Universitat Autònoma de Barcelona

ON THE STRATIFICATION OF SMOOTH PLANE CURVES BY AUTOMORPHISM GROUPS

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On the stratification of smooth plane curves by automorphism groups

by

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CERTIFICO que aquesta memòria ha estat realitzada per Eslam Essam Ebrahim Farag sota la direcció del Dr. Francesc Bars Cortina.

Bellaterra, 12 Juliol 2017

Signature: Dr. Francesc Bars Cortina

To my beloved ones,,,

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Notations and conventions

By $\overline{k}$ we mean a fixed algebraic closure of the field k of characteristic $p \geq 0$. We use ζ_n for a fixed primitive n-th root of unity inside $\overline{k}$ when the characteristic of k is coprime with n.

A smooth curve C over k is a projective, non-singular and geometrically irreducible curve defined over k, and it will be denoted by C/k or simply by C when understood. As usual $\overline{C}$, Aut$(\overline{C})$ and g denote $C \otimes_k \overline{k}$, the automorphism group of $\overline{C}$, and its geometric genus, respectively. We assume, once and for all, that $g \geq 2$.

The (coarse) moduli space of smooth curves over $\overline{k}$ of genus g is denoted by $\mathcal{M}_g$. For a finite non-trivial group G, we set $\mathcal{M}_g(G)$ for the stratum of $\overline{k}$-isomorphism classes of smooth curves $\overline{C}$ of genus g, where G is isomorphic to a subgroup of $\mathrm{Aut}(\overline{C})$, and $\widetilde{\mathcal{M}_g}(G)$ for the substratum of $\mathcal{M}_g(G)$ representing smooth curves $\overline{C}$ such that $G \simeq \mathrm{Aut}(\overline{C})$. In particular, $\widetilde{\mathcal{M}_g}(G) \subseteq \mathcal{M}_g(G) \subseteq \mathcal{M}_g$.

Let $g \geq 3$ be an integer. We use the symbol $\mathcal{M}_g^{Pl}$ for the substratum of $\mathcal{M}_g$, representing smooth plane curves over $\overline{k}$ of genus g. Similarly, we define the substrata

$$\mathcal{M}_g^{Pl}(G) := \mathcal{M}_g^{Pl} \cap \mathcal{M}_g(G) \text{ and } \widetilde{\mathcal{M}_g^{Pl}}(G) := \widetilde{\mathcal{M}_g}(G) \cap \mathcal{M}_g^{Pl}.$$

Hence, $\widetilde{\mathcal{M}_g^{Pl}}(G) \subseteq \mathcal{M}_g^{Pl}(G) \subseteq \mathcal{M}_g(G) \subseteq \mathcal{M}_g$.

The $n-1$-dimensional projective space over an algebraically closed field L is denoted by $\mathbb{P}_L^{n-1}$, and its automorphism group is the projective general linear group $\mathrm{PGL}_n(L)$. A projective linear transformation $A = (a_{i,j})$ of $\mathbb{P}_L^2$ is often written as

$$[a_{1,1}X + a_{1,2}Y + a_{1,3}Z : a_{2,1}X + a_{2,2}Y + a_{2,3}Z : a_{3,1}X + a_{3,2}Y + a_{3,3}Z],$$

where $\{X, Y, Z\}$ are the homogenous coordinates of $\mathbb{P}_L^2$.

Definition. By *a smooth $\overline{k}$-plane curve C over k* we mean a smooth curve over k, that is $\overline{k}$-isomorphic to a non-singular plane model $F_{\overline{C}}(X,Y,Z) = 0$ in $\mathbb{P}^2_{\overline{k}}$, where $F_{\overline{C}}(X,Y,Z)$ is a homogenous polynomial of degree $d \geq 4$ with coefficients in $\overline{k}$. In this case, we say that C/k admits a non-singular plane model of degree d over $\overline{k}$.

We note that any other non-singular plane model for $\overline{C}$ over $\overline{k}$ has the form $F_{\phi^{-1}\overline{C}}(X,Y,Z) = 0$ for some $\phi \in \mathrm{PGL}_3(\overline{k})$, where $F_{\phi^{-1}\overline{C}}(X,Y,Z) := F_{\overline{C}}(\phi(X,Y,Z))$. Moreover, the automorphism group $\mathrm{Aut}(F_{\phi^{-1}\overline{C}})$ of $F_{\phi^{-1}\overline{C}}(X,Y,Z)=0$ is a finite subgroup of $\mathrm{PGL}_3(\overline{k})$, and it is equal to $\phi^{-1}\mathrm{Aut}(F_{\overline{C}})\phi$. For $\phi, \psi \in \mathrm{PGL}_3(\overline{k})$, the natural map of smooth plane curves over $\overline{k}$:

$$\overline{C} \xrightarrow{\phi^{-1}} \phi^{-1}\overline{C} \xrightarrow{\psi^{-1}} \psi^{-1}\phi^{-1}\overline{C} = (\phi\psi)^{-1}\overline{C}$$

corresponds to

$$\{F_{\overline{C}}(X,Y,Z)=0\} \xrightarrow{\phi^{-1}} \{F_{\phi^{-1}\overline{C}}(X,Y,Z)=0\} \xrightarrow{\psi^{-1}} \{F_{(\phi\psi)^{-1}\overline{C}}(X,Y,Z)=0\}.$$

Given a smooth $\overline{k}$-plane curve C over k, we say that C admits a non-singular plane model over L with $k \subseteq L \subseteq \overline{k}$, if there exists $\phi \in \mathrm{PGL}_3(\overline{k})$ with $F_{\phi^{-1}\overline{C}}(X,Y,Z) \in L[X,Y,Z]$, and such that $C \otimes_k L$ and $F_{\phi^{-1}\overline{C}}(X,Y,Z)=0$ are isomorphic over L.

If a smooth $\overline{k}$-plane curve C over k admits a non-singular plane model $F_{\phi^{-1}\overline{C}}(X,Y,Z)=0$ over k, then we call C *a smooth plane curve over* k, and we identify, by an abuse of notation, C with the plane model $F_{\phi^{-1}\overline{C}}(X,Y,Z) = 0$ and $\mathrm{Aut}(\overline{C})$ with $\mathrm{Aut}(F_{\phi^{-1}\overline{C}})$ as a fixed finite subgroup of $\mathrm{PGL}_3(\overline{k})$.

- The group of diagonal matrices in $\mathrm{PGL}_3(\overline{k})$ is denoted by $\mathrm{D}(\overline{k})$, and by $T_X(\overline{k})$, we mean its subgroup of all 3×3 projective linear matrices of the shape $[\lambda X : Y : Z]$ for some $\lambda \in \overline{k}$. Symmetrically, one defines $T_Y(\overline{k})$ and $T_Z(\overline{k})$.

- The Hessian groups: The group Hess_9 of order 9 generated by $T := [Y : Z : X]$ and $S := \mathrm{diag}(1, \zeta_3, \zeta_3^2)$.

  The group Hess_{18} of order 18 generated by $R := [X : Z : Y]$ and Hess_9.

  The group Hess_{36} of order 36 generated by Hess_{18} and

$$V := \begin{pmatrix} 1 & 1 & 1 \\ 1 & \zeta_3 & \zeta_3^2 \\ 1 & \zeta_3^2 & \zeta_3 \end{pmatrix}.$$

The group Hess_{72} of order 72 generated by Hess_{36} and UVU^{-1}, where $U := \mathrm{diag}(1, 1, \zeta_3)$.

The group Hess_{216} of order 216 generated by Hess_{72} and U.

- Alternating groups: The group A_5 of order 60 generated by $E_1 := \mathrm{diag}(1, \zeta_5^4, \zeta_5)$, $E_2 := [X : Z : Y]$, and
$$E_3 := \begin{pmatrix} 1 & 1 & 1 \\ 1 & \zeta_5^2 + \zeta_5^{-2} & \zeta_5 + \zeta_5^{-1} \\ 1 & \zeta_5 + \zeta_5^{-1} & \zeta_5^2 + \zeta_5^{-2} \end{pmatrix}.$$
The group A_6 of order 360 generated by E_1, E_2, E_3, and
$$E_4 := \begin{pmatrix} 1 & \nu_1 & \nu_1 \\ 2\nu_2 & \zeta_5^2 + \zeta_5^{-2} & \zeta_5 + \zeta_5^{-1} \\ 2\nu_2 & \zeta_5 + \zeta_5^{-1} & \zeta_5^2 + \zeta_5^{-2} \end{pmatrix},$$
where $\nu_1 := \frac{1}{4}(-1 + \sqrt{-15})$ and $\nu_2 := \frac{1}{4}(-1 - \sqrt{-15})$.

- The Klein group: The group $\mathrm{PSL}_2(\mathbb{F}_7)$ of order 168 generated by $F_1 := \mathrm{diag}(\zeta_7, \zeta_7^2, \zeta_7^4)$, $F_2 := [Y : Z : X]$, and
$$\begin{pmatrix} a' & b' & c' \\ b' & c' & a' \\ c' & a' & b' \end{pmatrix},$$
where $a' := \zeta_7^4 - \zeta_7^3$, $b' := \zeta_7^2 - \zeta_7^5$, and $c' := \zeta_7 - \zeta_7^6$.

- The image of the natural inclusion $\mathrm{GL}_2(\overline{k}) \hookrightarrow \mathrm{PGL}_3(\overline{k})$ given by
$$A \mapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix},$$
is denoted by $\mathrm{GL}_{2,X}(\overline{k})$. In the same way, we can define $\mathrm{GL}_{2,Y}(\overline{k})$ and $\mathrm{GL}_{2,Z}(\overline{k})$.

- Finally, the following subgroups of $\mathrm{PGL}_3(\overline{k})$; $\tilde{\mathrm{S}}_3 := \langle [X : Z : Y], [Z : X : Y] \rangle$, $G_{03} := \langle [X : Z : Y], [X : \zeta_3 Y : Z] \rangle$, and $G_{05} := \langle [Z : Y : X], [X : Y : \zeta_5 Z] \rangle$ are also considered.

We write Gal(L/k) for the Galois group of L/k, where L is an extension of k inside $\overline{k}$, and also we consider left actions. The Galois cohomology sets of a Gal(L/k)-group G, where L/k is Galois, are denoted by $\mathrm{H}^i(\mathrm{Gal}(L/k), G)$ with $i \in \{0, 1\}$ respectively. For the particular case $L = \overline{k}$ and k is perfect, we use G_k instead of $\mathrm{Gal}(\overline{k}/k)$ and $\mathrm{H}^1(k, G)$ instead of $\mathrm{H}^1(\mathrm{Gal}(\overline{k}/k), G)$.

We use the formal GAP library notations "GAP(n, m)" to refer the finite group of order n, appearing in the mth position of the atlas for small finite groups [Gro].

The abbreviation "CSA" means a central simple algebra. The set of all Brauer equivalence classes of CSAs over k is called the Brauer group of k and is denoted by $\mathrm{Br}(k)$. The set of all equivalence classes of CSAs of dimension n^2 over k modulo k-algebras isomorphisms is denoted by Az_n^k. The n-torsion of $\mathrm{Br}(k)$ is $\mathrm{Br}(k)[n]$.

INTRODUCTION

Smooth projective curves over a field k with non-trivial automorphism group are always of deep interest in the literature. For instance, in algebraic geometry, the structure of the automorphism groups of smooth curves of genus $g \geq 2$ defined over an algebraically closed field $\overline{k}$ is an old subject of research. One finds a lot of work trying to understand the different strata $\mathcal{M}_g(G)$ of the moduli space $\mathcal{M}_g$, representing smooth curves C of genus g which have a finite non-trivial group G as a subgroup of automorphisms. We mention, for example, the most famous universal bound, the so-called *Hurwitz bound* (see Theorem 1.1), given by Hurwitz [Hur92] as an application of Riemann-Hurwitz formula, which turns out to be sharp for infinitely many genera. Oikawa [Oik56, Theorem 1] and Arakawa [Ara00, Theorem 3] gave even better upper bounds when the automorphism group fixes (not necessarily pointwise) finite subsets of points on the curve. These bounds become very useful in our study of smooth plane curves. Also, we may ask about irreducibility of $\mathcal{M}_g(G)$ as a subset of the moduli space $\mathcal{M}_g$ or even about the existence of universal families to recover information on its points. In arithmetic geometry, Fermat and Klein curves are quite well known examples of smooth curves with non-trivial automorphism group, and so many interesting arithmetic properties, see for example [CHW17, CM88, FGL16, HS13, JSW07, LM08, MT03, Sch86, Shi88] for Fermat curves, and [BN10, Elk99, Far10, KFR00, Tze04] for Klein curves. On the other hand, the term *dessin d'enfant*, which appears in a set of notes in Alexander Grothendieck's *Esquisse d'un Programme* written and circulated in 1984 but not published until 1997, received so much attention from the mathematical community during the last 30 years. In this sense, complex smooth curves (i.e. compact Riemann surfaces) with many automorphisms are also of arithmetic interests, viewed as dessins over a number field, see [MR194, JW16].

Following the philosophy of Diophantine equations theory, the simplest case is to consider smooth plane curves over a field k of geometric genus $g \geq 3$. That is, smooth projective curves C over k, which is k- isomorphic to the zero locus in $\mathbb{P}^2_k$ of a homogenous polynomial $F(X,Y,Z)\in k[X,Y,Z]$ of degree $d\geq 4$ without singularities. In particular, the curve $\overline{C}=C\otimes_k\overline{k}$ has a g^2_d-linear system that allows us to embed $\overline{C}\overset{g^2_d}{\hookrightarrow}\mathbb{P}^2_{\overline{k}}$. By elementary algebraic geometry (Riemann-Hurwitz formula, Bézout's theorem,...etc), one can show that $\overline{C}$ is non-hyperelliptic of genus $g=(d-1)(d-2)/2$. Furthermore, the g^2_d-linear system is unique up to conjugation in $\mathrm{Aut}(\mathbb{P}^2_{\overline{k}})=\mathrm{PGL}_3(\overline{k})$, the 3-dimensional projective general linear group. Hence, any two non-singular plane models (there are infinitely many) for $\overline{C}$ in $\mathbb{P}^2_{\overline{k}}$ are isomorphic via a change of variables in $\mathrm{PGL}_3(\overline{k})$, and the corresponding automorphism groups are conjugate. That is to say $\mathrm{Aut}(\overline{C})$ can always be viewed as a finite subgroup of $\mathrm{PGL}_3(\overline{k})$, fixing a certain non-singular plane model of $\overline{C}$ in $\mathbb{P}^2_{\overline{k}}$.

In the thesis, we study the stratification of smooth plane curves by their automorphism groups, and we deal with both algebraic and arithmetic geometry aspects. We are going now to detail a bit the study and the contributions resulted herein:

Automorphism groups and normal forms

The structure of the automorphism group is quite explicit for hyperelliptic curves ([BEMn87, BGG93, Sha03, SS07]). For non-hyperelliptic curves, it seems that we still have a lack of knowledge about the structure, except for some special cases. For example, the cases of low genus ([Bre00, Hen76, KK86, KK90b, KK90a] and Hurwitz curves, i.e. smooth curves that attains the Hurwitz bound. This in turns motivated us for more investigations in this direction, and we restrict ourselves to the case of smooth plane curves of degree $d\geq 4$.

Consider the stratum $\mathcal{M}^{Pl}_g(G)\subset\mathcal{M}_g$, consisting of the $\overline{k}$-isomorphism classes of smooth plane curves $\overline{C}$ of genus $g=\frac{1}{2}(d-1)(d-2)\geq 3$ such that $\mathrm{Aut}(\overline{C})$ contains a subgroup isomorphic to G. Similarly, we write $\widetilde{\mathcal{M}^{Pl}_g}(G)$ when $\mathrm{Aut}(\overline{C})$ is itself isomorphic to G, in particular, $\widetilde{\mathcal{M}^{Pl}_g}(G)\subseteq\mathcal{M}^{Pl}_g(G)$.

The next two classical questions appears naturally for $\widetilde{\mathcal{M}^{Pl}_g}(G)$;

Question A. *Let G be a finite non-trivial group. Which are the genera $g\geq 3$ so that the strata $\widetilde{\mathcal{M}^{Pl}_g}(G)\neq\emptyset$, i.e. there exists a smooth plane curve $\overline{C}$ over $\overline{k}$ of genus g with $\mathrm{Aut}(\overline{C})\simeq G$?*

For example, by the work of S. Crass in [Cra99, p.28], we know that $\widetilde{\mathcal{M}_g^{Pl}}(\mathrm{A}_6) \neq \emptyset$ exactly for $g = 10$, $g = 55$ and $g = 406$, where A_6 is the alternating group of order 6.

Reversely, we might ask:

Question B. *Fix an integer $g \geq 3$. How does it look like the stratification by non-trivial automorphism groups of $\mathcal{M}_g^{Pl}$, representing the $\overline{k}$-isomorphism classes of smooth plane curve $\overline{C}$ of genus g? More precisely, to describe the possible list of the finite non-trivial groups G, for which $\widetilde{\mathcal{M}_g^{Pl}}(G) \neq \emptyset$, and also to give families which helps to recover information on the $\overline{k}$-points of $\widetilde{\mathcal{M}_g^{Pl}}(G)$ or even of $\mathcal{M}_g^{Pl}(G)$.*

P. Henn in [Hen76] and Komiya-Kuribayashi in [KK79], obtained the answer for Question B when $g = 3$ (or equivalently, when $d = 4$) and $\overline{k}$ has zero characteristic, see Theorem 2.2.1.

Definition C. *We define the* associated normal forms to $\mathcal{M}_g^{Pl}(G)$ *to be a finite set of homogenous equations in X, Y, Z, such that any $\overline{k}$-point $\overline{C}$ of $\mathcal{M}_g^{Pl}(G)$ is $\overline{k}$-isomorphic to a non-singular plane model $F_{\overline{C}}(X, Y, Z) = 0$ through a specialization of the parameters in one of these equations, and vice versa. Similarly, we define the* associated normal forms to $\widetilde{\mathcal{M}_g^{Pl}}(G)$*; see Definition 2.4.*

The recent work of Lercier-Ritzenthaler-Rovetta-Sisling in [LRRS14, §2] helps to understand more (geometrically) the terminology of normal forms. *In their language, the associated normal forms to $\widetilde{\mathcal{M}_g^{Pl}}(G)$ is a* geometrically complete family over k for $\widetilde{\mathcal{M}_g^{Pl}}(G)$.

In particular, for $g = 3$ the stratum $\widetilde{\mathcal{M}_3^{Pl}}(G)$ is always described by a single normal form, whenever it is non-empty. In general, we have an obvious union decomposition for $\widetilde{\mathcal{M}_g^{Pl}}(G)$ defined in the following way: Fix an injective representation $G \overset{\varrho}{\hookrightarrow} \mathrm{PGL}_3(\overline{k})$. Next, define $\varrho(\mathcal{M}_g^{Pl}(G))$ to be the component of $\mathcal{M}_g^{Pl}(G)$, consisting of all $\overline{k}$-points $\overline{C} \in \mathcal{M}_g^{Pl}(G)$ such that $\varrho(G)$ acting as a subgroup of automorphisms of some non-singular plane model $F_{\overline{C}}(X, Y, Z) = 0$ for $\overline{C}$ in $\mathbb{P}^2_{\overline{k}}$. In the same way, one defines $\varrho(\widetilde{\mathcal{M}_g^{Pl}}(G))$ when $\varrho(G)$ is the full automorphism group; see Definition 2.2. It is easy to see that $\varrho(\widetilde{\mathcal{M}_g^{Pl}}(G))$ is always given by a single normal form, and moreover

$$\widetilde{\mathcal{M}_g^{Pl}}(G) = \bigsqcup_{[\varrho] \in A_G} \varrho(\widetilde{\mathcal{M}_g^{Pl}}(G)),$$

where A_G denotes the set of all injective representations of G inside $\mathrm{PGL}_3(\overline{k})$, modulo conjugation and $[\varrho]$ is the equivalence class of ϱ in A_G, see Lemma 2.2.4.

Using these notations, Henn, Komiya-Kuribayashi proved that any $\widetilde{\mathcal{M}_3^{Pl}}(G)$ which is not empty, coincides with $\varrho(\widetilde{\mathcal{M}_3^{Pl}}(G))$ for some $\varrho \in A_G$. By the work of Cornalba [Cor87] and Catanese [Cat12], it becomes sensible to think about any non-empty stratum $\varrho(\widetilde{\mathcal{M}_3^{Pl}}(G))$ as a subset of an irreducible set of smooth curves of genus $g = 3$, admitting Galois covers with a prescribed ramification data (see §2.2). Thus, it is probably an irreducible subset of the moduli space $\mathcal{M}_g$.

In chapter 2, we aim to give a wide study for $\widetilde{\mathcal{M}_g^{Pl}}(G)$, where we restrict $\overline{k}$ to have characteristic $p = 0$ or $p > 2g + 1$ to ensure that p does not divide the order of the full automorphism group. For instance, we prove:

Theorem D. *Fix an integer $g = (d-1)(d-2)/2$ with $d \geq 4$. We give a way to describe all the pairs $(\varrho,\, G = \mathbb{Z}/m\mathbb{Z})$, where $\mathbb{Z}/m\mathbb{Z}$ is the cyclic group of order $m > 1$ and $\mathbb{Z}/m\mathbb{Z} \overset{\varrho}{\hookrightarrow} \mathrm{PGL}_3(\overline{k})$, such that $\varrho(\mathcal{M}_g^{Pl}(G))$ might be non-empty. Also, we associate a generic single normal form for each pair $(\varrho,\, G = \mathbb{Z}/m\mathbb{Z})$, where $\varrho(\mathbb{Z}/m\mathbb{Z})$ acts on its members. In particular, m should divide one of the integers: $d(d-1)$, $d^2 - 3d + 3$, $(d-1)^2$ or $d(d-2)$.*

Theorem E. *Let $\overline{C}$ be a smooth plane curve of degree $d \geq 5$ over $\overline{k}$. Suppose that there exists an automorphism $\sigma \in \mathrm{Aut}(\overline{C})$ of exact order $m = d(d-1)$, $d^2 - 3d + 3$, $(d-1)^2$ or $d(d-2)$. Then, $\mathcal{M}_g^{Pl}(\mathbb{Z}/m\mathbb{Z})$ is an irreducible set formed by a single point, such that one of the following situations holds:*

1. *If σ has order $d(d-1)$, then $\mathrm{Aut}(\overline{C}) = \langle \sigma \rangle$ and $\overline{C}$ is $\overline{k}$-isomorphic to*

$$X^d + Y^d + XZ^{d-1} = 0.$$

Moreover,

$$\widetilde{\mathcal{M}_g^{Pl}}(\mathbb{Z}/d(d-1)\mathbb{Z}) = \mathcal{M}_g^{Pl}(\mathbb{Z}/d(d-1)\mathbb{Z}) = \varrho(\mathcal{M}_g^{Pl}(\mathbb{Z}/d(d-1)\mathbb{Z})),$$

with $\varrho(\mathbb{Z}/d(d-1)\mathbb{Z}) = \langle \mathrm{diag}(1, \zeta_{d(d-1)}^{d-1}, \zeta_{d(d-1)}^{d}) \rangle$.

2. *If σ has order $(d-1)^2$, then $\mathrm{Aut}(\overline{C}) = \langle\sigma\rangle$ and $\overline{C}$ is $\overline{k}$-isomorphic to*

$$X^d + Y^{d-1}Z + XZ^{d-1} = 0.$$

Moreover,

$$\widetilde{\mathcal{M}_g^{Pl}}(\mathbb{Z}/(d-1)^2\mathbb{Z}) = \mathcal{M}_g^{Pl}(\mathbb{Z}/(d-1)^2\mathbb{Z}) = \varrho(\mathcal{M}_g^{Pl}(\mathbb{Z}/(d-1)^2\mathbb{Z})),$$

with $\varrho(\mathbb{Z}/(d-1)^2\mathbb{Z}) = \langle\mathrm{diag}(1, \zeta_{(d-1)^2}, \zeta_{(d-1)^2}^{(d-1)(d-2)})\rangle$.

3. *If σ has order $d(d-2)$, then $\overline{C}$ is $\overline{k}$-isomorphic to*

$$X^d + Y^{d-1}Z + YZ^{d-1} = 0$$

and when $d \neq 6$, we obtain

$$\mathrm{Aut}(\overline{C}) \simeq H_d := \langle \sigma, \tau \,:\, \tau^2 = \sigma^{d(d-2)} = 1,\ and\ \tau\sigma\tau = \sigma^{-(d-1)}\rangle.$$

Moreover,

$$\widetilde{\mathcal{M}_g}(H_d) = \mathcal{M}_g^{Pl}(\mathbb{Z}/d(d-2)\mathbb{Z}) = \varrho(\mathcal{M}_g^{Pl}(\mathbb{Z}/d(d-2)\mathbb{Z})),$$

with $\varrho(\mathbb{Z}/d(d-2)\mathbb{Z}) = \langle\mathrm{diag}(1, \zeta_{d(d-2)}, \zeta_{d(d-2)}^{-(d-1)})\rangle$.

4. *If σ has order d^2-3d+3, then $\overline{C}$ is $\overline{k}$-isomorphic to the Klein curve*

$$K_d : X^{d-1}Y + Y^{d-1}Z + Z^{d-1}X = 0$$

and

$$\mathrm{Aut}(\overline{C}) \simeq H_{K_d} := \langle \sigma, \tau | \sigma^{d^2-3d+3} = \tau^3 = 1\, and\ \sigma\tau = \tau\sigma^{-(d-1)}\rangle.$$

Moreover,

$$\widetilde{\mathcal{M}_g}(H_{K_d}) = \mathcal{M}_g^{Pl}(\mathbb{Z}/(d^2-3d+3)\mathbb{Z}) = \varrho(\mathcal{M}_g^{Pl}(\mathbb{Z}/(d^2-3d+3)\mathbb{Z})),$$

with $\varrho(\mathbb{Z}/(d^2-3d+3)\mathbb{Z}) = \langle\mathrm{diag}(1, \zeta_{d^2-3d+3}, \zeta_{d^2-3d+3}^{-(d-2)})\rangle$.

We also studied in section §2.4.2 the cases when $\overline{C}$ has an automorphism σ of exact order ℓd, $\ell(d-1)$ $\ell(d-2)$ for some $\ell \geq 2$.

In contrast to the degree $d = 4$ case, the stratum $\widetilde{\mathcal{M}_g^{Pl}}(G)$ for $g = \frac{1}{2}(d-1)(d-2)$ with $d \geq 5$ may decompose into more than one component of the form $\varrho(\widetilde{\mathcal{M}_g^{Pl}}(G))$. For example, we show:

Theorem F. *For any odd degree $d \geq 5$, the stratum $\widetilde{\mathcal{M}_g^{Pl}}(\mathbb{Z}/(d-1)\mathbb{Z})$ is a disjoint union decomposition of at least two components $\varrho_i(\widetilde{\mathcal{M}_g^{Pl}}(G))$, for $i = 1, 2$, where $\varrho_i \in A_G$ with $G = \mathbb{Z}/(d-1)\mathbb{Z}$. The same is true when $d = 6$ and $G = \mathbb{Z}/3\mathbb{Z}$, that is for $\widetilde{\mathcal{M}_{10}^{Pl}}(\mathbb{Z}/3\mathbb{Z})$.*

We go further, in section §4.1, by turning our attention to the Question B when $d = 5$. In particular, we get:

Theorem G. *A complete determination of the pairs (ϱ, G), such that $\widetilde{\mathcal{M}_6^{Pl}}(G) \neq \emptyset$, is given. The associated normal forms to each non-empty stratum is also provided.*

In our way to do this, we observed a new interesting phenomenon, which does not appear for $g = 3$: Let $\mathcal{F}_{\varrho,G,g}(X, Y, Z) = 0$ be a normal form that describes the stratum $\varrho(\mathcal{M}_g^{Pl}(G))$. One could expect that by adding restrictions to its parameters, one get bigger automorphism groups until obtaining a zero-dimensional stratum. This happens for all the families of degree $d = 5$ except for one. For this family each restriction in the parameters providing a bigger automorphism group yields a singular curve. If this is the case for some $\varrho(\mathcal{M}_g^{Pl}(G))$, then we call it *final*. This phenomenon can be explained very well by using the family of the canonical models in $\mathbb{P}^{g-1}_{\overline{k}}$ as we will see later in section §4.2. Moreover, we prove:

Theorem H. *For any integer $g = \frac{1}{2}(d-1)(d-2)$ with $d \geq 5$ and $d \equiv 1 \,(\mathrm{mod}\, 4)$, the stratum $\varrho(\mathcal{M}_g^{Pl}(\mathbb{Z}/(d-1)\mathbb{Z}))$ is non-zero dimensional final, where $\varrho(\mathbb{Z}/(d-1)Z) = \langle \mathrm{diag}(1, -1, \zeta_{d-1}) \rangle$.*

Plane-models fields of definition and Twists

Let $\overline{k}$ be a fixed algebraic closure of perfect field k. By a smooth $\overline{k}$-plane curve C over k, we mean a smooth projective curve C defined over k, such that $\overline{C} = C \otimes_k \overline{k}$ is a smooth plane curve. We aim to study fields of definition for non-singular plane models of C and also of its twists over k by considering the embedding $\mathrm{Aut}(\overline{C}) \hookrightarrow \mathrm{PGL}_3(\overline{k})$ instead of the one given by the canonical model, see [LG14, Chp. 1] or Appendix B.

Recall that the set of all twists of a quasi-projective variety V over k, denoted by $\mathrm{Twist}_k(V)$, is in bijection with the first Galois cohomology set $\mathrm{H}^1(\mathrm{Gal}(\overline{k}/k), \mathrm{Aut}(V \otimes_k \overline{k}))$. Furthermore, from the work of Roé-Xarles in [RX14] and since $\overline{C}$ has a (unique) g^2_d-linear system, then there exists a Brauer-Severi surface D defined over k (i.e. a twist of $\mathbb{P}^2_k$), together with a k-morphism $g : C \hookrightarrow D$, such that $g \otimes_k \overline{k} : \overline{C} \to \mathbb{P}^2_{\overline{k}}$ coincides with $\overline{C} \overset{g^2_d}{\hookrightarrow} \mathbb{P}^2_{\overline{k}}$. Therefore, we have a natural map of sets,

$$\Sigma : \mathrm{Twist}_k(C) = \mathrm{H}^1(\mathrm{Gal}(\overline{k}/k), \mathrm{Aut}(\overline{C})) \to \mathrm{Twist}_k(D) = \mathrm{H}^1(\mathrm{Gal}(\overline{k}/k), \mathrm{Aut}(B \times_k \overline{k}) = \mathrm{PGL}_3(\overline{k})).$$

This approach leads to two natural questions:

Question I. *The first one, given a smooth $\overline{k}$-plane curve C defined over a perfect field k, is it a smooth plane curve over k?; and secondly, if the answer is yes, is every twist of C over k also a smooth plane curve over k?,*

For both questions the answer is no in general, it is not. The next result concerns the negative general answer, where the full details can be found in chapter 3:

Theorem J. *Let us consider $\mathbb{Q}_f$ the splitting field of the irreducible polynomial $f(t) = t^3 + 12t^2 - 64$ over $\mathbb{Q}$, and denote the roots of f by a, b, c in a fixed algebraic closure of $\mathbb{Q}$. The smooth plane curve over $\mathbb{Q}_f$*

$$C :\ 64Z^6 + abY^6 + aX^6 + 8Y^3Z^3 + \frac{ab}{8}X^3Y^3 + aZ^3X^3 = 0,$$

has $\mathbb{Q}$ as a field of definition, but it does not admit a non-singular plane model over $\mathbb{Q}$.

Also, we obtain results for the curves for which the above questions always have an affirmative answer:

Theorem K. *Let C be a smooth $\overline{k}$-plane curve of degree $d \geq 4$ defined over a perfect field k. Then, C is a smooth plane curve over k if one of the following conditions holds:*

1. *C has a k-rational point,*

2. *the degree d is coprime with 3,*

3. *the 3-torsion $\mathrm{Br}(k)[3]$ of the Brauer group $\mathrm{Br}(k)$ of k is trivial.*

Moreover, there always exists a field extension L/k inside $\overline{k}$ of index $[L:k]$ dividing 3, such that $C\otimes_k L$ is a smooth plane curve over L.

We proceed with the second part of Question I:

Theorem L. *Assume that the curve C, or any of its twists over a perfect field k, is a smooth plane curve over k. Then, we have an embedding of $\mathrm{Gal}(\overline{k}/k)$-groups for its automorphisms group into PGL$_3(\overline{k})$. In particular, $\Sigma^{-1}([\mathbb{P}^2_k])$ corresponds to the set of all twists of C, which are smooth plane curves over k, where $[\mathbb{P}^2_k]$ denotes the class of the trivial twist of $\mathbb{P}^2_k$. This allows us to construct in chapter 3 (explicit) twists of C that are not smooth plane curves over k, and living inside a non-trivial Brauer-Severi surface.*

Now, assume that C is a smooth curve over k with a plane non-singular model over k such that the image of $\mathrm{Twist}_k(C)=\mathrm{H}^1(\mathrm{Gal}(\overline{k}/k),\mathrm{Aut}(\overline{C}))$ under the map Σ is trivial. In such case, all the twist admits a non-singular plane model over k. Therefore, to compute equations for the twists it is enough to look for isomorphisms in $\mathrm{GL}_3(\overline{k})$ instead of $\mathrm{GL}_g(\overline{k})$. As in [LG14, LG17] the elements to reach for solutions in $\mathrm{GL}_g(\overline{k})$ or $\mathrm{GL}_3(\overline{k})$ is quite hard except that we have a control of the matrix that could appear. In this direction, we prove:

Proposition M. *Let $C:\ F_{\overline{C}}(X,Y,Z)=0$ be a smooth plane curve over a perfect field k. Assume that $\mathrm{Aut}(F_{\overline{C}})\subseteq\mathrm{PGL}_3(\overline{k})$ is a non-trivial cyclic group of order n (relatively prime with the characteristic of k), generated by an automorphism $\alpha=\mathrm{diag}(1,\zeta_n^a,\zeta_n^b)$ for some $a,b\in\mathbb{N}$.*

Then all the twists of C are diagonal, i.e. the elements of $\mathrm{Twist}_k(C)$ are given by non-singular plane equations of the form $F_{D^{-1}\overline{C}}(X,Y,Z)=0$ with $F_{D^{-1}\overline{C}}(X,Y,Z)\in k[X,Y,Z]$ and D is a 3×3 projective linear matrix of diagonal shape.

Complete and representative families

It is well known that the (coarse) moduli spaces $\mathcal{M}_g$ are algebraic varieties whose geometric points give a classification of isomorphism classes of smooth curves of genus g over $\overline{k}$. The existence of universal families for a moduli space helps to recover the information on its points and allows to write down the attached objects to a point of this space. However, universal families do not exist for the moduli space $\mathcal{M}_g$. Lercier-Ritzenthaler-Rovetta-Sisling in [LRRS14, §2] introduced three good substitutes for the notion of universal family in our case: complete, finite and representative families. Among the three substitutes, the representative families (if exist) are the best. The authors of [LRRS14] give explicit representative families over a perfect field k of characteristic $p = 0$ or $p \geq 7$ for all the different strata $\mathcal{M}_3^{Pl}(G)$ of smooth $\overline{k}$-plane curves over k, except for $G = \mathbb{Z}/2\mathbb{Z}$. For the remaining situation, when $|G| = 2$, they prove that representative families for the stratum of $\mathbb{Z}/2\mathbb{Z}$ fail to exist even if k is a finite field.

In section 4.3, we start with the classification already obtained in section §4.1 for the different strata of smooth $\overline{k}$-plane curves of genus $g = 6$ of the form $\varrho(\mathcal{M}_6^{Pl}(G))$. After, we mimic the techniques in [LRRS14] to give explicit descriptions (when possible) for representative families over a perfect field k. In particular, we prove:

Theorem N. *Let k be a perfect field of characteristic $p = 0$ or $p > 13$. Then, any non-empty stratum $\varrho(\mathcal{M}_6^{Pl}(G))$ has a representative family over k. Furthermore, these families are explicit for all strata, except when $G = \mathbb{Z}/5\mathbb{Z}$.*

The field of moduli versus fields of definition

Let C be a smooth projective curve of genus g defined over a field k. The field of moduli of C, denoted by k_C, is the intersection over all fields of definition of the base extension $C \otimes_k \overline{k}$ (see Definition 5.1.2). There is another definition for the field of moduli, relative to a given field extension L/k, which is commonly used (see Definition 5.1.3); Given a smooth curve C/L, the field of moduli $M_{L/k}(C)$ of C relative to L/k is the fixed subfield of L by the subgroup

$$U_{L/k}(C) := \{\sigma \in \mathrm{Gal}(L/k) \,|\, C \text{ isomorphic over } L \text{ to } {}^{\sigma}C\}.$$

Due to S. Koizumi [Koi72, Proposition 2.3-(ii)] we know that, $M_{\overline{k}/k_0}(C)$ is a purely inseparable extension of k_C, where k_0 is the prime field of k.

In the case that C is a $\overline{k}$-smooth plane curve of genus $g \geq 3$, where k is a perfect field k of characteristic $p \neq 2$, B. Huggins in [Hug05] showed that the field of moduli $M_{\overline{k}/k}(C)$ for C, relative to the Galois extension $\overline{k}/k$, is a field of definition if $\mathrm{Aut}(\overline{C})$ is not $\mathrm{PGL}_3(\overline{k})$-conjugate to a diagonal subgroup of $\mathrm{PGL}_3(\overline{k})$, one of the Hessian groups Hess_* with $* \in \{18, 36\}$, or to a semidirect product of a finite diagonal subgroup of $\mathrm{PGL}_3(\overline{k})$ and a non-trivial p-group consisting entirely of elements of specific shapes. Moreover, an example of a smooth plane curve over $\overline{k}$ that is not definable over its field of moduli is also given for each subcase, see [Hug05, Chps. 6, 7].

Because of the above results, we were motivated in chapter 5 to answer the next question:

Question O. *Let C be a smooth $\overline{k}$-plane curve of genus $g \geq 3$, where k is a field of characteristic $p = 0$ or $p > 2g+1$, such that $\varrho(\mathrm{Aut}(\overline{C})) \leq \mathrm{PGL}_3(\overline{k})$ is made exclusively of diagonal 3×3 projective matrices, for some injective representation ϱ. When $M_{\overline{k}/k}(C)$ is a field of definition for C?*

We introduce in chapter 5 further improvements of the work of B. Huggins in [Hug05], relating to the particular situation in Question O. They can be used as a constructive source of examples of smooth plane curves with diagonal automorphism groups, not definable over their field of moduli.

Mainly, we show:

Theorem P. *Let C be a smooth $\overline{k}$-plane curve of genus $g = \frac{1}{2}(d-1)(d-2) \geq 3$, where k is a field of characteristic $p = 0$ or $p > 2g+1$, such that $\varrho(\mathrm{Aut}(\overline{C})) \leq \mathrm{PGL}_3(\overline{k})$ is made exclusively of diagonal 3×3 projective matrices, for some injective representation ϱ. Then,*

1. *if $\mathrm{Aut}(\overline{C})$ contains a non-homology of order $n > 1$ (see Definition 1.2.6), then the field of moduli $M_{\overline{k}/k}(C)$ is always a field of definition, unless n divides one of the integers d, $d-1$ or $d(d-2)$.*

2. *if $\mathrm{Aut}(\overline{C})$ does not contain a non-homology, then it is either cyclic of order dividing d or $d-1$, or it is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Moreover, if $\mathrm{Aut}(\overline{C})$ is cyclic of order dividing*

d with d odd or dividing $d-1$, then again $M_{\overline{k}/k}(C)$ needs to be a field of definition for C.

We also provide a geometrically complete family in each subcase where the field of moduli might not be a field of definition. Finally, we construct (explicit) examples of smooth plane curves over the complex field $\mathbb{C}$, whose field of moduli relative to the Galois extension $\mathbb{C}/\mathbb{R}$ is $\mathbb{R}$, but it is not a field of definition.

Contents of the chapters

We aim in chapter 1 to survey the most well known results in the literature about the classification of automorphism groups of smooth plane curves over algebraically closed fields, which will be useful for our purposes for smooth plane curves. The structure of this chapter is as follows: In section 1.1, we recall a (scheme theoretic) proof of the fact $\mathrm{Aut}(\mathbb{P}^{n-1}_{\overline{k}}) = \mathrm{PGL}_n(\overline{k})$, that is automorphisms of the $n-1$-dimensional projective space $\mathbb{P}^{n-1}_{\overline{k}}$ are linear; Theorem 1.1.1. In particular, any isomorphism between two smooth plane curves of degree $d \geq 4$ over $\overline{k}$ is induced by a 3×3 projective linear matrix; Theorem 1.1.5. Next, in section 1.2, we recall the determination of the finite subgroups of $\mathrm{PGL}_3(\overline{k})$, which is well understood in the subject. We start with the classification made by H. Mitchell [Mit11], based entirely on geometrical methods; Theorem 1.2.1. A detailed study of this geometric classification would lead to an extended version, including a very good description of all possible finite subgroups of $\mathrm{PGL}_3(\overline{k})$; Theorem 1.2.4. The notion of *Galois points* for smooth plane curves is presented in section 1.3. As far as we know, it was first introduced by H. Yoshihara in 1996, see [Fuk09, MY00, Yos01]. Galois points (Definition 1.3.4) is a bit useful tool when one wants to compute the full automorphism group in some cases of smooth plane curves; Theorem 1.3.12. We end up this chapter with section 1.4, where the classification of the automorphism groups of smooth plane curves of degree $d \geq 4$ over algebraically closed fields of zero characteristic is given by T. Harui, in his unpublished paper in arXiv [Har13]; Theorem 1.4.4. The result still true for positive characteristic $p > (d-1)(d-2)+1$, as will be seen at the end of this section.

Dolgachev in [Dol12] determined the $\varrho's$ and $m's$ for which $\varrho(\mathcal{M}_3^{Pl}(\mathbb{Z}/m\mathbb{Z})) \neq \emptyset$. The defining equation of each non-empty $\varrho(\mathcal{M}_3^{Pl}(\mathbb{Z}/m\mathbb{Z}))$ is also given. On the other hand, P.

Henn in [Hen76] and Komiya-Kuribayashi in [KK79], provided the list of $\varrho' s$ and $G' s$ such that $\varrho(\mathcal{M}_3^{Pl}(G))$ and $\varrho(\widetilde{\mathcal{M}_3^{Pl}}(G))$ are non-empty. Moreover, the associated *normal forms* to each non-empty $\widetilde{\mathcal{M}_3^{Pl}}(G)$ are determined (Theorem 2.2.1). See also E. Lorenzo García's PhD thesis [LG14, § 2.1 and § 2.2] and Lercier-Ritzenthaler-Rovetta-Sisling [LRRS14], in order to fix some minor details. In section 2.1, we follow the same technique as Dolgachev [Dol12] to give the list of $\varrho' s$ and $m's$ where $\varrho(\mathcal{M}_g^{Pl}(\mathbb{Z}/m\mathbb{Z})) \neq \emptyset$, for any $g \geq 3$; Theorem 2.1.3 and Corollary 2.1.6. We introduce in section 2.2 the concept of ES-irreducibility of $\widetilde{\mathcal{M}_g^{Pl}}(G)$, motivated by the next observation occurred in Henn Table (Theorem 2.2.1): Given a finite non-trivial group G such that $\widetilde{\mathcal{M}_3^{Pl}}(G) \neq \emptyset$, there exists a single *normal form*, that describes the stratum $\widetilde{\mathcal{M}_3^{Pl}}(G)$, up to $\mathrm{PGL}_3(\overline{k})$-conjugation. In this situation, we call the stratum $\widetilde{\mathcal{M}_3^{Pl}}(G)$ to be ES-Irreducible and so is any $\widetilde{\mathcal{M}_g^{Pl}}(G)$ satisfying this property (see Definition 2.2.6 for a precise statement). This would be a weaker concept than the irreducibility of $\widetilde{\mathcal{M}_g^{Pl}}(G)$ inside the moduli space $\mathcal{M}_g$, in the sense that the number of ES-irreducible components is a lower bound of the number of its irreducible components in $\mathcal{M}_g$. We will show, in section 2.3, examples of strata of the form $\widetilde{\mathcal{M}_g^{Pl}}(\mathbb{Z}/m\mathbb{Z})$, which are not ES-Irreducible for infinitely many genera $g \geq 6$. Section 2.4 characterizes the stratum $\mathcal{M}_g^{Pl}(G)$ when G has elements of order $d^2-3d+3, (d-1)^2, d(d-2), d(d-1), md$, or $m(d-1)$ with $m \geq 2$, to be always defined by a single *normal form*. In particular, $\widetilde{\mathcal{M}_g^{Pl}}(G)$, in this case, is ES-Irreducible, if it is non-empty.

The structure of chapter 3 is as follows. In section 3.1, we collect the most necessary results, known in the literature, about central simple algebras (CSAs), and the connection with Brauer-Severi varieties, which will be used in this chapter. For more details, we refer, for example, to [Jah, GS06]. Section 3.2 is devoted to the study of the minimal field L where there exists a non-singular model over L for a smooth $\overline{k}$-plane curve C defined over k, i.e. that C is L-isomorphic to $F_{Q^{-1}\overline{C}}(X,Y,Z) = 0$ for some $Q \in \mathrm{PGL}_3(\overline{k})$ with $F_{Q^{-1}\overline{C}} \in L[X,Y,Z]$. We prove that if the degree of a non-singular $\overline{k}$-plane model of C is coprime with 3, or C has a k-rational point or the 3-torsion of the Brauer group of k is trivial (in particular, if k is a finite field), then the curve C is a smooth plane over k (i.e. admits a k-model): Theorem 3.2.8 and Corollaries 3.2.1, 3.2.2. Moreover, we prove that a smooth plane model of C always exists in a finite extension of k of degree dividing 3, see Theorem 3.2.4. Section 3.2 ends with an explicit example of a smooth $\overline{\mathbb{Q}}$-plane curve over $\mathbb{Q}$ which is not a smooth plane curve over $\mathbb{Q}$; however, we construct a smooth plane model over a degree 3 extension of $\mathbb{Q}$. In Section 3.3, we assume that C is a smooth plane curve over k. We obtain Theorem 3.3.2 characterizing the twists of C which are also smooth plane curves over k. Moreover, we construct a family of examples over $k = \mathbb{Q}$ for which a twist of C does not admit a non-singular plane model over $\mathbb{Q}$. This construction is not explicit because we do not provide equations of such twists. Section 3.4 details an explicit example of a smooth $\overline{\mathbb{Q}(\zeta_3)}$-plane curve over $\mathbb{Q}(\zeta_3)$ having a twist that does not possess such a model in the field $\mathbb{Q}(\zeta_3)$, where ζ_3 is a primitive 3rd root of unity. Interestingly, we find the already mentioned explicit equations for a non-trivial Brauer-Severi variety. In Section 3.5, we study the twists for smooth plane curve C over k, such that $\mathrm{Aut}(\overline{C})$ is a cyclic group. We prove that if $\mathrm{Aut}(F_{P^{-1}\overline{C}})$ is represented in $\mathrm{PGL}_3(\overline{k})$ by a diagonal matrix, (where $F_{P^{-1}\overline{C}}(X,Y,Z)$ is k-isomorphic to C) then all the twists are diagonal, i.e. of the form $F_{(PD)^{-1}\overline{C}}(X,Y,Z) = 0$ with D a diagonal matrix, Theorem 3.5.2. We apply this result to some special families of curves, see Corollary 3.5.4. We also construct an example of a curve C that being $\mathrm{Aut}(F_{P^{-1}\overline{C}})$ cyclic (but not diagonal) has all the twists not diagonal.

Chapter 4, section 4.1 is devoted to the study of the stratification by automorphism group of smooth $\overline{k}$-plane curves of genus 6, i.e. the different strata of $\mathcal{M}_6^{Pl}$, where k has characteristic $p = 0$ or $p > 2g + 1 = 13$. A full description of the automorphism groups and the associated normal forms is given in Theorem 4.1.12. The diagram in Figure 4.1 shows how looks like the stratification by automorphism groups of non-singular plane quintic curves. In section 4.2, we explain an interesting phenomenon, which appears in Figure 4.1; the existence of a final stratum of plane curves whose dimension is not zero. By a final stratum we mean a stratum not containing any other proper stratum. One could expect that by adding restrictions in the parameters of a family defining a stratum with a given automorphisms group, one could get bigger automorphism groups until obtaining a zero-dimensional stratum. This happens for all the families except for one. For this family each restriction in the parameters providing a bigger automorphism group yields a singular curve. We find an explanation for this fact: this family can be embedded in a family of curves of genus 6 with the same automorphism group for which we can carry out the previous operation without getting singular curves, the key point is they are not plane curves anymore: Proposition 4.2.1, Corollary 4.2.2. Moreover, we prove that this may happen in general for higher genera: Theorem 4.2.4. In section 4.3, we refine the classification given in Theorem 4.1.12, since it is not representative or even complete over k (see Remark 4.3.4): Theorem 4.3.6. We end up this chapter with section 4.4, in which a full description of the set $\mathrm{Twist}_k(C)$ of twists of a smooth $\overline{k}$-plane curve of genus 6 defined over k can be found.

In chapter 5, we fix a non-singular plane model $F_{\overline{C}}(X,Y,Z)=0$ over $\overline{k}$ for $\overline{C}$ in one of the families mentioned in Theorem 2.1.3, and such that $\mathrm{Aut}(F_{\overline{C}})\leq \mathrm{PGL}_3(\overline{K})$ is diagonal, that is made entirely of 3×3 projective matrices of diagonal shapes. We first show that if $\mathrm{Aut}(F_{\overline{C}})$ contains a non-homology of order $n>1$ (Definition 1.2.6), then $M_{\overline{k}/k}(\overline{C})$ is always a field of definition, unless n divides one of the integers d, $d-1$ or $d(d-2)$. We also give a geometrically complete family over k and describe the automorphism group in each subcase as well: Theorem 5.4.4. Secondly, if $\mathrm{Aut}(F_{\overline{C}})$ is made entirely of homologies, then it is either a cyclic group of order dividing d or $d-1$, or it is isomorphic to $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$: Lemma 5.4.8. In the case that $\mathrm{Aut}(F_{\overline{C}})$ is cyclic generated by an homology of order $n>1$, that divides d with d odd or divides $d-1$, then again $M_{\overline{K}/K}(\overline{C})$ needs to be a field of definition: Theorem 5.4.14 and Theorem 5.4.15. Otherwise, we construct explicit examples of smooth plane curves over $\mathbb{C}$, whose field of moduli relative to the Galois extension $\mathbb{C}/\mathbb{R}$ is $\mathbb{R}$, but it is not a field of definition; Proposition 5.4.2, Theorem 5.4.6, Theorem 5.4.16 and Proposition 5.4.20.

Appendix A contains the tables of all cyclic subgroups of automorphisms and the associated defining equations, obtained for low degrees through manipulating Theorem 2.1.3 in chapter 2. In appendix B, we briefly explain the algorithm for computing $\mathrm{Twist}_k(C)$ of a non-hyperelliptic curve C of genus $g\geq 3$ developed in [LG14, Chp.1] and [LG17]. Alternatively, we use the modified algorithm resulted by chapter 3 to compute the twists over k for the smooth plane curves defined over k by $X^5+Y^5+XZ^4=0$ and $X^5+Y^4Z+XZ^4=0$, where k is a field of zero characteristic or positive characteristic $>(5-1)(5-2)+1=13$. Finally, we generate in appendix C the full list of geometric fibers for the stratum of smooth plane curves of genus $g=6$ with automorphism group $\mathbb{Z}/5\mathbb{Z}$, which are isomorphic over $\overline{k}$.

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1

CHAPTER

Automorphism groups of smooth plane curves: A Literature review

It is classically well-known that the full automorphism group $\mathrm{Aut}(\overline{C})$ of a smooth curve $\overline{C}$ of genus $g \geq 2$ defined over $\overline{k}$ is a finite group (Schmid (1938), Iwasawa-Tamagawa (1951), Roquette (1952), Rosentlich (1955), Garcia (1993)). Moreover, if k has characteristic zero characteristic, then it has order at most $84(g-1)$. This bound is known as the *Hurwitz bound* [Hur92] on $\mathrm{Aut}(\overline{C})$:

Theorem 1.1 (Hurwitz (1892), Roquette (1970))**.** *Let G be a subgroup of automorphisms of a smooth curve $\overline{C}$ of genus $g \geq 2$ over $\overline{k}$ of zero characteristic. Then $|G| \leq 84(g-1)$, more precisely*

$$\frac{|G|}{g-1} = 84,\ 48,\ 40,\ 36,\ 30,\ \frac{132}{5},\ \textit{or} \leq 24.$$

The same bound holds over positive characteristic $p > g+1$, with one exception, namely the hyperelliptic curve $Y^2Z^{p-2} = X^p - XZ^{p-1}$, which has $p = 2g+1$ and $2p(p^2-1)$ automorphisms.

Such a bound turns out to be sharp for infinitely many genera [Mac61]. The lowest genus example is the Klein quartic curve given by the equation $X^3Y + Y^3Z + Z^3X = 0$, whose automorphism group is the unique simple group of order 168 namely, the Klein group $\mathrm{PSL}_2(\mathbb{F}_7)$.

On the other hand, if $p \,|\, |\mathrm{Aut}(\overline{C})|$, then a much larger automorphism group compared to g could happen as was first pointed out by P. Roquette in [Roq70]. For example, if $\overline{C}$ is birational equivalent to a Hermitian curve $\mathrm{H}(q)$, i.e. to a smooth plane curve of the form $Y^qZ + YZ^q - X^{q+1} = 0$ for some $q = p^n \geq 3$, then $g = \frac{1}{2}(q^2-q)$ and $|\mathrm{Aut}(\mathrm{H}(q))| =$

$q^3(q^3+1)(q^2-1)$. More precisely, $\mathrm{Aut}(\mathrm{H}(q))$ is isomorphic to the *projective unitary group*[1] denoted by $\mathrm{PGU}(3,q)$. Furthermore, any smooth curve $\overline{C}$ satisfying $|\,\mathrm{Aut}(\overline{C})| \geq 16g^4$ is birational equivalent to a Hermitian curve $\mathrm{H}(q)$ as was proved by Stichtenoth [Sti73a, Sti73b]. A substantial improvement of the last bound to $16g^3+24g^2+g$ was a consequence of Henn classification in [Hen78] and later on to $3(2g^2+g)(3+\sqrt{8g+1})$ for the case of smooth plane curves by Anbar-Bartoli-Fanali-Giulietti in [ABFG13].

§1.1 Linearity of isomorphism between smooth curves

Given an invertible $(n+1)\times(n+1)$ matrix $\phi=(a_{ij})$ defined over $\overline{k}$, the rule $x_i' := \sum_j a_{ij}x_j$ determines an automorphism of the polynomial ring $\overline{k}[x_0,...,x_n]$ and also an automorphism of the n-dimensional projective space $\mathbb{P}^n_{\overline{k}}$. One easily checks that $\lambda\phi=(\lambda a_{ij})$ produces the same action on $\mathbb{P}^n_{\overline{k}}$, for any non-zero $\lambda\in\overline{k}$. So one is led to consider the action of $\mathrm{PGL}_{n+1}(\overline{k}) = \mathrm{GL}_{n+1}(\overline{k})\,/\,\overline{k}^*$, which acts faithfully as a subgroup of automorphisms of $\mathbb{P}^n_{\overline{k}}$. The converse is also true by the next result, see Example 7.1.1 in [Har77]:

Theorem 1.1.1. *Any $\overline{k}$-automorphism of $\mathbb{P}^n_{\overline{k}}$ is linear, i.e. it can be viewed as an element of* $\mathrm{PGL}_{n+1}(\overline{k})$. *In particular,* $\mathrm{Aut}(\mathbb{P}^n_{\overline{k}})=\mathrm{PGL}_{n+1}(\overline{k})$.

Before we present the proof of Theorem 1.1.1, we need the following facts and terminologies: Given a sheaf of rings $\mathcal{F}$ on a topological space X and an open subset U of X, the set $\Gamma(U,\mathcal{F}) := \mathcal{F}(U)$ refers to the *sections* of $\mathcal{F}$ on U. When $U=X$ then $\Gamma(X,\mathcal{F})$ are the *global sections* of $\mathcal{F}$ on X.

A *ringed space* is a pair $(X,\mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings on X. A *morphism* of ringed spaces from $(X,\mathcal{O}_X)$ to $(Y,\mathcal{O}_Y)$ is a pair $(f,f^\sharp)$ of a continuous map $f:X\to Y$ and a map $f^\sharp:\mathcal{O}_Y\to f_*\mathcal{O}_X$ of sheaves of rings on Y, where $f_*\mathcal{O}_X$ is the *direct image* sheaf on Y by $f_*\mathcal{O}_X(V)=\mathcal{O}_X(f^{-1}(V))$ for any open subset $V\subseteq Y$.

[1]A *unital* in $\mathbb{P}^2_{\mathbb{F}_{q^2}}$ is a set $\mathcal{U}$ of q^3+1 points meeting every line in $\mathbb{P}^2_{\mathbb{F}_{q^2}}$ in either 1 or $q+1$ points. It is called *classical* if it is preserved by a cyclic linear collineation group of order q^2-q+1. For more information on unitals on projective planes, see for example [BE08, CEK00].
The linear collineation group preserving a classical unital $\mathcal{U}$ is called the *projective unitary group*. See for example [Blo67, Hof72].

An *invertible sheaf* $\mathcal{F}$ on a ringed space $(X, \mathcal{O}_X)$ is defined to be a locally free $\mathcal{O}_X$-module of rank 1. The *Picard group* of $(X, \mathcal{O}_X)$, denoted by $\mathrm{Pic}(X)$, is the set of all isomorphism classes of invertible sheaves on X.

Proposition 6.12 in [Har77] shows that in fact $\mathrm{Pic}(X)$ is a group under $\otimes$ of sheaves. A morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of ringed spaces always induces a group morphism

$$f^* : \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$$

defined by the rule $f^*(\mathcal{F}) = f^{-1}(\mathcal{F}) \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$, where $f^*(\mathcal{F})$ is called the *inverse image* of the invertible sheaf $\mathcal{F}$ on Y under f.

Example 1.1.2 (The twisting sheaf of Serre)**.** *Any invertible sheaf on $\mathbb{P}^n_{\overline{k}}$ is of the form $\mathcal{O}(\ell)$ for some $\ell \in \mathbb{Z}$, i.e. a twisting of $\mathcal{O}(1)$ by ℓ, where $\mathcal{O}(1)$ is* the twisted sheaf of Serre *defined on $\mathbb{P}^n_{\overline{k}}$. The basic way to think about is that the global sections $\Gamma(\mathbb{P}^n_{\overline{k}}, \mathcal{O}(\ell))$ of $\mathcal{O}(\ell)$ on $\mathbb{P}^n_{\overline{k}}$ is precisely the $\overline{k}$-vector space of homogeneous polynomials of degree ℓ. So we can use it to talk about homogeneous polynomials in a more geometric way. We address the reader to [Har77, II, §5] for more details.*

Recall that two divisors D and D' are said to be equivalent, written $D \sim D'$ if $D - D'$ is a principal divisor. The group $\mathrm{Div}(\mathbb{P}^n_{\overline{k}})$ of all divisors on $\mathbb{P}^n_{\overline{k}}$ divided by the subgroup of principal divisors is called the *divisor class group*, and is denoted by $\mathrm{Cl}(\mathbb{P}^n_{\overline{k}})$.

Proof. (of Theorem 1.1.1) Let f be any automorphism of $\mathbb{P}^n_{\overline{k}}$. Then f^* is an automorphism of $\mathrm{Pic}(\mathbb{P}^n_{\overline{k}})$. We know form Corollaries 6.16 and 6.17 in [Har77] that $\mathrm{Pic}(\mathbb{P}^n_{\overline{k}}) \simeq \mathrm{Cl}(\mathbb{P}^n_{\overline{k}}) \overset{\deg}{\simeq} \mathbb{Z}$. Consequently, $\mathrm{Cl}(\mathbb{P}^n_{\overline{k}})$ is generated by a hyperplane, which in turns corresponds to $\mathcal{O}(1)$ as a generator of $\mathrm{Pic}(\mathbb{P}^n_{\overline{k}})$. Thus $f^*(\mathcal{O}(1))$ must be a generator of $\mathrm{Pic}(\mathbb{P}^n_{\overline{k}})$, hence it isomorphic to either $\mathcal{O}(1)$ or $\mathcal{O}(-1)$, since any invertible sheaf on $\mathbb{P}^n_{\overline{k}}$ is of the form $\mathcal{O}(\ell)$ for some $\ell \in \mathbb{Z}$, see Example 1.1.2. However, $\mathcal{O}(-1)$ has no global sections, but $\mathcal{O}(1)$ does, therefore $f^*(\mathcal{O}(1)) = \mathcal{O}(1)$. In particular, f^* induces an automorphism of the $\overline{k}$-vector space $\Gamma(\mathbb{P}^n_{\overline{k}}, \mathcal{O}(1))$. Furthermore, the global sections $x_0, ..., x_n$ of $\mathcal{O}(1)$ on $\mathbb{P}^n_{\overline{k}}$ forms a basis of $\Gamma(\mathbb{P}^n_{\overline{k}}, \mathcal{O}(1))$, see Proposition 5.13 and Theorem 5.19 in [Har77]. Thus, the pull-backs $s_i := f^*(x_i)$ must be another basis of the vector space $\Gamma(\mathbb{P}^n_{\overline{k}}, \mathcal{O}(1))$, and we may write $s_i := \sum_i a_{ij}x_j$, where (a_{ij}) is an invertible

$(n+1)\times(n+1)$ matrix over $\overline{k}$. In this case, f is uniquely determined by the s_i, and it coincides with the automorphism (a_{ij}) as an element of $\mathrm{PGL}_{n+1}(\overline{k})$. □

Definition 1.1.3. By a smooth plane curve $\overline{C}$ of degree $d \geq 4$ over $\overline{k}$, we mean a smooth projective curve $\overline{C}$ that is $\overline{k}$-isomorphic to a non-singular plane model $F_{\overline{C}}(X,Y,Z)=0$ in $\mathbb{P}^2_{\overline{k}}$, where $F_{\overline{C}}(X,Y,Z)$ is a homogenous polynomial of degree d with coefficient in $\overline{k}$. In this case, $\overline{C}$ admits a g^2_d-linear system allowing us to embed

$$\overline{C} = C\otimes_k \overline{k} \overset{g^2_d}{\hookrightarrow} \mathbb{P}^2_{\overline{k}},$$

where g is the genus of $\overline{C}$.

Lemma 1.1.4. *Let $\overline{C}\overset{g^2_d}{\hookrightarrow}\mathbb{P}^2_{\overline{k}}$ be a smooth plane curve of degree $d\geq 4$ over $\overline{k}$. Then, it is non-hyperelliptic of genus $g=(d-1)(d-2)/2$.*

Proof. Let $H\cap\overline{C}\subset\mathbb{P}^2_{\overline{k}}$ be a hyperplane section of $\overline{C}$, i.e. $H\cap\overline{C}$ is the intersection of $\overline{C}$ with a hyperplane H in $\mathbb{P}^2_{\overline{k}}$. In particular, the canonical divisor $\mathcal{K}_{\overline{C}}$ of $\overline{C}$ is equivalent to $(d-3)(H\cap\overline{C})$, see [Har77, Example 8.20.3]. By Bézout's theorem $H\cap\overline{C}$ has degree exactly d. Therefore, one reads Riemann-Hürwitz formula as

$$2g-2=\deg(\mathcal{K}_{\overline{C}})=(d-3)d,$$

that is $g=(d-1)(d-2)/2$.

Next, if $f(x,y)=0$ is the affine equation of a smooth plane curve $\overline{C}$ of degree $d\geq 4$, then

$$\{\frac{x^ry^s}{f_y}\,|\,0\leq r+s\leq d-3\}$$

is a basis of the space of regular differentials on $\overline{C}$. Therefore, the canonical map $\overline{C}\to\mathbb{P}^{g-1}_{\overline{k}}$ can be seen as the map

$$(x:y:1)\mapsto(x^ry^s\,|\,0\leq r+s\leq d-3).$$

In particular, when $d=4$, this map is exactly the identity map, and hence is an embedding. That is, a smooth plane curve $\overline{C}$ of degree $d=4$ over $\overline{k}$ is non-hyperelliptic.

Now, assume that $d\geq 5$ and $\overline{C}$ is hyperelliptic. Hence, it has a hyperelliptic involution ι of order 2, which fixes exactly $2g+2=(d-4)(d+1)$ points on $\overline{C}$. Thinking about ι, up to $\mathrm{PGL}_3(\overline{k})$-conjugation, as the automorphism $[X : Y : -Z]$, gives at most d fixed points on $F_{\overline{C}}(X,Y,Z) = 0$, since ι leaves invariant in $\mathbb{P}^2_{\overline{k}}$, the line $Z = 0$, the point $(0 : 0 : 1) \notin \overline{C}$ and no other points. That is, $(d-4)(d+1) \leq d$, a contradiction!. Therefore, $\overline{C}$ must be non-hyperelliptic. $\square$

Given a divisor D on $\overline{C}$, the set of all rational functions g on $\overline{C}$ such that $D + \mathrm{div}(g) \geq 0$ forms a $\overline{k}$-vector space $\mathcal{L}(D)$ of finite dimension $\ell(D)$. If $\mathcal{K}_{\overline{C}}$ is the canonical divisor of $\overline{k}$. Then, Riemann-Roch theorem states that

$$\ell(D) - \ell(\mathcal{K}_{\overline{C}} - D) = \deg(D) + 1 - g.$$

Theorem 1.1.5 (Theorem 1, [Cha78]). *Any isomorphism between smooth plane curves of degree $d \geq 4$ over $\overline{k}$ is induced by a projective linear transformation of $\mathbb{P}^2_{\overline{k}}$. In particular, a smooth plane curve $\overline{C}$ of degree $d \geq 4$ over $\overline{k}$ has a unique g^2_d linear system, up to $\mathrm{PGL}_3(\overline{k})$-conjugation.*

Proof. Let $\overline{C}$ be a smooth plane curve of degree d over $\overline{k}$, and let $H \cap \overline{C}$ be a hyperplane section of $\overline{C}$. We can assume through a g^2_d-linear system that $H \cap \overline{C} = P_1 + \ldots + P_d$, for some pairwise distinct points P_i of $\overline{C}$. By Riemann-Roch theorem and Lemma 1.1.4, we obtain

$$\ell(H \cap \overline{C}) - \ell(\mathcal{K}_{\overline{C}} - H \cap \overline{C}) = d + 1 - (d-1)(d-2)/2.$$

On the other hand, $\mathcal{L}(\mathcal{K}_{\overline{C}})$ is exactly the vector space of homogenous polynomials of degree $d-3$, so its dimension equals to $\binom{d-1}{2}$, see [Har77, Example 8.20.3]. Hence, the members of $\mathcal{L}(\mathcal{K}_{\overline{C}})$, cutting out $H \cap \overline{C}$ are exactly polynomials with a fixed linear factor (recall that $H \cap \overline{C}$ has degree $d > d-3$, and lies on a line in $\mathbb{P}^2_{\overline{k}}$). Consequently, $\ell(\mathcal{K}_{\overline{C}} - H \cap \overline{C}) = \binom{d-2}{2}$.

Next, let $\phi : \overline{C}' \to \overline{C}$ be an isomorphism as in the theorem, where $\overline{C}'$ is a smooth plane curve of degree d over $\overline{k}$. It suffices to show that the divisor $\phi^*(H \cap \overline{C})$ lies on a line in $\mathbb{P}^2_{\overline{k}}$, or equivalently, hyperplane sections of $\overline{C}$ are mapped to hyperplane sections of $\overline{C}'$. That is, ϕ sends collinear points in $\mathbb{P}^2_{\overline{k}}$ to collinear points[2]. Because $\phi^*(\mathcal{K}_{\overline{C}} - H \cap \overline{C}) = \mathcal{K}_{\overline{C}'} - \phi^*(H \cap \overline{C})$,

[2] $\mathrm{PGL}_3(\overline{k})$ sends collinear points of $\mathbb{P}^2_{\overline{k}}$ to collinear points of $\mathbb{P}^2_{\overline{k}}$.

then $\ell(\mathcal{K}_{\overline{C}'} - \phi^*(H \cap \overline{C})) = \ell(\mathcal{K}_{\overline{C}} - H \cap \overline{C})$. In particular,

$$\ell(\mathcal{K}_{\overline{C}'}) - \ell(\mathcal{K}_{\overline{C}'} - \phi^*(H \cap \overline{C})) = \binom{d-1}{2} - \binom{d-2}{2} = d-2.$$

So there exist $d-2$ points $P'_1, ..., P'_{d-2}$ on $\phi^*(H \cap \overline{C})$, such that any curve of degree $d-3$ passing them will contain the whole $\phi^*(H \cap \overline{C})$. Now, suppose that $\phi^*(H \cap \overline{C}) = P'_1 + ... + P'_{d-2} + P'_{d-1} + P'_d$ is not on a line. Denote the line which joins P'_i and P'_j by $L_{i,j}$ with $i \neq j$. Therefore, there is one point, say P'_1, which is not on $L_{d-1,d}$, and $\{P'_{d-1}, P'_d\} \nsubseteq L_{1,2}$. For each $3 \leq i \leq d-2$, draw a line L_i through P'_i, but missing the points P'_{d-1} and P'_d. Then $L_{1,2}L_3...L_{d-2}$ is a curve of degree $d-3$ passing through $P'_1, ..., P'_{d-2}$, but does not contain $\phi^*(H \cap \overline{C})$, a contradiction. □

Corollary 1.1.6. *Let $\overline{C}$ be a smooth plane curve of degree $d \geq 4$ defined over an algebraically closed field $\overline{k}$ of characteristic $p \geq 0$. Then, $\mathrm{Aut}(\overline{C})$ is a finite subgroup inside $\mathrm{PGL}_3(\overline{k})$, fixing a certain non-singular plane model $F_{\overline{C}}(X,Y,Z) = 0$ of $\overline{C}$ in $\mathbb{P}^2_{\overline{k}}$.*

Proof. Since $\overline{C}$ has genus $g = (d-1)(d-2)/2 \geq 3$, the full automorphism group $\mathrm{Aut}(\overline{C})$ is finite. Moreover, $\overline{C}$ is a smooth plane curve over $\overline{k}$, then it has a g^2_d-linear system, which is also unique, up to conjugation in $\mathrm{PGL}_3(\overline{k})$, from the proof of Theorem 1.1.5. In particular, $\mathrm{Aut}(\overline{C})$ admits an injective representation inside $\mathrm{Aut}(\mathbb{P}^2_{\overline{k}}) = \mathrm{PGL}_3(\overline{k})$, characterized by leaving invariant a fixed non-singular plane model say, $F_{\overline{C}}(X,Y,Z) = 0$ in $\mathbb{P}^2_{\overline{k}}$. □

Theorem 1.1.7. *Let $\overline{C}_i$, for $i = 1,2$, be two isomorphic smooth non-hyperelliptic curves of genus g over $\overline{k}$, canonically embedded in $\mathbb{P}^{g-1}_{\overline{k}}$. Then any isomorphism between the $\overline{C}_i$ is linear. In particular, $\mathrm{Aut}(\overline{C}_i)$ can be seen as a finite subgroup of $\mathrm{PGL}_g(\overline{k})$.*

Proof. The idea can be rephrased in terms of linear series, we refer for example to [HKT08, Chp. 11]. Let $\mathcal{L}_i$, for $i = 1,2$, denotes the canonical linear series of $\overline{C}_i$ respectively. By Theorem 6.72-(i) in [HKT08], the canonical series is the unique series of dimension $g-1$ and order $2g-2$. Therefore, an isomorphism $f : \overline{C}_1 \to \overline{C}_2$ can always be identified with its action between the $\mathcal{L}_i$, see Theorem 11.18 and Lemma 11.19 in [HKT08]. In other words, f naturally induces an isomorphism $\widetilde{f} : \mathcal{K}_1 \to \mathcal{K}_2$ between the canonical models $\mathcal{K}_i$ for $\overline{C}_i$ in $\mathbb{P}^{g-1}_{\overline{k}}$, so it is linear. Consequently, $\mathrm{Aut}(\overline{C}_i)$ is embedded, as a finite subgroup, into $\mathrm{Aut}(\mathbb{P}^{g-1}_{\overline{k}}) = \mathrm{PGL}_g(\overline{k})$. □

Remark 1.1.8. Let $\overline{C}$ be a smooth curve of genus $g \geq 2$ defined over an algebraically closed field $\overline{k}$. The automorphisms of $\overline{C}$ are induced by automorphisms of the ringed space $(\overline{C}, \mathcal{O}_{\overline{C}})$, where $\mathcal{O}_{\overline{C}}$ denotes the ring of regular functions of $\overline{C}$. This coincides with the set of all $\overline{k}$-automorphisms of the algebraic rational function field $\overline{k}(\overline{C})$ of $\overline{C}$, denoted by $\mathrm{Gal}(\overline{k}(\overline{C})/\overline{k})$. In particular, if $\overline{C}$ is a smooth plane curve of degree $d \geq 4$, then $\mathrm{Gal}(\overline{k}(\overline{C})/\overline{k})$ can be viewed as a finite subgroup of $\mathrm{PGL}_3(\overline{k})$, by using Remark 1.1.6.

§1.2 Finite subgroups of the 3-dimensional projective general linear group

Assume that $\overline{C}$ has a smooth plane model $F_{\overline{C}}(X,Y,Z) = 0$ of degree $d \geq 4$ over $\overline{k}$. That is, the genus $g = (d-1)(d-2)/2 \geq 3$. Also, the g^2_d-linear system is unique, up to conjugation in $\mathrm{PGL}_3(\overline{k})$, see [HKT08, Lemma 11.28], thus we can think about $\mathrm{Aut}(\overline{C})$ as a finite subgroup of $\mathrm{PGL}_3(\overline{k})$.

The determination of the finite subgroups of $\mathrm{PGL}_3(\overline{k})$ is quite well understood in the subject. For instance, we recall this one made by H. Mitchell [Mit11, §1-10], which is based entirely on geometrical methods. H. Mitchell [Mit11] proved that G fixes a point, a line or a triangle unless it is primitive[3] and conjugate to some group in a specific list. However, as a consequence of Maschke's theorem in group representation theory, the first two cases are equivalent, in the sense that if G fixes a point (resp. a line) then it also fixes a line not passing through the point (resp. a point not lying the line). In particular, we have the next result, which can also be read in its present form in [DI09, Theorem 4.8]:

Theorem 1.2.1. *Let G be a finite subgroup of $\mathrm{PGL}_3(\overline{k})$ such that $p \nmid |G|$. Then G satisfies one of the following conditions:*

1. *it fixes a line in $\mathbb{P}^2_{\overline{k}}$ and a point Q off this line,*

[3] A subgroup H of a group G is termed *core-free* if $\bigcap_{x\in G} xHx^{-1}$ is trivial. A group G is said to be *primitive* if it has a core-free maximal subgroup.

2. *it fixes a triangle (i.e. a set of three non-concurrent lines),*

3. *it is conjugate to one of the finite primitive subgroups*[4] *of* $\mathrm{PGL}_3(\overline{k})$ *namely, the Klein group* $\mathrm{PSL}(2,7)$*, the icosahedral group* A_5*, the alternating group* A_6*, or to one of the Hessian groups* Hess_* *with* $* \in \{36,\ 72,\ 216\}$.

Definition 1.2.2. An element of $\mathrm{PGL}_3(\overline{k})$ is called *intransitive* if it has the matrix shape

$$\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We use the notation $\mathrm{PBD}(2,1)$ for the subgroup of $\mathrm{PGL}_3(\overline{k})$ of all intransitive elements. A subgroup of $\mathrm{PBD}(2,1)$ is also called *intransitive*.

Obviously, there is a natural map $\Lambda : \mathrm{PBD}(2,1) \to \mathrm{PGL}_2(\overline{k})$ given by

$$\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{PBD}(2,1) \mapsto \begin{pmatrix} * & * \\ * & * \end{pmatrix} \in \mathrm{PGL}_2(\overline{k}).$$

The next result gives the list of finite subgroups of the 2-dimensional projective general linear group $\mathrm{PGL}_2(\overline{k})$. See [Suz77, Chapter 3] and [Web98, §§71-74], or [Hug05, Lemma 2.2.1, I], for more details.

Theorem 1.2.3. *Let* k *be a field of of characteristic* $p = 0$ *or* $p > 2$*. Any finite subgroup* G *of* $\mathrm{PGL}_2(\overline{k})$*, such that* $p = 0$ *or* $p > 0$ *with* $p \nmid |G|$*, is conjugate to one of the following groups:*

1. *The cyclic group* $\mathbb{Z}/n\mathbb{Z} = \langle \mathrm{diag}(\zeta_n, 1)\rangle$ *of order* n*,* $n \geq 1$*,*

2. *The dihedral group* $\mathrm{D}_{2n} = \langle \mathrm{diag}(\zeta_n, 1), \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle$ *of order* $2n$*,* $n > 1$*,*

[4] See Theorem 1.2.4-(4) or the notations at the beginning of the memoir.

3. *The alternating group* A_4*, consisting of the transformations*

$$\begin{pmatrix} \pm 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & \pm 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \zeta_4^\nu & \zeta_4^\nu \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} \zeta_4^\nu & -\zeta_4^\nu \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \zeta_4^\nu \\ 1 & -\zeta_4^\nu \end{pmatrix}, \begin{pmatrix} -1 & -\zeta_4^\nu \\ 1 & -\zeta_4^\nu \end{pmatrix}$$

   where $\nu = 1, 3$*,*

4. *The symmetry group* S_4*, consisting of the transformations*

$$\begin{pmatrix} \zeta_4^\nu & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & \zeta_4^\nu \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \zeta_4^\nu & -\zeta_4^{\nu+\nu'} \\ 1 & \zeta_4^{\nu'} \end{pmatrix},$$

   where $\nu, \nu' = 0, 1, 2, 3$*,*

5. *The alternating group* A_5*, consisting of the transformations*

$$\begin{pmatrix} \zeta_5^\nu & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & \zeta_4^\nu \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} \lambda\zeta_5^\nu & \zeta_5^{\nu-\nu'} \\ 1 & -\lambda\zeta_5^{-\nu'} \end{pmatrix}, \begin{pmatrix} \overline{\lambda}\zeta_5^\nu & \zeta_5^{\nu-\nu'} \\ 1 & -\overline{\lambda}\zeta_5^{-\nu'} \end{pmatrix},$$

   where $\lambda := \frac{1}{2}(-1+\sqrt{5})$*,* $\overline{\lambda} := \frac{1}{2}(-1-\sqrt{5})$*, and* $\nu, \nu' = 0, 1, 2, 3, 4$*.*

The list of finite groups G of $\mathrm{PGL}_3(\overline{k})$, where $\overline{k}$ has characteristic $p = 0$, are explicitly given in [MBD61, Chapter VII]. Using Corollary 2.3.6 in [Hug05], one also obtains the list when $p > 2$ is relatively prime with $|G|$. In particular, we have the next theorem, which is Lemma 2.3.7, case I in [Hug05]:

Theorem 1.2.4. *Let* G *be a finite subgroup of* $\mathrm{PGL}_3(\overline{k})$*, where* k *has characteristic* $p \neq 2$ *such that* $p \nmid |G|$*. Then,* G *is conjugate to one of the following groups:*

1. *Type I: An intransitive subgroup of* $\mathrm{PGL}_3(\overline{k})$ *whose natural image under* Λ *in* $\mathrm{PGL}_2(\overline{k})$ *is equal to one of the groups in Theorem 1.2.3,*

2. *Type II: A group generated by* $T := [Y : Z : X]$ *and a finite group generated by the image in* $\mathrm{PGL}_3(\overline{k})$ *of diagonal matrices.*

   The group of order 9 *generated by* T *and* $S := \mathrm{diag}(1, \zeta_3, \zeta_3^2)$ *will be called* Hess_9*,*

3. *Type III: A group generated by* $R := [X : Z : Y]$ *and a group of Type II.*

   The group generated by Hess_9 *and* R *will be called* Hess_{18}*,*

4. *Type IV: One of the Hessian groups; The group* Hess_{36} *of order* 36 *generated by* Hess_{18} *and*

$$V := \begin{pmatrix} 1 & 1 & 1 \\ 1 & \zeta_3 & \zeta_3^2 \\ 1 & \zeta_3^2 & \zeta_3 \end{pmatrix},$$

the group Hess_{72} *of order* 72 *generated by* Hess_{36} *and* UVU^{-1}*, where* $U := \mathrm{diag}(1, 1, \zeta_3)$*, or the group* Hess_{216} *of order* 216 *generated by* Hess_{72} *and* U*.*

The alternating group A_5 *of order* 60 *generated by* $E_1 := \mathrm{diag}(1, \zeta_5^4, \zeta_5)$*,* $E_2 := [X : Z : Y]$*, and*

$$E_3 := \begin{pmatrix} 1 & 1 & 1 \\ 1 & \zeta_5^2 + \zeta_5^{-2} & \zeta_5 + \zeta_5^{-1} \\ 1 & \zeta_5 + \zeta_5^{-1} & \zeta_5^2 + \zeta_5^{-2} \end{pmatrix},$$

or the alternating group A_6 *of order* 360 *generated by* E_1*,* E_2*,* E_3*, and*

$$E_4 := \begin{pmatrix} 1 & \nu_1 & \nu_1 \\ 2\nu_2 & \zeta_5^2 + \zeta_5^{-2} & \zeta_5 + \zeta_5^{-1} \\ 2\nu_2 & \zeta_5 + \zeta_5^{-1} & \zeta_5^2 + \zeta_5^{-2} \end{pmatrix},$$

where $\nu_1 := \frac{1}{4}(-1 + \sqrt{-15})$ *and* $\nu_2 := \frac{1}{4}(-1 - \sqrt{-15})$*.*

The Klein group $\mathrm{PSL}_2(\mathbb{F}_7)$ *of order* 168 *generated by* $F_1 := \mathrm{diag}(\zeta_7, \zeta_7^2, \zeta_7^4)$*,* $F_2 := [Y : Z : X]$*, and*

$$\begin{pmatrix} a' & b' & c' \\ b' & c' & a' \\ c' & a' & b' \end{pmatrix},$$

where $a' := \zeta_7^4 - \zeta_7^3$*,* $b' := \zeta_7^2 - \zeta_7^5$*, and* $c' := \zeta_7 - \zeta_7^6$*.*

Remark 1.2.5. Theorem 1.2.4 could be viewed as an extended version of the classification given in Theorem 1.2.1. First, given any a line $L \subset \mathbb{P}^2_{\overline{k}}$ and a point $P \in \mathbb{P}^2_{\overline{k}} \setminus L$, one can consider a transformation $A \in \mathrm{PGL}_3(\overline{k})$ that moves L to the reference line $Z = 0$ and the point P to the reference point $(0 : 0 : 1)$. In this way, any finite subgroup G of $\mathrm{PGL}_3(\overline{k})$ that fixes a line and a point off this line is conjugate to a group of *Type I*. Second, for any finite subgroup $G \subset \mathrm{PGL}_3(\overline{k})$ that fixes a triangle Δ, we may assume up to conjugation that Δ has vertices

$(1 : 0 : 0)$, $(0 : 1 : 0)$, and $(0 : 0 : 1)$ respectively. In particular, such groups are classified according to its action on the vertices, i.e with respect to a subgroup of the permutation group $\langle [X : Z : Y], [Y : Z : X] \rangle$ modulo a finite group made entirely of diagonal matrices. So we are again in *Type I*, *Type II*, or *Type III*. Finally, it is straightforward that the groups of *Type IV* coincides with the finite primitive subgroups of $\mathrm{PGL}_3(\overline{k})$ given in Theorem 1.2.1-(3).

Definition 1.2.6. By an homology of period $n \in \mathbb{Z}_{\geq 1}$ coprime with p, we mean a projective linear transformation of the plane $\mathbb{P}^2_{\overline{k}}$, which acts, up to conjugation in $\mathrm{PGL}_3(\overline{k})$, as

$$(X : Y : Z) \mapsto (\zeta_n X : Y : Z), \tag{1.1}$$

where ζ_n is a primitive nth root of unity.

Such a transformation fixes pointwise a line (its axis) and a point off this line (its center). For example, a homology in the canonical shape (1.1) has axis $X = 0$ and center $(1 : 0 : 0)$.

One easily can see the following observation:

Lemma 1.2.7. *Let $\sigma \in \mathrm{PGL}_3(\overline{k})$ be a non-trivial planar projective transformation of finite order coprime with p.*

1. *If σ is a homology, then the fixed points of σ consists entirely of its center and all points on its axis. In particular, every triangle whose set of vertices is pointwise fixed by σ contains its center as a vertex.*

2. *If σ is a non-homology, then it fixes exactly three points. In particular, there is a unique triangle whose vertices are pointwise fixed by σ.*

The following results turns out to be very useful in hand when one wants to determine the automorphism group of a smooth plane curve over $\overline{k}$. See [Mit11, Theorems 6,8, and 9] and [Mit11, Theorem 4], respectively:

Theorem 1.2.8 (Mitchell, [Mit11])**.** *Let G be a finite group of $\mathrm{PGL}_3(\overline{k})$, where k has characteristic $p \neq 2$ such that $p \nmid |G|$. If G contains an homology of period $n \geq 4$, then it fixes a point, a line or a triangle. Moreover, the Hessian group Hess_{216} is the only finite subgroup of $\mathrm{PGL}_3(\overline{k})$ that contains homologies of period $n = 3$, and does not leave invariant a point, a line or a triangle.*

Proposition 1.2.9 (Mitchell, [Mit11]). *Let G be a finite group of $\mathrm{PGL}_3(\overline{k})$, where where k has characteristic $p \neq 2$ such that $p \nmid |G|$. Inside G, a transformation, which leaves invariant the center of an homology, must leave invariant its axis and vice versa.*

§1.3 Galois points for smooth plane curves

Let K be an algebraic function field in one variable over $\overline{k}$.

Definition 1.3.1 (see [Nam84]). The gonality $\mathrm{Gon}(K)$ of K, is the minimum of the degree extension $[K : \overline{k}(t)]$ where t runs over the transcendental elements of K.

Definition 1.3.2 (see [MY00]). A *maximal rational subfield* K_m of K is a maximal subfield among the ones that are purely transcendental extensions of $\overline{k}$. By Lüroth's theorem, any subfield K' satisfying $\overline{k} \neq K' \subset K_m$ is rational.

A maximal rational subfield K_m satisfying $[K : K_m] = \mathrm{Gon}(K)$ is called a *g-maximal rational subfield of K*.

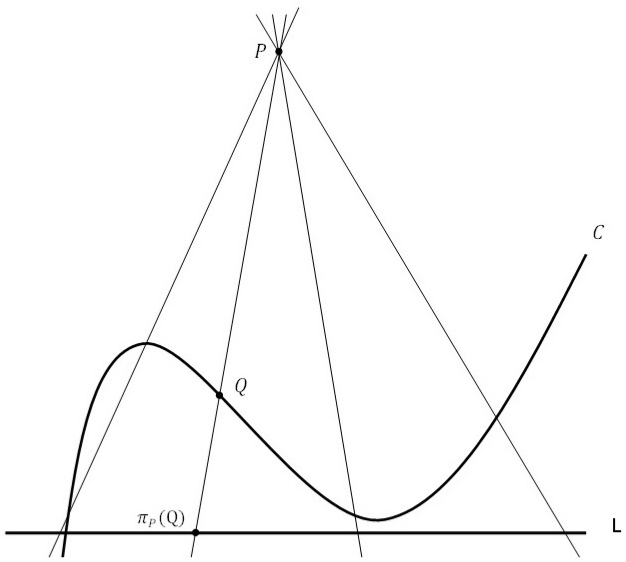
Example 1.3.3. *In the case where $K = \overline{k}(\overline{C})$ is the rational function field of a smooth plane curve $\overline{C}$ over $\overline{k}$ of degree $d \geq 4$, we have the following facts (see Theorem 5.3.17 in [Nam84]):*

1. *If K_m is a g-maximal rational subfield of K, then $[K : K_m] = d - 1$. The extension K/K_m is obtained by $\pi_P^* : \overline{k}(L) \cong \overline{k}(\mathbb{P}^1_{\overline{k}}) \hookrightarrow \overline{k}(\overline{C})$, where π_P is the projection from $\overline{C}$ to a line L with a center $P \in \overline{C}$.*

2. *If one considers a projection π_P from $\overline{C}$ to a line L with a center $P \notin \overline{C}$, then we get an extension of fields $\pi_P^* : \overline{k}(L) \cong \overline{k}(\mathbb{P}^1_{\overline{k}}) \hookrightarrow \overline{k}(\overline{C})$ such that $[\overline{k}(\overline{C}) : \overline{k}(L)] = d$. In this case, $\overline{k}(L)$ is a maximal rational subfield of $\overline{k}(\overline{C})$, but not a g-maximal one.*

In both cases, we get a maximal rational subfield. Therefore we may consider the natural point projection map $\pi_P : C \to \mathbb{P}^1_{\overline{k}}$, i.e. for an arbitrary point $Q \in \overline{C}$, the point $\pi_P(Q)$ is the intersection point of L with the line $\overline{PQ}$, joining P and Q.

The field extension does not depend on the line L, but on whether the point $P \in \mathbb{P}^2_{\overline{k}}$ lies on $\overline{C}$ or not, see [Yos01]. So the function field $\overline{k}(L)$ is denoted by K_P, and the extension is considered from a geometric point of view.



The notion of *Galois points*, as far as we know, was first introduced by H. Yoshihara in 1996, see [Fuk09, MY00, Yos01].

Definition 1.3.4. Following the notations above, a point $P \in \mathbb{P}^2_{\overline{k}}$ is a *Galois point* for $\overline{C}$, if the function field extension $\pi_P^*: K_P \hookrightarrow K$, induced by π_P, is Galois.

A Galois point P is an inner (resp. outer) Galois point for $\overline{C}$, if $P \in \overline{C}$ (resp. $P \notin \overline{C}$). The number of inner (resp. outer) Galois points for $\overline{C}$ is denoted by $\delta(\overline{C})$ (resp. $\delta'(\overline{C})$).

Example 1.3.5. *Let $\overline{C}$ be the smooth plane curve over $\overline{k}$ defined by $X^3Z+Y^4+Z^4=0$, where k is a field of characteristic $p \neq 2,3$. The point $P := (1:0:0)$ is an $P := (1:0:0)$ is an inner Galois point for $\overline{C}$. The natural point projection from P, $\pi_P: C \to \mathbb{P}^1_{\overline{k}}$ is defined by $(X:Y:Z) \in C \mapsto (Y:Z)$. So $K/K_P = \overline{k}(x,y)/\overline{k}(y): x^3+y^4+1=0$ is cyclic extension, in particular is Galois.*

Example 1.3.6. *Let $\overline{C}$ be the smooth plane curve defined by $X^pZ+XZ^p-Y^{p+1}=0$ over $\overline{k}$, where k is a field of characteristic $p \geq 3$. The point $P := (1:0:0)$ is an inner Galois point for $\overline{C}$, since $K/K_P = \overline{k}(x,y)/\overline{k}(y): x^p+x-y^{p+1}=0$ is an Artin-Schreier extension, which is well-known to be Galois.*

Example 1.3.7. *Let $\overline{C}$ be the smooth plane curve of degree $d \geq 5$ defined by $X^d+Y^{d-1}Z+YZ^{d-1}=0$ over $\overline{k}$, where k is a field of characteristic $p=0$ or $p > (d-1)(d-2)+1$. The* *point $P := (1 : 0 : 0)$ is the unique outer Galois point for $\overline{C}$, see Proposition 2.4.10 and its proof.*

H. Yoshihara in [MY00, Yos01] classified the numbers $\delta(\overline{C})$ and $\delta'(\overline{C})$ when k has zero characteristic. For positive characteristic $p > 0$, M. Homma in [Hom06] determined $\delta(\overline{C})$ and $\delta'(\overline{C})$ when $\overline{C}$ is a Fermat curve of degree $d = p^\ell + 1$. S. Fukasawa in [Fuk06, Fuk08, Fuk14b] introduced the number $\delta(\overline{C})$ when $p > 2$ or $d - 1$ is not a power of 2, and $\delta'(\overline{C})$ when $p \nmid d$, $d = p$ or $d = 2^\ell$ with $p = 2$. In [Fuk13], he investigated the remaining cases for $\delta(\overline{C})$ and $\delta'(\overline{C})$. More precisely, a complete answer was given to the following problems: Find and classify smooth plane curves of degree $d = 2^\ell + 1$ with $\ell \geq 2$, $p = 2$ and $\delta(\overline{C}) = d$. Second, let $p > 0$, $e \geq 1$, $d = p^e\ell$ with $p \nmid \ell$. Then determine $\delta'(\overline{C})$ when $(p^e, \ell) \notin \{(p, 1), (2^e, 1)\}$. Summing up, we have the following classification theorem of smooth plane curves by the numbers $\delta(\overline{C})$ and $\delta'(\overline{C})$:

Theorem 1.3.8 (Yoshihara, Homma, Fukasawa)**.** *Let $\overline{C}$ be a smooth plane curve of degree $d \geq 4$ over $\overline{k}$, where k is a field of characteristic $p \geq 0$. Then,*

1. $\delta(\overline{C}) = 0, 1, d$ *or* $(d-1)^3 + 1$*. Furthermore, we have:*

   (a) $\delta(\overline{C}) = (d-1)^3 + 1$ *if and only if* $p > 0$, $d = p^e + 1$ *for some* $e \in \mathbb{N}$*, and* $\overline{C}$ *is isomorphic to the Fermat curve of degree* d*.*

   (b) $\delta(\overline{C}) = d \geq 5$ *if and only if* $p = 2$, $d = 2^e + 1$*, and* $\overline{C}$ *is isomorphic to a curve defined by*
   $$cY^{2^e+1} + \prod_{\alpha \in \mathbb{F}_{2^e}} (X + \alpha Y + \alpha^2 Z) = 0,$$
   for some $c \in \overline{k} \setminus \{0, 1\}$*.*

   (c) $\delta(\overline{C}) = d = 4$ *if and only if* $p \neq 2, 3$ *and* $\overline{C}$ *is isomorphic to the curve*
   $$X^3Z + Y^4 + Z^4 = 0.$$

2. $\delta'(\overline{C}) = 0, 1, 3, 7$ *or* $(d-1)^4 - (d-1)^3 + (d-1)^2$*. More precisely,*

   (i) $\delta'(\overline{C}) = (d-1)^4 - (d-1)^3 + (d-1)^2$ *if and only if* $p > 0$, $d - 1$ *is a power of* p*, and* $\overline{C}$ *is isomorphic to the Fermat curve of degree* d*.*

(ii) $\delta'(\overline{C}) = 7$ *if and only if* $p = 2,\ d = 4$, *and* $\overline{C}$ *is isomorphic to the Klein quartic curve.*

(iii) $\delta'(\overline{C}) = 3$ *and three Galois points are not contained in a common line if and only if* $p \nmid d,\ d-1$ *is not a power of* p, *and* $\overline{C}$ *is isomorphic to the Fermat curve of degree* d.

(iv) $\delta'(\overline{C}) = 3$ *and three Galois points are contained in a common line if and only if* $p = 2, d = 4$, *and* $\overline{C}$ *is isomorphic to a plane curve defined by*

$$(X^2+XZ)^2 + (X^2+XZ)(Y^2+YZ) + (Y^2+Y)^2 + cZ^4 = 0,$$

for some $c \in \overline{k} \setminus \{0,1\}$.

Remark 1.3.9. The assumption $c \neq 0,1$ is to avoid singular points on $\overline{C}$. For example, when $\delta(\overline{C}) = d \geq 5$, there are exactly d points on $\overline{C} \cap \{Y = 0\}$. Therefore, singular points should lie on $Y \neq 0$. Moreover, by [Fuk13, Lemma 5], $(X_0 : 1 : Z_0)$ is a singular point only if it is $\mathbb{F}_{2^e}$-rational such that $c + h(X_0, Y_0) = 0$, where $h(X,Z) := \prod_{\alpha \in \mathbb{F}_{2^e}} (X + \alpha + \alpha^2 Z) = 0$. However, we have by [Fuk13, Lemma 6] that $\{h(X,Z) \,:\, X, Z \in \mathbb{F}_{2^e}\} = \{0,1\}$. Consequently, $c = 0$ or 1 are discarded.

As a consequence of Theorem 1.3.8, one has:

Theorem 1.3.10. *(Yoshihara, Theorems 4,4' and Propositions 5,5', [Yos01]) Let* $\overline{C}$ *be a smooth plane curve of degree* $d \geq 4$ *over* $\overline{k}$, *where* k *is a field of characteristic* $p = 0$ *or* $p > (d-1)(d-2)+1$. *Then* $\delta'(\overline{C}) = 0, 1,$ *or* 3, *and moreover* $\delta'(\overline{C}) = 3$ *if and only if* $\overline{C}$ *is isomorphic to the Fermat curve of degree* d. *On the other hand, if* $d = 4$ *then* $\delta(\overline{C}) = 0, 1,$ *or* 4, *and similarly the curve with* $\delta(\overline{C}) = 4$ *is unique and is isomorphic to* $YZ^3 + X^4 + Y^4 = 0$. *On the contrary, for* $d \geq 5$, *one gets* $\delta(\overline{C}) = 0$ *or* 1.

Remark 1.3.11. The full automorphism groups of smooth plane curves with at least two Galois points have already been investigated. Fermat, Klein curves and the curve $X^3Z + Y^4 + Z^4 = 0$ are quite well understood and were studied by many authors (see for example [HKT08, Hur03, KS96, Rit04]). The curve characterized by $\delta'(\overline{C}) = 3$ and three Galois points are contained in a common line has automorphism group isomorphic to the symmetry group S_4, while the curve characterized by $\delta(\overline{C}) = d \geq 5$ has automorphism group isomorphic to $\mathrm{PGL}_2(\mathbb{F}_{2^e})$ provided that $e \geq 2$ (see [Fuk14a, Theorem 1, Theorem 2].

Finally, we end up this subsection by proving the next statement, see [Har13, Lemma 3.7]:

Proposition 1.3.12 (Harui, [Har13])**.** *Let $\overline{C}$ be a smooth plane curve of degree $d \geq 5$ over $\overline{k}$, where k is a field of characteristic $p = 0$. A cyclic group G of automorphisms of $\overline{C}$ automorphisms of $\overline{C}$ has order at most $d(d-1)$. Furthermore, if G is generated by an homology with center P, then $|G| \,|\, d$ when $P \notin \overline{C}$ (resp. $d-1$ when $P \in \overline{C}$). The equality $|G| = d$ (resp. $|G| = d-1$) holds if and only if P is an outer (resp. inner) Galois point for $\overline{C}$.*

Oikawa [Oik56] and Arakawa [Ara00] gave, possibly stronger, upper bounds than *Hurwitz bound* (Theorem 1.1) when G fixes finite subsets of $\overline{C}$ (not necessarily pointwise). As an application of Riemann-Hurwitz formula, one gets the next results. We address the reader to [Oik56, Theorem 1] and [Ara00, Theorem 3] or [Har13, Theorem 3.2] for the complete details.

Theorem 1.3.13. *Let $\overline{C}$ be a smooth plane curve of genus $g \geq 3$ defined over an algebraically closed field of characteristic zero, and let G be a subgroup of $\mathrm{Aut}(\overline{C})$. Then*

1. *(Oikawa's inequality) If G fixes a finite subset S of $\overline{C}$ with $|S| = j \geq 1$, then $|G| \leq 12(g-1) + 6j$.*

2. *(Arakawa's inequality) If G fixes three distinct finite subsets S_i $(i = 1, 2, 3)$ of $\overline{C}$ with $|S_i| = j_i \geq 1$, then $|G| \leq 2(g-1) + j_1 + j_2 + j_3$.*

Proof. (of Proposition 1.3.12) We may assume, without loss of generality, that σ is a generator of G, represented by a diagonal shape matrix. In particular, G fixes each of the three reference lines $L_1 : X = 0$, $L_2 : Y = 0$, and $L_3 : Z = 0$, and each of the three reference points $P_1 := (1 : 0 : 0)$, $P_2 := (0 : 1 : 0)$, and $P_3 := (0 : 0 : 1)$. Set $S_i = \overline{C} \cap L_i$ for $i = 1, 2, 3$, hence each S_i is a non-empty subset of $\overline{C}$ of cardinality at most d, and is fixed by G.

We distinguish between the different situations of $\overline{C} \cap V$, where $V = \{P_1, P_2, P_3\}$:

(i) If $|\overline{C} \cap V| \geq 2$, say $P_1, P_2 \in \overline{C}$, then at least one of the subsets $S_1 \setminus \{P_2\}$, $S_2 \setminus \{P_1\}$ and $S_3 \setminus \{P_1, P_2\}$ is non-empty with cardinality at most $d-1$ (otherwise, $S_1 = \{P_2\}$, $S_2 = \{P_1\}$ and $S_3 = \{P_1, P_2\}$. So L_1 and L_2 intersects $\overline{C}$ at P_2 and P_1, respectively with multiplicity d. Hence the defining equation for $\overline{C}$ becomes $Z^d + XYG(X, Y, Z) = 0$ for some homogenous polynomial of degree $d-2$. But also L_3 intersects $\overline{C}$ only at P_1

and P_2, that is $G(X,Y,Z) = X^jY^{d-2-j}$ for some $0 \leq j \leq d-2$, a contradiction to non-singularity). One applies Arakawa's inequality (Theorem 1.3.13) for such a subset with the two subsets $\{P_1\}$, $\{P_2\}$ to obtain

$$|G| \leq 2(g-1) + (d-1) + 1 + 1 = (d-1)^2 < d(d-1).$$

(ii) If $|\overline{C} \cap V| = 1$, say $P_1 \in \overline{C}$, then either $S_2 \setminus \{P_1\}$ or $S_3 \setminus \{P_1\}$ is a non-empty subset of $\overline{C}$ with cardinality at most $d-1$. Using Arakawa's inequality (Theorem 1.3.13) again for such a subset with the two subsets $\{P_1\}$ and S_1, we then have

$$|G| \leq 2(g-1) + (d-1) + 1 + d = d(d-1).$$

(iii) If $\overline{C} \cap V = \emptyset$ then $\overline{C} : X^d + Y^d + Z^d +$ lower order terms. This implies that $\sigma = \mathrm{diag}(\zeta_d^m, \zeta_d^n, 1)$ for some integers m, n where $0 \leq m, n \leq d-1$. Hence $\sigma^d = 1$ and $|G|$ divides d, in particular, $|G| \leq d(d-1)$.

Second, suppose that σ is a homology. Then, up to conjugation in $\mathrm{PGL}_3(\overline{k})$, one can take $\sigma = \mathrm{diag}(1, 1, \zeta)$, where ζ is a root of unity. That is, its center is $P_3 := (0 : 0 : 1)$ and its axis is $L_3 : Z = 0$. Let $\pi_{P_3} : \overline{C} \to \mathbb{P}^1_{\overline{k}} : (X : Y : Z) \mapsto (X : Y)$ be the natural point projection map from P_3, and $\pi : \overline{C} \to \overline{C}/G$ be the natural quotient map. Hence

$$\psi : \overline{C}/G \to \mathbb{P}^1_{\overline{k}} : G.(X : Y : Z) \mapsto (X : Y)$$

is well-defined, and the following diagram is commutative

$$\begin{tikzcd} \overline{C} \arrow[rr, "\pi"] \arrow[dr, "\pi_{P_3}"'] & & \overline{C}/G \arrow[dl, "\psi"] \\ & \mathbb{P}^1_{\overline{k}}. & \end{tikzcd}$$

In particular, $|G| = \deg(\pi)$ is a factor of $\deg(\pi_{P_3})$, which is $d-1$ if $P_3 \in \overline{C}$ and d otherwise. Furthermore, if $|G| = \deg(\pi_{P_3})$, then π_{P_3} coincides with the quotient map π, which implies that P_3 is a Galois point for $\overline{C}$ and G is the Galois group at P_3. $\square$

§1.4 The classification for smooth plane curves: Harui's work

Proposition 1.4.1. *Let $\overline{C}$ be a smooth curve of genus $g \geq 2$ over $\overline{k}$, where k is a field of characteristic $p \geq 0$. The stabilizer $G_{\mathcal{P}}$ of a place $\mathcal{P}$ of the function field $\overline{k}(\overline{C})$ of $\overline{C}$ is always of the form $N \rtimes \mathbb{Z}/m\mathbb{Z}$, such that*

(i) if $p = 0$, then N is trivial.

(ii) if $p > 0$, then N is a nilpotent group consisting of all elements of $G_{\mathcal{P}}$ whose order is a power of p, and m is relatively prime to p.

Proof. We only sketch the proof, and we address the reader to Lemma 11.44 and Theorem 11.49 in [HKT08] for more details. Choose a uniformizer element z of $\overline{k}(\overline{C})$ at $\mathcal{P}$. The key point is that for every $\alpha \in G_{\mathcal{P}}$, there exists a unique non-zero constant $c_\alpha \in \overline{k}$, such that $\mathrm{ord}_{\mathcal{P}}(\alpha(z) - c_\alpha z) > 1$. Moreover, c_α is independant of the choice of the uniformizing element z and any eigenvalue of α is a power of c_α. The mapping $\phi : G_{\mathcal{P}} \to \overline{k} : \alpha \mapsto c_\alpha$ is a group homomorphism. So $G_{\mathcal{P}}/\ker(\phi)$ is isomorphic to a finite multiplicative subgroup of $\overline{k}$. Thus it is cyclic of order m coprime with p, since any finite multiplicative subgroup of $\overline{k}$ is the group of m-th roots of unity for a suitable m with $\gcd(m, p) = 1$.

Now if $\alpha \in \ker(\phi)$, then any eigenvalue of α is equal to 1. Hence $\ker(\phi)$ consists of all those α, for which it associates a lower-triangular matrix A_α whose main diagonal consists entirely of 1's. Since any $\overline{k}$-automorphism of $\overline{k}(\overline{C})$ is of finite order, we deduce that

- if $p = 0$, then A_α must be the identity matrix and thus α is the identity $\overline{k}$-automorphism,
- if $p > 0$, then A_α has order a power of p , and $\ker(\phi)$ is nilpotent, being isomorphic to a subgroup of matrices of the shape A_α.

$\square$

Corollary 1.4.2. *Let $\overline{C}$ be a smooth curve of genus $g \geq 2$ defined over $\overline{k}$, where k is a field of characteristic $p \geq 0$. Any subgroup G of automorphisms of order coprime with p, that fixes a point on $\overline{C}$, is cyclic.*

Definition 1.4.3. For a non-zero monomial $cX^{i_1}Y^{i_2}Z^{i_3}$ with $c \in \overline{k}\backslash\{0\}$, its exponent is defined to be $\max\{i_1, i_2, i_3\}$. For a homogenous polynomial $F(X, Y, Z)$, the core of it is defined to be the sum of all terms of F with the greatest exponent. Now, let $\overline{C}_0$ be a smooth plane curve over $\overline{k}$, a pair $(\overline{C}, H)$ with $H \leq \mathrm{Aut}(\overline{C})$ is said to be a descendant of $\overline{C}_0$ if $\overline{C}$ is defined by a homogenous polynomial whose core is a defining polynomial of $\overline{C}_0$ and H acts on $\overline{C}_0$ under a suitable change of the coordinates system, i.e. H is conjugate to a subgroup of $\mathrm{Aut}(\overline{C}_0)$.

Recently, T. Harui, in his unpublished paper in arXiv [Har13], provided a classification of the automorphism groups of smooth plane curves of degree $d \geq 4$ over algebraically closed fields of zero characteristic. We detail the statement and its proof next.

Theorem 1.4.4 (Harui, Theroem 2.1, [Har13])**.** *Let k be a field of characteristic $p = 0$, and let G be a subgroup of automorphisms of a smooth plane curve $\overline{C}$ over $\overline{k}$ of degree $d \geq 4$. Then one of the following situations holds:*

1. *G fixes a point on $\overline{C}$ and then it is cyclic.*

2. *G fixes a point not lying on $\overline{C}$ and we always think in the following commutative diagram, with exact rows and vertical injective morphisms:*

$$\begin{tikzcd}
1 \arrow[r] & \overline{k}^* \arrow[r] & \mathrm{PBD}(2,1) \arrow[r, "\Lambda"] & \mathrm{PGL}_2(\overline{k}) \arrow[r] & 1 \\
1 \arrow[r] & N \arrow[r] \arrow[u, hook] & G \arrow[r] \arrow[u, hook] & G' \arrow[r] \arrow[u, hook] & 1
\end{tikzcd}$$

where N is a cyclic group of order dividing the degree d and G' is a subgroup of $\mathrm{PGL}_2(\overline{k})$, which is conjugate to a cyclic group $\mathbb{Z}/m\mathbb{Z}$ of order m with $m \leq d-1$, a Dihedral group D_{2m} of order $2m$ with $|N| = 1$ or $m|(d-2)$, one of the alternating groups A_4, A_5, or to the symmetry group S_4.

3. *G is conjugate to a subgroup of $\mathrm{Aut}(F_d)$, where F_d is the Fermat curve $X^d + Y^d + Z^d = 0$. In particular, $|G|$ divides $\mathrm{Aut}(F_d)| = 6d^2$, and $(\overline{C}, G)$ is a descendant of F_d.*

4. *G is conjugate to a subgroup of $\mathrm{Aut}(K_d)$, where K_d is the Klein curve curve $XY^{d-1} + YZ^{d-1} + ZX^{d-1}$. In this case, $|G|$ divides $|\mathrm{Aut}(K_d)| = 3(d^2 - 3d + 3)$, and $(\overline{C}, G)$ is a*

descendant of K_d.

5. *G is conjugate to a finite primitive subgroup of $\mathrm{PGL}_3(\overline{k})$ mentioned before in Theorem 1.2.1-(3).*

Proof. From Theorem 1.2.1 there are three cases:

(i) G fixes a line $L \subset \mathbb{P}^2_{\overline{k}}$ and point $P \notin L$. If $P \in \overline{C}$ then G is cyclic by Corollary 1.4.2, and it has order at most $d(d-1)$ by Proposition 1.3.12. Otherwise, we can assume that $L : Z = 0$ and $P = (0 : 0 : 1) \notin \overline{C}$, and thus G is an intransitive finite subgroup of $\mathrm{PGL}_3(\overline{k})$ by the virtue of Theorem 1.2.4, *Type I*. In particular, we can think about G in a short exact sequence of the form $1 \longrightarrow N \longrightarrow G \stackrel{\Lambda}{\longrightarrow} G' \longrightarrow 1$, where $N = \mathrm{Ker}(\Lambda|_G)$ and $G' = \mathrm{Img}(\Lambda|_G)$. Here Λ is the natural embedding appeared in Definition 1.2.2. In other words, N could be viewed as the part of G acting on the variable Z and fixing the other variables, while G' is the part acting on X, Y and fixing Z. This gives us the embedding $N \hookrightarrow \overline{k}^*$ and $G' \hookrightarrow \mathrm{PBD}(2,1)$ in the statement. Moreover, every automorphism η in N is represented by a unique diagonal matrix $\mathrm{diag}(1,1,\zeta)$ for some root of unity ζ. Thus the embedding $N \hookrightarrow \overline{k}^* : \eta \mapsto \zeta$ is injective, and N is isomorphic to a subgroup of $\overline{k}^*$. Hence N is cyclic generated by a homology η, and the assertion on the order of N follows from Proposition 1.3.12. On the other hand, the possibilities for G' as a subgroup of $\mathrm{PGL}_2(\overline{k})$ is given by Theorem 1.2.3. It remains to give an upper bound for m when G' is isomorphic to $\mathbb{Z}/m\mathbb{Z}$ or D_{2m}: In both cases, there exists an automorphism $\sigma \in G$ whose image $\sigma' := \Lambda(\sigma)$ is of order m. Up to a change of variables in $\mathrm{PBD}(2,1)$, in particular preserving the assumptions on the line L and the point P, we may take $\sigma = \mathrm{diag}(\alpha, \beta, 1)$ such that $\frac{\alpha}{\beta}$ is a primitive mth root of unity. Recall that $P_1 = (1 : 0 : 0)$ and $P_2 = (0 : 1 : 0)$ are the fixed points of σ on L, so when $G' \cong \mathbb{Z}/m\mathbb{Z}$, i.e. $G = \langle \eta, \sigma \rangle$, then P_1 and P_2 are the fixed points by G on L. Moreover, in case of D_{2m}, there must be $\tau \in G$ such that $\tau' := \Lambda(\tau)$ and σ' generate G' with $\tau'^2 = 1$ and $\tau'\sigma'\tau' = \sigma'^{-1}$. That is $G = \langle \eta, \sigma, \tau \rangle$ and we may assume that $\tau = [\gamma Y : \gamma X : Z]$.

Let $\overline{C} : F(X, Y, Z) = 0$ be the defining equation for $\overline{C}$ with respect to the above assumptions, and e_j the intersection multiplicity of $\overline{C}$ and L at P_j for $j = 1, 2$. We first note that

$e_1 = e_2$ if $G' \cong \mathrm{D}_{2m}$, because of the automorphism τ. As a first insight we show the the next observation:

Observation 1. If $e_1 \geq 2$ or $e_2 \geq 2$, then N is trivial.

Proof. Suppose that $e_1 \geq 2$, then $\overline{C}$ is defined by $F(X,Y,Z) = X^{d-1}Z + X^{e_1}Y^{e_2}G(X,Y)+$ lower order terms in X, where $G(X,Y)$ is a homogenous polynomial in X,Y of degree $d-e_1-e_2$, such that neither X nor Y is a factor of it. Since $\eta = \mathrm{diag}(1,1,\zeta)$ is an automorphism of $\overline{C}$, then $F(\eta(X,Y,Z)) = \lambda F(X,Y,Z)$ for some $\lambda \in \overline{k}^*$. Hence $\zeta = 1$ and N is trivial. Similarly we deal the situation when $e_2 \geq 2$. $\square$

We need to treat each of the following subcases:

(a) $\overline{C} \cap L \subseteq \{P_1, P_2\}$: If G' is isomorphic to $\mathbb{Z}/m\mathbb{Z}$, then G fixes each of P_1 and P_2 and at least one of them belongs to $\overline{C}$. So G itself is cyclic and we go back to the former situation at the beginning (Theorem 1.4.4-(1)). If G' is isomorphic to D_{2m}, then $e_1 = e_2 = \frac{d}{2} \geq 2$, and N is trivial by Observation 1. Furthermore, $\overline{C}$ is smooth at P_1 and P_2, hence the defining equation of $\overline{C}$ becomes $(X^{d-1}+Y^{d-1})Z+$ lower order terms in X and Y. Since $\sigma \in \mathrm{Aut}(C)$, $\alpha^{d-1} = \beta^{d-1}$, and $\sigma'^{d-1} = 1$. Thus m divides $d-1$, in particular $m \leq d-1$.

(b) $\overline{C} \cap L$ contains a point Q distinct from P_1 and P_2: We show the following observation:

Observation 2: The order m of σ' divides $d - e_1 - e_2$. Moreover, if $m = d$ then $(\overline{C}, G)$ is a descendant of the Fermat curve F_d.

Proof. Suppose that σ^j fixes Q for some integer $j \geq 1$. Then $\sigma^j \in N$, since it fixes three points on L, namely, Q, P_1 and P_2, in particular it fixes L pointwise. Therefore $\sigma'^j = 1$ and m divides j. Moreover, it is obvious that σ^m fixes Q, then the orbit of Q by $H := \langle \sigma \rangle$ equals $|H/\langle \sigma^m \rangle| = m$. In other words, we can write $(\overline{C} \cap L) \setminus \{P_1, P_2\}$ as a disjoint union of orbits of m points. Now, Bézout's theorem for $\overline{C}$ reads as $d = e_1 + e_2 + \sum_{i=3}^{r} m \cdot e_i$ for some positive integers e_i, hence m divides $d - e_1 - e_2$. In particular, if $m = d$, then $e_1 = e_2 = 0$, and neither P_1

nor P_2 lies on $\overline{C}$. Hence the core of $F(X,Y,Z)$ must be $X^d+Y^d+Z^d$, but also $G=\langle\eta,\sigma\rangle$ when $G'\cong\mathbb{Z}/m\mathbb{Z}$ and $\langle\eta,\sigma,\tau\rangle$ when $G'\cong \mathrm{D}_{2m}$. In both situation the core of $F(X,Y,Z)$ is invariant under the action of G, which implies that $(\overline{C},G)$ is a descendant of the Fermat curve F_d. □

It is clear now, by Observation 2, that when $G'\cong\mathbb{Z}/m\mathbb{Z}$ then $m\leq d-1$ or $(\overline{C},G)$ is a descendant of the Fermat curve F_d. On the other hand, $e_1=e_2$ when $G'\cong\mathrm{D}_{2m}$, and moreover

- $(\overline{C},G)$ is a descendant of the Fermat curve if $e_1=e_2=0$.
- $m|d-2$ if $e_1=e_2=1$.
- $m\leq d-4$ and N is trivial if $e_1=e_2\geq 2$.

That is to say, Theorem 1.4.4, (1), (2) and (3) follows in this case.

(ii) G fixes a trianlge Δ and neither a line nor a point is leaved invariant by G. Up to projective equivalence, we may assume that Δ consists of the three reference lines $L_1:X=0$, $L_2:Y=0$, and $L_3:Z=0$. So the set of vericies of δ is $V=\{P_1,\ P_2,\ P_3\}$, where G acts transitively on V, by our assumptions that neither a line nor a point is leaved invariant by G. Hence either $\overline{C}$ and V are disjoint or $V\subset\overline{C}$. Also, each element of G gives a permutation of the set $\{X,Y,Z\}$ of the corrdinate functions, up to a constant.

In the case that $\overline{C}$ contains V, we denote by T_i the tanget line to $\overline{C}$ at P_i, for $i=1,2,3$. By the assumptions on G, these lines are not concurrent, pairwise distinct, and G fixes the set $\{T_1,T_2,T_3\}$ and acts on it transitivley.

We treat each of the following subcases:

(a) $\overline{C}$ and V are disjoint: The core of $F(X,Y,Z)$ should be $X^d+Y^d+Z^d$, and G is then a subgroup of $\mathrm{Aut}(F_d)$. Hence $(\overline{C},G)$ is a descendant of the Fermat curve in this subcase.

(b) $V\subset\overline{C}$ and each of T_i's, for $i=1,2,3$, is an edge of Δ: We may assume that $T_1:Z=0$, $T_2:X=0$, and $T_3:Y=0$. Then the core of $F(X,Y,Z)$ is

$XY^{d-1}+YZ^{d-1}+ZX^{d-1}$, which is fixed by G. So $(\overline{C}, G)$ is a descendant of the Klein curve K_d.

(c) $V \subset \overline{C}$ and non of T_i's, for $i = 1, 2, 3$, is an edge of Δ: We show that this subcase does not occur.

Let P'_{i_1} be the intersection point of T_{i_2} and T_{i_3}, where $\{i_1, i_2, i_3\} = \{1, 2, 3\}$. Then they are pairwise distinct because otherwise T_1, T_2, and T_3 are concurrent and their intersection point is fixed by G, a contradiction. Therefore, we have a triangle Δ' fixed by G, whose edges are formed by $T'_i s$ and the set $V' := \{P'_1, P'_2, P'_3\}$ represents its vertices. We also note that V and V' are disjoint by assumption.

There is a natural group homomorphism $\varphi : G \to \mathrm{S}_3$ given by

$$\sigma = [\alpha X_{i_1} : \beta X_{i_2} : \gamma X_{i_3}] \in G \mapsto (i_1 i_2 i_3),$$

where $\alpha, \beta, \gamma \in \overline{k}^*$ with $\{i_1, i_2, i_3\} = \{1, 2, 3\}, X_1 = X, X_2 = Y$, and $X_3 = Z$. Since neither a line nor a point is fixed by G, $\mathrm{Im}(\varphi)$ is isomorphic to $\mathbb{Z}/3\mathbb{Z}$ or S_3. Furthermore, any $\sigma \in \mathrm{Ker}(\varphi)$ is written in the shape $\mathrm{diag}(\alpha, \beta, 1) \in \mathrm{PGL}_3(\overline{k})$. Hence it fixes V pointwise, which implies that it fixes each T_i, for $i = 1, 2, 3$. In particular, it fixes V' pointwise, and it follows by Lemma 1.2.7 that σ is trivial. We then conclude that G is isomorphic to $\mathrm{Im}(\varphi)$, which is $\mathbb{Z}/3\mathbb{Z}$ or S_3. If $G \cong \mathbb{Z}/3Z$, then it fixes a line, which conflicts our assumptions on G. On the other hand, if $G \cong \mathrm{S}_3$, then it is G is generated by $\sigma = [Y : Z : X]$ and some τ of order 2 such that $\tau\sigma\tau = \sigma^{-1}$. After a suitable change of coordinates, if necessary, we may take $\tau = [\zeta_3 Y : \zeta_3^{-1} X : Z]$. Thus G fixes the point $(1 : \zeta_3^{-1} : \zeta_3)$, which contradicts again the restrictions on G.

(iii) G is conjugate to a finite primitive subgroup of $\mathrm{PGL}_3(\overline{k})$. This leads us to the statement (5) in Theorem 1.4.4.

This completes the proof. $\square$

We also need Theorem 2.3 in [Har13]:

Theorem 1.4.5 (Harui, [Har13]). *Let $\overline{C}$ be a smooth plane curve of degree $d \geq 5$ with $d \neq 6$ over $\overline{k}$, where k is a field of characteristic $p = 0$. Then $|\operatorname{Aut}(C)| \leq 6d^2$.*

Proof. Suppose first that $\operatorname{Aut}(\overline{C})$ is not primitive, then it fixes a line or a triangle, by Theorem 1.2.1. If it fixes a line L, then $S := \overline{C} \cap L$ is a non-empty set of cardinality at most d, which is also fixed by $\operatorname{Aut}(\overline{C})$. Apply Oikawa's inequality (Theorem 1.3.13-(1)) to obtain that

$$|\operatorname{Aut}(\overline{C})| \leq 12(g-1) + 6|S| \leq 6d(d-3) + 6d = 6d(d-2) < 6d^2.$$

Similarly, if $\operatorname{Aut}(\overline{C})$ a triangle Δ, then $\overline{C} \cap \Delta$ is non-empty set of cardinality at most $3d$, which is fixed by $\operatorname{Aut}(\overline{C})$. So we have the same inequality $|\operatorname{Aut}(\overline{C})| \leq 6d^2$ by the same argument above.

Second, if $\operatorname{Aut}(\overline{C})$ is primitive, then $|\operatorname{Aut}(\overline{C})| \leq 360$. Hence $|\operatorname{Aut}(\overline{C})| \leq 6d^2$ for any $d \geq 8$. If $d = 5$ or 7, then we still have the inequality $|\operatorname{Aut}(\overline{C})| \leq 6d^2$ except for the pairs $(d, |\operatorname{Aut}(\overline{C})|) = (5, 168), (5, 216), (5, 360)$ or $(7, 360)$ again by Theorem 1.2.1. However, these four exceptional cases do not occur following Theorem 1.1. $\square$

The previous results; Theorem 1.4.4, Theorem 1.4.5 still true when the characteristic p is positive and big enough. For example, it does for $p > (d-1)(d-2) + 1 = 2g + 1$, and we justify this next (see also Badr-Bars [BB16c, §6]):

Fix a prime $p > 0$ and let k be a field of characteristic $p > 2$. Consider a smooth plane curve $\overline{C}$ of genus $g = (d-1)(d-2)/2 \geq 3$ over $\overline{k}$, and suppose that $|\operatorname{Aut}(\overline{C})|$, $|\operatorname{Aut}(F_d)|$, $|\operatorname{Aut}(K_d)|$, and $d(d-1)$ are relatively prime with p, and $p \geq 7$, where F_d and K_d are the Fermat curve and the Klein curve of degree d, respectively. Then the techniques appeared in [Har13] hold.

Consider the p-torsion of the degree 0 Picard group of $\overline{C}$, which is a finitely generated $\mathbb{Z}/p\mathbb{Z}$-module of dimension γ. The integer γ is called the p-rank of $\overline{C}$, and always $\gamma \leq g$, where g is the genus of $\overline{C}$. For a point $\mathcal{P}$ of $\overline{C}$, we mean by $\operatorname{Aut}(\overline{C})_{\mathcal{P}}$, the subgroup of $\operatorname{Aut}(\overline{C})$ fixing the place $\mathcal{P}$.

Lemma 1.4.6. *Assume that $|\operatorname{Aut}(\overline{C})_{\mathcal{P}}|$ is prime to p, for any point $\mathcal{P}$ of $\overline{C}$ and that the p-rank of $\overline{C}$ is trivial. Then $|\operatorname{Aut}(\overline{C})|$ is prime to p.*

Proof. Let $\sigma \in \mathrm{Aut}(\overline{C})$ be of order p. Then the extension $\overline{k}(\overline{C})/\overline{k}(\overline{C})^{\sigma}$ is a finite extension of degree p, and is unramified everywhere (if it ramifies at a place $\mathcal{P}$, then σ is an element of $\mathrm{Aut}(\overline{C})_{\mathcal{P}}$, which conflicts our assumption on $|\mathrm{Aut}(\overline{C})_{\mathcal{P}}|$). Now, if $\gamma = 0$, i.e the p-rank is trivial for $\overline{C}$, then using Deuring-Shafarevich formula [HKT08, Theorem 11.62], we get $\frac{\gamma-1}{\gamma'-1} = p$, where γ' is the p-rank for $\overline{C}/\langle\sigma\rangle$, which is not possible. Therefore such an extension does not exist. □

Lemma 1.4.7. *Let $\overline{C}$ be a smooth plane curve of degree $d \geq 4$ over $\overline{k}$. If $p > 2g + 1$, then $\mathrm{Aut}(\overline{C})_{\mathcal{P}}$ is coprime with p, for any point $\mathcal{P}$ of $\overline{C}$.*

Proof. By [HKT08, Theorem 11.78], the maximal order of the p-subgroup of $\mathrm{Aut}(\overline{C})_{\mathcal{P}}$ is at most $\frac{4p}{(p-1)^2g^2}$. Hence, with $g = \frac{1}{2}(d-1)(d-2)$ and assuming that $p > \frac{4pg^2}{(p-1)^2}$, we obtain the result. □

Lemma 1.4.8. *Let $\overline{C}$ be a smooth curve of genus $g \geq 2$ over an algebraically closed field $\overline{k}$ of characteristic $p > 0$. Suppose that $\overline{C}$ has a separable unramified subcover $\Phi : \overline{C} \to \overline{C}'$ of degree p. Then, $\overline{C}'$ has genus ≥ 2, $g \equiv 1 (\mathrm{mod}\ p)$ and $\gamma \equiv 1(\mathrm{mod}\ p)$. In particular, one needs to restrict $p < g$, for the existence of such a subcover.*

Proof. The Hurwitz formula for Φ gives the equality $(2g-2) = p(2g'-2)$, where g' is the genus of $\overline{C}'$. First $g' \neq 0, 1$, since $g \geq 2$. So $g' \geq 2$ and $g - 1 \equiv 0(\mathrm{mod}\ p)$. Now, consider the Deuring-Shafaravich formula, which could be read as $\gamma - 1 = p(\gamma' - 1)$ in such an unramified extension, where γ' the p-rank of $\overline{C}'$. If $\gamma = 1$, then there is nothing to prove, and if $\gamma > 1$, then the congruence is clear. Finally, the situation $\gamma = 0$ does not occur. □

Corollary 1.4.9. *Let $\overline{C}$ be a smooth plane curve of genus $g = \frac{1}{2}(d-1)(d-2) \geq 2$ over $\overline{k}$. Suppose that the characteristic p satisfies $p > (d-1)(d-2)+1 > g$. Then $\mathrm{Aut}(\overline{C})$ is coprime with p.*

Proof. If $\sigma \in \mathrm{Aut}(\overline{C})$ is of order p, then $\overline{k}(\overline{C})/\overline{k}(\overline{C})^{\sigma}$ is a separable degree p extension, and by Lemma 1.4.7, it is unramified everywhere. By Lemma 1.4.8, we conclude that such extensions do not exist. □

2

CHAPTER

ES-Irreducibility vs "large" and "very large" automorphisms groups

We consider, up to $\overline{k}$-isomorphism, a smooth plane curve $\overline{C}$ of degree $d \geq 4$ defined over $\overline{k}$, that is $\overline{C} \in \mathcal{M}_g^{Pl}$ with genus $g = \frac{1}{2}(d-1)(d-2) \geq 3$. It corresponds to $\overline{C}$ a set of infinitely many non-singular plane models in $\mathbb{P}^2_{\overline{k}}$ of degree d, where any two of them are isomorphic over $\overline{k}$ and their automorphism groups are $\mathrm{PGL}_3(\overline{k})$-conjugate. By the uniqueness of the linear series g^2_d ([HKT08, Lemma 11.28]), any isomorphism ϕ between two such non-singular plane models of $\overline{C}$ is given by an automorphism of the projective plane $\mathbb{P}^2_{\overline{k}}$, i.e. we can take $\phi \in \mathrm{PGL}_3(\overline{k})$, by Theorem 1.1.5. In other words, if $F_{\overline{C}}(X,Y,Z) = 0$ is a non-singular plane model of $\overline{C}$ in $\mathbb{P}^2_{\overline{k}}$, then any other non-singular plane model of $\overline{C}$ over $\overline{k}$ is given by an equation of the form $F_{\overline{C}}(\phi(X,Y,Z)) = 0$, for some change of variables $\phi \in \mathrm{PGL}_3(\overline{k})$. We use the notation

$$^{\phi}C : F_{\phi^{-1}\overline{C}}(X,Y,Z) := F_{\overline{C}}(\phi(X,Y,Z)) = 0.$$

Hence $\mathrm{Aut}(F_{\overline{C}})$ and $\mathrm{Aut}(F_{\phi^{-1}\overline{C}})$ are finite subgroups of $\mathrm{PGL}_3(\overline{k})$, and $\mathrm{Aut}(F_{\phi^{-1}\overline{C}}) = \phi^{-1}\,\mathrm{Aut}(F_{\overline{C}})\phi$.

Definition 2.1. *Given a smooth plane curve $\overline{C}$ over $\overline{k}$, any two non-singular plane models of $\overline{C}$ over $\overline{k}$ are said to be* $\overline{k}$-projectively equivalent *or* $\overline{k}$-isomorphic.

Now, let G be a finite non-trivial group. Recall that if $\overline{C} \in \mathcal{M}_g^{Pl}(G)$, then $\varrho(G) \leq \mathrm{Aut}(F_{\overline{C}})$ for some injective representation $\varrho : G \hookrightarrow \mathrm{PGL}_3(\overline{k})$. Also, $\overline{C} \in \widetilde{\mathcal{M}_g^{Pl}}(G)$ if and only if $\varrho(G) = \mathrm{Aut}(F_{\overline{C}})$, for some ϱ.

Definition 2.2. *Let $\varrho : G \hookrightarrow \mathrm{PGL}_3(\overline{k})$ be an injective representation of a finite non-trivial group G inside $\mathrm{PGL}_3(\overline{k})$. The stratum of smooth plane curves $\overline{C}$ over $\overline{k}$ modulo $\overline{k}$-isomorphism, such that $\varrho(G) \leq \mathrm{Aut}(F_{\overline{C}})$ for some non-singular plane model $F_{\overline{C}}(X,Y,Z) = 0$ of $\overline{C}$ over $\overline{k}$, is denoted by $\varrho(\mathcal{M}_g^{Pl}(G))$. Similarly, we define $\varrho(\widetilde{\mathcal{M}_g^{Pl}}(G))$ when $\varrho(G) = \mathrm{Aut}(F_{\overline{C}})$.*

Remark 2.3. *We note that if $\varrho_i : G \hookrightarrow \mathrm{PGL}_3(\overline{k})$, for $i = 1, 2$, are $\mathrm{PGL}_3(\overline{k})$-conjugated, then $\varrho_1(\mathcal{M}_g^{Pl}(G)) = \varrho_2(\mathcal{M}_g^{Pl}(G))$.*

Definition 2.4 (Normal forms)**.** *Given a finite non-trivial group G, the associated* normal forms *to the stratum $\mathcal{M}_g^{Pl}(G)$ is a finite set of homogenous equations $\{\mathcal{N}_{1,G}, \ldots, \mathcal{N}_{m,G}\}$ in X, Y, Z, each one of them is equipped with parameters, under some algebraic restrictions, as the coefficients of its monomials. Moreover, any specialization of the parameters in $\overline{k}$ of an $\mathcal{N}_{i,G}$ corresponds to some $\overline{C} \in \mathcal{M}_g^{Pl}(G)$. Conversely, any $\overline{C} \in \mathcal{M}_g^{Pl}(G)$ is $\overline{k}$-isomorphic to a non-singular plane model over $\overline{k}$ given by a specialization of the parameters in $\overline{k}$ of some $\mathcal{N}_{i,G}$.*

In the same way, one defines the associated normal forms *to the stratum $\widetilde{\mathcal{M}_g^{Pl}}(G)$. However, in this case, a specialization of the parameters of two distinct forms $\mathcal{N}_{i_1,G}$ and $\mathcal{N}_{i_1,G}$ gives two non-singular plane models over $\overline{k}$, which in turns relate to two non-isomorphic smooth plane curves in $\widetilde{\mathcal{M}_g^{Pl}}(G)$, see Lemma 2.2.4.*

Following the above notations, it becomes very natural to investigate the next question:

Question 2.5. *For a fixed degree d, list the $\varrho's$ and the groups $G's$ such that $\varrho(\mathcal{M}_g^{Pl}(G))$ is non-empty. Next, determine the associated normal forms to the stratum $\mathcal{M}_g^{Pl}(G)$ for each such G. The same problem is also rephrased for the different strata $\varrho(\widetilde{\mathcal{M}_g^{Pl}}(G))$ and for $\widetilde{\mathcal{M}_g^{Pl}}(G)$.*

For a cyclic group $\mathbb{Z}/m\mathbb{Z}$ of order m, Dolgachev in [Dol12] determined the $\varrho's$ and $m's$ for which $\varrho(\mathcal{M}_3^{Pl}(\mathbb{Z}/m\mathbb{Z})) \neq \emptyset$. The defining equation of each non-empty $\varrho(\mathcal{M}_3^{Pl}(\mathbb{Z}/m\mathbb{Z}))$ is also given. On the other hand, P. Henn in [Hen76] and Komiya-Kuribayashi in [KK79], provided the list of $\varrho's$ and $G's$ such that $\varrho(\mathcal{M}_3^{Pl}(G))$ and $\varrho(\widetilde{\mathcal{M}_3^{Pl}}(G))$ are non-empty. Moreover, the associated *normal forms* to each non-empty $\widetilde{\mathcal{M}_3^{Pl}}(G)$ are determined (Theorem 2.2.1). See also E. Lorenzo's PhD thesis [LG14, § 2.1 and § 2.2] and R. Lercier, et al. [LRRS14], in order to fix some minor details.

The structure of this chapter is as follows: In section 2.1, we follow the same technique as Dolgachev [Dol12] to give the list of $\varrho's$ and $m's$ where $\varrho(\mathcal{M}_g^{Pl}(\mathbb{Z}/m\mathbb{Z}))\neq\emptyset$, for any $g\geq 3$; Theorem 2.1.3 and Corollary 2.1.6. Next, one sees in Henn Table (Theorem 2.2.1) the following phenomenon, occurring for $g=3$: Given a finite non-trivial group G such that $\widetilde{\mathcal{M}_3^{Pl}}(G)\neq\emptyset$, there exists at most a single *normal form*, that describes the stratum $\widetilde{\mathcal{M}_3^{Pl}}(G)$, up to $\mathrm{PGL}_3(\overline{k})$-conjugation. This motivates us to define the concept of "ES-irreducibility" of $\widetilde{\mathcal{M}_g^{Pl}}(G)$ in section 2.2. Roughly speaking, the stratum $\widetilde{\mathcal{M}_g^{Pl}}(G)$ is called ES-Irreducible if it is defined by a single normal form (see Definition 2.2.6 for the precise statement). This would be a weaker concept than the irreducibility of $\widetilde{\mathcal{M}_g^{Pl}}(G)$ inside the moduli space $\mathcal{M}_g$, in the sense that the number of ES-irreducible components is a lower bound of the number of its irreducible components in $\mathcal{M}_g$. In section 2.3, we show examples of non ES-Irreducible strata of the form $\widetilde{\mathcal{M}_g^{Pl}}(\mathbb{Z}/m\mathbb{Z})$ for infinitely many genera $g\geq 6$. Finally, we characterize, in Section 2.4, the stratum $\mathcal{M}_g^{Pl}(G)$, where G has elements of order d^2-3d+3, $(d-1)^2$, $d(d-2)$, $d(d-1)$, md, or $m(d-1)$ with $m\geq 2$, to be always defined by a single *normal form*. In particular, a non-empty $\widetilde{\mathcal{M}_g^{Pl}}(G)$, in this case, is ES-Irreducible.

We shall deal with the following items:

2.1. Cyclic automorphism subgroups of smooth plane curves.

2.2. Union decomposition of $\mathcal{M}_g^{Pl}(G)$ and "ES-irreducibility".

2.3. Strata of smooth plane curves not ES-irreducible.

2.4 On smooth plane curves, admitting "large" or "very large" automorphisms.

The main results of sections §2.1 and 2.4 have been published in [BB16b], whereas those of sections §2.2 and 2.3 have been published in [BB16c].

§2.1 Cyclic automorphism subgroups of smooth plane curves

Fix an integer $g=\frac{1}{2}(d-1)(d-2)\geq 3$ and a finite non-trivial group G.

Lemma 2.1.1. *Let $\overline{C}: F(X,Y,Z)$ be a smooth plane curve of degree d defined over $\overline{k}$. Then the defining equation $F(X,Y,Z)=0$ must have degree at least $d-1$ in each variable.*

Proof. For example, if $F(X,Y,Z)=0$ has degree $\leq d-2$ in Z, then $F(X,Y,Z)=$

$\sum_{i=0}^{d-2} \beta_i Z^i L_{d-i}(X,Y)$, where $L_{d-i}(X,Y)$ is a homogenous polynomial of degree $d-i$ in X and Y. Hence, $(0:0:1)$ is a projective solution of the system

$$F(X,Y,Z) = \frac{\partial F}{\partial X}(X,Y,Z) = \frac{\partial F}{\partial Y}(X,Y,Z) = \frac{\partial F}{\partial Z}(X,Y,Z) = 0.$$

That is, it is a singular point of $F(X,Y,Z) = 0$. $\square$

Definition 2.1.2. Let $F_{\overline{C}}(X,Y,Z) = 0$ be a non-singular plane model of degree d over $\overline{k}$ for $\overline{C} \in \mathcal{M}_g^{Pl}$, where k is a field of characteristic $p = 0$ or $p > 2g+1$. Then, $\mathrm{Aut}(F_{\overline{C}}) = \varrho(\mathrm{Aut}(\overline{C}))$, for some $\varrho : G \hookrightarrow \mathrm{PGL}_3(\overline{k})$. If $\sigma \in \mathrm{Aut}(\overline{C})$ is an element of exact order m, then by a change of variables in $\mathbb{P}^2_{\overline{k}}$ (that is, changing the non-singular plane model of $\overline{C}$ to a $\overline{k}$-projectively equivalent one), we may assume that $\varrho(\sigma)$ acts on $F_{\overline{C}}(X,Y,Z) = 0$ as the automorphism

$$(X:Y:Z) \mapsto (X:\zeta_m^a Y:\zeta_m^b Z),$$

where ζ_m is a primitive mth root of unity in K and a, b are integers, such that $0 \leq a < b < m$. Moreover, if $ab \neq 0$, then m and $\gcd(a,b)$ are relatively prime (we can reduce to $\gcd(a,b) = 1$) and if $a = 0$, then $\gcd(b,m) = 1$.

We write $\varrho_{a,b,m}(\mathbb{Z}/m\mathbb{Z})$ for $\langle \mathrm{diag}(1, \zeta_m^a, \zeta_m^b) \rangle$ in $\mathrm{PGL}_3(\overline{k})$. Furthermore, we call $\overline{C}$ of *Type* $m, (a,b)$ and $\overline{C} \in \varrho_{a,b,m}(\mathcal{M}_g^{Pl}(\mathbb{Z}/m\mathbb{Z}))$.

Our aim here is to investigate, which cyclic groups could appear inside $\mathrm{Aut}(\overline{C})$ and the associated normal forms to each case as well. Therefore, to determine all possible *Types* $m, (a,b)$, for which the stratum $\varrho_{a,b,m}(\mathcal{M}_g^{Pl}(\mathbb{Z}/m\mathbb{Z}))$ might be non-empty. We follow a similar approach as Dolgachev in [Dol12], that deals with the same question for $d = 4$ (see also [Bar12, §2.1]).

Before we state our main result for this section, we need the following notations, which will be used through the sequel.

Notations.

- We say that *Type* $m, (a,b)$ is a generator of $\varrho(\mathbb{Z}/m\mathbb{Z})$, for some $\varrho : \mathbb{Z}/m\mathbb{Z} \hookrightarrow \mathrm{PGL}_3(\overline{k})$ when $\varrho(\mathbb{Z}/m\mathbb{Z}) = \varrho_{a,b,m}(\mathbb{Z}/m\mathbb{Z})$.

- $L_{i,*}$ is homogeneous polynomial in $\overline{k}[X,Y,Z]$ of degree i, such that the variable $* \in \{X,Y,Z\}$ does not appear.

For simplicity, we also define the next index sets:

- $S(u)_m := \{u \leq j \leq d-1 \,|\, d-j \equiv 0 \,(\mathrm{mod}\; m)\}$;
- $S_u^{d,X}{}_{m,(a,b)} := \{u \leq i \leq d-u \,|\, ai + (d-i)b \equiv 0 \,(\mathrm{mod}\; m)\}$;
- $S_u^{d-1,X}{}_{m,(a,b)} := \{1 \leq i \leq d-u \,|\, ai + (d-1-i)b \equiv 0 \,(\mathrm{mod}\; m)\}$;
- $S(1)^{j,X}_{m,(a,b)} := \{0 \leq i \leq j \,|\, ai + (j-i)b \equiv a \,(\mathrm{mod}\; m)\}$;
- $S(2)^{j,X}_{m,(a,b)} := \{0 \leq i \leq j \,|\, ai + (j-i)b \equiv 0 \,(\mathrm{mod}\; m)\}$;
- $S^{j,Y}_{m,(a,b)} := \{0 \leq i \leq j \,|\, bi + (d-j)a \equiv a \,(\mathrm{mod}\; m)\}$;
- $S^{j,Z}_{m,(a,b)} := \{0 \leq i \leq j \,|\, ai + (d-j)b \equiv a \,(\mathrm{mod}\; m)\}$;
- $\Gamma_m := \{(a,b) \in \mathbb{N}^2 \,|\, \gcd(a,b) = 1,\; 1 \leq a \neq b \leq m-1\}$.
- The three reference points in $\mathbb{P}^2_{\overline{k}}$ are $P_1 := (1:0:0),\; P_2 := (0:1:0)$ and $P_3 := (0:0:1)$, respectively.

Here u, j, m, d, a and b are all non-negative integers.

Theorem 2.1.3 (Badr-Bars, Theorem 7, [BB16b]). *Let $\overline{C} \in \mathcal{M}_g^{Pl}$ be a smooth plane curve of degree $d \geq 4$ over $\overline{k}$, where k is a field of characteristic $p = 0$ or $p > (d-1)(d-2)+1$, such that $\mathrm{Aut}(\overline{C})$ is not trivial. Then, $\overline{C} \in \varrho_{a,b,m}(\mathcal{M}_g^{Pl}(\mathbb{Z}/m\mathbb{Z}))$ for some a, b, m as in the list $(1)-(6)$ below. Moreover, each component $\varrho_{a,b,m}(\mathcal{M}_g^{Pl}(\mathbb{Z}/m\mathbb{Z}))$ is associated to a single normal form $\mathcal{F}_{\varrho_{a,b,m}}(X,Y,Z) = 0$:*

1. *The curve $\overline{C} \in \varrho_{m,0,1}(\mathcal{M}_g^{Pl}(\mathbb{Z}/m\mathbb{Z}))$ with $m|d-1$ and $\mathcal{F}_{\varrho_{a,b,m}}(X,Y,Z)$ is defined by*
$$Z^{d-1}Y + \sum_{j \in S(2)_m} Z^{d-j} L_{j,Z} + L_{d,Z}.$$
2. *The curve $\overline{C} \in \varrho_{m,0,1}(\mathcal{M}_g^{Pl}(\mathbb{Z}/m\mathbb{Z}))$ with $m|d$, and $\mathcal{F}_{\varrho_{a,b,m}}(X,Y,Z)$ is given by the form*
$$Z^d + \sum_{j \in S(1)_m} Z^{d-j} L_{j,Z} + L_{d,Z}.$$

3. *All reference points lie on $\mathcal{F}_{\varrho_{a,b,m}}(X,Y,Z)=0$: The curve $\overline{C}\in\varrho_{m,a,b}(\mathcal{M}_g^{Pl}(\mathbb{Z}/m\mathbb{Z}))$ with $m\,|\,(d^2-3d+3)$ and $(a,b)\in\Gamma_m$, such that $a=(d-1)a+b=(d-1)b\,(\mathrm{mod}\ m)$. In this case,*

$$\begin{aligned}\mathcal{F}_{\varrho_{a,b,m}}(X,Y,Z) &:= X^{d-1}Y+Y^{d-1}Z+Z^{d-1}X+\sum_{j=2}^{d-2}\sum_{i\in S(1)^{j,X}_{m,(a,b)}}\beta_{j,i}X^{d-j}Y^iZ^{j-i}+\\ &+\sum_{i\in S^{j,Y}_{m,(a,b)}}\alpha_{j,i}X^{j-i}Y^{d-j}Z^i+\sum_{i\in S^{j,Z}_{m,(a,b)}}\gamma_{j,i}X^{j-i}Y^iZ^{d-j}.\end{aligned}$$

4. *Exactly two of the reference points lie on $\mathcal{F}_{\varrho_{a,b,m}}(X,Y,Z)=0$: One of the following subcases occurs:*

   (4.1) $\overline{C}\in\varrho_{m,a,b}(\mathcal{M}_g^{Pl}(\mathbb{Z}/m\mathbb{Z}))$ for some $m\,|\,d(d-2)$ and $(a,b)\in\Gamma_m$, such that $(d-1)a+b\equiv 0\,(\mathrm{mod}\ m)$ and $a+(d-1)b\equiv 0\,(\mathrm{mod}\ m)$. Moreover, $\mathcal{F}_{\varrho_{a,b,m}}(X,Y,Z)$ is

$$X^d+Y^{d-1}Z+YZ^{d-1}+\sum_{j=2}^{d-1}\sum_{i\in S(2)^{j,X}_{m,(a,b)}}\beta_{j,i}X^{d-j}Y^iZ^{j-i}+\sum_{i\in S_2^{d,X}{}_{m,(a,b)}}\beta_{d,i}Y^iZ^{d-i},$$

   (4.2) $\overline{C}\in\varrho_{m,a,b}(\mathcal{M}_g^{Pl}(\mathbb{Z}/m\mathbb{Z}))$ for some $m|(d-1)^2$, and $(a,b)\in\Gamma_m$, such that $(d-1)a+b\equiv 0\,(\mathrm{mod}\ m)$ and $(d-1)b\equiv 0\,(\mathrm{mod}\ m)$. We thus have

$$\begin{aligned}\mathcal{F}_{\varrho_{a,b,m}}(X,Y,Z) &:= X^d+XZ^{d-1}+Y^{d-1}Z+\sum_{i\in S_2^{d,X}{}_{m,(a,b)}}\beta_{d,i}Y^iZ^{d-i}+\\ &+\sum_{j=2}^{d-2}\sum_{i\in S(2)^{j,X}_{m,(a,b)}}\beta_{j,i}X^{d-j}Y^iZ^{j-i}+\sum_{i\in S_1^{d-1,X}{}_{m,(a,b)}}\beta_{(d-1),i}XY^iZ^{d-1-i},\end{aligned}$$

   (4.3) $\overline{C}\in\varrho_{m,a,b}(\mathcal{M}_g^{Pl}(\mathbb{Z}/m\mathbb{Z}))$ for some $m|(d-1)$ and $(a,b)\in\Gamma_m$. In this situation, $\mathcal{F}_{\varrho_{a,b,m}}(X,Y,Z)$ is defined by

$$\begin{aligned}X^d &+ X(Y^{d-1}+Z^{d-1}+\sum_{i\in S_2^{d-1,X}{}_{m,\,(a,b)}}\beta_{(d-1),i}Y^iZ^{d-1-i})+\\ &+\sum_{i\in S_2^{d,X}{}_{m,\,(a,b)}}\beta_{d,i}Y^iZ^{d-i}+\sum_{j=2}^{d-2}\sum_{i\in S(2)^{j,X}_{m,(a,b)}}\beta_{j,i}X^{d-j}Y^iZ^{j-i}.\end{aligned}$$

5. *Only one reference point lies on $\mathcal{F}_{\varrho_{a,b,m}}(X,Y,Z)=0$: Then $\overline{C}\in\varrho_{m,a,b}(\mathcal{M}_g^{Pl}(\mathbb{Z}/m\mathbb{Z}))$,*

where $m|\, d(d-1)$ *and* $(a,b) \in \Gamma_m$*, such that* $da \equiv 0 \,(\mathrm{mod}\ m)$ *and* $(d-1)b \equiv 0 \,(\mathrm{mod}\ m)$*. Moreover,* $\mathcal{F}_{\varrho_{a,b,m}}(X,Y,Z)$ *reduces to*

$$X^d + Y^d + X(Z^{d-1} + \sum_{i \in S_1^{d-1,X}{}_{m,\,(a,b)}} \beta_{(d-1),i} Y^i Z^{d-1-i}) + \\ + \sum_{j=2}^{d-2} \sum_{i \in S(2)^{j,X}_{m,(a,b)}} \beta_{j,i} X^{d-j} Y^i Z^{j-i} + \sum_{i \in S_1^{d,X}{}_{m,\,(a,b)}} \beta_{d,i} Y^i Z^{d-i}.$$

6. *None of the reference points lie on* $\mathcal{F}_{\varrho_{a,b,m}}(X,Y,Z) = 0$*: Then* $\overline{C} \in \varrho_{m,a,b}(\mathcal{M}_g^{Pl}(\mathbb{Z}/m\mathbb{Z}))$ *with* $m|d$ *and* $(a,b) \in \Gamma_m$*, and* $\mathcal{F}_{\varrho_{a,b,m}}(X,Y,Z)$ *has the form*

$$X^d + Y^d + Z^d + \sum_{j=2}^{d-1} \sum_{i \in S(2)^{j,X}_{m,(a,b)}} \beta_{j,i} X^{d-j} Y^i Z^{j-i} + \sum_{i \in S_1^{d,X}{}_{m,\,(a,b)}} \beta_{d,i} Y^i Z^{d-i}.$$

Remark 2.1.4. We warn the reader because it may happen that a projective equation obtained for some *Type* $m(a,b)$, is not geometrically irreducible or is not smooth for any specialization of the parameters. Hence, $\varrho_{a,b,m}(\mathcal{M}_g^{Pl}(\mathbb{Z}/m\mathbb{Z}))$ is empty and should be discarded from the list. Finally, for some reason, a repetition of monomials might happen in Theorem 2.1.3- case (3), so one needs to unify those repeated terms, by renaming the parameters.

Proof. We show the result when k has characteristic $p = 0$. The same argument holds for $p > 2g+1 = (d-1)(d-2)+1$, since this restriction implies that $|\operatorname{Aut}(\overline{C})|$ is coprime with p, we address the reader to the discussion after Theorem 1.4.5.

Without loss of generality, we consider a non-singular plane model $F_{\overline{C}}(X,Y,Z) = 0$ of $\overline{C}$ over $\overline{k}$, such that the cyclic element order m acts as the diagonal matrix $\operatorname{diag}(1, \zeta_m^a, \zeta_m^b)$. In other words, if σ is a generator of order m, then we can choose coordinates, so that $\varrho_{a,b,m}(\sigma)$ is represented by $(X:Y:Z) \mapsto (X : \zeta_m^a Y : \zeta_m^b Z)$, where a,b are integers with $0 \leq a \neq b < m$. Moreover, one can assume that $a < b$ with $\gcd(b,m) = 1$ if $a = 0$, and with $\gcd(a,b) = 1$, otherwise.

Case I: Suppose first that $a = 0$ and write: $F_{\overline{C}}(X,Y,Z) = \lambda Z^d + \sum_{j=1}^{d-1} Z^{d-j} L_{j,Z} + L_{d,Z}$.

If $\lambda = 0$, then, by non-singularity (Lemma 2.1.1), $L_{1,Z} \neq 0$ and $(d-1)b \equiv 0 \,(\mathrm{mod}\ m)$. Hence, $m|d-1$ and we can take a generator $(a,b) = (0,1)$. By checking each monomial's invariance, we obtain that $L_{j,Z} \neq 0$ only if $j \in S(2)_m$ and we therefore recover *Types* $m,\ (0,1)$

of in Theorem 2.1.3, case (1), after transforming $L_{1,Z}$ to be Y through a change of the variables X, Y, which in turns preserves the shape of $\varrho_{a,b,m}(\sigma)$.

If $\lambda \neq 0$, then $db \equiv 0 \,(\mathrm{mod}\ m)$. From this we get $m|d$ and $(a,b) = (0,1)$ is again generator for each m. Also, we use the same argument as above, and we obtain *Types* m, $(0,1)$ of the form $Z^d + \sum_{j\in S(1)_m} Z^{d-j}L_{j,Z} + L_{d,Z}$, which proves Theorem 2.1.3, case (2).

Case II : Suppose that $a \neq 0$, then, necessarily, $m > 2$ and we distinguish between the different subcases related to how many reference points lies on $F_{\overline{C}}(X,Y,Z) = 0$:

- If all reference points lie on $F_{\overline{C}}(X,Y,Z) = 0$, then the possibilities for the defining equation are now:

$$\overline{C} : \sum_{j=1}^{d-2} (X^{d-j}L_{j,X} + Y^{d-j}L_{j,Y} + Z^{d-j}L_{j,Z}).$$

  Because $a \neq b$ with $a \neq 0$, we can reduce to

$$\overline{C} : X^{d-1}Y + Y^{d-1}Z + Z^{d-1}X + \sum_{j=2}^{d-2} (X^{d-j}L_{j,X} + Y^{d-j}L_{j,Y} + Z^{d-j}L_{j,Z}).$$

  The first three factors implies that $a = (d-1)a+b = (d-1)b \,(\mathrm{mod}\ m)$, so $m|d^2-3d+3$. The normal form in Theorem 2.1.3-(3) now follows, by checking monomials' invariance in each $L_{j,B}$. For example, rewrite $L_{j,X}$ as $\sum_{i=0}^{j} \beta_{j,i}Y^iZ^{j-i}$, hence $\beta_{j,i} = 0$, if $m \nmid ai + (j-i)b$ or equivalently $i \notin S(1)_{m,(a,b)}^{j,x}$), since $\mathrm{diag}(1, \zeta_m^a, \zeta_m^b) \in \mathrm{Aut}(F_{\overline{C}})$.

- If two reference points lie on $F_{\overline{C}}(X,Y,Z) = 0$, then by re-scaling the matrix $\varrho_{a,b,m}(\sigma)$ and permuting the coordinates, we can assume that $(1:0:0) \notin \{F_{\overline{C}}(X,Y,Z) = 0\}$. The equation then is $X^d + X^{d-2}L_{2,X} + X^{d-3}L_{3,X} + \ldots + XL_{d-1,X} + L_{d,X} = 0$, since $L_{1,x}$ is not invariant by $\varrho_{a,b,m}(\sigma)$ because $ab \neq 0$. Moreover, Z^d and Y^d are not in $L_{d,X}$, as $(0:1:0)$ and $(0:0:1)$ are in $F_{\overline{C}}(X,Y,Z) = 0$. Suppose first that $Y^{d-1}Z$ and YZ^{d-1} are in $L_{d,X}$. Then $(d-1)a+b \equiv 0 \,(\mathrm{mod}\ m)$ and $a+(d-1)b \equiv 0 \,(\mathrm{mod}\ m)$. In particular, $m|\, d(d-2)$ and for each such *Type* m, (a,b), the equation becomes

$$X^d + Y^{d-1}Z + YZ^{d-1} + \sum_{j=2}^{d-1}\sum_{i=0}^{j} \beta_{j,i}X^{d-j}y^iz^{j-i} + \sum_{i=2}^{d-2} \beta_{d,i}Y^iZ^{d-i} = 0.$$

  It is straightforward to see that if $i \notin S(2)_{m,(a,b)}^{j,x}$ (resp. $i \notin S_2^{d,x}\ {}_{m,(a,b)}$), then $\beta_{j,i} = 0$

(resp. $\beta_{di} = 0$). This shows Theorem 2.1.3, subcase (4.1). Second, assume that $Y^{d-1}Z \in L_{d,X}$ and $YZ^{d-1} \notin L_{d,X}$. Then, by non-singularity (Lemma 2.1.1), Z^{d-1} is in $L_{d-1,X}$, and $(d-1)a + b \equiv 0 \,(\mathrm{mod}\ m)$ and $(d-1)b \equiv 0 \,(\mathrm{mod}\ m)$. Therefore, $m|\,(d-1)^2$, and we get the equation

$$\begin{aligned} X^d \quad &+ \quad XZ^{d-1} + Y^{d-1}Z + \sum_{j=2}^{d-2}\sum_{i=0}^{j} \beta_{j,i} X^{d-j} Y^i Z^{j-i} \\ &+ \quad \sum_{i=1}^{d-1} \beta_{(d-1),i} XY^i Z^{d-1-i} + \sum_{i=2}^{d-2} \beta_{d,i} Y^i Z^{d-i} = 0. \end{aligned}$$

Consequently, by monomials' invariance, we conclude that

(i) if $i \notin S(2)^{j,x}_{m,(a,b)}$, then $\beta_{j,i} = 0$,

(ii) if $i \notin S_1^{d-1,x}{}_{m,(a,b)}$, then $\beta_{(d-1),i} = 0$,

(iii) if $i \notin S_2^{d,x}{}_{m,(a,b)}$, then $\beta_{d,i} = 0$,

and Theorem 2.1.3, subcase (4.2) is deduced.

Up to a permutation of Y and Z, it remains to consider the subcase, for which $Y^{d-1}Z$ and YZ^{d-1} are not in $L_{d,x}$. Again, by non-singularity, Z^{d-1} and Y^{d-1} must appear in $L_{d-1,X}$. Consequently, $(d-1)b \equiv 0 \,(\mathrm{mod}\ m)$ and $(d-1)a \equiv 0 \,(\mathrm{mod}\ m)$, and thus $m|(d-1)$. Hence, the defining equation is reduced to

$$\begin{aligned} X^d + XZ^{d-1} + XY^{d-1} \quad &+ \quad \sum_{j=2}^{d-2}\sum_{i=0}^{j} \beta_{j,i} X^{d-j} Y^i Z^{j-i} \\ &+ \quad \sum_{i=2}^{d-2} \beta_{d,i} Y^i Z^{d-i} + \sum_{i=1}^{d-2} \beta_{(d-1),i} XY^i Z^{d-1-i} = 0. \end{aligned}$$

Theorem 2.1.3, subcase (4.3) is now obvious by noticing that $\beta_{j,i} = 0$ when $m \nmid ai + (j-i)b$.

- If one reference point lie on $F_{\overline{C}}(X,Y,Z) = 0$, then by normalizing the matrix $\varrho_{a,b,m}(\sigma)$ and permuting the coordinates, we may take $(1:0:0)$, $(0:1:0) \notin \{F_{\overline{C}}(X,Y,Z) = 0\}$. We thus can write

$$\overline{C} : X^d + Y^d + X^{d-2}L_{2,X} + X^{d-3}L_{3,X} + ... + XL_{d-1,X} + L_{d,X} = 0,$$

such that $Z^d \notin L_{d,X}$. By non-singularity (Lemma 2.1.1), we have $Z^{d-1} \in L_{d-1,X}$. In particular, $da \equiv 0 \,(\mathrm{mod}\; m)$, $(d-1)b \equiv 0 \,(\mathrm{mod}\; m)$, and $m|\, d(d-1)$. The above equation turns out to be

$$X^d+Y^d+XZ^{d-1}+\sum_{j=2}^{d-2}\sum_{i=0}^{j}\beta_{j,i}X^{d-j}Y^iZ^{j-i}+\sum_{i=1}^{d-1}\beta_{d,i}Y^iZ^{d-i}+\sum_{i=1}^{d-1}\beta_{(d-1),i}XY^iZ^{d-1-i}=0$$

From which we conclude Theorem 2.1.3, case (5).

- If none of the reference points lie in $F_{\overline{C}}(X,Y,Z)=0$, then

$$\overline{C}: X^d+Y^d+Z^d+\sum_{j=2}^{d-1}X^{d-j}L_{j,X}+L_{d,X}=0,$$

where $L_{1,X}$ does not appear, since $ab\neq 0$ and $L_{1,X}$ is not invariant under $\varrho_{a,b,m}(\sigma)$. Clearly, $da = db = 0\,(\mathrm{mod}\; m)$ and therefore $m|d$. Moreover, we check monomials' invariance to obtain

$$\overline{C}:\; X^d+Y^d+Z^d+\sum_{j=2}^{d-1}\sum_{i\in S(2)^{j,X}_{m,(a,b)}}\beta_{j,i}X^{d-j}Y^iZ^{j-i}+\sum_{i\in S_1^{d,X}{}_{m,\,(a,b)}}\beta_{d,i}Y^iZ^{d-i}=0.$$

This shows Theorem 2.1.3, case (6), and therefore completes the proof. □

Remark 2.1.5. Theorem 2.1.3 and its proof lead to an algorithm for listing all cyclic groups that could appear together with a normal form, for any arbitrary degree $d\geq 4$. The detailed implementation in SAGE is valid at the link http://mat.uab.cat/∼eslam/CAGPC.sagews. We also refer the reader to Appendix A, for the complete determination of the different types appearing up to degree 9. In other words, to the possible non-trivial strata $\varrho_{m,a,b}(\mathcal{M}_g^{Pl}(\mathbb{Z}/m\mathbb{Z}))$, with their associated normal forms, once the degree is fixed.

We conclude from Theorem 2.1.3:

Corollary 2.1.6. *Let H be a non-trivial cyclic subgroup of $\mathrm{Aut}(\overline{C})$, where $\overline{C}\in\mathcal{M}_g^{Pl}$ with $d\geq 4$. Then, the order of H divides one of the integers*

$$d-1,\; d,\; d^2-3d+3,\; (d-1)^2,\; d(d-2),\; d(d-1).$$

In particular, all automorphisms of $\overline{C}$ have orders at most $d(d-1)$.

Corollary 2.1.7. *Let G be a finite non-trivial group, embedded into $\mathrm{PGL}_3(\overline{k})$ via some ϱ, such that $\varrho(\mathcal{M}_g^{Pl}(G))$ is non-empty, where $g = \frac{1}{2}(d-1)(d-2) \geq 3$ and k has characteristic $p = 0$ or $p > 2g+1$. Then, there exists a single normal form $\mathcal{F}_{\varrho,G}(X,Y,Z) = 0$ of degree d over $\overline{k}$, which defines the stratum $\varrho(\mathcal{M}_g^{Pl}(G))$. The same is also true for the stratum $\varrho(\widetilde{\mathcal{M}_g^{Pl}}(G))$. In this case, the single normal form is denoted by $\mathcal{F}_{\varrho,G,*}(X,Y,Z) = 0$.*

Proof. Let $\sigma \in G$ be an automorphism of maximal order $m > 1$. Up to $\mathrm{PGL}_3(\overline{k})$-conjugation, we may take $\varrho(\sigma)$ diagonal $\mathrm{diag}(1, \zeta_m^a, \zeta_m^b)$, where $0 \leq a < b$ and ζ_m a primitive mth root of unity in K. One follows the same idea in the proof of Theorem 2.1.3, to associate a single normal form $\mathcal{F}_{\varrho(\sigma)}(X,Y,Z) = 0$ to the stratum $\varrho(\mathcal{M}_g^{Pl}(\langle\sigma\rangle))$. For example, if $0 < a < b < m$ and all reference points $\{(1:0:0), (0:1:0), (0:0:1)\}$ satisfy the equation $\mathcal{F}_{\varrho(\sigma)}(X,Y,Z) = 0$, then we reduces to Theorem 2.1.3, case (3).

Now, to move from $\varrho(\langle\sigma\rangle)$ to $\varrho(G)$, we assume an element τ of G, which does not belong to $\langle\sigma\rangle$. Since $\varrho(\tau)$ should retain invariant the equation $\mathcal{F}_{\varrho(\sigma)}(X,Y,Z) = 0$, one obtains extra algebraic relations between the parameters of $\mathcal{F}_{\varrho(\sigma)}(X,Y,Z)$. In this way, $\mathcal{F}_{\varrho,G}(X,Y,Z)$ is constructed from $\mathcal{F}_{\varrho(\sigma)}(X,Y,Z)$ by repeating the procedure for each such τ and gluing together all the algebraic restrictions needed for non-singularity, irreducibility,...etc.

Similarly, we get $\mathcal{F}_{\varrho,G,*}(X,Y,Z)$ from $\mathcal{F}_{\varrho,G}(X,Y,Z)$. In fact, for a finite group H such that $\varrho(G) \leq H \leq \mathrm{PGL}_3(\overline{k})$, and for which there exists a smooth plane curve of genus g whose automorphism group is isomorphic to H, we need to apply the process above for the generators of H not in $\varrho(G)$. Hence, we only need to consider a complement of certain algebraic constraints, so that $\mathcal{F}_{\varrho,G,*}(X,Y,Z) = 0$ does not have a bigger automorphism group than H. □

Remark 2.1.8. It could happen that two different specializations of $\mathcal{F}_{\varrho,G}$ in $\overline{k}$ give two non-singular plane model over $\overline{k}$ for the same curve $\overline{C} \in \varrho(\mathcal{M}_g^{Pl}(G))$. This happens if there exists an isomorphism ϕ between the two models of $\overline{C}$, such that $\phi^{-1}\varrho(G)\phi = \varrho(G)$, and $\phi^{-1}\varrho(\langle\sigma\rangle)\phi = \varrho(\langle\sigma\rangle)$. If this is the case for some $\varrho(\mathcal{M}_g^{Pl}(G))$, then the family $\mathcal{F}_{\varrho,G}$ is said to be *geometrically complete over* k for the stratum $\varrho(\mathcal{M}_g^{Pl}(G))$. Otherwise, it is called a *geometrically representative over* k for $\varrho(\mathcal{M}_g^{Pl}(G))$. The same holds for $\mathcal{F}_{\varrho,G,*}$. One can reads [LRRS14] for more details (see also section §4.3 in chapter 4).

§2.2 Union decomposition of $\mathcal{M}_{\mathrm{g}}^{\mathrm{Pl}}(\mathrm{G})$ and "ES-irreducibility"

The motivation of this section comes from the well-known Henn table, which classifies the strata of smooth non-hyperelliptic plane curves of genus 3 over $\mathbb{C}$, by their automorphism groups and the associated normal forms: We use the formal GAP library notations "GAP(n,m)" to refer the finite group of order n, appearing in the mth position of the atlas for small finite groups [Gro]. We also use $\mathbb{Z}/m\mathbb{Z}$ for the cyclic group of order m. The field k is always of characteristic $p=0$ or p a big enough prime.

Theorem 2.2.1 (P. Henn [Hen76] and Komiya-Kuribayashi [KK79])**.** *The following table determines completely the set of $\mathbb{C}$-isomorphism classes of smooth plane quartic curves over $\mathbb{C}$, together with their automorphism groups:*

Table 2.1: Henn's Table

$\mathrm{Aut}(\overline{C})$	Model	Restrictions
$\mathbb{Z}/2\mathbb{Z}$	$X^4+X^2L_2(Y,Z)+L_4(Y,Z)$	$L_2(Y,Z)\neq 0$, *not below*
$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$	$X^4+Y^4+Z^4+aX^2Y^2+bY^2Z^2+cZ^2X^2$	$a\neq\pm b\neq c\neq\pm a$
$\mathbb{Z}/3\mathbb{Z}$	$Z^3Y+X(X-Y)(X-aY)(X-bY)$	*not below*
$\mathbb{Z}/6\mathbb{Z}$	$Z^3Y+X^4+aX^2Y^2+Y^4$	$a\neq 0$
S_3	$X^3Z+Y^3Z+X^2Y^2+aXYZ^2+bZ^4$	$a\neq b,\ ab\neq 0$
D_4	$X^4+Y^4+Z^4+aX^2Y^2+bXYZ^2$	$b\neq 0,\pm\frac{2a}{\sqrt{1-a}}$
$\mathbb{Z}/9\mathbb{Z}$	$X^4+XY^3+YZ^3$	$-$
$\mathrm{GAP}(16,13)$	$X^4+Y^4+Z^4+aX^2Y^2$	$\pm a\neq 0,2,6,2\sqrt{-3}$
S_4	$X^4+Y^4+Z^4+a(X^2Y^2+Y^2Z^2+Z^2X^2)$	$a\neq 0,\frac{-1\pm\sqrt{-7}}{2}$
$\mathrm{GAP}(48,33)$	$X^4+Y^4+XZ^3$	$-$
$\mathrm{GAP}(96,64)$	$X^4+Y^4+Z^4$	$-$
$\mathrm{PSL}_2(\mathbb{F}_7)$	$X^3Y+Y^3Z+Z^3X$	$-$

The algebraic restrictions for the parameters (in the last column) are taken so that the defining equation is non-singular and has no bigger automorphism group. For example, the term "not below" is equivalent to assume more restrictions, so that no larger automorphism group occurs.

It appears in Table 2.1 the phenomenon: There exists a single normal form describing any of the strata $\widetilde{\mathcal{M}_3^{Pl}}(G)$, where G is one of the groups appearing in Henn Table. Roughly speaking, if this phenomenon holds for some genus g, we say that the stratum $\widetilde{\mathcal{M}_g^{Pl}}(G)$ is ES-Irreducible (or Strongly Equation Irreducible). This would be a weaker condition than the irreducibility inside the moduli space $\mathcal{M}_g$.

In this language, we can formulate the main result in [Hen76] and Komiya-Kuribayashi [KK79] as follows:

Theorem 2.2.2 (Henn, Komiya-Kuribayashi)**.** *If G is a non-trivial group that appears as the full automorphism group of a smooth plane curve of genus $g = 3$ over $\mathbb{C}$, then $\widetilde{M_3^{Pl}}(G)$ is ES-Irreducible.*

Remark 2.2.3. P. Henn in [Hen76], observed that $\mathcal{M}_3^{Pl}(\mathbb{Z}/3\mathbb{Z})$ admits already two irreducible equation components. The first component corresponds to $\varrho_{0,1,3}(\mathbb{Z}/3\mathbb{Z}) = \langle \mathrm{diag}(1,1,\zeta_3)\rangle$, and is defined by the normal form

$$Z^3Y + L_{4,Z} = 0,$$

whereas the second corresponds to $\varrho_{1,2,3}(\mathbb{Z}/3\mathbb{Z}) = \langle \mathrm{diag}(1,\zeta_3,\zeta_3^2)\rangle$. Its defining normal form is

$$X^4 + X(Y^3 + Z^3) + \alpha X^2YZ + \beta X(YZ)^2 = 0.$$

However, the second one has always a bigger automorphism group, the symmetry group S_3.

Now, we introduce the precise definition of ES-Irreducibility of the stratum $\widetilde{\mathcal{M}_g^{Pl}}(G)$, see Definition 2.2.6:

Denote by A_G the quotient set $\{\varrho : G \hookrightarrow \mathrm{PGL}_3(\overline{k})\}/\sim$, where $\varrho_1 \sim \varrho_2$ if and only if $\exists \phi \in \mathrm{PGL}_3(\overline{k})$, such that $\varrho_1(G) = \phi^{-1}\varrho_2(G)\phi$.

Clearly, $\mathcal{M}_g^{Pl}(G) = \bigcup_{[\varrho]\in A_G} \varrho(\mathcal{M}_g^{Pl}(G))$, where $[\varrho]$ is the equivalence class of ϱ in A_G.

Lemma 2.2.4. *We have* $\widetilde{\mathcal{M}_g^{Pl}}(G) = \bigsqcup_{[\varrho]\in A_G} \varrho(\widetilde{\mathcal{M}_g^{Pl}}(G))$

Proof. By definition, $\widetilde{\mathcal{M}_g^{Pl}}(G) = \bigcup_{[\varrho]\in A_G} \varrho(\widetilde{\mathcal{M}_g^{Pl}}(G))$. Moreover, $\overline{C} \in \varrho_1(\widetilde{\mathcal{M}_g^{Pl}}(G)) \cap \varrho_2(\widetilde{\mathcal{M}_g^{Pl}}(G))$ means that it has a non-singular plane model $F_{\overline{C}}(X,Y,Z) = 0$ over $\overline{k}$ such that $\mathrm{Aut}(F_{\overline{C}}) = \phi_1^{-1}\varrho_1(G)\phi_1 = \phi_2^{-1}\varrho_2(G)\phi_2$ for some $\phi_1, \phi_2 \in \mathrm{PGL}_3(\overline{k})$. That is $\varrho_1 \sim \varrho_2$ and the union is disjoint. $\square$

If $\overline{C} \in \varrho_1(\mathcal{M}_g^{Pl}(G)) \cap \varrho_2(\mathcal{M}_g^{Pl}(G))$ with $[\varrho_1] \neq [\varrho_2] \in A_G$, and $F_{\overline{C}}(X,Y,Z) = 0$ is a non-singular plane model over $\overline{k}$ of $\overline{C}$, then $\mathrm{Aut}(F_{\overline{C}}) \leq \mathrm{PGL}_3(\overline{k})$ should have two non-conjugate subgroups isomorphic to G. A detailed study of the work of Blichfeldt [Bli10] would give the list of Gs for which the decomposition $\mathcal{M}_g^{Pl}(G) = \bigcup_{[\varrho]\in A_G} \varrho(\mathcal{M}_g^{Pl}(G))$ may not be disjoint. For instance, we address the reader to Example 2.2.5 below due to B. Huggins [Hug05]:

Example 2.2.5 (B. Huggins, Lemma 7.1.1, [Hug05]). *Take $m, r \in \mathbb{Z}^+$ such that $2mr > 5$ and r is odd when m does. Let z^c be the complex conjugate of z for any $z \in \mathbb{C}$. Consider a binary form $G(X,Y) \in \mathbb{C}[X,Y] \setminus \mathbb{R}[X,Y]$ given by*

$$G(X,Y) := \prod_{i=1}^{r} (X^m - a_i Y^m)(X^m + a_i^c Y^m),$$

for some $a_1, ..., a_r \in \mathbb{C}$ such that the next conditions hold: $G(X,1)$ has no repeated zeros, the map $[\alpha : \beta] \mapsto [\beta : \alpha]$ does not map the zero set of $G(X,Y)$ into itself, for any root of unity ζ we should have $\{a_i, -1/a_i^c\} \neq \{\zeta a_i, -\zeta/a_i^c\}$, and when $n = 3$, the map $[\alpha : \beta] \mapsto [-\alpha + (1+\sqrt{3})\beta : (1+\sqrt{3})\alpha + \beta]$ does not map the zero set of $G(X,Y)$ into itself.

Now, the equation

$$F_{\overline{C}}(X,Y,Z) := Z^{2mr} - G(X,Y) = 0$$

defines a smooth plane plane curve $\overline{C}$ of degree $d = 2mr > 5$ over $\mathbb{C}$, whose automorphism group is $\mathrm{PGL}_3(\mathbb{C})$-conjugate to

$$\langle \mathrm{diag}(\zeta_m, 1, 1),\ \mathrm{diag}(1, \zeta_m, 1),\ \mathrm{diag}(1, 1, \zeta_{2mr}) \rangle.$$

Therefore, for $m > 2$, $\overline{C} \in \varrho_1(\mathcal{M}_g^{Pl}(\mathbb{Z}/m\mathbb{Z})) \cap \varrho_2(\mathcal{M}_g^{Pl}(\mathbb{Z}/m\mathbb{Z}))$ where $\varrho_1(\mathbb{Z}/m\mathbb{Z}) := \langle \mathrm{diag}(1,1,\zeta_m) \rangle$ and $\varrho_2(\mathbb{Z}/m\mathbb{Z}) := \langle \mathrm{diag}(1, \zeta_m, \zeta_m^2) \rangle$. In particular, $[\varrho_1] \neq [\varrho_2] \in A_G$, since $\mathrm{diag}(1,1,\zeta_m)$ and $\mathrm{diag}(1,\zeta_m,\zeta_m^2)$ are in different conjugacy classes of $\mathrm{PGL}_3(\overline{k})$.

Fix a $[\varrho] \in A_G$, then we can associate infinitely many non-singular plane models over $\overline{k}$ for $\overline{C} \in \varrho(\mathcal{M}_g^{Pl}(G))$, which are pairwise $\overline{k}$-isomorphic through a change of variables $\phi \in \mathrm{PGL}_3(\overline{k})$. However, it is suffices to work with the models such that G is identified with $\varrho(G) \leq \mathrm{PGL}_3(\overline{k})$, for some ϱ in $[\varrho] \in A_G$ as a subgroup of automorphisms. Under this restriction, $\overline{C}$ is associated with a non-empty family of non-singular plane model over $\overline{k}$, where any two of them are $\overline{k}$-isomorphic via an isomorphism ϕ in the normalizer of $\varrho(G)$ in $\mathrm{PGL}_3(\overline{k})$, i.e. $\phi^{-1}\varrho(G)\phi = \varrho(G)$.

Second, Lemma 2.1.1 assures that the defining equation of any non-singular model of $\overline{C}$ over $\overline{k}$ must have degree at least $d-1$ in each variable. Moreover, we use a change of variables, so that a non-singular plane model, whenever it exists, has only monic monomials in its core (Definition 1.4.3). Consequently, we reduce the situation to a set of $\overline{k}$-projectively equivalent non-singular plane models for $\overline{C}$, such that $\varrho(G)$ retains invariant the defining equation of each of them, and the core is formed entirely of monic monomial terms.

Definition 2.2.6. Write $\mathcal{M}_g^{Pl}(G)$ as $\bigcup_{[\varrho]\in A_G}\varrho(\mathcal{M}_g^{Pl}(G))$, we define the number of the equation components of $\mathcal{M}_g^{Pl}(G)$ to be the number of elements $[\varrho] \in A_G$ such that $\varrho(\mathcal{M}_g^{Pl}(G))$ is not empty. We say that $\mathcal{M}_g^{Pl}(G)$ is equation irreducible if it is not empty and $\mathcal{M}_g^{Pl}(G) = \varrho(\mathcal{M}_g^{Pl}(G))$ for a certain $[\varrho] \in A_G$. A similar notion arises for the stratum $\widetilde{\mathcal{M}_g^{Pl}}(G) = \bigsqcup_{[\varrho]\in A_G}\varrho(\widetilde{\mathcal{M}_g^{Pl}}(G))$. We define the number of the strongly equation irreducible components of $\widetilde{\mathcal{M}_g^{Pl}}(G)$ to be the number of the elements $[\varrho] \in A_G$ such that $\varrho(\widetilde{\mathcal{M}_g^{Pl}}(G))$ is not empty. We say that $\widetilde{\mathcal{M}_g^{Pl}}(G)$ is equation strongly irreducible (or simply, ES-irreducible) if it is not empty and $\widetilde{\mathcal{M}_g^{Pl}}(G) = \varrho(\widetilde{\mathcal{M}_g^{Pl}}(G))$, for some $[\varrho] \in A_G$.

Of course, if $\widetilde{\mathcal{M}_g^{Pl}}(G)$ is not ES-irreducible then it is not irreducible and the number of the strongly equation irreducible components of $\widetilde{\mathcal{M}_g^{Pl}}(G)$ is a lower bound for the number of its irreducible components inside $\mathcal{M}_g$.

To finish this section, we state some natural questions concerning the stratum $\varrho(\mathcal{M}_g^{Pl}(G))$, and similar questions are also formulated for $\varrho(\widetilde{\mathcal{M}_g^{Pl}}(G))$ with different strata of the moduli space $\mathcal{M}_g$:

Question 2.2.7. *Is it true that, for all the elements $\overline{C}$ of $\varrho(\mathcal{M}_g^{Pl}(G))$, the corresponding Galois covers $\overline{C} \to \overline{C}/G$ have a fixed ramification data?*

We believe that the answer to this question for $\overline{k} = \mathbb{C}$ (i.e. for the case of Riemann surfaces) should always be true from the work of Breuer [Bre00]. See also the next section 2.2.1 for the explicit Galois subcover and the ramification data of the strata $\varrho(\mathcal{M}_6^{Pl}(\mathbb{Z}/8\mathbb{Z}))$ and $\varrho(\mathcal{M}_6^{Pl}(\mathbb{Z}/4\mathbb{Z}))$.

Question 2.2.8. *Is $\varrho(\mathcal{M}_g^{Pl}(G))$ an irreducible set when G is a cyclic group?*

We note that when $\overline{k} = \mathbb{C}$, Cornalba [Cor87], for a cyclic group G of prime order, and Catanese [Cat12], for a general order, obtained that the stratum of smooth, projective, genus g curves with a cyclic Galois subcover of a group that is isomorphic to G and a prescribed ramification is irreducible.

Concerning the irreducibility question, we prove in §2.4 that, if G has an element of order $(d-1)^2$, $d(d-1)$, $d(d-2)$ or d^2-3d+3, then $\varrho(\mathcal{M}_g^{Pl}(G))$ has at most one element. Therefore, it is irreducible. In the next subsection (§2.2.1), we deal with the irreducibility of the ES-Irreducible stratum $\mathcal{M}_6^{Pl}(\mathbb{Z}/8\mathbb{Z})$, where the single normal form has only one parameter.

Moreover, Catanese, Lönne and Perroni in [CLP15, §2] define a topological invariant for the strata $\mathcal{M}_g(G)$, which is trivial if it is irreducible.

Question 2.2.9. *Consider a non-trivial group G such that the set A_G is given by one element (see Example 2.2.10 below for such a groups). Is it a necessary condition that the topological invariant in [CLP15, §2] is trivial in order to be irreducible? Is it true that the strata $\mathcal{M}_g^{Pl}(G)$ are irreducible?*

Example 2.2.10 (The Hessian group)**.** *The representations of the Hessian group Hess_{216} of order 216 inside $\mathrm{PGL}_3(\overline{k})$ are unique up to conjugation, see H. Mitchell [Mit11, p. 217]. For example, $\mathrm{Hess}_{216} = \langle S, T, U, V\rangle$, where*

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3^2 \end{pmatrix},\ U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta_3 \end{pmatrix},\ V = \frac{1}{\zeta_3 - \zeta_3^2}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \zeta_3 & \zeta_3^2 \\ 1 & \zeta_3^2 & \zeta_3 \end{pmatrix},\ T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Also, we consider the primitive Hessian subgroups of order 36, Hess_{36} (one of them is $\langle S, T, V\rangle$), and the primitive subgroup of order 72, $\mathrm{Hess}_{72} = \langle S, T, V, UVU^{-1}\rangle$.

For the above fixed representation, there are exactly three primitive subgroups of order 36 (see [Gro06]), which are also normal in Hess_{72}. Moreover, the Hessian subgroup Hess_{72} is normal in Hess_{216}. We recall, by Grove in [Gro06, §23,p.25] and by Blichfeldt in [Bli10] (see also [HL88, §1] for the statement of Blichfeldt's result of our interest) that any representation of Hess_{216} corresponds geometrically to a certain subgroup fixing four triangles (having 18 *elements), and the alternating group A_4 acting in such four triangles. Furthermore, any representation of the primitive subgroups of order 36 or 72 is obtained by the group of 18 elements fixing the four triangles together with certain permutations on the four triangles (equivalently, with certain subgroups of A_4). On the other hand, it follows, by Blichfeldt (see [HL88, §1, on type (E),(F),(G)]), that such Hessian groups are represented in $\mathrm{PGL}_3(\overline{k})$, up to conjugation, with respect to the representation described above. Therefore, any injective representation of Hess_{36} or Hess_{72} in $\mathrm{PGL}_3(\overline{k})$ extends to an injective representation of Hess_{216}, and moreover the three different subgroups of Hess_{36} in any representation are conjugate to $\langle S, T, V\rangle$. Consequently, one concludes that the representations of Hess_* with $* \in \{36, 72, 216\}$ inside $\mathrm{PGL}_3(K)$ forms a unique set up to conjugation.*

Thus, for any of the Hessian groups Hess_* with $* \in \{36, 72, 216\}$, the stratum $\widetilde{M_g^{Pl}}(\mathrm{Hess}_*)$ is ES-Irreducible, when it is not empty, since the set A_{Hess_*} is trivial.

Our aim interest in investigating whether the $\widetilde{M_g^{Pl}}(G)$ is ES-irreducible or not, and the classical result of Klein concerning the uniqueness (up to conjugation) of the finite subgroups of $\mathrm{PGL}_2(\overline{k})$, encourage us to ask the following question in group theory:

Question 2.2.11. *Is it true that there exists a non-cyclic finite subgroup $\underline{G}$ of $\mathrm{PGL}_3(\overline{k})$, such that the set $A_{\underline{G}}$ has at least two elements?*

2.2.1 The strata $\mathcal{M}_6^{\mathbf{Pl}}(\mathbb{Z}/8\mathbb{Z})$ and $\widetilde{\mathcal{M}_6^{\mathbf{Pl}}}(\mathbb{Z}/8\mathbb{Z})$

Let k be a field of characteristic $p = 0$ or $p > 13$. Consider an element $\overline{C}$ in the moduli space $\mathcal{M}_6$, which has a non-singular plane model over $\overline{k}$ with an effective action of the cyclic group of order 8, that is $\overline{C} \in \mathcal{M}_6^{Pl}(\mathbb{Z}/8\mathbb{Z})$. More concretely, we have $\mathcal{M}_6^{Pl}(\mathbb{Z}/8\mathbb{Z}) = \varrho(\mathcal{M}_6^{Pl}(\mathbb{Z}/8\mathbb{Z}))$ (we will justify this later in chapter 4) with $\varrho(\mathbb{Z}/8\mathbb{Z}) = \langle \mathrm{diag}(1, \zeta_8, -1)\rangle$, where ζ_8 is an 8th primitive root of unity in $\overline{k}$. Furthermore, such a stratum is described over $\overline{k}$ by the single normal form

$$\mathcal{F}_{\varrho,\mathbb{Z}/8\mathbb{Z}}(X, Y, Z) := X^5 + Y^4Z + XZ^4 + \beta X^3Z^2 = 0$$

with a parameter β, taking values in $\overline{k}\backslash\{\pm 2\}$, for non-singularity (see Table 4.1). Therefore, we can always associate to $\overline{C}$ a non-singular plane model of the form $X^5 + Y^4Z + XZ^4 +$

$\beta_{\overline{C}} X^3Z^2 = 0$ over $\overline{k}$ for some $\beta_{\overline{C}} \neq \pm 2$. In the language of [LRRS14] (or see Remark 2.1.8 and section §4.3), the family $\mathcal{F}_{\varrho,\mathbb{Z}/8\mathbb{Z}}(X, Y, Z) = 0$ is a geometrically complete over k for $\mathcal{M}_6^{Pl}(\mathbb{Z}/8\mathbb{Z})$. However, it is not geometrically representative over k, since a curve in the family $\mathcal{F}_{\varrho,\mathbb{Z}/8\mathbb{Z}}(X, Y, Z) = 0$ with parameter β is isomorphic to the curve with parameter $-\beta$ through $\phi = \mathrm{diag}(1, \zeta_{16}^{-1}, \zeta_{16}^4)$, where ζ_{16} is a primitive 16th primitive root of unity. So, for any $\overline{C} \in \mathcal{M}_6^{Pl}(\mathbb{Z}/8\mathbb{Z})$, there is at least two non-singular plane models of $\overline{C}$ over $\overline{k}$ in the family $\mathcal{F}_{\varrho,\mathbb{Z}/8\mathbb{Z}}(X, Y, Z) = 0$.

Now, let us compute all the non-singular plane models of the form $X^5 + Y^4Z + XZ^4 + \beta X^3Z^2 = 0$, which can be associated to the fixed curve $\overline{C}$: These models are obtained by a change of the variables $\phi \in \mathrm{PGL}_3(\overline{k})$ such that $\phi^{-1}\langle \mathrm{diag}(1, \zeta_8, -1)\rangle\phi = \langle \mathrm{diag}(1, \zeta_8, -1)\rangle$, and the new model has a similar defining equation of the form $X^5 + Y^4Z + XZ^4 + \beta' X^3Z^2 = 0$. Without any loss of generality, we can suppose that $\phi^{-1}\,\mathrm{diag}(1, \zeta_8, -1)\phi = \mathrm{diag}(1, \zeta_8, -1)$. Hence, in order to have the same eigenvalues which are pairwise distinct, we may assume that ϕ is a diagonal matrix, say $\phi = \mathrm{diag}(1, \lambda_2, \lambda_3)$. Therefore, we get an equation of the form $X^5 + \lambda_2^4\lambda_3 Y^4Z + \lambda_3^4 XZ^4 + \beta_{\overline{C}}\lambda_3^2 X^3Z^2 = 0$. So, we must have $\lambda_2^4\lambda_3 = \lambda_3^4 = 1$, and thus λ_3^2 is 1 or -1. This means that the number of non-singular plane models isomorphic to $\overline{C} \in \mathcal{M}_6^{Pl}(\mathbb{Z}/8\mathbb{Z})$ over $\overline{k}$ in the family $\mathcal{F}_{\varrho,\mathbb{Z}/8\mathbb{Z}}(X, Y, Z) = 0$ is exactly two. In this case, we get a map

$$f : \mathcal{F}_{\varrho,\mathbb{Z}/8\mathbb{Z}} \to \mathbb{A}^1_{\overline{k}} \setminus \{-2, 2\},$$

which is finite and has degree 2, where $\mathbb{A}^1_{\overline{k}}$ is the affine line over $\overline{k}$. Consequently, the map

$$g : \mathcal{M}_6^{Pl}(\mathbb{Z}/8\mathbb{Z}) \to \mathbb{A}^1_{\overline{k}} \setminus \{-2, 2\}/\sim$$

$$C \mapsto [\beta_{\overline{C}}] = \{\beta_{\overline{C}}, -\beta_{\overline{C}}\},$$

where $a \sim b \Leftrightarrow b = a \text{ or } a = -b$, is bijective. Moreover, as we will see in chapter 4, $X^5 + Y^4Z + XZ^4 + \beta X^3Z^2 = 0$ has a larger group of automorphisms than $\mathbb{Z}/8\mathbb{Z}$ if and only if $\beta = 0$. Then, we still have a bijective map

$$\tilde{g} : \widetilde{\mathcal{M}_6^{Pl}}(\mathbb{Z}/8\mathbb{Z}) \to \mathbb{A}^1_{\overline{k}} \setminus \{-2, 0, 2\}/\sim$$

$$C \mapsto [\beta_{\overline{C}}] = \{\beta_{\overline{C}}, -\beta_{\overline{C}}\}.$$

For example, when $\overline{k} = \mathbb{C}$, the right hand side in both situations is irreducible.

On the other hand (about an example for Question 2.2.7), if we consider the Galois cyclic cover of degree 8, given by the action of $\varrho(\mathbb{Z}/8\mathbb{Z})$ on $X^5 + Y^4Z + XZ^4 + \beta X^3Z^2 = 0$, we get that it ramifies at the points $(0 : 1 : 0)$ and $(0 : 0 : 1)$ with ramification index 8, and at the four points $(1 : 0 : h)$, where $1 + h^4 + \beta h^2 = 0$ with ramification index 2, if $\beta \neq \pm 2$. Hence, $\mathcal{M}_6^{Pl}(\mathbb{Z}/8\mathbb{Z})$ is inside the stratum of smooth curves of the moduli space $\mathcal{M}_6$, which have a cyclic Galois subcover of degree 8 to a genus zero curve, and also ramify at six points (two of them are with ramification index 8, and the other four points are with ramification index 4).

§2.3 Strata of smooth plane curves not ES-irreducible

We construct certain strata $\widetilde{\mathcal{M}_g^{Pl}}(G)$, which are not ES-Irreducible (see Definition 2.2.6). We first ask for a group G such that there exist at least two non-conjugated injective representations $\varrho_i : G \hookrightarrow \mathrm{PGL}_3(\overline{k})$ with $i = 1, 2$, i.e. $\nexists \phi \in \mathrm{PGL}_3(\overline{k})$ with $\phi^{-1}\varrho_1(G)\phi = \varrho_2(G)$ (more details are included in the previous section §2.2). Because of the zoo of the groups that could appear for smooth plane curves (Theorem 1.4.4), we only consider G, a cyclic group of order m. Second, one needs to prove the existence of two smooth plane curves over $\overline{k}$ whose automorphism groups are conjugate to $\varrho_i(G)$, for each $i = 1, 2$ respectively.

In this section, we prove that the stratum $\widetilde{\mathcal{M}_g^{Pl}}(\mathbb{Z}/(d-1)\mathbb{Z})$ is not ES-irreducible, for any odd degree $d \geq 5$, and it has at least two irreducible components. In particular, when $d = 5$, we will see in chapter 4 that $G = \mathbb{Z}/4\mathbb{Z}$ is the only group such that $\widetilde{\mathcal{M}_6^{Pl}}(G)$ is not ES-Irreducible. Moreover, for even degree d, we prove that $\widetilde{\mathcal{M}_{10}^{Pl}}(\mathbb{Z}/3\mathbb{Z})$ is not ES-irreducible. More generally, we may conjecture, by our work in §2.4, that the stratum $\widetilde{\mathcal{M}_g^{Pl}}(\mathbb{Z}/m\mathbb{Z})$ might be not be ES-Irreducible only if m divides d or $d - 1$, which is true up to degree 9, at least. See the tables in Appendix A.

The above construction of non-irreducible strata holds when k is a field of characteristic $p = 0$ or $p > (d - 1)(d - 2) + 1$, by the virtue of the discussion at the end of chapter 1.

2.3.1 The stratum $\widetilde{\mathcal{M}_{\mathbf{g}}^{\mathbf{Pl}}}(\mathbb{Z}/(\mathbf{d}-\mathbf{1})\mathbb{Z})$

Consider Table A.3. One finds find that $\mathcal{M}_6^{Pl}(\mathbb{Z}/m\mathbb{Z})$ is not empty only for $m = 2, 3, 4, 5, 8, 10, 13, 15, 16$, and 20. Moreover $\mathcal{M}_6^{Pl}(\mathbb{Z}/m\mathbb{Z}) = \varrho(\mathcal{M}_6^{Pl}(\mathbb{Z}/m\mathbb{Z}))$ when $m \neq 4, 5$, where ϱ is determined by $\varrho(\mathbb{Z}/m\mathbb{Z}) = \langle \mathrm{diag}(1, \zeta_m^a, \zeta_m^b)\rangle$. Hence, the strata $\widetilde{\mathcal{M}_6^{Pl}}(\mathbb{Z}/m\mathbb{Z})$, for $m \neq 4, 5$, are expected to be ES-Irreducible, if they are non-empty.

Now, we consider the remaining situation $\widetilde{\mathcal{M}_6^{Pl}}(\mathbb{Z}/m\mathbb{Z})$ with $m = 4$ and 5: obviously, the *Type* $5, (1, 2)$ always have a bigger automorphism group by permuting X and Z. Therefore, there is at most one normal form, defining smooth plane curves over $\overline{k}$ of genus 6, whose full automorphism group is isomorphic to $\mathbb{Z}/5\mathbb{Z}$ (observe that the number of the conjugacy classes of representations of $\mathbb{Z}/5\mathbb{Z}$ in $\mathrm{PGL}_3(\overline{k})$ is three). In particular, $\widetilde{\mathcal{M}_6^{Pl}}(\mathbb{Z}/5\mathbb{Z})$ is also ES-Irreducible, when it is non-empty. More precisely, $\widetilde{\mathcal{M}_6^{Pl}}(\mathbb{Z}/5\mathbb{Z}) = \varrho(\mathcal{M}_6^{Pl}(\mathbb{Z}/5\mathbb{Z}))$, where $\varrho(\mathbb{Z}/5\mathbb{Z}) = \langle \mathrm{diag}(1, 1, \zeta_5)\rangle$. On the other hand, for the cyclic groups of order 4, we have: *Type* $4, (1, 3)$ is not irreducible, since it decomposes as $X\,.\,G(X, Y, Z)$. Hence, it is singular, and will be out of the scope of our purposes. Then, we obtain $\mathcal{M}_6^{Pl}(\mathbb{Z}/4\mathbb{Z}) = \varrho_1(\mathcal{M}_6^{Pl}(\mathbb{Z}/4\mathbb{Z})) \cup \varrho_2(\mathcal{M}_6^{Pl}(\mathbb{Z}/4\mathbb{Z}))$, where ϱ_1 corresponds to *Type* $4, (0, 1)$ and ϱ_2 to *Type* $4, (1, 2)$ respectively.

On Type $4, (0, 1)$

Consider the one parameter family $\overline{\mathcal{C}}_{\varrho_1}$ of smooth plane curves over $\overline{k}$ defined by an equation of the form: $\mathcal{F}_{\varrho_1}(X, Y, Z) := X^5 + Y^5 + XZ^4 + \beta X^3Y^2 = 0$, where $\beta \neq 0$. Since $\eta := \mathrm{diag}(1, 1, \zeta_4) \in \mathrm{Aut}(\mathcal{F}_{\varrho_1})$ is an homology of order 4, with axis, the reference line $L_3 : Z = 0$ and center, the reference point $P_3 = (0 : 0 : 1)$, then $\mathrm{Aut}(\mathcal{F}_{\varrho_1})$ should fix a point, a line or a triangle (Theorem 1.2.8).

If $\mathrm{Aut}(\mathcal{F}_{\varrho_1})$ fixes a triangle and neither a line nor a point is leaved invariant, then $\mathcal{F}_{\varrho_1}(X, Y, Z)$ must be a descendant of the Fermat curve $F_5 : X^5 + Y^5 + Z^5 = 0$ or the Klein curve $K_5 : X^4Y + Y^4Z + Z^4X = 0$ (we refer to the proof of Theorem 1.4.4). However, this is impossible because $4 \nmid |Aut(F_5)|(= 150)$, and $4 \nmid |Aut(K_5)|(= 39)$. So $\mathrm{Aut}(\mathcal{F}_{\varrho_1})$ should fix a line and a point off that line. By Proposition 1.3.12, the center P_3 is an inner Galois point for $\mathcal{F}_{\varrho_1}(X, Y, Z)$, and it is unique by Theorem 1.3.10. Therefore, it must be fixed by the full automorphism group $\mathrm{Aut}(\mathcal{F}_{\varrho_1})$. In particular, $\mathrm{Aut}(\mathcal{F}_{\varrho_1})$ is cyclic, by Corollary 1.4.2, and the axis

$Z = 0$ is also fixed due to Proposition 1.2.9. Consequently, any automorphism of $\mathcal{F}_{\varrho_1}(X, Y, Z)$ is of the shape

$$\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and we reduce to $\mathrm{diag}(1, \lambda, \mu)$, because of the term XZ^4 and since $\beta \neq 0$. This in turns yields $\lambda^5 = \lambda^2 = \mu^4 = 1$, so $\lambda = 1$ and t is a 4th root of unity. This shows that $\mathrm{Aut}(\mathcal{F}_{\varrho_1})$ is isomorphic to $\mathbb{Z}/4\mathbb{Z}$.

By the above discussion, we conclude:

Proposition 2.3.1. *The substratum* $\varrho_1(\widetilde{\mathcal{M}_6^{Pl}}(\mathbb{Z}/4\mathbb{Z}))$ *is non-empty.*

On Type $4, (1, 2)$

Similarly, we study the one parameter family $\overline{\mathcal{C}}_{\varrho_2}$ of smooth plane curves over $\overline{k}$ defined by $\mathcal{F}_{\varrho_2}(X, Y, Z) := X^5 + X(Y^4 + Z^4) + \beta Y^2 Z^3 = 0$, where $\beta \neq 0$. The family admits a cyclic subgroup of automorphisms generated by $\widehat{\eta} := diag(1, \zeta_4, -1)$. For the same reason as before, i.e $4 \nmid |\mathrm{Aut}(K_5)|$ and $|\mathrm{Aut}(F_5)|$, $\mathcal{F}_{\varrho_2}(X, Y, Z)$ is not a descendant of the Fermat curve F_5 or the Klein curve K_5. Moreover, $\mathrm{Aut}(\mathcal{F}_{\varrho_2})$ is not conjugate to an icosahedral group A_5, since it contains no elements of order 4. Also, we exclude the groups: the Klein group $\mathrm{PSL}(2, 7)$, the Hessian group Hess_{216}, and the alternating group A_6, using Theorem 1.4.5 for $d = 5$. Second, the next lemma shows that $\mathrm{Aut}(\mathcal{F}_{\varrho_2})$ is not conjugate to any of the Hessian subgroups Hess_*, for $* = 36, 72$, and hence it should fix a line and a point off this line:

Lemma 2.3.2. *There is no smooth plane curve* $\overline{C}$ *over* $\overline{k}$ *of genus* 6*, whose automorphism group is conjugate to* Hess_**, for any* $* \in \{36, 72, 216\}$.

Proof. Let $F_{\overline{C}}(X, Y, Z) = 0$ be a non-singular plane model for $\overline{C}$ of degree 5 over $\overline{k}$, and suppose on the contrary that $\mathrm{Aut}(F_{\overline{C}})$ is conjugate, through some $\phi \in \mathrm{PGL}_3(\overline{k})$, to Hess_*. Then $\mathrm{Aut}(F_{\phi^{-1}\overline{C}})$ is given by the usual representation inside $\mathrm{PGL}_3(\overline{k})$ of the above Hessian groups. In particular, it has always the five automorphisms: $[Z : Y : X]$, $[X : Z : Y]$, $[Y : X : Z]$, $[Y : Z : X]$, and $[X : \zeta_3 Y : \zeta_3^2 Z]$. Because $F_{\phi^{-1}\overline{C}}(X, Y, Z) = 0$ is invariant under the action of $[Z : Y : X]$, $[X : Z : Y]$, $[Y : X : Z]$, and $[Y : Z : X]$, it must be of the form: $u(X^5+Y^5+Z^5)+a(X^4Z+X^4Y+Y^4X+Y^4Z+Z^4X+Z^4Y)+G(X,Y,Z)$, for some $u,a\in\overline{k}$, and $G(X,Y,Z)$ is a homogenous polynomial of degree at most three in each variable. Now, acting by the fifth automorphism $\mathrm{diag}(1,\zeta_3,\zeta_3^2)$, we get $u=a=0$, which in turns conflicts non-singularity. $\square$

Since $\widehat{\tau}\in\mathrm{Aut}(\mathcal{F}_{\varrho_2})$ is a non-homology in its canonical form, i.e. $\mathrm{diag}(1,a,b)$ with $1,a,b$ (resp. $1,a^3,b^3$) are pairwise distinct, then the fixed line is one of the reference lines $B=0$ with $B\in\{X,Y,Z\}$ and the fixed point is one of the reference points, characterized by $B\neq0$. Equivalently, all automorphisms of $\mathcal{F}_{\varrho_2}(X,Y,Z)$ are of one of the shapes $\widehat{\varphi}_1:=[X:vY+wZ:sY+tZ]$, $\widehat{\varphi}_2:=[vX+wZ:Y:sX+tZ]$, or $\widehat{\varphi}_3:=[vX+wY:sX+tY:Z]$. If $\widehat{\varphi}_1\in\mathrm{Aut}(\mathcal{F}_{\varrho_2})$, then $s=0=w$ (Coefficients of Y^5 and Z^5), and we have the same conclusion for $\widehat{\varphi}_2$ and $\widehat{\varphi}_3$, by the coefficients of X^3Y^2, Y^4Z and Z^3X, YZ^4, respectively. So any automorphism is diagonal say $\mathrm{diag}(1,\lambda,\mu)$, hence $\lambda^4=\mu^4=\lambda^2\mu^3=1$, and $\lambda=\zeta_4^r$, $s=\zeta_4^{r'}$, for $(r,r')\in\{(0,0),\ (2,0),\ (1,2),\ (3,2)\}$. That is $\mathrm{Aut}(\mathcal{F}_{\varrho_2})$ is isomorphic to $\mathbb{Z}/4\mathbb{Z}$, which was to be shown.

The analogue of Proposition 2.3.1 is:

Proposition 2.3.3. *The substratum $\varrho_2(\widetilde{\mathcal{M}_6^{Pl}}(\mathbb{Z}/4\mathbb{Z}))$ is non-empty.*

Summarizing all the discussion made in this subsection, we can write:

Corollary 2.3.4. *Suppose that the stratum $\widetilde{\mathcal{M}_6^{Pl}}(\mathbb{Z}/m\mathbb{Z})$ is non-empty. Then it is ES-Irreducible if and only if $m\neq4$. Furthermore, for $m=4$, it has exactly two irreducible equation components, and hence the number of its irreducible components is at least two.*

Remark 2.3.5. For any element $\overline{C}$ in $\varrho_1(\mathcal{M}_6^{Pl}(\mathbb{Z}/4\mathbb{Z}))$, the Galois cover of degree 4 corresponding to

$$\{\mathcal{F}_{\varrho_1}(X,Y,Z)=0\}\to\{\mathcal{F}_{\varrho_1}(X,Y,Z)=0\}/\langle\mathrm{diag}(1,1,\zeta_4)\rangle,$$

where $\mathcal{F}_{\varrho_1}(X,Y,Z):=Z^4Y+L_{5,Z}$, is ramified exactly at six points with ramification index 4: indeed, the fixed points of η^i, for $i=1,2,3,4$ in $\mathbb{P}^2(\overline{k})$ are all the same set, where $\eta=\mathrm{diag}(1,1,\zeta_4)$. Therefore, we only need to consider the ramification points of η. In particular, the ramification index is always 4. By Hurwitz formula, we get $10 = 4(2g_0 - 2) + 3s$, where g_0 is the genus of $\mathcal{F}_{\varrho_1}(X, Y, Z)/\langle \operatorname{diag}(1, 1, \zeta_4)\rangle$. Hence $g_0 = 0$ and $s = 6$.

On the other hand, for any element $\overline{C}$ in $\varrho_2(\mathcal{M}_6^{Pl}(\mathbb{Z}/4\mathbb{Z}))$, the Galois cover

$$\{\mathcal{F}_{\varrho_2}(X, Y, Z) = 0\} \to \{\mathcal{F}_{\varrho_2}(X, Y, Z) = 0\}/\langle \operatorname{diag}(1, \zeta_4, -1)\rangle,$$

where $\mathcal{F}_{\varrho_2}(X, Y, Z) := X^5 + X(Y^4 + Z^4) + \beta_{2,0}X^3Z^2 + \beta_{3,2}X^2Y^2Z + \beta_{5,2}Y^2Z^3$, is ramified at the points $(0 : 1 : 0)$, and $(0 : 0 : 1)$ with ramification index 4, and at the 4 points $(1 : 0 : h)$, where $1 + h^4 + \beta_{2,0}h^2 = 0$ with ramification index 2 provided that $\beta_{2,0} \neq \pm 2$. We exclude the situation $\beta_{2,0} = \pm 2$ so that the defining equation is non-singular.

The above results, Propositions 2.3.1, 2.3.3 and Corollary 2.3.4, are generalized as follows:

Theorem 2.3.6. *Let $d \geq 5$ be an odd integer, and consider $g = \frac{1}{2}(d-1)(d-2)$ as usual. Then $\widetilde{\mathcal{M}_g^{Pl}}(\mathbb{Z}/(d-1)\mathbb{Z})$ is not ES-Irreducible, and it has at least two irreducible components.*

Proof. The above argument for the concrete families of *Type* $4, (0, 1)$ and *Type* $4, (1, 2)$ still valid, for any odd degree $d \geq 5$, and the proof is quite similar. In other words, let $\mathcal{F}_{\varrho_1}(X, Y, Z)$ and $\mathcal{F}_{\varrho_2}(X, Y, Z)$ be the two families of smooth plane curves over $\overline{k}$ of *Type* $d-1, (0, 1)$ and *Type* $d-1, (1, 2)$ defined by the normal forms $X^d + Y^d + Z^{d-1}X + \beta X^{d-2}Y^2 = 0$, and $X^d + X(Y^{d-1} + Z^{d-1}) + \beta Y^2Zd - 2 = 0$ respectively, and such that $\beta \neq 0$. Then $\operatorname{Aut}(\mathcal{F}_{\varrho_1})$ and $\operatorname{Aut}(\mathcal{F}_{\varrho_2})$ are non-conjugate cyclic groups of order $d-1$, and generated by $\eta := \operatorname{diag}(1, 1, \zeta_{d-1})$ and $\widehat{\eta} := \operatorname{diag}(1, \zeta_{d-1}, \zeta_{d-1}^2)$, respectively. Therefore, they belong to two different $[\varrho]'s$.

On Type $\mathbf{d-1, (0, 1)}$: Again with an homology η of period $d - 1 \geq 4$ inside $\operatorname{Aut}(\mathcal{F}_{\varrho_1})$, the group $\operatorname{Aut}(\mathcal{F}_{\varrho_1})$ fixes a point, a line or a triangle (Theorem 1.2.8). Moreover, the center $P_3 = (0 : 0 : 1)$ of η is an inner Galois point (Proposition 1.3.12), and also it is unique (Theorem 1.3.10). Hence, it should be fixed by $\operatorname{Aut}(\mathcal{F}_{\varrho_1})$, and its axis $Z = 0$ is then leaved invariant (Proposition 1.2.9). So $\operatorname{Aut}(\mathcal{F}_{\varrho_1})$ is cyclic (Corollary 1.4.2), and automorphisms of $\mathcal{F}_{\varrho_1}(X, Y, Z) = 0$ are all of diagonal shapes $\operatorname{diag}(1, v, t)$, such that $v^d = t^{d-1} = v = 1$. This shows that $|\operatorname{Aut}(\mathcal{F}_{\varrho_1})| = d - 1$.

On Type $\mathbf{d-1,(1,2)}$**:** First, $\mathrm{Aut}(\mathcal{F}_{\varrho_2})$ fixes a line and a point off this line: No loss of generality to assume $d\geq 7$ (for $d=5$, we refer Proposition 2.3.3). The alternating group A_6 has no elements of order $d-1\geq 6$. The Klein group $\mathrm{PSL}(2,7)$, which is the only simple group of order 168 (up to isomorphism), has no elements of order ≥ 8, and also there are no elements of order 6 inside (for more details, we refer to [Vis]). Therefore, the primitive groups A_5, A_6, and $\mathrm{PSL}(2,7)$ do not appear as the full automorphism group. Furthermore, the elements of the Hessian group $\mathrm{Hess}_{216}\cong\mathrm{GAP}(216,153)$ have orders $1,2,3,4$, and 6. Then Hess_* with $*\in\{36,72,216\}$ do not appear as the full automorphism group, except possibly for $d=7$. On the other hand, $d-1\nmid 3(d^2-3d+3)$, thus $\mathcal{F}_{\varrho_2}(X,Y,Z)=0$ is not a descendant of the Klein curve K_d. Also, $\mathcal{F}_{\varrho_2}(X,Y,Z)=0$ is not a descendant of the Fermat curve F_d, since $d-1\nmid 6d^2$ (except for $d=7$). Finally, it remains to deal with the degree $d=7$ to exclude the Hessian groups and being a descendant of the Fermat curve F_7: By the same line of argument as we did for *Type* $4,(1,2)$, one shows that none of the Hessian groups could appear as the automorphism group of a smooth plane curve of degree 7. Furthermore, automorphisms of F_7 are of the shapes $[X:\zeta_7^aY:\zeta_7^bZ]$, $[\zeta_7^bZ:\zeta_7^aY:X]$, $[X:\zeta_7^bZ:\zeta_7^aY]$, $[\zeta_7^aY:X:\zeta_7^bZ]$, $[\zeta_7^aY:\zeta_7^bZ:X]$, $[\zeta_7^bZ:X:\zeta_7^aY]$. However, none of them is of order 6, and the claim follows.

Now, the full automorphism group should fix a line and a point off this line. Due to the presence of $\widehat{\eta}$ in $\mathrm{Aut}(\mathcal{F}_{\varrho_2})$, we obtain all automorphisms to be of one of the shapes $[X:vY+wZ:sY+tZ]$, $[vX+wZ:Y:sX+tZ]$, or $[vX+wY:sX+tY:Z]$. If $[X:vY+wZ:sY+tZ]\in\mathrm{Aut}(\mathcal{F}_{\varrho_2})$, then $s=w=0$ (coefficient of Y^d and Z^d), and the same holds for $[vX+wZ:Y:sX+tZ]$ (resp. $[vX+wY:sX+tY:Z]$) from the coefficients of $X^{d-2}Y^2$, $Y^{d-1}Z$ (resp. $Z^{d-2}X^2$, YZ^{d-1}). Hence, all automorphisms reduces to diagonal shapes say $\mathrm{diag}(1,v,s)$. Hence $v^{d-1}=s^{d-1}=v^2s^{d-2}=1$, and $v=\zeta_{d-1}^r$, $s=\zeta_{d-1}^{r'}$ with $d-1|2r-r'$. Therefore, $\mathrm{Aut}(\mathcal{F}_{\varrho_2})$ is cyclic of order $d-1$. $\square$

2.3.2 The stratum $\widetilde{\mathcal{M}_{10}^{\mathbf{Pl}}}(\mathbb{Z}/3\mathbb{Z})$.

From Table A.4 of Appendix A, we get the following normal forms for $\varrho(\mathcal{M}_{10}^{Pl}(\mathbb{Z}/3\mathbb{Z}))$:

Table 2.2: Normal forms for $\varrho(\mathcal{M}_{10}^{Pl}(\mathbb{Z}/3\mathbb{Z}))$

$3,(0,1)$	$Z^6+Z^3L_{3,Z}+L_{6,Z}$
$3,(1,2)$	$X^5Y+Y^5Z+Z^5X+\mu_1X^4Z^2+\mu_2X^2Y^4+\mu_3Y^2Z^4+$ $+\mu_4X^3Y^2Z+\mu_5XY^3Z^2+\mu_6X^2YZ^3$

where μ_i, are parameters that assumes values in $\overline{k}$, so that the associated plane model over $\overline{k}$ of the respective stratum $\varrho(\mathcal{M}_{10}^{Pl}(\mathbb{Z}/3\mathbb{Z}))$ are non-singular.

On Type $\mathbf{3},(\mathbf{1},\mathbf{2})$

Proposition 2.3.7. *Let $\overline{C}\in\mathcal{M}_{10}^{Pl}(\mathbb{Z}/3\mathbb{Z})$, such that $\overline{C}$ admits a non-singular plane model $F_{\overline{C}}(X,Y,Z)=0$ over $\overline{k}$ of the form*

$$X^5Y+Y^5Z+Z^5X+\mu_1X^4Z^2+\mu_2X^2Y^4+\mu_3Y^2Z^4+\mu_4X^3Y^2Z+\mu_5XY^3Z^2+\mu_6X^2YZ^3=0.$$

Then, $\mathrm{Aut}(F_{\overline{C}})$ should fix a line, a point or a triangle.

Proof. It suffices to show that $\mathrm{Aut}(F_{\overline{C}})$ is not conjugate to any of the finite primitive subgroups inside $\mathrm{PGL}_3(\overline{k})$, and the result follows by Theorem 1.2.1. Before we go further, we recall that $S=\mathrm{diag}(1,\zeta_3,\zeta_3^2)$ is an automorphism.

Let $S'\in\mathrm{Aut}(F_{\overline{C}})$ be of order 2 such that $S'SS'=S^{-1}$. Then S' must have the shape $[X:\beta Z:\beta^{-1}Y]$, $[\beta Y:\beta^{-1}X:Z]$, or $[\beta Z:Y:\beta^{-1}X]$, for some $\beta\in\overline{k}\setminus\{0\}$. However, non of these transformations retains $F_{\overline{C}}(X,Y,Z)=0$, hence $\mathrm{Aut}(F_{\overline{C}})$ does not contain an S_3 as a subgroup. In particular, it is not conjugate to A_5 or A_6. Similarly, we exclude the Klein group $\mathrm{PSL}(2,7)$, since it contains an octahedral group of order 24 (but not an isocahedral group of order 60), and because all elements of order 3 inside forms a single conjugacy class (we refer to [Vis]). Lastly, if $\mathrm{Aut}(F_{\overline{C}})$ is conjugate, through a transformation ϕ, to the Hessian group Hess_* with $*\in\{36,72,216\}$, then, we may assume $\phi^{-1}S\phi=S$, as we did not fix a plane model over $\overline{k}$ for a smooth plane curve whose automorphism group is Hess_*. In particular, ϕ has the shape; $[Y:\gamma Z:\beta X]$, $[Z:\gamma X:\beta Y]$, or $[X:\gamma Y:\beta Z]$. Clearly, non of them transforms $F_{\overline{C}}(X,Y,Z)=0$ to $F_{\phi^{-1}\overline{C}}(X,Y,Z)=0$ with $\{[X:Z:Y],[Y:X:Z],[Z:Y:X]\}\subseteq\mathrm{Aut}(F_{\phi^{-1}\overline{C}})$. Therefore, $\mathrm{Aut}(F_{\overline{C}})$ is not conjugate to Hess_*, for any $*\in\{36,72,216\}$, and we are done. $\square$

Notation. Let $\mathcal{A}$ be the subset of $\overline{k}^3 \setminus \{(0,0,0)\}$, consisting of all solutions of the following system of polynomial equations

$$\Psi_1(\varsigma_1,\varsigma_2,\varsigma_3) = 1,\ \Psi_2(\varsigma_1,\varsigma_2,\varsigma_3) = \Psi_3(\varsigma_1,\varsigma_2,\varsigma_3) = \zeta_9^4\Psi_4(\varsigma_1,\varsigma_2,\varsigma_3),$$

where

$$\begin{aligned}
\Psi_1(\varsigma_1,\varsigma_2,\varsigma_3) &:= \varsigma_3\varsigma_2^5 + (\varsigma_1\varsigma_3^3+1)\varsigma_2 + \varsigma_3^5,\\
\Psi_2(\varsigma_1,\varsigma_2,\varsigma_3) &:= \zeta_9^2\left((5\zeta_9^3+1)\varsigma_3\varsigma_2^5 + (5\zeta_9^6+\zeta_9^3+(2\zeta_9^6+\zeta_9^3+3)\varsigma_1\varsigma_3^3)\varsigma_2 + (\zeta_9^6+5)\varsigma_3^5\right),\\
\Psi_3(\varsigma_1,\varsigma_2,\varsigma_3) &:= \zeta_9^5\left((\zeta_9^6+5)\varsigma_3\varsigma_2^5 + (5\zeta_9^3+(3\zeta_9^6+2\zeta_9^3+1)\varsigma_1\varsigma_3^3+1)\varsigma_2 + \zeta_9^3(5\zeta_9^3+1)\varsigma_3^5\right),\\
\Psi_4(\varsigma_1,\varsigma_2,\varsigma_3) &:= \zeta_9\left(\zeta_9^4(5\zeta_9^3+1)\varsigma_3\varsigma_2^5 + \zeta_9(\zeta_9^6+(\zeta_9^6+3\zeta_9^3+2)\varsigma_1\varsigma_3^3+5)\varsigma_2 + \zeta_9(5\zeta_9^3+1)\varsigma_3^5\right),
\end{aligned}$$

and ζ_9 is a primitive 9th root of unity. Define $(\mathcal{A})_1$ to be the set of all values that appear in the 1st coordinate of the 3-tuples in $\mathcal{A}$, which (by a computation) is a finite subset of $\overline{k}^*$.

Now, we state and prove the main result for this part:

Theorem 2.3.8. *Consider $\overline{C} \in \mathcal{M}_{10}^{Pl}(\mathbb{Z}/3\mathbb{Z})$ that has a non-singular plane model $F_{\overline{C}}(X,Y,Z)=0$ over $\overline{k}$ of the form $X^5Y+Y^5Z+Z^5X+\mu X^2YZ^3=0$ with $\mu \notin (\mathcal{A})_1\cup\{0\}$. The full automorphism group of $F_{\overline{C}}(X,Y,Z)=0$ is isomorphic to $\mathbb{Z}/3\mathbb{Z}$, and it is generated by $S:\ (X:Y:Z)\mapsto(X:\zeta_3Y:\zeta_3^2Z)$.*

Proof. A priori, $\mathrm{Aut}(F_{\overline{C}})$ fixes a line and a point off that line or it fixes a triangle using Proposition 2.3.7. We treat each of the two subcases:

(A) If it fixes a line and a point off this line, then the line must be one of the reference lines $B=0$, with $B\in\{X,Y,Z\}$, and the point is one of the reference points, given by $B\neq 0$ (recall that S is an automorphism, which is non-homology). Therefore, $\mathrm{Aut}(F_{\overline{C}})$ is cyclic, since all reference points lie on $F_{\overline{C}}(X,Y,Z)=0$. In particular, any automorphism should be in the normalizer of $\langle S\rangle$ in $\mathrm{PGL}_3(\overline{k})$, which is generated by the set of all diagonal matrices together with the set of all permutations of $\{X,Y,Z\}$ in $\mathrm{PGL}_3(\overline{k})$. Since $\mu\neq 0$, all automorphisms of $F_{\overline{C}}(X,Y,Z)=0$ reduces to diagonal shapes. Moreover, if $\mathrm{diag}(1,v,t)$ is an automorphism, then $tv^4=1=t^3$, and $t^5=v$. Hence $t=\zeta_9^{2r}=\zeta_3^r$ and $v=\zeta_9^{4r}=\zeta_3^{2r}$ for some integer r, which implies that $\mathrm{Aut}(F_{\overline{C}})=\langle S\rangle$.

(B) If it fixes a triangle and neither a line nor a point is leaved invariant, then $F_{\overline{C}}(X,Y,Z)=0$ is a descendant of the Fermat curve $F_6:\ X^6+Y^6+Z^6=0$ or it is a descendant of the Klein curve $K_6:\ X^5Y+Y^5Z+Z^5X=0$. Moreover, $\mathrm{Aut}(F_{\overline{C}})$ is conjugate to a subgroup of one of the following groups (see [Har13, §5]):

$$\begin{aligned}\mathrm{Aut}(F_6) &= \langle[\zeta_6X:Y:Z],\,[X:\zeta_6Y:Z],\,[Y:Z:X],\,[X:Z:Y]\rangle,\\ \mathrm{Aut}(K_6) &= \langle[Z:X:Y],\,[X:\zeta_{21}Y:\zeta_{21}^{-4}Z]\rangle,\end{aligned}$$

respectively. We deal with each of these subcases:

(i) Suppose first that $\mathrm{Aut}(F_{\overline{C}})$ is conjugate, through some ϕ, to a subgroup of $\mathrm{Aut}(F_6)$. It is enough to assume $\phi^{-1}S\phi\in\{S,\,[Y:Z:X],\,[Y:\zeta_6Z:X],\,[Y:\zeta_6^2Z:X]\}$ where ζ_6 is a primitive 6th root if unity, since any automorphism of order 3 in $\mathrm{Aut}(F_6)$, which is not an homology, is conjugate to one of these inside $\mathrm{Aut}(F_6)$.

If $\phi^{-1}S\phi=S$, then ϕ is again in the normalizer of $\langle S\rangle$ in $\mathrm{PGL}_3(\overline{k})$. So, it belongs to the subgroup generated by diagonal matrices and the symmetry group on $\{X,Y,Z\}$. Obviously, non of them produces the core $X^6+Y^6+Z^6$ from $F_{\overline{C}}(X,Y,Z)=0$.

If $\phi^{-1}S\phi=[Y:\nu Z:X]$ for some $\nu\in\{1,\zeta_6,\zeta_6^2\}$, then ϕ has reduces to the shape

$$\begin{pmatrix}\lambda & 1 & \lambda^2\\ \zeta_3\lambda\beta_2 & \beta_2 & \zeta_3^2\lambda^2\beta_2\\ \zeta_3^2\lambda\beta_3 & \beta_3 & \lambda^2\zeta_3\beta_3\end{pmatrix},$$

where $\lambda^3=\nu$. We thus get $F_{\phi^{-1}\overline{C}}(X,Y,Z)$ of the form

$$\Psi_1(\mu,\beta_2,\beta_3)\left(\nu^2\zeta_3X^6+Y^6+\nu^4\zeta_3^2Z^6\right)+\text{ lower terms.}$$

Hence, $\Psi_1(\mu,\beta_2,\beta_3)=1$, $\nu=\zeta_3$, and $F_{\phi^{-1}\overline{C}}(X,Y,Z)$ has the form

$$X^6+Y^6+Z^6+\left(\Psi_2(\mu,\beta_2,\beta_3)X^5Y+\Psi_4(\mu,\beta_2,\beta_3)Y^5Z+\Psi_3(\mu,\beta_2,\beta_3)XZ^5\right)+\ldots$$

Moreover, $[Y:\zeta_6^2Z:X]$ is an automorphism of $F_{\phi^{-1}\overline{C}}(X,Y,Z)=0$, which is impossible because $\mu\notin(\mathcal{A})_1$. Therefore, $F_{\overline{C}}(X,Y,Z)=0$ is not a descendant of the Fermat curve F_6.

(ii) In the same way, if $F_{\overline{C}}(X,Y,Z)=0$ is a descendant of the Klein curve K_6 through $\phi\in\mathrm{PGL}_3(\overline{k})$, then we may impose

$$\phi^{-1}S\phi\in\{S,\,S^{-1},\,[\zeta_{21}^{r}Y:\zeta_{21}^{-4r}Z:X],\,[\zeta_{21}^{-4r}Z:X:\zeta_{21}^{r}Y]\}$$

with $r=0,1,2$, and ζ_{21} is a primitive 21th root of unity. If $\phi^{-1}S\phi=[\zeta_{21}^{r}Y:\zeta_{21}^{-4r}Z:X]$, then ϕ has the shape

$$\begin{pmatrix}\lambda\zeta_{21}^{-r} & 1 & \lambda^2\zeta_{21}^{-r}\\ \lambda\zeta_{21}^{-r}\zeta_3\beta_2 & \beta_2 & \lambda^2\zeta_{21}^{-r}\zeta_3^2\beta_2\\ \lambda\zeta_{21}^{-r}\zeta_3^2\beta_3 & \beta_3 & \lambda^2\zeta_{21}^{-r}\zeta_3\beta_3\end{pmatrix},$$

with $\lambda^3=\zeta_{21}^{-3r}$. To get the core of the Klein curve, we, particularly, need to specialize the parameters $\mu,\,\beta_2,\,\beta_3$, so that the monomials $X^6,\,Y^6,\,Z^6\,X^5Z,\,XY^5$, and YZ^5 disappear from $F_{\phi^{-1}\overline{C}}(X,Y,Z)=0$. This is impossible, unless $\mu=0$, which is not the situation. Similarly, we exclude $[\zeta_{21}^{-4r}Z:X:\zeta_{21}^{r}Y]$, and therefore, ϕ is in the normalizer of $\langle S\rangle$, that also produces the core $X^5Y+Y^5Z+Z^5X$. In this case, we get the defining equation $\mu_0(X^5Y+Y^5Z+Z^5X)+\mu_1G(X,Y,Z)$, for some $G(X,Y,Z)\in\{X^2YZ^3,\,X^3Y^2Z,\,XZ^2Y^3\}$. In particular, $\mathrm{Aut}(F_{\phi^{-1}\overline{C}})\leq\langle[X:\zeta_{21}Y:\zeta_{21}^{-4}Z]\rangle$. Checking monomials' invariance of $F_{\phi^{-1}\overline{C}}(X,Y,Z)$ under this action, we only get the automorphisms $[X:\zeta_{21}^{r'}Y:\zeta_{21}^{-4r'}Z]$ with $7|r'$, and the automorphism group is $\mathbb{Z}/3\mathbb{Z}$.

□

On Type 3, $(0,1)$

Proposition 2.3.9. *If $\overline{C}\in\mathcal{M}_{10}^{Pl}(\mathbb{Z}/3\mathbb{Z})$ has a non-singular plane model $F_{\overline{C}}(X,Y,Z)=0$ over $\overline{k}$ of the form $Z^6+Z^3L_{3,Z}+L_{6,Z}=0$, then $\mathrm{Aut}(F_{\overline{C}})$ is conjugate to the Hessian group Hess_{216}, or it leaves invariant a point, a line or a triangle.*

Proof. The result follows directly from Theorem 1.2.8, since $U=\mathrm{diag}(1,1,\zeta_3)\in\mathrm{Aut}(F_{\overline{C}})$ is an homology of period 3, and Hess_{216} is the only primitive group, which contains such automorphisms and does not leave invariant a point, a line or a triangle. □

The main result of this part is stated below:

Theorem 2.3.10. *The automorphisms group of any $\overline{C} \in \mathcal{M}_{10}^{Pl}(\mathbb{Z}/3\mathbb{Z})$, having a non-singular plane model $F_{\overline{C}}(X,Y,Z) = 0$ over $\overline{k}$ of the form $Z^6 + X^5Y + XY^5 + \mu X^3Z^3 = 0$, with $\mu \neq 0$ is cyclic of order 3, and generated by $U : (X : Y : Z) \mapsto (X : Y : \zeta_3 Z)$.*

Proof. First, assume a change of the variables ϕ such that $\phi^{-1}U\phi = U$. Then ϕ should be of the shape

$$\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and it is clear that $\{[Z : Y : X], [X : Z : Y]\} \nsubseteq \mathrm{Aut}(F_{\phi^{-1}\overline{C}})$. In particular, Hess_{216} does not occur as the full automorphism group, and $\mathrm{Aut}(F_{\overline{C}})$ therefore fixes a point, a line or a triangle (Proposition 2.3.9).

We treat each of the following subcases:

(A) First, assume that $\mathrm{Aut}(F_{\overline{C}})$ fixes a line and a point off that line. If $F_{\overline{C}}(X,Y,Z) = 0$ admits a larger non cyclic automorphism group than $\mathbb{Z}/3\mathbb{Z}$, then $\mathrm{Aut}(F_{\overline{C}})$ satisfies a short exact sequence of the shape $1 \to \mathbb{Z}/3\mathbb{Z} \to \mathrm{Aut}(F_{\overline{C}}) \to G \to 1$, where G is conjugate to $\mathbb{Z}/m\mathbb{Z}$ with $m \in \{2,3,4\}$, D_{2m} with $m = 2$, or 4, A_4, S_4, or to A_5. For G, conjugate to $\mathbb{Z}/3\mathbb{Z}$, A_4, S_4, or to A_5, there exists, by Sylow's theorem, a subgroup H of automorphisms of $F_{\overline{C}}(X,Y,Z) = 0$ of order 9. Hence, H is $\mathbb{Z}/9\mathbb{Z}$ or $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, however non of them happens: one easily excludes $\mathbb{Z}/9\mathbb{Z}$, since $9 \nmid d-1$, d, $(d-1)^2$, $d(d-2)$, $d(d-1)$, or d^2-3d+3 with $d = 6$ (see Corollary 2.1.6). Moreover, if $H = \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, then we have an automorphism η of order 3, which commutes with U. So, $\eta = [vX + wY : sX + tY : Z]$, and from the monomials X^6 and Z^3Y^3, we obtain $w = 0 = s$, and $v^5t = vt^5 = v^3 = 1$. That is $\eta \in \langle U \rangle$, a contradiction.

By a similar argument, one shows that G is not conjugate to $\mathbb{Z}/4\mathbb{Z}$, or D_{2m}, because for any $\mathrm{GAP}(6m, j)$, there must be an automorphism η of order 2 or 4, which also commutes with U. On the other hand, if G is conjugate to $\mathbb{Z}/2\mathbb{Z}$, then there should be an automorphism η of order 2 with $\eta U \eta = U^{-1}$. And, such an automorphism does not exist, as U and U^{-1} are in different conjugacy classes in $\mathrm{PGL}_3(\overline{k})$

Now, we conclude by the above discussion that $\mathrm{Aut}(F_{\overline{C}})$ is cyclic (so, it is abelian). Hence, it can not be of order > 3, since we then have to recognize our curve to be of *Type* $m,\ (0,1)$ for some m, divisible by 3. By Table A.4 in Appendix A, this is not possible, and the automorphism group is exactly $\mathbb{Z}/3\mathbb{Z}$.

(B) Second, if a triangle is fixed by $\mathrm{Aut}(F_{\overline{C}})$, and neither a point nor a line is leaved invariant, then as we mentioned earlier (Theorem 1.4.4), $F_{\overline{C}}(X,Y,Z) = 0$ is a descendant of the Fermat sextic curve F_6 or the Klein sextic curve K_6. However the Klein curve is not an option, as it does not have automorphisms of order 3 whose Jordan form looks like an homology. Also, the set of automorphisms of order 3 of the Fermat curve, which are homologies forms exactly two conjugacy classes in $\mathrm{Aut}(F_6)$, represented by U and U^2 respectively. But also U and U^{-1} are in different conjugacy classes of $\mathrm{PGL}_3(\overline{k})$, therefore $F_{\overline{C}}(X,Y,Z) = 0$ is a descendant of the Fermat curve through a change of variables ϕ, with $\phi^{-1}U\phi = U$. Thus, $\phi = [\alpha_1 X + \alpha_2 Y : \beta_1 X + \beta_2 Y : Z]$, and $F_{\phi^{-1}\overline{C}}(X,Y,Z) = 0$ is given by the equation

$$\varepsilon_0 X^6 + \varepsilon_1 Y^6 + Z^6 + \mu(\alpha_1 X + \alpha_2 Y)^3 Z^3 + \varepsilon_2 X^5 Y + \varepsilon_3 X^4 Y^2 + \varepsilon_4 X^3 Y^3 + \varepsilon_5 X^2 Y^4 + \varepsilon_6 X Y^5,$$

where $\varepsilon_0 := \alpha_1\beta_1(\alpha_1^4 + \beta_1^4)\,(= 1)$, and $\varepsilon_1 := \alpha_2\beta_2(\alpha_2^4 + \beta_2^4)\,(= 1)$. In particular, $(\alpha_1\beta_1)(\alpha_2\beta_2) \neq 0$, and hence $[X : \lambda Z : \delta Y]$, $[\lambda Z : \delta Y : X]$, $[\lambda Y : \delta Z : X]$, and $[\lambda Z : X : \mu Y] \notin \mathrm{Aut}(F_{\phi^{-1}\overline{C}})$, for any $\lambda, \mu \in \overline{k}^*$ (for instance, due to the monomial XY^2Z^3). Furthermore, $[\lambda Y : X : \delta Z] \in \mathrm{Aut}(F_{\phi^{-1}\overline{C}})$ only if $\alpha_1 = \alpha_2$ and $\lambda = \delta^3 = 1$, and $F_{\phi^{-1}\overline{C}}(X,Y,Z)$ becomes

$$Z^6 + \mu\alpha_1^3(X+Y)^3 Z^3 + \alpha_1(X+Y)(\beta_1 X + \beta_2 Y)\left(\alpha_1^4(X+Y)^4 + (\beta_1 X + \beta_2 Y)^4\right).$$

Then $\beta_1 = \beta_2$, since we are assuming $[Y : X : \delta Z] \in \mathrm{Aut}(F_{\phi^{-1}\overline{C}})$. In this case, ϕ is not invertible, a contradiction. Finally, if $\mathrm{diag}(1, \zeta_6^r, \zeta_6^{r'})$ retains $F_{\phi^{-1}\overline{C}}(X,Y,Z) = 0$, for some integers $0 \leq r, r' < 6$, then $r = 0$ and r' is even (recall that $\alpha_1\alpha_2 \neq 0$). So, being a descendant of the Fermat curve yields also that $\mathrm{Aut}(F_{\phi^{-1}\overline{C}})$ is $\mathbb{Z}/3\mathbb{Z}$.

$\square$

Corollary 2.3.11. *The stratum $\widetilde{\mathcal{M}_{10}^{Pl}}(\mathbb{Z}/3\mathbb{Z})$ is not ES-Irreducible, and it has at least two irreducible components.*

§2.4 On smooth plane curves, admitting "large" or "very large" automorphisms

There is a lot of interest on smooth curves having a large automorphism group: For $\overline{k} = \mathbb{C}$, a smooth curve $\overline{C} \in \mathcal{M}_g$ has large automorphism group if it has a neighborhood (with respect to the complex topology) in $\mathcal{M}_g$, such that any other smooth curve inside the neighborhood has a smaller automorphism group. For such situations $\overline{C}$ admits a model defined over $\mathbb{Q}$, $C/\operatorname{Aut}(\overline{C})$ corresponds to the projective line and the Galois cover $\overline{C} \to \overline{C}/\operatorname{Aut}(\overline{C})$ is a Belyi morphism, in particular it is ramified exactly at 3 points. The last property of a Belyi morphism that is ramified at three points and is a Galois cover, characterizes smooth curves with large automorphism group. For more details, we refer to Wolfart [Wol97]. Another notion in the literature for $\overline{C}$ to be of large automorphism group is when $|\operatorname{Aut}(\overline{C})| > 4(g-1)$. In particular, for $\overline{C} \in \mathcal{M}_g^{Pl}$, it means that $|\operatorname{Aut}(\delta)| > 2(d^2-3d+2)-4$. In this case $C \to C/\operatorname{Aut}(\overline{C})$ is a map from $\overline{C}$ to a projective line, which is ramified at 3 or 4 points, see [FK92, p.258-260].

The above definitions of large automorphism group are very restrictive to our proposes for smooth plane curves $\overline{C} \in \mathcal{M}_g^{Pl}$ in this chapter.

Definition 2.4.1. We say that an automorphism $\eta \in \operatorname{Aut}(\overline{C})$ is "very large", if its order is d^2-3d+3, $(d-1)^2$, $d(d-2)$, or $d(d-1)$, and it is "large", if its order is md, $m(d-1)$ or $m(d-2)$ for some integer $m \geq 2$.

We devote this section to study $\mathcal{M}_g^{Pl}(G)$ when G has elements of "large" or "very large" order. Recall that the irreducibility of the strata $\widetilde{\mathcal{M}_g^{Pl}}(\mathbb{Z}/m\mathbb{Z})$ is a deep problem (see section §2.2). In this section, we will show that $\mathcal{M}_g^{Pl}(G)$ is irreducible when G has an element of order $(d-1)^2$, $d(d-1)$, $d(d-2)$ or d^2-3d+3, since the stratum in this case is a single point. On the other hand, $\mathcal{M}_g^{Pl}(G)$ is ES-irreducible when G has an element of order md, $m(d-1)$ or $m(d-2)$ for some integer $m \geq 2$.

2.4.1 Strata of smooth plane curves, having a "very large" automorphism

Take, as usual, a non-singular plane model $F_{\overline{C}}(X,Y,Z) = 0$ of degree $d \geq 4$ for $\overline{C} \in \mathcal{M}_g^{Pl}(\mathbb{Z}/m\mathbb{Z})$ over $\overline{k}$, where k is a field of characteristic $p = 0$ or $p > 2g+1$. Assume moreover that $\eta \in \operatorname{Aut}(F_{\overline{C}})$ is of order m that acts on $F_{\overline{C}}(X,Y,Z) = 0$ as the automorphism $(X:Y:Z) \mapsto (X:\zeta_m^a Y:\zeta_m^b Z)$. In particular, m must divide one of the integers $d-1$, d, d^2-3d+3, $(d-1)^2$, $d(d-2)$, or $d(d-1)$, by Corollary 2.1.6.

The stratum $\mathcal{M}_{\mathbf{g}}^{\mathbf{Pl}}(\mathbb{Z}/\mathbf{d}(\mathbf{d}-\mathbf{1})\mathbb{Z})$

The following results (Proportions 2.4.2 and 2.4.3) are well-known in the literature, see for example [Har13, Proposition 3.8] when k has characteristic $p=0$ and the same result follows by our discussion in chapter 1 when $p > 2g+1$:

Proposition 2.4.2. *For any $d \geq 5$, $\overline{C} \in \mathcal{M}_g^{Pl}(\mathbb{Z}/d(d-1)\mathbb{Z})$ if and only if $F_{\overline{C}}(X,Y,Z) = 0$ is $\overline{k}$-isomorphic to $X^d+Y^d+XZ^{d-1} = 0$. In particular, $\mathcal{M}_g^{Pl}(\mathbb{Z}/d(d-1)\mathbb{Z})$ is irreducible with a single element. In fact, $\mathcal{M}_g^{Pl}(\mathbb{Z}/d(d-1)\mathbb{Z}) = \varrho(\mathcal{M}_g^{Pl}(\mathbb{Z}/d(d-1)\mathbb{Z}))$, where $\varrho(\mathbb{Z}/d(d-1)\mathbb{Z}) = \langle \operatorname{diag}(1, \zeta_{d(d-1)}^{d-1}, \zeta_{d(d-1)}^{d})\rangle$.*

Proof. If $\overline{C} : X^d + Y^d + XZ^{d-1} = 0$ is a non-singular plane model for $\overline{C}$, then $\overline{C} \in \mathcal{M}_g^{Pl}(\mathbb{Z}/d(d-1)\mathbb{Z})$, since $\operatorname{diag}(1, \zeta_{d(d-1)}^{d-1}, \zeta_{d(d-1)}^{d})$ is an automorphism of order $d(d-1)$. Conversely, suppose that $\overline{C} \in \mathcal{M}_g^{Pl}(\mathbb{Z}/d(d-1)\mathbb{Z})$ and fix a non-singular plane model $F_{\overline{C}}(X,Y,Z) = 0$ for $\overline{C}$. Since $d(d-1)$ does not divide any of the integers $d-1$, d, d^2-3d+3, $d(d-2)$, and $(d-1)^2$, then $F_{\overline{C}}(X,Y,Z) = 0$ is $\overline{k}$-projectively equivalent to *Type* $d(d-1)$, (a,b) of the form Theorem 2.1.3-(5), for some $(a,b) \in \Gamma_{d(d-1)}$, such that $da = 0 \mod d(d-1)$ and $(d-1)b \equiv 0 \mod d(d-1)$. In particular, we can take $a = d-1$ and $b = d$ as a generator of these types: indeed, $a \equiv 0 \mod d-1$ and $b \equiv 0 \mod d$, and we also have

$$\operatorname{diag}(1, \zeta_{d(d-1)}^{(d-1)m}, \zeta_{d(d-1)}^{dm'}) = \operatorname{diag}(1, \zeta_{d(d-1)}^{d-1}, \zeta_{d(d-1)}^{d})^{d(m'-m)+m}.$$

Hence, the index sets $S(2)_{m,(a,b)}^{j,X}$, $S_1^{d,X}{}_{m,(a,b)}$ and $S_1^{d-1,X}{}_{m,(a,b)}$ are all the empty set, for all $j = 2, ..., d-2$: we justify this by the computations;

$$
\begin{aligned}
S(2)^{j,X}_{d(d-1),(d-1,d)} &:= \{0 \le i \le j \,|\, (d-1)i + (j-i)d \equiv 0 \bmod d(d-1)\} \\
&= \{0 \le i \le j \,|\, dj - i \equiv 0 \bmod d(d-1)\} \\
&= \emptyset, \text{ since } 0 < dj - i < d(d-1), \\
S_1^{d,X}{}_{d(d-1),(d-1,d)} &:= \{1 \le i \le d-1 \,|\, (d-1)i + (d-i)d \equiv 0 \bmod d(d-1)\} \\
&= \{1 \le i \le d-1 \,|\, d - i \equiv 0 \bmod d(d-1)\} \\
&= \emptyset, \text{ since }, 0 < d - i < d(d-1), \\
S_1^{d-1,X}{}_{d(d-1),(d-1,d)} &:= \{1 \le i \le d-1 \,|\, (d-1)i + (d-1-i)d \equiv 0 \bmod d(d-1)\} \\
&= \{1 \le i \le d-1 \,|\, i \equiv 0 \bmod d(d-1)\} \\
&= \emptyset.
\end{aligned}
$$

Now, if we substitute in the normal form of Theorem 2.1.3, case (5), then one finds that $F_{\overline{C}}(X,Y,Z) = 0$ reduces to $X^d + Y^d + XZ^{d-1} = 0$, which was to be shown first. $\square$

Proposition 2.4.3. *The full automorphism group of $\overline{C} : X^d + Y^d + XZ^{d-1} = 0$ with $d \ge 5$ is cyclic of order $d(d-1)$. Hence $\widetilde{\mathcal{M}_g^{Pl}}(\mathbb{Z}/d(d-1)\mathbb{Z}) = \mathcal{M}_g^{Pl}(\mathbb{Z}/d(d-1)\mathbb{Z})$ is also irreducible.*

Proof. Since $\operatorname{diag}(1, \zeta_{d(d-1)}^{d-1}, \zeta_{d(d-1)}^{d})^d \in \operatorname{Aut}(\overline{C})$ is a homology of order $d-1$, then its center $P_3 = (0:0:1)$ is an inner Galois point of $\overline{C}$ (Proposition 1.3.12). Moreover, it is also unique (Theorem 1.3.8). Therefore, it must be fixed by $\operatorname{Aut}(\overline{C})$, which implies that $\operatorname{Aut}(\overline{C})$ is cyclic by the virtue of Corollary 1.4.2, and thus $|\operatorname{Aut}(\overline{C})| \le d(d-1)$ by Corollary 2.1.6. But also $\operatorname{diag}(1, \zeta_{d(d-1)}^{d-1}, \zeta_{d(d-1)}^{d}) \in \operatorname{Aut}(\overline{C})$ is of order $d(d-1)$. Consequently, $\operatorname{Aut}(\overline{C}) = \langle \operatorname{diag}(1, \zeta_{d(d-1)}^{d-1}, \zeta_{d(d-1)}^{d}) \rangle$. $\square$

Remark 2.4.4. Recall that for $d = 4$, the automorphism group of $\overline{C} : X^4 + Y^4 + XZ^3 = 0$ is isomorphic to $\mathbb{Z}/4\mathbb{Z} \circledcirc \mathrm{A}_4$, given by

$$\{(\kappa, g) \in \mu_{12} \times H : \kappa^4 = \chi(g)\}/\pm 1,$$

where μ_n is the group of nth roots of unity, H is the group A_4 and χ is the character $\chi : H \to \mu_3$ defined by $\chi(S) = 1$ and $\chi(T) = \zeta_3$, where S, T are generators of H of order 2 and 3 respectively with the representation $H = \langle S, T | S^2 = 1, T^3 = 1, ...\rangle$, and ζ_3 is a primitive 3rd-root of unity, see [Hen76] (or also [Bar12]).

Corollary 2.4.5. *If G is a non-cyclic automorphism subgroup of a smooth plane curve of order divisible by $d(d-1)$ with $d \geq 5$, then it does not contain any automorphism of order order $d(d-1)$.*

The stratum $\mathcal{M}_{\mathbf{g}}^{\mathbf{Pl}}(\mathbb{Z}/(\mathbf{d}-\mathbf{1})^{\mathbf{2}}\mathbb{Z})$

Proposition 2.4.6. *For $d \geq 4$, $\overline{C} \in \mathcal{M}_g^{Pl}(\mathbb{Z}/(d-1)^2\mathbb{Z})$ if and only if it is $\overline{k}$-isomorphic to $X^d + Y^{d-1}Z + XZ^{d-1} = 0$. In particular, the stratum $\mathcal{M}_g^{Pl}(\mathbb{Z}/(d-1)^2\mathbb{Z})$ is irreducible and contains only a single element. More precisely, $\mathcal{M}_g^{Pl}(\mathbb{Z}/(d-1)^2\mathbb{Z}) = \varrho(\mathcal{M}_g^{Pl}(\mathbb{Z}/(d-1)^2\mathbb{Z}))$, where $\varrho(\mathbb{Z}/(d-1)^2\mathbb{Z}) = \langle \mathrm{diag}(1, \zeta_{(d-1)^2}, \zeta_{(d-1)^2}^{(d-1)(d-2)})\rangle$.*

Proof. If $\overline{C}\ :\ X^d + Y^{d-1}Z + XZ^{d-1} = 0$ is a non-singular plane model for $\overline{C}$, then $\overline{C} \in \mathcal{M}_g^{Pl}(\mathbb{Z}/(d-1)^2\mathbb{Z})$, since $\mathrm{diag}(1, \zeta_{(d-1)^2}, \zeta_{(d-1)^2}^{(d-1)(d-2)})$ is an automorphism of order $(d-1)^2$. Conversely, suppose that $\overline{C} \in \mathcal{M}_g^{Pl}(\mathbb{Z}/(d-1)^2\mathbb{Z})$. Because $(d-1)^2 \nmid d-1,\ d,\ d^2-3d+3,\ d(d-2)$, and $d(d-1)$, we can consider, up to $\overline{k}$-isomorphism, a non-singular plane model $F_{\overline{C}} = 0$ of *Type* $(d-1)^2, (a,b)$ of the form Theorem 2.1.3, subcase (4.2). In particular $(a,b) \in \Gamma_{(d-1)^2}$ such that $(d-1)a + b \equiv 0 \bmod (d-1)^2$, $(d-1)b \equiv 0 \bmod (d-1)^2$, and $a = (d-1)m - m'$, $b = (d-1)m'$ for some integers m and m'. Moreover, $\mathrm{diag}(1, \zeta_{(d-1)^2}^{(d-1)m-m'}, \zeta_{(d-1)^2}^{(d-1)m'}) = \mathrm{diag}(1, \zeta_{(d-1)^2}, \zeta_{(d-1)^2}^{(d-1)(d-2)})^{(d-1)m-m'}$. So we take $a = 1$ and $b = (d-1)(d-2)$ as a generator of such types of smooth plane curves. Consequently, the index sets $S_2^{d,X}{}_{(d-1)^2,(a,b)}$, $S_1^{d-1,X}{}_{(d-1)^2,(a,b)}$, and $S(2)_{(d-1)^2,(a,b)}^{j,X}$, for $j = 2, ..., d-2$, become all empty;

$$
\begin{aligned}
S(2)_{(d-1)^2,(a,b)}^{j,X} &:= \{0 \leq i \leq j \,|\, i + (j-i)(d-1)(d-2) \equiv 0 \bmod (d-1)^2\} \\
&= \{0 \leq i \leq j \,|\, j(d-1) - di \equiv 0 \bmod (d-1)^2\} \\
&= \emptyset.
\end{aligned}
$$

We get the last equality, because $(d-1)^2 \,|\, j(d-1) - di$ gives $d-1|i$, and thus $i = 0$. This in turns implies that $d-1 \mid j$, which is not possible as $0 < j < d-1$. Also, we we obtain

$$\begin{aligned}
S_2^{d,X}\ {}_{(d-1)^2,\,(a,b)} &:= \{2 \leq i \leq d-2 \mid i + (d-i)(d-1)(d-2) \equiv 0 \bmod (d-1)^2\} \\
&\subseteq \{2 \leq i \leq d-2 \mid d-1 \mid i\} \\
&= \emptyset, \\
S_1^{d-1,X}\ {}_{(d-1)^2,\,(a,b)} &:= \{1 \leq i \leq d-1 \mid i + (d-1-i)(d-1)(d-2) \equiv 0 \bmod (d-1)^2\} \\
&= \{1 \leq i \leq d-1 \mid (d-1)^2 \mid di\} \\
&= \emptyset.
\end{aligned}$$

Lastly, we substitute into equation Theorem 2.1.3, subcase (4.2) to obtain the prescribed defining equation $X^d + Y^{d-1}Z + XZ^{d-1} = 0$, and we have done. $\square$

Proposition 2.4.7. *The full automorphism group of $\overline{C}: X^d + Y^{d-1}Z + XZ^{d-1} = 0$, for any $d \geq 4$ is cyclic of order $(d-1)^2$. Hence $\widetilde{\mathcal{M}_g^{Pl}}(\mathbb{Z}/(d-1)^2\mathbb{Z}) = \mathcal{M}_g^{Pl}(\mathbb{Z}/(d-1)^2\mathbb{Z})$ is also irreducible.*

Proof. Since $\operatorname{diag}(1, \zeta_{d-1}, 1) \in \operatorname{Aut}(\overline{C})$ is a homology of order $d-1$, we can deduce that $\operatorname{Aut}(\overline{C})$ is cyclic of order at most $d(d-1)$ and is also divisible by $(d-1)^2$ (one follows the same argument for Proposition 2.4.3). Thus $\operatorname{Aut}(\overline{C})$ is exactly $\mathbb{Z}/(d-1)^2\mathbb{Z}$. $\square$

Corollary 2.4.8. *If G is a non-cyclic subgroup of automorphisms of a smooth plane curve of order divisible by $(d-1)^2$ with $d \geq 4$, then it does not contain any automorphism of order order $(d-1)^2$.*

The stratum $\mathcal{M}_{\mathbf{g}}^{\mathbf{Pl}}(\mathbb{Z}/\mathbf{d}(\mathbf{d}-\mathbf{2})\mathbb{Z})$

Definition 2.4.9. (central extension) Let N, E and G be three groups. We call G *an extension of E by N* if there is a short exact sequence

$$1 \to N \to G \to E \to 1$$

An extension is called *a central extension* if the subgroup N lies in the center of G.

The next result (Proposition 2.4.10) is [Har13, Proposition 6.2] when k has characteristic $p = 0$, and the same is true when $p > 2g+1$ by our discussion in chapter 1.

Proposition 2.4.10. *Consider the smooth plane curve $\overline{C}$ of degree $d \geq 4$ defined by the form $\overline{C} : X^d + Y^{d-1}Z + YZ^{d-1} = 0$ over $\overline{k}$. The full automorphism group H_d is completely determined as follows:*

1. *For $d \neq 4, 6$, it is the central extension*

$$\langle \sigma,\, \tau | \;\; \sigma^2 = \tau^{d(d-2)} = 1 \;\; and \;\; \sigma\tau\sigma = \tau^{-(d-1)} \rangle$$

of the dihedral group $\mathrm{D}_{2(d-2)}$ of order $2(d-2)$ by $\mathbb{Z}/d\mathbb{Z}$. In particular, $\mathrm{Aut}(\overline{C})$ is of order $2d(d-2)$.

2. *For $d = 6$, it is a central extension of S_4 by $\mathbb{Z}/6\mathbb{Z}$, and its order is 144.*

3. *For $d = 4$, $\overline{C}$ is $\overline{k}$-isomorphic to the Fermat quartic curve $F_4 : X^4 + Y^4 + Z^4 = 0$. Hence $\mathrm{Aut}(\overline{C}) \cong (\mathbb{Z}/4\mathbb{Z})^2 \rtimes \mathrm{S}_3$.*

Proof. For $d = 4$, the change of variables $[X : Y + \zeta_4 Z : Y - \zeta_4 Z]$ transforms $\overline{C} : X^4 + Y^3Z + YZ^3 = 0$ to $X^4 + 2(Y^4 - Z^4) = 0$, which is clearly the Fermat quartic curve, up to a rescaling the variables Y and Z. The automorphism group in this case is already quite well known, see [Hen76] or [Bar12].

For $d \geq 5$, we have $\sigma := \mathrm{diag}(1, \zeta_{d(d-2)}, \zeta_{d(d-2)}^{-(d-1)}) \in \mathrm{Aut}(\overline{C})$ of order $d(d-2) > 2d$. Hence $\overline{C}$ can not be a descendant of the Fermat curve F_d. But also $\sigma^{d-2} \in \mathrm{Aut}(\overline{C})$ is a homology of order d with center $P_1 = (1 : 0 : 0)$ and axis $L : X = 0$. Thus P_1 is an outer Galois point for $\overline{C}$ (Proposition 1.3.12), and it is unique (Theorem 1.3.8). Therefore, P_1 should be fixed by $\mathrm{Aut}(\overline{C})$, and the same for the axis $L : X = 0$ (Proposition 1.2.9). So we $\mathrm{Aut}(\overline{C})$ is a subgroup of $\mathrm{PBD}(2, 1)$, and we can think about it in a short exact sequence of the form

$$1 \longrightarrow \mathrm{Ker}(\Lambda|_{\mathrm{Aut}(\overline{C})}) \longrightarrow \mathrm{Aut}(\overline{C}) \xrightarrow{\Lambda} \mathrm{Im}(\Lambda|_{\mathrm{Aut}(\overline{C})}) \longrightarrow 1,$$

where $\mathrm{Ker}(\Lambda|_{\mathrm{Aut}(\overline{C})}) = \langle \sigma^{d-2} \rangle$ by using Theorem 1.4.4-(2). Furthermore, $\mathrm{Im}(\Lambda|_{\mathrm{Aut}(\overline{C})})$ contains an $\mathrm{D}_{2(d-2)}$ coming from the images of σ and $\tau := [X : Z : Y]$. Consequently $\mathrm{Im}(\Lambda|_{\mathrm{Aut}(\overline{C})}) = \mathrm{D}_{2(d-2)}$, A_4, S_4 or A_5, again by Theorem 1.4.4-(2).

If $d = 6$, then $\mathrm{Im}(\Lambda|_{\mathrm{Aut}(\overline{C})})$ has a subgroup of order 8, and hence can not be A_4. Moreover, for some suitable $\alpha, \beta \in \overline{k}^*$, $[\alpha X : Y + \beta\zeta_4 Z : \beta^{-1}Y - \zeta_4 Z] \in \mathrm{PGL}_3(\overline{k})$ defines an automorphism of $\overline{C} : X^6 + YZ^5 + Y^5Z = 0$, whose image in $\mathrm{Im}(\Lambda|_{\mathrm{Aut}(\overline{C})})$ has order 3. In particular, $\mathrm{Im}(\Lambda|_{\mathrm{Aut}(\overline{C})}) = \mathrm{S}_4$, and $\mathrm{Aut}(\overline{C})$ is a central extension of S_4 by $\mathbb{Z}/6\mathbb{Z}$.

Next assume that $d \geq 5$ and $d \neq 6$. We apply Oikawa's inequality (Theorem 1.3.13) to the the set $S := C \cap L$, which is a non-empty set of $\overline{C}$ of cardinality at most the degree d, to obtain $|\mathrm{Aut}(\overline{C})| \leq 6d(d-2)$. Moreover, $\Lambda(\sigma) \in \mathrm{Im}(\Lambda|_{\mathrm{Aut}(\overline{C})})$ is of order $d-2$, and an element of A_4 or S_4 (resp. A_5) has order at most 4 (resp. 5). So if $\mathrm{Im}(\Lambda|_{\mathrm{Aut}(\overline{C})}) = \mathrm{A}_4$ or S_4 (resp. A_5), then $d = 5$ (resp. $d \leq 7$). On the other hand, if $\mathrm{Im}(\Lambda|_{\mathrm{Aut}(\overline{C})}) = \mathrm{A}_5$, then $60d = |\mathrm{Aut}(\overline{C})| \leq 6d(d-2)$, which gives $d \geq 12$, a contradiction to $d \leq 7$. If $d = 5$ and $\mathrm{Im}(\Lambda|_{\mathrm{Aut}(\overline{C})}) = \mathrm{S}_4$, then $24 \cdot 5 = |\mathrm{Aut}(\overline{C})| \leq 6 \cdot 5 \cdot 3$, which is impossible. Lastly, if $d = 5$ and $\mathrm{Im}(\Lambda|_{\mathrm{Aut}(\overline{C})}) = \mathrm{A}_4$, then $\langle \Lambda(\sigma), \Lambda(\tau) \rangle = \mathrm{D}_6$ must be a subgroup of A_4 of index two, which is also absurd. Consequently, $\mathrm{Im}(\Lambda)$ is conjugate to $\mathrm{D}_{2(d-2)}$, and $\mathrm{Aut}(\overline{C}) = \langle \sigma, \tau \rangle$. □

Proposition 2.4.11. *For $d \geq 4$, $\overline{C} \in \mathcal{M}_g^{Pl}(\mathbb{Z}/d(d-2)\mathbb{Z})$ if and only if it is $\overline{k}$-isomorphic to $X^d + Y^{d-1}Z + YZ^{d-1} = 0$. Hence $\mathcal{M}_g^{Pl}(\mathbb{Z}/d(d-2)\mathbb{Z})$ is irreducible and consists of a single point of $\mathcal{M}_g$. Furthermore, $\widetilde{\mathcal{M}_g^{Pl}}(\mathrm{H}_d) = \mathcal{M}_g^{Pl}(\mathbb{Z}/d(d-1)\mathbb{Z}) = \varrho(\mathcal{M}_g^{Pl}(\mathbb{Z}/d(d-1)\mathbb{Z}))$, where $\varrho(\mathbb{Z}/d(d-2)\mathbb{Z}) = \langle \mathrm{diag}(1, \zeta_{d(d-2)}, \zeta_{d(d-2)}^{-(d-1)}) \rangle$.*

Proof. It suffices to prove the "only if" part, since the other parts are immediate consequences of Proposition 2.4.10. Since $d(d-2)$ does not divide $d-1$, d, d^2-3d+3, $(d-1)^2$ and $d(d-1)$, we may consider an $F_{\overline{C}}(X,Y,Z) = 0$ to be of *Type* $d(d-2), (a,b)$ defined by the normal form Theorem 2.1.3, subcase (4.1). Hence $(a,b) \in \Gamma_{d(d-2)}$ with $(d-1)a + b \equiv 0 \bmod d(d-2)$ and $a + (d-1)b \equiv 0 \bmod d(d-2)$. In particular, $a = m$ and $b = dm' + m$ for some integers m and m', such that m and $dm'+m$ are relatively prime and $d-2|m+m'$. Consequently, $m = 1$ and $m' = d-3$ is a generator because

$$\mathrm{diag}(1, \zeta_{d(d-2)}, \zeta_{d(d-2)}^{d(d-3)+1})^m = \mathrm{diag}(1, \zeta_{d(d-2)}^m, \zeta_{d(d-2)}^{dm'+m}).$$

Therefore,

$$\begin{aligned} S(2)^{j,X}_{d(d-2),\,(1,d-3)} &:= \{0 \leq i \leq j \,|\, i + (j-i)\,(d(d-3)+1) \equiv 0 \bmod d(d-2)\} \\ &= \{0 \leq i \leq j \,|\, j(d-1) - di \equiv 0 \bmod d(d-2)\} \\ &= \emptyset,\ \forall j = 2, ..., d-2, \end{aligned}$$

as $d(d-2)\,|\,j(d-1)-di$ gives $d|j$, a contradiction.

$$\begin{aligned} S_{2\ \ d(d-2),\,(1,d-3)}^{d,X} &:= \{2\leq i\leq d-2\,|\,i+(d-i)\,(d(d-3)+1)\equiv 0 \bmod d(d-2)\}\\ &\subseteq \{2\leq i\leq d-2\,|\,d-1-i\equiv 0 \bmod d-2\}\\ &= \emptyset. \end{aligned}$$

This implies that $\overline{C}$ is $\overline{k}$-isomorphic to $X^d+Y^{d-1}Z+YZ^{d-1}=0$. $\square$

The stratum $\mathcal{M}_{\mathbf{g}}^{\mathbf{Pl}}(\mathbb{Z}/(\mathbf{d^2-3d+3})\mathbb{Z})$

The next result is well-known in the literature, see for example [Har13, Propostion 3.5] when k has characteristic $p=0$, and the same is true when $p>2g+1$ by our discussion in chapter 1:

Proposition 2.4.12. *The automorphism group of the Klein curve $K_d\,:\,X^{d-1}Y+Y^{d-1}Z+Z^{d-1}X=0$ of degree $d\geq 5$ is a semidirect product of $\mathbb{Z}/3\mathbb{Z}$ by $\mathbb{Z}/(d^2-3d+3)\mathbb{Z}$. More precisely, it is isomorphic to*

$$\langle \tau,\,\sigma|\ \tau^{d^2-3d+3}=\sigma^3=1\ \ and\ \ \tau\sigma=\sigma\tau^{-(d-1)}\,\rangle.$$

In particular, $\mathrm{Aut}(K_d)$ *is of order* $3(d^2-3d+3)$.

Proof. The group $H:=\langle\sigma,\tau\rangle$, where $\sigma:=[Z:X:Y]$ and $\tau:=\mathrm{diag}(1,\zeta_{d^2-3d+3},\zeta_{d^2-3d+3}^{-(d-2)})$ is a subgroup of $\mathrm{Aut}(K_d)$, which is a semidirect product of $\mathbb{Z}/3\mathbb{Z}$ acting on $\mathbb{Z}/(d^2-3d+3)\mathbb{Z}$. Hence $|\operatorname{Aut}(K_d)|$ is a multiple of $|H|=3(d^2-3d+3)$. On the other hand, K_d has exactly three $(d-3)$-inflection points namely the three reference points P_i, for $i=1,2,3$, see [Kat01, Lemma 2.3]. Therefore, these distinguished points constitute a set S of cardinality 3, which is fixed by $\mathrm{Aut}(K_d)$. Using again Oikawa's inequality (Theorem 1.3.13), we obtain $|\operatorname{Aut}(K_d)|\leq 12(g-1)+6.3=6(d^2-3d+3)$. Now it remains to show that $|\operatorname{Aut}(K_d)|$ is odd: assume on the contrary that K_d admits an involution η, then it fixes at least one of the three $(d-3)$-inflection points, say P_3. Hence, the set $\{P_1,P_2\}$ of the remaining two $(d-3)$-inflection points must also be fixed, therefore η has the shape $\mathrm{diag}(\alpha,\beta,1)$ with $(\alpha,\beta)\in\{(1,-1),(-1,1),(-1,-1)\}$, or $[\gamma Y:\gamma^{-1}X:Z]$ for some $\gamma\neq 0$. Obviously, non of these transformations retains K_d, and the result follows. $\square$

Remark 2.4.13. The automorphism group of the Klein quartic curve is isomorphic to $\mathrm{PSL}_2(\mathbb{F}_7)$, the unique simple group of order 168 (see [Hen76]). This completes the result for all degrees $d \geq 4$.

The next result should also be well-known in the literature, we write it for completeness:

Proposition 2.4.14. *For $d \geq 5$, $\overline{C} \in \mathcal{M}_g^{Pl}(\mathbb{Z}/(d^2-3d+3)\mathbb{Z})$ if and only if $\overline{C}$ is $\overline{k}$-isomorphic to the Klein curve $K_d : X^{d-1}Y + Y^{d-1}Z + Z^{d-1}X = 0$. Hence $\mathcal{M}_g^{Pl}(\mathbb{Z}/(d^2-3d+3)\mathbb{Z})$ is irreducible, being a set of a single point. Moreover,*

$$\widetilde{\mathcal{M}_g^{Pl}}(\mathrm{Aut}(K_d)) = \mathcal{M}_g^{Pl}(\mathbb{Z}/(d^2-3d+3)\mathbb{Z}) = \varrho(\mathcal{M}_g^{Pl}(\mathbb{Z}/(d^2-3d+3)\mathbb{Z})),$$

where $\varrho(\mathbb{Z}/(d^2-3d+3)\mathbb{Z}) = \langle \mathrm{diag}(1, \zeta_{d^2-3d+3}, \zeta_{d^2-3d+3}^{-(d-2)})\rangle$.

Proof. Again we only need to show the "only if" statement, and the other parts are consequences of Proposition 2.4.12. Since $d^2-3d+3 \nmid d-1,\, d,\, d(d-1),\, d(d-2),\, (d-1)^2$ for any $d \geq 5$, then $\overline{C}$ has a non-singular plane model $F_{\overline{C}}(X,Y,Z) = 0$ over $\overline{k}$ of *Type* d^2-3d+3, (a,b) of the form Theorem 2.1.3, case (3), for some $(a,b) \in \Gamma_{d^2-3d+3}$, such that $a = (d-1)a + b = (d-1)b \,(\mathrm{mod}\ d^2-3d+3)$. In particular, every solution is of the shape $a = m$ and $b = (d^2-3d+3)m' - (d-2)m$ for some integers m and m'. Because $\mathrm{diag}(1, \zeta_{d^2-3d+3}, \zeta_{d^2-3d+3}^{-(d-2)})^m = \mathrm{diag}(1, \zeta_{d^2-3d+3}^m, \zeta_{d^2-3d+3}^{(d^2-3d+3)m'-(d-2)m})$, we can take $a \equiv 1 \bmod d^2-3d+3$ and $b = -(d-2) \bmod d^2-3d+3$ as a generator of these types of curves. Consequently, for all $j = 2, ..., d-2$

$$\begin{aligned} S(1)^{j,X}_{d^2-3d+3,\,(1,-(d-2))} &:= \{0 \leq i \leq j \,|\, i-(j-i)(d-2) \equiv 1 \bmod (d^2-3d+3)\} \\ &= \{0 \leq i \leq j \,|\, j(d-2) - i(d-1) + 1 \equiv 0 \bmod (d^2-3d+3)\} \\ &= \emptyset. \end{aligned}$$

The last equality comes from the fact $|j(d-2)-i(d-1)+1| < d^2-3d+3$. Then $j(d-2) - i(d-1)+1 = 0$, which in turns gives $d-1|j-1$. This is impossible, as $0 < j-1 < d-1$.

Also,

$$\begin{aligned}
S^{j,Z}_{d^2-3d+3,\,(1,-(d-2))} &:= \{0 \le i \le j \,|\, i + (d-j)(d^2-4d+5) \equiv 1 \bmod (d^2-3d+3)\} \\
&= \{0 \le i \le j \,|\, (d-j)(d-2) - i + 1 \equiv 0 \bmod (d^2-3d+3)\} \\
&= \emptyset, \text{ since } 0 < (d-j)(d-2) - i + 1 < d^2 - 3d + 3. \\
S^{j,Y}_{d^2-3d+3,\,(1,-(d-2))} &:= \{0 \le i \le j \,|\, -(d-2)i + (d-j) \equiv 1 \bmod (d^2-3d+3)\} \\
&= \emptyset,
\end{aligned}$$

since $|(d-j)-(d-2)i-1| < d^2-3d+3$, and if $(d-j)-(d-2)i-1 = 0$, then $d-2|j-1$, a contradiction.

Therefore $\overline{C}$ is $\overline{k}$-isomorphic to $X^{d-1}Y + Y^{d-1}Z + Z^{d-1}X = 0$. The full automorphism group $\mathrm{Aut}(\overline{C})$ is then determined explicitly by Proposition 2.4.12. $\square$

2.4.2 Strata of smooth plane curves, having a "large" automorphism: Galois points

In the previous subsection (§2.4.1) we showed that if m is "very large", then the stratum $\mathcal{M}_g^{Pl}(\mathbb{Z}/m\mathbb{Z})$ is given, up to $\overline{k}$-isomorphism, by a single point of $\mathcal{M}_g$. Therefore, it is an irreducible stratum. In general it is difficult, for an arbitrary m, to decide whether $\mathcal{M}_g^{Pl}(\mathbb{Z}/m\mathbb{Z})$ is irreducible or not. We have seen in §2.2 a weaker concept than the irreducibility that we call *ES-irreducibility* (see Definition 2.2.6). For instance, the strata $\mathcal{M}_g^{Pl}(\mathbb{Z}/m\mathbb{Z})$ ES-irreducible when m is "very large". However, it is not true that $\mathcal{M}_g^{Pl}(\mathbb{Z}/m\mathbb{Z})$ is always ES-irreducible, and we already have seen such examples in §2.3. In this case, the stratum, which is not ES-irreducible, is also not irreducible subset of $\mathcal{M}_g$.

Here we show that the stratum $\mathcal{M}_g^{Pl}(\mathbb{Z}/m\mathbb{Z})$ for m "large" remains ES-irreducible, and moreover we obtain further details about them. The cases where m is a multiple of d or $d-1$ are strongly related to inner and outer Galois points (see §1.3 of chapter 1), which will help a lot in determining, more precisely, the automorphism groups of these strata in some cases.

One can read Henn [Hen76] or Bars [Bar12] for the well-known results in the literature on smooth plane quartic curves. So, we may assume that $d \ge 5$.

We are interested in smooth plane curves $\overline{C} \in \mathcal{M}_g^{Pl}$ of an arbitrary, but a fixed degree $d \geq 5$, and whose automorphism groups contain homologies of period d (resp. $d-1$). When a homology η of period d or $d-1$ is present inside $\mathrm{Aut}(\overline{C})$, the genus of $\overline{C}/\langle\eta\rangle$ is zero and $\overline{C}$ has an outer (resp. inner) Galois point P (see Proposition 1.3.12). Moreover, $\tau(P)$ remains an outer (resp. inner) Galois point for $\overline{C}$, for any $\tau \in \mathrm{Aut}(\overline{C})$. Consequently, if $\overline{C}$ has, for example, a unique inner Galois point, then it should be fixed by the full automorphism group $\mathrm{Aut}(\overline{C})$. In particular, $\mathrm{Aut}(\overline{C})$ is cyclic when the characteristic $p = 0$ or sufficiently big (see [HKT08, Lemma 11.44] and Corollary 1.4.9).

The stratum $\mathcal{M}_{\mathbf{g}}^{\mathbf{Pl}}(\mathbb{Z}/\mathbf{m}(\mathbf{d}-\mathbf{1})\mathbb{Z})$ for $\mathbf{2} \leq \mathbf{m} \leq \mathbf{d}$.

We start with the following observation:

Lemma 2.4.15. *The stratum $\mathcal{M}_g^{Pl}(\mathbb{Z}/m(d-1)\mathbb{Z})$ with $2 \leq m \leq d$ is not empty only if $d \equiv 0$ or $1 \mod m$.*

Proof. Since $m(d-1)$ does not divide $d-1,\ d,\ d^2-3d+3$, and $d(d-2)$, it must then divides $d(d-1)$ or $(d-1)^2$, by Corollary 2.1.6. $\square$

Proposition 2.4.16. *For $d \geq 5$ and $2 \leq m \leq d$, such that $d \equiv 0 \mod m$, $\overline{C} \in \mathcal{M}_g^{Pl}(\mathbb{Z}/m(d-1)\mathbb{Z})$ if and only if $\overline{C}$ has a non-singular plane model $F_{\overline{C}}(X,Y,Z) = 0$ over $\overline{k}$ of the form*

$$X^d + Y^d + XZ^{d-1} + \sum_{2 \leq mj \leq d-2} \beta_j X^{d-mj} Y^{mj}. \tag{2.1}$$

In particular, $\mathrm{Aut}(\overline{C})$ is a cyclic group of order divisible by $m(d-1)$.

Proof. If $\overline{C}$ is a smooth plane curve of the form (2.1), then $\eta := \mathrm{diag}(1, \zeta_{m(d-1)}^{d-1}, \zeta_{m(d-1)}^{m})$ is an automorphism of order $m(d-1)$. Hence, $\overline{C} \in \mathcal{M}_g^{Pl}(\mathbb{Z}/m(d-1)\mathbb{Z})$. Moreover, $F_{\overline{C}}(X,Y,Z) = 0$ can not be a descendant of the Klein curve K_d, since $m(d-1) \nmid 3(d^2-3d+3)$. Also $F_{\overline{C}}(X,Y,Z) = 0$ is not a descendant of the Fermat curve F_d, as $2(d-1) \nmid 6d^2$, and $m(d-1) > 2d$ for $m \geq 3$, while $\mathrm{Aut}(F_d)$ has elements of order at most $2d$. On the other hand, $\eta^m = \mathrm{diag}(1, 1, \zeta_{m(d-1)}^{m^2})$ is a homology of period $d-1 \geq 4$, with center $P_3 = (0:0:1)$ and axis $L_3 : Z = 0$. Therefore, the point $(0:0:1)$ is an inner Galois point for $\overline{C}$ (Proposition 1.3.12), and it is unique (Theorem 1.3.8). Hence it should be fixed by the full automorphism group $\mathrm{Aut}(\overline{C})$. Consequently, $\mathrm{Aut}(\overline{C})$ is a cyclic group (Corollary 1.4.2) of order divisible by $m(d-1)$.

Conversely, if $\overline{C} \in \mathcal{M}_g^{Pl}(\mathbb{Z}/m(d-1)\mathbb{Z})$, then $F_{\overline{C}}(X,Y,Z) = 0$ can be taken to be of *Type* $m(d-1),(a,b)$ of the form Theorem 2.1.3, case (5), because $m(d-1) \nmid d-1,\ d,\ d^2-3d+3,\ (d-1)^2$, and $d(d-2)$. So $(a,b) \in \Gamma_{m(d-1)}$ such that $m(d-1)|da$, and $m(d-1)|(d-1)b$. In particular, $a = (d-1)t$ and $b = mt'$ for some integers t and t'. If we consider any integer s with $t = s \pmod{m}$, then $\eta^{(t'-s)(d-1)+t'} = \mathrm{diag}(1, \zeta_{m(d-1)}^{(d-1)t}, \zeta_{m(d-1)}^{mt'})$, and we can take $t = 1 = t'$. Thus we get

$$\begin{aligned}
S_1^{d,X}{}_{m(d-1),\,(d-1,m)} &:= \{1 \le i \le d-1 \,|\, (d-1)i + (d-i)m \equiv 0 \bmod m(d-1)\} \\
&= \{1 \le i \le d-1 \,|\, (d-1)i - (i-1)m \equiv 0 \bmod m(d-1)\} \\
&\subseteq \{1 \le i \le d-1 \,|\, (i-1) \equiv 0 \bmod d-1\} \\
&= \{1\}.
\end{aligned}$$

But also $m(d-1)$ does not divide $(d-1)(m+1)$, so $1 \notin S_1^{d,X}{}_{m(d-1),\,(d-1,m)}$, and $S_1^{d,X}{}_{m(d-1),\,(d-1,m)} = \emptyset$. Similarly, we conclude that $S_1^{d-1,X}{}_{m(d-1),\,(d-1,m)} = \emptyset$. Furthermore,

$$\begin{aligned}
S(2)^{j,X}{}_{m(d-1),\,(d-1,m)} &:= \{0 \le i \le j \,|\, (d-1)i + (j-i)m \equiv 0 \bmod m(d-1)\} \\
&\subseteq \{0 \le i \le j \,|\, d-1|j-i\} \\
&= \{j\}
\end{aligned}$$

By assumption, $\eta \in \mathrm{Aut}(F_{\overline{C}})$, therefore $S(2)^{j,X}{}_{m(d-1),\,(d-1,m)} = \emptyset$ when $m \nmid j$, and $\{j\}$, otherwise. We substitute into equation Theorem 2.1.3, case (5) in order to obtain the defining form (2.1). $\square$

Proposition 2.4.17. *For $d \ge 5$ and $2 \le m \le d$, such that $d \equiv 1 \bmod m$, $\overline{C} \in \mathcal{M}_g^{Pl}(\mathbb{Z}/m(d-1)\mathbb{Z})$ if and only if $\overline{C}$ has a non-singular plane model $F_{\overline{C}}(X,Y,Z) = 0$ over $\overline{k}$ of the form*

$$X^d + Y^{d-1}Z + XZ^{d-1} + \sum_{2 \le mj \le d-2} \beta_j X^{d-mj} Z^{mj} \tag{2.2}$$

In this case, $\mathrm{Aut}(\overline{C})$ is again cyclic of order divisible by $m(d-1)$.

Proof. We first modify η of Proposition 2.4.16 to be $\mathrm{diag}(1, \zeta_{m(d-1)}, \zeta_{m(d-1)}^{(m-1)(d-1)})$. Then, following the same technique, one deduces that smooth plane curves $\overline{C}$ of the form (2.2) are in the stratum $\mathcal{M}_g^{Pl}(\mathbb{Z}/m(d-1)\mathbb{Z})$, and also the full automorphism group is cyclic of order divisible by $m(d-1)$.

Conversely, $\overline{C} \in \mathcal{M}_g^{Pl}(\mathbb{Z}/m(d-1)\mathbb{Z})$ with $d \equiv 1 \bmod m$ implies that $F_{\overline{C}}(X,Y,Z) = 0$ is of *Type* $m(d-1)$, (a,b) of the form Theorem 2.1.3, subcase (4.2). In particular, $(a,b) \in \Gamma_{m(d-1)}$, such that $m(d-1) \,|\, (d-1)a+b, (d-1)b$. Hence $a = mt - t'$ and $b = (d-1)t'$ for some integers t and t'. Similarly, it suffices to assume $t = 1$ and $t' = m-1$, since $\eta^{mt-t'} = \mathrm{diag}(1, \zeta^a_{m(d-1)}, \zeta^b_{m(d-1)})$. Thus

$$
\begin{aligned}
S_1^{d-1,X}{}_{m(d-1),\,(a,b)} &:= \{1 \leq i \leq d-1 \,|\, i + (d-1-i)(m-1)(d-1) \equiv 0 \bmod m(d-1)\}\\
&= \{1 \leq i \leq d-1 \,|\, m(d-1) \,|i\}\\
&= \emptyset,\\
S_2^{d,X}{}_{m(d-1),\,(a,b)} &:= \{2 \leq i \leq d-2 \,|\, i + (d-i)(m-1)(d-1) \equiv 0 \bmod m(d-1)\}\\
&= \{2 \leq i \leq d-2 \,|\, m(d-1)|di - (d-1)\}\\
&\subseteq \{2 \leq i \leq d-2 \,|\, d-1|di\}\\
&= \emptyset.
\end{aligned}
$$

Lastly, for all $2 \leq j \leq d-2$,

$$
\begin{aligned}
S(2)^{j,X}{}_{m(d-1),\,(a,b)} &:= \{0 \leq i \leq j \,|\, i + (j-i)(m-1)(d-1) \equiv 0 \bmod m(d-1)\}\\
&= \{0 \leq i \leq j \,|\, di - j(d-1) \equiv 0 \bmod m(d-1)\}\\
&\subseteq \{0 \leq i \leq j \,|\, d-1|di\}\\
&= \{0\}.
\end{aligned}
$$

Notice that $i = 0$ gives $m|j$, so, we obtain the form (2.2) after we substitute into Theorem 2.1.3, subcase (4.2). □

The following corollaries are consequences of Propositions 2.4.16 and 2.4.17:

Corollary 2.4.18. *The stratum $\mathcal{M}_g^{Pl}(\mathbb{Z}/m(d-1)\mathbb{Z})$, with $2 \leq m \leq d$ and $d \geq 5$ are either empty or ES-irreducible, given by a single normal form.*

Corollary 2.4.19. *The full automorphism group of any $\overline{C} \in \mathcal{M}_g^{Pl}(\mathbb{Z}/m(d-1)\mathbb{Z})$, for some $2 \leq m \leq d$ is cyclic and always contains a homology of period $d-1$. In particular, $\overline{C}$ has a unique inner Galois point.*

Remark 2.4.20. The converse of Corollary 2.4.19 is also true. That is to say, if $\overline{C}$ is a smooth plane curve of degree $d \geq 5$, such that $\mathrm{Aut}(\overline{C})$ contains a homology η of order $d-1$, then $\overline{C}$ has an inner Galois point (Proposition 1.3.12), and moreover it is unique (Theorem 1.3.8). This point should be fixed by $\mathrm{Aut}(\overline{C})$, which in turns implies that $\mathrm{Aut}(\overline{C})$ is cyclic (Corollary 1.4.2).

The stratum $\mathcal{M}_{\mathbf{g}}^{\mathbf{Pl}}(\mathbb{Z}/\mathbf{md}\mathbb{Z})$ for $\mathbf{2 \leq m \leq d-1}$.

Lemma 2.4.21. *The stratum $\mathcal{M}_g^{Pl}(\mathbb{Z}/md\mathbb{Z})$ with $2 \leq m \leq d-1$ is not empty only if $d = 1$ or $2 \bmod m$.*

Proof. The result follows again by Corollary 2.1.6, since md is not a divisor of $d-1$, d, $d^2 - 3d+3$, and $(d-1)^2$. □

Proposition 2.4.22. *For $d \geq 5$ and $3 \leq m \leq d-1$, such that $d \equiv 1 \bmod m$, $\overline{C} \in \mathcal{M}_g^{Pl}(\mathbb{Z}/md\mathbb{Z})$ if and only if $\overline{C}$ has a non-singular plane model $F_{\overline{C}}(X,Y,Z) = 0$ over $\overline{k}$ of the form*

$$X^d + Y^d + XZ^{d-1} + \sum_{2 \leq mj \leq d-2} \beta_j X^{d-mj} Z^{mj} \tag{2.3}$$

In this case, $\mathrm{Aut}(F_{\overline{C}})$ should fix a line in $\mathbb{P}^2(\overline{k})$ and a point off this line. In particular, all automorphisms of $F_{\overline{C}}(X,Y,Z) = 0$ are of the shape

$$\begin{pmatrix} * & 0 & * \\ 0 & 1 & 0 \\ * & 0 & * \end{pmatrix}.$$

Proof. Any smooth plane curve $\overline{C}$ of the form (2.3) is in $\mathcal{M}_g^{Pl}(\mathbb{Z}/md\mathbb{Z})$, since $\eta := \mathrm{diag}(1, \zeta_{md}^m, \zeta_{md}^d) \in \mathrm{Aut}(F_{\overline{C}})$ is of order md. Moreover η^m is a homology of period $d > 4$ with center $P_2 = (0:1:0)$ and axis $L_2 : Y = 0$. Therefore, $\mathrm{Aut}(\overline{C})$ fixes a projective line and a point off that line or it fixes a triangle (Theorem 1.2.8). Suppose first that it fixes a triangle and neither a point nor line is leaved invariant, then $F_{\overline{C}}(X,Y,Z)=0$ must be a descendant of the Fermat curve K_d or the Klein curve K_d, which is impossible, as $md\nmid 3(d^2-3d+3)$, and automorphisms of F_d have orders at most $2d<md$. Consequently, a line and a point off that line is leaved invariant. Furthermore, P_2 is an outer Galois point for $\overline{C}$ (Proposition 1.3.12), and it is unique due to Theorem 1.3.10 and because $\overline{C}$ is not $\overline{k}$-isomorphic to the Fermat curve F_d. Hence such a point should be fixed by $\operatorname{Aut}(F_{\overline{C}})$, and so does the axis $Y=0$ (Proposition 1.2.9). So we deduce the shapes of automorphisms of $F_{\overline{C}}(X,Y,Z)=0$.

Conversely, one may follow the same line of argument, that we did in Proposition 2.4.16 to conclude that $\overline{C}$ is of *Type* $md,\ (mt,dt')$ of the form Theorem 2.1.3, case (5), and to also figure out that we can assume $t=1=t'$ as a generator, since $\eta^{(t'-s)d+t}=\operatorname{diag}(1,\zeta_{md}^{mt},\zeta_{md}^{dt'})$, for any s satisfying $t=s \bmod m$. In particular, the index sets $S_1^{d,X}{}_{md,(m,d)}$ and $S_1^{d-1,X}{}_{md,(m,d)}$ are empty, and moreover $i\in S(2)^{j,X}{}_{md,(m,d)}$, for some j if and only if $md|mi-(j-i)d$, thus $d|i$ and $i=0$. Then $S(2)^{j,X}{}_{md,(m,d)}\neq\emptyset$ only if $m|j$, which completes the proof. $\square$

Remark 2.4.23. For $m=2$, Proposition 2.4.22 still true with the same proof, if we assume that $F_{\overline{C}}(X,Y,Z)=0$ is not a descendent of the Fermat curve F_d of degree d.

There is a similar statement to the previous results when $d\equiv 2 \bmod m$. We only state the result, since the proof can be obtained through similar techniques:

Proposition 2.4.24. *For $d\geq 5$ and $2\leq m\leq d-1$, with $d\equiv 2 \bmod m$, $\overline{C}\in\mathcal{M}_g^{Pl}(\mathbb{Z}/md\mathbb{Z})$ if and only if $\overline{C}$ has a non-singular plane model $F_{\overline{C}}(X,Y,Z)=0$ over $\overline{k}$ of the form*

$$X^d+Y^{d-1}Z+YZ^{d-1}+\sum_{2\leq i\equiv 1 \bmod m\leq d-2}\beta_iY^iZ^{d-i}=0. \tag{2.4}$$

In such a case, $F_{\overline{C}}(X,Y,Z)=0$ is a descendant of the Fermat curve F_d (only if $m=2$), or $\operatorname{Aut}(F_{\overline{C}})$ fixes a line and a point off this line. So, for $m>2$, all automorphisms of $F_{\overline{C}}(X,Y,Z)=0$ have the shapes

$$\begin{pmatrix}1&0&0\\0&*&*\\0&*&*\end{pmatrix}$$

Remark 2.4.25. Unfortunately, it might happen here that different kinds, i.e. non-cyclic, of groups occurs as the full automorphism of $\overline{C}\in\mathcal{M}_g^{Pl}(\mathbb{Z}/md\mathbb{Z})$. For example, when $d=6$ and

$m = 2$, the defining equation for the stratum $\mathcal{M}_{10}^{Pl}(\mathbb{Z}/12\mathbb{Z})$ reduces to

$$X^6 + Y^5Z + YZ^5 + \beta_{6,3}Y^3Z^3.$$

In particular, $\mathrm{diag}(1, \zeta_{12}, \zeta_{12}^7)$ is an automorphism of order 12, but the automorphism group is not cyclic, since we have the extra automorphism $[X : Z : Y]$.

Now, we end up this part with the following corollaries:

Corollary 2.4.26. *The strata $\mathcal{M}_g^{Pl}(\mathbb{Z}/md\mathbb{Z})$, for $2 \leq m \leq d-1$ and $d \geq 5$, are either empty or ES-irreducible.*

As we mentioned earlier that if $\mathrm{Aut}(\overline{C})$ contains a homology of period d then $\overline{C}$ has an outer Galois point. Moreover, if $\overline{C}$ is isomorphic to the Fermat curve of degree d, then it has two more outer Galois points and it is unique, otherwise. See Theorem 1.3.8 and Proposition 1.3.12 for more details. Finally, we conclude:

Corollary 2.4.27. *For any $\overline{C} \in \mathcal{M}_g^{Pl}(\mathbb{Z}/md\mathbb{Z})$ with $3 \leq m \leq d-1$, $\mathrm{Aut}(\overline{C})$ always contains a homology of period d. In particular, $\overline{C}$ has a unique outer Galois point.*

2.4.3 More cases: The stratum $\mathcal{M}_{\mathbf{g}}^{\mathbf{Pl}}(\mathbb{Z}/\mathbf{m}(\mathbf{d}-\mathbf{2})\mathbb{Z})$

We investigate the finite groups G that contain cyclic subgroups of order $m(d-2)$, and for which the stratum $\mathcal{M}_g^{Pl}(G)$ is non-empty. This question is completely solved when $g = 3$ (see [Hen76]) and we solve it in chapter 4 when $g = 6$. Therefore, we take $d \geq 6$ and $m \geq 2$.

Lemma 2.4.28. *The stratum $\mathcal{M}_g^{Pl}(\mathbb{Z}/m(d-2)\mathbb{Z})$, for $d \geq 6$ and $m \geq 2$ is non-empty only if $d \equiv 0 \mod m$.*

Proof. Since $d \geq 6 > 2 + \frac{2}{m-1}$, then $m(d-2) > d$, and $m(d-2)$ does not divide d and $d-1$. Moreover, $(d-1)^2 \equiv 1 \mod d-2$, $d^2 - 3d + 3 \equiv 1 \mod d-2$, and $d(d-1) \equiv d \mod d-2$, so $m(d-2)$ does not divide $(d-1)^2$, $d(d-1)$, and $d^2 - 3d + 3$. Using Corollary 2.1.6, we deduce the result. $\square$

We first treat the situation when m is even:

Proposition 2.4.29. *For any even integer $m \geq 2$, dividing the degree $d \geq 6$, any $\overline{C} \in \mathcal{M}_g^{Pl}(\mathbb{Z}/m(d-2)\mathbb{Z})$ has a non-singular plane model $F_{\overline{C}}(X,Y,Z) = 0$ over $\overline{k}$ of the form*

$$X^d + Y^{d-1}Z + YZ^{d-1} + \sum_{j=1}^{\lfloor \frac{d}{2m} \rfloor} \beta_j X^{d-2mj}(YZ)^{mj} = 0 \tag{2.5}$$

In particular, $\mathcal{M}_g^{Pl}(\mathbb{Z}/m(d-2)\mathbb{Z})$ is ES-irreducible, in this case.

Proof. It suffices to prove the result for $\mathcal{M}_g^{Pl}(\mathbb{Z}/2(d-2)\mathbb{Z})$, since $\mathcal{M}_g^{Pl}(\mathbb{Z}/m(d-2)\mathbb{Z}) \subseteq \mathcal{M}_g^{Pl}(\mathbb{Z}/2(d-2)\mathbb{Z})$: indeed, if η is an automorphism of $\overline{C}$ of order $m(d-2)$, then $\eta^{\frac{m}{2}}$ is also an automorphism, but it is of order $2(d-2)$.

It follows, by Lemma 2.4.28, that $F_{\overline{C}}(X,Y,Z) = 0$ is considered to be of *Type* $2(d-2),(a,b)$ of the form Theorem 2.1.3, subcase (4.1), for some $(a,b) \in \Gamma_{2(d-2)}$ such that $2(d-1)$ is dividing both $(d-1)a+b$, and $a+(d-1)b$. First, we show that $a=1$ and $b=d-3$ is a generator of these types of curves: We already have $(1,d-3) \in \Gamma_{2(d-2)}$, and moreover $2|a-b$ and $d-2|a+b$. So we can write $a = t + (\frac{d-2}{2})t'$, and $b = -t + (\frac{d-2}{2})t'$, for some integers t and t'. In particular, we get $2|\pm t + (\frac{d}{2})t'$, and

$$\mathrm{diag}(1, \zeta_{2(d-2)}, \zeta_{2(d-2)}^{d-3})^{t+(\frac{d-2}{2})t'} = \mathrm{diag}(1, \zeta_{2(d-2)}^{a}, \zeta_{2(d-2)}^{b}),$$

which proves the claim on a and b. Second, the associated sets $S_2^{d,X}{}_{2(d-2),(1,d-3)}$ and $S(2)^{j,X}{}_{2(d-2),(1,d-3)}$ for $j = 2, ..., d-1$ are computed as follows:

$$\begin{aligned} S_2^{d,X}{}_{2(d-2),(1,d-3)} &:= \{2 \leq i \leq d-2 \,|\, i + (d-i)(d-3) \equiv 0 \bmod 2(d-2)\} \\ &\subseteq \{2 \leq i \leq d-2 \,|\, 2(i-1) \equiv 0 \bmod (d-2)\} \\ &= \{\frac{1}{2}d\}, \end{aligned}$$

since $0 < 2(i-1) < 2(d-2)$, and thus $2(i-1) = d-2$. Moreover, we have

$$\begin{aligned} S(2)^{j,X}{}_{2(d-2),(1,d-3)} &:= \{0 \leq i \leq j \,|\, i + (j-i)(d-3) \equiv 0 \bmod 2(d-2)\} \\ &\subseteq \{0 \leq i \leq j \,|\, j - 2i \equiv 0 \bmod (d-2)\}. \end{aligned}$$

But $|j-2i| \leq d-1$, therefore $j-2i = 0$ or $\pm(d-2)$. In particular, $S(2)^{j,X}{}_{2(d-2),(1,d-3)} = \emptyset$, if j is odd and $\{\frac{j}{2}, \frac{j\pm(d-2)}{2}\}$, if j is even. Furthermore, always $0 \leq i \leq j$, thus when j is even and $< d-2$, $S(2)^{j,X}{}_{2(d-2),(1,d-3)} = \{\frac{j}{2}\}$, and when $j = d-2$, $S(2)^{d-2,X}{}_{2(d-2),(1,d-3)} = \{0, \frac{d-2}{2}, d-2\}$. Consequently, we obtain the normal form

$$X^d + Y^{d-1}Z + YZ^{d-1} + X^2\left(\beta_{d-2,0}Z^{d-2} + \beta_{0,d-2}Y^{d-2}\right) + \sum_{j=2,4,\ldots,d-2,d} \beta_j X^{d-j}(YZ)^{\frac{j}{2}} = 0.$$

Now, $\beta_{d-2,0} = \beta_{0,d-2} = 0$, because $\mathrm{diag}(1, \zeta_{2(d-2)}, \zeta_{2(d-2)}^{d-3}) \in \mathrm{Aut}(F_{\overline{C}})$. Also $\beta_j = 0$, if $2m \nmid j$. To deal with the case when $m > 2$ even, one just need to impose more restrictions on the parameters β_j, appearing in (2.5), to ensure that $\overline{C}$ is also of *Type* $m(d-2),(a,b)$. $\square$

The full automorphism group of $\overline{C} \in \mathcal{M}_g^{Pl}(\mathbb{Z}/m(d-2)\mathbb{Z})$, with an even $m \geq 2$, is described by the next proposition[1]:

Proposition 2.4.30. *Let G be a finite subgroup of $\mathrm{PGL}_3(\overline{k})$, then for an even $m \geq 2$, dividing $d \geq 6$, $\overline{C} \in \mathcal{M}_g^{Pl}(\mathbb{Z}/m(d-2)\mathbb{Z}) \cap \widetilde{\mathcal{M}_g^{Pl}}(G)$ only if one of the following situations occurs:*

1. *$d = 6$ and G is isomorphic to a central extension of S_4 by $\mathbb{Z}/6\mathbb{Z}$. So G is of order 144, and $\widetilde{\mathcal{M}_g^{Pl}}(G)$ is irreducible set of one element, defined by $X^6 + Y^5Z + YZ^5 = 0$.*

2. *$d > 6$ and G is isomorphic to $\langle \sigma, \tau | \tau^2 = \sigma^{d(d-2)} = 1, \tau\sigma\tau = \sigma^{-(d-1)} \rangle$, a central extension of order $2d(d-2)$ of the dihedral group $\mathrm{D}_{2(d-2)}$ of order $2(d-2)$ by $\mathbb{Z}/d\mathbb{Z}$. Again, $\widetilde{\mathcal{M}_g^{Pl}}(G)$ is an irreducible set of one element given by the equation $X^d + Y^{d-1}Z + YZ^{d-1} = 0$.*

3. *G is a central extension of a dihedral group D_{2s} of order $2s$ by a cyclic group N of order, dividing d and also divisible by m, such that $s = \frac{d-2}{2}$ if $4 \nmid d-2$, and $s = d-2$ otherwise. Furthermore, we can think about G as an intransitive subgroup of $\mathrm{PGL}_3(\overline{k})$, whose elements are all of the shape*

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix},$$

[1] The statement rectifies [BB16b, Proposition 36, (3)-(5)] due to some small gaps.

which contains a subgroup isomorphic to

$$\langle \sigma, \tau |\, \tau^2 = \sigma^{m(d-2)} = 1\, \textit{and}\, \tau\sigma\tau = \sigma^{-(m-1)(d-2)-1} \rangle,$$

with $\sigma = \mathrm{diag}(1, \zeta_{m(d-2)}, \zeta_{m(d-2)}^{d-3})$ *and* $\tau = [X : Z : Y]$. *Finally, any element of* $\widetilde{\mathcal{M}_g^{Pl}}(G)$ *has a non-singular plane model through the form* (2.5) *of Proposition 2.4.29 such that* $\beta_j \neq 0$, *for some* $j \in \{1, ..., \lfloor \frac{d}{2m} \rfloor\}$.

Proof. By Proposition 2.4.29, it suffices to study the normal form defined by equation (2.5). If $\beta_j = 0$ for all $j = 1, ..., \lfloor \frac{d}{2\ell} \rfloor$, then the form reduces to $X^d + Y^{d-1}Z + YZ^{d-1} = 0$. The full automorphism group in such case is well-known by Proposition 2.4.10, which proves (1) and (2). Second, suppose that $\beta_j \neq 0$, for some j. We note that the form (2.5) of Proposition 2.4.29 admits always a bigger automorphism group through permuting the variables Y and Z. More precisely, $\langle \sigma, \tau |\, \tau^2 = \sigma^{m(d-2)} = 1,\, and\, \tau\sigma\tau = \sigma^{-(m-1)(d-2)-1} \rangle$ is a subgroup of automorphisms, where $\sigma = \mathrm{diag}(1, \zeta_{m(d-2)}, \zeta_{m(d-2)}^{d-3})$ and $\tau = [X : Z : Y]$. Consequently, $\mathrm{Aut}(\overline{C})$ is not cyclic, since $\langle \eta, \eta' \rangle$ does. Also $F_{\overline{C}}(X, Y, Z) = 0$ is not a descendant of the Klein curve K_d because $|\langle \sigma, \tau \rangle| \nmid 3(d^2 - 3d + 3)$. Moreover, $\mathrm{Aut}(F_{\overline{C}})$ is not conjugate to any of the finite primitive subgroups of $\mathrm{PGL}_3(\overline{k})$, because $m(d-2) \geq 8$ and non of these groups contains elements of order > 7 (in fact, the Klein group $\mathrm{PSL}(2, 7)$ is the only one with elements of order 7). On the other hand, $F_{\overline{C}}(X, Y, Z) = 0$ is not a descendant of the Fermat curve F_d, since $m(d-2) > 2d$, for all even $m > 2$, and automorphisms F_d have orders at most $2d$, also for $m = 2$, $|\langle \sigma, \tau \rangle| = 4(d-2)$ does not divide $|Aut(F_d)| = 6d^2$ (recall that $d \geq 6$ and is even).

We therefore conclude by the above discussion that $\mathrm{Aut}(F_{\overline{C}})$ should fix a line and a point off this line, moreover the fixed point does not belong to $F_{\overline{C}}(X, Y, Z) = 0$ (otherwise, $\mathrm{Aut}(F_{\overline{C}})$ is cyclic). Through the subgroup $\langle \sigma, \tau \rangle$ of automorphisms of $F_{\overline{C}}(X, Y, Z) = 0$, we obtain that the line must be $X = 0$, and the point is $(1 : 0 : 0)$. In particular, all automorphisms of $F_{\overline{C}}(X, Y, Z) = 0$ are of the shape

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix},$$

and we can think about $\mathrm{Aut}(F_{\overline{C}})$ in a short exact sequence

$$1 \to N \to \mathrm{Aut}(F_{\overline{C}}) \to \Lambda(\mathrm{Aut}(F_{\overline{C}})) \to 1$$

with $N = \langle \mathrm{diag}(\zeta_\ell, 1, 1) \rangle$, a cyclic group of order ℓ, dividing d, and $\Lambda(\mathrm{Aut}(F_{\overline{C}}))$ is conjugate to a cyclic group $\mathbb{Z}/s\mathbb{Z}$ of order $s \leq d-1$, a Dihedral group D_{2s} of order $2s$ with $s|(d-2)$ (recall that $\mathrm{diag}(-1,1,1) \in N$), one of the alternating groups A_4, A_5, or to the symmetry S_4. As mentioned earlier, we just need to consider the case when $m = 2$, since $\mathcal{M}_g^{Pl}(\mathbb{Z}/m(d-2)\mathbb{Z}) \subseteq \mathcal{M}_g^{Pl}(\mathbb{Z}/2(d-2)\mathbb{Z})$. Hence $\Lambda(\mathrm{Aut}(\overline{C}))$ contains the elements $\Lambda(\tau) = [Z : Y]$ of order 2 and $\Lambda(\sigma) = \mathrm{diag}(1, \zeta_{2(d-2)}^{d-4})$ of order $n = \frac{d-2}{gcd(d-2,\frac{d-4}{2})}$ (hence $n = \frac{d-2}{2}$ if $4 \nmid d-2$, and $n = d-2$, otherwise). In particular, $\Lambda(\mathrm{Aut}(F_{\overline{C}}))$ always contains a dihedral subgroup of order $d-2 \geq 6$, when $4 \nmid d-2$ and $2(d-2) \geq 8$, when $4|d-2$. Then it is not isomorphic to $\mathbb{Z}/m\mathbb{Z}$ and A_4. Furthermore, for $d \neq 6, 8$ we exclude the group S_4 and, for $d \neq 8, 12$, we also exclude A_5. In this case, i.e, when $d \neq 6, 8, 12$, $\Lambda(\mathrm{Aut}(F_{\overline{C}}))$ is conjugate to D_{2s} where $s = \frac{d-2}{2}$, if $4 \nmid d-2$, and $s = d-2$. That is $|\,\mathrm{Aut}(F_{\overline{C}})| = 2s|N|$ is divisible by $|\langle \sigma, \tau \rangle| = 2m(d-2)$, hence m divides $|N|$.

Finally, we claim to show that $\Lambda(\mathrm{Aut}(F_{\overline{C}}))$ is not conjugate to S_4 and A_5, for $d = 6, 8, 12$: We first mention that our dihedral group D_{2s}, if exists inside S_4 or A_5, forms a single conjugacy class in S_4 and A_5 respectively. In other words, it could only be isomorphic to an D_8 inside S_4, when $d = 6$, D_6 inside S_4 or A_5, when $d = 8$, and to D_{10} inside A_5, when $d = 12$. In all situations, it is unique up to conjugation inside S_4 and A_5, respectively. So, up to change of the variables Y and Z in $\langle \mathrm{diag}(1, \zeta_{2s}),\, [Z : Y] \rangle$, we may consider S_4 and A_5, if happens, to be the same as in Lemma 2.2.1, (d)-(e) of [Hug05]. We refer to [Hug05, Lemma 2.2.1 and Lemma 2.2.3] for the details. Consequently, if the defining equation for some curve $\overline{C}$ contains a monomial $X^{d-2mj}(YZ)^{mj}$ for some j with $d - 2mj > 0$, then all automorphisms of $F_{\overline{C}}(X, Y, Z) = 0$, when restricting on Y and Z are of the shapes $\mathrm{diag}(\lambda, \mu)$ or $[\lambda Z : \mu Y]$. So $\Lambda(\mathrm{Aut}(F_{\overline{C}})) \neq \mathrm{S}_4,\ \mathrm{A}_5$, in this case. Otherwise, by [Hug05, Lemma 6.2.1, (d)-(e)], we always ask for the binary form $\lambda Y^{d-1}Z + \lambda^{d-1}YZ^{d-1} + \lambda^{\frac{d}{2}}(YZ)^{\frac{d}{2}}$, where λ is a $2s$-th root of unity, to be in the ideal generated by

- For $\Lambda(\mathrm{Aut}(F_{\overline{C}})) = \mathrm{S}_4$:

$$Y^{12} - 33Y^8Z^4 - 33Y^4Z^8 + Z^{12},\ Y^8 + 14Y^4Z^4 + Z^8,\ YZ(Y^4 - Z^4),$$

- For $\Lambda(\mathrm{Aut}(F_{\overline{C}})) = \mathrm{A}_5$:

$$YZ(Y^{10} + 11Y^5Z^5 - Z^{10}),$$

$$-(Y^{20} + Z^{20}) + 228(Y^{15}Z^5 - Y^5Z^{15}) - 494Y^{10}Z^{10},$$

$$(Y^{30} + Z^{30}) + 522(Y^{25}Z^5 - Y^5Z^{25}) - 10005(Y^{20}Z^{10} + Y^{20}Z^{10}).$$

For $d = 6$, we have the monomial $(XYZ)^2$, so both groups are already excluded. For $d = 8$ and 12, it is not possible to express the polynomial $\lambda Y^{d-1}Z + \lambda^{d-1}YZ^{d-1} + \lambda^{\frac{d}{2}}(YZ)^{\frac{d}{2}}$ as an element of the ideals above. Hence, both groups can not also occur for $d = 8$ and 12.

This completes the proof. $\square$

As a corollary of Proposition 2.4.30, we have:

Corollary 2.4.31. *The stratum $\widetilde{\mathcal{M}_g^{Pl}}(\mathbb{Z}/m(d-2)\mathbb{Z})$, for any even integer $m \geq 2$ is always empty.*

Now, we handle the situation when $d \equiv 0 \mod m$ and $m > 2$ is odd:

Proposition 2.4.32. *For any odd integer $m > 2$, dividing the degree $d \geq 6$, any $\overline{C} \in \mathcal{M}_g^{Pl}(\mathbb{Z}/m(d-2)\mathbb{Z})$ has a non-singular plane model $F_{\overline{C}}(X,Y,Z) = 0$ over $\overline{k}$ of the form*

$$X^d + Y^{d-1}Z + YZ^{d-1} + \sum_{j=1}^{t} \beta_j X^{d-2mj}(YZ)^{mj} = 0, \tag{2.6}$$

where $t = \frac{d}{2m}$ when d is even, and $t = \lfloor\frac{d-1}{2m}\rfloor$, otherwise. In this case, the stratum $\mathcal{M}_g^{Pl}(\mathbb{Z}/m(d-2)\mathbb{Z})$ is ES-irreducible.

Proof. Similarly, as Proposition 2.4.29, we consider an $F_{\overline{C}}(X,Y,Z) = 0$ to be of *Type* $m(d-2)$, (a,b) of the form Theorem 2.1.3, subcase (4.1), for some $(a,b) \in \Gamma_{m(d-2)}$ such that $m(d-1)|(d-1)a+b,\ a+(d-1)b$. In particular, $2a = (d-2)t_0'+mt_0$ and $2b = (d-2)t_0'-mt_0$, for some integers t_0 and t_0', and we distinguish between whether d is even or odd as follows: If d is even, then so is t_0 and $a = mt + (\frac{d-2}{2})t'$, $b = -mt + (\frac{d-2}{2})t'$, for some integers t and t'. Moreover, m divides $\frac{d}{2}t'$, since $m(d-2)|(d-1)a+b$ and consequently,

$\mathrm{diag}(1, \zeta_{m(d-2)}, \zeta_{m(d-2)}^{(m-1)(d-2)-1})^{mt+(\frac{d-2}{2})t'} = \mathrm{diag}(1, \zeta_{m(d-2)}^{a}, \zeta_{m(d-2)}^{b})$. Therefore, we set $a = 1$ and $b = (m-1)(d-2) - 1$ as a generator of these curves. As usual, it remains to determine the index sets $S_2^{d,X}{}_{m,(a,b)}$ and $S(2)^{j,X}{}_{m,(a,b)}$, for $j = 2, ..., d-1$. In fact, these sets are the same as those of Proposition 2.4.29, and the rest will be typical, except possibly we use the automorphism $\mathrm{diag}(1, \zeta_{m(d-2)}, \zeta_{m(d-2)}^{-(d-1)})$ instead of $\mathrm{diag}(1, \zeta_{m(d-2)}, \zeta_{m(d-2)}^{d-3})$, to obtain the prescribed equation in the statement. If d is odd then t_0 and t_0' have the same parity, thus $a = \frac{mt_0+t_0'(d-2)}{2}$, $b = \frac{-mt_0+t_0'(d-2)}{2}$. Moreover, $2|\pm t_0 + \frac{d}{m}t_0'$, since $m(d-2)|(d-1)a+b,\ a+(d-1)b$, and in particular, we can replace t_0 by $2t - (\frac{d}{m})t_0'$, for some integer t. So $\zeta_{m(d-2)}^{b} = \zeta_{m(d-2)}^{-(d-1)a}$, and $\mathrm{diag}(1, \zeta_{m(d-2)}, \zeta_{m(d-2)}^{-(d-1)})^a = \mathrm{diag}(1, \zeta_{m(d-2)}^{a}, \zeta_{m(d-2)}^{b})$. Hence, we can set again $a = 1$ and $b = (m-1)(d-2) - 1$ as a generator. Finally, the sets $S_2^{d,X}{}_{m,(a,b)}$ and $S(2)^{j,X}{}_{m,(a,b)}$, for $j = 2, ..., d-1$ are given below:

$$\begin{aligned} S_2^{d,X}{}_{m,(a,b)} &:= \{2 \le i \le d-2 \,|\, i + (d-i)((m-1)(d-2)-1) \equiv 0 \bmod m(d-2)\} \\ &\subseteq \{2 \le i \le d-2 \,|\, 2(i-1) \equiv 0 \bmod (d-2)\} \\ &= \emptyset. \end{aligned}$$

The last equality follows because $0 < 2(i-1) < 2(d-2)$, so $2(i-1) = d-2$, which is not possible since d is odd. On the other hand, for $2 \le j \le d-1$

$$\begin{aligned} S(2)^{j,X}{}_{m,(a,b)} &:= \{0 \le i \le j \,|\, i + (j-i)((m-1)(d-2)-1) \equiv 0 \bmod m(d-2)\} \\ &= \{0 \le i \le j \,|\, (d-1)j - di \equiv 0 \bmod m(d-2)\} \\ &\subseteq \{0 \le i \le j \,|\, j - 2i \equiv 0 \bmod (d-2)\} \end{aligned}$$

Because $|j-2i| \le j \le d-1$, then $j - 2i = 0, \pm(d-2)$, and so

$$S(2)^{j,X}{}_{m,(a,b)} \subseteq \begin{cases} \emptyset, & \text{if } j \in \{1, 3, ..., d-4\} \\ \{0, d-2\}, & \text{if } j = d-2 \\ \{\frac{j}{2}\} & \text{otherwise} \end{cases}$$

In particular, we obtain the normal form

$$X^d + Y^{d-1}Z + YZ^{d-1} + X^2\big(\alpha Z^{d-2} + \beta Y^{d-2}\big) + \sum_{j=2,4,...,d-1} \beta_j X^{d-j}(YZ)^{\frac{j}{2}} = 0,$$

for which we need to impose more the condition $\mathrm{diag}(1, \zeta_{m(d-2)}, \zeta_{m(d-2)}^{-(d-1)}) \in \mathrm{Aut}(\overline{C})$. Then $\alpha = \beta = 0$, and moreover $\beta_j = 0$ when $m \nmid \frac{j}{2}$. Lastly, rename j to be $2mj$ in order to get the mentioned defining equation.

This completes the proof. $\square$

The full automorphism groups of the elements inside $\mathcal{M}_g^{Pl}(\mathbb{Z}/m(d-2)\mathbb{Z})$, for an odd $m > 2$, that divides $d \geq 6$ are also investigated[2]:

Proposition 2.4.33. *Let G be a finite subgroup of $\mathrm{PGL}_3(\overline{k})$, then for an odd $m > 2$, dividing $d \geq 6$, $\overline{C} \in \mathcal{M}_g^{Pl}(\mathbb{Z}/m(d-2)\mathbb{Z}) \cap \widetilde{\mathcal{M}_g^{Pl}}(G)$ only if one of the following situations occurs:*

1. *$d = 6$ and G is isomorphic to a central extension of S_4 by $\mathbb{Z}/6\mathbb{Z}$. So G is of order 144, and $\widetilde{\mathcal{M}_g^{Pl}}(G)$ is irreducible set of one element, defined by $X^6 + Y^5Z + YZ^5 = 0$.*

2. *$d > 6$ and G is isomorphic to $\langle \sigma, \tau | \tau^2 = \sigma^{d(d-2)} = 1, \tau\sigma\tau = \sigma^{-(d-1)} \rangle$, a central extension of order $2d(d-2)$ of the dihedral group $\mathrm{D}_{2(d-2)}$ of order $2(d-2)$ by $\mathbb{Z}/d\mathbb{Z}$. Again, $\widetilde{\mathcal{M}_g^{Pl}}(G)$ is irreducible set of one element given by the equation $X^d + Y^{d-1}Z + YZ^{d-1} = 0$.*

3. *$m \neq 3$ or $d > 6$, and G is a central extension of D_{2s} by a cyclic group N of order dividing d and also divisible by m, such that $s = \frac{d-2}{2}$ if $4 \nmid d-2$, and $s = d-2$ otherwise. Furthermore, we can think about G as an intransitive subgroup of $\mathrm{PGL}_3(\overline{k})$, whose elements are all of the shape*

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix},$$

which contains a subgroup isomorphic to

$$\langle \sigma, \tau |\, \tau^2 = \sigma^{m(d-2)} = 1 \,\mathit{and}\, \tau\sigma\tau = \sigma^{-(m-1)(d-2)-1} \rangle,$$

[2]The statement is a refined version of [BB16b, Proposition 39]. The proof is even simpler, especially when we treat the case $m = 3$ and $d = 6$, such that $\overline{C}$ is a descendant of the Fermat curve.

with $\sigma = \mathrm{diag}(1, \zeta_{m(d-2)}, \zeta_{m(d-2)}^{d-3})$ *and* $\tau = [X : Z : Y]$. *Finally, any element of* $\widetilde{\mathcal{M}_g^{Pl}}(G)$ *has a non-singular plane model through the form* (2.6) *of Proposition* 2.4.32 *such that* $\beta_j \neq 0$*, for some* $j \in \{1, ..., t\}$.

Proof. One applies the first part of our argument for Proposition 2.4.30, to conclude the following:

- Case (1) and (2) are equivalent to the situation when $\beta_j = 0$ for all $j \in \{1, 2, ..., t\}$.
- The normal form (2.6), always admits a bigger automorphism group $\langle\sigma, \tau\rangle$ of order $2m(d-2)$, generated by $\sigma := \mathrm{diag}(1, \zeta_{m(d-2)}, \zeta_{m(d-2)}^{d-3})$ and $\tau := [X : Z : Y]$. In particular, $\mathrm{Aut}(\overline{C})$ is not cyclic.
- $\overline{C}$ is not a descendant of the the Klein curve and also $\mathrm{Aut}(\overline{C})$ is not conjugate to any of the finite primitive groups inside $\mathrm{PGL}_3(\overline{k})$.
- If $m \neq 3$ or $d > 6$, then $\overline{C}$ is not a descendant of the Fermat curve F_d, as well.

Therefore, when $m \neq 3$ or $d \neq 6$, we can think again about $\mathrm{Aut}(F_{\overline{C}})$ in a short exact sequence $1 \to N \to \mathrm{Aut}(F_{\overline{C}}) \to \Lambda(\mathrm{Aut}(F_{\overline{C}})) \to 1$, where $\Lambda(\mathrm{Aut}(F_{\overline{C}}))$ contains again $\Lambda(\tau) = [Z : Y]$ and $\Lambda(\sigma) = \mathrm{diag}(1, \zeta_{m(d-3)}^{d-4})$. So we follow the same line of discussion in order to deduce (3) of Proposition 2.4.33.

Finally, we treat the case when $m = 3$ and $d = 6$, and $F_{\overline{C}}(X, Y, Z) = 0$ is a descendant of the Fermat curve $F_6 : X^6 + Y^6 + Z^6 = 0$, through a projective linear transformation $\phi \in \mathrm{PGL}_3(\overline{k})$: The normal form reduces to $\overline{C} : X^6 + Y^5Z + YZ^5 + \beta Y^3Z^3 = 0$. Recall that $\sigma^4 = \mathrm{diag}(1, \zeta_3, 1)$ is an automorphism for $F_{\overline{C}}(X, Y, Z) = 0$ of order 3, which is a homology. It is also know that homologies of order 3 in $\mathrm{Aut}(F_6)$ forms two conjugacy classes represented by σ^4 and σ^8 respectively. Moreover, both σ^4 and σ^8 lies in a different conjugacy classes in $\mathrm{PGL}_3(\overline{k})$. Therefore, we may assume $\phi^{-1}\sigma^4\phi = \sigma^4$, thus $\phi = [X : \mu_2Y + \mu_3Z : \gamma_2Y + \gamma_3Z]$ and $F_{\overline{C}}(X, Y, Z) = 0$ is transformed to $F_{\phi^{-1}\overline{C}}(X, Y, Z) = 0$ of the form

$$X^6 + \nu_0Y^6 + \nu_1Z^6 + G(Y, Z),$$

where $\nu_0 := \gamma_2\mu_2\,(\gamma_2^4 + \beta\mu_2^2\gamma_2^2 + \mu_2^4) = 1$ and $\nu_1 := \gamma_3\mu_3\,(\gamma_3^4 + \beta\mu_3^2\gamma_3^2 + \mu_3^4) = 1$. In particular, $(\gamma_2\mu_2)(\gamma_3\mu_3) \neq 0$ and $[\zeta_6^bY : \zeta_6^aX : Z]$, $[\zeta_6^bZ : Y : \zeta_6^aX] \notin \mathrm{Aut}(\overline{C}_P)$. However,

$|\operatorname{Aut}(F_{\phi^{-1}\overline{C}})| > 3$ and $\sigma^4 \in \operatorname{Aut}(F_{\phi^{-1}\overline{C}})$, so it follows by the next observation (see [Har13, Proposition 3.3 and Lemma 6.5]), that $F_{\phi^{-1}\overline{C}}$ is the Fermat curve itself, a contradiction.

Observation 3: Let $(\overline{C}, G)$ be a descendant of the Fermat curve F_d of degree $d \geq 4$, where $G = \operatorname{Aut}(\overline{C})$. Then there exists a commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z} \arrow[r] & \operatorname{Aut}(F_d) \arrow[r, "\varrho"] & \mathrm{S}_3 \arrow[r] & 1 \\ 1 \arrow[r] & \operatorname{Ker}(\varrho|_G) \arrow[r] \arrow[u, hook] & G \arrow[r] \arrow[u, hook] & \operatorname{Im}(\varrho|_G) \arrow[r] \arrow[u, hook] & 1 \end{tikzcd}$$

Denote by η_1, η_2 and η_3 the three automorphisms $\operatorname{diag}(\zeta_d, 1, 1)$, $\operatorname{diag}(1, \zeta_d, 1)$ and $\operatorname{diag}(1, 1, \zeta_d)$ respectively. Hence, if G contains two of the three $\eta_i's$, then it contains the other and $\overline{C}$ is projectively equivalent to the Fermat curve F_d. If $|\operatorname{Im}(\varrho|_G)| \geq 3$ and G contains one of the $\eta_i's$, then it contains all of them and again $\overline{C}$ is $\overline{k}$-projectively equivalent to F_d.

Proof. Using a quite similar argument like the one made for the Klein curve (Proposition 2.4.12), one shows that $\operatorname{Aut}(F_d)$ is a semidirect product of $\mathrm{S}_3 = \langle [Y:Z:X], [X:Z:Y] \rangle$ acting on $\mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z} = \langle \operatorname{diag}(\zeta_d, 1, 1), \operatorname{diag}(1, \zeta_d, 1) \rangle$ (or see [Har13, Proposition 3.3] for the complete details). In particular, $\operatorname{Aut}(F_d)$ lives in a short exact sequence of the form $1 \to \mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z} \to \operatorname{Aut}(F_d) \to \mathrm{S}_3 \to 1$, from which the mentioned diagram comes from. On the other hand, it is obvious that any two of the $\eta_i's$ generate the third one. Therefore, if G contains at least two of them, then it must have the other one, and moreover, if $F_{\overline{C}}(X, Y, Z) = 0$ is a plane model of $\overline{C}$ over $\overline{k}$ whose core is $X^d + Y^d + Z^d$, then it must be invariant under the action of η_i, for $i = 1, 2, 3$. This only can happen when $F_{\overline{C}}(X, Y, Z) = 0$ is the Fermat curve itself. Lastly, if $|\operatorname{Im}(\varrho|_G)| \geq 3$ and G contains one of the $\eta_i's$ say η_1, then $\operatorname{Im}(\varrho|_G)$ is isomorphic to either $\mathbb{Z}/3\mathbb{Z}$ or S_3, and thus G must contain an element of order 3. We may assume it to be of the shape $[\zeta_d^a Y : \zeta_d^b Z : X]$, for some integers $0 \leq a, b < d$ Thus $\eta_1^{-a}[\zeta_d^a Y : \zeta_d^b Z : X] = [Y : \zeta_d^b Z : X] \in G$, which in turns gives $\eta_1^{-b}[Y : \zeta_d^b Z : X]^2 \eta_1^{-b} = [Z : X : Y] \in G$. In particular, $[Z : X : Y]\eta_1[Z : X : Y] = \eta_2 \in G$, and $\overline{C}$ is again $\overline{k}$-projectively equivalent to the Fermat curve F_d. □

This finishes the proof. □

3

CHAPTER

Fields of definition of non-singular plane models of smooth curves

Given a smooth curve C defined over a field k that admits a non-singular plane model of degree $d \geq 4$ over k^{sep}, a fixed separable algebraic closure of k, it does not necessarily have a non-singular plane model defined over the field k. We determine under which conditions this happens and we show an example of such phenomenon: a curve defined over k admitting plane models but none defined over k. Now, even assuming that such a smooth plane model exists, we wonder about the existence of non-singular plane models over k for its twists:

Definition 3.1. *Let V be a smooth quasi-projective variety over k. A variety V' defined over k is called a twist of V over k if there is a k^{sep}-isomorphism*

$$\phi : \overline{V'} := V' \otimes_k k^{\text{sep}} \to \overline{V} := V \otimes_k k^{\text{sep}}.$$

A twist V' is called trivial if V and V' are k-isomorphic. The set of all twists of V modulo k-isomorphisms is denoted by $\text{Twist}_k(V)$.

Example 3.2 (Example 2, [MT10])**.** *Consider the Fermat quartic curve*

$$C : X^4 + Y^4 + Z^4 = 0.$$

Over the field $k = \mathbb{F}_{13}$, it has 32 points, while the curve

$$C' : X^4 + 4Y^4 - X^2Y^2 + 7Z^4 = 0$$

has 8 points. Therefore, C and C' are not $\mathbb{F}_{13}$-isomorphic. However, they do over $\mathbb{F}_{13}(\alpha)$ where

$\alpha \in \overline{\mathbb{F}}_{13}$ such that $\alpha^2 = 2$, through the isomorphism

$$\phi : (X : Y : Z) \mapsto (X + \alpha Y : X - \alpha Y : Z).$$

That is C' is a twist of C over $\mathbb{F}_{13}$, which is non-trivial.

We characterize twists possessing such models and we also show an example of a twist not admitting any non-singular plane model over k. As a consequence, we get explicit equations for a non-trivial Brauer-Severi surface. Finally, we obtain a theoretical result to describe all the twists of smooth plane curves with cyclic automorphism group having a model defined over k whose automorphism group is generated by a diagonal matrix.

The structure of this chapter is as follows. In section 3.1, we collect the most necessary results, known in the literature, about central simple algebras (or briefly CSA's), and the connection with Brauer-Severi varieties, which will be used in this chapter. For more details, we refer, for example, to [GS06, Jah].

Definition 3.3. *By a smooth k^{sep}-plane curve C over k, we mean a smooth curve over k admitting a non-singular plane model $F_{\overline{C}}(X,Y,Z) = 0$ over k^{sep} of degree $d \geq 4$. We say that C is a smooth plane curve over k if C as a smooth curve defined over k is also k-isomorphic to a non-singular plane model $F(X,Y,Z) = 0$ in $\mathbb{P}^2_k$.*

Section 3.2 is devoted to the study of the minimal field L where there exists a non-singular model over L for a smooth k^{sep}-plane curve C defined over k, i.e. that C is L-isomorphic to $F_{Q^{-1}\overline{C}}(X,Y,Z) = 0$ for some $Q \in \mathrm{PGL}_3(k^{\text{sep}})$ with $F_{Q^{-1}\overline{C}} \in L[X,Y,Z]$. We prove that if the degree of a non-singular k^{sep}-plane model of C is coprime with 3, or C has a k-rational point or the 3-torsion of the Brauer group of k is trivial (in particular, if k is a finite field), then the curve C is a smooth plane over k (i.e. admits a k-model): Theorem 3.2.8 and Corollaries 3.2.1, 3.2.2. Moreover, we prove that a smooth plane model of C always exists in a finite extension of k of degree dividing 3, see Theorem 3.2.4. Section 3.2 ends with an explicit example of a smooth $\overline{\mathbb{Q}}$-plane curve over $\mathbb{Q}$ which is not a smooth plane curve over $\mathbb{Q}$; however, we construct a smooth plane model over a degree 3 extension of $\mathbb{Q}$. In Section 3.3, we assume that C is a smooth plane curve over k. We obtain Theorem 3.3.2 characterizing the twists of C which are also smooth plane curves over k. Moreover, we construct a family of examples over $k = \mathbb{Q}$ for which a twist of C does not admit a non-singular plane model over $\mathbb{Q}$. This construction is not explicit because we do not provide equations of such twists. Section 3.4 details an explicit example of a smooth $\overline{\mathbb{Q}(\zeta_3)}$-plane curve over $\mathbb{Q}(\zeta_3)$ having a twist that does not possess such a model in the field $\mathbb{Q}(\zeta_3)$, where ζ_3 is a primitive 3rd root of unity. Interestingly, we find the already mentioned explicit equations for a non-trivial Brauer-Severi variety. In Section 3.5, we study the twists for smooth plane curve C over k, such that $\mathrm{Aut}(\overline{C})$ is a cyclic group. We prove that if $\mathrm{Aut}(F_{P^{-1}\overline{C}})$ is represented in $\mathrm{PGL}_3(k^{\mathrm{sep}})$ by a diagonal matrix, (where $F_{P^{-1}\overline{C}}(X,Y,Z)$ is k-isomorphic to C) then all the twists are diagonal, i.e. of the form $F_{(PD)^{-1}\overline{C}}(X,Y,Z) = 0$ with D a diagonal matrix, Theorem 3.5.2. We apply this result to some special families of curves, see Corollary 3.5.4. We also construct an example of a curve C that being $\mathrm{Aut}(F_{P^{-1}\overline{C}})$ cyclic (but not diagonal) has all the twists not diagonal.

We shall deal with the following items:

3.1. Brauer-Severi varieties and Central simple algebras.

3.2. The field of definition of a non-singular plane model.

3.3. On twists of plane models defined over k.

3.4. An explicit non-trivial Brauer-Severi variety.

3.5. Twists of smooth plane curves with diagonal cyclic automorphism group.

The main results in this chapter are resulted into the arXiv preprint [BBLG16].

§3.1 Brauer-Severi varieties and Central simple algebras

We aim to collect the most necessary results, known in the literature, about central simple algebras (CSA's), and the connection with Brauer-Severi varieties, which will be used in this chapter. For more details, we refer, for example, to [GS06, Jah].

The connection between CSA and Brauer-Severi varieties was first observed by E. Witt in [Wit35] and H. Hasse, in the particular case of quaternion algebras and plane conics. To that connection in its general form there are several approaches; the most elementary one was promoted by J.-P. Serre in his books Corps locaux [Ser68, X, §5 and §6] and Cohomologie

Galoisienne [Ser94, Remarque III.1.3.1]. This approach is based on non-abelian group cohomology. The main observation is that CSA's of dimension n^2 over a field k as well as $(n-1)$-dimensional Brauer-Severi varieties over k can both be described by classes in one and the same cohomology set $\mathrm{H}^1(\mathrm{Gal}(k^{\mathrm{sep}}/k), \mathrm{PGL}_n(k^{\mathrm{sep}}))$, see Definition 3.1.18.

Definition 3.1.1. A central simple algebra (or simply CSA) over a field k is a finite dimensional associative algebra over k, which is simple, i.e contains no non-trivial (two sided) ideal and the multiplication operation is not uniformly zero , and for which the center is exactly k.

Remark 3.1.2. Any simple algebra can be viewed as a CSA over its center. A division algebra is a CSA such that all non-zero elements are invertible, see [GS06, Example 2.1.1].

Theorem 3.1.3 (Wedderburn)**.** *Given a CSA A over k, there exists a division ring D over k and a positive integer n, such that A is isomorphic to $M_n(D)$. Moreover the division algebra D is unique up to isomorphism.*

Corollary 3.1.4. *The dimension of any CSA over k is always a square.*

Definition 3.1.5. The degree of a CSA, A over k is defined to be the square root of $\dim_k(A)$.

Definition 3.1.6. Let A be a CSA over k, a field extension L/k is said to be a splitting field of A if $A_L := A \otimes_k L \cong M_n(L)$ for some n, and we say that L/k splits A. For any CSA A, there exists a finite Galois extension $k \subset L$ that splits A.

Example 3.1.7 (Example 4.2, [Ten09])**.** *$L = \overline{k}$ is always a splitting field of any CSA A over a field k.*

Theorem 3.1.8 (Theorem 4.4, [Ten09])**.** *Let A be a CSA over k of degree n. If L/k is a field extension of k of index n that is contained in A, then L splits A.*

Wedderburn's Theorem gives a strict relation between central simple algebras and division algebras, and suggests the introduction of the following relation: Two central simple algebras A and B over the same field k are equivalent if there are positive integers m, n such that $M_m(A) \cong M_n(B)$. Equivalently, A and B are equivalent if A and B are matrix algebras over the same division algebra. The equivalence class of the central simple algebra A over k is denoted by by $[A]$, and is called a Brauer class.

Definition 3.1.9. Let L/k be a finite Galois extension, the set of all Brauer equivalence classes of central simple algebras over k, which split by L, is denoted by $\mathrm{Br}(L/k)$, and is called the Brauer group of k relative to L. While, the union of the sets $\mathrm{Br}(L/k)$, for all finite Galois extension is denoted by $\mathrm{Br}(k)$, and is called the Brauer group of k.

By Remarks 2.4.7 in [GS06], we know that $\mathrm{Br}(L/k)$ classifies division algebras splits by L, up to isomorphism, since each Brauer equivalence class contains a unique (up to isomorphism) division algebra. So, by Wedderburn's Theorem, if A and B are two Brauer equivalent k-algebras of the same dimension, then $A \cong B$. Moreover, the sets $\mathrm{Br}(L/k)$ and $\mathrm{Br}(k)$, equipped with the tensor product of k-algebras, are abelian groups, see Proposition 2.4.8 in [GS06].

Example 3.1.10. *In the following cases, every division algebra over a field k is k itself, so that the Brauer group $\mathrm{Br}(k)$ is trivial:*

(i) k is an algebraically closed field (Example 3.1.7).

(ii) k is a finite field (Wedderburn's Little theorem), see [Ser79, page 162].

(iii) k is the function field of an algebraic curve over an algebraically closed field (Tsen's Theorem, see [GS06, Theorem 6.2.8]). More generally, the Brauer group vanishes for any quasi-algebraically closed field.

(iv) k is an algebraic extension of $\mathbb{Q}$, containing all roots of unity, see [Ser79, page 162].

Example 3.1.11. *The Brauer group $\mathrm{Br}(\mathbb{R})$ is the cyclic group of order two. There are just two non-isomorphic real division algebras with center $\mathbb{R}$: $\mathbb{R}$ itself and the quaternion algebra $\mathbb{H}$. Since $\mathbb{H} \otimes \mathbb{H} \cong M_4(\mathbb{R})$, the class of $\mathbb{H}$ has order two in the Brauer group.*

Example 3.1.12. *Let k be a local field, meaning that k is complete under a discrete valuation with finite residue field. Then $\mathrm{Br}(k)$ is isomorphic to $\mathbb{Q}/\mathbb{Z}$, see [Ser79, page 193].*

Definition 3.1.13. The n-torsion $\mathrm{Br}(k)[n]$ of the Brauer group $\mathrm{Br}(k)$ is the set of all elements of $\mathrm{Br}(k)$ of order, at most n.

Definition 3.1.14. The period of a CSA A over k is defined to be its order as an element of the Brauer group $\mathrm{Br}(k)$. Define the index of A to be the degree of the division algebra that is Brauer equivalent to A.

We particularly have:

Corollary 3.1.15 (e.g. [dJ04]). *The period of a CSA over k divides its index, and hence is finite.*

For any natural number $n \in \mathbb{N}$, we consider the set Az_n^k of all CSA's of dimension n^2 over k, modulo k-algebras isomorphism. Similarly, one constructs the set $\mathrm{Az}_n^{L/k}$ of all isomorphism classes of CSA's of dimension n^2 over k, which splits by L, for any field extension L/k. In particular,

$$\mathrm{Az}_n^k = \bigcup_{L/k} \mathrm{Az}_n^{L/k}\,.$$

Theorem 3.1.16. *For $n \in \mathbb{N}$, and a finite Galois extension L/k, there is a natural bijection of pointed sets*

$$a_n^{L/k} :\ \mathrm{Az}_n^{L/k} \leftrightarrow \mathrm{H}^1(\mathrm{Gal}(L/k), \mathrm{PGL}_n(L)).$$

Moreover, this is inflated to a unique natural bijection

$$a_n^k :\ \mathrm{Az}_n^k \leftrightarrow \mathrm{H}^1(\mathrm{Gal}(k^{\mathrm{sep}}/k), \mathrm{PGL}_n(k^{\mathrm{sep}})),$$

such that $a^k_{n|_{\mathrm{Az}_n^{K/k}}} = a_n^{K/k}$, for each finite Galois extension K/k inside k^{sep}.

The bijections $a_n^{L/k}$ are defined in the following way: Given a CSA $A \in \mathrm{Az}_n^{L/k}$ and an isomorphism $\phi : A \otimes_k L \to M_n(L)$, the class of A is mapped to the class of the 1-cocylce

$$f(\sigma) := \phi \circ \sigma \circ \phi^{-1} \circ \sigma^{-1} \in \mathrm{Aut}_L(M_n(L)) = \mathrm{PGL}_n(L), \text{ for } \sigma \in \mathrm{Gal}(L/k).$$

In particular, one gets the following commutative diagram

$$\begin{tikzcd}
A \otimes_k L \arrow[r, "\phi"] & M_n(L) \\
A \otimes_k L \arrow[r, "\phi"] \arrow[u, "\sigma"] & M_n(L) \arrow[u, "f(\sigma)\circ\sigma"']
\end{tikzcd}$$

The inverse map associates to each 1-cocycle $f \in Z^1(\mathrm{Gal}(L/K), \mathrm{PGL}_n(L))$, the k-subalgebra of $M_n(L)$ given by

$$\{M \in M_n(L) \,|\, f(\sigma) \circ {}^{\sigma}M = M \text{ for all } \sigma \in \mathrm{Gal}(L/k)\}.$$

Definition 3.1.17. (G-sets) Let G be a finite group. A G-set E is a set equipped with a G-operation from the left. We will use the notation ${}^g x := g \cdot x$ for $x \in E$ and $g \in G$. A morphism of G-sets, or simply a G-morphism, is a map $\iota : E \to F$ of G-sets such that the diagram

$$\begin{tikzcd} G \times F \arrow[r, "\cdot"] & F \\ G \times_k E \arrow[u, "id\times\iota"] \arrow[r, "\cdot"] & E \arrow[u, "\iota"'] \end{tikzcd}$$

commutes.

Given a G-set E, one puts $\mathrm{H}^0(G, E) := E^G$, i.e. the *zeroth cohomology set of G with coefficients in E* is just the subset of G-invariants in E. If E is a G-group then $\mathrm{H}^0(G, E)$ is a group.

Definition 3.1.18. If A is a G-group then a cocycle from G to A is a map $a : g \in G \mapsto a_g \in A$, such that $a_{gg'} = a_g \cdot {}^g a_{g'}$ for each $g, g' \in G$. Two cocycles a, a' are *cohomologous* if there exists some $b \in A$ where $a'_g = b^{-1} \cdot a_g \cdot {}^g b$ for every $g \in G$. This is an equivalence relation and the quotient set, the first cohomology set of G with coefficients in A, is denoted by $\mathrm{H}^1(G, A)$. This is a pointed set as the map $g \mapsto e$ defines a cocycle, the so-called *trivial cocycle*.

Definition 3.1.19. (The inflation map) Let $h : G' \to G$ be a homomorphism of finite groups. Then, for an arbitrary G-set E, one has a natural pull-back map $h^* : \mathrm{H}^0(G, E) \to \mathrm{H}^0(G', E)$. If E is a G-group then the pull-back map is a group homomorphism. For an arbitrary G-group A there is the natural pull-back map $h^* : \mathrm{H}^1(G, A) \to \mathrm{H}^1(G', A)$, which is a morphism of pointed sets[1]. If h is the canonical projection on a quotient group then $\inf_G^{G'} := h^*$ is called the *inflation map*. The composition of $\inf_G^{G'}$ with some extension of the G'-set E (the G'-group A) is usually called the inflation, as well. We simply use the notation $\inf$ if G and G' are well understood.

Remark 3.1.20. Non-abelian group cohomology $\mathrm{H}^i(G, A)$, for $i = 1, 2$, can be extended to the case where G is a profinite group and A is a discrete G-set (respectively G-group) on which G

[1]The image $h^*(a)$ of $a \in \mathrm{H}^1(G, A)$ is defined by $g' \in G' \mapsto a_{h(g')}$

operates continuously. Put

$$\mathrm{H}^i(G, A) := \varinjlim_{G'} \mathrm{H}^i(G/G', A^{G'})$$

where the direct limit is taken over the inflation maps and G' runs through the normal open subgroups G' of G such that the quotient G/G' is finite.

The bijection a_n^k, in Theorem 3.1.16 above, is then obtained from the bijections $a_n^{L/k}$ using the inflation map. We address the reader to Theorem 3.6 and Corollary 3.8 in [Jah] and Theorem 5.4 in [Ten09], for the complete details.

3.1.1 Cyclic algebras

Definition 3.1.21. Let L/k be a cyclic extension of degree n, that is, A Galois cyclic field extension of k. Fix a character χ : Gal$(L/k) \xrightarrow{\simeq} \mathbb{Z}/n\mathbb{Z}$, i.e. choose a specific generator σ of $\mathrm{Gal}(L/k)$ characterized by $\chi(\sigma) = \overline{1}$. Given $a \in k^*$, we consider a k-algebra (χ, a) as follows: As an additive group, (χ, a) is an n-dimensional vector space over L with basis $1, e, \ldots, e^{n-1}$:

$$(\chi, a) := \bigoplus_{0 \leq i < n} Le^i.$$

Multiplication is given by the relations: $e \,.\, \lambda = \sigma(\lambda) \,.\, e$ for $\lambda \in L$, and $e^n = a$.

Some computations shows that (χ, a) becomes a CSA of dimension n^2 over k. Moreover, by the proof of Theorem 2.2 in [Ten09], the map $\phi : (\chi, a) \otimes_k L \to M_n(L)$ defined by

$$\phi(e \otimes 1) := \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ & & & \vdots & & \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ a & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}$$

and

$$\phi(\lambda \otimes 1) := \mathrm{diag}(\lambda, \sigma(\lambda), ..., \sigma^{n-1}(\lambda)), \text{ for } \lambda \in L$$

is an isomorphism of L-algebras. That is (χ, a) splits by L. It is called the *cyclic algebra* associated to the character χ and the element $a \in k$.

Example 3.1.22 (Example 5.5, [Ten09])**.** *Let L/k be a cyclic extension of degree n with $\mathrm{Gal}(L/k) = \langle\sigma\rangle$. Then an element of $\mathrm{H}^1(\mathrm{Gal}(L/k), \mathrm{PGL}_n(L))$ represented by a 1-cocycle $f : \mathrm{Gal}(L/k) \to \mathrm{PGL}_n(L)$ is completely determined by the value of $f(\sigma)$, which is subject to*

$$f(\sigma) \cdot {}^{\sigma}(f(\sigma)) \cdot {}^{\sigma^2}(f(\sigma)) \cdot \ldots \cdot {}^{\sigma^{n-1}}(f(\sigma)) = 1. \tag{3.1}$$

For instance, let $a \in k^*$ and consider the matrix

$$C_a := \begin{pmatrix} 0 & 0 & \ldots & 0 & a \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix}.$$

Define a 1-cocycle f by setting $f(\sigma) = C_a \bmod L^*$. Hence

$$f(\sigma) \cdot {}^{\sigma}(f(\sigma)) \cdot {}^{\sigma^2}(f(\sigma)) \cdot \ldots \cdot {}^{\sigma^{n-1}}(f(\sigma)) = C_a^n \mod L^* = aI \mod L^* = I \mod L^*,$$

where I is the identity matrix. That is, the condition (3.1) is indeed verified. Moreover, according to Theorem 3.1.16, the k-algebra corresponds to this 1-cocycle is the set of matrices of $M \in M_n(L)$ satisfying $f(\sigma^i) \circ {}^{\sigma^i}M = M \Leftrightarrow C_a^i\, {}^{\sigma^i}M\, C_a^{-i} = M$ for all i, which amounts to $C_a\, {}^{\sigma}M\, C_a^{-1} = M$. Clearly, $I, C_a, ..., C_a^{n-1}$ satisfy the latter identity. Moreover, since conjugation by C_a is "almost a cyclic permutation", it is not difficult to verify that the matrices

$$S_b := \mathrm{diag}(b, \sigma(b), ..., \sigma^{n-1}b), \text{ for } b \in L$$

also satisfy the identity as well. That is, $S_bC_a = C_a\, {}^{\sigma}S_b = C_aS_{\sigma(b)}$, or equivalently, $S_{\sigma^{-1}(b)}C_a = C_aS_b$, for all $b \in L$. Therefore, $A := \bigoplus_{0\le i<n} S_bC_a^i$ is a k-subalgebra of the correct dimension n^2, hence it must be the algebra defined by the 1-cocycle f above. Obviously, A is isomorphic to (χ, a), the cyclic algebra given by a and the character $\chi(\sigma) = -1 \bmod n$.

Theorem 3.1.23 (Wedderburn, Theorem III, [Wed21])**.** *The elements of Az_3^k are given by cyclic algebras of the form (χ, a) with $n = 3$. In particular, each of them splits by a cyclic cubic* *extension L of k.*

Remark 3.1.24. Using the inflation map in Galois cohomology ([Jah, Lemma 3.7]), one deduces a bijection can be given by

$$(\chi, a) \in \mathrm{Az}_3^k \mapsto \inf(f : \sigma \mapsto C_a) \in \mathrm{H}^1(\mathrm{Gal}(k^{\mathrm{sep}}/k), \mathrm{PGL}_3(k^{\mathrm{sep}})).$$

Proposition 3.1.25 (e.g. §2.1, [Han07])**.** *We have $(\chi, a) \in \mathrm{Az}_3^k$ is the trivial CSA if and only if a is the norm of an element of L.*

3.1.2 Brauer-Severi varieties

Definition 3.1.26. A Brauer-Severi variety D over k of dimension n is a smooth projective variety, such that the base extension $\overline{D} = D \otimes_k k^{\mathrm{sep}}$ is k^{sep}-isomorphic to the n-dimensional projective space $\mathbb{P}^n_{k^{\mathrm{sep}}}$ over k^{sep}. In other words, it is a twist of $\mathbb{P}^n_k$ over k (Defintion 3.1).

The set of all isomorphism classes of Brauer-Severi varieties of dimension n over k is denoted by BS_n^k.

The next result is [Jah, Corollary 4.7] (see also section §3.3 for the general statement for quasi-projective varieties):

Corollary 3.1.27. *The set BS_n^k is in bijection with $\mathrm{H}^1(\mathrm{Gal}(k^{\mathrm{sep}}/k), \mathrm{PGL}_{n+1}(k^{\mathrm{sep}}))$.*

Remark 3.1.28. By Corollary 3.1.15, Example 3.1.22, Theroem 3.1.23 and Corollary 3.1.27, one deduces that a Brauer-Severi surface corresponds to a CSA of dimension 9 and period dividing 3. Hence, to an element of $\mathrm{Br}(k)[3]$.

Moreover, we have by F. Severi, cf. J.-P. Serre [Ser68, X, §6, Excercise 1]:

Proposition 3.1.29 (Severi)**.** *A Brauer-Severi variety of dimension n over k, with a k-rational point is isomorphic over k to $\mathbb{P}^n_k$, i.e. it is a trivial twist of $\mathbb{P}^n_k$.*

J. Roé and X. Xarles in [RX14, Corollary 6] proved the following result:

Theorem 3.1.30 (Roé-Xarles)**.** *Let C be a smooth k^{sep}-plane curve defined over k of degree $d \geq 4$, and let $\Upsilon : \overline{C} \hookrightarrow \mathbb{P}^2_{k^{\mathrm{sep}}}$ be a morphism given by (the unique) g^2_d-linear system over k^{sep}.*

Then there exists a Brauer-Severi variety D (of dimension two) defined over k, together with a k-morphism $g : C \hookrightarrow D$, such that $g \otimes_k k^{\text{sep}} : \overline{C} \to \mathbb{P}^2_{k^{\text{sep}}}$ is equal to Υ.

The idea of the proof of Theorem 3.1.30, that is also used in the next section (§3.2.1), is: A k^{sep}-plane model of the curve C defines a 1-cocycle $f \in \mathrm{H}^1(\mathrm{Gal}(k^{\text{sep}}/k), \mathrm{PGL}_3(k^{\text{sep}}))$ by the g^2_d-linear series over k^{sep}. Therefore, the corresponding twist $\iota_f : \mathbb{P}^2_{k^{\text{sep}}} \to D \otimes_k k^{\text{sep}}$, maps the k^{sep}-plane model of C into a smooth curve defined over k, which lives inside the Brauer-Severi variety D over k.

§3.2 The field of definition of a non-singular plane model

This section is devoted to the study of the minimal field L where there exists a non-singular model over L for a smooth k^{sep}-plane curve C defined over k, i.e. that C is L-isomorphic to $F_{Q^{-1}\overline{C}}(X, Y, Z) = 0$ for some $Q \in \mathrm{PGL}_3(k^{\text{sep}})$ with $F_{Q^{-1}\overline{C}} \in L[X, Y, Z]$.

One deduces some remarkable consequences from Theorem 3.1.30:

Corollary 3.2.1. *Let C be a smooth k^{sep}-plane curve over k. Assume that C has a k-rational point, i.e. $C(k)$ is not-empty. Then C admits a non-singular plane model over k.*

Proof. By Proposition 3.1.29, a Brauer-Severi variety over k of dimension n with a k-rational point is isomorphic over k to $\mathbb{P}^n_k$. Therefore, by Theorem 3.1.30, the map $g : C/k \to D \cong \mathbb{P}^2_k$, which is defined over k, gives a non-singular plane model for C over k. □

Corollary 3.2.2. *Let k be a field for which $\mathrm{Br}(k)[3]$ is trivial. Hence, any smooth k^{sep}-plane curve C over k admits a non-singular plane model over k, and so does every twist of C over k.*

Proof. We mention earlier (Remark 3.1.28) that a non-trivial Brauer-Severi surface over k corresponds to a non-trivial 3-torsion element of $\mathrm{Br}(k)$. Therefore, if $\mathrm{Br}(k)[3]$ is trivial, then the g^2_d-system factors through $g : C/k \hookrightarrow \mathbb{P}^2_k$ and, by Theorem 3.1.30, all is defined over k. In particular, a non-singular plane model of C (hence, of any of its twists) over k exists. □

Example 3.2.3. *It is well-known that $\mathrm{Br}(k)[3]$ is trivial when $k = \mathbb{F}_q$ and $k = \mathbb{R}$. In particular, any smooth k^{sep}-plane curve over such a field k always has a non-singular plane model over k.*

Theorem 3.2.4. *Let C be a smooth k^{sep}-plane curve defined over k, then it admits a non-singular plane model over an L such that $[L:k] \mid 3$, i.e. $\exists P \in \mathrm{PGL}_3(k^{\text{sep}})$ for which $F_{P^{-1}\overline{C}}(X,Y,Z) \in L[X,Y,Z]$ and such that C and $F_{P^{-1}\overline{C}}(X,Y,Z) = 0$ are L-isomorphic.*

Proof. First, we have a k-morphism of C to a Brauer-Severi surface D over k (Theorem 3.1.30). Second, by Theorem 3.1.16 and Theorem 3.1.23, D corresponds to an element of Az_3^k, i.e. a central simple algebra over k of dimension 9 that splits (if it is not trivial) by a degree 3 Galois extension L/k. Moreover, $D \otimes_k L$ corresponds to the trivial element in $\mathrm{H}^1(\mathrm{Gal}(k^{\text{sep}}/L), \mathrm{PGL}_3(k^{\text{sep}}))$. In particular, $D \otimes_k L \cong \mathbb{P}^2_L$ over L, and

$$g \otimes_k L : C \otimes_k L \hookrightarrow \mathbb{P}^2_L$$

are all defined over L. In this way, we obtain a non-singular plane model of C over L. Third, any non-singular plane models of C over k^{sep} is of the form $F_{P^{-1}\overline{C}}(X,Y,Z) = 0$ for some $P \in \mathrm{PGL}_3(k^{\text{sep}})$, so one gets the second part of the statement. $\square$

Let D be a Brauer-Severi variety of dimension $n-1$ over k. By Corollary 3.1.27, it corresponds to an element of $\mathrm{H}^1(k, \mathrm{PGL}_n(k^{\text{sep}}))$, hence to a CSA of dimension n^2 by Theorem 3.1.16. Therefore, we always can identify D with its image in the the Brauer group $\mathrm{Br}(k)$.

Now, if V be an algebraic variety over k. The natural inclusion $k \subset k(V)$, where $k(V)$ is the algebraic function field of V, induces a map

$$\mathrm{Br}(k) \xrightarrow{r_V} \mathrm{Br}(k(V))$$

given by mapping the class of a Brauer-Severi D over k to the class of the variety $D \otimes_k k(V)$. In particular, this applies to $V = D$. In this case, the base extension $D \otimes_k k(D)$ has a $k(D)$-rational point coming from the generic point of D. Hence, by Châtelet's theorem (Theorem 5.1.3 in [GS06]), the class of D in $\mathrm{Br}(k)$ lies in the kernel of the map r_D. The following famous theorem shows that this construction already describes the kernel.

Theorem 3.2.5 (Amitsur, Theorem 5.4.1, [GS06])**.** *Let D be a Brauer-Severi variety defined over a field k. Then, the kernel of the restriction map $\mathrm{Br}(k) \xrightarrow{r_D} \mathrm{Br}(k(D))$ is a cyclic group generated by the class of D in $\mathrm{Br}(k)$.*

Theorem 3.2.6 (Lichtenbaum, Theorem 5.4.10, [GS06]). *Let D be a Brauer-Severi variety over k. Then, there is an exact sequence*

$$0 \longrightarrow \mathrm{Pic}(D) \longrightarrow \mathrm{Pic}(D \otimes_k k^{\mathrm{sep}}) \cong \mathbb{Z} \xrightarrow{\delta} \mathrm{Br}(k) \xrightarrow{r_D} \mathrm{Br}(k(D)). \tag{3.2}$$

The map δ sends 1 to the Brauer class corresponding to D. Here $\mathrm{Pic}(D)$ is the Picard group of D.

Corollary 3.2.7 (Remark 5.4.11, [GS06]). *If the class of D has order ℓ in the Brauer group, then there is a divisor class of degree ℓ on D. The associated linear system defines the ℓ-dimensional embedding of D over a splitting field L.*

The following result is a particular case of an argument by J. Roé and X. Xarles in [RX14] following Châtelet [Ch4]:

Theorem 3.2.8 (Roé-Xarles). *Let C be a smooth k^{sep}-plane curve defined over k of degree d coprime with 3. Then C is a smooth plane curve over k.*

Proof. Recall that a Brauer-Severi surface D over k corresponds to a CSA of period dividing 3, hence its class in the Brauer group $\mathrm{Br}(k)$ has order dividing 3, say m. On the other hand, by Theorem 3.2.6 and Corollary 3.2.7, there exists a divisor class on D of the same order m, which generates $\mathrm{Pic}(D)$. Now, let C be a curve over k in $\mathrm{Pic}(D)$ such that $\overline{C} = C \otimes_k k^{\mathrm{sep}}$ has a non-singular plane model of degree d. Then its image in $\mathrm{Pic}(D \otimes_k k^{\mathrm{sep}}) \cong \mathbb{Z}$ equals d. Consequently, if d is coprime with 3, then so does m and hence $m = 1$. That is, D is the projective plane $\mathbb{P}^2_k$.

We address the reader to [RX14, Theorem 13] for a more general statement on hypersurfaces in Brauer-Severi varieties. □

Corollary 3.2.9. *Let C be a smooth k^{sep}-plane curve defined over k of degree d coprime with 3. Then, any twist $C' \in \mathrm{Twist}_k(C)$ is a smooth plane curve over k.*

Proof. By our assumption, any twist of C over k is also a smooth k^{sep}-plane curve of degree d, coprime with 3. Then a non-singular plane model over k exists for each twist, by using Theorem 3.2.8. □

3.2.1 An example of a smooth $\overline{\mathbb{Q}}$-plane curve over $\mathbb{Q}$, which is not a smooth plane curve over $\mathbb{Q}$

Following the proof of Theorem 3.1.30, in order to construct a smooth k^{sep}-plane curve C over k, which is not a smooth plane curve over k, we need to construct a non-trivial 1-cocycle in $\mathrm{H}^1(\mathrm{Gal}(k^{\text{sep}}/k), \mathrm{PGL}_3(k^{\text{sep}}))$ corresponding to C.

Theorem 3.2.10 (Weil, [Wei56]). *Let C be a smooth curve defined over a field F, and let F/K be a Galois extension. Suppose that for every $\sigma \in \mathrm{Gal}(F/K)$, there exists an F-isomorphism $\phi_\sigma : {}^\sigma C \mapsto C$ such that*

$$\phi_\sigma \circ {}^\sigma\phi_\tau = \phi_{\sigma\tau} \text{ for all } \sigma, \tau \in \mathrm{Gal}(F/K).$$

Then there exists a curve C' over K and an F-isomorphism $\phi : C' \otimes_K F \to C$ such that $\phi_\sigma \circ {}^\sigma\phi = \phi$ for all $\sigma \in \mathrm{Gal}(F/K)$.

We now construct the example: let us consider $\mathbb{Q}_f$, the splitting field of the polynomial $f(t) = t^3 + 12t^2 - 64$. It is an irreducible polynomial and the discriminant of f is $(2^6 3^2)^2$, then $\mathrm{Gal}(\mathbb{Q}_f/\mathbb{Q}) \cong \mathbb{Z}/3\mathbb{Z}$, moreover, as we can check with SAGE [ea], the discriminant of the field $\mathbb{Q}_f$ is a power of 3, and the prime 2 becomes inert in $\mathbb{Q}_f$. Let us denote the roots of f by a, b, c in a fixed algebraic closure of $\mathbb{Q}$, and let us call σ the element in the Galois group that acts by sending $a \to b \to c$.

Definition 3.2.11. (Fields of definition) Given a smooth curve C/F over F, then C is *defined* over $k \subset F$ if and only if there is a curve C'/k defined over k, that is isomorphic over F to C. In such case, K is called a *field of definition* of C.

Proposition 3.2.12. *The smooth plane curve over $\mathbb{Q}_f$*

$$C : \; 64Z^6 + abY^6 + aX^6 + 8Y^3Z^3 + \frac{ab}{8}X^3Y^3 + aZ^3X^3 = 0,$$

has $\mathbb{Q}$ as a field of definition, but it does not admit a non-singular plane model over $\mathbb{Q}$.

Proof. The matrix

$$\phi = \begin{pmatrix} 0 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

defines an isomorphism $\phi : {}^{\sigma}C \to C$, satisfying the Weil's cocycle condition (Theorem 3.2.10): $\phi_{\sigma^3} = \phi_{\sigma}^3 = 1$. We therefore obtain that the curve is defined over $\mathbb{Q}$, and that there exists an isomorphism $\varphi_0 : C_{\mathbb{Q}} \to C$, where $C_{\mathbb{Q}}$ is a rational model such that $\phi = \varphi_0 \circ {}^{\sigma}\varphi_0^{-1} \in \mathrm{PGL}_3(\mathbb{Q})$. The assignation $\phi_{\tau} := \varphi_0 \circ {}^{\tau}\varphi_0^{-1}$ defines an element of $\mathrm{H}^1(\mathrm{Gal}(\mathbb{Q}_f/\mathbb{Q}), \mathrm{PGL}_3(\mathbb{Q}_f))$. By Proposition 3.1.25, this cohomology element is non-trivial, since 2 is not a norm of an element of $\mathbb{Q}_f$ (recall that 2 is inert in $\mathbb{Q}_f$). Therefore φ_0 is not given by an element of $\mathrm{PGL}_3(\mathbb{Q}_f)$, or of $\mathrm{PGL}_3(\overline{\mathbb{Q}})$ because the cohomology class by the inflation map is not trivial, as well. Thus the curve C over $\mathbb{Q}$ does not admit a non-singular plane model over $\mathbb{Q}$, as if there is a non-singular plane model over $\mathbb{Q}$, such a model would be of the form $F_{(PQ)^{-1}\overline{C}}(X, Y, Z) = 0$, for some $P \in \mathrm{PGL}_3(\overline{\mathbb{Q}})$ where $F_{Q^{-1}\overline{C}}(X, Y, Z) = 0$ a non-singular model over $\mathbb{Q}_f$, hence φ_0 would be given by $P \in \mathrm{PGL}_3(\overline{\mathbb{Q}})$, a contradiction. $\square$

Remark 3.2.13. We have just seen an example of a curve defined over a field k, not admitting a particular model (a plane one) over the same field. For hyperelliptic models, we find such examples after Proposition 4.14 in [LR12]. In [Hug05, chapters 5,7], there are also examples of hyperelliptic curves and smooth plane curves where the field of moduli is not a field of definition, so, in particular, there are not such models defined over the fields of moduli.

§3.3 On twists of smooth plane curves over k

Let C be a smooth curve over a field k and let $\mathrm{Twist}_k(C)$ be the set of isomorphism classes of twists of C over k. One can read chapter III of [Ser94] for a proof of the following theorem:

Theorem 3.3.1. *Let V be a quasi-projective algebraic variety over k. The set $\mathrm{Twist}_k(V)$ is in one to one correspondence with the first Galois cohomology set $\mathrm{H}^1(\mathrm{Gal}(k^{\mathrm{sep}}/k), \mathrm{Aut}(\overline{V}))$ given by $[V'] \mapsto \xi : \tau \mapsto \xi_{\tau} := \phi \circ {}^{\tau}\phi^{-1}$, for $\tau \in \mathrm{Gal}(k^{\mathrm{sep}}/k)$, where*

$$\phi : \overline{V'} = V' \otimes_k k^{\mathrm{sep}} \to \overline{V} = V \otimes_k k^{\mathrm{sep}}$$

is a fixed k^{sep}-isomorphism.

Given a cocycle $\xi \in \mathrm{H}^1(\mathrm{Gal}(k^{\mathrm{sep}}/k), \mathrm{Aut}(\overline{C}))$, the idea behind the computation of equations for the twist, is finding a $\mathrm{Gal}(k^{\mathrm{sep}}/k)$-modulo isomorphism between the subgroup generated by the image of ξ in $\mathrm{Aut}(\overline{C})$ and a subgroup of a general linear group $\mathrm{GL}_n(k^{\mathrm{sep}})$. After that, by making explicitly Hilbert's Theorem 90, we can compute an isomorphism $\phi : \overline{C'} \to \overline{C}$, and hence, we obtain equations for the twist. For non-hyperelliptic curves, see a description in [LG14] (or Appendix B), the canonical model gives a natural $\mathrm{Gal}(k^{\mathrm{sep}}/k)$-inclusion $\mathrm{Aut}(\overline{C}) \hookrightarrow \mathrm{PGL}_g(k^{\mathrm{sep}})$, but we can go further, the action gives a $\mathrm{Gal}(k^{\mathrm{sep}}/k)$-inclusion $\mathrm{Aut}(\overline{C}) \hookrightarrow \mathrm{GL}_g(k^{\mathrm{sep}})$ which allows us to compute the twists. For hyperelliptic curves, we refer to [LLG16], where an efficient algorithm to compute equations of twists of hyperelliptic curves of arbitrary genus over any separable field (of characteristic different from 2) is given.

In this section, we assume that C is a smooth plane curve over k, that is, that C is given by an equation $F_{\overline{C}}(X,Y,Z) = 0$ with $F_{\overline{C}}(X,Y,Z) \in k[X,Y,Z]$. We give a characterization of the twists of C which are also smooth plane curves over k.

Theorem 3.3.2. *Let C be a smooth plane curve over k and identify it with a fixed non-singular plane model $F_{\overline{C}}(X,Y,Z) = 0$ with $F_{\overline{C}}[X,Y,Z] \in k[X,Y,Z]$. Then there exists a natural map*

$$\Sigma : \mathrm{H}^1(\mathrm{Gal}(k^{\mathrm{sep}}/k), \mathrm{Aut}(F_{\overline{C}})) \to \mathrm{H}^1(k, \mathrm{PGL}_3(k^{\mathrm{sep}})),$$

defined by the inclusion $\mathrm{Aut}(F_{\overline{C}}) \subseteq \mathrm{PGL}_3(k^{\mathrm{sep}})$ as G_k-groups. The kernel of Σ is the set of all twists of C that are smooth plane curves over k. Moreover, any such twist is obtained through an automorphism of $\mathbb{P}^2_{k^{\mathrm{sep}}}$, that is, the twist is k-isomorphic to

$$F_{M^{-1}\overline{C}}(X,Y,Z) := F_{\overline{C}}(M(X,Y,Z)) \in k[X,Y,Z],$$

for some $M \in \mathrm{PGL}_3(k^{\mathrm{sep}})$.

Proof. The map is clearly well-defined. Second if a twist C' admits a non-singular plane model $F_{\overline{C'}}(X,Y,Z) = 0$ over k, then $F_{\overline{C'}}(X,Y,Z) = 0$ and $F_{\overline{C}}(X,Y,Z) = 0$ are isomorphic through an $M \in \mathrm{PGL}_3(k^{\mathrm{sep}})$, since any isomorphism between two non-singular plane curves of degrees > 3 is given by a linear transformation in $\mathbb{P}^2_{k^{\mathrm{sep}}}$. Hence, the corresponding 1-cocycle $\sigma \mapsto$

$M \circ {}^{\sigma}M^{-1} \in \mathrm{Aut}(F_{\overline{C}})$ becomes trivial in $\mathrm{H}^1(\mathrm{Gal}(k^{\mathrm{sep}}/k), \mathrm{PGL}_3(k^{\mathrm{sep}}))$. Conversely, if the image of a twist C' of C over k under Σ is trivial, then it must be given by a k^{sep}-isomorphism $\varphi : F_{\overline{C}} \to C'$ that is defined by some $M_\phi \in \mathrm{PGL}_3(k^{\mathrm{sep}})$. Such an M_ϕ produces a non-singular plane model over k for C'. In this case, both C' and its model are k-isomorphic by definition. $\square$

Remark 3.3.3. We can reinterpret the map Σ in Theorem 3.3.2 as the map that sends a twist C' to the Brauer-Severi variety D in Theorem 3.1.30.

Remark 3.3.4. In order to define a natural map $\Sigma' : \mathrm{Twist}_k(C) \to \mathrm{H}^1(\mathrm{Gal}(k^{\mathrm{sep}}/k), \mathrm{PGL}_3(k^{\mathrm{sep}}))$ for a smooth k^{sep}-plane curve C over k, we need that $\mathrm{Aut}(\overline{C})$ has a natural inclusion in $\mathrm{PGL}_3(k^{\mathrm{sep}})$ as $\mathrm{Gal}(k^{\mathrm{sep}}/k)$-groups. For instance, this is possible when there exists $P \in \mathrm{PGL}_3(k^{\mathrm{sep}})$ where $F_{P^{-1}\overline{C}}(X, Y, Z) \in k[X, Y, Z]$. Indeed, in this situation the inclusion $\mathrm{Aut}(F_{P^{-1}\overline{C}}) \subseteq \mathrm{PGL}_3(k^{\mathrm{sep}})$ is of $\mathrm{Gal}(k^{\mathrm{sep}}/k)$-groups and defines a map

$$\mathrm{Twist}_k(C) = \mathrm{H}^1(\mathrm{Gal}(k^{\mathrm{sep}}/k), \mathrm{Aut}(F_{P^{-1}\overline{C}})) \to \mathrm{H}^1(k, \mathrm{PGL}_3(k^{\mathrm{sep}})).$$

Remark 3.3.5. Consider a smooth plane curve C defined over k of degree d coprime with 3 or such that $\mathrm{Br}(k)[3]$ is trivial. Then Σ in Theorem 3.3.2 is the trivial map by Corollary 3.2.9 and Corollary 3.2.2.

Remark 3.3.6. Theorem 3.3.2 can be used to improve the algorithm for computing twists for non-hyperelliptic curves, see [LG17] or [LG14, Chp.1], for the special case of non-singular plane curves. If Σ is trivial in Theorem 3.3.2, then we can work with matrices in $\mathrm{GL}_3(k^{\mathrm{sep}})$ instead of in $\mathrm{GL}_g(k^{\mathrm{sep}})$.

We use this improvement to compute the twists of some particular families of smooth plane curves over k, in section §3.5.

3.3.1 Twists of a smooth plane curve over k which are not smooth plane curves over k

We construct a family of smooth plane curves over $\mathbb{Q}$ but some of its twists are not smooth plane curves over $\mathbb{Q}$. This construction is not explicit in the sense that we do not construct the equations of the twist and the Brauer-Severi surface where the twist lives (Remark 3.3.3). Nevertheless, in the next section (§3.4), we provide an explicit concrete construction giving defining equations.

Consider the family $\mathcal{C}_{\alpha_0,a}$ of $\overline{\mathbb{Q}}$-plane curves defined by the equation

$$\mathcal{C}_{\alpha_0,a}:\ X^6+\frac{1}{\alpha_0^2}Y^6+\frac{1}{\alpha_0^4}Z^6+\frac{a}{\alpha_0^3}(\alpha_0^2X^3Y^3+\alpha_0X^3Z^3+Y^3Z^3)=0,$$

with $\alpha_0, a\in\overline{\mathbb{Q}}$ such that $a\neq -10,\pm 2,-1,0,\frac{1}{2}(-1\pm\sqrt{5})$.

Proposition 3.3.7. *Let C be a $\overline{\mathbb{Q}}$-plane curve in the family $\mathcal{C}_{\alpha_0,a}$ as above (in particular, it corresponds to a certain $\alpha_0, a\in\overline{\mathbb{Q}}$). Then C is a smooth $\overline{\mathbb{Q}}$-plane curve with automorphism group isomorphic to* $\mathrm{GAP}(54,5)$*, which is generated by the automorphisms* $\mathrm{diag}(1,\zeta_3,\zeta_3^2)$, $\mathrm{diag}(1,1,\zeta_3)$, $[\alpha_0^{-1}Z:X:Y]$ *and* $[\sqrt[3]{\alpha_0}X:Z:\sqrt[3]{\alpha_0^2}Y]$.

Proof. For simplicity, we work with the $\overline{\mathbb{Q}}$-isomorphic model ${}^{\phi}C$ obtained by a change of variables of the shape $\phi:=[X:\sqrt[3]{\alpha_0}Y:\sqrt[3]{\alpha_0^2}Z]$. Hence ${}^{\phi}C$ is defined by the equation

$$F(X,Y,Z):=X^6+Y^6+Z^6+a(X^3Y^3+X^3Z^3+Y^3Z^3)=0.$$

We first show that ${}^{\phi}C$ is smooth. Since $a\neq\pm 2$, the polynomial $F(X,0,Z)=X^6+Z^6+aX^3Z^3$ has no repeated zeros, in particular the system $F(X,0,Z)=F_X(X,0,Z)=0$ has no solutions. On the other hand, a point $(X_0:1:Z_0)\in\mathbb{P}^2_{\overline{\mathbb{Q}}}$ with $X_0Z_0\neq 0$ satisfies $F(X_0,1,Z_0)=F_X(X_0,1,Z_0)=F_Z(X_0,1,Z_0)=0$ only if $a^3+2a^2-1=0$, which conflicts our assumption that $a\neq -1,\frac{1}{2}(-1\pm\sqrt{5})$. Moreover, if $X_0=0$ or $Z_0=0$, then $a^2-4=0$, which is also absurd.

Second, we prove the claim on $\mathrm{Aut}(\overline{C})$. Let G be the subgroup of automorphisms of $F(X,Y,Z)=0$ generated by $S:=\mathrm{diag}(1,\zeta_3,\zeta_3^2)$, $U:=\mathrm{diag}(1,1,\zeta_3)$, $T:=[Z:X:Y]$,and $W:=[X:Z:Y]$. Thus G is conjugate to $\mathrm{GAP}(54,5)$, since $SU=US$, $ST=TS$, $WSW=S^{-1}$, $UT=STU$, $WTW=T^{-1}$ and $WUW=SU$. Consequently, $\mathrm{Aut}(\overline{C})$ is not conjugate to any of the following groups: a cyclic group, the Klein group $\mathrm{PSL}(2,7)$, the icosahedral group A_5, the alternating group A_6 , the Hessian groups Hess_* with $*\in\{36,72\}$. Also $F(X,Y,Z)=0$ can not be a descendant of the Klein curve K_6 of degree 6, since $|G|$ does not divide $|\mathrm{Aut}(K_6)|=63$. Furthermore, G fixes no points in the projective plane $\mathbb{P}^2_{\overline{\mathbb{Q}}}$, then so does $\mathrm{Aut}(\overline{C})$. Therefore, by the aid of Theorem 1.4.4, we just need to investigate whether $F(X, Y, Z) = 0$ is a descendant of the Fermat curve F_6 of degree 6 or $\mathrm{Aut}(F_{\overline{\phi C}})$ is conjugate to the Hessian group Hess_{216}. We saw in Example 2.2.10 that the representations of Hess_{216} forms a unique set, up to conjugation in $\mathrm{PGL}_3(\overline{\mathbb{Q}})$. Hence there is no loss of generality to suppose that this fixed representation in Example 2.2.10 acts on $F(X, Y, Z) = 0$. More generally, any non-singular plane curve of degree 6 whose automorphism group is the Hessian group $\mathrm{Hess}_{216} = \langle S, T, U, V\rangle$ should be of the form $X^6 + Y^6 + Z^6 + a'(X^3Y^3 + X^3Z^3 + Y^3Z^3) = 0$ for some $a' \in \overline{\mathbb{Q}}$, since its defining equation must be invariant under the action of $[Z : Y : X]$, $[X : Z : Y]$, $[Y : X : Z]$, $[Y : Z : X]$ and $\mathrm{diag}(1, \zeta_3, \zeta_3^2)$. Now,

$$V = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \zeta_3 & \zeta_3^2 \\ 1 & \zeta_3^2 & \zeta_3 \end{pmatrix} \in \mathrm{Aut}({}^{\phi}C)$$

only if $a = -10$, which is not allowed by our assumptions on a. Next, one can easily check that $\mathrm{Aut}(F_6)$ is isomorphic to $\mathrm{GAP}(216, 92)$, thus it contains a unique subgroup of order 54, up to conjugation inside $\mathrm{Aut}(F_6)$ itself. Therefore, we may assume that $F(X, Y, Z) = 0$ is a descendant of the Fermat curve F_6 through a projective transformation $\psi \in \mathrm{PGL}_3(\overline{\mathbb{Q}})$ such that $\psi^{-1}G\psi = G$. Then again the transformed equation should be of the form $C' : X^6 + Y^6 + Z^6 + a'(X^3Y^3 + X^3Z^3 + Y^3Z^3)$ for some $a' \in \overline{\mathbb{Q}}$. In particular, $F(X, Y, Z) = 0$ admits no more automorphisms in $\mathrm{Aut}(F_6)$ (recall that $a \neq 0$).

This shows the result. $\square$

Theorem 3.3.8. *Consider the subfamily $\mathcal{C}_{p,a}$ of smooth plane curves over $\mathbb{Q}$ given by*

$$\mathcal{C}_{p,a} : X^6 + \frac{1}{p^2}Y^6 + \frac{1}{p^4}Z^6 + \frac{a}{p^3}(p^2X^3Y^3 + pX^3Z^3 + Y^3Z^3) = 0,$$

with $a \in \mathbb{Q} \setminus \{-10, \pm 2, -1, 0\}$ and $p \equiv 3$ or 5 (mod 7) a prime number. Given p and a as before, or equivalently a smooth plane curve C over $\mathbb{Q}$ in $\mathcal{C}_{p,a}$, then there exists a twist $C' \in \mathrm{Twist}_{\mathbb{Q}}(C)$ which does not admit a non-singular plane model over $\mathbb{Q}$.

Proof. Consider the Galois extension $M/\mathbb{Q}$ with $M = \mathbb{Q}(cos(2\pi/7), \zeta_3, \sqrt[3]{p})$, where all the automorphisms of $\mathrm{Aut}(\overline{C})$ are defined (Proposition 3.3.7). Let σ be a generator of the cyclic Galois group $\text{Gal}(\mathbb{Q}(cos(2\pi/7))/\mathbb{Q})$. We define a 1-cocycle on $\text{Gal}(M/\mathbb{Q}) \cong \text{Gal}(\mathbb{Q}(cos(2\pi/7))/\mathbb{Q}) \otimes \text{Gal}(\mathbb{Q}(\zeta_3, \sqrt[3]{p})/\mathbb{Q})$ to $\text{Aut}(\overline{C})$ by mapping $(\sigma, id) \mapsto [Y : Z : pX]$ and $(id, \tau) \mapsto id$. This defines an element of $\text{H}^1(\text{Gal}(M/\mathbb{Q}), \text{Aut}(\overline{C}))$. It remains to show that its image under Σ is not trivial inside $\text{H}^1(\text{Gal}(M/\mathbb{Q}), \text{PGL}_3(M))$, and the conclusion is then an immediate consequence of Theorem 3.3.2: By Theorem 3.1.16, $\text{H}^1(\text{Gal}(M/\mathbb{Q}), \text{PGL}_3(M))$ is the set of CSA's over $\mathbb{Q}$ of dimension 9. Moreover, each of these algebras splits by a degree 3 field extension of $\mathbb{Q}$ inside M, by the virtue of Example 3.1.22. We know from [Was82, Theorem 2.13] that (p) is prime in $\mathbb{Q}(cos(2\pi/7))/\mathbb{Q}$, hence p is not a norm of an element of $\mathbb{Q}(\cos(2\pi/7))$. In particular, the image of our 1-cocycle is not trivial $\text{H}^1(\text{Gal}(\mathbb{Q}(cos(2\pi/7))/\mathbb{Q}), \text{PGL}_3(\mathbb{Q}(cos(2\pi/7))))$ (it is trivial if and only if p is a norm of an element of $\mathbb{Q}(cos(2\pi/7))/\mathbb{Q}$). Then so does its image in $\text{H}^1(\text{Gal}(M/\mathbb{Q}), \text{PGL}_3(M))$, which was to be shown. □

§3.4 An explicit non-trivial Brauer-Severi variety

This section details, following the ideas in §3.3, an explicit example of a smooth plane curve over $\mathbb{Q}(\zeta_3)$ having a twist that does not possess such a model in the field $\mathbb{Q}(\zeta_3)$, where ζ_3 is a primitive 3rd root of unity. Interestingly, we find the already mentioned explicit equations for a non-trivial Brauer-Severi variety. As far as we know, this is the first time that this kind of equations are exhibited.

Let us consider the curve $C_a : X^6 + Y^6 + Z^6 + a(X^3Y^3 + Y^3Z^3 + Z^3X^3) = 0$ defined over a number field $k \supseteq \mathbb{Q}(\zeta_3)$ where $a \in k$. For $a \in k \setminus \{-10, \pm 2, -1, 0, \frac{1}{2}(-1 \pm \sqrt{5})\}$, it is a non-hyperelliptic, smooth plane curve of genus $g = 10$ and its automorphism group is the group of order 54 determined in the previous section (Proposition 3.3.7).

The algorithm in [LG17], allows us to compute all the twists of C_a, previous computation of its canonical model in $\mathbb{P}^9_k$. We follow such algorithm, since this time we will see that Σ is not trivial, so we cannot use the improvements in Remark 3.3.6.

3.4.1 A canonical model of C_a in $\mathbb{P}^9_{\overline{k}}$

Let us denote by α_i, $i \in \{1, ..., 6\}$, the six different roots of the polynomial $T^6 + aT^3 + 1 = 0$, and define the points on C_a: $P_i = (0 : \alpha_i : 1)$, $Q_i = (\alpha_i : 0 : 1)$ and $\infty_i = (\alpha_i : 1 : 0)$ for $i \in \{1, ..., 6\}$. The divisor of the function $x = X/Z$ is $\text{div}(x) = \sum P_i - \sum \infty_i$ and the function $y = Y/Z$ is $\text{div}(y) = \sum Q_i - \sum \infty_i$. The ramification data of the morphism $x : (X_0 : Y_0 : 1) \in C_a \mapsto (Y_0 : 1) \in \mathbb{P}^1_{\overline{k}}$ gives the zeros of dx, and the poles of dx are exactly these of x but with order increased by 1. Since x is ramified at $P = (X_0 : Y_0 : 1) \in C_a$ if and only if $F_y(x, y, 1) = 0$ at P. Equivalently, if $T^6 + a(X_0^3 + 1)T^3 + X_0^6 + aX_0^3 + 1 = 0$ has double roots. That is, if $X_0^6 + aX_0^3 + 1 = 0$ or $4(X_0^6 + aX_0^3 + 1) = a^2(X_0^3 + 1)^2$. Let us denote by β_i, $i \in \{1, ..., 6\}$, the six different roots of the polynomial $T^6 + \frac{2a}{a+2}T^3 + 1 = 0$ and denote by $V_{ij} = (\beta_i : \zeta_3^{j+1}\sqrt[3]{-\frac{a}{2}(\beta_i^3 + 1)} : 1)$ where $j \in \{1, 2, 3\}$. We finally get

$$\text{div}(dx) = 2\sum Q_i + \sum V_{i,j} - 2\sum \infty.$$

In particular, dx is not a regular differential on C_a. However, any of the differentials ω_i, for $i = 1, 2, ..., 10$, where

$$\begin{aligned}
\omega_1 &= \frac{xdx}{y(2y^3 + a(x^3 + 1))}, \; \omega_2 = \frac{x^2}{y}\omega_1, \; \omega_3 = \frac{y^2}{x}\omega_1, \; \omega_4 = \frac{1}{xy}\omega_1 \\
\omega_5 &= x\omega_1, \; \omega_6 = \frac{y}{x}\omega_1, \; \omega_7 = \frac{1}{y}\omega_1, \; \omega_8 = y\omega_1, \; \omega_9 = \frac{x}{y}\omega_1, \; \omega_{10} = \frac{1}{x}\omega_1.
\end{aligned}$$

is regular on C_a, since $\text{div}(2y^3 + a(x^3 + 1)) = \sum V_{i,j} - 3\sum \infty_i$. We list the divisors of these differentials below.

$$\begin{aligned}
\text{div}(\omega_1) &= \sum P_i + \sum Q_i + \sum \infty_i, \; \text{div}(\omega_2) = 3\sum P_i, \; \text{div}(\omega_3) = 3\sum Q_i, \\
\text{div}(\omega_4) &= 3\sum \infty_i, \; \text{div}(\omega_5) = 2\sum P_i + \sum Q_i, \; \text{div}(\omega_6) = 2\sum Q_i + \sum \infty_i, \\
\text{div}(\omega_7) &= \sum P_i + 2\sum \infty_i, \; \text{div}(\omega_8) = \sum P_i + 2\sum Q_i, \; \text{div}(\omega_9) = 2\sum P_i + \sum \infty_i, \\
\text{div}(\omega_{10}) &= \sum Q_i + 2\sum \infty_i.
\end{aligned}$$

The space of regular differential on C_a is isomorphic to the space of cubic tangents to C_a with basis $\{1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3\}$. In particular, $\{\omega_i\}_i$ forms a basis of the space of the regular differentials on C_a, since it coincides with the set

$$\{(1/F_y)dx, (x/F_y)dx, (y/F_y)dx, ..., (x^3/F_y)dx, (x^2y/F_y)dx, (xy^2/F_y)dx, (y^3/F_y)dx\}.$$

Lemma 3.4.1. *The ideal of the canonical model of C_a in $\mathbb{P}^9_{\overline{k}}[\omega_1, ..., \omega_{10}]$ is generated by the polynomials*

$$\omega_4\omega_9 = \omega_7^2,\ \omega_4\omega_6 = \omega_{10}^2,\ \omega_4\omega_1 = \omega_7\omega_{10},\ \omega_4\omega_5 = \omega_9\omega_{10},\ \omega_4\omega_8 = \omega_6\omega_7,\ \omega_4\omega_2 = \omega_7\omega_9,\ \omega_4\omega_3 = \omega_6\omega_{10},$$

$$\omega_3\omega_{10} = \omega_6^2,\ \omega_2\omega_7 = \omega_9^2,\ \omega_6\omega_9 = \omega_1^2,\ \omega_3\omega_5 = \omega_8^2,\ \omega_2\omega_3 = \omega_5\omega_8,\ \omega_2\omega_8 = \omega_5^2,$$

$$\omega_2^2 + \omega_3^2 + \omega_4^2 + a(\omega_5\omega_8 + \omega_6\omega_{10} + \omega_7\omega_9) = 0.$$

We denote by $\mathcal{C}_a$ this canonical model.

Proof. If $\omega_4 \neq 0$, then the des-homogenization of this ideal with respect to ω_4 gives the affine curve C_a for $Z = 1$. If $\omega_4 = 0$, then $\omega_7 = \omega_{10} = 0$, so $\omega_6 = \omega_9 = 0$ and $\omega_1 = 0$, so if $\omega_3 \neq 0$ we recover the part at infinity ($Z = 0$) of C_a. If $\omega_4 = \omega_3 = 0$, then all the variables are equal to zero which produces a contradiction.

To check that it is non-singular, we need to see if the rank of the matrix of partial derivatives of the previous generating functions has rank equal to $8 = \dim(\mathbb{P}^9_{\overline{k}}) - \dim(C)$ at every point, that is, that the tangent space has codimension 1. If $\omega_4 \neq 0$, then the partial derivatives of the first seven equation plus the last one produce linearly independent vectors in the tangent space. If $\omega_4 = 0$, we have already seen that $\omega_3 \neq 0$ and by equivalent arguments, neither it is ω_2. Then the $6th$, $7th$, $8th$, $9th$ equations plus the last four equations produce 8 linearly independent vectors. □

Remark 3.4.2. The canonical ideal for a non-hyperelliptic C is generated, at worst, by quadrics and cubics. In fact, cubics are only needed for trigonal curves and plane curves of degree 5, see [Swi11, page 3].

Remark 3.4.3. The canonical embedding of C_a in $\mathbb{P}^{g-1}_{\overline{k}} = \mathbb{P}^9_{\overline{k}}$ coincides with the composition of the g^2_d-linear system of C_a with the Veronese embedding given by:

$$\mathbb{P}^2_{\overline{k}} \hookrightarrow \mathbb{P}^9_{\overline{k}} : (x : y : z) \to (xyz : x^3 : y^3 : z^3 : x^2y : y^2z : z^2x : xy^2 : x^2z : yz^2).$$

In particular, we get that the ideal defining the projective space $\mathbb{P}^2_{\overline{k}}$ in $\mathbb{P}^9_{\overline{k}}$ by the Veronese embedding is generated by the polynomials defined in Lemma 3.4.1 after removing the last one.

This is true in general for any smooth plane curves C over k of degree $d \geq 4$. Because the sheaves $\Omega^1(C)$ and $\mathcal{O}(d-3)|_C$ are isomorphic (e.g. [Har77, Example 8.20.3]), then $\mathrm{H}^0(\mathbb{P}^2, \mathcal{O}(d-3)) \longrightarrow \mathrm{H}^0(C, \Omega^1)$ is an isomorphism. That is, the canonical embedding of C is isomorphic to the composition $C \xrightarrow{\iota} \mathbb{P}^2_k \xrightarrow{\mathrm{Ver}_{d-3}} \mathbb{P}^{g-1}_k$, where ι comes from the (unique) g^2_d-linear system and Ver_{d-3} is the $(d-3)$-Veronese map, all are defined over k.

3.4.2 The automorphism group of C_a in $\mathbb{P}^9_{\overline{k}}$

Let us consider the automorphisms of the curve C_a given by $R = [Y : X : Z], T = [Z : X : Y]$ and $U = \mathrm{diag}(1, 1, \zeta_3)$. By Lemma 3.3.7, we easily check that $\mathrm{Aut}(C_a) = \langle R, T, U\rangle$. For instance, $W := [X : Z : Y] = TRT^{-1} \in \langle R, T, U\rangle$ and $S := \mathrm{diag}(1, \zeta_3, \zeta_3^2) = U^2WUW \in \langle R, T, U\rangle$. Thus $\langle R, T, U\rangle \leq \mathrm{Aut}(C_a) = \mathrm{GAP}(54, 5) = \langle S, U, T, W\rangle \leq \langle R, T, U\rangle$.

Second, we obtain that the pullbacks $R^*(\omega_1) = -\omega_1$, $T^*(\omega_1) = \omega_1$ and $U^*(\omega_1) = \zeta_3^2\omega_1$. So, in the canonical model, these automorphisms look like

$$R \to -\mathcal{R} := \left(\begin{array}{c|ccc|ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right), T \to \mathcal{T} := \left(\begin{array}{c|ccc|ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right)$$

and $U \to \zeta_3^2\mathrm{diag}(1, \zeta_3^2, \zeta_3^2, \zeta_3^2, \zeta_3^2, 1, \zeta_3, \zeta_3^2, 1, \zeta_3) := \zeta_3^2\mathcal{U}$. We define the faithful linear representation $\mathrm{Aut}(C_a) \hookrightarrow \mathrm{GL}_{10}(\overline{k})$ by sending $R, T, U \to \mathcal{R}, \mathcal{T}, \mathcal{U}$. Moreover, it preserves the action of the Galois group G_k.

3.4.3 An explicit twist over $k = \mathbb{Q}(\zeta_3)$ of C_a without a non-singular plane model over k

Now, let us consider the curve C_a defined over $k = \mathbb{Q}(\zeta_3)$, and the field extension $L = k(\sqrt[3]{7})$ with Galois group $\mathrm{Gal}(L/k) = \langle \sigma \rangle \cong \mathbb{Z}/3\mathbb{Z}$, where $\sigma(\sqrt[3]{7}) = \zeta_3\sqrt[3]{7}$. We define the cocycle $\xi \in \mathrm{Z}^1(G_k, \mathrm{Aut}(C_a)) \hookrightarrow \mathrm{Z}^1(G_k, \mathrm{PGL}_{10}(\overline{k}))$ given by $\xi_\sigma = \mathcal{T}\mathcal{U}$.

Lemma 3.4.4. *The image of the cocycle ξ by the map*

$$\Sigma : \mathrm{H}^1(k, \mathrm{Aut}(\overline{C_a})) \to \mathrm{H}^1(k, \mathrm{PGL}_3(\overline{k}))$$

is not trivial.

Proof. By construction, the image of the cocycle ξ in $\mathrm{H}^1(k, \mathrm{PGL}_3(\overline{k}))$ coincides with the inflation of the cocycle in $\mathrm{H}^1(\mathrm{Gal}(L/k), \mathrm{PGL}_3(L))$ where $\xi_\sigma = TU$. Now by Theorem 3.1.16 we conclude, since ζ_3 is not a norm in L/k (no new primitive root of unity appears in L than k and ζ_3 is not a norm of an element of L). $\square$

In order to compute equations defining the twist $\mathcal{C}'_a$ associated to the cocycle ξ (and the Brauer-Severi surface that contains such a twist), we need to find a matrix $\phi \in \mathrm{PGL}_{10}(\overline{k})$ such that $\xi_\sigma = \phi \circ {}^\sigma\phi^{-1}$. We can then take

$$\phi = \left(\begin{array}{c|ccc|ccc|ccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & \sqrt[3]{7} & \sqrt[3]{7^2} & 7 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \sqrt[3]{7} & \zeta_3\sqrt[3]{7^2} & 7\zeta_3^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \sqrt[3]{7} & \zeta_3^2\sqrt[3]{7^2} & 7\zeta_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 1 & \sqrt[3]{7} & \zeta_3\sqrt[3]{7^2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & \zeta_3\sqrt[3]{7} & \sqrt[3]{7^2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \zeta_3 & \sqrt[3]{7} & \sqrt[3]{7^2} & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \zeta_3\sqrt[3]{7} & \zeta_3\sqrt[3]{7^2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_3 & \zeta_3\sqrt[3]{7} & \sqrt[3]{7^2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_3 & \sqrt[3]{7} & \zeta_3\sqrt[3]{7^2}
\end{array}\right)$$

Lemma 3.4.5. *Let $f_0, f_1, f_2 \in k[x_1, ..., x_n]$, and define $g_0 = f_0 + \sqrt[3]{7}f_1 + \sqrt[3]{7^2}f_2$, $g_1 = f_0 + \zeta_3\sqrt[3]{7}f_1 + \zeta_3^2\sqrt[3]{7^2}f_2$, $g_2 = f_0 + \zeta_3^2\sqrt[3]{7}f_1 + \zeta_3\sqrt[3]{7^2}f_2$. Then the ideals in $L[x_1, ..., x_n]$ generated by $\langle g_0, g_1, g_2\rangle$ and $\langle f_0, f_1, f_2\rangle$ are equal.*

Proof. Clearly, we have the inclusion $\langle g_0, g_1, g_2\rangle \subseteq \langle f_0, f_1, f_2\rangle$. The reverse inclusion can be checked by writing $3f_0 = g_0 + g_1 + g_2$, $(\zeta_3 - 1)\sqrt[3]{7}f_1 = g_1 - \zeta_3 g_2 + (\zeta_3 - 1)f_0$ and $\sqrt[3]{7^2}f_2 = g_0 - f_0 - \sqrt[3]{7}f_1$. $\square$

Proposition 3.4.6. *The equations in $\mathbb{P}^9_{\overline{k}}$ of the non-trivial Brauer-Severi surface B over k constructed as in Theorem 3.3.2 from the cocycle ξ above are*

$$\begin{aligned}
\omega_1\omega_2 &= \zeta_3\omega_5\omega_9 + \zeta_3\omega_6\omega_8 + 7\zeta_3\omega_7\omega_{10}, & \omega_2^2 - 7\omega_3\omega_4 &= \zeta_3\omega_5\omega_{10} + \zeta_3\omega_7\omega_8 + \zeta_3\omega_6\omega_9,\\
\omega_1\omega_3 &= \omega_5\omega_{10} + \zeta^2\omega_7\omega_8 + \zeta_3\omega_6\omega_9, & 7\omega_3^2 - 7\zeta_3\omega_2\omega_4 &= \omega_5\omega_9 + \zeta_3^2\omega_6\omega_8 + 7\zeta_3\omega_7\omega_{10},\\
7\omega_1\omega_4 &= \zeta_3\omega_5\omega_8 + 7\omega_6\omega_{10} + 7\zeta_3^2\omega_7\omega_9, & 49\omega_4^2 - 7\zeta_3^2\omega_2\omega_3 &= \omega_5\omega_8 + 7\zeta_3\omega_6\omega_{10} + 7\zeta_3^2\omega_7\omega_9,
\end{aligned}$$

$$\begin{aligned}
\omega_5^2 + 14\zeta_3\omega_6\omega_7 &= 7\zeta_3\omega_2\omega_{10} + 7\omega_4\omega_8 + 7\zeta_3\omega_3\omega_9, & \omega_5^2 - 7\zeta_3\omega_6\omega_7 &= 7\omega_2\omega_{10} + 7\omega_4\omega_8 + 7\zeta_3^2\omega_3\omega_9,\\
\omega_6^2 + 2\zeta_3\omega_5\omega_7 &= \zeta_3\omega_2\omega_9 + \omega_3\omega_8 + 7\zeta_3\omega_4\omega_{10}, & \omega_6^2 - \zeta_3\omega_5\omega_7 &= \omega_2\omega_9 + \zeta_3\omega_3\omega_8 + 7\zeta_3\omega_4\omega_{10},\\
7\omega_7^2 + 2\zeta_3\omega_5\omega_6 &= \zeta_3\omega_2\omega_8 + 7\zeta_3^2\omega_3\omega_{10} + 7\zeta_3^2\omega_4\omega_9, & 7\omega_7^2 - \zeta_3\omega_5\omega_6 &= \omega_2\omega_8 + 7\omega_3\omega_{10} + 7\zeta_3^2\omega_4\omega_9,
\end{aligned}$$

$$\begin{aligned}
\omega_8^2 + 14\zeta_3\omega_9\omega_{10} &= 7\zeta_3^2\omega_2\omega_7 + \omega_4\omega_5 + 7\zeta_3^2\omega_3\omega_6, & \omega_8^2 - 7\zeta_3^2\omega_9\omega_{10} &= 7\zeta_3^2\omega_2\omega_7 + 7\zeta_3^2\omega_4\omega_5 + 7\omega_3\omega_6,\\
\omega_9^2 + 14\zeta_3^2\omega_8\omega_{10} &= \zeta_3\omega_2\omega_6 + \zeta_3^2\omega_3\omega_5 + 7\zeta_3\omega_4\omega_7, & \omega_9^2 - 7\zeta_3^2\omega_8\omega_{10} &= \zeta_3^2\omega_2\omega_6 + \zeta_3\omega_3\omega_5 + 7\zeta_3\omega_4\omega_7,\\
7\omega_{10}^2 + 2\zeta_3^2\omega_8\omega_9 &= \zeta_3^2\omega_2\omega_5 + 7\omega_3\omega_7 + 7\omega_4\omega_6, & 7\omega_{10}^2 - \zeta_3^2\omega_8\omega - 9 &= \zeta_3^2\omega_2\omega_5 + 7\zeta_3^2\omega_3\omega_7 + 7\omega_4\omega_6,
\end{aligned}$$

Proof. We only need to plug the equations of the isomorphism ϕ into the equations defining $\mathcal{C}_a$. We will get equations for $\mathcal{C}'_a$. However, even defining a curve over k, these equations are defined over $L = k(\sqrt[3]{7})$. In order to get generators of the ideal defined over k, we need to apply Lemma 3.4.5. $\square$

In order to get the equations of the twisted curve, we only need to add the equation that we get by plugging ϕ in $\omega_2^2 + \omega_3^2 + \omega_4^2 + a(\omega_5\omega_8 + \omega_6\omega_{10} + \omega_7\omega_9) = 0$, and apply Lemma 3.4.5 again.

Proposition 3.4.7. *The curve C'_a is a twist over k of the curve C_a for $a \in k \setminus \{-10, \pm 2, -1, 0, \frac{1}{2}(-1 \pm \sqrt{5})\}$ which does not admits a non-singular plane model over k, i.e. is not a smooth plane curve over k, and the defining equations of C'_a in $\mathbb{P}^9$ are the ones given in Proposition 3.4.6 plus the extra equation:*

$$\omega_2^2 + 14\omega_3\omega_4 + a(\omega_2^2 - 7\omega_3\omega_4) = 0$$

§3.5 Twists of smooth plane curves with diagonal cyclic automorphism group

In this section, we prove a theoretical result, by which we obtain directly all the twists for smooth plane curves C over a perfect field k having the extra property: C is isomorphic over k to a plane k-model $F_{\overline{C}}(X,Y,Z)=0$, such that $\mathrm{Aut}(F_{\overline{C}})$ is cyclic and generated by an automorphism $\psi\in\mathrm{PGL}_3(\overline{k})$ of diagonal shape. In this case, we show that any twist in $\mathrm{Twist}_k(F_{\overline{C}}(X,Y,Z)=0)$ is represented by a non-singular plane model of the form $F_{\phi^{-1}\overline{C}}(X,Y,Z)=0$ for some diagonal $\phi\in\mathrm{PGL}_3(\overline{k})$. We apply this result to some particular families of smooth plane curves over k with large automorphism group, different from the Fermat curve and the Klein curve, see section §2.4 of chapter 2 for such families.

The condition that ψ is a diagonal matrix is necessary, and we will provide examples when ψ is not diagonal, such that not all the twists are diagonal ones.

Definition 3.5.1. Consider a smooth plane curve C over k given by $F_{\overline{C}}(X,Y,Z)=0$. We say that $[C']\in\mathrm{Twist}_k(C)$ is a diagonal twist of C, if there exists an $M\in\mathrm{PGL}_3(k)$ and a diagonal $D\in\mathrm{PGL}_3(\overline{k})$, such that C' is k-isomorphic to $F_{(MD)^{-1}\overline{C}}(X,Y,Z)=0$.

3.5.1 Diagonal cyclic automorphism group: all twists are diagonal

Theorem 3.5.2. *Let $C\,:\,F_{\overline{C}}(X,Y,Z)=0$ be a smooth plane curve over a perfect field k. Assume that $\mathrm{Aut}(F_{\overline{C}})\subseteq\mathrm{PGL}_3(\overline{k})$ is a non-trivial cyclic group of order n (relatively prime with the characteristic of k), generated by an automorphism $\psi=\mathrm{diag}(1,\zeta_n^a,\zeta_n^b)$ for some $a,b\in\mathbb{N}$.*

Then all the twists in $\mathrm{Twist}_k(C)$ are given by plane equations of the form $F_{D^{-1}\overline{C}}(X,Y,Z)=0$ with $F_{D^{-1}\overline{C}}(X,Y,Z)\in k[X,Y,Z]$ and D is a diagonal matrix. In particular, the map Σ is trivial.

Proof. We just need to notice that the embedding $\mathrm{Aut}(\overline{C})\hookrightarrow\mathrm{PGL}_3(\overline{k})$ factors through $\mathrm{GL}_3(\overline{k})$. Thus the map Σ in Theorem 3.3.2 factors as follows:

$$\Sigma:\ \mathrm{H}^1(k,\mathrm{Aut}(F_{\overline{C}}))\to\mathrm{H}^1(k,\mathrm{GL}_3(\overline{k}))\to\mathrm{H}^1(k,\mathrm{PGL}_3(\overline{k})).$$

Moreover, $\mathrm{H}^1(k, \mathrm{GL}_3(\overline{k})) = 1$, so the map Σ is trivial. By Theorem 3.3.2 any twist has a non-singular plane model $F_{P^{-1}\overline{C}}(X, Y, Z) = 0$ over k, for some $P \in \mathrm{PGL}_3(\overline{k})$. Since $P \circ {}^{\sigma}(P^{-1}) \in \mathrm{Aut}(F_{\overline{C}}) = \langle \mathrm{diag}(1, \zeta_n^a, \zeta_n^b) \rangle$ for any $\sigma \in G_k$, then ${}^{\sigma}P = P \circ \mathrm{diag}(1, v, w)$ for some n-th roots of unity v, w. Writing $P = (a_{i,j})$, one easily deduces that $\sigma(a_{i,j}) = u_j a_{i,j}$ with $u_1 = 1, u_2 = v, u_3 = w$. Consequently, for any fixed integer j, we have $\sigma(a_{i,j})a_{i,j}^{-1} = \sigma(a_{i',j})a_{i',j}^{-1}$. That is $a_{i,j}a_{i',j}^{-1}$ is a G_k-invariant, which in turns gives that $a_{i,j} = m_i a_{i',j}$ for some $m_i \in k$. In particular, P reduces to MD for some D a diagonal projective 3×3 matrix and $M \in \mathrm{PGL}_3(k)$. This proves that all the twists are diagonal. However, the plane model $F_{(MD)^{-1}\overline{C}}(X, Y, Z) = 0$ over k is k-isomorphic through M to $F_{D^{-1}\overline{C}}(X, Y, Z) = 0$. Hence $F_{D^{-1}\overline{C}}(X, Y, Z) = 0$ defines a non-singular plane model over k for the twist. $\square$

Remark 3.5.3. More generally, suppose that C is a smooth plane curve over k, identified with $F_{\overline{C}}(X, Y, Z) = 0$, and having a twist $[C'] \in \mathrm{Twist}_k(C)$ with a non-singular plane model $F_{Q^{-1}\overline{C}}(X, Y, Z) = 0$ over k for some $Q \in PGL_3(\overline{k})$, such that $\mathrm{Aut}(F_{Q\overline{C}}) = \langle \mathrm{diag}(1, \zeta_n^a, \zeta_n^b) \rangle$. Then, any other twist $[C''] \in \mathrm{Twist}_k(C)$ is represented by a model $F_{(QD)^{-1}\overline{C}}(X, Y, Z) = 0$ over k through some diagonal $D \in \mathrm{PGL}_3(\overline{k})$.

Now, we apply Theorem 3.5.2 to some particular smooth plane curves of degree $d \geq 5$ with cyclic automorphism group in order to obtain all of their twists: let k be a perfect field of characteristic $p = 0$ or $p > (d-1)(d-2)$, and consider the smooth $\overline{k}$-plane curves

$$\begin{aligned} C &: X^d + Y^d + XZ^{d-1} = 0, \\ C' &: X^d + Y^{d-1}Z + XZ^{d-1} = 0. \end{aligned}$$

Both curves are defined over k with cyclic diagonal automorphism groups of orders $d(d-1)$ and $(d-1)^2$, generated by $\mathrm{diag}(1, \zeta_{d(d-1)}^{d-1}, \zeta_{d(d-1)}^{d})$, and $\mathrm{diag}(1, \zeta_{(d-1)^2}, \zeta_{(d-1)^2}^{(d-1)(d-2)})$ respectively (see subsection §2.4.1). Furthermore, applying the Theorem, we obtain:

Corollary 3.5.4. *Let k be a perfect field of characteristic $p = 0$ or $p > (d-1)(d-2)+1$. For $d \geq 5$, any twist of $C : X^d + Y^d + XZ^{d-1} = 0$ over k is given by $X^d + aY^d + bXZ^{d-1} = 0$ for some $a, b \in k^*$. Moreover, two twists $\{a, b\}$ and $\{a', b'\}$ are k-isomorphic if and only if $a = a' \mod k^{*^5}$ and $b = b' \mod k^{*^4}$.*

Similarly, for C' : $X^d + Y^{d-1}Z + XZ^{d-1} = 0$, any twist is given by $X^d + aY^{d-1}Z + bXZ^{d-1} = 0$ for some $a, b \in k^*$, where the twist $\{a, b\}$ and $\{a', b'\}$ are k-isomorphic if and only if $a' = M^4a$, $b' = MN^4b$ for some $M, N \in k^*$.

Proof. By Theorem 3.5.2, any twist of C has a non-singular plane model over k, which can be obtained through a diagonal change of variables in $\mathrm{PGL}_3(\overline{k})$ of the shape $\mathrm{diag}(1, \lambda, \mu)$. Hence, one could think about the defining equation as $X^d + \lambda^d Y^d + \mu^{d-1}XZ^{d-1} = 0$ such that $\lambda^d, \mu^{d-1} \in k$. So $\lambda = \sqrt[d]{a}$ and $\mu = \sqrt[d-1]{b}$ for some $a, b \in k^*$, and the defining equation for the twist is $X^d + aY^d + bXZ^{d-1} = 0$. On the other hand, two twists $\{a, b\}$ and $\{a', b'\}$ are equivalent if and only if there exists an $\psi \in \mathrm{PGL}_3(k)$ and an automorphism α of C such that $\alpha \circ \phi = \phi' \circ \psi$, where $\phi = \mathrm{diag}(1, \sqrt[d]{a}, \sqrt[d-1]{b})$ and $\phi = \mathrm{diag}(1, \sqrt[d]{a'}, \sqrt[d-1]{b'})$, see Remark 1.3.1 in [LG14]. This is equivalent to write $\psi = \mathrm{diag}(1, q, q')$ for some $q, q' \in k^*$, such that $a' = aM^5$ and $b' = bN'^4$ for some $M, N \in k^*$, which was to be shown in this situation.

In the same way, one shows the result for C'. $\square$

3.5.2 Aut(C) cyclic does not imply diagonal twists

Let C be a smooth plane curve over k, a field of characteristic $p \geq 0$, and identify C with its model $F_{\overline{C}}(X, Y, Z) = 0$ over k. Suppose also that $\mathrm{Aut}(F_{\overline{C}}) \subseteq \mathrm{PGL}_3(\overline{k})$ is a cyclic group of order n, generated by a matrix σ, such that the conjugacy class of σ in $\mathrm{PGL}_3(k)$ contains no elements of a diagonal shape. Then the twists of C mapped to zero by Σ (i.e., those ones that admits a smooth plane curve over k), are not necessarily represented by diagonal twists.

Proposition 3.5.5. *Let C be the smooth plane curve over $\mathbb{Q}$ by:*

$$F_{\overline{C}}(X, Y, Z) = X^4Y + Y^4Z + XZ^4 + (X^3Y^2 + Y^3Z^2 + X^2Z^3) = 0.$$

Then $\mathrm{Aut}(F_{\overline{C}}) = \mathbb{Z}/3\mathbb{Z}$, and generated by $[Y : Z : X]$ in $\mathrm{PGL}_3(\overline{\mathbb{Q}})$.

Proof. Because $\sigma := [Y : Z : X] \in \mathrm{Aut}(F_{\overline{C}})$ is of order 3, then $\mathrm{Aut}(F_{\overline{C}})$ is conjugate to one of the automorphism groups appearing in [BB16a, Table 2] (or see Table 4.2 in subsection §4.1 of the next chapter), with 3 dividing its order.

Assume first that $\tau \in \mathrm{Aut}(F_{\overline{C}})$ is of order 2 with $\tau\sigma\tau = \sigma^{-1}$. For simplicity, we consider the $\overline{\mathbb{Q}}$-equivalent model $F_{P^{-1}\overline{C}}(X,Y,Z) = 0$ defined by

$$4X^5+20X^3YZ+\left((-5-9i\sqrt{3})Y^3+(-5+9i\sqrt{3})Z^3\right)X^2-6XY^2Z^2-4YZ(Y^3+Z^3)=0,$$

which is obtained via the change of variables P of the shape

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & \zeta_3 & \zeta_3^2 \\ 1 & \zeta_3^2 & \zeta_3 \end{pmatrix}.$$

Hence $P^{-1}\sigma P = \mathrm{diag}(1,\zeta_3,\zeta_3^2) \in \mathrm{Aut}(F_{P^{-1}\overline{C}})$, and $P^{-1}\tau P \in \mathrm{Aut}(F_{P^{-1}\overline{C}})$ should be $[X : aZ : a^{-1}Y]$ for some $a \in \overline{\mathbb{Q}}$. One easily checks that $F_{P^{-1}\overline{C}}(X,Y,Z) = 0$ can not have automorphisms of this shape. Consequently, the symmetry group $\mathrm{S}_3 = \langle \eta, \rho \rangle$ does not happen as a bigger subgroup of automorphisms. Then so are $\mathrm{GAP}(30,1)$ and $\mathrm{GAP}(150,5)$ in [BB16a, Table 2], since both groups contain an S_3 and a single conjugacy class of elements of order 3.

Second, any automorphism of order 3 of the $\mathrm{GAP}(39,1)$ in [BB16a, Table 2] is conjugate to either σ or σ^{-1}. Therefore, if $\mathrm{Aut}(F_{\overline{C}})$ is conjugate, through some $P \in \mathrm{PGL}_3(\overline{\mathbb{Q}})$, to $\mathrm{GAP}(39,1)$, then we may suppose that $P^{-1}\sigma P = \sigma$. Thus P reduces to

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3^2 \end{pmatrix}^s \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_3 & \alpha_1 & \alpha_2 \\ \alpha_2 & \alpha_3 & \alpha_1 \end{pmatrix} \in \mathrm{PGL}_3(\overline{\mathbb{Q}}),$$

for some $s \in \{0,1,2\}$, and the transformed defining equation of $F_{P^{-1}\overline{C}}(X,Y,Z)$ must be $X^4Y + Y^4Z + Z^4X = 0$. For $s = 0$, we need to terminate the coefficients of $Y^5, Y^4X, Y^3X^2, Y^3Z^2$, and Y^3XZ, which conflicts the assumption that P is invertible. For $s = 1$ or 2, we impose that the monomials X^5, Y^5 and Z^5 do not appear, so P is of diagonal shape and $F_{P^{-1}\overline{C}}(X,Y,Z) = 0$ is not $X^4Y + Y^4Z + Z^4X = 0$. Consequently, $\mathrm{Aut}(F_{\overline{C}})$ is not conjugate to $\mathrm{GAP}(39,1)$.

By all the above argument, we conclude by Table 4.2 that $\mathrm{Aut}(F_{\overline{C}})$ must be of order 3. $\square$

Proposition 3.5.6. *Let C be the smooth plane curve over $\mathbb{Q}$ as before (Proposition 3.5.5). Then C admits a twist over $\mathbb{Q}$, which is not diagonal.*

Proof. The defining equation $F_{\overline{C}}(X,Y,Z)=0$ has degree 5, coprime with 3. Then, by Corollary 3.2.9, any twist of $\overline{C}=C\otimes_{\mathbb{Q}}\overline{\mathbb{Q}}$ is also a smooth plane curve over $\mathbb{Q}$.

We construct the twist following the algorithm in [LG17] (see Appendix B) and Theorem 3.3.2 because Σ is trivial: By Proposition 3.5.5, all automorphisms of are defined over $K=\mathbb{Q}$, and the twisted product $\Gamma=\mathrm{Aut}(\overline{C})\rtimes\mathrm{Gal}(K/k)\simeq\mathbb{Z}/3\mathbb{Z}$. That is, for each cyclic field extension $L/\mathbb{Q}$ of degree 3, there exist (exactly two) non-trivial twists of the curve $F_{\overline{C}}(X,Y,Z)=0$ over k, which have L as their splitting field. Since the set of such extensions is not empty, the curve $\overline{C}$ has a non-trivial twist. However, it is easy to check, that a twist of $F_{\overline{C}}(X,Y,Z)=0$ through a diagonal isomorphism $D\in\mathrm{PGL}_3(\overline{\mathbb{Q}})$ is always the trivial one. Therefore, any non-trivial twist of C must be a non-diagonal twist. $\square$

Remark 3.5.7. By our discussion in [BB16c, §6] (or see section §1.4), Proposition 3.5.5 extends to any perfect field k of characteristic $p>13$. And, we ask for $\zeta_3\notin k$ in order to construct a non-trivial twist as in Proposition 3.5.6. Degree 5 is the smallest degree for which such an example exists, see [LG14] to discard degree 4 exceptions.

Remark 3.5.8. In Appendix B, we apply the algorithm in [LG17] to the simplest degree 5 examples. This shows the improvements of Theorem 3.5.2 and Corollary 3.5.4 to compute the twists inside $\mathbb{P}^2_{\overline{k}}$ not in $\mathbb{P}^{g-1}_{\overline{k}}$.

4

CHAPTER

Arithmetic aspects of smooth plane curves of genus 6

Let $\overline{k}$ be a fixed algebraic closure of a field k of characteristic $p \geq 0$. By a smooth $\overline{k}$-plane curve C of genus g, we mean a smooth projective curve C that admits a non-singular plane model $\{F_C(X,Y,Z) = 0\} \subseteq \mathbb{P}^2_{\overline{k}}$ over $\overline{k}$ of degree d, in this case, the genus g equals $\frac{1}{2}(d-1)(d-2)$. The first genus for which there exist smooth $\overline{k}$-plane curves are: $0, 1, 3, 6, ...$ The curves of genus 0 are isomorphic to the projective line, and the curves of genus 1 are elliptic ones, which are quite well understood. For genus 3, we always get plane quartic curves, and different arithmetics properties have been investigated by many people around. We mention, for example, a classification up to isomorphism with good properties that can be found in [LRRS14, LG14], or the study of their twists in [LG14, LG16]. For genus 6, the dimension of the (coarse) moduli space $\mathcal{M}_6$ of smooth curves of genus $g = 6$ over $\overline{k}$ is equal to $3g - 3 = 15$. The stratum $\mathcal{M}_6^{Pl}$ of smooth $\overline{k}$-plane curves of genus 6 has dimension equal to $21 - 9 = 12$, since there are 21 monic monomial of degree 5 in 3 variables and all the isomorphisms are given by projective matrices of size 3×3. In particular, this dimension is larger than the dimension of the hyperelliptic locus, which is $2g - 1 = 11$.

The existence of universal families for a moduli space helps to recover the information on its points and allows to write down the attached objects to a point of this space. It becomes difficult to deal with a moduli space when a universal family does not exist. R. Lercier, C. Ritzenthaler, F. Rovetta and J. Sijsling in [LRRS14, §2] introduced three good substitutes for the notion of universal family: complete, finite and representative families.

The aim of this chapter is to study the existence of a so-called representative classification for the different strata in $\mathcal{M}_6^{Pl}$ by their automorphism group. In particular, we look for complete and representative families over k, which tends to be suitable substitutes to *universal families* of (coarse) moduli spaces, especially when the spaces have no extra structures.

The structure of this chapter is as follows. Section 4.1 is devoted to the study of the stratification by automorphism group of smooth $\overline{k}$-plane curves of genus 6, i.e. the different strata of $\mathcal{M}_6^{Pl}$, where k has characteristic $p = 0$ or $p > 2g + 1 = 13$. A full description of the automorphism groups and the associated normal forms is given in Theorem 4.1.12. The diagram in Figure 4.1 shows how looks like the stratification by automorphism groups of non-singular plane quintic curves. In section 4.2, we explain an interesting phenomenon, which appears in Figure 4.1; the existence of a final stratum of plane curves whose dimension is not zero. By a final stratum we mean a stratum not containing any other proper stratum. One could expect that by adding restrictions in the parameters of a family defining a stratum with a given automorphisms group, one get bigger automorphism groups until obtaining a zero-dimensional stratum. This happens for all the families except for one. For this family each restriction in the parameters providing a bigger automorphism group yields a singular curve. We find an explanation for this fact: this family can be embedded in a family of curves of genus 6 with the same automorphism group for which we can carry out the previous operation without getting singular curves, the key point is they are not plane curves anymore: Proposition 4.2.1, Corollary 4.2.2. Moreover, we prove that this may happen in general for higher genera: Theorem 4.2.4. In section 4.3, we refine the classification given in Theorem 4.1.12, since it is not representative or even complete over k (see Remark 4.3.4): Theorem 4.3.6. We end up this chapter with section 4.4, in which a full description of the set $\mathrm{Twist}_k(C)$ of twists of a smooth $\overline{k}$-plane curve of genus 6 defined over k can be found.

We shall deal with the following items:

4.1. Stratification by automorphism group of $\mathcal{M}_6^{Pl}$.

4.2. Final families: A canonical interpretation.

4.3. Complete and representative families.

4.4. Twists of smooth pane curves of genus 6.

The main results of section §4.1 have been published in [BB16a]. The results of sections

§4.2 and §4.3 are resulted into the manuscript [BLG17].

§4.1 Stratification by automorphism group of $\mathcal{M}_6^{\mathrm{Pl}}$

Given any finite non-trivial group G, it is classically known from Greenberg [Gre74, Theorem 4] that one can construct a Riemann surface R whose conformal automorphism group $\mathrm{Aut}(R)$ is isomorphic to G.

This section is concerned with the following question:

Question 4.1.1. *Let k be a field of characteristic $p \geq 0$. Once the genus $g \geq 3$ is fixed, determine the finite non-trivial groups G (up to isomorphism), for which the stratum $\widetilde{\mathcal{M}_g^{Pl}}(G)$ over $\overline{k}$ is non-empty. That is, the groups G such that there exists a smooth $\overline{k}$-plane curve C of genus g whose automorphism group is isomorphic to G.*

Henn in [Hen76] and Komiya-Kuribayashi in [KK79] solved the question for $g = 3$ over the complex field $\mathbb{C}$. We solve it for $g = 6$, in order to get a compact table as Henn Table (Theorem 2.2.1), but for smooth $\overline{k}$-plane curves of degree 5.

We start with the next result, which is a consequence of Theorem 1.4.5 for characteristic $p = 0$, and the discussion after, at the end of chapter 1, for $p > 13$:

Corollary 4.1.2. *Let C be a smooth $\overline{k}$-plane curve of degree $d = 5$, where k is a field of characteristic $p = 0$ or $p > 13$. Then, the full automorphism group $\mathrm{Aut}(C)$ of C (seen as a smooth plane curve over $\overline{k}$) is not conjugate to the Hessian group Hess_{216}, the Klein group $\mathrm{PSL}(2,7)$ and the alternating group A_6.*

Starting with the results in section §2.4, we conclude:

Corollary 4.1.3. *Let C be a smooth $\overline{k}$-plane curve of degree $d = 5$, where k is a field of characteristic $p = 0$ or $p > 13$. Then, we have (up to $\overline{k}$-isomorphism):*

1. *The cyclic group $\rho(\mathbb{Z}/20\mathbb{Z}) = \langle \mathrm{diag}(1, \zeta_{20}^4, \zeta_{20}^5) \rangle$ appears as $\mathrm{Aut}(C)$ inside $\mathrm{PGL}_3(\overline{k})$, where C is $\overline{k}$-isomorphic to $X^5 + Y^5 + XZ^4 = 0$.*

2. *The cyclic group $\rho(\mathbb{Z}/16\mathbb{Z}) = \langle \mathrm{diag}(1, \zeta_{16}, \zeta_{16}^{12}) \rangle$ appears as $\mathrm{Aut}(C)$ inside $\mathrm{PGL}_3(\overline{k})$, where C is $\overline{k}$-isomorphic to $X^5 + Y^4Z + XZ^4 = 0$.*

3. *The group* $\mathrm{GAP}(30,1) = \langle \sigma, \tau | \, \tau^2 = \sigma^{15} = (\tau\sigma)^2\sigma^3 = 1\rangle$ *appears as* $\mathrm{Aut}(C)$ *inside* $\mathrm{PGL}_3(\overline{k})$ *with* $\sigma := \mathrm{diag}(1, \zeta_{15}, \zeta_{15}^{11})$ *and* $\tau := [X : Z : Y]$. *In this case,* C *is* $\overline{k}$-*isomorphic to* $X^5 + Y^4Z + YZ^4 = 0$.

4. *The group* $\mathrm{GAP}(39,1) = \langle \tau, \sigma | \sigma^{13} = \tau^3 = 1, \, \sigma\tau = \tau\sigma^3\rangle$ *appears as* $\mathrm{Aut}(C)$ *inside* $\mathrm{PGL}_3(\overline{k})$ *with* $\sigma := \mathrm{diag}(1, \zeta_{13}, \zeta_{13}^{10})$ *and* $\tau := [Y : Z : X]$. *Moreover,* C *is* $\overline{k}$-*isomorphic to the Klein curve* $X^4Y + Y^4Z + Z^4X = 0$.

5. *The cyclic group* $\rho(\mathbb{Z}/8\mathbb{Z}) = \langle \mathrm{diag}(1, \zeta_8, -1)\rangle$ *appears as* $\mathrm{Aut}(C)$ *inside* $\mathrm{PGL}_3(\overline{k})$, *where* C *is* $\overline{k}$-*isomorphic to* $X^5 + Y^4Z + XZ^4 + \beta_{2,0}X^3Z^2$, *for some* $\beta_{2,0} \neq 0, \pm 2$.

Before we prove Corollary 4.1.3, we reproduce here Table A.3, representing the different types of cyclic subgroups of automorphisms of smooth $\overline{k}$-plane curves of degree 5, which was a consequence of the work in chapter 2.

Table 4.1: degree 5

Type: $m, (a, b)$	$F(X, Y, Z)$
$20, (4, 5)$	$X^5 + Y^5 + XZ^4$
$16, (1, 12)$	$X^5 + Y^4Z + XZ^4$
$15, (1, 11)$	$X^5 + Y^4Z + YZ^4$
$13, (1, 10)$	$X^4Y + Y^4Z + Z^4X$
$10, (2, 5)$	$X^5 + Y^5 + XZ^4 + \beta_{2,0}X^3Z^2$
$8, (1, 4)$	$X^5 + Y^4Z + XZ^4 + \beta_{2,0}X^3Z^2$
$5, (1, 2)$	$X^5 + Y^5 + Z^5 + \beta_{3,1}X^2YZ^2 + \beta_{4,3}XY^3Z$
$5, (0, 1)$	$Z^5 + L_{5,Z}$
$4, (1, 2)$	$X^5 + X\left(Z^4 + Y^4\right) + \beta_{2,0}X^3Z^2 + \beta_{3,2}X^2Y^2Z + \beta_{5,2}Y^2Z^3$
$4, (0, 1)$	$Z^4Y + L_{5,Z}$
$3, (1, 2)$	$X^5 + Y^4Z + YZ^4 + \beta_{2,1}X^3YZ + X^2\left(\beta_{3,0}Z^3 + \beta_{3,3}Y^3\right) + \beta_{4,2}XY^2Z^2$
$2, (0, 1)$	$Z^4L_{1,Z} + Z^2L_{3,Z} + L_{5,Z}$

Proof. (of Corollary 4.1.3) One just needs to apply Propositions 2.4.3, 2.4.7, 2.4.11, 2.4.14, and 2.4.17 when $d = 5$, by the aid of Table 4.1.

It remains, for the last case, to observe that if $\mathrm{Aut}(C)$ is bigger, then it is always cyclic (Proposition 2.4.17 when $d=5$ and $m=2$), and then should be the cyclic group of order 16 (Table A.3 above). Therefore, $\beta_{2,0}\neq 0$ is the only restriction to impose, so that the curve has automorphism group exactly $\mathbb{Z}/8\mathbb{Z}$. While the restrictions $\beta_{2,0}\neq\pm 2$ comes form non-singularity. □

4.1.1 Automorphism groups having small cyclic subgroups

By the aid of Corollary 4.1.3, it remains to describe $\mathrm{Aut}(C)$, where C is a smooth $\overline{k}$-plane curve of degree 5, which appears in Table 4.1 and such that the maximal order for any element inside $\mathrm{Aut}(C)$ is exactly $2d=10$ or at most $d=5$.

Proposition 4.1.4. *Let C be a smooth $\overline{k}$-plane curve of degree 5, with $\sigma\in\mathrm{Aut}(C)$ of order 10 as an automorphism of maximal order. Then, we reduce up to $\overline{k}$-isomorphism to $C: X^5+Y^5+XZ^4+\beta_{2,0}X^3Z^2=0$ for some $\beta_{2,0}\neq 0$, and σ acts on C as $\mathrm{diag}(1,\zeta_{10}^2,-1)$. Moreover, one of the following subcases occurs:*

1. *If $\beta_{2,0}^2=20$, then C is $\overline{k}$-isomorphic to the Fermat curve $F_5: X^5+Y^5+Z^5=0$ and $\mathrm{Aut}(C)$ is isomorphic to $\mathrm{GAP}(150,5)$.*

2. *If $\beta_{2,0}^2\neq 20$, then $\mathrm{Aut}(C)$ is isomorphic to $\mathbb{Z}/10\mathbb{Z}$, and we can think about C as a descendant of the Fermat curve of the form*

$$C_P: X^5+Y^5+Z^5+u\left(\xi_{10}^6Y^4Z+YZ^4\right)+u'\left(\xi_{10}^2Y^3Z^2+Y^2Z^3\right)=0,$$

for some $(u,u')\in\overline{k}^2\setminus\{(0,0)\}$.

Proof. Since the maximal order is 10, we have by, Table 4.1, a non-singular $\overline{k}$-plane model of C via the normal form $X^5+Y^5+XZ^4+\beta_{2,0}X^3Z^2=0$ for some $\beta_{2,0}\neq 0$. In particular, σ acts on such a model as the automorphism $\mathrm{diag}(1,\zeta_{10}^2,-1)$. We know, by Proposition 2.4.22 and the Remark after it, that if C is not a descendant of the Fermat curve, then $\mathrm{Aut}(C)$ fixes the line $Y=0$ and the point $(0:1:0)$.

Assume first that C is not a descendent of the Fermat quintic curve. Then, by Theorem 1.4.4, $\mathrm{Aut}(C)$ satisfies a short exact sequence $1\to\mathbb{Z}/5\mathbb{Z}\to\mathrm{Aut}(C)\to^{\Lambda} G\to 1$, where

$\mathbb{Z}/5\mathbb{Z}$ is generated by σ^2, and G (as a subgroup of $\mathrm{PGL}_2(\overline{k})$) contains $\Lambda(\sigma) = \mathrm{diag}(1,-1)$ of order 2. Then G is $\mathrm{PGL}_2(\overline{k})$-conjugate to $\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}, \mathrm{S}_3, \mathrm{A}_4, \mathrm{S}_4$, or to A_5. However, $\mathbb{Z}/5\mathbb{Z}$ is contained in the center of $\mathrm{Aut}(C)$ (Theorem 1.4.4-(2)), so if G has an element $\Lambda(\tau)$ of order 3 or 4, then so does $\mathrm{Aut}(C)$, since τ must be of order divisible by 3 or 4, respectively. In particular, $\mathrm{Aut}(C)$ has an element of order 15 or 20, which contradicts the maximality of the order of σ. Therefore, $G = \mathbb{Z}/2\mathbb{Z}$ and $|\mathrm{Aut}(C)| = 10$, in other words, $\mathrm{Aut}(C) = \mathbb{Z}/10\mathbb{Z}$.

Secondly, if C is a descendant of the Fermat curve of degree 5, then this happens through a change of variables $P \in \mathrm{PGL}_3(\overline{k})$, such that

$$P^{-1}\sigma P \in \{[X : \zeta_{10}^{2b}Z : \zeta_{10}^{2a}Y],\ [\zeta_{10}^{2a}Y : X : \zeta_{10}^{2b}Z],\ [\zeta_{10}^{2b}Z : \zeta_{10}^{2a}Y : X] \,|\, 5 \nmid (a+b)\}.$$

In any of these situations, we always obtain a Fermat descendant of the form

$$C_P:\ X^5+Y^5+Z^5+u(\zeta_{10}^{a'}A^4B+AB^4)+u'(\zeta_{10}^{b'}A^3B^2+A^2B^3)=0,$$

where $\{A, B\} \subset \{X, Y, Z\}$. Furthermore, C_P is the Fermat curve only if $\beta_{2,0}^2 = 20$, and $\mathrm{Aut}(C_P)$ is cyclic of order 10 otherwise. For example, if $P^{-1}\sigma P = \lambda[X : \zeta_{10}^{2b}Z : \zeta_{10}^{2a}Y]$, then $\lambda = \zeta_{10}^2$, $5|a+b+2$, and P reduces to

$$\begin{pmatrix} 0 & \zeta_{10}^{2a+2}\alpha_3 & \alpha_3 \\ 1 & 0 & 0 \\ 0 & -\zeta_{10}^{2a+2}\gamma_3 & \gamma_3 \end{pmatrix} \in \mathrm{PGL}_3(\overline{k}).$$

Therefore, C is transformed into C_P of the form

$$X^5+Y^5+Z^5+u(\zeta_{10}^{6(a+1)}Y^4Z+YZ^4)+u'(\zeta_{10}^{2(a+1)}Y^3Z^2+Y^2Z^3)=0,$$

by setting $\alpha_3(\alpha_3^4+\beta_{2,0}\gamma_3^2\alpha_3^2+\gamma_3^4)=1$. Now, C_P is the Fermat curve only if $u=u'=0$, or equivalently, $5\alpha_3^4+\beta_{2,0}\gamma_3^2\alpha_3^2-3\gamma_3^4 = 5\alpha_3^4-\beta_{2,0}\gamma_3^2\alpha_3^2+\gamma_3^4=0$. Thus $\beta_{2,0}^2=20$ (for instance, one can take $\alpha_3 = -\frac{(-1)^{3/5}}{2^{4/5}}$, and $\gamma_3 = \frac{(-1)^{3/5}\sqrt[4]{5}}{2^{4/5}}$ when $\beta_{2,0}=2\sqrt{5}$, and $\alpha_3 = \frac{1}{2^{4/5}}$, and $\gamma_3 = -\frac{i\sqrt[4]{5}}{2^{4/5}}$ when $\beta_{2,0} = -2\sqrt{5}$). Otherwise, i.e. $u \neq 0$ or $u' \neq 0$, one may assume that $a = 0$ and $b = 3$, since any $[X : \zeta_{10}^{2b}Z : \zeta_{10}^{2a}Y] \in \mathrm{Aut}(C_P)$ is conjugate inside $\mathrm{Aut}(F_5)$ to $[X : \zeta_{10}^6 Z : Y]$ through $\mathrm{diag}(1, \zeta_{10}^t, \zeta_{10}^s)$ for some integers s and t. Hence C_P admits no more automorphisms inside $\mathrm{Aut}(F_5)$ and $\mathrm{Aut}(C_P)$, as a subgroup of $\mathrm{Aut}(F_5)$, is again cyclic of order 10. Finally, we note that, for any other value of $P^{-1}\sigma P$, one can reduce to some concrete (a, b) and obtain exactly the same system to solve involving $\beta_{2,0}$ as before. Thus, we always get the same conclusion as above. $\square$

Remark 4.1.5. Recall that $\mathrm{Aut}(F_5)$ is generated by $\eta_1 := [X : Z : Y]$, $\eta_2 := [Y : Z : X]$, $\eta_3 := \mathrm{diag}(\zeta_5, 1, 1)$, and $\eta_4 := \mathrm{diag}(1, \zeta_5, 1)$ of orders $2, 3, 5$, and 5 respectively. Moreover,

$$(\eta_1\eta_2)^2 = (\eta_1\eta_3)(\eta_3\eta_1)^{-1} = (\eta_3\eta_4)(\eta_4\eta_3)^{-1} = \eta_1\eta_4^2\eta_1(\eta_3\eta_4)^{-3} = \eta_2\eta_3\eta_2^{-1}(\eta_3\eta_4)^{-4} = 1.$$

Consequently $\mathrm{Aut}(F_5) = \mathrm{GAP}(150, 5)$.

The following lemma is very useful to discard all the groups, that contains a $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, for smooth plane curves of degree 5.

Lemma 4.1.6. *Let k be a field of characteristic $p = 0$ or $p > 13$. Then there is no smooth $\overline{k}$-plane curve C of degree 5, with $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \leq \mathrm{Aut}(C)$. In particular, the full automorphism group $\mathrm{Aut}(C)$ is not isomorphic to any of the groups: $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, A_4, S_4 and A_5.*

Proof. By Theorem 1.2.1 and Theorem 1.4.4, the group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ inside $\mathrm{PGL}_3(\overline{k})$, which gives invariant a smooth plane curve C of degree d, should fix a point not lying on C, or C is a descendant of either the Fermat or the Klein curve. For $d = 5$, it could not be a descendant of the Fermat curve and the Klein curve, since 4 is not a divisor of $|Aut(F_5)| = 150$ and $|Aut(K_5)| = 39$. Therefore, the automorphism subgroup $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ fixes a point not lying in C. Moreover, $2 \nmid d$, so we can think about it in a short exact sequence (see Theorem 1.4.4 and its proof):

$$1 \to N = 1 \to H \to H \to 1,$$

where H is conjugate to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ inside $\mathrm{PGL}_2(\overline{k})$. We can also assume that H acts only on the variables Y, Z because N is the subgroup of $\mathrm{Aut}(C)$ acting on X. Now, if $\sigma, \tau \in H \subseteq \mathrm{PGL}_2(K)$ are of order two, such that $\sigma\tau = \tau\sigma$, then we can suppose, up to a change of variables of $\mathbb{P}^2_{\overline{k}}$, that $\sigma = \mathrm{diag}(1, -1)$ and $\tau = [aY + bZ : cY - aZ] \neq \sigma$. Consequently, C has a model of *Type* $2, (0, 1)$. However, all possible τ does not retain invariant the equation of *Type* $2, (0, 1)$, for any choice of the parameters, and the result follows: indeed, τ commutes with σ, therefore $\tau = \mathrm{diag}(-1, 1)$ or $[bZ : cY]$, with $bc \neq 0$. Hence C has the expressions:

$Z^4L_{1,Z}+Z^2L_{3,Z}+L_{5,Z}$ and $Y^4L_{1,Y}+Y^2L_{3,Y}+L_{5,Y}$, simultaneously. In particular, it reduces to the form $X\cdot G(X,Y,Z)$, a contradiction to irreducibility. $\square$

Now, we handle the situation for smooth $\overline{k}$-plane curves of degree 5, whose automorphism groups has an automorphism of order 5 (resp. order 4), as an element of maximal order.

Proposition 4.1.7. *Let k be a field of characteristic $p=0$ or $p>13$. Suppose that C is a smooth $\overline{k}$-plane curve of degree 5, with an automorphism σ of maximal order 5. Then we reduce, up to $\overline{k}$-isomorphism, to one of the following subcases:*

1. $\mathrm{Aut}(C)=\mathbb{Z}/5\mathbb{Z}$*, and C is of* Type $5,(0,1)$ *of the form $Z^5+L_{5,Z}=0$,*
2. $\mathrm{Aut}(C)=\mathrm{D}_{10}$*, generated by $\sigma:=\mathrm{diag}(1,\zeta_5,\zeta_5^2)$ and $\tau:=[Z:Y:X]$. Moreover, C is defined by the form $X^5+Y^5+Z^5+\beta_{3,1}X^2YZ^2+\beta_{4,3}XY^3Z=0$ for some $(\beta_{3,1},\beta_{4,3})\in\overline{k}^2\setminus\{(0,0)\}$.*

Proof. We investigate smooth $\overline{k}$-plane curves of *Type* $5,(a,b)$, which appear in Table 4.1:

(A) Type $5,(1,2)$: First $\mathrm{Aut}(C)$ is not conjugate to Hess_* with $*=36,\ 72$, and also C is not a descendant of of the Klein curve, since Hess_* and $\mathrm{Aut}(K_5)$ do not have elements of order 5. On the other hand, C always admits a larger automorphism group isomorphic to D_{10}, through the extra automorphism $\tau=[Z:Y:X]$, in particular, it is not cyclic. Moreover, we use Lemma 4.1.6 to discard A_5. Consequently, $\mathrm{Aut}(C)$ fixes a line and a point off this line, or C is a descendant of the Fermat curve F_5 of degree 5.

We treat each of these subcases:

(i) If $\mathrm{Aut}(C)$ fixes a line and a point off this line, then it should be the line $Y=0$, and the point is $(0:1:0)$, since $\langle\sigma,\tau\rangle\leq\mathrm{Aut}(C)$. Hence we reduce to automorphisms of the shapes

$$\begin{pmatrix}\alpha_1&0&\alpha_3\\0&1&0\\\gamma_1&0&\gamma_3\end{pmatrix}\in\mathrm{PGL}_3(\overline{k}).$$

From the coefficients of $Y^3Z^2,\ Y^3X^2$ (resp. $X^4Y,\ YZ^4$), we must have $\alpha_1=\gamma_3=0$ (resp. $\alpha_3=\gamma_1=0$). Moreover, $\alpha_{3,1}^5=\gamma_{1,3}^5=1$ and $\alpha_{3,1}\gamma_{1,3}=1$, or $(\alpha_{3,1}\gamma_{1,3})^2=1$, since $(\beta_{3,1}, \beta_{4,3}) \neq (0,0)$. This implies that $|\operatorname{Aut}(C)| = 10$, and then is exactly the dihedral group D_{10}.

(ii) If C is a descendant of the Fermat curve F_5, through a change of variables $P \in \mathrm{PGL}_3(\overline{k})$, and neither a line nor a point is leaved invariant, then we may impose $P^{-1}\sigma P = \sigma$, since automorphisms of the Fermat curve of order 5, which are not homologies forms a single conjugacy class of $\operatorname{Aut}(F_5)$. So P has one of the shapes $\operatorname{diag}(1, \lambda, \mu)$, $[Y : \lambda Z : \mu X]$, or $[Z : \lambda X : \mu Y]$. However, it is straightforward to verify that C_P has no more automorphisms in $\operatorname{Aut}(F_5)$ than $P^{-1}\langle \sigma, \tau\rangle P$. So again $\operatorname{Aut}(C) = \mathrm{D}_{10}$.

(B) Type $5, (0,1)$: In this case, we have a homology $\sigma = \operatorname{diag}(1, 1, \zeta_5) \in \operatorname{Aut}(C)$ of order $d = 5$, with center $(0 : 0 : 1)$ and axis $Z = 0$. Then, by Proposition 1.3.12, $(0 : 0 : 1)$ is an outer Galois point for C, moreover it is unique, since C is not $\overline{k}$-isomorphic to the Fermat curve (recall that any automorphism of C has order ≤ 5). Thus, it should be fixed by $\operatorname{Aut}(C)$. Consequently, $\operatorname{Aut}(C)$ fixes also the line $Z = 0$ (Proposition 1.2.9). Therefore, $\operatorname{Aut}(C)$ satisfies a short exact sequence (Theorem 1.4.4-(2))

$$1 \to N \to \operatorname{Aut}(C) \to G \to 1,$$

where N is cyclic of order dividing 5, and G is conjugate to $\mathbb{Z}/m\mathbb{Z}$, D_{2m}, A_4, S_4, or to A_5 with $m \leq d-1 (= 4)$, and moreover $m \,|\, d-2 (= 3)$ or N is trivial, when $G = \mathrm{D}_{2m}$. If N is trivial, then $G = \mathrm{A}_5$, since it is the only option with elements of order 5 are present inside. Hence, $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is a subgroup of $\operatorname{Aut}(C)$, a contradiction to Lemma 4.1.6. Thus $N = \mathbb{Z}/5\mathbb{Z}$. Moreover, for any value of G (except possibly $\{1\}$, $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$ and A_4 such that $\operatorname{Aut}(C) = \mathrm{D}_{10}$, $\mathrm{GAP}(20, 3)$, A_5), there are automorphisms of order greater than 5, which conflicts our assumption. Again, by Lemma 4.1.6, we exclude $G = \mathrm{A}_4$. Furthermore, one verifies that there exists no $\tau \in \operatorname{Aut}(C)$ of order 2, such that $\tau\sigma\tau = \sigma^{-1}$, that is $G \neq \mathbb{Z}/2\mathbb{Z}$. Also, there are no $\tau \in \operatorname{Aut}(C)$ of order 4, such that $(\tau\sigma)^2 = 1$ and $\sigma\tau\sigma^{-1} = \tau\sigma$. Thus $G \neq \mathbb{Z}/4\mathbb{Z}$, and $\operatorname{Aut}(C)$ is cyclic of order 5.

This completes the proof. □

Proposition 4.1.8. *Let k be a field of characteristic $p = 0$ or $p > 13$. Suppose that C is a smooth $\overline{k}$-plane curve of degree 5, with an automorphism σ of maximal order 4. Then $\mathrm{Aut}(C) = \langle\sigma\rangle$ and we reduce, up to $\overline{k}$-isomorphism, to one of the following situations:*

1. *σ acts on C as the automorphism $\mathrm{diag}(1, 1, \zeta_4)$, and C is defined by an equation of the form $Z^4Y + L_{5,Z}(X, Y) = 0$, such that $L_{5,Z}(X, \zeta_m Y) \neq \zeta_m^n L_{5,Z}(X, Y)$ for any $(m, n) \in \{(8, 1), (16, 1), (20, 4)\}$,*

2. *σ acts on C as the automorphism $\mathrm{diag}(1, \zeta_4, -1)$, and C is defined by $X^5 + X(Y^4 + Z^4) + \beta_{2,0}X^3Z^2 + \beta_{3,2}X^2Y^2Z + \beta_{5,2}Y^2Z^3 = 0$ for some $\beta_{5,2} \neq 0$.*

Proof. We study the two *Types* $4, (a, b)$ mentioned in Table 4.1: In both cases, C can not be a descendant of the Fermat curve F_5 or the Klein curve K_5, since 4 does not divide $|\mathrm{Aut}(F_5)| = 150$ and $|\mathrm{Aut}(K_5)| = 39$. Also, $\mathrm{Aut}(C) \neq \mathrm{A}_5$, because it has no elements of order 4. Therefore, $\mathrm{Aut}(C)$ is conjugate to one of the Hessian subgroups Hess_*, with $* = 36$ or 72, or it fixes a line and a point off this line (Theorem 1.4.4 and Corollary 4.1.2). Moreover, for the last case, we need to consider a short exact sequence $1 \to N = 1 \to \mathrm{Aut}(C) \to G \to 1$, where G must contain an element of order 4. So G could only be conjugate to $\mathbb{Z}/4\mathbb{Z}$, or D_8 (Lemma 4.1.6).

We treat now each of the subcases:

(A) *Type* $4, (0, 1)$: Similarly to *Type* $d - 1, (0, 1)$ in chapter 2, §2.3.1, there is a unique inner Galois point for C, and hence it should be fixed by $\mathrm{Aut}(C)$. Consequently, $\mathrm{Aut}(C)$ is cyclic. The algebraic restrictions on the binary form $L_{5,Z}$ are to avoid C to be with larger cyclic automorphism group, more precisely, to be of *Type* $8, (1, 4)$, *Type,* $16, (1, 12)$, or *Type* $20, (4, 5)$.

(B) *Type* $4, (1, 2)$: Clearly $\beta_{5,2} \neq 0$, or C decomposes to $X.G(X, Y, Z) = 0$, in particular it is singular. Second, we claim to show that non of Hess_* for $* = 36, 72$ occurs as the automorphism group of a smooth $\overline{k}$-plane curve of degree 5: We know that both groups contain reflections, but no four groups, hence all reflections in the group will be conjugate to $[Z : Y : X]$, see [Mit11, Theorem 11]. Therefore, we can take $P \in \mathrm{PGL}_3(\overline{k})$ with $P^{-1}\sigma^2 P = [Z : Y : X]$, where $\mathrm{Aut}(C_P) \hookrightarrow \mathrm{PGL}_3(\overline{k})$ is described by the usual representation of the above Hessian groups. In particular, $\mathrm{Aut}(C_P)$ always have the five automorphisms: $[Z:Y:X]$, $[X:Z:Y]$, $[Y:X:Z]$, $[Y:Z:X]$, and $\mathrm{diag}(1,\zeta_3,\zeta_3^2)$. Since the defining equation for C_P is invariant under the action of $[Z:Y:X]$, $[X:Z:Y]$, $[Y:X:Z]$, and $[Y:Z:X]$, C_P is defined by an equation of the form $u(X^5+Y^5+Z^5)+a(X^4Z+X^4Y+Y^4X+Y^4Z+Z^4X+Z^4Y)+H(X,Y,Z)$, for some $u,a\in K$ and $H(X,Y,Z)$, a homogenous polynomial of degree 5, such that the degree of any of the variables is at most three. Then consider the action of $\mathrm{diag}(1,\zeta_3,\zeta_3^2)$ to get $u=0$ and $a=0$, a contradiction to non-singularity (Lemma 2.1.1). This shows the claim.

On the other hand, because N is trivial and $\sigma\in\mathrm{Aut}(C)$ is a non-homology, we can suppose that the fixed line is $X=0$ and the point is $(1:0:0)$. That is, all automorphisms of C are of the shape $[X:vY+wZ:sY+tZ]\in\mathrm{PGL}_3(K)$. One checks that there is no automorphism τ of this shape of order 2, with $\tau\sigma\tau=\sigma^{-1}$. Hence $\mathrm{Aut}(C)$ is not conjugate to D_8, and therefore it is the cyclic group of order 4.

This finishes the proof. □

Now it remains the study of smooth $\overline{k}$-plane curves C of degree 5, such that their automorphisms are of orders at most 3. So $\mathrm{Aut}(C)$ is not conjugate to A_5 and Hess_* with $*=36,72$, since in each case, automorphisms of order >3 exist. Therefore $\mathrm{Aut}(C)$ fixes a line and a point off this line, or it is conjugate to a subgroup of $\mathrm{Aut}(F_5)$ or $\mathrm{Aut}(K_5)$.

Proposition 4.1.9. *Let k be a field of characteristic $p=0$ or $p>13$, and let C be a smooth $\overline{k}$-plane curve of degree 5 of* Type $3,(1,2)$, *such that automorphisms of C have orders, at most 3. Then C is defined by the normal form*

$$X^5+Y^4Z+YZ^4+\beta_{2,1}X^3YZ+X^2(\beta_{3,0}Z^3+\beta_{3,3}Y^3)+\beta_{4,2}XY^2Z^2=0.$$

Moreover, $\mathrm{Aut}(C)=\mathbb{Z}/3\mathbb{Z}=\langle\mathrm{diag}(1,\zeta_3,\zeta_3^2)\rangle$ *when $\beta_{3,0}\neq\beta_{3,3}$, and* $\mathrm{Aut}(C)=\mathrm{S}_3$ *via the extra automorphism* $[X:Z:Y]$ *otherwise.*

Proof. Because $|\mathrm{Aut}(K_5)|=3\cdot 13$, then if C is a descendant of the Klein curve K_5, then $\mathrm{Aut}(C)$ is exactly a $\mathbb{Z}/3\mathbb{Z}$ inside $\mathrm{Aut}(K_5)$, since otherwise an automorphism of C of order

> 3 should exist by Sylow's theorem. Second, if C is a descendant of the Fermat curve F_5, then $\mathrm{Aut}(C)$ is a $\mathbb{Z}/3\mathbb{Z}$ or S_3 inside $\mathrm{Aut}(F_5)$: indeed, $|\mathrm{Aut}(F_5)| = 2{\cdot}3{\cdot}5^2$, hence any subgroup of order > 3 is conjugate to S_3 (remark that $\mathrm{Aut}(F_5)$ has no elements of order 6), or it contains elements of order $5 > 3$. Moreover, if $\mathrm{Aut}(C) = \mathrm{S}_3$, then there exists $\tau \in \mathrm{Aut}(C)$ of order 2 such that $\tau\sigma\tau = \sigma^{-1}$. This reduces τ to be of the shape $[X : \beta Z : \beta^{-1}Y]$, which retains the defining equation for C if and only if $\beta^3 = 1$ and $\beta_{3,0} = \beta_{3,3}$. Third, if $\mathrm{Aut}(C)$ fixes a line and a point of this line, then (Theorem 1.4.4-(2) and Lemma 4.1.6) $\mathrm{Aut}(C)$ satisfies a short exact sequence of the form $1 \to N = 1 \to \mathrm{Aut}(C) \to G \to 1$, where G is conjugate to $\mathbb{Z}/3\mathbb{Z}$ or S_3, which was to be shown. □

Proposition 4.1.10. *Let k be a field of characteristic $p = 0$ or $p > 13$, and let C be a smooth $\overline{k}$-plane curve of degree 5 of* Type $2, (0, 1)$*, such that any automorphism of C has order, at most 2. Then C is $\overline{k}$-isomorphic to $Z^4Y + Z^2L_{3,Z} + L_{5,Z} = 0$, and* $\mathrm{Aut}(C) = \mathbb{Z}/2\mathbb{Z} = \langle \mathrm{diag}(1, 1, -1)\rangle$.

Proof. Clearly C is not a descendant of the Klein curve, since $|\mathrm{Aut}(K_5)| = 39$ is odd. Also, if it is a descendant of the Fermat curve, then $\mathrm{Aut}(C)$ is a $\mathbb{Z}/2\mathbb{Z}$ in $\mathrm{Aut}(F_5)$, as $|Aut(F_5)| = 2 \cdot 3 \cdot 5^2$, so subgroups of order > 2 obviously have elements of order > 2. Lastly, if $\mathrm{Aut}(C)$ fixes a line and a point off this line, then we think about it in a short exact sequence $1 \to N = 1 \to \mathrm{Aut}(C) \to G \to 1$, where G contain an element of order 2 and no higher orders happens. Therefore $\mathrm{Aut}(C)$ should be $\mathbb{Z}/2\mathbb{Z}$, or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. However, we exclude the latter case by Lemma 4.1.6. □

We need to assure the existence of a smooth $\overline{k}$-plane curve C, through some specializations of the parameters, for which the maximal order of its automorphisms is exactly m, for $m \leq 5$. This is some sort of tedious computations, since we do not know a priori the dimension of the strata $\widetilde{\mathcal{M}_6}(G)$. We already saw the situation when $m = 4$ in chapter 2, §2.3, also when $m = 3$ and $G = \mathbb{Z}/3\mathbb{Z}$. To treat the case when $m \neq 4$, we can apply similar arguments, which will not be reproduced here (nevertheless, we know all the possible groups and the representations that could appear such that m divides their orders). This in turns simplifies the computations, in order to conclude:

Lemma 4.1.11. *Let k be a field of characteristic $p = 0$ or $p > 13$. Consider a degree 5*

homogenous equation $F(X,Y,Z)=0$ over $\overline{k}$, associated to some Type $m,(a,b)$ *in Table 4.1, where $m\leq 5$ and $m\neq 4$. Then, there exists a smooth $\overline{k}$-plane curve C obtained by a concrete specialization of the parameters, such that all of its automorphisms are of order, at most m. Moreover, for* Type $3,(1,2)$, *we have smooth $\overline{k}$-plane curves with this property, where some of them satisfy $\beta_{3,0}\neq\beta_{3,3}$ and others also satisfy $\beta_{3,0}=\beta_{3,3}$.*

4.1.2 How looks like the stratification by automorphism groups?

We sum up all the results obtained previously in section §4.1.

Theorem 4.1.12 (Badr-Bars, [BB16a])**.** *Let $\overline{k}$ be a fixed algebraic closure of a field k of characteristic p with $p=0$ or $p>13$. The following table gives the complete list of automorphism groups of non-singular plane curves of degree 5 over $\overline{k}$, along with normal forms (or geometrically complete families over k using Remark 2.1.8 and Definition 4.3.2) for the associated strata. We denote by $L_{i,B}$ a homogeneous polynomial of degree i in the variables $\{X,Y,Z\}\setminus\{B\}$.*

Case	G	$\varrho(G)$	$F_{\varrho(G)}(X,Y,Z)$
1	$\mathrm{GAP}(150,5)$	$[\zeta_5X:Y:Z],[X:\zeta_5Y:Z]$ $[X:Z:Y]$, $[Y:Z:X]$	$X^5+Y^5+Z^5$
2	$\mathrm{GAP}(39,1)$	$[X:\zeta_{13}Y:\zeta_{13}^{10}Z],[Y:Z:X]$	$X^4Y+Y^4Z+Z^4X$
3	$\mathrm{GAP}(30,1)$	$[X:\zeta_{15}Y:\zeta_{15}^{11}Z],[X:Z:Y]$	$X^5+Y^4Z+YZ^4$
4	$\mathbb{Z}/20\mathbb{Z}$	$[X:\zeta_{20}^4Y:\zeta_{20}^5Z]$	$X^5+Y^5+XZ^4$
5	$\mathbb{Z}/16\mathbb{Z}$	$[X:\zeta_{16}Y:\zeta_{16}^{12}Z]$	$X^5+Y^4Z+XZ^4$
6	$\mathbb{Z}/10\mathbb{Z}$	$[X:\zeta_{10}^2Y:\zeta_{10}^5Z]$	$X^5+Y^5+XZ^4+\beta_{2,0}X^3Z^2$
7	D_{10}	$[X:\zeta_5Y:\zeta_5^2Z]$, $[Z:Y:X]$	$X^5+Y^5+Z^5+\beta_{3,1}X^2YZ^2+\beta_{4,3}XY^3Z$
8	$\mathbb{Z}/8\mathbb{Z}$	$[X:\zeta_8Y:\zeta_8^4Z]$	$X^5+Y^4Z+XZ^4+\beta_{2,0}X^3Z^2$
9	S_3	$[X:\zeta_3Y:\zeta_3^2Z]$ $[X:Z:Y]$	$X^5+Y^4Z+YZ^4+\beta_{2,1}X^3YZ+\beta_{3,3}X^2(Z^3+Y^3)+$ $+\beta_{4,2}XY^2Z^2$
10	$\mathbb{Z}/5\mathbb{Z}$	$[X:Y:\zeta_5Z]$	$Z^5+L_{5,Z}$
11	$\mathbb{Z}/4\mathbb{Z}$	$[X:\zeta_4Y:\zeta_4^2Z]$	$X^5+X(Z^4+Y^4)+\beta_{2,0}X^3Z^2+\beta_{3,2}X^2Y^2Z+\beta_{5,2}Y^2Z^3$
12	$\mathbb{Z}/4\mathbb{Z}$	$[X:Y:\zeta_4Z]$	$Z^4L_{1,Z}+L_{5,Z}$
13	$\mathbb{Z}/3\mathbb{Z}$	$[X:\zeta_3Y:\zeta_3^2Z]$	$X^5+Y^4Z+YZ^4+\beta_{2,1}X^3YZ+$ $+X^2(\beta_{3,0}Z^3+\beta_{3,3}Y^3)+\beta_{4,2}XY^2Z^2$
14	$\mathbb{Z}/2\mathbb{Z}$	$[X:Y:\zeta_2Z]$	$Z^4L_{1,Z}+Z^2L_{3,Z}+L_{5,Z}$
15	$\{1\}$	$[X:Y:Z]$	$L_5(X,Y,Z)$

Table 4.2: Geometrically complete families over k

The algebraic restrictions on the coefficients, so that each family is smooth, geometrically irreducible, and has no larger automorphism group are not given for seek of simplicity.

Table 4.2 can be seen as the Henn Table (Theorem 2.2.1), but for smooth $\overline{k}$-plane curves of degree 5. The following diagram shows how looks like the stratification by automorphism groups of non-singular plane quintic curves (we will justify the computation of the dimensions later):

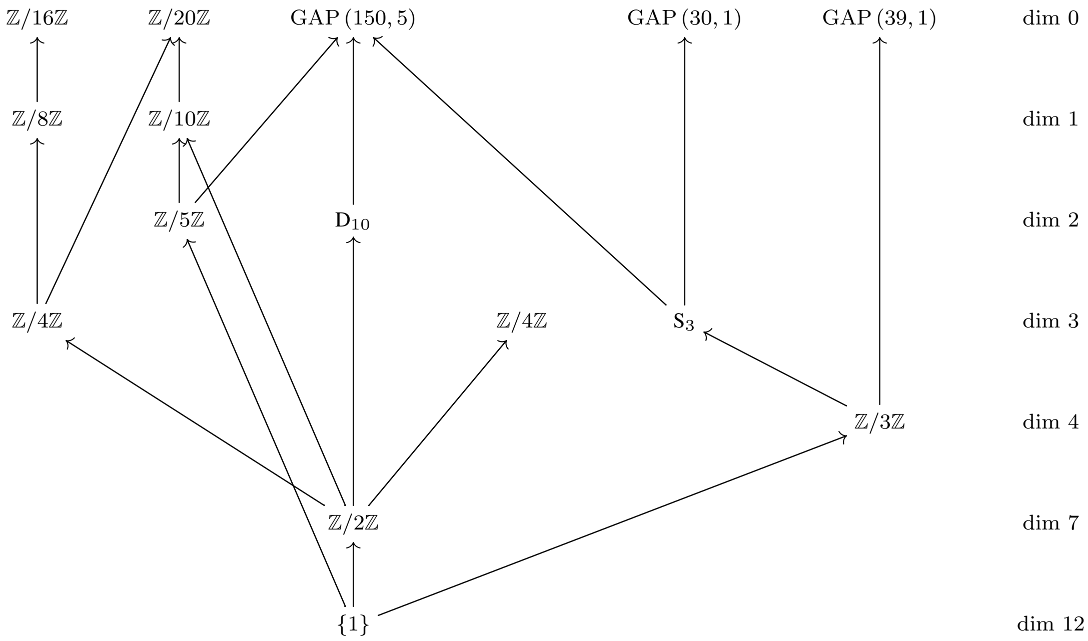


Figure 4.1: Stratification by automorphisms group

Remark 4.1.13. Table 4.2 confirms that $G = \mathbb{Z}/4\mathbb{Z}$ is the only group, such that $\widetilde{\mathcal{M}_6^{Pl}}(G)$ is not ES-irreducible, as we mentioned before in chapter 2, §2.3. In other words, for $\mathbb{Z}/4\mathbb{Z}$, we obtain two ρ's, where their normal forms $F_\rho(X, Y, Z) = 0$ are not $\overline{k}$-isomorphic, and corresponds to a disjoint decomposition of $\widetilde{\mathcal{M}_6^{Pl}}(\mathbb{Z}/4\mathbb{Z})$.

§4.2 Final families: A canonical interpretation

We can see in Figure 4.1 a phenomenon that does not happen for degree 4. We define a final stratum by automorphism groups to be a stratum that does not properly contain any other stratum. There is a final stratum in $\mathcal{M}_6^{Pl}$ of dimension greater than zero. This may sound odd since we could expect that by adding conditions in the parameters we would get bigger automorphisms groups. However, we will see that this is a normal situation for higher degrees.

4.2.1 A canonical interpretation

The family for the stratum with $\rho(G) = \langle \mathrm{diag}(1, \zeta_4, -1) \rangle$ can be described by the equation[1]

$$\mathcal{S}_{A,B,C} : \; X^5 + AX^3Z^2 + BX^2Y^2Z + CXY^4 + XZ^4 + Y^2Z^3 = 0.$$

This stratum has dimension 3 and it is a final stratum: no restriction of the parameters give a larger automorphism group. Now, it is easy to see that making $A = B = C = 0$ we get a larger automorphism group. For instance, we get the new automorphism $\mathrm{diag}(1, \zeta_8, \zeta_4)$. However, the plane curve defined by this equation is singular.

We next show an explanation of this final stratum not having dimension zero. We will regard the family $\mathcal{S}_{A,B,C}$ in $\mathcal{M}_6^{Pl}$ inside a family in $\mathcal{M}_6$ that is not final. When we add restrictions there, we get extra symmetries and the curve is not plane anymore.

Let us start by computing the family $\mathcal{K}_{A,B,C}$ of canonical models of $\mathcal{S}_{\mathcal{A},\mathcal{B},\mathcal{C}}$ in $\mathbb{P}^{g-1}_{\overline{k}} = \mathbb{P}^5_{\overline{k}}$. We define the functions $x = X/Z$ and $y = Y/Z$.

$$\begin{aligned}
\mathrm{div}(x) &= 2(0:0:1) + 2(0:1:0) - \sum_{s=1}^{4} (\zeta_4^s \sqrt[4]{-C} : 1 : 0) \\
&:= 2P + 2Q - \sum_{i=1}^{4} R_i, \\
\mathrm{div}(y) &= (0:0:1) - (0:1:0) - \sum_{s=1}^{4} (\zeta_4^s \sqrt[4]{-C} : 1 : 0) + (1:0:\pm\sqrt{\tfrac{-A\pm\sqrt{A^2-4}}{2}}) \\
&:= P - Q - \sum_{i=1}^{4} R_i + \sum_{i=1}^{4} T_i.
\end{aligned}$$

In order to compute $\mathrm{div}(dx)$, we work with the affine form

$$F(x, y, 1) = x^5 + Ax^3 + Bx^2y^2 + Cxy^4 + x + y^2.$$

[1] One starts with the defining family equation in Table 4.1, case 11. By non-singularity, the coefficient of Y^2Z^3 must be non-zero, so we can re-scale Y to get $\mathcal{S}_{A,B,C}$.

The differential dx is an uniformizer for all points except for P and the T_i's because the tangent space to the curve at these points have equation $x - \alpha$ for some $\alpha \in \overline{k}(A, B, C)$ (we have used $d(x - \alpha) = dx$). Then, for those points, we have to work with the expression

$$dx = -\frac{y(2Bx^2 + 4Cxy^2 + 2)}{5x^4 + 3Ax^2 + 2Bxy^2 + Cy^4 + 1}dy.$$

We finally get

$$\text{div}(dx) = P + Q + \sum_{i=1}^{4} R_i + \sum_{i=1}^{4} T_i,$$

and a basis of regular differentials is given by

$$\begin{aligned}\omega_0 &= \frac{dx}{y},\ \omega_1 = \frac{xdx}{y},\ \omega_2 = dx,\\ \omega_3 &= \frac{x^2dx}{y},\ \omega_4 = xdx,\ \omega_5 = ydx.\end{aligned}$$

Proposition 4.2.1. *The ideal of the canonical model of $\mathcal{S}_{A,B,C}$ in $\mathbb{P}^5[\omega_0, ..., \omega_5]$ is generated by the polynomials*

$$\omega_0\omega_3 = \omega_1^2,\ \omega_0\omega_4 = \omega_1\omega_2,\ \omega_0\omega_5 = \omega_2^2,\ \omega_4^2 = \omega_3\omega_5,\ \omega_1\omega_5 = \omega_2\omega_4,\ \omega_1\omega_4 = \omega_2\omega_3,$$

$$\omega_1\omega_3^2 + A\omega_1^3 + B\omega_0\omega_4^2 + C\omega_2\omega_4\omega_5 + \omega_0^2\omega_1 + \omega_0^2\omega_5 = 0,$$

$$\omega_3^3 + A\omega_0\omega_3^2 + B\omega_1\omega_3\omega_5 + C\omega_3\omega_5^2 + \omega_0^2\omega_3 + \omega_0\omega_1\omega_5 = 0,$$

$$\omega_4\omega_3^2 + A\omega_0\omega_3\omega_4 + B\omega_2\omega_3\omega_5 + C\omega_4\omega_5^2 + \omega_0^2\omega_4 + \omega_0\omega_2\omega_5 = 0.$$

We denote it by $\mathcal{K}_{A,B,C}$.

Proof. If $\omega_0 \neq 0$, then the des-homogenization of this ideal with respect to ω_0 gives

$$\omega_1^5 + A\omega_1^3 + B\omega_1^2\omega_2^2 + C\omega_1\omega_2^4 + \omega_1^4 + \omega_2^2,$$

and we recover the affine curve $\mathcal{S}_{A,B,C}$ for $Z = 1$. If $\omega_0 = 0$, then $\omega_1 = \omega_2 = 0$, $\omega_3\omega_4 = \omega_5^2$, $\omega_3(\omega_3^2 + \omega_5^2) = 0$ and we recover the points at infinity for $\mathcal{S}_{A,B,C}$: Q, R_i's.

To check that it is non-singular, we need to see if the rank of the matrix of partial derivatives of the previous generating functions has rank equal to dim$(\mathbb{P}^5)$ − dim$(\mathcal{K}_{A,B,C}) = 4$ at every point, that is, the tangent space has codimension 4. If $\omega \neq 0$, the partial derivatives of the first three equations plus the equation in the second line produce 4 linearly independent vectors in the tangent space. If $\omega_0 = 0$, then ω_5 is non-zero, and, the 3rd, 4th, and the 6th equations plus the equation in the last line provide 4 linearly independent vectors. Moreover, this is independent of the choice of the parameters A, B, C. $\square$

Corollary 4.2.2. *If we specialize the parameters to $A = B = C = 0$, then we get a smooth curve of genus 6, whose full automorphism group has order multiple of 8 and contains*

$$\mathrm{diag}(\zeta_8^7, \zeta_8^5, \zeta_8^6, \zeta_8^3, \zeta_8^4, \zeta_8^5).$$

This curve does not admit a non-singular plane model over $\overline{k}$. So the 11th stratum of plane curves of genus 6 in Table 4.2 that is final as a plane stratum, is indeed living inside a stratum of smooth curves of genus 6 which is not final.

Proof. We only need to check that the curve $\mathcal{K}_{0,0,0}$ is not isomorphic to any one in the family 8th in Table 4.2 with automorphism group $\mathbb{Z}/8\mathbb{Z}$. In order to check that we just look at the automorphism $[X : \zeta_8 Y : -Z]$ of order 8 in this family acting in its canonical model. We mimic the previous computations and we get the matrix $\mathrm{diag}(\zeta_8^5, \zeta_8^6, \zeta_8, \zeta_8^7, \zeta_8^2, \zeta_8^5)$. The group generated by this matrix is clearly non conjugated to the group generated by the one in the statement of the corollary. So, the curve $\mathcal{K}_{0,0,0}$ does not have a smooth plane model. $\square$

We reinterpret the existence of these special kind of families of plane curves in terms of g_d^r linear series as follows: suppose that $\mathcal{C}$ is such a family describing a stratum of plane curves, and let D be a divisor in $\mathrm{Div}(\mathcal{C})$ that defines a g_d^2 linear series for $\mathcal{C}$. In particular, D is of degree d, and the vector space $\mathcal{L}(D)$ has dimension 2. For some specializations of the parameters in the canonical family $\mathcal{K}$, given by the canonical embedding $\Phi : \mathcal{C} \hookrightarrow \mathbb{P}^{g-1}(\overline{k})$, one gets more automorphisms. Hence more symmetries in the defining equations, which in turns produce more meromorphic functions with poles bounded above by $\mathcal{D} = \Phi(D)$. Therefore, $dim(\mathcal{L}(\mathcal{D})) > 2$, and we do not get a smooth $\overline{k}$-plane model anymore.

In our example the divisor D generating the g_2^d-linear system is $D = Q + \sum_{i=1}^4 R_i$, and $\mathcal{L}(\mathcal{D})$ is generated by $1, \frac{\omega_1}{\omega_0}, \frac{\omega_2}{\omega_0}$. For the special choice of the parameters $A = B = C = 0$, we get $D = 5Q$ and $\mathcal{L}(\mathcal{D})$ contains the linearly independent functions $1, \frac{\omega_1}{\omega_0}, \frac{\omega_2}{\omega_0}, \frac{\omega_3}{\omega_0}$, so the (projective) dimension of $\mathcal{L}(\mathcal{D})$ is greater than 2 and it does not define a g_2^d linear system anymore. This why, for this choice of the parameters we do not get a smooth plane model anymore, and it is due to the extra symmetries of the curve.

4.2.2 Non-zero dimensional final strata for higher odd degrees

In this subsection we show examples of non-zero dimensional final strata in $\mathcal{M}_g^{Pl}$ for infinitely many g's. Throughout this subsection, k has characteristic $p = 0$ or $p > 2g + 1$.

To prove that they are final strata, that is, that there cannot be more automorphisms, we use similar techniques to the ones used in section §2.3. In particular, we need the next Theorem, which follows when one combines Theorem 1.2.1, and the proof of Theorem 1.4.4

Theorem 4.2.3 (Mitchell, Harui)**.** *Let G be a subgroup of automorphisms of a smooth $\overline{k}$-plane curve C of degree $d \geq 4$. Then one of the following situations holds:*

1. *G fixes a line and a point off this line.*
2. *G fixes a triangle and neither line nor a point is leaved invariant. In this case, (C, G) is a descendant of the the Fermat curve $F_d : X^d + Y^d + Z^d = 0$ or the Klein curve $K_d : XY^{d-1} + YZ^{d-1} + ZX^{d-1} = 0$.*
3. *G is conjugate to a finite primitive subgroup of $\mathrm{PGL}_3(\overline{k})$ namely, the Klein group $\mathrm{PSL}(2,7)$, the icosahedral group A_5, the alternating group A_6, the Hessian group Hess_* with $* \in \{36, 72, 216\}$.*

We prove:

Theorem 4.2.4. *Let $\mathcal{C}$ be a family of smooth $\overline{k}$-plane curves of an odd degree $d \geq 7$ with $d \equiv 1 \pmod 4$, defined by an equation of the form*

$$\begin{aligned} X^d + XY^{d-1} \;&+\; aXZ^{d-1} + Y^{(d+1)/2}Z^{(d-1)/2} + \\ &+ \sum_{\substack{j \text{ odd} \\ 1 \leq j \leq (d-3)/2}} b_j X^{(d+1)/2-j} Y^j Z^{(d-1)/2} + \sum_{\substack{j \text{ even} \\ 2 \leq j \leq d-3}} c_j X^{d-j} Y^j = 0. \end{aligned}$$

Then, it is a non-zero dimensional final stratum with automorphism group $\langle \mathrm{diag}(1, -1, \zeta_{d-1}) \rangle$.

Proof. We first note that $\eta:\ (X:Y:Z)\mapsto(X:-Y:\zeta_{d-1}Z)$ defines an automorphism of $\mathcal{C}$ of order $d-1$, and also $a\neq 0$ by non-singularity. Because $\eta^2=\mathrm{diag}(1,1,\zeta_{(d-1)/2})\in\mathrm{Aut}(\mathcal{C})$ is a homology of period $(d-1)/2\geq 3$, then by Theorem 1.2.8, $\mathrm{Aut}(\mathcal{C})$ is conjugate to the Hessian group Hess_{216} or it fixes a point, a line, or a triangle.

Any element of Hess_{216} has order at most 6. Hence if $\mathrm{Aut}(\mathcal{C})$ is the Hessian group then $d=7$. However, applying the same argument that we did for Lemma 2.3.2, we deduce that there exists no smooth $\overline{k}$-plane curve of degree 7, whose automorphism group is conjugate to Hess_{216}. That is, $\mathrm{Aut}(\mathcal{C})$ fixes a point, a line, or a triangle. On the other hand, $\mathcal{C}$ can not be a descendant of the Klein curve K_d, since $d-1$ does not divide $|\,\mathrm{Aut}(K_d)|=3(d^2-3d+3)$. Similarly, it is not a descendant of the Fermat curve, except possibly if $d=7$. Fix $d=7$, and recall that the automorphisms of $F_7\,:\,X^7+Y^7+Z^7=0$ are of the shapes $[X:\zeta_7^sY:\zeta_7^tZ]$, $[\zeta_7^sZ:\zeta_7^tY:X]$, $[X:\zeta_7^sZ:\zeta_7^tY]$, $[\zeta_7^sY:X:\zeta_7^tZ]$, $[\zeta_7^sY:\zeta_7^tZ:X]$, or $[\zeta_7^sZ:X:\zeta_7^tY]$ for some integers s,t. Non of these transformations has order 6, so $\mathcal{C}$ with $d=7$ is not a descendant of the Fermat curve F_7 of degree 7. Consequently, $\mathrm{Aut}(\mathcal{C})$ in $\mathrm{PGL}_3(\overline{k})$ must fix a line and a point off this line. In particular, the fixed line is one of the reference lines $B=0$ with $B\in\{X,Y,Z\}$, and the point is one of the reference points P_i, for $i=1,2,3$, since $\eta\in\mathrm{Aut}(\mathcal{C})$ does. Consequently, all automorphisms of $\mathcal{C}$ are all of one of the next shapes; $[X:vY+wZ:sY+tZ]$, $[vX+wZ:Y:sX+tZ]$, or $[vX+wY:sX+tY:Z]$. In any case, we always get $s=w=0$ through the term $Y^{(d+1)/2}Z^{(d-1)/2}$, and since Y^d, Z^d does not appear in the defining equation for $\mathcal{C}$. Consequently, any automorphism of $\mathcal{C}$ is diagonal, say $\mathrm{diag}(1,\lambda,\mu)$, where $\lambda^{d-1}=\mu^{d-1}=\lambda^{(d+1)/2}\mu^{(d-1)/2}=1$. So $\lambda=\pm1$ and $\mu=\zeta_{d-1}^{2s+1}$ for some integer s. In other words, $|\,\mathrm{Aut}(\mathcal{C})|=d-1$ and $\mathrm{Aut}(\mathcal{C})=\langle\mathrm{diag}(1,-1,\zeta_{d-1})\rangle$.

Finally, we show that our family is final. Given a smooth plane curve C over $\overline{k}$ of an odd degree $d\geq 7$ with $d\equiv 1\,(\mathrm{mod}\,4)$, such that $\mathrm{diag}(1,-1,\zeta_{d-1})$ is an automorphism, then C must be defined by an equation of the form

$$\begin{aligned}X^d+XY^{d-1}\ &+\ XZ^{d-1}+\\ &+\sum_{\substack{j\ \mathrm{odd}\\ 1\leq j\leq (d+1)/2}} b_jX^{(d+1)/2-j}Y^jZ^{(d-1)/2}+\sum_{\substack{j\ \mathrm{even}\\ 2\leq j\leq d-3}} c_jX^{d-j}Y^j=0.\end{aligned}$$

By non-singularity, we must have $b_{(d+1)/2} \neq 0$, or C decomposes as $X \cdot G(X, Y, Z)$ for some homogenous polynomial of degree $d-1$, and it becomes singular. Now, rescale the varaible Z in order to make $Y^{(d+1)/2}Z^{(d-1)/2}$ has coefficient 1, and rename the parameters after to get a non-singular plane model in the family of Theorem 4.2.4. $\square$

§4.3 Complete and representative families

R. Lercier, C. Ritzenthaler, F. Rovetta and J. Sijsling in [LRRS14] explicitly constructed normal forms, which are complete and representative families over k (Definition 4.3.2) for smooth plane quartics with automorphism group of order > 2. These kind of families are used to determine unique representatives for the isomorphism classes of smooth plane quartics over finite fields. We also refer to the PhD thesis [LG14, Ch. 2] for such families of smooth plane quartic curves over number fields. We start with a classification already obtained in section §4.1 and we mimic the techniques in [LRRS14] and [LG14].

First, for convenience, we recall the definitions of complete and representative families over a field k of characteristic $p = 0$ or $p > 2g + 1$. For more details, see [LRRS14, §2].

Definition 4.3.1. Let $\mathtt{S}$ be a scheme over k. A family of smooth curves of genus $g \geq 2$ over $\mathtt{S}$ is a morphism of schemes $\mathtt{C} \to \mathtt{S}$ that is proper and smooth with geometrically irreducible fibers of dimension 1 and genus g.

Definition 4.3.2. [complete, finite, representative family] Let $\mathtt{C}$ be a family of smooth curves over a scheme $\mathtt{S}$ over k, and assume that each geometric fiber of $\mathtt{C}$ corresponds to a point of a fixed stratum $S \subseteq \mathcal{M}_g$. We then get a morphism $f_{\mathtt{C}} : \mathtt{S} \to S$ over k. The family $\mathtt{C} \to \mathtt{S}$ is *complete* (resp. *representative*) *over* k for the stratum S if $f_{\mathtt{C}}$ is surjective (resp. bijective) on F-points for every algebraic extension F/k. If the family is complete over k and all the fibers of $f_{\mathtt{C}}$ are finite and with bounded cardinality, we say that the family is *finite* over k. In particular, if a family is finite, the dimension of the family is equal to the dimension of the scheme $\mathtt{S}$.

The family $\mathtt{C} \to \mathtt{S}$ is *geometrically complete* (*representative*) over k if it is complete (*representative*) after extending the scalers to $\overline{k}$.

Lemma 4.3.3 (Lemma 2.2, [LRRS14]). *If a family defined over a perfect field k is geometrically representative, then it is representative and complete over k. In this case, all the curves in the family are defined over the field of moduli (see section §5.1), since they cannot be isomorphic to its conjugates.*

Remark 4.3.4. In the above sense, the families given in Theorem 4.1.12 are not necessarily representative or even complete over a field k of characteristic $p = 0$ or $p > 13$. For example, the smooth plane curve C defined over $\mathbb{Q}$ by

$$C : X^5 + Y^5 + \frac{1}{2} XZ^4 + X^3Z^2 = 0$$

is isomorphic through $\phi_\lambda = \mathrm{diag}(1, 1, \lambda\sqrt[4]{2})$, where $\lambda = 1$ or ζ_4, to

$$X^5 + Y^5 + XZ^4 \pm \sqrt{2}X^3Z^2 = 0,$$

respectively. In particular, it has two representatives in Table 4.2 with automorphism group isomorphic to $\mathbb{Z}/10\mathbb{Z}$. However, non of them is defined over $\mathbb{Q}$.

We start with the families in Theorem 4.1.12, which are geometrically complete over k for each of the strata $\varrho(\widetilde{\mathcal{M}_6^{Pl}}(G))$. Isomorphisms between two curves in the same family, in particular with identical automorphism group $\varrho(G)$ in $\mathrm{PGL}_3(\overline{k})$, are clearly given by 3×3 projective matrices in the normalizer $N_{\varrho(G)}(\overline{k})$ over $\overline{k}$.

Theorem 4.3.5. *Let $\varrho(G)$ be one of the automorphism groups given by Theorem 4.1.12, such that $\widetilde{\mathcal{M}_6^{Pl}}(\varrho(G))$ is not 0-dimensional. The normalizer $N_{\varrho(G)}(\overline{k})$ of $\varrho(G)$ in $\mathrm{PGL}_3(\overline{k})$ is generated by:*

- $\mathrm{N}_{\{1\}}(\overline{k}) = \mathrm{PGL}_3(\overline{k})$;
- $\mathrm{N}_{\varrho(\mathbb{Z}/2\mathbb{Z})}(\overline{k}) = \mathrm{GL}_{2,Z}(\overline{k})$;
- $\mathrm{N}_{\varrho(\mathbb{Z}/3\mathbb{Z})}(\overline{k}) = \langle \mathrm{D}(\overline{k}), \tilde{S}_3 \rangle$;
- $\mathrm{N}_{\varrho(\mathbb{Z}/4\mathbb{Z})}(\overline{k}) = \mathrm{GL}_{2,Z}(\overline{k})$;
- $\mathrm{N}_{\varrho(\mathbb{Z}/4\mathbb{Z})}(\overline{k}) = \langle \mathrm{D}(\overline{k}), [Z : Y : X] \rangle$;
- $\mathrm{N}_{\varrho(\mathbb{Z}/5\mathbb{Z})}(\overline{k}) = \mathrm{GL}_{2,Z}(\overline{k})$;
- $\mathrm{N}_{\varrho(\mathrm{S}_3)}(\overline{k}) = \langle T_X(\overline{k}), G_{03} \rangle$;
- $\mathrm{N}_{\varrho(\mathbb{Z}/8\mathbb{Z})}(\overline{k}) = \langle \mathrm{D}(\overline{k}), [Z : Y : X] \rangle$;
- $\mathrm{N}_{\varrho(\mathrm{D}_{10})}(\overline{k}) = \langle T_Y(\overline{k}),\ G_{05} \rangle$;
- $\mathrm{N}_{\varrho(\mathbb{Z}/10\mathbb{Z})} = \mathrm{D}(\overline{k})$.

Proof. The Theorem is a straightforward implication from the well-known result that says that two non-singular matrices commute if and only if there is a common basis in which both of them diagonalize or one is a multiple of the identity. As an example, we prove the cases $\varrho(\mathbb{Z}/3\mathbb{Z})$ and $\varrho(S_3)$ simultaneously, and the remaining situations are proven in the same way: if $\phi \in \mathrm{N}_{\varrho(\mathbb{Z}/3\mathbb{Z})}(\overline{k})$, then $\phi^{-1}\,\mathrm{diag}(1,\zeta_3,\zeta_3^2)\phi = \mathrm{diag}(1,\zeta_3,\zeta_3^2)$, or $\mathrm{diag}(1,\zeta_3^2,\zeta_3)$. Hence, ϕ is diagonal or a permutation of the variables, up to a re-scaling. In particular, ϕ is a product of an element of $\mathrm{D}(\overline{k})$ and an element of $\tilde{S}_3$, which gives the situation for $\varrho(\mathbb{Z}/3\mathbb{Z})$. On the other hand $\mathrm{N}_{\varrho(S_3)}(\overline{k}) \subseteq \mathrm{N}_{\varrho(\mathbb{Z}/3\mathbb{Z})}(\overline{k})$. Furthermore, if $\phi \in \mathrm{N}_{\varrho(\mathbb{Z}/3\mathbb{Z})}(\overline{k})$ such that $\phi^{-1}[X:Z:Y]\phi$ is of order 2 in $\varrho(\mathrm{S}_3) = \langle \mathrm{diag}(1,\zeta_3,\zeta_3^2),\ [X:Z:Y]\rangle$, then

$$\phi \in \{\mathrm{diag}(a,\zeta_3^r,1), [aX:\zeta_3^r Z:Y]\,|\, a\in\overline{k} \text{ and } 0\le r\le 2\}.$$

Rewriting $\mathrm{diag}(a,\zeta_3^r,1)$ as $\mathrm{diag}(a,1,1)\,\mathrm{diag}(a,\zeta_3,1)^r$, and $[aX : \zeta_3^r Z : Y]$ as $\mathrm{diag}(a,\zeta_3^r,1)[X:Z:Y]$, gives the conclusion for $\varrho(\mathrm{S}_3)$. □

Theorem 4.3.6 (Representative families). *The following table shows representative families over a perfect field k of characteristic $p = 0$ or $p > 13$, for each stratum of smooth $\overline{k}$-plane curves of genus 6, with non-trivial automorphism group of order $\neq 5$. For $\widetilde{\mathcal{M}_6^{Pl}}(\varrho(\mathbb{Z}/5\mathbb{Z}))$ a geometrically complete family is shown.*

Case	G	$F_{\varrho(G)}(X;Y;Z)$	*Parameters restrictions*
1	$\mathrm{GAP}(150,5)$	$X^5+Y^5+Z^5$	-
2	$\mathrm{GAP}(39,1)$	$X^4Y+Y^4Z+Z^4X$	-
3	$\mathrm{GAP}(30,1)$	$X^5+Y^4Z+YZ^4$	-
4	$\mathbb{Z}/20\mathbb{Z}$	$X^5+Y^5+XZ^4$	-
5	$\mathbb{Z}/16\mathbb{Z}$	$X^5+Y^4Z+XZ^4$	-
6	$\mathbb{Z}/10\mathbb{Z}$	$\mathbf{X^5+Y^5+aXZ^4+X^3Z^2}$	$a\neq 0,1/4$
7	D_{10}	$\mathbf{Y^5+a(X^5+Z^5)+X^2YZ^2+bXY^3Z}$, $\mathbf{Y^5+c(X^5+Z^5)+XY^3Z}$	$a\neq 0,\ b\neq 1$ $c^3\neq -3^3 5^{-5}$
8	$\mathbb{Z}/8\mathbb{Z}$	$\mathbf{X^5+Y^4Z+aXZ^4+X^3Z^2}$	$a\neq 0,1/4$
9	S_3	$\mathbf{a^3X^5+Y^4Z+YZ^4+a^2X^3YZ+abX^2(Z^3+Y^3)+cXY^2Z^2}$, $\mathbf{d^2X^5+Y^4Z+YZ^4+dX^2(Z^3+Y^3)+eXY^2Z^2}$, $\mathbf{f^4X^5+Y^4Z+YZ^4+fXY^2Z^2}$	*n.s* *n.s* $f\neq 0,-\frac{3125}{16}$
10	$\mathbb{Z}/5\mathbb{Z}$	$\mathbf{Z^5+XY(X+Y)(X+aY)(X+bY)}$	$ab(a-1)(b-1)(a-b)\neq 0,\ n.b$
11	$\mathbb{Z}/4\mathbb{Z}$	$\mathbf{X^5+X^3Z^2+Y^2Z^3+aX^2Y^2Z+X(bY^4+cZ^4)}$, $\mathbf{X^5+X^2Y^2Z+X(dY^4+eZ^4)+Y^2Z^3}$, $\mathbf{X^5+f(Y^2Z^3+X(Y^4+Z^4))}$	$bc\neq 0,\ c\neq -\frac{7}{20}$ $de\neq 0$ $f^3\neq -(\frac{3}{4})^3, 0$

12	$\mathbb{Z}/4\mathbb{Z}$	$\mathbf{X^5 + c(X^3Y^2 + aX^2Y^3 + bcXY^4 + c^2Y^5 + Z^4Y)}$ $\mathbf{X^5 + s(XY^4 + Y^5 + Z^4Y)}$ $\mathbf{X^5 + e(X^2Y^3 + fXY^4 + eY^5 + Z^4Y)}$, $\mathbf{X^5 + g(X^2Y^3 + XY^4 + Z^4Y)}$	*n.s, n.b*
13	$\mathbb{Z}/3\mathbb{Z}$	$\mathbf{a^3X^5 + Y^4Z + YZ^4 + a^2X^3YZ + aX^2(bY^3 + cZ^3) + dXY^2Z^2}$, $\mathbf{e^2X^5 + Y^4Z + YZ^4 + X^2(eY^3 + fZ^3) + gXY^2Z^2}$, $\mathbf{h^2X^5 + Y^4Z + YZ^4 + hX^2Z^3 + sXY^2Z^2}$, $\mathbf{t^2X^5 + Y^4Z + YZ^4 + tX^2Z^3}$	 *n.s,* $e \neq f$ *n.s* $t \neq 0, \frac{3125}{1024}$
14	$\mathbb{Z}/2\mathbb{Z}$	$\mathbf{Z^4Y + Z^2(X^3 + XY^2 + aY^3) + L_{5,Z}}$	*n.s, n.b*

Table 4.3: Representative families over k

The families that are modified respect to the ones in Table 4.2 are highlighted. The automorphism groups remain the same one than in Table 4.2.

The parameters restrictions come from avoiding singular equations and larger automorphism groups. We use the abbreviations "n.s" and "n.b" for non-singularity and no bigger automorphism group, when it is tedious to write down the restrictions.

Proof. Clearly, the zero dimensional strata families are representative over k, since each represents a single point in the (coarse) moduli space $\mathcal{M}_6$. For the rest of cases, except for the case with $G \simeq \mathbb{Z}/5\mathbb{Z}$ (see section 4.3.1), we will use the same techniques used in [LRRS14] and [LG14].

(i) **The cases G $\simeq \mathbb{Z}/10\mathbb{Z}$ and $\mathbb{Z}/8\mathbb{Z}$.** We first see that the family for $\mathbb{Z}/10\mathbb{Z}$ is geometrically complete, since the matrix $\phi = \text{daig}(1, 1, 1/\sqrt{\beta_{2,0}})$ gives an isomorphism between this family and the one in Theorem 4.1.12. Next, since the automorphism group is made of diagonal matrices with different eigenvalues, any isomorphism between two curves in this family should also be given by a diagonal matrix, see Theorem 4.3.5. It is easy to check that such isomorphism should look like $\text{daig}(1, \zeta_5^{\epsilon}, \alpha)$, but then $\alpha^2 = 1$ and we get an automorphism. In particular, a curve in the new family is only isomorphic to itself, not to its conjugates, so the family is complete and representative.

Symmetrically, one treat the situation when $G \simeq \mathbb{Z}/8\mathbb{Z}$.

(ii) **The case G $\simeq$ $\mathrm{D}_{\mathbf{10}}$:** We start with the more general family than in Theorem 4.1.12: $aY^5 + b(X^5 + Z^5) + cXY^3Z + dX^2YZ^2 = 0$. We know $a, b \neq 0$ to avoid getting the singular points $(1:0:0)$ and $(0:1:0)$. So, after rescaling and renaming variables, we obtain the geometrically complete family: $X^5+Y^5+Z^5+aXY^3Z+bX^2YZ^2=0$.

If $b=0$, then $a\neq 0$ to avoid having a bigger automorphism group, so again, after rescaling and renaming variables, we can work with the family: $Y^5+c(X^5+Z^5)+XY^3Z=0$. Now, it is easy to check that all the matrices in $\mathrm{N}_{\rho(\mathrm{D}_{10})}$ (see Theorem 4.3.5) carrying equations in this family into equations again in the family, leave the equation invariant. In other words, a curve with parameter a in this family is only isomorphic to itself, which implies that the component in the family is representative over k.

If $b\neq 0$, we work with the family $Y^5+a(X^5+Z^5)+bXY^3Z+X^2YZ^2=0$. Again any matrix in $\mathrm{N}_{\rho(\mathrm{D}_{10})}$ leaving the family invariant, fixes each equation.

So, the family given by these two components is representative over k for the stratum with D_{10}.

(iii) **The cases G $\simeq$ S_3 and $\mathbb{Z}/3\mathbb{Z}$.** By Theorem 4.1.12, the family $X^5+Y^4Z+YZ^4+a_1X^3YZ+a_2X^2\big(Z^3+Y^3\big)+a_3XY^2Z^2=0$ is geometrically complete over k when $G\simeq\mathrm{S}_3$. One easily checks that it it is not geometrically representative. Moreover, at least one of the $a_i's$ is non-zero or we get a larger automorphism group isomorphic to $\mathrm{GAP}(30,1)$. We split up into three disjoint subfamilies by the rule: if $a\neq 0$, then we use the isomorphism $\mathrm{diag}(a_1^3,1,1)$ and after rename the parameters to get the subfamily $a^3X^5+Y^4Z+YZ^4+a^2X^3YZ+abX^2\big(Z^3+Y^3\big)+cXY^2Z^2=0$. It is geometrically representative over k, since a curve in the subfamily is only isomorphic to itself using Theorem 4.3.5. Similarly, we obtain the subfamily $d^2X^5+Y^4Z+YZ^4+dX^2\big(Z^3+Y^3\big)+eXY^2Z^2=0$ when $a_1=0$ and $a_2\neq 0$, and $f^4X^5+Y^4Z+YZ^4+fXY^2Z^2=0$ when $a_1=a_2=0$ and $a_3\neq 0$.

The stratum of $\mathbb{Z}/3\mathbb{Z}$ is handled in the same way.

(iv) **The cases G $\simeq$ $\mathbb{Z}/4\mathbb{Z}$.** As above, we need to split up the stratum into different pieces. For the one with automorphism group isomorphic to $\mathbb{Z}/4\mathbb{Z}$ in case 11, the $3rd$ component was originally given by $X^5+X(hY^4+Z^4)+Y^2Z^3=0$, and on which the normalizer (see Theorem 4.3.5) acts non-trivially. More precisely, the geometric fibers over $\pm h$ are isomorphic via $\phi = \mathrm{diag}(1, \pm\sqrt{\zeta_4}, \pm\zeta_4)$. So one asks to trivialize the action in order to obtain a geometrically representative family over k. Equivalently, we need to solve a Galois descent problem by descending our subfamily to $K := \overline{k}(h^2)$, the fixed subfield of $L := \overline{k}(h)$ under the automorphism $\sigma : h \mapsto -h$. This could be easily done through the change of variables $\mathrm{diag}(1, \sqrt[4]{h}, \sqrt{h})$ to get the prescribed component in the Theorem, after renaming the parameter.

The stratum of $\mathbb{Z}/4\mathbb{Z}$ in case 12 is handled in the same way, but we ask to solve more than one Galois descent problem: First, $X^5 + a_1X^3Y^2 + a_2X^2Y^3 + a_3XY^4 + a_4Y^5 + Z^4Y = 0$ is geometrically complete over k, where a_3 or a_4 is not zero (by non-singularity). Second, we split up as follows: If $a_1 \neq 0$, re-scale Y and Z, and rename the parameters to obtain the subfamily $X^5 + X^3Y^2 + aX^2Y^3 + bXY^4 + cY^5 + Z^4Y = 0$. It is not geometrically representative over k, since the isomorphism $\mathrm{diag}(1, -1, \zeta_8)$ produces isomorphic geometric fibers over (a, b, c) and $(-a, b, -c)$. Change the variables by $\phi = \mathrm{diag}(1, c, \sqrt[4]{c})$ to get the family $X^5 + c^2X^3Y^2 + (ac)c^2X^2Y^3 + bc^4XY^4 + c^6Y^5 + c^2Z^4Y = 0$ over $K := \overline{k}(c^2, b, ac)$, a fixed subfield of $L := \overline{k}(a, b, c)$ under the action of the automorphism $a \mapsto -a$, $b \mapsto b$, and $c \mapsto -c$. In particular, it is geometrically representative over k, and we rename the parameters to get the component $X^5 + c(X^3Y^2 + aX^2Y^3 + bcXY^4 + c^2Y^5 + Z^4Y) = 0$. Now we distinguish between the following subcases when $a_1 = 0$: if $a_2 = 0$, then, necessarily, $a_3a_4 \neq 0$, or we have a larger automorphism group through the extra automorphism $\mathrm{diag}(1, \zeta_5^{-1}, \zeta_{20})$ or $\mathrm{diag}(1, \zeta_4^{-1}, \zeta_{16})$ respectively. Therefore, after re-scaling Y, Z and renaming the parameters, the subfamily $X^5 + XY^4 + dY^5 + Z^4Y = 0$ is a geometrically complete subfamily over k with isomorphic geometric fibers over d and $\zeta_4^s d$ for any integer $0 \leq s \leq 3$. In this case, we descend to $K := \overline{k}(d^4)$ via the isomorphism $\phi = \mathrm{diag}(1, d^{-1}, \sqrt[4]{d^{-3}})$, and we get a geometrically representative over k for our substratum defined by $X^5 + s(XY^4 + Y^5 + Z^4Y) = 0$. If $a_2 \neq 0$, the subfamily $X^5 + X^2Y^3 + cXY^4 + dY^5 + Z^4Y = 0$ is geometrically complete over k, with isomorphic geometric fibers over (c, d) and $(\zeta_3^s c, \zeta_3^{-s} d)$ through $\mathrm{diag}(1, \zeta_3^s, \zeta_{12}^{-s})$. Use the isomorphism $\phi = \mathrm{diag}(1, d, \sqrt[4]{d^2})$, for $d \neq 0$, in order to solve the Galois descent problem to get a subfamily over $K := \overline{k}(cd, d^3)$. In particular, $X^5 + e(X^2Y^3 + fXY^4 + eY^5 + Z^4Y) = 0$, is therefore geometrically representative over k. Lastly, if $d = 0$ (thus $c \neq 0$), we use the isomorphism $\phi = \operatorname{diag}(1, c^{-1}, \sqrt{c^{-2}})$, and we obtain a family over $\overline{k}(c^3)$ by $X^5 + g(X^2Y^3 + XY^4 + Z^4Y) = 0$, which is geometrically representative over k. This completes the discussion for these strata.

(v) **The case G $\simeq \mathbb{Z}/2\mathbb{Z}$.** The family $Z^4Y + Z^2(X^3 + XY^2 + aY^3) + L_{5,Z} = 0$ is geometrically complete over k for $\widetilde{\mathcal{M}_6^{Pl}}(\varrho(\mathbb{Z}/3\mathbb{Z}))$ (and finite, so the dimension of these stratum is 7). It is even geometrically representative over k: assume that we have an isomorphism between two curves in the family. Then $Z \to Z$ and $Y \to bY$ and $X \to eX + fY$. Hence, $X^3 + XY^2 + aY^3 \to e^3X^3 + 3e^2fX^2Y + (3ef^2 + eb^2)XY^2 + (f^3 + fb^2 + ab^3)Y^3$, which means $e^3/b = 1$, $f/b = 0$ and $eb = 1$, so the isomorphism is an automorphism.

□

4.3.1 The stratum for $G \simeq \mathbb{Z}/5\mathbb{Z}$

Before proving the last part of Theorem 4.3.6, we need some previous results.

Lemma 4.3.7. *The family $\mathcal{C}_{(a,b)} : Z^5 + XY(X+Y)(X+aY)(X+bY) = 0$ is a geometrically complete family over k for the stratum of smooth $\overline{k}$-plane curves of genus 6, with automorphism group isomorphic to $\mathbb{Z}/5\mathbb{Z}$. In particular, the associated scheme has dimension 2.*

Proof. The family $Z^5 + L_{5,Z} = 0$ is a geometrically complete family over k for the stratum, by Theorem 4.1.12. Moreover, $L_{5,Z}$ should factored in $\overline{k}[X, Y]$ into pairwise distinct linear factors, otherwise, it will be singular. Now, up to $\overline{k}$-isomorphism, we change X and Y, separately, to make one of the factors equals to $X = 0$ and another to $Y = 0$. Second, re-scale X, Y and Z simultaneously to get the factor $(X + Y)$ in the factorization of $L_{5,Z}$. Now, we can write the family as $\mathcal{C}_{(a,b)} : Z^5 + XY(X + Y)(X + aY)(X + bY) = 0$. This family is geometrically complete over k for $\widetilde{\mathcal{M}_6^{Pl}}(\varrho(\mathbb{Z}/5\mathbb{Z}))$, and finite (we justify this next), so the dimension is 2.

Isomorphisms from the curve $\mathcal{C}_{(a,b)}$ to another curve in this family come from transformations

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : t \mapsto \frac{\alpha t + \beta}{\gamma t + \delta},$$

sending the set $\{0, 1, \infty, a, b\}$ to a set $\{0, 1, \infty, c, d\}$. The set $\mathcal{T}$ of such transformations is a group and it is isomorphic to S_5. Moreover, it is generated by

$$\tau_1(a,b) = (a, \frac{a(b-1)}{b-a}), \quad \tau_2(a,b) = (\frac{1}{b}, \frac{a}{b}), \quad \tau_3(a,b) = (b,a).$$

The latest does not properly define a transformation of the curve in the family since switching the parameters a, b does not change the equation. The first two satisfy the relations $\tau_1^2 = \tau_2^3 = (\tau_1\tau_2)^5 = 1$ generating a group isomorphic to A_5.

The family defined in this way $\mathsf{C} \to \mathsf{S}$ is finite and the fibers of $f_{\mathsf{C}} : \mathsf{S} \to S$ have cardinality 120. Another way of checking the cardinality is starting with a generic curve $Z^5 + \prod_{i=0}^{5}(X + \alpha_i Y)$ and counting the $(5 \cdot 4 \cdot 3) \cdot 2$ ways of choosing the 3 roots going to $\infty, 0, 1$ and getting the parameters a and b. See Appendix C. $\square$

The family $\mathcal{C}_{(a,b)}$ is defined over $\overline{k}(a,b)$. We are ideally looking for a family (with two parameters since we know the dimension is two) defined over $L = \overline{k}(a,b)^{\mathcal{T}}$. Hence, we look for the Galois descent from $\overline{k}(a,b)$ to L. This the idea behind the ad-hoc method used in [LRRS14], see the proof of [LRRS14, Theorem 3.3].

The following asserts that the analogue of Lüroth's theorem [Har77, Chapter IV, 2.5.5] holds in dimension 2.

Theorem 4.3.8 (Chapter V, Theorem 6.2 and Remark 6.2.1, [Har77])**.** *Let L/K be a subfield extension of a purely transcendental extension $K(a,b)/K$, where K is an algebraically closed field. If $K(a,b)$ is a finite separable extension of L, then L is also a pure transcendental extension of K.*

We can now prove:

Claim. There exists a representative family over k, for the stratum $\widetilde{\mathcal{M}_6^{Pl}}(\varrho(\mathbb{Z}/5\mathbb{Z}))$ of smooth $\overline{k}$-plane curves of genus 6, and the proof of Theorem 4.3.6 is finished.

Proof. Consider the family $\mathcal{C}_{(a,b)}$, which is geometrically complete over k (see Lemma 4.3.7). It is defined over $\overline{k}(a,b)$, or to be more precise, it is already defined over the subextension $\overline{k}(c,d)$ where $c = a+b$ and $d = ab$. We want to descend it to the invariant subfield L under the action of the symmetric group S_5 on $\overline{k}(a,b)$, or equivalently of A_5 on $\overline{k}(c,d)$. In this way, we will get a representative family over k: if such a family exists, it is purely transcendental of dimension 2 by Theorem 4.3.8, and hence given by 2 parameters.

On the other hand, Weil's cocycle criterion [Wei56] (see Theorem 3.2.10) states that if there exists a family of isomorphisms

$$\{\phi_\sigma : {}^\sigma\mathcal{C}_{(a,b)} \to \mathcal{C}_{(a,b)}\}_{\sigma\in\mathrm{Gal}(\overline{k}(a,b)/L)},$$

satisfying the cocycle condition $\phi_{\sigma_1\sigma_2} = \phi_{\sigma_1}\,{}^{\sigma_1}\phi_{\sigma_2}$ for all σ_1 and σ_2, then there exist a descend of the curve $\mathcal{C}_{(a,b)}$ over L. That is an isomorphism $\phi : \mathcal{C} \to \mathcal{C}_{(a,b)}$ of smooth curves, satisfying $\phi \circ {}^\sigma\phi^{-1} = \phi_\sigma$ and where $\mathcal{C}$ is defined over $L = \overline{k}(\alpha,\beta)^{\mathcal{T}}$.

The Galois group $\mathrm{Gal}(\overline{k}(a,b)/L) = \mathrm{S}_5$ and generated by τ_1, τ_2 and τ_3. We define $\phi_{\tau_1} = [\lambda(-X+Y) : \lambda(\frac{-1}{a}X+Y) : Z]$, $\phi_{\tau_2} = [\sqrt[5]{b}X : \frac{\sqrt[5]{b}}{b}Y : Z]$ and $\phi_{\tau_3} = [X : Y : Z]$ where $\lambda = \sqrt[5]{(a-1)^{-2}(a-b)^{-1}a^3}$, and the family $\{\phi_\sigma\}_{\sigma\in\mathrm{Gal}(\overline{k}(a,b)/L)}$ by extending with the cocycle condition. This gives a well-defined family of isomorphisms satisfying the Weil cocycle condition. Hence, the mentioned descend $\mathcal{C}$ exists.

A priori, the curve $\mathcal{C}$ does not need to be a smooth plane curve over L, even if the isomorphisms ϕ_σ are (projective) matrices and $\mathcal{C}_{(a,b)}$ are smooth plane curves, see Proposition 2.19 in [BBLG16]. But, Theorem 3.2.8 says that a smooth $\overline{L}$-plane curve defined over L has a non-singular plane model over L when the degree is coprime to 3, so even we can find a non-singular plane model of $\mathcal{C}$ over L. □

Remark 4.3.9. In this case, we conclude that the isomorphism ϕ is also defined by a matrix, since all matrices ϕ_σ can be seen, not only as matrices in $\mathrm{PGL}_3(\overline{k}(a,b))$, but as matrices in $\mathrm{GL}_2(\overline{k}(a,b))$, since all of them will have the shape

$$\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now, Hilbert's Theorem 90 implies the existence of a matrix ϕ satisfying $\phi \circ {}^\sigma\phi^{-1} = \phi_\sigma$. Indeed, we can construct an explicit matrix ϕ by taking a sufficiently general matrix M, and making Hilbert's Theorem 90 explicit:

$$\phi = \sum_{\tau \in Gal(\overline{k}(a,b)/L)} \phi_\tau \, {}^\tau M,$$

The meaning of a sufficiently general matrix M is that the matrix ϕ constructed in that way is invertible. It is a straightforward computation to check that $\phi_\sigma = \phi \circ {}^\sigma\phi^{-1}$.

4.3.2 Detecting representatives in the list

Given a smooth $\overline{k}$-plane curve C of genus 6 with known non-trivial automorphism group, we can find its representative over $\overline{k}$ in the classification in Theorem 4.3.6 by using the remarks at the end of chapter 2 in [LG14]. Roughly speaking, if $\phi : \widehat{C} \to C$ is an isomorphism between C and its representative $\widehat{C}$ in the list, then $\mathrm{Aut}(\widehat{C}) = \phi^{-1}\,\mathrm{Aut}(C)\phi$. The idea is to find a suitable projective 3×3 matrix ϕ such that the equality holds. Moreover, the families provided in Theorem 4.3.6 are geometrically representative over k, that is a curve in the family is only isomorphic to itself, not to its conjugates. Therefore, if C is defined over k, then its representative must be defined over k as well, by the virtue of Weil's criterion of decent (Theorem 3.2.10).

(i) $G \simeq \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}$ (case 11), or $\mathbb{Z}/5\mathbb{Z}$: Let η be a generator of $\mathrm{Aut}(C)$, then it is conjugate in $\mathrm{PGL}_3(\overline{k})$ to $\mathrm{diag}(1, 1, \zeta_n)$ with $n = 2, 4$ or 5 respectively. Let $\underline{v}_1, \underline{v}_2$ be two eigenvectors of η with respect to the two equal eigenvalues, and let $\underline{v}_3$ be an eigenvector of η with respect to the other eigenvalue. We can take

$$\phi = (\lambda_1\underline{v}_1 + \lambda_2\underline{v}_2 \,|\, \lambda_3\underline{v}_1 + \lambda_4\underline{v}_2 \,|\, \underline{v}_3)\,.$$

It now remains to adjust the scalers λ_i so that $\widehat{C}$ is defined over k.

(ii) $G \simeq \mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}$ (case 12), $\mathbb{Z}/8\mathbb{Z}, \mathbb{Z}/10\mathbb{Z}, \mathbb{Z}/16\mathbb{Z}, \mathbb{Z}/20\mathbb{Z}$: A generator η of G is $\mathrm{PGL}_3(\overline{k})$-conjugate to $\mathrm{diag}(1, a, b)$, for some a, b, where $1, a, b$ are pairwise distinct. Let $\underline{v}_i$ for $i = 1, 2, 3$ be three eigenvectors associated to the three distinct eigenvalues. Thus, we can take

$$\phi = (\lambda_1\underline{v}_1 \,|\, \lambda_2\underline{v}_2 \,|\, \underline{v}_3)\,,$$

and then choose the scalers $\lambda_i's$ properly to get $\widehat{C}$ defined over k.

(iii) $G \simeq \mathrm{S}_3, \mathrm{D}_{10}, \mathrm{GAP}(30,1)$ or $\mathrm{GAP}(39,1)$: consider an element η of G of order 3 when $G \simeq \mathrm{S}_3$ and 15 when $G \simeq \mathrm{GAP}(30,1)$. Then, as the previous case, η is $\mathrm{PGL}_3(\overline{k})$-conjugate to $\mathrm{diag}(1,a,b)$, for some a,b, such that $1,a,b$ are pairwise distinct. Moreover, we can take

$$\varphi = (\lambda_1\underline{v}_1 \,|\, \lambda_2\underline{v}_2 \,|\, \underline{v}_3)\,,$$

where $\underline{v}_i$, for $i = 1,2,3$, denotes a three eigenvectors associated to the three distinct eigenvalues. Since elements of order 2 in G forms a single conjugacy classes, we may further assume that $\phi^{-1}[X:Z:Y]\phi = \eta'$ where $\mathrm{Aut}(C) = \langle \eta, \eta' \rangle$, and such that $\widehat{C}$ is defined over k. We follow the same method for $G \simeq \mathrm{D}_{10}$, by modifying η to be of order 5 and replacing $[X:Z:Y]$ with $[Z:Y:X]$. Lastly, for $G \simeq \mathrm{GAP}(39,1)$, η has order 13 and $[X;Z;Y]$ is replaced with either $[Y:Z:X]$ or $[Z:X:Y]$, because elements of order 3 in $\mathrm{GAP}(39,1)$ forms two conjugacy classes represented by P and P^{-1} respectively.

(iv) $G \simeq \mathrm{GAP}(150,5)$: $\exists!$ element of order 2 (resp. order 3), up to conjugation in $\mathrm{Aut}(C)$, say η_1 (resp. η_2). This η_1 (resp. η_2) is $\mathrm{PGL}_3(\overline{k})$-conjugate to $[X:Z:Y]$ (resp. $[Y:Z:X]$). Also, there exist two homologies η_3 and η_4 of order 5 in $\mathrm{Aut}(C)$, which are conjugate to $\mathrm{diag}(\zeta_5,1,1)$, and $\mathrm{diag}(1,\zeta_5,1)$ respectively. Moreover,

$$(\eta_1\eta_3)(\eta_3\eta_1)^{-1} = (\eta_3\eta_4)(\eta_4\eta_3)^{-1} = \eta_1\eta_4^2\eta_1(\eta_3\eta_4)^{-3} = \eta_2\eta_3\eta_2^{-1}(\eta_3\eta_4)^{-4} = 1.$$

Let $\underline{v}_1$ be an eigenvector of η_1 associated to the eigenvalue different from the other two. Similarly, $\underline{v}_3$ for η_3 and $\underline{v}_4$ for η_4. Thus

$$\varphi = (\lambda_3\underline{v}_3 \,|\, \lambda_4\underline{v}_4 \,|\, \mu_1\underline{v}_1 + \mu_4\underline{v}_4)\,.$$

Secondly, adjust the scalers so that $\phi^{-1}\eta_2\phi = [Y:Z:X]$, and $\widehat{C}$ is defined over k.

Example 4.3.10. *Consider the smooth plane curve C defined over k by the equation*

$$C: 2Y^5+Y^3(X-Z)^2-Y^2(X+Z)^3+4Y(X+Z)^2(X^2+XZ+Z^2)-4XZ(X+Z)(X-Z)^2 = 0$$

It is clear that $[Z : Y : X]$ is an automorphism. For simplicity, we assume that C has no more automorphisms. Since $\underline{v}_1 := (1 : 0 : 1)$, $\underline{v}_2 := (0 : 1 : 0)$ are two eigenvectors associated to the eigenvalue 1, and $\underline{v}_3 := (1 : 0 : -1)$ is an eigenvector associated to the eigenvalue -1, we may take ϕ of the shape

$$\begin{pmatrix} \lambda_1 & \lambda_3 & 1 \\ \lambda_2 & \lambda_4 & 0 \\ \lambda_1 & \lambda_3 & -1 \end{pmatrix} \in \mathrm{PGL}_3(\overline{k}).$$

We then need to adjust the scalers to get something of the form

$$Z^4Y + Z^2(X^3 + XY^2 + aY^3) + L_{5,Z}.$$

For instance, if $\lambda_1 = 0$, then $\lambda_3 \neq 0$, by invertibility of ϕ. Hence, the transformed equation using the reduced ϕ in this situation becomes

$$32\lambda_3 Z^4Y + Z^2(-32\lambda_3^3Y^3 + 16\lambda_3^2(\lambda_2X + \lambda_4Y)Y^2 + 4(\lambda_2X + \lambda_4Y)^3) + L_{5,Z} = 0.$$

Thus, $\lambda_2 \neq 0$ in order to get the monomial X^3Z^2, and also $\lambda_4 = 0$ to avoid the monomial X^2YZ^2. Consequently, we get

$$Z^4Y + Z^2(-\lambda_3^2Y^3 + \frac{\lambda_2\lambda_3}{2}XY^2 + \frac{\lambda_2^3}{8\lambda_3}X^3) + L_{5,Z} = 0.$$

In particular, we ask for λ_2 and λ_3 in $\overline{k}$ such that $\lambda_2\lambda_3 = 2$ and $\lambda_2^3 = 8\lambda_3$. So, ϕ should be

$$\phi = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix},$$

and the representative of C over k in Theorem 4.3.6 turns out to be

$$\widehat{C} : Z^4Y + Z^2(X^3 + XY^2 - Y^3) + X(2X^4 - XY^3 + 3Y^4) = 0.$$

§4.4 Twists of smooth plane curves of genus 6

Let C be a smooth $\overline{k}$-plane curve of degree 5, and assume that k is a perfect field of zero characteristic or $p > 13$. We compute equations of all twists of C over k, except for the Fermat and the Klein curves [2], by using the parameterizations obtained in Theorem 4.3.6. However, we do not give a big emphasize computing them modulo k-equivalence for the cases with non-cyclic automorphism group [3]. The idea is that we can find an unique non-singular plane model $F(X,Y,Z)=0$ over $\overline{k}$ in one of these families (in Theorem 4.3.6), representing $\overline{C}=C\otimes_k\overline{k}$, in particular $\mathrm{Aut}(\overline{C})$ equals to one of the fixed group representations $\varrho(G)$ in Theorem 4.1.12. However, $\overline{C}$ descends to k also as a smooth plane curve over k (Theorem 3.2.8), then so does $F(X,Y,Z)=0$. In particular, it is isomorphic to its conjugates, which is not possible, since the families in Theorem 4.3.6 are geometrically representative over k (in particular, a curve in the family is only isomorphic to itself). Consequently, $F(X,Y,Z)=0$ should be defined over k by the virtue of Weil's criterion of decent (Theorem 3.2.10). In other words, computing the twists of $F(X,Y,Z)=0$ over k is the same as computing them for C over k.

For computing equations for twists, we mainly use the improved method in [LG14] and [LG17], given in chapter 3 for smooth plane curves. For some families, we directly use Theorem 3.5.2, and for the other cases, the following observation is useful:

Remark 4.4.1 (Remark 3.3, [LG16]). Let C_i/k for $i\in\{1,2\}$ be two curves such that there exists an inclusion of automorphism groups $\iota:\mathrm{Aut}(C_1)\to\mathrm{Aut}(C_2)$, compatible with the action of G_k, that is, such that $\sigma(\iota(\alpha))=\iota(\sigma(\alpha))$ for all $\sigma\in G_k$ and all $\alpha\in\mathrm{Aut}(C_1)$. Then, there is a natural inclusion of the set of cocycles of the first Galois cohomology groups $\mathrm{Z}^1(\mathrm{G}_k,\mathrm{Aut}(C_1))\hookrightarrow\mathrm{Z}^1(\mathrm{G}_k,\mathrm{Aut}(C_2))$. The inclusion does not lift to an inclusion of cohomology sets

In what follows, we use $\sqrt[\ell]{\alpha}$ to denote a fixed ℓ-th root of $\alpha\in k$ in the algebraic closure $\overline{k}$ of k, where we assumed all the time that k is perfect.

[2]For both cases we only inspect how are the twists following what is done in Fermat and Klein quartics.

[3]A reader who is interested to get a complete classification of twists for non-cyclic cases may mimic the techniques [LG14] for genus 3 curves. Our aim is just to measure how much our representative classification in Theorem 4.3.6 would be helpful to compute the twists.

4.4.1 Cyclic cases

We deduce by Theorem 3.5.2 that the set $\mathrm{Twist}_k(C)$ is made exclusively of diagonal twists, for any when $\mathrm{Aut}(C\otimes_k\overline{k})$ is cyclic. We detail the description for each such subcase.

Proposition 4.4.2. *Let C/k be a smooth $\overline{k}$-plane curve of genus 6. Hence,*

1. *if $\mathrm{Aut}(C\otimes_k\overline{k})\simeq\mathbb{Z}/20\mathbb{Z}$, then from Theorem 4.3.6 C is $\overline{k}$-isomorphic to $X^5+Y^5+XZ^4=0$. In particular, any twist for C over k is given by $\phi=\mathrm{diag}(1,\sqrt[5]{m},\sqrt[4]{n})$, and has the form*
$$X^5+mY^5+nXZ^4=0,$$
for some $m,n\in k^*$. Two twists $\{m,n\}$ and $\{m',n'\}$ are equivalent if and only if $m=m' \mod {k^*}^5$ and $n=n' \mod {k^*}^4$.

2. *if $\mathrm{Aut}(C\otimes_k\overline{k})\simeq\mathbb{Z}/16\mathbb{Z}$, then from Theorem 4.3.6 C is $\overline{k}$-isomorphic to $X^5+XZ^4+Y^4Z=0$. In particular, any twist for C over k is given by $\phi=\mathrm{diag}(1,\sqrt[4]{n/\sqrt[4]{m}},\sqrt[4]{m})$, and has the form*
$$X^5+mXZ^4+nY^4Z=0,$$
for some $m,n\in k^*$. The twists $\{m,n\}$ and $\{m',n'\}$ are equivalent if and only if $m'=q^4m$, $n'=qq'^4n$ for some $q,q'\in k^*$.

3. *if $\mathrm{Aut}(C\otimes_k\overline{k})\simeq\mathbb{Z}/10\mathbb{Z}$, then from Theorem 4.3.6 C is given by a single choice of the parameter a in k of the family $X^5+Y^5+aXZ^4+X^3Z^2=0$. In particular, any twist for C over k is given by $\phi=\mathrm{diag}(1,\sqrt[5]{m},\sqrt{nm})$, and has the form*
$$X^5+mY^5+a(nm)^2XZ^4+nmX^3Z^2=0,$$
for some $m,n\in k^*$. The twists $\{m,n\}$ and $\{m',n'\}$ are equivalent if and only if $m'=q^5m$ and $n'=q^{-5}q'^2n$ for some $q,\,q'\in k^*$.

4. *if $\mathrm{Aut}(C\otimes_k\overline{k})\simeq\mathbb{Z}/8\mathbb{Z}$, then from Theorem 4.3.6 C is given by a single choice of the parameter a in k of the family $X^5+Y^4Z+aXZ^4+X^3Z^2=0$. In particular, any twist*

for C over k is given by $\phi = \mathrm{diag}(1, \sqrt[4]{n/\sqrt{m}}, \sqrt{m})$, and has the form

$$X^5 + nY^4Z + am^2XZ^4 + mX^3Z^2 = 0,$$

for some $m, n \in k^*$. Two twists $\{m, n\}$ and $\{m', n'\}$ are equivalent if and only if $m' = q^2m$ and $n' = qq'^4n$ for some q, $q' \in k^*$.

5. *if $\mathrm{Aut}(C \otimes_k \overline{k}) \simeq \mathbb{Z}/5\mathbb{Z}$, then from Theorem 4.3.6 C is given by a choice of the parameters a, b in k, not necessarily unique, of the family $Z^5 + XY(X+Y)(X+aY)(X+bY) = 0$. In particular, any twist for C over k is given by $\phi = \mathrm{diag}(1, 1, \sqrt[5]{m})$, and has the form*

$$mZ^5 + XY(X+Y)(X+aY)(X+bY) = 0,$$

for some $m \in k^*$. Two twists $\{m\}$ and $\{m'\}$ are equivalent if and only if $m' = m \bmod {k^*}^5$.

6. *if $\mathrm{Aut}(C \otimes_k \overline{k}) \simeq \mathbb{Z}/4\mathbb{Z}$ as case 11 in Theorem 4.3.6, then from Theorem 4.3.6 C is $\bar{k}$-isomorphic to a non-singular plane model, for a single choice of the parameters, in one of the following families*

$$X^5 + X^3Z^2 + Y^2Z^3 + aX^2Y^2Z + X(bY^4 + cZ^4),$$
$$X^5 + X^2Y^2Z + X(dY^4 + eZ^4) + Y^2Z^3,$$
$$X^5 + f(Y^2Z^3 + X(Y^4 + Z^4)), \text{ respectively.}$$

Moreover, any twist for C over k is given by $\phi = \mathrm{diag}(1, \sqrt{n/\sqrt{m}}, \sqrt{m})$, and has the form

$$X^5 + mX^3Z^2 + mnY^2Z^3 + anX^2Y^2Z + X(b(n^2/m)Y^4 + cm^2Z^4),$$
$$X^5 + nX^2Y^2Z + X(d(n^2/m)Y^4 + em^2Z^4) + mnY^2Z^3,$$
$$X^5 + f(mnY^2Z^3 + X((n^2/m)Y^4 + m^2Z^4)), \text{ respectively,}$$

for some $m, n \in k^*$. The twists $\{m, n\}$ and $\{m', n'\}$ are equivalent if and only if $m' =$

mq^2, $n' = nqq'^2$ for some $q, q' \in k^*$.

7. *if $\mathrm{Aut}(C \otimes_k \overline{k}) \simeq \mathbb{Z}/4\mathbb{Z}$ as case 12 in Theorem 4.3.6, then from Theorem 4.3.6 C is $\bar{k}$-isomorphic a non-singular plane model, for a single choice of the parameters, in one of the following families*

$$X^5 + c(X^3Y^2 + aX^2Y^3 + bcXY^4 + c^2Y^5 + Z^4Y),$$
$$X^5 + e(X^2Y^3 + fXY^4 + eY^5 + Z^4Y),$$
$$X^5 + s(XY^4 + Y^5 + Z^4Y),$$
$$X^5 + g(X^2Y^3 + XY^4 + Z^4Y), \text{ respectively.}$$

In this case, any twist for C over k is given by $\phi = \mathrm{diag}(1, 1, \sqrt[4]{m})$, and has the form

$$X^5 + c(X^3Y^2 + aX^2Y^3 + bcXY^4 + c^2Y^5 + mZ^4Y),$$
$$X^5 + s(XY^4 + Y^5 + mZ^4Y),$$
$$X^5 + e(X^2Y^3 + fXY^4 + eY^5 + mZ^4Y),$$
$$X^5 + g(X^2Y^3 + XY^4 + mZ^4Y), \text{ respectively.}$$

Two twists $\{m\}$ and $\{m'\}$ are equivalent if and only if $m' = m \mod k^{*^4}$.

8. *if $\mathrm{Aut}(C \otimes_k \overline{k}) \simeq \mathbb{Z}/3\mathbb{Z}$, then from Theorem 4.3.6 C is $\bar{k}$-isomorphic to a non-singular plane model, for a single choice of the parameters, in one of the following families*

$$a^3X^5 + Y^4Z + YZ^4 + a^2X^3YZ + aX^2\big(bY^3 + cZ^3\big) + dXY^2Z^2,$$
$$e^2X^5 + Y^4Z + YZ^4 + X^2\big(eY^3 + fZ^3\big) + gXY^2Z^2,$$
$$h^2X^5 + Y^4Z + YZ^4 + hX^2Z^3 + sXY^2Z^2,$$
$$t^2X^5 + Y^4Z + YZ^4 + tX^2Z^3, \text{ respectively.}$$

Thus any twist for C over k is given by $\phi = \operatorname{diag}(1, \sqrt[3]{m}, \sqrt[3]{m^2})$, and has the form

$$a^3X^5 + m^2Y^4Z + m^3YZ^4 + a^2mX^3YZ + amX^2\big(bY^3 + cmZ^3\big) + dm^2XY^2Z^2,$$
$$e^2X^5 + m^2Y^4Z + m^3YZ^4 + mX^2\big(eY^3 + fmZ^3\big) + gm^2XY^2Z^2,$$
$$h^2X^5 + m^2Y^4Z + m^3YZ^4 + hm^2X^2Z^3 + sm^2XY^2Z^2,$$
$$t^2X^5 + m^2Y^4Z + m^3YZ^4 + tm^2X^2Z^3, \text{ respectively,}$$

for some $m \in k^*$. Two twists $\{m\}$ and $\{m'\}$ are equivalent if and only if $m' = m \bmod {k^*}^3$.

9. *if $\operatorname{Aut}(C \otimes_k \overline{k}) \simeq \mathbb{Z}/2\mathbb{Z}$, then from Theorem 4.3.6 C is $\bar{k}$-isomorphic to a non-singular plane model, for a single choice of the parameters, in the family $Z^4Y + Z^2(X^3 + XY^2 + aY^3) + L_{5,Z} = 0$. Any any twist for C over k is given by $\phi = \operatorname{diag}(1, 1, \sqrt{m})$, and has the form*

$$m^2Z^4Y + mZ^2(X^3 + XY^2 + aY^3) + L_{5,Z} = 0,$$

for some $m \in k^*$. Two twists $\{m\}$ and $\{m'\}$ are equivalent if and only if $m' = m \bmod {k^*}^2$.

Proof. For any of the above cases, twists of C over k are all diagonal of the shape $\operatorname{diag}(1, \lambda, \mu) \in \operatorname{PGL}_3(\overline{k})$. We just need to adjust the scalers $\lambda, \mu \in \overline{k}$ properly, so that the transformed equation for C under ϕ is defined over k.

We show, for example, the 6th case when $\operatorname{Aut}(C \otimes_k \overline{k}) \simeq \mathbb{Z}/4\mathbb{Z}$. By Theorem 4.3.6, we get C in one of the following families

$$X^5 + X^3Z^2 + Y^2Z^3 + aX^2Y^2Z + X(bY^4 + cZ^4),$$
$$X^5 + X^2Y^2Z + X(dY^4 + eZ^4) + Y^2Z^3,$$
$$X^5 + f(Y^2Z^3 + X(Y^4 + Z^4)), \text{ respectively,}$$

for a single choice of the parameters. By non-singularity, we know that XY^4 and XZ^4 occurs with non-zero coefficients. Hence $\lambda^4, \mu^4 \in k$, and moreover $\lambda^2\mu^3 \in k$ by the aid of the term Y^2Z^3. Consequently, $\mu = \sqrt{m}$ and $\lambda = \sqrt[4]{n^2/m} = \sqrt{n/\sqrt{m}}$ for some $n, m \in k^*$.

Next, substitute into the above equations to obtain the defining form for the twist over k as in the statement. Finally, two twists $\{m,n\}$ and $\{m',n'\}$ are equivalent if and only if there exists an $\psi\in\mathrm{PGL}_3(k)$ and an automorphism α of C such that $\alpha\circ\phi=\phi'\circ\psi$, where $\phi=\mathrm{diag}(1,\sqrt{n/\sqrt{m}},\sqrt{m})$ and $\phi=\mathrm{diag}(1,\sqrt{n'/\sqrt{m'}},\sqrt{m'})$, see Remark 1.3.1 in [LG14]. This is equivalent to write $\psi=\mathrm{diag}(1,q,q')$ for some $q,q'\in k^*$, such that $m'=mq^2$ and $n'=nq^2q'$, which was to be shown in this situation.

In the same way, one can treats any of the other cases. □

4.4.2 Non-cyclic cases

We handle the situation for which the automorphism group of $C\otimes_k\overline{k}$ is not cyclic.

Proposition 4.4.3. *Let C/k be be a smooth $\overline{k}$-plane curve of genus 6 such that $\mathrm{Aut}(C\otimes_k\overline{k})\simeq\mathrm{GAP}(30,1)$. From Theorem 4.3.6 C is $\overline{k}$-isomorphic to $X^5+Y^4Z+YZ^4=0$, and the set $\mathrm{Twist}_k(C)$ is formed by twists of one of the following form:*

1. *Almost-diagonal twists of the form*

$$X^5+2rY^5+6smY^4Z+4mrY^3Z^2-2sm^2Y^2Z^3-6rm^2YZ^4-2sm^3Z^5=0,\ \ r,s,m\in k.$$

2. *Diagonal twists of the form*

$$X^5+mY^4Z+m^3YZ^4=0,\ \ m\in k^*.$$

Proof. By Lemma B.1, we know that every twist of C is given by an isomorphism ϕ of the shape $[X:a_{11}Y+a_{12}Z:a_{21}Y+a_{22}Z]:C'\to C$, where we can assume that $a_{11}a_{21}\neq 0$, and then we write $a_{11}=\alpha, a_{12}=\alpha\beta, a_{21}=\gamma$ and $a_{22}=\gamma\delta$. Making the substitution

$Y \to \alpha(Y+\beta Z)$ and $Z \to \gamma(Y+\delta Z)$ in the equation of C, we get

$$AB \in k$$
$$ABM + 3AH \in k$$
$$4AHM - 2ABN \in k$$
$$4AHM^2 - 4ABMN + AHN \in k$$
$$AHM^3 - ABNM^2 + 2AHNM - 3ABN^2 \in k$$
$$AHM^2N - ABMN^2 - AHN^2 \in k$$

where $A = \alpha\gamma$, $B = \alpha^3 + \gamma^3$, $H = \alpha^3\beta + \gamma^3\delta$, $M = \beta + \delta$ and $N = \beta\delta$. A computations shows that $M, N, AH, AB \in k$. In particular, β and δ are the two roots of a polynomial of degree 2 over k. If β and δ are both k-rational numbers, then we get a diagonal twist of the form $X^5 + mY^4Z + m^3YZ^4$ through $\phi = \mathrm{diag}(1, \sqrt[15]{m}, \sqrt[15]{m^{11}})$, for some $m \in k^*$. Otherwise, we can assume $\beta = \sqrt{m} = -\delta$, and we get $\alpha^4\gamma + \alpha\gamma^4 \in k$ and $(\alpha^4\gamma - \alpha\gamma^4)\sqrt{m} \in k$. Hence, $\alpha^4\gamma = r + s\sqrt{m}$ and $\alpha\gamma^4 = r - s\sqrt{m}$ for some $r, s \in k$. We therefore obtain an almost-diagonal twist as in the statement, since

$$\alpha = \sqrt[15]{\frac{(r+s\sqrt{m})^4}{r-s\sqrt{m}}}, \text{ and } \gamma = \sqrt[15]{\frac{(r-s\sqrt{m})^4}{r+s\sqrt{m}}}.$$

$\square$

Proposition 4.4.4. *Let C/k be a smooth $\overline{k}$-plane curve of genus 6 such that $\mathrm{Aut}(C \otimes_k \overline{k}) \simeq \mathrm{S}_3$. From Theorem 4.3.6 C is $\overline{k}$-isomorphic to a curve in one of the next families for a single choice of the parameters*

$$a^3X^5 + Y^4Z + YZ^4 + a^2X^3YZ + abX^2\big(Z^3 + Y^3\big) + cXY^2Z^2 = 0,$$
$$d^2X^5 + Y^4Z + YZ^4 + dX^2\big(Z^3 + Y^3\big) + eXY^2Z^2 = 0,$$
$$f^4X^5 + Y^4Z + YZ^4 + fXY^2Z^2 = 0.$$

The set of twists $\mathrm{Twist_k}(C)$ is formed by twists of one of the following forms:

1. *Diagonal twists defined by an equation of the form*

$$a^3X^5+m^2(Y^4Z+mYZ^4+a^2m^{-1}X^3YZ+abX^2\left(Z^3+m^{-1}Y^3\right)+cXY^2Z^2)=0,$$
$$d^2X^5+m^2(Y^4Z+mYZ^4+dX^2\left(Z^3+m^{-1}Y^3\right)+eXY^2Z^2)=0,$$
$$f^4X^5+m^2(Y^4Z+mYZ^4+fXY^2Z^2)=0,\ \textit{respectively}.$$

through $\phi=\operatorname{diag}(1,\sqrt[3]{m},\sqrt[3]{m^2})$*, for* $m\in k^*$.

2. *Almost-diagonal twists parameterized by elements* $A,B,m\in k$ *such that there exist* $N\in k$ *with* $A^2-mB^2=N^3$*, where we can take an isomorphism*

$$\phi=\begin{pmatrix}1&0&0\\0&\sqrt[3]{A+B\sqrt{m}}&\sqrt{m}\sqrt[3]{A+B\sqrt{m}}\\0&\sqrt[3]{A-B\sqrt{m}}&-\sqrt{m}\sqrt[3]{A-B\sqrt{m}}\end{pmatrix}.$$

Therefore, an almost-diagonal twist $\{A,B,m,N\}$ *is defined by an equation of the form*

$$2N(Y^2-mZ^2)\left(mBZ(3Y^2+mZ^2)+A(Y^3+3mZ^2Y)\right)+cN^2(Y^2-mZ^2)^2X+$$
$$+2ab\left(mBZ(3Y^2+mZ^2)+A(Y^3+3mZ^2Y)\right)X^2+a^2N(Y^2-mZ^2)X^3+a^3X^5=0,$$
$$d^2X^5+2d\left(mBZ(3Y^2+mZ^2)+A(Y^3+3mYZ^2)\right)X^2+eN^2(Y^2-mZ^2)^2X+$$
$$+2N(Y^2-mZ^2)\left(mBZ(3Y^2+mZ^2)+A(Y^3+3mZ^2Y)\right)=0,$$
$$2N(Y^2-mZ^2)\left(mBZ(3Y^2+mZ^2)+A(Y^3+3mZ^2Y)\right)+$$
$$+f^4X^5+fN^2(Y^2-mZ^2)^2X=0,\ \textit{respectively}.$$

Proof. Since $\operatorname{Aut}(C\otimes_k\overline{k})\leq\operatorname{GAP}(30,1)$, we can apply Remark 4.4.1. In particular, by Proposition 4.4.3, any twist C' of C over k is either a diagonal twist of the shape $\phi=\operatorname{diag}(1,\sqrt[15]{q},\sqrt[15]{q^{11}})$ or it is an almost-diagonal twist of the shape

$$\phi=\begin{pmatrix}1&0&0\\0&\sqrt[15]{\frac{(r+s\sqrt{m})^4}{r-s\sqrt{m}}}&\sqrt{m}\sqrt[15]{\frac{(r+s\sqrt{m})^4}{r-s\sqrt{m}}}\\0&\sqrt[15]{\frac{(r-s\sqrt{m})^4}{r+s\sqrt{m}}}&-\sqrt{m}\sqrt[15]{\frac{(r-s\sqrt{m})^4}{r+s\sqrt{m}}}\end{pmatrix},$$

where $m, q \in k^*$. Consider, for example, C of the form

$$f^4X^5 + Y^4Z + YZ^4 + fXY^2Z^2 = 0.$$

When C' is a diagonal twist, the monomial term XY^2Z^2 restricts $q \in {k^*}^5$ and the twist reduces to $\phi = \mathrm{diag}(1, \sqrt[3]{m}, m\sqrt[3]{m^2})$ for some $m \in k^*$, which in turns is equivalent to $\phi = \mathrm{diag}(1, \sqrt[3]{m}, \sqrt[3]{m^2})$ in the statement.

When C' is almost-diagonal, the coefficient of XY^4 in C' is $f\sqrt[5]{(r^2 - ms^2)^2}$ should be in k^*. Therefore, $r \pm s\sqrt{m} = (A \pm B\sqrt{m})^5$ for some $A, B \in k$. Moreover, the term $X(Y^2 - mZ^2)^2$ occurs in C' with coefficient $f\sqrt[3]{(A^2 - B^2m)^2}$. Consequently, $A^2 - B^2m \in {k^*}^3$, and we deduce the result in this situation.

In the same way, we can handle the remaining situations when C lies in any of the other components. $\square$

Proposition 4.4.5. *Let C/k be a smooth $\overline{k}$-plane curve of genus 6 such that $\mathrm{Aut}(C \otimes_k \overline{k}) \simeq \mathrm{D}_{10}$. From Theorem 4.3.6 C is $\overline{k}$-isomorphic to a curve in one of the two next families for a single choice of the parameters*

$$Y^5 + a(X^5 + Z^5) + X^2YZ^2 + bXY^3Z = 0$$
$$Y^5 + c(X^5 + Z^5) + XY^3Z = 0.$$

The set of twists $\mathrm{Twist}_k(C)$ is then formed by twists of one of the following forms:

1. *Diagonal twists of the form*

$$Y^5 + a(mX^5 + m^{-1}Z^5) + X^2YZ^2 + bXY^3Z = 0$$
$$Y^5 + c(mX^5 + m^{-1}Z^5) + XY^3Z = 0 \text{ respectively.}$$

2. *Almost-diagonal twists parameterized by elements $A, B, m \in k$ such that there exists $N \in k$*

with $A^2 - mB^2 = N^5$, where an equation is given by

$$2aAX^5 + X^4(N^2Y + 10amBZ) + 20amAX^3Z^2 + 2mX^2Z^2(10amBZ - N^2Y) + bcX^2Y^3 + \\ +10am^2AXZ^4 + Z^2(2am^3BZ^3 + (mN)^2YZ^2 - bmNY^3) + Y^5 = 0,$$

$$2c(AX^5 + 5mBX^4Z + 10mAX^3Z^2 + 10m^2BX^2Z^3 + 5m^2AXZ^4 + m^3BZ^5) + Y^5 + \\ +N(X^2 - mZ^2)Y^3 = 0 \text{ respectively.}$$

Proof. We know from Lemma B.1, that any twist of C is given by an isomorphism ϕ of the shape $[a_{11}X + a_{13}Z : Y : a_{31}X + a_{33}Z] : C' \to C$. Now, we can proceed exactly as in Proposition 4.4.3. In particular, C' is either a diagonal twist coming from an isomorphism of the shape $\phi = \mathrm{diag}(\sqrt[5]{\alpha}, 1, \sqrt[5]{\beta})$ or it is an almost-diagonal twist by an isomorphism

$$\phi = \begin{pmatrix} \sqrt[5]{A + B\sqrt{m}} & 0 & \sqrt{m}\sqrt[5]{A + B\sqrt{m}} \\ 0 & 1 & 0 \\ \sqrt[5]{A - B\sqrt{m}} & 0 & -\sqrt{m}\sqrt[5]{A - B\sqrt{m}} \end{pmatrix}.$$

Moreover, the monomial terms XY^3Z and X^2YZ^2 restricts $\alpha = m, \beta = m^{-1}$ for some $m \in k^*$, when C' is a diagonal twist, and $A^2 - mB^2 \in k^{*^5}$ when C' is almost-diagonal. □

The Klein and the Fermat curves

In this part, we inspect the determination of the set of twists of a smooth $\overline{k}$-plane curve C/k of genus 6 such that $\mathrm{Aut}(C \otimes_k \overline{k}) \simeq \mathrm{GAP}(39, 1)$ or $\mathrm{GAP}(150, 5)$.

Here we detail our inspections for each case;

(A) If $\mathrm{Aut}(C \otimes_k \overline{k}) \simeq \mathrm{GAP}(39, 1)$, then C is $\overline{k}$-isomorphic to the Klein quintic

$$C_K : X^4Y + Y^4Z + Z^4X = 0.$$

It is remarkable, that the study of the twists of the Klein quartic is considerably more complicated than the study of the twists of the Klein quintic. The explanation is that in the degree 5 case, we do not have the extra involution s (see Section 6 in [LG16]) in the automorphism group. As a consequence of this simplicity, is that the splitting field L of a twist of C_K over k always contains a unique cyclic extension of L_0/k of degree 3, that is

$L_0 = k(\alpha, \beta, \gamma)$ where α, β, γ are the three roots of a cubic polynomial over k. Moreover, there are exactly two non-equivalent twists with the same splitting field L. Therefore, one expects the set of twists $\mathrm{Twist}_k(C_K)$ to be in two-to-one correspondence with cyclic Galois extensions of k of degree 3: $k(\alpha, \beta, \gamma)$, where α, β, γ are the three roots of a cubic polynomial over k such that $\alpha\beta\gamma \in k^{*^{13}}$.

Furthermore, if one follows the same method in [LG17, Proposition 4.1] and [LG16, Section 6], for the Klein quartic curve, and mimic the techniques there, then it is also expected that any twist comes with splitting field $L = k(\zeta_{13}, \sqrt[13]{\alpha}, \sqrt[13]{\beta}, \sqrt[13]{\gamma})$ with $\alpha\beta\gamma \in k^{*^{13}}$. In particular, we can always take an isomorphism $\phi : C'_K \to C_K$ of the shape

$$\begin{pmatrix} \sqrt[13]{\alpha} & \alpha \sqrt[13]{\alpha} & \alpha^2 \sqrt[13]{\alpha} \\ \sqrt[13]{\beta} & \beta \sqrt[13]{\beta} & \beta^2 \sqrt[13]{\beta} \\ \sqrt[13]{\gamma} & \gamma \sqrt[13]{\gamma} & \gamma^2 \sqrt[13]{\gamma} \end{pmatrix},$$

to get a representative twist C'_K of C_K over k with splitting field L. The other twist is obtained by switching α, β, γ by α^{12}, β^{12}, γ^{12}.

(B) Let C/k be a smooth $\overline{k}$-plane curve of genus 6 such that $\mathrm{Aut}(C \otimes_k \overline{k}) \simeq \mathrm{GAP}(150, 5)$. Then, C is $\overline{k}$-isomorphic to the Fermat quintic

$$C_F : X^5 + Y^5 + Z^5 = 0.$$

Definition 4.4.6 (Definition 4.7, [LG16])**.** The set $\mathrm{Pol}_3^n(k)$ is defined to be the set of separable polynomials of degree 3 with coefficients in k and whose independent coefficient is in $-1 \cdot k^{*n}$, i.e. equals to $-\alpha$ for some $\alpha \in k^{*^n}$.

Given $P(T) = (T-\alpha)(T-\beta)(T-\gamma) \in \mathrm{Pol}_3^{10}(k)$, we can attach to it the twist

$$C' : \sum_{\substack{i_1+i_2+i_3=5 \\ i_1, i_2, i_3 \geq 0}} \binom{5}{i_1}\binom{5-i_1}{i_2} S_{1+i_2+2i_3} X^{i_1} Y^{i_2} Z^{i_3} = 0, \tag{4.1}$$

where $S_j = \alpha^j + \beta^j + \gamma^j$ for $j \in \mathbb{N}$, and the isomorphism $\phi : C' \to C_F$ is given by

$$\phi = \begin{pmatrix} \sqrt[5]{\alpha} & \alpha\sqrt[5]{\alpha} & \alpha^2\sqrt[5]{\alpha} \\ \sqrt[5]{\beta} & \beta\sqrt[5]{\beta} & \beta^2\sqrt[5]{\beta} \\ \sqrt[5]{\gamma} & \gamma\sqrt[5]{\gamma} & \gamma^2\sqrt[5]{\gamma} \end{pmatrix}.$$

whose splitting field is $L = k(\zeta_5, \sqrt[5]{\alpha}, \sqrt[5]{\beta}, \sqrt[5]{\gamma})$.

Now, if we mimic the computations in [LG14, §3.1] or [LG16, §4], for the Fermat quartic curve, and apply Lemma B.1, then it becomes expected that any other twist of C_F lies in one of the two categories:

(i) A *diagonal twist* of the form $aX^5 + bY^5 + Z^5 = 0$ via an isomorphism $\phi = \mathrm{diag}(\sqrt[5]{a}, \sqrt[5]{b}, 1)$ where $1 \neq a \neq b \neq 1$. After right multiplication by a suitable rational matrix, say

$$\begin{pmatrix} 1 & qa & (qa)^2 \\ 1 & qb & (qb)^2 \\ 1 & q & q^2 \end{pmatrix}$$

with $q = (ab)^3$, we obtain the equivalent twist given by the isomorphism

$$\phi = \begin{pmatrix} \sqrt[5]{a} & qa\sqrt[5]{a} & (qa)^2\sqrt[5]{a} \\ \sqrt[5]{b} & qb\sqrt[5]{b} & (qb)^2\sqrt[5]{b} \\ 1 & q & q^2 \end{pmatrix} = \begin{pmatrix} \sqrt[5]{qa} & qa\sqrt[5]{qa} & (qa)^2\sqrt[5]{qa} \\ \sqrt[5]{qb} & qb\sqrt[5]{qb} & (qb)^2\sqrt[5]{qb} \\ \sqrt[5]{q} & q\sqrt[5]{q} & q^2\sqrt[5]{q} \end{pmatrix},$$

which has the form (4.1) with $\alpha = qa$, $\beta = qb$, and $\gamma = q$. Thus, it corresponds to $P(T) = (T-\alpha)(T-\beta)(T-\gamma) \in \mathrm{Pol}_3^{10}(k)$.

(ii) An *almost-diagonal twist* given by an isomorphism

$$\phi = \begin{pmatrix} \sqrt[5]{c} & \sqrt{m}\sqrt[5]{c} & 0 \\ \sqrt[5]{\bar{c}} & -\sqrt{m}\sqrt[5]{\bar{c}} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $c = a + b\sqrt{m}$ and $\bar{c} = a - b\sqrt{m}$ and $m \in k^*$. We can assume that $b \neq 0$, after right multiplication, if necessary, by the rational matrix

$$\begin{pmatrix} 1 & m & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We take the equivalent twist

$$\phi = \begin{pmatrix} \sqrt[5]{qc} & qc\sqrt[5]{qc} & (qc)^2\sqrt[5]{qc} \\ \sqrt[5]{q\bar{c}} & q\bar{c}\sqrt[5]{q\bar{c}} & (q\bar{c})^2\sqrt[5]{q\bar{c}} \\ \sqrt[5]{q} & q\sqrt[5]{q} & q^2\sqrt[5]{q} \end{pmatrix},$$

where $q = (a^2 - b^2m)^3$, which is again of the form (4.1) with $\alpha = qc$, $\beta = q\bar{c}$, and $\gamma = q$. Thus, it corresponds to $P(T) = (T - \alpha)(T - \beta)(T - \gamma) \in \mathrm{Pol}_3^{10}(k)$.

5

CHAPTER

The field of moduli and fields of definition for smooth plane curves

Let $\overline{C}$ be a smooth plane curve of genus $g \geq 3$ over $\overline{k}$, where k is a perfect field of characteristic $p = 0$ or $p > 2g+1$. The field of moduli of $\overline{C}$, relative to the Galois extension $\overline{k}/k$, is denoted by $M_{\overline{k}/k}(C)$ (see Definition 5.1.3). It has been proven by B. Huggins, in her PhD [Hug05], that $M_{\overline{k}/k}(\overline{C})$ is always a field of definition for $\overline{C}$ unless $\mathrm{Aut}(\overline{C})$ is $\mathrm{PGL}_3(\overline{k})$-conjugate to a diagonal subgroup of $\mathrm{PGL}_3(\overline{k})$, or to one of the Hessian groups Hess_* with $* \in \{18, 36\}$.

We aim in this chapter to investigate the next question:

Question. *Given a smooth plane curve $\overline{C}$ over $\overline{k}$ such that $\mathrm{Aut}(\overline{C})$ is $\mathrm{PGL}_3(\overline{k})$-conjugate to a diagonal subgroup of $\mathrm{PGL}_3(\overline{k})$, when the field of moduli $M_{\overline{k}/k}(\overline{C})$ needs to be a field of definition?*

To answer this question, we fix a non-singular plane model $F_{\overline{C}}(X,Y,Z) = 0$ over $\overline{k}$ for $\overline{C}$ in one of the families of Theorem 2.1.3, such that $\mathrm{Aut}(F_{\overline{C}}) \leq \mathrm{PGL}_3(\overline{k})$ is diagonal, that is made entirely of 3×3 projective matrices of diagonal shapes. We first show that if $\mathrm{Aut}(F_{\overline{C}})$ contains a non-homology of order $n > 1$ (Definition 1.2.6), then $M_{\overline{k}/k}(\overline{C})$ is always a field of definition, unless n divides one of the integers d, $d-1$ or $d(d-2)$. We also give a geometrically complete family over k and describe the automorphism group in each subcase as well; see Theorem 5.4.4. Secondly, if $\mathrm{Aut}(F_{\overline{C}})$ is made entirely of homologies, then it is either a cyclic group of order dividing d or $d-1$, or it is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$; see Lemma 5.4.8. In the case that $\mathrm{Aut}(F_{\overline{C}})$ is cyclic generated by an homology of order $n > 1$, dividing d with d odd or divides $d-1$, then again $M_{\overline{k}/k}(\overline{C})$ is a field of definition; see Theorem 5.4.14 and Theorem 5.4.15. In the remaining situations, we construct explicit examples of smooth plane curves over $\mathbb{C}$, whose field of moduli relative to the Galois extension $\mathbb{C}/\mathbb{R}$ is $\mathbb{R}$, but it is not a field of definition; see Proposition 5.4.2, Theorem 5.4.6, Theorem 5.4.16 and Proposition 5.4.20.

We shall deal with the following items:

5.1. The field of moduli and fields of definition.

5.2. Débes-Emsalem: The canonical model for $\overline{C}/\operatorname{Aut}(\overline{C})$.

5.3. On the field of moduli of smooth curves with odd signature.

5.4. Smooth plane curves with diagonal automorphism groups.

§5.1 The field of moduli and fields of definition

Definition 5.1.1. (Fields of definition) Let $k \subseteq L \subseteq \overline{L}$ be fields, where $\overline{L}$ is a fixed algebraic closure of L. Given a smooth curve C/L, then C is *defined* over k if and only if there is a curve C'/k that is isomorphic over L to C. In such case, k is called a *field of definition* of C.

We say that C is *definable* over k if there is a curve C'/k such that C and C' are isomorphic, viewed as smooth curves over $\overline{L}$.

Definition 5.1.2. (The field of moduli) Let C/k be a smooth curve over k. The *field of moduli* of C, denoted by k_C, is the intersection of all fields of definition of $\overline{C} := C \otimes_k \overline{k}$.

There is another definition for the field of moduli which is commonly used and is defined relative to a given field extension L/k:

Definition 5.1.3 (Definition 1.2, [AQ12]). Let C/L be a smooth curve and let L/k be a field extension. The *field of moduli of C relative to the extension L/k*, denoted by $M_{L/k}(C)$, is the subfield of L fixed by the subgroup

$$U_{L/k}(C) := \{\sigma \in \operatorname{Gal}(L/k) : C \cong_L {}^{\sigma}C\}.$$

We recall Weil's condition of decent, which gives necessary and sufficient conditions for a field k to be a field of definition for C:

Theorem 5.1.4 (Weil, [Wei56]). *Let C be a smooth curve defined over a field L, and let L/k be a Galois extension. Suppose that for every $\sigma \in \operatorname{Gal}(L/k)$, there exists an L-isomorphism*

$\phi_\sigma : {}^\sigma C \mapsto C$ such that

$$\phi_\sigma \circ {}^\sigma\phi_\tau = \phi_{\sigma\tau} \text{ for all } \sigma, \tau \in \mathrm{Gal}(L/k).$$

Then, one gets a curve C' over k and an L-isomorphism $\varphi : C' \otimes_k L \to C$ such that $\phi_\sigma \circ {}^\sigma\varphi = \varphi$, for all $\sigma \in \mathrm{Gal}(L/k)$.

If $\mathrm{Aut}(C)$ is trivial, then Weil's condition of descent becomes trivially true and so the field of moduli needs to be a field of definition. On the other hand (see for example the Introduction in [DE99]), a smooth curve C/k of genus $g = 0$ is k^{sep}-isomorphic to the projective line $\mathbb{P}^1$, which is defined over the prime field k_0 of k. Moreover, if $g = 1$, the field of moduli is $k_0(j)$, where j is the modular invariant of C, and it is known that for characteristic $p \neq 2, 3$, C is k^{sep}-isomorphic to a model defined over $k(j)$ (see [Sil09, Chp. III, Proposition 1.4]).

The real difficulty happens for $g \geq 2$ and non-trivial automorphism groups, since the Weil's criterion of decent is not easily checked.

Proposition 5.1.5 (Débes-Emsalem, Proposition 2.1, [DE99])**.** *Let C be a smooth curve over L and let L/k be a Galois extension. The group $U_{L/k}(C)$ is a closed subgroup of* $\mathrm{Gal}(L/k)$ *with respect to the Krull topology. In particular,*

$$U_{L/k}(C) = \mathrm{Gal}(L/M_{L/k}(C)).$$

The field of moduli $M_{L/k}(C)$ of C relative to the extension L/k is contained in each field of definition of C between k and L. Hence if the field of moduli is a field of definition, it is the smallest field of definition between k and L. Finally, if $F := M_{L/k}(C)$, then the field of moduli of C relative to the extension L/F is exactly F.

Remark 5.1.6. The final observation of Proposition 5.1.5 that the field of moduli relative to the extension $L/M_{L/k}(C)$ equals $M_{L/k}(C)$ generally allows one to reduce to the situation where the base field k is the field of moduli of the given curve C, relative to L/k, by extending the scalars from k to $M_{L/k}(C)$.

Due to S. Koizumi [Koi72, Proposition 2.3-(ii)], Theorem 1.5.8 in [Hug05] shows that, $M_{\overline{k}/k_0}(C)$ is a purely inseparable extension of k_C, where k_0 is the prime field of k.

Corollary 5.1.7 (Corollary 1.5.9, [Hug05])**.** *Let C be a smooth curve over a field k. Then, C is definable over a finite separable extension of its field of moduli k_C.*

The main relation between the field of moduli in Definition 5.1.2 and Definition 5.1.3 is the following theorem:

Theorem 5.1.8 (B. Huggins, Theorem 1.6.9, [Hug05])**.** *Let C be a smooth curve over a field k. Then, C is definable over its field of moduli k_C if and only if given any algebraically closed field $F \supseteq k$, and any subfield $L \subseteq F$ with F/L Galois, $C \otimes_k F$ can be defined over its field of moduli $M_{F/L}(C)$, relative to the extension F/L.*

5.1.1 The field of moduli for smooth plane curves

Definition 5.1.9. (Diagonal groups) The group of all 3×3 projective linear matrices of diagonal shapes over an algebraically closed field $\overline{k}$ of characteristic $p \geq 0$ is denoted by $\mathrm{D}(\overline{k})$. A finite non-trivial group G is called diagonal if it can be viewed as a subgroup of $\mathrm{D}(\overline{k})$, i.e. if there is an injective representation $\varrho : G \hookrightarrow \mathrm{D}(\overline{k})$.

Example 5.1.10. *Of course, cyclic groups of order relatively prime with the characteristic p are diagonal. The converse is not true, for example, the group*

$$\langle \mathrm{diag}(-1, 1, 1), \mathrm{diag}(1, -1, 1)\rangle$$

is diagonal, but not cyclic, since it is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

It has been proven in [Hug05] the next result for smooth plane curves:

Theorem 5.1.11 (B. Huggins, Theorem 6.4.8, [Hug05])**.** *Let k be a perfect field of characteristic p not equal to 2, and let $\overline{C}$ be a smooth plane curve of genus $g \geq 3$ defined over $\overline{k}$. The field of moduli $M_{\overline{k}/k}(\overline{C})$ of $\overline{C}$, relative to the Galois extension $\overline{k}/k$, is a field of definition, unless $\mathrm{Aut}(\overline{C})$ is $\mathrm{PGL}_3(\overline{k})$-conjugate to a diagonal subgroup of $\mathrm{PGL}_3(\overline{k})$, or to one of the Hessian groups Hess_* with $* \in \{18, 36\}$, or to a semidirect product $\mathcal{B} \rtimes \mathcal{A}$ for some finite diagonal subgroup $\mathcal{A}$ of $\mathrm{PGL}_3(\overline{k})$ and a non-trivial p-group $\mathcal{B}$ consisting entirely of elements* *of the shape*

$$\begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ \beta & \gamma & 1 \end{pmatrix}.$$

Remark 5.1.12. We are interested in smooth plane curves $\overline{C}$ over $\overline{k}$, where k is a perfect field of characteristic $p = 0$ or $p > 2g + 1$. In particular, $\mathrm{Aut}(\overline{C})$ has order coprime with p (see section §1.4 in chapter 2). Thus, by Theorem 5.1.11, one just needs to study the cases where the automorphism group is diagonal or $\mathrm{PGL}_3(\overline{k})$-conjugate to Hess_{18} or Hess_{36}.

On the stratum $\mathcal{M}_{\mathbf{g}}^{\mathbf{pl}}(\mathbf{Hess}_*)$

Recall that $\mathrm{Hess}_{18} = \langle S, T, R\rangle$, where $S := \mathrm{diag}(1, \zeta_3, \zeta_3^2)$, $T := [Y : Z : X]$ and $R := [X : Z : Y]$, and $\mathrm{Hess}_{36} = \langle \mathrm{Hess}_{18}, V\rangle$ where $V := [X+Y+Z : X+\zeta_3 Y+\zeta_3^2 Z : X+\zeta_3^2 Y+\zeta_3 Z]$.

Lemma 5.1.13. *The Hessian groups* Hess_**, for* $* \in \{18, 36\}$ *above are not diagonal in the sense of Definition 5.1.9.*

Proof. Assume on Hess_* is $\mathrm{PGL}_3(\overline{k})$-conjugate to a diagonal subgroup $G \leq \mathrm{PGL}_3(\overline{k})$. By definition G should contain a non-homology ϕ of order 3, and we may write it as $\mathrm{diag}(1, \zeta_3, \zeta_3^2)$. Moreover, there should be another element ψ of order 3, such that $\psi\phi = \phi\psi$. Hence ψ should be of the shape $\{[Y : Z : X], [Z : X : Y]\}$ modulo $\mathrm{D}(\overline{k})$. Thus $\psi \notin \mathrm{D}(\overline{k})$, which contradicts the assumption on G. □

Consider the stratum $\mathcal{M}_g^{pl}(\mathrm{Hess}_{18})$ of smooth plane curves $\overline{C}$ defined over $\overline{k}$ such that $\mathrm{Hess}_{18} \leq \mathrm{Aut}(\overline{C})$. A geometrically complete family over k (Definition 4.3.2) is given below:

Proposition 5.1.14. *Let k be a perfect field of characteristic $p = 0$ or $p > 2g + 1$. The stratum* $\mathcal{M}_g^{pl}(\mathrm{Hess}_{18})$ *is not empty only if the degree d is divisible by 3. In this case, the family*

$$\sum_{\substack{m,n \in \mathbb{N} \\ 2m+n=0}}^{d/3} \alpha_{m,n}(X^3+Y^3+Z^3)^{(d/3)-(2m+n)}((XY)^3+(YZ)^3+(XZ)^3)^m(XYZ)^n = 0, \quad (5.1)$$

with $\alpha_{0,0} \neq 0$ is geometrically complete over k for $\mathcal{M}_g^{pl}(\mathrm{Hess}_{18})$.

Remark 5.1.15. We note that $\mathcal{M}_g^{pl}(\mathrm{Hess}_{36}) \subseteq \mathcal{M}_g^{pl}(\mathrm{Hess}_{18})$. Therefore, if there is a smooth curve $\overline{C} \in \mathcal{M}_g^{pl}(\mathrm{Hess}_{36})$, then it is defined by an equation of the form (5.1), probably with more algebraic restrictions on the parameters a_j, b_j, c_j via the extra automorphism V. That is, the family (5.1) would define a geometrically complete family over k for $\mathcal{M}_g^{pl}(\mathrm{Hess}_{36})$ as well.

Proof. (of Proposition 5.1.14) Since Hess_{18} contains an element of order 3, which is not a homology, we may consider a non-singular plane model $F_{\overline{C}}(X, Y, Z) = 0$ over $\overline{k}$ of degree d, such that $\phi := \mathrm{diag}(1, \zeta_3, \zeta_3^2) \in \mathrm{Aut}(F_{\overline{C}})(\leq \mathrm{Hess}_*)$. In particular, any monomial term in the defining equation for $F_{\overline{C}}(X, Y, Z) = 0$ should be of in the ideal

$$\langle X, Y^3, Z^3, YZ\rangle \subseteq \overline{k}[X, Y, Z].$$

That is, each monomial term of $F_{\overline{C}}(X, Y, Z)$ is of the form $X^sY^{3t}Z^{3u}(YZ)^{d-(s+3t+3u)}$, for some $s, t, u \in \mathbb{N}$. Moreover, there should be two automorphisms ψ and ϑ of orders 3 and 2 respectively, such that $\psi\phi\psi^{-1} = \phi$ and $\vartheta\phi\vartheta = \phi^{-1}$. Some computations shows that there must be $D, D' \in \mathrm{D}(\overline{k})$, $j \in \{1, 2\}$ and $A \in \{[X : Z : Y], [Y : X : Z], [Z : Y : X]\}$, where $\psi = DT^j$ and $\vartheta = D'A$. Thus we may take $\psi = T = [Y : Z : X]$ and $\vartheta = R = [X : Z : Y]$. The defining equation $F_{\overline{C}}(X, Y, Z) = 0$ is then formed by terms in the ideal

$$\langle X^3 + Y^3 + Z^3, XYZ, (XY)^3 + (YZ)^3 + (XZ)^3\rangle.$$

By non-singularity, $F_{\overline{C}}(X, Y, Z)$ should have degree $\geq d - 1$ in each variable. Consequently, the core of $F_{\overline{C}}(X, Y, Z)$ is $X^d + Y^d + Z^d$ and $3 \,|\, d$, since $\langle S, T, R\rangle \leq \mathrm{Aut}(\overline{C})$. Thus the form (5.1) is geometrically complete over k. □

B. Huggins in [Hug05, Chp. 7, §2 and 3] constructed examples of smooth plane curves of genus 10 not definable over their field of moduli, and whose automorphism groups conjugate to Hess_*, for $* = 18$, 36.

Definition 5.1.16. A quaternion extension of a field K is a Galois extension F/K such that $\mathrm{Gal}(F/K)$ is isomorphic to the quaternion group of order 8.

Definition 5.1.17 (Lemma 7.2.3, [Hug05])**.** A field K is of level 2 if -1 is not a square in K, but it is a sum of two squares in K.

Lemma 5.1.18 (Lemma 7.2.3, [Hug05]). *Let K be a field of level 2. Then, for $u, v \in K^*\backslash(K^*)^2$ such that $uv \notin (K^*)^2$, $K(\sqrt{u}, \sqrt{v})$ is embeddable into a quaternion extension of K if and only if $-u$ is a norm from $K(\sqrt{-v})$ to K (i.e. $-u = x^2 + vy^2$ for some $x, y \in K$).*

For instance, the field $K := \mathbb{Q}(\zeta_3)$ is of level 2, since $(\zeta_3^2)^2 + \zeta_3^2 = -1$ and $\sqrt{-1} \notin K$. It is easily shown that ± 2 are not norms from $K(\sqrt{-13})$ to K. So neither $K(\sqrt{2}, \sqrt{13})$ nor $K(\sqrt{-2}, \sqrt{13})$ are embeddable into a quaternion extension of K.

Now fix K to be the field $\mathbb{Q}(\zeta_3)$, and define the following:

$$\begin{aligned}
\natural &:= XYZ,\\
\flat &:= X^3 + Y^3 + Z^3,\\
\sharp &:= (XY)^3 + (YZ)^3 + (XZ)^3.
\end{aligned}$$

Suppose that $\alpha_1, \alpha_2, \alpha_3, u, v \in \mathbb{Q}^*$, such that $L := K(\sqrt{u}, \sqrt{v})$ is a $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ extension of K that can not be embedded into a quaternion extension of K. Let

$$\begin{aligned}
c_{\natural^2} &:= \alpha_1\zeta_3\sqrt{u} + \alpha_2\sqrt{v} + \alpha_3\zeta_3^2\sqrt{uv},\\
c_{\natural\flat} &:= \alpha_1\zeta_3^2\sqrt{u} + \alpha_2\sqrt{v} + \alpha_3\zeta_3\sqrt{uv},\\
c_{\flat^2} &:= \alpha_1\sqrt{u} + \alpha_2\sqrt{v} + \alpha_3\sqrt{uv} - \frac{1}{12}.
\end{aligned}$$

Theorem 5.1.19 (B. Huggins, Lemma 7.2.5 and Proposition 7.2.6, [Hug05]). *Following the above notations, let*

$$F_{\sqrt{u},\sqrt{v}}(X, Y, Z) := c_{\natural^2}\natural^2 - 6c_{\natural\flat}\natural\flat - 18c_{\flat^2}\flat^2 + \sharp.$$

Then the equation $F_{\sqrt{u},\sqrt{v}}(X, Y, Z) = 0$ such that $F_{\sqrt{u},\sqrt{v}}(X, 1, 1)$ is square free, defines a smooth plane curve C over $\overline{\mathbb{Q}}$, with automorphism group Hess_{18}. The field of moduli $M_{\overline{\mathbb{Q}}/\mathbb{Q}}(C)$ is $\mathbb{Q}(\zeta_3)$, but it is not a field of definition.

Remark 5.1.20. It has been mentioned by B. Huggins [Hug05, page 136] that smooth plane curves C over $\mathbb{C}$ with automorphism group Hess_{18} are always definable over its field of moduli, relative to $\mathbb{C}/\mathbb{R}$.

Theorem 5.1.21 (B. Huggins, Lemma 7.3.2 and Proposition 7.3.3, [Hug05]). *Following the* *above notations. For $a = \alpha i \in \mathbb{C}$ with $\alpha \in \mathbb{R}^*$, the equation*

$$F_a(X, Y, Z) := \sharp - 18\natural^2 + (a - \frac{1}{12})\flat^2 - 6a\natural\flat = 0$$

gives a smooth plane curve C over $\mathbb{C}$, with automorphism group Hess_{36}. Moreover, the field of moduli $M_{\mathbb{C}/\mathbb{R}}(C) = \mathbb{R}$, and it is not a field of definition.

§5.2 Débes-Emsalem: The canonical model for $\overline{\mathrm{C}}/\mathrm{Aut}(\overline{\mathrm{C}})$

Let $\overline{C}$ be a smooth curve of genus $g \geq 2$ with non-trivial automorphism group over $\overline{k}$, where k is a perfect field of characteristic $p = 0$ or $p > 2g + 1$. Using Proposition 5.1.5 and Remark 5.1.6, we may take k as the field of moduli $M_{\overline{k}/k}(\overline{C})$ of $\overline{C}$, relative to the Galois extension $\overline{k}/k$. Consider a family of $\overline{k}$-isomorphisms $\{\phi_\sigma : {}^\sigma\overline{C} \to \overline{C}\}_{\sigma \in G_k}$. Each isomorphism ϕ_σ induces an isomorphism $\widetilde{\phi}_\sigma : {}^\sigma\overline{C}/\operatorname{Aut}({}^\sigma\overline{C}) \to \overline{C}/\operatorname{Aut}(\overline{C})$ such that the following diagram is commutative:

$$\begin{tikzcd}
{}^\sigma\overline{C} \arrow[r, "\phi_\sigma"] \arrow[d, "{}^\sigma\pi"] & \overline{C} \arrow[d, "\pi"] \\
{}^\sigma\overline{C}/\operatorname{Aut}({}^\sigma\overline{C}) \arrow[r, "\widetilde{\phi}_\sigma"] & \overline{C}/\operatorname{Aut}(\overline{C})
\end{tikzcd}$$

Next we compose with the canonical isomorphism $i_\sigma : {}^\sigma(\overline{C}/\operatorname{Aut}(\overline{C})) \to {}^\sigma\overline{C}/\operatorname{Aut}({}^\sigma\overline{C})$, that sends

$${}^\sigma P\,.\,{}^\sigma(\operatorname{Aut}(\overline{C})) \in {}^\sigma(\overline{C}/\operatorname{Aut}(\overline{C})) \mapsto {}^\sigma P\,.\,\operatorname{Aut}({}^\sigma\overline{C}) \in {}^\sigma\overline{C}/\operatorname{Aut}({}^\sigma\overline{C}),$$

to get a family of isomorphisms $\{\overline{\phi}_\sigma := i_\sigma \circ \widetilde{\phi}_\sigma : {}^\sigma(\overline{C}/\operatorname{Aut}(C)) \to \overline{C}/\operatorname{Aut}(\overline{C})\}_{\sigma \in G_k}$, satisfying the Weil's cocycle condition of descent (see [DE99, Theorem 3.1]). Therefore, by Theorem 5.1.4, there exists a k-model B_k of $\overline{C}/\operatorname{Aut}(\overline{C})$, and an isomorphism $\theta : B_k \otimes_k \overline{k} \to \overline{C}/\operatorname{Aut}(\overline{C})$ over $\overline{k}$, such that $\theta \circ {}^\sigma\theta^{-1} = \overline{\phi}_\sigma$. In other words, we obtain the next commutative diagram (Figure 5.1) The model B_k is called the *canonical model of $C/\operatorname{Aut}(C)$ over k.*

Proposition 5.2.1 (Débes-Emsalem, Corollary 4.3-(c), [DE99])**.** *Following all the above notations. The curve $\overline{C}$ is definable over its field of moduli k, if the canonical model B_k has a k-point.*

$$
\begin{tikzcd}
{}^{\sigma}\overline{C} \arrow[rr, "\phi_\sigma"] \arrow[d, "{}^{\sigma}\pi"] & & \overline{C} \arrow[d, "\pi"] \\
{}^{\sigma}(\overline{C}/\operatorname{Aut}(\overline{C})) \arrow[rr, "\overline{\phi}_\sigma"] & & \overline{C}/\operatorname{Aut}(\overline{C}) \\
& B_k \otimes_k \overline{k} \arrow[ul, "{}^{\sigma}\theta"] \arrow[ur, "\theta"'] &
\end{tikzcd}
$$

Figure 5.1: The canonical model for $\overline{C}/\operatorname{Aut}(\overline{C})$

§5.3 On the field of moduli of smooth curves with odd signature

Definition 5.3.1 (signature)**.** Let $\Phi : \overline{C} \to \overline{C}/G$ be a branched Galois covering between smooth curves defined over an algebraically closed field, where G is a finite group. Let $y_1, ..., y_r$ be its branch points, that is $\Phi^{-1}(y_i)$ has cardinality $< |G|$. The signature of Φ is defined as $(g_0; m_1, ..., m_r)$, where g_0 is the genus of $\overline{C}/G$ and m_i is the ramification index of any point in $\Phi^{-1}(y_i)$. The *branch divisor* of Φ, denoted by $\mathcal{D}(\Phi)$ is the divisor of $\overline{C}/\operatorname{Aut}(\overline{C})$ defined by $\mathcal{D}(\Phi) := \sum_{i=1}^{r} m_i \cdot y_i$.

R. Hidalgo [Hid12] considered complex curves $\overline{C}$ such that the natural cover $\pi_{\overline{C}} : \overline{C} \to \overline{C}/\operatorname{Aut}(\overline{C})$ has signature of the form $(0; m_1, m_2, m_3, m_4)$, proving that $\overline{C}$ can be defied over its field of moduli if $m_4 \notin \{m_1, m_2, m_3\}$. Artebani-Quispe in [AQ12] extended such a result to smooth curves of odd signature:

Definition 5.3.2 (odd Signature)**.** A smooth curve $\overline{C}$ defined over an algebraically closed field of genus $g \geq 2$ has odd signature if the signature of the natural covering $\pi_{\overline{C}} : \overline{C} \to \overline{C}/\operatorname{Aut}(\overline{C})$ is of the form $(0; m_1, ..., m_r)$ where some m_i appears exactly an odd number of times.

Theorem 5.3.3 (Artebani-Quispe, Theorem 2.5, [AQ12])**.** *Let $\overline{C}$ be a smooth curve of genus $g \geq 2$ defined over an algebraically closed field F. Let L be a subfield of F, such that F/L is Galois. If $\overline{C}$ is an odd signature curve, then $M_{F/L}(\overline{C})$ is a field of definition for $\overline{C}$.*

To prove Theorem 5.3.3, we need some well-known results, we refer, for example, to Lemma 2.3 and Lemma 2.4 in [AQ12]:

Lemma 5.3.4. *Let B be a smooth curve of genus 0 defined over an infinite field L, and suppose that B has an L-rational divisor $\mathcal{D}$ of odd degree. Then B has infinitely many L-rational points.*

Lemma 5.3.5. *Given a Galois branched covering $\Phi : \overline{C} \to \overline{C}/G$ defined over an algebraically closed field F, we have $\mathcal{D}({}^{\sigma}\Phi) = {}^{\sigma}(\mathcal{D}(\Phi))$ for any $\sigma \in \mathrm{Gal}(F/L)$.*

Proof. (of Theorem 5.3.3) By Proposition 5.1.5 and Remark 5.1.6, we may assume that $M_{F/L}(\overline{C}) = L$. Following the notations of section §5.2, we set $\Phi := \theta^{-1} \circ \pi_{\overline{C}}$. Since Φ_{σ} is an isomorphism, $\mathcal{D}(\Phi) = \mathcal{D}({}^{\sigma}\Phi) = {}^{\sigma}(\mathcal{D}(\Phi))$ for any $\sigma \in \mathrm{Gal}(F/L)$, by the aid of Lemma 5.3.5. That is, $\mathcal{D}(\Phi)$ is an L-rational divisor of $\overline{C}/\operatorname{Aut}(\overline{C})$. Moreover, θ is an isomorphism, so $\mathcal{D}(\Phi) = \theta^{-1}(\mathcal{D}(\pi_{\overline{C}}))$. In particular, Φ and $\pi_{\overline{C}}$ have the same signature. Because Φ has an odd signature, then we can take $y_{i_1}, ..., y_{i_{2s+1}}$ to be the points on the support of $\mathcal{D}(\Phi)$ with the same coefficient say m_j. The divisor $y_{i_1} + ... + y_{i_{2s+1}}$ is an L-rational divisor of $\overline{C}/\operatorname{Aut}(\overline{C})$ of odd degree. If L is infinite, then Lemma 5.3.4 implies that B has an L-rational point and $\overline{C}$ can be defined over L by Corollary 5.2.1. Otherwise, the result follows by Corollary $1.6.6$ in [Hug05]. □

From Theorem 5.1.8 and Theorem 5.3.3, it follows:

Theorem 5.3.6 (Artebani-Quispe, Theorem 0.1, [AQ12])**.** *Let C be a smooth curve of genus $g \geq 2$ defined over a field k. If $C \otimes_k \overline{k}$ is an odd signature curve, then C is definable over k_C.*

Remark 5.3.7. A smooth curve $\overline{C}$ defined over $\mathbb{C}$ is called *pseudo-real* if the field of moduli, relative to the extension $\mathbb{C}/\mathbb{R}$, is $\mathbb{R}$, but it is not a field of definition. The idea of having *odd signature* is used among other techniques by Artebani-Quispe-Reyes, in [AQR17], to show that a smooth plane curve $\overline{C}$ of genus 6 over $\mathbb{C}$ is pseudo-real, only if $\operatorname{Aut}(\overline{C})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/4\mathbb{Z}$. However, by Theorem 4.3.6, we know that representative families over $\mathbb{R}$ exist for these two particular stratum. Hence, Lemma 4.3.3 implies that the field of moduli relative to $\mathbb{C}/\mathbb{R}$ is always a field of definition. That is smooth plane curves over $\mathbb{C}$ of genus 6, which are pseudo-real, do not exist.

§5.4 Smooth plane curves with diagonal automorphism group

Let $\mathcal{C} : \mathcal{F}(X, Y, Z) = 0$ be a family of smooth $\overline{k}$-plane curves associated to the stratum $\varrho(\widetilde{\mathcal{M}_g^{Pl}}(G))$ (see Definition 2.2), where k is a perfect field of characteristic $p = 0$ or $p > 2g+1$. Isomorphisms between smooth $\overline{k}$-plane curves in the same family $\mathcal{C}$ (in particular with identical automorphism group $\varrho(G) \leq \mathrm{PGL}_3(\overline{k})$) are given be elements in the normalizer $N_{\varrho(G)}(\overline{k})$ of their automorphism group in $\mathrm{PGL}_3(\overline{k})$. Thus the following lemma is useful for computations:

Lemma 5.4.1 (Normalizer). *Let $\varrho : G \hookrightarrow \mathrm{PGL}_3(\overline{k})$ be a diagonal finite non-trivial group, such that $p \nmid |G|$ when $p > 0$. Hence*

1. *If $\varrho(G)$ contains a non-homology $\phi = \mathrm{diag}(\zeta_n^a, \zeta_n^b, 1)$, then $N_{\varrho(G)}(\overline{k}) = \langle \mathrm{D}(\overline{k}), H\rangle$ for some $H \leq \tilde{\mathrm{S}}_3$*

2. *If $\varrho(G)$ is generated by an homology $\phi = \mathrm{diag}(1, 1, \zeta_n)$ for some $n \in \mathbb{Z}_{\geq 2}$, then $N_{\varrho(G)}(\overline{k}) = \mathrm{GL}_{2,Z}(\overline{k})$.*

Proof. Using Lemma 1.2.7 and the assumption that $\varrho(G)$ is diagonal, we deduce that there is always a unique set V, which is fixed pointwise by $\varrho(G)$. It is $\{P_1 = (1 : 0 : 0),\ P_2 = (0 : 1 : 0),\ P_3 = (0 : 0 : 1)\}$ when a non-homology is present inside $\varrho(G)$, while it is formed by all points of the line $L_3 : Z = 0$ and the point P_3 otherwise. Therefore, V is also fixed by $N_{\varrho(G)}(\overline{k})$, and the computations becomes too straightforward. □

We motivate this section by the next example due to B. Huggins in [Hug05]. Take $m, r \in \mathbb{N}$ such that $2mr > 5$ and r is odd when m does. Let z^c be the complex conjugate of z for any $z \in \mathbb{C}$. Consider a binary form $G(X, Y) \in \mathbb{C}[X, Y] \setminus \mathbb{R}[X, Y]$ given by

$$G(X, Y) := \prod_{i=1}^{r} (X^m - a_i Y^m)(X^m + a_i^c Y^m),$$

for some $a_1, ..., a_r \in \mathbb{C}$ such that the next conditions hold: $G(X, 1)$ has no repeated zeros, the map $[\alpha : \beta] \mapsto [\beta : \alpha]$ does not map the zero set of $G(X, Y)$ into itself, for any root of unity ζ we should have $\{a_i, -1/a_i^c\} \neq \{\zeta a_i, -\zeta/a_i^c\}$, and when $m = 3$, the map $[\alpha : \beta] \mapsto [-\alpha + (1 + \sqrt{3})\beta : (1 + \sqrt{3})\alpha + \beta]$ does not map the zero set of $G(X, Y)$ into itself.

Proposition 5.4.2 (B. Huggins, Chapter 7, §1, [Hug05]). *Following the above notations, let $\overline{C}$ be a smooth plane curve of degree > 5 defined over $\mathbb{C}$ by an equation of the form*

$$F_{\overline{C}}(X,Y,Z) := Z^{2mr} - G(X,Y) = 0$$

Then, the automorphism group $\mathrm{Aut}(F_{\overline{C}})$ is diagonal and equals

$$\langle \mathrm{diag}(\zeta_m, 1, 1),\ \mathrm{diag}(1, \zeta_m, 1),\ \mathrm{diag}(1, 1, \zeta_{2mr})\rangle.$$

Moreover, the field of moduli $M_{\mathbb{C}/\mathbb{R}}(\overline{C})$ is $\mathbb{R}$, but it is not a field of definition.

5.4.1 Diagonal automorphism groups, containing a non-homology

Consider the moduli space $\mathcal{M}_g$ of smooth plane curves $\overline{C}$ of genus $g = \frac{1}{2}(d-1)(d-2) \geq 3$ defined over $\overline{k}$, where k is a perfect field of characteristic $p = 0$ or $p > 2g+1$.

Definition 5.4.3. For arbitrary integer $n \geq 2$, let $\widetilde{(\mathcal{M}_g^{Pl})}^{nh}_{n,\mathrm{diag}}$ be the substratum of $\mathcal{M}_g$ of smooth plane curves $\overline{C}$, satisfying

1. $\mathrm{Aut}(\overline{C})$ is diagonal, i.e. $\varrho(\mathrm{Aut}(\overline{C})) \leq \mathrm{D}(\overline{k})$ for some injective representation ϱ.

2. $\mathrm{Aut}(\overline{C})$ contains a non-homology of maximal order $n > 1$. Being maximal means that any other non-homology in $\mathrm{Aut}(\overline{C})$ is of order at most n.

Fix a non-singular plane model $F_{\overline{C}}(X,Y,Z) = 0$ of $\overline{C}$ over $\overline{k}$ of degree d, where $\mathrm{Aut}(F_{\overline{C}}) = \varrho(\mathrm{Aut}(\overline{\mathrm{C}}))$. Hence $N_{\varrho(\mathrm{Aut}(\overline{C}))}(\overline{k})$ equals $\langle \mathrm{D}(\overline{k}),\ H\rangle$ for some $H \leq \widetilde{\mathrm{S}}_3$, by Lemma 5.4.1-(1). Now we state our main result for this section, improving the results of B. Huggins (Theorem 5.1.11) for smooth plane curves that possess a diagonal automorphism group.

Theorem 5.4.4. *Following the above notations, let k be a perfect field of characteristic $p = 0$ or $p > 2g+1$, $\overline{C} \in \widetilde{(\mathcal{M}_g^{Pl})}^{nh}_{n,\mathrm{diag}}$, and moreover assume that $M_{\overline{k}/k}(\overline{C}) = k$. Then, k is always a field of definition for $\overline{C}$, except possibly when one of the following cases occur:*

1. *First, $n|d$ and $F_{\overline{C}}(X,Y,Z) = 0$ is defined over $\overline{k}$ by an equation of the form*

$$X^d + Y^d + Z^d + \sum_{j=2}^{d-1} \big(X^{d-j} \sum_{i \in S(2)^{j,X}_{n,(a,b)}} \beta_{j,i} Y^i Z^{j-i}\big) + \sum_{i \in S_1^{d,X}{}_{n,\,(a,b)}} \beta_{d,i} Y^i Z^{d-i} = 0.$$

In particular, $\mathrm{Aut}(F_{\overline{C}}) \leq \langle \mathrm{diag}(\zeta_d, 1, 1),\ \mathrm{diag}(1, \zeta_d, 1),\ \mathrm{diag}(1, 1, \zeta_d)\rangle$, *and so any automorphism of* $\overline{C}$ *is of order dividing the degree* d.

2. *Second,* $n|d-1$ *and* $F_{\overline{C}}(X,Y,Z) = 0$ *is defined over* $\overline{k}$ *by an equation of the form*

$$X^d + X\big(Z^{d-1} + Y^{d-1} + \sum_{\substack{r_1, r_2 \in \mathbb{N} \\ 2r_1 + r_2 n = d-1}} (YZ)^{r_1}\big(\alpha_{r_1,r_2} Y^{r_2 n} + \beta_{r_1,r_2} Z^{r_2 n}\big)\big) + $$
$$+ \sum_{j=2}^{d-2} \sum_{\substack{r_5, r_6 \in \mathbb{N} \\ 2r_5 + r_6 n = j}} X^{d-j}(YZ)^{r_5}\big(\mu_{r_5,r_6} Y^{r_6 n} + \lambda_{r_5,r_6} Z^{r_6 n}\big) + $$
$$+ \sum_{\substack{r_3, r_4 \in \mathbb{N} \\ 2r_3 + r_4 n = d}} (YZ)^{r_3}\big(\gamma_{r_3,r_4} Y^{r_4 n} + \delta_{r_3,r_4} Z^{r_4 n}\big) = 0.$$

with $\gamma_{r_3,r_4}\delta_{r_3,r_4} \neq 0$ *and* $\gamma_{r_3 r_4}^2 \neq \nu\delta_{r_3,r_4}^2$ *for some* r_3, r_4, *where* ν *is a* $(d-1)/n$*-th root of unity* ν. *Moreover,* $\alpha_{r_1 r_2} = 0$ *iff* $\beta_{r_1 r_2} = 0$, $\gamma_{r_3 r_4} = 0$ *iff* $\delta_{r_3 r_4} = 0$ *and* $\mu_{r_5 r_6} = 0$ *iff* $\lambda_{r_5 r_6} = 0$. *In this case,* $\mathrm{Aut}(F_{\overline{C}}) = \langle \mathrm{diag}(1, \zeta_n, \zeta_n^{-1})\rangle$.

3. *Third,* $n = mm'$ *for some positive integers* m *and* $m' > 1$ *such that* $m \,|\, d$ *and* $m' \,|\, d-2$, *where* $F_{\overline{C}}(X,Y,Z) = 0$ *is defined over* $\overline{k}$ *by an equation of the form*

$$X^d + Y^{d-1}Z + YZ^{d-1} + $$
$$+ \sum_{\substack{i, j, \ell_j \in \mathbb{N} \\ m \,|\, 2i + jm'}} \beta_j X^{d-2i-jm'}(YZ)^i(Y^{jm'} + \zeta_{2(d-2)}^{jm'\ell_j} Z^{jm'}) = 0,$$

such that $\beta_j \neq 0$ *for some* j. *Moreover,* $\mathrm{Aut}(F_{\overline{C}}) = \langle \mathrm{diag}(1, \zeta_n, \zeta_n^{-(d-1)})\rangle$.

Proof. Since we have a non-homology inside $\mathrm{Aut}(\overline{C})$, it suffices to consider $\overline{C}$ of *Type* $n, (a,b)$ for some n as in Theorem 2.1.3, cases (3)-(6). Hence, for all situations, except when $n \,|\, d$ with respect to case (6) in Theorem 2.1.3, we have at least one of the $P_i's$, the three reference points, lies on $F_{\overline{C}}(X,Y,Z) = 0$. But also $\mathrm{Aut}(F_{\overline{C}})$ is made entirely of diagonal 3×3 projective matrices by assumptions, so $\mathrm{Aut}(F_{\overline{C}})$ fixes a point on $F_{\overline{C}}(X,Y,Z) = 0$. Therefore, it is cyclic (Corollary 1.4.2), generated by some $\mathrm{diag}(1, \zeta_n^a, \zeta_n^b)$ with $ab \neq 0$. For $\overline{C}$ in the family of Theorem 2.1.3, case (6), non of the reference points lies on $F_{\overline{C}}(X,Y,Z) = 0$. In such case, $\mathrm{Aut}(F_{\overline{C}})$ does not need to be cyclic, see for example, Proposition 5.4.2. More concretely, the core of the defining equation for $\overline{C}$ is $X^d+Y^d+Z^d$, thus $\mathrm{Aut}(F_{\overline{C}})$ (being diagonal) lives inside

$$\langle \mathrm{diag}(\zeta_d,1,1),\ \mathrm{diag}(1,\zeta_d,1),\ \mathrm{diag}(1,1,\zeta_d)\rangle,$$

which shows the result in the statement of Theorem 5.4.4-(1).

On the other hand, the idea of the proof applied to the different families in Theorem 2.1.3, cases (3)-(5) is to study the action of the normalizer $N_{\mathrm{Aut}(F_{\overline{C}})}(\overline{k}) \leq \langle \mathrm{D}(\overline{k}), \widetilde{\mathrm{S}}_3\rangle$ (Lemma 5.4.1). Once we are able to reduce to a situation where there exists a set of isomorphisms

$$({}^{\sigma}F_{\overline{C}})(X,Y,Z)=0 \xrightarrow{\phi_\sigma} F_{\overline{C}}(X,Y,Z)=0,$$

for $\sigma\in G_k$, living in $\mathrm{D}(\overline{k})$, we deduce that the canonical model B_k over k for $\overline{C}/\,\mathrm{Aut}(\overline{C})$ has k-points, and therefore k is a field of definition of $\overline{C}$, by using Proposition 5.2.1.

We distinguish between the following cases appeared in Theorem 2.1.3, (3)-(5):

(I) Theorem 2.1.3-(3): This case is distinguished by involving all reference points P_i, for $i=1,2,3$, lying on $F_{\overline{C}}(X,Y,Z)=0$. Moreover, $\overline{C}$ is of *Type* n, (a,b) for some $(a,b)\in\Gamma_n$ such that $n\,|\,(d^2-3d+3)$ and $a\equiv(d-1)a+b\equiv(d-1)b \,(\mathrm{mod}\ n)$. We may take $a\equiv 1\,(\mathrm{mod}\ n)$ and $b\equiv -(d-1)\,(\mathrm{mod}\ n)$ as a generator of $\varrho(\mathbb{Z}/n\mathbb{Z})$, since we have $b\equiv -(d-1)a\,(\mathrm{mod}\ n)$, and so

$$\mathrm{diag}(1,\zeta_n^a,\zeta_n^b)=\mathrm{diag}(1,\zeta_n,\zeta_n^{-(d-1)})^a\in\langle\mathrm{diag}(1,\zeta_n,\zeta_n^{-(d-1)})\rangle.$$

The defining equation in this case has core $X^{d-1}Y+Y^{d-1}Z+Z^{d-1}X$, hence the action of $\langle \mathrm{D}(\overline{k}),\widetilde{\mathrm{S}}_3\rangle$ is trivial except possibly an isomorphism of $\langle \mathrm{D}(\overline{k}),[Y:Z:X]\rangle$. However, $[Y:\lambda Z:\mu X]\notin N_{\mathrm{Aut}(F_{\overline{C}})}(\overline{k})$, since otherwise $\mathrm{diag}(1,\zeta_n^{d-1},\zeta_n^{d})\in \mathrm{Aut}(F_{\overline{C}})=\langle\mathrm{diag}(1,\zeta_n,\zeta_n^{-(d-1)})\rangle$. That is, $2(d-1)\equiv 0\,(\mathrm{mod}\ n)$, and n should divide $\gcd(2(d-1),d^2-3d+3)=1$, a contradiction. Therefore, by Lemma 5.4.1, the normalizer $N_{\mathrm{Aut}(F_{\overline{C}})}(\overline{k})\leq \mathrm{D}(\overline{k})$. In particular, we always can take

$$({}^{\sigma}F_{\overline{C}})(X,Y,Z)=0 \xrightarrow{\phi_\sigma} F_{\overline{C}}(X,Y,Z)=0$$

in $\mathrm{D}(\overline{k})$.

Next, the three reference points $P_i's$ are common k-points of $F_{\overline{C}}(X,Y,Z)=0$ and

$({}^{\sigma}F_{\overline{C}})(X,Y,Z) = 0$, which also pointwise fixed by $\mathrm{Aut}(F_{\overline{C}}) = \mathrm{Aut}({}^{\sigma}F_{\overline{C}})$, and $\phi'_{\sigma}s$, since all are 3×3 projective matrices of diagonal shapes. Following the notations of section 5.2, the images $\overline{P}_i := \pi(P_i) = ({}^{\sigma}\pi)(P_i)$ become k-points of $F_{\overline{C}}/\,\mathrm{Aut}(F_{\overline{C}})$ and ${}^{\sigma}(F_{\overline{C}}/\,\mathrm{Aut}(F_{\overline{C}}))$ simultaneously, and also pointwise fixed by the isomorphism $\overline{\phi}_{\sigma}$ via the commutativity of Figure (5.1). Hence, each of the points $\theta^{-1}(\overline{P}_i) \in B_k \otimes_k \overline{k}$, for $i = 1,2,3$ satisfies

$$\theta^{-1}(\overline{P}_i) = ({}^{\sigma}\theta^{-1}\circ\overline{\phi}_{\sigma})(\overline{P}_i) = ({}^{\sigma}\theta^{-1})(\overline{P}_i) = {}^{\sigma}(\theta^{-1}({}^{\sigma^{-1}}\overline{P}_i)) = {}^{\sigma}(\theta^{-1}(\overline{P}_i)),$$

for all $\sigma \in G_k$. In particular, they are k-points of $B_k \otimes_k \overline{k}$, and so of B_k.

(II) Theorem 2.1.3-(4): In this situation, the two reference points P_2 and P_3 lie on $\overline{C}$: $F_{\overline{C}}(X,Y,Z) = 0$, where $\overline{C}$ is of *Type* n, (a,b) for some $(a,b) \in \Gamma_n$, according to one of the following subcases:

(i) Theorem 2.1.3-(4.1): $\overline{C}$ is of *Type* n, (a,b) for some $n\,|\,d(d-2)$, such that $(d-1)a + b \equiv 0\,(\mathrm{mod}\ n)$ and $a + (d-1)b \equiv 0\,(\mathrm{mod}\ n)$. Since

$$\mathrm{diag}(1,\zeta_n^a,\zeta_n^b) = \mathrm{diag}(1,\zeta_n^a,\zeta_n^{-(d-1)a}) = \mathrm{diag}(1,\zeta_n,\zeta_n^{-(d-1)})^a,$$

we can take $a \equiv 1\,(\mathrm{mod}\ n)$ and $b \equiv -(d-1)\,(\mathrm{mod}\ n)$ as a generator of these curves. Moreover, if we write $n = mm'$ for some $m\,|\,d$ and $m'\,|\,d-2$, then $m' > 1$ (otherwise, $n\,|\,d$, and the automorphism $\mathrm{diag}(1,\zeta_n,\zeta_n^{-(d-1)})$ reduces to $\mathrm{diag}(1,\zeta_n,\zeta_n)$, i.e. it is a homology, a contradiction). Therefore, $\mathrm{diag}(\zeta_m,1,1)$ and $\mathrm{diag}(1,\zeta_{m'},\zeta_{m'}^{-1})$ are also automorphisms of $\mathrm{Aut}(F_{\overline{C}})$. This in turns restricts the defining equation $\mathcal{F}_{\varrho_{a,b,m}}(X,Y,Z) = 0$ in Theorem 2.1.3-(4.1) to be

$$X^d + Y^{d-1}Z + YZ^{d-1} + \sum_{\substack{i,j\,\in\,\mathbb{N} \\ m\,|\,2i+jm'}} X^{d-2i-jm'}(YZ)^i(\alpha_{i,j}Y^{jm'} + \beta_{i,j}Z^{jm'}) = 0.$$

A priori, one does not need to worry about smooth curves in this family, which are isomorphic to their conjugates through a family of isomorphisms in $\mathrm{D}(\overline{k})$ (in such a situation, k is a field of definition, since the canonical model has k-points arising from the two reference points P_2 and P_3 on $\mathcal{F}_{\varrho_{a,b,m}}(X,Y,Z) = 0$). So, we just pay our attention to the subfamily $\overline{\mathcal{C}}_0 : \mathcal{F}_{0,\overline{\mathcal{C}}}(X,Y,Z) = 0$, where $[X : \lambda Z : \mu Y]$ (or equivalently, $[X : Z : Y]$) may act non-trivially and might contains smooth curves not having k as a field of definition. This subfamily is characterized by the next property: For any $i, j \in \mathbb{N}$, $\alpha_{i,j} = 0$ iff $\beta_{i,j} = 0$, and also $\alpha_{i_0,j_0}\beta_{i_0,j_0} \neq 0$ for some $i_0, j_0 \in \mathbb{N}$. For each $i, j \in \mathbb{N}$ with $m \,|\, 2i + jm'$, we define the subfamily $\overline{\mathcal{C}}_0^{(i,j)}$ by the equation $\mathcal{F}^{(i,j)}_{0,\overline{\mathcal{C}}}(X,Y,Z) = 0$ of the form

$$\begin{aligned} X^d + Y^{d-1}Z + YZ^{d-1} \;&+\; X^{d-2i-jm'}(YZ)^i(a_jY^{jm'} + b_jZ^{jm'}) + \\ &+ \sum_{\substack{i',\, j' \in \mathbb{N} \\ m \,|\, 2i' + j'm'}} X^{d-2i'-j'm'}(YZ)^{i'}(a_{j'}Y^{j'm'} + b_{j'}Z^{j'm'}) = 0, \end{aligned}$$

with $a_jb_j \neq 0$, $a_j \neq b_j$, and $a_{j'} = 0$ iff $b_{j'} = 0$. This gives us a union decomposition for the family $\overline{\mathcal{C}}_0$ as $\bigcup_{i,j} \overline{\mathcal{C}}_0^{(i,j)}$. Furthermore, the action of $[X : Z : Y]$ on the component $\mathcal{F}^{(i,j)}_{0,\overline{\mathcal{C}}}(X,Y,Z) = 0$ can always be trivialized when $b_j \neq -a_j$, through the isomorphism

$$\phi_{i,j} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_j & b_j \\ 0 & b_j & a_j \end{pmatrix} : {}^{\phi_{i,j}}\{\mathcal{F}^{(i,j)}_{0,\overline{\mathcal{C}}}(X,Y,Z) = 0\} \to \{\mathcal{F}^{(i,j)}_{0,\overline{\mathcal{C}}}(X,Y,Z) = 0\}.$$

To see this we notice that $[X : Z : Y]$ acts non-trivially on $\mathcal{F}^{(i,j)}_{0,\overline{\mathcal{C}}}(X,Y,Z) = 0$ iff it does for the family

$$X^d + Y^{d-1}Z + YZ^{d-1} + X^{d-2i-jm'}(YZ)^i(a_jY^{jm'} + b_jZ^{jm'}) = 0.$$

Moreover, $\phi_{i,j}$ satisfies the Weil's cocycle condition of descent

$$[X : Z : Y] \circ {}^{\sigma_0}\phi_{i,j} = \phi_{i,j},$$

where σ_0 is the automorphism of $\overline{k}(a_j, b_j)$ mapping $a_j \mapsto b_j$ and $b_j \mapsto a_j$. The new component

$${}^{\phi_{i,j}}\overline{\mathcal{C}}_0^{(i,j)} : \mathcal{F}^{(i,j)}_{0,\overline{\mathcal{C}}}(\phi_{i,j}^{-1}(X,Y,Z)) = 0$$

for the transformed family ${}^{\phi_{i,j}}\overline{\mathcal{C}}_0 : \mathcal{F}_{0,\overline{\mathcal{C}}}(\phi_{i,j}^{-1}(X,Y,Z)) = 0$, which has less isomorphic geometric fibers, satisfies the property that any of its curves ${}^{\phi_{i,j}}F_{\overline{C}}(X,Y,Z) = F_{\overline{C}}(\phi_{i,j}^{-1}(X,Y,Z)) = 0$ is isomorphic to its conjugates $({}^{\sigma}F_{\overline{C}})({}^{\sigma}\phi_{i,j}^{-1}(X,Y,Z))) = 0$ through $\widehat{\phi}_{\sigma} = {}^{\sigma}\phi_{i,j}^{-1} \circ \phi_{\sigma} \circ \phi_{i,j}$ for some $\phi_{\sigma} \in \mathrm{D}(\overline{k})$. Furthermore, $\sigma(\frac{a_j}{b_j})$ must be either $\frac{a_j}{b_j}$ or $\frac{b_j}{a_j}$, since the two points $P_2^{i,j} := \phi_{i,j}^{-1}(P_2) = (0:1:\frac{-b_j}{a_j})$ and $P_3^{i,j} := \phi_{i,j}^{-1}(P_3) = (0:1:\frac{-a_j}{b_j})$ are the only points of $F_{\overline{C}}(\phi_{i,j}^{-1}(X,Y,Z)) = 0$, which are fixed by $\mathrm{Aut}({}^{\phi_{i,j}}F_{\overline{C}}) = \phi_{i,j}^{-1} \circ \mathrm{Aut}(F_{\overline{C}}) \circ \phi_{i,j}$ (recall that $\mathrm{Aut}(F_{\overline{C}})$ is diagonal, containing a non-homology by assumption. Thus P_2 and P_3 are the fixed points of $\mathrm{Aut}(F_{\overline{C}})$ on $F_{\overline{C}}(X,Y,Z) = 0$). But if $\phi_{\sigma} = \mathrm{diag}(1,\lambda_{\sigma},\mu_{\sigma})$ acts on $\mathcal{F}_{0,\mathcal{C}}^{(i,j)}(X,Y,Z) = 0$, then $\lambda_{\sigma}^{d-1}\mu_{\sigma} = \lambda_{\sigma}\mu_{\sigma}^{d-1} = 1, \sigma(a_j) = (\lambda_{\sigma}\mu_{\sigma})^i\lambda_{\sigma}^{jm'}a_j$ and $\sigma(b_j) = (\lambda_{\sigma}\mu_{\sigma})^i\mu_{\sigma}^{jm'}b_j$. Hence $\sigma(\frac{a_j}{b_j}) = (\frac{a_j}{b_j})\zeta_{d-2}^{jm'\ell_j}$, for some integer ℓ_j. Consequently, if $b_j^2 \neq a_j^2\zeta_{d-2}^{jm'\ell_j}$, we always get a k-point on the canonical model of ${}^{\phi_{i,j}}F_{\overline{C}}/\mathrm{Aut}({}^{\phi_{i,j}}F_{\overline{C}})$. For example, the two points $\theta^{-1}(\overline{P_2^{i,j}})$ and $\theta^{-1}(\overline{P_2^{i,j}})$ are such k-points, where $\overline{P_2^{i,j}}$ and $\overline{P_2^{i,j}}$ are the images of $P_2^{i,j}$ and $P_3^{i,j}$ under the action of π. We justify this for $\overline{P_2^{i,j}}$ and similarly for $\overline{P_3^{i,j}}$;

$$\theta^{-1}(\overline{P_2^{i,j}}) = ({}^{\sigma}\theta^{-1} \circ \overline{\phi}_{\sigma})(\overline{P_2^{i,j}}) = ({}^{\sigma}\theta^{-1})(\overline{P_2^{i,j}}) = {}^{\sigma}(\theta^{-1}({}^{\sigma^{-1}}\overline{P_2^{i,j}})) = {}^{\sigma}(\theta^{-1}(\overline{P_2^{i,j}}),$$

for any $\sigma \in G_k$. Finally, we conclude by our discussion that curves $\overline{C}$ in the original family, which might not be definable over their field of moduli are in one of the components $\overline{\mathcal{C}}_0^{(i,j)}$ where $b_j^2 = a_j^2\zeta_{d-2}^{jm'\ell_j}$. This reduced the equation to

$$\begin{aligned} X^d + Y^{d-1}Z \;&+\; YZ^{d-1} + \\ &+ \sum_{\substack{i,\,j,\,\ell_j\,\in\,\mathbb{N} \\ m\,|\,2i+jm'}} \beta_j X^{d-2i-jm'}(YZ)^i(Y^{jm'} + \zeta_{2(d-2)}^{jm'\ell_j}Z^{jm'}) = 0, \end{aligned}$$

such that $\beta_j \neq 0$ for some j. However, the canonical model always has the two points $\theta^{-1}(\overline{P_2^{i,j}})$ and $\theta^{-1}(\overline{P_2^{i,j}})$, where

$${}^{\sigma}\{\theta^{-1}(\overline{P_2^{i,j}}),\, \theta^{-1}(\overline{P_2^{i,j}})\} = \{\theta^{-1}(\overline{P_2^{i,j}}),\, \theta^{-1}(\overline{P_2^{i,j}})\},$$

for all $\sigma \in G_k$. In particular, both points are definable over at most a quadratic extension of k. This shows case (3) in the statement of the theorem.

(ii) Theorem 2.1.3-(4.2): $\overline{C}$ is of *Type* $n,\ (a,b)$ for some $n\,|\,(d-1)^2$ with $(d-1)a+b\equiv 0\,(\mathrm{mod}\ n)$ and $(d-1)b\equiv 0\,(\mathrm{mod}\ n)$. Obviously, the core $X^d+XZ^{d-1}+Y^{d-1}Z$ is not retained by any permutation of the coordinates functions $\{X,Y,Z\}$. So the group $\widetilde{S}_3$ gives no non-trivial isomorphic geometric fibers in the family. In particular, $F_{\overline{C}}(X,Y,Z)=0$ is isomorphic to its conjugates through isomorphisms in $\mathrm{D}(\overline{k})$, by using again Lemma 5.4.1. Similarly as before, the canonical model B_k for $F_{\overline{C}}/\operatorname{Aut}(F_{\overline{C}})$ over k has at least two k-points, which shows the result in this subcase.

(iii) Theorem 2.1.3-(4.3): $\overline{C}$ is of *Type* $n,\ (a,b)$ for some $n\,|\,(d-1)$. We observe that curves in such a family are classified to be of one of the types: *Type* $n,\ (1,b)$, *Type* $n,\ (a,1)$ for some $a,b\in\{2,3,...,n-1\}$, or *Type* $mm',\ (m',m)$ where $m,m'>1$ are relatively prime. Indeed, if $\gcd(a,n)=1$ (resp. $\gcd(b,n)=1$), then $ab'\equiv 1\,(\mathrm{mod}\ n)$ (resp. $a'b\equiv 1\,(\mathrm{mod}\ n)$) for some b' (resp. a'). So, rename $\zeta_n:=\zeta_n^a$ (resp. ζ_n^b), and $b:=bb'$ (resp. $a:=aa'$) to obtain $\mathrm{diag}(1,\zeta_n^a,\zeta_n^b)=\mathrm{diag}(1,\zeta_n^a,\zeta_n^{bb'a})$ (resp. $\mathrm{diag}(1,\zeta_n^{ba'a},\zeta_n^b)):=\mathrm{diag}(1,\zeta_n,\zeta_n^b)$ (resp. $\mathrm{diag}(1,\zeta_n^a,\zeta_n)$), and we get $F_{\overline{C}}(X,Y,Z)=0$ of *Type* $n,\ (1,b)$ or *Type* $n,\ (1,a)$. Otherwise $\gcd(a,n),\ \gcd(b,n)>1$ and $\mathrm{diag}(1,\zeta_n^a,\zeta_n^b)=\mathrm{diag}(1,\zeta_m^{a'},\zeta_{m'}^{b'})$, where $m:=\frac{n}{\gcd(a,n)}$, $m'=\frac{n}{\gcd(b,n)}$, $a'=\frac{a}{\gcd(a,n)}$, and $b'=\frac{b}{\gcd(b,n)}$. Thus $m,\ a'$ and $m',\ b'$ are relatively prime. Rename $\zeta_m^{a'}:=\zeta_m$ and $\zeta_{m'}^{b'}:=\zeta_{m'}$ to recover the last types.

On the other hand, the geometrically complete family over k, defining the whole stratum in this subcase has core $X^d+X(Z^{d-1}+Y^{d-1})$, which restricts the normalizer $N_{\operatorname{Aut}(F_{\overline{C}})}(\overline{k})$ to be a subgroup of $\langle\mathrm{D}(\overline{k}),[X:Z:Y]\rangle$, by the aid of Lemma 5.4.1. Therefore, we just need to characterize those curves in the family, for which the action of $[X:\lambda Z:\mu Y]\in\mathrm{PGL}_3(\overline{k})$ is not trivial, since otherwise we can take $\{\phi_\sigma\}\subset\mathrm{D}(\overline{k})$, which in turns gives an existence of k-points on the canonical model B_k over k for $F_{\overline{C}}/\operatorname{Aut}(F_{\overline{C}})$, using the same discussions as before. By non-singularity, the index set $S_2^{d,X}{}_{n,(a,b)}$ is non-empty or the curve is reduced to $X.\,G(X,Y,Z)$. Moreover, if $[X:\lambda Z:\mu Y]$ provides isomorphic geometric fibers in the family, then $d-i\in S_2^{d,X}{}_{n,(a,b)}$ whenever an i does. We treat the situation for the different types mentioned earlier in our substratum:

For $F_{\overline{C}}(X,Y,Z)=0$ of *Type* $mm',\ (m',\,m)$ with m and m' relatively prime, $i(m'-m)+m\equiv 0\,(\mathrm{mod}\ mm')$ and $i(m'-m)-m'\equiv 0\,(\mathrm{mod}\ mm')$. Hence $m+m'\equiv 0\,(\mathrm{mod}\ mm')$, which is not possible. For $F_{\overline{C}}(X,Y,Z)=0$ of *Type* $n,\ (1,b)$, we get $-i(b-1)+b\equiv 0\,(\mathrm{mod}\ n)$ and $i(b-1)+1\equiv 0\,(\mathrm{mod}\ n)$. Hence $b\equiv -1\,(\mathrm{mod}\ n)$, $2i\equiv 1\,(\mathrm{mod}\ n)$. Consequently, $\mathrm{Aut}(F_{\overline{C}})=\langle \mathrm{diag}(1,\zeta_n,\zeta_n^{-1})\rangle$, and the defining equation, in Theorem 2.1.3-(4.3), for $F_{\overline{C}}(X,Y,Z)=0$ is reduced to the prescribed form in the statement of Theorem 5.4.4-(2). Lastly, the restrictions on the parameters arises from the fact that $[X:\lambda Z:\mu Y]$ acts non-trivially on $F_{\overline{C}}(X,Y,Z)=0$. Similarly, we handle the situation for *Type* $n,\ (a,1)$.

(III) Theorem 2.1.3-(5): We get $\overline{C}$ of *Type n, (a,b)* for some $n\,|\,d(d-1)$ and $(a,b)\in\Gamma_n$, such that $da\equiv 0\,(\mathrm{mod}\ n)$ and $(d-1)b\equiv 0\,(\mathrm{mod}\ n)$. Moreover, exactly one reference point lies on $F_{\overline{C}}(X,Y,Z)=0$. If we look at the core $X^d+Y^d+XZ^{d-1}$ of the family describing our stratum, and noticing that $1\notin S_1^{d,X}$, then we recognize that the action of the normalizer $N_{\mathrm{Aut}(F_{\overline{C}})}(\overline{k})$ is trivial except possibly an element of $\mathrm{D}(\overline{k})$, see Lemma 5.4.1. Hence, as we explain many times, one obtains a k-point on the canonical model for $F_{\overline{C}}/\,\mathrm{Aut}(F_{\overline{C}})$, and $\overline{C}$ is definable over k, its field of moduli.

This completes the proof. □

Remark 5.4.5. Suppose that a smooth plane curve $\overline{C}$ over $\overline{k}$, as in Theorem 5.4.4, descends to its field of moduli k, relative to the extension $\overline{k}/k$, where k is perfect. Then there is no guarantee, in this case, that the curve $\overline{C}$ has a non-singular plane model over k. This is not true in general, however it does when the degree d is coprime with 3, the curve has a k-point, or the 3-torsion $\mathrm{Br}[3](k)$ of the Brauer group $\mathrm{Br}(k)$ is trivial; Corollaries 3.2.1, 3.2.2 and Theorem 3.2.8.

The general question concerning the existence of non-singular plane models over fields of definition of a smooth plane curve $\overline{C}$ over $\overline{k}$ has already been addressed in chapter 3 of this memoir.

We already have seen an example of (1) at the beginning of this section (Proposition 5.4.2

with $m, r > 2$). We construct next an explicit example for case (2) in Theorem 5.4.4:

Theorem 5.4.6. *For any degree $d = nm + 1$ with $n > 10$ even and $4 \nmid nm$, let $\overline{C}$ be a smooth plane curve over $\mathbb{C}$ of degree d defined by an equation $F_{\overline{C}}(X, Y, Z) = 0$ of the form*

$$\begin{aligned} X^d \;&+ \sum_{\substack{r_3,\, r_4 \,\in\, \mathbb{N} \\ 2r_3 \,+\, r_4 n \,=\, d}} (YZ)^{r_3}\big(\gamma_{r_3,r_4} Y^{r_4 n} + (-1)^{r_3}\overline{\gamma}_{r_3,r_4} Z^{r_4 n}\big) + \\ &+ \sum_{j=2}^{d-2} \sum_{\substack{r_5,\, r_6 \,\in\, \mathbb{N} \\ 2r_5 \,+\, r_6 n \,=\, j}} X^{d-j}(YZ)^{r_5}\big(\mu_{r_5,r_6} Y^{r_6 n} + (-1)^{r_5}\overline{\mu}_{r_5,r_6} Z^{r_6 n}\big) + \\ &+ X\Big(Z^{d-1} + Y^{d-1} + \sum_{\substack{r_1,\, r_2 \,\in\, \mathbb{N} \\ 2r_1 \,+\, r_2 n \,=\, d-1}} (YZ)^{r_1}\big(\alpha_{r_1,r_2} Y^{r_2 n} + (-1)^{r_1}\overline{\alpha}_{r_1,r_2} Z^{r_2 n}\big)\Big) = 0, \end{aligned}$$

such that $\gamma_{r_3,r_4} \neq 0$, $(-1)^{r_3}$, $(-1)^{r_3}\zeta_{d-1}^{j}\overline{\gamma}_{r_3,r_4}$ for some $r_3, r_4 \in \mathbb{N}$ with $2r_3 + r_4 n = d$ and $\gcd(r_4, m) = 1$. *Then* $\mathrm{Aut}(F_{\overline{C}})$ *is cyclic, generated by* $\mathrm{diag}(1, \zeta_n, \zeta_n^{-1})$. *Moreover, the field of moduli $M_{\mathbb{C}/\mathbb{R}}(\overline{C})$ is $\mathbb{R}$, but it is not a field of definition.*

Remark 5.4.7. The non-singularity restrictions for $F_{\overline{C}}(X, Y, Z) = 0$ are too tedious to be explicitly written down. However, we warn the reader that a plane curve over $\mathbb{C}$ defined by such a form, which is also smooth, may not exist.

Proof. (of Theorem 5.4.6) Since $\mathrm{diag}(1, \zeta_n, \zeta_n^{-1})$ is an automorphism of $F_{\overline{C}}(X, Y, Z) = 0$ of order $n > 10$, $\mathrm{Aut}(F_{\overline{C}})$ can not be conjugate to any of the finite primitive subgroups of $\mathrm{PGL}_3(\mathbb{C})$ mentioned in Theorem 4.2.3-(3). Moreover, $(F_{\overline{C}}, G)$ is not a descendant of the Fermat curve $X^d + Y^d + Z^d = 0$ or the Klein curve $X^{d-1}Y + Y^{d-1}Z + Z^{d-1}X = 0$ with $G = \mathrm{Aut}(F_{\overline{C}})$, because $n = \frac{d-1}{m}$ does not divide any of the integers $6d^2$ and $3(d^2 - 3d + 3)$. Consequently, we can think about $\mathrm{Aut}(F_{\overline{C}})$ in a short exact sequence (Theorem 1.4.4) of the form

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{C}^* \arrow[r] & \mathrm{GL}_{2,Z}(\mathbb{C}) \arrow[r, "\Lambda"] & \mathrm{PGL}_2(\mathbb{C}) \arrow[r] & 1 \\ 1 \arrow[r] & H \arrow[r] \arrow[u, hook] & \mathrm{Aut}(F_{\overline{C}}) \arrow[r] \arrow[u, hook] & G \arrow[r] \arrow[u, hook, "\varrho"] & 1 \end{tikzcd}$$

where H is a cyclic group of order dividing the degree d, and G is conjugate to one of the following group: $\mathbb{Z}/t\mathbb{Z}$, D_{2t} with $t \leq d-1$, A_4, A_5 or to S_4. Furthermore, H can be thought as the subgroup of automorphisms of $F_{\overline{C}}(X,Y,Z)=0$ acting trivially on Y and Z, whereas G acts trivially on X. Hence H is trivial because of the core $X^d + X(Y^{d-1}+Z^{d-1})$ of the defining equation for $F_{\overline{C}}(X,Y,Z) = 0$. Also, G always has the element $\varrho(\mathrm{diag}(1,\zeta_n,\zeta_n^{-1})) = \mathrm{diag}(\zeta_n^2,1)$ of order $\frac{n}{\gcd(n,2)} > 5$, so it should be cyclic or dihedral. In both cases, $\langle \mathrm{diag}(1,\zeta_n,\zeta_n^{-1})\rangle$ is normal in $\mathrm{Aut}(C)$, and therefore $\mathrm{Aut}(F_{\overline{C}}) \leq \langle D(\mathbb{C}), [X:Z:Y]\rangle$ using Lemma 5.4.1. More concretely, any automorphism of $F_{\overline{C}}(X,Y,Z) = 0$ is of the shape $\mathrm{diag}(1,\zeta_{d-1}^f,\zeta_{d-1}^{f'})$ or $[X : \zeta_{d-1}^{f'}Z : \zeta_{d-1}^{f}Y]$ for some $0 \leq f, f' < d-1$. The condition $\gamma_{r_3,r_4} \neq 0, (-1)^{r_3}, (-1)^{r_3}\zeta_{d-1}^{j}\overline{\gamma}_{r_3,r_4}$ for some r_3 and r_4 ensures that $\mathrm{Aut}(F_{\overline{C}})$ is diagonal. Hence it is cyclic, since it fixes the two reference points P_2 and P_3 on $F_{\overline{C}}(X,Y,Z) = 0$. Lastly the same kind of argument as we did in the proof of Theorem 5.4.4-(II) works for $F_{\overline{C}}(X,Y,Z) = 0$, and one deduces that $\mathrm{Aut}(F_{\overline{C}}) = \langle \mathrm{diag}(1,\zeta_{ns},\zeta_{ns}^{-1})\rangle$ for some s dividing m (recall that $d-1=nm$). So, the monomial terms in defining equation for C are in the ideal (X, YZ, Y^s, Z^s). However, the restriction $\gcd(r_4,m)=1$ for some r_4 restricts $s=1$, and thus $\mathrm{Aut}(F_{\overline{C}})$ is generated by $\mathrm{diag}(1,\zeta_n,\zeta_n^{-1})$.

On the other hand, $F_{\overline{C}}(X,Y,Z) = 0$ is isomorphic to its complex conjugate $({}^{\sigma}F_{\overline{C}})(X,Y,Z))$ through the isomorphism $\phi = [X : -Z : Y]$. Hence $\mathbb{R}$ is the field of moduli of $F_{\overline{C}}(X,Y,Z) = 0$, relative to $\mathbb{C}/\mathbb{R}$. Moreover, any isomorphism $\phi' : ({}^{\sigma}F_{\overline{C}})(X,Y,Z) \to F_{\overline{C}}(X,Y,Z)$ is of the shape $\eta\circ\phi$ for some $\eta \in \mathrm{Aut}(F_{\overline{C}})$. That is $\phi' = [X : -\zeta_n^{-f}Z : \zeta_n^{f}Y]$ for some integer $0 \leq f < n$. One easily checks that such a ϕ' does not satisfy Weil's cocycle condition of descent ($\phi'\circ\overline{\phi'} = 1$), since nm is not divisible by 4: indeed, $\phi'\circ\overline{\phi'} = \mathrm{diag}(-1,\zeta_n^{2f},\zeta_n^{-2f})$, so we ask for $\zeta_n^{2f} = -1$, which in turns gives $4f = 0$, n, $2n$, or $3n$. Since $4 \nmid n$, $\phi' = [X : -Z : Y]$ or $[X : Z : -Y]$, and $\phi^2 \neq 1$ in both situations. Consequently $\mathbb{R}$ is not a field of definition for C. $\square$

5.4.2 Diagonal automorphism groups containing only homologies

The following lemma classifies the diagonal groups in $\mathrm{PGL}_3(\overline{k})$, which are made entirely of homologies. Here k is a perfect field of characteristic $p \geq 0$.

Lemma 5.4.8. *Let $\varrho : G \hookrightarrow \mathrm{PGL}_3(\overline{k})$ be a diagonal finite non-trivial group, such that $p \nmid |G|$*

when $p > 0$. If $\varrho(G)$ is made entirely of homologies (Definition 1.2.6), then $\varrho(G)$ is either cyclic or it is conjugate to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} := \langle \mathrm{diag}(1, -1, 1), \mathrm{diag}(1, 1, -1)\rangle$.

Proof. Assume that $\varrho(G) \neq \langle \mathrm{diag}(1, -1, 1), \mathrm{diag}(1, 1, -1)\rangle$. Hence there must be an homology $\phi \in \varrho(G)$ of order $m > 2$, since $\langle \mathrm{diag}(1, -1, 1), \mathrm{diag}(1, 1, -1)\rangle$ is the unique non-trivial diagonal subgroup whose elements have orders at most 2. There is no loss of generality to assume that $\phi = \mathrm{diag}(1, 1, \zeta_m)$, in particular its axis is the reference line $L_3 : Z = 0$ and its center is the reference point $P_1 = (1 : 0 : 0)$. If $\varrho(G) \setminus \langle\phi\rangle = \emptyset$, then $\varrho(G)$ is cyclic and there is nothing to prove further. Otherwise, we can take $\psi \in \varrho(G) \setminus \langle\phi\rangle$. Moreover, if ψ has a different axis from L_3, then $\phi^s\psi \in \varrho(G)$ is a non-homology for a suitable choice of the integer s because $m > 2$: For example, write ψ as $\mathrm{diag}(1, \zeta_{m'}, 1)$ for some integer $m' > 1$, hence $\phi^s\psi = \mathrm{diag}(1, \zeta_{m'}, \zeta_m^s)$. So when $m \neq m'$, we can take $s = 1$, and $s = 2$ otherwise. In both cases, $\phi^s\psi$ is a non-homology in $\varrho(G)$, which conflicts our assumption that $\varrho(G)$ is made entirely of homologies (Definition 1.2.6). Therefore, all elements of $\varrho(G)$ admit the same axis and the same center, i.e. each is of the shape $\mathrm{diag}(1, 1, \zeta_n)$ for some $n \in \mathbb{N}$. Consequently, $\varrho(G)$ is contained in the cyclic group generated by $\mathrm{diag}(1, 1, \zeta_{n_0})$, where n_0 is the least common multiple of the orders of the elements of $\varrho(G)$. Thus $\varrho(G)$ is cyclic. □

Definition 5.4.9. The substratum of $\mathcal{M}_g$, representing smooth plane curves $\overline{C}$ of genus $g = \frac{(d-1)(d-2)}{2} \geq 3$ over $\overline{k}$, whose automorphism group $\mathrm{Aut}(\overline{C})$ is cyclic generated by an homology of a fixed order $n > 1$, is denoted by $\widetilde{(\mathcal{M}_g^{Pl})}^h_{n,\mathrm{diag}}$.

As usual, fix a non-singular plane model $F_{\overline{C}}(X, Y, Z) = 0$ of $\overline{C}$ over $\overline{k}$ of degree d, such that $\mathrm{Aut}(F_{\overline{C}}) = \langle \mathrm{diag}(1, 1, \zeta_n)\rangle$. Hence $N_{\mathrm{Aut}(F_{\overline{C}})}(\overline{k})$ equals $\mathrm{GL}_{2,Z}(\overline{k})$, by Lemma 5.4.1-(2).

A necessary condition on n so that the stratum $\widetilde{(\mathcal{M}_g^{Pl})}^h_{n,\mathrm{diag}}$ might be non-empty is concluded directly from Theorem 2.1.3-(1), (2), where we follow the notations and conventions of chapter 2:

Proposition 5.4.10. *The stratum $\widetilde{(\mathcal{M}_g^{Pl})}^h_{n,\mathrm{diag}}$ is non-empty only if n divides d or $d - 1$. Moreover, the family*

$$\mathcal{C}_1 : Z^d + \sum_{j \in S(1)_n} Z^{d-j} L_{j,Z} + L_{d,Z} = 0, \tag{5.2}$$

is geometrically complete over k when $n|d$, whereas the family

$$\mathcal{C}_2 : Z^{d-1}Y + \sum_{j\in S(2)_n} Z^{d-j}L_{j,Z} + L_{d,Z} = 0, \tag{5.3}$$

does when $n|d-1$. That is, for any $\overline{C} \in \widetilde{(\mathcal{M}_g^{Pl})}^h_{n,\text{diag}}$, an $F_{\overline{C}}(X,Y,Z) = 0$ with $\langle \text{diag}(1,1,\zeta_n)\rangle \leq \text{Aut}(F_{\overline{C}})$ is given by a specialization of the parameters in (5.2) or (5.3) over $\overline{k}$.

Remark 5.4.11. By non-singularity, the homogenous binary form $L_{d,Z}(X,Y)$ in Proposition 5.4.10 can not have any repeated linear factors for any specializations of the parameters in $\overline{k}$; otherwise, we may assume (up to $\overline{k}$-equivalence) that the repeated factor is $X = 0$, and $L_{d,Z}(X,Y)$ reduces to $X^2L_{d-2,Z}(X,Y)$. But also $d-1 \notin S(u)_n$ for $u = 1,2$, since $n > 1$. Hence $F_{\overline{C}}(X,Y,Z) = 0$ in the family is defined by $Z^2G(X,Y,Z) + X^2L_{d-2,Z}(X,Y) = 0$, which in turns implies singularity at the reference point $P_2 = (0:1:0)$.

Curves of $\widetilde{(\mathcal{M}_g^{Pl})}^h_{n,\text{diag}}$ having odd signature

We characterize the situation when the stratum $\widetilde{(\mathcal{M}_g^{Pl})}^h_{n,\text{diag}}$ contains smooth plane curves over $\overline{k}$ of odd signature.

Proposition 5.4.12. *A smooth curve $\overline{C} \in \widetilde{(\mathcal{M}_g^{Pl})}^h_{n,\text{diag}}$ has an odd signature if and only if either d is odd and $n = d$, or d is even and $n = d-1$. In this case, i.e. when $\overline{C}$ is of odd signature, $F_{\overline{C}}(X,Y,Z) = 0$ is given, up to $\overline{k}$-isomorphism, by a specialization of the parameters of the form $Z^d + L_{d,Z} = 0$ when $n = d$ for an odd d, and $Z^{d-1}Y + L_{d,Z} = 0$ when $n = d-1$ for an even d.*

Proof. By Remark 5.4.11, we know that the binary form $L_{d,Z}$ factors into d distinct factors associated to d distinct roots, say $(a_i : b_i) \in \mathbb{P}^1(\overline{k})$, for $i = 1, 2, ..., d$. Since $\text{Aut}(F_{\overline{C}}) = \langle \text{diag}(1,1,\zeta_n)\rangle$, the covering $\pi_{F_{\overline{C}}} : F_{\overline{C}} \mapsto F_{\overline{C}}/\text{Aut}(F_{\overline{C}})$ is ramified exactly at the d points $\{(a_i : b_i : 0)\}$ when $n|d$, plus the extra point $P_3 = (0:0:1)$ when $n|d-1$. This gives d branch points (resp. $d+1$) each is of ramification index n when $n|d$ (resp. $n|d-1$). Consequently, $F_{\overline{C}}(X,Y,Z) = 0$ has odd signature only if $n|d$ and d odd or if $n|d-1$ and d even. The

Riemann-Hurwitz formula reads as

$$(d-1)(d-2)-2 = n(2g_0 - 2 + d(1-\frac{1}{n})),$$

when $n|d$ and d odd, and as

$$(d-1)(d-2)-2 = n(2g_0 - 2 + (d+1)(1-\frac{1}{n})),$$

when $n|d-1$ and d even, where g_0 is the geometric genus of the quotient curve $F_{\overline{C}}/\operatorname{Aut}(F_{\overline{C}})$. Setting $g_0 = 0$ and solving for n, we obtain $n = d$ (resp. $n = d-1$). This proves the "if and only if" statement.

Lastly, for $n = d$ (resp. $n = d-1$), the index set $S(1)_d = \{1 \leq j \leq d-1 : d-j \equiv 0 \bmod d\}$ (resp. $S(2)_{d-1} = \{2 \leq j \leq d-1 : d-j \equiv 0 \bmod d-1\}$) is obviously empty. $\square$

The stratum $\widetilde{(\mathcal{M}_g^{Pl})}^h_{n,\text{diag}}$ with $n|\, d-1$

Suppose that the stratum $\widetilde{(\mathcal{M}_g^{Pl})}^h_{n,\text{diag}}$ is non-empty for some fixed integer $n > 1$ with $n|d-1$. Let $\overline{C} \in (\mathcal{M}_g^{Pl})^h_{n,\text{diag}}$, and moreover assume that k is perfect of characteristic $p = 0$ or $p > 2g+1$. We will see that when $k = M_{\overline{k}/k}(\overline{C})$, then k becomes a field of definition of $\overline{C}$. The idea is to split up the family $\mathcal{C}_2$, of Proposition 5.4.10, into at most four components with an extra property. We then show that the canonical model B_k for $F_{\overline{C}}/\operatorname{Aut}(F_{\overline{C}})$ always has a k-rational point, and the field of moduli k therefore is a field of definition, by the aid of Proposition 5.2.1.

Proposition 5.4.13. *Consider the subfamilies $\mathcal{C}_2^{(s)}$ of $\mathcal{C}_2$, for each $s = 1,2,3,4$, which is given by*

$$\mathcal{C}_2^{(1)} \;:\; Z^{d-1}Y + \sum_{j\in S(2)_n} Z^{d-j}L_{j,Z} + X^d + X^{d-2}Y^2 + \sum_{j=3}^{d} a_jX^{d-j}Y^j = 0,$$

$$\mathcal{C}_2^{(2)} \;:\; Z^{d-1}Y + \sum_{j\in S(2)_n} Z^{d-j}L_{j,Z} + X^d + Y^d + \sum_{j=3}^{d-1} a_jX^{d-j}Y^j = 0,$$

$$\mathcal{C}_2^{(3)} \;:\; Z^{d-1}Y + \sum_{j\in S(2)_n} Z^{d-j}L_{j,Z} + X^d + XY^{d-1} + \sum_{j=3}^{d-2} a_jX^{d-j}Y^j = 0,$$

$$\mathcal{C}_2^{(4)} \;:\; Z^{d-1}Y + \sum_{j\in S(2)_n} Z^{d-j}L_{j,Z} + X^{d-1}Y + \sum_{j=3}^{d} a_jX^{d-j}Y^j = 0.$$

Then $\bigcup_{s=1}^{4}\mathcal{C}_2^{(s)}$ defines a geometrically complete over k for the stratum $\widetilde{(\mathcal{M}_g^{Pl})}^h_{n,\mathrm{diag}}$ when $n|d-1$. Moreover, the index set $S(2)_n$ is always empty when $n=d-1$, and the stratum $\widetilde{(\mathcal{M}_g^{Pl})}^h_{d-1,\mathrm{diag}}$ is only described via the three components $\mathcal{C}_2^{(s)}$, for $s=1,2,3$.

Proof. By non-singularity, X^d or $X^{d-1}Y$ should appear in $L_{d,Z}$. Hence, up to rescaling the variable X and then renaming the parameters, we can split up $\mathcal{C}_2$, in Proposition 5.4.10, into two components defined by the forms

$$\begin{aligned}
1st\ &:\ Z^{d-1}Y+\sum_{j\in S(2)_n}Z^{d-j}L_{j,Z}+X^{d-1}Y+a_2X^{d-2}Y^2+...+a_{d-1}XY^{d-1}+a_dY^d=0,\\
2nd\ &:\ Z^{d-1}Y+\sum_{j\in S(2)_n}Z^{d-j}L_{j,Z}+X^{d}+a_1X^{d-1}Y+...+a_{d-1}XY^{d-1}+a_dY^d=0.
\end{aligned}$$

We always can assume $a_2=0$ in the first component, by a change of variables of the shape $X\mapsto X-\frac{a_2}{d-1}Y$ and then renaming the parameters. This in turns gives the fourth component $\mathcal{C}_2^{(4)}$ in the statement. Similarly, we may take $a_1=0$ in the second component via $X\mapsto X-\frac{a_1}{d}Y$ and renaming after. Moreover, if $a_2=0$, then we split it up with respect to Y^d, if it appears or not (if it does not appear, then XY^{d-1} does, by non-singularity). Therefore, we get the substrata $\mathcal{C}_2^{(2)}$ and $\mathcal{C}_2^{(3)}$, up to rescaling Y and Z. Finally, for $a\neq 0$, we rescale Y and Z to get $\mathcal{C}_2^{(1)}$.

A priori, the index set $S(2)_n$ is empty if and only if $\mathrm{diag}(1,1,\zeta_{d-1})\in\mathrm{Aut}(\mathcal{C}_2)$. In this case, $n=d-1$ and the subfamily $\mathcal{C}_2^{(4)}$ is not irreducible anymore, since it factors as $Y.G(X,Y,Z)$. For this reason we exclude $\mathcal{C}_2^{(4)}$. ☐

The main result for this subsection is now stated:

Theorem 5.4.14. *Following the above notations, let k be a perfect field of characteristic $p=0$ or $p>(d-1)(d-2)+1$ with $d\geq 4$. Let $\overline{C}\in\widetilde{(\mathcal{M}_g^{Pl})}^h_{n,\mathrm{diag}}$ with $n,g>1$ such that n divides $d-1$. If $k=M_{\overline{k}/k}(\overline{C})$, then it is also a field of definition for $\overline{C}$.*

Proof. It suffices to consider an $F_{\overline{C}}(X,Y,Z)=0$ in the family $\bigcup_{s=1}^{4}\mathcal{C}_2^{(s)}$ (Proposition 5.4.13), since it is a geometrically complete family over k for our stratum. Because of the monomial term $Z^{d-1}Y$ in the defining equation $F_{\overline{C}}(X,Y,Z)=0$, the action of the normalizer $N_{\mathrm{Aut}(F_{\overline{C}})}(\overline{k})$, which is $\mathrm{GL}_{2,Z}(\overline{k})$ by Lemma 5.4.1, is possibly not trivial only for an isomorphism of the shape $[\alpha X+\beta Y:\gamma Y:Z]$. Moreover, the components $\mathcal{C}_2^{(s)}$, for $s=1,2,3,4$, are well-defined up to $\overline{k}$-isomorphism, which means that even $[\alpha X+\beta Y:\gamma Y:Z]$ does not define an isomorphism between two curves in two distinct components. Now, it is straightforward to check that $\beta=0$ in any case, so $F_{\overline{C}}(X,Y,Z)=0$ is isomorphic to its conjugates $\{{}^{\sigma}(F_{\overline{C}})(X,Y,Z)=0\}_{\sigma\in G_k}$ through a set of isomorphisms $\{\phi_\sigma\}\subset \mathrm{D}(\overline{k})$, i.e ϕ_σ is diagonal: For example, consider an $F_{\overline{C}}(X,Y,Z)=0$ in the subfamily $\mathcal{C}_2^{(1)}$. Since $X^{d-1}Y$ does not belong to the defining equation for $F_{\overline{C}}(X,Y,Z)=0$, we must impose $d\alpha^{d-1}\beta=0$. Because $[\alpha X+\beta Y:\gamma Y:Z]$ is invertible, then $\beta=0$, which was to be shown.

Hence, as explained previously in section §5.4.1, the reference point $P_3=(0:0:1)$ on $F_{\overline{C}}(X,Y,Z)=0$ shall produce a k-point on the canonical model B_k for $F_{\overline{C}}/\operatorname{Aut}(F_{\overline{C}})$ over k. Thus k is a field of definition for $\overline{C}$ by Proposition 5.2.1. $\square$

The stratum $\widetilde{(\mathcal{M}_g^{Pl})}^h_{n,\mathrm{diag}}$ with $n|\,d$ and d odd

Theorem 5.4.15. *Let $\overline{C}\in\widetilde{(\mathcal{M}_g^{Pl})}^h_{n,\mathrm{diag}}$, where $n>1$ is a fixed integer dividing the odd degree $d\geq 5$. As usual, assume that $M_{\overline{k}/k}=k$ and k is perfect of characteristic $p=0$ or $p>2g+1$. Then k is a field of definition for $\overline{C}$.*

Proof. By Theorem 5.3.3 and Proposition 5.4.12, one gets the result when $\overline{C}\in\widetilde{(\mathcal{M}_g^{Pl})}^h_{d,\mathrm{diag}}$, i.e. when $n=d$. Therefore, we take $1<n<d$ divides d. Since the automorphism group $\operatorname{Aut}(F_{\overline{C}})=\langle\mathrm{diag}(1,1,\zeta_n)\rangle$, we get by Proposition 5.4.10 that the family

$$\mathcal{C}_1':\,Z^d+\sum_{1\leq f\leq \frac{d}{n}-1}Z^{d-fn}L_{fn,Z}+L_{d,Z}=0,$$

such that $L_{fn,Z}\neq 0$ for some $1\leq f\leq\frac{d}{n}-1$, is a geometrically complete family over k for $\widetilde{(\mathcal{M}_g^{Pl})}^h_{n,\mathrm{diag}}$. Moreover, the normalizer $N_{\operatorname{Aut}(F_{\overline{C}})}(\overline{k})=\mathrm{GL}_{2,Z}(\overline{k})$ by using Lemma 5.4.1.

We first show the next observation:

Observation. Denote by $\mathcal{C}_{1,0}'$ the family defined by $Z^d+L_{d,Z}=0$. Then, for any $\sigma\in G_k$ and any isomorphism $\phi_\sigma:{}^{\sigma}\mathcal{C}_{1,0}'\to\mathcal{C}_{1,0}'$, there always exists an $\eta_\sigma\in\langle\mathrm{diag}(\zeta_d^{-1},\zeta_d^{-1},1)\rangle$ and an isomorphism $\widetilde{\phi}_\sigma:{}^{\sigma}\mathcal{C}_1'\to\mathcal{C}_1'$ such that $\eta_\sigma\circ\phi_\sigma$ and $\widetilde{\phi}_\sigma$, as elements of $\mathrm{GL}_{2,Z}(\overline{k})$, give the same action on $\mathcal{C}_1'$.

Proof. Clearly an element $\phi\in\mathrm{GL}_{2,Z}(\overline{k})$, which acts non-trivially on the family $\mathcal{C}_{1,0}'$, acts also non-trivially on the family $\mathcal{C}'_1$. The converse still true, unless $\phi \in \langle \mathrm{diag}(\zeta_d^{-1}, \zeta_d^{-1}, 1)\rangle$. Therefore, the number of geometric fibers, which are isomorphic and given by the action of $\mathrm{GL}_{2,Z}(\overline{k})$ on the family $\mathcal{C}'_{1,0}$ is exactly the same number of isomorphic geometric fibers of the family $\mathcal{C}'_1$ arisen by the action of $\mathrm{GL}_{2,Z}(\overline{k}) \setminus \langle \mathrm{diag}(\zeta_d^{-1}, \zeta_d^{-1}, 1)\rangle$. Consequently, for any $\sigma \in G_k$, the action of any isomorphism $\phi_\sigma : {}^\sigma\mathcal{C}'_{1,0} \to \mathcal{C}'_{1,0}$ can always be extended to an action $\widetilde{\phi}_\sigma : {}^\sigma\mathcal{C}'_1 \to \mathcal{C}'_1$. In particular, the composition $\widetilde{\phi}_\sigma \circ \phi_\sigma^{-1}$ acts trivially on $Z^d + L_{d,Z} = 0$, that is $\widetilde{\phi}_\sigma \circ \phi_\sigma^{-1} = \eta_\sigma \in \langle \mathrm{diag}(\zeta_d^{-1}, \zeta_d^{-1}, 1)\rangle$. $\square$

Next, by the virtue of Theorem 5.3.3 and Proposition 5.4.12, we may consider a family of isomorphisms $\{\phi_\sigma : {}^\sigma\mathcal{C}'_{1,0} \to \mathcal{C}'_{1,0}\}_{\sigma\in G_k}$, satisfying the Weil's cocycle criterion of descent (Theorem 5.1.4). Using the above observation, we also have a set of isomorphism $\{\widetilde{\phi}_\sigma := \eta_\sigma \circ \phi_\sigma : {}^\sigma\mathcal{C}'_1 \to \mathcal{C}'_1\}_{\sigma\in G_k}$, where $\eta_\sigma := \mathrm{diag}(\epsilon_\sigma^{-1}, \epsilon_\sigma^{-1}, 1)$ for some dth root of unity ϵ_σ. Hence, it satisfies

$$({}^\sigma L_{fn,Z})(X,Y) = L_{fn,Z}(\widetilde{\phi}_\sigma(X,Y)) = \epsilon_\sigma^{-fn} L_{fn,Z}(\phi_\sigma(X,Y)),$$

for all $\sigma \in G_k$, and all $1 \leq f \leq \frac{d}{n} - 1$. Take any $\sigma, \tau \in G_k$, then

$$\begin{aligned}
L_{fn,Z}(\widetilde{\phi}_{\sigma\tau}(X,Y)) &= ({}^{\sigma\tau}L_{fn,Z})(X,Y) = {}^\sigma({}^\tau L_{fn,Z})(X,Y) ={}^\sigma (\epsilon_\tau^{-fn} L_{fn,Z}(\phi_\tau(X,Y)))\\
&= \sigma(\epsilon_\tau^{-fn})\,{}^\sigma(L_{fn,Z})({}^\sigma\phi_\tau(X,Y)) := \sigma(\epsilon_\tau^{-fn})\,{}^\sigma(L_{fn,Z})(X',Y')\\
&= \sigma(\epsilon_\tau^{-fn})\epsilon_\sigma^{-fn} L_{fn,Z}(\phi_\sigma(X',Y'))\\
&= (\epsilon_\sigma\,\sigma(\epsilon_\tau))^{-fn} L_{fn,Z}((\phi_\sigma \circ {}^\sigma\phi_\tau)(X,Y))\\
&= L_{fn,Z}(((\eta_\sigma \circ {}^\sigma\eta_\tau) \circ (\phi_\sigma \circ {}^\sigma\phi_\tau))(X,Y))\\
&= L_{fn,Z}(((\eta_\sigma \circ \phi_\sigma) \circ ({}^\sigma\eta_\tau \circ {}^\sigma\phi_\tau))(X,Y))\\
&= L_{fn,Z}((\widetilde{\phi}_\sigma \circ {}^\sigma\widetilde{\phi}_\tau)(X,Y)).
\end{aligned}$$

So the family $\{\widetilde{\phi}_\sigma\}_{\sigma\in G_k}$ satisfies the Weil's condition of descent, and k is a field of definition for $\mathcal{C}'_1$. $\square$

The stratum $\widetilde{(\mathcal{M}_g^{Pl})}^h_{n,\mathrm{diag}}$ with $n \mid d$ and d even

There is no smooth plane curve $\overline{C}$ of degree 4 over $\overline{k}$ with automorphism group conjugate to $\langle \mathrm{diag}(1,1,\zeta_4)\rangle$. Hence, the stratum $\widetilde{(\mathcal{M}_3^{Pl})}^h_{4,\mathrm{diag}}$ is empty, and we have nothing to say in this case. See P. Henn [Hen76] or F. Bars [Bar12] for more details.

The next result shows an example of a smooth plane curve in the stratum $\widetilde{(\mathcal{M}_g^{Pl})}^h_{d,\text{diag}}$ over $\mathbb{C}$, for any even degree $d = 2(2s+1) \geq 6$, whose field of moduli is $\mathbb{R}$, relative to the Galois extension $\mathbb{C}/\mathbb{R}$. However, it is not a field of definition for $\overline{C}$.

Theorem 5.4.16. *For an arbitrary, but a fixed even degree d of the form $2(2s+1) \geq 6$, the stratum $\widetilde{(\mathcal{M}_g^{Pl})}^h_{d,\text{diag}}$ is non-empty, in the sense that, there exists a smooth plane curve $\overline{C}$ of genus $g = \frac{1}{2}(d-1)(d-2)$ over $\mathbb{C}$ with automorphism group $\langle \text{diag}(1,1,\zeta_d)\rangle$. Moreover, the field of moduli $M_{\mathbb{C}/\mathbb{R}}(\overline{C})$ is $\mathbb{R}$, but it is not a field of definition.*

In particular, representative families over $\mathbb{R}$ do not exist for the stratum $\widetilde{(\mathcal{M}_g^{Pl})}^h_{d,\text{diag}}$.

Proof. For example, take $\overline{C}$ to be a smooth plane curve over $\mathbb{C}$ defined as Proposition 5.4.2 with $m = 1$ and $r = 2s+1$. Then, $\text{Aut}(\overline{C}) = \langle \text{diag}(1,1,\zeta_d)\rangle$. Also $\mathbb{R}$ is the field of moduli for $\overline{C}$, relative to $\mathbb{C}/\mathbb{R}$, but it is not a field of definition.

On the other hand, if a representative family over k exist for some stratum of $\mathcal{M}_g$, then the field of moduli needs to be a field of definition for every curve C in this stratum (Lemma 4.3.3). Thus by the above counter example, we deduce that such a family over $\mathbb{R}$ does not exist for $\widetilde{(\mathcal{M}_g^{Pl})}^h_{d,\text{diag}}$. □

It remains yet the study of $\widetilde{(\mathcal{M}_g^{Pl})}^h_{n,\text{diag}}$ when the degree $d \geq 4$ is even and n divides d properly. In this case, the field of moduli does not need to be a field of definition as well. The first example appears for genus 3 curves, and we refer to the work of Artebani-Quispe, [AQ12, §4, Lemma 4.2, Proposition 4.3], for a smooth plane quartic curve over $\mathbb{C}$, not definable over its field of moduli $\mathbb{R}$, and whose automorphism group is the cyclic one of order 2. We generalize this example taking into account the next example.

For an arbitrary, but a fixed integer $d = 4m \geq 12$, consider a plane curve $\overline{C}$ over $\mathbb{C}$ defined by an equation of the form

$$F_{\overline{C}}(X,Y,Z) := Z^d + Z^{\frac{d}{2}} g(X,Y) - f(X,Y) = 0, \tag{5.4}$$

where

$$\begin{aligned} g(X,Y) &:= \prod_{i=1}^{\frac{d}{4}}(X-a_iY)(X+\frac{1}{a_i}Y) \\ f(X,Y) &:= \prod_{i=1}^{\frac{d}{2}}(X-b_iY)(X+\frac{1}{b_i^c}Y), \end{aligned}$$

such that $a_i \in \mathbb{R}^*$ for all $1 \leq i \leq \frac{d}{4}$, hence $g(X,Y)$ is a binary form in $\mathbb{R}[X,Y]$. Assume also that $f(X,Y)$ and $g(X,Y)$ has no repeated zeros. We also choose the $a_i's$ in the way that $g(X,Y)$ is not $\psi_{\overline{C}}$-invariant or $\psi_{a,b}$-invariant for any $\psi_c : (X:Y) \mapsto (Y:cX)$ and $\psi_{a,b} : (X:Y) \mapsto (X+aY : bX-Y)$ in $\mathrm{PGL}_2(\mathbb{C})$.

Remark 5.4.17. The last condition on the zero set of $g(X,Y)$, not to be invariant under any $\psi_{\overline{C}}$ or $\psi_{a,b}$, is not strong. One just need to impose finitely many conditions on the $a_i's$: For instance, an $\psi_{\overline{C}}$ acts as a product of pairwise disjoint 2-cycles on the set $\{(a_i:1)\}_i$ since it has order 2 in $\mathrm{PGL}_2(\mathbb{C})$. So if $\psi_{\overline{C}} : (a_s:1) \leftrightarrow (a_t:1)$ (resp. $(-1/a_t:1)$) for some s,t, then $c = a_s/a_t$ (resp. $-a_t/a_s$). Therefore, it suffices to choose the $a_i's$ such that $\{a_i, \frac{-1}{a_i}\}_i \neq \{\frac{\pm a_t}{a_s a_i}, \frac{\mp a_t a_i}{a_s}\}_i$ for any s,t. In this case, $g(X,Y)$ is not ψ_c-invariant for any $\psi_c \in \mathrm{PGL}_2(\mathbb{C})$.

The action of an $\psi_{a,b}$ is treated in the same way. However, it is a bit more tedious.

Lemma 5.4.18. *A plane curve defined by an equation of the form (5.4) over $\mathbb{C}$ as above is always smooth.*

Proof. Since $F_{\overline{C}}(X,0,Z) = Z^d + (XZ)^{\frac{d}{2}} - X^d = 0$ has no repeated zeros, the common zeros of $F_{\overline{C}}(X,0,Z)$ and $(F_{\overline{C}})_X(X,0,Z)$ do not exist. Moreover, $(F_{\overline{C}})_X(X,1,Z) = Z^{\frac{d}{2}}g'(X,1) - f'(X,1)$ and $(F_{\overline{C}})_Z(X,1,Z) = \frac{d}{2}Z^{\frac{d}{2}-1}(2Z^{\frac{d}{2}} + g(X,1))$. But $f(X,Y)$ is square free, then $(X:1:0)$ gives no singular points on $F_{\overline{C}}(X,Y,Z) = 0$. Furthermore, if we substitute $g(X,1) = -2Z^{\frac{d}{2}}$ into $F_{\mathcal{S}}(X,1,Z) = \frac{\partial F_{\mathcal{S}}}{\partial X}(X,1,Z) = 0$, we get that $\mathcal{S}$ is singular only if $f(X,1)g'(X,1)^2 = -f'(X,1)^2$, that is when $f(X,1)$ has repeated zeros, a contradiction. So the equation is smooth. $\square$

Proposition 5.4.19. *The stratum $\widetilde{(\mathcal{M}_g^{Pl})}^h_{\frac{d}{2},\mathrm{diag}}$ is not empty for any $d = 4m \geq 12$. That is, there exist a smooth plane curve over $\mathbb{C}$ of degree $d = 4m \geq 12$, for any $m \in \mathbb{Z}_{\geq 1}$, such that its automorphism group is cyclic of order $\frac{d}{2}$, generated in $\mathrm{PGL}_3(\mathbb{C})$ by $\mathrm{diag}(1,1,\zeta_{\frac{d}{2}})$.*

Proof. Consider a plane curve $\overline{C}$ of degree d, given by an equation $F_{\overline{C}}(X,Y,Z)=0$ of the form (5.4) over $\mathbb{C}$ as above. Hence $\overline{C}$ is smooth by Lemma 5.4.18.

Second we show the claim on $\mathrm{Aut}(F_{\overline{C}})$ to be the cyclic group $\langle\mathrm{diag}(1,1,\zeta_{\frac{d}{2}})\rangle$ under a suitable choice of the $b_i's$. Obviously, $\psi:=\mathrm{diag}(1,1,\zeta_{d/2})\in\mathrm{Aut}(F_{\overline{C}})$ is a homology of order $d/2\geq 4$. Therefore, $\mathrm{Aut}(F_{\overline{C}})$ fixes a point, a line or a triangle, by Theorem 1.2.8. In particular, it is not conjugate to any of the finite primitive group mentioned in Theorem 4.2.3-(iii). Now, we treat each of the following subcases:

(i) A line $L\subset\mathbb{P}^2_{\mathbb{C}}$ and a point $P\notin L$ are leaved invariant: By Theorem 1.4.4, we can think about $\mathrm{Aut}(F_{\overline{C}})$ in a short exact sequence

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{C}^* \arrow[r] & \mathrm{GL}_{2,Y}(\mathbb{C}) \arrow[r, "\Lambda"] & \mathrm{PGL}_2(\mathbb{C}) \arrow[r] & 1 \\
1 \arrow[r] & \langle\psi\rangle \arrow[r] \arrow[u, hook] & \mathrm{Aut}(F_{\overline{C}}) \arrow[r] \arrow[u, hook] & G \arrow[r] \arrow[u, hook] & 1
\end{tikzcd}$$

where G is conjugate to a cyclic group $\mathbb{Z}/m\mathbb{Z}$ of order $m\leq d-1$, a Dihedral group D_{2m} of order $2m$ with $m|(d-2)$, one of the alternating groups A_4, A_5, or to the symmetry group S_4. Any such G, which is not cyclic, contains an element of order 2. Let $\psi'\in\mathrm{Aut}(F_{\overline{C}})$ such that $\varrho(\psi')$ has order 2. Then, $\varrho(\psi')$ has the shape ψ_c or $\psi_{a,b}$ for some $a,b,c\in\mathbb{C}\setminus\{0\}$, which is absurd by our assumptions on $g(X,Y)$. Consequently, $G=\varrho(\mathrm{Aut}(F_{\overline{C}}))$ is cyclic, generated by the image of a specific $\psi_G\in\mathrm{GL}_{2,Y}(\mathbb{C})$. This would lead to a polynomial expression of $b_i's$ in terms of the $a_i's$, hence we still have infinitely many possibilities to choose the $b_i's$ such that $f(X,Y)$ not $\langle\varrho(\psi_G)\rangle$-invariant. In particular, $|G|=1$ and $\mathrm{Aut}(F_{\overline{C}})$ is $\mathrm{PGL}_3(\mathbb{C})$-conjugate to $\langle\mathrm{diag}(1,1,\zeta_{d/2})\rangle$.

(ii) A triangle Δ is fixed by $\mathrm{Aut}(\mathrm{F}_{\overline{C}})$ and neither a line nor a point is leaved invariant: It follows by Theorem 1.4.4 and its proof that $(F_{\overline{C}},\mathrm{Aut}(\mathrm{F}_{\overline{C}}))$ must be a descendant of the Fermat curve $F_d: X^d+Y^d+Z^d=0$ or the Klein curve $K_d: X^{d-1}Y+Y^{d-1}Z+Z^{d-1}X=0$. Since $d/2$ does not divide $|\mathrm{Aut}(K_d)|=3(d^2-3d+3)$, $(F_{\overline{C}},\mathrm{Aut}(\mathrm{F}_{\overline{C}}))$ is not a descendant of K_d. Hence $\exists P\in\mathrm{PGL}_3(\mathbb{C})$ such that $H:=P^{-1}\,\mathrm{Aut}(\mathrm{F}_{\overline{C}})\,P$ is a subgroup of $\mathrm{Aut}(F_d)$, which is a semidirect product of $\mathrm{S}_3=\langle T:=[Y:Z:X],R:=[X:Z:Y]\rangle$

acting on $(\mathbb{Z}/d\mathbb{Z})^2 = \langle \mathrm{diag}(\zeta_d, 1, 1), \mathrm{diag}(1, \zeta_d, 1)\rangle$, see [Har13, Proposition 3.3]. That is, any element of H has the shape DR^iT^j, for some $0 \leq i \leq 1$ and $0 \leq j \leq 2$ and $D \in \mathrm{D}(\overline{k})$. It is straightforward to check that any DT^j and DRT^j with $j \neq 0$ has order $3 < \frac{d}{2}$. Thus $P^{-1}\psi P \in \mathrm{D}(\overline{k})$, and then we may assume that P in the normalizer of $\langle\psi\rangle$, up to a change of variables in $\mathrm{Aut}(F_d)$. In this case, we can think about H in the commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & (\mathbb{Z}/d\mathbb{Z})^2 \arrow[r] & \mathrm{Aut}(F_d) \arrow[r, "\varrho"] & \mathrm{S}_3 \arrow[r] & 1 \\
1 \arrow[r] & \mathrm{Ker}(\varrho|_H) = \langle\psi\rangle \arrow[r] \arrow[u, hook] & H \arrow[r] \arrow[u, hook] & G := \mathrm{Im}(\varrho|_H) \arrow[r] \arrow[u, hook] & 1
\end{tikzcd}$$

We note that Z in the transformed defining equation through P appears exactly as the original equation. Hence G is at most cyclic of order 2, since $P^{-1}\,\mathrm{Aut}(F_{\overline{C}})P$ should have an element of the shape $[\zeta_d^sY : \zeta_d^tZ : X]$ or $[\zeta_d^sZ : \zeta_d^tX : Y]$ for some integers s, t otherwise, which is not possible. For the same reason, G is then generated by an $\varrho([\zeta_d^sY : \zeta_d^tX : Z])$, and again it enough to restrict $f(\phi(X,Y))$ not to be $\langle\varrho([\zeta_d^sY : \zeta_d^tX : Z])\rangle$-invariant, where ϕ is the restriction of P on the ideal (X, Y).

$\square$

Proposition 5.4.20. *Let $\overline{C}$ be a smooth plane curves defined over $\mathbb{C}$ by an equation $F_{\overline{C}}(X,Y,Z) = 0$ of degree $d = 4m \geq 12$ with m odd, of the form (5.4) as in Lemma 5.4.18 and Proposition 5.4.19 with the extra condition $\prod_{i=1}^{\frac{d}{2}} b_i \in \mathbb{R}$. Then $\overline{C} \in \widetilde{(\mathcal{M}_g^{Pl})}^h_{\frac{d}{2},\mathrm{diag}}$, moreover the field of moduli for $\overline{C}$ relative to the Galois extension $\mathbb{C}/\mathbb{R}$ is $\mathbb{R}$, but it is not a field of definition.*

Proof. Such a curve is isomorphic to its complex conjugate $(\overline{F_{\overline{C}}})(X,Y,Z) = 0$ through $\phi := [-Y : X : \zeta_d]$. Hence $\mathbb{R}$ is the field of moduli for C relative to $\mathbb{C}/\mathbb{R}$. However, it is not a field of definition for C. To see this, let $\phi' : {}^{\sigma}F_{\overline{C}} \to F_{\overline{C}}$ be any isomorphism. Then $\phi\circ\phi'^{-1} \in \mathrm{Aut}(F_{\overline{C}})$, and so $\phi' = \phi \circ \mathrm{diag}(1, 1, \zeta_{\frac{d}{2}})^r$ for some integer $0 \leq r < \frac{d}{2}$. Any such ϕ' does not satisfy Weil's condition of descent (Theorem 5.1.4): $\overline{\phi'} \circ \phi' = 1$, since $\overline{\phi'} \circ \phi' = \mathrm{diag}(1, 1, -1) \neq 1$. Therefore $\mathbb{R}$ is not a field of definition for $\overline{C}$. $\square$

On the stratum $\mathcal{M}_g^{pl}(\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z})$

The following generalizes Lemma 4.1.6:

Proposition 5.4.21. *Smooth plane curve $\overline{C}$ over $\overline{k}$ of odd degree $d\geq 5$ with $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\leq \mathrm{Aut}(C)$ do not exist, where $\overline{k}$ is a field of characteristic $p=0$ or $p>(d-1)(d-2)+1$. Hence $\mathrm{Aut}(\overline{C})$ is not conjugate to any of the groups $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$, A_4, S_4 or A_5 inside $\mathrm{PGL}_3(\overline{k})$.*

Proof. Following the work of H. Mitchell [Mit11] and T. Harui [Har13], the group $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\subset\mathrm{PGL}_3(\overline{k})$, giving invariant a smooth plane curve $\overline{C}$ of degree $d\geq 4$, should fix a point not lying on $\overline{C}$ or $\overline{C}$ must be a descendant of the Fermat curve $F_d\,:\,X^d+Y^d+Z^d=0$ or the Klein curve $K_d\,:\,X^{d-1}Y+Y^{d-1}Z+Z^{d-1}X=0$. But also, for an odd degree $d\geq 5$, 4 does not divide $|\,\mathrm{Aut}(F_d)|=6d^2$ and $|\,\mathrm{Aut}(K_d)|=3(d^2-3d+3)$. In particular, $\overline{C}$ can not be a descendant of F_d or K_d, and we can think about $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$, in a short exact sequence of the form $1\to N=1\to H\to H\to 1$, where H is $\mathrm{PGL}_2(\overline{k})$-conjugate to $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ (Theorem 1.4.4). Let $H=\langle\eta_1,\eta_2\rangle\leq\mathrm{PGL}_2(\overline{k})$ acts on the variables Y,Z, then we can assume, up to conjugation of groups in $\mathrm{PGL}_2(\overline{k})$, that $\eta_1=\mathrm{diag}(1,-1)$ and $\eta_2=[aY+bZ:cY-aZ]$. Because $\eta_1\eta_2=\eta_2\eta_1$, we get $\eta_2=\mathrm{diag}(-1,1)$ or $[bZ:cY]$ for some $bc\neq 0$. Being of *Type* $2,(0,1)$ with $2\nmid d$, $\overline{C}$ should have defining equation of the form $Z^{d-1}L_{1,Z}+Z^{d-3}L_{3,Z}+...+Z^2L_{d-2,Z}+L_{d,Z}=0$, and $Y^{d-1}L_{1,Y}+Y^{d-3}L_{3,Y}+...+Y^2L_{d-2,Y}+L_{d,Y}=0$ simultaneously. This reduces $\overline{C}$ to $X\,.\,G(X,Y,Z)$ for some homogenous polynomial of degree $d-1$, a contradiction to non-singularity. So such a smooth curve does not exist. The second part is clear, since any of these group contains a subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$. □

Now, fix an injective representation $\varrho:\,\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\hookrightarrow\mathrm{PGL}_3(\overline{k})$ with

$$\varrho(\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z})=\langle\mathrm{diag}(1,-1,1),\,\mathrm{diag}(1,1,-1)\rangle,$$

and let $d\geq 4$ be an even integer.

Proposition 5.4.22. *Let $\overline{C}$ be a smooth plane curve of even degree $d\geq 4$, such that $\varrho(\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z})$ acts on a non-singular plane model $F_{\overline{C}}(X,Y,Z)=0$ of C over $\overline{k}$, i.e. $\overline{C}\in\mathcal{M}_g^{pl}(\mathbb{Z}/2\mathbb{Z}\times$*

$\mathbb{Z}/2\mathbb{Z})$*, where* k *is a field of characteristic* $p = 0$ *or* $p > (d-1)(d-2)+1$*. Then* $F_{\overline{C}}(X, Y, Z) = 0$ *is given by a specialization in* $\overline{k}$ *of the parameters* $\alpha_{s,t,u} \in \overline{k}$ *of the family defined by* $X^d + Y^d + Z^d + \sum_{s,t,u} \alpha_{s,t,u}(X^sY^tZ^u)^2 = 0$*, where the sum is taken over* $0 \leq s, t, u \leq \frac{d-2}{2}$ *with* $s + t + u = \frac{d}{2}$.

For $d = 4$, $\varrho(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$ appears as the full automorphism group of some smooth plane curves of genus $g = 3$ over $\overline{k}$. The family

$$\mathcal{C}_{a,b,c} : X^4 + Y^4 + Z^4 + aX^2Y^2 + bX^2Z^2 + cY^2Z^2 = 0,$$

with a^2, b^2, c^2 are pairwise distinct, and $a^2 + b^2 + c^2 - abc, a^2, b^2, c^2 \neq 4$ is geometrically complete over k for the stratum $\widetilde{\mathcal{M}_3^{pl}}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$, where k is perfect of characteristic $p = 0$ or $p > (d-1)(d-2) + 1$.

Let G be the group acting on the triples $(a, b, c) \in \mathbb{C}^3$, generated by

$$g_1(a, b, c) := (b, a, c),\ g_2(a, b, c) := (b, c, a),\ g_3(a, b, c) := (-a, -b, c),\ g_4(a, b, c) := (a, -b, -c).$$

E. W. Howe [How01, Proposition 2] observed that any isomorphism between $\mathcal{C}_{a,b,c}$ and $\mathcal{C}_{g(a,b,c)}$, for $g \in G$, is defined over $\mathbb{Q}(i)$. Moreover, if F is a subfield of $\mathbb{C}$ containing $\mathbb{Q}(i)$, then $\mathcal{C}_{a,b,c}$ is isomorphic to $\mathcal{C}_{a',b',c'}$ if and only if $g(a, b, c) = (a', b', c')$ for some $g \in G$ (Proposition 4.4 in [AQ12]).

Theorem 5.4.23 (Artebani-Quispe, §4, [AQ12])**.** *Following the notations above, let* $C_{a,b,c}$ *be a smooth plane curve over* $\overline{k}$ *in the family* $\mathcal{C}_{a,b,c}$*. Then*

(i) If $k = \mathbb{R}$ *is the field of moduli for* $C_{a,b,c}$*, relative to* $\mathbb{C}/\mathbb{R}$*, then it is also a field of definition.*

(ii) If F/k *is a Galois extension with* $\mathbb{Q}(i) \subset F \subset \mathbb{C}$ *and* $a, b, c \in F$*, then the field of moduli for* $C_{a,b,c}$*, relative to* F/k *is* $k(abc, a^2 + b^2 + c^2, a^4 + b^4 + c^4)$.

(iii) If F/k *is a general Galois extension with* $a, b, c \in F$*, then* $C_{a,b,c}$ *is isomorphic to* $C_{a',b',c'}$ *over* F *if and only if* $g(a, b, c) = (a', b', c')$ *for some* $g \in \langle g_1, g_2\rangle$*. Moreover, the field of moduli for* $C_{a,b,c}$*, relative to* F/k *equals* $k(a + b + c, a^2 + b^2 + c^2, a^3 + b^3 + c^3)$.

(iv) Any $\sigma \in \mathrm{Gal}(\mathbb{Q}(a, b, c)/\mathbb{Q}(abc, a^2 + b^2 + c^2, a^4 + b^4 + c^4)$ *acts as some* $g_\sigma \in G$*. Hence*

we have a natural injective homomorphism of groups

$$\psi : \mathrm{Gal}(\mathbb{Q}(a,b,c)/\mathbb{Q}(abc, a^2+b^2+c^2, a^4+b^4+c^4) \to G : \sigma \mapsto g_\sigma.$$

Moreover, if $\mathbb{Q}(i) \subset \mathbb{Q}(a,b,c)$ *and* $\mathrm{Im}(\psi) \subset \langle g_1, g_2 \rangle$*, then the field of moduli* $\mathbb{Q}(abc, a^2 + b^2 + c^2, a^4 + b^4 + c^4)$ *for* $C_{a,b,c}$ *relative to the Galois extension* $\mathbb{Q}(a,b,c)/\mathbb{Q}(abc, a^2 + b^2 + c^2, a^4 + b^4 + c^4)$ *is a field of definition.*

Appendices

A

APPENDIX

Types of cyclic groups of automorphisms for low degrees

We list down all cyclic subgroups of automorphisms and the associated defining equations, obtained for low degrees through manipulating Theorem 2.1.3 in Chapter 2. In other words, we determine the possible *Type* $m,\ (a,b)$, for which the locus $\rho_{m,a,b}(\mathcal{M}_g^{Pl}(\mathbb{Z}/m\mathbb{Z}))$ might be non-empty, and we also associate a normal form $\mathcal{F}_{\varrho_{a,b,m}}(X,Y,Z)=0$ describing the corresponding stratum.

For a fixed degree d, there is no relation between the notations for the parameters from one family to another: For example, we use, by an abuse of notation, $\beta_{i,j}$ as the parameter of the monomial $X^{d-j}Y^iZ^{j-i}$ in any normal form.

It might happen that two families *Type* $m,(a,b)$ and *Type* $m,(a',b')$ are isomorphic through a permutation of the variables, or $\mathcal{F}_{\varrho_{a,b,m}}(X,Y,Z)=0$ always decomposes as $X\,.\,\mathcal{G}_{\varrho_{a,b,m}}(X,Y,Z)=0$. Compiling the code in SAGE and removing such situations yields the next tables (see the programm in http://mat.uab.cat/∼eslam/CAGPC.sagews)

Table A.1: degree 4

Type $m,(a,b)$	$F(X,Y,Z)$
$12,(3,4)$	$X^4+Y^4+XZ^3$
$9,(1,6)$	$X^4+Y^3Z+XZ^3$
$8,(1,5)$	$X^4+Y^3Z+YZ^3$
$7,(1,5)$	$X^3Y+Y^3Z+Z^3X$
$6,(3,4)$	$X^4+Y^4+XZ^3+\beta_{2,2}X^2Y^2$
$4,(1,2)$	$X^4+Y^4+Z^4+\beta_{2,0}X^2Z^2+\beta_{3,2}XY^2Z$
$4,(0,1)$	$Z^4+L_{4,Z}$

Table A.2: degree 4 (continued)

$3, (1, 2)$	$X^4 + X\big(Z^3 + Y^3\big) + \beta_{2,1}X^2YZ + \beta_{4,2}Y^2Z^2$
$3, (0, 1)$	$Z^3Y + L_{4,Z}$
$2, (0, 1)$	$Z^4 + Z^2L_{2,Z} + L_{4,Z}$

Table A.3: degree 5

Type: $m, (a,\, b)$	$F(X, Y, Z)$
$20, (4, 5)$	$X^5 + Y^5 + XZ^4$
$16, (1, 12)$	$X^5 + Y^4Z + XZ^4$
$15, (1, 11)$	$X^5 + Y^4Z + YZ^4$
$13, (1, 10)$	$X^4Y + Y^4Z + Z^4X$
$10, (2, 5)$	$X^5 + Y^5 + XZ^4 + \beta_{2,0}X^3Z^2$
$8, (1, 4)$	$X^5 + Y^4Z + XZ^4 + \beta_{2,0}X^3Z^2$
$5, (1, 2)$	$X^5 + Y^5 + Z^5 + \beta_{3,1}X^2YZ^2 + \beta_{4,3}XY^3Z$
$5, (0, 1)$	$Z^5 + L_{5,Z}$
$4, (1, 2)$	$X^5 + X\big(Z^4 + Y^4\big) + \beta_{2,0}X^3Z^2 + \beta_{3,2}X^2Y^2Z + \beta_{5,2}Y^2Z^3$
$4, (0, 1)$	$Z^4Y + L_{5,Z}$
$3, (1, 2)$	$X^5 + Y^4Z + YZ^4 + \beta_{2,1}X^3YZ + X^2\big(\beta_{3,0}Z^3 + \beta_{3,3}Y^3\big) + \beta_{4,2}XY^2Z^2$
$2, (0, 1)$	$Z^4L_{1,Z} + Z^2L_{3,Z} + L_{5,Z}$

Table A.4: degree 6

Type: $m, (a,\, b)$	$F(X, Y, Z)$
$30, (5, 6)$	$X^6 + Y^6 + XZ^5$
$25, (1, 20)$	$X^6 + Y^5Z + XZ^5$
$24, (1, 19)$	$X^6 + Y^5Z + YZ^5$
$21, (1, 17)$	$X^5Y + Y^5Z + XZ^5$
$15, (5, 6)$	$X^6 + Y^6 + XZ^5 + \beta_{3,3}X^3Y^3$
$12, (1, 7)$	$X^6 + Y^5Z + YZ^5 + \beta_{6,3}Y^3Z^3$
$10, (5, 6)$	$X^6 + Y^6 + XZ^5 + \beta_{2,2}X^4Y^2 + \beta_{4,4}X^2Y^4$
$8, (1, 3)$	$X^6 + Y^5Z + YZ^5 + \beta_{4,2}X^2Y^2Z^2$
$7, (1, 3)$	$X^5Y + Y^5Z + XZ^5 + \alpha_{4,2}X^2Y^2Z^2$
$6, (1, 2)$	$X^6 + Y^6 + Z^6 + \beta_{3,0}X^3Z^3 + \beta_{4,2}X^2Y^2Z^2 + \beta_{5,4}XY^4Z$
$6, (1, 3)$	$X^6 + Y^6 + Z^6 + \beta_{2,0}X^4Z^2 + \beta_{6,3}Y^3Z^3 + X^2\big(\beta_{4,0}Z^4 + \beta_{4,3}Y^3Z\big)$
$6, (0, 1)$	$Z^6 + L_{6,Z}$
$5, (1, 2)$	$X^6 + XZ^5 + XY^5 + \beta_{3,1}X^3YZ^2 + \beta_{4,3}X^2Y^3Z + \beta_{6,2}Y^2Z^4$
$5, (1, 4)$	$X^6 + XZ^5 + XY^5 + \beta_{2,1}X^4YZ + \beta_{4,2}X^2Y^2Z^2 + \beta_{6,3}Y^3Z^3$
$5, (0, 1)$	$Z^5Y + L_{6,Z}$
$4, (1, 3)$	$X^6 + Y^5Z + YZ^5 + \beta_{6,3}Y^3Z^3 + \beta_{2,1}X^4YZ + X^2\big(\beta_{4,0}Z^4 + \beta_{4,2}Y^2Z^2 + \beta_{4,4}Y^4\big)$

Table A.5: degree 6 (continued)

$3,(1,2)$	$X^5Y+Y^5Z+XZ^5+\alpha_{2,0}X^2Y^4+\alpha_{3,2}XY^3Z^2+\beta_{2,0}X^4Z^2+\beta_{3,2}X^3Y^2Z+\gamma_{2,2}Y^2Z^4$ $+\gamma_{3,1}X^2YZ^3$
$3,(0,1)$	$Z^6+Z^3L_{3,Z}+L_{6,Z}$
$2,(0,1)$	$Z^6+Z^4L_{2,Z}+Z^2L_{4,Z}+L_{6,Z}$

Table A.6: degree 7

Type: $m,(a,\,b)$	$F(X,Y,Z)$
$42,(6,7)$	$X^7+Y^7+XZ^6$
$36,(1,30)$	$X^7+Y^6Z+XZ^6$
$35,(1,29)$	$X^7+Y^6Z+YZ^6$
$31,(1,26)$	$X^6Y+Y^6Z+XZ^6$
$21,(3,7)$	$X^7+Y^7+XZ^6+\beta_{3,0}X^4Z^3$
$18,(1,12)$	$X^7+Y^6Z+XZ^6+\beta_{3,0}X^4Z^3$
$14,(2,7)$	$X^7+Y^7+XZ^6+\beta_{2,0}X^5Z^2+\beta_{4,0}X^3Z^4$
$12,(1,6)$	$X^7+Y^6Z+XZ^6+\beta_{2,0}X^5Z^2+\beta_{4,0}X^3Z^4$
$9,(1,3)$	$X^7+Y^6Z+XZ^6+\beta_{3,0}X^4Z^3+\beta_{5,3}X^2Y^3Z^2$
$7,(1,2)$	$X^7+Y^7+Z^7+\beta_{4,1}X^3YZ^3+\beta_{5,3}X^2Y^3Z^2+\beta_{6,5}XY^5Z$
$7,(1,3)$	$X^7+Y^7+Z^7+\beta_{3,1}X^4YZ^2+\beta_{5,4}X^2Y^4Z+\beta_{6,2}XY^2Z^4$
$7,(0,1)$	$Z^7+L_{7,Z}$
$6,(1,2)$	$X^7+XZ^6+XY^6+\beta_{3,0}X^4Z^3+\beta_{4,2}X^3Y^2Z^2+\beta_{5,4}X^2Y^4Z+\beta_{7,2}Y^2Z^5$
$6,(2,3)$	$X^7+XZ^6+XY^6+\beta_{2,0}X^5Z^2+\beta_{3,3}X^4Y^3+\beta_{4,0}X^3Z^4+\beta_{5,3}X^2Y^3Z^2$ $+\beta_{7,3}Y^3Z^4$
$6,(0,1)$	$Z^6Y+L_{7,Z}$
$5,(1,4)$	$X^7+Y^6Z+YZ^6+\beta_{2,1}X^5YZ+\beta_{4,2}X^3Y^2Z^2+\beta_{6,3}XY^3Z^3$ $+X^2\big(\beta_{5,0}Z^5+\beta_{5,5}Y^5\big)$
$4,(1,2)$	$X^7+Y^6Z+XZ^6+\beta_{2,0}X^5Z^2+\beta_{3,2}X^4Y^2Z+\beta_{5,2}X^2Y^2Z^3+\beta_{6,4}XY^4Z^2$ $\beta_{7,2}Y^2Z^5+X^3\big(\beta_{4,0}Z^4+\beta_{4,4}Y^4\big)$
$3,(1,2)$	$X^7+XZ^6+XY^6+\beta_{2,1}X^5YZ+\beta_{4,2}X^3Y^2Z^2+\beta_{6,3}XY^3Z^3+\beta_{7,2}Y^2Z^5$ $+\beta_{7,5}Y^5Z^2+X^4\left(\beta_{3,0}Z^3+\beta_{3,3}Y^3\right)+X^2\big(\beta_{5,1}YZ^4+\beta_{5,4}Y^4Z\big)$
$3,(0,1)$	$Z^6Y+Z^3L_{4,Z}+L_{7,Z}$
$2,(0,1)$	$Z^6Y+Z^4L_{3,Z}+Z^2L_{5,Z}+L_{7,Z}$

Table A.7: degree 8

Type: $m,(a,\,b)$	$F(X,Y,Z)$
$56,(7,8)$	$X^8+Y^8+XZ^7$
$49,(1,42)$	$X^8+Y^7Z+XZ^7$
$48,(1,41)$	$X^8+Y^7Z+YZ^7$
$43,(1,37)$	$X^7Y+Y^7Z+XZ^7$
$28,(7,8)$	$X^8+Y^8+XZ^7+\beta_{4,4}X^4Y^4$

Table A.8: degree 8 (continued)

$24,(1,17)$	$X^8+Y^7Z+YZ^7+\beta_{8,4}Y^4Z^4$
$16,(1,9)$	$X^8+Y^7Z+YZ^7+\beta_{8,5}Y^5Z^3+\beta_{8,3}Y^3Z^5$
$14,(7,8)$	$X^8+Y^8+XZ^7+\beta_{2,2}X^6Y^2+\beta_{4,4}X^4Y^4+\beta_{6,6}X^2Y^6$
$12,(1,5)$	$X^8+Y^7Z+YZ^7+\beta_{8,4}Y^4Z^4+\beta_{4,2}X^4Y^2Z^2$
$8,(1,2)$	$X^8+Y^8+Z^8+\beta_{4,0}X^4Z^4+\beta_{5,2}X^3Y^2Z^3+\beta_{6,4}X^2Y^4Z^2+\beta_{7,6}XY^6Z$
$8,(1,3)$	$X^8+Y^8+Z^8+\beta_{4,2}X^4Y^2Z^2+\beta_{8,4}Y^4Z^4+X^2\big(\beta_{6,1}YZ^5+\beta_{6,5}Y^5Z\big)$
$8,(1,4)$	$X^8+Y^8+Z^8+\beta_{2,0}X^6Z^2+\beta_{4,0}X^4Z^4+\beta_{5,4}X^3Y^4Z+\beta_{6,0}X^2Z^6$ $+\beta_{7,4}XY^4Z^3$
$8,(0,1)$	$Z^8+L_{8,Z}$
$7,(1,2)$	$X^8+XZ^7+XY^7+\beta_{4,1}X^4YZ^3+\beta_{5,3}X^3Y^3Z^2+\beta_{6,5}X^2Y^5Z+\beta_{8,2}Y^2Z^6$
$7,(1,3)$	$X^8+XZ^7+XY^7+\beta_{3,1}X^5YZ^2+\beta_{5,4}X^3Y^4Z+\beta_{6,2}X^2Y^2Z^4+\beta_{8,5}Y^5Z^3$
$7,(1,6)$	$X^8+XZ^7+XY^7+\beta_{2,1}X^6YZ+\beta_{4,2}X^4Y^2Z^2+\beta_{6,3}X^2Y^3Z^3+\beta_{8,4}Y^4Z^4$
$7,(0,1)$	$Z^7Y+L_{8,Z}$
$6,(1,5)$	$X^8+Y^7Z+YZ^7+\beta_{2,1}X^6YZ+\beta_{4,2}X^4Y^2Z^2+\beta_{8,4}Y^4Z^4$ $+X^2\big(\beta_{6,0}Z^6+\beta_{6,3}Y^3Z^3+\beta_{6,6}Y^6\big)$
$4,(0,1)$	$Z^8+Z^4L_{4,Z}+L_{8,Z}$
$3,(1,2)$	$X^8+Y^7Z+YZ^7+\beta_{8,4}Y^4Z^4+\beta_{2,1}X^6YZ+\beta_{4,2}X^4Y^2Z^2$ $+X^5\big(\beta_{3,0}Z^3+\beta_{3,3}Y^3\big)+X^3\big(\beta_{5,1}YZ^4+\beta_{5,4}Y^4Z\big)$ $+X^2\big(\beta_{6,0}Z^6+\beta_{6,3}Y^3Z^3+\beta_{6,6}Y^6\big)+X\big(\beta_{7,2}Y^2Z^5+\beta_{7,5}Y^5Z^2\big)$
$2,(0,1)$	$Z^8+Z^6L_{2,Z}+Z^4L_{4,Z}+Z^2L_{6,Z}+L_{8,Z}$

Table A.9: degree 9

Type: $m,(a,\,b)$	$F(X,Y,Z)$
$72,(8,9)$	$X^9+Y^9+XZ^8$
$64,(1,56)$	$X^9+Y^8Z+XZ^8$
$63,(1,55)$	$X^9+Y^8Z+YZ^8$
$57,(1,50)$	$X^8Y+Y^8Z+XZ^8$
$36,(4,9)$	$X^9+Y^9+XZ^8+\beta_{4,0}X^5Z^4$
$32,(1,24)$	$X^9+Y^8Z+XZ^8+\beta_{4,0}X^5Z^4$
$24,(8,9)$	$X^9+Y^9+XZ^8+\beta_{3,3}X^6Y^3+\beta_{6,6}X^3Y^6$
$21,(1,13)$	$X^9+Y^8Z+YZ^8+\beta_{6,3}X^3Y^3Z^3$
$19,(1,12)$	$X^8Y+Y^8Z+XZ^8+\alpha_{6,3}X^3Y^3Z^3$
$18,(2,9)$	$X^9+Y^9+XZ^8+\beta_{2,0}X^7Z^2+\beta_{4,0}X^5Z^4+\beta_{6,0}X^3Z^6$
$16,(1,8)$	$X^9+Y^8Z+XZ^8+\beta_{2,0}X^7Z^2+\beta_{4,0}X^5Z^4+\beta_{6,0}X^3Z^6$
$12,(4,9)$	$X^9+Y^9+XZ^8+\beta_{3,3}X^6Y^3+\beta_{4,0}X^5Z^4+\beta_{6,6}X^3Y^6$ $+\beta_{7,3}X^2Y^3Z^4$
$9,(1,2)$	$X^9+Y^9+Z^9+\beta_{5,1}X^4YZ^4+\beta_{6,3}X^3Y^3Z^3+\beta_{7,5}X^2Y^5Z^2$ $+\beta_{8,7}XY^7Z$
$9,(1,3)$	$X^9+Y^9+Z^9+\beta_{3,0}X^6Z^3+\beta_{5,3}X^4Y^3Z^2+\beta_{6,0}X^3Z^6+\beta_{7,6}X^2Y^6Z$ $+\beta_{8,3}XY^3Z^5$
$9,(0,1)$	$Z^9+L_{9,Z}$

Table A.10: degree 9 (continued)

$8,(1,2)$	$X^9+XZ^8+XY^8+\beta_{4,0}X^5Z^4+\beta_{5,2}X^4Y^2Z^3+\beta_{6,4}X^3Y^4Z^2+\beta_{7,6}X^2Y^6Z$ $+\beta_{7,6}X^2Y^6Z+\beta_{9,2}Y^2Z^7$
$8,(1,4)$	$X^9+XZ^8+XY^8+\beta_{2,0}X^7Z^2+\beta_{4,0}X^5Z^4+\beta_{5,4}X^4Y^4Z+\beta_{6,0}X^3Z^6$ $+\beta_{7,4}X^2Y^4Z^3+\beta_{9,4}Y^4Z^5$
$8,(1,6)$	$X^9+XZ^8+XY^8+\beta_{3,2}X^6Y^2Z+\beta_{4,0}X^5Z^4+\beta_{6,4}X^3Y^4Z^2+\beta_{7,2}X^2Y^2Z^5$ $+\beta_{9,6}Y^6Z^3$
$8,(0,1)$	$Z^8L_{1,Z}+L_{9,Z}$
$7,(1,6)$	$X^9+Y^8Z+YZ^8+\beta_{2,1}X^7YZ+\beta_{4,2}X^5Y^2Z^2+\beta_{6,3}X^3Y^3Z^3$ $+\beta_{8,4}XY^4Z^4+X^2\left(\beta_{7,0}Z^7+\beta_{7,7}Y^7\right)$
$6,(2,3)$	$X^9+Y^9+XZ^8+\beta_{2,0}X^7Z^2+\beta_{3,3}X^6Y^3+\beta_{4,0}X^5Z^4+\beta_{5,3}X^4Y^3Z^2$ $+\beta_{7,3}X^2Y^3Z^4+\beta_{7,3}Y^3Z^6+\beta_{8,6}Y^6Z^3+X^3\left(\beta_{6,0}Z^6+\beta_{6,6}Y^6\right)$
$4,(1,2)$	$X^9+XZ^8+XY^8+\beta_{2,0}X^7Z^2+\beta_{3,2}X^6Y^2Z+\beta_{5,2}X^4Y^2Z^3+$ $+\beta_{8,4}XY^4Z^4+\beta_{9,2}Y^2Z^7+\beta_{9,6}Y^6Z^3+X^5\left(\beta_{4,0}Z^4+\beta_{4,4}Y^4\right)$ $+X^3\left(\beta_{6,0}Z^6+\beta_{6,4}Y^4Z^2\right)+X^2\left(\beta_{7,2}Y^2Z^5+\beta_{7,6}Y^6Z\right)$
$4,(0,1)$	$Z^8Y+Z^4L_{5,Z}+L_{9,Z}$
$3,(1,2)$	$X^8Y+Y^8Z+XZ^8+\alpha_{2,0}X^2Y^7+\beta_{5,3}X^4Y^3Z^2+\alpha_{5,3}X^2Y^4Z^3$ $+\alpha_{3,2}XY^6Z^2+\gamma_{5,2}X^3Y^2Z^4+\beta_{6,2}X^3Y^2Z^4+\alpha_{6,2}X^4Y^3Z^2+\beta_{7,4}X^2Y^4Z^3$ $+\beta_{2,0}X^7Z^2+\beta_{3,2}X^6Y^2Z+\gamma_{2,2}Y^2Z^7+\left(\beta_{4,1}YZ^3+\beta_{4,4}Y^4\right)X^5$ $+\gamma_{3,1}X^2YZ^6+\left(\alpha_{4,1}X^3Z+\alpha_{4,4}Z^4\right)Y^5+\left(\gamma_{4,0}X^4+\gamma_{4,3}XY^3\right)Z^5$
$3,(0,1)$	$Z^9+Z^6L_{3,Z}+Z^3L_{6,Z}+L_{9,Z}$
$2,(0,1)$	$Z^8L_{1,Z}+Z^6L_{3,Z}+Z^4L_{5,Z}+Z^2L_{7,Z}+L_{9,Z}$

B

APPENDIX

The algorithm on twist for smooth curves: Explicit examples

The algorithm for computing $\mathrm{Twist}_k(C)$ of a non-hyperelliptic curve C of genus $g \geq 3$ developed in [LG14, Chp.1] and [LG17] has three main steps:

(i) Find a canonical model: Take a basis of the space of the regular differentials $\Omega^1(C)$, and compute a canonical model $\mathcal{C}$ over k via a canonical embedding $C \hookrightarrow \mathbb{P}^{g-1}$ that we can take also defined over k. Hence, C and $\mathcal{C}$ belong to the same class in $\mathrm{Twist}_k(\mathcal{C})$ and $\mathrm{Twist}_k(\mathcal{C}) = \mathrm{Twist}_k(C)$. Furthermore, the automorphism group $\mathrm{Aut}(\mathcal{C})$ can be naturally viewed as a subgroup of $\mathrm{PGL}_g(\overline{k})$. Also, any isomorphism $\phi : \mathcal{C}' \to C$ can be also viewed as a matrix in $\mathrm{PGL}_g(\overline{k})$.

(ii) *Galois embedding problem* (see for example [NSW08, §9.4]): Given a field k and a finite group G, one may ask the following question, the so called *Inverse Galois problem*: does there exist a Galois extension F/k such that $\mathrm{Gal}(F/k) \simeq G$? The Galois embedding problem is a generalization of this. It asks whether a given Galois extension K/k can be embedded into another Galois extension F/k in such a way that the restriction map between the corresponding Galois groups is given in advance. In other words, a Galois embedding problem is a diagram:

$$\begin{array}{ccccc} & & \mathrm{Gal}(F/k) & & \\ & & \downarrow \pi & & \\ G & \xrightarrow{f} & \mathrm{Gal}(K/k) & \longrightarrow & 1 \end{array}$$

where π is the natural projection and f is an epimorphism. A solution to this embedding problem is a morphism $\psi : \mathrm{Gal}(F/k) \to G$ such that next diagram is commutative:

$$\begin{tikzcd} & \mathrm{Gal}(F/k) \arrow[dl, dotted, "\psi"'] \arrow[d, two heads, "\pi"] & \\ G \arrow[r, "f"] & \mathrm{Gal}(K/k) \arrow[r] & 1 \end{tikzcd}$$

A solution is called *proper* if it is surjective.

Now given a smooth curve C over a field k, let K be the field over which all the automorphisms of C are defined. Denote by Γ the twisted product $\mathrm{Aut}(\overline{C}) \rtimes \mathrm{Gal}(K/k)$, where $\mathrm{Gal}(K/k)$ acts naturally on $\mathrm{Aut}(\overline{C})$ and the multiplication rule is $(\eta, \sigma)(\rho, \tau) = (\eta\, {}^{\sigma}\rho, \sigma\tau)$.

A group homomorphism $\psi : G_k \to \Gamma$ is said to be an epi$_2$-morphism if the composition $\pi \circ \psi : G_k \to \Gamma \to \mathrm{Gal}(K/k)$ is surjective, where π is the natural projection on the second component of Γ. It is known that the set $\mathrm{Twist}_k(C)$ is in one-to-one correspondence with the set

$$\widetilde{\mathrm{Hom}}(G_k, \Gamma) := \{\psi : G_k \to \Gamma : \ \psi \text{ epi}_2 - \text{morphism}\}/\sim,$$

where two epi$_2$-morphisms ψ and ψ' are equivalent if there exists $(\varphi, 1) \in \Gamma$ such that $\psi'(\sigma) = (\varphi, 1) \circ \psi(\sigma) \circ (\varphi, 1)^{-1}$ for all $\sigma \in G_k$, and we write $\psi \sim \psi'$. This correspondence sends a twist $\phi : C' \to C$ to $\psi_\phi : G_k \to \Gamma : \sigma \mapsto (\phi \circ {}^{\sigma}\phi^{-1}, \pi(\sigma))$.

Let $\psi \in \widetilde{\mathrm{Hom}}(G_k, \Gamma)$ and let L be its splitting field. We have $\psi(\mathrm{Gal}(\overline{K}/K)) \simeq \mathrm{Gal}(L/K)$ and $\psi(G_k) \simeq \mathrm{Gal}(L/k)$. Then ψ can be seen as a proper solution to the Galois embedding problem:

$$\begin{tikzcd} & & & G_k \arrow[dl, dotted, "\psi"'] \arrow[d, two heads] & \\ 1 \arrow[r] & \psi(\mathrm{Gal}(\overline{K}/K)) \arrow[r] & \psi(G_k) \arrow[r, "\pi"] & \mathrm{Gal}(K/k) \arrow[r] & 1 \end{tikzcd}$$

As it was noticed in [LG14, §1.1], $\mathrm{Gal}(L/k) \simeq \mathrm{Im}(\psi) \leq \Gamma$ and $\mathrm{Gal}(L/K) \simeq \psi(\mathrm{Gal}(\overline{K}/K)) \leq \mathrm{Aut}(\overline{C}) \rtimes \{1\}$. Hence, in order to compute $\widetilde{\mathrm{Hom}}(G_k, \Gamma)$ we should compute all the pairs (G, H) where $G \leq \Gamma$, $H = G \cap \mathrm{Aut}(\overline{C}) \rtimes \{1\}$ and $[G : H] = |\operatorname{Gal}(K/k)|$ up to conjugacy by elements $(\varphi, 1) \in \Gamma$. Then, we will find all proper solutions (and thus the corresponding splitting fields L) to the Galois embedding problems:

$$\begin{tikzcd} & & & G_k \arrow[dl, dotted, two heads, "\psi"'] \arrow[d, two heads] & \\ 1 \arrow[r] & H \arrow[r] & G \arrow[r, two heads, "\pi"] & \operatorname{Gal}(K/k) \arrow[r] & 1 \end{tikzcd}$$

Every such a solution can be lifted to a solution to the Galois embedding problem:

$$\begin{tikzcd} & & & G_k \arrow[dl, dotted, two heads, "\psi"'] \arrow[d, two heads] & \\ 1 \arrow[r] & \mathrm{Aut}(\overline{C}) \arrow[r, hook] & \Gamma \arrow[r, two heads, "\pi"] & \operatorname{Gal}(K/k) \arrow[r] & 1 \end{tikzcd}$$

Conversely, every solution ψ of the above embedding problem provides a twist over k of C.

(iii) Explicit equation of Twists: the idea behind the computation of equations for the twist, is finding a G_k-modulo isomorphism between the subgroup in $\mathrm{Aut}(\overline{C})$ associated to a pair (G, H) as above and a subgroup of a general linear group $\mathrm{GL}_n(\overline{k})$. After that, by making explicitly Hilbert's Theorem 90, we can compute an isomorphism $\phi : C' \to C$, and hence, we obtain equations for the twist.

Assume that C is a smooth curve over k with a plane non-singular model over k such that Σ in Theorem 3.3.2 is trivial, in such case all the twist admits a plane non-singular model over k, see Theorem 3.3.2. Now instead of computing a canonical model of C, we can consider a plane model over k associated to C, modifying point (i) of the algorithm. The point (ii) is independent of the embedding of C inside a projective space. In point (iii), the algorithm of [LG17] requires to investigate the solutions in $\mathrm{GL}_g(\overline{k})$ using [LG14, Lemma 1.1.3]. Now, in the modified algorithm, it is enough to look for solutions in $\mathrm{GL}_3(\overline{k})$. As in [LG14, LG17] the elements to reach for solutions in $\mathrm{GL}_g(\overline{k})$ or $\mathrm{GL}_3(\overline{k})$ is quite hard except that we have a control of the matrix that could appear. For example, we can apply the next result in some situations, which can ne proved in the same way as Lemma 2.2.2 in [LG14] for $g = 3$:

Lemma B.1. *Assume that C is a smooth curve over k with a plane non-singular model over k such that Σ in Theorem 3.3.2 is trivial. Let $\xi \in \mathrm{H}^1(k, \mathrm{Aut}(\overline{C}))$ be a cocycle with splitting field L. Assume that the elements of $\xi(\mathrm{Gal}(L/k))$, as matrices in $\mathrm{GL}_3(L)$, have the form as block matrices*

$$\begin{pmatrix} A & 0 \\ 0 & a \end{pmatrix} \tag{B.1}$$

where $A \in \mathrm{GL}_2(L)$ and $a \in L$. Then, there exists an isomorphism $\phi : C' \to C$ associated to ξ has the form as the block matrix (B.1). In the particular case, in which $\xi(\mathrm{Gal}(L/k))$ is made of diagonal matrices, we can take $\phi : C' \to C$ also a diagonal matrix.

We use the above modified algorithm to compute the twists over k for the smooth plane curves defined over k by $X^5 + Y^5 + XZ^4 = 0$ and $X^5 + Y^4Z + XZ^4 = 0$, where k is a field of zero characteristic or positive characteristic $> (5-1)(5-2)+1 = 13$. Here we recover the result obtained for these curves in Theorem 3.5.4.

Example B.2. *Let C be the smooth plane curve $X^5 + Y^5 + XZ^4 = 0$ over a field k of characteristic $p = 0$ or $p > 13$. The full automorphism group $\mathrm{Aut}(\overline{C}) \simeq \mathrm{GAP}(20, 2)$ is generated by $S := \mathrm{diag}(1, 1, \zeta_4)$ and $T := \mathrm{diag}(1, \zeta_5, 1)$, so it is defined over $K = k(\zeta_4, \zeta_5)$.*

We assume the generic case in which $\zeta_4, \zeta_5 \notin k$. Then $[K : k] = 8$ and the Galois group $\mathrm{Gal}(K/k)$ is generated by $\tau_1 : \zeta_4 \mapsto -\zeta_4$, $\zeta_5 \mapsto \zeta_5$ and $\tau_2 : \zeta_4 \mapsto \zeta_4$, $\zeta_5 \mapsto \zeta_5^2$ of order 2 and 4 respectively where $\tau_2\tau_1 = \tau_1\tau_2$. In particular, $\mathrm{Gal}(K/k) \simeq \mathrm{GAP}(8, 2)$.

The group $\mathrm{Gal}(K/k)$ acts naturally on $\mathrm{Aut}(\overline{C})$ as follows: $\tau_1 : S \mapsto S^3, T \mapsto T$ and $\tau_2 : S \mapsto S, T \mapsto T^2$. The twisted product $\Gamma := \mathrm{Aut}(\overline{C}) \rtimes \mathrm{Gal}(K/k)$ is isomorphic to $\mathrm{GAP}(160, 207)$, and generated by the elements $\vartheta_1 := (ST, 1), \vartheta_2 := (1, \tau_1)$ and $\vartheta_3 := (1, \tau_2)$, where $\vartheta_1^{20} = \vartheta_2^2 = \vartheta_3^4 = 1$, $\vartheta_2\vartheta_1\vartheta_2 = \vartheta_1^{11}$, $\vartheta_1\vartheta_3 = \vartheta_3\vartheta_1^{13}$ and $\vartheta_2\vartheta_3\vartheta_2 = \vartheta_3$.

The degree of the defining equation of C is coprime with 3, thus, by Corollary 3.2.9, every twist of C has a non-singular plane model over k. Consequently, by Theorem 3.3.2, the map Σ is trivial. In particular, we only look for solutions of the Galois embedding problems inside $\mathrm{GL}_3(\overline{k})$ not in $\mathrm{GL}_6(\overline{k})$. One finds that all the twists of C over k are covered by diagonal *matrices, and they are of the form* $aX^5+Y^5+bXZ^5$ *for some* $a, b \in k$ *through an isomorphism of the shape* $\mathrm{diag}(\alpha, 1, \beta)$ *in* $\mathrm{GL}_3(\overline{k})$.

The Galois embedding problems for C *are given by the pairs* (G, H) *appears in the 2nd and the 3rd columns of Table B.1 in* GAP *notations. This means that a twist* $\varphi : C' \to C$ *over* k *has a splitting field* L *such that* $\mathrm{Gal}(L/k) \simeq G$ *and* $\mathrm{Gal}(L/K) \simeq H$ *for some pair* (G, H) *in the list. The pairs* (G, H) *can be generated via a slight modification of [LG14, Table 5.5] in MAGMA [BCP97]. A set of generators of both* G *and* H *are given in the* 4^{th}, 5^{th} *and* 6^{th} *columns:* G *is generated by the elements* $h \rtimes 1 \in H \rtimes 1$ *and* $g_i \rtimes \tau_i$ *with* $i = 1, 2$. *The integer* $n_{(G,H)}$ *that appears in the* 7^{th} *column is the number of non-equivalent twists of* C *with the same splitting field* L. *By the aid of [LG14, Proposition 4.1], we find solutions to these Galois embedding problems as described in the* 8^{th} *column. In the remaining part of the table, we give the associated set of non-equivalent twists which are defined by equations of the form* $aX^d + Y^d + bXZ^{d-1} = 0$ *through an isomorphism of the shape* $\mathrm{diag}(\alpha, 1, \beta)$ *whose splitting field is* L.

We thus collect the computations into the following result:

Theorem B.3. *Following all the above notations, the set* $\mathrm{Twist}_k(C)$ *is completely determined by Table B.1 and Table B.2 below.*

Table B.1: The pairs (G, H), and Twists

	$ID(G)$	$ID(H)$	gen (H)	g_1	g_2	$n_{(G,H)}$	L	a	b
1	$\mathrm{GAP}(8,2)$	$\mathrm{GAP}(1,1)$	1	S	1	1	K	1	-4
2				S	S^2				-100
3				1	S^2				25
4				1	1				1
5	$\mathrm{GAP}(16,10)$	$\mathrm{GAP}(2,1)$	S^2	S	1	2	$K(\sqrt{n})$	1	$-4rn^2$
6	$\mathrm{GAP}(16,3)$			S	S		$K(\sqrt[4]{5n^2})$		$-20rn^2$
7	$\mathrm{GAP}(16,3)$			1	S				$5rn^2$
8	$\mathrm{GAP}(16,10)$			1	1		$K(\sqrt{n})$		rn^2
9	$\mathrm{GAP}(40,12)$	$\mathrm{GAP}(5,1)$	T	1	S^2	4	$K(\sqrt[5]{m})$	m^s	$25m^s$
10				1	1				m^s
11				S	S^2				$-100m^s$
12				S	1				$-4m^s$

Table B.2: The pairs (G, H), and Twists (continued)

	$ID(G)$	$ID(H)$	gen (H)	g_1	g_2	$n_{(G,H)}$	L	a	b
13	$\mathrm{GAP}(32,25)$	$\mathrm{GAP}(4,1)$	S	1	1	8	$K(\sqrt[4]{n})$	1	$(-4)^{j_1}(25)^{j_2}n^{2j_3+1}$
14	$\mathrm{GAP}(80,50)$			S	1		$K(\sqrt{n},\, \sqrt[5]{m})$		$-4rn^2m^s$
15	$\mathrm{GAP}(80,34)$	$\mathrm{GAP}(10,2)$	$S^2,\, T$	1	S	8	$K(\sqrt[4]{5n^2},\, \sqrt[5]{m})$	m^s	$125rn^2m^s$
16	$\mathrm{GAP}(80,34)$			S	S				$-500rn^2m^s$
17	$\mathrm{GAP}(80,50)$			1	1		$K(\sqrt{n},\, \sqrt[5]{m})$		rn^2m^s
18	$\mathrm{GAP}(160,207)$	$\mathrm{GAP}(20,2)$	$S,\, T$	1	1	32	$K(\sqrt[4]{n},\, \sqrt[5]{m})$	m^s	$(-4)^{j_1}(25)^{j_2}n^{2j_3+1}m^s$

We obtain the equation of the twist via the isomorphism $\mathrm{diag}(\alpha, 1, \beta)$*, where* $\alpha = \sqrt[5]{a}$ *and* $\beta = \sqrt[4]{b}$ *for the cases (1)-(8),* $\beta = \sqrt[4]{\frac{b}{m^s}}\sqrt[5]{m^s}$ *for the cases (9)-(12) and* $\beta = \sqrt[4]{b}\sqrt[5]{m^s}$ *for the cases (13)-(18), for some* $n \in k \setminus k^2$ *and* $m \in k \setminus k^5$*, with* $s \in \{1,2,3,4\}$, $r \in \{1, 25\}$ *and* $j_i \in \{0,1\}$*, for* $i = 1,2,3$.

Proof. As an example, we show the computations for the 9th case. Let m be an element of $k \setminus k^5$, and set $L := K(\sqrt[5]{m})$. Then L/k is a Galois extension with Galois group isomorphic to $\mathrm{GAP}(40,12)$: we have $\mathrm{Gal}(L/k) = \langle h_0, h_1, h_2\rangle$ where $h_0 : \zeta_5 \mapsto \zeta_5,\, \zeta_4 \mapsto \zeta_4,\, \sqrt[5]{M} \mapsto \zeta_5\sqrt[5]{M}$, $h_1 : \zeta_5 \mapsto \zeta_5,\, \zeta_4 \mapsto -\zeta_4,\, \sqrt[5]{M} \mapsto \sqrt[5]{M}$, and $h_2 : \zeta_5 \mapsto \zeta_5^2,\, \zeta_4 \mapsto \zeta_4,\, \sqrt[5]{M} \mapsto \sqrt[5]{M}$. In particular, $h_0^5 = h_1^2 = h_2^4 = 1$, moreover $h_1h_2 = h_2h_1, h_1h_0h_1 = h_0$, and $h_0h_2 = h_2h_0^3$. Hence $\mathrm{Gal}(L/k) \simeq \mathrm{GAP}(40,12)$. This proves that solutions to this Galois embedding problems already exist. Second we are looking for isomorphisms $\varphi_s = \mathrm{diag}(\alpha_s, 1, \beta_s)$, for $s = 1,2,3,4$ whose splitting field is L and produce the defining equations for the four non-equivalent twists over k. This can be done by applying the 1-cocylce condition: $\varphi_s \circ {}^{h_{1,2}}\varphi_s^{-1} = g_{1,2}$. Therefore, we can take $\varphi_s = \mathrm{diag}(\sqrt[5]{m^s}, 1, \sqrt{5}\sqrt[5]{m^s})$, and the twists are defined by $m^sX^5 + Y^5 + 25m^sXZ^4 = 0$, for $s = 1,2,3$ and 4. $\square$

Example B.4. *Consider the smooth plane curve* C *defined over a field* k *of characteristic* $p = 0$ *or* $p > 13$ *by the equation* $X^5 + Y^4Z + XZ^4 = 0$*. The full automorphism group* $\mathrm{Aut}(\overline{C})$ *is isomorphic to* $\mathrm{GAP}(16,1)$*, and is generated by* $S := \mathrm{diag}(1, \zeta_{16}, \zeta_{16}^{12})$*. Assuming that* $\zeta_{16} \notin k$*, then* $K = k(\xi_{16})$ *and* $[K : k] = 8$*. Moreover,* $\mathrm{Gal}(K/k)$ *is generated by* $\tau_1 : \zeta_{16} \mapsto \zeta_{16}^3$ *(of order* 4*) and* $\tau_2 : \zeta_{16} \mapsto \zeta_{16}^7$ *(of order* 2*) with* $\tau_2\tau_1\tau_2 = \tau_1$*, in particular* $Gal(K/k)$ *is isomorphic* $\mathrm{GAP}(8,2)$*. The action of* $\mathrm{Gal}(K/k)$ *on* $\mathrm{Aut}(\overline{C})$ *is defined by* $\tau_1(S) = S^3$ *and* $\tau_2(S) = S^7$*. Now, the group* $\Gamma := \mathrm{Aut}(\overline{C}) \rtimes \mathrm{Gal}(K/k)$ *is generated by the elements* $\vartheta_1 := (s,1)$*,* $\vartheta_2 :=$

$(1, \tau_1)$ *and* $\vartheta_3 := (1, \tau_2)$ *of orders* $16, 4$ *and* 2 *respectively. Thus*

$$\Gamma \cong \langle \vartheta_1, \vartheta_2, \vartheta_3 \mid \vartheta_1^{16} = \vartheta_2^4 = \vartheta_3^2 = 1,\ \vartheta_2\vartheta_1 = \vartheta_1^3\vartheta_2,\ \vartheta_3\vartheta_1\vartheta_3 = \vartheta_1^7,\ \vartheta_3\vartheta_2\vartheta_3 = \vartheta_2 \rangle = \mathrm{GAP}(128, 913),$$

where $\mathrm{Aut}(\overline{C})$ *is identified with* $lat[42]$ *inside the subgroups lattice of* Γ.

Similarly, as the previous example, the computations can be collected into the following result:

Theorem B.5. *Following all the above notations, the set* $\mathrm{Twist}_k(C)$ *is completely determined by the following tables: The pairs* (G, H) *and their splitting fields as solutions of the associated Galois embedding problems are included in the next table.*

Table B.3: The pairs (G, H) and their splitting fields

	$ID(G)$	$ID(H)$	gen (H)	g_1	g_2	$n_{(G,H)}$	L
1	$\mathrm{GAP}(8,2)$	$\mathrm{GAP}(1,1)$	1	1	1	1	K
2				S^8	1		
3	$\mathrm{GAP}(16,5)$	$\mathrm{GAP}(2,1)$	S^8	S	S^3	2	$K(\sqrt[16]{-n^8})$
4	$\mathrm{GAP}(16,10)$			1	1		$K(\sqrt{n})$
5	$\mathrm{GAP}(16,6)$			S^3	S^5		$K(\sqrt[16]{-16n^8})$
6	$\mathrm{GAP}(16,3)$			S^2	S^2		$K(\sqrt[4]{2n^2})$
7	$\mathrm{GAP}(32,9)$	$\mathrm{GAP}(4,1)$	S^4	S^2	S^8	4	$K(\sqrt[16]{4n^4})$
8	$\mathrm{GAP}(32,25)$			1	1		$K(\sqrt[4]{n})$
9	$\mathrm{GAP}(32,38)$			S	S^3		$K(\sqrt[16]{-n^4})$
10	$\mathrm{GAP}(32,11)$			S	S^5		$K(\sqrt[16]{-4n^4})$
11	$\mathrm{GAP}(64,41)$	$\mathrm{GAP}(8,1)$	S^2	S^2	S	8	$K(\sqrt[16]{-2n^2})$
12	$\mathrm{GAP}(64,42)$			S	1		$K(\sqrt[16]{2n^2})$
13	$\mathrm{GAP}(64,123)$			1	1		$K(\sqrt[8]{n})$
14	$\mathrm{GAP}(64,125)$			S	S		$K(\sqrt[16]{-n^2})$
15	$\mathrm{GAP}(128,913)$	$\mathrm{GAP}(16,1)$	S	1	1	16	$K(\sqrt[16]{n})$

The equations of each twist of C *over* k *is defined by an equation of the form* $X^5 + aY^4Z + bXZ^4$ *for some* $a, b \in k$ *through an isomorphism of the shape* $\mathrm{diag}(1, \lambda, \mu)$.

Table B.4: Equations of Twists

	$ID(G)$	$n_{(G,H)}$	a	b	λ	μ
1	$\mathrm{GAP}(8,2)$	1	1	1	1	1
2			4		$\sqrt{2}$	
3	$\mathrm{GAP}(16,5>$	2	$-n^2, -4n^2$	-1	$\zeta_{16}^7\ \sqrt[16]{-n^8}, -\zeta_{16}^3\sqrt{2}\ \sqrt[16]{-n^8}$	ζ_{16}^{-6}
4	$\mathrm{GAP}(16,10>$		$n^2, 4n^2$	1	$\sqrt{n}, \sqrt{2}\sqrt{n}$	1
5	$\mathrm{GAP}(16,6>$		$-2n^2, -8n^2$	-1	$\zeta_4\ \sqrt[16]{-16n^8}, -\sqrt{2}\ \sqrt[16]{-16n^8}, \gamma=\zeta_4, -\sqrt{2}$	ζ_{16}^6
6	$\mathrm{GAP}(16,3>$		$-2n^2, -8n^2$	1	$\zeta_{16}^5\ \sqrt[4]{2n^2}, \zeta_{16}^5\sqrt{2}\ \sqrt[4]{2n^2}$	ζ_4
7	$\mathrm{GAP}(32,9>$	4	$2n, 4n^3$	4	$\zeta_4\ \sqrt[16]{4n^4},\ \zeta_4(\sqrt[16]{4n^4})^3$	$\sqrt{2}$
			$8n, n^3$		$\sqrt[4]{-4}\ \sqrt[16]{4n^4},\ \frac{\sqrt[4]{-4}(\sqrt[16]{4n^4})^3}{2}$	$-\sqrt{2}$
8	$\mathrm{GAP}(32,25>$		n, n^3	1	$\sqrt[4]{n},\ \zeta_4\ \sqrt[4]{n^3}$	1
			$4n, 4n^3$		$(1-\zeta_4)\sqrt[4]{n},\ (1+\zeta_4)\sqrt[4]{n^3}$	-1
9	$\mathrm{GAP}(32,38>$		n, n^3	-1	$\zeta_{16}^7\ \sqrt[16]{-n^4},\ \zeta_{16}^2(\sqrt[16]{-n^4})^3$	ζ_{16}^2
			$4n, 4n^3$		$(\zeta_{16}^3-\zeta_{16}^7)\ \sqrt[16]{-n^4},\ \zeta_4\sqrt{2}(\sqrt[16]{-n^4})^3$	$-\zeta_{16}^2$
10	$\mathrm{GAP}(32,11>$		$2n, 4n^3$	-4	$i\ \sqrt[16]{-4n^4},\ \zeta_{16}^3(\sqrt[16]{-4n^4})^3$	$-\zeta_4\sqrt[4]{-4}$
			$8n, n^3$		$\sqrt[4]{-4}\ \sqrt[16]{-4n^4},\ \frac{\zeta_{16}^3\ \sqrt[4]{-4}(\sqrt[16]{-4n^4})^3}{2}$	$\zeta_4\sqrt[4]{-4}$
11	$\mathrm{GAP}(64,41>$	8	$4, 8n^2$	$-2n^2$	$\frac{1}{n}(\sqrt[16]{-2n^2})^7,\ (1-\zeta_4)(\sqrt[16]{-2n^2})^3$	$(\sqrt[16]{-2n^2})^4$
			$1, 2n^2$		$\frac{1}{2n}\sqrt[4]{-4}(\sqrt[16]{-2n^2})^7,\ (\sqrt[16]{-2n^2})^3$	$-(\sqrt[16]{-2n^2})^4$
			$2n, 4n^3$	$-8n^2$	$\zeta_{16}^7\ \sqrt[16]{-2n^2},\ \zeta_{16}(\sqrt[16]{-2n^2})^5$	$\sqrt{2}(\sqrt[16]{-2n^2})^4$
			$8n, n^3$		$\zeta_{16}\sqrt[4]{-4}\ \sqrt[16]{-2n^2},\ \frac{1}{2}\zeta_{16}\sqrt[4]{-4}(\sqrt[16]{-2n^2})^5$	$-\sqrt{2}(\sqrt[16]{-2n^2})^4$
12	$\mathrm{GAP}(64,42>$		$4, 8n^2$	$2n^2$	$\frac{1}{n}\zeta_{16}^2(\sqrt[16]{2n^2})^7,\ (1-\zeta_4)(\sqrt[16]{2n^2})^3$	$-(\sqrt[16]{2n^2})^4$
			$1, 2n^2$		$\frac{1}{2n}\zeta_{16}^2\sqrt[4]{-4}(\sqrt[16]{2n^2})^7,\ (\sqrt[16]{2n^2})^3$	$(\sqrt[16]{2n^2})^4$
			$2n, 4n^3$	$8n^2$	$\sqrt[16]{2n^2},\ (\sqrt[16]{2n^2})^5$	$\sqrt{2}(\sqrt[16]{2n^2})^4$
			$8n, n^3$		$\sqrt[4]{-4}\ \sqrt[16]{2n^2},\ \frac{1}{2}\sqrt[4]{-4}(\sqrt[16]{2n^2})^5$	$-\sqrt{2}(\sqrt[16]{2n^2})^4$
13	$\mathrm{GAP}(64,123>$	8	n, n^2 $n^3,\ 1$	n^2	$\sqrt[8]{n},\ (\sqrt[8]{n})^3$ $(\sqrt[8]{n})^5,\ \frac{1}{n}(\sqrt[8]{n})^7$	$(\sqrt[8]{n})^4$
			$4n, 4n^2$ $4n^3, 4$		$\sqrt[4]{-4}\sqrt[8]{n},\ \sqrt[4]{-4}(\sqrt[8]{n})^3$ $\sqrt[4]{-4}(\sqrt[8]{n})^5,\ \frac{1}{n}\sqrt[4]{-4}(\sqrt[8]{n})^7$	$-(\sqrt[8]{n})^4$
14	$\mathrm{GAP}(64,125>$		n, n^2 $n^3, 1$	$-n^2$	$\zeta_{16}^3\ \sqrt[16]{-n^2},\ \zeta_{16}^2(\sqrt[16]{-n^2})^3$ $\zeta_{16}(\sqrt[16]{-n^2})^5,\ \frac{1}{n}(\sqrt[16]{-n^2})^7$	$(\sqrt[16]{-n^2})^4$
			$4n, 4n^2$ $4n^3, 4$		$\zeta_{16}^3\sqrt[4]{-4}\ \sqrt[16]{-n^2},\ \sqrt[4]{-4}(\sqrt[16]{-n^2})^3$ $\zeta_{16}\sqrt[4]{-4}(\sqrt[16]{-n^2})^5,\ \frac{1}{n}\sqrt[4]{-4}(\sqrt[16]{-n^2})^7$	$-(\sqrt[16]{-n^2})^4$
15	$\mathrm{GAP}(128,913>$	16	rn, rn^2	n^3	$\gamma\ \sqrt[16]{n},\ \gamma\ \sqrt[16]{n^5}$ $\gamma\ \sqrt[16]{n^9},\ \gamma\frac{\sqrt[16]{n^{13}}}{n}$ $\gamma=1\ for\ r=1$ $\gamma=\sqrt[4]{-4}\ for\ r=4$	$(-1)^{r-1}\ \sqrt[16]{n^{12}}$
			rn^3, r	n		$(-1)^{r-1}\ \sqrt[16]{n^4}$

C

APPENDIX

Isomorphic geometric fibers for the stratum $\widetilde{\mathcal{M}_{\mathbf{6}}^{\mathrm{Pl}}}(\mathbb{Z}/\mathbf{5}\mathbb{Z})$

We saw in section §4.3 that the family $\mathcal{C}_{(a,b)}$ defined by $Z^5+XY(X+Y)(X+aY)(X+bY)=0$ is geometrically complete over k for the stratum $\widetilde{\mathcal{M}_6^{Pl}}(\mathbb{Z}/5\mathbb{Z})$, where k is a field of characteristic $p=0$ or $p>13$. Isomorphisms from the curve $\mathcal{C}_{(a,b)}$ to another curve in this family come from transformations

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} : t\mapsto \frac{\alpha t+\beta}{\gamma t+\delta},$$

sending the set $\{0,1,\infty,a,b\}$ to a set $\{0,1,\infty,c,d\}$. The set $\mathcal{T}$ of such transformations is a group and it is isomorphic to S_5. Moreover, it is generated by

$$\tau_1(a,b)=(a,\frac{a(b-1)}{b-a}),\quad \tau_2(a,b)=(\frac{1}{b},\frac{a}{b}),\quad \tau_3(a,b)=(b,a).$$

The latest does not properly define a transformation of the curve in the family since switching the parameters a,b does not change the equation. The first two satisfy the relations $\tau_1^2=\tau_2^3=(\tau_1\tau_2)^5=1$ generating a group isomorphic to A_5. We generate here the full list of geometric fibers over (a,b), which are isomorphic over $\overline{k}$. For each situation (a',b'), we associate an isomorphism between the fibers (a,b) and (a',b'). The 5th column determine the order of the field automorphism $\sigma_i:\ (a,b)\mapsto(a',b')$.

Table C.1: Isomorphic fibers

	a'	b'	ϕ_{σ_i}	$O(\sigma_i : (a,b) \mapsto (a',b'))$
1	$\frac{1}{a}$	$\frac{b}{a}$	$\mathrm{diag}(1, \frac{1}{a}, \frac{1}{\sqrt[5]{a}})$	2
2	$\frac{b}{a}$	$\frac{1}{a}$		3
3	$\frac{1}{b}$	$\frac{a}{b}$	$\mathrm{diag}(1, \frac{1}{b}, \frac{1}{\sqrt[5]{b}})$	3
4	$\frac{a}{b}$	$\frac{1}{b}$		2
5	$\frac{a}{a-1}$	$\frac{b}{b-1}$	$[X : -(X+Y) : \sqrt[5]{(1-a)(1-b)}Z]$	2
6	$\frac{b}{b-1}$	$\frac{-a}{1-a}$		2
7	$\frac{1}{1-a}$	$\frac{b}{b-a}$	$[X : \frac{-1}{a}(X+Y) : \sqrt[5]{\frac{(a-1)(a-b)}{a^3}}Z]$	3
8	$\frac{b}{b-a}$	$\frac{1}{1-a}$		4
9	$\frac{1}{1-b}$	$\frac{a}{a-b}$	$[X : \frac{-1}{b}(X+Y) : \sqrt[5]{\frac{(b-1)(b-a)}{b^3}}Z]$	4
10	$\frac{a}{a-b}$	$\frac{1}{1-b}$		3
11	$\frac{a-1}{a}$	$\frac{(1-a)b}{(1-b)a}$	$[X : -X + (\frac{1-a}{a})Y : \sqrt[5]{\frac{-(1-a)^2(1-b)}{a}}Z]$	3
12	$\frac{(1-a)b}{(1-b)a}$	$\frac{a-1}{a}$		4
13	$\frac{b-1}{b}$	$\frac{(1-b)a}{(1-a)b}$	$[X : -X + (\frac{1-b}{b})Y : \sqrt[5]{\frac{-(1-b)^2(1-a)}{b}}Z]$	4
14	$\frac{(1-b)a}{(1-a)b}$	$\frac{b-1}{b}$		3
15	$1-a$	$\frac{(a-1)b}{a-b}$	$[X : \frac{-1}{a}(X+(1-a)Y) : \sqrt[5]{\frac{-(1-a)^2(a-b)}{a^3}}Z]$	2
16	$\frac{(a-1)b}{a-b}$	$1-a$		3
17	$\frac{b-a}{b}$	$\frac{a-b}{b(a-1)}$	$[X : \frac{-1}{a}(X+(\frac{b-a}{a})Y) : \sqrt[5]{\frac{(1-a)(a-b)^2}{a^4}}Z]$	2
18	$\frac{a-b}{b(a-1)}$	$\frac{b-a}{b}$		4
19	$1-b$	$\frac{(1-b)a}{a-b}$	$[X : \frac{-1}{b}(X+(1-b)Y) : \sqrt[5]{\frac{(1-b)^2(a-b)}{b^3}}Z]$	3
20	$\frac{(1-b)a}{a-b}$	$1-b$		2
21	$\frac{a-b}{a}$	$\frac{b-a}{a(b-1)}$	$[X : \frac{-1}{b}(X+(\frac{a-b}{b})Y) : \sqrt[5]{\frac{(1-b)(a-b)^2}{b^4}}Z]$	4
22	$\frac{b-a}{a(b-1)}$	$\frac{a-b}{a}$		2
23	$\frac{1}{a}$	$\frac{1}{b}$	$[Y : X : \sqrt[5]{ab}Z]$	2
24	$\frac{1}{b}$	$\frac{1}{a}$		2
25	a	$\frac{a}{b}$	$[Y : \frac{1}{a}X : \sqrt[5]{\frac{b}{a^3}}Z]$	2
26	$\frac{a}{b}$	a		6
27	b	$\frac{b}{a}$	$[Y : \frac{1}{b}X : \sqrt[5]{\frac{a}{b^3}}Z]$	6
28	$\frac{b}{a}$	b		2
29	$\frac{a-1}{a}$	$\frac{b-1}{b}$	$[Y : -(X+Y) : \sqrt[5]{ab}Z]$	3
30	$\frac{b-1}{b}$	$\frac{a-1}{a}$		6
31	$1-a$	$\frac{b-a}{b}$	$[Y : \frac{-1}{a}(X+Y) : \sqrt[5]{\frac{b}{a^3}}Z]$	4
32	$\frac{b-a}{b}$	$1-a$		5
33	$1-b$	$\frac{a-b}{a}$	$[Y : \frac{-1}{b}(X+Y) : \sqrt[5]{\frac{a}{b^3}}Z]$	5
34	$\frac{a-b}{a}$	$1-b$		4
35	$\frac{a}{a-1}$	$\frac{a(1-b)}{b(1-a)}$	$[Y : \frac{1-a}{a}X - Y : \sqrt[5]{(ab)(\frac{a-1}{a})^4}Z]$	4
36	$\frac{a(1-b)}{b(1-a)}$	$\frac{a}{a-1}$		5
37	$\frac{b}{b-1}$	$\frac{b(1-a)}{a(1-b)}$	$[Y : \frac{1-b}{b}X - Y : \sqrt[5]{(ab)(\frac{b-1}{b})^4}Z]$	5
38	$\frac{b(1-a)}{a(1-b)}$	$\frac{b}{b-1}$		4
39	$\frac{1}{1-a}$	$\frac{b-a}{b(1-a)}$	$[Y : \frac{1}{a}((a-1)X - Y) : \sqrt[5]{\frac{b(1-a)^4}{a^3}}Z]$	3
40	$\frac{b-a}{b(1-a)}$	$\frac{1}{1-a}$		4
41	$\frac{b}{b-a}$	$\frac{b(a-1)}{a-b}$	$[Y : \frac{1}{ab}((a-b)X - bY) : \sqrt[5]{\frac{(a-b)^4}{(ab)^3}}Z]$	6
42	$\frac{b(a-1)}{a-b}$	$\frac{b}{b-a}$		5
43	$\frac{1}{1-b}$	$\frac{a-b}{a(1-b)}$	$[Y : \frac{b-1}{b}(X + (\frac{1}{1-b})Y) : \sqrt[5]{\frac{a(1-b)^4}{b^3}}Z]$	4
44	$\frac{a-b}{a(1-b)}$	$\frac{1}{1-b}$		3

Table C.2: Isomorphic fibers (continued)

	a'	b'	ϕ_{σ_i}	$O(\sigma_i : (a,b) \mapsto (a',b'))$
45	$\frac{a}{a-b}$	$\frac{a(1-b)}{a-b}$	$[Y : \frac{b-a}{ab}(X + (\frac{a}{a-b})Y) : \sqrt[5]{\frac{(a-b)^4}{(ab)^3}}Z]$	5
46	$\frac{a(1-b)}{a-b}$	$\frac{a}{a-b}$		6
47	$\frac{1}{1-a}$	$\frac{1}{1-b}$	$[X + Y : -X : \sqrt[5]{(a-1)(1-b)}Z]$	3
48	$\frac{1}{1-b}$	$\frac{1}{1-a}$		6
49	$\frac{a}{a-1}$	$\frac{a}{a-b}$	$[X + Y : \frac{-1}{a}X : \sqrt[5]{\frac{(1-a)(a-b)}{a^3}}Z]$	4
50	$\frac{a}{a-b}$	$\frac{a}{a-1}$		5
51	$\frac{b}{b-a}$	$\frac{b}{b-1}$	$[X + Y : \frac{-1}{b}X : \sqrt[5]{\frac{(1-b)(b-a)}{b^3}}Z]$	4
52	$\frac{b}{b-1}$	$\frac{b}{b-a}$		5
53	$1-a$	$1-b$	$[X + Y : -Y : \sqrt[5]{-1}Z]$	2
54	$1-b$	$1-a$		2
55	$\frac{a-1}{a}$	$\frac{a-b}{a}$	$[X + Y : \frac{-1}{a}Y : \sqrt[5]{\frac{-1}{a}}Z]$	3
56	$\frac{a-b}{a}$	$\frac{a-1}{a}$		4
57	$\frac{b-1}{b}$	$\frac{b-a}{b}$	$[X + Y : \frac{-1}{b}Y : \sqrt[5]{\frac{-1}{b}}Z]$	4
58	$\frac{b-a}{b}$	$\frac{b-1}{b}$		3
59	$\frac{b-a}{a(b-1)}$	$\frac{1}{a}$	$[X + Y : -(X + \frac{1}{a}Y) : \sqrt[5]{\frac{(a-1)^2(1-b)}{a}}Z]$	5
60	$\frac{1}{a}$	$\frac{b-a}{a(b-1)}$		4
61	$\frac{1}{b}$	$\frac{a-b}{b(a-1)}$	$[X + Y : -(X + \frac{1}{b}Y) : \sqrt[5]{\frac{(b-1)^2(1-a)}{b}}Z]$	5
62	$\frac{a-b}{b(a-1)}$	$\frac{1}{b}$		4
63	a	$\frac{a(b-1)}{b-a}$	$[X + Y : \frac{-1}{a}X - Y : \sqrt[5]{\frac{(a-1)^2(a-b)}{a^3}}Z]$	2
64	$\frac{a(b-1)}{b-a}$	a		6
65	$\frac{a(1-b)}{b(1-a)}$	$\frac{a}{b}$	$[X + Y : -(\frac{1}{a}X + \frac{1}{b}Y) : \sqrt[5]{\frac{(a-1)^2(a-b)}{a^3b}}Z]$	5
66	$\frac{a}{b}$	$\frac{a(1-b)}{b(1-a)}$		6
67	$\frac{b(a-1)}{a-b}$	b	$[X + Y : \frac{-1}{b}X - Y : \sqrt[5]{\frac{(b-1)^2(b-a)}{b^3}}Z]$	2
68	b	$\frac{b(a-1)}{a-b}$		6
69	$\frac{b}{a}$	$\frac{b(1-a)}{a(1-b)}$	$[X + Y : -(\frac{1}{b}X + \frac{1}{a}Y) : \sqrt[5]{\frac{(b-1)(b-a)^2}{ab^3}}Z]$	5
70	$\frac{b(1-a)}{a(1-b)}$	$\frac{b}{a}$		6
71	$1-a$	$\frac{a-1}{b-1}$	$[X + (1-a)Y : -X : \sqrt[5]{(1-a)^2(b-1)}Z]$	4
72	$\frac{a-1}{b-1}$	$1-a$		5
73	$\frac{b-1}{a-1}$	$1-b$	$[X + (1-b)Y : -X : \sqrt[5]{(a-1)(1-b)^2}Z]$	4
74	$1-b$	$\frac{b-1}{a-1}$		5
75	$\frac{a-1}{a}$	$\frac{a-1}{a-b}$	$[X + \frac{a-1}{a}Y : \frac{-1}{a}X : \sqrt[5]{\frac{(a-1)^2(b-a)}{a^4}}Z]$	3
76	$\frac{a-1}{a-b}$	$\frac{a-1}{a}$		4
77	$\frac{a-b}{a}$	$\frac{a-b}{a-1}$	$[X + \frac{a-b}{a}Y : \frac{-1}{a}X : \sqrt[5]{\frac{(1-a)(a-b)^2}{a^4}}Z]$	6
78	$\frac{a-b}{a-1}$	$\frac{a-b}{a}$		5
79	$\frac{b-1}{b-a}$	$\frac{b-1}{b}$	$[X + \frac{b-1}{b}Y : \frac{-1}{b}X : \sqrt[5]{\frac{(b-1)^2(a-b)}{b^4}}Z]$	3
80	$\frac{b-1}{b}$	$\frac{b-1}{b-a}$		4
81	$\frac{b-a}{b-1}$	$\frac{b-a}{b}$	$[X + \frac{b-a}{b}Y : \frac{-1}{b}X : \sqrt[5]{\frac{(1-b)(b-a)^2}{b^4}}Z]$	6
82	$\frac{b-a}{b}$	$\frac{b-a}{b-1}$		5
83	a	$\frac{a-b}{1-b}$	$[X + aY : -(X + Y) : \sqrt[5]{(1-a)^2(1-b)}Z]$	2
84	$\frac{a-b}{1-b}$	a		6
85	$\frac{b-a}{1-a}$	b	$[X + bY : -(X + Y) : \sqrt[5]{(1-a)(1-b)^2}Z]$	2
86	b	$\frac{b-a}{1-a}$		6

Table C.3: Isomorphic fibers (continued)

	a'	b'	ϕ_{σ_i}	$O(\sigma_i : (a,b) \mapsto (a',b'))$
87	$\frac{1-b}{a-b}$	$\frac{1}{a}$	$[X+\frac{1}{a}Y : \frac{-1}{a}(X+Y) : \sqrt[5]{\frac{(1-a)^2(a-b)}{a^4}}Z]$	5
88	$\frac{1}{a}$	$\frac{1-b}{a-b}$		4
89	$\frac{b}{a}$	$\frac{b-1}{a-1}$	$[X+\frac{b}{a}Y : \frac{-1}{a}(X+Y) : \sqrt[5]{\frac{(a-1)(a-b)^2}{a^4}}Z]$	5
90	$\frac{b-1}{a-1}$	$\frac{b}{a}$		6
91	$\frac{1}{b}$	$\frac{a-1}{a-b}$	$[X+\frac{1}{b}Y : \frac{-1}{b}(X+Y) : \sqrt[5]{\frac{(1-b)^2(b-a)}{b^4}}Z]$	5
92	$\frac{a-1}{a-b}$	$\frac{1}{b}$		4
93	$\frac{a-1}{b-1}$	$\frac{a}{b}$	$[X+\frac{a}{b}Y : \frac{-1}{b}(X+Y) : \sqrt[5]{\frac{(b-1)(b-a)^2}{b^4}}Z]$	5
94	$\frac{a}{b}$	$\frac{a-1}{b-1}$		6
95	$\frac{1}{1-a}$	$\frac{1-b}{1-a}$	$[X+\frac{Y}{1-a} : \frac{Y}{a-1} : \frac{1}{\sqrt[5]{a-1}}Z]$	3
96	$\frac{1-b}{1-a}$	$\frac{1}{1-a}$		4
97	$\frac{1}{1-b}$	$\frac{a-1}{b-1}$	$[X+\frac{Y}{1-b} : \frac{Y}{b-1} : \frac{1}{\sqrt[5]{b-1}}Z]$	4
98	$\frac{a-1}{b-1}$	$\frac{1}{1-b}$		3
99	$\frac{a-b}{a-1}$	$\frac{a}{a-1}$	$[X+\frac{aY}{a-1} : \frac{Y}{1-a} : \frac{1}{\sqrt[5]{1-a}}Z]$	3
100	$\frac{a}{a-1}$	$\frac{a-b}{a-1}$		2
101	$\frac{a}{a-b}$	$\frac{a-1}{a-b}$	$[X+\frac{aY}{a-b} : \frac{Y}{b-a} : \frac{1}{\sqrt[5]{b-a}}Z]$	2
102	$\frac{a-1}{a-b}$	$\frac{a}{a-b}$		4
103	$\frac{b}{b-1}$	$\frac{a-b}{1-b}$	$[X+\frac{bY}{b-1} : \frac{Y}{1-b} : \frac{1}{\sqrt[5]{1-b}}Z]$	3
104	$\frac{a-b}{1-b}$	$\frac{b}{b-1}$		2
105	$\frac{1-b}{a-b}$	$\frac{b}{b-a}$	$[X+\frac{bY}{b-a} : \frac{Y}{a-b} : \frac{1}{\sqrt[5]{a-b}}Z]$	2
106	$\frac{b}{b-a}$	$\frac{1-b}{a-b}$		4
107	$\frac{b-1}{b-a}$	$\frac{a(1-b)}{a-b}$	$[X+\frac{a(1-b)Y}{a-b} : -X+\frac{(b-1)Y}{a-b} : \sqrt[5]{\frac{(1-b)^2(1-a)^2}{a-b}}Z]$	5
108	$\frac{a(1-b)}{a-b}$	$\frac{b-1}{b-a}$		6
109	$\frac{b(1-a)}{b-a}$	$\frac{a-1}{a-b}$	$[X+\frac{b(1-a)Y}{b-a} : -X+\frac{(a-1)Y}{b-a} : \sqrt[5]{\frac{(1-b)^2(1-a)^2}{b-a}}Z]$	5
110	$\frac{a-1}{a-b}$	$\frac{b(1-a)}{b-a}$		6
111	$\frac{a-b}{a(1-b)}$	$\frac{a-b}{1-b}$	$[X+\frac{a-b}{a(1-b)}Y : \frac{-1}{a}X+\frac{b-a}{a(1-b)}Y : \sqrt[5]{\frac{(a-b)^2(a-1)^2}{a^4(1-b)}}Z]$	5
112	$\frac{a-b}{1-b}$	$\frac{a-b}{a(1-b)}$		6
113	$\frac{1-a}{1-b}$	$\frac{b(1-a)}{a(1-b)}$	$[X+\frac{b(a-1)}{a(b-1)}Y : \frac{-1}{a}X+\frac{1-a}{a(b-1)}Y : \sqrt[5]{\frac{(a-b)^2(a-1)^2}{a^4(b-1)}}Z]$	2
114	$\frac{b(1-a)}{a(1-b)}$	$\frac{1-a}{1-b}$		4
115	$\frac{b-a}{1-a}$	$\frac{b-a}{b(1-a)}$	$[X+\frac{b-a}{b(1-a)}Y : \frac{-1}{b}X+\frac{a-b}{b(1-a)}Y : \sqrt[5]{\frac{(b-a)^2(b-1)^2}{b^4(1-a)}}Z]$	5
116	$\frac{b-a}{b(1-a)}$	$\frac{b-a}{1-a}$		6
117	$\frac{a(1-b)}{b(1-a)}$	$\frac{1-b}{1-a}$	$[X+\frac{a(b-1)}{b(a-1)}Y : \frac{-1}{b}X+\frac{1-b}{b(a-1)}Y : \sqrt[5]{\frac{(b-a)^2(b-1)^2}{b^4(a-1)}}Z]$	2
118	$\frac{1-b}{1-a}$	$\frac{a(1-b)}{b(1-a)}$		4
119	a	b	$diag(1, 1, \sqrt[5]{-1})$	1
120	b	a		2

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