Auctions, Mechanisms and Uncertainty

by

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Szüleimnek.

Contents

Chapter 1

Introduction

Mechanisms through which individuals interact may have important impact on the outcomes of this interaction. The economic theory of mechanism design is concerned with the design of social decision procedures for situations in which economic agents own relevant private information and behave, use it, strategically.

As an example, consider the case in which the central authority of a country is studying the possibility of declaring national reserve a given geographic area. In order to come up with the optimal decision, that for instance maximizes social welfare, it should be conditioned on the related information owned by cities, states, or individuals. They might be asked directly for their opinion on the underlying problem, but will not report their information truthfully unless proper incentives are given to them through monetary transfers or some other instruments controlled by the authority. In other words, mechanism design theory is concerned with the harmonization of incentives that must be applied to a set of agents that interact in order to get those agents to exhibit some desired behavior, i.e. in order the schemes to work as intended. The central authority, or social planner, of this example who acts on behalf of the whole society can also be replaced by an imaginary social goal or by a principal who is pursuing his own interest.

The formalization of this problem can be find in the seminal work by Hurwicz (1972). Nevertheless, one of the first applications that can be considered as from the theory of mechanism design is due to Hayek who started to study the limitations on the amount of information that central planners can acquire in the early 1920s. He considered a large scale problem focusing his attention on the free market mechanism. He fiercely opposed to the socialist system from every angle and described the main problem in his 1945 paper as follows:

> "If we possess all the relevant information, if we can start out from a given system of preferences, and if we command complete knowledge of available means, the problem which remains is purely one of logic. [...] The peculiar character of the problem of a rational economic order is determined precisely by the fact that the knowledge of the circumstances of which we must make use never exists in concentrated or integrated form but solely as the dispersed bits of incomplete and frequently contradictory knowledge which all the separate individuals possess."

A similar application is the design of a constitution that determines the actions that agents may take (strategy space) and the electoral rules that transform votes into decisions (outcome function). Along with the literature on the ways of reducing market failures, on optimal taxation and public good theory, the design of auctions is also subject of the field of mechanism design.

This thesis dissertation is divided into three chapters that present self-contained studies of economic situations in which private information, uncertainty, plays an important role. In deriving the results game theoretic tools and the approach taken by the mechanism design literature are used. The following part of the Introduction is devoted to highlight more details about the main insights of each chapter and to locate them in the literature.

1.1 Multibidding Game under Uncertainty

Economic agents often have to take a common decision, or choose a joint project, in situations where their preferences may be very different from one another. Imagine, for instance, that a noxious recycling center has to be built according to some political plan and the final decision is to choose between two geographic areas (projects) taking into account its implications on social welfare. The recycling center may not only affect the population in its host town, but a larger set of people at the same time as it might influence social welfare across state and country borders. In general, one can consider any number of affected parties whose well-being depends on the decision in question. In this situation there exists a natural tendency to exaggerate the positive/negative consequences of the projects and agents try to free ride.

Pérez-Castrillo and Wettstein (2002) address this type of problems in environments where agents have symmetric information about everybody's preferences. They propose a simple one-stage multibidding mechanism in which agents bid for the projects. The mechanism determines both the project to be implemented and a system of budget-balanced transfer payments to possibly compensate those who are not pleased with the chosen project. Pérez-Castrillo and Wettstein (2002) show that the multibidding mechanism always generates an efficient decision in Nash (and strong Nash) equilibrium. In this chapter I propose the use of the multibidding game and study its theoretical properties in economies in which agents withhold important information related to the problem in focus at the moment of decision making.

The consequences of the switch between the certain and the uncertain case are often surprisingly unexpected. It is shown with numerous examples by McAffe (1992) who studies simple mechanism. Except for efficiency the attractive features of the multibidding game, such as its simplicity, budget-balancedness, a special kind of individual rationality and incentive compatibility, are immune to private information. Considering risk neutral players, two alternative projects and a continuum of possible private valuations, I show that the multibidding game under uncertainty (in its symmetric Bayes-Nash equilibria) is always efficient in the two-player case if the prior belief distributions are symmetric or players are asymmetric[1], while efficiency is tied to more conditions when there are more players. Namely, the number of agents must be large or (with a similar intuition behind) uncertainty must be large with zero expected value, in order to achieve efficient outcomes. This analysis introduces the bidding game into the group of simple mechanisms for public decision making considered by McAfee (1992).

1.2 Multibidding Game Experiment

As the step to a situation with uncertainty (from another without) may result in drastic changes in the features of a mechanism, the step from theory to laboratory may not be less interesting. Nevertheless, the implementation literature has paid little attention to empirical evidences yet. It has generated a large number of

[1]The asymmetry of agents refers to the case in which agent prefer different projects and their valuations for the preferred project follow the same probabilistic distribution.

mechanisms that have to be evaluated, compared and ranked through empirical work. There is an ample room for tests and experiments. Even if the existing empirical results mainly present hurdles for implementation theory, they can help to induce some feedback and add guidance to it. With their help one can check whether usual criticisms of unnatural features of mechanism are supported or not by empirical evidence. As the first on this paths, Cabrales, Charness and Corchón (1998) present an experiment on the canonical mechanism by Maskin (1999) that focuses on two of these unnatural features, namely integer games and the existence of mixed strategy equilibria.

Chapter 3, a joint work with David Pérez-Castrillo, follows this path and reports experimental results from the laboratory on the multibidding game under uncertainty. Confident in its simple rules and its theoretical equilibrium properties explored in Chapter 2, we took the multibidding game into the experimental laboratory. Chapter 3 reports empirical results based on the data we gathered across four treatments. With the help of computers our subjects were assigned random private valuations, were grouped and were asked to make a joint decision over a public project and its alternative using the multibidding game (in its theoretical form without any modification). We find that it succeeds in extracting private information from agents, though not all participants followed the Bayes-Nash equilibrium predicted by theory. The mechanism gave rise to ex post efficient outcomes in almost 3/4 of the cases across the treatments. Apart from the expected utility maximizing Bayes-Nash behavior we could identify bidding behavior according to the safe maximin strategies in one of our sessions.

Next to our encouraging experimental results, it is important to point out that a considerable fraction of participants bid (significantly) less aggressively than expected in theory. Since they did well in monetary terms among all participants and did not harm the ex post efficiency of the common choice, we suggest to obtain theoretical results for economies in which there are several groups (types) of agents: some play maximin strategies, some Bayes-Nash, etc. Beside the expansion of theoretical work on the multibidding game, undoubtedly also more empirical research is needed to explore its empirical performance.

1.3 Fairness under Uncertainty

The fourth chapter of the thesis contains theoretical work dealing with implementation problems and mechanisms, and it brings fairness considerations into spotlight.[2] I adopt the definition of fairness based on the notions of Pareto efficiency and envy-freeness. In the economies studied here a set of indivisible objects is to be distributed to a group of agents such that individuals consume at most one object: for example the empty rooms of a flat rented jointly by a group of students. In general envy-free allocations might not exist, but when a proper amount of perfectly divisible good - typically money - is available in the economy the set of envy-free allocations is not empty and indeed can be quite large. In my example money is a natural feature that allows for paying and sharing the rent. Alkan, Demange, Gale (1991) and Aragonés (1995) study these economies, the existence of envy-free allocations and how the amount of the divisible good affects the existence results. It is shown that for a sufficiently large amount of money the set of envy-free allocations is not empty. It is well known that in this environment envy-freeness implies Pareto efficiency and therefore envy-free allocations can be considered as fair ones,

[2]Mechanisms often fail to have a unique equilibrium and the usual approach in mechanism design does not account for all equilibria. It is the implementation literature that keeps track of this problem: if the equilibrium outcomes of a mechanism coincide with the outcomes of the social choice correspondence, then we say that the mechanism implements it. Check Jackson (2001a) and (2002b) for more.

too. Also some nice features of the envy-free set are proper to the indivisible case, as for example its lattice structure.

This chapter of the thesis continues studies a model similar to the above described one introducing uncertainty to it. A distinction is made among ex-ante, interim and ex-post stages and according to that different envy-free, efficiency and fairness notions are defined. The (most restrictive) intersection between the ex-ante Pareto optimal and ex-post envy-free sets is particularly interesting and is considered ex-ante intertemporally fair. Moulin (1997) point out that fairness from an ex-ante point of view can be seen as a concept of procedural justice, while ex-post fairness can be interpreted as endstate justice that deals with a particular utility or judgement profile and a particular endstate in a given state of the nature. The definition used here takes into account both judgement concepts.

In general, little is known about the generalization of the fairness literature to environments with uncertainty. This is the reason why before proceeding to general implementation matters existence results on fairness are derived and the structure of the fair set is explored. I deliver a necessary and sufficient condition for the non-emptiness of the set of ex-ante intertemporally fair social choice functions. A further section discusses implementation matters. Relying on results in Palfrey and Srivastava (1987), for implementability the condition of non-exclusive information is introduced and a mechanism is defined that implements the set of non-wasteful ex-post envy-free social choice functions in Bayes-Nash equilibrium.

Considering the above described part of the chapter, it is a self-contained study based on the axiomatically accepted notion of intertemporal fairness that embodies envy-freeness. The literature on distributive justice usually follows a similar path and does not deal with the problematic of choosing fairness criteria. However, I include an extra section that considers the aspiration function as an appropriate tool for studying fairness without restricting our attention on a particular concept. Corchón and Iturbe-Ormaetxe (2001) offers a detailed study of fairness in a generalized set-up. Our results here can be seen as the adaptation of some very few definitions from Corchón and Iturbe-Ormaetxe (2001) to the uncertainty case with indivisibilities. The most important point in that part of the chapter is the generalization of the existence result. Under the conditions stated for the envy-free case, and under some restrictions on personal aspirations, it is shown that an intertemporally fair social choice function exists. I conclude with a positive result: a necessary condition (on the fairness concept) is derived for Bayesian monotonicity, i.e. for Bayesian implementation of the set of the generalized intertemporally fair social choice functions.

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Chapter 2

Multibidding Game under Uncertainty

2.1 Introduction

The presence of external effects and/or public goods in the economy makes the market mechanism unreliable for allocating resources efficiently. Inefficiency appears in the form of unexploited gains that can be eliminated by side payments and rearrangements in the distribution of goods. However, it is usually unclear which mechanism to use for implementing the suggested improvements. In the present paper I study situations in which externalities and/or public goods exist and the members of the society hold important private information related to the problem that is undisclosed to the others. I propose the use of a concrete mechanism for the considered family of problems and argue that with it, under some conditions, efficiency as social goal can be achieved. Let us first see an example of the type of situations that form this family.

Imagine that a noxious recycling center has to be built according to some political plans. The feasibility studies have already identified two potential areas that are suitable for hosting the site. The decision to be made by the government is to choose between these two areas (projects) trying to take into account its implications on social welfare. In particular, the government's goal is to locate the recycling center where its aggregate positive(/negative) impact is the highest(/lowest). Supposing that parties hold private information (private valuations) on the effects of the recycling center, it is in the best interest of the authority to find out as much as possible about individual private valuations. In order to do so, it can force the affected parties to take part in a *procedure* or mechanism that may make reduce the informational asymmetries.

As for the impact of the site on their surroundings, one can consider the following two scenarios: In the first one the recycling center only affects people in its closest area, i.e. in the settlement that is located closest to it. This reduces the number of interested parties in the problem to two (plus the central government whose unique objective is to reach a socially efficient decision) and causes positive or negatives changes in the welfare of at most two parties. In the second possible scenario the recycling center not only affects the population in its host town, but a larger set of people at the same time as it might influence social welfare across state and country borders. Because of the different nature of the problem the cases in which there are two and more than two parties will be discussed separately.

Problems of the type described above have already been analyzed in the literature. Under complete information, when parties have precise information on how the others value the projects, the multibidding game proposed and studied by Pérez-Castrillo and Wettstein (2002) can be used efficiently. Without formal definitions, in cases of choices between two projects this mechanism operates as follows:

- Strategies: each participant (each of the affected parties) announces two bids, one for each of the available projects such that these bids sum up to zero.
- Outcomes: the planner sums the bids for every project and chooses the project with the highest aggregate bid as the winner. In case of a tie some device is used to choose the winner among the projects with the highest aggregate bid. The winner project is carried out, the bids related to it are paid and the surplus (the aggregated bid) is shared among all the agents in equal parts.

Note that the mechanism has a unique (bidding) stage and each agent is asked to bid for all the available projects. Besides each agent is forced to pay her bid given for the project that has been chosen winner. Since the revenue raised by the bidding is given back entirely to participants in equal shares, the multibidding game is budget-balanced. In the complete-information setting Pérez-Castrillo and Wettstein (2002) showed that in every Nash equilibrium of the bidding the winning project is efficient, and that any Nash equilibrium of the multibidding mechanism is also a strong Nash equilibrium. For its appealing properties under complete information, its simplicity and feasibility in a wide range of problems, I propose the use of the multibidding mechanism under uncertainty; i.e. incomplete information.

In this paper, I study how the *multibidding mechanism* performs when agents hold private information and are uninformed about others' preferences. I consider ex ante identical risk neutral players and a continuum of possible private valuations, i.e. the continuous case, and study the theoretical properties of the multibidding mechanism with two alternatives. By its definition the mechanism is safe both to run and to participate, because it is budget-balanced and individually rational once supposed that agents can not escape from the effects of the chosen public project.

In the multibidding game bids must sum up to zero for every participant. This feature aims at extracting individual private information on the relative valuations between the projects. The mechanism succeeds in it, as at the symmetric Bayes-Nash equilibria participants' bids depend on the difference between private valuations for the alternatives. The equilibrium bidding function is strictly increasing and continuos. Its curvature is determined by the underlying uncertainty that also involves the number of agents.

I show that under uncertainty the multibidding mechanism is always efficient in the two-player two-project case if the prior distributions are symmetric or players are antagonistically asymmetric[1], while efficiency is tied to more conditions when there are more players. Namely, the number of agents must be large or - with a similar intuition behind - uncertainty must be large with zero expected value, in order to achieve efficient outcomes.

The two-player, two-project case has been widely analyzed in the auction literature. McAfee (1992) studies simple mechanisms, explores their properties under uncertainty and presents results for an environment with constant absolute risk aversion. He finds that the winner's bid auction reaches (allocative) efficiency in the chosen set-up. As for the multibidding mechanism, it is important to point out that private valuations are now attached to projects and not only to the object in question. This feature makes the model a slightly more general in this aspect even in the two-agent case. Normally, both parties are eager to win the object and feel bad if it is their opponent who does so. Normalization of payoffs can get us back to the situation studied in the auction literature where players receive zero pay-off when not winning the auction. There also exist problems in which the ob-

[1]Symmetry of distibution means symmetry of the density function around zero. Asymmetry of players refers to situations in which players tend to prefer different projects and form prior beliefs in the opposite way. That is player 1 is identical to player 2 with switched project labels.

ject is bad and both wish that the other one will get it. These situations can be efficiently coped with using, for example, a first-price sealed-bid auction with the proper definition for *bids* and *winner*. However one might imagine situations in which agents share the same opinion and, for instance, both wish that agent 1 get the object. Under these circumstances the first-price sealed-bid auction is a feasible mechanism once we generalize it, allowing for both negative and positive bids. The multibidding mechanism can be used without modification in this case, too.

An important part of the environments considered here has been studied in the literature that deals with the problem of siting noxious facilities. Several sealed-bid mechanisms have been proposed for the problem. The first to suggest an auction to this situation were Kunreuther and Kleindorfer (1986). They showed that outcomes realized by min-max strategies in a low-bid auction are efficient as long as the non-hosting participants are indifferent between all outcomes. For the case of two cities, O'Sullivan (1993) proved that symmetric Bayes-Nash equilibria of the modified low bid game[2] yield an efficient outcome when private valuations are independently drawn. He argues that min-max strategies deliver problematic equilibria in which beliefs may be inconsistent. The rationality of participation, however, is conditional on the compensation for the host city.

Ingberman (1995) analyzed the siting problem with costs depending on the distance from the noxious site and using a majority vote approach. He concluded that decisions reached in this manner would not be efficient, as markets would produce an excessive number of noxious facilities and place them in the wrong sites. Rob (1989) modelled the problem between a pollution-generating firm and the residents as a mechanism design approach for the siting problem. Notice that

[2] It is a voluntary auction under which the city submitting the low bid hosts the region's noxious facility and receives the high bid as compensation.

my model is different in that I suppose that the planner is not interested in revenue-raising. There is a binary decision to be made (accept or reject the construction of a pollution-generating plant) and compensatory payments should be determined. The outcomes of the resulting mechanism are sometimes inefficient. In contrast to the equilibrium outcomes of the multibidding mechanism inefficiencies become rampant when there are many residents affected by pollution and the degree of uncertainty is large.

Jehiel et al. (1996) analyzed a similar model in which external effects appear as the value of a project to an agent depends on the identity of who carries it out. Their setup includes a seller who wants to sell an object to one of n agents and they characterize the individually rational and incentive compatible mechanisms that maximize the seller's revenue. Revenue maximization is not in my interest in this paper and there are other important assumptions that I do not make. For example, in Jehiel et al. (1996) agents not only know their own valuation, but also the externality they impose on other players.

The well-known Vickrey-Clarke-Groves mechanisms are designed for similar problems, to choose a public project to carry out, under uncertainty and for them truthtelling is a dominant strategy. Therefore, these mechanisms result in efficient outcomes, however they are not budget-balanced. The surplus generated by payments is a loss for the agents.

D'Aspremont and Gérard-Varet (1979) proposed a mechanism that works in a public good set-up under uncertainty with independent types. That mechanism works similarly to the Vickrey-Clarke-Groves schemes, but it substitutes dominant strategy incentive compatibility with Bayesian incentive compatibility. This helps to overcome budget-balance problems and still ex post efficiency is guaranteed. However, there still exists a problematic issue, namely the one of voluntary participation or individual rationality that cannot be reached with the proposed mechanism in their set-up.[3]

The rest of this paper is organized as follows. The next section introduces the mechanism formally and starts studying its theoretical properties with symmetric underlying distributions modelling uncertainty. The analysis is done separately in different sections for the two-player and n-player case because of the differences in the techniques and results. I comment on the consequences of asymmetric distributions in Section 5, and relate the multibidding game to a special problem that frequently arises in the literature: a dissolving partnership. Section 6 concludes. Proofs are presented in the appendix.

2.2 Multibidding game under uncertainty

Consider a set of alternatives $P = \{1, 2\}$ and a set of risk neutral agents $N = \{1, \ldots, i, \ldots, n\}$ whose utility depends on the alternative carried out. I shall denote by $x_i^j \in X \subset \mathbb{R}$ the utility that player i enjoys when project j is the winning project. These values are private information and will be treated as random draws from some underlying common distribution with density $f_{x^j}(x)$ and cumulative distribution function $F_{x^j}(x)$. Agents are identical ex ante, i.e. these functions do not vary across agents, but may do so across projects. I also make the usual assumption of these being common knowledge. The variables x_i^j are considered as continuous random variables here, though my results apply also in the discrete case with the proper adaptation of the concepts to the discrete environment.

A mechanism is called *ex post efficient* if it picks out efficient projects for every

[3] A more detailed review on the topic including the Vickrey-Clarke-Groves mechanisms can be found in Jackson (2001).

possible private valuation profile. Project j is (ex post Pareto) efficient if $\sum_{i\in N} x_i^j \geq \sum_{i\in N} x_i^k$ for all $k \in P$. With this, the social planner's objective is identified.

The *multibidding mechanism* can be formally defined as follows:

In the unique stage of the game agents simultaneously submit a vector of two real numbers, one for each available project, that sum up to zero. These numbers are called bids and B_i^j denotes agent i's bid for project j.

The project with the highest aggregated bid is chosen winner, where the aggregated bid B_N^j for project j is defined as $B_N^j = \sum_{i\in N} B_i^j$. In case of a tie, it is randomly selected.

Once chosen, the winning project is carried out and agents enjoy the utility that it delivers. They also must pay/receive their bids submitted for the winning project and they are returned the aggregated winning bid in equal shares. For example, if project j has obtained the largest aggregated bid then player i receives the following pay-off:

$$V_i^j = x_i^j - B_i^j + \frac{1}{n} B_N^j.$$

Note that, since by the rules of the multibidding game $B_i^1 = -B_i^2$ must hold for every i, bids may be negative, but the aggregated winning bid B_N^j is always non-negative.

The multibidding game achieves *budget balance* by construction, because the raised revenue by bids is entirely given back to participants. The social planner or some central authority does not need any positive or negative amount of money to operate it, therefore it is safe.

The other properties of the mechanism are studied assuming that agents behave strategically and form their bids as to maximize their expected payoff based on the information available to them. They being ex ante identical, the symmetric Bayes-Nash equilibria (SBNE) of the game are considered. Therefore, the bid for a given project j is represented by $B^j\left(x_i^1, x_i^2\right)$ as a function of the personal characteristics whose form does not depend on the identity of the player. The expected utility for player i is defined as the expected value of V_i^j. The bidding function that maximizes players' expected utility will be called *optimal*.

Since submitted bids must add up to zero, agents are forced to report on their relative preferences between the two projects. The optimal bidding behavior of agents taking part in the multibidding game satisfies an appealing and intuitive property: it depends only on the difference between their private valuations for the two projects. That is, at equilibrium agents do report truthfully on their relative valuation of the projects.

Lemma 1 *In the SBNE of the multibidding game, the optimal bidding function depends only on the difference between private valuations for the two projects.*

Taking into account the result from Lemma 1, one can reformulate the problem at hand. For that, some more pieces of notation are needed. Let the difference between player i's private valuations be d_i with the following definition: $d_i = x_i^1 - x_i^2$. This new variable is random in general, since it is defined by the difference between two other random variables. Abusing a bit of the notation, denote its density by $f\left(d\right)$ and its cumulative distribution function by $F\left(d\right)$. Due to presentational considerations, first I study problems in which $f\left(d\right)$ is symmetric to the origin.[4] There does not appear any subindex on these objects, because they are common to every agent and correspond to a central variable.

[4] This assumption on the symmetry of the distribution is not crutial for all of my results, but makes explanations simpler. I comment on the consequencies of asymmetry in a separate section.

With the bidding function for project j being $B^j(d_i)$ for every player i, the payoff that player i receives if project j obtains the largest aggregated bid can be rewritten as:

$$V_i^j\left(x_i^j, d_1, \ldots, d_n, B^j\right) = x_i^j - B^j(d_i) + \frac{1}{n}\sum_{i\in N} B^j(d_i).$$

Player 1's expected utility, when she happens to value project 1 by x_1^1, d_1 utility units more than project 2, and bids as if this difference were of a value y_1, can be written in the following form:

$$\begin{aligned} & v_1\left(x_1^1, x_1^2, d_1, y_1, B^1, B^2\right) = \\ = & \int \cdots \int_{\substack{(d_2,\ldots,d_n)\text{ such that}\\ \text{project 1 wins}}} V_1^1\left(x_1^1, y_1, \ldots, d_n, B^1\right)\cdot f(d_2)\cdot \ldots \cdot f(d_n)\, dd_2 \ldots dd_n + \\ + & \int \cdots \int_{\substack{(d_2,\ldots,d_n)\text{ such that}\\ \text{project 2 wins}}} V_1^2\left(x_1^2, y_1, \ldots, d_n, B^2\right)\cdot f(d_2)\cdot \ldots \cdot f(d_n)\, dd_2 \ldots dd_n. \end{aligned}$$

For simplicity I shall write player i's expected utility as $v_i\left[x_i^1, d_i, B(y_i)\right]$, because x_i^1 and d_i give the individual valuations for both projects and by Lemma 1, given the bidding function, it is d_i that determines bids. Also, Lemma 1 combined with the complementarity of bids makes that a single function B can characterize the bidding behavior. This notation will be very helpful in the following analysis and for this reason let me reiterate the meaning of the above symbols. Player 1, exactly as the other $(n-1)$ players in the game, considers two possible results of the social decision procedure: either project 1 or project 2 will be carried out. The first one delivers x_1^1 units of utility to player 1 who must pay her bid, $B^1(y_1)$, for project 1 and will receive the nth part of the aggregated bid, $\frac{1}{n}\left[B^1(y_1) + \sum_{j\in N\setminus\{1\}} B^1(d_j)\right]$. Note that $B^1(y_1)$ can perfectly be a negative number, nevertheless I shall use the term *pay* when referring to monetary transactions according to bids. The expression for the expected utility involves $(n-1)$ integrals, because every agent is faced with the uncertainty captured by $(n-1)$ random variables, the differences between others' private valuations. The second term is to be interpreted in a similar way.

The characterization of the optimal bidding function can be enriched by some general results on its smoothness and increasing nature. The proof behind these intuitive facts uses standard arguments, to be found for example in Fudenberg and Tirole (1991), adapted to the multibidding game.

Lemma 2 *In the SBNE of the multibidding game, the optimal bidding function is continuous and strictly increasing.*

Thanks to the assumption on the symmetry of the underlying distribution, the optimal bidding function is also symmetric as it is shown in Lemma 3.

Lemma 3 *In the SBNE of the multibidding game, the optimal bidding function satisfies the following symmetry property:*

$$B^j(-d_i) = -B^j(d_i) \text{ for every } j \text{ and } d_i.$$

This result has a key role in deriving ex post efficient outcomes. It simplifies proofs and helps to compare the multibidding game to other mechanisms in the literature. Its impact is studied carefully in the next sections.

Taking into account the situations in the above enumerated examples, it seems natural to suppose that agents might abstain from participating in the bidding (decision making), but can not escape from the externalities, if such external effects exist. For example, villages and towns affected by the public project may wish not to exert influence on the choice of the project, but with this decision of their they accept both the positive and negative consequences of the others' decision. The multibidding mechanism, however, has another appealing property which assures that agents cannot do better by staying out of the decision making process.

Proposition 1 *The multibidding mechanism is individually rational.*

The intuition behind the above result is that not participation, as for bids and the collective choice of the project to be carried out, is equivalent to bidding zero. This bid, of course, will not be optimal in general. Moreover, the abstaining agent looses her part from the aggregated bid that is always non-negative in this mechanism.

Now, I start analyzing the efficiency properties of the multibidding mechanism with private information. For two players one can compute the explicit form of the optimal bidding function in the multibidding mechanism that is always efficient in the present set-up. If there are more than two players in the game, efficiency is not guaranteed in general. However the problem of inefficient decisions diminishes with a large number of players or a large degree of uncertainty.

2.3 The two-player case

Consider the situation in which a casino has to be located in one of two cities; and suppose that these cities have no precise information on how the other values the project of building the casino. When cities are asked individually for their preferences they have incentives to exaggerate and no to report it truthfully. The multibidding mechanism can help to overcome this problem in the decision making. In this example the following interpretation is given to the previously defined variables:

- Project i: city i builds the casino.
- The differences between private valuations, d_i, show how city i's utility changes when city 1 gets the right to build the casino. Let $B(d_i)$ denote the optimal bidding function determining city i's bid for project 1.

Note that if both x_1^1 and x_2^2 are positive, and $x_1^2 = x_2^1 = 0$ we are in the case in which a desired object has to be allocated between two agents who experience no regret or envy when loosing. I shall refer to this case as the *classical case.*[5]

Now city 1, that experiences x_1^1 and d_1, and bids according to some function B at point y_1, has to maximize the following expression:

$$v_1\left[x_1^1, d_1, B(y_1)\right] = \int_{B(y_1)+B(d_2)\geq 0} \left\{x_1^1 - B(y_1) + \frac{1}{2}\left[B(y_1) + B(d_2)\right]\right\} \cdot f(d_2)\, dd_2 +$$
$$+ \int_{B(y_1)+B(d_2)\leq 0} \left\{x_1^2 + B(y_1) - \frac{1}{2}\left[B(y_1) + B(d_2)\right]\right\} \cdot f(d_2)\, dd_2.$$

The multibidding mechanism in this situation generates ex post efficient outcomes, i.e. it chooses efficient projects that are socially optimal. In the classical case, it means that it assigns the object to the player that values it most.

Proposition 2 *In its SBNE with two players, the multibidding mechanism is efficient.*

[5]Note that the classical case enters in my setup if x_1^1 and x_2^2 have the same symmetric distribution, while x_1^2 and x_2^1 are degenerate random variables.

This result is a direct consequences of the symmetry property of the optimal bidding function, the fact that it is strictly increasing and that the winning project is chosen taking into account the largest aggregated bid. Intuitively it is due to the complementary bids of the multibidding mechanism that extract information from participants on their relative private valuations between the projects. Since one of the two projects must be carried out by assumption, the absolute social impact of the projects is irrelevant for efficiency. Social welfare is maximized taking into account the sum of individual relative impacts that are revealed truthfully in the equilibrium aggregated bids.

The multibidding game is secure for participants too, because they can secure for themselves a minimum payoff by bidding the half of the difference between their private valuations for the two projects. Doing so, since the aggregate bid for the winning project is always non-negative, the utility level that players enjoy ex post is never less than the personal average of private valuations. The bidding function represented by a line with slope $\frac{1}{2}$ corresponds to these maximin strategies.

Efficiency, budget balance and individual rationality are appealing properties, but one also might be interested in the explicit form of the optimal bidding function. This could be used in empirical work when one recovers private valuations from data on observed bids. Denote by d_M the median difference[6], defined by the difference that solves the following equality $F(d_M) = \frac{1}{2}$.

Proposition 3 *In the SBNE of the multibidding game with two players, the opti-*

[6]Since the distribution of d_i is symmetric here, the median coincides with the expected value. But this is not the case in general as I discuss it in Section 5.

mal bidding function can be written as

$$B(d_i) = \begin{cases} \frac{1}{2}d_i + \frac{1}{2}\left[1-2F(d_i)\right]^{-2} \cdot \int_{d_i}^{d_M} \left[1-2F(t)\right]^2 dt & \text{if } d_i < d_M \\ \frac{d_i}{2} & \text{if } d_i = d_M \\ \frac{1}{2}d_i - \frac{1}{2}\left[1-2F(d_i)\right]^{-2} \cdot \int_{d_M}^{d_i} \left[1-2F(t)\right]^2 dt & \text{if } d_i > d_M \end{cases}. \qquad (2.3.1)$$

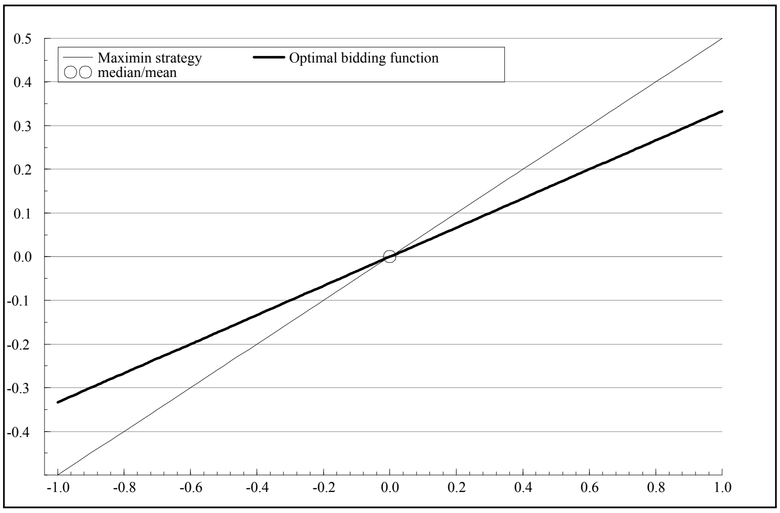
When considering SBNE, Proposition 3 shows that the above described maximin bidding behavior is only optimal at the median difference d_i. For d_i's above the median it is optimal to bid less *aggressively*, because bidding truthfully - according to the optimal bidding function - balances the probability of the preferred project to win and the utility loss due to paying bids. This maximizes agent i's expected utility, because with d_i increasing above the median level the population that agent i should outbid in order to achieve a favorable outcome for herself is getting smaller. The intuition for values below the median is very similar and it also follows from the symmetry property of the optimal bidding function.

Before deriving result for the general n-player case, I consider some numerical examples which involve computing and plotting the optimal bidding function for two concrete distributions - uniform and normal. The uniform and the normal distributions, apart from their practical importance, play a crucial role in the general case.

Example 1 *The uniform distribution: agents only attach the same likelihood to each value in the interval from which the differences between private valuations come. When differences are distributed uniformly,* $d_i \sim U[a;b]$*, the mathematical form of the optimal bidding function can be simplified to*

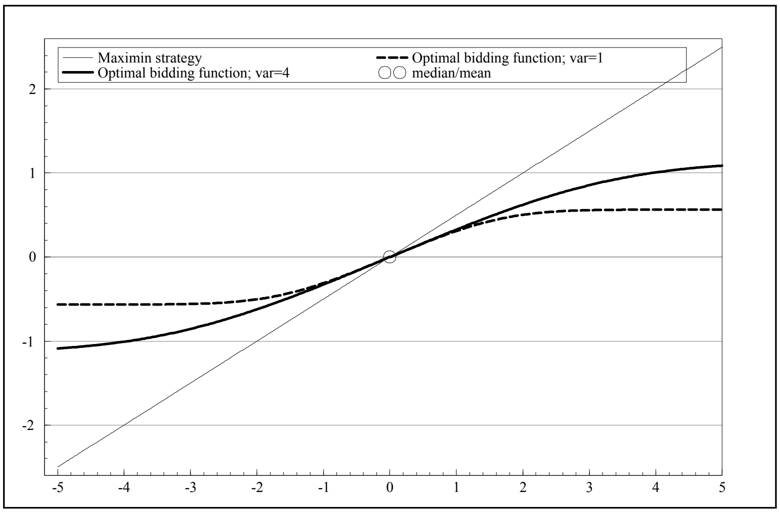
$$B(d_i) = \frac{1}{3}d_i + \frac{a+b}{12}, \qquad d_i \in [a;b].$$

Note that the above function is linear. This feature is proper to the uniform distribution, because when player i increases her bid from $B(d_i)$ with one unit she outbids the same number of players independently on the original bid, $B(d_i)$. If the uniform distribution is symmetric to 0 the optimal bidding function is proportional, and independently on the limits of the interval of possible differences the slope is equal to $\frac{1}{3}$ of the experienced difference. Graph 1 plots the optimal bidding function in the $U[-1;1]$ case. For reference the picture contains the $\frac{1}{2}d_i$ maximin line.



(Graph 1. Optimal bidding function with uniform distribution and maximin strategies.)

Example 2 *The normal distribution. In this example I consider the standard normal distribution and an other normal with zero mean and variance equal to four. The optimal bidding functions cannot be put in a simple explicit form as in the previous example, therefore I solely represent them graphically. Graph 2 also contains the $\frac{1}{2}d_i$ maximin line for reference. As one can observe in both cases, the optimal bidding function equals zero when the difference between private valuations is zero, its slope increases and it gets closer to linear as the variance (uncertainty)* *increases.*



(Graph 2. Optimal bidding function with normal distributions and maximin strategies.)

2.4 Large groups

The construction of a casino may affect the welfare of a whole community formed by many agents. Therefore, it is important to explore the properties of the multibidding game in the presence of groups with cardinality larger than two. It turns out that whenever there are more then two participants in the bidding the characteristics of the SBNE of the mechanism related to efficiency change.

Lemma 4 *In its SBNE with $n > 2$, the multibidding mechanism is ex post efficient if and only if the optimal bidding function is proportional, i.e. $B(d_i) = \beta \cdot d_i$ with some parameter $\beta > 0$ for all $i \in N$.*

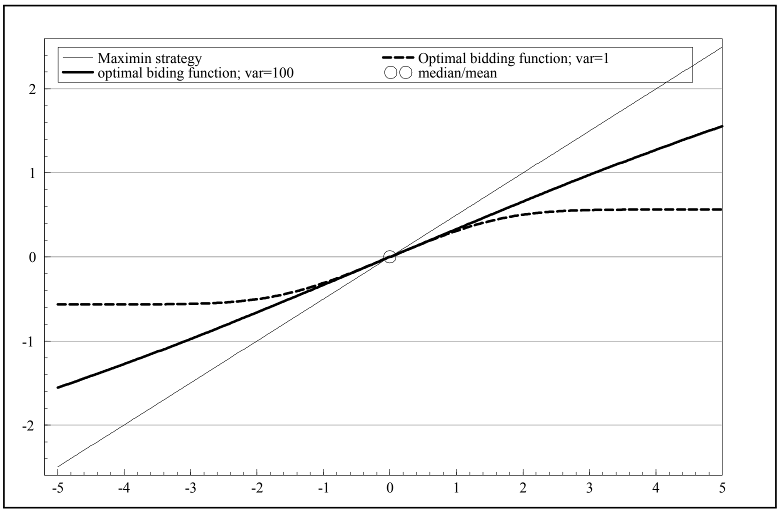
Efficiency of the multibidding mechanism can not be guaranteed in general, for any number of players. In the case of large groups the efficiency requirement puts an important restriction on the admissible bidding function in equilibrium: it must be proportional to d_i.

Even if proportional functions are intuitive and easy to deal with it turns out that they are not optimal in general. The reason behind this finding can be vaguely described as follows. Participants in the n-player case are facing an aggregate of bids that can be considered as the bid of an imaginary player with a difference between her private valuations defined by $D = \sum_{j \in N \setminus \{i\}} d_i$. Knowing $f(d_i)$ the distribution of this aggregate can be characterized, being the sum of $(n-1)$ iid random variables whose density I shall denote by $f_D(D)$. With a proportional bidding function this imaginary player bids $\beta \cdot D$ for project 1. For example, if each d_i is drawn from the normal distribution, then D will be distributed normally, too. And we have seen in the previous section that in that case the optimal bidding function is not proportional, not even linear.

Nevertheless, when n gets large the distribution of D can be characterized by a very flat density function, since the variances of d_i add up. This distribution can also be considered as very close to a uniform. When this distribution can be approximated by a uniform distribution that is symmetric to zero, the multibidding mechanism can approximate ex post efficiency. Therefore a proportional bidding function is not a *bad* choice whenever the number of participants is large enough. Proposition 4 and its proof make the above argument more rigorous.

Proposition 4 *In the SBNE of the multibidding game, if n is large the optimal bidding function is close to a proportional function with slope $\frac{n}{4n-2}$.*

Graphs 3 delivers the graphical argument behind Proposition 4. It plots the optimal bidding function for the case with two players when the distribution of differences is normal with a large variance (100).[7] For reference it also contains the $\frac{1}{2}d_i$ line and the optimal bidding function computed with a standard normal distribution. One can observe that with the increase of the variance the bidding function in equilibrium gets close to linear, in particular to a proportional function with slope $\frac{1}{3}$.



(Graph 3. Optimal bidding function with normal distribution and maximin strategies.)

The intuition behind the result can be described in the following way: As the number of participants gets larger each agent faces higher uncertainty, because the sum of everybody else's bid, D, can obtain values from a larger set. In statistical terms, the variance of D is getting larger. Instead of computing the exact distribution of D, agents might find satisfactory to approximate it by a uniform distribution. In the proof of Proposition 4 I show that the error of this approximation can be as small as one may require if the number of agents can grow large. In the case of a uniform distribution that is symmetric to zero the optimal bidding function is proportional.

[7]The normal distribution is considered here, because by the central limit theorem the distribution of D gets close to normal with growing variance as n increases.

Once Proposition 4 and Lemma 4 are combined, it is shown that the multibidding mechanism recovers efficiency if the number of affected parties, i.e. participants in the bidding is large. On the efficiency properties of the mechanism I state the following two propositions.

Proposition 5 *In its SBNE if n is large, the multibidding mechanism is close to be efficient.*

Proposition 6 offers a result similar to the ones in Proposition 4 and Proposition 5 without the condition on n, the number of participants, being large, but with individual uncertainty of a very high degree. Technically speaking this means that the variance of the d_i is large, therefore the variance of the aggregate D is also very large. With this the multibidding mechanism can approximate efficiency also in cases with a small number of players that face big uncertainty.

Proposition 6 *In its SBNE if uncertainty is of a high degree, the multibidding mechanism is close to be efficient.*

Before further theoretical remarks, a few comments on two practical features of the n-player model are in order. The efficiency of the mechanism is obtained only in the limit, but in empirical situations one hardly finds an infinite number of participants. The following three points give support for the possible existence of efficient outcomes and suggest a method that agents might use in order to compute their *almost* optimal bidding function.

- Consider a finite number of participants. As shown in the proof of Proposition 4, if the optimal bidding function is linear, $B(d_i) = \beta \cdot d_i$, the slope coefficient,

β, should solve the following equality for all d_i

$$\frac{-d_i \cdot f_D(-d_i)}{\left(\frac{1}{n}-1\right)+2\left(1-\frac{1}{n}\right)\cdot F_D(-d_i)-2d_i\cdot f_D(-d_i)}=\beta, \tag{2.4.1}$$

where the symbol $F_D(\cdot)$ stands for the accumulative distribution function of D. This is clearly impossible in general, that is why the multibidding mechanism only reaches efficiency on the limit. Nevertheless for a large n agents might bid proportionally, since the error they make decreases with n. On the other hand, the proportional bidding function is easy to apply and cope with, and as I show it now it is not difficult to compute its only parameter β.
For simplicity let us denote the left-hand side of equation 2 by $b(d_i)$. Let us denote the largest and the smallest possible value of d_i by $d_{\max}$ and $d_{\min}$ respectively.[8] Now agent i can find the value for β that minimizes the mean squared error (MSE), defined below:

$$MSE=\int_{d_{\min}}^{d_{\max}}\left[b(d_i)-\beta\right]^2\cdot f(d_i)\,dd_i.$$

The minimization problem $\min_\beta \int_{d_{\min}}^{d_{\max}}\left[b(d_i)-\beta\right]^2 f(d_1)\,dd_1$ such that $\beta>0$ gives the following result.

$$\beta=\int_{d_{\min}}^{d_{\max}} b(d_i)\cdot f(d_i)\,dd_i=E\left[b(d_i)\right],$$

where $E[\cdot]$ is the expected value operator.

- The proof of Proposition 4 shows that the error made by approximating the optimal bidding function by a proportional one diminishes as the number of participants grows, and that for efficiency a large number of participants is

[8] The limits, $d_{\max}$ and $d_{\min}$, may very well be infinite.

needed. However it is natural to ask how large is large. Even though I can not deliver an explicit formula for the optimal bidding function in the general n-player case, simulations have been performed and their results answer the above question.[9] Numerical simulations of the multibidding game also suggest that efficiency increases with the number of bidders (above two) in a continuos way. For the case in which uncertainty is captured by the uniform distribution, $U[-1;1]$, Table 1 shows the number of efficient decision as a function of the number of bidders.

n	2	3	5	10	20
efficient decisions	100%	98.6%	99.1%	99.5%	99.6%

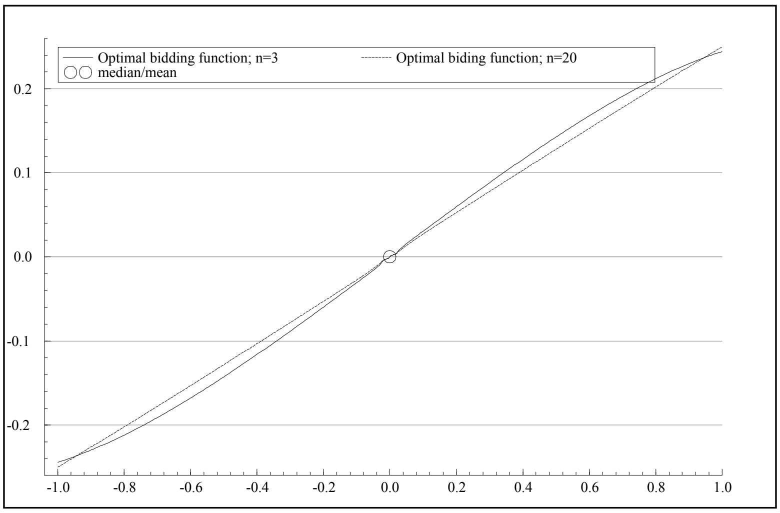
(Table 1. Number of efficient decisions as a function of group size in the $U[-1;1]$ case.)

One can observe that even in the 3-player case, that is the one with the highest number of inefficient decisions, approximately 98.6% of the decisions will maximize social welfare. Graph 4 plots the simulated optimal bidding function for the 3-player and 20-player cases in this example. It illustrates how the function looses curvature and gets proportional with the increasing number of participants.

- One can also argue that the interpretation of the above assumptions can be changed in the following way: agents' prior beliefs might not coincide with the underlying true distributions. As long as they are symmetric and identical for every participant in the model, the results hold. This argument gives more field for the efficiency result in the n-player case: when agents expect in a symmetric manner that every state of the world is equally likely to occur the distribution of D will be symmetric and uniform. In this case agents will

[9] The simulation results have been generated using Ox version 2.20 (see Doornik, 1999), and are based on theoretical results that are presented in a subsection by the end of the appendix.

bid according to a proportional function in equilibrium. Therefore ex post efficiency will be achieved.



(Graph 4. Simulated optimal bidding functions for the $U[-1;1]$ case with 3 and 20 players.)

2.5 Asymmetries and a dissolving partnership

The literature on mechanism design has discussed extensively the problem of dissolving partnerships. The problem is a classical one which has been offered efficient solutions in a fairly general set-up. For a broad summary of the performance of simple mechanisms that one might use in such situations under uncertainty check McAfee (1992). The multibidding mechanism widens this list, and offering efficient solution for the two-player problem serves as a reference point for further generalization. Moreover this example will be useful in order to illustrate the rule of symmetry in prior beliefs in the multibidding game. The assumption on the symmetry of the distribution of the difference between private valuations, d_i, is now relaxed and its consequences are studied.[10]

When a marriage or, in general, a partnership breaks down there are usually indivisible objects to be allocated among two agents. The literature on mechanism design, and closer the literature on auction theory, typically considers a single object due to technical reasons. Using now this nomenclature there are two parties and there exist two projects: one according to which party 1 receives the object, and an other according to which party 2 gets it. I shall assume that players have private valuations over these projects and the social planner wishes to allocate the object taking into account social welfare and is not interested in raising revenue.

Let me now consider two parties and an indivisible good that has to be allocated among them. In this section, we shall use the multibidding mechanism to solve the problem. For this reason, the following interpretation is given to the variables:

- Project i: player i receives the object.

As for the differences between private valuations one can proceed in two ways in the case with two players. These will be called the symmetric case and the asymmetric one due to the different meaning of the bidding function in them. I introduce the following piece of notation: f^* is a density function such that $f(-d) = f^*(d)$ for all d. The respective cumulative distribution function is F^*. $B^*(d_i)$ denotes the optimal bidding function in the case of $f^*(d)$ being the density of the underlying distribution and $F^*(d)$ its distribution function. In other words, if $B(\cdot)$ represents bids for project 1 then $B^*(\cdot)$ denotes bids for its alternative computed in the problem where project names are reversed, and vice versa.

[10]Nevertheless I keep the assumption of the symmetry of the support of this distribution. The lack of this assumption would bring us to the case that is known as asymmetric auctions in the literature. At this point of the study of the multibidding game I wish to concentrate on other features of the mechanism and keep this topic for further research.

Lemma 5 *In the SBNE of the multibidding game, the optimal bidding function satisfies the following property:*

$$B^* (-d_i) = -B (d_i) \text{ for every } d_i.$$

- The asymmetric case arises once one defines the differences between private valuations in the following way: $d_1 = x_1^1 - x_1^2$ and $d_2 = x_2^2 - x_2^1$. With this, d_i shows how agent i's utility changes when she gets the object. Therefore the optimal bidding function $B (d_i)$ can be interpreted as player i's bid for having the object. I shall assume that the distributions of these two differences coincide and can be characterized by functions $f (d)$ and $F (d)$. However with this players value the projects in an asymmetric, in fact opposite, way. The bidding function (1) presented in Proposition 3 is the optimal bidding function in the asymmetric case for any underlying distribution characterizing uncertainty. With this ex post efficiency is guaranteed in general.

- The symmetric case follows from the model specification according to which $d_1 = x_1^1 - x_1^2$ and $d_2 = x_2^1 - x_2^2$. With this the optimal bidding function $B (d_i)$ can be interpreted as player i's bid for the first project in equilibrium. Similarly to the asymmetric case, consider situations in which the distributions of d_1 and d_2 coincide, and can be characterized by the density function $f (d)$ and the cumulative distribution function $F (d)$. The name *symmetric* is due to the latter assumption, since now players value the projects in the same manner, according to the same underlying distribution that does not need to be symmetric. The symmetry of prior belief on d_i is crucial for ex post efficiency in this case. If prior beliefs follow an asymmetric distribution inefficient decisions may occur in the symmetric case. Proposition 7 and 8 analyze this problem.

In the symmetric case, players tend to prefer the same project and seem not to be as antagonistically opposed as in the asymmetric case. This situation may arise, for example, when the two affected parties share the same opinion on the allocation of the indivisible object in question. That is, they tend to value the projects in the same way, according to the same underlying distribution. Based on Lemma 3 it is easy to derive the explicit form of the optimal bidding function, and I can state the symmetric version of Proposition 3.

Proposition 7 *In the SBNE of the multibidding game with two players, the optimal bidding function can be written as*

$$B(d_1) = \left\{ \begin{array}{cc} \frac{1}{2}d_1 + \frac{1}{2}\left[1 - 2F^*(d_1)\right]^{-2} \cdot \int_{d_1}^{d_M^*} \left[1 - 2F^*(t)\right]^2 dt & \text{if } d_1 < d_M^* \\ \frac{d_1}{2} & \text{if } d_1 = d_M^* \\ \frac{1}{2}d_1 - \frac{1}{2}\left[1 - 2F^*(d_1)\right]^{-2} \cdot \int_{d_M^*}^{d_1} \left[1 - 2F^*(t)\right]^2 dt & \text{if } d_1 > d_M^* \end{array} \right\}.$$

Remember that by definition $F^*(d_M^*) = \frac{1}{2}$. Note that the distinction between the symmetric and the asymmetric cases becomes superfluous whenever the underlying distribution of differences in valuations is symmetric. This intuitive fact makes that the bidding functions presented in Graphs 1-3 are optimal both in the symmetric and asymmetric set-up.

The result on the optimal bidding function in the multibidding game shares some interesting features with the cake-cutting mechanism (CCM)[11] studied in McAfee (1992). In the CCM players bid their true valuations at the median. In the multibidding game, at the median players bid half of the difference between their valuations. This not being the whole truth can be intuitively explained by

[11] In the cake-cutting mechanism one party proposes a division and the other party chooses one of the parts of the division. This mechanism can be adapted to the indivisible case when money is available in the economy. For more details check McAfee (1992).

the rules of the multibidding mechanism, because players are forced to bid over two projects and bids must sum up to zero. Below the median value players overbid in the sense that $B(d_i)$ is larger than the half of the difference between the private valuations. While above the median they underbid. Nevertheless, there is an important difference between the CCM and the multibidding mechanism, namely that the latter treats players symmetrically and precisely because of its feature ex post efficiency can be achieved. The CCM, distinguishing the roles of proposer and chooser, turns out to be "ex post inefficient, and in an unusual way" [McAfee (1992)].

In the (symmetric) case in which players bid for the same project according to the same bidding function and this fact may cause the loss of ex-post social efficiency. As shown previously, this problem is absent when players bid for opposite projects using the same bidding function. The next proposition states that for ex-post efficiency, in the symmetric case, a certain condition on the symmetry of the optimal bidding function must hold. This condition requires the symmetry of the distribution of the prior beliefs.

Proposition 8 *In its SBNE with two players, the multibidding mechanism is efficient if and only if the prior distribution is symmetric, that is if and only if the following condition holds:*

$$B(-d_i) = -B(d_i) \text{ for every } d_i \text{ and every } i.$$

Section 4 showed that in situations with more than two players the multibidding game can only deliver ex post efficient decisions if players bid according to a proportional function in equilibrium. Once the original assumption of symmetry of the underlying density function is relaxed, an extra condition is needed to ensure proportionality in the n-player case. The increasing number of bidders increases uncertainty and makes the optimal bidding function flatter, close to be linear in the model. With this the number of ex post efficient decisions also increases. However a constant term in the bidding function works against this improvement and makes inefficient decisions persist even with very large number of players. As shown in the proofs of the propositions for the n-player case the expected value of the aggregate D must be zero for results to hold. This condition is satisfied when the distribution of d is symmetric, i.e. when agents value the two project equal in expected terms, since this implies that the expected value of d - and also D - is zero.

2.6 Conclusions

I treated the problem of choosing an efficient project by a group of agents, and have studied the theoretical performance of the multibidding mechanism in situations in which agents may hold private information. My analysis is embedded in the general setup with any number, n, of players and any number, m, of projects that shows technical complexity of high degree. Therefore, in the present work, I determined the properties of the equilibria in the case of two available projects and risk neutral players. The complexity that arises with more than two projects or risk aversion is due to the fact that by the rules of the multibidding game expected utilities depend on more than one variable. With two available projects agents' expected utility depends on the two private valuations, too, but the dimension of the problem can be reduced by one. As has been shown, it is enough to know the difference between those private valuations in order to be able to determine the optimal bidding behavior. The multibidding mechanism is always efficient in the two-player two-project case with the above restriction, and with the symmetry of prior distributions or asymmetry of players, while efficiency is tied to more conditions when there are more players. Namely, the number of agents must be large or (with a similar intuition behind) uncertainty must be large with zero expected value, in order to achieve efficient outcomes. Because of presentational considerations, a continuum of possible valuations has been used, but the results, with the proper modification, hold in the discrete case too.

It is important to bear in mind that in the analysis attention has been focused on symmetric Bayes-Nash equilibria; i.e. agents face the same uncertainty and act according to the same optimal bidding function. The appealing features of the multibidding mechanism without uncertainty, and under uncertainty with two projects and risk neutral agents make it a powerful tool for choosing an efficient project by some set of players in the presence of a public good and/or externalities. The mechanism is simple and can be easily understood by agents even in the most general $n \times m$ case. Determining the properties of its equilibria in the general case is a topic for further research.

Beside its theoretical performance both with and without uncertainty, the multibidding game has also appealing empirical properties. Pérez-Castrillo and Veszteg (2004) report results from the experimental laboratory on the mechanism presented here. In terms of efficiency, the multibidding game picked out the ex post efficient project in roughly three quarters of the cases across four experimental treatments. In line with the theoretical predictions, the number of efficient decisions was larger when individuals were paired than when they formed groups of larger size. Also, the largest part of the subject pool formed their bids according to the theoretical Bayes-Nash bidding behavior.

2.7 Appendix

The appendix contains the formal proof of all the results in the paper in the order as they appear in the text.

Proof. [Lemma 1] Consider the following notation: agent 1 experiences (x_1^1, x_1^2) and bids for project 1 according to some function $B^1 = -B^2$ at (y_1^1, y_1^2). The other agents have private valuations $\left(x_{-1}^1, x_{-1}^2\right) = \left[\left(x_2^1, x_3^1, \ldots\right), \left(x_2^2, x_3^2, \ldots\right)\right]$ and bid truthfully using the same function B^1. The distribution of the vector x_{-1}^j can be characterized by the density f_j which is the joint density of the others' valuations for project j. The expected utility for agent 1 can be written as:

$$v_1\left[x_1^1, x_1^2, B^1\left(y_1^1, y_1^2\right)\right] =$$

$$= \int\int_{\substack{\left(x_{-1}^1, x_{-1}^2\right) \text{ such that} \\ \text{project1 wins}}} \left[x_1^1 - B^1\left(y_1^1, y_1^2\right) + \frac{1}{n}B^1\left(y_1^1, y_1^2\right) + \frac{1}{n}\sum_{i \in N \setminus \{1\}} B^1\left(x_i^1, x_i^2\right)\right] \cdot$$

$$\cdot f_1\left(x_{-1}^1\right) \cdot f_2\left(x_{-1}^2\right) dx_{-1}^1 dx_{-1}^2 +$$

$$+ \int\int_{\substack{\left(x_{-1}^1, x_{-1}^2\right) \text{ such that} \\ \text{project2 wins}}} \left[x_1^2 + B^1\left(y_1^1, y_1^2\right) - \frac{1}{n}B^1\left(y_1^1, y_1^2\right) - \frac{1}{n}\sum_{i \in N \setminus \{1\}} B^1\left(x_i^1, x_i^2\right)\right] \cdot$$

$$\cdot f_1\left(x_{-1}^1\right) \cdot f_2\left(x_{-1}^2\right) dx_{-1}^1 dx_{-1}^2$$

Now consider the case in which agent 1's private values are $(x_1^1 + \delta, x_1^2 + \delta)$ where δ has a constant real value. In order to prove Lemma 1 it is enough to show that

$$\frac{\partial v_1\left[x_1^1, x_1^2, B^1\left(y_1^1, y_1^2\right)\right]}{\partial y_1^j} = \frac{\partial v_1\left[x_1^1 + \delta, x_1^2 + \delta, B^1\left(y_1^1, y_1^2\right)\right]}{\partial y_1^j} \tag{2.7.1}$$

for $j = 1, 2$. Note that

$$v_1 \left[x_1^1 + \delta, x_1^2 + \delta, B^1 \left(y_1^1, y_1^2 \right) \right] =$$

$$= \int \int_{\substack{\left(x_{-1}^1, x_{-1}^2 \right) \text{ such that} \\ \text{project1 wins}}} \left[x_1^1 + \delta - B^1 \left(y_1^1, y_1^2 \right) + \frac{1}{n} B^1 \left(y_1^1, y_1^2 \right) + \frac{1}{n} \sum_{i \in N \backslash \{1\}} B^1 \left(x_i^1, x_i^2 \right) \right] \cdot$$

$$\cdot f_1 \left(x_{-1}^1 \right) \cdot f_2 \left(x_{-1}^2 \right) dx_{-1}^1 dx_{-1}^2 +$$

$$+ \int \int_{\substack{\left(x_{-1}^1, x_{-1}^2 \right) \text{ such that} \\ \text{project2 wins}}} \left[x_1^2 + \delta + B^1 \left(y_1^1, y_1^2 \right) - \frac{1}{n} B^1 \left(y_1^1, y_1^2 \right) - \frac{1}{n} \sum_{i \in N \backslash \{1\}} B^1 \left(x_i^1, x_i^2 \right) \right] \cdot$$

$$\cdot f_1 \left(x_{-1}^1 \right) \cdot f_2 \left(x_{-1}^2 \right) dx_{-1}^1 dx_{-1}^2 =$$

$$= v_1 \left[x_1^1, x_1^2, B^1 \left(y_1^1, y_1^2 \right) \right] + \delta.$$

Taking into account the first and the last expression in the equality above (3) follows immediately. ■

Proof. [Lemma 2] Let us prove first that the optimal bidding function is increasing.

Note that for project 1 to be the winning project I must have a non-negative aggregated bid for project 1, i.e.

$$B(y_1) + \sum_{i \in N \backslash \{1\}} B(d_i) \geq 0.$$

Player 1's expected utility can be written in general as

$$v_1\left[x_1^1, d_1, B(y_1)\right] = \int \cdots \int_{\substack{(d_2,\ldots,d_n)\text{ such that}\\ \text{project1 wins}}} \left[x_1^1 - B(y_1) + \frac{1}{n}B(y_1) + \frac{1}{n}\sum_{i\in N\setminus\{1\}} B(d_i)\right]\cdot$$
$$\cdot f(d_2)\cdot\ldots\cdot f(d_n)\,dd_2\ldots dd_n +$$
$$+\int \cdots \int_{\substack{(d_2,\ldots,d_n)\text{ such that}\\ \text{project2 wins}}} \left[x_1^2 + B(y_1) - \frac{1}{n}B(y_1) - \frac{1}{n}\sum_{i\in N\setminus\{1\}} B(d_i)\right]\cdot$$
$$\cdot f(d_2)\cdot\ldots\cdot f(d_n)\,dd_2\ldots dd_n.$$

Since B is the optimal bidding function, for any d_1 and d_1^* such that $d_1 > d_1^*$ I have that

$$\begin{aligned} v_1\left[x_1^1, d_1, B(d_1)\right] &\geq v_1\left[x_1^1, d_1, B(d_1^*)\right]; \\ v_1\left[x_1^1, d_1^*, B(d_1^*)\right] &\geq v_1\left[x_1^1, d_1^*, B(d_1)\right]. \end{aligned}$$

And therefore

$$v_1\left[x_1^1, d_1, B(d_1)\right] - v_1\left[x_1^1, d_1^*, B(d_1)\right] \geq v_1\left[x_1^1, d_1, B(d_1^*)\right] - v_1\left[x_1^1, d_1^*, B(d_1^*)\right]. \tag{2.7.2}$$

For the sake of this proof let us normalize player 1's private valuation such that $d_1 = x_1^1$, $(0 = x_1^2)$ and $d_2 = x_2^2$, $(0 = x_2^1)$. This will not effect the generality of my results since this normalization can be done by adding/subtracting the same constant from both sides in inequality 4. Let us substitute the expected utilities with their form in integrals and simplify the result.

$$\int \cdots \int_{B(d_1)+\sum_{i\in N\setminus\{1\}} B(d_i)\geq 0} (d_1 - d_1^*) \cdot f(d_2) \cdot \ldots \cdot f(d_n)\, dd_2 \ldots dd_n \geq$$
$$\geq \int \cdots \int_{B\left(d_1^*\right)+\sum_{i\in N\setminus\{1\}} B(d_i)\geq 0} (d_1 - d_1^*) \cdot f(d_2) \cdot \ldots \cdot f(d_n)\, dd_2 \ldots dd_n$$

For this inequality to hold I must have $B(d_1) \geq B(d_1^*)$ and this completes the first part of the proof.

Strict monotonicity and continuity can be proven using a standard indirect argument following Fudenberg and Tirole (1991). I only explain the idea of the proof here.

Strict monotonicity: suppose that there is an atom at b in the bidding function, that is $pr\,[B(d_j) = b] > 0$ for some agent j. In this case agent i would assign probability 0 to the interval $[b - \varepsilon; b)$ for some $\varepsilon > 0$, and she bids just above b. But then agent j with a difference d_j such that $B(d_j) = b$, would be better off bidding $b - \varepsilon$, as this does not reduce the probability of winning, but does reduce cost. Therefore there cannot be an atom at b.

Continuity: if B is discontinuous there exist b' and $b''(> b')$ such that

$$pr\,\{B(d_j) \in [b'; b'']\} = 0,$$

while there exist d_j^* and $\varepsilon \geq 0$ for which $B\left(d_j^*\right) = b'' + \varepsilon$. In this case, agent i strictly prefers bidding b' to any other bid in $(b'; b'')$, since doing so does not reduce the probability of winning, but does reduce cost. But then agent j's choice of quitting at b'', or just beyond, is not optimal when she experiences d_j^*. Therefore B is continuous. ■

Proof. [Lemma 3] I shall omit the superindex from the optimal bidding function in the proof, since $B^1(d_i) = -B^2(d_i)$ holds for every d_i. Suppose that agent i experiences private valuations with a difference of $d_i = x_i^1 - x_i^2$. Her bid for project 1 in the equilibrium can be computed according to the optimal bidding function and will be equal to $B(d_i)$. Due to the rules of the multibidding mechanism, in particular to the fact that bids must sum up to zero, with this her bid for project 2 is $-B(d_i)$. Now I can consider situations in which for player 1 it is more convenient to compute her bid for project 2 first, i.e. to take into account $d_i^* = x_i^2 - x_i^1 = -d_i$. Of course, equilibrium bids can not change with the above technicality, therefore $B^*(-d_i) = B^*(d_i^*) = -B(d_i)$. Since by symmetry the density functions of d_i and d_i^* coincide, we have for bidding functions that $B = B^*$. That is $B(-d_i) = -B(d_i)$. ■

Proof. [Proposition 1] Consider agent i's expected payoff when her type is d_i. If she bids according to the optimal bidding function this quantity is equal to $v_i[x_i^1, d_i, B(d_i)]$. When agent i does not wish to influence the choice of the winning project she can bid 0, and with it obtain $v_i(x_i^1, d_i, 0)$ in expected terms. For any d_i by definition I have that $v_i[x_i^1, d_i, B(d_i)] \geq v_i(x_i^1, d_i, 0)$. With zero bid agent i does not affect the choice of the winning project, but does receive her part from the aggregated winning bid that is non-negative by the rules of the multibidding mechanism. If $v_i^a(x_i^1, d_i)$ is agent i's expected utility when she stays out of the process, then for any d_i I must have $v_i[x_i^1, d_i, B(d_i)] \geq v_i^a(x_i^1, d_i)$. That is the multibidding mechanism is individually rational. ■

Proof. [Proposition 2] This is a direct consequence of the fact that the optimal bidding function is strictly increasing. To see this, consider the following table that describes a two-player situation in general with the notation introduced in the text before.

	project 1	project 2	d_i
player 1	x_1^1	x_1^2	$x_1^1 - x_1^2$
player 2	x_2^1	x_2^2	$x_2^1 - x_2^2$
$\sum$	$x_1^1 + x_2^1$	$x_1^2 + x_2^2$	$*$

Note that for ex post efficiency I need project 1 to win if and only if $x_1^1+x_2^1 \geq x_1^2+x_2^2$. That is $x_1^1 - x_1^2 + x_2^1 - x_2^2 \geq 0$, or $d_1 + d_2 \geq 0$. The above requirement is met since the optimal bidding function is strictly increasing and symmetric: $d_1 + d_2 \geq 0 \leftrightarrow d_1 \geq -d_2 \leftrightarrow B(d_1) \geq B(-d_2) \leftrightarrow B(d_1) \geq -B(d_2) \leftrightarrow B(d_1) + B(d_2) \geq 0 \leftrightarrow$ Project 1 wins. ■

Proof. [Proposition 3] By result from Lemma 2 project 1 wins if $B(d_1) \geq B(-d_2)$, that is $d_1 \geq -d_2$. Therefore project 1 wins with probability $pr(d_1 \geq -d_2) = pr(-d_1 \leq d_2) = 1 - F(-d_1)$. Due to the assumption on the symmetry of the underlying density function the density of d_2 and $-d_2$ coincide. Now let us find the expected utility for player 1 that experiences $d_1\,(= x_1^1 - x_1^2)$ and bids according to y_1 using the function B:

$$v_1\left[x_1^1, d_1, B(y_1)\right] = \int_{x_L}^{y_1} \left\{x_1^1 - B(y_1) + \frac{1}{2}\left[B(y_1) + B(d_2)\right]\right\} f(-d_2)\, d(-d_2) + \\ + \int_{y_1}^{x_H} \left\{x_1^2 + B(y_1) - \frac{1}{2}\left[B(y_1) + B(d_2)\right]\right\} f(-d_2)\, d(-d_2).$$

By the symmetry property of the optimal bidding function one can write:

$$v_1\left[x_1^1, d_1, B(y_1)\right] = \int_{x_L}^{y_1} \left\{x_1^1 - \frac{1}{2}B(y_1) - \frac{1}{2}B(-d_2)\right\} f(-d_2)\, d(-d_2) + \\ + \int_{y_1}^{x_H} \left\{x_1^2 + \frac{1}{2}B(y_1) + \frac{1}{2}B(-d_2)\right\} f(-d_2)\, d(-d_2).$$

In order to simplify the above expression let us use the following notation: $d_2^* = -d_2$.This will also help to interpret the proof in the case when I relax the assumption on the symmetry of f in Section 6.

$$v_1\left[x_1^1, d_1, B(y_1)\right] = \int_{x_L}^{y_1}\left[x_1^1 - \frac{1}{2}B(y_1) - \frac{1}{2}B(d_2^*)\right] f(d_2^*)\, dd_2^* + \\ + \int_{y_1}^{x_H}\left[x_1^2 + \frac{1}{2}B(y_1) + \frac{1}{2}B(d_2^*)\right] f(d_2^*)\, dd_2^* \quad (2.7.3)$$

Agents are supposed to maximize their expected utility in the bidding. The first order condition of the problem is derived below.

$$\frac{\partial}{\partial y_1} v_1\left[x_1^1, d_1, B(y_1)\right] = -\frac{1}{2}\int_{x_L}^{y_1} B'(y_1) f(d_2^*)\, dd_2^* + \left[x_1^1 - \frac{1}{2}B(y_1) - \frac{1}{2}B(y_1)\right] f(y_1) + \\ +\frac{1}{2}\int_{y_1}^{x_H} B'(y_1) f(d_2^*)\, dd_2^* - \left[x_1^2 + \frac{1}{2}B(y_1) + \frac{1}{2}B(y_1)\right] f(y_1) = \\ = -\frac{1}{2}B'(y_1) F(y_1) + \frac{1}{2}B'(y_1)\left[1 - F(y_1)\right] + \left[x_1^1 - x_1^2 - B(y_1) - B(y_1)\right] f(y_1) = \\ = \left[\frac{1}{2} - F(y_1)\right] B'(y_1) - 2B(y_1) f(y_1) + d_1 f(y_1) = 0$$

The optimal bidding function must solve the above differential equation for $y_1 = d_1$.

$$\left[\frac{1}{2} - F(d_1)\right] B'(d_1) - 2B(d_1) f(d_1) + d_1 f(d_1) = 0 \quad (2.7.4)$$

The solution for equation 6:

If $F(d_1) = \frac{1}{2}$ then $B(d_1) = \frac{d_1}{2}$.

If $F(d_1) \neq \frac{1}{2}$

$$B'(d_1) - 2\frac{f(d_1)}{\frac{1}{2} - F(d_1)} B(d_1) + d_1 \frac{f(d_1)}{\frac{1}{2} - F(d_1)} = 0.$$

Let us introduce the notation $A(d_1) = \frac{f(d_1)}{\frac{1}{2} - F(d_1)}$ and $x \in [x_L, x_H]$. The latter identifies the lowest and the largest admissible value for x. The differential equation and its general solution can be written now as

$$B'(d_1) - 2A(d_1) B(d_1) + d_1 A(d_1) = 0,$$

$$B(d_1) = \exp\left[2\int_x^{d_1} A(t)\, dt\right] \cdot \left(\eta - \int_x^{d_1} \left\{tA(t) \cdot \exp\left[-2\int_x^t A(s)\, ds\right]\right\} dt\right).$$

Note that the integrals in the solution might include a difference such that $A(d_1)$ is not defined, therefore x must be carefully chosen. This parameter along with η can be fixed taking into account that the optimal bidding function must be continuous and strictly increasing.

One can check that the following function is the optimal bidding function in this problem:

$$B(d_1) = \left\{\begin{matrix} \frac{1}{2}d_1 + \frac{1}{2}[1 - 2F(d_1)]^{-2} \cdot \int_{d_1}^{d_M} [1 - 2F(t)]^2\, dt & \text{if } d_1 < d_M \\ \frac{d_1}{2} & \text{if } d_1 = d_M \\ \frac{1}{2}d_1 - \frac{1}{2}[1 - 2F(d_1)]^{-2} \cdot \int_{d_M}^{d_1} [1 - 2F(t)]^2\, dt & \text{if } d_1 > d_M \end{matrix}\right\}. \qquad (2.7.5)$$

To do so note that the following holds. For $d_1 > d_M$ fix some $x > d_M$ and choose η such that $B'(d_1) > 0$.

$$B(d_1) = \frac{1}{2}d_1 + [1 - 2F(d_1)]^{-2} \cdot \left\{\eta - \frac{1}{2}x_H + \frac{1}{2}\int_{d_1}^{x_H} [1 - 2F(t)]^2\, dt\right\}$$

Take $\eta = \frac{1}{2}x_H - \frac{1}{2}\int_{d_M}^{x_H} [1 - 2F(t)]^2\, dt$. It exists, it is finite, it does not depend on d_1 and guarantees the properties that I require from $B(d_1)$. In particular, the optimal bidding function needs to be continuous, therefore the above proposed value for η

is unique. To see this note that according to (7) discontinuity may occur at the median, and also that η is a constant shifting parameter that allows us to move the optimal bidding function for all $d_1 < d_M$ in order to reach continuity at d_M. For $d_1 < d_M$ fix some $x < d_M$ and choose η such that $B'(d_1) > 0$.

$$B(d_1) = \frac{1}{2}d_1 + [1 - 2F(d_1)]^{-2} \cdot \left\{ \eta - \frac{1}{2}x_L - \frac{1}{2}\int_{x_L}^{d_1} [1 - 2F(t)]^2 \, dt \right\}$$

Now take $\eta = \frac{1}{2}x_L + \frac{1}{2}\int_{x_L}^{d_M} [1 - 2F(t)]^2 \, dt$. It exists, it is finite, it does not depend on d_1 and guarantees the properties that I require from $B(d_1)$. As in below the median, the proposed value for η is unique here, too. These parameter values give the expression in equation 7 that completes the proof. ■

Proof. [Lemma 4] For ex post efficiency I need the aggregated optimal bid function to be a increasing strictly monotone function of the aggregated true valuations.

If the optimal bid function is proportional, $B(d_i) = \beta d_i$, this is the case, since $\sum_{i \in N} B(d_i) = \beta \sum_{i \in N} d_i$ holds. I already know that $\beta > 0$, since the optimal bidding function is strictly increasing.

In order to show the other implication consider the following. Suppose that I have $\sum_{i \in N} B(d_i) = B$ for some vector d with $\sum_{i \in N} d_i = A$ where A and B are some real numbers. Now let the valuation change for some players i_1 and i_2 such that $d^*_{i_1} = d_{i_1} + \Delta$, while $d^*_{i_2} = d_{i_2} - \Delta$. Therefore $\sum_{i \in N} d^*_i = A$. For the result to be ex post efficient I need the aggregated bid to remain unchanged. To see this consider the following inequalities implied by the ex post efficiency requirement:

$$\begin{aligned} \sum_{i \in N} d_i &\geq \sum_{i \in N} d^*_i = \sum_{i \in N} d_i \Leftrightarrow \sum_{i \in N} B(d_i) \geq \sum_{i \in N} B(d^*_i), \\ \sum_{i \in N} d_i &\leq \sum_{i \in N} d^*_i = \sum_{i \in N} d_i \Leftrightarrow \sum_{i \in N} B(d_i) \leq \sum_{i \in N} B(d^*_i), \end{aligned}$$

that is

$$\sum_{i\in N} d_i = \sum_{i\in N} d_i^* \Leftrightarrow \sum_{i\in N} B(d_i) = \sum_{i\in N} B(d_i^*).$$

Having the above results I can write

$$\begin{aligned}
B\left(d_{i_1}^*\right) + B\left(d_{i_2}^*\right) + \sum_{i\in N\setminus\{i_1,i_2\}} B(d_i) - \sum_{i\in N} B(d_i) &= 0, \\
B(d_{i_1}+\Delta) + B(d_{i_2}-\Delta) + \sum_{i\in N\setminus\{i_1,i_2\}} B(d_i) - \sum_{i\in N} B(d_i) &= 0, \\
B(d_{i_1}+\Delta) - B(d_{i_1}) + B(d_{i_2}-\Delta) - B(d_{i_2}) &= 0, \\
\frac{B(d_{i_1}+\Delta) - B(d_{i_1})}{\Delta} &= \frac{B(d_{i_2}) - B(d_{i_2}-\Delta)}{\Delta},
\end{aligned}$$

for d_{i_1} and d_{i_2}, and all Δ. I can consider $\Delta \to 0$. The above requirement then says that $B'(d_{i_1}) = B'(d_{i_2})$ for d_{i_1} and d_{i_2}. Precisely this means that the optimal bidding function must be linear. Now let us argument that the constant term in this linear function must be equal to zero. If the mechanism is ex post efficient then $\sum_{i\in N} B(d_i) = n\alpha + \beta\sum_{i\in N} d_i \geq 0$ iff $\sum_{i\in N} d_i \geq 0$. ■

Proof. [Proposition 4] Consider Player 1's expected utility with $B(d_i) = \beta d_i$. Since Project 1 is chosen if $\sum_{i\in N\setminus\{1\}} d_i = D \geq -y_1$,

$$\begin{aligned}
v_1\left[x_1^1, d_1, B(y_1)\right] = &\int_{D\geq -y_1} \left[x_1^1 + \left(\frac{1}{n}-1\right)\beta y_1 + \frac{1}{n}\beta D\right] \cdot f_D(D)\, dD + \\
&+ \int_{D<-y_1} \left[x_1^2 + \left(1-\frac{1}{n}\right)\beta y_1 - \frac{1}{n}\beta D\right] \cdot f_D(D)\, dD,
\end{aligned}$$

where $f_D(D)$ is the density function of the aggregate D. Note that $D \in [D_{\min}; D_{\max}]$

with some lower, $D_{\min}$, and upper bound, $D_{\max}$, therefore:

$$v_1\left[x_1^1, d_1, B(y_1)\right] = \int_{-y_1}^{D_{\max}} \left[x_1^1 + \left(\frac{1}{n} - 1\right)\beta y_1 + \frac{1}{n}\beta D\right] \cdot f_D(D)\, dD + \\ + \int_{D_{\min}}^{-y_1} \left[x_1^2 + \left(1 - \frac{1}{n}\right)\beta y_1 - \frac{1}{n}\beta D\right] \cdot f_D(D)\, dD.$$

$$\frac{\partial}{\partial y_1} v_1\left[x_1^1, d_1, B(y_1)\right] = \\ = \left(\frac{1}{n} - 1\right)\beta \cdot \int_{-y_1}^{D_{\max}} f_D(D)\, dD + \left(1 - \frac{1}{n}\right)\beta \cdot \int_{D_{\min}}^{-y_1} f_D(D)\, dD + \\ + \left[d_1 + 2\left(\frac{1}{n} - 1\right)\beta y_1 - \frac{2}{n}\beta y_1\right] \cdot f_D(-y_1).$$

In the equilibrium I require the following equality to hold for every d_1:

$$\left(\frac{1}{n} - 1\right)\beta \cdot \int_{-d_1}^{D_{\max}} f_D(D)\, dD + \left(1 - \frac{1}{n}\right)\beta \cdot \int_{D_{\min}}^{-d_1} f_D(D)\, dD + \\ + \left[d_1 + 2\left(\frac{1}{n} - 1\right)\beta d_1 - \frac{2}{n}\beta d_1\right] \cdot f_D(-d_1) = 0.$$

The expression can be put in a different way.

$$\beta\left[\left(\frac{1}{n} - 1\right) + 2\left(1 - \frac{1}{n}\right) \cdot F_D(-d_1) - 2d_1 \cdot f_D(-d_1)\right] + d_1 \cdot f_D(-d_1) = 0.$$

This expression, in general for any F_D and f_D, cannot be set to be equal to zero for all values of d_1 by choosing a constant value for β. I already know that the optimal bidding function is strictly increasing which can be translated into a strictly positive β in the proportional case. Nevertheless, if the functions F_D and f_D belonged to the uniform distribution over some interval $[-a; a]$ I would have

$$\beta\left[\left(\frac{1}{n}-1\right)+2\left(1-\frac{1}{n}\right)\cdot\frac{-d_1+a}{2a}-2d_1\cdot\frac{1}{2a}\right]+d_1\cdot\frac{1}{2a} = 0, \tag{2.7.6}$$
$$\beta = \frac{n}{4n-2}.$$

In other words, if the distribution of D is uniform with expected value zero, the optimal bidding function is proportional, hence ex post efficiency is achieved. This requirement is met in the special case of symmetric distributions. The interval, $[-a; a]$, is symmetric to zero by assumption, since D must have expected value zero. Since D is the sum of iid random variables as n gets very large it converges to a normally distributed variable whose expected value is zero and whose variance tends to infinity. Now let us argue that, when n is large, agents do not make a big mistake if taking into account the uniform distribution instead of the normal.
In order to keep expressions simple I consider the normal distribution with variance n. If there are n agents the distribution of the sum of the differences of their private valuations will typically have a variance of $(n-1)\,\sigma^2$. This simplification does not affect the generality of my results. Consider the squared error of the approximation:

$$SQE\,(a,n) = \int_{-a}^{a}\left(\frac{1}{\sqrt{2\pi n}}e^{-\frac{x^2}{2n}}-\frac{1}{2a}\right)^2 dx +$$
$$+\int_{-\infty}^{-a}\left(\frac{1}{\sqrt{2\pi n}}e^{-\frac{x^2}{2n}}\right)^2 dx+\int_{a}^{\infty}\left(\frac{1}{\sqrt{2\pi n}}e^{-\frac{x^2}{2n}}\right)^2 dx$$

One can show that the above expression can be written as

$$SQE\,(a,n) = \frac{1}{2\sqrt{\pi n}}-\frac{1}{a}\left[2\Phi_n\,(a)-\frac{3}{2}\right],$$

where Φ_n denotes the cumulative distribution function of the normal distribution with zero mean and variance equal to n. As the parameters a and n increase the squared error decreases towards zero. That is, for any $\varepsilon > 0$ one can find $\delta > 0$ such that with any $a, n > \delta$ I have $SQE(a, n) < \varepsilon$. ■

Proof. [Proposition 5] This result follows immediately from Lemma 4 and Proposition 4. ■

Proof. [Proposition 6] The result in Proposition 4 relies on the fact that the variance of D can be any large whenever the number of participants is large enough. Naturally the large variance of D can be due to the large variance of every single d_i, too. ■

Proof. [Lemma 5] Suppose that agent i experiences private valuations with a difference of $d_1 = x_1^1 - x_1^2$. Her bid for project 1 in the equilibrium can be computed according to the optimal bidding function and will be equal to $B(d_1)$. Due to the rules of the multibidding mechanism, in particular to the fact that bids must sum up to zero, with this her bid for project 2 is $-B(d_1)$. Now I can consider situations in which for player 1 it is more convenient to compute her bid for project 2 first, i.e. to take into account $d_1^* = x_1^2 - x_1^1 = -d_1$. Of course, equilibrium bids can not change with the above technicality, therefore $B^*(-d_1) = B^*(d_1^*) = -B(d_1)$. Since the support of f and f^* coincides both bidding functions, B and B^* are well-defined. ■

Proof. [Proposition 7] Proposition 7 follows from Proposition 3 and Lemma 3. The expected utility player 1 has to maximize in the symmetric set-up can be written as

$$v_1\left[x_1^1, d_1, B(y_1)\right] = \int_{-y_1}^{x_H}\left[x_1^1 - \frac{1}{2}B(y_1) + \frac{1}{2}B(d_2)\right] f(d_2)\,dd_2 + \\ + \int_{x_L}^{-y_1}\left[x_1^2 + \frac{1}{2}B(y_1) - \frac{1}{2}B(d_2)\right] f(d_2)\,dd_2.$$

In the next steps I shall transform the above expression in order to get (5) that will allow us to use the solution from Proposition 3. Now let us introduce the following change in the variables: $-d_2^* = d_2$. Note that since the support of f and f^* is the same I have that $x_L = x_H^*$ and $x_H = x_L^*$.

$$\begin{aligned} v_1\left[x_1^1, d_1, B(y_1)\right] &= -\int_{y_1}^{x_L}\left[x_1^1 - \frac{1}{2}B(y_1) + \frac{1}{2}B(-d_2^*)\right] f(-d_2^*)\,dd_2^* + \\ &- \int_{x_M}^{y_1}\left[x_1^2 + \frac{1}{2}B(y_1) - \frac{1}{2}B(-d_2^*)\right] f(-d_2^*)\,dd_2^* = \\ &= \int_{x_L}^{y_1}\left[x_1^1 - \frac{1}{2}B(y_1) - \frac{1}{2}B^*(d_2^*)\right] f^*(d_2^*)\,dd_2^* + \\ &+ \int_{y_1}^{x_M}\left[x_1^2 + \frac{1}{2}B(y_1) + \frac{1}{2}B^*(d_2^*)\right] f^*(d_2^*)\,dd_2^*. \end{aligned} \tag{2.7.7}$$

If $B(\cdot)$ represents player 1's bid (bidding function) for project 1 in equilibrium, $B^*(\cdot)$ in the above expression can be interpreted as player 2's bid for the alternative project 2. The variables these functions depend on once again have the same distribution, i.e. I am back in the asymmetric case. Proposition 7 can be derived from (8) applying the solution from Proposition 3. ■

Proof. [Proposition 8] From Proposition 3 and Proposition 7 we have that the optimal bidding function is symmetric,

$$B(-d_i) = -B(d_i) \text{ for every } d_i \text{ and every } i,$$

if and only if the prior distribution is symmetric around 0. Let me show first that symmetry implies ex-post efficiency. To see that note that $x_1^1 + x_2^1 \geq x_2^2 + x_2^2 \rightarrow d_1 \geq -d_2$. By strict monotonicity of the bidding function $B(d_1) \geq B(-d_2)$, that implies ex-post efficiency if the symmetry condition holds:

$$B(d_1) \geq -B(d_2) \rightarrow B(d_1) + B(d_2) \geq 0.$$

For the reverse implication note first that by efficiency

$$\begin{aligned} d_1 + d_2 &= 0 \leftrightarrow B(d_1) + B(d_2) = 0, \\ -d_1 + d_1 &= 0 \leftrightarrow B(-d_1) + B(d_1) = 0, \end{aligned}$$

that gives the symmetry condition of $B(-d_1) = -B(d_1)$. ■

2.7.1 Simulation

This subsection contains theoretical results that have been used in the simulation process for the uniform, $U[-1;1]$, example with more than two bidders. Final results of the simulation are resumed in Table 1. in the main text. In order to analyze the general n-player as a special case with only two players, e.g. player 1 and the rest of the agents, the following pieces of notations are introduced: $D_{-1} = B^{-1}\left[\sum_{i=2}^{n} B(d_i)\right]$.

The distributions of the random variables in question are $d_i \sim iiF_d$ and $B(d_i) \sim iiF_{B(d)}$, and by the central limit theorem $\sum_{i=2}^{n} B(d_i) \overset{a}{\sim} N(\mu; \sigma^2)$. Now I can state a symmetry result on the optimal bidding function in the general n-player case.

Lemma 6 *If n is large, the distribution of D_{-1} is symmetric if and only if B is* *symmetric, i.e.*

$$B(-d_i) = -B(d_i) \;\; \textit{for every } d_i.$$

Proof. Note that the following relations hold between distribution and density functions.

$$\begin{aligned}
&\sum_{i=2}^{n} B(d_i) \sim F_{\Sigma}, f_{\Sigma}; \\
F_D(x) &= pr[D_{-1} \leq x] = pr\left[\sum_{i=2}^{n} B(d_i) \leq B(x)\right] = F_{B(d)}[B(x)] = F_{\Sigma}[B(x)]; \\
f_D(x) &= \frac{\partial F_D(x)}{\partial x} = \frac{\partial F_{\Sigma}[B(x)]}{\partial x} = f_{\Sigma}[B(x)] \cdot B'(x).
\end{aligned}$$

Now let me consider the first implication in the proposition with the following equalities: $f_D(-x) = f_{\Sigma}[B(-x)] \cdot B'(-x)$. If the optimal bidding function B is symmetric we also have that $f_{\Sigma}[-B(x)] \cdot B'(x) = f_{\Sigma}[B(x)] \cdot B'(x) = f_D(x)$. That is the underlying distribution is symmetric.

In order to prove the proposition in the opposite direction, suppose that the distribution characterized by F_D is symmetric. Now one has that

$$\begin{aligned}
F_D(-x) &= 1 - F_D(x); \\
F_{\Sigma}[B(-x)] &= 1 - F_{\Sigma}[B(x)]; \\
B(-x) &= -B(x).
\end{aligned}$$

That is the optimal bidding function B is symmetric. ■

Even if I can not compute the optimal bidding function in the general case, I can deliver a mathematical expression for its explicit form that is useful in the simulation.

Proposition 9 *The optimal bidding function in the case of n bidders can be written as*

$$B(d_1) = \begin{cases} \frac{1}{2}d_1 + \frac{1}{2}\left[1-2F(d_1)\right]^{-\frac{n}{n-1}} \cdot \int_{d_1}^{d_M} \left[1-2F(t)\right]^{\frac{n}{n-1}} dt & \text{if } d_1 < d_M \\ \frac{d_1}{2} & \text{if } d_1 = d_M \\ \frac{1}{2}d_1 - \frac{1}{2}\left[2F(d_1)-1\right]^{-\frac{n}{n-1}} \cdot \int_{d_M}^{d_1} \left[2F(t)-1\right]^{\frac{n}{n-1}} dt & \text{if } d_1 > d_M \end{cases}$$

where F is the cumulative distribution function of $D_{-1} = B^{-1}\left[\sum_{i=2}^{n} B(d_i)\right]$.

Proof. Let me define $D_{-1} = B^{-1}\left[\sum_{i=2}^{n} B(d_i)\right] \sim F, f$. Now project 1 wins if $B(d_1) + \sum_{i=2}^{n} B(d_i) \geq 0$, that is when $B^{-1}\left[-B(d_1)\right] \leq D_{-1}$. Since the distribution of D_{-1} is symmetric by the previous lemma we have that the optimal bidding function is also symmetric. With this, project 1 wins if $-d_1 \leq D_{-1}$.The expected utility for agent 1 can be written as

$$v_1\left[x_1^1, d_1, B(y_1)\right] = \int_{-d_1}^{x_H} \left\{x_1^1 - B(y_1) + \frac{1}{n}\left[B(y_1) + B(D_{-1})\right]\right\} f(D_{-1})\, dD_{-1} +$$
$$+ \int_{x_L}^{-d_1} \left\{x_1^2 + B(y_1) - \frac{1}{n}\left[B(y_1) + B(D_{-1})\right]\right\} f(D_{-1})\, dD_{-1}.$$

Agents are supposed to maximize their expected utility in the bidding. The first order condition of the problem is $\frac{\partial}{\partial y_1} v_1\left[x_1^1, d_1, B(y_1)\right] = 0$ that gives the following results: the optimal bidding function must solve the differential equation below for $y_1 = d_1$.

$$\left[\left(1 - \frac{1}{n}\right) - 2\left(1 - \frac{1}{n}\right) F(d_1)\right] B'(d_1) - 2B(d_1) f(d_1) + d_1 f(d_1) = 0$$

The solution for the differential equation is:

If $\left(1-\frac{1}{n}\right)-2\left(1-\frac{1}{n}\right)F\left(d_1\right)=0$, i.e. $F\left(d_1\right)=\frac{1}{2}$ then $B\left(d_1\right)=\frac{d_1}{2}$.

If $F\left(d_1\right)\neq\frac{1}{2}$ then

$$B'\left(d_1\right)-2\frac{f\left(d_1\right)}{\left(1-\frac{1}{n}\right)-2\left(1-\frac{1}{n}\right)F\left(d_1\right)}B\left(d_1\right)+d_1\frac{f\left(d_1\right)}{\left(1-\frac{1}{n}\right)-2\left(1-\frac{1}{n}\right)F\left(d_1\right)}=0.$$

Let me introduce the notation $A\left(d_1\right)=\frac{f\left(d_1\right)}{\left(1-\frac{1}{n}\right)-2\left(1-\frac{1}{n}\right)F\left(d_1\right)}$ and $x\in\left[x_L,x_H\right]$. The latter identifies the lowest and the largest admissible value for x. The differential equation and its general solution can be written now as

$$B'\left(d_1\right)-2A\left(d_1\right)B\left(d_1\right)+d_1A\left(d_1\right)=0,$$
$$B\left(d_1\right)=\exp\left(2\int_x^{d_1}A\left(t\right)dt\right)\cdot\left\{\eta-\int_x^{d_1}\left[tA\left(t\right)\cdot\exp\left(-2\int_x^t A\left(s\right)ds\right)\right]dt\right\}.$$

Note that the integrals in the solution might include a difference such that $A\left(d_1\right)$ is not defined, therefore x must be carefully chosen. This parameter along with η can be fixed taking into account that the optimal bidding function must be continuous and strictly increasing.

For $d_1>d_M$ fix some $x>d_M$ and choose η such that $B'\left(d_1\right)>0$.

$$\int_{x_H}^{d_1}\left[tA\left(t\right)\cdot\exp\left(-2\int_{x_H}^t A\left(s\right)ds\right)\right]dt=$$
$$=-\frac{1}{2}d_1\cdot\left[2F\left(d_1\right)-1\right]^{\frac{n}{n-1}}+\frac{1}{2}x_H+\frac{1}{2}\int_{x_H}^{d_1}\left[2F\left(t\right)-1\right]^{\frac{n}{n-1}}dt$$

$$B\left(d_1\right)=\frac{1}{2}d_1+\left[2F\left(d_1\right)-1\right]^{-\frac{n}{n-1}}\cdot\left[\eta-\frac{1}{2}x_H+\frac{1}{2}\int_{d_1}^{x_H}\left[2F\left(t\right)-1\right]^{\frac{n}{n-1}}dt\right]$$

Let us choose $\eta=\frac{1}{2}x_H-\frac{1}{2}\int_{d_M}^{x_H}\left[2F\left(t\right)-1\right]^{\frac{n}{n-1}}dt$.

One can solve similarly for the opposite case, $d_1 < d_M$. Finally one gets that

$$B(d_1) = \left\{ \begin{array}{cc} \frac{1}{2}d_1 + \frac{1}{2}\left[1 - 2F(d_1)\right]^{-\frac{n}{n-1}} \cdot \int_{d_1}^{d_M} \left[1 - 2F(t)\right]^{\frac{n}{n-1}} dt & \text{if } d_1 < d_M \\ \frac{d_1}{2} & \text{if } d_1 = d_M \\ \frac{1}{2}d_1 - \frac{1}{2}\left[2F(d_1) - 1\right]^{-\frac{n}{n-1}} \cdot \int_{d_M}^{d_1} \left[2F(t) - 1\right]^{\frac{n}{n-1}} dt & \text{if } d_1 > d_M \end{array} \right\}.$$

■

The problem with the above result is that the formula for $B(d_1)$ implicitly contains the inverse of the optimal bidding function, because the distribution function F is defined in $D_{-1} = B^{-1}\left[\sum_{i=2}^{n} B(d_i)\right] \sim F, f$. That is $\sum_{i=2}^{n} B(d_i) \overset{a}{\sim} N(\mu; \sigma^2)$ and $F_D(x) = F_\Sigma[B(x)]$. But we can use these results for simulating the optimal bidding function and computing a measure for its efficiency. The optimal bidding function is determined according to the following iterative procedure:

1. Take as given $\sum_{i=2}^{n} B(d_i) \sim F_\Sigma$, possibly some $N(\mu; \sigma^2)$, and compute with it $B_{F_1}(d_1)$.

2. Compute $F_2(d_1) = F_\Sigma[B_{F_1}(d_1)]$.

3. Using the resulting distribution function $F_2(d_1)$ from the previous point compute $B_{F_2}(d_1)$.

...

n. Repeat the procedure until the result converges, that is

$$\max_{d_1} \left| B_{F_{n-1}}(d_1) - B_{F_n}(d_1) \right| < \varepsilon$$

for some predefined $\varepsilon > 0$.

In the example presented in the paper $\varepsilon = 10^{-5}$ and I have used 501 evaluation points in the $[-1; 1]$ interval in order to plot the optimal bidding function. The number of ex post efficient decisions has been approximated by a Monte Carlo experiment with 50000 draws. Results are presented in Table 1. in the main text of the paper.

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Chapter 3

Experimental Evidence on the Multibidding Mechanism

3.1 Introduction

Economic agents often have to make a common decision, or choose a joint project, in situations where their preferences may be very different from one another. Decisions involving public goods (or public "bads") belong to this class of situations. We can consider the examples of several municipalities deciding on the location and quality of a common hospital, several states deciding on the location of a nuclear reactor, or several countries choosing the identity of the leader for an international organization. In these situations the natural tendency for the agents is to try to free ride on the others by exaggerating the benefits and/or losses of a particular decision while, at the same time, minimizing their willingness to pay.

Pérez-Castrillo and Wettstein (2002) address the problem of making this type of decision in environments where the agents have *symmetric information* about everybody's preferences. They propose a simple one-stage *multibidding mechanism* in which each agent submits a bid for each project with the restriction that bids must sum up to zero for each participant. Hence, agents are asked to report on their relative valuations among the projects. The mechanism determines both the project that will be implemented (the one that most bids receives) and a system of (budget-balanced) transfer payments to possibly compensate those agents who are not pleased with the chosen project. Pérez-Castrillo and Wettstein (2002) show that the multibidding mechanism always generates an efficient decision in Nash (and strong Nash) equilibrium.

Veszteg (2004) analyzes the working of the multibidding mechanism in environments where agents hold *private information* regarding their valuation of the projects. He characterizes the symmetric Bayes-Nash equilibrium strategies for the agents, when they have to choose between two projects. He shows that the equilibrium outcomes are always individually rational (i.e. agents have incentives to participate in the mechanism). Veszteg (2004) further proves that, when the decision only concerns two agents, the project chosen at equilibrium is always efficient. Moreover, the number of inefficient decisions diminishes and it approaches zero as the number of agents or uncertainty gets large.

The multibidding mechanism is very simple: its rules are easy to explain, the action that each agent must take is simple, and the outcome is a straightforward function of the actions taken by the agents. Moreover, as we have pointed out, it induces the agents to make, at equilibrium, efficient decisions in a variety of environments. Therefore, we could advocate its use in real economic situations.[1]

[1]In environments with private information, we could also use the (more complex) mechanism proposed by d'Aspremont and Gérard-Varet (1979), that is inspired by the Vickrey-Clarke-Groves schemes. The Bayesian equilibrium outcomes of their mechanism are budget-balanced and efficient. However, it is not necessarily individually rational, some agents may prefer to stay outside

In this paper, we want to further support the use of the multibidding mechanism by providing and analyzing evidence of its functioning in laboratory experiments.[2]

We report the results of two sessions of experiments which have been designed to test the practical tractability and effectiveness of the multibidding mechanism in environments where agents hold private information concerning their valuation of the projects. We implemented two treatments in each session. The first treatment involved decisions by groups of two agents, while we arranged the agents in larger groups for the second treatment. In both treatments, the agents had to choose between two projects. We test the theoretical predictions of the paper by Veszteg (2004).

The first property that we check is to what extend the agents' bids reflect their relative valuations of the projects. According to the rules of the multibidding mechanism, agents are asked to report their relative valuations, and any agent's Bayesian equilibrium bids do indeed only depend on the difference between her valuation for the first and second project. The bids submitted by the individuals in the experiments also follow this pattern. Hence, the mechanism does a good job at extracting the information concerning agents' relative valuation.

Second, the analysis of the joint results also indicates that agents' bidding behavior is close to the theoretical equilibrium prediction. The individual analysis of

of the game. Moreover, if we were to design a mechanism that is Bayesian incentive compatible, (ex post) efficient and balanced we would end up with a mechanism of the d'Aspremont and Gérard-Varet type that is not individually rational in general. For more on this literature check the survey by Jackson (2001). It is worth noting that in this paper we use voluntary participation conditional on the impossibility of avoiding external effects. Even with this definition, it is easy to show that some agents may prefer not to participate in the d'Aspremont and Gérard-Varet mechanisms.

[2] As Ledyard (1995) points out when he discusses the behavior of individuals in public goods environments: "We need not rely on voluntary contribution approaches but can instead use new organizations... Experiments will provide the basic empirical description of behavior which must be understood by the mechanism designer, and experiments will provide the test-bed in which the new organizations will be tested before implementation."

bidding allows however to identify four types of players. More than half of the individuals were bidding according to the equilibrium. Also, another 20% of them bid in a similar manner, although in a less aggressive way. A third group of individuals (identified in one of the sessions, it accounts for 15% of the people in this session) followed a very safe strategy, by bidding according to maximin strategies. Finally, we could not explain the bidding of around 20% of the individuals participating in the experiments.

In terms of efficiency, the multibidding game picked out the ex post efficient project in roughly three quarter of the cases across the four experimental treatments. In line with the theoretical predictions, efficiency was larger when the individuals were paired than when they formed groups of larger size.

Our work follows the line of research that includes papers as, for example, Smith (1979 and 1980), and Falkinger *et al.* (2000), that advocate for the use of experiments to provide evidence on the empirical properties of mechanisms in public good environments. The characteristics of the multibidding mechanism, and the fact that the experiments were conducted in an environment where individuals hold private information, place our paper in close relationship with the extensive literature about experiments in auctions; in particular, with experimental papers on independent private-values auctions (see, for example, the early work of Coppinger *et al.*, 1980, and Cox *et al.*, 1982). This literature shows that equilibrium bidding theory correctly predicts the directional relationships between bids and valuations (see Kagel, 1995). Our results show that when a (multi)bidding mechanism is used to make a joint decision (and not to sell an object), theory is still a good predictor of the individuals' behavior.

The paper proceeds as follows: Section 2 introduces the environment and the mechanism to be studied empirically. Section 3 presents the experimental design and Section 4 the empirical results. Finally, Section 5 concludes and offers further directions of research.

3.2 The environment and the mechanism

Consider an economy where a set of agents $\mathcal{N} = \{1, ..., n\}$ has to choose between two public projects, the set of projects is denoted $\mathcal{K} = \{1, 2\}$. The agents are risk neutral and their utilities depend on the alternative carried out. We shall denote by $x_i^q \in X \subset \mathbb{R}$ the utility that player $i \in \mathcal{N}$ enjoys when project $q \in \mathcal{K}$ is chosen. These values are private information and are treated as random draws from some underlying common distribution. The latter, that characterizes uncertainty, is common knowledge.

The socially desirable outcome is the one that maximizes social welfare. We shall call project q *efficient* if:

$$\sum_{i \in \mathcal{N}} x_i^q = \max \left\{ \sum_{i \in \mathcal{N}} x_i^1, \sum_{i \in \mathcal{N}} x_i^2 \right\}.$$

The presence of external effects in the economy makes the market mechanism unreliable for taking the public decision efficiently. For these environments, Veszteg (2004) proposes the use of a multibidding mechanism, previously suggested by Pérez-Castrillo and Wettstein (2002), to provide a simple incentive scheme for the agents to reveal their private information. The *multibidding mechanism* is a one-stage game and it can be formally defined as follows:

Each agent $i \in \mathcal{N}$ submits a vector of two real numbers that sum up to zero.[3]

[3]Two is the number of available projects in the experiment.

That is, agent i announces B_i^1 and B_i^2, such that $B_i^2 = -B_i^1$. Agents submit their bids simultaneously.

The project with the highest aggregated bid will be carried out, where the aggregated bid $B_{\mathcal{N}}^q$ for project q is defined as:

$$B_{\mathcal{N}}^q = \sum_{i \in \mathcal{N}} B_i^q.$$

In case of a tie, the winning project is randomly chosen from the available ones in the tie.

Once the winning project is determined, players enjoy the utility that it delivers, they pay the bids submitted for this project, and they are returned the aggregated winning bid in equal shares.[4] That is, if project q has obtained the largest aggregated bid, then player i receives the payoff V_i^q, where

$$V_i^q = x_i^q - B_i^q + \frac{1}{n} B_{\mathcal{N}}^q.$$

A key property of the multibidding mechanism is that it can be operated without any positive or negative amount of money by the social planner, i.e., it is safe for the central government or for the authority entitled to carry out a social project. *Budget-balance* is achieved by construction since funds raised through the bidding process are entirely given back to participants in equal shares.

Moreover, the multibidding mechanism is safe for bidders, too. Once we suppose that members of the economy may abstain from participating in the bidding, but cannot avoid external effects, the mechanism assures that agents cannot do better

[4]The aggregate bid for the winning project is always non-negative, since bids of each agent sum up to zero.

by staying out of the decision-making process. Bidding exactly half of the difference between her private valuations, any agent can secure for herself a final payoff that is never less than the average of the private valuations. That is, if agent i takes the decisions of bidding:

$$B_i^1 = \frac{x_i^2 - x_i^1}{2} \text{ and } B_i^2 = -B_i^1,$$

then her payoff V_i is at least

$$\frac{x_i^2 + x_i^1}{2},$$

independently on whether project 1 or 2 is chosen. We shall refer to this bidding behavior as bidding according to *maximin strategies.*

Maximin strategies are not equilibrium strategies, an agent can typically obtain a higher expected payoff if she follows a different strategy. Hence, it is more interesting to consider the *Bayes-Nash equilibria* of the multibidding game. In particular, we concentrate on *symmetric Bayes-Nash equilibria.* The bidding behavior in these equilibria is substantially different for different types/degrees of uncertainty that individuals face in their decision making. That means that the optimal bidding function depends both on the underlying probability distribution and the number of agents in the economy. We offer a brief summary of the theoretical results related to the empirical problem studied in the experimental sessions. For more general results and formal proofs, we refer to Veszteg (2004).

We denote by $B_i(x_i^1, x_i^2)$ the equilibrium bid by agent i on project 1, when the utility levels that this agent enjoys for the two projects are x_i^1 and x_i^2. Given the restriction on the bids, agent i shall bid $-B_i(x_i^1, x_i^2)$ on project 2.

In the multibidding game, players must submit bids that add up to zero, that is, they are asked to report their relative preferences over the alternative projects. The first important result we highlight is that the optimal bidding behavior also reflects only relative preferences:

Proposition 1 *The bidding function in symmetric Bayes-Nash equilibria depends only on the difference between the true private valuations. That is,* $B_i\left(x_i^1, x_i^2\right) = B_i\left(\widehat{x}_i^1, \widehat{x}_i^2\right)$ *whenever* $x_i^1 - x_i^2 = \widehat{x}_i^1 - \widehat{x}_i^2$.

This result allows for an important simplification in the notation and in the numerical analysis of the problem. Let the difference between player i's private valuations be d_i with the following definition: $d_i = x_i^1 - x_i^2$. We denote by $f(d)$ the density and by $F(d)$ the cumulative distribution functions of the difference d for both agents. Also, we denote d_M the median of the distribution. The next proposition states the optimal bidding function when there are two agents:

Proposition 2 *In the case of two agents and symmetric distributions, the symmetric Bayes-Nash bidding function is given by the following expression:*

$$B_i(d_i) = \left\{ \begin{array}{cc} \frac{1}{2}d_i + \frac{1}{2}\left[1 - 2F(d_i)\right]^{-2} \cdot \int_{d_i}^{d_M} \left[1 - 2F(t)\right]^2 dt & \text{if } d_i < d_M \\ \frac{1}{2}d_i & \text{if } d_i = d_M \\ \frac{1}{2}d_i - \frac{1}{2}\left[1 - 2F(d_i)\right]^{-2} \cdot \int_{d_M}^{d_i} \left[1 - 2F(t)\right]^2 dt & \text{if } d_i > d_M \end{array} \right\} \tag{3.2.1}$$

The optimal bidding behavior coincides with the maximin strategy at the median, d_M. Due to the strategic behavior that takes into account the distribution of valuations in the economy, below this value agents submit higher bids, while under the median they bid less aggressively.

In the experiments, we used the uniform distribution from the interval $[0; 300]$ to assign private valuations to each subject and for each project. With this choice, the variable of the difference between private valuations follows a symmetric triangular distribution over the interval $[-300; 300]$. By the continuity of the underlying distribution and the rules of the multibidding game, the optimal bidding function is continuous and strictly increasing in d_i. Graph 1 plots the optimal bidding function according to Bayes-Nash (thick line) and to maximin strategies (thin line) for the triangular distribution over the interval $[-300; 300]$. Calculations are to be found in Appendix A.

The explicit formula of the optimal bidding function for economies with more than two players is not available. Veszteg (2004) shows that it can be approximated with a proportional function, the slope of which depends on the number of bidders, n:

Proposition 3 *If the number of agents is large, the symmetric Bayes-Nash bidding function is close to a proportional function:*

$$B_i(d_i) \approx \frac{n}{4n-2} d_i. \tag{3.2.2}$$

According to the rules of the multibidding game, the project to be carried out is the one that receives the highest aggregate bid. Taking into account our experimental setup, i.e., uncertainty is characterized by a symmetric triangular distribution, theory predicts ex post efficient public decisions in case of two bidders. If there are more than two participants, inefficiencies may appear. Although we do not have analytical results for the latter case, simulation shows that one can expect around 99% of the public decisions taken to be ex post efficient when the number of agents is larger than 5.

3.3 Experimental design

To investigate the empirical properties of the multibidding game, computerized sessions were conducted at the Universitat Jaume I in Castellón and at the Universitat Pompeu Fabra in Barcelona. We have invited 20 and 16 participants, respectively, to take part in the experiment. Sessions lasted less than two hours and the average net pay, including EUR 3 show-up fee, was about EUR 20 per subject and session.

The experiment was programmed and conducted with the software z-Tree (Fischbacher, 1999). We implemented two treatments in each session. At the beginning of each treatment, printed instructions were given to subjects and were read aloud to the entire room. Instructions explained all rules to determine the resulting payoff for each participant. They were written is Spanish, contained a numerical example to illustrate how the program works, and presented pictures of each screen to show up. The English translation of the instructions, without pictures, can be found in Appendix C.

At the start of each round the computer randomly assigned subjects to groups. We applied stranger treatment, that is participants were not informed about who the other members of their group were. Also, the assignment was done every period, hence participants knew that the groups were typically different from period to period. Subjects were not allowed to communicate among themselves, the only information given to them in this respect was the size of the group. In the first treatment of each session groups of two were formed, while in the second treatment groups of ten (in Castellón) and eight (in Barcelona) were constituted.

Private valuations for the two projects at each round were assigned to subjects by the computer in a random manner. We used the built-in function of z-Tree to generate random draws from the $U\,[0;300]$ uniform distribution. For this reason, valuations for the alternatives were typically different in each round and for each subject. Treatments consisted in 3 practice and 20 paying rounds. Table 1 summarizes the features of the four treatments.

For computational convenience, numbers (valuations, bids, and gains) used in the experiment were rounded to integers. Since our objective had been to verify theoretical results on the multibidding game in an environment where agents hold private information and common prior beliefs, we dedicated a paragraph in the instructions to explain the nature of the uniform distribution.[5]

In each round, participants were asked to enter their bids over the two projects. Taking into account the rules of the multibidding game, the winning project was determined and payoffs were calculated automatically by the computer.[6] At the end of each round, subjects received on-screen information about the aggregated bid of other players in the same group; and also detailed information about the determining components of the personal final payoff. The history of personal earnings was always visible on screen during the experiment.

At the end of each session participants were paid individually and privately. Final profits were computed according to a simple conversion rule, based on the personal gains in experimental monetary units during the whole session.

[5] Although theoretical results are provided for a wide range of probability distributions, we had chosen the uniform. We thought that this one would be the most intuitive and simplest to explain to subjects who are not familiar with probability theory. We followed the example of Binmore *et al.* (2002) in the instructions.

[6] In case of a tie, the program breaks the tie choosing the project randomly assigning equal probability to the alternatives.

3.4 Results

3.4.1 Efficiency

The multibidding game achieved efficiency in the large majority of decision problems, as it picked out the ex post efficient public projects roughly in 3/4 of the cases across the four experimental treatments. Table 2 contains detailed information on efficiency for each treatment and also presents the 90% confidence intervals around the data. Theoretical result on the multibidding game refer to the number (or proportion) of ex post efficient public decisions when talking about efficiency. Nevertheless, a different measure can be constructed to capture the empirical efficiency of the mechanism that also takes into account the magnitude of the efficiency loss when an inefficient project is chosen. We call it realized efficiency (RE) and define it as

$$RE = \frac{\sum_{i \in \mathcal{N}} x_i^{winning\ project}}{\max\left\{\sum_{i \in \mathcal{N}} x_i^1, \sum_{i \in \mathcal{N}} x_i^2\right\}} \cdot 100 \text{ percent.}$$

Table 2c reports the realized efficiency with the 90% confidence interval. The point estimates are above 90% in all of our treatments.

In order to extend the efficiency analysis we estimated a Logit model for each treatment, trying to establish some empirical relation between the probability of an efficient decision and the absolute difference between the two projects. Table 2d contains the estimation results, and shows that the larger the difference between the projects, the higher the probability of an efficient decisions in treatments B1 and C1. That is, observed inefficiencies tended to occur in cases in which the projects were similar, causing a relatively small drop in realized efficiency. In the remaining two treatments we can not identify any significant relationship of the above type.

Due to the small number of experimental sessions, we can not establish the empirical ranking of group sizes according to efficiency. Nevertheless, it is worth to note that our treatments with groups of two do not give statistically different results on efficiency. Operating with groups of eight, in treatment B2, the multibidding game performed significantly better than with groups of ten in C2. Simulation results suggest that the efficiency change caused by an increase of group size from eight to ten is minimal in theory. And according to theory we would expect this minimal change to be a gain, in spite of the loss we observed in the experiments. We explain this observed feature with differences in the subject pool and attribute it to different individual bidding behavior from one session to the other. Before starting with the empirical analysis of the bidding function, let us point out that we could not identify any significant linear time trend in the evolution of efficiency over the 20 paying periods (see Table 3).

3.4.2 Bidding behavior

We have used experimental data from two sessions and a simple linear model to estimate the empirical bidding function. The linear approximation is the strongest available theoretical result for the bidding behavior with decision problems that involve more than two agents. In case of groups of two, the optimal bidding function shows pronounced curvature at the extremes of the support. Since we generated private valuation according to a uniform distribution in the experiments, i.e., the theoretical distribution of the difference was triangular, we do not have many observations on the positive and negative ends of the support and could not give significant estimates for the curvature. Moreover, the linear specification allows for a single expression that approximates optimal bids as a function of group size and the difference between private valuations. In Appendix A we show that the first-order Taylor-approximation - around zero - of the optimal bidding function has slope 1/3. For this reason, in this analysis we shall treat Expression (3.2.2) as the theoretical optimal bidding function in all of our treatments. The maximin bidding behavior can also be characterized by a linear function in the multibidding game. That function has slope 1/2 independently from the group size.

We have estimated two linear specifications of the bidding function:

$$\widehat{B}_i = \widehat{\alpha}_1 + \widehat{\beta}_1 x_i^1 + \widehat{\beta}_2 x_i^2, \tag{3.4.1}$$

$$\widehat{B}_i = \widehat{\alpha}_2 + \widehat{\beta} d_i. \tag{3.4.2}$$

Equation (3.4.1) represents a linear bidding function that does not force bids to depend solely on the difference between private valuations, while equation (3.4.2) does. The dependent variable in both specification is the bid submitted for project 2. Recall that in the original theoretical model the function B stands for bids for project 1. This switch is due to the following: theoretical models deal with positive bids as amounts that agents are willing to pay; nevertheless in the experiments we asked subjects to type in a negative number in case they were willing to pay for a given project and a positive one in the opposite case. Since the multibidding game operates both with positive and negative bids we thought that in this way concepts might be more intuitive for people participating in the sessions.

Tables 4a through 8a contain the OLS estimates (with indexes for significance of the results) of the empirical bidding functions both individually for each subject and jointly for the subject pool across different treatments.

When considering treatments globally, at 5% significance level we can not reject the hypothesis that subjects decide their bids taking into consideration only the difference between their private valuations for the two public projects. That is, the empirical results are in accordance with Proposition 1, in the sense that individuals seem to "report" (through the bids) their relative valuations of the two projects. Moreover, this is a robust result, since it holds in each of the four treatments that we implemented.

We now turn our attention to the fit between the experimental data and propositions 2 and 3, which state the expressions for the equilibrium bids as function of the distance between the valuations. Table 4a provides the estimated bidding functions. For treatments C1 and C2, these functions are not different statistically from the theoretical ones, i.e. they are proportional with slopes (statistically) equal to 1/3 and 0.26 respectively. Estimates for the other two treatments, B1 and B2, are more precise in that we obtained a better fit with smaller variance of the estimates. For B2 the constant term turns out to be significantly different from zero at 5%, but its estimated absolute value, 1.45, is very small compared to the magnitude of the private valuations, $[0; 300]$, used in the experiment. In the latter two treatments, subjects seem to have bid according to a linear function, though more conservatively than predicted by theory: the small variance of the estimates confirms that bidding behavior can be approximated by a simple linear function with slopes 0.22 for B1 and 0.20 for B2, significantly less than 1/3 and 0.27, respectively.

The individual analysis of bidding offers a deeper insight into the above pooled results and their consequences on the number of ex post efficient public decisions. We have estimated the two linear models in Equation (3.4.1) and (3.4.2) for each subject separately, and performed the same tests that we have done for the subject pools. Detailed estimation results are to be found in Appendix B. In order to have a structured summary of the subject pool we have grouped agents into three groups based on the estimated slope coefficient of the empirical bidding function. Table 9 shows that the largest part of our subjects falls into the two strategy groups studied by theory, i.e. maximin with slope 1/2 and Bayes-Nash with slope either 1/3 or

$n/(4n-2)$.

Bidding in treatment B1, in spite of being the most efficient among the four, seems to be difficult to explain at first sight with the latter two types of strategies. As mentioned before, in B1 subject formed their bids linearly, but less aggressively than predicted by theory in Bayes-Nash equilibrium. This is why we split the residual category of Other in Table 9 into two: linear bidding behavior based on the difference between private valuations and other kind of behavior we can not account for.[7] The distinction clearly improves statistics presented in Table 10a for our Barcelona session, and leaves at most 30% of the subjects as *irrational*.

Different reasons may explain why a sensible share of the subjects bid less aggressively than predicted by the bayes-Nash equilibrium. Agents, for example, may form their bids according to the symmetric Bayes-Nash equilibria of the multi-bidding game, but perceive uncertainty in a biased way. Therefore, the bidding function $B(d_i) = 0.2 \cdot d_i$ may be optimal. It turns out that this is the case under uncertainty characterized by the distribution function $\widetilde{F}(d_i) = \frac{1}{2} \cdot \left[1 + \left(\frac{d_i}{300}\right)^{\frac{1}{3}}\right]$ over the interval $[-300; 300]$.[8] The comparison of this and the underlying true triangular distribution, presented numerically in Table 12, shows that participants possibly overweighted high-probability events and underweighted the low-probability ones. The distribution defined by $\widetilde{F}(d_i)$ is symmetric around zero, has a smaller standard deviation than the triangular and it is more peaked around zero.[9]

[7] The latter category includes some subject that handed in their bids independently from the difference between their private valuations, and some that we have estimated negative slope coefficient for. Table 10 has been built at 1% significance level, but results do not change in the Other category if we move to 5%, either.

[8] We do not provide the proof of this result here, but it is available upon request.

[9] This finding is in line with those presented by Harbaugh *et al.* (2002) who examine how risk attitudes change with age. The ages of participant in their experiments range from 5 to 64. They observe that young people's choices are consistent with the underweighting of low-probability events and the overweighting of high-probability ones, and that this tendency diminishes with age. Participants in our sessions were university student with approximately 20 years of age.

Interestingly, these subjects who bid in a linear way, but did not follow the Bayes-Nash strategy, did very well in terms of (ex post) profits in every treatment. Table 10c shows the mean payments in four bidding categories. Subjects in the Other category were the ones who gained less, even though the difference between the first three and the fourth category is not significant statistically in our Barcelona treatments, at any usual significance level.

Bidding less aggressively is also self-consistent in the following sense: if players, in groups of two, are applying linear bidding functions and believe that their opponents bid according to $B(d_i) = 0.2 \cdot d_i$, they maximize their expected payoff by bidding slightly more for any given d_i. The best response in this example is $B(d_i) \approx 0.222 \cdot d_i$ and very well may explain the observed behavior.[10] In order to understand and provide further support for a more conservative empirical bidding function in B1 and B2, we have also studied whether subjects could have been better off applying the Bayes-Nash bidding function against the others' observed behavior. Taking into account those who bid in a linear way, but significantly different from the predicted one by theory, we get that the Bayes-Nash bidding function (ceteris paribus) could have improved their gains only moderately: by 1.23% in B1 and 2.56% in B2. That is, facing the others' bids these participants did not have enough incentives to abandon their bidding function and play Bayes-Nash instead.

It is important to point out that in Castellón we encountered subjects bidding safe according to maximin strategies. This feature of the observed behavior, along with more conservative bidding in B1 and B2, gives partial explanation for the reported efficiency rates, too. The available theoretical results deal with symmetric equilibria. An important part of the ex post efficiency of the multibidding game

[10]This numerical result follows directly from the expected utility maximization problem with the triangular distribution.

is due to the fact that agents bids according to the same theoretical function. The heterogeneity of the subject pool in C1 and C2 appears also in the observed efficiency loss. In treatments B1 and B2, even though subjects do not play strictly Bayes-Nash, the number of ex post efficient decisions is larger because the subject pool was more homogeneous.[11]

We have discussed above that the available data set is not large enough to deliver empirical evidence for the curvature of the bidding function. This curvature is responsible, as theory predicts, for the occurrence of ex post inefficient decisions once the group size is larger than two. Unfortunately we can not present empirical proofs for this feature, nevertheless we can explore statistically how bidding behavior changes when the group size (and with it uncertainty) increases. As a response to this, according to theory, the slope of the Bayes-Nash bidding function should decrease. We can verify a change in this direction looking at the estimated bidding function for the whole subject pool both in Barcelona and Castellón. This drop is significant at 5% in Castellón, while it is not in Barcelona.

Table 11 offers a summary of the individual data in this respect. The estimated slope coefficient of the individual bidding function decreases in 45% of the cases in Castellón, and in 63% in Barcelona. Though, the vast majority of these estimated changes is not significant individually at 5% or 10%.[12]

Subjects were asked to decide over public projects and their alternative in 20 paying rounds. Even though private valuations were different from round to round, according to the underlying uniform probability distribution, one might expect that

[11] A measure for homogeneity could be the (length of the) range of our estimates for the slope coefficient of the bidding function according to Equation 3.4.2: C1 - $[-0.43; 1.82]$; C2 - $[-0.85; 1.18]$; B1 - $[0.05; 0.036]$; B2 - $[0.02; 0.35]$.

[12] When fixing the significance level at 15% the only change in Table 11 is that a difference into the unexpected direction becomes significant for a subject in Castellón.

participants get trained and gain experience in each treatment. In order to show possible learning effects we have split every data set into two,[13] and estimated the individual bidding function (according to Equation 3.4.2) separately for the subsamples. Tables 4b through 8b offer the estimation results. For a quick view consider Table 10b in which we repeated the categorization of bidding behavior taking into account four groups. Unexpected bidding behavior, i.e., frequencies in the Other category, barely or do not change from the first to the last 10 playing rounds. In treatment C1 three subjects, while in treatments C2 and B1 one and one subject seem to adjust their bidding behavior to the one predicted by theory.

3.5 Conclusion

In this paper we have studied the empirical properties of the multibidding game under uncertainty described by Veszteg (2004). The results of our four treatments, with two projects to choose between, show that the mechanism performs well in the laboratory. We find that the one-shot multibidding game with its simple rules succeeds in extracting private information from agents, as the observed bids were formed taking into account relative private valuations between two projects.

Though not all participants followed the Bayes-Nash equilibrium predicted by theory, the mechanism gave rise to ex post efficient outcomes in almost 3/4 of the cases across the treatments. Apart from the expected utility maximizing Bayes-Nash behavior we could identify bidding behavior according to the safe maximin strategies in one of our sessions. Unfortunately our sample size, due to feasibility constraints in the laboratory, does not allow for verifying theoretical predictions for

[13] We wanted to form two independent data set for each subject and treatment. Taking into account our relatively small sample size we decided no to eliminate any observation from the analysis.

large groups in a significant way. More subjects and more repetitions are needed to possibly reduce the observed variance of the data and study those effects.

Our data set does not contain any significant linear trend in time. Neither if we consider global efficiency or in the case of individual bidding behavior. A longer time series would also be able to show whether the rules of the multibidding game are simple enough to understand, or learning indeed plays an important role in the performance of the mechanism.

It is important to point out that a considerable fraction of participants (especially in the Barcelona treatments) applied linear bidding function based on their relative valuations, though they bid less aggressively than expected in theory. Since they did well in monetary term among all the participants and did not harm ex post efficiency, we suggest to obtain theoretical results for economies in which there are several groups (types) of agents: some play maximin strategies, some Bayes-Nash. Beside the expansion of theoretical work on the multibidding game, undoubtedly also more empirical research is needed to explore its empirical performance. We think that further experiments can help to identify features that allow for designing successful practical mechanisms.

3.6 Appendix A. Optimal bidding behavior

The triangular distribution over the interval $[-300; 300]$ of our experimental design can be characterized by the following density function:

$$f(x) = \left\{ \begin{array}{cc} 0 & x \notin [-300; 300] \\ \frac{1}{90000}x + \frac{1}{300} & x \in [-300; 0] \\ -\frac{1}{90000}x + \frac{1}{300} & x \in [0; 300] \end{array} \right\},$$

and cumulative density function:

$$F(x) = \left\{ \begin{array}{cc} 0 & x \in (-\infty; -300) \\ \frac{1}{180000}x^2 + \frac{1}{300}x + \frac{1}{2} & x \in [-300; 0] \\ -\frac{1}{180000}x^2 + \frac{1}{300}x + \frac{1}{2} & x \in [0; 300] \\ 1 & x \in (300; \infty) \end{array} \right\}.$$

It is symmetric to the origin and for this reason its median is zero. By substituting the above function into equation (3.2.1), we have that the optimal bidding function in our example can be written as:

$$B_i(d_i) = \left\{ \begin{array}{cc} \frac{1}{2}d_i + \frac{-600\,000d_i - 1500d_i^2 - d_i^3}{12\,000d_i + 10d_i^2 + 3600\,000} & \text{if } d_i < 0 \\ 0 & \text{if } d_i = 0 \\ \frac{1}{2}d_i + \frac{-600\,000d_i + 1500d_i^2 - d_i^3}{10d_i^2 - 12\,000d_i + 3600\,000} & \text{if } d_i > 0 \end{array} \right\}.$$

If we consider the first-order Taylor-approximation of this resulting bidding function around zero, we have $B_T(d_i) = \frac{1}{3}d_i$.

3.7 Appendix B. Results

Treatment	Number of groups	Group size	Uncertainty	Practice periods	Paying periods
C1	10	2	$U\,[0;300]$	3	20
C2	2	10	$U\,[0;300]$	3	20
B1	8	2	$U\,[0;300]$	3	20
B2	2	8	$U\,[0;300]$	3	20

Table 1. Treatment summary.

Treatment	C1	C2	B1	B2
Efficient decisions	72%	58%	81%	70%
Upper bound	77%	71%	86%	82%
Lower bound	67%	44%	76%	58%

Table 2a. Proportion of efficient decisions with 90% confidence interval.

Treatment	C1	C2	B1	B2
First 10 - Efficient decisions	73%	45%	84%	75%
First 10 - Upper bound	80%	64%	91%	91%
First 10 - Lower bound	66%	26%	77%	59%
Last 10 - Efficient decisions	71%	70%	79%	65%
Last 10 - Upper bound	79%	87%	86%	83%
Last 10 - Lower bound	63%	53%	71%	47%

Table 2b. Proportion of efficient decisions with 90% confidence interval for the first and last 10 rounds.

Treatment	C1	C2	B1	B2
Realized efficiency	91%	94%	96%	95%
Upper bound	94%	100%	98%	100%
Lower bound	88%	87%	93%	89%

Table 2c. Realized efficiency with 90% confidence interval.

Treatment	C1	C2	B1	B2
Constant term	0.30	0.57^{*}	0.23	0.99^{**}
$\lvert d_i \rvert$	0.005^{*}	0.00	0.01^{*}	0.00

*Stat. significant at 5%; **Stat. significant at 10%; ***Stat. significant at 15%.

Table 2d. Estimated coefficients of the impact of the absolute difference between private valuation on the probability of an efficient decision (Logit)

	Treatment			
Round	C1	C2	B1	B2
1	60%	50%	88%	100%
2	60%	50%	100%	0%
3	70%	50%	75%	100%
4	80%	50%	88%	100%
5	80%	100%	63%	50%
6	80%	0%	75%	100%
7	70%	50%	75%	50%
8	70%	100%	100%	100%
9	70%	0%	88%	50%
10	90%	0%	88%	100%
11	60%	50%	88%	0%
12	70%	100%	88%	0%
13	90%	50%	75%	50%
14	60%	100%	63%	100%
15	50%	0%	88%	100%
16	70%	50%	75%	100%
17	70%	100%	63%	50%
18	80%	50%	75%	100%
19	80%	100%	100%	50%
20	80%	100%	75%	100%

Table 3. Proportion of efficient decisions per round.

Treatment†	Constant term	Slope coefficient		Constant term	Slope coefficient
		Project 1	Project 2		
C1	-2.22	0.37*	-0.36*	-0.84	0.37*
C2**	-22.80*	0.30*	-0.18*	-5.95***	0.24*
B1	-0.32	0.21*	-0.22*	-0.80	0.22*
B2	1.12	0.20*	-0.20	1.45*	0.20*

*Stat. significant at 5%; **Stat. significant at 10%; ***Stat. significant at 15%.

†Diff. between the absolute value of slope coefficients for the two projects stat. significant.

Table 4a. Bidding functions (OLS).

	First 10	First 10	Last 10	Last 10
Treatment	Constant term	Slope coefficient	Constant term	Slope coefficient
C1	-7.94	0.37*	6.29	0.36*
C2	-7.63	0.22*	-4.11	0.27*
B1	0.18	0.22*	-1.79	0.21*
B2	-0.15	0.19*	3.31**	0.21*

*Stat. significant at 5%; **Stat. significant at 10%; ***Stat. significant at 15%.

Table 4b. Bidding functions for the first and last 10 rounds (OLS).

Treatment C1					
Subject†	Constant term	Slope coefficient		Constant term	Slope coefficient
		Project 1	Project 2		
1	-5.53	0.08	-0.28	-32.68**	0.22
2***	-48.14***	0.57*	-0.27*	-3.37	0.40*
3	15.72	0.03	-0.22	-9.97	0.12
4	21.05	0.05	-0.21***	-1.39	0.14***
5	-6.85	0.62*	-0.54*	1.79	0.57*
6	-16.78	-0.39*	0.48*	-3.25	-0.43*
7	37.67	0.62*	-0.66*	31.57*	0.64*
8	-35.39	0.47*	-0.20***	3.71	0.31*
9	34.36	1.19*	-1.31*	15.39	1.26*
10	9.10	-0.06	-0.01	-0.37	-0.02
11	12.00	-0.41	0.19	-17.47	-0.31**
12	-7.50	0.38*	-0.39*	-8.58	0.39*
13**	-92.31**	0.13	0.40*	-7.14	-0.20
14	4.98	0.16*	-0.22*	-4.79	0.19*
15	127.87	1.41*	-2.24*	-10.73	1.82*
16	4.72	0.13**	-0.15**	1.50	0.14*
17	-35.71	0.50*	-0.05	31.22***	0.28***
18	-13.7	0.25***	-0.24***	-11.29	0.24*
19	10.84	0.19***	-0.24**	3.28	0.22*
20	-6.14	0.17	-0.08	5.87	0.12

*Stat. significant at 5%; **Stat. significant at 10%; ***Stat. significant at 15%.

†Diff. between the absolute value of slope coefficients for the two projects stat. significant.

Table 5a. Individual bidding functions (OLS).

Treatment C1				
	First 10	First 10	Last 10	Last 10
Subject	Constant term	Slope coefficient	Constant term	Slope coefficient
1	-13.12	0.00	-29.26	0.48**
2	-16.96	0.43*	14.37	0.33*
3	-22.88	0.04	3.06	0.24**
4	8.60	0.08	-18.98	0.30**
5	4.80	0.60*	2.44	0.55*
6	6.85	-0.55*	-10.25	-0.40*
7	43.76***	0.80*	31.56***	0.53*
8	-0.97	0.32***	8.55	0.31*
9	-9.98	1.18*	42.04	1.23*
10	0.22	-0.21**	0.67	0.21**
11	-23.09	-0.36***	-13.93	-0.26
12	-11.83	0.42*	0.26	0.25*
13	19.45	-0.06	-36.91	-0.28
14	-6.20	0.21*	-3.36	0.15***
15	40.11	1.78*	-66.13	2.15*
16	9.75	0.10	-5.41	0.15*
17	-0.46	0.21*	52.00	0.57
18	-46.61*	-0.06	26.98**	0.46*
19	-13.69	0.16	19.63	0.25**
20	30.48***	0.52*	-0.25	0.03

*Stat. significant at 5%; **Stat. significant at 10%; ***Stat. significant at 15%.

Table 5b. Individual bidding functions for the first and last 10 rounds (OLS).

Treatment C2					
Subject†	Constant term	Slope coefficient		Constant term	Slope coefficient
		Project 1	Project 2		
1	20.01	0.07	-0.21	-1.40	0.12
2*	-43.94**	0.71*	-0.38*	5.30	0.55*
3***	34.48	0.43*	-0.72*	-8.46	0.60*
4	-29.39	-0.68**	0.98*	12.95	-0.85*
5	24.00	0.40*	-0.67*	-20.14***	0.52*
6	-11.13	0.19*	-0.04	12.91***	0.12*
7	40.86	0.23*	-0.47*	-0.73	0.36*
8*	-100.79*	0.79*	-0.22***	-1.01	0.52*
9	0.79	0.13**	-0.14**	-0.79	0.14*
10*	36.20***	0.16**	-0.49*	-12.42	0.28*
11*	-39.60	0.38*	-0.13**	-2.50	0.28*
12	2.33	0.18**	-0.17**	4.04	0.18*
13	-22.9**	0.13*	-0.02	-7.46	0.07
14*	-121.01**	1.51*	-0.70*	-5.48	1.18*
15	-84.13**	0.40*	0.01	-30.58***	0.28**
16	-0.14	0.22*	-0.23*	-2.23	0.22*
17*	84.87	-0.19	-0.76*	-51.86*	0.36
18	-30.75	0.45**	-0.26	-1.26	0.35**
19	-67.73	0.03	0.50*	7.87	-0.27*
20	2.00	0.21*	-0.25*	-3.23	0.23*

*Stat. significant at 5%; **Stat. significant at 10%; ***Stat. significant at 15%.

†Diff. between the absolute value of slope coefficients for the two projects stat. significant.

Table 6a. Individual bidding functions (OLS).

Treatment C2				
	First 10	First 10	Last 10	Last 10
Subject	Constant term	Slope coefficient	Constant term	Slope coefficient
1	-5.74	-0.11	-5.40	0.46
2	3.21	0.37***	21.09**	0.72*
3	-12.47	0.64*	-6.49***	0.57*
4	21.44	-1.07*	-3.05	-0.66*
5	-20.93	0.55*	-20.54	0.50*
6	20.18	0.13	4.51	0.08**
7	-7.63	0.50*	8.55	0.24*
8	0.02	0.52*	-2.12	0.51*
9	3.00	0.17**	-5.56	0.10*
10	-14.11	0.35*	-8.96	0.22***
11	2.17	0.28*	-7.06	0.27*
12	0.26	0.01	1.94	0.28*
13	-7.27	0.14	-9.93	0.02
14	-49.76	0.97*	32.22	1.15*
15	-64.64*	0.37**	3.89	0.39***
16	2.36	0.23*	-7.01	0.21*
17	-104.40*	0.13	-8.45	0.37
18	27.09	0.43**	-37.06	0.42
19	6.91	-0.38**	-0.49	-0.09
20	-2.09	0.22*	-5.38	0.25*

*Stat. significant at 5%; **Stat. significant at 10%; ***Stat. significant at 15%.

Table 6b. Individual bidding functions for the first and last 10 rounds (OLS).

Treatment B1					
Subject†	Constant term	Slope coefficient		Constant term	Slope coefficient
		Project 1	Project 2		
1	1.53	0.11	-0.14**	-3.60	0.13*
2	9.29	0.20*	-0.23*	5.44	0.22*
3	-10.76	0.07	-0.04	-6.25	0.05
4	1.78	0.26*	-0.26*	2.09	0.26*
5	2.79	0.17*	-0.17*	1.74	0.17*
6	6.52	0.22*	-0.25*	1.00	0.23*
7	-1.48	0.25*	-0.22*	0.59	0.23*
8	2.02	0.35*	-0.37*	-1.71	0.36*
9	3.97	0.20*	-0.19*	4.58	0.19*
10***	-24.41	0.36*	-0.22*	-3.57	0.29*
11	-4.82	0.24*	-0.22*	-2.39	0.22*
12	8.46	0.23*	-0.36*	-11.00***	0.31*
13	10.42	0.10	-0.11	8.58	0.10***
14	10.67	0.17**	-0.28*	-3.61	0.23*
15	-14.14	0.31*	-0.31*	-14.82	0.31*
16	3.03	0.07	-0.07	3.18	0.07

*Stat. significant at 5%; **Stat. significant at 10%; ***Stat. significant at 15%.

†Diff. between the absolute value of slope coefficients for the two projects stat. significant.

Table 7a. Individual bidding functions (OLS).

Treatment B1				
	First 10	First 10	Last 10	Last 10
Subject	Constant term	Slope coefficient	Constant term	Slope coefficient
1	-13.66	0.22*	3.45	0.08
2	10.52	0.15*	2.92	0.35*
3	-5.01	0.00	-6.40	0.11***
4	2.80	0.25*	-1.33	0.29*
5	-0.04	0.16*	3.93	0.21*
6	2.52	0.22*	0.47	0.23*
7	7.14	0.22*	-5.16	0.28*
8	2.01	0.38*	-5.58**	0.36*
9	-0.21	0.29*	4.65***	0.13*
10	-5.28	0.31*	-0.26	0.25*
11	-0.56	0.20*	-4.16	0.23*
12	3.95	0.32*	-27.36*	0.36*
13	10.37	0.27**	3.08*	0.02*
14	-0.84	0.23*	-6.57	0.24*
15	-4.16	0.26*	-26.86	0.30***
16	-2.43	0.15*	8.28	0.01

*Stat. significant at 5%; **Stat. significant at 10%; ***Stat. significant at 15%.

Table 7b. Individual bidding functions for the first and last 10 rounds (OLS).

Treatment B2					
Subject†	Constant term	Slope coefficient		Constant term	Slope coefficient
		Project 1	Project 2		
1	3.90	0.21*	-0.20*	6.46	0.21*
2	0.24	0.19*	-0.19*	0.99	0.19*
3	5.98	0.03	-0.05	1.17	0.04***
4	2.07	0.25*	-0.26*	-0.52	0.26*
5	-0.99	0.26*	-0.24*	1.65	0.25*
6	-0.40	0.14*	-0.14*	0.58	0.14*
7	-16.10	0.30*	-0.19*	-0.77	0.25*
8	0.53	0.36*	-0.35*	2.51	0.35*
9	-4.85	0.30*	-0.23*	4.21	0.27*
10	19.06	0.16*	-0.26*	3.87	0.21*
11*	18.53	0.12	-0.35*	-17.90*	0.28*
12	-7.61	0.25*	-0.21*	-1.47	0.23*
13	8.34	0.03***	-0.07*	2.12	0.05*
14	-25.17**	0.33*	-0.19*	-3.41	0.25*
15	-10.25	0.25*	-0.13**	8.70***	0.21*
16***	-9.24	0.05**	0.00	-2.31	0.02

*Stat. significant at 5%; **Stat. significant at 10%; ***Stat. significant at 15%.

†Diff. between the absolute value of slope coefficients for the two projects stat. significant.

Table 8a. Individual bidding functions (OLS).

Treatment B2				
	First 10	First 10	Last 10	Last 10
Subject	Constant term	Slope coefficient	Constant term	Slope coefficient
1	-0.22	0.18**	13.15	0.23*
2	-0.67	0.25*	1.17	0.18*
3	-0.33	0.04***	2.63	0.03
4	7.74	0.30*	-7.53	0.22*
5	2.32	0.24*	-1.29	0.28*
6	-0.50	0.15*	-1.59	0.13*
7	-0.41	0.24*	-0.87	0.26*
8	1.11	0.39*	2.63	0.33*
9	-1.19	0.21*	10.02	0.32*
10	-4.37	0.15*	11.47	0.29*
11	-8.15	0.17	-25.99**	0.38*
12	0.93	0.30*	-2.70	0.19*
13	1.77	0.05*	2.50	0.05**
14	-18.41**	0.28*	10.56***	0.27*
15	17.84**	0.16**	1.07	0.21*
16	-4.79	0.01	1.77	0.06***

*Stat. significant at 5%; **Stat. significant at 10%; ***Stat. significant at 15%.

Table 8b. Individual bidding functions for the first and last 10 rounds (OLS).

	C1	C2	B1	B2
Maximin	10%	20%	0%	0%
Bayes-Nash	50%	55%	50%	63%
Other	40%	25%	50%	37%
Total	100%	100%	100%	100%

Table 9. Observed strategies grouped into theoretical categories at 1% significance level.

	C1	C2	B1	B2
Maximin	10%	20%	0%	0%
Bayes-Nash	50%	55%	50%	63%
Other linear bidding	10%	5%	38%	19%
Other	30%	20%	13%	19%
Total	100%	100%	100%	100%

Table 10a. Observed strategies grouped into four theoretical categories at 1% significance level.

	C1		C2		B1		B2	
	First	Last	First	Last	First	Last	First	Last
Maximin	25%	25%	30%	35%	0%	0%	0%	0%
Bayes-Nash	35%	50%	50%	35%	69%	56%	69%	69%
Other linear bidding	5%	5%	5%	10%	31%	38%	13%	13%
Other	35%	20%	15%	20%	0%	6%	19%	19%
Total	100%	100%	100%	100%	100%	100%	100%	100%

Table 10b. Observed strategies grouped into four theoretical categories at 1% significance level for the first and last 10 rounds.

	C1	C2	B1	B2
Maximin	8.89	8.93	*	*
Bayes-Nash	8.87	7.99	8.38	8.17
Other linear bidding	10.28	8.73	9.42	9.02
Other	6.92	4.90	7.94	8.02

Table 10c. Mean payment (without show-up fee in EUR) according to the four theoretical equilibrium categories.

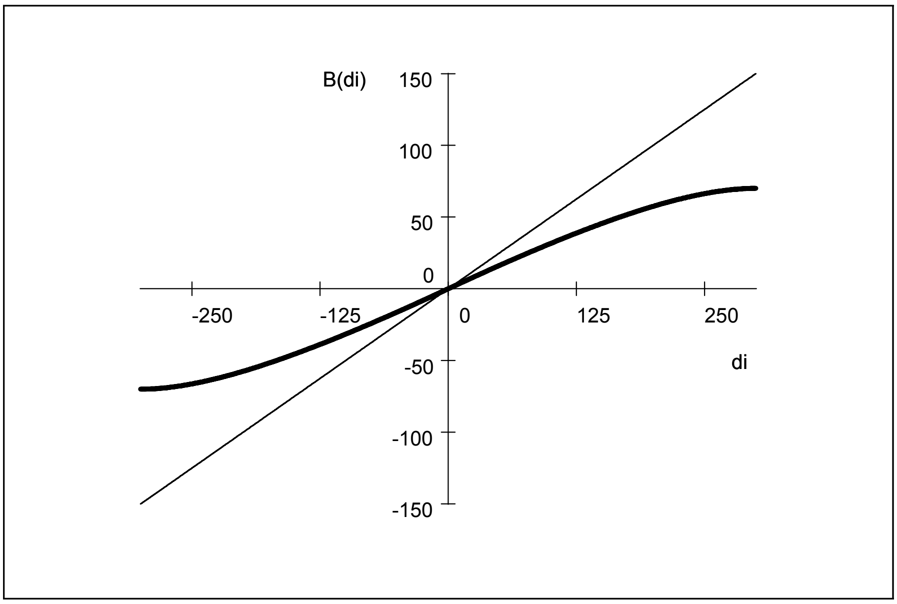
	Change in expected direction	Direction of significant* change		Not significant* change
		expected	unexpected	
Castellón	45%	25%	20%	55%
Barcelona	63%	6%	6%	88%

*Significance at 5% and 10%.

Table 11. Change in the slope of the empirical bidding function due to group size.

Distribution	Mean	St.deviation	Kurtosis	Skewness
$\widetilde{F}$	0	113.39	3.78	0
Triangular $[-300; 300]$	0	212.13	0.27	0

Table 12. Comparison between the triangular distribution and $\widetilde{F}$



Graph 1. Optimal Bayes-Nash and maximin bidding function for groups of two.

3.8 Appendix C. Instructions

3.8.1 First treatment

Thank you for participating in the experiment.

This session has 3 practice periods and other 20 that will determine a part of the amount of money that you will receive by the end of the experiment.

In each game groups of two will be formed in a random manner. Your task is to make decisions on your own and for this reason you are not allowed to talk to other participants. Games have a unique stage in which you will have to choose between two projects (project 1 and project 2). The resulting choice will influence the benefit you obtain in each period.

The first screen will inform you about the value each project has for you. The table on the left, in this example, shows that if project 1 is chosen you receive 33 monetary units; while if project 2 is chosen you receive 128 monetary units. These values are integer numbers between 0 and 300, and are assigned randomly in each game, such that every number has the same probability to be picked out. For this reason, these values are typically different for each player and for each project.

The other player in your group receives similar information on the values that each project has for him/her. You do not know the value of the projects for the other player, not even which project he/she prefers. He/she does not know the value of the project for you either. The only information in this aspect is the following:

The value of each project is an integer number between 0 and 300 (including limits) for each player. Each value within the limits occurs with the same probability. A common question is: what does it mean that each value occurs with the same probability? Suppose that we have a roulette wheel with 301 slots of equal size, numbered from 0 to 300. The ball in this case will stop with equal probability at each slot. In the experiment, the four values – for project 1 and project 2 for both players – are assigned using a similar method, with the help of the computer.

In our example, chance has assigned the values 250 and 102 for projects 1 and 2, respectively, for the other player.

The project is chosen through an auction especially designed for this occasion, according to which you have to decide how many monetary units you are willing to pay for project 1, for example, to be chosen. It is also possible that you prefer project 2 and for this reason if project 1 is chosen you wish to receive some amount of compensation. In the auction you have to choose two bids (one for each project) that must sum up to zero. Negative numbers will indicate the amounts you are willing to pay, while positive numbers the amounts that you wish to receive. Suppose that you are willing to pay 10 units if project 1 is chosen and you would like to receive 10 if project 2 is chosen. In this case you have to type the number -10 and 10 in the purple cells of the table on the right hand side; and after that click on the "OK" button to continue.

Let us suppose that the other player decides to bid -25 for project 1 and 25 for project 2. With this project 1 receives a total of -35 bids, while project 2 gets 35.

The project with more negative bids is chosen to carry out. In case of a tie the result is determined randomly. Bids for the chosen project will be paid / received and the aggregated bid will be given back to the members of the group in equal shares. When all of you have chosen your bids, a screen appears with the results.

The right side of the screen with the results informs you about the other player's bids. In our example project 1 has received $(-10) + (-25) = (-35)$ bids, while project 2 has received $10 + 25 = 35$. Project 1 is chosen. Your profit in the game appears on the left part of the results screen. In this case it is computed as follows:

- you receive 33 units, because project 1 has been chosen,
- you have to pay your bid for this project, that is 10 units, and
- you receive half of the aggregated bid, 17.5 units

Summing up: $33 - 10 + 17.5 = 40.5$ monetary units. The other player in the example earns $250 - 25 + 17.5 = 242.5$ units.

If you click on the "OK" button of the results screen, the game ends.

A table down on the left hand side keeps you informed about your profit obtained during the whole session. 400 monetary units are equal to 1 euro. For any computation you might want to perform, you may use the Windows calculator by clicking on its icon next to the "OK" button.

3.8.2 Second treatment

In this session, we will use the game from the previous session, but with one modification. The groups that form randomly in each game will have 8 members (not 2 as in the first session). Each group of 8 will choose a common project.

The auction to be used is the same. Your task is to make decisions on your own and for this reason you are not allowed to talk to other participants. Your principal task is to choose a project between two alternatives. The value of each project for each player is assigned in a random manner, therefore these values can be equal to any integer number between 0 and 300 (including limits), and each occurs with the same probability.

There will be 3 practice periods and other 20 that will determine a part of the amount of money that you will receive by the end of the experiment

The computer screens you will see are identical to the ones you have seen before except for one detail. On the results screen the aggregated bid of the other players in your group will appear.

The table on the left informs you about bids in the auction. Following the example in the instructions, let us suppose that you are willing to pay 10 monetary units if project 1 is chosen, and wish to receive 10 units if project 2 is chosen. The column of the other players' bid in this example indicates that the bids of the other 7 members of your group or project 1 sum up to -135 monetary units. The seven bids for project 2 sum up to 135.

Taking into account your bids, the aggregated bid for the projects are -145 and 145, respectively. For this reason, project 1 is chosen and you earn 37.5 monetary units: 33 (the value of project 1 for you) -10 (your bid for project 1) $+14.5$ (your share from the aggregated bid).

400 monetary units are equal to 1 euro. For any computation you might want to perform you may use the Windows calculator by clicking on its icon next to the "OK" button.

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Chapter 4

Fairness under Uncertainty with Indivisibilities

4.1 Introduction

Envy-free allocations are allocations for which every agent prefers his own bundle to the ones assigned to other agents. In the economies I deal with, a set of indivisible objects is to be distributed among a group of agents such that individuals consume at most one object. In general envy-free allocations might not exist, but when a proper amount of a perfectly-divisible good, typically money, is available in the economy, the set of envy-free allocations is not empty and indeed can be quite large. Alkan, Demange and Gale (1991) and Aragonés (1995) study these economies, the existence of envy-free allocations and how the amount of the divisible good affects the existence results. It is shown that for a sufficiently large amount of money the set of envy-free allocations is not empty. When negative distribution of money is not allowed there exists a minimum level of money that guarantees non-emptiness and implies a unique way to combine objects and money such that these bundles give rise to an envy-free allocation. Based on this result refinements can be defined that reduce the size of the set down to a singleton. It is the case of the so-called Rawlsian solutions proposed by Alkan, Demange and Gale (1991) and the one in Aragonés (1995) that in fact coincide. It is well known that in this environment envy-freeness implies Pareto efficiency and therefore envy-free allocations can be considered as fair ones, too. Also, some nice features of the envy-free set are proper to the indivisible case, as for example its lattice structure.

There are many situations in which this type of models can be useful. Nevertheless there are numerous cases that can not be handled due to the presence of uncertainty. The present work deals with the study of the latter. I study economies with any number of objects and individuals participating in the distribution as long as there are at least as many agents as objects. I review the above mentioned results taking into account economies in which information is not complete in some timing stages. A distinction is made among ex ante, interim and ex post stages; and according to that different envy-free, efficiency and fairness notions are defined. The intersection between the sets of ex ante Pareto optimal and ex post envy-free is particularly interesting and will be called ex ante intertemporally fair. Moulin (1997) pointed out that fairness from an ex ante point of view can be seen as a concept of procedural justice. It is a characteristic of the mechanism or game form itself and is independent of the way the game is played by the agents in the future. Ex post fairness can be interpreted as endstate justice that deals with a particular utility or judgement profile and a particular endstate in a given state of the nature. I take into account both judgement concepts, since my model considers the most restrictive definition for fairness that allows both for ex ante and ex post justice.

I keep the assumption that individuals can consume at most one indivisible object in a given state, but do not constrain the amount of money that can appear in consumption bundles. The former is clearly a restriction, nevertheless my model can account for numerous real-life problems and is accepted as a common assumption in the discrete literature. Two of the classical real-life situations covered by my framework appear in the following examples.

Example 1 *A group of students decides to share a flat. There are as many rooms in the flat as students and it is agreed upon that everyone will have a private room and noone leaves or enters the group later. Rooms are of different characteristics (size, quietness, etc.) and the students have their own private valuations over them. These valuations might be unknown at the time they enter in the flat as they might have never lived in a similar situation before. The problem to be solved - before (ex ante), immediately after (interim) or after (ex post) entering the flat - is to assign a room to every students and decide the share of the rent each of them must pay in a fair way.*

Example 2 *A number of villages has to participate in a flood-protection project. There is a certain number of tasks to be executed and some amount of money available for the realization. Tasks are of different characteristics and the villages are supposed to have their own valuations over them. For example, some tasks might be or might look easier to carry out for a village than for the others, etc. Nevertheless, these villages might have never participated in a similar cooperation and therefore could have some uncertainty when evaluating future possibilities, e.g. for example the ones of success or failure in their tasks. The problem to be solved by the central authority is to assign a task to every village and with it a share of the fixed budget in a possibly fair way.*

As is well known, competitive equilibrium theory runs into difficulties when considering indivisibilities. However, there exists a special case that has been studied in the literature that is tractable. Here that framework with several indivisible objects and a perfectly divisible one (usually thought of as money) is adopted. I prove that ex ante intertemporally fair social choice functions exist whenever certain condition on prior beliefs and preferences holds. Beside of the constructive nature of the proof, the importance of that condition is shown in two simple numeric examples and through identifying an intuitive special case of the condition: If agents have the same prior beliefs and preferences show constant marginal utility of money among agents for a given state, then the set of ex ante intertemporally fair state-contingent allocations is not empty. Some fairness results under uncertainty without indivisibilities are discussed in Gajdos and Tallon (2001). They prove existence in the adopted perfectly divisible framework. I obtain similar results to those in Gajdos and Tallon (2001) according to which the existence of intertemporally fair allocations depends on agents' prior beliefs, for a given state they must be the same for every agent. In contrast to Gajdos and Tallon (2001), here utilities are state dependent, and the necessary and sufficient condition for existence is slightly less restrictive.

After considering the problem of existence I proceed to implementation matters. The literature under certainty offers the characterization of fair allocations, and gives methods to find them once the social planner (or some central government) learns the individuals' preferences. A constructive and very elegant way to find them is presented in Su (1999) that is based on a simple combinatorial lemma due to Sperner in 1928. The leading example in Su (1999) is flat sharing as in Example 1. This paper studies the case in which agents behave strategically, the social planner is not informed about the preferences, and players are not completely informed either. Relying on results in Palfrey and Srivastava (1987), for Bayesian implementability the condition of non-exclusive information is introduced and a mechanism that implements the set of non-wasteful ex post envy-free social choice functions in Bayes-Nash equilibrium is defined. A subsection deals with the problem of implementation at the ex ante stage in which information is symmetric. Due to this fact I conclude that simple mechanisms of the "divide-and-permute" or "cake-cutting" type can be used to implement the set of ex ante intertemporally social choice functions. This result is presented beside of the ones in McAfee (1992) according to which the cake-cutting mechanism produces efficient results under symmetric information, but under asymmetric information it is ex post inefficient in an unusual way.

The fair-division literature has already examined the above implementation problem under certainty with indivisibilities and two players. Crawford and Heller (1979), for example, showed that a modified version of the divide-and-choose mechanism performs well in the adopted set-up.

The present paper is organized as follows: Section 2 introduces the formal model and defines the basic concepts of fairness that are studied, while Section 3 deals with the question of existence. A subsection presents a generalization of the Rawlsian refinement proposed by Alkan, Demange and Gale (1991) and Aragonés (1995). On Section 4 I discuss implementation matters.

Considering the first chapters of this work, it is a self-contained study based on the axiomatically accepted notion of intertemporal fairness that embodies envy-freeness. The literature on distributive justice usually follows a similar path and does not deal with the problematic of choosing fairness criteria. However, an extra section (Section 5) is included that considers the aspiration function as an appropriate tool for studying fairness without restricting attention on a particular concept.

Corchón and Iturbe-Ormaetxe (2001) offers a detailed study of fairness in a generalized set-up. Section 5 here can be seen as the adaptation of some very few definitions from Corchón and Iturbe-Ormaetxe (2001) to the uncertainty case with indivisibilities. The most important point in that part of the paper is the generalization of the existence result. I find that under the conditions stated for the envy-free case, and under some restrictions on personal aspirations, an intertemporally fair social choice function exists. Intertemporally fair is now a broader concept that allows among others for envy-free and for the egalitarian solutions. The condition is sufficient and necessary here, too. At the end of the paper I study the problem of implementation of the set of the generalized intertemporally fair social choice functions. I conclude with a positive result: a condition (on the fairness concept) that is necessary for Bayesian monotonicity, i.e. for Bayesian implementation is derived.

4.2 The model

Let N be a finite set of agents, O a finite set of indivisible objects and S a finite set of possible states of nature. The typical elements of the sets are i, o and s respectively. For simplicity I shall denote the cardinality of the sets by the same symbols N, O and S. There is also a perfectly divisible good in the economy called money, the total available amount of which is $M \in \mathbb{R}$. Each agent can consume at most one indivisible object and any amount of money.

For simplicity and presentational considerations assume that the set of agents and the set of objects have the same cardinality. This is a standard assumption in the literature. If there are at least as many agents as objects one can achieve this situation by introducing worthless null objects. Hence, the analysis holds for any economy with more agents than objects, too. The reverse case in which there are more objects than agents is tractable, too. Alkan, Demange and Gale (1991) present an argument in a set-up without uncertainty. It requires fictitious agents that only value money, and also different definitions for efficiency, envy-freeness, etc. To keep things simple this case is not considered here.

In the economy there is uncertainty concerning the state of the nature. As for timing, I distinguish three stages: In the ex ante stage information is symmetric, but agents have to cope with uncertainty as they do not know which state of nature will occur. The interim stage is the one in which a given state of nature has already occurred, but agents cannot observe it perfectly and may own private information. This informational asymmetry is dissolved at the next, ex post stage when agents are completely informed about the state. There is no aggregate risk in my model, as the list of objects and the amount of money if fixed across all states of the world.

In general, information available to agent i is given by a partition Π^i of the set S, where the event in partition Π^i that contains the state s is denoted by $E^i(s)$. From an ex ante point of view a prior probability distribution can be defined over states, that for agent i and state s will be denoted by $q^i(s) > 0$. I shall assume that the set S does not contain any redundant elements, that is $\cap_i E^i(s) = \{s\}$ for all $s \in S$.

Allocations in this economy will be represented by vectors in $A = (O \times \mathbb{R})^N$. Let $A^i = O \times \mathbb{R}$ denote player i's set of allocation. Now the set of allocations can be expressed as a Cartesian product $A = A^1 \times \ldots \times A^N$. For example, an allocation is given by

$$a = \left(a^1, ..., a^N\right) = \left[\left(o^1, m^1\right), ..., \left(o^N, m^N\right)\right]$$

where a^i stands for the bundle that agent i consumes in which she receives object

o^i that may be as well the empty set. The amount of money that agent i enjoys in the given allocation is m^i. The set of feasible allocations is defined as

$$A^f = \left\{ a \in A : o^i \neq o^j \text{ for any } i \neq j \text{ with } o^i \neq \varnothing,\, o^j \neq \varnothing, \text{ and } \sum_{i=1}^{N} m^i \leq M \right\}.$$

An allocation will be called non-wasteful if every agent has an object and the money shares sum up to M. Formally,

$$A^{fnw} = \left\{ a \in A^f : \bigcup_{i=1}^{N} \left\{ o^i \right\} = O, \text{ and } \sum_{i=1}^{N} m^i = M \right\}.$$

Non-wasteful allocations are those feasible ones in which every object finds an owner and the money shares sum up to the total available amount, M.

Let $X = \left\{ x : S \longrightarrow A^f \right\}$ be the set of feasible state-contingent allocations, that also will be referred to as social choice functions.[1] A social choice set is a subset $F \subset X$. The sets of non-wasteful social choice functions (X^{fnw}) and social choice sets (F^{fnw}) are defined in an similar way. Agents' preferences are represented by state-dependent utility functions, $u^i\left(x^i\left(s\right), s\right)$. I shall suppose preferences are quasi-linear in money. If $\phi^i\left(s\right)$ represents agent i's marginal utility of money in state s then the utility function can be written as

$$u^i\left(x^i\left(s\right), s\right) = u^i\left[o^{xi}\left(s\right), m^{xi}\left(s\right), s\right] = u^i\left[o^{xi}\left(s\right), s\right] + \phi^i\left(s\right) \cdot m^{xi}\left(s\right)$$

with $\phi^i\left(s\right) > 0$ finite for all i and s, where $o^{xi}\left(s\right)$ denotes the indivisible object and $m^{xi}\left(s\right)$ the money that agent i consumes in state s according to the social choice

[1]Note that this definition differs from the one generally used in the social choice literature, since now the social choice function is defined for a given economy, precisely over the set of possible states in that economy. However our definition is a standard one in the Bayesian implementation literature.

function x. Assume that no indivisible object is infinitely desirable or undesirable as compared to money. That is $u^i\left(o^{xi}\left(s\right),s\right)$ is bounded for every i and s. An economy is represented by a list

$$\mathcal{E}=\left(N,O,S,M,\left[q^i\left(s\right)\right]_{i\in N},\left[u^i\left(s\right)\right]_{i\in N}\right).$$

More notation is introduced in the text when needed.

Before continuing with fairness concepts, recall Example 1 and identify the ex ante, interim and ex post stages. Now states of nature can be defined as utility profiles taking into account how students value the different rooms available in the flat and, in comparison with them, money. Before moving into the new flat, i.e. ex ante, students are not supposed to know how they value the rooms, because they have never lived in the flat before nor have any information about their characteristics. They deal with uncertainty of the same type in a symmetric way. At the interim stage, when they arrive and can have a first look around they are able to observe the characteristics of the flat, therefore can tell how they personally value the rooms. Nevertheless, they can not identify the state of the nature, since private valuations may not be announced truthfully or observable. Therefore, students in this stage have to cope with uncertainty in an asymmetric way. Uncertainty disappears at the ex post stage in which, after some time of living together, students know how their flat-mates think about the flat and value its rooms.

4.2.1 Fairness concepts

In order to analyze fairness in this model I introduce some useful concepts by the following definitions. They consider widely used fairness notions from the literature and continue with the distinction among the three timing stages.

Definition 1 *A non-wasteful social choice function* x *is ex post Pareto optimal if there is no non-wasteful social choice function* y *such that*

$$u^i\left[y^i(s), s\right] \geq u^i\left[x^i(s), s\right]$$

for all i *in* N *and all* s *in* S*, and with strict inequality for at least one* i *and one* s*. Let* P_p *denote the set of ex post Pareto optimal social choice functions.*

Definition 2 *A non-wasteful social choice function* x *is interim Pareto optimal if there is no non-wasteful social choice function* y *such that*

$$\sum_{s\in E^i(s^*)} q^i\left[s \mid E^i(s^*)\right] \cdot u^i\left[y^i(s), s\right] \geq \sum_{s\in E^i(s^*)} q^i\left[s \mid E^i(s^*)\right] \cdot u^i\left[x^i(s), s\right]$$

for all i *in* N *and* s^* *in* S*, with strict inequality for at least one* i *and one* s^**. Let* P_i *denote the set of interim Pareto optimal social choice functions.*[2]

Definition 3 *A non-wasteful social choice function* x *is ex ante Pareto optimal if there is no non-wasteful social choice function* y *such that*

$$\sum_{s\in S} q^i(s) \cdot u^i\left[y^i(s), s\right] \geq \sum_{s\in S} q^i(s) \cdot u^i\left[x^i(s), s\right]$$

for all i *in* N *and with strict inequality for at least one* i*. Let* P_a *denote the set of ex ante Pareto optimal social choice functions.*

Non-wastefulness has been included in the definitions for Pareto efficiency, but it is not a requirement for envy-freeness.

[2] $q^i\left[s \mid E^i(s^*)\right]$ is the probability that agent i assigns to state s conditional on the information she owns after that state s^* has occured.

Definition 4 *A social choice function* x *is ex post envy-free if*

$$u^i\left[x^i(s), s\right] \geq u^i\left[x^j(s), s\right]$$

for all i, j *in* N *and* s *in* S. *Let* EF_p *denote the set of ex post envy-free social choice functions.*

Definition 5 *A social choice function* x *is interim envy-free if*

$$\sum_{s\in E^i(s^*)} q^i\left[s \mid E^i(s^*)\right] \cdot u^i\left[x^i(s), s\right] \geq \sum_{s\in E^i(s^*)} q^i\left[s \mid E^i(s^*)\right] \cdot u^i\left[x^j(s), s\right]$$

for all i, j *in* N *and* s^* *in* S. *Let* EF_i *denote the set of interim envy-free social choice functions.*

Definition 6 *A social choice function* x *is ex ante envy-free if*

$$\sum_{s\in S} q^i(s) \cdot u^i\left[x^i(s), s\right] \geq \sum_{s\in S} q^i(s) \cdot u^i\left[x^j(s), s\right]$$

for all i *and* j *in* N. *Let* EF_a *denote the set of ex ante envy-free social choice functions.*

The literature offers some general results on the structure of these sets. They are summarized in the following propositions.

Proposition 1 *If a social choice function is ex ante Pareto efficient, then it is also interim Pareto efficient. If a social choice function is interim Pareto efficient, then it is also ex post Pareto efficient. That is* $P_a \subset P_i \subset P_p$.

Proof. See Laffont (1981), Mas-Colell, Whinston and Greene (1995) and also check Holmstrom and Myerson (1983). ■

Proposition 2 *If a social choice function is ex post envy-free, then it is also interim envy-free. If a social choice function is interim envy-free, then it is also ex ante envy-free. That is* $EF_a \supset EF_i \supset EF_p$.

Proof. See Gajdos and Tallon (2001). ■

The referred papers give proofs for the perfectly divisible case that can be adapted directly to the present framework. For this reason specific proofs are not included here.

Now fairness can be defined. I use the definition adopted in the literature and call ex ante intertemporally fair those social choice functions that are both ex ante Pareto optimal and also ex post envy-free. The main reason for adopting this definition is that allocations in the intersection cannot be criticized from any point of view and/or any timing stage considered in this paper, as this intersection is the most restrictive among all the possible ones formed by the introduced sets. I emphasize the ex ante point of view in this definition in order to make contrast to the similar interim fairness concept that is introduced later.

Definition 7 *A social choice function* x *is ex ante intertemporally fair if* $x \in P_a \cap EF_p$.

4.3 Existence

In this section I focus on the existence of ex ante intertemporally fair allocations. As for the case of economies with uncertainty, but without indivisibilities it is shown in

Gajdos and Tallon (2001) that an ex ante intertemporally fair social choice function exists whenever agents' prior beliefs coincide. Alkan, Demange and Gale (1991) study an economy with indivisible goods in a framework without uncertainty, but in other features very similar to ours. They study whether there exist allocations that are both efficient and envy-free. They reach the existence result across two steps: first is shown that non-wasteful envy-free allocations are also Pareto optimal in this specific environment. Then the existence of envy-free allocations is proved, that combined with the previous result establishes non-emptiness for the set of fair allocations. The proof is constructive and can be used to give an algorithm for finding a fair allocation in the indivisible case. In this paper I shall follow a similar path. Unfortunately the set of intertemporally fair social choice functions may be empty. Proposition 3 gives a necessary and sufficient condition under which the set of ex ante intertemporally fair social choice functions in not empty. The basic idea of the proof is that under that condition ex post envy-free social choice functions are also Pareto optimal. More discussion of the result is presented after the proof.

Proposition 3 *Consider the economy*

$$\mathcal{E} = \left\{ N, O, S, M, \left[q^i(s)\right]_{i\in N}, \left[u^i(s)\right]_{i\in N} \right\}.$$

An ex ante intertemporally fair social choice function exists if and only if

$$\text{for all } s,\ s' \text{ in } S \text{ there exists } \gamma(s,s') \in \mathbb{R}^+ \text{ such that}$$
$$q^i(s)\cdot\phi^i(s) = \gamma(s,s')\cdot q^i(s')\cdot\phi^i(s') \text{ for all } i \text{ in } N. \qquad \text{(Condition 1)}$$

Proof. With mathematical terms the proposition says that $EF_p \cap P_a \neq \varnothing$ if and only if Condition 1 holds. The proof is presented in two parts according to the two directions of implication.

1. The if part: I know that the set EF_p is not empty (see for example see Alkan, Demange and Gale (1991)). Therefore it is enough to prove that under Condition 1 any non-wasteful ex post envy-free efficient social choice function is ex ante Pareto efficient. That is for non-wasteful social choice functions I have the inclusion $EF_p \subset P_a$.

Let x be a non-wasteful ex post envy free social choice function, $x^i(s) = [o^{xi}(s), m^{xi}(s)]$ and $y\,(\neq x)$, $y^i(s) = [o^{yi}(s), m^{yi}(s)]$ any non-wasteful social choice function. Let us suppose that y ex ante Pareto dominates x. This means that for every i

$$\sum_{s\in S} q^i(s) \cdot \left\{u^i\left[o^{yi}(s), s\right] + \phi^i(s) \cdot m^{yi}(s)\right\} \geq \sum_{s\in S} q^i(s) \cdot \left\{u^i\left[o^{xi}(s), s\right] + \phi^i(s) \cdot m^{xi}(s)\right\} \tag{4.3.1}$$

holds and with strict inequality for some i_0. Since x is ex post envy-free for all i I have:

$$\sum_{s\in S} q^i(s) \cdot \left\{u^i\left[o^{xi}(s), s\right] + \phi^i(s) \cdot m^{xi}(s)\right\} \geq \sum_{s\in S} q^i(s) \cdot \left\{u^i\left[o^{yi}(s), s\right] + \phi^i(s) \cdot m^{x\rho(i)}(s)\right\}, \tag{4.3.2}$$

where $\rho(i)$ is the agent who receives object $o^{yi}(s)$ under x in state s. I let $\rho(i)$ denote that agent without making reference to state s, since I do not need to specify it for the proof. The bundle with $o^{yi}(s)$ also consisted of $m^{x\rho(i)}(s)$ units of money besides the object. Finding $\rho(i)$ is like permuting the agents among themselves; and therefore summing for $\rho(i)$ is like summing up for i in every state, and vice versa. Now let us multiply Equations 1 and 2 by an appropriate positive number, $\lambda^i \in \mathbb{R}^+$ for every i, where λ^i is defined such that $\lambda^i \cdot q^i(s_1) \cdot \phi^i(s_1) = 1$ for all i. After that sum up Equations 1 and 2 on both sides for all agents, and take the right end and the left end of the resulting expression

$$\sum_{i=1}^{N}\sum_{s\in S}\lambda^{i}\cdot q^{i}\left(s\right)\cdot u^{i}\left[o^{yi}\left(s\right),s\right]+\phi^{i}\left(s\right)\cdot m^{yi}\left(s\right)>$$
$$>\sum_{i=1}^{N}\sum_{s\in S}\lambda^{i}\cdot q^{i}\left(s\right)\cdot u^{i}\left[o^{yi}\left(s\right),s\right]+\phi^{i}\left(s\right)\cdot m^{x\rho(i)}\left(s\right)$$

That is, $\sum_{i=1}^{N}\sum_{s\in S}\lambda^{i}\cdot q^{i}\left(s\right)\cdot\phi^{i}\left(s\right)\cdot m^{yi}\left(s\right)>\sum_{i=1}^{N}\sum_{s\in S}\lambda^{i}\cdot q^{i}\left(s\right)\cdot\phi^{i}\left(s\right)\cdot m^{x\rho(i)}\left(s\right)$.
Note that fixing s_1 by Condition 1 for any i and s' I have that $\lambda^{i}\cdot q^{i}\left(s'\right)\cdot\phi^{i}\left(s'\right)=\frac{1}{\gamma(s_1,s')}$. Therefore the last inequality can be rewritten in the following form.

$$\sum_{i=1}^{N}\sum_{s\in S}\frac{m^{yi}\left(s\right)}{\gamma\left(s_1,s\right)}>\sum_{i=1}^{N}\sum_{s\in S}\frac{m^{xi}\left(s\right)}{\gamma\left(s_1,s\right)}$$

By non-wastefullness of x I have $\sum_{i=1}^{N}m^{yi}\left(s\right)=\sum_{i=1}^{N}m^{xj}\left(s\right)=M$ for all s. Now what is left over from the previous inequality is $M\cdot\sum_{s\in S}\frac{1}{\gamma(s_1,s)}>M\cdot\sum_{s\in S}\frac{1}{\gamma(s_1,s)}$ that is clearly impossible. I have reached a contradiction, hence x is ex ante Pareto optimal.

2. The only if part: Also this part of the proof is by contradiction. I shall show that any non-wasteful social choice function that is ex post envy-free cannot be ex ante Pareto efficient if Condition 1 does not hold. As before, take λ^{i} such that $\lambda^{i}\cdot q^{i}\left(s_1\right)\cdot\phi^{i}\left(s_1\right)=1$ for all i with $\lambda^{i}\in\mathbb{R}^{+}$. Let $\delta^{i}\left(s\right)=\frac{1}{\lambda^{i}\cdot q^{i}(s)\cdot\phi^{i}(s)}$. Now take s^{*} such that $\delta^{i}\left(s^{*}\right)\neq\delta^{j}\left(s^{*}\right)$ for some agents $i\neq j$. Note that such a state s^{*} always exists if Condition 1 does not hold. For simplicity suppose that I have that inequality for agents i_1 and i_2, and also that $\delta^{i_1}\left(s^{*}\right)>\delta^{i_2}\left(s^{*}\right)$. Take any non-wasteful ex post envy-free social choice function, x, and consider the following transfers (distortioning x) among agents: if s^{*} occurs agent i_1 pays one monetary

unit to agent i_2; if s_1 occurs agent i_2 pays $\frac{1}{\delta^{i_1}(s^*)}$ monetary units to agent i_2. With this the expected utilities will change in the following manner.
For agent i_1:

$$q^{i_1}(s_1) \cdot \phi^{i_1}(s_1) \cdot \frac{1}{\delta^{i_1}(s^*)} + q^{i_1}(s^*) \cdot \phi^{i_1}(s^*) \cdot (-1) = 0$$

For agent i_2:

$$q^{i_2}(s_1) \cdot \phi^{i_2}(s_1) \cdot \left(-\frac{1}{\delta^{i_1}(s^*)}\right) + q^{i_2}(s^*) \cdot \phi^{i_2}(s^*) =$$
$$= -\frac{\lambda^{i_2} \cdot \beta}{\delta^{i_1}(s^*)} + \frac{\lambda^{i_2} \cdot \beta}{\delta^{i_2}(s^*)} > 0$$

Clearly, this means an ex ante Pareto improvement that concludes the proof.
■

Condition 1 contains as a special case an intuitive restriction on the economy in order to guarantee the existence of ex ante intertemporally fair social choice functions. It is stated in the following corollary.

Corollary 1 *If agents have the same prior beliefs and preferences show constant marginal utility of money among agents for a given state, then there exist social choice functions that are ex ante intertemporally fair, that is,* $EF_p \cap P_a \neq \varnothing$.

Proof. The result is a direct consequence of Proposition 3, since if agents have the same prior beliefs and preferences show constant marginal utility of money among agents for a given state, then Condition 1 is satisfied. ■

The proof of Proposition 3 is based on the idea that ex post envy-freeness can be sacrificed in order to reach some ex ante Pareto improvement whenever Condition 1 does not hold. The following two examples contain numerical illustration for this in a simple economy with two possible states and one valuable object - plus $(N-1)$ null-objects. Example 3 considers an ex post envy-free social choice function and allow for money transfers between agents with different marginal utility of money. The resulting state-dependent allocation represents an ex ante Pareto improvement, but it is not ex post envy-free anymore.

Example 3 *Take the case in which* $M = 10$, *preferences show different marginal utility for money for some agents, there are only two states of nature,* s_1 *and* s_2, *and only one indivisible good complemented by* $(N-1)$ *null objects. Let us suppose that* $u^{i_0}\left[o^{i_0}(s), m^i(s), s\right] = \begin{Bmatrix} 20o^{i_0} + m^i \text{ if } s = s_1 \\ 10o^{i_0} + 2m^i \text{ if } s = s_2 \end{Bmatrix}$ *and* $u^i\left(o^i(s), m^i(s), s\right) = \begin{Bmatrix} 20o^i + 2m^i \text{ if } s = s_1 \\ 20o^i + m^i \text{ if } s = s_2 \end{Bmatrix}$ *for all* $i \in N \backslash \{i_0\}$. *Consider also that* $q^i(s) = \begin{Bmatrix} 0.4 \text{ if } s = s_1 \\ 0.6 \text{ if } s = s_2 \end{Bmatrix}$ *for all* $i \in N$. *The following social choice function is non-wasteful and ex post envy free: in every state give the object to* i_0 *and also* $\left(\frac{20}{N} - 10\right)$ *units of money, let each of the other agents receive* $\left(\frac{20}{N}\right)$ *units of money. This social choice function is ex ante Pareto dominated by the following one: if* $s = s_1$, *then give the object to* i_0 *and also* $\left(\frac{20}{N} - 11\right)$ *units of money, let* $i_1 \neq i_0$ *get* $\left(\frac{20}{N} + 1\right)$ *and each of the other agents receive* $\left(\frac{20}{N}\right)$ *units of money; if* $s = s_2$, *then give the object to* i_0 *with* $\left(\frac{20}{N} - 9\right)$ *units of money, give* $\left(\frac{20}{N} - 1\right)$ *money to* $i_1 \neq i_0$ *and let each of the other agents receive* $\frac{20}{N}$ *units of money. The latter social choice function is clearly not ex post envy-free.*

Note that in the same way one can ex ante Pareto improve any ex post envy-free state-contingent allocation whenever there is different marginal utility for money for some agents in the same state and the following condition does not hold:

For all s, s' $\phi^i(s) = \gamma(s, s') \cdot \phi^i(s')$ for all i with $\gamma(s, s') \in \mathbb{R}^+$. (Condition 2)

Example 4 considers a similar economy to the one in Example 3, but now agents will not share a common prior distribution and this will allow for ex ante Pareto improvements in the case of any ex post envy-free social choice function.

Example 4 *Take the case in which $M = 10$, preferences show the same marginal utility of money for every agent, there are only two states of nature, s_1 and s_2, and only one indivisible good complemented by $(N-1)$ null objects. Let us suppose that $u^{i_0}[o^{i_0}(s), s] = \begin{Bmatrix} 20o^{i_0} \text{ if } s = s_1 \\ 10o^{i_0} \text{ if } s = s_2 \end{Bmatrix}$ and $u^i[o^i(s), s] = \begin{Bmatrix} 10o^i \text{ if } s = s_1 \\ 20o^i \text{ if } s = s_2 \end{Bmatrix}$ for all $i \in N \setminus \{i_0\}$.*
Consider also that $q^{i_0}(s) = \begin{Bmatrix} 0.2 \text{ if } s = s_1 \\ 0.8 \text{ if } s = s_2 \end{Bmatrix}$ and $q^i(s) = \begin{Bmatrix} 0.8 \text{ if } s = s_1 \\ 0.2 \text{ if } s = s_2 \end{Bmatrix}$ for all $i \in N \setminus \{i_0\}$. The following social choice function is non-wasteful and ex post envy free: if $s = s_1$, then give the object to i_0 and also $\left(\frac{20}{N} - 10\right)$ units of money, let each of the other agents receive $\left(\frac{20}{N}\right)$ units of money; if $s = s_2$, then give the object to $i_1 \neq i_0$ with $\left(\frac{30}{N} - 20\right)$ units of money and let each of the other agents receive $\left(\frac{30}{N}\right)$ units of money. This social choice function is ex ante Pareto dominated by the following one: if $s = s_1$, then give the object to i_0 and also $\left(\frac{20}{N} - 11\right)$ units of money, let $i_1 \neq i_0$ get $\left(\frac{20}{N} + 1\right)$ and each of the other agents receive $\left(\frac{20}{N}\right)$ units of money; if $s = s_2$, then give the object to $i_1 \neq i_0$ with $\left(\frac{30}{N} - 21\right)$ units of money, give $\left(\frac{30}{N} + 1\right)$ money to i_0 and let each of the other agents receive $\frac{30}{N}$ units of money. The latter social choice function is clearly not ex post envy-free.

Corollary 2 *Under Condition 2 I have that $EF_a \cap P_a \neq \varnothing$, $EF_i \cap P_i \neq \varnothing$, $EF_p \cap P_p \neq \varnothing$, $EF_p \cap P_a \neq \varnothing$, $EF_p \cap P_i \neq \varnothing$.*

Proof. The results are direct consequences of previous propositions and the inclusion results among the sets in question. ■

I have not put any restriction on the sign of the amount of money contained in the bundles. The possibility of negative distribution of money might be essential for the existence of intertemporally fair social choice functions. As discussed in Alkan, Demange and Gale (1991) and Aragonés (1995) in the certainty case, if distribution of money is restricted to be positive, then for the existence result to hold one must be sure that there is enough money to be distributed in the economy. This finding can be easily presented for the uncertainty case as well. Supposing that Condition 1 holds, the amount of money in the economy M, that is not state-dependent, should be large enough to be able to assure existence in every state of the nature.

Note that until this point I have been following an ex ante approach, because I have been dealing with a symmetric situation, considering the uncertainty of each agent not knowing which state of nature from S will occur. This is the reason for putting the qualification ex ante before intertemporally fair social choice functions. Nevertheless, fairness for the interim stage can defined in a similar way. It is interesting that if I move to the interim stage, that is I consider for example that state s_1 has occurred then there are only degenerated cases in which ex post envy-freeness combined with non-wastefulness implies interim Pareto optimality. For the formal definition of interim efficiency and more details on the statement check Appendix A.[3]

Condition 1 plays a decisive role in the existence of intertemporally fair social choice functions, as it is required to ensure ex ante Pareto efficiency. I have shown in a formal proof and also illustrated with two examples that without it one can always

[3]We state the following proposition (Proposition 10) in the appendix, because the interim considerations do not constitute the main objective of this paper.

find an ex ante Pareto improvement. If I were to define intertemporal fairness with the help of ex post Pareto efficiency I could do without Condition 1. According to the fairness literature with certainty an (ex post) envy-free and (ex post) Pareto efficient social choice function exists. Since this result holds for every state s I have existence with all the possible definitions of intertemporal fairness that deals with ex post efficiency.

4.3.1 Structure of the fair set

The set of envy free allocations, in a set-up with indivisible goods without uncertainty, has a nice structure as was shown by Alkan, Demange and Gale (1991). My next results generalize this finding for the case of uncertainty supposed that Condition 1 holds. In particular I show that the set of ex post envy-free social choice functions owns the lattice property. In order to do so some pieces of notation have to be introduced.

Let x and y be two social choice functions and let

$$\overline{u}_x(s) = \left[u_x^1(s), \ldots, u_x^N(s)\right]$$

denote the vector of utility levels that players enjoy according to x in state s. The owner of object o in state s will be referred to as i^o. An other vector $\overline{u}_y(s)$ is defined similarly. Now consider the following sets

$$\begin{aligned}
N_x^s &= \left\{i \in N : u_x^i(s) > u_y^i(s)\right\}, \; O_x^s = \left\{o \in O : m^{xi^o}(s) > m^{yi^o}(s)\right\}, \\
N_y^s &= \left\{i \in N : u_y^i(s) > u_x^i(s)\right\}, \; O_y^s = \left\{o \in O : m^{yi^o}(s) > m^{xi^o}(s)\right\}, \\
N_0^s &= \left\{i \in N : u_x^i(s) = u_y^i(s)\right\}, \; O_0^s = \left\{o \in O : m^{xi^o}(s) = m^{yi^o}(s)\right\}.
\end{aligned}$$

The social choice function x induces a mapping between N and O for every state s. It attaches to every agent in N an object from O, precisely the object that the agent receives according to x in state s. Alkan, Demange and Gale (1991) in a set-up without uncertainty about the state of nature show that for any state s two ex post envy-free social choice functions x and y are indeed bijections between N_x^s and O_x^s, N_y^s and O_y^s, N_0^s and O_0^s. A consequence of this result is the lattice property for which the following operators are defined. Given vectors a and b

$$\begin{aligned} a \vee b &= c, \text{ where } c_i = \min(a_i, b_i), \\ a \wedge b &= c, \text{ where } c_i = \max(a_i, b_i). \end{aligned}$$

Let $z = x \wedge y$ be a social choice function defined as follows

$$\text{a) for every state } s \text{ in } S,\ o^{zi}(s) = \left\{ \begin{array}{c} o^{xi}(s) \text{ if } i \in N_x^s \\ o^{yi}(s) \text{ if } i \in N_y^s \cup N_0^s \end{array} \right\};$$

$$\text{b) for every state } s \text{ in } S,\ m^{zi}(s) = m^{xi}(s) \wedge m^{yi}(s).$$

Proposition 4 (Lattice property) *If x and y are ex post envy-free social choice functions, then the social choice function $x \wedge y$ is ex post envy-free.*

Proof. I omit the proof as, using the above introduced notation, it is very similar to the one presented in Alkan, Demange and Gale (1991). ■

As shown in Alkan, Demange and Gale (1991), similar result holds with the minimum operator. This result is useful when defining refinements on the set of ex post envy-free social choice functions that can be very large in general. For a given social choice function x and state s I write $\overline{u}_x^{\min}(s) = \min_{i \in N} \overline{u}_x(s)$ and $\overline{u}_x^{\max}(s) = \max_{i \in N} \overline{u}_x(s)$.

Definition 8 *A non-wasteful ex post envy-free social choice function x is called Rawlsian if $\overline{u}_x^{\min}(s) \geq \overline{u}_y^{\min}(s)$ for all non-wasteful ex post envy-free social choice function y and state s.*

The set of Rawlsian ex post envy-free social choice functions is well-defined thanks to the lattice property. A Rawlsian social choice function gives a Rawlsian allocation in every possible state of nature. As in the case without uncertainty one can show that the elements of the set of Rawlsian ex post envy-free social choice functions induce the same utility level profile. The result is trivial if one considers that in a given state s utility levels are the same for all Rawlsian allocations in that state.

4.4 Implementation

Now that non-emptiness of the ex ante intertemporally fair set is assured under Condition 1, I can turn my attention to implementation matters. I shall suppose that Condition 1 holds and will concentrate on the implementation of the set of non-wasteful ex post envy-free social choice functions. First implementation at the interim stage is considered, i.e. after the occurrence of a given state when information is asymmetric. In the interim set-up Bayesian implementation is the adequate tool. ex ante implementation is studied later in a separate subsection. Now I introduce some extra notation that will be used in this section.

A mechanism for an economy is a pair (Σ, g), where $\Sigma = \Sigma^1 \times \ldots \times \Sigma^N$, $g : \Sigma \to A^f$. A strategy for agent i is $\sigma^i : \Pi^i \to \Sigma^i$. A deception for agent i is a mapping $\alpha^i : \Pi^i \to \Pi^i$, $\alpha = \left(\alpha^1, ..., \alpha^N\right)$.

The following definition of implementation comes from Jackson (1991).

Definition 9 *A mechanism* (Σ, g) *implements in Bayes-Nash equilibrium a social choice set,* F *if:*
a) for any $x \in F$ *there exists a Bayes-Nash equilibrium* σ *with* $g\{\sigma[E^i(s)]\} = x(s)$ *for all* s*, and*
b) for any Bayes-Nash equilibrium σ *there exists* $x \in F$ *with* $g\{\sigma[E^i(s)]\} = x(s)$ *for all* s*.*

As shown in Jackson (1991) Bayesian incentive compatibility is needed for Bayesian implementability.

Definition 10 *A social choice set F satisfies Bayesian Incentive Compatibility is for all* $x \in F$*,* i*,* s *and* $E^i \in \Pi^i$*,*

$$\sum_{s \in E^i(s)} q^i\left[s \mid E^i(s)\right] \cdot u^i\left[x^i(s), s\right] \geq \sum_{s \in E^i(s)} q^i\left[s \mid E^i(s)\right] \cdot u^i\left[x^i_{E^i}(s), s\right]$$

with

$$x_{E^i}(s) = \left\{ \begin{array}{c} x\left\{\left[\underset{j \neq i}{\cap} E^j(s)\right] \cap E^i\right\} \text{ if the argument is not empty} \\ 0 \text{ otherwise} \end{array} \right\}.$$

Unfortunately Bayesian Incentive Compatibility is not guaranteed in general in the model. Some restrictions on the structure of information owned by agents have to be introduced. Palfrey and Srivastava (1987) consider Bayesian implementation in a set-up in which information is non-exclusive. They prove that if there are at least three agents, information is non-exclusive and the social choice set $F \neq \varnothing$ to be implemented satisfies Bayesian monotonicity, then F is indeed implementable. It is not difficult to show that the set of non-wasteful ex post envy-free social choice functions satisfy the requirement of Bayesian monotonicity. In order to assure implementability from now on let us suppose that information is non-exclusive, that is

$$E^i(s) \supset \bigcap_{j \neq i} E^j(s) \text{ for all } i \text{ in } N \text{ and } s \text{ in } S. \tag{NEI}$$

Note that the former assumption of no redundant states combine with NEI delivers the fact that $(N-1)$ agents can identify without uncertainty the state that has occurred, i.e. $\bigcap_{j \neq i} E^j(s) = \{s\}$ for all i and s. Now let me introduce the following notation that will be useful in defining a mechanism:

$$D(\sigma) = \left\{ s^* \in S : \bigcap_{i \in N \setminus \{j\}} \sigma^i = \{s^*\} \text{ for some } j \in N \right\}.$$

The punishment that dissuades agents from deviation in some cases is $\overline{F} \in \mathbb{R}$. It can be interpreted as a fine that players must pay when they fail to reach some agreement to be specified later with the mechanism. As for the example of flat-mates it could be seen as monetary equivalent of all the inconveniences that the lack of agreement can cause, for example the cost of looking for a new flat or flat-mates, or the utility loss due to the persistence of envy. $\overline{F}$ can be found with the help of the following inequality:

$$u^i\left[x^i(s), s\right] > u^i\left(o, \frac{M}{N}, s\right) - \overline{F} \text{ for all } i,\ s,\ o \text{ and non-wasteful}$$
$$\text{ex post envy free social choice function } x.$$

$\overline{F}$ is well-defined since no object is infinitely desirable, and the set of states and the one of objects are both finite. Let us define the mechanism as follows.

Definition 11 ($\mathcal{M}$) *Every agent has to announce (simultaneously) some elements from the partition* Π^i *denoted by* e^i. *Players also have to choose a permutation* p^i *over the set* N *and a non-negative integer. The message space for agent* i *is then* $\Sigma^i = \Pi^i \times P^i \times \mathbb{Z}_{0+}$ *with a typical element in the form of* (e^i, z^i). *The outcome of the mechanism,* g, *is defined as follows.*
a) If $\#D = 1$ *and there are at least* $(N-1)$ *zeros among the* z^i, *then the outcome is* $\left(p^1 \circ \ldots \circ p^N\right)[x(s^0)]$ *where* $\{s^0\} = D$ *and* x *is some non-wasteful ex post envy-free social choice function. In this case, after the first stage the planner offers a non-wasteful ex post envy-free allocation for* s^0, *i.e.* $x(s^0) \in EF_p \cap F^{fnw}$.
b) If $\#D > 1$ *and there are at least* $(N-1)$ *zeros among the* z^i, *then let assign objects to agents in a random way (for example in such a way that agent* i *can get any of the objects with the same probability) and allocate money equally, giving* $\frac{M}{N}$ *to everyone. Agents in this case are forced to pay a fine of the amount* $\overline{F}$ *each.*
c) In any other case let the agent with the smallest index among those who have announced the largest z^i *receive the object of her choice and* $\max\{0, M\}$ *amount of money. The other players receive a random object from the ones that have been left over and the following amount of money* $\min\left\{0, \frac{M}{N-1}\right\}$.

In what follows I show that this mechanism can be used to implement the set of ex-post envy-free social choice functions. It is point a) in the above definition according to which the Bayes-Nash equilibrium outcomes of the game are determined. Points b) and c) introduce incentives for reporting the state of nature truthfully be optimal for agents. Under b) agents are severely punished by a fine that amounts $\overline{F}$. In point c) the mechanism contains an integer game that gives incentives to participants to send messages that give rise to situations that fall under point a). Proposition 5 states the formal result and is followed by the formal proof.

Proposition 5 *Let x be a non-wasteful ex post envy-free social choice function. If $N \geq 3$ and information is non-exclusive the mechanism $\mathcal{M}$ implements x in Bayes-Nash equilibrium.*

Proof. Suppose that the conditions of the proposition hold. Let us prove first that for any state of the world every equilibrium outcome of $\mathcal{M}$ is a non-wasteful ex post envy-free allocation.

1.1. Note that there can not exist any equilibrium under c) or b). The first one is the case of an integer game in which, given the others' choice, every player i has incentives to announce a larger integer above the level of

$$max\left\{z^1, \ldots, z^{i-1}, z^{i+1}, \ldots, z^N\right\}.$$

1.2. Under b) agents are severely punished and have incentives to change their announcements. If there are at least $(N-2)$ zeros among the announced z^i by setting $e^i = \emptyset$ and $z^i = 0$, and inducing case a). Or, if there are less then $(N-1)$ zeros among the announced z^i by the others any agent can switch to case c) by announcing a sufficiently large integer.

1.3. Therefore only case a) can support an equilibrium. Its allocation must be ex post envy-free due to the enclosed permutation game.

In the second part I shall show that any non-wasteful envy-free social choice function can be supported as an equilibrium of the mechanism $\mathcal{M}$.

2. In order to do so let us show that $\sigma^{i*} = \left[E^i(s), p^{id}, 0\right]$ for all i constitutes an equilibrium of the mechanism $\mathcal{M}$ where p^{id} stands for the identity permutation. Note that no agent has incentives to switch to case b), since in that one all agents are seriously punished by a fine that makes them worse off than in any possible result under a). Case c) might be a tempting possibility but for a single player it

is impossible to switch directly from a) to c) when the others are playing according to σ^*. ■

In first place the mechanism $\mathcal{M}$ is designed to extract true information from agents. Joining that, the planner is able to find out which state of nature has occurred and her task is to find a particular non-wasteful ex post envy-free social choice function. There exist numerous algorithms that can be used to find envy-free allocations for every state, hence for constructing the ex post envy-free social choice function. For examples check Alkan, Demange and Gale (1991), Aragonés (1995) and Su(1999).

Note that the mechanism is not wasteful in equilibrium and the indivisible objects are always allocated according to the rules of the economy, however it contains threats under case b) that are not budget balanced. These are needed, because even if $(N-1)$ players are able to identify the occurred state, the planner can not always identify the deviator and therefore has to punish everyone to avoid deviations.

4.4.1 Implementation ex ante

Let us study more carefully the ex ante situation when information is symmetric, i.e. agents have prior beliefs about the occurrence of the states and these beliefs are known to everyone. I do not have to deal with social choice functions or set anymore, but with allocations.[4] Since agents do not know which state will occur, they value these bundles according to their expected utility function. For simplicity

[4]Nevertheless there might exist problems in which the ex-ante implementation of a social choice function is interesting. A "divide and permute" type mechanism can be used in those cases, too. The only modification required is that the first two players have to announce a non-wasteful social choice function instead of an allocation.

I shall define a utility function for this case: $v^i(a^i) = \sum_{s \in S} q^i(s) \cdot u^i(a^i, s)$. Note that previous results hold, meaning that an envy-free allocation is also Pareto optimal in this economy. For v^i it is useless to distinguish between ex ante, interim or ex post concepts, but it is worth to point out that for example its envy-freeness is closely related to the ex ante envy-freeness notion that I had before. The most important change is that before I had social choice functions and now I am working with allocations that do not change with the states of nature - they are no longer state-contingent. For this reason I only consider constant social choice functions in this subsection that simply will be called allocations. Now with redefining my envy-freeness and Pareto-optimality concepts with allocations I have the following result.[5]

Proposition 6 *If Condition 1 holds any non-wasteful ex ante envy-free feasible allocation is ex ante Pareto optimal.*

Proof. Just like in Proposition 3. ■

Now let us turn my attention to implementation matters. Taking into account the previous notes and assuming that Condition 1 holds I have that the well-known "divide and permute"[6] mechanism implements (in Nash equilibrium) the set of ex ante envy free allocations that are also ex ante Pareto optimal according to the last proposition.

Definition 12 *The "divide and permute" mechanism. The message space for agent i is* $\Sigma^i = \left\{ \begin{array}{c} A^{fnw} \times p \text{ if } i = 1,2 \\ p \text{ otherwise} \end{array} \right\}$ *where p denotes the set of all possible permutations*

[5]Note that in this context only notions corresponding to the earlier ex-ante concepts have meanings.

[6]For details check Thomson (1995).

in N. The outcome function is g with the following definition[7]*:*

$$g(\sigma) = \left\{ \begin{array}{c} (p^n \circ \ldots \circ p^1)(a^1) \;\; \textit{if } a^{f1} = a^{f2} \\ \left(o^{\text{rand}}, \frac{M}{N} - \overline{F}\right)^N \;\; \textit{otherwise} \end{array} \right\}.$$

The proof of this implementation result is not included here, since my ex ante implementation problem is technically identical to the Nash implementation of envy-free allocations problem studied in the literature, for instance in Thomson (1995). The "divide and permute" or cake-cutting mechanisms that was designed for the two-player divisible case, have many variants and generalizations for situations with more participants and indivisibilities.[8] Unfortunately they do not perform well under uncertainties. McAfee (1992) points out that the cake-cutting mechanism produces efficient results under symmetric information, but under asymmetric information it is ex post inefficient in an unusual way.

4.5 Existence with generalized fairness concept

The literature on distributive justice usually does not deal with the problematic of choosing fairness criteria. Concepts are very often axiomatically justified, and/or their use is made acceptable intuitively. In this section I enlarge my focus and study some generalized fairness concepts. This allows for more judgements on fairness and does not exclusively deal with envy-freeness.

In order to do so, following the idea in Corchón and Iturbe-Ormaetxe (2001) I define for every agent i a function $\psi^i : A^f \times S \longrightarrow A^i$ which I call state-dependent aspiration function, or simply aspiration function. Let $\psi = \left(\psi^1, \ldots, \psi^N\right)$. The

[7] The symbol o^{rand} stands for a random object from O.

[8] For examples check Brams and Taylor (1996).

expression $\psi^i[x(s), s]$ denotes the personal aspiration of agent i in state s when the social choice function is x.[9] It can be interpreted as the list of bundles for agent i that she thinks are fair in each state, when bundles are assigned according to the social choice function x in the population. Note that the personal aspirations may perfectly be unfeasible together. The next definition identifies the feasible aspiration correspondences.

Definition 13 *Given the social choice function x, the aspiration function ψ is feasible if*

$$\psi^i[x(s), s] \in A^f \text{ for all } i \text{ and } s.$$

I can generalize the fairness concepts with the help of the aspiration function also in the uncertainty case. Envy-free social choice functions, for example, will be a special case of the satisfactory ones defined below. For this to be true, one should think about personal aspirations, for a given state and social choice function, as the best bundle owned by any agent in the given state and according to the given social choice function.[10]

Definition 14 *A feasible social choice function x, given ψ, is ex post satisfactory if*

$$u^i\left[x^i(s), s\right] \geq u^i\left\{\psi^i[x(s), s], s\right\} \text{ for all } i \text{ and } s.$$

Definition 15 *A feasible social choice function x, given ψ, is ex ante satisfactory*

[9]The notation $\psi^i(x(s), s)$ is redundant, since $x(s) \in A^f$ gives the allocation for the whole set of agents in state s. Therefore it is clear that we are dealing with aspirations for state s and we could simply write $\psi^i(x(s))$. However the notation in longer form is more in line with the formal definition and for this reason is kept.

[10]For more explanation, intuition and results under certainty check Corchón, Iturbe-Ormaetxe (2001).

if

$$\sum_{s \in S} q^i(s) \cdot u^i\left[x^i(s), s\right] \geq \sum_{s \in S} q^i(s) \cdot u^i\left\{\psi^i\left[x(s), s\right], s\right\} \text{ for all } i.$$

The widely used egalitarian solution can be captured also as a special case with the following definitions of adequate social choice functions.

Definition 16 *A feasible social choice function x, given ψ, is ex post adequate if*

$$u^i\left[x^i(s), s\right] = u^i\left\{\psi^i\left[x(s), s\right], s\right\} \text{ for all } i \text{ and } s.$$

Definition 17 *A feasible social choice function x, given ψ, is ex ante adequate if*

$$\sum_{s \in S} q^i(s) \cdot u^i\left[x^i(s), s\right] = \sum_{s \in S} q^i(s) \cdot u^i\left\{\psi^i\left[x(s), s\right], s\right\} \text{ for all } i.$$

The above definitions are natural generalizations of the ones in Corchón and Iturbe-Ormaetxe (2001). The generalized version of Proposition 2, describing the relation between ex post and ex ante terms, holds using either the satisfactory or the adequate fairness concepts.

Proposition 7 *If a social choice function is ex post satisfactory (adequate), then it is also ex ante satisfactory (adequate).*

Proof. If one weights the inequalities (equalities) in the definition for ex post satisfactory (adequate) social choice functions by the prior probabilities and sum the results up for every possible state, one gets the requirement stated in the definition for ex ante satisfactory (adequate) social choice function. ■

Naturally I could define the corresponding interim concepts as well, and state the inclusion result in a similar proposition. Since I am not particularly interested in the interim stage in this section these parts are omitted.

Intertemporal fairness now can be captured by ex post satisfactory (or adequate) social choice functions that are ex ante Pareto efficient. This point of view allows for the same arguments as presented before for the envy-free case. Unfortunately with the general form of aspirations I cannot prove existence of the satisfactory nor the adequate social choice functions, neither ex post or ex ante. Therefore let us introduce the concept of unbiased social choice functions that do exist under some mild assumptions on preferences and the feasible consumption set in the certain case, as shown in Corchón and Iturbe-Ormaetxe (2001).

Definition 18 *A feasible social choice function x, given ψ, is ex post unbiased if for any state s any of the following statements holds:*
a) $u^i\left[x^i(s), s\right] \geq u^i\left\{\psi^i\left[x(s), s\right], s\right\}$ for all i, or
b) $u^i\left[x^i(s), s\right] < u^i\left\{\psi^i\left[x(s), s\right], s\right\}$ for all i.

Note that I can not define ex post biasedness in such a way that requires inequality a) or inequality b) to hold for all possible s. Vaguely speaking this is because aspirations are now state-dependent. There might be states in which aspirations are too high to inequality a) to hold, while in some other might be so low that b) is impossible. The next example, even if it is an extreme case, illustrates this statement.

Example 5 *Suppose that in state s_1, independently from the social choice function, every agent is satisfied with the indivisible object she has been assigned to, but aspires to some extra amount of money, $\varepsilon > 0$. This falls clearly under case b) in Definition*

18, as personal aspirations can not be reached. If s_1 *was the only possible state of nature, I would have unbiasedness. Let us suppose that in some other state* s_2 *aspirations are humble in the sense that every agent is satisfied with her indivisible object and does not aspires to any amount of money. This and* $M > 0$ *give case a) in Definition 18.*

With my definition an ex post unbiased and ex ante Pareto efficient social choice functions exists if and only if Condition 1 holds. This is a direct consequence of the results for the certain case in Corchón and Iturbe-Ormaetxe (2001) and the one that I discussed before according to which Condition 1 is needed for ex ante Pareto efficiency.

Definition 19 *A feasible social choice function* x*, given* ψ*, is ex ante unbiased if for any state* s *any of the following statements holds:*
a) $\sum_{s \in S} q^i(s) \cdot u^i[x^i(s), s] \geq \sum_{s \in S} q^i(s) \cdot u^i\{\psi^i[x(s), s], s\}$ *for all* i*, or*
b) $\sum_{s \in S} q^i(s) \cdot u^i[x^i(s), s] < \sum_{s \in S} q^i(s) \cdot u^i\{\psi^i[x(s), s], s\}$ *for all* i*.*

There is no relation between the above ex post and ex ante unbiasedness concepts of the inclusion type, like I had before for the envy-free case. Therefore the question whether an ex ante unbiased and ex ante efficient social choice function exists is not trivial. The following propositions states that in fact, under Condition 1, there exist social choice functions that are ex ante unbiased and ex ante Pareto efficient.

Proposition 8 *Given the aspiration function* ψ*, there exists an ex ante unbiased and ex ante Pareto social choice function if and only if Condition 1 holds.*

Proof. This proof for the uncertainty case I present here goes parallel with the proof for certainty from Corchón and Iturbe-Ormaetxe (2001), and it is tailored to the specific set-up I study. For the sake of this proof let us introduce a technical change in the definition of the consumption set and extend the set of possible money consumption in the bundles to the set of extended real numbers: $A = (O \times \mathbb{R}^*)^N$ where $\mathbb{R}^*$ denotes the set of extended real numbers that is compact, non-empty and convex. Let S^{n-1} be the $(n-1)$ dimensional simplex. In my set-up a social choice function is ex ante Pareto efficient if for a given $\lambda \in S^{n-1}$ it solves the following maximization problem:

$$\max_{x \in X} \sum_{i \in N} \lambda^i \cdot \sum_{s \in S} q^i(s) \cdot u^i\left[x^i(s), s\right].$$

I can split the above problem into two part: finding the way of distributing the indivisible objects among agents and then the distribution of the perfectly divisible one.

$$\max_{x \in X} \sum_{i \in N} \lambda^i \cdot \sum_{s \in S} q^i(s) \cdot u^i\left[o^{xi}(s), s\right] + \sum_{i \in N} \lambda^i \cdot \sum_{s \in S} q^i(s) \cdot \phi^i(s) \cdot m^{xi}(s),$$
$$\max_{o^x} \sum_{i \in N} \lambda^i \cdot \sum_{s \in S} q^i(s) \cdot u^i\left[o^{xi}(s), s\right] + \max_{m^x} \sum_{i \in N} \lambda^i \cdot \sum_{s \in S} q^i(s) \cdot \phi^i(s) \cdot m^{xi}(s).$$

Since the sets S, N, O are finite the first maximization problem has a solution with some finite value. The second one has a solution too, because $m^{xi}(s) \in \mathbb{R}^*$ for all i and s, and $q^i(s) \cdot \phi^i(s) \cdot m^{xi}(s)$ is continuous, strictly increasing in $m^{xi}(s)$. Let the solution of the second maximization problem be $\rho : S^{n-1} \rightarrow (\mathbb{R}^*)^{N \cdot S}$. It is convex-valued ($\mathbb{R}^*$ is convex and utilities are quasilinear in money) and upper hemicontinuous (by Berge's Maximum Theorem). Now let us define

$$D^i[x(s), s] = D^i\left\{x^i(s), \psi^i[x(s), s], s\right\} = u^i\left\{\psi^i[x(s), s], s\right\} - u^i\left[x^i(s), s\right]$$

that can be separated into two parts: utility difference from the indivisible goods and the difference from money. Note that $D^i[x(s), s]$ is continuous in money. Consider now the following maximization problem for a fixed social choice function x:

$$\max_{\lambda \in S^{n-1}} \sum_{i \in N} \lambda^i \cdot \sum_{s \in S} q^i(s) \cdot D^i\left\{x^i(s), \psi^i[x(s), s], s\right\}.$$

I define $\varphi : (\mathbb{R}^*)^{N \cdot S} \rightarrow S^{n-1}$ as the solution function for the money allocation part of the above maximization problem. This correspondence is also convex-valued and upper hemicontinuous. Therefore the mapping $\varphi \circ \rho : S^{n-1} \rightarrow S^{n-1}$ has a fixed point λ^* with $m^* \in \lambda^*$. And there is some x^* that belongs to that m^*. Now let us show that I must have ex ante unbiasedness by contradiction. Suppose that there are i and j such that

$$\begin{aligned}\sum_{s \in S} q^i(s) \cdot D^i[x^*(s), s] &\geq 0, \\ \sum_{s \in S} q^j(s) \cdot D^j[x^*(s), s] &< 0.\end{aligned}$$

Then I must have $\lambda^{*j} = 0$, and in the ex ante Pareto program (according to x^*) agent j will be assigned the amount of $-\infty$ of money (the worst possible amount) in every state. But that, with my assumptions would be in contradiction with $\sum_{s \in S} q^j(s) \cdot D^j[x^*(s), s] < 0$. This completes the proof. ■

Condition 1 turns out to be crucial for existence results, because I have required Pareto efficiency for every fairness concept. Condition 1, in fact, arises due to this fact, and it is possible to show that any non-wasteful social choice function that induces an ex post optimal assignment of indivisible objects is ex ante Pareto efficient if and only if Condition 1 holds. For the formal proof check Appendix B.

When considering implementation of some social choice set an important topic of monotonicity arises. In the certainty case Maskin monotonicity can be guaranteed by some rationality requirements on aspirations.[11] my goal now is to study Bayesian monotonicity, because that is the property needed for Bayesian implementability. In order to do so more concepts and some pieces of notation that generalize those from Corchón and Iturbe-Ormaetxe (2001) for the uncertainty case have to be introduced .

Definition 20 *The social choice set F is attainable in an ex ante satisfactory way if there is ψ such that the social choice function x belongs to F if and only if x is an ex ante satisfactory social choice function for ψ.*

Let $Z^i : X \longrightarrow X^i$ be a correspondence. The set $Z^i(x)$ is interpreted as the set of state-contingent allocations that agent i thinks she is entitled to, given the social choice function x.

Definition 21 *The aspiration function ψ is called ex ante rational if, for all i,*

$$\psi^i = \arg\max \sum_{s \in S} q^i(s) \cdot u^i\left[x^i(s), s\right]$$

with $x^i \in Z^i(x)$.

The above maximization problem need not to have a single solution, but ties can be handled by some arbitrary rule. ex ante rationality now requires utility

[11]For more details see Corchón and Iturbe-Ormaetxe (2001).

maximization ex ante. For a given social choice function, every agent should choose her personal aspiration in such a way that maximizes her expected utility over the set $Z^i(x)$.

I consider compatible deceptions and Bayesian monotonicity in the form as they appear in Palfrey and Srivastava (1987). I take the definitions from there and present them in order to have a self-contained study of the problem.

Definition 22 *A collection of functions* $\alpha = \left(\alpha^1, ..., \alpha^N\right)$, *with* $\alpha^i : \Pi^i \to \Pi^i$, *is a deception compatible with* $\{\Pi^i\}$ *if for all* $\left(E^1, ..., E^N\right)$ *such that* $E^i \in \Pi^i$ *for all* i, $\underset{i \in N}{\cap} E^i \neq \varnothing$ *implies* $\underset{i \in N}{\cap} \alpha\left(E^i\right) \neq \varnothing$.

Let us introduce the following short-hand notation:

$$\alpha(s) = \underset{i \in N}{\cap} \alpha^i\left[E^i(s)\right], \; x_\alpha(s) = x\left[\alpha(s)\right], \; x_\alpha = \left[x_\alpha(s_1), x_\alpha(s_2), ...\right].$$

It will simplify the following definition of Bayesian monotonicity.

Definition 23 *The social choice set* F *satisfies Bayesian monotonicity if for all* α *compatible with* $\{\Pi^i\}$ *if*

a) $x \in F$,

b) for all agent i, *state* s^* *and social choice function* y,

$$\sum_{s \in E^i(\alpha(s^*))} q^i\left\{s \mid E^i\left[\alpha(s^*)\right]\right\} \cdot u^i\left[x^i(s), s\right] \geq \sum_{s \in E^i(\alpha(s^*))} q^i\left\{s \mid E^i\left[\alpha(s^*)\right]\right\} \cdot u^i\left[y^i(s), s\right]$$
$$\Downarrow$$
$$\sum_{s \in E^i(s^*)} q^i\left[s \mid E^i(s^*)\right] \cdot u^i\left[x^i_\alpha(s), s\right] \geq \sum_{s \in E^i(s^*)} q^i\left[s \mid E^i(s^*)\right] \cdot u^i\left[y^i_\alpha(s), s\right]$$

then $x_\alpha \in F$.

Now I can state the following positive result on Bayesian monotonicity in my set-up. It implies that imposing an extra condition on the social choice function and aspirations, the set of intertemporally fair social choice functions can be implemented at the interim stage, i.e. by Bayesian implementations.

Proposition 9 *Let F be a social choice set that is attainable in an ex ante satisfactory way with ex ante rational aspirations. F is Bayesian monotonic if*

$$y^i \in Z^i(x) \Longrightarrow y^i_\alpha \in Z^i(x_\alpha) \text{ for all } i. \qquad \text{(Condition 3)}$$

Proof. Let $x \in F$, where F is attainable in an ex ante satisfactory way with rational aspirations. Then for all i and s^* I have that

$$\sum_{s \in E^i(\alpha(s^*))} q^i \left\{ s \mid E^i[\alpha(s^*)] \right\} \cdot u^i \left[x^i(s), s \right] \geq \sum_{s \in E^i(\alpha(s^*))} q^i \left\{ s \mid E^i[\alpha(s^*)] \right\} \cdot u^i \left[y^i(s), s \right]$$
$$\text{for all } y^i \in Z^i(x).$$

By the implication in the definition of Bayesian monotonicity I also have that the above implies

$$\sum_{s \in E^i(s^*)} q^i \left[s \mid E^i(s^*) \right] \cdot u^i \left[x^i_\alpha(s), s \right] \geq \sum_{s \in E^i(s^*)} q^i \left[s \mid E^i(s^*) \right] \cdot u^i \left[y^i_\alpha(s), s \right]$$
$$\text{for all } y^i \in Z^i(x).$$

If Condition 3 holds the latter shows that x_α is satisfactory, because then I have the inequality for all $y_\alpha \in Z^i(x_\alpha)$. Hence $x_\alpha \in F$ and therefore F is Bayesian monotonic. ■

As a special case, the ex post envy-free social choice set satisfies Condition 3.

To see this in an intuitive way consider the following. With the concept of envy-freeness, and some social choice function x, the set $Z^i(x)$ contains those state-contingent allocations that in a given state s^* have the other agents' consumption bundles, from the same state s^*, as components. For x to be ex post envy-free every agent i has to choose the best one among these, and that for every state in M. Now let us introduce compatible deceptions, α. With this $Z^i(x_\alpha)$ will contain those state-contingent allocations that in state $\alpha(s^*)$ have the others' consumption bundles, from state $\alpha(s^*)$. Note that $\alpha(s^*)$ contains elements from S. Therefore the implication in Condition 3 is straightforward.

4.6 Concluding remarks

I have considered the problem of allocating indivisible goods and money among members of an economy in which agents are not perfectly informed on the others' preferences. The set of intertemporally fair social choice functions have been studied that are defined as ex-post envy-free and ex-ante Pareto efficient, as this intersection is the most restrictive among all the possible ones. The appealing features of envy-free allocations explored in the literature on economies without uncertainties extend to the economy with uncertainty. These are the lattice structure of the intertemporally fair set, the consistency and monotonicity results (not studied here in detail) and the fact that envy-freeness implies Pareto efficiency. It is the latter in the ex-ante stage that might make the intertemporally fair set be empty. I have derived a necessary and sufficient condition for existence (non-emptiness) that in economies, in which the marginal utility of money is the same for every agent, requires prior beliefs to be the same for everyone.

Under this conditions and the one of nonexclusiveness of information the implementation of the intertemporally fair set has been studied both in the interim and ex-ante stage. Concrete mechanisms have been proposed to achieve full implementation.

I have also proposed a generalized version of intertemporal fairness based on the aspiration function and Pareto efficiency. Due to the presence of Pareto efficiency the condition on beliefs derived in the first part of the paper for existence remains necessary and sufficient. In the concluding result a condition for Bayesian monotonicity has been derived, i.e. for Bayesian implementation of the generalized intertemporally fair set.

4.7 Appendix A

Definition 24 *Suppose that agents have reached the interim stage and some state of nature, s_1, has occurred. A social choice function x is interim intertemporally fair if $x \in P_i \cap EF_p$.*

I keep the notation for simplicity, but point out that the definition of P_i slightly differs now, because I only consider the state that in fact has occurred, i.e. when for instance s_1 has occurred. From an interim point of view I shall use the following definition.

Definition 25 *From an interim point of view a non-wasteful social choice function x is interim Pareto optimal if there is no non-wasteful social choice function y such that*

$$\sum_{s \in E^i(s_1)} q^i \left[s \mid E^i(s_1)\right] \cdot u^i \left[y^i(s), s\right] \geq \sum_{s \in E^i(s_1)} q^i \left[s \mid E^i(s_1)\right] \cdot u^i \left[x^i(s), s\right]$$

for all i in N, with strict inequality for at least one i. Let P_i denote the set of interim Pareto optimal social choice functions.

Proposition 10 *Suppose that agents have reached the interim stage, i.e. a given state has occurred. $EF_p \cap P_i \neq \varnothing$ if and only if agents have no uncertainty about the state of nature at the interim stage.*

Proof. The if part has been already shown before, since if there is no uncertainty I am dealing with the intersection of $EF_p \cap P_p$ that is known to be not empty in my set-up with indivisibilities.
For the only if part suppose that s_1 has occurred. Then the result can be proven similarly as Proposition 3. From that proof the condition for non-emptiness that arises is

$$q^i(s \mid s_1) \cdot \phi^i(s) = \gamma(s') \cdot q^i(s' \mid s_1) \cdot \phi^i(s')$$

for all s, $s' \in E^i(s_1)$ and i with $\gamma \in \mathbb{R}^+$. This implies that $E^i(s_1) = E(s_1)$ for all i that is only compatible with the assumption of no redundant states if $E^i(s_1) = E(s_1) = \{s_1\}$ for all i. And of course this should hold for any particular s_1. ■

4.8 Appendix B

Definition 26 *The assignment of the indivisible objects under the non-wasteful social choice function x is ex post optimal whenever $\sum_{i \in N} u^i[o^{xi}(s), s] \geq \sum_{i \in N} u^i[o^{yi}(s), s]$ for every s, and any non-wasteful social choice function y.*

Proposition 11 *Any non-wasteful social choice function that induces an ex post* *optimal assignment of indivisible objects is ex ante Pareto efficient if and only if Condition 1 holds.*

Proof. The if part: Note that the optimal assignment of the indivisible objects is a necessary condition for any kind of Pareto efficiency. Taking into account the previous proposition it is enough to prove that under Condition 1 ex post unbiasedness implies ex ante Pareto efficiency. Consider the social choice function x that is supposed to be non-wasteful and ex post unbiased, and the following maximization problem whose solutions give the ex ante efficient money transfers.

$$\max_{y \in X^{fnw}} \sum_{i \in N} \tau^i \cdot \sum_{s \in S} q^i(s) \cdot \phi^i(s) \cdot m^{yi}(s)$$
$$\text{with } \tau^i \in (0,1) \text{ for all } i, \text{ and } \sum_{i \in N}^{i} \tau^i = 1 \tag{4.8.1}$$

If I can find some weights $\overline{\tau^i}$ such that x solves the above problem then I am done. Now let consider a non-wasteful social choice function y and the positive numbers, $\lambda^i \in \mathbb{R}^+$ for every i, where λ^i is defined such that $\lambda^i \cdot q^i(s_1) \cdot \phi^i(s_1) = \beta$ for all i with $\beta \in \mathbb{R}^+$. Note that fixing s_1 by Condition 1 for any i and s' I have that $\lambda^i \cdot q^i(s') \cdot \phi^i(s') = \frac{\beta}{\gamma(s_1,s')}$

$$\frac{1}{\sum_{i \in N} \lambda^i} \cdot \lambda^i \cdot \sum_{s \in S} q^i(s) \cdot \phi^i(s) \cdot m^{yi}(s) =$$
$$= \frac{1}{\sum_{i \in N} \lambda^i} \cdot \beta \cdot \left[m^{yi}(s_1) + \frac{m^{yi}(s_2)}{\gamma(s_1,s_2)} + \ldots \right]$$

Now let us sum up the above equality for all players.

$$\sum_{i\in N}\frac{1}{\sum\limits_{i\in N}\lambda^i}\cdot\lambda^i\cdot\sum_{s\in S}q^i(s)\cdot\phi^i(s)\cdot m^{yi}(s)=$$

$$=\frac{1}{\sum\limits_{i\in N}\lambda^i}\cdot\beta\cdot\left[\sum_{i\in N}m^{yi}(s_1)+\frac{m^{yi}(s_2)}{\gamma(s_1,s_2)}+\ldots\right]=$$

$$=\frac{1}{\sum\limits_{i\in N}\lambda^i}\cdot\beta\cdot M\cdot\sum_{i\in N}\left[1+\frac{1}{\gamma(s_1,s_2)}+\ldots\right]=const.$$

The above expression does not depend on the chosen social choice function y, therefore I can take

$$\overline{\tau^i}=\frac{\lambda^i}{\sum\limits_{i\in N}\lambda^i}$$

that will guarantee that the social choice function x solves the maximization problem in (7).

The only if part: The proof is like in Proposition 3. By contradiction one could consider an ex ante Pareto efficient social choice function with optimal assignment and suppose that Condition 1 does not hold. Using the same argument now I can conclude that in this case the social choice function in question can not be ex ante Pareto efficient. It is possible to improve somebody's expected utility without harming anyone else in the way I did for the proof of Proposition 3. ■

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