_1, \qquad m \geq 1$$

and that $\Phi^{(m)}$ is analytic on $\overline{D^{(m)}} := \nu_{-m}\left(\overline{\mathcal{D}_{\widetilde{R}_0}}\right)$. Moreover, since the origin is fixed by ν it turns out that it belongs to $\overline{D^{(m)}}$, for any $m \geq 1$. These sets $\overline{D^{(m)}}$ become thinner and thinner deformations (hyperbolae-like) of the initial domain $\overline{\mathcal{D}_{\widetilde{R}_0}}$ and evolve along the stable invariant manifold $\mathcal{W}^s$ of the origin in system (4.48).

Let $N \geq 1$ be a number satisfying that $\overline{D^{(N)}}$ has crossed the symmetry line $\{\zeta_1 = \zeta_2\}$ and contains a rectangle of length bigger than ε starting at the line and going forwards. Notice that this number N depends on λ_0 and $\widetilde{R}_0$ but is independent of μ and ε.

Hence, following [21], we consider the (finite) *extended normal form transformation*

$$\Phi^* := \bigcup_{m=0}^{N} \Phi^{(m)},$$

defined on the region

$$\overline{D^*} := \bigcup_{m=0}^{N} \overline{D^{(m)}}.$$

By construction, Φ^* is analytic on $\overline{D^*}$ and preserves $\mathfrak{R}'$-reversibility. Moreover, it is well defined and constitutes an extension of the normalizing transformation $\widetilde{\Phi}$ provided by the BNF-Theorem (4.3). Indeed, since $\widetilde{\Phi}$ transforms system (4.37) into (4.48) it conjugates (analytically) their associated flows ψ_t and ν_t, that is, $\widetilde{\Phi}^{-1} \circ \psi_t \circ \widetilde{\Phi} = \nu_t$ and, therefore, $\psi_t \circ \widetilde{\Phi} \circ \nu_{-t} = \widetilde{\Phi} \quad \forall t$. Thus, having this property into account, it follows, for any $j \geq m \geq 0$ and in any region where both $\Phi^{(j)}$, $\Phi^{(m)}$ converge, that

$$\Phi^{(j)} = \psi_{-j} \circ \widetilde{\Phi} \circ \nu_j = \psi_{-j}\left(\psi_{j-m} \circ \widetilde{\Phi} \circ \nu_{-(j-m)}\right) \circ \nu_j = \psi_{-j+j-m} \circ \widetilde{\Phi} \circ \nu_{m-j+j} = \psi_{-m} \circ \widetilde{\Phi} \circ \nu_m = \Phi^{(m)}.$$

The BNF-flow $\nu_t(\zeta_1^0, \zeta_2^0)$ evolves following the hyperbolae $\zeta_1 \zeta_2 = \zeta_1^0 \zeta_2^0$ so, the width of the domains $\overline{D^{(m)}}$ decreases with an approximative ratio

$$\mathrm{e}^{-mF'(\zeta_1^0\zeta_2^0)} = \mathrm{e}^{-m(F^{(0)})'(\zeta_1^0\zeta_2^0)} + O(\mu\varepsilon^{p+2}).$$

Therefore, for $m = N$ the width of the last set $\overline{D^{(N)}}$ is close to $\mathrm{e}^{-N(F^{(0)})'(\widetilde{R}_0^2)}\widetilde{R}_0$.

□

Remark 19 *The main consequence of this Proposition is that we can extend our normal form up to a region which contains a* rectangle *of length bigger than* $2\pi\varepsilon$ *and width of order* $O(1)$*, starting from the symmetry line and going forwards. This will be the region where we will measure the size of the splitting of the separatrices.*

§3.1.5 The $\mathfrak{R}$-reversible normal form

In order to recover again the initial $\mathfrak{R}$-reversibility we perform the linear change $\xi = \Omega^{-1}\zeta$, where the matrix Ω has been defined at (4.39). Applying it onto system (4.46) we obtain the system

$$\text{(4.50)} \qquad \left\{ \begin{array}{rcl} \dot{\zeta}_1 & = & F'\left((\zeta_1^2 - \zeta_2^2)/2\right)\, \zeta_2 \\ \dot{\zeta}_2 & = & F'\left((\zeta_1^2 - \zeta_2^2)/2\right)\, \zeta_1 \end{array} \right. ,$$

where the vector field F is the one provided by the BNF-Proposition (4.3). Composing all the transformations performed along these sections, we obtain the following expression for the transformation leading system (4.2) into system (4.50):

$$x = \Psi(\zeta, t/\varepsilon) = \Psi(\zeta, t/\varepsilon, \mu, \varepsilon)$$

where

$$\text{(4.51)} \qquad \Psi(\zeta, \theta) := \Pi\left(\bar{\gamma}_p(\theta) + \Omega\, A(\theta)\, \Phi(\Omega^{-1}\zeta, \theta)\ ,\ \theta\right).$$

Let us check that it verifies $\mathfrak{R}\,\Psi(\mathfrak{R}\,\zeta,-\theta)=\Psi(\zeta,\theta)$. Therefore, this implies that it preserves the $\mathfrak{R}$-reversibility. Indeed, from properties (4.40) and (4.44) it follows that

$$\Phi(\Omega^{-1}\mathfrak{R}\,\zeta,-\theta)=\Phi(\mathfrak{R}\,\Omega^{-1}\zeta,-\theta)=\mathfrak{R}'\,\Phi(\Omega^{-1}\zeta,\theta).$$

On the other hand, from (4.35) one has that $\mathfrak{R}'\,A(\theta)\mathfrak{R}'=A(-\theta)$, and from (4.25) it is derived that

$$\mathfrak{R}\,\Pi(\mathfrak{R}\,\bar{x},-\theta,\mu,\varepsilon)=\Pi(\bar{x},\theta,\mu,\varepsilon).$$

Thus, having in mind all these identities, it follows that

$$\begin{aligned}\mathfrak{R}\,\Psi(\mathfrak{R}\,\zeta,-\theta)&=\mathfrak{R}\,\Pi\left(\bar{\gamma}_p(-\theta)+\Omega\,A(-\theta)\,\Phi(\Omega^{-1}\mathfrak{R}\,\zeta,-\theta)\ ,\ -\theta\right)=\\ &\mathfrak{R}\,\Pi\left(\mathfrak{R}\,\bar{\gamma}_p(\theta)+\Omega\,\mathfrak{R}'\,A(\theta)\,\mathfrak{R}'\,\mathfrak{R}'\,\Phi(\Omega^{-1}\zeta,\theta)\ ,\ -\theta\right)=\\ &\mathfrak{R}\,\Pi\left(\mathfrak{R}\left(\bar{\gamma}_p(\theta)+\Omega\,A(\theta)\,\Phi(\Omega^{-1}\zeta,\theta)\right)\ ,\ -\theta\right)=\\ &\Pi\left(\bar{\gamma}_p(\theta)+\Omega\,A(\theta)\,\Phi(\Omega^{-1}\zeta,\theta)\ ,\ \theta\right)=\Psi(\zeta,\theta),\end{aligned}$$

where it has been also used that $\bar{\gamma}_p(-\theta)=\mathfrak{R}\bar{\gamma}_p(\theta)$, that $\Omega\,\mathfrak{R}'=\mathfrak{R}\,\Omega$ and $\mathfrak{R}^2=\mathrm{id}$. In particular, from expression (4.51) and the fact that

$$\bar{\gamma}_p(\theta)=O(\mu\varepsilon^{p+2}),\quad A(\theta)=A_0+O(\mu\varepsilon^{p+2}),\Phi=\Phi^{(0)}+O(\mu\varepsilon^{p+2}),$$

it turns out that

$$x=\Psi(\zeta,\theta)=\Psi^{(0)}(\zeta)+\mu\varepsilon^{p+1}\Pi\left(\Psi^{(0)}(\zeta),\theta\right)+O(\mu\varepsilon^{p+2}),\tag{4.52}$$

where $x=\Psi^{(0)}(\zeta)$ leads system (4.1) into the normal form

$$\left\{\begin{array}{lcl}\dot{\zeta}_1&=&(F^{(0)})'\left((\zeta_1^2-\zeta_2^2)/2\right)\,\zeta_2\\ \dot{\zeta}_2&=&(F^{(0)})'\left((\zeta_1^2-\zeta_2^2)/2\right)\,\zeta_1,\end{array}\right.,$$

is $\mathfrak{R}$-symmetric and satisfies that

$$\Psi^{(0)}(\zeta)=\Omega\,A_0\,\Phi^{(0)}(\Omega^{-1}\zeta)=\Omega\,A_0\Omega^{-1}\zeta+O(\zeta^2).\tag{4.53}$$

§3.2 Proof of the Local parameterization Corollary

From the normalized system (4.7) it is clear that

$$\frac{d}{dt}(\zeta_1+\zeta_2)=F'\left((\zeta_1^2-\zeta_2^2)/2\right)\,(\zeta_1+\zeta_2)\,,\qquad \frac{d}{dt}(\zeta_1-\zeta_2)=-F'\left((\zeta_1^2-\zeta_2^2)/2\right)\,(\zeta_1-\zeta_2)\,.$$

Therefore, it follows that

$$(\zeta_1+\zeta_2)\,(t)=\left(\zeta_1^0+\zeta_2^0\right)\mathrm{e}^{tF'((\zeta_1^2-\zeta_2^2)/2)},\qquad (\zeta_1-\zeta_2)\,(t)=\left(\zeta_1^0-\zeta_2^0\right)\mathrm{e}^{-tF'((\zeta_1^2-\zeta_2^2)/2)},$$

where ζ_1^0 and ζ_2^0 are positive constants. Since $F'(I)=\lambda+O(I)$ with $\lambda>0$, and taking $\zeta_1^0>0$, $\zeta_2^0=-\zeta_1^0$, we get that $\zeta_1(t)=-\zeta_2(t)\ \forall t>0$ and

$$\xi_1(t)=\zeta_1^0\mathrm{e}^{-tF'(0)}=\zeta_1^0\mathrm{e}^{-\lambda t}.$$

Thus, the local stable manifold is given by

$$\zeta(t) = \left(\zeta_1^0 \mathrm{e}^{-\lambda t}, -\zeta_1^0 \mathrm{e}^{-\lambda t}\right) = (1, -1)\ \zeta_1^0 \mathrm{e}^{-\lambda t},$$

for $\zeta_1^0 > 0$. Introducing $s + c_s := -(1/\lambda) \log \zeta_1^0$, the following parameterization for the *local stable invariant manifold* of system (4.7) is derived

$$\zeta^s(t, s) = (1, -1)\ \mathrm{e}^{-\lambda(t+s+c_s)}. \tag{4.54}$$

Proceeding in a similar way, one obtains that the *local unstable invariant manifold* can be parameterizated by

$$\zeta^u(t, s) = (1, 1)\ \mathrm{e}^{\lambda(t+s+c_u)}, \tag{4.55}$$

where $s + c_u := (1/\lambda) \log \zeta_1^0$. The constants c_s and c_u are arbitrary and will be fixed later.

Since system (4.7) is $\mathfrak{R}$-reversible and the origin is fixed by $\mathfrak{R}$, we know that the involution $\mathfrak{R}$ transforms its stable and unstable invariant manifolds one in each other plus a reversion in time. Therefore, it is natural to extend such reversibility to their parameterizations and ask them to satisfy $\mathfrak{R}\,\zeta^s(-t, -s) = \zeta^u(t, s)$. Imposing it, we obtain that $c_u = -c_s =: c$ and then

$$\zeta^s(t, s) = \left(\mathrm{e}^{-\lambda(t+s+c)}, -\mathrm{e}^{-\lambda(t+s+c)}\right), \qquad \zeta^u(t, s) = \left(\mathrm{e}^{\lambda(t+s-c)}, \mathrm{e}^{\lambda(t+s-c)}\right). \tag{4.56}$$

Once we have parameterizated the local invariant manifolds of system (4.7), a parameterization for the corresponding ones of the original system (4.2) can be achieved applying the transformation Ψ (4.6). Imposing $\left(\mathrm{e}^{-\lambda(t+s+c)}, -\mathrm{e}^{-\lambda(t+s+c)}\right)$ to belong to the extended normal form domain $\overline{D_{R_*}} = \overline{B_{R_*}} \cup \overline{V_{R_*}}$ defined by (4.9) and (4.10), one obtains

$$x^s(t, s) = \Psi\left(\mathrm{e}^{-\lambda(t+s+c)}, -\mathrm{e}^{-\lambda(t+s+c)}, t/\varepsilon\right), \qquad \text{for } \Re(t + s + c) \geq -T_1, \tag{4.57}$$

where $T_1 := -(1/(2\lambda)) \log(R_{*h}^2/2)$. Analogously, for the local unstable invariant manifold we impose that $\left(\mathrm{e}^{\lambda(t+s-c)}, \mathrm{e}^{\lambda(t+s-c)}\right)$ belongs to $\overline{B_{R_*}}$ and get the parameterization

$$x^u(t, s) = \Psi\left(\mathrm{e}^{\lambda(t+s-c)}, \mathrm{e}^{\lambda(t+s-c)}, t/\varepsilon\right), \qquad \text{for } \Re(t + s - c) \leq -T_0, \tag{4.58}$$

for $T_0 := -(1/(2\lambda)) \log(R_0^2/2)$. Since Ψ is $2\pi\varepsilon$-periodic in t, it follows straightforwardly that

$$x^s(t + 2\pi\varepsilon, s) = x^s(t, s + 2\pi\varepsilon), \qquad x^u(t + 2\pi\varepsilon, s) = x^u(t, s + 2\pi\varepsilon).$$

Moreover, from the $\mathfrak{R}$-reversibility of Ψ and definitions (4.57), (4.58) it follows these parameterizations satisfy

$$\mathfrak{R}\,x^u(-t, -s) = x^s(t, s), \qquad \mathfrak{R}\,x^s(-t, -s) = x^u(t, s).$$

We know that our system has a homoclinic solution $x^{(0)}(u)$ when we take $\mu = 0$. Therefore, it seems natural that, when we consider the case $\mu = 0$, the parameterizations $x^s(t, s)$ and $x^u(t, s)$ give rise to a parameterization of $x^{(0)}$. Indeed, from (4.53) we have that

$$x = \Psi^{(0)}(\zeta) = \Omega A_0 \Omega^{-1} \zeta + O(\zeta^2),$$

where

$$\Omega = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \qquad \text{and} \qquad A_0 = c_f\begin{pmatrix} 1+\lambda_0 & 1-\lambda_0 \\ 1-\lambda_0 & 1+\lambda_0 \end{pmatrix},$$

have been defined at (4.39) and (4.36), respectively. This implies that the parameterization provided by $x^s(t,s)$ and $x^u(t,s)$ in the case $\mu = 0$ are

$$x^{s,0}(t,s) = \left(2c_f \mathrm{e}^{-\lambda_0(t+s+c)}, -2c_f\lambda_0 \mathrm{e}^{-\lambda_0(t+s+c)}\right) + O\left(\mathrm{e}^{-2\lambda_0(t+s+c)}\right)$$

and

$$x^{u,0}(t,s) = \left(2c_f \mathrm{e}^{\lambda_0(t+s-c)}, 2c_f\lambda_0 \mathrm{e}^{-\lambda_0(t+s-c)}\right) + O\left(\mathrm{e}^{2\lambda_0(t+s-c)}\right).$$

On the other hand, having in mind that the origin is a hyperbolic point of system (4.1), the asymptotic approximation of the separatrix $x^{(0(t+s))}$ around this equilibrium point are given by

$$x^{(0)}(t+s) = \left(k\mathrm{e}^{\lambda_0(t+s)}, \lambda_0\, k\, \mathrm{e}^{\lambda_0(t+s)}\right) + O\left(\mathrm{e}^{2\lambda_0(t+s)}\right), \qquad t + \Re s \longrightarrow -\infty$$

and

$$x^{(0)}(t+s) = \left(k\mathrm{e}^{-\lambda_0(t+s)}, -\lambda_0\, k\, \mathrm{e}^{-\lambda_0(t+s)}\right) + O\left(\mathrm{e}^{-2\lambda_0(t+s)}\right), \qquad t + \Re s \longrightarrow +\infty.$$

Imposing them to coincide with $x^{s,0}(t,s)$ and $x^{u,0}(t,s)$, it yields to $2\,c_f \mathrm{e}^{-\lambda_0\, c} = k$. Choosing, for instance, $c = 0$ it follows that $c_f = k/2$.

Coming back to the perturbed system, we know that

$$\Psi(\zeta,\theta) = \Psi^{(0)}(\zeta) + \mu\varepsilon^{p+1}P\left(\Psi^{(0)}(\zeta),\theta\right) + O(\mu\varepsilon^{p+2}).$$

Therefore, taking into account that $c = 0$, it follows that

$$\begin{aligned} x^u(t,s) = \Psi\left(\zeta^u(t,s), t/\varepsilon\right) = \Psi^{(0)}\left(\mathrm{e}^{\lambda(t+s)}, \mathrm{e}^{\lambda(t+s)}\right) + \\ \mu\varepsilon^{p+1}P\left(\Psi^{(0)}\left(\mathrm{e}^{\lambda(t+s)}, \mathrm{e}^{\lambda(t+s)}\right), t/\varepsilon\right) + O(\mu\varepsilon^{p+2}). \end{aligned}$$

Since $\lambda = \lambda_0 + O(\mu\varepsilon^{p+2})$, we can assume that $|\lambda - \lambda_0| \le 1$ and then, restricting μ_0 if necessary, to obtain that

$$\left\| \left(\mathrm{e}^{\lambda(t+s)}, \mathrm{e}^{\lambda(t+s)}\right) - \left(\mathrm{e}^{\lambda_0(t+s)}, \mathrm{e}^{\lambda_0(t+s)}\right) \right\| < c'|\lambda - \lambda_0|\mathrm{e}^{t+\Re s},$$

for $t + \Re s \le -T_0$ and being c' a positive constant depending on T_0. Applying this bound to the expression of $x^u(t,s)$ above it yields to

$$\begin{aligned} x^u(t,s) &= \Psi^{(0)}\left(\mathrm{e}^{\lambda_0(t+s)}, \mathrm{e}^{\lambda_0(t+s)}\right) + \mu\varepsilon^{p+1}P\left(\Psi^{(0)}\left(\mathrm{e}^{\lambda_0(t+s)}, \mathrm{e}^{\lambda_0(t+s)}\right), t/\varepsilon\right) + O(\mu\varepsilon^{p+2}) = \\ & x^{(0)}(t+s) + \mu\varepsilon^{p+1}P\left(x^{(0)}(t+s), t/\varepsilon\right) + O(\mu\varepsilon^{p+2}). \end{aligned}$$

Proceeding in a similar way, an equivalent expression for $x^s(t,s)$ is derived and, if we take into account that $\Psi(0,\theta) = \gamma_p(\theta)$, also can be derived 0 (4.11) and (4.12).

§3.3 Proof of the Flow-Box Theorem

It is clear that the change

$$\begin{array}{llcl} \Upsilon : & U_* & \longrightarrow & \mathcal{V} = \Upsilon(U_*) \subseteq \mathbb{C}^2 \\ & (\zeta_1,\zeta_2) & \longmapsto & (S,E) = \left(\dfrac{-\log\left(\frac{\zeta_1-\zeta_2}{2}\right)}{F'\left(\frac{\zeta_1^2-\zeta_2^2}{2}\right)} \ , \ F\left(\frac{\zeta_1^2-\zeta_2^2}{2}\right) \right) \end{array} \tag{4.59}$$

where $U_* = \overline{D_*} \cap \{\zeta_1 - \zeta_2 > 0\}$ and $\overline{D_*}$ defined at (4.9)–(4.10), transforms system (4.7) into the flow-box equations $\dot{S} = 1$, $\dot{E} = 0$.

Remark 20 *This change of variables satisfies that*

$$\det D\Upsilon(\zeta_1,\zeta_2) = -1$$

for any $\zeta = (\zeta_1,\zeta_2) \in U_*$, so it preserves measure but reverses orientation. In fact, if we consider the change

$$(E,S) = \left(F\left(\frac{\zeta_1^2-\zeta_2^2}{2}\right), \frac{-\log\left(\frac{\zeta_1-\zeta_2}{2}\right)}{F'\left(\frac{\zeta_1^2-\zeta_2^2}{2}\right)} \right)$$

instead of the change (4.59)*, we have that this new transformation preserves measure and is $\mathfrak{R}$-symmetric.*

Composing now this change Υ with Ψ, the transformation provided by the Normal Form Theorem, we obtain the transformation (4.14)

$$(\mathcal{S}(x,t/e),\mathcal{E}(x,t/\varepsilon)) = \Upsilon\left(\Psi^{-1}(x,t/\varepsilon)\right).$$

Since $F(I) = F^{(0)}(I) + O(\mu\varepsilon^{p+2})$ we have that $(S,E) = \Upsilon(\zeta_1,\zeta_2)$ is $= O(\mu\varepsilon^{p+2})$-close to $(S^{(0)},E^{(0)}) = \Upsilon(\zeta_1^{(0)},\zeta_2^{(0)})$, the corresponding flow-box transformation for the unperturbed case $\mu = 0$. As a consequence, estimates in (i) are derived. Concerning the inverse of change (4.14), we consider first

$$\begin{array}{llcl} \Xi : & \mathcal{V} & \longrightarrow & U_* \\ & (S,E) & \longmapsto & (\zeta_1,\zeta_2) \end{array}$$

defined by

$$\zeta_1 = 2F^{-1}(E)\,\mathrm{e}^{SF'(F^{-1}(E))} + \mathrm{e}^{-SF'(F^{-1}(E))}, \qquad \zeta_2 = 2F^{-1}(E)\,\mathrm{e}^{SF'(F^{-1}(E))} - \mathrm{e}^{-SF'(F^{-1}(E))}.$$

Composing Ξ with Ψ we obtain $\chi(S,E,t/\varepsilon) = \Psi\left(\Xi(S,E),t/\varepsilon\right)$ which clearly satisfies property (ii). Finally, we want to compute the value of these flow-box functions $\mathcal{S}$ and $\mathcal{E}$ onto the parameterization $x^s(t,s)$ of the stable manifold $W^s(\gamma_p)$. Indeed, we have

$$\left(\mathcal{S}\left(x^s(t,s),t/\varepsilon\right),\mathcal{E}\left(x^s(t,s),t/\varepsilon\right)\right) = \mathcal{V}\left(\Psi^{-1}\left(x^s(t,s),t/\varepsilon\right)\right).$$

From equation (4.57) we know that

$$x^s(t,s) = \Psi\left(\mathrm{e}^{-\lambda(t+s+c)}, -\mathrm{e}^{-\lambda(t+s+c)}, t/\varepsilon\right)$$

so, applying it onto the expression above and using the definition of Υ, it yields to

$$\begin{aligned}&(\mathcal{S}\left(x^s(t,s),t/\varepsilon\right),\mathcal{E}\left(x^s(t,s),t/\varepsilon\right)) = \\ &\quad \mathcal{V}\left(\Psi^{-1}\left(x^s(t,s),t/\varepsilon\right)\right) = \Upsilon\left(\mathrm{e}^{-\lambda(t+s+c)}, -\mathrm{e}^{-\lambda(t+s+c)}, t/\varepsilon\right) = (t+s,0).\end{aligned}$$

Remark 21 *During the Normal Form Theorem and this Flow-Box Theorem, the region considered has evolved along the global stable invariant manifold $W^s(\gamma_p)$. However, because of the reversibility of our system, the same would work if we wanted to define these flow-box around the unstable invariant manifold. In particular, it is straightforward to check that the transformations*

$$\widetilde{\Upsilon}(\zeta) := -\mathfrak{R}\,\Upsilon\left(\mathfrak{R}\,\zeta\right),$$

given explicitly by

$$(\widetilde{S},\widetilde{E}) = \left(\frac{\log\left(\frac{\zeta_1+\zeta_2}{2}\right)}{F'\left(\frac{\zeta_1^2-\zeta_2^2}{2}\right)}\ ,\ F\left(\frac{\zeta_1^2-\zeta_2^2}{2}\right)\right),$$

defined for $\zeta_1+\zeta_2>0$, leads system (4.50) *into the flow-box system*

$$\dot{\widetilde{S}} = 1, \qquad \dot{\widetilde{E}} = 0.$$

Composing $\widetilde{\Upsilon}$ with Ψ^{-1}, we obtain that

$$\left(\widetilde{\mathcal{S}}(x,t/\varepsilon),\widetilde{\mathcal{E}}(x,t/\varepsilon)\right) := \widetilde{\Upsilon}\left(\Psi^{-1}(x,t/\varepsilon)\right)$$

leads system (4.2) *into flow-box coordinates in a region evolving along the unstable invariant manifold $W^u(\gamma_p)$. Moreover, since Ψ^{-1} is $\mathfrak{R}$-reversible (see Lemma* 4.1*(ii)), it turns out that*

$$\begin{aligned}&\left(\widetilde{\mathcal{S}}(x,t/\varepsilon),\widetilde{\mathcal{E}}(x,t/\varepsilon)\right) = \widetilde{\Upsilon}\left(\Psi^{-1}(x,t/\varepsilon)\right) = \\ &\quad -\mathfrak{R}\,\Upsilon\left(\mathfrak{R}\,\Psi^{-1}(x,t/\varepsilon)\right) = -\mathfrak{R}\,\Upsilon\left(\Psi^{-1}(\mathfrak{R}\,x,-t/\varepsilon)\right) = \\ &\quad -\mathfrak{R}\left(\mathcal{S}(\mathfrak{R}\,x,-t/\varepsilon),\mathcal{E}(\mathfrak{R}\,x,-t/\varepsilon)\right) = \left(-\mathfrak{R}\,\mathcal{S}(\mathfrak{R}x,-t/\varepsilon),-\mathfrak{R}\,\mathcal{E}(\mathfrak{R}\,x,-t/\varepsilon)\right).\end{aligned}$$

§3.4 Proof of the Extension Theorem

The proof of this theorem is divided in several subsections and it follows almost exactly the one given by Delshams and Seara at [24].

§3.4.1 Set up

In order to compare the solution $x(t,s)$ of the full system with the homoclinic orbit $x^{(0)}(t+s)$, we introduce the variable

$$\xi(t) = \xi(t,s) := x(t,s) - x^{(0)}(t+s),$$

with $\xi(t,s) = (\xi_1(t,s),\xi_2(t,s))$. Thus, the system of differential equations satisfied by the new variable $\xi(t,s)$ is

$$\left\{\begin{array}{lcl}\dot{\xi}_1 &=& \xi_2 + \mu\varepsilon^p g_1\left(x^{(0)}(t+s)+\xi,t/\varepsilon\right) \\ \dot{\xi}_2 &=& f\left({x_1}^{(0)}(t+s)+\xi_1\right) - f\left({x_1}^{(0)}(t+s)\right) + \mu\varepsilon^p g_2\left(x^{(0)}(t+s)+\xi,t/\varepsilon\right)\end{array}\right. ,$$

whose components can be expanded as

$$\dot{\xi}_1 = \xi_2 + \mu\varepsilon^p g_1\left(x^{(0)}(t+s), t/\varepsilon\right) + \\ \mu\varepsilon^p\left(g_1\left(x^{(0)}(t+s)+\xi, t/\varepsilon\right) - g_1\left(x^{(0)}(t+s), t/\varepsilon\right)\right)$$

and

$$\dot{\xi}_2 = f'\left({x_1}^{(0)}(t+s)\right)\xi_1 + \left(f\left({x_1}^{(0)}(t+s)+\xi_1\right) - f\left({x_1}^{(0)}(t+s)\right) - f'\left({x_1}^{(0)}(t+s)\right)\xi_1\right) + \\ \mu\varepsilon^p\left(g_2\left(x^{(0)}(t+s), t/\varepsilon\right) + \left(g_2\left(x^{(0)}(t+s)+\xi, t/\varepsilon\right) - g_2\left(x^{(0)}(t+s), t/\varepsilon\right)\right)\right).$$

Defining

$$A(u) := \begin{pmatrix} 0 & 1 \\ f'({x_1}^{(0)}(u)) & 0 \end{pmatrix} \tag{4.60}$$

and

$$F(\xi, u, t/\varepsilon) = \begin{pmatrix} 0 \\ f\left({x_1}^{(0)}(u)+\xi_1\right) - f\left({x_1}^{(0)}(u)\right) - f'\left({x_1}^{(0)}(u)\right)\xi_1 \end{pmatrix} + \\ \mu\varepsilon^p \begin{pmatrix} g_1\left(x^{(0)}(u)+\xi, t/\varepsilon\right) - g_1\left(x^{(0)}(u), t/\varepsilon\right) \\ g_2\left(x^{(0)}(u)+\xi, t/\varepsilon\right) - g_2\left(x^{(0)}(u), t/\varepsilon\right) \end{pmatrix} = \begin{pmatrix} F_1(\xi, u, t/\varepsilon) \\ F_2(\xi, u, t/\varepsilon) \end{pmatrix} \tag{4.61}$$

this system admits the following expression

$$\dot{\xi} = A(t+s)\xi + \mu\varepsilon^p G\left(x^{(0)}(t+s), t/\varepsilon\right) + F(\xi, t+s, t/\varepsilon), \tag{4.62}$$

where F depends also on $\mu\varepsilon^p$. Therefore, our problem consists on looking for the solution $\xi(t)$ of system (4.62) with initial condition

$$\xi(t_0) = \mu\varepsilon^{p+1}P\left(x^{(0)}(t_0), t_0/\varepsilon\right) + O(\mu\varepsilon^{p+2}), \tag{4.63}$$

with P defined from equations (4.30). To solve it we proceed in the standard way: we first seek for a solution of the corresponding homogeneous system

$$\frac{d\xi}{du} = A(u)\xi. \tag{4.64}$$

It is clear that $\dot{x}^{(0)}(u)$ is a solution. Therefore, another independent solution can be obtained of the form $\xi_1(u) = {x_2}^{(0)}(u) \cdot W(u)$, $\xi_2 = \xi_1' := d\xi_1/du$ with

$$W(u) = \int_b^u \frac{dv}{{x_2}^{(0)}(v)^2}, \tag{4.65}$$

where $b \in \mathbb{C}$ is a suitable constant to determine later. Introducing now

$$\begin{aligned} \Psi(u) &= {x_2}^{(0)}(u) = \dot{x}_1^{(0)}(u), \\ \Phi(u) &= {x_2}^{(0)}(u)W(u) = \Psi(u)W(u), \end{aligned}$$

it is straightforward to check that

$$M(u)=\begin{pmatrix}\Psi(u) & \Phi(u)\\ \Psi'(u) & \Phi'(u)\end{pmatrix}$$

is a fundamental matrix of system (4.64), verifying that $\det M(u)=1$. Consequently, the fundamental solution $\varphi(u,\sigma)$ of (4.64) satisfying that $\varphi(u,u)=\mathrm{id}$ is given by $\varphi(u,\sigma)=M(u)M(\sigma)^{-1}$, being

$$M(u)^{-1}=\begin{pmatrix}\Phi'(u) & -\Phi(u)\\ -\Psi'(u) & \Psi(u)\end{pmatrix}.$$

Thus, the solution of the non-homogeneous system (4.62) with initial condition (4.63) can be written as

$$\xi(t)=\xi_p(t)+M(t+s)\int_{t_0}^{t}M(\sigma+s)^{-1}F(\xi(\sigma),\sigma+s,\sigma/\varepsilon)\,d\sigma, \tag{4.66}$$

with

$$\xi_p(t):=M(t+s)\left(M(t_0+s)^{-1}\xi(t_0)+\mu\varepsilon^p\int_{t_0}^{t}M(\sigma+s)^{-1}G(x^{(0)}(\sigma+s),\sigma/\varepsilon)\,d\sigma\right).$$

Again, as in [24], since G has zero mean, by hypothesis, it follows the existence of $P(x,\theta)$ (already introduced at (4.30)), with zero mean, and $\mathcal{G}(x,\theta)$ such that they verify

$$\partial_\theta P=G \qquad \text{and} \qquad \partial_\theta\mathcal{G}=P.$$

Thus, defining

$$m(u,\theta):=M(u)^{-1}\mathcal{G}(x^{(0)}(u),\theta), \tag{4.67}$$

the particular solution $\xi_p(t)$ admits the equivalent expression

$$\xi_p(t)=M(t+s)\left(M(t_0+s)^{-1}\xi(t_0)+\mu\varepsilon^p\int_{t_0}^{t}\partial_\theta^2 m(\sigma+s,\sigma/\varepsilon)\,d\sigma\right).$$

Denoting $m'=\partial m/\partial u$ and $\partial_\theta m(u,\theta)=M(u)^{-1}P(x^{(0)}(u),\theta)$, the identity

$$\partial_\theta^2 m(\sigma+s,\sigma/\varepsilon)-\varepsilon^2 m''(\sigma+s,\sigma/\varepsilon)=\frac{d}{d\sigma}\left(\varepsilon\partial_\theta m(\sigma+s,\sigma/\varepsilon)-\varepsilon^2 m'(\sigma+s,\sigma/\varepsilon)\right)$$

is derived. Taking it into account, it turns out the following expression for $\xi_p(t)$,

$$\begin{aligned}\xi_p(t)={}&\mu\varepsilon^{p+1}P\left(x^{(0)}(t+s),t/\varepsilon\right)+\\ &M(t+s)\left(M(t_0+s)^{-1}\left(\xi(t_0)-\mu\varepsilon^{p+1}P\left(x^{(0)}(t_0+s),t_0/\varepsilon\right)\right)-\right.\\ &\left.\mu\varepsilon^{p+2}\left(m'(t+s,t/\varepsilon)-m'(t_0+s,t_0/\varepsilon)-\int_{t_0}^{t}m''(\sigma+s,\sigma/\varepsilon)\,d\sigma\right)\right).\end{aligned} \tag{4.68}$$

§3.4.2 Technical lemmas and estimates

The standard procedure to solve equations of type (4.66) consist on performing an iterative scheme starting from the particular solution $\xi_p(t)$. The convergence of such scheme is based essentially on the natural assumption that the first iterate ξ_p dominates any other approximated solution. This hypothesis is usually supported on numerical simulations.

From a practical point of view, to get such convergence we need to control the size of the vector fields appearing in formula (4.66). Namely, it is important to have a well behaved fundamental matrix $M(u)$ specially when we evolve close to the singularity of $x^{(0)}(u)$. We know by hypothesis, that ${x_2}^{(()}u)$ has a pole of order $r \geq 1$ at the points $u = \pm a\,\mathrm{i}$ so, close to them, it behaves like

$$(4.69) \qquad {x_2}^{(0)}(u) = \dot{x}_1^{(0)}(u) = \frac{C}{(u \mp a\,\mathrm{i})^r}\,(1 + O(u \mp a\,\mathrm{i}))\,, \qquad C \neq 0.$$

This fact lead us to consider our problem restricted to the complex strip $0 \leq |\Im u| \leq a$. For simplicity, let us consider first the case $0 \leq \Im u \leq a$ and leave the case $-a \leq \Im u < 0$ to be discussed later. We will see, in fact, that both situations are analogous. Under this assumption on the domain, estimate (4.69) reads

$$(4.70) \qquad {x_2}^{(0)}(u) = \dot{x}_1^{(0)}(u) = \frac{C}{(u - a\,\mathrm{i})^r}\,(1 + O(u - a\,\mathrm{i}))\,, \qquad C \neq 0.$$

This pole, placed now at $u = a\mathrm{i}$ motivates the choice of b, the undetermined constant appearing at the definition (4.65) of $W(u)$, as $b = a\mathrm{i}$. Thus, $W(u)$ has a zero of multiplicity $2r+1$ at $u = a\mathrm{i}$, ${x_2}^{(0)}(u)W(u)$ has a zero of multiplicity $r+1$ at $u = a\mathrm{i}$ and, therefore,

$$\begin{aligned}
\Psi(u) &= \frac{C}{(u - a\,\mathrm{i})^r}\,(1 + O(u - a\,\mathrm{i})) \\
\Phi(u) &= \frac{1}{(2r+1)C}(u - a\,\mathrm{i})^{r+1}\,(1 + O(u - a\,\mathrm{i}))\,.
\end{aligned}$$

Consequently, close to the pole $u = a\,\mathrm{i}$, it follows that

$$M(u) \sim \begin{pmatrix} \dfrac{C}{(u - a\,\mathrm{i})^r} & \dfrac{1}{(2r+1)C}(u - a\,\mathrm{i})^{r+1} \\ \dfrac{-rC}{(u - a\,\mathrm{i})^{r+1}} & \dfrac{r+1}{(2r+1)C}(u - a\,\mathrm{i})^{r} \end{pmatrix}.$$

In order to express more precisely this behavior, let us introduce before some notation and a suitable norm. First, we recall that t is the real time and s is a complex parameter both ranging over $-T \leq t + \Re s \leq 0$ and $0 < \Im s \leq a - \varepsilon$. We will denote

$$0 < \tau := |t + s - a\,\mathrm{i}| = \left((t + \Re s)^2 + (\Im s - a)^2\right)^{1/2}$$

as well as $\tau_* = |t_* + s - a\,\mathrm{i}|$. Moreover, given $\tau > 0$ and $v(t) = (v_1(t), v_2(t)) \in \mathbb{C}^2$, we define the norm

$$(4.71) \qquad |v(t)|_\tau := |v_1(t)| + |v_2(t)|\,\tau.$$

After these considerations, the results above can be rewritten more precisely as follows.

Lemma 4.6 *For $-T \le t + \Re s,\ t_* + \Re s \le 0$ and $0 \le \Im s < a$ the following bounds hold:*

$$\left|\Psi^{(k)}(t+s)\right| \le \frac{K}{\tau^{r+k}}, \qquad \left|\Phi^{(k)}(t+s)\right| \le K\tau^{r+1-k}, \qquad k = 0,1,2,$$

$$|M(t+s)v(t)|_\tau \le K\left(\frac{|v_1(t)|}{\tau^r} + \tau^{r+1}|v_2(t)|\right), \tag{4.72}$$

$$\left|M(t+s)M(t_*+s)^{-1}v(t_*)\right|_\tau \le K\,|v(t_*)|_{\tau_*}\left(\left(\frac{\tau_*}{\tau}\right)^r + \left(\frac{\tau}{\tau_*}\right)^{r+1}\right), \tag{4.73}$$

for every $v(t) = (v_1(t), v_2(t)) \in \mathbb{C}^2$ and where $K = K(a,T,t_0)$ denotes a generic positive constant independent of μ and ε.

To get bounds for f, g_1 and g_2 in a neighborhood of the singularity, we need previously bounds for these functions on the unperturbed homoclinic. Such a estimates rely strongly on the hypotheses concerning f and the perturbation $G = (g_1, g_2)$. Thus, by the trigonometric polynomial character of f and since $f({x_1}^{(0)}(u)) = \ddot{x_1}^{(0)}(u)$ has a pole of order $r+1$ at $u = a\mathrm{i}$, we obtain for $-T \le t + \Re s \le 0$ and $0 \le \Im s \le a$

$$\left|f^{(m)}\left({x_1}^{(0)}(t+s)\right)\right| \le \frac{K}{\tau^{r+1-m(r-1)}} = \frac{K}{\tau^{2-(m-1)(r-1)}}, \qquad m \ge 0. \tag{4.74}$$

In a similar way, g_1 and g_2 are assumed to be trigonometric polynomials in x_1 and polynomials in x_2. Moreover, from the definition of ℓ_1 and ℓ_2, all the monomials in x of g_1, g_2, when evaluated on $x = x^{(0)}(u)$ have at most a pole of order ℓ_1 and ℓ_2 at $u = a\,\mathrm{i}$, respectively. Therefore, for (t,s) ranging over $-T \le t + \Re s \le 0$, $0 \le \Im s \le a$, one has

$$\left|\partial_1^{m_1}\partial_2^{m_2} g_j\left(x^{(0)}(t+s), t/\varepsilon\right)\right| \le \frac{K}{\tau^{\ell_j - m_1(r-1) - m_2 r}}, \tag{4.75}$$

for $j = 1,2$ and $m_1, m_2 \ge 0$. Using these estimates and applying Taylor's Theorem, the following bounds are readily obtained.

Lemma 4.7 *Let us consider $\xi(t)$, $\xi^*(t) \in \mathbb{C}^2$ such that $|\xi(t)|_\tau, |\xi^*(t)|_\tau \le \eta(\tau) \le \delta/\tau^{r-1}$ with $0 < \delta < 1$. Then, for $-T \le t + \Re s \le 0$, $0 \le \Im s \le a$ and $\varepsilon \le \tau = |t + s - a\,\mathrm{i}| \le T$, the following bounds are satisfied:*

$$\begin{aligned}
\left|f\left({x_1}^{(0)}(t+s) + \xi_1\right) - f\left({x_1}^{(0)}(t+s) + \xi_1^*\right)\right| &\le \frac{K\,|\xi_1 - \xi_1^*|}{\tau^2} \\
\left|g_j\left(x^{(0)}(t+s) + \xi, t/\varepsilon\right) - g_j\left(x^{(0)}(t+s) + \xi^*, t/\varepsilon\right)\right| &\le \frac{K\,|\xi - \xi^*|_\tau}{\tau^{\ell_j - r + 1}} \\
\left|G\left(x^{(0)}(t+s) + \xi, t/\varepsilon\right) - G\left(x^{(0)}(t+s) + \xi^*, t/\varepsilon\right)\right|_\tau &\le \frac{K\,|\xi - \xi^*|_\tau}{\tau^{\ell - r + 1}} \\
\left|F\left(\xi, t+s, t/\varepsilon\right) - F\left(\xi^*, t+s, t/\varepsilon\right)\right|_\tau &\le K\left(\frac{\mu\varepsilon^p}{\tau^{\ell - r + 1}} + \frac{\eta(\tau)}{\tau^{2-r}}\right)|\xi - \xi^*|_\tau,
\end{aligned}$$

where $j = 1,2$ and $\ell = \max\{\ell_1, \ell_2\}$.

Proof. We proceed separately. Thus,

- Expanding in Taylor series up to first order and taking into account estimate (4.74) it yields

$$\left|f\left({x_1}^{(0)}(t+s)+\xi_1\right)-f\left({x_1}^{(0)}(t+s)+\xi_1^*\right)\right|\le \left|f'\left({x_1}^{(0)}(t+s)+\alpha\xi+(1-\alpha)\xi^*\right)\right||\xi_1-\xi_1^*|\le\frac{K|\xi_1-\xi_1^*|}{\tau^2},$$

where $\alpha\in(0,1)$.

- Dealing as before,

$$\left|g_j\left(x^{(0)}(t+s)+\xi,t/\varepsilon\right)-g_j\left(x^{(0)}(t+s)+\xi^*,t/\varepsilon\right)\right|\le \left|\partial_1 g_j\left(x^{(0)}(t+s)+\alpha_1\xi+(1-\alpha_1)\xi^*,t/\varepsilon\right)\right||\xi_1-\xi_1^*|+ \left|\partial_2 g_j\left(x^{(0)}(t+s)+\alpha_2\xi+(1-\alpha_2)\xi^*,t/\varepsilon\right)\right||\xi_2-\xi_2^*|\le \frac{K|\xi_1-\xi_1^*|}{\tau^{\ell_j-(r-1)}}+\frac{K|\xi_2-\xi_2^*|}{\tau^{\ell_j-r}}=\frac{K|\xi_1-\xi_1^*|}{\tau^{\ell_j-(r-1)}}+\frac{K|\xi_2-\xi_2^*|\tau}{\tau^{\ell_j-r+1}}=\frac{K|\xi-\xi^*|_\tau}{\tau^{\ell_j-r+1}},$$

with $\alpha_1,\alpha_2\in(0,1)$ and ∂_1, ∂_2 meaning $\partial/\partial x_1$ and $\partial/\partial x_2$, respectively.

- From the previous bound it follows

$$\left|G\left(x^{(0)}(t+s)+\xi,t/\varepsilon\right)-G\left(x^{(0)}(t+s)+\xi^*,t/\varepsilon\right)\right|_\tau= \left|g_1\left(x^{(0)}(t+s)+\xi,t/\varepsilon\right)-g_1\left(x^{(0)}(t+s)+\xi^*,t/\varepsilon\right)\right|+ \left|g_2\left(x^{(0)}(t+s)+\xi,t/\varepsilon\right)-g_2\left(x^{(0)}(t+s)+\xi^*,t/\varepsilon\right)\right|\tau\le \frac{K|\xi-\xi^*|_\tau}{\tau^{\ell_1-r+1}}+\frac{K|\xi-\xi^*|_\tau}{\tau^{\ell_2-r+1}}\tau\le K\left(\frac{T^{\ell_2-r}+T^{\ell_1-r+1}}{\tau^{\ell-r+1}}\right)|\xi-\xi^*|_\tau\le\frac{K|\xi-\xi^*|_\tau}{\tau^{\ell-r+1}}.$$

- With respect to the first component, using the previous bounds, one has

$$|F_1(\xi,t+s,t/\varepsilon)-F_1(\xi^*,t+s,t/\varepsilon)|= \mu\varepsilon^p\left|\left(g_1\left(x^{(0)}(t+s)+\xi,t/\varepsilon\right)-g_1\left(x^{(0)}(t+s),t/\varepsilon\right)\right)-\left(g_1\left(x^{(0)}(t+s)+\xi^*,t/\varepsilon\right)-g_1\left(x^{(0)}(t+s),t/\varepsilon\right)\right)\right|= \mu\varepsilon^p\left|g_1\left(x^{(0)}(t+s)+\xi,t/\varepsilon\right)-g_1\left(x^{(0)}(t+s)+\xi^*,t/\varepsilon\right)\right|\le\mu\varepsilon^p\frac{K|\xi-\xi^*|_\tau}{\tau^{\ell_1-r+1}}.$$

Proceeding analogously for the second one, it follows that

$$|F_2(\xi,t+s,t/\varepsilon)-F_2(\xi^*,t+s,t/\varepsilon)|\le\frac{K|\xi_1-\xi_1^*|}{\tau^2}+\mu\varepsilon^p\frac{K|\xi-\xi^*|_\tau}{\tau^{\ell_2-r+1}}.$$

Therefore,

$$
\begin{aligned}
&|F(\xi,t+s,t/\varepsilon)-F(\xi^*,t+s,t/\varepsilon)|_\tau = \\
&\quad |F_1(\xi,t+s,t/\varepsilon)-F_1(\xi^*,t+s,t/\varepsilon)|+|F_2(\xi,t+s,t/\varepsilon)-F_2(\xi^*,t+s,t/\varepsilon)|\,\tau \le \\
&\quad \mu\varepsilon^p\frac{K\,|\xi-\xi^*|_\tau}{\tau^{\ell_1-r+1}}+\left(\frac{K\,|\xi_1-\xi_1^*|}{\tau^2}+\mu\varepsilon^p\frac{K\,|\xi-\xi^*|_\tau}{\tau^{\ell_2-r+1}}\right)\tau \le \\
&\quad K\mu\varepsilon^p\left(\frac{1}{\tau^{\ell_1-r+1}}+\frac{1}{\tau^{\ell_2-r}}\right)|\xi-\xi^*|_\tau+K\frac{|\xi_1-\xi_1^*|}{\tau} \le \\
&\quad K\mu\varepsilon^p\left(\frac{1}{\tau^{\ell-r+1}}\right)|\xi-\xi^*|_\tau+K\frac{|\xi_1-\xi_1^*|}{\tau}
\end{aligned}
$$

Using that

$$
K\frac{|\xi_1-\xi_1^*|}{\tau}\le K\frac{|\xi_1-\xi_1^*|_\tau}{\tau}=\frac{K}{\delta}\;\frac{|\xi_1-\xi_1^*|_\tau}{\tau^{2-r}}\;\frac{\delta}{\tau^{r-1}}=K\,\frac{\eta(\tau)}{\tau^{2-r}}\;|\xi_1-\xi_1^*|_\tau\,,
$$

the previous expression is bounded by

$$
K\left(\frac{\mu\varepsilon^p}{\tau^{\ell-r+1}}+\frac{\eta(\tau)}{\tau^{2-r}}\right)|\xi-\xi^*|_\tau\,,
$$

as the lemma claims.

□

Finally, we present the last technical lemma which will be used later.

Lemma 4.8 *For t,t_0,β real and s complex, such that*

$$
0\le\Im s<a,\qquad -T\le t_0+\Re s\le t+\Re s\le 0,
$$

let us denote

$$
\rho^{-\beta}_{[t_0,t]}(s):=\begin{cases}\sup\dfrac{1}{|\sigma+s-a\mathrm{i}|^\beta} & \textit{if } \beta\ne 0\\ \sup|\log(|\sigma+s-a\mathrm{i}|)| & \textit{if } \beta=0\end{cases},
$$

where the supremum is taken on $\sigma\in[t_0,t]$. Then, there exists $K=K(a,t_0,T,\beta)>0$ such that the following inequalities hold

$$
\int_{t_0}^{t}\frac{d\sigma}{|\sigma+s-a\mathrm{i}|^\beta}\ \le\ K\,\rho^{-(\beta-1)}_{[t_0,t]}(s) \tag{4.76}
$$

$$
\left|M(t+s)\int_{t_0}^{t}M(\sigma+s)^{-1}v(\sigma)\,d\sigma\right|_\tau\ \le\ KC\left(\frac{\rho^{-(\beta-r-1)}_{[t_0,t]}(s)}{\tau^r}+\tau^{r+1}\rho^{-(\beta+r)}_{[t_0,t]}(s)\right), \tag{4.77}
$$

where we recall that $\tau=|t+s-a\mathrm{i}|$ and $v(t)\in\mathbb{C}^2$ verifies that

$$
|v(t)|_\tau\le\frac{C}{\tau^\beta}. \tag{4.78}
$$

As it also mentioned in [24], the proof of (4.76) is straightforward and can be found in [23, Lemma 7.1] for the case $\beta=3$ and $a=\pi/2$. Estimate (4.77) follows from the third bound in Lemma 4.6 and bound (4.76), provided (4.78) is satisfied.

§3.4.3 Convergence of the recurrent scheme

Let us define

$$\mathcal{D} = \{(t,s) : \; -T \le t + \Re s \le 0, \; 0 \le \Im s \le a - \varepsilon\}.$$

We will prove (with very small changes from [24]) the convergence in this domain of an extended solution $\xi(t)$ of equation (4.62) with initial condition (4.63). Precisely, we have

Proposition 4.5 [[24, Prop. 4.4]] *Given $s \in \mathbb{C}$ such that $0 \le \Im s \le a - \varepsilon$, let $\xi(t) = \xi(t,s)$ be a solution of system* (4.62) *with initial condition* (4.63) *on $t_0 = -T - \Re s$. Then, if $\gamma = p - \ell \ge -1$ (where $\ell = \max\{\ell_1, \ell_2\}$) this solution $\xi(t)$ can be extended for $t \in [t_0, -\Re s]$, satisfying there the following estimates:*

$$\left|\xi(t) - \mu\varepsilon^{p+1} P(x^{(0)}(t+s), t/\varepsilon)\right|_\tau \le K \frac{\mu\varepsilon^{\gamma+r+1}}{\tau^r} \tag{4.79}$$

$$|\xi(t)|_\tau \le K\mu\varepsilon^{\gamma+1}. \tag{4.80}$$

Proof. The idea is to perform a recurrent scheme providing successive approximations. We start with $\xi^{(0)}(t)$ and obtain the following iterates from

$$\xi^{(n+1)}(t) = \xi_p(t) + M(t+s) \int_{t_0}^{t} M(\sigma+s)^{-1} F(\xi^{(n)}(\sigma), \sigma+s, \sigma/\varepsilon)\, d\sigma. \tag{4.81}$$

Notice that $\xi^{(1)}(t) = \xi_p(t)$. As it has been already mentioned, the idea of the proof is to check that the first iterate $\xi^{(1)}(t) = \xi_p(t)$ satisfies the bound (4.79) and, later, that it dominates the rest of the iterates in such a way that the bound remains valid.

Indeed, let us recall formula (4.68), where an explicit expression for $\xi_p(t)$ was given,

$$\begin{aligned}\xi_p(t) = \mu\varepsilon^{p+1} P\left(x^{(0)}(t+s), t/\varepsilon\right) + \\ M(t+s)\left(M(t_0+s)^{-1}\left(\xi(t_0) - \mu\varepsilon^{p+1}P\left(x^{(0)}(t_0+s), t_0/\varepsilon\right)\right) - \right.\\ \left.\mu\varepsilon^{p+2}\left(m'(t+s,t/\varepsilon) - m'(t_0+s,t_0/\varepsilon) - \int_{t_0}^{t} m''(\sigma+s,\sigma/\varepsilon)\,d\sigma\right)\right).\end{aligned}$$

We proceed to estimate it piece to piece. First, from the initial condition (4.63) and using that $t_0 + s = -T + \Im s$ is far from the singularity, it follows that

$$M(t_0+s)^{-1}\left(\xi(t_0) - \mu\varepsilon^{p+1}P\left(x^{(0)}(t_0+s), t_0/\varepsilon\right)\right) = O(\mu\varepsilon^{p+2})$$

and, taking into account estimate (4.72), that

$$\left|M(t+s)M(t_0+s)^{-1}\left(\xi(t_0) - \mu\varepsilon^{p+1}P\left(x^{(0)}(t_0+s), t_0/\varepsilon\right)\right)\right|_\tau \le K\mu\varepsilon^{p+2}\left(\frac{1}{\tau^r} + \tau^{r+1}\right).$$

With respect to

$$\mu\varepsilon^{p+2}M(t+s)\left(m'(t+s,t/\varepsilon) - m'(t_0+s,t_0/\varepsilon) - \int_{t_0}^{t} m''(\sigma+s,\sigma/\varepsilon)\,d\sigma\right)$$

we use definition (4.67), that is $m(u,\theta) := M(u)^{-1}\mathcal{G}(x^{(0)}(u),\theta))$, and the fact that $(M^{-1})' = -M^{-1}A$ (since M is fundamental matrix of $\xi' = d\xi/du = A(u)\xi$). Therefore, we have $m' = M^{-1}(\mathcal{G}' - A\mathcal{G}) =: M^{-1}\mathcal{G}_1$ and $m'' = M^{-1}(\mathcal{G}_1' - A\mathcal{G}_1) =: M^{-1}\mathcal{G}_2$. Using that P and $\mathcal{G}$ verify the same estimates of type Lemma 4.7 as G and having in mind the definitions of $\mathcal{G}_1$, $\mathcal{G}_2$ and A (in (4.60)), the following estimates can be derived

$$|P|_\tau \le \frac{K}{\tau^\ell}, \quad |\mathcal{G}|_\tau \le \frac{K}{\tau^\ell}, \quad |\mathcal{G}_1|_\tau \le \frac{K}{\tau^{\ell+1}}, \quad |\mathcal{G}_2|_\tau \le \frac{K}{\tau^{\ell+2}}, \tag{4.82}$$

for $-T \le t + \Re s \le 0$, $0 \le \Im s < a$ and where K means a generic positive constant independent of μ and ε. Thus, using (4.82) as well as estimates (4.73) and (4.77) (for $\beta = \ell + 2$ and $C = \mu\varepsilon^{p+2}$), it follows that

$$\begin{aligned} &\left|\xi_p(t) - \mu\varepsilon^{p+1}P(x^{(0)}(t+s), t/\varepsilon)\right|_\tau \le \\ &\quad K\mu\varepsilon^{p+2}\left(\frac{1}{\tau^r} + \tau^{r+1} + \frac{1}{\tau^{\ell+1}} + \frac{1}{\tau^r} + \tau^{r+1} + \frac{\rho_0^{-(\ell-r+1)}}{\tau^r} + \tau^{r+1}\rho_0^{-(\ell+r+2)}\right)\end{aligned} \tag{4.83}$$

where the functions $\rho_0^{-\beta} = \rho_{[t_0,t]}^{-\beta}(s)$ have been defined at Lemma 4.8. Their bounds depend on the value of the exponent β so, in our case, we have

$$\begin{aligned} \rho_0^{-(\ell+r+2)} &\le \tau^{-(\ell+r+2)} \\ \rho_0^{-(\ell-r+1)} &\le \begin{cases} \tau^{-(\ell-r+1)} & \text{if } \ell - r + 1 > 0 \\ |\log\tau| & \text{if } \ell - r + 1 = 0 \\ K & \text{if } \ell - r + 1 < 0 \end{cases}\end{aligned} \tag{4.84}$$

Thus, inequality (4.83) becomes

$$\begin{aligned} &\left|\xi_p(t) - \mu\varepsilon^{p+1}P(x^{(0)}(t+s), t/\varepsilon)\right|_\tau \tau^r \le \\ &\quad K\mu\varepsilon^{p+2}\left(1 + \frac{1}{\tau^{\ell-r+1}} + \rho_0^{-(\ell-r+1)} + \tau^{2r+1}\rho_0^{-(\ell+r+2)}\right).\end{aligned} \tag{4.85}$$

Using that $\tau \ge \varepsilon$ and estimates (4.84) we obtain that

$$\rho_0^{-(\ell-r+1)} \le |\log\varepsilon|, \qquad \tau^{2r+1}\rho_0^{-(\ell+r+2)} \le \tau^{2r+1}\tau^{-(\ell+r+2)} = \frac{1}{\tau^{\ell-r+1}} \le \frac{1}{\varepsilon^{\ell-r+1}},$$

which applied onto (4.85) yield to

$$\begin{aligned} &\left|\xi_p(t) - \mu\varepsilon^{p+1}P(x^{(0)}(t+s), t/\varepsilon)\right|_\tau \tau^r \le \\ &\quad K\mu\varepsilon^{p+2}\left(1 + |\log\varepsilon| + \frac{1}{\varepsilon^{\ell-r+1}}\right) \le K\mu\varepsilon^{p+2-\ell+r-1} = K\mu\varepsilon^{\gamma+r+1},\end{aligned} \tag{4.86}$$

where $\gamma = p - \ell$. If we introduce the norm

$$\|\xi\|_r := \sup|\xi(t)|_\tau \tau^r,$$

where the supremum is taken for $t \in [t_0, -\Re s]$, the previous expression can we written as

$$\left\|\xi_p - \mu\varepsilon^{p+1}P\right\|_r \le K\mu\varepsilon^{\gamma+r+1}. \tag{4.87}$$

Lemma 4.9 *Let us consider the sequence* $\left\{\xi^{(n)}\right\}_n$ *of approximative solutions provided by the formula* (4.81)*. Assume that* $\xi^{(1)} = \xi_p$ *satisfies the bound* (4.87)*. Then, if* $\gamma = p - \ell \geq -1$ *and* μ *small enough, it follows that* $\left\{\xi^{(n)}\right\}_n$ *converges uniformly to the solution* ξ *of* (4.62) *in* $[t_0, -\Re s]$*. Moreover, in this domain the following bound holds:*

$$\left\|\xi - \mu\varepsilon^{p+1}P\right\|_r \leq K\mu\varepsilon^{\gamma+r+1}.$$

Proof. We will prove first that

$$\left\|\xi^{(n)} - \mu\varepsilon^{p+1}P\right\|_r \leq K\mu\varepsilon^{\gamma+r+1}$$

holds for any $n \geq 1$. We proceed inductively. For $n = 1$ is clear from equation (4.86), so let us assume that for $k = 1, 2, \ldots, n$ we have

$$\left\|\xi^{(k)} - \mu\varepsilon^{p+1}P\right\|_r \leq K\mu\varepsilon^{\gamma+r+1}.$$

Using that $|P|_\tau \leq K/\tau^\ell$ it follows that

$$\begin{aligned}\left|\xi^{(k)}\right|_\tau &\leq \left|\xi^{(k)} - \mu\varepsilon^{p+1}P\right|_\tau + \mu\varepsilon^{p+1}\left|P\right|_\tau \leq \\ &\frac{1}{\tau^r}\left\|\xi^{(k)} - \mu\varepsilon^{p+1}P\right\|_r + \mu\varepsilon^{p+1}\left|P\right|_\tau \leq \frac{K\mu\varepsilon^{\gamma+r+1}}{\tau^r} + \frac{K\mu\varepsilon^{p+1}}{\tau^\ell} =: \eta(\tau).\end{aligned}$$

Notice that, since $\tau \geq \varepsilon$ one obtains

$$\eta(\tau) = \frac{K\mu\varepsilon^{\gamma+1}}{\tau^r} + \frac{K\mu\varepsilon^{p+1}}{\tau^\ell} = \frac{K\mu}{\tau^{r-1}}\left(\frac{\varepsilon^{\gamma+r+1}}{\tau} + \frac{\varepsilon^{p+1}}{\tau^{\ell-r+1}}\right) \leq \frac{K\mu\varepsilon^{\gamma+r}}{\tau^{r-1}}.$$

So, therefore, $\eta(\tau) \leq \delta/\tau^{r-1}$ with $\delta = K\mu\varepsilon^{\gamma+r} < 1$ and we can apply the last estimate of Lemma 4.7 for $\xi = \xi^{(n)}$ and $\xi^* = \xi^{(n-1)}$. Thus,

$$\begin{aligned}&\left|F\left(\xi^{(n)}(t), t+s, t/\varepsilon\right) - F\left(\xi^{(n-1)}(t), t+s, t/\varepsilon\right)\right|_\tau \leq \\ &K\mu\left(\frac{\varepsilon^p}{\tau^{\ell-r+1}} + \frac{\varepsilon^{p+1}}{\tau^{\ell+(2-r)}} + \frac{\varepsilon^{\gamma+r+1}}{\tau^{r+(2-r)}}\right)\left|\xi^{(n)} - \xi^{(n-1)}\right|_\tau \leq \\ &K\mu\left(\frac{\varepsilon^p}{\tau^{\ell+1}} + \frac{\varepsilon^{p+1}}{\tau^{\ell+2}} + \frac{\varepsilon^{\gamma+r+1}}{\tau^{r+2}}\right)\left\|\xi^{(n)} - \xi^{(n-1)}\right\|_r \leq \\ &K\mu\left(\frac{\varepsilon^p}{\tau^{\ell+1}} + \frac{\varepsilon^{\gamma+r+1}}{\tau^{r+2}}\right)\left\|\xi^{(n)} - \xi^{(n-1)}\right\|_r.\end{aligned}$$

Applying twice Lemma 4.8 for $\beta = \ell + 1$ and $C = \varepsilon^p$ and $\beta = r + 2$ and $C = \varepsilon^{\gamma+r+1}$, respectively, we get

$$\begin{aligned}&\left|M(t+s)\int_{t_0}^{t} M(\sigma+s)^{-1}\left(F\left(\xi^{(n)}(t), t+s, t/\varepsilon\right) - F\left(\xi^{(n-1)}(t), t+s, t/\varepsilon\right)\right)d\sigma\right|_\tau \leq \\ &\left(K\mu\varepsilon^p\left(\frac{\rho_0^{(\ell-r)}}{\tau^r} + \tau^{r+1}\rho_0^{-(\ell+r+1)}\right) + \right. \\ &\left. K\mu\varepsilon^{\gamma+r+1}\left(\frac{\rho_0^{-(1)}}{\tau^r} + \tau^{r+1}\rho_0^{-(2r+2)}\right)\right)\left\|\xi^{(n)} - \xi^{(n-1)}\right\|_r.\end{aligned}$$

Multiplying by τ^r, taking the supremum on $[t_0, -\Re s]$, using estimates of type (4.84) and that $\gamma \geq -1$ (and, therefore, $\gamma + r \geq 0$), it follows

$$\begin{aligned}
&\left\|\xi^{(n+1)} - \xi^{(n)}\right\|_r \leq \\
&\quad \left\|M(t+s)\int_{t_0}^{t} M(\sigma+s)^{-1}\left(F\left(\xi^{(n)}(t), t+s, t/\varepsilon\right) - F\left(\xi^{(n-1)}(t), t+s, t/\varepsilon\right)\right) d\sigma\right\|_r \leq \\
&\quad K\mu\left(\varepsilon^p\left(|\log\varepsilon| + \frac{1}{\varepsilon^{\ell-r}}\right) + \varepsilon^{\gamma+r+1}\frac{1}{\varepsilon}\right)\left\|\xi^{(n+1)} - \xi^{(n)}\right\|_r \leq \\
&\quad K\mu\left(\varepsilon^{p-(\ell-r)} + \varepsilon^{\gamma+r}\right)\left\|\xi^{(n+1)} - \xi^{(n)}\right\|_r \leq K\mu\varepsilon^{\gamma+r}\left\|\xi^{(n+1)} - \xi^{(n)}\right\|_r.
\end{aligned}$$

Taking $\mu_0 > 0$ small enough, it follows that for $n \geq 1$ and $|\mu| \leq \mu_0$,

$$\begin{aligned}
\left\|\xi^{(n+1)} - \xi^{(n)}\right\|_r &\leq \frac{1}{2}\left\|\xi^{(n)} - \xi^{(n-1)}\right\|_r \\
\left\|\xi^{(n)} - \mu\varepsilon^{p+1}P\right\|_r &\leq 2\left\|\xi^{(1)} - \mu\varepsilon^{p+1}P\right\|_r \leq K\mu\varepsilon^{\gamma+r+1}.
\end{aligned}$$

As a consequence, the sequence $\left\{\xi^{(n)}\right\}_n$ converges uniformly on $[t_0, -\Re s]$ to the solution $\xi(t)$ of (4.62) with initial condition (4.63). Moreover, in such domain, the estimate

$$\left\|\xi - \mu\varepsilon^{p+1}P\right\|_r \leq K\mu\varepsilon^{\gamma+r+1}$$

holds and, therefore, also (4.79). The bound (4.80) comes from

$$\begin{aligned}
|\xi(t)|_r &\leq \frac{K\mu\varepsilon^{p+1}}{\tau^\ell} + \frac{K\mu\varepsilon^{\gamma+r+1}}{\tau^r} \leq \\
&\quad K\mu\varepsilon^{p+1}\varepsilon^{-\ell} + K\mu\varepsilon^{\gamma+r+1}\varepsilon^{-r} \leq K\mu\varepsilon^{\gamma+1}.
\end{aligned}$$

□

This proves the Extension Theorem for $0 \leq \Im s \leq a - \varepsilon$. The proof for $-a + \varepsilon \leq \Im s \leq 0$ it works analogously provided we choose $b = -a\,\mathrm{i}$ at the definition of $W(u)$.

§3.5 Proof of Proposition 4.1

(*i*) From Corollary (4.2) we know that

$$\mathcal{E}(x^s(t,s), t/\varepsilon) = 0, \qquad \mathcal{E}(x,\theta) = \mathcal{E}^{(0)}(x) + O(\mu\varepsilon^{p+1}) = h^{(0)}(x) + O(\mu\varepsilon^{p+1})$$

so, therefore, it follows that

$$\begin{aligned}
\mathcal{E}^u(s) &= \mathcal{E}(x^u(t,s), t/\varepsilon) = \\
&\quad \mathcal{E}(x^u(t,s), t/\varepsilon) - \mathcal{E}(x^s(t,s), t/\varepsilon) = \\
&\quad h^{(0)}(x^u(t,s)) - h^{(0)}(x^s(t,s)) + O(\mu\varepsilon^{p+1}),
\end{aligned} \tag{4.88}$$

for $-T_1 \leq t + \Re s \leq 0$, $|\Im s| \leq a - \varepsilon$ and $x^u(t,s)$, $x^s(t,s)$ belonging to $\mathcal{R}$. Since $\mathcal{E}^u(s)$, does not depend on t, we can choose any value in the previous domain. In our case, for a given $s \in \mathbb{C}$ we consider $T_s := -\Re s$, defined in such a way that $T_s + \Re s = 0$ (that is, we are at the symmetric homoclinic point). As in [24], we consider the functions

$$\begin{aligned}\Delta^u(t,s) &:= h^{(0)}(x^u(t,s)) - h^{(0)}(\gamma_p(t/\varepsilon)),\\ \Delta^s(t,s) &:= h^{(0)}(x^s(t,s)) - h^{(0)}(\gamma_p(t/\varepsilon)).\end{aligned}$$

Lemma 4.10 *The functions Δ^u and Δ^s defined above satisfy the following properties:*

(a) $\lim_{t\mapsto+\infty} \Delta^s(t,s) = 0$ *and* $\lim_{t\mapsto-\infty} \Delta^u(t,s) = 0$.

(b)

$$\begin{aligned}\partial_t\Delta^u(t,s) &= \mu\varepsilon^p \left(\mathcal{L}_G h^{(0)}(x^u(t,s), t/\varepsilon) - \mathcal{L}_G h^{(0)}(\gamma_p(t/\varepsilon), t/\varepsilon)\right),\\ \partial_t\Delta^s(t,s) &= \mu\varepsilon^p \left(\mathcal{L}_G h^{(0)}(x^s(t,s), t/\varepsilon) - \mathcal{L}_G h^{(0)}(\gamma_p(t/\varepsilon), t/\varepsilon)\right),\end{aligned}$$

where we recall that

$$\mathcal{L}_G f(x,\theta) = (\nabla f(x,\theta))^\top G(x,\theta)$$

is the Lie derivative of the function f with respect to the vector field G.

(c) $h^{(0)}(x^u(t,s)) - h^{(0)}(x^s(t,s)) = \Delta^u(t,s) - \Delta^s(t,s)$.

Proof. Since (a) and (c) are clear, we will just deal with (b). Indeed, using that $x^u(t,s)$ and $\gamma_p(t/\varepsilon)$ are solutions of system (4.2), we have that

$$\begin{aligned}&\partial_t\Delta^u(t,s) = \partial_t h^{(0)}(x^u(t,s)) - \partial_t h^{(0)}(\gamma_p(t/\varepsilon)) =\\ &\quad \nabla h^{(0)}(x^u(t,s))\,\partial_t x^u(t,s) - \nabla h^{(0)}(\gamma_p(t/\varepsilon))\,\partial_t\left(\gamma_p(t/\varepsilon)\right) =\\ &\quad \left(\nabla h^{(0)}(x^u(t,s))\right)^\top \left(J\nabla h^{(0)}(x^u(t,s)) + \mu\varepsilon^p G(x^u(t,s), t/\varepsilon)\right) -\\ &\quad \left(\nabla h^{(0)}(\gamma_p(t/\varepsilon))\right)^\top \left(J\nabla h^{(0)}(\gamma_p(t/\varepsilon)) + \mu\varepsilon^p G(\gamma_p(t/\varepsilon), t/\varepsilon)\right) =\\ &\quad \mu\varepsilon^p\left(\left(\nabla h^{(0)}(x^u(t,s))\right)^\top G(x^u(t,s), t/\varepsilon) - \left(\nabla h^{(0)}(\gamma_p(t/\varepsilon))\right)^\top G(\gamma_p(t/\varepsilon), t/\varepsilon)\right) =\\ &\quad \mu\varepsilon^p\left(\mathcal{L}_G h^{(0)}(x^u(t,s), t/\varepsilon) - \mathcal{L}_G h^{(0)}(\gamma_p(t/\varepsilon), t/\varepsilon)\right).\end{aligned}$$

In a similar way, one obtains that

$$\partial_t\Delta^s(t,s) = \mu\varepsilon^p\left(\mathcal{L}_G h^{(0)}(x^s(t,s), t/\varepsilon) - \mathcal{L}_G h^{(0)}(\gamma_p(t/\varepsilon), t/\varepsilon)\right).$$

□

Taking $t = T_s$ and applying this lemma onto (4.88) we have

$$\begin{aligned}&h^{(0)}(x^u(T_s,s))h^{(0)}(x^s(T_s,s)) =\\ &\quad \Delta^u(T_s,s) - \Delta^s(T_s,s) \int_{-\infty}^{T_s} \partial_t\Delta^u(t,s)\,dt + \int_{T_s}^{\infty} \partial_t\Delta^s(t,s)\,dt =\\ &\quad \mu\varepsilon^p\left(\int_{-\infty}^{T_s}\left(\mathcal{L}_G h^{(0)}(x^u(t,s), t/\varepsilon) - \mathcal{L}_G h^{(0)}(\gamma_p(t/\varepsilon), t/\varepsilon)\right)dt + \right.\\ &\quad \left.\int_{T_s}^{+\infty}\left(\mathcal{L}_G h^{(0)}(x^s(t,s), t/\varepsilon) - \mathcal{L}_G h^{(0)}(\gamma_p(t/\varepsilon), t/\varepsilon)\right)dt.\right)\end{aligned}$$

Adding and subtracting the Melnikov function

$$M(s,\varepsilon) = \int_{-\infty}^{+\infty} \mathcal{L}_G h^{(0)}(x^{(0)}(t+s), t/\varepsilon)\, dt$$

the last expression becomes

$$\begin{aligned} &h^{(0)}(x^u(T_s,s))h^{(0)}(x^s(T_s,s)) = \\ &\quad \mu\varepsilon^p \int_{-\infty}^{T_s} \left(\mathcal{L}_G h^{(0)}(x^u(t,s), t/\varepsilon) - \mathcal{L}_G h^{(0)}(\gamma_p, t/\varepsilon) - \mathcal{L}_G h^{(0)}(x^{(0)}(t+s), t/\varepsilon) \right) dt + \\ &\quad \mu\varepsilon^p \int_{T_s}^{+\infty} \left(\mathcal{L}_G h^{(0)}(x^s(t,s), t/\varepsilon) - \mathcal{L}_G h^{(0)}(\gamma_p, t/\varepsilon) - \mathcal{L}_G h^{(0)}(x^{(0)}(t+s), t/\varepsilon) \right) dt + \\ &\quad \mu\varepsilon^p M(s,\varepsilon). \end{aligned}$$

Let us bound them. Using Corollary 4.1(*iv*), the second integral is $O(\mu\varepsilon^{p+1})$. With respect to the first one, it can be divided between the integral over $(-\infty, -T_0]$ and $[-T_0, T_s]$, where T_0 was introduced in the Corollary 4.1. Following again the same point (*iv*), it follows that the first integral, considered over $(-\infty, -T_0]$ is $O(\mu\varepsilon^{p+1})$. In order to estimate the integral over $[-T_0, T_s]$ we need the following result.

Lemma 4.11 *For $s \in \mathbb{C}$ such that $|\Im a| \leq a - \varepsilon$ and $T_s = -\Re s$, we have that*

$$\int_{-T_0}^{T_s} \left(\mathcal{L}_G h^{(0)}(x^u, t/\varepsilon) - \mathcal{L}_G h^{(0)}(\gamma_p, t/\varepsilon) - \mathcal{L}_G h^{(0)}(x^{(0)}, t/\varepsilon) \right) dt = O\left(\mu\varepsilon^{\gamma-\ell}, \mu\varepsilon^{p+1} \right),$$

where x^u, $x^{(0)}$ and γ_p denote $x^u(t,s)$, $x^{(0)}(t+s)$ and $\gamma_p(t/\varepsilon)$, respectively.

Proof. We will bound this integral in two parts:

$$\int_{-T_0}^{T_s} \left(\mathcal{L}_G h^{(0)}(x^u, t/\varepsilon) - \mathcal{L}_G h^{(0)}(x^{(0)}, t/\varepsilon) \right) dt$$

and

$$\int_{-T_0}^{T_s} \mathcal{L}_G h^{(0)}(\gamma_p, t/\varepsilon)\, dt.$$

Concerning the first one, we have that

$$\mathcal{L}_G h^{(0)}(x, t/\varepsilon) = \left(\nabla h^{(0)}(x) \right)^\top G(x, t/\varepsilon) = -f(x_1) g_1(x, t/\varepsilon) + x_2 g_2(x, t/\varepsilon).$$

Thus, we can write

$$\begin{aligned} &\mathcal{L}_G h^{(0)}(x^u(t,s), t/\varepsilon) - \mathcal{L}_G h^{(0)}(x^{(0)}(t+s), t/\varepsilon) = \\ &\quad -f(x_1^u) g_1(x^u, t/\varepsilon) + x_2^u g_2(x^u, t/\varepsilon) + f(x_1^{(0)}) g_1(x^{(0)}, t/\varepsilon) - x_2^{(0)} g_2(x^{(0)}, t/\varepsilon) = \\ &\quad -f(x_1^u) \left(g_1(x^u, t/\varepsilon) - g_1(x^{(0)}, t/\varepsilon) \right) + g_1(x^{(0)}, t/\varepsilon) \left(f(x_1^{(0)}) - f(x_1^u) \right) + \\ &\quad x_2^u \left(g_2(x^u, t/\varepsilon) - g_2(x^{(0)}, t/\varepsilon) \right) + g_2(x^{(0)}, t/\varepsilon) \left(x_2^u - x_2^{(0)} \right). \end{aligned}$$

Defining $\xi := x^u - x^{(0)}$, we have from the Extension Theorem that $|\xi(t)|_\tau \le K\mu\varepsilon^{\gamma+1}$. Having these bounds in mind and using (4.74), (4.75) and Lemma 4.7, it yields to

$$\begin{aligned}
&\left|\mathcal{L}_G h^{(0)}(x^u, t/\varepsilon) - \mathcal{L}_G h^{(0)}(x^{(0)}, t/\varepsilon)\right| \le \\
&K\left(\frac{1}{\tau^{r+1}} + \frac{\mu\varepsilon^{\gamma+1}}{\tau^2}\right)\frac{\mu\varepsilon^{\gamma+1}}{\tau^{\ell-r+1}} + \frac{K}{\tau^\ell}\left(\frac{\mu\varepsilon^{\gamma+1}}{\tau^2}\right) + \\
&K\left(\frac{1}{\tau^r} + \mu\varepsilon^{\gamma+1}\right)\frac{\mu\varepsilon^{\gamma+1}}{\tau^{\ell-r+1}} + \frac{K}{\tau^{\ell+2}}\left(K\mu\varepsilon^{\gamma+1}\right) \le \\
&K\left(\frac{1}{\tau^{r+1}}\right)\frac{\mu\varepsilon^{\gamma+1}}{\tau^{\ell-r+1}} + \frac{K\mu\varepsilon^{\gamma+1}}{\tau^{\ell+2}} + \left(\frac{K}{\tau^r}\right)\frac{\mu\varepsilon^{\gamma+1}}{\tau^{\ell-r+1}} + \frac{K\mu\varepsilon^{\gamma+1}}{\tau^{\ell+2}} \le \\
&\frac{K\mu\varepsilon^{\gamma-r+1}}{\tau^{\ell+2}}.
\end{aligned}$$

Applying formula (4.76) for $\beta = \ell+2$ and taking into account that $\rho_0^{-(\ell+1)} \le \tau^{-(\ell+1)} \le 1/\varepsilon^{\ell+1}$, it follows that

$$\int_{-T_0}^{T_s}\left|\mathcal{L}_G h^{(0)}(x^u, t/\varepsilon) - \mathcal{L}_G h^{(0)}(x^{(0)}, t/\varepsilon)\right| dt \le K\mu\varepsilon^{\gamma-\ell}.$$

On the other hand, with respect to the integral of $\mathcal{L}_G h^{(0)}(\gamma_p, t/\varepsilon)$, we have that

$$\mathcal{L}_G h^{(0)}(\gamma_p, t/\varepsilon) = -f(\gamma_1)g_1(\gamma_p, t/\varepsilon),$$

where $\gamma_p = (\gamma_1, \gamma_2)$. Since $f(0) = 0$ by hypothesis and $\gamma_p = O(\mu\varepsilon^{p+1})$, by the Normal Form Theorem, it turns out that $\mathcal{L}_G h^{(0)}(\gamma_p, t/\varepsilon) = O(\mu\varepsilon^{p+1})$ and, therefore, also its integral over $[-T_0, T_s]$.

□

Joining all these estimates, the following bound is derived

$$h^{(0)}(x^u(T_s, s)) - h^{(0)}(x^s(T_s, s)) = \mu\varepsilon^p M(s,\varepsilon) + O\left(\mu\varepsilon^p\mu\varepsilon^{\gamma-\ell}, \mu^2\varepsilon^{2p+1}\right)$$

and, finally, equation (4.88) becomes

$$\mathcal{E}^u(s) = \mu\varepsilon^p M(s,\varepsilon) + O\left(\mu\varepsilon^p\mu\varepsilon^{\gamma-\ell}, \mu\varepsilon^{p+1}\right).$$

(*ii*) Since $\mathcal{E}^u(s)$ is a $2\pi\varepsilon$-periodic function, we can express it in Fourier series

$$\mathcal{E}^u(s) = \sum_{k\in\mathbb{Z}} \mathcal{E}_k^u \mathrm{e}^{\mathrm{i}ks/\varepsilon}.$$

Therefore, using estimate (*i*) and shifting along complex lines $\Im u = \pm(a-\varepsilon)$, it follows that

$$\begin{aligned}
\mathcal{E}_k^u = \frac{\mathrm{e}^{-|k|(a-\varepsilon)/\varepsilon}}{2\pi\varepsilon}\int_0^{2\pi\varepsilon} \mathcal{E}^u(\sigma \pm \mathrm{i}(a-\varepsilon))\mathrm{e}^{-\mathrm{i}k\sigma/\varepsilon}\, d\sigma = \\
\mu\varepsilon^p M_k(\varepsilon) + O\left(\mu\varepsilon^p\mu\varepsilon^{\gamma-\ell}, \mu\varepsilon^{p+1}\right)\mathrm{e}^{-|k|a/\varepsilon},
\end{aligned}$$

for $k \ne 0$ and where $M_k(\varepsilon)$ is the k-th Fourier coefficient of the Melnikov function. The claimed bound follows readily.

(*iii*) It follows exactly from applying the same argument given in [24, Prop. 2.6].

§3.6 Proof of Proposition 4.2

The analyticity and periodicity of ψ follows directly from the fact that $\mathcal{E}^u$ and $\mathcal{S}^u - s$ are analytic and $2\pi\varepsilon$-periodic.

(*i*) From Corollary 4.1 and for real s, we know that the stable and unstable invariant curves C^s, C^u of the Poincaré map $\mathcal{P}$ associated to system (4.2),

$$\mathcal{P}(x_p) = x(x_p, 2\pi\varepsilon),$$

are given by

$$\begin{aligned} C^s &= \{x^s(2\pi n\varepsilon, s) = x^s(0, s + 2\pi n\varepsilon)\}_{n\in\mathbb{N}}, \\ C^u &= \{x^u(2\pi n\varepsilon, s) = x^u(0, s + 2\pi n\varepsilon)\}_{n\in\mathbb{N}}. \end{aligned}$$

Since system (4.2) is $\mathfrak{R}$-reversible, we know that $\mathcal{P}$ is also $\mathfrak{R}$-reversible and that the invariant curves associated to the new symmetric hyperbolic point, intersect each other at a point belonging to the symmetry line Fix $\mathfrak{R}$. Let x^h be this homoclinic symmetric point. Moreover, by construction, the parameterizations $x^u(t,s)$ and $x^s(t,s)$ are defined in such a way that for $t + \Re s = t + s = 0$ they are placed at Fix $\mathfrak{R}$. This implies that the homoclinic point x^h satisfies that $x^h = x^s(0,0) = x^u(0,0)$. As a consequence, we are in the domain of definition of the flow-box coordinates and it follows that $\mathcal{S}(x^h, 0) = \mathcal{S}(x^s(0,0), 0) = 0 + 0 = 0$ and $\mathcal{E}(x^h, 0) = \mathcal{E}(x^s(0,0), 0) = 0$. Moreover, from the definition of $\mathcal{S}^u$, we have

$$0 = \mathcal{S}(x^s(0,0), 0) - 0 = \mathcal{S}(x^u(0,0), 0) = \mathcal{S}^u(0).$$

These equalities and the definition (4.21) of ψ yield to

$$\psi(0) = \psi(\mathcal{S}^u(0)) = \mathcal{E}^u(0) = \mathcal{E}(x^u(0,0), 0) = \mathcal{E}(x^s(0,0), 0) = \mathcal{E}(x^h, 0) = 0.$$

Since ψ is $2\pi\varepsilon$-periodic, it follows that $\psi(h_n) = 0$, for $h_n = 2\pi\varepsilon$, with $n \in \mathbb{N}$.

Using again the periodicity of ψ, to know the value of $\psi'(h_n)$, for $n \in \mathbb{N}$, it will be enough to compute $\psi'(0)$. Indeed, differentiating the formula (4.22)

$$\psi(S) = \mathcal{E}(\widetilde{x}^u(t, S), t/\varepsilon)$$

and evaluating it at $S = 0$, $t = 0$, we obtain

$$\psi'(0) = \partial_1\mathcal{E}(\widetilde{x}^u(0,0), 0) \cdot \frac{\partial \widetilde{x}^u_1}{\partial S}(0,0) + \partial_2\mathcal{E}(\widetilde{x}^u(0,0), 0) \cdot \frac{\partial \widetilde{x}^u_2}{\partial S}(0,0)$$

or, equivalently,

$$\psi'(0) = \partial_1\mathcal{E}(x^h, 0) \cdot \frac{\partial \widetilde{x}^u_1}{\partial S}(0,0) + \partial_2\mathcal{E}(x^h, 0) \cdot \frac{\partial \widetilde{x}^u_2}{\partial S}(0,0), \tag{4.89}$$

where $\widetilde{x}^u = (\widetilde{x}^u_1, \widetilde{x}^u_2)$ and ∂_1, ∂_2 denote the derivatives with respect to x_1 and x_2, respectively. We look for an expression writing $\partial_1\mathcal{E}(x^h, 0)$ and $\partial_2\mathcal{E}(x^h, 0)$ in terms of $\partial\widetilde{x}^s_1/\partial S$ and $\partial\widetilde{x}^s_2/\partial S$. To do it, let us differentiate equations (4.15)

$$\mathcal{S}(x^s(t,s), t/\varepsilon) = t + s, \qquad \mathcal{E}(x^s(t,s), t/\varepsilon) = 0$$

with respect to $s = S$ and evaluate them at $(t, s) = (0, 0)$. Thus, regarding the first equation, it follows that

$$\partial_1 \mathcal{S}(x^h, 0) \cdot \frac{\partial x_1^s}{\partial S}(0,0) + \partial_2 \mathcal{S}(x^h, 0) \cdot \frac{\partial x_2^s}{\partial S}(0,0) = 1. \tag{4.90}$$

Proceeding analogously, form the second equation, it is derived that

$$\partial_1 \mathcal{E}(x^h, 0) \cdot \frac{\partial x_1^s}{\partial S}(0,0) + \partial_2 \mathcal{E}(x^h, 0) \cdot \frac{\partial x_2^s}{\partial S}(0,0) = 0. \tag{4.91}$$

Besides, from Remark 20 we know that the change $(S, E) = \Upsilon(\zeta_1, \zeta_2)$ given at (4.59) preserves measure but reverses orientation, that is, $\det D\Upsilon(\zeta_1, \zeta_2) = -1$ for $\zeta_1 - \zeta_2 > 0$. Therefore, since $(\mathcal{S}(x, t/e), \mathcal{E}(x, t/\varepsilon))$ are $O(\mu\varepsilon^{p+1})$-close to the flow-box functions $\big(\mathcal{S}^{(0)}(x), \mathcal{E}^{(0)}(x)\big)$ of the unperturbed system (see Corollary 4.2), it follows that $\Delta = -\Delta^{(0)} + O(\mu\varepsilon^{p+1})$ where

$$\Delta := \begin{vmatrix} \partial_1 \mathcal{S}(x^h, 0) & \partial_2 \mathcal{S}(x^h, 0) \\ \partial_1 \mathcal{E}(x^h, 0) & \partial_2 \mathcal{E}(x^h, 0) \end{vmatrix}, \quad \Delta^{(0)} := \begin{vmatrix} \partial_1 \mathcal{S}^{(0)}(x^h) & \partial_2 \mathcal{S}^{(0)}(x^h) \\ \partial_1 \mathcal{E}^{(0)}(x^h) & \partial_2 \mathcal{E}^{(0)}(x^h) \end{vmatrix}.$$

Taking into account that the transformation

$$(x_1, x_2) \longmapsto (S, E) = \left(\mathcal{S}^{(0)}(x), \mathcal{E}^{(0)}(x)\right),$$

leading system (4.1) into a flow-box system, can be chosen canonical, one has that the solution of system (4.90)–(4.91) is given by

$$\frac{\partial x_1^s}{\partial S}(0,0) = -\partial_2 \mathcal{E}^{(0)}(x^h), \quad \frac{\partial x_2^s}{\partial S}(0,0) = \partial_1 \mathcal{E}^{(0)}(x^h),$$

or, equivalently,

$$\partial_2 \mathcal{E}(x^h, 0) = -\frac{\partial x_1^s}{\partial S}(0,0), \qquad \partial_1 \mathcal{E}(x^h, 0) = \frac{\partial x_2^s}{\partial S}(0,0)$$

plus terms of order $O(\mu\varepsilon^{p+1}$. Therefore $\Delta + 1 = O(\mu\varepsilon^{p+1})$ and for real values of S, it turns out that $\Delta = -1 + O(\mu\varepsilon^{p+1})\mathrm{e}^{-a/\varepsilon}$. Substituting these formulas above in the expression for $\psi'(0)$ it follows that

$$\psi'(0) = \left(\frac{\partial x_2^s}{\partial S}(0,0) \cdot \frac{\partial \widetilde{x}_1^u}{\partial S}(0,0) - \frac{\partial x_1^s}{\partial S}(0,0) \cdot \frac{\partial \widetilde{x}_2^u}{\partial S}(0,0)\right) =$$

$$\left(\frac{\partial \widetilde{x}^u}{\partial S}(0,0) \wedge \frac{\partial x^s}{\partial S}(0,0)\right) = \left\|\frac{\partial \widetilde{x}^u}{\partial S}(0,0)\right\| \left\|\frac{\partial x^s}{\partial S}(0,0)\right\| \sin\alpha + O(\mu\varepsilon^{p+1})\mathrm{e}^{-a/\varepsilon}.$$

where $u \wedge v$ denotes the exterior product of the vectors u and v, and α is the angle between $\widetilde{x}^u(0,0)$ and $x^s(0,0)$. As mentioned before, from the periodicity of x^s and $\widetilde{x}^u$ one derives that the same formula apply for $\psi'(h_n)$, $n \in \mathbb{N}$.

(*ii*) From the definition of ψ and Proposition 4.1(*ii*), it follows that

$$\psi(S) = \mathcal{E}_0^u + \mu\varepsilon^p M(s^u(S), \varepsilon) + O(\mu^2\varepsilon^{p+\gamma-\ell}, \mu\varepsilon^{p+1})\,\mathrm{e}^{-a/\varepsilon}.$$

An straightforward computation shows that $\mu\varepsilon^p M'(S,\varepsilon) = O(\mu\varepsilon^\gamma \mathrm{e}^{-a/\varepsilon})$. Applying this bound and Taylor's Theorem it turns out that

$$\mu\varepsilon^p M(s^u(S),\varepsilon) = \mu\varepsilon^p M(S + O(\mu\varepsilon^{p+1}),\varepsilon) = \mu\varepsilon^p M(S,\varepsilon) + O(\mu^2\varepsilon^{p+\gamma+1}\mathrm{e}^{-a/\varepsilon}).$$

Since $\ell \geq 0$ we have that $p+\gamma+1 \geq p+\gamma-\ell$ and, therefore, it yields

$$\psi(S) = \mathcal{E}_0^u + \mu\varepsilon^p M(S,\varepsilon) + O(\mu^2\varepsilon^{p+\gamma-\ell}, \mu\varepsilon^{p+1})\,\mathrm{e}^{-a/\varepsilon}.$$

On the other hand, since $\mathcal{E}^u(0) = 0$ it follows that

$$\mathcal{E}_0^u = O(\mu^2\varepsilon^{p+\gamma-\ell}, \mu\varepsilon^{p+1})\,\mathrm{e}^{-a/\varepsilon}.$$

Substituting this bound into the expression for $\psi(S)$ we obtain that

$$\begin{aligned}\psi(S) = \mu\varepsilon^p M(S,\varepsilon) + O(\mu^2\varepsilon^{p+\gamma-\ell}, \mu\varepsilon^{p+1})\,\mathrm{e}^{-a/\varepsilon} = \\ \mu\varepsilon^p M(S,\varepsilon) + O(\mu^2\varepsilon^{2\gamma}, \mu\varepsilon^{p+1})\,\mathrm{e}^{-a/\varepsilon},\end{aligned}$$

as the proposition claims.

Chapter 5

Open problems and future work

There are a lot of interesting problems related to the techniques employed in this memory. Here we list some of them that we would like to investigate in a close future.

1. *The center-focus problem.*

   Although it has been mentioned in Chapter 2, let us recall it very briefly. Thus, let us consider a planar system

$$\left\{ \begin{array}{rcrcl} \dot{x} & = & -y & + & \widehat{P}(x,y) \\ \dot{y} & = & x & + & \widehat{Q}(x,y) \end{array} \right. \tag{5.1}$$

   where $\widehat{P}$, $\widehat{Q}$ are polynomials in x, y. Moreover, from now on, $\widehat{\ }$ will denote that the corresponding function (vector field) starts with terms of order at least 2 in the spatial variables. It is clear that the origin is an equilibrium point of system (5.1), which is a *linear center*. It is well-known (see, for instance [14]) that in this case the Birkhoff Normal Form (BNF in short) is convergent if and only if the origin is a center. On the contrary, if it is not convergent, we have in fact a weak focus on this point. The *center-focus problem* (which is a local version of 16th Hilbert's problem) consists on giving conditions on the coefficients of $\widehat{P}(x,y)$ and $\widehat{Q}(x,y)$ ensuring that the equilibrium point is a center. This apparently simple problem has not yet been solved and just some partial results are known (see, for instance [50]).

   In other words, this problem is devoted to seek for conditions on $\widehat{P}(x,y)$, $\widehat{Q}(x,y)$ in such a way that system (5.1) can be led into BNF by means of a convergent transformation. From this point of view, this is closely related with the problem of the convergence of the ΨNF (introduced in Chapter 1). In that Chapter it was proved that the ΨNF-procedure was convergent, giving rise to a BNF-vector field N and a *residual* vector field B (depending on an scalar analytic function $b(r)$). This analytic function $b(r)$ contains the obstructions for the (local) integrability of this system and, as it was noted in Chapter 2, it seems to have a close relation with the *Lyapunov constants* (invariant constants related to the fact that the equilibrium is a center). We think that it could be interesting to apply this tool of the ΨNF, in some known examples, in order to understand which could the algebraic relations between the coefficients appearing in b and the Lyapunov constants associated to this system.

Moreover, we know that the (isolated, of course) zeroes of the scalar analytic function $b(r)$ give rise to limit cycles close to the equilibrium. It could be nice to find interesting examples of system where this kind of cycles could be obtained using this technique.

2. *Convergence of the* ΨNF *for quasi-periodic systems around a hyperbolic torus*

   This could be the following natural step after proving the convergence of the ΨNF of a periodic system in the vicinity of a hyperbolic periodic orbit. Quite briefly, the problem would be the following: let us consider an analytic system

$$\dot{z} = \frac{dz}{dt} = F(z) = \Lambda\, z + \widehat{F}(z), \tag{5.2}$$

   where $z \in \mathbb{C}^2$ and Λ is a diagonal matrix with values $\{\pm\lambda\}$, $\lambda > 0$. Suppose that it can be led into BNF by means of a convergent transformation

$$z = \Phi^{(0)}(\zeta) = \zeta + \widehat{\Phi}^{(0)}(\zeta).$$

   Let us consider now the following perturbation of system (5.2)

$$\dot{z} = F(z) + \mu\varepsilon^p G(z, \theta, \mu, \varepsilon) \tag{5.3}$$

   where G is analytic in z and analytic and periodic in θ, with

$$\theta = (\theta_1, \theta_2, \ldots, \theta_n) = (\omega_1\, t, \omega_2\, t, \ldots, \omega_n\, t)$$

   and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ being a vector in $\mathbb{R}^n$ satisfying some Diophantine condition. μ and ε are small parameters. The problem consists on investigate if there exists a (convergent) analytic in ζ and θ transformation

$$z = \Phi(\zeta) = \Phi^{(0)}(\zeta) + \mu\varepsilon^p \Phi^{(1)}(\zeta, \theta, \mu, \varepsilon)$$

   leading system (5.3) into ΨNF. How big is its domain of convergence ? Does ΨNF become BNF in the case that system (5.3) is Hamiltonian or reversible ? Can be this method useful to find *limit tori* in non (locally) conservative systems ? Can we obtain some interesting examples exhibiting this behavior ?

   The formal approach to this ΨNF would be quite similar to the one presented at Chapter 3 but introducing a quadratic scheme in the perturbative parameter ε which would allow us to control the loss of the domain in the angular variable θ.

3. *Study of the splitting of the separatrices of a whiskered torus under a fast linear reversible quasi-periodic perturbation.*

   This is problem intimately related to the previous item and to the Chapter 4. It is concerned to measure the size of the splitting of the invariant whiskers of a hyperbolic torus when we add a linear reversible quasi-periodic perturbation. The model to follow would be the paper of Delshams, Gelfreich, Jorba and Seara [25], where they deal with this problem in a Hamiltonian setting. Like in that paper, the first approach could be done for the frequency vector ω/ε being of the form $\omega = (1, \gamma)$, where $\gamma = (\sqrt{5}+1)/2$ is the *golden mean number* (the most irrational number).

4. In [27], Delshams, Gutiérrez and Seara, introduce tools to study the splitting of the invariant manifolds associated to a weakly hyperbolic torus of an integrable Hamiltonian system when an small perturbation is considered. The invariant torus of the unperturbed system is assumed to have fast frequencies $\omega/\sqrt{\varepsilon}$ and coincident whiskers. Precisely, they deal with a Hamiltonian system (with $n+1 \geq 3$ degrees of freedom) that in canonical coordinates $(x, y, \varphi, I) \in \mathbb{T} \times \mathbb{R} \times \mathbb{T}^n \times \mathbb{R}^n$ takes the form

$$H(x, y, \varphi, I) = H_0(x, y, I) + \mu H_1(x, \varphi)$$

where $\mu = \varepsilon^p$ and

$$\begin{aligned} H_0(x, y, I) &= \langle \frac{\omega}{\sqrt{\varepsilon}}, I \rangle + \frac{1}{2}\langle \Lambda I, I \rangle + \frac{y^2}{2} + \cos x - 1 \\ H_1(x, \varphi) &= h(x) f(\varphi). \end{aligned}$$

Using suitable flow-box coordinates (defined on a piece of the stable whisker but excluding the torus), extending solutions and defining the corresponding splitting function and splitting potential, they obtain exponentially small upper bounds for this splitting function when considered on the real domain.

We would like to investigate an analogous situation of this singular problem when considering reversible systems instead of Hamiltonian ones.

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