1

Introduction

1.1 A bit of history

The flow of a homogeneous viscous incompressible fluid in a long tube of circular cross section was studied first by Hagen (1839) and Poiseuille (1840) in what today is known as the Hagen–Poiseuille flow. Jean Léonard Marie Poiseuille (Paris 1797–1869) worked on a variety of fields including engineering, physics, medicine, and biology. He was very interested in the study of the flow of liquids in small diameter glass capillaries where he considered the effects of pressure drop, tube length, tube diameter, and temperature. Gotthilf Heinrich Ludwig Hagen (1797–1884) was a German hydraulic engineer who published a paper in 1839 on the flow of water in cylindrical tubes, and obtained similar results to those of Poiseuille, but less extensive and less accurate. Hagen and Poiseuille established empirically the Hagen–Poiseuille law or Poiseuille's law, as is more commonly known. This law provides a formula to calculate the mass flux of fluid Q, for the laminar flow as $Q = \pi\Delta p R^4/(8\nu l)$, i.e. it is proportional to the drop of pressure Δp and the fourth power of the radius R of the tube, where l represents its length and ν the kinematical viscosity of the fluid. Sir George Gabriel Stokes (1813–1903) of Cambridge University, apparently solved the problem of Poiseuille's law as an application of the Navier–Stokes equations which he derived in Stokes (1845). However, he did not publish the result because he was unsure of the boundary condition of zero velocity at the tube wall. Stokes (1845) also discussed the flow in channels (which is called *plane Poiseuille flow*, and is the main subject of this work) and pointed out the similarity to pipe flow under gravity at constant pressure.

One of the interesting uses of Poiseuille's law was to furnish evidence as to the correct no-slip boundary condition for a viscous flow at a solid boundary. It can be also used for the experimental determination of the viscosity by measuring the rate of flow and the pressure drop across a fixed portion of a capillary tube of known radius. These and other details on Poiseuille's law can be enlarged in Sutera & Skalak (1993).

1.2 Hydrodynamic stability

The theory of hydrodynamic stability is one of the main topics in fluid mechanics. Their essential problems were recognized and formulated in the nineteenth century, notably by Helmholtz, Kelvin, Rayleigh and Reynolds. The origin of turbulence and accompanying transition from laminar to turbulent flow is also of fundamental importance in fluid mechanics. The incidence of turbulence was first recognized in relation to flows through straight pipes and channels: Hagen–Poiseuille flow and plane Poiseuille flow (see § 1.5) respectively. These are two examples of test problems where it is possible the evaluation of different analytical and numerical methods, due essentially to the simplicity of their geometry. Reynolds (1883) studied the stability of flow in a pipe by means of experiments and showed that the laminar flow breaks down when the dimensionless number Re, which after him is called the Reynolds number, exceeds a critical value, and that turbulence quickly ensues. For the case of Hagen–Poiseuille flow, Reynolds's estimate of the critical value of Re was about 6400. About the same time, Rayleigh (1880) studied the stability characteristics of inviscid plane-parallel channel flows to infinitesimal perturbations. His results, though, cannot explain experimentally observed transitions. Rayleigh's necessary condition for inviscid linear instability states that the laminar flow must have an inflexion point interior to the channel. This condition applied to plane Poiseuille flow immediately indicates global stability. Therefore viscosity should be the cause of a linear instability.

One of the main steps in understanding the stability of viscous flows was taken by Orr (1907) and Sommerfeld (1908), who derived the celebrated equation that now bears their names. The Orr–Sommerfeld equation arose from the research of the stability to infinitesimal disturbances in a linear approximation of plane Poiseuille flow (see § 3.1). Much of the subsequent work on stability of viscous flows is based on it. For instance, as it is known, plane Poiseuille flow is stable for small Reynolds numbers, but Heisenberg (1924) was the first to propose that it is unstable for large Reynolds numbers. He calculated four points of the neutral curve of stability by an heuristic method of approximating the solution of the Orr–Sommerfeld equation. Nevertheless Heisenberg did not arrive at a critical value beyond which instability begins. The first numerical solution of the Orr–Sommerfeld equation was obtained by Thomas (1953) trying to clarify the controversies concerning the existing asymptotic methods of approximation and confirming the instability of plane Poiseuille flow. Using a finite difference method to approximate derivatives, he obtained the critical Reynolds number for instability at $Re = 5780$ for $\alpha = 1.026$. Calculations of Lin (1944) and Shen (1954) on the neutral curve of the Orr-Sommerfeld equation by means of analytical methods, found the minimum critical Re to be 5300. Subsequently the equation has been studied by other authors as Lin (1955), Orszag (1971), Drazin & Reid (1982) and Maslowe (1985) among many others, and it is well understood. The critical Reynolds number of the linear theory, $Re_{cr} = 5772.22$ for the wavenumber $\alpha = 1.02056$, has been obtained accurately in Orszag (1971) by this approach.

However, as noted in Orszag & Patera (1983*a*), the maximum linear growth rate for plane Poiseuille flow is approximately 0.0076 at $Re \approx 8000$. For a normal mode perturbation to grow by a factor of 10, requires the disturbance wave to travel around 50 channel widths, which is clearly inconsistent with the explosive instabilities seen experimentally. In addition, as experiments of Carlson, Widnall & Peeters (1982), Nishioka & Asai (1985), and Alavyoon, Henningson & Alfredsson (1986) showed, transition to turbulence is observed for Reynolds number ≈ 1000, what motivates that finite-amplitude disturbances originate the transition. The understanding of the transition to turbulence has been conjectured by Saffman (1983) to depend on intermediate vortical states and turbulence takes place due to their three-dimensional instability. Examples of vortical states are periodic flows in time or space, among which can be mentioned: two-dimensional travelling waves, secondary flows in two or three dimensions (for them the flow rate and the pressure gradient are constants) and quasi-periodic flows. Ehrenstein & Koch (1991) discovered a new family of secondary bifurcation branches in dimension 3 which contains only even spanwise Fourier modes and it reduces the critical Reynolds number to $Re_{Q_m} \approx 1000$ (defined in terms of the averaged velocity across the channel) as observed in experiments.

Two-dimensional disordered motion is associated with the large scales of some turbulent flows, so there probably exist attractors for those two-dimensional flows. Besides, two-dimensional and three-dimensional states can compete and coexist in the final flow (cf. Jiménez 1987, and the references therein). In spite of the fact that transition to turbulence is a three-dimensional phenomenon, there are many properties of the two-dimensional flows observed in fully turbulent three-dimensional flows such as wall sweeps, ejections, intermittency and bursting, as Jiménez (1990) showed. The two-dimensional case has attracted the attention of many authors but it is not completely understood as the problem of two-dimensional transition to turbulence proves. Due to Squire's (1933) theorem, to every three-dimensional perturbation of the linearized Navier–Stokes equations for a given Re, α, it corresponds a two-dimensional one (for detailed formulas see for instance Drazin & Reid 1982 p. 155) for some $\tilde{\alpha} \geqslant \alpha$ and $\tilde{Re} \leqslant Re$, so the critical Re for the linear theory must be attained by a two-dimensional flow. However the theorem does not imply that the most unstable mode for $Re > Re_{cr}$ is necessarily two-dimensional. Squire's theorem has been one of the main reasons to firstly try to understand the two-dimensional case, apart from the obvious easiness of computations compared to the three-dimensional situation. In addition, some of the properties obtained from the two-dimensional case can also provide new insight for three-dimensional flows.

1.3 Some works on finite-amplitude solutions

Following we summarize some papers concerning finite-amplitude solutions of plane Poiseuille flow in two and three dimensions, which have been taken as a reference in the course of the present work.

- The first attempt to compute them was carried out by Noether (1921), who expanded equilibrium waves disturbances in Fourier series, considering only one Fourier mode in the periodic direction (the so called *mean-field*).
- Using also the mean-field, Meksyn & Stuart (1951), by means of asymptotic expansions, solved simultaneously the Orr–Sommerfeld equation and a non-linear equation of mean motion, to find periodic solutions of finite amplitude. They found for each Reynolds number Re the corresponding value of the amplitude and the wavenumber for which Re is the minimum value, yielding a curve of periodic solutions which borns subcritically from the basic flow at $Re \approx 5000$, attains a minimum $Re \approx 2900$ and then grows with Re. The critical wavenumber for periodic flows at $Re \approx 2900$ was found $\alpha = 1.20$ to be larger than for infinitesimal disturbances at $Re \approx 5000$, which is $\alpha = 1.12$.
- Based on the perturbation theory of Joseph & Sattinger (1972), Chen & Joseph (1973) determine the form of the time-periodic solutions which bifurcate from plane Poiseuille flow. They expand a flow, periodic in time and streamwise direction, as a power series and, by means of

Runge–Kutta integration of the resulting equations of vorticity, they solve for the first two order in the series. From these calculations they present the neutral curve of stability of the linear theory and schematic sketch of the surface of periodic solutions. From the theory of Joseph & Sattinger (1972), they showed that the only time-periodic solution which bifurcates from laminar Poiseuille flow is a two-dimensional wave, which is unstable for the lowest Reynolds and small values of the amplitude. This instability makes perturbations of laminar flow snap through the unstable time-periodic flow to solutions of larger amplitudes.

- Zahn, Toomre, Spiegel & Gough (1974) constructed a numerical integrator in time and space, using one and two Fourier modes in the streamwise coordinate x in two-dimensions and only one Fourier mode in three-dimensions. The cross-stream variable z was transformed to improve accuracy on boundary layers: their derivatives were approximated by means of finite differences. The time discretization was implemented by an implicit scheme. Using a constant mean pressure gradient, they compute travelling waves as steady flows in an appropriate Galilean reference. For the two-dimensional case they found the region in the Re_p-α plane where there exist travelling waves and in turn the instability region of the linear theory. They obtained a surface of the energy of disturbances for each pair Re, α, given rise to an upper and lower branch of solutions. By perturbing these two sets of solutions and following them in time, they estimated their stability to find that the lower branch is unstable and the upper stable, at least to the disturbances employed.
- Herbert (1976) employed a spectral method to approximate the vorticity equation for plane Poiseuille flow in two dimensions. For the stream variable he used Galerkin–Fourier with $N \leqslant 4$ modes, and for the cross-stream variable τ–Chebyshev and collocation–Chebyshev with $K \leqslant 64$ modes. He imposed the solution to be a periodic secondary flow with even or odd Fourier harmonics according to the parity of its order, to obtain a finite system of algebraic equations. The mean pressure gradient was assumed to be constant. For this discretization he found the neutral surface of periodic flows and the minimum value of Re_p, pointing out the slow convergence of the Fourier series. He also made some comparisons between experimental data and stable periodic solutions, with good overall agreement.
- Rozhdestvensky & Simakin (1984) implemented a numerical integrator to simulate the flow over different time intervals. By observing the flow rate and the pressure gradient they obtain secondary flows. They found secondary flows non-periodic in time in two dimensions for long wave-lengths ($\alpha \approx 0.3$) and also in three dimensions for different wave-lengths.
- Jiménez (1987) proposed to study the existence of bifurcations leading to limit cycles in plane Poiseuille flow as possible early steps in the appearance of disorder. At $Re_Q = 5000$ (always $\alpha = 1.0$) he followed the upper branch of time-periodic orbits and found that for $Re_Q \approx 5000$-6000 the periodic flow shed a limit cycle. The number of Fourier–Chebyshev modes used, varies from 7×33 to 15×65. For $Re_Q > 9000$, the limit cycle becomes disordered and bifurcates into new solutions that include tori and, later, chaos.
- Jiménez (1990) does full numerical simulation of spatially periodic channels with fairly large longitudinal aspect ratios. He finds travelling waves, one and two frequency tori and chaos for several values of Re_Q and α. Computations were mostly performed using 41×85 Fourier–Chebyshev modes. With this approach he was only able to obtain attracting solutions.
- Soibelman & Meiron (1991) studied the stability of the basic flow by analysing the Orr–Sommerfeld equation. They found the marginal curve of stability and the critical $Re = 5772.22$. Next they compute steady two-dimensional travelling waves in the streamwise direction with phase speed c. They considered both boundary conditions by prescribing the average flux or pressure

gradient and calculated the critical Re of travelling waves. To analyse the stability of travelling waves they set up an eigenvalue problem in finite dimensions. All the calculations were performed with $\alpha = 1.1$, $N = 1$–4 (Fourier modes in the streamwise direction), and $K = 70$ (Chebyshev modes in the cross-stream direction). For constant flux the lower branch is unstable with a stability transition occurring at the limit point of the bifurcation curve. Two Hopf bifurcations were found on the upper branch. For constant pressure they detected a first Hopf bifurcation on the upper branch, together with other two further on this branch. Branches of quasi-periodic solutions which bifurcate from the two-dimensional travelling waves were found in a frame of reference moving with speed c. The main results were obtained for $\alpha = 1.1$ but they also calculated branches for $\alpha = 1.15, 1.21$. The maximum number of Fourier modes used were $N = 2$, $M = 4$ (temporal modes), and $K = 70$. For constant pressure they found that the first Hopf bifurcation is subcritical and therefore locally stable, but they do not implement a Floquet analysis due to memory requirements. This branch of quasi-periodic orbits reaches a limit point above the limit point of two-dimensional travelling waves for all the wavenumbers studied. Similar results were obtained for the other Hopf bifurcations even for constant flux: Re increases with increasing amplitude so the quasi-periodic flows are locally stable. Instead, the periods of the orbits decreases with increasing amplitude. The time scale of these orbits is of the same order as three-dimensional flows and they exhibit phenomena which are reminiscent of “bursting”.

- Drissi, Net & Mercader (1999) analysed superharmonic and subharmonic instabilities of two-dimensional shear travelling waves, contained in boxes of a given periodicity. They extended to any value of α the studies of Herbert (1976), Pugh & Saffman (1988) and Soibelman & Meiron (1991). They considered sequences of subharmonic bifurcations of the wave train that led to stable wave packets. They also found that for some values of Re and α there exist uniform wave trains for long boxes.

1.4 Purpose of the work and results

In this work we try to analyse the dynamics of an easily treatable problem without domain complexities as is the case of the two-dimensional plane Poiseuille flow. Different levels of bifurcation to respective vortical states are considered, starting at the basic parabolic profile. From it, a family of travelling waves is born subcritically for a certain range of wavenumbers $\alpha \approx 1$. There are many papers concerning this kind of waves as was presented in §1.3.

We also reproduce the calculations to find the travelling waves for several values of α. Jiménez (1987, 1990) and Soibelman & Meiron (1991) obtained the next level of bifurcation to quasi-periodic solutions. As was pointed in §1.3, employing full numerical simulation of the Navier–Stokes equations, Jiménez (1987, 1990) computed different attracting flows with a moderate number of Chebyshev and Fourier modes. On the other hand Soibelman & Meiron (1991) implemented an algebraic approach to capture stable and unstable quasi-periodic flows, but the number of modes used were not enough to give good results and they were not able to carry out the stability analysis. The method implemented in the present work combines both: We solve a stationary problem to compute travelling waves for an observer moving at an appropriate speed, whereas the quasi-periodic flows are found by means of full numerical integration of the Navier–Stokes equations. We have used a Poincaré section of the flow in order to obtain also unstable quasi-periodic solutions from the numerical integrator. These unstable intermediate states of the flow provide a highly useful insight into the transition process, as exemplified by secondary bifurcations in shear flows. The spatio-temporal symmetries of the channel allows the reduction of two-frequencies quasi-periodic flows to periodic flows in the appropriate Galilean reference. The quasi-periodic solutions found in this work correspond to the first two Hopf bifurcations of travelling waves, for the case of pressure drop through the channel held constant and the first Hopf bifurcation when the max flux is held constant. The property of behaving as time-periodic flows if we take a suitable Galilean reference, simplifies enormously the search of this kind of solutions. In this case, the associated return time to the Poincaré section is roughly 10000 at the first Hopf bifurcation for constant pressure, what makes the temporal integration very costly. The employed numerical procedure utilizes a parallel algorithm to evaluate the different columns of a Jacobian matrix, needed in the application of Newton's method to the continuation of quasi-periodic solutions. We find that on the analysed Hopf bifurcations for both constant pressure and constant flux, there exist quasi-periodic flows with increasing Re, so the bifurcations are all supercritical. On the first bifurcation for constant pressure all the quasi-periodic solutions found are unstable. On the remaining bifurcations, there are stable quasi-periodic solutions to disturbances with the same wavenumber α and likewise, we have obtained unstable solutions.

- Once we have situated the different studies concerning plane parallel flows, in the next sections of **this chapter** we pose the concrete terms that define the plane Poiseuille problem in two-dimensions, together with their equations and the most common boundary conditions used to drive the fluid: mean constant pressure gradient or constant flux through the channel. We also establish the relationship between those two settings.

- Next in **chapter 2** we give the details of the direct numerical solution of the full two-dimensional, time-dependent, incompressible Navier–Stokes equations. Unlike other authors we have considered the classical formulation in terms of primitive variables for velocity and pressure. We also describe the approach adopted to eliminate the pressure and the cross-stream component of the velocity, obtaining thus a reduced system of ordinary differential equations from an original system of differential-algebraic equations. This is translated to a reduction of two thirds in the dimension of the system and in addition, it allows us to study the stability of fixed points by means of the analytical Jacobian matrix. Likewise in this chapter are included other implemented numerical tools, some imported from dynamical system theory, like Poincaré sections or continuation methods.

- **Chapter 3** is devoted to the computation of travelling waves and its stability to superharmonic disturbances. We begin by reviewing some results of the Orr–Sommerfeld equation which serve as a starting point to obtain the bifurcating solutions of time-periodic flows for several values of α. In turn, we also calculate several Hopf bifurcations that appear on the branch of periodic flows, for both cases of imposed constant flux or pressure. Likewise for each unstable periodic flow we study the connection of its unstable manifold to other attracting solutions.

- Starting at the Hopf bifurcations found in chapter 3, we analyse in **chapter 4** the bifurcating branches of quasi-periodic solutions at the two first Hopf bifurcations for the case of imposed constant pressure and the first one for constant flux. Those solutions are found as fixed points of an appropriate Poincaré map, since, by the symmetry of the channel, they are modulated waves (cf. § A.3). We also study their stability by analysing the linear part of the Poincaré map. In the case of constant flux we have found a branch of quasi-periodic solutions, which on increasing the Reynolds number changes from stable to unstable, giving rise to an attracting family of quasi-periodic flows with 3 frequencies. The results of this chapter referring to the first Hopf bifurcation for constant pressure, are not in qualitative agreement with those of Soibelman & Meiron (1991), which yield a different bifurcation picture and stability properties of the obtained quasi-periodic flows. From the computed unstable flows we follow their unstable invariant manifold and describe what new attracting solution they are conducted to.

- Finally in **chapter 5** we point out some conclusions and comments about the employed techniques, its advantages and drawbacks, and future work.

1.5 Poiseuille flow formulation

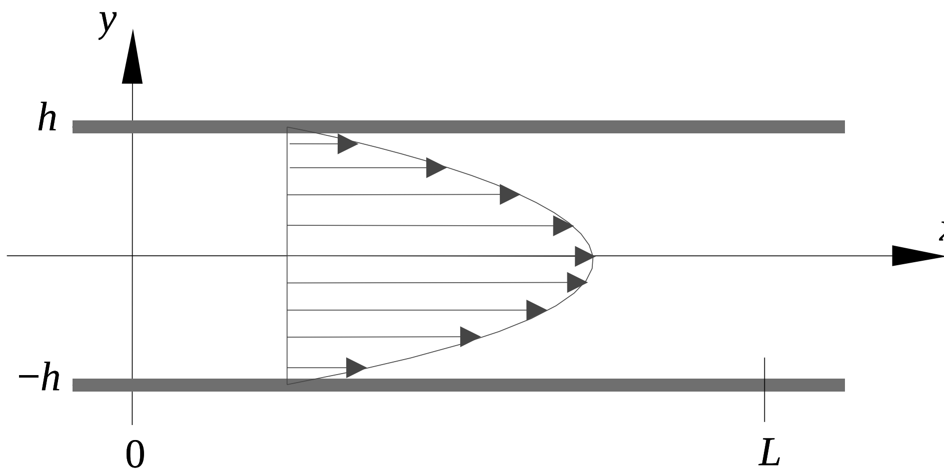

FIGURE 1.1. Sketch of Poiseuille flow. The fluid moves between the vertical plates at $y = \pm h$ and is considered periodic in x of period L.

The Poiseuille problem is described as the flow of a viscous incompressible fluid, in a channel between two infinite parallel plates. In this work we consider this problem in two dimensions as shown in figure 1.1. We suppose the fluid governed by the Navier–Stokes equations together with the incompressibility condition

$$\rho\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}\right] = -\nabla p + \mu\boldsymbol{\Delta}\mathbf{u}, \qquad \boldsymbol{\nabla}\cdot\mathbf{u} = 0,$$

or expressed in coordinates

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{1.1a}$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{1.1b}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1.1c}$$

where $\mathbf{u} = \mathbf{u}(x,y,t) = (u,v)(x,y,t)$, are the components of the velocity, $p = p(x,y,t)$ the pressure and ρ, μ the constant density and viscosity respectively. As boundary conditions we suppose no-slip on the channel walls at $y = \pm h$ and, at artificial boundaries in the stream direction x, a fixed period L, i.e.

$$\left.\begin{array}{r} u(x,\pm h,t) = v(x,\pm h,t) = 0 \\ (u,v,p')(x+L,y,t) = (u,v,p')(x,y,t) \end{array}\right\} \quad x \in \mathbb{R}, \quad y \in [-h,h], \quad t \geqslant 0, \tag{1.2}$$

being $p' = p + Gx$, for $G = G(t)$ the mean pressure gradient on the channel length L in the streamwise direction. In terms of the change in pressure, $\Delta p = p(L,y) - p(0,y)$, supposed uniform in y, we can obtain G as

$$\begin{aligned} -\frac{\Delta p}{L} &= -\frac{1}{2hL}\int_{-h}^{h} \Delta p \,\mathrm{d}y = \frac{1}{2hL}\int_{-h}^{h}\int_{0}^{L}\left[-\frac{\partial p}{\partial x}\right] \mathrm{d}x\,\mathrm{d}y \\ &= \frac{1}{2hL}\int_{-h}^{h}\int_{0}^{L}\left(G - \frac{\partial p'}{\partial x}\right) \mathrm{d}x\,\mathrm{d}y = G. \end{aligned} \tag{1.3}$$

The last equality holds from boundary conditions (1.2) in p'.

The laminar solution. For system (1.1) there is a time-independent solution known as the basic or laminar flow. We deduce it by imposing $\mathbf{u} = (u_b(x,y), 0)$ to be a solution of (1.1). By (1.1c) we have

$$\frac{\partial u}{\partial x} = 0 \quad \Longrightarrow \quad u_b = u_b(y),$$

and from (1.1b)

$$0 = -\frac{\partial p}{\partial y} \quad \Longrightarrow \quad p = p(x).$$

Finally (1.1a) drives us to

$$0 = -p'(x) + \mu u_b''(y) \quad \Longrightarrow \quad p'(x) = \mu u_b''(y) = -G,$$

where G has the same meaning as in (1.3) and in this case is constant, because a function of x can coincide with a function of y, only if both functions are constant. Solving this last equation for $u_b(y)$ and imposing no-slip boundary conditions, it turns out a parabolic profile of velocities, namely

$$u_b(y) = U_c\left[1 - \left(\frac{y}{h}\right)^2\right], \quad v_b = 0, \quad \nabla p_b = (-G, 0), \tag{1.4}$$

where $U_c = Gh^2/(2\mu)$ is the centreline velocity.

Non-dimensional equations. For given values of viscosity μ, density ρ, the channel half-length h and the centre velocity of the basic flow U_c, we scale lengths as $\bar{x} = x/h,\ \bar{y} = y/h$, velocity as $\bar{\mathbf{u}} = \mathbf{u}/U_c$, time as $\bar{t} = U_c t/h$ and pressure as $\bar{p} = p/(\rho U_c^2)$. Applying this rescaling to (1.1) we arrive at

$$\begin{cases} \dfrac{\partial \bar{u}}{\partial t} + \bar{u}\dfrac{\partial \bar{u}}{\partial x} + \bar{v}\dfrac{\partial \bar{u}}{\partial y} = -\dfrac{\partial \bar{p}}{\partial x} + \dfrac{1}{Re}\left(\dfrac{\partial^2 \bar{u}}{\partial x^2} + \dfrac{\partial^2 \bar{u}}{\partial y^2}\right), \\ \dfrac{\partial \bar{v}}{\partial t} + \bar{u}\dfrac{\partial \bar{v}}{\partial x} + \bar{v}\dfrac{\partial \bar{v}}{\partial y} = -\dfrac{\partial \bar{p}}{\partial y} + \dfrac{1}{Re}\left(\dfrac{\partial^2 \bar{v}}{\partial x^2} + \dfrac{\partial^2 \bar{v}}{\partial y^2}\right), \\ \dfrac{\partial \bar{u}}{\partial x} + \dfrac{\partial \bar{v}}{\partial y} = 0, \end{cases} \tag{1.5}$$

where $Re = hU_c/\nu$ is the Reynolds number for $\nu = \mu/\rho$, the kinematic viscosity. The boundary conditions are analogous to (1.2)

$$\left.\begin{aligned} \bar{u}(x,\pm 1,t) = \bar{v}(x,\pm 1,t) = 0 \\ (\bar{u},\bar{v},\bar{p}')(x+L,y,t) = (\bar{u},\bar{v},\bar{p}')(x,y,t) \end{aligned}\right\} \quad x \in \mathbb{R}, \quad y \in [-1,1], \quad t \geqslant 0, \tag{1.6}$$

and the basic flow in non-dimensional form is written as

$$\bar{u}_b(y) = 1 - y^2, \quad \bar{v}_b = 0, \quad \nabla \bar{p}_b = (-\frac{2}{Re}, 0).$$

Moving observer. We will justify later that periodic conditions at artificial boundaries in the stream direction x, yield a great simplification in the structure of the flow: quasi-periodic solutions may be viewed as periodic flows, and periodic solutions as stationary flows, if the observer moves at an adequate speed c in the stream direction. Consequently we write system (1.5) performing the change of variable $\tilde{x} = x - ct$, and in this way we define the transformed velocity

$$\tilde{u}(\tilde{x},y,t) \stackrel{\text{def}}{=} \bar{u}(\tilde{x}+ct,y,t).$$

We obtain the following relations among the respective derivatives

$$\frac{\partial \tilde{u}}{\partial t}(\tilde{x},y,t) = c\frac{\partial \bar{u}}{\partial x}(\tilde{x}+ct,y,t) + \frac{\partial \bar{u}}{\partial t}(\tilde{x}+ct,y,t),$$
$$\frac{\partial^k \tilde{u}}{\partial \tilde{x}^k}(\tilde{x},y,t) = \frac{\partial^k \bar{u}}{\partial x^k}(\tilde{x}+ct,y,t), \quad \frac{\partial^k \tilde{u}}{\partial y^k}(\tilde{x},y,t) = \frac{\partial^k \bar{u}}{\partial y^k}(\tilde{x}+ct,y,t), \quad k = 1,2.$$

Analogous formulae hold for $\tilde{v}$ and $\tilde{p}$. Substituting these derivatives, (1.5) becomes

$$\left\{\begin{aligned} &\frac{\partial \tilde{u}}{\partial t} + (\tilde{u}-c)\frac{\partial \tilde{u}}{\partial \tilde{x}} + \tilde{v}\frac{\partial \tilde{u}}{\partial y} = -\frac{\partial \tilde{p}}{\partial \tilde{x}} + \frac{1}{Re}\left(\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{u}}{\partial y^2}\right), \\ &\frac{\partial \tilde{v}}{\partial t} + (\tilde{u}-c)\frac{\partial \tilde{v}}{\partial \tilde{x}} + \tilde{v}\frac{\partial \tilde{v}}{\partial y} = -\frac{\partial \tilde{p}}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 \tilde{v}}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{v}}{\partial y^2}\right), \\ &\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial y} = 0, \end{aligned}\right. \tag{1.7}$$

together with boundary conditions identical to (1.6) in terms of the new variable $\tilde{x}$. We can recover (1.5) by simply putting $c = 0$ in (1.7).

1.6 Different flow conditions: Constant pressure gradient or constant flux

As we have seen in (1.4), for given μ, ρ and L, varying U_c, we have a family of laminar flows, solutions of (1.1)–(1.2), where $G = 2\mu U_c/h^2$. We avoid this lack of uniqueness in the basic flow by fixing typical quantities associated to the fluid such as the total flux Q or the mean pressure gradient G through the channel. For each choice let us show that there is only one value U_c which defines the basic flow.

Given a profile of velocities $\mathbf{u} = (u, v)$ for Poiseuille flow, the flux Q through the channel is obtained by

$$Q = \int_{-h}^{h} u(x,y)\,\mathrm{d}y.$$

Due to the incompressibility condition (1.1c), Q does not depend on x for

$$\frac{\partial Q}{\partial x} = \int_{-h}^{h} \frac{\partial u}{\partial x}(x,y)\,\mathrm{d}y = -\int_{-h}^{h} \frac{\partial v}{\partial y}(x,y)\,\mathrm{d}y = v(x,-h) - v(x,h) = 0.$$

The last step is consequence of the no-slip boundary conditions (1.2). In this way if for $\alpha = 2\pi/L$ we expand $u(x,y)$ as

$$u(x,y) = \sum_{k\in\mathbb{Z}} \hat{u}_k(y)\mathrm{e}^{\mathrm{i}k\alpha x} \quad \Longrightarrow \quad Q = \int_{-h}^{h} \hat{u}_0(y)\,\mathrm{d}y. \tag{1.8}$$

On the other hand we can compute the mean pressure gradient G on the channel by

$$\begin{aligned}
G =& \frac{1}{2hL}\int_0^L \int_{-h}^{h} \left[-\frac{\partial p}{\partial x}\right] \mathrm{d}y\,\mathrm{d}x \\
\overset{1}{=}& \frac{1}{2hL}\int_0^L \int_{-h}^{h} \left[\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) - \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\right] \mathrm{d}y\,\mathrm{d}x \\
\overset{2}{=}& \frac{1}{2hL}\left\{\rho\frac{\partial}{\partial t}\int_0^L \int_{-h}^{h} u\,\mathrm{d}y\,\mathrm{d}x + \rho\int_{-h}^{h}\left[\frac{u^2}{2}\right]_0^L \mathrm{d}y - \rho\int_0^L\int_{-h}^{h} u\frac{\partial v}{\partial y}\,\mathrm{d}y\,\mathrm{d}x \right.\\
&\left. -\mu\int_{-h}^{h}\left[\frac{\partial u}{\partial x}\right]_0^L \mathrm{d}y - \mu\int_0^L \left[\frac{\partial u}{\partial y}\right]_{-h}^{h} \mathrm{d}x\right\} \\
\overset{3}{=}& \frac{1}{2hL}\left\{\rho\frac{\partial}{\partial t}\int_0^L Q\,\mathrm{d}x + \rho\int_0^L\int_{-h}^{h} u\frac{\partial u}{\partial x}\,\mathrm{d}y\,\mathrm{d}x - \mu\int_0^L \left[\frac{\partial u}{\partial y}\right]_{-h}^{h} \mathrm{d}x\right\} \\
\overset{4}{=}& \frac{\rho}{2h}\int_{-h}^{h}\frac{\partial \hat{u}_0}{\partial t}\,\mathrm{d}y - \frac{\mu}{2h}\left[\frac{\partial \hat{u}_0}{\partial y}\right]_{-h}^{h}.
\end{aligned} \tag{1.9}$$

Step 1: By substitution according to the momentum equation (1.1a).

Step 2: Derivating under the integral sign, integrating, integrating by parts, and integrating respectively at each term, bearing in mind boundary conditions (1.2).

Step 3: Due to the incompressibility condition (1.1c).
Step 4: Using the Fourier expansion in x for u.

Let us now suppose that a constant flux Q_0 is imposed through the channel. For the laminar flow u_b, the total flux is

$$Q = \int_{-h}^{h} u_b(y)\,\mathrm{d}y = \int_{-h}^{h} U_c\left[1 - \left(\frac{y}{h}\right)^2\right]\,\mathrm{d}y = \frac{4}{3}hU_c.$$

In order to obtain flux Q_0 we set $U_c = 3Q_0/(4h)$. According to (1.9) we derive the mean pressure gradient G for the basic flow as

$$G = -\frac{\mu}{2h}\,[u_b']_{-h}^{h} = \frac{3}{2}\frac{\mu Q_0}{h^3}.$$

Finally we calculate Re_Q, the Reynolds number

$$Re_Q = hU_c/\nu = \frac{3Q_0}{4\nu}.$$

Analogously if we impose a mean constant pressure gradient G_0 we find the centreline velocity for the basic flow $U_c = G_0h^2/(2\mu)$ and the flux $Q = 2h^3G_0/(3\mu)$. Now for the Reynolds number Re_p we have

$$Re_p = hU_c/\nu = \frac{G_0h^3\rho}{2\mu^2}.$$

For a given laminar flow, i.e. if we fix U_c, then both definitions of the Reynolds number coincides with $Re = hU_c/\nu$. That is not the case for secondary flows, defined as the ones for which the flux and mean pressure gradient through the channel are kept constant. We consider the case of constant flux Q and the associated laminar flow $u_b^Q = U_Q(1 - y^2/h^2)$ with $U_Q = 3Q/(4h)$. Let us suppose that u^Q is a secondary flow given by (we ignore for the moment the time-dependence)

$$u^Q(x,y) = \sum_{k\in\mathbb{Z}} \hat{u}_k^Q(y)\mathrm{e}^{\mathrm{i}k\alpha x},$$

which has constant flux Q and constant mean pressure gradient

$$G_Q = -\frac{\mu}{2h}\left[\frac{\partial \hat{u}_0^Q}{\partial y}\right]_{-h}^{h},$$

in view of (1.9). Taking the laminar flow which attain G_Q as its mean pressure gradient, the centerline velocity U_p has the expression

$$U_p = \frac{G_Qh^2}{2\mu} = -\frac{h}{4}\left[\frac{\partial \hat{u}_0^Q}{\partial y}\right]_{-h}^{h}.$$

imposed	Q	G
flux	$\frac{4}{3}$	$\frac{-d}{2Re_Q}$
pressure	$\frac{-16}{3d}$	$\frac{-8}{d\,Re_Q}$

TABLE 1.2. Expressions of Q and G for non-dimensional secondary flows in cases where the flux or the average pressure gradient is held constant. In this formulas d and the relation between Re_Q and Re_p is given in (1.10).

Writing this last expression in non-dimensional form we have

$$\frac{Re_p}{Re_Q} = \frac{U_p}{U_Q} = -\frac{h}{4U_Q}\left[\frac{\partial \hat{u}_0^Q}{\partial y}\right]_{-h}^{h} = -\frac{1}{4}\left[\frac{\partial \hat{\bar{u}}_0^Q}{\partial \bar{y}}\right]_{-1}^{1},$$

being $\hat{\bar{u}}_0^Q$, $\bar{y}$, non-dimensional magnitudes, so the relation between both Reynolds numbers can be set as

$$Re_p = -\frac{Re_Q}{4}d \qquad d = \left[\frac{\partial \hat{\bar{u}}_0^Q}{\partial y}\right]_{-1}^{1}. \tag{1.10}$$

We remark that if we put $\hat{\bar{u}}_0^Q = 1 - y^2$ in (1.10) we get $Re_p = Re_Q$ and therefore both definitions of Reynolds number coincides for laminar flows as stated previously.

Conversely if u^p is a secondary flow for fixed G over the channel, let us consider its constant flux

$$Q_p = \int_{-h}^{h} u^p(x,y)\,\mathrm{d}y.$$

The laminar flow associated to this flux attains a centreline velocity $U_Q = 3Q_p/(4h)$, and analogously let U_p be the centreline velocity for the laminar flow which preserves G. The relation between U_Q and U_p gives the ratio of the Reynolds numbers as

$$\frac{Re_Q}{Re_p} = \frac{U_Q}{U_p} = \frac{3Q_p}{4hU_p} = \frac{3}{4}\int_{-h}^{h}\frac{u^p(x,y)}{U_p}\frac{\mathrm{d}y}{h} = \frac{3}{4}\int_{-1}^{1}\bar{u}^p(\bar{x},\bar{y})\,\mathrm{d}\bar{y}. \tag{1.11}$$

In the last integral we have changed the integrand to non-dimensional form. In this way (1.11) establishes the ratio between Re_Q and Re_p as proportional to the flux of the non-dimensional flow $\bar{u}^p(\bar{x},\bar{y})$.

Because centreline velocities U_Q and U_p for basic flows associated to secondary flows are different, each of them give rise to a distinct scaling of the same flow. In terms of dimensional variables a secondary flow can be equally expressed from both points of view. Indeed if the same flow $u(x,y) = u^p(x,y) = u^Q(x,y)$ is non-dimensionalized using two different centerline velocities U_Q, U_p, then we have

$$\frac{u^p(x,y)}{U_p} = \frac{U_Q}{U_p}\frac{u^Q(x,y)}{U_Q} \iff \bar{u}^p(\bar{x},\bar{y}) = \frac{Re_Q}{Re_p}\bar{u}^Q(\bar{x},\bar{y}), \tag{1.12}$$

being $\bar{u}^Q(\bar{x},\bar{y})$, $\bar{u}^p(\bar{x},\bar{y})$ the non-dimensional velocities in the scales of U_Q and U_p respectively. Therefore, if $\bar{u}^Q(\bar{x},\bar{y})$ represents a non-dimensional secondary flow for Re_Q, then $\bar{u}^p(\bar{x},\bar{y}) = Re_Q/Re_p\bar{u}^Q(\bar{x},\bar{y})$ is also a secondary flow for Re_p and the relation between Re_Q and Re_p is given in (1.10) or (1.11). The different possibilities for Q and G in both cases are presented in table 1.2. Following the same procedure we find the relation between pressures $p(x,y) = p^p(x,y) = p^Q(x,y)$ in dimensional coordinates, which yields

$$\frac{p^p(x,y)}{\rho U_p^2} = \frac{U_Q^2}{U_p^2}\frac{p^Q(x,y)}{\rho U_Q^2} \quad \Longleftrightarrow \quad \bar{p}^p(\bar{x},\bar{y}) = \frac{Re_Q^2}{Re_p^2}\bar{p}^Q(\bar{x},\bar{y}). \tag{1.13}$$